International Game Theory Review
Vol. 17, No. 1 (2015) 1540002 (31 pages)

DOI: 10.1142/S0219198915400022

Nearest Neighbor Based Estimation Technique for Pricing Bermudan Options

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Received 25 December 2012
Revised 12 August 2013
Accepted 12 September 2014
Published 2 March 2015

Bermudan option is an option which allows the holder to exercise at pre-specified time instants where the aim is to maximize expected payoff upon exercise. In most practical cases, the underlying dimensionality of Bermudan options is high and the numerical methods for solving partial differential equations as satisfied by the price process become inapplicable. In the absence of analytical formula a popular approach is to solve the Bermudan option pricing problem approximately using dynamic programming via estimation of the so-called continuation value function. In this paper we develop a nearest neighbor estimator based technique which gives biased estimators for the true option price. We provide algorithms for calculating lower and upper biased estimators which can be used to construct valid confidence intervals. The computation of lower biased estimator is straightforward and relies on suboptimal exercise policy generated using the nearest neighbor estimate of the continuation value function. The upper biased estimator is similarly obtained using likelihood ratio weighted nearest neighbors. We analyze the convergence properties of mean square error of the lower biased estimator. We develop order of magnitude relationship between the simulation parameters and computational budget in an asymptotic regime as the computational budget increases to infinity.

Keywords: Bermudan option; dynamic programming; nearest neighbor estimation; Hoeffding's inequality.

Subject Classification: 65C05, 65C50, 91B28

1. Introduction

Option is a financial instrument which gives its buyer the right, but not the obligation to engage in a transaction while the seller incurs the corresponding obligation, in case the option is "exercised" by the counter-party. An option which conveys the right to buy an underlying asset at a specific price is called a call option and an option which conveys the right to sell an underlying asset at a specific price is called a put option. Over the last 30 years, financial markets have witnessed considerable increase in option trading both in terms of volume and value of the trades. A vast majority of the traded options is European-style option, an option which may be exercised only at the fixed expiry date, i.e., at a single pre-defined point in time. To illustrate an example, on 24 December 2012, call option on the stock of Tata Consultancy Services Limited with strike price INR 1250 and maturity 31 January 2013, was quoted at price of INR 58.45 after close of trading. The underlying stock was quoted at INR 1263.20 on the same day.

Black and Scholes [1973] derived a partial differential equation, now called the Black–Scholes equation, which governs the price of a European option over time. The authors argued that under few regularity conditions, an option payoff can be perfectly replicated by a portfolio in which the underlying asset and money market account are always traded in just the right way. There is no exogenous infusion or withdrawal of money from the portfolio and such a trading strategy is called *self-financing*. In the absence of arbitrage, when no positive profit can be made without any downside risk, financial instruments providing the same payoff cannot have different prices and hence, the price of the option is given as the price of the *self-financing* portfolio. The price of the portfolio is shown to satisfy the Black–Scholes equation which is then also satisfied by the price of European option. Later, Harrison and Kreps [1979] showed that the price of such option can also be expressed as an expectation of the discounted payoff under an appropriate probability measure called the *risk-neutral measure*.

The other commonly traded options in financial markets are American-style options. An American option may be exercised any time up to the expiry date and hence the value of such an option is the value achieved by exercising optimally. The pricing entails solving for the optimal exercise rule such that the expected discounted payoff upon exercise is maximized under this rule. A variant of American option, where the exercise dates are restricted to a finite set of fixed exercise opportunities, is also traded in financial markets. Such an option with a fixed set of exercise dates is called a Bermudan option. If the dimensionality of the underlying process is less than four, the price of an American option can be approximated using finite difference methods for partial differential equations. But high-dimensional swaptions and multi-asset basket options with American-style exercise are quite popular (e.g., interest rate Bermudan swaptions with dimensionality 20–40 are common). Due to the curse of dimensionality, the price of such high-dimensional American options can be approximated only by using dynamic programming. With the advent of technology and faster computers, a popular approach to solve the approximate dynamic program has been through the use of simulation methods. Simulation methods are flexible in the sense that they allow underlying state variables to follow general stochastic processes and to be multi-dimensional. In general, the simulation techniques for pricing American option restrict the exercise dates to a finite set of fixed exercise opportunities. Thus, solving the approximate dynamic program using simulation methods can be used to obtain both, an approximation for Bermudan option price, and an approximation for the American option price. In this paper, we devise a new method using nearest neighbor estimator to approximately solve the dynamic program associated with Bermudan option pricing.

In the setting of Bermudan option with set of exercise dates $\mathcal{T} := \{0, 1, \ldots, T\}$, $T \in \mathrm{N}_+$, we consider an underlying Markov process X valued in $\mathbb{R}^d$ on a filtered measurable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F} := (\mathcal{F}_l)_{l=0,1,\ldots,T}$. The option payoff function $g_l : \mathbb{R}^d \to \mathbb{R}_+, l \in \{0, 1, \ldots, T\}$, is a set of measurable functions. We use this general notation for payoff function which easily incorporates a deterministic discount factor at each exercise opportunity. In the case of stochastic discount rate, we can augment the underlying process vector $X = \{X_0, X_1, \ldots, X_T\}$ and still assume a deterministic discount factor to use the same notation. Under the assumption that $\mathbb{P}$ corresponds to the risk-neutral probability measure, the price of a Bermudan option with payoff function $\{g_l\}_{l\in\mathcal{T}}$ can be represented as a solution to the following optimal stopping problem

$$V_0(X_0) = \sup_{\tau\in\mathcal{T}} \mathbb{E}[g_\tau(X_\tau)],$$

where τ is $\mathcal{F}$-stopping time taking values in the set $\mathcal{T}$. According to the dynamic programming principle, the option value function V_l satisfies the following backward recursion

$$\begin{aligned} V_T(X_T) &= g_T(X_T), \\ V_l(X_l) &= \max(g_l(X_l), C_l(X_l)), \quad l = 0, 1, \ldots, T-1, \end{aligned} \tag{1}$$

where $C_l(X_l) := \mathbb{E}[V_{l+1}(X_{l+1})|X_l]$ is called the continuation value of the option at time l and the optimal stopping time can be shown to equal (see pp. 56–57 in Duffie [1996])

$$\tau^* := \min\{l \in \mathcal{T} : g_l(X_l) \geq C_l(X_l)\}. \tag{2}$$

In finite time horizon optimal stopping problems, analytical form of the continuation value function cannot be evaluated and hence efficient estimates are sought after. We calculate the estimates $\hat{C}_l$ of continuation value function C_l at each exercise opportunity to calculate a suboptimal stopping policy

$$\hat{\tau} := \min\{l \in \mathcal{T} : g_l(X_l) \geq \hat{C}_l(X_l)\}.$$

This policy is evaluated on independent sample paths of X to obtain an estimator for the true option price.

Many estimation techniques have been proposed to approximately price Bermudan option via the estimation of continuation value function. Broadie and Glasserman [1997] introduced nested simulation method in which the continuation value function is estimated as average of the option value on inner step sample paths. The main drawback of the method is that it estimates continuation value function at every exercise opportunity using nested simulation and the computational cost scales up exponentially with the increase in number of exercise opportunities. Basis function regression based approach resolves the issue of exponentially increasing computational cost and was first proposed by Carriere [1996]. In this method, continuation value function is modeled as a linear combination of the chosen basis functions. Hence, the option pricing problem is reduced to the problem of finding optimal coefficients in the linear combination. This method was later more successfully implemented by Tsitsiklis and Van Roy [2001] and Longstaff and Schwartz [2001]. Although very popular and efficient, the essential drawback in regression based method is that the choice of basis functions is specific to the option pricing problem. In order to overcome this issue, more recently Kohler *et al.* [2010] proposed the use of neural networks to estimate continuation value function. It is well known that neural network based estimation methods are computationally expensive which is further evident from the results of Kohler *et al.* [2010]. In the similar spirit of non-parametric regression, Belomestny [2011] proposed the use of local polynomial regression where different observations are assigned a weight with respect to a kernel function.

We propose to use nearest neighbor method to estimate the continuation value function for option pricing. Similar idea has been used by Hong *et al.* [2013] for portfolio risk measurement. The nearest neighbor method can be represented as a weighted average of the observations with a special weight structure. Hence, the method belongs to the class of stochastic mesh estimators introduced by Broadie and Glasserman [2004]. We analyze theoretical convergence properties of this method and validate them with the help of a numerical study. The main idea of the paper is to simulate a set of independent trajectories of the underlying process in Phase 1 and use them to estimate the continuation value function using the nearest neighbor method. Once we know the exercise decision at each exercise opportunity, we generate another set of independent sample paths in Phase 2 to exercise the suboptimal policy calculated from Phase 1. The option payoff on each of the Phase 2 paths is used to estimate the value of the option. This procedure gives a lower biased estimator. Furthermore, we propose a single phase method using likelihood ratio weighted continuation value estimates which provides an upper biased estimator for the true option price. Our main contributions are:

(1) Development of nearest neighbor estimation based pricing algorithm which provides lower and upper biased option price estimators.
(2) Derivation of asymptotic bias and variance for lower biased option price estimator.
(3) Asymptotic mean square error (MSE) analysis to find the optimal order of magnitude relationship between the number of Phase 1 and Phase 2 paths and total computational budget.
(4) Analysis of the impact of underlying dimensionality on the convergence rate of MSE.
(5) Numerical validation of theoretical results for the lower biased estimator.

The rest of the paper is organized as follows. We discuss our method and main results in Secs. 2 and 3. The proofs are presented in Appendix A. We conclude with directions for further research after presenting the numerical study in Sec. 4.

2. Option Pricing with Nearest Neighbor Method

The nearest neighbor method has been used in the field of machine learning for the estimation of probability density function and conditional expectation. We consider this method for estimating the continuation value function defined in (1). Let M independent, identically distributed (i.i.d.) samples of $(X, Y), X \in \mathbb{R}^d, Y \in \mathbb{R}$ be denoted by $\{(X_i, Y_i), i = 1, \ldots, M\}$ such that $\mathbb{E}|Y_i| < \infty$. The nearest neighbor estimator for conditional expectation

$$C(x) := \mathbb{E}[Y|X = x]$$

is then defined as

$$\hat{C}_M(x) := \frac{\frac{1}{Mh^d}\sum_{i=1}^{M} Y_i K(\frac{X_i - x}{h})}{\frac{1}{Mh^d}\sum_{i=1}^{M} K(\frac{X_i - x}{h})}, \tag{3}$$

where $K(\cdot)$ is called the kernel function and h is the bandwidth parameter satisfying $h \searrow 0$ and $Mh^d \nearrow \infty$ as $M \nearrow \infty$. In the case of non-vanishing probability density at reference value $X = x$, the expected number of neighbors can be seen as proportional to Mh^d. In the estimation literature, nearest neighbor estimator for conditional expectation as defined in (3) is more commonly known as Nadaraya–Watson estimator. In order to quantify the quality of an estimator, we typically calculate its MSE which is given as

$$\begin{aligned} \mathbb{E}[(\hat{C}_M - C)^2] &= \mathbb{E}[(\hat{C}_M - \mathbb{E}[\hat{C}_M] + \mathbb{E}[\hat{C}_M] - C)^2] \\ &= \mathbb{E}[(\hat{C}_M - \mathbb{E}[\hat{C}_M])^2] + (\mathbb{E}[\hat{C}_M] - C)^2 \\ &= \mathrm{Var}(\hat{C}_M) + \mathrm{Bias}^2(\hat{C}_M), \end{aligned}$$

where $\mathrm{Var}(\cdot)$ and $\mathrm{Bias}(\cdot)$ denote variance and bias of the estimator respectively.

In this paper, we use Landau's notation for describing limiting behavior of functions. For given functions $s : \mathbb{R}_+ \to \mathbb{R}_+$ and $u : \mathbb{R}_+ \to \mathbb{R}_+$, we write $s(x) = O(u(x))$ if there exist $b_1 > 0$ and x_1 large enough such that $s(x) \le b_1 u(x)$ for all $x > x_1$; and $s(x) = \Omega(u(x))$ if there exist $b_2 > 0$ and x_2 large enough such that $s(x) \ge b_2 u(x)$ for all $x > x_2$. We use $s(x) = o(u(x))$ if $s(x)/u(x) \to 0$ and $s(x) \sim u(x)$ if $s(x)/u(x) \to 1$, as $x \to \infty$.

From Pagan and Ullah [1999] and Bosq [1998], we know that the following result for Nadaraya–Watson estimator holds.

Lemma 1. *Let* $\sigma^2(x) = \mathbb{E}[Y^2|X = x] - C^2(x), \mu_K = \int_{\mathbb{R}^d} u^T u K(u)du$ *and* $c_K = \int_{\mathbb{R}^d} K^2(u)du$. *Then under few regularity conditions on* $f(\cdot)$ *and* $C(\cdot)$,

$$\mathrm{Bias}(\hat{C}_M(x)) = B(x)h^2 + o(h^2),$$

$$\mathrm{Var}(\hat{C}_M(x)) = \frac{V(x)}{Mh^d} + \frac{o(1)}{Mh^d}.$$

where

$$B(x) = \frac{\mu_K}{2f(x)}\left[\mathrm{tr}\left\{\frac{\partial}{\partial x}\frac{\partial}{\partial x^T}[C(x)f(x)]\right\} - C(x)\mathrm{tr}\left\{\frac{\partial}{\partial x}\frac{\partial}{\partial x^T}f(x)\right\}\right],$$

$$V(x) = c_K\sigma^2(x)/f(x).$$

Here $f(x)$ *is the density function of* X, *the notation* A^T *and* $\mathrm{tr}(A)$ *denote the transpose and trace of a matrix* A *respectively.*

Under the risk-neutral measure $\mathbb{P}$, let us suppose for $l = 0, 1, \ldots, T$, that conditional on $X_l = x$, X_{l+1} has density $f_l(x,\cdot)$ and let $f_l(\cdot)$ denote the marginal density of X_l with X_0 fixed. Then similar to Lemma 1, we make the following regularity assumptions.

Assumption 1. There exist positive constants c_0 and c_1 which satisfy

$$c_0 \le \inf_{x\in\mathbb{R}^d} f_l(x) \quad \text{and} \quad \sup_{x\in\mathbb{R}^d} f_l(x) \le c_1, \quad \text{for } l = 1, \ldots, T.$$

Assumption 2. For $l = 1, \ldots, T, X_l$ is distributed with probability density function $f_l(\cdot)$ which is twice continuously differentiable.

Assumption 3. For $l = 0, \ldots, T-1$, the continuation value function $C_l(\cdot)$ is twice continuously differentiable with bounded third-order derivative.

For simplicity of presentation, in this paper we use the practically important d-dimensional uniform kernel function $K(x) = \prod_{i=1}^d 1_{\{|x_i|\le 1\}}$ where $x = (x_1, \ldots, x_d)$ and the norm used is L_∞ norm.

2.1. *Lower biased estimator*

In this subsection, we show how to use the nearest neighbor estimation technique to calculate a lower biased estimate of the true option price.

In Phase 1 of the algorithm, we generate a set of M independent sample paths of the underlying Markov process X starting from x_0 at time 0 and denote them as $(X_0^{(j)}, X_1^{(j)}, \ldots, X_T^{(j)})_{j=1}^M$. On each of these sample paths, we aim to estimate the option value at every exercise opportunity. We denote the option value estimate by $\hat{V}_{l,M}(\cdot)$ and the continuation value estimate by $\hat{C}_{l,M}(\cdot)$ for $l = 0, \ldots, T$. At time T, the option value estimate $\hat{V}_{T,M}(X_T^{(j)})$ is equal to the payoff value from immediate exercise $g_T(X_T^{(j)})$ and $\hat{C}_{T,M}(X_T^{(j)}) = C_T(X_T^{(j)}) = 0$. For exercise opportunities $l < T$, we first estimate the continuation value function using nearest neighbor technique. At any reference value x, we search for the neighbors in Phase 1 sample paths at time l. We use the option value estimate $\hat{V}_{l+1,M}(\cdot)$ for corresponding neighbor and calculate the continuation value estimator $\hat{C}_{l,M}(\cdot)$ (see Fig. 1) defined as

$$\hat{C}_{l,M}(x) := \frac{\sum_{k=1}^M \hat{V}_{l+1,M}(X_{l+1}^{(k)})\cdot 1_{\{|X_l^{(k)}-x|<h\}}}{\sum_{k=1}^M 1_{\{|X_l^{(k)}-x|<h\}}}. \tag{4}$$

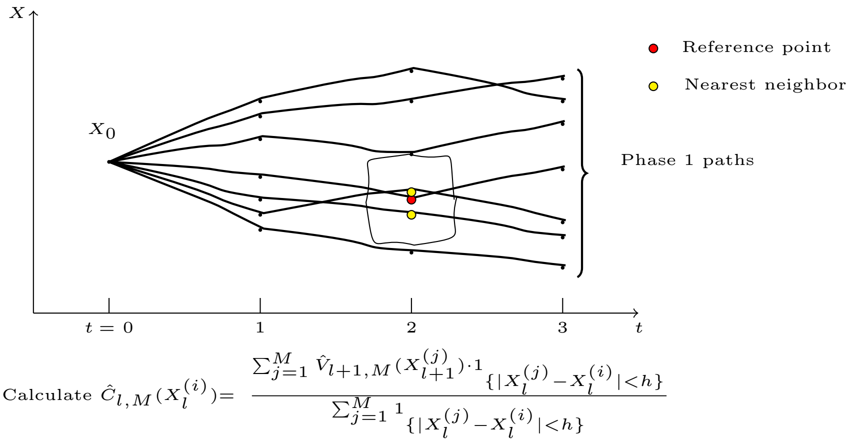


Fig. 1. Phase 1: M independent sample paths are generated and continuation value estimates are calculated using nearest neighbor method.

Next, we compare the continuation value estimate with the option payoff from immediate exercise. Finally, the option value estimate is set equal to the maximum of two quantities. The backward recursion is summarized below:

$$\hat{V}_{T,M}(X_T^{(j)}) = g_T(X_T^{(j)}),$$
$$\hat{V}_{l,M}(X_l^{(j)}) = \max\{g_l(X_l^{(j)}), \hat{C}_{l,M}(X_l^{(j)})\}, \quad l = 0, \dots, T-1.$$

Once we know the value function estimates at each exercise opportunity on Phase 1 sample paths, we generate a new set of independent sample paths and denote them as $(Z_0^{(i)}, Z_1^{(i)}, \dots, Z_T^{(i)})_{i=1}^N$. The suboptimal stopping policy calculated from the Phase 1 sample paths is evaluated on each of these paths. For a given path of the underlying process $(Z_0, Z_1, \dots, Z_T)$, the policy is given as

$$\hat{\tau} := \min\{l \in \{1, \dots, T\} : g_l(Z_l) \geq \hat{C}_{l,M}(Z_l)\}, \tag{5}$$

where

$$\hat{C}_{l,M}(Z_l) = \frac{\sum_{k=1}^{M} \hat{V}_{l+1,M}(X_{l+1}^{(k)}) \cdot 1_{\{|X_l^{(k)} - Z_l| < h\}}}{\sum_{k=1}^{M} 1_{\{|X_l^{(k)} - Z_l| < h\}}}.$$

The payoff from exercise on these sample paths (see Fig. 2) is used to calculate the option price estimator

$$\hat{v}_{0,M}(x_0) = \frac{1}{N} \sum_{i=1}^{N} g_{\hat{\tau}_i}(Z_{\hat{\tau}_i}^{(i)}), \tag{6}$$

where the suboptimal policy on each Phase 2 sample path is defined as

$$\hat{\tau}_i := \min\{l \in \{1, \dots, T\} : g_l(Z_l^{(i)}) \geq \hat{C}_{l,M}(Z_l^{(i)})\}.$$

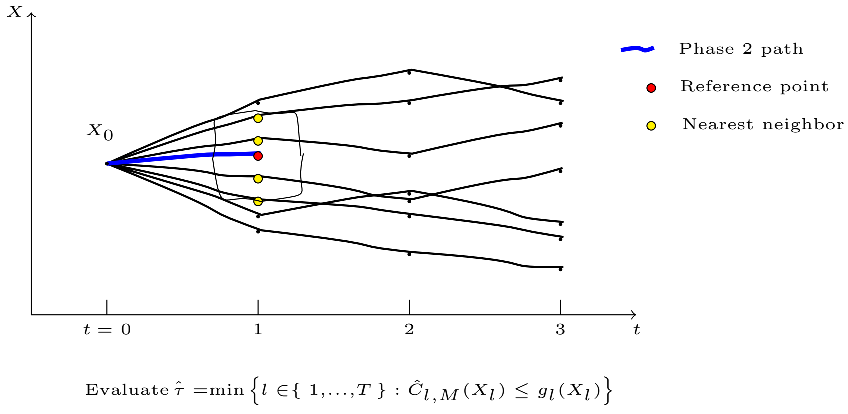


Fig. 2. Phase 2: N independent sample paths are generated and suboptimal stopping policy calculated from Phase 1 continuation value estimates is evaluated.

As we use an approximation to the optimal stopping policy, the proposed option price estimator $\hat{v}_{0,M}$ is lower biased, i.e., $V_0 - \mathbb{E}[\hat{v}_{0,M}] \ge 0$. We summarize the complete algorithm below:

Step 1. Generate $(X_0^{(j)}, X_1^{(j)}, \ldots, X_T^{(j)})_{j=1}^M$.

Step 2. For $j = 1, \ldots, M$, calculate the backward recursion:

$$\hat{V}_{T,M}(X_T^{(j)}) = g_T(X_T^{(j)}),$$

$$\hat{V}_{l,M}(X_l^{(j)}) = \max\{g_l(X_l^{(j)}), \hat{C}_{l,M}(X_l^{(j)})\}, \quad l = 0, \ldots, T-1.$$

Step 3. Generate $(Z_0^{(i)}, Z_1^{(i)}, \ldots, Z_T^{(i)})_{i=1}^N$.

Step 4. For $i = 1, \ldots, N$, exercise $\hat{\tau}_i := \min\{l \in \{1, \ldots, T\} : g_l(Z_l^{(i)}) \ge \hat{C}_{l,M}(Z_l^{(i)})\}$.

Step 5. Calculate $\hat{v}_{0,M}(x_0) = \frac{1}{N}\sum_{i=1}^N g_{\hat{\tau}_i}(Z_{\hat{\tau}_i}^{(i)})$.

We further make the following assumptions to aid in convergence analysis of $\hat{v}_{0,M}$.

Assumption 4. For any $\epsilon > 0$, the following condition holds for $l = 0, \ldots, T-1$,

$$\mathbb{P}(|C_l(X_l) - g_l(X_l)| \le \epsilon) \le K_l \epsilon,$$

where $K_l > 0$.

The above assumption provides a characterization of the behavior of the continuation values $\{C_l\}$ and payoffs $\{g_l\}$ close to the exercise boundary $\partial\mathcal{E}$ with

$$\mathcal{E} = \{(l, x) : g_l(x) \ge C_l(x)\}.$$

In problems of practical interest, typically the random variable $C_l(X_l) - g_l(X_l)$ has a smooth density function and the assumption holds. A more general form of this assumption is discussed in Belomestny [2011].

Assumption 5. There exists a compact set $\mathcal{A} \subset \mathbb{R}^d$ such that $\mathbb{P}(X_l \in \mathcal{A}) = 1$ for all $l \in T$.

This assumption is also justified in Belomestny [2011].

2.1.1. *Asymptotic bias*

The bias of an estimator is one of the contributing terms in its MSE. The use of suboptimal policy results in a biased estimator due to the error incurred by a wrong decision at any exercise opportunity. For a given set of M Phase 1 paths, suboptimal policy $\hat{\tau}_i$ on Phase 2 path $Z^{(i)}$ makes an error at exercise opportunity l, when $Z_l^{(i)} \in \mathcal{E}_{1,l} \cup \mathcal{E}_{2,l}$, where

$$\mathcal{E}_{1,l} := \{x \in \mathbb{R}^d : \hat{C}_{l,M}(x) \le g_l(x) < C_l(x)\},$$

$$\mathcal{E}_{2,l} := \{x \in \mathbb{R}^d : C_l(x) \le g_l(x) < \hat{C}_{l,M}(x)\}.$$

For fixed M, we can see from Lemma 1, that the continuation value estimator $\hat{C}_{l,M}(\cdot)$ is a noisy and biased estimator of $C_l(\cdot)$. Therefore, if the continuation value function value $C_l(x)$ is close to the value from option payoff $g_l(x)$, the probability of making a wrong decision using the estimator $\hat{C}_{l,M}(x)$ is relatively high. Hence, in order to understand the properties of asymptotic bias of the estimator $\hat{v}_{0,M}$, we first calculate the bound on probability of error of $\hat{C}_{l,M}(x)$.

We study the properties of option price estimator $\hat{v}_{0,M}$ under the asymptotic limit of simulation parameters: $h \searrow 0$ and $Mh^d \nearrow \infty$ as $M \nearrow \infty$. Recall from Assumption 1, that positive constants c_0 and c_1 satisfy:

$$c_0 \le \inf_{x \in \mathbb{R}^d} f_l(x) \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} f_l(x) \le c_1, \quad \text{for } l = 1, \ldots, T.$$

The following observation is useful for calculating the probability of error bound for continuation value estimator.

Lemma 2. *Let $X_l, X_l^{(1)}, X_l^{(2)}, \ldots, X_l^{(M)} \in \mathbb{R}^d$ be i.i.d. random variables with probability density function f_l with $l = 1, \ldots, T$. Suppose that Assumption 1 is satisfied. Then, for all γ with $0 < \gamma \le c_0$ and for small enough $h > 0$, there exists constant c with $0 < c < c_0$ such that*

$$\mathbb{P}\left(\frac{1}{Mh^d} \sum_{j=1}^{M} 1_{\{|X_l^{(j)} - X_l| < h\}} \le \gamma \,\middle|\, X_l \right) \le \exp\left(-\frac{1}{2} \frac{(2c_1 + c_0 - \gamma)^2}{(2c_0 - c)} Mh^d \right) \quad a.s. \tag{7}$$

Next, we prove the probability of error bound for continuation value estimator $\hat{C}_{l,M}(\cdot)$. The following proposition is one of the main results of this paper and is used to derive the asymptotic bias of $\hat{v}_{0,M}$.

Proposition 1. *Let* $g_l : \mathbb{R}^d \to \mathbb{R}_+$ *be bounded function for* $l = 1, \ldots, T$. *Then, under Assumptions* 1–5, *there exist constants* γ, B_1, B_2 $(B_1 \neq B_2)$ *such that if*

$$h = O(M^{-\frac{1}{4+d}}), \quad M \nearrow \infty,$$

then for large enough $M > 0$ *and any* $\epsilon > 0$,

$$Prr(|\hat{C}_{l,M}(X_l) - C_l(X_l)| \geq \epsilon(Mh^d)^{-1/2} \mid X_l) \leq \exp\left(-\frac{2\gamma\epsilon^2}{(B_2 - B_1)^2}\right) \quad a.s. \tag{8}$$

Finally, bias of the lower biased option price estimator $\hat{v}_{0,M}$ is given as follows.

Proposition 2. *Under Assumption* 1–5, *if* $h = O(M^{-\frac{1}{4+d}})$ *as* $M \nearrow \infty$, *then, for large enough* $M > 0$, *we have,*

$$0 \leq V_0 - \mathbb{E}[\hat{v}_{0,M}] \leq D_0 \left(\sum_{l=0}^{T-1} K_l\right) h^4, \tag{9}$$

where D_0 *is a positive constant.*

The above proposition can be proved in a similar way as Belomestny [2011]. Here, we discuss the main idea of the proof.

Remark 1. First, by making use of an induction argument, the bias of option price estimator $\hat{v}_{0,M}$ is expressed as an expectation of the difference of continuation and payoff values over an error set where exercise error happens due to using continuation value estimator. That is,

$$V_0 - \mathbb{E}[\hat{v}_{0,M}] \leq \mathbb{E}\left[\sum_{l=0}^{T-1} |C_l(X_l) - g_l(X_l)|(1_{\{\hat{C}_{l,M}(X_l) \leq g_l(X_l) < C_l(X_l)\}}\right.$$
$$\left. + 1_{\{C_l(X_l) \leq g_l(X_l) < \hat{C}_{l,M}(X_l)\}})\right].$$

Next, it is argued using Proposition 1, that the exercise error, denoted by indicator functions in the above equation, occurs in a region of order $(Mh^d)^{-1/2}$ around the exercise boundary. Hence, by using conditioning arguments the bias calculation is reduced to calculating the product of probability of underlying process being in $O((Mh^d)^{-1/2})$ with probability of exercise error. Thus, the main term to be calculated is

$$\mathbb{E}[\mathbb{P}(|\hat{C}_{l,M}(X_l) - C_l(X_l)| \geq (Mh^d)^{-1/2} \mid X_l) \times 1_{\{0 < |C_l(X_l) - g_l(X_l)| < (Mh^d)^{-1/2}\}}],$$

which can be calculated from Proposition 1 and Assumption 4.

2.1.2. *Asymptotic variance*

Recall that the lower biased option price estimator is given as

$$\hat{v}_{0,M}(x_0) = \frac{1}{N} \sum_{i=1}^{N} g_{\hat{\tau}_i}(Z_{\hat{\tau}_i}^{(i)}),$$

where $\hat{\tau}_i = \min\{l \in \{1, \dots, T\} : g_l(Z_l^{(i)}) \geq \hat{C}_{l,M}(Z_l^{(i)})\}$ for $i = 1, \dots, N$. Then, variance of the estimator $\hat{v}_{0,M}$ equals

$$\mathrm{Var}(\hat{v}_{0,M}) = \frac{1}{N}\mathrm{Var}(g_{\hat{\tau}_1}(Z_{\hat{\tau}_1}^{(1)})) + \left(1 - \frac{1}{N}\right)\mathrm{Cov}(g_{\hat{\tau}_i}(Z_{\hat{\tau}_i}^{(i)}), g_{\hat{\tau}_j}(Z_{\hat{\tau}_j}^{(j)})), \quad i \neq j. \tag{10}$$

We analyze each term in the right-hand side of (10) separately.

$\mathrm{Cov}(g_{\hat{\tau}_i}(Z_{\hat{\tau}_i}^{(i)}), g_{\hat{\tau}_j}(Z_{\hat{\tau}_j}^{(j)}))$: The suboptimal stopping time $\hat{\tau}_i$ and $\hat{\tau}_j$ on independent Phase 2 paths i and j respectively, are calculated by using nearest neighbor estimators of the continuation value. The two decisions are dependent when the Phase 1 paths used for nearest neighbor estimator are common. But as $M \nearrow \infty$, the bandwidth parameter $h \searrow 0$ and the probability of two decisions being dependent decreases. We make use of this observation and show in the following lemma that the covariance term is negligibly small and decays as $O(h^8)$ as $M \nearrow \infty$.

Lemma 3. *Under Assumption* 1–5, *if* $h = O(M^{-\frac{1}{4+d}})$ *as* $M \nearrow \infty$, *then, for large enough* $M > 0$, *we have,*

$$\mathrm{Cov}(g_{\hat{\tau}_i}(Z_{\hat{\tau}_i}^{(i)}), g_{\hat{\tau}_j}(Z_{\hat{\tau}_j}^{(j)})) \leq D_1 h^8, \tag{11}$$

where D_1 *is a constant.*

$\mathrm{Var}(g_{\hat{\tau}}(Z_{\hat{\tau}}))$: Intuitively, we can see by application of dominated convergence theorem that as the bias of estimator $\hat{v}_{0,M} \to 0$ as $M \nearrow \infty, \mathrm{Var}(g_{\hat{\tau}}(Z_{\hat{\tau}})) \to \mathrm{Var}(g_\tau(Z_\tau))$ where τ is the optimal stopping time. We present here the result which validates our intuition.

Lemma 4. *Under Assumption* 1–5, *if* $h = O(M^{-\frac{1}{4+d}})$ *as* $M \nearrow \infty$, *then, for large enough* $M > 0$, *we have,*

$$\mathrm{Var}(g_{\hat{\tau}}(Z_{\hat{\tau}})) = \mathrm{Var}(g_\tau(Z_\tau)) + O(h^4), \quad M \nearrow \infty.$$

The following result then follows for the asymptotic variance of $\hat{v}_{0,M}$.

Proposition 3. *Under Assumption* 1–5, *there exist constants* D_1 *and* σ *such that if*

$$h = O(M^{-\frac{1}{4+d}}), \quad M \nearrow \infty,$$

then, for large enough M, $N > 0$, *we have,*

$$\mathrm{Var}(\hat{v}_{0,M}) \leq \frac{\sigma}{N} + D_1 h^8.$$

Remark 2. Our main aim is to study the convergence properties of optimal MSE of the lower biased estimator $\hat{v}_{0,M}$. In the above sections, we have derived upper bounds for the asymptotic bias and variance of $\hat{v}_{0,M}$. We have shown that as $M \nearrow \infty$, then for $h = O(M^{-\frac{1}{4+d}})$, bias varies as $O(h^4)$ and variance varies as $O(\frac{1}{N}) + O(h^8)$. In order to derive the order of magnitude relationship between the number of Phase 1 paths M and Phase 2 paths N, we also require the corresponding lower bounds. We observe that in the nearest neighbor continuation value estimator $\hat{C}_{l,M}(x)$, the individual terms in the numerator are bounded. It can be verified that Lindeberg's condition (see Feller [2008]) is satisfied which gives a central limit theorem for $\hat{C}_{l,M}(x)$:

$$(Mh^d)^{-1/2}(\hat{C}_{l,M}(x) - C_l(x)) \Longrightarrow \mathcal{N}(0, \sigma^2),$$

where σ is some constant. Then, for any $\delta > 0$, as $M \nearrow \infty$

$$\begin{aligned}\mathbb{P}(|\hat{C}_{l,M}(X_l) - C_l(X_l)| &\geq \delta (Mh^d)^{-1/2} \mid X_l) \\ &\sim 2\int_\delta^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}\, dy \\ &\geq \frac{1}{\sqrt{2\pi}}\left(\frac{\sigma}{\delta} - \frac{\sigma^3}{\delta^3}\right) e^{-\delta^2/2\sigma^2} \quad \text{a.s.}\end{aligned}$$

Typically, the random variable $C_l(X_l) - g_l(X_l)$ has a smooth density function for $l = 1, \ldots, T$ and with mild assumption on the probability density function, for any $\delta > 0$, we get

$$\mathbb{P}(|C_l(X_l) - g_l(X_l)| \leq \delta) \geq k_l \delta,$$

where k_l is some constant. Therefore, using the same proof methodology as in the previous section, we can show that the estimator bias is similarly lower bounded by terms which have the same order relationship as the upper bound. We can use similar ideas to lower bound the covariance term and show that

$$\mathrm{Var}(\hat{v}_{0,M}) \geq \frac{\theta}{N}$$

for some constant $\theta > 0$.

2.2. *Upper biased estimator*

To construct valid confidence intervals for the estimate of Bermudan option price, we need both lower and upper biased estimators. In this section, we propose a likelihood ratio weighted nearest neighbor approach which allows us to calculate an upper biased option price estimator. In the single phase method, we generate M sample paths of the underlying process X starting from x_0 at time 0 and denote them as $(X_0^{(j)}, \ldots, X_T^{(j)})_{j=1}^M$. Each observation on these sample paths $j = 1, \ldots, M$ is then assigned an appropriate likelihood ratio weight such that if the true value function $V_l(\cdot)$ is known, the continuation value estimator is unbiased. For this purpose, let us recall that under the risk-neutral measure $\mathbb{P}$, conditional on $X_l = x$, the density of X_{l+1} is denoted by $f_l(x, \cdot)$ for $l = 0, \ldots, T-1$. The backward recursion for calculating the upper biased estimator $\tilde{V}_{0,M}(\cdot)$ is given as follows:

$$\tilde{V}_{T,M}(X_T^{(j)}) = V_T(X_T^{(j)}) = g_T(X_T^{(j)}), \tag{12}$$

$$\tilde{V}_{k,M}(X_l^{(j)}) = \max\{g_l(X_l^{(j)}), \tilde{C}_{l,M}(X_l^{(j)})\}, \quad l = 0, \ldots, T-1, \tag{13}$$

where

$$\tilde{C}_{l,M}(X_l^{(j)}) := \frac{\sum_{k=1}^{M} \tilde{V}_{l+1,M}(X_{l+1}^{(k)}) \cdot \frac{f_l(X_l^{(j)}, X_{l+1}^{(k)})}{f_l(X_l^{(k)}, X_{l+1}^{(k)})} 1_{\{|X_l^{(k)} - X_l^{(j)}| < h\}}}{\sum_{k=1}^{M} 1_{\{|X_l^{(k)} - X_l^{(j)}| < h\}}}. \tag{14}$$

As discussed in Broadie and Glasserman [2004], the likelihood ratio weights chosen in the upper biased estimator $\tilde{V}_{0,M}(\cdot)$ leads to variance build-up. To eliminate this risk, the average density method is proposed. For the choice of average density likelihood ratio weights, the modified upper biased estimator is calculated using the following backward recursion:

$$\bar{V}_{T,M}(X_T^{(j)}) = V_T(X_T^{(j)}) = g_T(X_T^{(j)}),$$

$$\bar{V}_{k,M}(X_l^{(j)}) = \max\{g_l(X_l^{(j)}), \bar{C}_{l,M}(X_l^{(j)})\}, \quad l = 0, \ldots, T-1,$$

where

$$\bar{C}_{l,M}(X_l^{(j)}) = \frac{\sum_{k=1}^{M} \bar{V}_{l+1,M}(X_{l+1}^{(k)}) \cdot \frac{f_l(X_l^{(j)}, X_{l+1}^{(k)})}{\frac{1}{M}\sum_{i=1}^{M} f_l(X_l^{(i)}, X_{l+1}^{(k)})} 1_{\{|X_l^{(k)} - X_l^{(j)}| < h\}}}{\sum_{k=1}^{M} 1_{\{|X_l^{(k)} - X_l^{(j)}| < h\}}}.$$

The following result shows that both the estimators $\tilde{V}_{0,M}(\cdot)$ and $\bar{V}_{0,M}(\cdot)$ are upper biased.

Proposition 4. *Suppose that Assumptions* 1 *and* 2 *are satisfied. Then*,

$$\mathbb{E}[\tilde{V}_{0,M}(X_0)] \geq V_0(X_0) \quad \text{and} \quad \mathbb{E}[\bar{V}_{0,M}(X_0)] \geq V_0(X_0). \tag{15}$$

3. Optimal Allocation of Computational Budget

In this section, we focus on finding the optimal allocation of a fixed computational budget to Phase 1 sample paths M and Phase 2 sample paths N such that the MSE of $\hat{v}_{0,M}$ is minimized. Recall from our earlier discussion that the MSE of the lower biased estimator $\hat{v}_{0,M}$ of true option price V_0 is given as

$$\mathbb{E}[(\hat{v}_{0,M} - V_0)^2] = \mathrm{Var}(\hat{v}_{0,M}) + \mathrm{Bias}^2(\hat{v}_{0,M}).$$

Suppose that the computational effort to generate a sample path of X on an average equals a_1, then the computational effort required to generate M sample paths for the Phase 1 on an average equals $a_1 M$. Now, assume that the effort required to calculate the continuation value estimator on an average approximately equals $a_2 M^\beta$ where $0 \leq \beta \leq 1$. If finding the neighbors in a fixed region is computationally much cheaper than generating sample paths of the underlying process, then $\beta = 0$ is a reasonable assumption. On the other hand, if all the sample paths are required to calculate the estimator it is reasonable to take $\beta = 1$. The naive implementation of the nearest neighbor method, where at every exercise opportunity the value of each sample path is compared with the reference value, will have

$\beta = 1$. Then, average computational effort for calculating continuation value estimator on M Phase 1 sample paths approximately equals $a_1 M + a_2 M^{1+\beta}$. Similarly, the total effort over N replications in Phase 2 on an average is approximately equal to $a_3 N + a_4 N M^{\beta}$ for some positive constants a_3 and a_4.

Suppose that the overall computational budget is fixed at Γ. We then solve the problem of choosing optimal (M, N) such that the MSE of the option price estimator $\hat{v}_{0,M}$ is minimized subject to Γ equals $a_2 M^{1+\beta} + a_4 N M^{\beta}$. From Proposition 2, Proposition 3 and later discussion, we have for sufficiently large M, N,

$$A_1 M^{-\frac{4}{4+d}} \le \mathrm{Bias}(\hat{v}_{0,M}) \le A_2 M^{-\frac{4}{4+d}},$$

$$\frac{\theta_1}{N} \le \mathrm{Var}(\hat{v}_{0,M}) \le \frac{\theta_2}{N} + D_2 M^{-\frac{8}{4+d}}$$

for some positive constants $A_1, A_2, \theta_1, \theta_2$ and D_2. As $M, N \nearrow \infty$, this suggests that $\mathrm{Bias}(\hat{v}_{0,M})$ and $\mathrm{Var}(\hat{v}_{0,M})$ have the same order relationship with M and N as their respective bounds. Therefore, for some constants $\bar{\theta}$ and D, we consider the following MSE minimization problem

$$\min_{M,N \ge 0} \frac{\bar{\theta}}{N} + D M^{-\frac{8}{4+d}}$$

$$\text{subject to } a_2 M^{1+\beta} + a_4 N M^{\beta} = \Gamma,$$

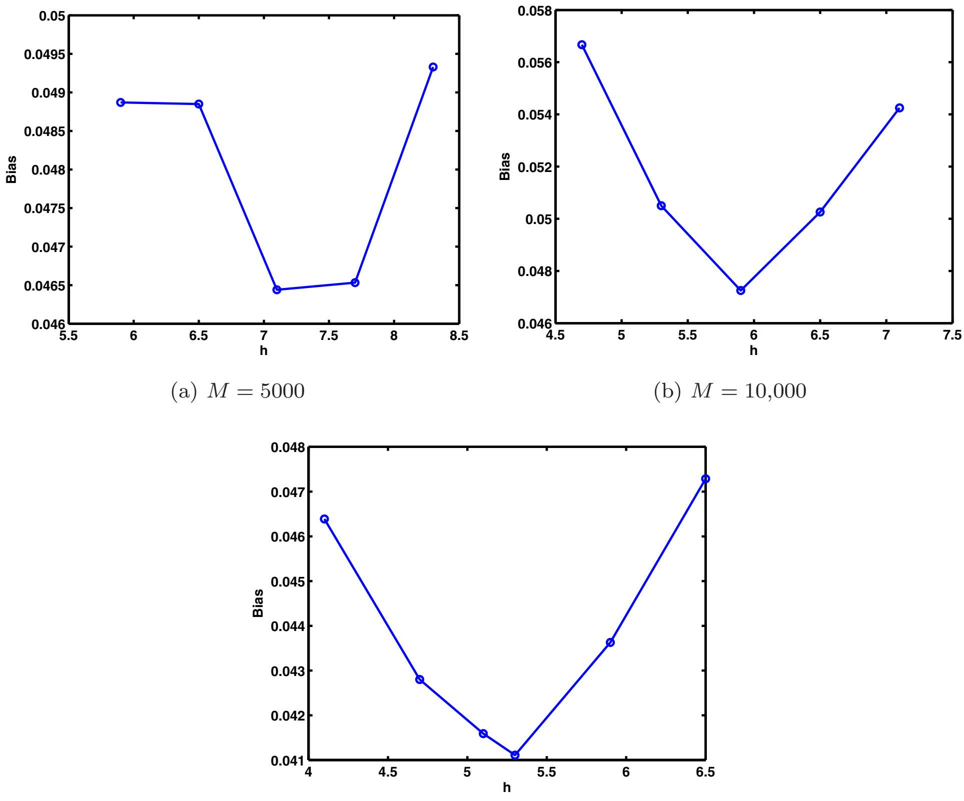
as $\Gamma \nearrow \infty$. After simple analysis, we can see that the optimal M, N and MSE are given as in Table 1. Note that for optimal M and N, squared bias and variance of the estimator $\hat{v}_{0,M}$ decay at the same rate as optimal MSE.

3.1. *For $d \le 4$*

For $\beta = 0$, the case when generating sample paths is computationally more intensive than finding the nearest neighbors, for example, generating high-dimensional forward interest rate paths or underlying paths in the stochastic volatility setting, we can see that the optimal MSE decays at the rate of $O(\Gamma^{-1})$ which is the best possible achievable rate of decay of error. When $d < 4$, it is obvious from the results in Table 1 that the number of Phase 2 paths, N, should be of a higher-order of the computational budget than the number of Phase 1 paths M. This implies that more computational effort should be utilized in Phase 2 of the algorithm to minimize the

Table 1. Order relationship of optimal M, N and MSE with computational budget Γ as $\Gamma \nearrow \infty$.

	$d < 4$	$d = 4$	$d > 4$
M	$O_\Gamma(\Gamma^{\frac{4+d}{8+\beta(4+d)}})$	$O_\Gamma(\Gamma^{\frac{1}{1+\beta}})$	$O_\Gamma(\Gamma^{\frac{1}{1+\beta}})$
N	$O_\Gamma(\Gamma^{\frac{8}{8+\beta(4+d)}})$	$O_\Gamma(\Gamma^{\frac{1}{1+\beta}})$	$O_\Gamma(\Gamma^{\frac{12+d}{2(1+\beta)(4+d)}})$
MSE	$O_\Gamma(\Gamma^{-\frac{8}{8+\beta(4+d)}})$	$O_\Gamma(\Gamma^{-\frac{1}{1+\beta}})$	$O_\Gamma(\Gamma^{-\frac{8}{(1+\beta)(4+d)}})$



(a) $M = 5000$

(b) $M = 10{,}000$

(c) $M = 20{,}000$

Fig. 3. Plot of average bias of lower biased estimator $\hat{v}_{0,M}$ for different values of bandwidth parameter h and number of Phase 1 sample paths M. Number of Phase 2 sample paths $N = 80{,}000$.

MSE. Whereas, for $d = 4$, the number of Phase 1 and Phase 2 sample paths should be of the same order.

3.2. *For* $d > 4$

As the dimensionality increases, the performance of the estimator deteriorates which is expected as the nearest neighbor method suffers from the curse of dimensionality. For $\beta = 0$, it is easy to see that MSE decays at the rate $O(\Gamma^{-\frac{8}{4+d}})$. Also, the formula supports our intuition that in order to estimate the option price accurately, the number of Phase 1 paths M should be of higher-order than Phase 2 sample paths N. This implies that as the method suffers from the curse of dimensionality, more computational effort is required in Phase 1 to get reasonably accurate estimates of the continuation value function. The rate of convergence of MSE can be further improved by reducing the bias using jackknifing technique which is discussed by Hong *et al.* [2013] in the setting of portfolio risk measurement.

4. Numerical Examples

To test the performance of the method and validate the theoretical results derived for lower biased estimator, we consider a Bermudan max-call option on two correlated assets with strike price K where the payoff function is given as

$$(\max(X^1, X^2) - K)^+.$$

We assume that under the risk-neutral measure, the two assets follow correlated geometric Brownian motion processes,

$$dX_t^i = (r - \delta)X_t^i dt + \sigma X_t^i dW_t^i, \quad i = 1, 2$$

with risk-less interest rate $r = 5\%$, dividend yield $\delta = 10\%$, and volatility $\sigma = 20\%$. The instantaneous correlation between standard Brownian motion process W^1 and W^2 is $\rho = 0.3$. The option expires in $T = 1$ year and can be exercised at three equally spaced dates $t_j = j/3, j = 1, 2, 3$. We take $X_0^1 = X_0^2 = 90$ and $K = 100$. The true value of this option is 4.077 which can be calculated from the formula in Geske and Johnson [1984].

In order to price this option, we need to set the number of Phase 1, Phase 2 paths and the bandwidth parameter h. We compare various combinations and report the results in Table 2 for lower biased estimator defined in (6) and upper biased estimator defined in (12) and (13). We replicate these estimators 50 times to calculate the MSE.

Next, we show that the relationship between asymptotic bias minimizing bandwidth parameter h^* and number of Phase 1 sample paths M for the lower biased estimator is as chosen in Sec. 2.1.1. We set the number of Phase 2 sample paths $N = 80{,}000$ and plot the average bias of lower biased estimator $\hat{v}_{0,M}$ for different values of M using different h (see Fig. 3). Average bias is calculated using 50 independent iterations of the estimator. It is observed in Fig. 4 that h^* appears to closely follow the relationship of $h = O(M^{-1/6})$ for $d = 2$, with the number of Phase 1 sample paths M. We use corresponding h^* for different values of M to calculate the bias of $\hat{v}_{0,M}$ with $N = 80{,}000$. In Fig. 4, we plot the average bias from 50 independent samples with respect to the number of Phase 1 sample paths M. It is observed (see Fig. 5) that for $d = 2$, the average bias closely resembles the relationship, $\text{Bias}(\hat{v}_{0,M}) = O(M^{-2/3})$.

Table 2. Price estimates for Bermudan option on two assets. The number of Phase 2 paths is $N = 80{,}000$. The lowest MSE achieved with different bandwidth parameter h is shown.

M	Lower estimate				Upper estimate			
	$h = 5.0$	$h = 6.0$	$h = 7.0$	MSE	$h = 5.0$	$h = 6.0$	$h = 7.0$	MSE
5000	4.009	4.016	4.018	0.0044	4.236	4.218	4.210	0.0307
10,000	4.029	4.034	4.029	0.0026	4.173	4.163	4.158	0.0129
20,000	4.046	4.045	4.038	0.0018	4.133	4.126	4.123	0.0046

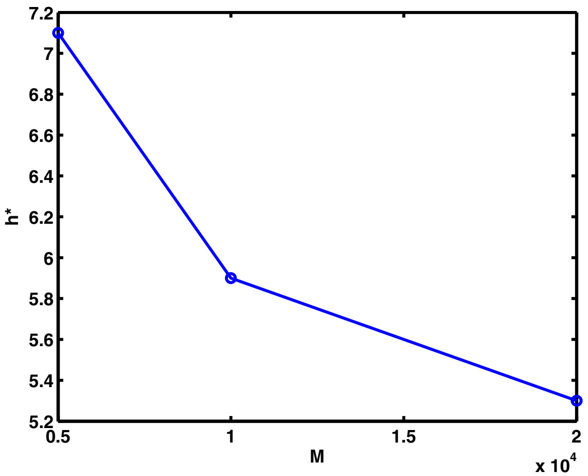


Fig. 4. Asymptotic bias minimizing bandwidth parameter h^* varies as $O(M^{-1/6})$ with the number of Phase 1 sample paths M.

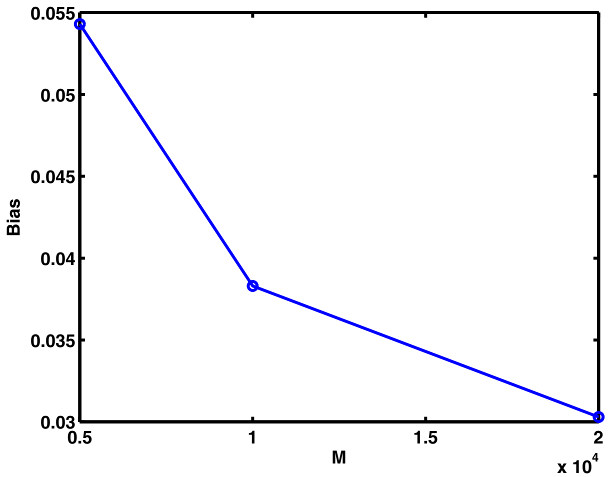


Fig. 5. Average bias of $\hat{v}_{0,M}$ varies as $O(M^{-2/3})$ with the number of Phase 1 sample paths M.

Finally, we test our method for Bermudan option pricing using the benchmark max-call option with nine exercise dates (Examples 8.6.1 in Glasserman [2004]). Again, we consider a Bermudan option with two underlying assets X^1 and X^2 modeled as geometric Brownian motion with drift $(r - \delta)$ and volatility σ. In this example, we set interest rate $r = 5\%$, dividend yield $\delta = 10\%$ and volatility $\sigma = 20\%$. The two assets are independent of each other and we set $X^1 = X^2 = 100$ and $K = 100$. The option expires in $T = 3$ years and can be exercised at nine equally spaced dates $t_j = j/3, j = 1, \ldots, 9$. The true value of this option is 13.90. The results are shown in Table 3.

Table 3. Price estimates for Bermudan option on two assets with nine exercise dates. The number of Phase 2 paths $N = 50{,}000$. The MSE of the best estimator is shown.

M	Lower estimate				Upper estimate	
	$h = 5.0$	$h = 6.0$	$h = 7.0$	MSE	$h = 7.0$	MSE
5000	13.492	13.582	13.646	0.0836	14.947	1.175
10,000	13.635	13.702	13.740	0.0360	14.549	0.461
20,000	13.740	13.778	13.786	0.0212	14.277	0.157

Table 4. Average computational time (in seconds) to complete Phase 1 of the nearest neighbor algorithm for a single exercise period.

M	$d = 2$			$d = 5$		
	$h = 5.0$	$h = 6.0$	$h = 7.0$	$h = 5.0$	$h = 6.0$	$h = 7.0$
5000	5	6	7	10	12.5	15
10,000	15	18	20	46.5	57.5	69
20,000	77	95	115	211	261	312

We observe that for the same computational budget of 46 min for a single iteration of the algorithm, the choice of $M = 20{,}000, N = 50{,}000$ with bandwidth parameter $h = 7$ gives a lower MSE than the estimator with the choice $M = 26{,}000, N = 20{,}000$. The estimator value obtained in the latter case is 13.778 with MSE of 0.0253. This supports our result in Table 1 which recommends for $d < 4$, the number of Phase 2 paths N should be of higher-order than number of Phase 1 paths M in order to minimize MSE. In Table 4, we have included the average computational time required to complete Phase 1 of the nearest neighbor method for different values of M, bandwidth parameter h and underlying dimension d.

5. Conclusions and Directions for Further Research

We proposed a new simulation method to price Bermudan options based on a nearest neighbor estimation technique. We provided algorithms for upper and lower biased estimators and established the rate of convergence of asymptotic bias and variance of the lower biased estimators. We further analyzed the MSE of the estimator and found the optimal relationship of the number of Phase 1 and Phase 2 sample paths with the computational budget Γ as $\Gamma \nearrow \infty$. A numerical example is used to validate the theoretical results derived for lower the biased estimator.

In this paper, we focused on pricing Bermudan options with the assumption that it is possible to generate exact sample paths of the underlying process. This is true, for example, for geometric Brownian motion, Ornstein–Uhlenbeck process, Cox–Ingersoll–Ross process and can be achieved for more general one-dimensional diffusion processes using the exact simulation approach proposed by Beskos and Roberts [2005] and Chen and Huang [2013]. However in high-dimension Bermudan option pricing problems, due to approximating the diffusion process via numerical schemes with a discretization step, an extra bias is unavoidable. There is a need to address this discretization error in the nearest neighbor method for option pricing and is an important direction for future research. Belomestny *et al.* [2013] address the contribution of discretization error in the final estimator for pricing American options and propose a multilevel approach to improve the order of computational complexity of the estimator. Another important research direction involves proposing a similar multilevel methodology for Bermudan option pricing.

Appendix A

To prove the main results of this paper, Proposition 1 and Lemma 3, we first prove a few key lemmas. The following theorem and its corollary help establish an upper bound on probability of the event that the denominator in the nearest neighbor estimator takes small values, the result in Lemma 2.

Theorem A.1 (Chernoff–Hoeffding Theorem, Dembo and Zeitouni [1998]). *Let $X_1, \dots, X_n$ be independent random variables taking values in the unit interval $[0,1]$ and let $S_n := X_1 + \cdots + X_n$. Let $p_i = \mathbb{E}[X_i], i = 1, \dots, n$ and let $p := (p_1 + \cdots + p_n)/n, q := 1 - p$. Then, for $t > 0$*

$$\mathbb{P}(S_n > (p+t)n) \leq \exp(-nH((p+t, q-t) \,|\, (p,q))),$$

$$\mathbb{P}(S_n < (p-t)n) \leq \exp(-nH((p+t, q-t) \,|\, (p,q))),$$

where H is the relative entropy,

$$H((p_1, \dots, p_r), (q_1, \dots, q_r)) := \sum_i p_i \log \frac{p_i}{q_i}.$$

Corollary A.1. *For $\epsilon > 0$,*

$$\mathbb{P}(S_n > (1+\epsilon)pn) < \exp\left(-\frac{\epsilon^2}{3}pn\right),$$

$$\mathbb{P}(S_n < (1-\epsilon)pn) < \exp\left(-\frac{\epsilon^2}{2}pn\right).$$

Proof of Lemma 2. For ease of notation, the calculations are shown only for $d = 1$. The result follows similarly for $d > 1$. Recall from Assumption 1, that positive constants c_0 and c_1 satisfy:

$$c_0 \leq \inf_{x \in \mathbb{R}^d} f_l(x) \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} f_l(x) \leq c_1, \quad \text{for } l = 1, \dots, T.$$

Let us define for $l = 1, \dots, T$ and $X_l = x$ in Eq. (7),

$$p_l(x) := \mathbb{P}(1_{\{|X_l^{(j)} - x| < h\}} = 1).$$

For small enough h, using Taylor series expansion, we can write

$$p_l(x) = \int_{x-h}^{x+h} f_l(y)\,dy = 2f_l(x)h + o(h),$$

and that there exists some constant c which satisfies

(i) $$0 < c < c_0,$$

(ii) $$|p_l(x) - 2f_l(x)h| \leq ch, \quad \forall\, x \in \mathbb{R}.$$

From Corollary A.1, for any γ such that $0 < \gamma \leq c_0$, we get

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{M} 1_{\{|X_l^{(j)} - x| < h\}} \leq \gamma M h\right) &\leq \exp\left(-\frac{1}{2}\frac{(p_l(x) - \gamma h)^2}{p_l(x)} M\right) \\ &\leq \exp\left(-\frac{1}{2}\frac{(2c_1 + c - \gamma)^2}{(2c_0 - c)} M h\right).\end{aligned}$$

The above result holds uniformly for any value of X_l. The general result for $d > 1$ can be argued similarly and we can show that for small enough h and $\gamma \leq c_0$, there exists some constant c with $0 < c < c_0$ such that

$$\mathbb{P}\left(\sum_{j=1}^{M} 1_{\{|X_l^{(j)} - X_l| < h\}} \leq \gamma M h^d \,\middle|\, X_l\right) \leq \exp\left(-\frac{1}{2}\frac{(2c_1 + c - \gamma)^2}{(2c_0 - c)} M h^d\right) \quad \text{a.s.}$$ □

The following lemma will be useful in proving the key result of Proposition 1.

Lemma A.1. *Let $\mathcal{A} \subset \mathbb{R}^d$ be a compact set and X, Y be i.i.d. random variables which take values in $\mathcal{A}$ with twice continuously differentiable probability density function $f(x) > 0, \forall\, x \in \mathcal{A}$. Then, for any twice continuously differentiable function $g : \mathbb{R}^d \to \mathbb{R}$ with bounded third-order derivative, we have*

$$\mathbb{E}[(g(Y) - g(X)) \,|\, |Y - X| < h, X] = B(X)h^2 + o(h^2) \quad \text{a.s.},$$

where $B(\cdot)$ is a bounded function.

Proof. Recall from Taylor series expansion,

$$g(y) = g(x) + \nabla g(x)^T(y - x) + \frac{1}{2}(y - x)^T\{\mathrm{H}g(z)\}(y - x),$$

where $\nabla g(x)$ is the gradient of g evaluated at x, $\mathrm{H}g(x)$ is the Hessian matrix and z lies between x and y. For ease of notation, the calculations are shown only for $d = 1$. The result follows similarly for $d > 1$. For given $X = x$, we can write,

$$\mathbb{E}[(g(Y) - g(x)) \,|\, |Y - x| < h] = \frac{\mathbb{E}[(g(Y) - g(x))1_{\{|Y - x| < h\}}]}{\mathbb{P}(|Y - x| < h)}. \tag{A.1}$$

The numerator in (A.1) can be expressed as,

$$\mathbb{E}[(g(Y) - g(x))1_{\{|Y - x| < h\}}] = \int_{x-h}^{x+h} (g(y) - g(x)) f(y)\, dy. \tag{A.2}$$

By Taylor series expansion of $g(y)$ and $f(y)$ at $y = x$, right-hand side in (A.2) equals

$$\frac{h^3}{3}\frac{d^2g}{dx^2}(x)f(x) + \frac{2h^3}{3}\frac{dg}{dx}(x)\frac{df}{dx}(x) + o(h^3). \tag{A.3}$$

Similarly, we can calculate $\mathbb{P}(|Y - x| < h)$ using Taylor series expansion to get

$$\mathbb{P}(|Y - x| < h) = 2hf(x) + o(h^3). \tag{A.4}$$

It is then easy to see from (A.3) and (A.4) that

$$\mathbb{E}[(g(Y) - g(x)) \,|\, |Y - x| < h] = \left(\frac{d^2g}{dx^2}(x) + \frac{1}{f(x)}\frac{dg}{dx}(x)\frac{df}{dx}(x)\right) h^2 + o(h^2).$$

Since, the functions in right-hand side of the above equation are continuous, they remain bounded on the compact set $\mathcal{A}$. Hence, the result follows for $d = 1$ and holds uniformly for all $x \in \mathcal{A}$. We can similarly argue for $d > 1$ that

$$\mathbb{E}[(g(Y) - g(X)) \,|\, |Y - X| < h, X] = B(X)h^2 + o(h^2) \quad \text{a.s.},$$

where $B(\cdot)$ is a bounded function. □

Proof of Proposition 1. We note that in continuation value estimator $\tilde{C}_{l,M}(x)$, the denominator could take zero value. In that case, we assign $\tilde{C}_{l,M}(x) = 0$. At exercise instant $T-1$, the nearest neighbor continuation value estimator at $X_{T-1} = x$ is given as

$$\hat{C}_{T-1,M}(x) = \frac{\frac{1}{Mh^d}\sum_{j=1}^M g_T(X_T^{(j)})1_{\{|X_{T-1}^{(j)} - x| < h\}}}{\frac{1}{Mh^d}\sum_{j=1}^M 1_{\{|X_{T-1}^{(j)} - x| < h\}}}.$$

Consider the decomposition

$$\hat{C}_{T-1,M}(x) - C_{T-1}(x) = S_{1,M}(x) + \alpha_{T-1}(x) - C_{T-1}(x), \tag{A.5}$$

where

$$S_{1,M}(x) := \frac{\sum_{j=1}^M (g_T(X_T^{(j)}) - \alpha_{T-1}(x))1_{\{|X_{T-1}^{(j)} - x| < h\}}}{\sum_{j=1}^M 1_{\{|X_{T-1}^{(j)} - x| < h\}}},$$

$$\alpha_{T-1}(x) := \mathbb{E}[C_{T-1}(X_{T-1}^{(1)}) \,|\, |X_{T-1}^{(1)} - x| < h].$$

According to the result in Lemma A.1, for small enough h, there exist constants b_1, b_2 such that

$$b_1h^2 \le \alpha_{T-1}(x) - C_{T-1}(x) \le b_2h^2, \quad j = 1, \ldots, M. \tag{A.6}$$

Using (A.5), (A.6), for some $\epsilon > 0$ which we choose later, we can write

$$\mathbb{P}(|\hat{C}_{T-1,M}(x) - C_{T-1}(x)| \ge \epsilon) \le \mathbb{P}(|S_{1,M}(x)| \ge \epsilon - b_1h^2). \tag{A.7}$$

Next, we choose $h = O(M^{-\frac{1}{4+d}})$. Therefore, for large enough M, if we set $\epsilon = t(Mh^d)^{-1/2}$ for sufficiently large $t > 0$, we can write

$$\epsilon - b_1h^2 \ge \nu(Mh^d)^{-1/2},$$

where $\nu > 0$ is some constant. Next, let us denote by $\lambda_M(\cdot)$ the denominator in continuation value estimator $\hat{C}_{T-1,M}(\cdot)$, that is $\lambda_M(x) := \sum_{j=1}^{M} 1_{\{|X_{T-1}^{(j)}-x|<h\}}$. From Lemma 2, we know that for any γ chosen such that $0 < \gamma \le c_0 < f_{T-1}(x)$ for all $x \in \mathcal{A}$, we have

$$\mathbb{P}(\lambda_M(x) \le \gamma M h^d) \le \exp\left(-\frac{1}{2}\frac{(2c_1 + c - \gamma)^2}{(2c_0 - c)} M h^d\right). \tag{A.8}$$

Then in Eq. (A.7), we can write

$$\begin{aligned}
&\mathbb{P}(|S_{1,M}(x)| \ge \epsilon - D_1 h^2) \\
&\quad \le \mathbb{P}(|S_{1,M}(x)| \ge \nu (M h^d)^{-1/2}) \\
&\quad \le \mathbb{P}(\lambda_M(x) \le \gamma M h^d) \\
&\qquad + \mathbb{P}(|S_{1,M}(x)| \ge \nu (M h^d)^{-1/2}, \lambda_M(x) > \gamma M h^d).
\end{aligned} \tag{A.9}$$

Let us define the event $A_n(x) := \{\lambda_M(x) = n\}$. We observe,

$$\begin{aligned}
&\mathbb{E}[g_T(X_T^{(j)}) 1_{\{|X_{T-1}^{(j)}-x|<h\}} \mid A_n(x)] \\
&\quad = \mathbb{E}[g_T(X_T^{(j)}) \mid |X_{T-1}^{(j)} - x| < h] \\
&\quad = \mathbb{E}[C_{T-1}(X_{T-1}^{(j)}) \mid |X_{T-1}^{(j)} - x| < h] \\
&\quad = \alpha_{T-1}(x).
\end{aligned} \tag{A.10}$$

In the first equality, we note that as $g_T(X_T^{(j)})$ depends only on the jth path, it allows the change in conditioning. In the second equality, we use $\mathbb{E}[g_T(X_T^{(j)}) \mid X_{T-1}^{(j)}] = C_{T-1}(X_{T-1}^{(j)})$. Let us denote by Y a random variable which is distributed as $g_T(X_T)$ conditioned on the event $\{|X_{T-1} - x| < h\}$. Thus, from the result in (A.10), we can write

$$\begin{aligned}
&\mathbb{P}(|S_{1,M}(x)| \ge \nu (M h^d)^{-1/2} \mid A_n(x)) \\
&\quad = \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(Y^{(i)} - \mathbb{E}[Y]) \ge \nu (M h^d)^{-1/2}\right),
\end{aligned} \tag{A.11}$$

where $Y^{(1)}, \ldots, Y^{(n)}$ are independent and i.i.d. as random variable Y, and $\mathbb{E}[Y] := \mathbb{E}[g_T(X_T) \mid |X_{T-1} - x| < h]$. Since $g_T(x)$ is a bounded function, there exist constants B_1, B_2 ($B_1 \ne B_2$), such that

$$B_1 \le g_T(x) \le B_2. \tag{A.12}$$

Then, using (A.12) we can apply conditional version of Hoeffding's inequality (see pp. 56 in Dembo and Zeitouni [1998]) in (A.11) to get

$$\mathbb{P}(|S_{1,M}(x)| \ge \nu (M h^d)^{-1/2} \mid A_n(x)) \le \exp\left(-\frac{2\nu^2 n}{M h^d (B_2 - B_1)^2}\right). \tag{A.13}$$

Therefore, from (A.8), (A.9) and (A.13) we get

$$\begin{aligned}
&\mathbb{P}(|S_{1,M}(x)| \geq \nu(Mh^d)^{-1/2}) \\
&\quad \leq \exp\left(-\frac{1}{2}\frac{(2c_1 + c - \gamma)^2}{(2c_0 - c)} Mh^d\right) \\
&\qquad + \sum_{n > \gamma Mh^d} \exp\left(-\frac{2\nu^2 n}{Mh^d(B_2 - B_1)^2}\right) \mathbb{P}(A_n(x)) \\
&\quad \leq \exp\left(-\frac{2\gamma\nu^2}{(B_2 - B_1)^2}\right).
\end{aligned}$$

This completes the argument. We can repeat the same method of proof to show that for large enough M, if we choose h which satisfies $h = O(M^{-\frac{1}{4+d}})$, then there exist constants γ, B_1, B_2 $(B_1 \neq B_2)$, such that for any $\epsilon > 0$,

$$\mathbb{P}(|\hat{C}_{l,M}(X_l) - C_l(X_l)| \geq \epsilon(Mh^d)^{-1/2} \mid X_l) \leq \exp\left(-\frac{2\gamma\epsilon^2}{(B_2 - B_1)^2}\right) \quad \text{a.s.} \qquad \square$$

Proof of Lemma 3. By adding and subtracting $g_{\tau_i}(Z^{(i)}_{\tau_i})$ to the original expression, we can write

$$\begin{aligned}
&\mathrm{Cov}(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}), g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})) \\
&\quad = \mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - \mathbb{E}[g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i})])(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - \mathbb{E}[g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})])] \\
&\quad = \mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i}))(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - \mathbb{E}[g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})])] \\
&\qquad + \mathbb{E}[(g_{\tau_i}(Z^{(i)}_{\tau_i}) - \mathbb{E}[g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i})])(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - \mathbb{E}[g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})])]. \qquad \text{(A.14)}
\end{aligned}$$

From the definition of optimal stopping time τ and suboptimal stopping time $\hat{\tau}$ in (2) and (5) respectively, we observe that for all Phase 2 paths $i \neq j, i, j \in \{1, \ldots, N\}, (Z^{(i)}_{\tau_i}, Z^{(j)}_{\hat{\tau}_j})$ are independent. Therefore,

$$\mathbb{E}[(g_{\tau_i}(Z^{(i)}_{\tau_i}) - \mathbb{E}[g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i})])(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - \mathbb{E}[g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})])] = 0.$$

Next, we add and subtract $g_{\tau_j}(Z^{(j)}_{\tau_j})$ to the remaining term in (A.14) to get

$$\begin{aligned}
&\mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i}))(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - \mathbb{E}[g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})])] \\
&\quad = \mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i}))(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - g_{\tau_j}(Z^{(j)}_{\tau_j}))] \\
&\qquad + \mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i}))(g_{\tau_j}(Z^{(j)}_{\tau_j}) - \mathbb{E}[g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})])].
\end{aligned}$$

Once again, we note that for independent Phase 2 paths $i \neq j$, $Z^{(i)}_{\hat{\tau}_i}$ and $Z^{(j)}_{\tau_j}$ are independent. Also, $Z^{(i)}_{\tau_i}$ and $Z^{(j)}_{\tau_j}$ are independent as the Phase 2 paths $Z^{(i)}, Z^{(j)}, i \neq j$, themselves are independent. Hence, we can write

$$\mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i}))(g_{\tau_j}(Z^{(j)}_{\tau_j}) - \mathbb{E}[g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})])]$$

$$= \mathbb{E}[g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i})] \cdot \mathbb{E}[g_{\tau_j}(Z^{(j)}_{\tau_j}) - g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j})] \leq \left(K\left(\sum_{l=0}^{T-1} K_l\right)\right)^2 h^8,$$

where the final inequality above is obtained by using the result on asymptotic bias in Proposition 2. Therefore, in order to complete the result for covariance of estimator $\hat{v}_{0,M}$, it suffices to find the upper bound for

$$\mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i}))(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - g_{\tau_j}(Z^{(j)}_{\tau_j}))].$$

We prove the result for $T = 2$ which can be similarly extended for $T > 2$. Consider the following decomposition for $k \in \{1, \ldots, N\}$

$$\xi_{1,k} = (g_1(Z^{(k)}_1) - g_2(Z^{(k)}_2))1_{\{\hat{C}_{1,M}(Z^{(k)}_1) \leq g_1(Z^{(k)}_1) < C_1(Z^{(k)}_1)\}},$$

$$\xi_{2,k} = (g_2(Z^{(k)}_2) - g_1(Z^{(k)}_1))1_{\{C_1(Z^{(k)}_1) \leq g_1(Z^{(k)}_1) < \hat{C}_{1,M}(Z^{(k)}_1)\}}.$$

For $T = 2$, we can write

$$\mathbb{E}[(g_{\hat{\tau}_i}(Z^{(i)}_{\hat{\tau}_i}) - g_{\tau_i}(Z^{(i)}_{\tau_i}))(g_{\hat{\tau}_j}(Z^{(j)}_{\hat{\tau}_j}) - g_{\tau_j}(Z^{(j)}_{\tau_j}))] = \mathbb{E}[(\xi_{1,i} + \xi_{2,i}) \cdot (\xi_{1,j} + \xi_{2,j})]. \tag{A.15}$$

We derive the upper bound for $\mathbb{E}[\xi_{1,i} \cdot \xi_{1,j}]$ in the following part and the other terms can be similarly shown to satisfy the same upper bound. Define

$$\mathcal{F}_{1,i,j,M} := \sigma(Z^{(i)}_1, Z^{(j)}_2, X^{(1)}, \ldots, X^{(M)}),$$

and

$$\mathcal{F}_{2,i,j,M} := \sigma(Z^{(j)}_1, Z^{(i)}_1, X^{(1)}, \ldots, X^{(M)}).$$

By first conditioning on the σ-algebra $\mathcal{F}_{1,i,j,M}$ and then $\mathcal{F}_{2,i,j,M}$, we can show

$$\begin{aligned}
\mathbb{E}[\xi_{1,i}\xi_{1,j}] &= \mathbb{E}[(g_1(Z^{(i)}_1) - \mathbb{E}[g_2(Z^{(i)}_2)|\mathcal{F}_{1,i,j,M}])1_{\{\hat{C}_{1,M}(Z^{(i)}_1) \leq g_1(Z^{(i)}_1) < C_1(Z^{(i)}_1)\}} \\
&\quad \times (g_1(Z^{(j)}_1) - g_2(Z^{(j)}_2))1_{\{\hat{C}_{1,M}(Z^{(j)}_1) \leq g_1(Z^{(j)}_1) < C_1(Z^{(j)}_1)\}}] \\
&\quad = \mathbb{E}[(g_1(Z^{(i)}_1) - C_1(Z^{(i)}_1))1_{\{\hat{C}_{1,M}(Z^{(i)}_1) \leq g_1(Z^{(i)}_1) < C_1(Z^{(i)}_1)\}} \\
&\quad \times (g_1(Z^{(j)}_1) - \mathbb{E}[g_2(Z^{(j)}_2) \,|\, \mathcal{F}_{2,i,j,M}])1_{\{\hat{C}_{1,M}(Z^{(j)}_1) \leq g_1(Z^{(j)}_1) < C_1(Z^{(j)}_1)\}}] \\
&= \mathbb{E}[(g_1(Z^{(i)}_1) - C_1(Z^{(i)}_1))1_{\{\hat{C}_{1,M}(Z^{(i)}_1) \leq g_1(Z^{(i)}_1) < C_1(Z^{(i)}_1)\}} \\
&\quad \times (g_1(Z^{(j)}_1) - C_1(Z^{(j)}_1))1_{\{\hat{C}_{1,M}(Z^{(j)}_1) \leq g_1(Z^{(j)}_1) < C_1(Z^{(j)}_1)\}}],
\end{aligned}$$

where we have used the result that for $T = 2$,

$$\mathbb{E}[g_2(Z^{(k)}_2)|Z^{(k)}_1] = C_1(Z^{(k)}_1), \quad \forall\, k \in \{1, \ldots, N\}.$$

The idea of proof in the remaining part is to use the observation that the suboptimal policy results in an error when the value of payoff function $g_1(x)$ and continuation value function $C_1(x)$ are close to each other. For $h = O(M^{-\frac{1}{4+d}})$, since probability of continuation value estimator taking deviations larger than $O((Mh^d)^{-1/2})$ decays super exponentially (by Proposition 1), the size of exercise error region is $O((Mh^d)^{-1/2})$. Equipped with this information, we consider the following decomposition where for ease of notation, we denote by $\alpha_M := Mh^d$:

$$\mathcal{Q} := \{x \in \mathbb{R}^d : \hat{C}_{1,M}(x) \leq g_1(x) < C_1(x)\},$$

$$\mathcal{S}_0 := \{x \in \mathbb{R}^d : 0 < C_1(x) - g_1(x) \leq \alpha_M^{-1/2}\},$$

$$\mathcal{S}_j := \{x \in \mathbb{R}^d : 2^{j-1}\alpha_M^{-1/2} < C_1(x) - g_1(x) \leq 2^j\alpha_M^{-1/2}\}.$$

We can use it to write

$$\begin{aligned}
\mathbb{E}[\xi_{1,i}\xi_{1,j}] &= \mathbb{E}[(C_1(Z_1^{(i)}) - g_1(Z_1^{(i)})) \cdot (C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{Q}\}}1_{\{Z_1^{(j)} \in \mathcal{Q}\}}] \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \mathbb{E}[(C_1(Z_1^{(i)}) - g_1(Z_1^{(i)})) \\
&\quad \cdot (C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{Q}\cap\mathcal{S}_l\}}1_{\{Z_1^{(j)} \in \mathcal{Q}\cap\mathcal{S}_m\}}] \\
&\leq \alpha_M^{-1}\mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} \cdot 1_{\{Z_1^{(j)} \in \mathcal{S}_0\}}] + \sum_{m=1}^{\infty} \alpha_M^{-1/2}\mathbb{E}[(C_1(Z_1^{(j)}) - g_1(Z_1^{(j)})) \\
&\quad \times 1_{\{Z_1^{(i)} \in \mathcal{S}_0\}}1_{\{Z_1^{(j)} \in \mathcal{Q}\cap\mathcal{S}_m\}}] \\
&\quad + \sum_{l=1}^{\infty} \alpha_M^{-1/2}\mathbb{E}[(C_1(Z_1^{(i)}) - g_1(Z_1^{(i)}))1_{\{Z_1^{(i)} \in \mathcal{Q}\cap\mathcal{S}_l\}}1_{\{Z_1^{(j)} \in \mathcal{S}_0\}}] \\
&\quad + \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \mathbb{E}[(C_1(Z_1^{(i)}) - g_1(Z_1^{(i)})) \\
&\quad \cdot (C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{Q}\cap\mathcal{S}_l\}}1_{\{Z_1^{(j)} \in \mathcal{Q}\cap\mathcal{S}_m\}}]. \qquad \text{(A.16)}
\end{aligned}$$

We will need the following three claims from which the proof easily follows.

Claim 1. For Phase 2 paths $i \neq j$, we have,

$$\mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} \cdot 1_{\{Z_1^{(j)} \in \mathcal{S}_0\}}] \leq K_1^2\alpha_M^{-1}. \qquad \text{(A.17)}$$

Claim 2. For Phase 2 paths $i \neq j$, we have,

$$\begin{aligned}
&\mathbb{E}[(C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{S}_0\}}1_{\{Z_1^{(j)} \in \mathcal{Q}\cap\mathcal{S}_m\}}] \\
&\quad \leq 2^{2m}K_1^2\alpha_M^{-3/2}\exp\left(-\frac{\gamma 2^{2m-1}}{(B_2 - B_1)^2}\right). \qquad \text{(A.18)}
\end{aligned}$$

Claim 3. For Phase 2 paths $i \neq j$, we have,

$$\mathbb{E}[(C_1(Z_1^{(i)}) - g_1(Z_1^{(i)})) \cdot (C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{Q} \cap \mathcal{S}_l\}} 1_{\{Z_1^{(j)} \in \mathcal{Q} \cap \mathcal{S}_m\}}] \\ \leq 2^{2l+2m} K_1^2 \alpha_M^{-2} \exp\left(-\frac{\gamma 2^{2l+2m-4}}{(B_2 - B_1)^2}\right). \tag{A.19}$$

Finally, in (A.16), we use the results shown in (A.17)–(A.19) to write

$$\begin{aligned} \mathbb{E}[\xi_{1,i}\xi_{1,j}] &= \mathbb{E}[(g_1(Z_1^{(i)}) - g_2(Z_2^{(i)}))1_{\{\hat{C}_{1,M}(Z_1^{(i)}) \leq g_1(Z_1^{(i)}) < C_1(Z_1^{(i)})\}} \\ &\quad \times (g_1(Z_1^{(j)}) - g_2(Z_2^{(j)}))1_{\{\hat{C}_{1,M}(Z_1^{(j)}) \leq g_1(Z_1^{(j)}) < C_1(Z_1^{(j)})\}}] \\ &\leq K_1^2 \alpha_M^{-2} + \sum_{m=1}^{\infty} 2^{2m} K_1^2 \alpha_M^{-2} \exp\left(-\frac{\gamma 2^{2m-1}}{(B_2 - B_1)^2}\right) \\ &\quad + \sum_{l=1}^{\infty} 2^{2l} K_1^2 \alpha_M^{-2} \exp\left(-\frac{\gamma 2^{2l-1}}{(B_2 - B_1)^2}\right) \\ &\quad + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} 2^{2l+2m} K_1^2 \alpha_M^{-2} \exp\left(-\frac{\gamma 2^{2l+2m-4}}{(B_2 - B_1)^2}\right) \\ &\leq \bar{K} \alpha_M^{-2}, \end{aligned}$$

where $\bar{K}$ depends on γ, K_1, B_1 and B_2. Proceeding in a similar manner we can show

$$\mathbb{E}[\xi_{1,i}\xi_{2,j}], \mathbb{E}[\xi_{2,i}\xi_{1,j}], \mathbb{E}[\xi_{2,i}\xi_{2,j}] \leq \bar{K} \alpha_M^{-2}.$$

Since, $h = O(M^{-\frac{1}{4+d}})$, we can write

$$\mathbb{E}[(g_{\hat{\tau}_i}(Z_{\hat{\tau}_i}^{(i)}) - g_{\tau_i}(Z_{\tau_i}^{(i)}))(g_{\hat{\tau}_j}(Z_{\hat{\tau}_j}^{(j)}) - g_{\tau_j}(Z_{\tau_j}^{(j)}))] \leq \bar{K} h^8.$$

This completes the proof. $\square$

Proof of Claim 2. From the definition, it is true that for $x \in \mathcal{Q}$,

$$0 < C_1(x) - g_1(x) \leq C_1(x) - \hat{C}_{1,M}(x). \tag{A.20}$$

We use the observation in (A.20) to check that for $m > 0$,

$$\begin{aligned} &\mathbb{E}[(C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} 1_{\{Z_1^{(j)} \in \mathcal{Q} \cap \mathcal{S}_m\}}] \\ &\quad \leq 2^m \alpha_M^{-1/2} \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} 1_{\{Z_1^{(j)} \in \mathcal{S}_m\}} 1_{\{C_1(Z_1^{(j)}) - \hat{C}_1(Z_1^{(j)}) \geq 2^{m-1}\alpha_M^{-1/2}\}}] \\ &\quad = 2^m \alpha_M^{-1/2} \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} 1_{\{Z_1^{(j)} \in \mathcal{S}_m\}} \mathbb{P}(C_1(Z_1^{(j)}) - \hat{C}_1(Z_1^{(j)}) \\ &\quad \geq 2^{m-1} \alpha_M^{-1/2} \mid Z_1^{(j)})]. \end{aligned} \tag{A.21}$$

From Proposition 1, we have

$$\mathbb{P}(C_1(Z_1^{(j)}) - \hat{C}_1(Z_1^{(j)}) \geq 2^{m-1} \alpha_M^{-1/2} \mid Z_1^{(j)}) \leq \exp\left(-\frac{\gamma 2^{2m-1}}{(B_2 - B_1)^2}\right).$$

Plugging the above result back in (A.21), we get

$$\mathbb{E}[(C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} 1_{\{Z_1^{(j)} \in \mathcal{Q} \cap \mathcal{S}_m\}}]$$
$$\le 2^m \alpha_M^{-1/2} \exp\left(-\frac{\gamma 2^{2m-1}}{(B_2 - B_1)^2}\right) \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} 1_{\{Z_1^{(j)} \in \mathcal{S}_m\}}]. \tag{A.22}$$

Next, we analyze the following term

$$\begin{aligned}
&\mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} 1_{\{Z_1^{(j)} \in \mathcal{S}_m\}}] \\
&\quad = \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} \mathbb{P}(Z_1^{(j)} \in \mathcal{S}_m | Z_1^{(i)})] \\
&\quad \le \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_0\}} \mathbb{P}(0 < C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}) \le 2^m \alpha_M^{-1/2})] \\
&\quad \le 2^m K_1 \alpha_M^{-1/2} \mathbb{P}(0 < C_1(Z_1^{(i)}) - g_1(Z_1^{(i)}) \le \alpha_M^{-1/2}) \quad \text{(from Assumption 4)} \\
&\quad \le 2^m K_1^2 \alpha_M^{-1} \quad \text{(from Assumption 4).}
\end{aligned} \tag{A.23}$$

We have used the independence of $Z^{(i)}, Z^{(j)}$ in the first inequality. Using the result from (A.23) back in (A.22) completes the proof. □

Proof of Claim 3. For $l, m > 0$, by using the definition of set $\mathcal{Q}$, we get

$$\begin{aligned}
&\mathbb{E}[(C_1(Z_1^{(i)}) - g_1(Z_1^{(i)})) \cdot (C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}))1_{\{Z_1^{(i)} \in \mathcal{Q} \cap \mathcal{S}_l\}} 1_{\{Z_1^{(j)} \in \mathcal{Q} \cap \mathcal{S}_m\}}] \\
&\quad \le 2^{l+m} \alpha_M^{-1} \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_l\}} 1_{\{Z_1^{(j)} \in \mathcal{S}_m\}} \mathbb{E}[1_{\{C_1(Z_1^{(i)}) - \hat{C}_1(Z_1^{(i)}) \ge 2^{l-1} \alpha_M^{-1/2}\}} \\
&\qquad \times 1_{\{C_1(Z_1^{(j)}) - \hat{C}_1(Z_1^{(j)}) \ge 2^{m-1} \alpha_M^{-1/2}\}} \mid Z_1^{(i)}, Z_1^{(j)}]].
\end{aligned} \tag{A.24}$$

We analyze each term in the right-hand side of (A.24). Applying Cauchy–Schwartz inequality, we get

$$\begin{aligned}
&\mathbb{E}[1_{\{C_1(Z_1^{(i)}) - \hat{C}_1(Z_1^{(i)}) \ge 2^{l-1} \alpha_M^{-1/2}\}} 1_{\{C_1(Z_1^{(j)}) - \hat{C}_1(Z_1^{(j)}) \ge 2^{m-1} \alpha_M^{-1/2}\}} \mid Z_1^{(i)}, Z_1^{(j)}] \\
&\quad \le (\mathbb{P}(C_1(Z_1^{(i)}) - \hat{C}_1(Z_1^{(i)}) \ge 2^{l-1} \alpha_M^{-1/2} \mid Z_1^{(i)}))^{1/2} \\
&\qquad \times (\mathbb{P}(C_1(Z_1^{(j)}) - \hat{C}_1(Z_1^{(j)}) \ge 2^{m-1} \alpha_M^{-1/2} \mid Z_1^{(j)}))^{1/2} \\
&\quad \le \exp\left(-\frac{\gamma 2^{2l+2m-4}}{(B_2 - B_1)^2}\right) \quad \text{(by Proposition 1).}
\end{aligned} \tag{A.25}$$

Next, by using the independence of Phase 2 sample paths $Z^{(i)}, Z^{(j)}, i \neq j$ and from Assumption 4, we check

$$\begin{aligned}
\mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_l\}} 1_{\{Z_1^{(j)} \in \mathcal{S}_m\}}] &= \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_l\}} \mathbb{P}(Z_1^{(j)} \in \mathcal{S}_m \mid Z_1^{(i)})] \\
&\le \mathbb{E}[1_{\{Z_1^{(i)} \in \mathcal{S}_l\}} \mathbb{P}(0 < C_1(Z_1^{(j)}) - g_1(Z_1^{(j)}) \le 2^m \alpha_M^{-1/2})] \\
&\le 2^m K_1 \alpha_M^{-1/2} \mathbb{P}(0 < C_1(Z_1^{(i)}) - g_1(Z_1^{(i)}) \le 2^l \alpha_M^{-1/2}) \\
&\le 2^{l+m} K_1^2 \alpha_M^{-1}.
\end{aligned} \tag{A.26}$$

Using the results from (A.25) and (A.26) back in (A.24) completes the proof. □

Proof of Lemma 4. We can write,

$$\begin{aligned}\mathrm{Var}(g_{\hat\tau}(Z_{\hat\tau})) &= \mathbb{E}[(g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})])^2]\\ &= \mathrm{Var}(g_\tau(Z_\tau)) + \mathbb{E}[(g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})])^2]\\ &\quad - \mathbb{E}[(g_\tau(Z_\tau) - \mathbb{E}[g_\tau(Z_\tau)])^2],\end{aligned}$$

where τ is the optimal stopping time. We can combine the last two terms in the right-hand side of above equation and use the relation $a^2 - b^2 = (a+b)(a-b)$ to write,

$$\mathbb{E}[(g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})])^2] - \mathbb{E}[(g_\tau(Z_\tau) - \mathbb{E}[g_\tau(Z_\tau)])^2] = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}I_1 &:= \mathbb{E}[(g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_\tau(Z_\tau)])(\mathbb{E}[g_\tau(Z_\tau)] - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})])],\\ I_2 &:= \mathbb{E}[(g_\tau(Z_\tau) - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})])(\mathbb{E}[g_\tau(Z_\tau)] - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})])],\\ I_3 &:= \mathbb{E}[(g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_\tau(Z_\tau)])(g_{\hat\tau}(Z_{\hat\tau}) - g_\tau(Z_\tau))],\\ I_4 &:= \mathbb{E}[(g_\tau(Z_\tau) - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})])(g_{\hat\tau}(Z_{\hat\tau}) - g_\tau(Z_\tau))].\end{aligned}$$

Consider,

$$I_1 = \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_\tau(Z_\tau)]](\mathbb{E}[g_\tau(Z_\tau)] - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})]).$$

Each individual term in the right-hand side above equals bias of the estimator $\hat v_{0,M}$. We can show using Proposition 2 that for some constant $\bar K$,

$$\begin{aligned}\mathbb{E}[g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_\tau(Z_\tau)]] &\le \bar K h^4,\\ \mathbb{E}[g_\tau(Z_\tau)] - \mathbb{E}[g_{\hat\tau}(Z_{\hat\tau})] &\le \bar K h^4.\end{aligned}$$

Therefore, we get

$$I_1 \le \bar K^2 h^8. \tag{A.27}$$

Similarly,

$$I_2 \le \bar K^2 h^8. \tag{A.28}$$

As payoff function g_l is bounded, we can further show using Proposition 2,

$$\begin{aligned}I_3 &= \mathbb{E}[(g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_\tau(Z_\tau)])(g_{\hat\tau}(Z_{\hat\tau}) - g_\tau(Z_\tau))]\\ &\le 2K_g\mathbb{E}[g_{\hat\tau}(Z_{\hat\tau}) - \mathbb{E}[g_\tau(Z_\tau)]] \le 2K_g \bar K h^4, \qquad \text{(A.29)}\end{aligned}$$

where $\bar K, K_g$ are some constants. Similarly,

$$I_4 \le 2K_g \bar K h^4. \tag{A.30}$$

Therefore, from (A.27)–(A.30), we conclude

$$\mathrm{Var}(g_{\hat\tau}(Z_{\hat\tau})) = \mathrm{Var}(g_\tau(Z_\tau)) + O(h^4).$$ □

Proof of Proposition 4. Here, we prove the result for the first choice of likelihood ratio weights in Sec. 2.2. The result for the choice of average density likelihood ratio weights can be proved similarly.

We note that in the continuation value estimator $\tilde{C}_{l,M}(x)$, the denominator could take zero value. In that case, we assign $\tilde{C}_{l,M}(x) = 0$. At any exercise opportunity l, the denominator of $\tilde{C}_{l,M}(x)$ is given as $\sum_{i=1}^{M} 1_{\{|X_l^{(i)} - x| < h\}}$. Note that

$$\mathbb{P}(|X_l - x| \le h) = f_l(x)h^d + o(h^d) \quad \text{as } h \to 0,$$

where $f_l(x)$ is bounded from Assumption 1. It then follows,

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{M} 1_{\{|X_l^{(i)} - x| < h\}} = 0\right) &= (1 - f_l(x)h^d - o(h^d))^M \\ &= \exp(-Mf_l(x)h^d + o(h^d)).\end{aligned}$$

Hence, as $Mh^d \to \infty$, the probability of the event that denominator of $\tilde{C}_{l,M}(x)$ equals zero decays exponentially fast to zero in Mh^d. In the algorithm to calculate the upper biased estimator, if for a sample path, the denominator of the continuation value estimator $\tilde{C}_{l,M}(x)$ equals zero at any exercise opportunity $l = 1, \ldots, T-1$, we assign a chosen upper bound to the value function estimate $\tilde{V}_{l,M}(\cdot)$ such that the final estimator remains upper biased.

For ease of presentation, we ignore this small probability event in the remaining discussion. To prove the result in (15), we use mathematical induction. For $l = T-1$, applying Jensen's inequality in (13), we get,

$$\mathbb{E}[\tilde{V}_{T-1,M}(X_{T-1}^{(j)}) \,|\, \mathcal{F}_{T-1,M}] \ge \max\{g(X_{T-1}^{(j)}), \mathbb{E}[\tilde{C}_{T-1,M}(X_{T-1}^{(j)}) \,|\, \mathcal{F}_{T-1,M}]\},$$

where $\mathcal{F}_{T-1,M}$ is the σ-algebra generated from Phase 1 sample paths

$$\mathcal{F}_{l,M} := \sigma(X_l^{(1)}, \ldots, X_l^{(M)}), \quad l = 1, \ldots, T-1.$$

We examine the conditional expectation in the right-hand side of the above equation. From (14), we have

$$\begin{aligned}&\mathbb{E}[\tilde{C}_{T-1,M}(X_{T-1}^{(j)}) \,|\, \mathcal{F}_{T-1,M}] \\ &\quad = \mathbb{E}\left[\left. \frac{\sum_{i=1}^{M} \tilde{V}_{T,M}(X_T^{(i)}) \frac{f_{T-1}(X_{T-1}^{(j)}, X_T^{(i)})}{f_{T-1}(X_{T-1}^{(i)}, X_T^{(i)})} 1_{\{|X_{T-1}^{(i)} - X_{T-1}^{(j)}| < h\}}}{\sum_{i=1}^{M} 1_{\{|X_{T-1}^{(i)} - X_{T-1}^{(j)}| < h\}}} \right| \mathcal{F}_{T-1,M} \right] \\ &\quad = \frac{\sum_{i=1}^{M} 1_{\{|X_{T-1}^{(i)} - X_{T-1}^{(j)}| < h\}} \mathbb{E}\left[V_T(X_T^{(i)}) \frac{f_{T-1}(X_{T-1}^{(j)}, X_T^{(i)})}{f_{T-1}(X_{T-1}^{(i)}, X_T^{(i)})} \middle| \mathcal{F}_{T-1,M} \right]}{\sum_{i=1}^{M} 1_{\{|X_{T-1}^{(i)} - X_{T-1}^{(j)}| < h\}}} \\ &\quad = \frac{\sum_{i=1}^{M} 1_{\{|X_{T-1}^{(i)} - X_{T-1}^{(j)}| < h\}} C_{T-1}(X_{T-1}^{(j)})}{\sum_{i=1}^{M} 1_{\{|X_{T-1}^{(i)} - X_{T-1}^{(j)}| < h\}}} = C_{T-1}(X_{T-1}^{(j)}).\end{aligned}$$

The first equality uses Eq. (12) and in the second equality we use the result

$$\mathbb{E}\left[V_T(X_T^{(i)})\frac{f_{T-1}(X_{T-1}^{(j)},X_T^{(i)})}{f_{T-1}(X_{T-1}^{(i)},X_T^{(i)})}\,\middle|\,\mathcal{F}_{T-1,M}\right]$$
$$= \int V_T(y)\frac{f_{T-1}(X_{T-1}^{(j)},y)}{f_{T-1}(X_{T-1}^{(i)},y)}f_{T-1}(X_{T-1}^{(i)},y)\,dy$$
$$= \int V_T(y)f_{T-1}(X_{T-1}^{(j)},y)\,dy = C_{T-1}(X_{T-1}^{(j)}).$$

Therefore, we get

$$\mathbb{E}[\tilde{V}_{T-1,M}(X_{T-1}^{(j)})\,|\,\mathcal{F}_{T-1,M}] \geq \max\{g(X_{T-1}^{(j)}), C_{T-1}(X_{T-1}^{(j)})\} = V_{T-1}(X_{T-1}^{(j)}).$$

Now we show that if

$$\mathbb{E}[\tilde{V}_{l,M}(X_l^{(j)})\,|\,\mathcal{F}_{l,M}] \geq V_l(X_l^{(j)}), \quad \text{for all } j = 1,\ldots,M$$

holds at $l = T'+1$, it holds at $l = T'$. Suppose that

$$\mathbb{E}[\tilde{V}_{T'+1,M}(X_{T'+1}^{(j)})\,|\,\mathcal{F}_{T'+1,M}] \geq V_{T'+1}(X_{T'+1}^{(j)}), \quad \text{for all } j = 1,\ldots,M. \tag{A.31}$$

Then, for $l = T'$, by applying Jensen's inequality, we get

$$\mathbb{E}[\tilde{V}_{T',M}(X_{T'}^{(j)})\,|\,\mathcal{F}_{T',M}] \geq \max\{g(X_{T'}^{(j)}), \mathbb{E}[\tilde{C}_{T',M}(X_{T'}^{(j)})\,|\,\mathcal{F}_{T',M}]\}.$$

Then in the right-hand side of above equation, we have

$$\mathbb{E}[\tilde{C}_{T',M}(X_{T'}^{(j)})\,|\,\mathcal{F}_{T',M}]$$
$$= \frac{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}\mathbb{E}\left[\tilde{V}_{T'+1,M}(X_{T'+1}^{(i)})\frac{f_{T'}(X_{T'}^{(j)},X_{T'+1}^{(i)})}{f_{T'}(X_{T'}^{(i)},X_{T'+1}^{(i)})}\,\middle|\,\mathcal{F}_{T',M}\right]}{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}}$$
$$= \frac{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}\mathbb{E}[\tilde{V}_{T'+1,M}(X_{T'+1}^{(j)})\,|\,\mathcal{F}_{T',M}]}{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}}$$
$$= \frac{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}\mathbb{E}[\mathbb{E}[\tilde{V}_{T'+1,M}(X_{T'+1}^{(j)})\,|\,\mathcal{F}_{T'+1,M}]\,|\,\mathcal{F}_{T',M}]}{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}}$$
$$\geq \frac{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}\mathbb{E}[V_{T'+1}(X_{T'+1}^{(j)})\,|\,\mathcal{F}_{T',M}]}{\sum_{i=1}^M 1_{\{|X_{T'}^{(i)}-X_{T'}^{(j)}|<h\}}} \quad \text{(from (A.31))}$$
$$= C_{T'}(X_{T'}^{(j)}).$$

Finally, we get

$$\mathbb{E}[\tilde{V}_{T',M}(X_{T'}^{(j)})\,|\,\mathcal{F}_{T',M}] \geq \max\{g(X_{T'}^{(j)}), C_{T'}(X_{T'}^{(j)})\} = V_{T'}(X_{T'}^{(j)})$$

which completes the induction argument. □

References

Belomestny, D. [2011] Pricing Bermudan options by nonparametric regression: Optimal rates of convergence for lower estimates, *Financ. Stoch.* **15**(4), 655–683.

Belomestny, D., Dickmann, F. and Nagapetyan, T. [2013] Pricing American options via multi-level approximation methods, preprint (2013), arXiv:1303.1334.

Beskos, A. and Roberts, G. O. [2005] Exact simulation of diffusions, *Ann. Appl. Probab.* **15**(4), 2422–2444.

Black, F. and Scholes, M. [1973] The pricing of options and corporate liabilities, *J. Polit. Econ.* **81**(3), 637–654.

Bosq, D. [1998] *Nonparametric Statistics for Stochastic Processes*: *Estimation and Prediction* (Springer-Verlag, Berlin).

Broadie, M. and Glasserman, P. [1997] Pricing American-style securities using simulation, *J. Econ. Dynam. Control* **21**(8), 1323–1352.

Broadie, M. and Glasserman, P. [2004] A stochastic mesh method for pricing high-dimensional American options, *J. Comput. Financ.* **21**, 35–72.

Carriere, J. F. [1996] Valuation of the early-exercise price for options using simulations and nonparametric regression, *Insur. Math. Econ.* **19**(1), 19–30.

Chen, N. and Huang, Z. [2013] Localization and exact simulation of Brownian motion-driven stochastic differential equations, *Math. Oper. Res.* **38**, 591–616.

Dembo, A. and Zeitouni, O. [1998] *Large Deviations Techniques and Applications* (Springer, Berlin).

Duffie, D. [1996] *Dynamic Asset Pricing Theory* (Princeton University Press, NJ, USA).

Feller, W. [2008] *An Introduction to Probability Theory and its Applications* (John Wiley & Sons, NJ, USA).

Geske, R. and Johnson, H. E. [1984] The American put option valued analytically, *J. Financ.* **39**(5) 1511–1524.

Glasserman, P. [2004] *Monte Carlo Methods in Financial Engineering* (Springer, Berlin).

Harrison, J. M. and Kreps, D. M. [1979] Martingales and arbitrage in multiperiod securities markets, *J. Econ. Theory* **20**(3) 381–408.

Hong, L. J., Juneja, S. and Liu, G. [2013] Kernel Smoothing for Nested Estimation with Application to Portfolio Risk Management, http://personal.cb.cityu.edu.hk/jeffhong/Papers/HongJunejaLiv_revised.pdf

Kohler, M., Krzyżak, A. and Todorovic, N. [2010] Pricing of high-dimensional American options by neural networks, *Math. Financ.* **20**(3), 383–410.

Longstaff, F. A. and Schwartz, E. S. [2001] Valuing American options by simulation: A simple least-squares approach, *Rev. Financ. Stud.* **14**(1), 113–147.

Pagan, A. and Ullah, A. [1999] *Nonparametric Econometrics* (Cambridge University Press, Cambridge, UK).

Tsitsiklis, J. N. and Van Roy, B. [2001] Regression methods for pricing complex American-style options, *IEEE Trans. Neural Networks* **12**(4), 694–703.