Solutions to Baby Rudin

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Contents

Chapter 1

The Real and Complex Number Systems

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution: We'll start with the proof that $r+x$ is irrational. After that, we'll move on to the proof that rx is irrational. Both proofs are very similar. Note that I will use the notation r^{-1} instead of $1/r$ for multiplicative inverses, since I think the notation is a lot cleaner this way.

$\underline{r + x \notin \mathbb{Q}}$: Assume that $r + x \in \mathbb{Q}$, for the sake of contradiction. If $r + x$ is rational, then for any $y \in \mathbb{Q}$, $(r + x) + y$ is also rational (since $\mathbb{Q}$ is a field), by the field axiom (A1). Note that since $r \in \mathbb{Q}$, then $-r \in \mathbb{Q}$ too, by axiom (A5). This means that

$$\begin{aligned}
(r + x) + (-r) &\in \mathbb{Q} && \text{by (A1)} \\
\Rightarrow r + (x + (-r)) &\in \mathbb{Q} && \text{by (A3)} \\
\Rightarrow r + (-r + x) &\in \mathbb{Q} && \text{by (A2)} \\
\Rightarrow (r + (-r)) + x &\in \mathbb{Q} && \text{by (A3)} \\
\Rightarrow 0 + x &\in \mathbb{Q} && \text{by (A5)} \\
\Rightarrow x &\in \mathbb{Q} && \text{by (A4)}
\end{aligned}$$

By assumption, x is irrational, which means that $x \notin \mathbb{Q}$. However, we just deduced that $x \in \mathbb{Q}$ from our assumptions. The only way to resolve this contradiction is to realize that our initial assumption was wrong. Hence, $r + x \notin \mathbb{Q}$. In other words, $r + x$ is irrational.

$\underline{rx \notin \mathbb{Q}}$: Similar to the last proof, assume that $rx \in \mathbb{Q}$ for the sake of contradiction. By axiom (M1), for any $y \in \mathbb{Q}$, it must be true that $(rx)y \in \mathbb{Q}$. Also, since $r \in \mathbb{Q}$ and $r \neq 0$, it follows that $r^{-1} \in \mathbb{Q}$ too, by axiom (M5). We can now deduce that

$$\begin{aligned}
(rx)r^{-1} &\in \mathbb{Q} && \text{by (M1)} \\
\Rightarrow r(xr^{-1}) &\in \mathbb{Q} && \text{by (M3)} \\
\Rightarrow r(r^{-1}x) &\in \mathbb{Q} && \text{by (M2)} \\
\Rightarrow (rr^{-1})x &\in \mathbb{Q} && \text{by (M3)} \\
\Rightarrow 1x &\in \mathbb{Q} && \text{by (M5)} \\
\Rightarrow x &\in \mathbb{Q} && \text{by (M4)}
\end{aligned}$$

You probably get the gist by now, but we cannot have $x \notin \mathbb{Q}$ by assumption and then deduce that $x \in \mathbb{Q}$ by applying the field axioms. The only possible way to move forward is to conclude that our initial assumption is wrong; that is, $rx \notin \mathbb{Q}$. In other words, rx is irrational. $\square$

2. Prove that there is no rational number whose square is 12.

Solution: The proof is rather similar to the proof that $\sqrt{2}$ is irrational. Assume that there is some $x \in \mathbb{Q}$ such that $x^2 = 12$. This means that $x = \frac{m}{n}$, with $m, n \in \mathbb{Z}$, such that at most one of m, n is divisible by 3. Observe that $x^2 = (\frac{m}{n})^2 = \frac{m^2}{n^2}$, meaning that $\frac{m^2}{n^2} = 12$. The facts above imply that

$$m^2 = 12n^2.$$

We will now proceed with a case split. Specifically, we will consider two specific cases: the case where m is divisible by 3 and the case in which it is not[1].

m is divisible by 3: First, suppose that m is not divisible by 3. Then clearly m^2 is not divisible by 3 either. However, $12n^2$ is definitely divisible by 3. This would imply that $m^2 \neq 12n^2$, which is clearly not true. So it can't be true that m is not divisible by 3.

m is not divisble by 3: So the only choice we have left is that m is divisible by 3. In this case, m^2 must be divisible by 9. This would imply that $12n^2$ must also be divisible by 9, since $m^2 = 12n^2$, per our work above. However, this cannot be the case. 12 is not divisible by 9, but it is divisible by 3. So we still have a factor of 3 left. This would imply that n^2 must be divisible by 3, but that itself would imply that n is divisible by 3. However, by assumption, at most one of m, n can be divisible by 3. This would lead us to conclude that $m^2 \neq 12n^2$, a clear contradiction.

We have exhausted all possible choices of $x \in \mathbb{Q}$. This means that there does not exist any $x \in \mathbb{Q}$ such that $x^2 = 12$. In other words, there is no rational number whose square is 12. In other words, 12 does not have a rational square root. $\square$

[1]We can choose whatever case split we want, as long as it benefits us in the proofwriting process. All we have to do is make sure that we do not accidentally exclude any values of m. Since m either is or is not divisible by 3 for all possible values of m, we can safely proceed with our case split.

3. Prove Proposition 1.15.

Solution: As a reminder, Proposition 1.15 states the following:

The axioms for multiplication imply the following statements.

(a) *If $x \neq 0$ and $xy = xz$ then $y = z$.*

(b) *If $x \neq 0$ and $xy = x$ then $y = 1$.*

(c) *If $x \neq 0$ and $xy = 1$ then $y = x^{-1}$.*

(d) *If $x \neq 0$ then $(x^{-1})^{-1} = x$.*

Let's prove each statement, one by one.

(a) Suppose $x \neq 0$ and $xy = xz$. Then

$$\begin{aligned} y &= 1y && \text{by (M4)} \\ &= (xx^{-1})y && \text{by (M5)} \\ &= x(x^{-1}y) && \text{by (M3)} \\ &= x(yx^{-1}) && \text{by (M2)} \\ &= (xy)x^{-1} && \text{by (M3)} \\ &= (xz)x^{-1} && \text{by assumption} \\ &= (zx)x^{-1} && \text{by (M2)} \\ &= z(xx^{-1}) && \text{by (M3)} \\ &= z1 && \text{by (M5)} \\ &= z && \text{by (M4)} \end{aligned}$$

By the transitivity of $=$, it follows that $y = z$.

(b) Suppose $x \neq 0$ and $xy = x$. Note that $x = x1$, by axiom (M4). So it must be true that $xy = x1$. Appealing to our work in part (a), we can deduce that $y = 1$.

(c) Suppose $x \neq 0$ and $xy = 1$. Then

$$\begin{aligned} y &= 1y && \text{by (M4)} \\ &= (xx^{-1})y && \text{by (M5)} \\ &= x(x^{-1}y) && \text{by (M3)} \\ &= x(yx^{-1}) && \text{by (M2)} \\ &= (xy)x^{-1} && \text{by (M3)} \\ &= 1x^{-1} && \text{by assumption} \\ &= x^{-1} && \text{by (M4)} \end{aligned}$$

So $y = x^{-1}$.

(d) Assume that $x \neq 0$. Then by axiom (M5), $x^{-1}x = 1$. So by our work in part (c), it must be true that $x = (x^{-1})^{-1}$.

And we are done! $\square$

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution: Suppose E is such a set, and let's give its ordered superset a name, say S. Given that α is a lower bound of E, then it must be true that

$$\forall x \ (x \in E \Rightarrow \alpha \leq x).$$

Similarly, since β is an upper bound of E, then it follows that

$$\forall x \ (x \in E \Rightarrow x \leq \beta).$$

We know that E is nonempty, meaning that there must exist some element $x^* \in E$. We will use this specific element to prove that $\alpha \leq \beta$. Recall that, by definition, $a \leq b$ means that either $a < b$ or $a = b$. So there are four possible cases that can apply to our situation:

(a) $\alpha = x^*$ and $x^* = \beta$,

(b) $\alpha = x^*$ and $x^* < \beta$,

(c) $\alpha < x^*$ and $x^* = \beta$, and

(d) $\alpha < x^*$ and $x^* = \beta$.

We will now show that we arrive at the same conclusion in all possible cases—that $\alpha \leq \beta$ in all possible scenarios.

(a) $\underline{\alpha = x^* \text{ and } x^* = \beta}$: In this case, we can simply appeal to the transitivity of $=$ and deduce that $\alpha = \beta$. If $\alpha = \beta$, then it clearly follows that $\alpha \leq \beta$.

(b) $\underline{\alpha = x^* \text{ and } x^* < \beta}$: Here, since $\alpha = x^*$, we can substitute α for x^* in the $x^* < \beta$ inequality. This tells us that $\alpha < \beta$, from which it follows that $\alpha \leq \beta$.

(c) $\underline{\alpha < x^* \text{ and } x^* = \beta}$: Since $\beta = x^*$, we can substitue β for x^* in the $\alpha < x^*$ inequality. This tells us that $\alpha < \beta$, so it follows that $\alpha \leq \beta$.

(d) $\underline{\alpha < x^* \text{ and } x^* < \beta}$: In this final case, we can appeal to the transitivity of the $<$ relation. After doing this, we get that $\alpha < \beta$, and it once more follows that $\alpha \leq \beta$.

As you can see, no matter which way we go, we arrive at the same conclusion: $\alpha \leq \beta$. This means that the statement holds in general, and we are done! $\square$

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Solution: The key to this exercise is to recall that $\mathbb{R}$ has the least-upper-bound property. This means that both A and $-A$, being subsets of $\mathbb{R}$, must have their infima in $\mathbb{R}$. Also, as shown in Theorem 1.11 in Rudin, the least-upper-bound property implies the greatest-lower-bound property. This means that the suprema of A and $-A$ must also exist in $\mathbb{R}$. Most relevant to the problem at hand, though:

$$\inf A \in \mathbb{R} \text{ and } \sup(-A) \in \mathbb{R}.$$

Our strategy will be to use the definition of inf and sup to show that $\inf A \leq -\sup(-A)$ and $\inf A \geq -\sup(-A)$. Note that if both of these are true at the same time, then $\inf A = -\sup(-A)$, which is what we want.

<u>$\inf A \leq -\sup(-A)$:</u> We will now show that $-\sup(-A)$ is a lower bound of A. By definition, $\sup(-A)$ is such that

$$\forall(-x) \in -A,\ -x \leq \sup(-A).$$

Equivalently,

$$\forall(-x) \in -A,\ -(-x) \geq -\sup(-A).$$

We've shown previously that $-(-x) = x$, and we also know that $x \in A$; this is how the set $-A$ was defined, after all. So it must be true that

$$\forall x \in A,\ -\sup(-A) \leq -x.$$

In other words, $-\sup(-A)$ is a lower bound of A. By the definition of $\inf A$, we must have that

$$\inf A \leq -\sup(-A).$$

<u>$\inf A \geq -\sup(-A)$:</u> Now we proceed to show that $-\inf A$ is an upper bound of $-A$. By definition, $\inf A$ is such that

$$\forall x \in A,\ \inf A \leq x.$$

Observe that this is equivalent to saying that

$$\forall x \in A,\ -\inf A \geq -x.$$

In other words,

$$\forall(-x) \in -A,\ -\inf A \geq -x.$$

This means that $-\inf A$ is an upper bound of $-A$. By the definition of $\sup(-A)$, it must be true that

$$-\inf A \leq \sup(-A).$$

But this is equivalent to saying that

$$\inf A \geq -\sup(-A).$$

<u>Conclusion:</u> The only time in which the statements $\inf A \leq -\sup(-A)$ and $\inf A \geq -\sup(-A)$ can both be true is when $\inf A = -\sup(-A)$, as desired. $\square$

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