1 Prove that $-(-v) = v$ for every $v \in V$.

Suppose $v \in V$.

$$v + (-v) = 0$$

$$-v + (-(-v)) = 0$$

$$\Rightarrow \quad v + (-v) = -v + (-(-v))$$

$$\Rightarrow \quad v + \cancelto{0}{(-v) + (-(-v))} = \cancelto{0}{-v + (-(-v))} + (-(-v))$$

$$\Rightarrow \quad v + 0 = 0 + (-(-v))$$

$$\Rightarrow \quad v = -(-v)$$

2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Assume $av = 0$, but, for the sake of contradiction, assume further that $a \neq 0$ and $v \neq 0$.

Then $av = 0$

So $av + av = 0 + av$

$\Rightarrow$ $a(v+v) = av$

$\Rightarrow$ So $v + v = v$

$\Rightarrow$ $v + v - v = v - v$

$\Rightarrow$ $v = 0$.

Alternatively, say $v \neq 0$.

Then $av + av = av$

$\Rightarrow$ $(a+a)v = av$

$\Rightarrow$ $a + a = a$

$\Rightarrow$ $a = 0$.

If $a = 0$ and $v = 0$, then this is clearly true.

3 Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Suppose there exist two distinct $x_1, x_2 \in V$ that satisfy this equation.

So $v + 3x_1 = w$ and $v + 3x_2 = w$.

Then $v + 3x_1 = v + 3x_2$

So $3x_1 = 3x_2$

Hence $x_1 = x_2$, a contradiction.

Therefore, any x that satisfies $v + 3x = w$ must be unique.

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

It fails to satisfy the additive identity property, which asserts that there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$. It fails to satisfy this property because there are no elements in $\emptyset$. All other properties are satisfied vacuously.

5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

My approach involves showing that these two conditions are equivalent, in the sense that they are either both true or both false. So I want to show that

There exists an element $0 \in V$ such that $v+0 = v$ for all $v \in V$

if and only if

$0v = 0$ for all $v \in V$.

($\Rightarrow$) Suppose $\exists\, 0 \in V$ s.t. $\forall v \in V$, $v+0 = v$.

Then $v+0 = v \Rightarrow v+(-v)+0 = v+(-v) \Rightarrow 0 = v-v$.

Then $0v = (v-v)v = vv - vv = 0$.

($\Leftarrow$) Suppose that $\forall v \in V$, $0v = 0$.

So $v+0 = v+0v = (1+0)v = 1v = v$. In short, $v+0 = v$.

So it happens that $\forall v \in V\ 0v = 0 \Leftrightarrow \forall v \in V,\ v+0 = v$. So we can effectively replace the additive identity condition with the one posed in this problem.

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty &= \infty + t = \infty + \infty = \infty, \\ t + (-\infty) &= (-\infty) + t = (-\infty) + (-\infty) = -\infty, \\ \infty + (-\infty) &= (-\infty) + \infty = 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

To be a vector space over $\mathbb{R}$, $\mathbb{R} \cup \{-\infty, \infty\}$ must satisfy all axioms of a vector space.

However, consider the operation $\infty + \infty - \infty$.

If we decide to parenthesize the operation as $(\infty + \infty) - \infty$, we get

$$\infty + \infty - \infty = (\infty + \infty) - \infty = \infty - \infty = 0.$$

However, if we parenthesize it as $\infty + (\infty - \infty)$, we get

$$\infty + (\infty - \infty) = \infty + 0 = \infty.$$

So $\mathbb{R} \cup \{-\infty, \infty\}$ fails to be associative, and thus cannot be a vector space.

* We use the shorthand $\infty - \infty$ to denote $\infty + (-\infty)$.

7* Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Key: use exercise 5

Let $u, v \in V^S$, $x \in S$, and $\lambda \in \mathbb{F}$.

addition in V

Define addition as $u+v = (u+v)(x) = u(x) + v(x)$.

Scalar multiplication is defined as $\lambda u = \lambda \cdot u(x)$.

Commutativity: $(u+v)(x) = u(x) + v(x) = v(x) + u(x) = (v+u)(x)$

Associativity: $(u+v)+w = (u(x)+v(x)) + w(x) = u(x) + (v(x) + w(x)) = u+(v+w)$

Zero multiplication: $0u = 0 \cdot u(x) = 0$.

Multiplicative identity: $1 \cdot u = 1 \cdot u(x) = u(x) = u$

Distributivity 1: $a(u+v) = a(u(x)+v(x)) = au(x) + av(x) = au + av$.

Distributivity 2: $(a+b)v = (a+b)v(x) = av(x) + bv(x) = av+bv$

V^S satisfies all properties, and therefore must constitute a vector space.

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.

Commutativity:

$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2) = (u_2 + u_1) + i(v_2 + v_1) = (u_2 + iv_2) + (u_1 + iv_1)$

Associativity:

$((u_1 + iv_1) + (u_2 + iv_2)) + (u_3 + iv_3) = ((u_1 + u_2) + i(v_1 + v_2)) + (u_3 + iv_3) = (u_1 + u_2 + u_3) + i(v_1 + v_2 + v_3) = (u_1 + iv_1) + ((u_2 + u_3) + i(v_2 + v_3))$

$= (u_1 + iv_1) + ((u_2 + iv_2) + (u_3 + iv_3))$

Additive identity: $(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv$

Additive inverse: $(u + iv) + (-u - iv) = (u - u) + (iv - iv) = 0 + 0i$

Distributivity 1: $a((u_1 + iv_1) + (u_2 + iv_2)) = a(u_1 + u_2 + iv_1 + iv_2) = au_1 + au_2 + aiv_1 + aiv_2 = a(u_1 + iv_1) + a(u_2 + iv_2)$

Distributivity 2: $(a+b)(u + iv) = (a+b)u + (a+b)iv = au + bu + iav + ibv = au + iav + bu + ibv = a(u + iv) + b(u + iv)$