1 Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Let $\alpha = a+bi$, $\beta = c+di$

$$\alpha+\beta = (a+bi) + (c+di) = (c+di) + (a+bi) = \beta + \alpha$$

2 Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Let $\alpha = a+bi$, $\beta = c+di$, $\lambda = e+fi$

$$(\alpha+\beta)+\lambda = ((a+bi)+(c+di))+(e+fi) = (a+bi)+((c+di)+(e+fi)) = \alpha+(\beta+\lambda)$$

3 Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

$\alpha := a+bi$

$\beta := c+di$

$\lambda := e+fi$

$$(\alpha\beta)\lambda = ((a+bi)(c+di))(e+fi) = (ac + adi + bic - bd)(e+fi)$$

$$= ace + adei + bcei - bde + acfi - adf - bcf - bdfi$$

$$= (ace - bde - adf - bcf) + (ade + bce + acf - bdf)i$$

$$\alpha(\beta\lambda) = (a+bi)((c+di)(e+fi)) = (a+bi)(ce + cfi + dei - df)$$

$$= ace + acfi + adei - adf + bcei - bcf - bde - bdfi$$

$$= (ace - adf - bde - bcf) + (acf + ade + bce - bdf)i$$

$$= (ace - bde - adf - bcf) + (ade + bce + ade - bdf)i$$

$$= (\alpha\beta)\lambda \quad \square$$

4 Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

$\alpha := a+bi$

$\beta := c+di$

$\lambda := e+fi$

$\lambda a = (e+fi)(a+bi) = ea + ebi + afi - bf$

$\lambda b = (e+fi)(c+di) = ec + edi + cfi - df$

$\lambda(\alpha+\beta) = (e+fi)(a+bi + c+di) = ea + ebi + ec + edi + afi - bf + cfi - df$

$= (ea + ebi + afi - bf) + (ec + edi + cfi - df)$

$= (e+fi)(a+bi) + (e+fi)(c+di)$

$= \lambda\alpha + \lambda\beta$

5 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Suppose towards a contradiction that there exist two such numbers, β_1 and β_2, and that $\beta_1 \neq \beta_2$.

So $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$

$\Rightarrow \alpha + \beta_1 = \alpha + \beta_2$

$\Rightarrow \cancelto{0}{(\beta_2 + \alpha)} + \beta_1 = \cancelto{0}{(\beta_2 + \alpha)} + \beta_2$

$\Rightarrow \beta_1 = \beta_2$

6 Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Assume there exist two such β_1 and β_2, and that $\beta_1 \neq \beta_2$.

Then $\alpha\beta_1 = 1$ and $\alpha\beta_2 = 1$

So $\alpha\beta_1 = \alpha\beta_2$

$\Rightarrow$ $\cancelto{1}{(\beta_2\alpha)}\beta_1 = \cancelto{1}{(\beta_2\alpha)}\beta_2$

$\Rightarrow$ $\beta_2 = \beta_2$

7 Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2}$$

$$= \frac{1-2\sqrt{3}i-3}{4} \cdot \frac{-1+\sqrt{3}i}{2}$$

$$= \frac{-2-2\sqrt{3}i}{4} \cdot \frac{-1+\sqrt{3}i}{2}$$

$$= \frac{\cancel{2}^{1}(-1-\sqrt{3}i)}{\cancel{4}^{2}} \cdot \frac{-1+\sqrt{3}i}{2}$$

$$= \frac{(-1-\sqrt{3}i)\cdot(-1+\sqrt{3}i)}{4}$$

$$= \frac{1+3}{4}$$

$$= 4/4$$

$$= 1$$

8 Find two distinct square roots of i.

$\alpha : a+bi$

$\alpha^2 = (a+bi)(a+bi) = a^2 + 2abi - b^2$

So $a^2 + 2abi - b^2 = i = 0 + 1 \cdot i$

$$\Rightarrow \begin{cases} a^2 - b^2 = 0 \\ 2abi = i \end{cases}$$

So $a^2 = b^2$ and $ab = 1/2$

$\Rightarrow a = 1/2b \Rightarrow a^2 = \frac{1}{4b^2} \Rightarrow \frac{1}{4b^2} = b^2 \Rightarrow b^4 = \frac{1}{4} \Rightarrow b = \pm \sqrt[4]{1/4}$

$\Rightarrow b = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$

$\Rightarrow a = \frac{\sqrt{2}}{2}$

So $\alpha_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $\alpha_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

$\alpha_1^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + 2\frac{2}{4}i - \left(\frac{\sqrt{2}}{2}\right)^2 = \cancel{\frac{2}{4} - \frac{2}{4}} + i = i$

$\alpha_2^2 = \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2 = \left(-\frac{\sqrt{2}}{2}\right)^2 + 2 \cdot \frac{2}{4}i + \left(\frac{\sqrt{2}}{2}i\right)^2 = \cancel{\frac{2}{4} - \frac{2}{4}} + i = i$

α_1 and α_2 are square roots of i.

9 Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

$x \in \mathbb{R}^4 \Rightarrow \quad x = (x_1, x_2, x_3, x_4)$

$$\Rightarrow \begin{cases} 4 + 2x_1 = 5 \\ -3 + 2x_2 = 9 \\ 1 + 2x_3 = -6 \\ 7 + 2x_4 = 8 \end{cases} \Rightarrow \begin{cases} 2x_1 = 1 \\ 2x_2 = 12 \\ 2x_3 = -7 \\ 2x_4 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 1/2 \\ x_2 = 6 \\ x_3 = -7/2 \\ x_4 = 1/2 \end{cases}$$

So $x = (1/2, 6, -7/2, 1/2)$.

10 Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i).$$

Let $\lambda := a+bi$

If such λ were to exist, we would have that

$$\begin{cases} \lambda(2-3i) = 12-5i \\ \lambda(5+4i) = 7+22i \\ \lambda(-6+7i) = -32-9i \end{cases}$$

By (1), we know that $\lambda = \frac{12-5i}{2-3i} = \frac{12-5i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{24+36i-10i+15}{4+9} = \frac{39+26i}{13} = 3+2i$

So (2) is $\lambda(5+4i) = (3+2i)(5+4i) = 15+12i+10i-8 = 7+22i$, which is fine.

However, the LHS of (3) is $\lambda(-6+7i) = (3+2i)(-6+7i) = -18+21i-12i-14 = -32+9i$.

But the RHS $= -32-9i$, which implies that $-32+9i = -32-9i$.

In other words, $9i = -9i$, or $i = -i$, a clear contradiction.

Since we arrived at a contradiction, we must conclude that no such $\lambda \in \mathbf{C}$ must exist.

11 Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

$x := (x_1, x_2, \dots, x_n)$, where $x_i \in \mathbb{F}$

$y := (y_1, y_2, \dots, y_n)$, where $y_i \in \mathbb{F}$

$z := (z_1, z_2, \dots, z_n)$, where $z_i \in \mathbb{F}$

$$
\begin{aligned}
(x+y)+z &= ((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)) + z \\
&= (x_1+y_1, x_2+y_2, \dots, x_n+y_n) + z \\
&= (x_1+y_1, x_2+y_2, \dots, x_n+y_n) + (z_1, z_2, \dots, z_n) \\
&= (x_1+y_1+z_1, x_2+y_2+z_2, \dots, x_n+y_n+z_n) \\
&= (x_1, x_2, \dots, x_n) + (y_1+z_1, y_2+z_2, \dots, y_n+z_n) \\
&= x + ((y_1, y_2, \dots, y_n) + (z_1, z_2, \dots, z_n)) \\
&= x + (y+z)
\end{aligned}
$$

12 Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

$$
\begin{aligned}
(ab)x &= ab(x_1, x_2, \ldots, x_n) \\
&= (abx_1, abx_2, \ldots, abx_n) \\
&= a(bx_1, bx_2, \ldots, bx_n) \\
&= a(bx)
\end{aligned}
$$

13 Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Let $x := (x_1, x_2, \ldots, x_n) \in \mathbb{F}$

Then
$$\begin{aligned} 1x &= 1 \cdot (x_1, x_2, \ldots, x_n) \\ &= (1 \cdot x_1, 1 \cdot x_2, \ldots, 1 \cdot x_n) \\ &= (x_1, x_2, \ldots, x_n) \\ &= x \end{aligned}$$

14 Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Let $x := (x_1, x_2, \dots, x_n)$, where $x_i \in \mathbb{F}$

and $y := (y_1, y_2, \dots, y_n)$, where $y_i \in \mathbb{F}$

Then
$$\begin{aligned}
\lambda(x+y) &= \lambda((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)) \\
&= \lambda(x_1+y_1, x_2+y_2, \dots, x_n+y_n) \\
&= (\lambda(x_1+y_1), \lambda(x_2+y_2), \dots, \lambda(x_n+y_n)) \\
&= (\lambda x_1 + \lambda y_1, \lambda x_2 + \lambda y_2, \dots, \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \lambda x_2, \dots, \lambda x_n) + (\lambda y_1, \lambda y_2, \dots, \lambda y_n) \\
&= \lambda(x_1, x_2, \dots, x_n) + \lambda(y_1, y_2, \dots, y_n) \\
&= \lambda x + \lambda y
\end{aligned}$$

15 Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Let $x := (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{F}$.

Then
$$\begin{aligned}
(a+b)x &= (a+b)(x_1, x_2, \ldots, x_n) \\
&= ((a+b)x_1, (a+b)x_2, \ldots, (a+b)x_n) \\
&= (ax_1 + bx_1, ax_2 + bx_2, \ldots, ax_n + bx_n) \\
&= (ax_1, ax_2, \ldots, ax_n) + (bx_1, bx_2, \ldots, bx_n) \\
&= a(x_1, x_2, \ldots, x_n) + b(x_1, x_2, \ldots, x_n) \\
&= ax + bx
\end{aligned}$$