Solutions to Baby Rudin

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Contents

Chapter 1

The Real and Complex Number Systems

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution: We'll start with the proof that $r+x$ is irrational. After that, we'll move on to the proof that rx is irrational. Both proofs are very similar. Note that I will use the notation r^{-1} instead of $1/r$ for multiplicative inverses, since I think the notation is a lot cleaner this way.

$\underline{r + x \notin \mathbb{Q}}$: Assume that $r + x \in \mathbb{Q}$, for the sake of contradiction. If $r + x$ is rational, then for any $y \in \mathbb{Q}$, $(r + x) + y$ is also rational (since $\mathbb{Q}$ is a field), by the field axiom (A1). Note that since $r \in \mathbb{Q}$, then $-r \in \mathbb{Q}$ too, by axiom (A5). This means that

$$
\begin{aligned}
(r + x) + (-r) &\in \mathbb{Q} && \text{by (A1)} \\
\Rightarrow r + (x + (-r)) &\in \mathbb{Q} && \text{by (A3)} \\
\Rightarrow r + (-r + x) &\in \mathbb{Q} && \text{by (A2)} \\
\Rightarrow (r + (-r)) + x &\in \mathbb{Q} && \text{by (A3)} \\
\Rightarrow 0 + x &\in \mathbb{Q} && \text{by (A5)} \\
\Rightarrow x &\in \mathbb{Q} && \text{by (A4)}
\end{aligned}
$$

By assumption, x is irrational, which means that $x \notin \mathbb{Q}$. However, we just deduced that $x \in \mathbb{Q}$ from our assumptions. The only way to resolve this contradiction is to realize that our initial assumption was wrong. Hence, $r + x \notin \mathbb{Q}$. In other words, $r + x$ is irrational.

$\underline{rx \notin \mathbb{Q}}$: Similar to the last proof, assume that $rx \in \mathbb{Q}$ for the sake of contradiction. By axiom (M1), for any $y \in \mathbb{Q}$, it must be true that $(rx)y \in \mathbb{Q}$. Also, since $r \in \mathbb{Q}$ and $r \neq 0$, it follows that $r^{-1} \in \mathbb{Q}$ too, by axiom (M5). We can now deduce that

$$
\begin{aligned}
(rx)r^{-1} &\in \mathbb{Q} && \text{by (M1)} \\
\Rightarrow r(xr^{-1}) &\in \mathbb{Q} && \text{by (M3)} \\
\Rightarrow r(r^{-1}x) &\in \mathbb{Q} && \text{by (M2)} \\
\Rightarrow (rr^{-1})x &\in \mathbb{Q} && \text{by (M3)} \\
\Rightarrow 1x &\in \mathbb{Q} && \text{by (M5)} \\
\Rightarrow x &\in \mathbb{Q} && \text{by (M4)}
\end{aligned}
$$

You probably get the gist by now, but we cannot have $x \notin \mathbb{Q}$ by assumption and then deduce that $x \in \mathbb{Q}$ by applying the field axioms. The only possible way to move forward is to conclude that our initial assumption is wrong; that is, $rx \notin \mathbb{Q}$. In other words, rx is irrational. $\square$

2. Prove that there is no rational number whose square is 12.

Solution: The proof is rather similar to the proof that $\sqrt{2}$ is irrational. Assume that there is some $x \in \mathbb{Q}$ such that $x^2 = 12$. This means that $x = \frac{m}{n}$, with $m, n \in \mathbb{Z}$, such that at most one of m, n is divisible by 3. Observe that $x^2 = (\frac{m}{n})^2$, meaning that $(\frac{m}{n})^2 = 12$. All of the above implies that

$$m^2 = 12n^2.$$

Let's now proceed by checking the different possible cases.

First, suppose that m is not divisible by 3. Then clearly m^2 is not divisible by 3 either. However, $12n^2$ is definitely divisible by 3. This would imply that $m^2 \neq 12n^2$, which is clearly not true. So it can't be true that m is not divisible by 3.

So the only choice we have left is that m is divisible by 3. In this case, m^2 must be divisible by 9. This would imply that $12n^2$ must also be divisible by 9, since $m^2 = 12n^2$, per our work above. However, this cannot be the case. 12 is not divisible by 9, but it is divisible by 3. So we still have a factor of 3 left. This would imply that n^2 must be divisible by 3, but that itself would imply that n is divisible by 3. However, by assumption, at most one of m, n can be divisible by 3. This would lead us to conclude that $m^2 \neq 12n^2$, a clear contradiction.

We have exhausted all possible choices of $x \in \mathbb{Q}$. This means that there does not exist any $x \in \mathbb{Q}$ such that $x^2 = 12$. In other words, there is no rational number whose square is 12. In other words, 12 does not have a rational square root. $\square$

3. Prove Proposition 1.15.

Solution: As a reminder, Proposition 1.15 states the following:

The axioms for multiplication imply the following statements.

(a) *If* $x \neq 0$ *and* $xy = xz$ *then* $y = z$.

(b) *If* $x \neq 0$ *and* $xy = x$ *then* $y = 1$.

(c) *If* $x \neq 0$ *and* $xy = 1$ *then* $y = x^{-1}$.

(d) *If* $x \neq 0$ *then* $(x^{-1})^{-1} = x$.

Let's prove each statement, one by one.

(a) Suppose $x \neq 0$ and $xy = xz$. Then

$$
\begin{aligned}
y &= 1y && \text{by (M4)} \\
&= (xx^{-1})y && \text{by (M5)} \\
&= x(x^{-1}y) && \text{by (M3)} \\
&= x(yx^{-1}) && \text{by (M2)} \\
&= (xy)x^{-1} && \text{by (M3)} \\
&= (xz)x^{-1} && \text{by assumption} \\
&= (zx)x^{-1} && \text{by (M2)} \\
&= z(xx^{-1}) && \text{by (M3)} \\
&= z1 && \text{by (M5)} \\
&= z && \text{by (M4)}
\end{aligned}
$$

So $y = z$.

(b) Suppose $x \neq 0$ and $xy = x$. Note that $x = x1$, by axiom (M4). So it must be true that $xy = x1$. Appealing to our work in part (a), we can deduce that $y = 1$.

(c) Suppose $x \neq 0$ and $xy = 1$. Then

$$
\begin{aligned}
y &= 1y && \text{by (M4)} \\
&= (xx^{-1})y && \text{by (M5)} \\
&= x(x^{-1}y) && \text{by (M3)} \\
&= x(yx^{-1}) && \text{by (M2)} \\
&= (xy)x^{-1} && \text{by (M3)} \\
&= 1x^{-1} && \text{by assumption} \\
&= x^{-1} && \text{by (M4)}
\end{aligned}
$$

So $y = x^{-1}$.

(d) Assume that $x \neq 0$. Then by axiom (M5), $x^{-1}x = 1$. So by our work in part (c), it must be true that $x = (x^{-1})^{-1}$.

And we are done! □

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution: Suppose E is such a set, and let's give its ordered superset a name, say S. Given that α is a lower bound of E, then it must be true that

$$\forall x \ (x \in E \Rightarrow \alpha \leq x).$$

Similarly, since β is an upper bound of E, then it follows that

$$\forall x \ (x \in E \Rightarrow x \leq \beta).$$

We know that E is nonempty, meaning that there must exist some element $x^* \in E$. We will use this specific element to prove that $\alpha \leq \beta$. Recall that, by definition, $a \leq b$ means that either $a < b$ or $a = b$. So there are four possible cases that can apply to our situation:

(a) $\alpha = x^*$ and $x^* = \beta$,

(b) $\alpha = x^*$ and $x^* < \beta$,

(c) $\alpha < x^*$ and $x^* = \beta$, and

(d) $\alpha < x^*$ and $x^* = \beta$.

We will now show that we arrive at the same conclusion in all possible cases—that $\alpha \leq \beta$ in all possible scenarios.

(a) $\underline{\alpha = x^* \text{ and } x^* = \beta}$: In this case, we can simply appeal to the transitivity of $=$ and deduce that $\alpha = \beta$. If $\alpha = \beta$, then it clearly follows that $\alpha \leq \beta$.

(b) $\underline{\alpha = x^* \text{ and } x^* < \beta}$: Here, since $\alpha = x^*$, we can substitute α for x^* in the $x^* < \beta$ inequality. This tells us that $\alpha < \beta$, from which it follows that $\alpha \leq \beta$.

(c) $\underline{\alpha < x^* \text{ and } x^* = \beta}$: Since $\beta = x^*$, we can substitue β for x^* in the $\alpha < x^*$ inequality. This tells us that $\alpha < \beta$, so it follows that $\alpha \leq \beta$.

(d) $\underline{\alpha < x^* \text{ and } x^* < \beta}$: In this final case, we can appeal to the transitivity of the $<$ relation. After doing this, we get that $\alpha < \beta$, and it once more follows that $\alpha \leq \beta$.

As you can see, no matter which way we go, we arrive at the same conclusion: $\alpha \leq \beta$. This means that the statement holds in general, and we are done! $\square$

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