Stochastic Order-Redistribution Problem with Integrated Vehicle Routing

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Problem Description

In this work, we address a commonly observed supply chain problem and propose new thinking approach towards solving it. A company operates a set of warehouses or ***distribution centers (DC)*** *located in multiple geographical locations.* ***Each DC has a recurring demand for a set of products in a multi-period time horizon, and the demand for any one product in any time period is stochastic****. For a given time period, the exact order amount demanded at every DC is realized or known at the very start of the time period only and must be fulfilled by the end of that period. From the view point of any DC, two types of suppliers are considered: (1)* ***primary suppliers*** *that are located in either far off places like international suppliers, or, the domestic ones that have large lead times for product delivery, and (2)* ***secondary suppliers****, for e.g. from the viewpoint of order fulfillment at a particular DC, other DCs within the same parent company can used as buffer points and the inventory from such can be re-directed. Conventionally, the importance of utilizing secondary suppliers is often overlooked, but it deserves more attention, especially when a shortfall of order fulfillment can not be compensated through any primary supplier within a short time span.*

For a given time period, the decision making is divided into two stages in order of sequence: (1) ***order phase****, and (2)* ***re-distribution phase****. During the order phase, a DC places a shipment order for an assortment of products (anticipating future demand) with a particular primary supplier, upon which the shipment is received at it after a couple of time periods. It is assumed here for the sake of simplicity that one supplier can deliver only a single shipment in a time period to its receiving DC. During the re-distribution phase, the demand for each product at its corresponding DC is realized or known exactly, and recourse decisions are made to compensate for any un-fulfilled demand at any DC in the network. This brings us to the role of secondary suppliers. A DC not only acts as a demand point but also serves as a supply point for other DCs in the network. Re-routing or re-distribution of products is accomplished using a fleet of vehicles. During the re-distribution phase, after deciding upon how much to pick and/or deliver at individual DCs, the routing decision making problem is what is known in literature as* ***Pickup and Delivery Vehicle Routing Problems with Time Windows (PDVRPTW)****.*

Practical considerations

From practical point of view, a couple of extra features added to the above problem description will render it more realistic, while at the same time increasing its tractability. Therefore, we discount them in current model formulation, and mention them here only in brief:

1. Accounting for one or more of transfer points might either be a necessary condition (like in multi-modal transportation), or even if not, a better strategy in PDVRPTW.
2. Multiple shipments could be arriving at any DC from various suppliers in any one time period.
3. The company might be using its own fleet of vehicles for transportation of some of the products form its primary suppliers which are domestic. In that case, vehicles can be used not just for internal redistribution between DCs, but also for transportation of products from external primary suppliers located domestically.
4. Note that, we have considered that all products arriving at various DCs are ready made ones, which are bought as it is, without consideration for their assembly (serial, parallel or mix of both) or related activities as is common in make-to-assemble or just-in-time systems. This will introduce additional complexity in the problem since the product specific processes will also have to be monitored, which may or may not be under the company's control, but none the less such integration will bring us closer to actual requirements in the industry and give us an opportunity to implement philosophies such as lot-streaming or deploying multiple sourcing for a product etc in a meaningful way.

We propose one integer programming model for the deterministic demand case, and a stochastic programming model for stochastic demand case.

Model Formulation

<u>Notation and Parameters</u>

Fundamental sets:
$\mathbf{I}$ $= \{1,2,\ldots,|\mathbf{I}|\}$, set of products
$\mathbf{J}$ $= \{1,2,\ldots,|\mathbf{J}|\}$, set of DCs
$\mathbf{T}$ $= \{1,2,\ldots,|\mathbf{T}|\}$, set of time periods
$\mathbf{V}$ $= \{1,2,\ldots,|\mathbf{V}|\}$, set of vehicles operating in each time period
$(i,\bar{t},j) \in \mathbf{ITJ}$ = set of triplets, representing whether product i can be shipped to DC j in time period $\bar{t}$ $(1 \leq \bar{t} \leq |\mathbf{T}|-1)$, from its primary supplier

Shipping from primary suppliers related:
$lt^j_{i,\bar{t}}$ lead time for shipment indexed as $(i,\bar{t},j) \in$ **ITJ** such that it arrives at the DC j at the start of time period $\bar{t} + lt^j_{i,\bar{t}}$, $lt^j_{i,\bar{t}} \geq 1$, and $\bar{t} + lt^j_{i,\bar{t}} \leq |\mathbf{T}|, \forall (i,\bar{t},j) \in$ **ITJ**
$sfc^j_{\bar{t}}$ fixed cost to make a shipment (of various products) to DC j in time period $\bar{t}$ such that, $\forall (i,\bar{t},j) \in$ **ITJ**
$svc^j_{i,\bar{t}}$ cost per unit volume for shipping product i to DC j in time period $\bar{t}$ such that, $\forall (i,\bar{t},j) \in$ **ITJ**
svu^j_i shipping volume utilization per unit of product i (same in any time period) to DC j such that, $\forall (i,\bar{t},j) \in$ **ITJ**
$svm^j_{i,\bar{t}}$ shipping maximum volume allowed for product i in time period $\bar{t}$ to DC j such that, $\forall (i,\bar{t},j) \in$ **ITJ**
$\widehat{svm}^j_{\bar{t}}$ shipping maximum volume allowed for all products in time period $\bar{t}$ to DC j such that, $\forall (i,\bar{t},j) \in$ **ITJ**
$swc^j_{i,\bar{t}}$ cost per unit weight for shipping product i to DC j in time period $\bar{t}$ such that, $\forall (i,\bar{t},j) \in$ **ITJ**
swu^j_i shipping weight utilization per unit of product i (same in any time period) to DC j such that, $\forall (i,\bar{t},j) \in$ **ITJ**
$swm^j_{i,\bar{t}}$ shipping maximum weight allowed for product i in time period $\bar{t}$ to DC j such that, $\forall (i,\bar{t},j) \in$ **ITJ**
$\widehat{swm}^j_{\bar{t}}$ shipping maximum weight allowed for all products in time period $\bar{t}$ to DC j such that, $\forall (i,\bar{t},j) \in$ **ITJ**
$d^j_{i,t}$ demand (zero or positive) of product i at DC j in time period t $(t \geq 2)$
$dlc^j_{i,t}$ cost for incurring the demand loss of a unit of product i at DC j in time period t $(t \geq 2)$

Inventory at DCs related:
$ivc^j_{i,t}$ inventory cost to carry unit volume of product i from time period t to $t+1$ for DC j
ivu^j_i inventory volume utilization per unit of product i (same in any time period) for DC j
$ivm^j_{i,t}$ inventory maximum volume allowed for product i in time period t for DC j
$\widehat{ivm}^j_t$ inventory maximum volume allowed for all products in time period t for DC j

Transportation among DCs (PDVRPTW) related:
P $= \{1,2,\ldots,n\}$, set of pickup nodes, $n = |\mathbf{J}|*|\mathbf{I}|*(|\mathbf{J}|\text{-}1)$
D $= \{n+1,n+2,\ldots,2n\}$, set of delivery nodes
O $= \{2n+1,2n+2,\ldots,2n+|\mathbf{V}|\}$, set of nodes representing depots for vehicles
N $= \mathbf{P} \cup \mathbf{D} \cup \mathbf{O}$
A $= \mathbf{N} \times \mathbf{N}$, set of arcs connecting different nodes in **N**, $((j,j) \notin \mathbf{A}\colon j \in \mathbf{N})$
vvu^k_v volume utilization on vehicle v by a unit of the product that is picked (or dropped) at node $k \in \mathbf{N}$ (note that only a unique product (or none) is assigned to be either picked up or dropped at any node $k \in \mathbf{N}$)
tfc_v fixed transportation cost incurred if vehicle v is operated in a time period
tvc^{jk}_v variable transportation cost to carry a unit volume of (mix of) products, if vehicle v travels on arc $(j,k) \in \mathbf{A}$
$t^{j,k}_v$ travel time on arc $(j,k) \in \mathbf{A}$ for vehicle v
st^j_v fixed service time required to load or unload a vehicle v at node $k \in \mathbf{N}$ $(st^j_v = 0, \forall j \in \mathbf{O}, \forall v \in \mathbf{V})$
ϵ_v minimum pickup quantity required from any node by vehicle $v \in \mathbf{V}$
q_v maximum vehicle capacity $v \in \mathbf{V}$
τ time length of any one period

Variables

$P^j_{i,\bar{t},t}$	$\forall(i,\bar{t},j) \in \mathbf{ITJ}, \forall t \in \mathbf{T}$	Units of product i ordered in period $\bar{t}$ and available at DC j at the start of period t
$Z^j_{\bar{t}}$	$\forall(i,\bar{t},j) \in \mathbf{ITJ}$	Binary variable indicating whether a shipment was ordered for DC j in time period $\bar{t}$ or not
$D^j_{1,i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Units of product i in current inventory at the start of period t, after its demand has been met by the inventory from the last period and/or direct shipments from various suppliers
$D^{j'}_{i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Demand shortage for product i that is met from redistribution of inventory among DCs, by the end of the same time period in which the demand was realized
$L^{j+}_{i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Units of product i sent to other DCs from DC j in time period t
$U^{j+}_{i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Binary variable, set to unity if $L^{j+}_{i,t}$ is positive
$D^j_{2,i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Units of product i left in current inventory after some of it has been sent to other DCs
$L^{j-}_{i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Units of product i coming to DC j from other DCs, by the end of time period t
$U^{j-}_{i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Binary variable, set to unity if $L^{j-}_{i,t}$ is positive (To make the model more realistic, in any time period, material can either be shipped from a DC to other DCs - when there is excess of it at the start of the time period, or received by a DC from other DCs - when it needs to increase its own inventory or fulfill any shortfall of current period demand, but not both)
$D^j_{3,i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Units of product i in current inventory after some of it has been received from other DCs
$D^{j''}_{i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	Lost demand for product i that could not be met even in the "redistribution phase" (Fulfilling of shortfall of demand from future is not considered in the model)
$S^j_{i,t}$	$\forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$	inventory of product i at DC j, at the end of time period t that is carried over to next time period (Note that initial inventories i.e. $S^j_{i,0}$ could be zero or positive, and is specified as a parameter. Also, stocking of inventory at any DC, at the end of a time period, is not allowed at the expense of lost demand for any product)
$L^{j,k+}_{i,t}$	$\forall i \in \mathbf{I}, \forall j,k \in \mathbf{J}, j \neq k, \forall t \in \mathbf{T}$	Units of product i sent to DC k from DC j in time period t
$L^{j,k-}_{i,t}$	$\forall i \in \mathbf{I}, \forall j,k \in \mathbf{J}, j \neq k, \forall t \in \mathbf{T}$	Units of product i received by DC j from DC k in time period t
$Y^k_{v,t}$	$\forall v \in \mathbf{V}, \forall k \in \mathbf{N}, \forall t \in \mathbf{T}$	Binary variable, set to unity if vehicle v covers node k in time period t, zero otherwise
$X^{j,k}_{v,t}$	$\forall v \in \mathbf{V}, \forall(j,k) \in \mathbf{A}, \forall t \in \mathbf{T}$	Binary variable, set to unity if vehicle v covers arc (j,k) in time period t, zero otherwise
$T^k_{v,t}$	$\forall v \in \mathbf{V}, \forall k \in \mathbf{N}, \forall t \in \mathbf{T}$	Time status at node k, by vehicle v in time period t
L^t_r	$\forall r \in \mathbf{P} \cup \mathbf{D}, \forall(j,k) \in \mathbf{A}, \forall t \in \mathbf{T}$	Units of a (corresponding) product to be either picked up from (if positive) or dropped at (if negative) node r in time period t
$V^k_{v,t}$	$\forall v \in \mathbf{V}, \forall k \in \mathbf{N}, \forall t \in \mathbf{T}$	Capacity status (in volume) of vehicle v at node k, in time period t
$W^{j,k}_{v,t}$	$\forall v \in \mathbf{V}, \forall(j,k) \in \mathbf{A}, \forall t \in \mathbf{T}$	$= V^k_{v,t} \times X^{j,k}_{v,t}$

Figure 1 illustrates various decisions taken during a time period at a DC.

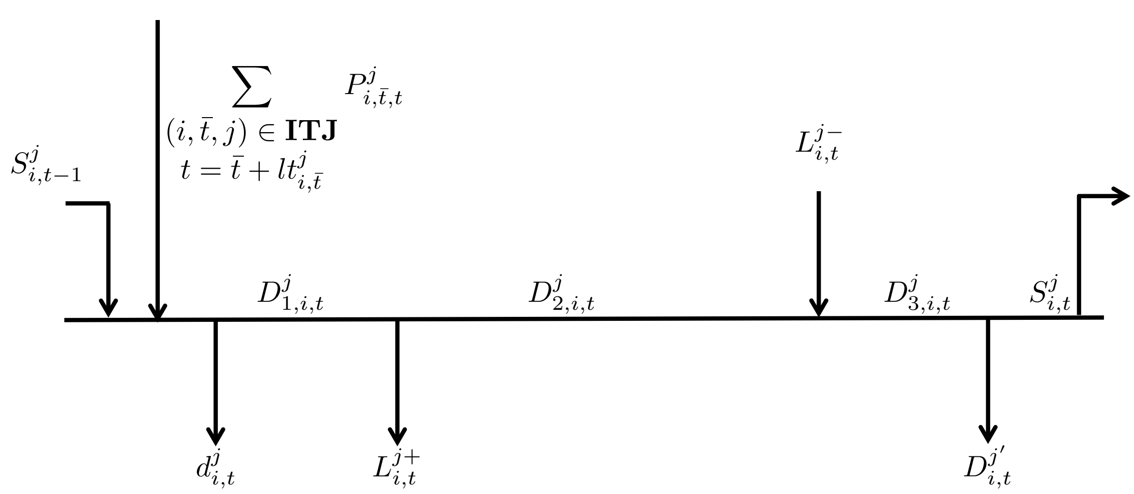

Figure 1: Change in inventory of product i in time period t for DC j

We now present model **M1** for the deterministic demand case.

Model **M1**:

$$\min \sum_{\substack{j \in \mathbf{J}, \bar{t} \in \mathbf{T}, \\ \exists (i,\bar{t},j) \in \mathbf{ITJ}}} sfc^j_{\bar{t}} Z^j_{\bar{t}} + \sum_{(i,\bar{t},j)\in \mathbf{ITJ}} (svc^j_{i,\bar{t}} svu^j_{i,\bar{t}} + swc^j_{i,\bar{t}} swu^j_{i,\bar{t}}) P^j_{i,\bar{t},\bar{t}} + \sum_{i\in \mathbf{I}, j\in \mathbf{J}, t\in \mathbf{T}} (dlc^j_{i,t} D^{j''}_{i,t} + ivc^j_{i,t} ivu^j_{i,t} S^j_{i,t})$$

$$+ \sum_{t\in \mathbf{T}, v\in \mathbf{V}} tfc_v Y^{2n+v}_{v,t} + \sum_{t\in \mathbf{T}, v\in \mathbf{V}, j\in \mathbf{J}, (j,k)\in \mathbf{A}} tvc^{j,k}_v W^{j,k}_{v,t}$$

$$P^j_{i,\bar{t},t} = 0 \qquad \forall 1 \le t \le \bar{t}-1, \forall \bar{t} + lt^j_{i,\bar{t}} + 1 \le t \le |\mathbf{T}|, \forall (i,\bar{t},j) \in \mathbf{ITJ} \tag{1}$$

$$P^j_{i,\bar{t},t} = P_{i,\bar{t},t-1} \qquad \forall \bar{t}+1 \le t \le \bar{t} + lt^j_{i,\bar{t}}, \forall (i,\bar{t},j) \in \mathbf{ITJ} \tag{2}$$

$$svu^j_{i,\bar{t}} P^j_{i,\bar{t},\bar{t}} \le svm^j_{i,\bar{t}} \qquad \forall (i,\bar{t},j) \in \mathbf{ITJ} \tag{3}$$

$$\sum_{(i,\bar{t},j)\in \mathbf{ITJ}} svu^j_{i,\bar{t}} P^j_{i,\bar{t},\bar{t}} \le \widehat{svm}^j_{\bar{t}} \qquad \forall j \in \mathbf{J}, \forall \bar{t} \in \mathbf{T} \tag{4}$$

$$swu^j_{i,\bar{t}} P^j_{i,\bar{t},\bar{t}} \le swm^j_{i,\bar{t}} \qquad \forall (i,\bar{t},j) \in \mathbf{ITJ} \tag{5}$$

$$\sum_{(i,\bar{t},j)\in \mathbf{ITJ}} swu^j_{i,\bar{t}} P^j_{i,\bar{t},\bar{t}} \le \widehat{swm}^j_{\bar{t}} \qquad \forall j \in \mathbf{J}, \forall \bar{t} \in \mathbf{T} \tag{6}$$

$$P^j_{i,\bar{t},\bar{t}} \le M Z^j_{\bar{t}} \qquad \forall (i,\bar{t},j) \in \mathbf{ITJ} \tag{7}$$

$$D^j_{0,i,t} = S^j_{i,t-1} + \sum_{\substack{(i,\bar{t},j) \in \mathbf{ITJ} \\ t = \bar{t} + lt^j_{i,\bar{t}}}} P^j_{i,\bar{t},t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{8}$$

$$D^j_{1,i,t} = \max(D^j_{0,i,t} - d^j_{i,t}, 0) \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{9}$$

$$D^j_{2,i,t} = D^j_{1,i,t} - L^{j+}_{i,t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{10}$$

$$D^j_{3,i,t} = D^j_{2,i,t} + L^{j-}_{i,t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{11}$$

$$D^{j'}_{i,t} = \max(d^j_{i,t} - D^j_{0,i,t}, 0) \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{12}$$

$$D^{j''}_{i,t} = \max(D^{j'}_{i,t} - D^j_{3,i,t}, 0) \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{13}$$

$$S^j_{i,t} = \max(D^j_{3,i,t} - D^{j'}_{i,t}, 0) \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{14}$$

$$L^{j+}_{i,t} = \sum_{k\in \mathbf{J}, k\neq j} L^{j,k+}_{i,t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{15}$$

$$L^{j-}_{i,t} = \sum_{k\in \mathbf{J}, k\neq j} L^{j,k-}_{i,t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{16}$$

$$L^{j,k+}_{i,t} = L^{k,j-}_{i,t} \qquad \forall i \in \mathbf{I}, \forall j,k \in \mathbf{J}, j \neq k, \forall t \in \mathbf{T} \tag{17}$$

$$L^{j+}_{i,t} \le c U^{j,+}_{i,t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{18}$$

$$L^{j-}_{i,t} \le c(1 - U^{j,+}_{i,t}) \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{19}$$

$$L^{j-}_{i,t} \le c U^{j,-}_{i,t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{20}$$

$$L^{j+}_{i,t} \le c(1 - U^{j,-}_{i,t}) \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{21}$$

where $c = (\max_{v\in \mathbf{V}} q_v) \min(|\mathbf{J}| - 1, |\mathbf{V}|)$

$$ivu^j_{i,t} S^j_{i,t} \le ivm^j_{i,t} \qquad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{22}$$

$$\sum_{i\in \mathbf{I}} ivu^j_{i,t} S^j_{i,t} \le \widehat{ivm}^j_t \qquad \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \tag{23}$$

$$\sum_{v\in \mathbf{V}} Y^r_{v,t} \le 1 \qquad \forall r \in \mathbf{N}, \forall t \in \mathbf{T} \tag{24}$$

$$Y^{2n+w}_{v,t} = 0 \qquad \forall v, w \in \mathbf{V}, w \neq v \tag{25}$$

$$\sum_{(j,k)\in \mathbf{A}} X^{j,k}_{v,t} = \sum_{(k,j)\in \mathbf{A}} X^{k,j}_{v,t} \qquad \forall v \in \mathbf{V}, \forall t \in \mathbf{T}, \forall k \in \mathbf{N} \tag{26}$$

$$\sum_{(j,k)\in \mathbf{A}} X^{j,k}_{v,t} = Y^k_{v,t} \qquad \forall v \in \mathbf{V}, \forall t \in \mathbf{T}, \forall k \in \mathbf{N} \tag{27}$$

$$\sum_{r\in \mathbf{P}} X^{2n+v,r}_{v,t} = \sum_{r\in \mathbf{D}} X^{r,2n+v}_{v,t} \qquad \forall v \in \mathbf{V}, \forall t \in \mathbf{T} \tag{28}$$

$$\sum_{r\in \mathbf{P}} X^{2n+v,r}_{v,t} = Y^{2n+v}_v \qquad \forall v \in \mathbf{V}, \forall t \in \mathbf{T} \tag{29}$$

$$Y^r_{v,t} = Y^{n+r}_{v,t} \qquad \forall v \in \mathbf{V}, \forall r \in \mathbf{P}, \forall t \in \mathbf{T} \tag{30}$$

$$T^r_{v,t} = 0 \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, r \in \mathbf{O} \tag{31}$$

$$T^k_{v,t} \geq T^j_{v,t} + (st^j_v + t^{j,k}_v)X^{j,k}_{v,t} - \tau(1 - X^{j,k}_{v,t}) \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, (j,k) \in \mathbf{A}, k \notin \mathbf{O} \tag{32}$$

$$T^j_{v,t} + (st^j_{v,t} + t^{j,2n+v}_{v,t})X^{j,2n+v}_{v,t} \leq \tau \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, j \in \mathbf{D} \tag{33}$$

A pseudo-code to define linking constraints between variables in "order phase" and "redistribution phase" is being presented in Algorithm 1.

Algorithm 1 Linking constraints between variables in "order phase" and "redistribution phase"

$r \leftarrow 1$
for $i = 1$ to $|\mathbf{I}|$, **do**
 for $j = 1$ to $|\mathbf{J}|$, **do**
  for $k = 1$ to $|\mathbf{J}|$, **do**
   if $k \neq j$ **then**

$$L^r_t = L^{j,k+}_{i,t} \tag{34}$$

    $r \leftarrow r + 1$
   end if
  end for
 end for
end for

In constraints [34], we have defined variable, $L^r_t, \forall r = 1, 2, \ldots, |\mathbf{J}| \times |\mathbf{I}| \times (|\mathbf{J}| - 1)$, i.e. $\forall r \in \mathbf{P}, \forall t \in \mathbf{T}$. We continue to define other constraints for model **M1**:

$$L^r_t = -L^{n+r}_t \qquad \forall r \in \mathbf{P}, \forall t \in \mathbf{T} \tag{35}$$

$$L^r_t \geq \epsilon_v Y^r_{v,t} \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, r \in \mathbf{P} \tag{36}$$

$$L^r_t \leq q_v Y^r_{v,t} \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, r \in \mathbf{P} \tag{37}$$

$$V^r_{v,t} = 0 \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, r \in \mathbf{O} \tag{38}$$

$$V^k_{v,t} \geq V^j_{v,t} + \nu v u^k_v L^k_t X^{j,k}_{v,t} - c^{'}(1 - X^{j,k}_{v,t}) \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, (j,k) \in \mathbf{A}, k \notin \mathbf{O} \tag{39}$$

where, $c^{'} = \max_{v \in \mathbf{V}} q_v$

Constraints [39] have a non-linear term $L^k_t X^{j,k}_{v,t}$, which can either be linearized by introducing another variable, or by some wise use of parameters themselves; we implement the latter and re-write it in constraints [40]:

$$V^k_{v,t} \geq V^j_{v,t} + \nu v u^k_v L^k_t - 2c^{'}(1 - X^{j,k}_{v,t}) \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, (j,k) \in \mathbf{A}, k \notin \mathbf{O} \tag{40}$$

$$V^k_{v,t} \leq q_v \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, k \in \mathbf{P} \tag{41}$$

$$W^{j,k}_{v,t} \geq 0 \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, (j,k) \in \mathbf{A} \tag{42}$$

$$W^{j,k}_{v,t} \leq q_v X^{j,k}_{v,t} \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, (j,k) \in \mathbf{A} \tag{43}$$

$$W^{j,k}_{v,t} \geq V^j_{v,t} - q_v(1 - X^{j,k}_{v,t}) \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, (j,k) \in \mathbf{A} \tag{44}$$

$$W^{j,k}_{v,t} \leq V^j_{v,t} \qquad \forall v \in \mathbf{V}, t \in \mathbf{T}, (j,k) \in \mathbf{A} \tag{45}$$

$$\left.\begin{array}{lll}
P^j_{i,\bar{t},t} & \geq 0 & \forall t \in \mathbf{T}, \forall (i,\bar{t},j) \in \mathbf{ITJ} \\
Z^j_{\bar{t}} & \in \{0,1\} & \forall (i,\bar{t},j) \in \mathbf{ITJ} \\
D^j_{2,i,t} & \geq 0 & \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \\
L^{j,k+}_{i,t} & \geq 0 & \forall i \in \mathbf{I}, \forall j,k \in \mathbf{J}, j \neq k, \forall t \in \mathbf{T} \\
L^{j,k-}_{i,t} & \geq 0 & \forall i \in \mathbf{I}, \forall j,k \in \mathbf{J}, j \neq k, \forall t \in \mathbf{T} \\
U^{j+}_{i,t} & \in \{0,1\} & \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \\
U^{j-}_{i,t} & \in \{0,1\} & \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T} \\
Y^r_{v,t} & \in \{0,1\} & \forall r \in \mathbf{N}, v \in \mathbf{V}, \forall t \in \mathbf{T} \\
X^{j,k}_{v,t} & \in \{0,1\} & \forall (j,k) \in \mathbf{A}, v \in \mathbf{V}, \forall t \in \mathbf{T} \\
V^k_{v,t} & \geq 0 & \forall v \in \mathbf{V}, t \in \mathbf{T}, k \in \mathbf{N}
\end{array}\right\} \tag{46}$$

The first term in the objective is the fixed cost associated with ordering shipments from suppliers; second and third term being the volume utilization and weight utilization costs to carry products in the shipments. The fourth term is the cost of losing demand and fifth term is cost of carrying inventories. The first five terms give us a cost for the "order phase" and the last two for "redistribution phase". The sixth term is the fixed cost of using the vehicles, and the last term is the variable cost dependent on loads (volume) carried between various nodes in PDVRPTW problem or "redistribution phase".

A small instance with only three DCs and single product is being illustrated in Figure 2, to understand (15) and (16) more clearly.

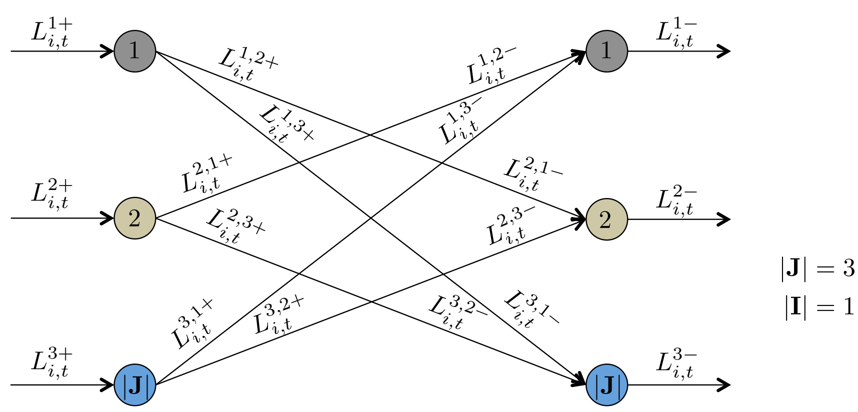

Figure 2: A small instance of redistribution of product i (and the only product)in time period t, represented by a flow diagram between three DCs

In Figure 3, we present the same network as shown in Figure 2, adapted for PDVRPTW, where pickup and demand points are shown and aid our understanding of (34) and (35).

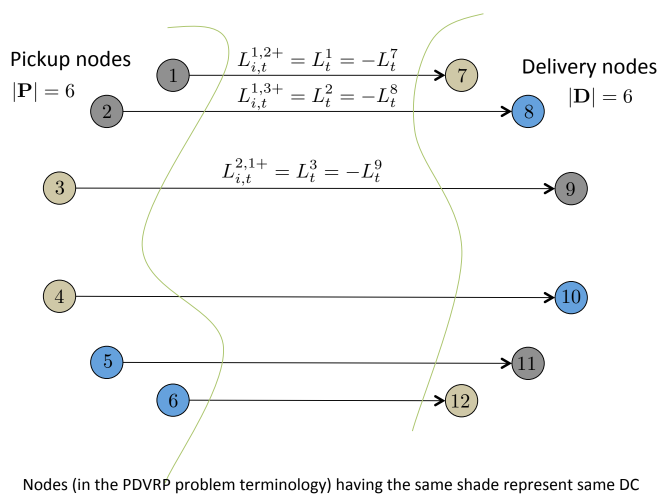

Figure 3: An equivalent graph for the graph shown in Fig 2, adapted for PDVRPTW problem

Let us show how to linearize nonlinear "max" function with equality sign, in (9) (rest of (12)-(14) follow the same):

$$D^j_{1,i,t} = \max(D^j_{0,i,t} - d^j_{i,t}, 0)$$
$$\equiv \begin{cases} D^j_{1,i,t} & \geq & D^j_{0,i,t} - d^j_{i,t} \\ D^j_{1,i,t} & \geq & 0 \\ D^j_{1,i,t} & \leq & D^j_{0,i,t} - d^j_{i,t} + M(1-\lambda^j_{i,t}) \\ D^j_{1,i,t} & \leq & M\lambda^j_{i,t} \\ D^j_{0,i,t} - d^j_{i,t} & \leq & M\lambda^j_{i,t} \\ 0 & \leq & D^j_{0,i,t} - d^j_{i,t} + M(1-\lambda^j_{i,t}) \\ \lambda^j_{i,t} & \in & \{0,1\} \end{cases} \tag{47}$$

Before we discuss the stochasticity in demands, let us first present a brief methodology for dealing with linear deterministic equivalent of stochastic programs. We refer the reader to [1] for this. In the model **M1**, the demands $d^j_{i,t}, \forall (i,\bar{t},j) \in \mathbf{ITJ}$, are deterministic, but our goal is to study the case when these are stochastic and spanning multiple time periods, and defined by individual demand scenarios.
Let us define sets $\mathbf{SI}^j_{i,t}$ and $\mathbf{ST}_t, \forall i \in \mathbf{I}, j \in \mathbf{J}, t = 2,3,\ldots,|\mathbf{T}|$, containing scenarios for demands in individual case and that for collective case for each time period, respectively.

$\mathbf{SI}^j_{i,t} = \{1,2,\ldots,|\mathbf{SI}^j_{i,t}|\}, \forall i \in \mathbf{I}, j \in \mathbf{J}, t = 2,3,\ldots,|\mathbf{T}|$. Each scenario, $s^j_{i,t} \in \mathbf{SI}^j_{i,t}$ is associated with a demand $d(s^j_{i,t}) \geq 0$ (equivalent of $d^j_{i,t}$, for the stochastic case), with probability of occurrence $0 < p(s^j_{i,t}) \leq 1$, with $\sum_{s^j_{i,t} \in \mathbf{SI}^j_{i,t}} p(s^j_{i,t}) = 1$. There can be two cases for $\mathbf{SI}^j_{i,t} = \{1\}$, one, when the demand exists but is deterministic, and two, when there is no demand at all, i.e. $(i,t,j) \notin \mathbf{ITJ}$; in either case, we have $p(s^j_{i,t}) = 1$.

$\mathbf{ST}_t = \{(x_1, x_2, \ldots, x_{|\mathbf{I}||\mathbf{J}|}) | x_{|\mathbf{J}|(i-1)+j} = s^j_{i,t} \in \mathbf{SI}^j_{i,t}, \forall i \in \mathbf{I}, j \in \mathbf{J}\}, \forall t = 2,\ldots,|\mathbf{T}|$. An element of this set, $\omega_t =$

$(x_1, x_2, \ldots, x_{|\mathbf{I}||\mathbf{J}|}) \in \mathbf{ST}_t$, represents a collective scenario for all demands in time period t. The probability associated with ω_t is $p(\omega_t) = \prod_{ij=1,\ldots,|\mathbf{I}||\mathbf{J}|} p(x_{ij})$. The number of such scenarios in time period t is given by $|\mathbf{ST}_t| = \prod_{i\in\mathbf{I}, j\in\mathbf{J}} |\mathbf{SI}^j_{i,t}|, \forall t = 2,\ldots,|\mathbf{T}|$. We will alternatively use $\tilde{\omega_t}$ to represent a random variable whose domain comes from the set $\mathbf{ST}_t$, and each instance is the element $\omega_t \in \mathbf{ST}_t$.

We break the time horizon in $|\mathbf{T}|$ stages and reformulate model **M1** for the deterministic equivalent of multi-stage "Stochastic Multi-Products Order-Redistribution Problem" with decisions in stage t represented collectively by $y_t^{\tilde{\omega_t},\ldots,\tilde{\omega_2}}, \forall t = 2,\ldots,|\mathbf{T}|$ and by y_1 for $t = 1$. We call this model as **M2**. Variables in stage 1 (i.e. $t = 1$) are not scenario dependent, but for the rest of the stages, they are decided as per the realization of sequence of random variables $(\tilde{\omega_t},\ldots,\tilde{\omega_2})$. Let us re-write all the variables that were mentioned earlier in model **M1**, which we now alternatively, and collectively refer to as $y_t^{\tilde{\omega_t},\ldots,\tilde{\omega_2}}, \forall t = 2,\ldots,|\mathbf{T}|$, or just, y_t:

$P^{j,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,\bar{t},t} \quad \forall(i,\bar{t},j) \in \mathbf{ITJ}, \forall t \in \mathbf{T}$
$Z^{j,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{\bar{t}} \quad \forall(i,\bar{t},j) \in \mathbf{ITJ}, \bar{t} = t$
$D^{j,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{1,i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$D^{j',(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$L^{j+,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$U^{j+,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$D^{j,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{2,i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$L^{j-,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$U^{j-,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$D^{j,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{3,i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$D^{j'',(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$
$S^{j,(\tilde{\omega_t},\ldots,\tilde{\omega_2})}_{i,t} \quad \forall i \in \mathbf{I}, \forall j \in \mathbf{J}, \forall t \in \mathbf{T}$ (and so on)

In addition to the above, y variables will also include necessary slack variables to reduce all the stage t constraints in the equality form as follows (A_t and B_{t-1} are the coefficient matrices and deterministic, whereas $b_t^{\tilde{\omega_t}}$ is the constant matrix which is dependent on scenario realization in stage t):

$$-B_{t-1} y_{t-1}^{\tilde{\omega_{t-1}},\ldots,\tilde{\omega_2}} + A_t y_t^{\tilde{\omega_t},\ldots,\tilde{\omega_2}} = b_t^{\tilde{\omega_t}}$$

Now , we can write model **M2** with the new notation discussed above.

Multi-stage Stochastic Model **M2**:

$$\min z = c_1 y_1 + E(c_2 y_2^{\tilde{\omega_2}} + \ldots + E(c_{|\mathbf{T}|} y_{|\mathbf{T}|}^{\tilde{\omega_{|\mathbf{T}|}},\ldots,\tilde{\omega_2}})\ldots)$$

$$\begin{array}{lllll}
A_1 y_1 & & & & = b_1 \\
-B_1 y_1 & +A_2 y_2^{\tilde{\omega_2}} & & & = b_2^{\tilde{\omega_2}} \\
 & -B_2 y_2^{\tilde{\omega_2}} & +A_3 y_3^{\tilde{\omega_3},\tilde{\omega_2}} & & = b_3^{\tilde{\omega_3}} \\
 & \ddots & \ddots & & \vdots \\
 & & -B_{|\mathbf{T}|-1} y_{|\mathbf{T}|-1}^{\tilde{\omega_{|\mathbf{T}|-1}},\ldots,\tilde{\omega_2}} & +A_{|\mathbf{T}|} y_{|\mathbf{T}|}^{\tilde{\omega_{|\mathbf{T}|}},\ldots,\tilde{\omega_2}} & = b_{|\mathbf{T}|}^{\tilde{\omega_{|\mathbf{T}|}}}
\end{array} \tag{48}$$

REFERENCES

[1] Dantzig, G. B., and Infanger, G. (1993). Multi-stage stochastic linear programs for portfolio optimization. *Annals of Operations Research*, 45(1), 59-76.

