Q. What is Numerical methods.

↳ A branch of mathematics in which numerical problems are sloved with the help of a machine (computer) and we get the solution in the form of Number.

In solving the equation $x^{10000} - 5x + 7 = 0$ ---- (1) in ordinary ways it may take a number of hours to solve equation (1) but by using computer, we can slove it within an hour. So we need numerical methods.

Error

↳ Let '$\bar{a}$' be approximate value of 'a' then error is defined as

Error = exact value - apporximate value

$$\boxed{e = a - \bar{a}}$$

* Classification of error

→ On the basis of quantity error is classified as below :

(i) Absolute error : The absolute value of error is called absolute error. i.e.

$$\text{absolute error} = |a - \bar{a}|$$

(ii) Relative error : Relative error is defined as

$$\text{relative error} = \frac{|a - \bar{a}|}{a}$$

(iii) Percentage of error : Percentage of error is defined as.

$$\text{percentage error} = \left|\frac{a - \bar{a}}{a}\right| \times 100\%$$

→ On the basis of source of error, error is classified as

(i) Truncation error: We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

If we take

$$e = 1 + \cancel{1} + \frac{1}{2!} + \frac{1}{3!}$$

$$\text{Truncation error} = \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots$$

Such type of error is called truncation error

(ii) Round of error: The error produce due to rounding of number is called round of error for example if we have

$$a = 23.41278$$

Rounding upto two decimal places

$$\bar{a} = 23.41$$

$$\text{Round of error} = 23.41278 - 23.41$$
$$= 0.00278$$

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(iii) Propagated error : The error arises as the process goes on is called propagated error.

(iv) Error in orignal data : The error arises due to input data is called error in orignal data.

(v) Blunder : The error due to human imperfection is called blunder.

Q Write short notes on error in numerical computing.

Q What are the use of numerical method in engineering study and explain different types of error.

Root of non-linear equation

let,

$$f(x) = 0 \quad ---- (1)$$

be an equation. Any number which satisfy (1) is called the root of equation.

(1) For example if we have

$$x^2 - 5x + 6 = 0 \quad ---- (ii)$$

here, $2^2 - 5 \times 2 + 6 = 0$

$0 = 0$

$\therefore$ 2 is a root of eq^n (ii)

$f(x) = x^2 - 5x + 6$
$f(2) = 0$
$f(3) = 0$
$\therefore x = 2, 3$ are the roots.

* Geometrical interpretation of roots

Let, $y = x^2 - 5x + 6$

x	0	2	2.5	3	4
y	6	0	-0.25	0	2

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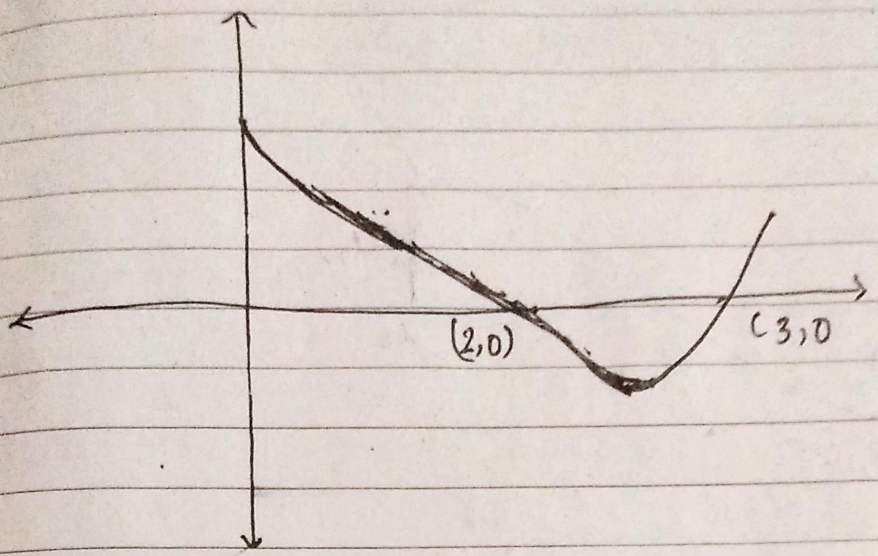


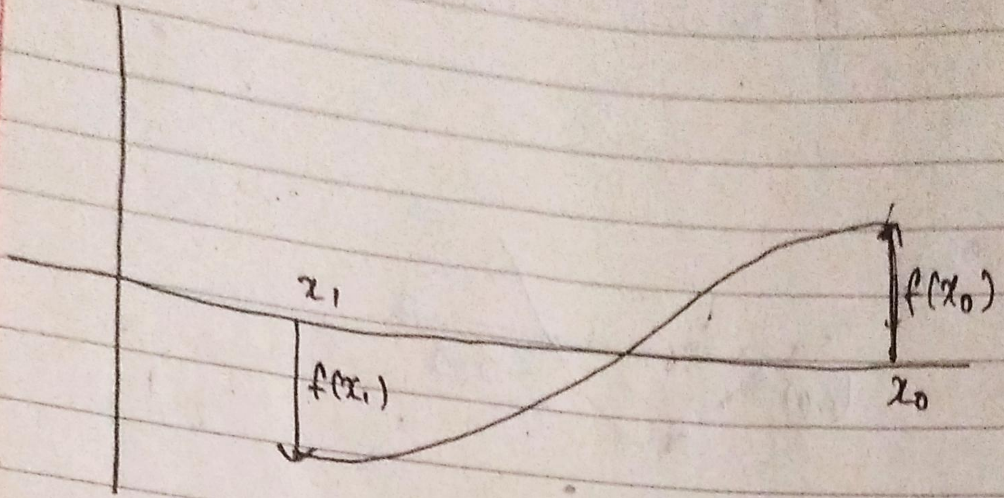
The points at which the function either cut or touches the axis are the root of the equation.

We can determine root of the equation by several methods.

* Bi-Section method

Let, $f(x)=0$ ----- ① be an equation

In this method we choose any two points x_0 and x_1 such that $f(x_0) \times f(x_1) < 0$



$x_2 = \frac{x_0 + x_1}{2}$ then find $f(x_2)$

The value of $f(x_2)$ may zero, +ve, -ve

if $(f(x_2)) = 0$

then x_2 is the root

If $f(x_2) * f(x_1) < 0$ then find $f(x_3)$

$x_3 = \frac{x_1 + x_2}{2}$

If $f(x_3) = 0$ then x_3 is the roots.

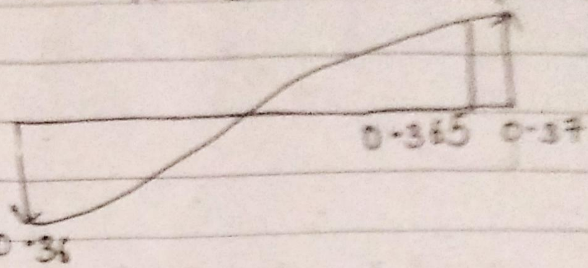
If $f(x_2) * f(x_3) < 0$ then $x_4 = \frac{x_2 + x_3}{2}$

find $f(x_4)$ if $f(x_4) = 0$ then x_4 is root.

Otherwise above process is repeated until we get the required roots. For example

Q. Solve, $3x + \sin x - e^x = 0$
by Bisection method correct to 2 decimal places

Soln



Let,
$f(x) = 3x + \sin x - e^x$

x	0	0.35	0.4	0.36	0.37
$f(x)$	-1	-0.026	0.097	-0.0010	0.023

Let, $x_0 = 0.36$, $x_1 = 0.37$
$f(x_0) = f(0.36) = -0.0010$
$f(x_1) = f(0.37) = 0.023$

here, $f(0.36) \times f(0.37) < 0$ $\qquad x_2 = \frac{0.36 + 0.37}{2} = 0.365$

$f(x)_2 = f(0.365) = 0.077$

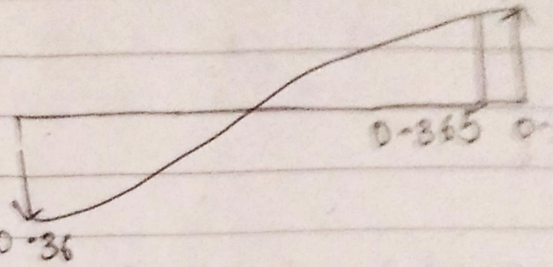
∴ Required root correct to two decimal place

$x = 0.36$

Q. Solve, $\log x - \cos x = 0$
by bisection method correct to 3 decimal places

Soln let
$f(x) = \log x - \cos x$

Q. Solve, $3x + \sin x - e^x = 0$
by Bisection method correct to 2 decimal places



Soln

Let,
$f(x) = 3x + \sin x - e^x$

x	0	0.35	0.4	0.36	0.37
$f(x)$	-1	-0.026	0.097	-0.0010	0.023

Let, $x_0 = 0.36$, $x_1 = 0.37$
$f(x_0) = f(0.36) = -0.0010$
$f(x_1) = f(0.37) = 0.023$

here, $f(0.36) \times f(0.37) < 0$ $\quad x_2 = \frac{0.36 + 0.37}{2} = 0.365$

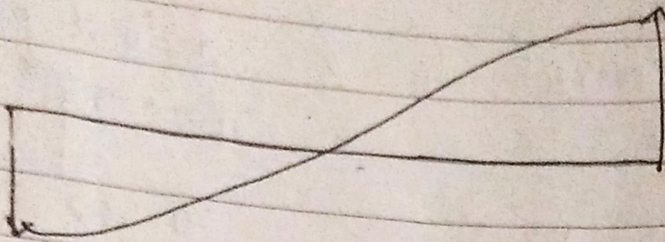
$f(x)_2 = f(0.365) = 0.011$

$\therefore$ Required root correct to two decimal place

$x = 0.36$

Q. Solve, $\log x - \cos x = 0$
by bisection method correct to 3 decimal place.

Soln let
$f(x) = \log x - \cos x$



let.

$x_0 = 1.3, \quad x_1 = 1.81$

$f_0 = -0.0051, \quad f_1 = 0.012$

$x_2 = 1.305 \quad f(x_2) = 0.0035$

$x_3 = 1.3025 \quad f(x_3) = -0.0008$

$x_4 = 1.30375 \quad f(x_4) = 0.002$

$x_5 = 1.30312 \quad f(x_5) = 0.00027$

$x_6 = 1.30281 \quad f(x_6) = 1.30281$

$\vdots$

$x_{10} = 1.30298 \quad f_{10} = 2.77 \times 10^{-5}$

$\therefore$ Required root correct to 3 decimal is

$x = 1.302$

↳ Limitations of Bisection method

- The determination of the roots by Bisection method is assured but the process is very slow. So, we need other methods.

* Secant Method

↳ In this method root is determined with the help of secants as it's name is.

let,

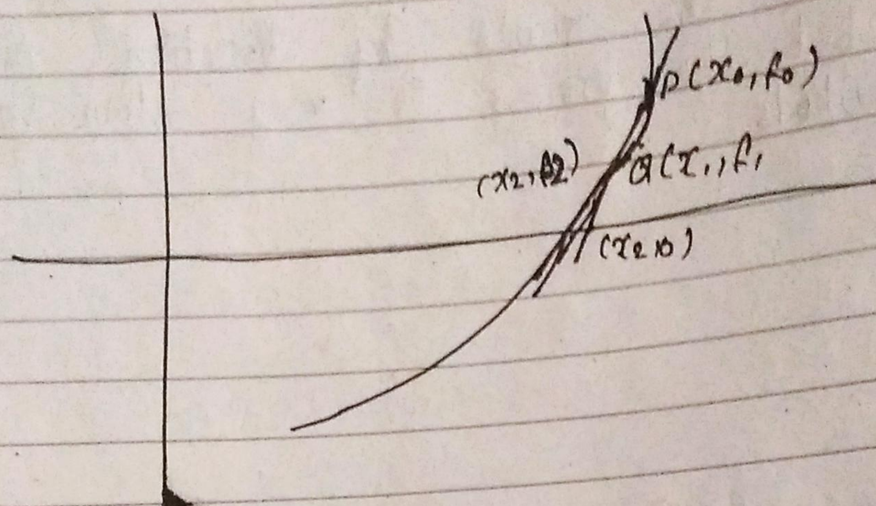
$f(x)=0$ ------ ①

be an equation. In this method we choose two points x_0 and x_1 near the roots as shown in fig.

A secant is drawn by joining (x_0, f_0) and (x_1, f_1) which has cut x-axis at $(x_2, 0)$

We know that,

Slope of RQ = Slope of QP



$$\frac{0 - f_1}{x_2 - x_1} = \frac{f_1 - f_0}{x_1 - x_0}$$

$$x_2 - x_1 = -\frac{f_1}{f_1 - f_0}(x_1 - x_0)$$

$$x_2 = x_1 - \frac{f_1}{f_1 - f_0}(x_1 - x_0)$$

Similarly, we can determine x_2, x_3 etc and hence the required root. The general formula for secant method is.

$$\boxed{x_{n+1} = x_n - \frac{f_n}{f_n - f_{n-1}}(x_n - x_{n-1})}$$

Q. Slove

(i) $3x + \sin x - e^x = 0$
(ii) $\log x - \cos x = 0$
(iii) $x^5 - 3x^2 - 100 = 0$
(iv) $x - 1.5\sin x - 2.5 = 0$
(v) $x\tan x - 1 = 0$
(vi) $x^4 + 3x^2 - 2x^2 - 12x - 8 = 0$

by secant method correct to 4 d.p.

(i) Soln

$3x + \sin x - e^x = 0$

$f(x) = 3x + \sin x - e^x$

x	0.5	0.3	0.36	0.37	0.365
$f(x)$	0.33	−0.1543	−0.001055	0.0238808	0.0114352

Let, $x_0 = 0.36$, $x_1 = 0.365$

$f_0 = -0.001055$, $f_1 = 0.011435$

we know that

$$x_2 = x_1 - \frac{f_1}{f_1 - f_0}(x_1 - x_0)$$

$$= 0.365 - \frac{0.011435}{0.011435 + 0.001055}(0.365 - 0.36)$$

$x_2 = 0.360423$

$f(x_2) = 0.0000003$

$$x_3 = x_2 - \frac{f_2}{f_2 - f_1}(x_2 - x_1)$$

Calculating we get,

$x_3 = 0.360422$

Therefore, required root correct to 4 dp is
$x = 0.3604$

(ii) $\log x - \cos x = 0$

let, $f(x) = \log x - \cos x$ ----- (i)

x	0.5	1.35	1.3	1.305	1.3025
$f(x)$	-1.57	0.081097	-0.005134	0.0035253	-0.0008036

now, we have

$x_0 = 1.305,\ x_1 = 1.3025$

$f_0 = 0.003525,\ f_1 = -0.000803$

$$x_{n+1} = x_n - \frac{f_n}{f_n - f_{n-1}}(x_n - x_{n-1})$$

$$x_2 = x_1 - \frac{f_1}{f_1 - f_0}(x_1 - x_0)$$

$$= 1.30296384$$

$$f_2 = -0.0000027919$$

$$x_3 = x_2 - \frac{f_2}{f_2 - f_1}(x_2 - x_1)$$

$$= 1.30296398$$

$\therefore$ Required root correct to 4 d.p is
$x = 1.3029$

(iii) soln $x^5 - 3x^2 - 100 = 0$

let, $f(x) = x^5 - 3x^2 - 100 = 0$

x	2.5	2.61	2.6075	2.6065
$f(x)$	-21.09	0.67998373	0.14016630	-0.07515030

$x_0 = 2.6075$, $x_1 = 2.6065$

$f_0 = 0.14016630$, $f_1 = -0.07515030$

We know that

$$x_{n+1} = x_n - \frac{f_n}{f_n - f_{n-1}}(x_n - x_{n-1})$$

$x_2 = 2.6068490$ $\qquad f(x_2) = f_2 = -4.43 \times 10^{-5}$

$x_3 = 2.6068492$

Therefore, required solution of the equation upto 4 decimal places

$x = 2.6068$

(iv) $x - 1.5\sin x - 2.5 = 0$

$f(x) = x - 1.5\sin x - 2.5$

x	3	2.85	2.859
$f(x)$	0.288319	-0.0812170	-0.05926960

$x_0 = 2.85, \quad x_1 = 2.859$

$f_0 = -0.0812170, \quad f_1 = -0.05926960$

We know that

$$x_{n+1} = x_n - \frac{f_n}{f_n - f_{n-1}}(x_n - x_{n-1})$$

$x_2 = 2.8833047 \qquad f_2 = 1.669 \times 10^{-4}$

$x_3 = 2.8832368 \qquad f_3 = -1.777835 \times 10^{-7}$

$x_4 = 2.88323687$

Therefore, upto 4 d.p $\quad x = 2.8832$

(i) $x\tan x - 1 = 0$

$f(x) = x\tan x - 1$

x	1	0.925	0.865	0.855
$f(x)$	0.55	0.2274538	0.014965598	-0.01685471

$x_0 = 0.855$, $x_1 = 0.865$

$f_0 = -0.01685471$, $f_1 = 0.01496559 8$

$$x_{n+1} = x_n - \frac{f_n}{f_n - f_{n-1}}(x_n - x_{n-1})$$

$x_2 = 0.86029684$ $\quad f_2 = -1.170397 \times 10^{-4}$

$x_3 = 0.8603333$ $\quad f_3 = -9.205301 \times 10^{-7}$

$x_4 = 0.8603335$

Therefore required soln of $f(x)$ upto 4 d.p

$x = 0.8603$

(vi) $x^4 + 3x^3 - 2x^2 - 12x - 8 = 0$

$f(x) = x^4 + 3x^3 - 2x^2 - 12x - 8 = 0$

x	1.9965	2.0005
$f(x)$	-0.16751047	0.024010

*

$x_0 = 1.9965$, $x_1 = 2.0005$

$f_0 = -0.16751047$, $f_1 = 0.024010$

$$x_{n+1} = x_n - \frac{f_n}{f_n - f_{n-1}} (x_n - x_{n-1})$$

$x_2 = 1.999998$ $\quad f_2 = -9.599984 \times 10^{-5}$

$x_3 = 1.9999999$

Therefore, soln of f(x) upto 4 decimal

$\alpha = 1.9999$

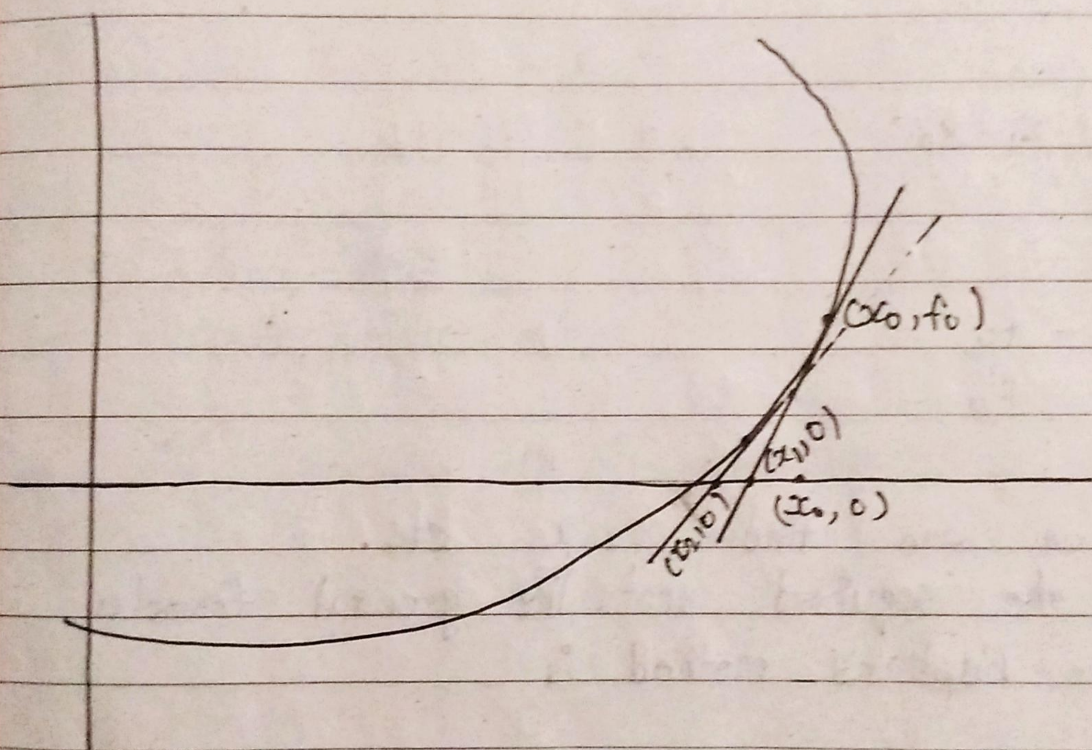
* Root of equation by Newton - Raphson method

Let,

$f(x) = 0$ ---- ①

be an equation

In this method root is determined with the help of tangents.



- We choose a point x_0 near the root of equation and a tangent is drawn from (x_0, f_0). Which has cut x-axis at $(x_1, 0)$

According to co-ordinate system,

slope of tangent AB is $m = \dfrac{0 - f_0}{x_1 - x_0}$ ---- ⑪

According to calculus slope of tangent at (x_0, f_0) is

$$m = f'(x_0)$$
$$m = f'_0 \quad \text{------} \quad (iii)$$

from (ii) and (iii)

$$\frac{-f_0}{x_1 - x_0} = f'_0$$

$$-\frac{f_0}{f'_0} = x_1 - x_0$$

$$x_1 = x_0 - \frac{f_0}{f'_0}$$

Similary, we can find x_2, x_3 etc and hence the required root. The general formula for Newton-Raphson's method is

$$\boxed{x_{n+1} = x_n - \frac{f_n}{f_n'}}$$

Q Solve

(i) $3x + \sin x - e^x = 0$
(ii) $\log x - \cos x = 0$
(iii) $x \tan x - 1 = 0$
(iv) $x^5 - 3x^2 - 100 = 0$
(v) $\sin x - 2x + 1 = 0$
(vi) $x \log_{10} x = 2$

by Newton-Raphson method correct four decimal places.

(i) soln

$3x + \sin x - e^x = 0$

Let, $f(x) = 3x + \sin x - e^x$ ----- (i)

diff. equation (i) w.r.t. x we get

$f'(x) = 3 + \cos x - e^x$ ----- (ii)

x	0.35	0.36
$f(x)$	-0.0261697	-0.00105518

So, $x_0 = 0.36$, $f_0 = -0.00105518$
$f'_0 = 2.5025674$

Also, we know

$$x_{n+1} = x_n - \frac{f_n}{f'_n} \quad \text{-----} \ (iii)$$

$$x_1 = 0.36 + \frac{0.00105518}{2.5025674}$$

$\cancel{x_1 = 0.359578}$ $x_1 = 0.360421$

$\cancel{f_1 = -0.00211142}$ $f_1 = -0.000001$

$\cancel{f'_1 = 2.503320}$ $f'_1 = 2.501815$

$$\cancel{x_2 = 0.359578 + \frac{0.00211142}{2.503320}}$$

$\cancel{x_2 = 0.358734}$ $x_2 = 0.360421$

$\cancel{f_2 = -0.0042248}$ $f_2 = 0.000001$

$\cancel{f'_2 = 2.504826}$ $f'_2 = 2.501815$

$$\cancel{x_3 = 0.358734 + \frac{0.0042248}{2.504826}}$$

Therefore, required soln of eqn 1

$\cancel{x_3 = 0.357047}$ is $x = 0.3604$

$\log_a x = \frac{1}{x} \log_a e$

$10^y = x$

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(vi) $x\log_{10} x = 2$

$f(x) = x\log_{10} x - 2$ ---- (1)

$f'(x) = \log_{10} x + \log_{10} e$

x	3.6
$f(x)$	0.002689

$x_0 = 3.6$, $f_0 = 0.002689$, $f'_0 = 0.990596$

$x_1 = 3.597285$, $f_1 = -0.00000002$, $f'_1 = 0.990269$

$x_2 = 3.597285$

∴ Solution of eqn (1) correct to 4 decimal places $\boxed{x = 3.5972}$

(v) $\sin x - 2x + 1 = 0$

$f(x) = \sin x - 2x + 1$ ---- (1)

diff eqn (1) we get

$f'(x) = \cos x - 2$

x	0.85	0.895
$f(x)$	0.0512804	-0.009790

$x_0 = 0.895$, $f_0 = -0.009790$, $f'_0 = -1.3744811$

$x_{n+1} = x_n - \frac{f_n}{f'_n}$

$x_1 = 0.887877$, $f_1 = -0.000020$, $f'_1 = -1.368939$

$x_2 = 0.887862$

Therefore, correct upto 4 d.P. $x = 0.8878$

(iv) $x^5 - 3x^2 - 100 = 0$

let, $f(x) = x^5 - 3x^2 - 100$

$f'(x) = 5x^4 - 6x$

x	2.5	2.6	2.6075
$f(x)$	−21.09375	−1.466	0.140166

$x_0 = 2.6075$, $f_0 = 0.140166$, $f'_0 = 215.490829$

we know that,

$$x_{n+1} = x_n - \frac{f_n}{f'_n}$$

$x_1 = 2.606849$, $f_1 = -0.000044$, $f'_1 = 215.263999$

$x_2 = 2.606849$

Therefore, soln of $f(x)$ correct to 4. d.p

$x = 2.6068$.

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(iii) $x\tan x - 1 = 0$

let, $f(x) = x\tan x - 1$

$f'(x) = x\sec^2 x + \tan x$

x	0.75	0.8625
$f(x)$	-0.30130	0.006922

$x_0 = 0.8625$, $f_0 = 0.006922$, $f'_0 = 3.2054732$

using newton Raphson's method

$x_1 = 0.860340$, $f_1 = 0.000020$, $f'_1 = 3.185073$

$x_2 = 0.860333$, $f_2 = 0.000001$, $f'_2 = 3.185007$

~~$x_3 = 0.860332$~~

Therefore, solution of $f(x)$ correct to 4 d.p

$x = 0.8603$

(ii) $\log x - \cos x = 0$

let, $f(x) = \log x - \cos x$

$f'(x) = \frac{1}{x} + \sin x$

x	1	1.3
$f(x)$	-0.54030	-0.005134

$x_0 = 1.3$, $f_0 = -0.005134$, $f'_0 = 1.732788$

$x_1 = 1.302962$, $f_1 = \cancel{0.010381}$ -0.000003, $f'_1 = 1.729880$

$x_2 = 1.302963$

$x = 1.3029$

* Limitation of Newton - Raphson's method

- If determination of derivative is difficult

- It is best method for the determination of the root but it has the following limitations:-

↳ if the function is difficult for finding derivative at that condition this method may not be useful for finding roots.

↳ if initial guessing is far from the roots as shown in figure at that condition root is not obtained by this method. So, we need other methods.

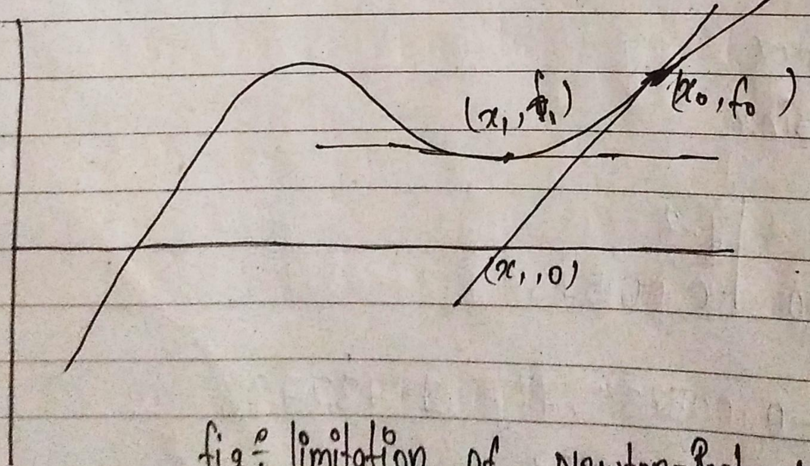

fig: limitation of Newton-Raphson's method

Algorithm for Newton-Raphson's method

1. Define $f(x)$
2. Read x_0 and e where x_0 = initial guessing
   e = allowed error
3. $f_0 = f(x_0)$
   $f_0' = f'(x_0)$

   $$x_1 = x_0 - \frac{f_0}{f_0'}$$

   if $|x_1 - x_0| < e$
   x_1 is the root
   else
   $x_0 = x_1$
   repeat step 3
4. Stop

* Fixed point of the function

A point 'p' is said to be fixed point of the function $f(x)$

if $f(p) = p$

For example,

(i) If $f(x) = x^2 - x + 1$
$f(1) = 1 - 1 + 1$
$f(1) = 1$

$\therefore x = 1$ is a fixed point of function $f(x) = x^2 - x + 1$

(ii) if $f(x) = \sqrt{5x - 6}$
$f(2) = \sqrt{5 \times 2 - 6} = 2$
$f(3) = \sqrt{5 \times 3 - 6} = 3$
$\therefore f(x) = \sqrt{5x - 6}$ has two fixed points $x = 2, 3$

(iii) if $f(x) = \sin x$
$f(0) = \sin 0$
$f(0) = 0$

here, $x = 0$ is fixed point.

(iv) if $f(x) = \cos x$

here, $f(x) = \cos x$ has no fixed point.

A function may have single fixed point, many fixed point or no fixed point

* Determination of root of the equation by fixed point iterative method

Let $f(x) = 0$ ----- (1)
be an equation.

In this method (1) is changed in the form of

$$x = g(x)$$

and set in iteration as

$$x_{n+1} = g(x_n)$$

and find fixed point of $g(x)$ which is root of equation (1).

For example

Solve, $x^2 - 5x + 6 = 0$

$x^2 = 5x - 6$

$x = \sqrt{5x-6}$

let, $g(x) = \sqrt{5x-6}$

$g(2) = \sqrt{5 \times 2 - 6}$

$= 2$

$g(3) = \sqrt{5 \times 3 - 6}$

$= 3$

Therefore, $x = 2, 3$.

Q Solve $2x = \cos x + 3$ by fixed point iterative method correct to 4. d.p.

let, $f(x) = 2x - \cos x - 3$

or, $x = \frac{1}{2}(\cos x + 3)$ ------ (1)

we have,

$x_{n+1} = g(x_n)$

x	1.5	1.525
$f(x)$	-0.0707	0.004219

let, $x_0 = 1.525$

$x_1 = g(x_0)$
$= 1.522890$

$x_2 = g(x_1)$
$= 1.523944$

$x_3 = g(x_2)$
$= 1.523417$

$x_4 = g(x_3)$
$= 1.523680$

$x_5 = g(x_4)$
$= 1.523549$

$x_6 = g(x_5)$
$= 1.523614$

$x_7 = g(x_6)$
$= 1.523582$

$x_8 = g(x_7)$
$= 1.523598$

Therefore, root of given function correct to 4 d.P.

$x = 1.5235$

Q. Solve

$$3x + \sin x - e^x = 0$$

by fixed point iterative method correct to 4 d.p.

let, $f(x) = 3x + \sin x - e^x$

$$3x = e^x - \sin x$$

$$x = \frac{e^x - \sin x}{3}$$

$$g(x) = \frac{e^x - \sin x}{3}$$

we have,

$$x_{n+1} = g(x_n)$$

x	0.4	0.3575
$f(x)$	0.09759	−0.007317

Let, $x_0 = 0.3575$

we know that,

$$x_1 = g(x_0)$$
$$= 0.359939$$

$$x_2 = g(x_1)$$
$$= 0.360341$$

$x_3 = g(x_2)$
$= 0.360408$

$x_4 = g(x_3)$
$= 0.360419$

Therefore, solution of $f(x)$ correct to 4 d.p.

$x = 0.3604$

Q. find square root of 7 by fixed point iterative method correct to 3. d.p.

Soln: let $x^2 = 7$ ---- (1) $x = ?$

$x = \frac{7}{x}$

$g(x) = \frac{7}{x}$

$x_{n+1} = g(x_n)$

let $x_0 = 2.62$
$x_1 = g(x_0)$
$= g(2.62)$
$= 2.671755$

$x_2 = g(x_1)$
$= g(2.671755)$
$= 2.670000$

From the expression of this $g(x)$, we can not determine roots. Rearranging (1)

$x^2 + x = 7 + x$
$x(x+1) = 7 + x$
$x = \frac{7+x}{x+1}$
$x = \frac{6}{x+1} + \frac{x+1}{x+1}$

$$x = 1 + \frac{6}{x+1}$$

$$g(x) = 1 + \frac{6}{x+1}$$

let $x_0 = 2.62$

$x_{n+1} = g(x_n)$

$x_1 = g(x_0)$
$= 2.637458$

$x_2 = g(x_1)$
$= 2.640483$

$x_3 = g(x_2)$
$= 2.648132$

$x_4 = g(x_3)$
$= 2.644677$

$x_5 = g(x_4)$
$= 2.646236$

$x_6 = g(x_5)$
$= 2.645532$

$x_7 = g(x_6)$
$= 2.64585$

$\therefore$ Required root correct to 3. d.p is
$x = 2.645$

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Q Solve $x^2 - x - 1 = 0$

ⓐ $x = x^2 - 1$

ⓑ $x = 1 + 2x - x^2$

ⓒ $x = \frac{1}{2}(1 + 3x - x^2)$

Starting with (I) $x_0 = 1$

(II) $x_0 = 2$

Discuss the result

↳ CASE I

a) $x = x^2 - 1$, $x_0 = 1$

$g(x) = x^2 - 1$

$x_{n+1} = g(x_n)$

$x_1 = g(x_0)$
$= 0$

$x_2 = g(x_1)$
$= -1$

$x_3 = g(x_2)$
$= 0$

In case I, we entered into infinite loop therefore we cannot determine root with this case.

↳ CASE II

a) $x = x^2 - 1$, $x_0 = 2$

$g(x) = x^2 - 1$

$x_{n+1} = g(x_n)$

$x_1 = g(x_0)$
$= 3$

$x_2 = g(x_1)$
$= 8$

$x_3 = (g(x_2))$
$= 63$

$x_4 = g(x_3) = 3968$

$x_5 = g(x_4) = 15745023$

$x_6 = g(x_5) = 2.4790 \times 10^{14}$

This function value is increasing exponentially therefore we can not obtain solution with case II.

↳ CASE III

b) $x = 1 + 2x - x^2$, $x_0 = 1$

$g(x) = 1 + 2x - x^2$
$x_{n+1} = g(x_n)$

$x_1 = g(x_0)$
$= 2$

$x_2 = g(x_1)$
$= 1$

$x_3 = g(x_2)$
$= 2$

In this case we entered into infinite loop so we cannot obtain solution in this case.

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↳ CASE IV

b) $x = 1+2x-x^2$, $x_{(0)} = 2$

$g(x) = 1+2x-x^2$

$x_{n+1} = g(x_n)$

$x_1 = g(x_0)$
$= 1$

$x_2 = g(x_1)$
$= 2$

In this case we entered into infinite loop so we cannot obtain solution in this case.

↳ CASE V

c) $x = \frac{1}{2}(1+3x-x^2)$, $x_0 = 1$

$g(x) = \frac{1}{2}(1+3x-x^2)$

$x_{n+1} = g(x_n)$

$x_1 = g(x_0)$
$= \frac{3}{2}$

$x_2 = g(x_1)$
$= \frac{13}{8}$
$= 1.625$

$x_3 = g(x_2)$
$= \frac{207}{128}$
$= 1.6171875$

$x_4 = 1.618133$, $x_5 = 1.618022$, $x_6 = 1.618035$

Correct to 4 d.p
$x = 1.6180$

→ CASE VI $\qquad x_0 = 2$

c) $x = \frac{1}{2}(1 + 3x - x^2)$

$g(x) = \frac{1}{2}(1 + 3x - x^2)$

$x_{n+1} = g(x_n)$

$x_1 = 1.5$, $x_2 = 1.625$, $x_3 = 1.6171875$,
$x_4 = 1.618133$, $x_5 = 1.618022$, $x_6 = 1.618035$

Correct to 4 d.p
$x = 1.6180$

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* Convergence of fixed point iterative method

[condition of getting root by fixed point iterative method.]

let $f(x) = 0$ ----- (1)

be an equation in fixed point iterative method (1) is changed in the form of

$$x = g(x) \quad ----- (II)$$

and set in iteration as

$$x_{n+1} = g(x_n) \quad ----- (III)$$

if 'P' be a fixed point of $g(x)$

$$g(P) = P \quad ------- (IV)$$

from (III) and (IV)

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$x_{n+1} - P = g(x_n) - g(P) \quad ---- (V)$$

By using lagrange mean value theorem on R.H.S. of (V)

$$x_{n+1} - P = (x_n - P)\, g'(c)$$

Taking modulus on both sides

$$|x_{n+1} - p| = |x_n - p| \, |g'(c)|$$

Where, c = constant

Now, in terms of error

$$e_{n+1} = e_n \cdot k$$

where, k = constant

or, $e_n = e_{n-1} \cdot k$ [Replacing n by $(n-1)$]

$$e_{n+1} = e_{n-1} k^2$$
$$e_{n+1} = e_{n-2} k^3$$
$$e_{n+1} = e_{n-3} k^4$$

$\vdots$ proceeding in the sameway

$$e_{n+1} = e_1 k^n$$

$$\boxed{e_{n+1} = e_0 k^{n+1}} \text{ ------- (vi)}$$

~~taking~~ taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty} e_0 k^{n+1}$$

$$\lim_{n\to\infty} e_{n+1} = 0 \quad \text{if } K<1$$

or $$\lim_{n\to\infty} |x_{n+1} - p| = 0 \quad \text{if } K<1$$

or $$\lim_{n\to\infty} (x_{n+1} - p) = 0 \quad \text{if } K<1$$

or $$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} p \quad \text{if } K<1$$

$$\boxed{\therefore \lim_{n\to\infty} x_{n+1} = p \quad \text{if } K<1} \text{ ---- (vii)}$$

Therefore, $\lim_{n\to\infty} x_{n+1} = p$ if $|g'(c)| < 1$

$= p$ if $|g'(x_0)| < 1$

Replacing c by x_0...

$\therefore$ Fixed point iterative method is convergent where $|g'(x_0)| < 1$

*

Q. Write short notes on convergence of fixed point iterative method.

→ A point P is said to be fixed point of $g(x)$ if

$$g(P) = P$$

...

Q. Write short notes on convergence of Newton - Raphson's method.

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Q. Write short notes on convergence of fixed point iterative method.

→ A point P is said to be fixed point of $g(x)$ if

$$g(P) = P$$

...

Q. Write short notes on convergence of Newton - Raphson's method.

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* Convergence of Newton Raphson's method

Let,

$f(x) = 0$ ----- (i)

be an equation.

by Newton-Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{----- (ii)}$$

Again, according to fixed point iterative method.

$$x_{n+1} = g(x_n) \quad \text{------ (iii)}$$

from (ii) & (iii)

$$g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

or, $g(x) = x - \dfrac{f(x)}{f'(x)}$ [Replacing x_n by x]

differentiating both sides w.r.t. x

$$g'(x) = 1 - \frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right)$$

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$$g'(x) = 1 - \left[\frac{f'(x) \times f'(x) - f(x) \cdot f''(x)}{\{f'(x)\}^2}\right]$$

$$g'(x) = \frac{f'(x)^2 - f'(x)^2 + f(x) f''(x)}{\{f'(x)\}^2}$$

$$g'(x) = \frac{f(x) f''(x)}{\{f'(x)\}^2}$$

$$\frac{f(x_0) f''(x_0)}{\{f'(x_0)\}^2} = g'(x_0)$$

taking modulus on both sides

$$\left|\frac{f(x_0) \; f''(x_0)}{\{f'(x_0)\}^2}\right| = |g'(x_0)|$$

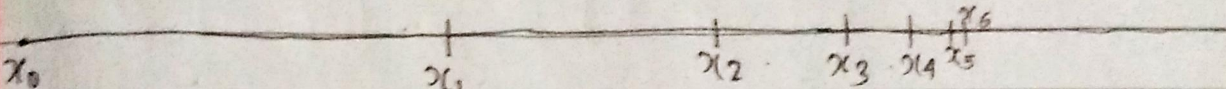
By fixed point iterative method. for convergence $|g'(x_0)| < 1$

$\therefore$ For convergence

$$\left|\frac{f(x_0) f''(x_0)}{\{f'(x_0)\}^2}\right| < 1$$

which is required condition for convergence of Newton Raphson's method.

It may be tedious to check the condition for the convergence of Newton Raphson method. So, for the convergence of iterative method we simply check the condition



$$|x_1 - x_0| > |x_2 - x_1|$$

$$|x_n - x_{n-1}| > |x_{n+1} \; x_n| \text{ ------- } (A)$$

If, condition (A) is true then convergence of any method is assured.

→ Any two method - in exam