Variance Reduction Techniques for Pricing American Options using Function Approximations

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Monte Carlo simulation techniques that use function approximations have been successfully applied to approximately price multi-dimensional American options. However, for many pricing problems the time required to get accurate estimates can still be prohibitive and this motivates the development of variance reduction techniques. In this paper, we describe a zero-variance or 'perfect' control variate and a zero variance or 'perfect' importance sampling distribution to price American options. We also observe the natural connection of the perfect control variate to additive duality and the perfect importance sampling distribution to multiplicative duality in American options. We then discuss how function approximations may be used to approximate the perfect control variate and the perfect importance sampling distribution. Empirically, we observe that both the techniques give significant variance reduction on examples of single and multi-dimension options.

1. Introduction

Accurate estimation of the price of an American option and the optimal exercise policy when the dimensionality of the underlying process is large remains an important problem in option pricing. Typically, this problem is simplified by restricting times at which the option can be exercised to a finite set of values so that it may be modeled as a discrete time Markov decision process (MDP), or more specifically, an optimal stopping problem in discrete time. Unfortunately, the well known numerical techniques for solving such MDP's suffer from a 'curse of dimensionality'. Recently, significant literature has developed that approximately solves the optimal stopping problem associated with the American option using Monte Carlo simulation (see, e.g., the overview in Chapter 8 of Glasserman 2004). These typically involve generating many sample paths (or more general sample trees) of the underlying assets until

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the expiry of the option and then using some form of backward induction to obtain an approximate solution. Carrière (1996), Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) propose regression based function approximations to aid in accurate and quick execution of this backward induction step. These methods involve approximating the 'continuation value' (the value of the option if not exercised) as a linear combination of certain easily evaluated and well-chosen basis functions.

In this paper, we develop efficient control variate and importance sampling based variance reduction techniques that build upon the regression based function approximation ideas to price American options. We first identify the form of zero variance or 'perfect' control variate in the American options setting. Here we use the results from Henderson and Glynn (2001) who show that for commonly used performance measures in Markovian settings, a martingale may be constructed that serves as a perfect control variate in the sense that the resultant performance measure estimator has zero variance. This martingale is typically unimplementable as it requires knowledge of measures that are at least as difficult to estimate as the performance measure of interest. In the American option setting, if our interest is in estimating the value of the option at a particular time at a particular value of the underlying assets, then (as we note later), a perfect control variate can be constructed that requires knowledge of the complete option price process, i.e., it requires knowledge of the price of option at each state of the underlying assets, at each future time period, up till option maturity.

From importance sampling viewpoint, we note that if the complete option price process is known, then by selecting the importance sampling probability density function (pdf) at any time and at any state to be proportional to the product of the original pdf at that time and state and appropriate option price, we get a zero-variance or perfect importance sampling measure.

Function approximation techniques proposed in Carrière (1996), Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) are useful as through regression based procedures they provide approximations to the option price process. These approximations are typically in the form of suitable linear combinations of selected basis functions. We note that for a well chosen set of basis functions these approximations can be used to develop easily implementable approximate perfect control variates and perfect importance sampling distributions. Empirically, we observe that the proposed variance reduction techniques provide significant variance reduction over naive simulation.

The additive duality in American options is well known (see Rogers 2002, Haugh and Kogan 2004, Andersen and Broadie 2004). Jamshidian (2003) and Bolia, Glasserman and Juneja (2004) propose multiplicative duality for American options (also see Chen and Glasserman 2007). In this paper, we make an interesting observation that the perfect control variate solves the additive duality problem and the perfect importance sampling estimator solves the multiplicative duality problem. Thus, the proposed approaches are also useful in deriving good upper bounds to option prices via duality results.

We view the pricing of an American option as a two-phase procedure: In the first phase an approximately optimal exercise policy is learned, while in the second phase, this policy is evaluated via the usual Monte-Carlo procedure. Our experiments suggest that with regression-based methods, the first phase is completed very quickly, even under naive simulation. Thus, our main focus is the second phase, i.e., development of variance reduction techniques for efficient evaluation of the approximately optimal exercise policy via Monte-Carlo simulation.

In Section 2, we develop the basic mathematical framework. We discuss the perfect control variate in Section 3. Here, we observe that it solves the well known additive duality problem. In Section 4, we discuss the perfect importance sampling distribution and show that it solves the multiplicative duality problem. In Section 5, we outline the proposed approximation methodology for the two techniques. Experimental results displaying the effectiveness of the proposed schemes are given in Section 6. Finally we conclude in Section 7 where we also discuss how our methodology specializes to some popular models. Significant portions of the analysis in this paper appeared in Bolia, Glasserman and Juneja (2004) and Bolia and Juneja (2005).

2.   MATHEMATICAL FRAMEWORK

We refer the reader to, e.g., Karatzas and Shreve (1998) and Duffie (1996) for a description of continuous time models used for asset price movements and for technical conditions under which the no-arbitrage condition implies the existence of a unique risk neutral measure so that price of a European option is an expectation of its discounted payoffs under the risk neutral measure. We assume that such conditions hold. Furthermore, we assume that the option under consideration can be exercised at a finite number of time points. In practice, options that provide fixed finite number of times at which they can be exercised are referred

to as Bermudan options. American options that expire at time T are well approximated by Bermudan options by dividing T into a large number of well spaced intervals. We assume that probability density functions are available to us so that we can generate samples of asset prices at finite number of time intervals where they can be exercised. In some cases, this may only be approximately true (see, e.g., Glasserman 2004), however, we do not dwell on this issue in the paper.

Specifically, suppose that the option can be exercised only at $N + 1$ times $0, 1, 2, ..., N$ (these times need not be integers or equally spaced, we do so for notational simplicity).

The underlying security prices at these times are modeled by a Markov process $(X_n \in \mathcal{X} : n \leq N)$, where $\mathcal{X}$ may be a multi-dimensional state space. The state description may include additional variables such as the value of stochastic interest rates and volatilities, and supplementary path dependent information, so that the resulting process $\{X_n\}$ is Markov. The value of the option at time n if exercised at that time, is denoted by $g_n : \mathcal{X} \to \Re^+$ (i.e., its exercise value or intrinsic value). Let $\mathcal{T}_n$ denote the set of stopping times taking values in $\{n, n + 1, \ldots, N\}$ (a random variable τ is a stopping time w.r.t. $\{X_n\}$ if the occurrence of $\{\tau = n\}$ can be ascertained by observing $(X_1, \ldots, X_n)$). Let

$$J_n(x) = \sup_{\tau \in \mathcal{T}_n} E[g_\tau(X_\tau)|X_n = x], \quad x \in \mathcal{X}, \tag{1}$$

where the expectation is taken under the risk neutral measure. Then $J_n(x)$ is the value of the option at time n given that the option is not exercised before time n. The initial state $X_0 = x_0$ is fixed and known. So, our pricing problem is to evaluate $J_0(x_0)$. Since no closed form solutions are available for the above except in the simplest cases, numerical and simulation methods are needed for pricing such multi-dimensional and multiple exercise opportunity options. This formulation is sufficiently general to include discounted payoffs through appropriate definition of the $\{X_n\}$ and $\{g_n\}$ (see Glasserman 2004, p.425), and hence these are not explicitly stated.

Further, we suppose that the pdf of X_{n+1} conditioned on $X_n = x$ evaluated at y is given by $f_n(x, y)$ under the risk-neutral measure. For any function $F : \mathcal{X} \to \Re$, define

$$E_n[F](x) = E[F(X_{n+1})|X_n = x] = \int_{\mathcal{X}} F(y) f_n(x, y) dy. \tag{2}$$

It is well known that the value functions $J = (J_n(x) : x \in \mathcal{X}, n \leq N)$ satisfy the following

backward recursions:

$$J_N(x) = g_N(x),$$
$$J_n(x) = \max(g_n(x), E_n[J_{n+1}](x)), \tag{3}$$

for $n = 0, 1, 2, ..., N - 1$, for each $x \in \mathcal{X}$. An alternative set of recursions based on the continuation value function $Q = (Q_n(x) : x \in \mathcal{X}, n \leq N - 1)$ may be developed, where

$$Q_n(x) = E_n[J_{n+1}](x). \tag{4}$$

These recursions are:

$$Q_{N-1}(x) = E_{N-1}[g_N](x),$$
$$Q_n(x) = E_n[\max(g_{n+1}, Q_{n+1})](x), \tag{5}$$

for $n = 0, 1, 2, ..., N - 2$. Note that $J_n(x) = \max(g_n(x), Q_n(x))$.

Let $\tau^* = \min\{n : g_n(X_n) = J_n(X_n)\}$. Then it is well known that τ^* is an optimal stopping time for our problem, i.e., $J_0(x_0) = E[g_{\tau^*}(X_{\tau^*})]$ (see, e.g., Duffie 1996).

As we observe in our experiments, this stopping time is easily and accurately estimated using functional approximations in the sense that if τ approximates τ^*, then $J_0(x_0) - E[g_\tau(X_\tau)]$ is positive, but typically small. Once an approximation τ to τ^* is known, $J_0(x_0)$ may be estimated by the usual Monte Carlo technique of generating independent samples of $g_\tau(X_\tau)$ and taking their average.

3. Perfect Control Variate

In this section we first identify a martingale corresponding to a perfect control variate for American options. We then show that it solves the additive duality problem. As mentioned in the introduction, to specify the perfect control variate we need to know the complete option price process. Next, we show that if a close approximation to the option price process exists and is appropriately used as a control variate, then the resultant simulation estimator has small variance.

3.1 Control Variate

Suppose that Y is a random variable that is a function of $(x_0, X_1, \ldots, X_\tau)$ whose mean equals zero. Then,

$$g_\tau(X_\tau) - Y \tag{6}$$

provides an unbiased estimator for $E[g_\tau(X_\tau)]$. In the control variate technique, one looks for such a Y that is highly positively correlated with $g_\tau(X_\tau)$ so that $g_\tau(X_\tau) - Y$ has a much smaller variance compared to $g_\tau(X_\tau)$. One way to arrive at such control variates is through the following martingale: For any collection of functions $(F_n : \mathcal{X} \to \Re, 1 \le n \le N)$ such that $E[|F_n(X_n)|] < \infty$ for all n, the process

$$M_n = \sum_{i=0}^{n-1} \Big(F_{i+1}(X_{i+1}) - E_i[F_{i+1}](X_i) \Big), \tag{7}$$

for $n = 1, \ldots, N$ is a zero mean martingale w.r.t. filteration $(\mathcal{F}_n : 1 \le n \le N)$, where $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ and $X_0 = x_0$. Since, $\tau \le N$, from the Martingale Stopping Time Theorem, $E[M_\tau] = 0$ (see, e.g., Williams 1991). Thus, M_τ may serve as a control variate for $g_\tau(X_\tau)$. Specializing the results from Henderson and Glynn (2001) to our setting, we note that the random variable $\tilde{M}_{\tau^*}$ associated with the zero mean martingale

$$\tilde{M}_n = \sum_{i=0}^{n-1} \Big(J_{i+1}(X_{i+1}) - E_i[J_{i+1}](X_i) \Big), \tag{8}$$

is the perfect control variate for estimating $J_0(x_0)$ in the sense that

$$g_{\tau^*}(X_{\tau^*}) - \tilde{M}_{\tau^*} = J_0(x_0) \tag{9}$$

a.s. This follows by simply observing that along the set $\{\tau^* > n\}$, we have $E_i[J_{i+1}](X_i) = Q_i(X_i) = J_i(X_i)$ for $i \le n$, so that

$$\tilde{M}_{\tau^*} = J_{\tau^*}(X_{\tau^*}) - J_0(x_0) = g_{\tau^*}(X_{\tau^*}) - J_0(x_0). \tag{10}$$

As mentioned in the introduction, in Section 5 we use regression based functional approximations of $J_n(\cdot)$ to approximate the martingale $(\tilde{M}_n : n \le N)$. First we observe the connection of this martingale with the well known additive duality in American options settings.

3.2   Additive Duality

Haugh and Kogan (2004) note that for any zero mean martingale $M = (M_n : 1 \le n \le N)$ w.r.t. the filteration $(\mathcal{F}_n : 1 \le n \le N)$,

$$\begin{aligned} J_0(x_0) &= E[g_{\tau^*}(X_{\tau^*})] & (11) \\ &= E[g_{\tau^*}(X_{\tau^*}) - M_{\tau^*}] & (12) \\ &\le E[\max_n (g_n(X_n) - M_n)]. & (13) \end{aligned}$$

Thus, average of independent samples of $\max_n(g_n(X_n) - M_n)$ provide an estimate of an upper bound to the option price. Furthermore,

$$J_0(x_0) \le \inf_M E[\max_n(g_n(X_n) - M_n)], \qquad (14)$$

where the infimum is over all zero mean martingales M. Haugh and Kogan (2004) also note that the inequality in (11) is tight if the martingale $(\tilde{M}_n : 1 \le n \le N)$ is considered in the upper bound. This can be seen by observing that $g_n(X_n) - \tilde{M}_n$ may be re-expressed as:

$$g_n(X_n) - J_n(X_n) + \sum_{i=0}^{n-1} (E_i[J_{i+1}](X_i) - J_i(X_i)) + J_0(x_0). \qquad (15)$$

This in turn is dominated by $J_0(x_0)$. To see this, note that $g_n(X_n) - J_n(X_n) \le 0$, and the well known fact that the process $(J_m(X_m) : 0 \le m \le N)$ is a super-martingale, so that for each i, $E_i[J_{i+1}](X_i) - J_i(X_i) \le 0$ (see, e.g., Duffie 1996).

This suggests that if we have an approximation $\hat{J}_n(\cdot)$ to the true option value $J_n(\cdot)$ (for $1 \le n \le N$), then average of independent samples of $\left[\max_n(g_n - \tilde{M}_n)\right]$, with

$$\hat{M}_n = \sum_{i=0}^{n-1} \left(\hat{J}_{i+1}(X_{i+1}) - E_i[\hat{J}_{i+1}](X_i)\right), \qquad (16)$$

can provide a good upper biased estimate of $J_0(x_0)$. We test this observation empirically in our experiments.

3.3 Characterizing Approximation Error

Consider an approximate value function process $\hat{J} = (\hat{J}_n(x) \ge 0, n \le N, x \in \mathcal{X})$ and the associated zero mean martingale $\hat{M} = (\hat{M}_n \in \Re : 1 \le n \le N)$, as in (16).

In the following proposition we note that when $\hat{J}$ is close to $J = (J_n(x) : n \le N, x \in \mathcal{X})$, the mean square error of the control variate estimator may not be very large. Let τ denote a stopping time. Then $g_\tau(X_\tau) - \hat{M}_\tau$ is an estimator of $J_0(x_0)$. Let

$$MSE = E\left(g_\tau(X_\tau) - \hat{M}_\tau - J_0(x_0)\right)^2$$

denote the associated mean square error.

Proposition 1. *Suppose that $E[J_n(X_n)]^2 < \infty$, $\forall n$, and there exists a positive $\epsilon > 0$ such that, for all n,*

$$E(\hat{J}_n(X_n) - J_n(X_n))^2 \le \epsilon^2. \qquad (17)$$

Then,

$$MSE \leq 4N^2\epsilon^2. \tag{18}$$

A sufficient condition for (17) is that the error

$$\sup_{n,x} |\hat{J}_n(x) - J_n(x)| \leq \epsilon. \tag{19}$$

Proof of Proposition 1   Without loss of generality, we consider the case $\tau > 0$ a.s. Throughout the proof, we suppress the use of X_n in expressing $J_n(X_n)$, $g_n(X_n)$, $E_n[\hat{J}_{n+1}](X_n)$ and $E_n[J_{n+1}](X_n)$ for ease of notation. Then by rearranging terms, $Y = g_\tau - \hat{M}_\tau$ can be re-expressed as

$$Y = g_\tau - \hat{J}_\tau + \sum_{k=1}^{\tau-1} (E_k[\hat{J}_{k+1}] - \hat{J}_k) + E_0[\hat{J}_1]. \tag{20}$$

Subtracting and adding $E_k[J_{k+1}]$ to each term in the sum in (20), and noting from (3) that $J_k \geq E_k[J_{k+1}]$ and $J_\tau \geq g_\tau$, we get

$$\begin{aligned} Y \leq \ & \sum_{k=0}^{\tau-1} (E_k[\hat{J}_{k+1}] - E_k[J_{k+1}]) \\ & + \sum_{k=1}^{\tau} (J_k - \hat{J}_k) + J_0(x_0). \end{aligned} \tag{21}$$

Then, $Y - J_0(x_0)$ may be further bound from above by

$$\sum_{k=0}^{N-1} |E_k[\hat{J}_{k+1}] - E_k[J_{k+1}]| + \sum_{k=1}^{N} |J_k - \hat{J}_k|. \tag{22}$$

To get the desired upper bound on MSE, $E(Y - J_0(x_0))^2$ note that since,

$$E(E_n[\hat{J}_{n+1}] - E_n[J_{n+1}])^2 \leq E(\hat{J}_{n+1} - J_{n+1})^2, \tag{23}$$

from (17) it follows that

$$E(E_n[\hat{J}_{n+1}] - E_n[J_{n+1}])^2 \leq \epsilon^2. \tag{24}$$

Now the desired upper bound follows from squaring (22) and applying Cauchy-Shwartz inequality to each term.

$\square$

4. Perfect Importance Sampling Measure

In this section we first introduce importance sampling in the American options setting and discuss the zero variance importance sampling probability measure. Then we show that this measure solves the multiplicative duality problem. Next we show that if a close approximation to the option price process exists and is appropriately used in importance sampling, then the resultant simulation estimator has small variance.

4.1 Importance Sampling

Let τ denote our best guess for τ^*, e.g., obtained via function approximation methods discussed later. For expository convenience assume that $\tau^* > 0$ a.s. Then, a naive estimate of $J_0(x_0) = Q_0(x_0)$ is obtained by taking an average of independent identically distributed samples of $g_\tau(X_\tau)$. Suppose, we generate these samples using the importance sampling pdf's $(\tilde{f}_n(x, \cdot) : x \in \mathcal{X}, n \leq N-1)$ such that $\tilde{f}_n(x, y) > 0$ whenever $f_n(x, y) > 0$ for each x, y and n. Let $\tilde{P}$ denote the resultant measure (let P denote the original measure). Let $\mathcal{P}$ denote the collection of all such probability measures. Then an importance sampling (IS) estimator of $J_0(x_0)$ is obtained by taking an average of independent, identically distributed samples of

$$\frac{f_0(x_0, X_1)}{\tilde{f}_0(x_0, X_1)} \frac{f_1(X_1, X_2)}{\tilde{f}_1(X_1, X_2)} \cdots \frac{f_{\tau-1}(X_{\tau-1}, X_\tau)}{\tilde{f}_{\tau-1}(X_{\tau-1}, X_\tau)} g_\tau(X_\tau). \tag{25}$$

(see, e.g., Juneja and Shahabuddin 2006).

Now suppose that the importance sampling distribution P^* corresponds to:

$$f_n^*(x, y) = \frac{f_n(x, y) J_{n+1}(y)}{Q_n(x)} \tag{26}$$

for each x, y and n so that $Q_n(x) > 0$. When, $Q_n(x) = 0$, we may set $f_n^*(x, y) = f_n(x, y)$. Equation (4) confirms the validity of $f_n^*(x, \cdot)$ as a pdf. Since $Q_n(X_n) = J_n(X_n)$ when $\tau^* > n$ and $J_{\tau^*}(X_{\tau^*}) = g_{\tau^*}(X_{\tau^*})$, it is easy to see that when $\tau = \tau^*$, P^* is a zero-variance measure as (25), with f_n^* replacing $\tilde{f}_n$, reduces to $Q_0(x_0) = J_0(x_0)$ a.s. (Such zero-variance measure are discussed, e.g., in Kollman et al. 1999; Ahamed, Borkar, and Juneja 2004).

4.2 Multiplicative Duality

Let $\tilde{L}_n$ denote the n step likelihood ratio of P w.r.t. $\tilde{P}$, i.e.,

$$\tilde{L}_n = \frac{f_0(x_0, X_1)}{\tilde{f}_0(x_0, X_1)} \frac{f_1(X_1, X_2)}{\tilde{f}_1(X_1, X_2)} \cdots \frac{f_{n-1}(X_{n-1}, X_n)}{\tilde{f}_{n-1}(X_{n-1}, X_n)}. \tag{27}$$

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(Define L_n^* similarly).

Let $\tilde{\tau}$ denote a stopping time. Then, $\tilde{L}_{\tilde{\tau}} g_{\tilde{\tau}}(X_{\tilde{\tau}})$ is upper bounded by

$$\max_{n \leq N} \tilde{L}_n g_n(X_n). \tag{28}$$

In particular,

$$E_{\tilde{P}} \tilde{L}_{\tilde{\tau}} g_{\tilde{\tau}}(X_{\tilde{\tau}}) \leq E_{\tilde{P}} \max_{n \leq N} \tilde{L}_n g_n(X_n). \tag{29}$$

It follows that

$$J_0(x_0) = sup_{\tilde{\tau}} E_{\tilde{P}} \tilde{L}_{\tilde{\tau}} g_{\tilde{\tau}}(X_{\tilde{\tau}}) \leq E_{\tilde{P}} \max_{n \leq N} \tilde{L}_n g_n(X_n). \tag{30}$$

Thus, the average of independent samples of (28) provides an unbiased estimator for an upper bound on $J_0(x_0)$. Recall that $\mathcal{P}$ denotes the collection of all valid importance sampling measures. It follows that

$$J_0(x_0) \leq \inf_{\tilde{P} \in \mathcal{P}} E_{\tilde{P}} \max_{n \leq N} \tilde{L}_n g_n(X_n). \tag{31}$$

The next proposition implies that P^*, solves the above dual problem for pricing American options.

Proposition 2. *Under P^*, with $\tau = \tau^*$,*

$$\max_{n \leq N} L_n^* g_n(X_n) = J_0(x_0),$$

i.e, the upper bound on the samples is constant and tight.

Proof   Note that

$$L_n^* g_n(X_n) = \frac{Q_0(x_0)}{J_1(X_1)} \frac{Q_1(X_1)}{J_2(X_2)} \cdots \frac{Q_{n-1}(X_{n-1})}{J_n(X_n)} g_n(X_n). \tag{32}$$

It follows that this is upper bounded by $Q_0(x_0) = J_0(x_0)$ since $J_i(x) = \max(g_i(x), Q_i(x))$. Thus, $\max_{n \leq N} L_n^* g_n(X_n) \leq J_0(x_0)$. The result follows as

$$\max_{n \leq N} L_n^* g_n(X_n) \geq L_{\tau^*}^* g_{\tau^*}(X_{\tau^*}) = J_0(x_0). \tag{33}$$

□

This result is an IS formulation of a multiplicative duality result in Jamshidian (2003). It suggests that if $\tilde{P}$ is close to P^*, then $\tilde{E}(\max_{n \leq N} \tilde{L}_n g_n(X_n))$ provides a close upper bound to $J_0(x_0)$ (where $\tilde{E}$ denotes the expectation operator under $\tilde{P}$). Also, note that

$$\tilde{E}(\max_{n \leq N} \tilde{L}_n g_n(X_n)) = E(\max_{n \leq N} \tilde{L}_n g_n(X_n) / \tilde{L}_N) \tag{34}$$

This provides two ways of estimating the upper bound associated with $\tilde{P}$: simulating under the IS distribution $\tilde{P}$ and using the expression on the left, or simulating under the original distribution and using the expression on the right. Jamshidian (2003) does the latter. Bolia et al. (2004) report that experiments using IS (with the approximate zero-variance distribution) to estimate the upper bound yield lower variance.

4.3 Characterizing Approximation Error

Consider $\tilde{J} = (\tilde{J}_n(x) > 0, n \le N - 1, \tilde{J}_N(x) \ge 0, x \in X)$. For each $n \le N - 1$ and $x \in \mathcal{X}$, set

$$\tilde{f}_n(x, y) = \frac{f_n(x, y)\tilde{J}_{n+1}(y)}{E_n[\tilde{J}_{n+1}](x)}, \quad y \in \mathcal{X}, \tag{35}$$

whenever $0 < E_n[\tilde{J}_{n+1}](x) < \infty$. Set $\tilde{f}_n(x, y) = f_n(x, y)$ otherwise, and denote the resultant probability measure by $\tilde{P}$. Let $\tilde{L}_n$ denote the corresponding likelihood ratio of P w.r.t. $\tilde{P}$ restricted to times 0 to n. Let τ denote a stopping time and let MSE denote the associated mean square error, i.e.,

$$MSE = \tilde{E}(\tilde{L}_\tau g_\tau(X_\tau) - J_0(x_0))^2.$$

Proposition 3. *Suppose there exists a positive $\epsilon < 1$ such that,*

$$(1 - \epsilon) \le \frac{\tilde{J}_n(x)}{J_n(x)} \le (1 + \epsilon) \tag{36}$$

for all n and x where $J_n(x) > 0$. Then,

$$\max_{n \le N} \tilde{L}_n g_n(X_n) \le J_0(x_0)\left(\frac{1+\epsilon}{1-\epsilon}\right)^N. \tag{37}$$

Furthermore,

$$MSE \le \left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^N - 1\right)^2 J_0(x_0)^2 = N^2 O(\epsilon^2). \tag{38}$$

    Proof   From (36), the fact that $Q_n(x) = E_n[J_{n+1}](x) \le J_n(x)$, it follows that

$$(1 - \epsilon)Q_n(x) \le E_n[\tilde{J}_{n+1}](x) \le (1 + \epsilon)J_n(x). \tag{39}$$

Note that

$$\tilde{L}_n g_n(X_n) = \tag{40}$$

$$\frac{E_0[\tilde{J}_1](x_0)}{\tilde{J}_1(X_1)} \frac{E_1[\tilde{J}_2](X_1)}{\tilde{J}_2(X_2)} \cdots \frac{E_{n-1}[\tilde{J}_n](X_{n-1})}{\tilde{J}_n(X_n)} g_n(X_n) \tag{41}$$

$$\le J_0(x_0)\left(\frac{1+\epsilon}{1-\epsilon}\right)^n. \tag{42}$$

From this, (37) follows. In particular,

$$\tilde{L}_\tau g_\tau(X_\tau) \leq J_0(x_0) \left(\frac{1+\epsilon}{1-\epsilon} \right)^N. \tag{43}$$

Therefore,

$$MSE \leq J_0(x_0)^2 \left(\left(\frac{1+\epsilon}{1-\epsilon} \right)^N - 1 \right)^2 \tag{44}$$

and the result follows. □

The above analysis suggests that if J_n can be closely approximated by $\tilde{J}_n$ a known quantity (e.g., a function of European options whose value is known), then the corresponding $\tilde{P}$ may provide a good proxy for the zero-variance measure and good simulation efficiency may be expected. However, there still remain issues concerning efficient generation of random variables from the distributions $\tilde{f}_n$. We address this in the next section.

5. Proposed Methodology

As mentioned in the introduction, we adopt a two-phase approach to pricing the option. In the first phase, we approximately estimate the optimal stopping policy. For this purpose we develop approximations $\hat{Q}$ for $Q = (Q_n(x), n \leq N-1, x \in \mathcal{X})$. Let $\phi_k : \mathcal{X} \rightarrow \Re$ for $1 \leq k \leq K$ denote a set of basis functions. Consider a parameterized value function $\hat{Q} : \mathcal{X} \times \Re^K \rightarrow \Re$ that assigns values $\hat{Q}(x, r)$ to state x, where $r = (r(1), \ldots, r(K)) \in \Re^K$ is a vector of free parameters and

$$\hat{Q}(x, r) = \sum_{k=1}^{K} \phi_k(x) r(k). \tag{45}$$

Using simulated paths, we find parameters $r_0^*, r_2^*, \ldots, r_{N-1}^*$ (each $\in \Re^K$) so that

$$\hat{Q}(x, r_n^*) \approx Q_n(x) \tag{46}$$

for each x and n. Then, at any state state x and time n we can compare $\hat{Q}(x, r_n^*)$ with $g_n(x)$ to determine whether to stop or to continue at that state so that we have a completely specified approximately optimal exercise policy. Tsitsiklis and Van Roy (2001) find that methods based on least squares regressions are better suited for estimating Q rather than $J = (J_n(x), n \leq N, x \in \mathcal{X})$, as the estimates of the former have better bias characteristics.

In the second phase we evaluate the approximately optimal exercise policy determined by $\hat{Q}$, using the variance reduction techniques involving approximating perfect control variate as

well as approximate perfect importance sampling measure. To implement these techniques we need to find the estimators $\hat{J}$ for J. Furthermore, we need an approximation $\hat{J}$ so that the integral $E_n[\hat{J}_{n+1}](x)$ can be computed easily. We now discuss how this is achieved using function approximations.

To compute the control variate, consider a parameterized value function $\hat{J} : \mathcal{X} \times \Re^K \to \Re$ that assigns values $\hat{J}(x, s)$ to state x, where again, $s = (s(1), \ldots, s(K)) \in \Re^K$, and

$$\hat{J}(x, s) = \sum_{k=1}^{K} \phi_k(x) s(k). \tag{47}$$

We choose each $\phi_k(\cdot)$ so that $E_n[\phi_k](x)$ can be explicitly evaluated. We estimate parameters $s_1^*, s_2^*, \ldots, s_N^*$ (each in $\Re^K$) simultaneously as we estimate $r_0^*, r_2^*, \ldots, r_{N-1}^*$ (the exact methodology is discussed later in Section 5.1) so that $\hat{J}(x, s_n^*) \approx J_n(x)$ for each x and n.

To conduct importance sampling, we again consider $\hat{J}$ as above. Here, the basis functions $(\phi_k : k \leq K)$ are selected to be non-negative and so that it is easy to generate samples from the probability density functions

$$\frac{f_n(x, y) \phi_k(y)}{E_n[\phi_k](x)}. \tag{48}$$

Unlike for the control variates, we restrict our search to non-negative parameters $s_1^*, s_2^*, \ldots, s_N^*$ in coming up with approximations $\hat{J}(x, s_n^*)$ to $J_n(x)$ for each x and n. As we discuss later, non-negativity of the estimated parameters facilitates generating samples from the resultant importance sampling probability measure although it may lead to poorer approximations of J and hence poorer performance of the importance sampling based algorithm.

Different basis functions can be used for $\hat{Q}$ and $\hat{J}$. These could further depend upon the time period n. The number of basis functions used could also be a function of n. However, to keep the notation simple we avoid these generalizations. We now discuss the procedure outlined above in detail.

5.1 Approximately Evaluating the Optimal Policy

As mentioned earlier, we follow a similar approach to that proposed in Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) to approximate the optimal stopping time τ^*. This involves generating L sample paths $(x_{m,n} : n \leq N, m \leq L)$ of the process $(X_n : n \leq N)$ using the densities $f_n(\cdot, \cdot)$.

The parameters $r_0^*, \ldots, r_{N-1}^*$ are found recursively:

$$r_{N-1}^* = \arg \min_{r \in \Re^K} \sum_{m=1}^L [g_N(x_{m,N}) - \sum_{k=1}^K \phi_k(x_{m,N-1}) r(k)]^2. \tag{49}$$

Here, $g_N(x_{m,N})$ is an unbiased sample of $Q_{N-1}(x_{m,N-1}) = E_{N-1}[g_N](x_{m,N-1})$. Then, using the approximation $\hat{Q}(\cdot, r_{N-1}^*)$ for $Q_{N-1}(\cdot)$ along each generated path m we can approximately evaluate when to exercise the option, given that we have not exercised it till time $N-2$. Call this time $\tau_{m,N-2}$.

Recursively, consider time n. Suppose that we know $\tau_{m,n}$, the time to exercise the option along path m, given that we have not exercised it till time n. Then, parameters r_n^* are found as a solution to the least squares problem:

$$r_n^* = \arg \min_{r \in \Re^K} \sum_{m=1}^L [g_{\tau_{m,n}}(x_{m,\tau_{m,n}}) - \sum_{k=1}^K \phi_k(x_{m,n}) r(k)]^2. \tag{50}$$

Note that if $\tau_{m,n}$ is a realization of the optimal stopping time along path m (given that the option has not been exercised till time n), then $g_{\tau_{m,n}}(x_{m,\tau_{m,n}})$ above is an unbiased sample of the continuation value $Q_n(x_{m,n})$ (and hence a reasonable proxy).

We modify this approach to determine the parameters $s_1^*, \ldots, s_N^*$. For control variate purposes we allow $s_1^*, \ldots, s_N^*$ to be unrestricted in sign. Set

$$s_N^* = \arg \min_{s \in \Re^K} \sum_{m=1}^L [g_N(x_{m,N}) - \sum_{k=1}^K \phi_k(x_{m,N}) s(k)]^2. \tag{51}$$

The parameters s_n^* for $n \leq N-1$ are found after parameters r_n^* have been determined. Knowing r_n^* allows us to determine whether to exercise the option at state $x_{m,n}$ or not by comparing $g_n(x_{m,n})$ and $\hat{Q}(x_{m,n}, r_n^*)$ for each m. Then, $\tau_{m,n-1}$ is known for each m. Set

$$s_n^* = \arg \min_s \sum_{m=1}^L [g_{\tau_{m,n-1}}(x_{m,\tau_{m,n-1}}) - \sum_{k=1}^K \phi_k(x_{m,n}) s(k)]^2. \tag{52}$$

The parameters r_n^* and s_n^* above are determined using the least squares method (see, e.g., Bertsekas and Tsitsiklis 1996). Again, if $\tau_{m,n-1}$ is a realization of the optimal stopping time (given that the option has not been exercised till time $n-1$), then $g_{\tau_{m,n-1}}(x_{m,\tau_{m,n}})$ above is an unbiased sample of $J_n(x_{m,n})$.

For importance sampling purposes, we restrict $s_1^*, \ldots, s_N^*$ to be non-negative and use non-negative least squares method to determine them (see, e.g., Lawson and Hanson 1974).

5.1.1 Second phase: Control variate

Once $(\hat{J}(x, s_n^*) : x \in \mathcal{X}, n \leq N)$ are known, we start the second phase of the algorithm using the control variate to evaluate the price of the option.

We define the martingale $\hat{M} = (\hat{M}_n \in \Re : 0 \leq n \leq N)$ as

$$\hat{M}_n = \sum_{i=0}^{n-1} \left(\hat{J}_{i+1}(X_{i+1}) - E_i[\hat{J}_{i+1}](X_i) \right). \tag{53}$$

As we discuss later in Section 6, the basis functions are selected so that terms $E_i[\hat{J}_{i+1}](X_i)$ can be explicitly computed in closed form.

We generate another set of independent paths of the underlying process that is also independent of the paths generated earlier to estimate the optimal policy. The martingale realization at time step n along the path m $(x_{m,n} : 1 \leq n \leq N)$ is then given by $\hat{M}_{m,n}$. The estimate of the option price is the average of $g_{\tau_{m,0}}(x_{\tau_{m,0}}) - \hat{M}_{m,\tau_{m,0}}$ over all m. The estimate of the upper bound on the option price is the average of $\max_n(g_n(x_{m,n}) - \hat{M}_{m,n})$ over all m. The stopping time $\tau_{m,0}$ along path m is found by comparing the $\hat{Q}_n$ approximated earlier with g_n at every time step n.

5.1.2 Second phase: Importance sampling

Once $(\hat{J}(x, s_n^*) : x \in \mathcal{X}, n \leq N)$ are known, we start the second phase of the algorithm involving importance sampling to evaluate the price of the option. The importance sampling probability densities are given by

$$\hat{f}_n(x, y) = \frac{f_n(x, y)\hat{J}(y, s_{n+1}^*)}{\int_{y \in \mathcal{X}} f_n(x, y)\hat{J}(y, s_{n+1}^*)dy}. \tag{54}$$

This may be re-expressed as

$$\hat{f}_n(x, y) = \sum_{k \leq K} p_k^*(x) \frac{f_n(x, y)\phi_k(y)}{E_n[\phi_k](x)}, \tag{55}$$

where

$$p_k^*(x) = \frac{s_{n+1}^*(k)E_n[\phi_k](x)}{\sum_{j \leq K} s_{n+1}^*(j)E_n[\phi_j](x)}. \tag{56}$$

Note that since non-negative least squares method is used to generate each $s_{n+1}^* = (s_{n+1}^*(k) : k \leq K)$, the associated $p_k^*(x) \geq 0$ and $\sum_{k \leq K} p_k^*(x) = 1$. Hence, if we can easily generate a sample from the pdf $\frac{f_n(x, \cdot)\phi_k(\cdot)}{E_n[\phi_k](x)}$, then this makes generation from $\hat{f}_n(x, \cdot)$ straightforward.

6. Numerical Results

We demonstrate the efficacy of the proposed control variate and importance sampling algorithms on three examples. We first consider a one dimensional option pricing problem. Here we price an American put option on a single underlying asset. Then we consider two multi-dimensional option pricing problem where we price three and five dimensional American max-call options.

In this section, any vector $x \in \Re^n$ is denoted by $(x(1), \ldots, x(n))$. Let $(X_t \in \Re^n : t \leq T)$ denote the asset price process of n underlying securities. We assume in all the examples that these securities follow geometric Brownian motion under the risk neutral measure. Then,

$$\frac{dX_t(j)}{X_t(j)} = (r - \delta(j))dt + \sigma(j)dW_t(j). \tag{57}$$

Here $W_t \in \Re^n$ is a standard Brownian motion process and the instantaneous correlation of $W_t(i)$ and $W_t(j)$ is ρ_{ij}. The parameters r, $\delta(i)$ and $\sigma(i)$ denote the risk free interest rate, dividend rate and volatility, respectively of the security i. The expiration time for the option is denoted by T and there are $N+1$ equally spaced discrete exercise opportunities time $\Delta t = T/N$ apart. For notational simplicity we use the subscript i to denote the time $i\Delta t$. Note that $(X_i : 0 \leq i \leq N)$ is a Markov process. Let $N(\mu, \Sigma)$ denote a n dimensional Gaussian random variable with mean $\mu \in \Re^n$ and a positive definite variance covariance matrix $\Sigma \in \Re^n \times \Re^n$. Observe that the element j of the vector X_i can be written as

$$X_i(j) = X_{i-1}(j) \exp[(r - \delta(j) - 0.5\sigma(j)^2) + \sqrt{\Delta t}N(0, \Sigma)(j)]. \tag{58}$$

Then, conditioned on $X_{i-1} = x$,

$$X_i = \exp[N(\hat{\mu}_i, \hat{\Sigma})] \tag{59}$$

where $\hat{\mu}_i(j) = (r - \delta(j) - 0.5 \ \sigma(j)^2) + \log x(j)$ for $1 \leq j \leq n$ and $\hat{\Sigma} = \sqrt{\Delta t}\Sigma$. Therefore, conditioned on $X_{i-1} = x$, X_i has a multi-variate lognormal distribution. Its pdf $f_i(x, y)$ has the form,

$$\frac{1}{(2\pi)^{n/2}|\hat{\Sigma}|^{1/2}y(1)\ldots y(n)} \ \exp(-1/2(\log y - \hat{\mu}_i)^t\hat{\Sigma}^{-1}(\log y - \hat{\mu}_i)) \tag{60}$$

where $(\cdot)^t$ represents the transpose of the given matrix or vector, and $|\cdot|$ denotes the determinant of the given matrix.

6.1 Choice of Basis Functions

As mentioned earlier, both for control variates and for importance sampling, we select basis functions so that each $E_i[\hat{J}_{i+1}](x)$ can be evaluated in closed form. For expository simplicity we present further analysis assuming same basis functions are used to obtain functional approximations for all exercise opportunities $0 \leq i \leq N$.

We use K basis functions of the form

$$\phi_k(y) = y(1)^{b_k(1)} \ldots y(n)^{b_k(n)} \exp(-0.5(\log y)^t \Sigma_k \log y), \tag{61}$$

where for all $k \leq K$, $\Sigma_k \in \Re^n \times \Re^n$ is symmetric and positive semi-definite, and $b_k \in \Re^n$. Observe that $\phi_k(y)$ may be re-expressed as

$$\exp[b_k^t \log y - 0.5(\log y)^t \Sigma_k \log y]. \tag{62}$$

Consider the expression,

$$\phi_k(y)f(x,y) = \frac{\exp[b_k^t \log y - 0.5(\log y)^t \Sigma_k \log y - 0.5(\log y - \hat{\mu})^t \hat{\Sigma}^{-1}(\log y - \hat{\mu})]}{(2\pi)^{n/2} |\hat{\Sigma}|^{1/2} (y(1) \ldots y(n))} \tag{63}$$

After algebraic manipulations we obtain

$$\phi_k(y)f(x,y) = \frac{|\tilde{\Sigma}|^{1/2}}{|\hat{\Sigma}|^{1/2}} \exp[0.5(\tilde{\mu})^t \tilde{\Sigma}^{-1} \tilde{\mu} - 0.5(\hat{\mu})^t \hat{\Sigma}^{-1} \hat{\mu}] \, D_{(\tilde{\mu},\tilde{\Sigma})}^{(k)}(y), \tag{64}$$

where $\tilde{\Sigma} = (\hat{\Sigma}^{-1} + \Sigma_k)^{-1}$, $\tilde{\mu} = (b_k^t + \hat{\mu}^t \hat{\Sigma}^{-1})\tilde{\Sigma}$ and $D_{(\tilde{\mu},\tilde{\Sigma})}^{(k)}(y)$ is the pdf of a $\exp[N(\tilde{\mu}, \tilde{\Sigma})]$ distributed random variable. To see this, observe that $\hat{\Sigma}^{-1}$ is positive definite and Σ_k is positive semi-definite. Hence, $\tilde{\Sigma}$ is positive definite. So,

$$\int \phi_k(y)f(x,y)dy = \frac{|\tilde{\Sigma}|^{1/2}}{|\hat{\Sigma}|^{1/2}} \exp[0.5(\tilde{\mu})^t \tilde{\Sigma}^{-1} \tilde{\mu} - 0.5(\hat{\mu})^t \hat{\Sigma}^{-1} \hat{\mu}]. \tag{65}$$

Recall that using a least squares regression we estimate $J_{i+1}(y) = \sum_{k \leq K} s_{i+1}^*(k)\phi_k(y)$. Let $\beta_i^*(k) = s_{i+1}^*(k)\frac{|\tilde{\Sigma}|^{1/2}}{|\hat{\Sigma}|^{1/2}} \exp[0.5(\tilde{\mu})^t \tilde{\Sigma}^{-1} \tilde{\mu} - 0.5(\hat{\mu})^t \hat{\Sigma}^{-1} \hat{\mu}]$. Closed form expressions for $E_i[\hat{J}_{i+1}](\cdot)$ can be expressed as,

$$E_i[\hat{J}_{i+1}](x) = \sum_{k \leq K} \beta_i^*(k). \tag{66}$$

Hence, we obtain an easily implementable control variate. Note that the basis functions may be set to sums or differences of terms as in RHS of (61), and even then closed form expressions for $E_i[\hat{J}_{i+1}](\cdot)$ are easily determined.

For importance sampling, set $p_i^*(k) = \frac{\beta_i^*(k)}{\sum_{k \leq K} \beta_i^*(k)}$. Then our importance sampling pdf's have the form

$$f_i^*(x, y) = \sum_{k \leq K} p_i^*(k) D_{\tilde{\mu}, \hat{\Sigma}}^{(k)}(y), \tag{67}$$

i.e., it is a non-negative mixture of multi-variate lognormal distributions. Hence, it is easy to generate samples using these density functions and easy to compute the associated likelihood ratio in a closed form.

In the next subsection we detail specific choice of parameters (b_k, Σ_k) for a one-dimension American put, followed by a section containing results for multi-dimensional American max-call.

6.2 Single Asset American Put Option

The payoff for a one dimensional American put is $g_i(X_i) = (K - X_i)^+$ where K denotes the strike price. Specific parameters for the underlying asset considered are $r = 6\%$, $\delta = 0$, $T = 1$, $K = \$40$ and volatility $\sigma = 0.2$. We report the results for different values of N and X_0. We use seven basis functions of the form (61). We set $\Sigma_k = [2]$ for all seven basis functions. $b_1 = [-3]$, $b_2 = [-2]$, $b_3 = [-1]$, $b_4 = [0]$, $b_5 = [1]$, $b_6 = [2]$, $b_7 = [3]$. Same basis functions are used for approximating the continuation value coefficients and the option value coefficients.

Variance reduction factor (VR) is calculated as the ratio of the estimated variance of the naive estimator and the estimated variance of the estimator using the variance reduction technique (control variate or importance sampling). Generating a path with a variance reduction technique takes more time than the time taken in generating a naive path. To account for this the VR is multiplied by the ratio of average path generation time for naive simulation and the average path generation time using the variance reduction technique to get the overall computation reduction factor (CR). This indicates the overall computational benefit of a variance reduction technique compared to naive simulation.

6.2.1 Control variate implementation

We generated 30,000 paths to estimate r^* and s^* for implementing control variate using ordinary least squares. It was observed in the experiments (not reported), that when the put was in the money and the number of exercise times were 20 or less, even 10,000 paths sufficed to give good estimates of r^* and s^* in the sense that having more paths did not

further improve the option value. When the put was out of money or the number of exercise times were 50, we needed about 30,000 paths to achieve stability in the option value. In the second phase, 100,000 paths are used to estimate the option value and upper bound so as to obtain a confidence interval that is approximately within 1% to 5% of the estimated value.

Results of simulations are displayed in Table 1. We observe that large amount of variance reduction is achieved using the control variate technique. The estimated upper bound is also quite close to the option value. This variance reduction is less significant when the option is out of money (when $X_0 = \$50$) and the upper bound is also not very close to the option value in this setting. This may be because the quality of approximations of the value functions (and hence of the control variate) is not very good when the option is out of money. Note that most paths generated to estimate the least square parameters with $X_0 = \$50$ are out of money. On close examination of these paths we found that approximations to the options price process are particularly inaccurate in these regions and they contribute the most to the increased variance of the estimate.

Table 1: Point Estimates and 95% CI half widths using naive simulation and control variate (CV). Here VR denotes the ratio of the estimated variance under naive simulation and under the control variate technique; CR denotes the computational reduction factor and equals VR times the ratio of the average time to generate a sample path under naive simulation and under the control variate technique.

$X_0 = 36$					
N	Naive Est.	CV Est.	VR	CR	UB
10	4.440(0.018)	4.437(0.001)	174.4	69.7	4.502(0.002)
20	4.472(0.017)	4.462(0.001)	166.5	66.6	4.552(0.010)
50	4.478(0.018)	4.475(0.001)	179.7	69.1	4.561(0.007)

$X_0 = 40$					
N	Naive Est.	CV Est.	VR	CR	UB
10	2.287(0.017)	2.288(0.002)	94.9	39.7	2.385(0.002)
20	2.307(0.017)	2.301(0.001)	97.8	32.6	2.432(0.002)
50	2.310(0.016)	2.310(0.001)	92.9	41.3	2.437(0.001)

$X_0 = 50$					
N	Naive Est.	CV Est.	VR	CR	UB
10	0.324(0.007)	0.321(0.000)	49.9	29.9	0.386(0.003)
20	0.322(0.007)	0.323(0.001)	43.8	24.6	0.389(0.003)
50	0.326(0.006)	0.323(0.001)	30.2	14.0	0.401(0.005)

6.2.2   Importance sampling implementation

For importance sampling, 10,000 trials of non negative least squares were used to estimate s^* (any further increase did not improve the quality of estimated parameters). In each case that we considered only one or two of the $s^*(n)$ assume a positive value; thus only one or two basis functions are used for conducting importance sampling. This suggests that that better implementable approximations of the option price process may lead to improved performance. The results are displayed in Table 2. Here, the VR is less than in the control variate case but it still provides significant improvement over naive simulation. This improves when the option is out of money. This may be because under the importance sampling measure, more in the money paths are generated that provide useful estimation information.

Table 2: Point Estimates and 95% CI half widths using naive simulation and importance sampling. Here VR denotes the ratio of the estimated variance under naive simulation and under the importance sampling technique; CR denotes the computational reduction factor and equals VR times the ratio of the average time to generate a sample path under naive simulation and under the importance sampling technique.

$X_0 = 36$					
N	Naive Est.	IS Est.	VR	CR	UB
10	4.438(0.018)	4.438(0.008)	5.4	2.1	4.945(0.004)
20	4.472(0.017)	4.467(0.007)	5.5	2.1	5.027(0.003)
50	4.478(0.018)	4.476(0.007)	5.4	2.4	5.067(0.003)

$X_0 = 40$					
N	Naive Est.	IS Est.	VR	CR	UB
10	2.287(0.017)	2.290(0.006)	6.5	4.8	2.662(0.005)
20	2.306(0.017)	2.302(0.006)	7.0	3.1	2.726(0.004)
50	2.319(0.016)	2.310(0.006)	6.8	3.3	2.777(0.004)

$X_0 = 50$					
N	Naive Est.	IS Est.	VR	CR	UB
10	0.325(0.007)	0.322(0.001)	16.5	6.6	0.373(0.001)
20	0.322(0.006)	0.323(0.001)	16.9	7.5	0.386(0.001)
50	0.329(0.006)	0.323(0.001)	17.5	9.6	0.399(0.001)

For this example, we also conducted experiments (not reported) where we learnt the coefficients of the basis functions using starting points different from X_0. This however, resulted in slight decline in the value of the policy evaluated and in the performance of our variance reduction techniques.

6.3 Multi-asset Call Options

We ran experiments on a Bermudan max call option on three and five securities. The payoff of the call option then is $g_i(X_i) = (\max(X_i(1), X_i(2), \ldots, X_i(n)) - K)^+$. The set of parameters and basis functions to estimate the exercise policy were same as those used in Andersen and Broadie (2004). Specifically, each asset price process is assumed to have an identical distribution independent of the other asset price processes. The associated parameter values considered are $r = 5\%$, $\delta = 10\%$, $\rho = 0$, and $\sigma = 0.2$. We take $T = 3$, $N = 9$ and $K = 100$. For estimating the continuation value functions we used 13 basis functions. These consisted of the largest and second largest asset prices, three polynomials of degree two (e.g., the squares of largest and second largest asset price and the square of the two), four polynomials of degree three, the value of a European max call option on the largest two assets, and the square and the cube of this value.

6.3.1 Control variate implementation

Recall that for approximating the perfect control variate we need to approximate the option price process using basis functions $(\phi_k : k \leq K)$ such that the expected value $E_i[\phi_k](x)$ has a closed form solution for each i, x and k. With this in mind, for ease of implementation we chose a set of polynomials as the set of basis functions to approximate the options price process. In particular, we used the first five hermite polynomials in each security value, pairwise product of each security, and product of all securities. Note that the first five hermite polynomials as a function of x are: $1, 2x, 4x^2 - 2, 8x^3 - 12x, 16x^4 - 48x^2 + 12$.

We used 30,000 paths to estimate r^* and s^* using ordinary least squares. In the second phase 200,000 paths were used to estimate the option value and the upper bound.

Results for initial stock prices (all assets are fixed at same initial value in our experiments) $X_0(i) = 90$, $X_0(i) = 100$ and $X_0(i) = 110$ for $i \leq n$, are listed in Table 3. The VR and the CR appear to decrease with the number of dimensions n. This may be because its difficult to identify basis functions that provide good approximation as the dimensionality of the space increases. Improving the quality of approximations in large dimensions to improve the performance of the control variate technique is an area that needs further research.

It is also noteworthy that the average time taken for simulations with the control variate technique in these examples was 1.2 to 1.3 times the average time for naive simulation while this was more than twice the naive simulation time in the one dimensional case. This is

Table 3: Point Estimates and 95% CI half widths using naive simulation and control variate (CV). Here VR denotes the ratio of the estimated variance under naive simulation and under the control variate technique; CR denotes the computational reduction factor and equals VR times the ratio of the average time to generate a sample path under naive simulation and under the control variate technique.

X_0	Naive Est.	CV Est.	VR	CR	UB
		$n = 3$			
90	11.274(0.061)	11.269(0.012)	24.8	19.8	12.102(0.010)
100	18.672(0.074)	18.665(0.014)	25.6	20.8	19.754(0.012)
110	27.531(0.086)	27.526(0.016)	27.3	21.5	28.831(0.013)
		$n = 5$			
X_0	Naive Est.	CV Est.	VR	CR	UB
90	16.601(0.071)	16.623(0.017)	16.1	11.7	17.910(0.016)
100	26.089(0.085)	26.125(0.021)	15.9	11.3	27.853(0.018)
110	36.666(0.096)	36.738(0.023)	16.2	11.2	38.824(0.020)

because the amount of effort in evaluating the control variate as a percentage of total effort goes down as the problem dimension increases.

Since these examples were also considered by Andersen and Broadie (2004) some comments are in order. Note that for policy evaluation in phase 1, we use identical basis functions as used by Andersen and Broadie (2004). Our naive estimates are comparable to their's once adjustments are made for the fact that they use about ten times more effort in the first as well as in the second phase. In the second phase, we use basis functions that allow explicit computation of $E_i[\hat{J}_{i+1}](x)$ for each i and x. This has the advantage of significant variance reduction at a relatively little computational overhead. As mentioned earlier, the computational effort increased by $20 - 30\%$ compared to naive simulation. In contrast, Andersen and Broadie (2004) report that the overall computational effort increased by 2.5 to 20 times in determining upper bounds. On the other hand, the upper bounds reported by them are far closer to the lower bounds than the ones that we report. This motivates further work to improve the quality of the upper bounds while controlling the implementation overheads (see also Broadie and Cao 2007).

6.3.2 Importance sampling implementation

The non-negative basis functions to implement importance sampling were determined as follows: We considered the set of all basis functions of the form

$$y(1)^{b_1} y(2)^{b_2} \ldots y(n)^{b_n} \tag{68}$$

for $b_i \leq 5$, $1 \leq i \leq n$. We found the coefficients associated with the basis functions using the non-negative least squares method for the last time period. Then we ignored the basis functions with zero coefficients and considered only those basis functions with positive coefficients in developing approximations for the value function in each time period. This is based on a reasonable view that at least when the time periods involved are few, the basis functions that approximate the value function in the last time period, also do a good job in approximating the value function at earlier time periods.

In the three dimensional setting, we found that only three basis functions had positive coefficients - fourth powers of each security. Similarly, for the five dimensional case, five basis functions had positive coefficients, again the fourth power of each security.

We used 10,000 paths to learn each s_i^*. Results are listed in Table 8. We observe VR of the order 5 and CR of order 4 in each case. Again, further research is needed to develop good quality approximations of the option price process to further improve the importance sampling performance.

To see how the proposed algorithms perform in a more realistic setting of correlated asset values, we repeated the experiments for different positive value of correlation between the Brownian motion of different assets, keeping all other parameters and the basis functions identical. Our observation was that the performance of the variance reduction techniques improved with increasing correlation. In Tables 5 and 6, we show the results for correlation $\rho = 0.5$.

7. Discussion and Conclusions

In this paper we showed how popular variance reduction techniques of control variate and importance sampling may be combined with the existing function approximation ideas to speed up the pricing of American options. In particular, we observed that the perfect control variate solves the well known additive duality problem and the perfect importance sampling measure solves the multiplicative duality problem. Our computational approach relied on

Table 4: Point Estimates and 95% CI half widths using naive simulation and importance sampling (IS). Here VR denotes the ratio of the estimated variance under naive simulation and under the importance sampling technique; CR denotes the computational reduction factor and equals VR times the ratio of the average time to generate a sample path under naive simulation and under the importance sampling technique.

$n = 3$					
X_0	Naive Est.	IS Est.	VR	CR	UB
90	11.282(0.061)	11.263(0.025)	5.9	4.5	13.163(0.020)
100	18.673(0.074)	18.654(0.030)	5.9	4.3	21.430(0.022)
110	27.529(0.086)	27.507(0.033)	6.5	4.6	31.034(0.023)

$n = 5$					
X_0	Naive Est.	IS Est.	VR	CR	UB
90	16.588(0.071)	16.640(0.032)	4.8	3.7	19.427(0.026)
100	26.076(0.085)	26.127(0.037)	5.1	5.1	29.735(0.027)
110	36.652(0.096)	36.742(0.040)	5.7	4.4	40.984(0.032)

Table 5: Point Estimates and 95% CI half widths using naive simulation and control variate (CV). The correlation coefficient between the Brownian motions is 0.5. Here VR denotes the ratio of the estimated variance under naive simulation and under the control variate technique; CR denotes the computational reduction factor and equals VR times the ratio of the average time to generate a sample path under naive simulation and under the control variate technique.

$n = 3$					
X_0	Naive Est.	CV Est.	VR	CR	UB
90	9.124(0.060)	9.135(0.012)	27.3	24.1	9.985(0.011)
100	15.091(0.075)	15.123(0.014)	30.8	25.2	16.242(0.011)
110	22.524(0.088)	22.543(0.015)	33.1	27.6	23.927(0.012)

$n = 5$					
X_0	Naive Est.	CV Est.	VR	CR	UB
90	12.083(0.071)	12.139(0.016)	18.8	14.1	13.520(0.019)
100	19.206(0.088)	19.279(0.019)	20.5	15.2	21.133(0.019)
110	27.610(0.102)	27.713(0.022)	21.9	16.6	29.924(0.019)

Table 6: Point Estimates and 95% CI half widths using naive simulation and importance sampling (IS). The correlation coefficient between the Brownian motions is 0.5. Here VR denotes the ratio of the estimated variance under naive simulation and under the importance sampling technique; CR denotes the computational reduction factor and equals VR times the ratio of the average time to generate a sample path under naive simulation and under the importance sampling technique.

X_0	Naive Est.	IS Est.	VR	CR	UB
		$n = 3$			
90	9.134(0.060)	9.124(0.022)	7.4	6.2	10.771(0.020)
100	15.100(0.075)	15.094(0.028)	7.2	5.6	17.727(0.023)
110	22.538(0.088)	22.524(0.032)	7.5	5.0	26.229(0.025)
		$n = 5$			
X_0	Naive Est.	IS Est.	VR	CR	UB
90	12.088(0.071)	12.121(0.027)	6.6	5.6	14.436(0.025)
100	19.212(0.088)	19.268(0.034)	6.7	5.4	22.788(0.029)
110	27.613(0.102)	27.708(0.040)	6.4	4.6	32.620(0.035)

selecting basis functions $\phi(\cdot)$ so that the conditional expectation $E_i[\phi(X_{i+1})](x)$ is explicitly known for each i and x. This then allows us to develop approximate perfect control variate techniques. The ability to sample from the distribution proportional to $f_i(x, y)\phi(y)$ further allows us to develop approximate perfect importance sampling techniques.

We showed empirically on a few examples that our control variate approximations perform better than the importance sampling ones and both perform better than naive procedures. Our theoretical analysis suggested that better the approximations to the value functions, better the performance of the proposed variance reduction techniques. Our empirical results further suggest that importance sampling techniques are more sensitive to inaccuracies in approximating the value function. The relative poor performance of importance sampling may also be due to the fact that we use poorer approximations of the value function in its implementation. Recall that to facilitate implementation we use the non-negative least squares method to find the approximate value function for importance sampling. Fortunately, we do not have this constraint for implementing control variate method where we use the ordinary least squares method that may provide a better approximation.

Again note that our approach to implementing approximately perfect control variates relies on selecting basis functions ϕ so that $E_i[\phi(X_{i+1})](x)$ is explicitly known for each i and x. It is easy to come up with such functions in practically important settings such

as those involving stochastic volatility and jump diffusion processes. To see this, consider the Heston's model (see Heston 1993) where the underlying asset and the volatility process $(S(t), v(t) : 0 \leq t \leq T)$ under the pricing measure follows the stochastic differential equation:

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dW_1(t)$$

and

$$dv(t) = \kappa^*(\theta^* - v(t))dt + \sigma\sqrt{v(t)}dW_2(t)$$

for specified constants $r, \kappa, \theta^*, \sigma$ and correlation ρ between between the two Brownian motions $(W_1(t), W_2(t) : 0 \leq t \leq T)$.

We may simulate the discretized version using the equations

$$S(t_{i+1}) = S(t_i)e^{(r-v(t_i)/2)(t_{i+1}-t_i)+\sqrt{v(t_i)(t_{i+1}-t_i)}N_1(0,1)}$$

and

$$v(t_{i+1}) = v(t_i) + \kappa^*(\theta^* - v(t_i))(t_{i+1} - t_i) + \sigma\sqrt{v(t_i)}N_2(0, 1)$$

where $N_1(0, 1)$ and $N_2(0, 1)$ are standard normal variates with correlation ρ (See Glasserman 2004). Therefore, conditioned on $S(t_i)$ and $v(t_i)$, $(\log S(t_{i+1}), v(t_{i+1}))$ have a bi-variate Normal distribution. Then, if the basis functions have the form $\phi(x, y) = e^{\alpha_1 x + \alpha_2 x^2 + \beta_1 y + \beta_2 y^2 + \gamma xy}$ for suitable constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ then $E[\phi(\log S(t_{i+1}), v(t_{i+1}))|S(t_i), v(t_i)]$ has an explicit closed form representation. With such basis functions, approximately perfect importance sampling is also easy to implement.

Also consider the setting where the asset pricing model involves jump processes. In popular discretization methods to generate samples of such processes, the asset price at time t_{i+1} equals the product of $\exp(X)\Pi_{i=0}^{N(t_{i+1}-t_i)}Y_i$ where X is Normally distributed (here $\Pi_{i=0}^0 Y_i = 1$), Y_i's are i.i.d., $(N(t) : t \geq 0)$ is a Poisson process and all the rv are independent of each other (see, e.g., Glasserman 2004) conditioned on the history up to time t_i. In this case, the basis functions that are product of the basis functions that we considered earlier for log-normal random variables and suitable moments of $\Pi_{i=0}^{N(t_{i+1}-t_i)}Y_i$ typically allow closed form computation of their conditional expectations.

Also note that when a European option, whose price at any time to maturity and any starting state has a closed form expression, is used as a basis function, its conditional expectation is again the price of the same option with a longer time to maturity and is known in closed form (see Ehrlichman and Henderson 2006).

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Finally, note that as better methodologies are developed to approximate basis functions, the proposed techniques should also show further improvements in computational benefits (see, for instance, Yu and Bertsekas 2009; for some initial work in learning good basis functions).

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