TIØ4317 Empirical and Quantitative Methods in Finance
Exercise #6 – Spring 2016

Sandaker, Kristian
krisanda@stud.ntnu.no

March 3, 2016

Problem 1

a) In order to take a short position, the investor must place a certain amount of funds c in a specific collateral bank account that has the risk-free rate. The investor's short positions will have a total weighting of max. γc, with the parameter $\gamma \geq 1$. If the investor liquidates the short position, the collateral is freed and returned to the investor.

We employ **Model 3.2.4** Mean-variance efficient portfolios with trading size limits (Zenios, p. 80), and do relevant adjustments to find a proper optimization model for the problem.

Key figure	**Name**	**Set relationship**	**Description**
Sets:	$l \in L$		The set of scenarios, indexed with l
	$i \in N$		The set of risky assets, indexed with i
Parameters:	$C_0 \geq 0$		The minimum amount of accepted collateral
	r_f		The risk-free rate
	$\xi_{il} \in \mathbf{\Xi}$	$\forall (i,j) \in L \times N$	Return of asset i in scenario l
	$\gamma \geq 1$		The shorting coefficient
	$\sigma_{ij}^2 \in \mathbf{\Sigma}$	$\forall (i,j) \in N \times N$	The covariance matrix of assets
	$p_l : \mathbf{1}^T \mathbf{p} = 1$	$\forall l \in L$	The probability of scenario l
	μ		The target return for the portfolio
	λ		Objective function weighting of variance to return
Variables:	$x_i \in \mathbf{x}$	$\forall i \in N$	Weighting of asset i in the portfolio
	x_i^+, x_i^-		
	$x_i^+ - x_i^- = x_i$	$\forall i \in N$	Variables for modeling short positions
	c		Weighting on collateral
	$\zeta \in \mathbb{B}$		Decision variable of whether or not to use collateral

Now, following the formulation as in **Model 3.2.4**, we have the following optimization problem:

$$
\begin{aligned}
\underset{\{\mathbf{x},c\}}{\text{minimize}} \quad & \lambda \mathbf{x}^T \mathbf{\Sigma} \mathbf{x} - (1-\lambda)[\mathbf{p}^T \mathbf{\Xi}^T \mathbf{x} + c r_f] && && (1) \\
\text{subject to} \quad & \xi_1{}^T \mathbf{x} + c r_f \geq \mu && \forall l \in L && (2) \\
& x_i = x_i^+ - x_i^- && \forall i \in N && (3) \\
& \sum_{i \in N} x_i + c = 1 && && (4) \\
& c \leq \zeta && && (5) \\
& c \geq C_0 \zeta && && (6) \\
& \sum_{i \in N} x_i^- \leq \gamma c && && (7) \\
& x_i^+, x_i^- \geq 0 && \forall i \in N && (8) \\
& \zeta \in \{0,1\} \equiv \mathbb{B} && && (9)
\end{aligned}
$$

Finally, we also note the expected return weighting in the objective's similarity to Zenios' model:

$$
(1-\lambda)\mathbf{p}^T \mathbf{\Xi}^T \mathbf{x} = (1-\lambda) \underbrace{\sum_{l \in L} p_l \sum_{i \in N} \xi_{il} x_i}_{R(x;\tilde{r})} \tag{10}
$$

b) We now formulate the model in case of constant and proportional transaction costs. In particular, for every asset $i \in N$, we model two cost regimes – as by Zenios (p. 81): Let $x_i^{0\{+,-\}}$ denote transactions up to $\bar{x}_i$, and $x_i^{1\{+,-\}}$ transactions above this threshold. Thus the total allocation of the i'th asset will be $x_i = x_i^{0+} + x_i^{0-} + x_i^{1+} + x_i^{1-}$. Let $z_i \in \mathbb{B}$ be as follows:

$$
z_i = \begin{cases} 1 & \text{if asset } i \text{ is included in the portfolio with fixed charge } c_0 \\ 0 & \text{otherwise} \end{cases}
$$

Making the necessary adjustments, our model will be as follows:

$$
\begin{aligned}
\underset{\{\mathbf{x},c\}}{\text{minimize}} \quad & \lambda \mathbf{x}^T \mathbf{\Sigma} \mathbf{x} - (1-\lambda) \Bigg[\Bigg\{ \sum_{l \in L} p_l \sum_{i \in N} [\bar{\xi}_{il}(x_i^{0+} - x_i^{0-}) \\
& \quad - c_0 z_i + (\bar{\xi}_{il} - c_1)(x_i^{1+} - x_i^{1-})] \Bigg\} + c r_f \Bigg] && && (11) \\
\text{subject to} \quad & x_i^{0+} + x_i^{0-} + x_i^{1+} + x_i^{1-} = x_i && \forall i \in N && (12) \\
& \bar{x}_i z_i \geq x_i^{0+} \geq 0 && \forall i \in N && (13) \\
& \bar{x}_i z_i \geq x_i^{0-} \geq 0 && \forall i \in N && (14) \\
& z_i \geq x_i^{1+} \geq 0 && \forall i \in N && (15) \\
& z_i \geq x_i^{1-} \geq 0 && \forall i \in N && (16) \\
& \sum_{i \in N} x_i + c = 1 && && (17) \\
& c \leq \zeta && && (18) \\
& c \geq C_0 \zeta && && (19) \\
& \sum_{i \in N} (x_i^{0-} + x_i^{1-}) \leq \gamma c && && (20) \\
& x_i^{0+}, x_i^{0-}, x_i^{1+}, x_i^{1-} \geq 0 && \forall i \in N && (21) \\
& \zeta \in \{0,1\} \equiv \mathbb{B} && && (22) \\
& z_i \in \mathbb{B} && \forall i \in N && (23)
\end{aligned}
$$

Problem 2

Historic VaR of portfolios: We select to arbitrary assets A_1 and A_2 from the data set of index assets. Let P_1 be the portfolio consisting of 100% of A_1, and P_2 be the portfolio consisting 100% of A_2. We now consider the portfolio $\lambda P_1 + (1-\lambda)P_2$, and calculate and draw the dependence of 95% VaR of this particular portfolio.

Recall the generic Value-at-Risk (VaR) optimization problem:

$$\text{VaR}(x;\alpha) \triangleq \min\{\zeta \in \mathbb{R} \;:\; \Psi(x,\zeta) \geq \alpha\}, \tag{24}$$

where

$$\Psi(x,\zeta) \triangleq \sum_{\{l\in\Omega \,\mid\, L(x,P_l)\leq\zeta\}} p^l \tag{25}$$

$$L(x;P_l) \triangleq V_0 - V(x;P_l) \equiv V_0 - \mathbf{x}^T\mathbf{P}_l. \tag{26}$$

Now, to limit the set $\{l \in \Omega \mid L(x,P_l) \leq \zeta\}$ to allow only for a (sub)set of valid states s.t. $l \subseteq \Omega$, we introduce binary variables $z_l \in \mathbb{B},\ \ \forall l \in \Omega$. Consequently, the VaR problem takes shape as the following linear program:

$$\begin{aligned}
&\underset{\{\zeta,\mathbf{z}\}}{\text{minimize}} && \zeta && && (27)\\
&\text{subject to} && \sum_{l\in\Omega} p_l z_l \geq \alpha && && (28)\\
& && \zeta - Mz_l \geq V_0 - \sum_{i\in N} x_i - M && \forall l \in \Omega && (29)\\
& && M = V_0 + \sum_{i\in N} \min\{p_{il}\ \forall l \in \Omega\} && && (30)\\
& && x_1 = \lambda \in [0,1] && && (31)\\
& && x_2 = 1-\lambda \in [0,1] && && (32)\\
& && z_l \in \mathbb{B} && \forall l \in \Omega, && (33)
\end{aligned}$$

that is, holdings x_1, x_2, of assets A_1, A_2, respectively, are parameterized by λ. Note that $p_{il} \in \mathbf{P}$, i.e. the price matrix for each asset i given state l; whereas $p_l \in \mathbf{p}$ is the probability vector over states $l \in \Omega$.

We simulate using the attached **R** code as by Listing 1 Value-at-Risk simulation code in `R`. Computing log returns for the two assets, `MSDUUS.Index`, `MSEUSAG.Index`, respectively, we sample an $n = 100$ day vector of these log returns and scale them up with the initial portfolio value $V_0 = 1$. We do this $|\Omega| \equiv \mathtt{L} = 100$ times, this representing our scenario set. Then the above VaR model is carried through. This simulation is repeated 5 times, yielding a relative VaR ζ as a function of the weighting λ by the respective assets. The dependence line for each simulation is as visualized in Figure 1 Plot of sample simulations – Relative Value-at-Risk ζ versus asset weighting λ.

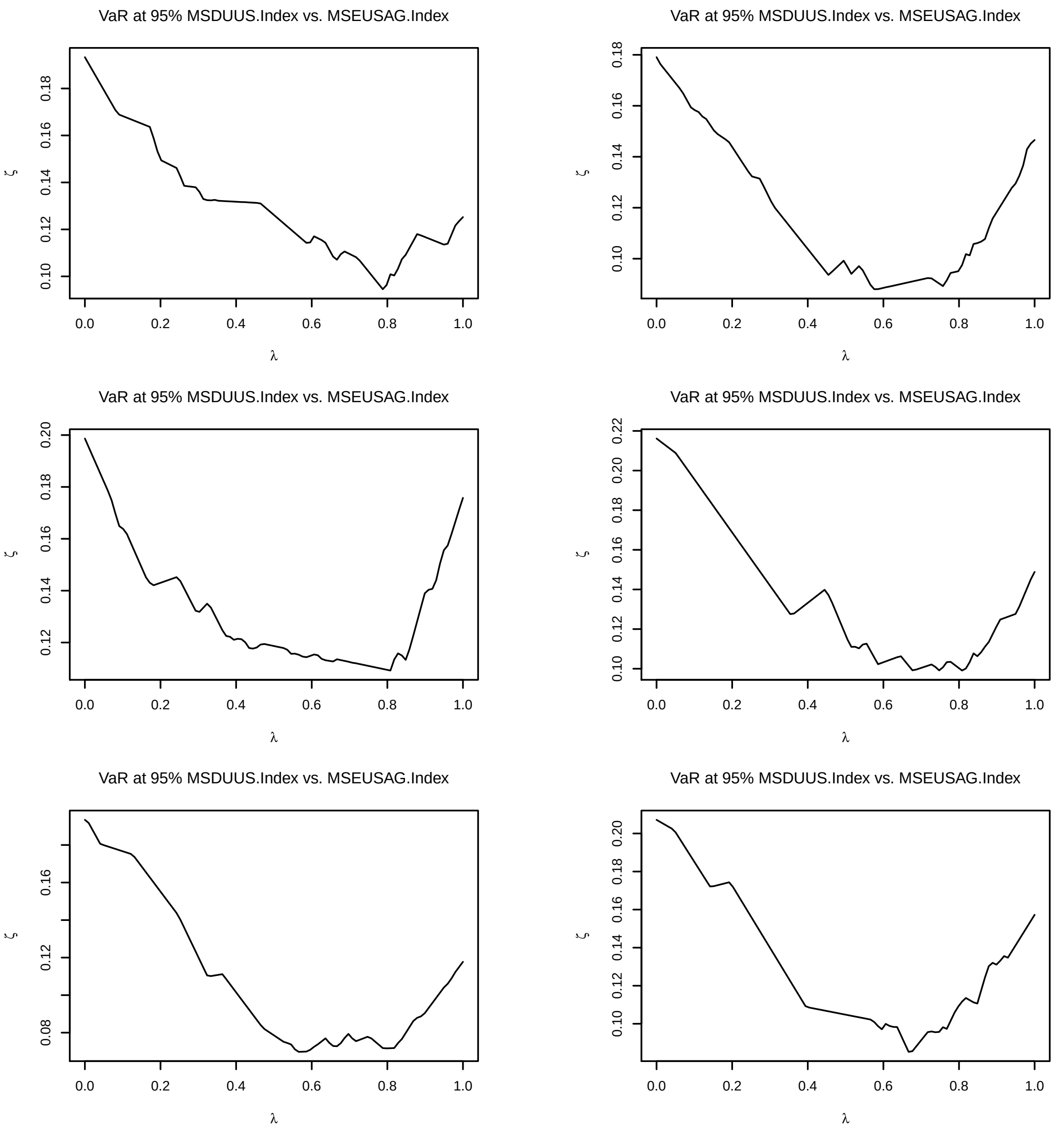

Figure 1: Plot of sample simulations – Relative Value-at-Risk ζ versus asset weighting λ

Listing 1: Value-at-Risk simulation code in R

```
library("Rsolnp")
library("lpSolve")
library("fitdistrplus")

# Set correct path in R
setwd("/Users/Kristian/Kristian/Dropbox/Studier/8. semester/TIO4317 Empirisk
    finans/Assignments/Assignment 6")

# Load the data set of stock and bond indices
df <- read.table("StocksBonds.csv", header=TRUE, sep=",")

### PARAMETRIC INPUT
assetNames <- c("MSDUUS.Index", "MSEUSAG.Index") # Assets to be compared
alpha <- 0.95 # Alpha level VaR
n_days <- 100 # Number of days to sample
L <- 100 # Number of scenarios
n_optimizations <- 6 # Number of optimizations and plots

### Model
```

```
P <- cbind(df[assetNames]) # Absolute price vector
r <- log(P[2:L,]) - log(P[1:(L-1),]) # Log returns
r[mapply(is.infinite, r)] <- 0 # Fix problematic values

# Do several optimizations
pdf(file = paste(getwd(), "/plots/var_plot_", n_optimizations, ".pdf", sep=""))
par(mfcol=c(n_optimizations/2,2)) # Plotting grid
par(mar=c(4,5,4,5))
par(cex=0.5)

for (k in 1:n_optimizations) {

  # Sample n-days log returns, L times, scale up
  R <- matrix(0, L, 2)
  for (i in 1:L) {
    r_1 <- sample(r[,1], n_days, replace = TRUE)
    r_2 <- sample(r[,2], n_days, replace = TRUE)
    R[i,] <- cbind(exp(sum(r_1)), exp(sum(r_2)))
  }


  # Lambda linspace, [0,1]
  Lambda <- seq(0,1, length = 100)

  # Set up the initial decision variable vector
  objective.in <- t(t(c(1, rep(0, L))))
  binary.vec <- t(seq(2,(L+1)))

  # Initialize V_0
  V_0 <- 1

  # Calculating the correct M
  M <- V_0 + V_0*sum(apply(R, 2, min))

  # Set up the coefficient matrix
  const.mat <- t(c(0, rep(1/L, L)))
  for (l in 1:L) {
    constrVec <- t(c(1, rep(0, L)))
    constrVec[l+1] <- -M
    const.mat <- rbind(const.mat, constrVec)
  }

  # Setting up the equality vector
  const.dir <- t(rep(">=", (L+1)))

  # Optimization loop for every lambda

  Zeta <- t(rep(0, length(Lambda)))

  for (i in 1:length(Lambda)) {
    lambda <- Lambda[i]
    # Set up the constant RHS
    const.rhs <- t(alpha)
    for (l in 1:L) {
      const.rhs <- rbind(const.rhs, t(V_0 - M
        - lambda*V_0*R[l,1] - (1-lambda)*V_0*R[l,2]))
    }
    zeta <- lp(direction = "min", objective.in = objective.in, const.mat =
        const.mat,
      const.dir = const.dir, const.rhs = const.rhs, binary.vec =
          binary.vec)$solution[1]
    Zeta[i] <- zeta
```

```
  }
  plot(c(Zeta) ~ c(Lambda), type = "l",
       main = bquote("VaR at "*.(alpha*100)*"% "*.(assetNames[1])*" vs.
            "*.(assetNames[2])),
       xlab = expression(lambda),
       ylab = expression(zeta))
  par(new=F)
}

dev.off()
```

References

[1] Alexei Gaivoronski, Georg Pflug (2004). *Value-at-Risk in Portfolio Optimization: Properties and Computational Approach*. Journal of Risk, Vol. 7, No. 2, pp. 1-31, Winter 2004-2005. From http://homepage.univie.ac.at/georg.pflug/science/technicalreports/GaivoronskiPflugNew.pdf.