Erratum: Optimal δ-Correct Best-Arm Selection for Heavy-Tailed Distributions

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In this erratum we address the errors that stemmed because of an incorrect assumption made in the proof of Lemma 32 and improper handling of the constant terms in Theorem 13 in the original paper [1]. We provide corrections for the following errors:

1. Equation (80) in the proof of Lemma 32 bounds the probability of empirical distribution not belonging to the Wasserstein-1 ball of radius ζ, centered at the original distributon. Here, we assumed that the term in the exponent, $\tilde{g}_i(T)$, is non-zero for each arm i, which may not be true. Similarly, for the exponent in Equation (81) in the proof of the Lemma.

2. In Theorem 13, the terms $\tilde{B}_a$ were mistaken to be constants, independent of the arm-distributions. This theorem was then used to specify the constant C (Equation (16)) that appears in the expression of the stopping threshold $\beta(n,\delta)$, resulting in $\beta(n,\delta)$ to depend on the parameters which are unknown to the algorithm. Whence, the threshold in (10) cannot be used in the current form.

We now provide corrections for these errors. For $\eta \in \mathcal{P}(\Re)$ define the projection onto $\mathcal{L}$ (denoted by Π) by

$$\Pi(\eta) \in \operatorname*{argmin}_{\kappa\in\mathcal{L}} d_K(\eta,\kappa), \quad \text{where} \quad d_K(\eta,\kappa) = \sup_{x\in\Re} \left|F_\eta(x) - F_\kappa(x)\right| \tag{1}$$

denotes the Kolmogorov distance between the two probability measures, and F_η and F_κ denote the CDF functions corresponding to η and κ. Furthermore, Lévy metric between η and κ is given by

$$d_L(\eta,\kappa) = \inf\left\{\epsilon > 0 : F_\eta(x-\epsilon) - \epsilon \le F_\kappa(x) \le F_\eta(x+\epsilon) + \epsilon,\ \forall x \in \Re\right\}.$$

To fix to 1 above, we make the following changes.

i) In the proof of sample complexity (Appendix C.6.2.), define the balls $B_\zeta(\mu_i)$ in terms of d_K instead of the Wasserstein-1 metric and use the standard DKW inequality to get an exponential decay.

ii) Observe that with the above definition of the ball B_ζ, as ζ reduces to 0, the distributions within the ball converge to the center in the Kolmogorov metric, and hence, in the Lévy metric. Sample complexity proof uses the fact that as the distributions in these balls converge to the original arm-distributions (center of the balls), the functions KL_{inf} and t^* also converge to their values at the original arm-distributions. Changing the metric above from Wasserstein-1 to Kolmogorov leads to following changes:

   a) In Lemma 4 and Theorem 5, the topology under consideration will be that generated by the Lévy metric, or equivalently, the topology of weak convergence. This update in the topology affects the proof of upper-semicontinuity of $\mathrm{KL}_{\inf}$ (Lemma 4), which follows exactly as in [2, Lemmas B.4, B.5]. The proofs of Theorem 5, and the remaining results in Lemma 4 stay unaffected. Observe that this change only guarantees continuity of $\mathrm{KL}_{\inf}(\cdot,\cdot)$ in the first argument, when restricted to $\mathcal{L}$, and hence, of t^* defined only as a function from $\mathcal{L}^K$, instead of from $(\mathcal{P}(\Re))^K$. Also, for $\eta \in \mathcal{P}(\Re) \setminus \mathcal{L}$, $\mathrm{KL}_{\inf}(\eta, \cdot)$ is lower-semicontinuous. This can be seen from the proof in [1, Page 19] or [2, Lemma B.3].

b) In view of a) above, t^* is now defined from $\mathcal{L}^K$ to Σ_K. However, empirical distribution may not belong to $\mathcal{L}^K$. To address this, we modify the sampling rule to compute t^* for the projected empirical distributions (projections defined using Kolmogorov metric on $\mathcal{P}(\Re)$, see (1)) instead of those for actual empirical distributions, as in [2]. Thus, the algorithm now computes $t^*\left(\Pi(\hat{\mu}(lm))\right)$, instead of $t^*(\hat{\mu}(lm))$. Notice that we use the projected empirical distribution only for the sampling rule.

c) Remark 6 is no longer needed.

d) Update $t^*(\mu')$ to $t^*\left(\Pi(\mu')\right)$, wherever applicable. This change does not affect the proofs of the results.

With these changes incorporated, the proof for sample complexity gives that the proposed algorithm is optimal asymptotically, as $\delta \to 0$. We now mention the steps from the original proof of sample complexity, that need to be modified.

- To address i), since t^* is a continuous function of the arguments (Theorem 5), and Π is a continuous map in the Kolmogorov metric, there exists $\zeta(\epsilon) \geq 0$ such that

$$\forall \mu' \in \mathcal{I}_\epsilon, \ \max_{a\in[K]} \left| t^*_a\left(\Pi\left(\mu'\right)\right) - t^*_a(\mu) \right| \leq \epsilon,$$

for $\mathcal{I}_\epsilon$ defined as:

$$\mathcal{I}_\epsilon \triangleq B_\zeta(\mu_1) \times \cdots \times B_\zeta(\mu_K),$$

where

$$B_\zeta(\mu_i) = \left\{ \kappa \in \mathcal{P}(\Re) : d_K(\kappa, \mu_i) \leq \zeta \right\}.$$

- For $\mu' \in \mathcal{P}(\Re)$, and $t' \in \Sigma_K$, define

$$g(\mu', t') \triangleq \max_{a\in[K]} \min_{b\neq a} \inf_{x\in[-f^{-1}(B), f^{-1}(B)]} \left(t'_a \,\mathrm{KL}_{\inf}(\mu'_a, x) + t'_b \,\mathrm{KL}_{\inf}(\mu'_b, x) \right).$$

Since $\mathrm{KL}_{\inf}$ is a jointly lower-semicontinuous function (from a)), it follows that $g(\mu, t)$ is a jointly lower-semicontinuous function of the (μ, t) (see, [2]).

- From Equation (68) we have that

$$\limsup_{\delta\to 0} \frac{\mathbb{E}_\mu(\tau_\delta)}{\log(1/\delta)} \leq \frac{(1+\tilde{e})}{C^*_\epsilon(\mu)} + \limsup_{\delta\to 0} \frac{m}{\log(1/\delta)}.$$

From lower-semicontinuity of $g(\mu', t')$ in (μ', t'), it follows that $\liminf_{\epsilon\to 0} C^*_\epsilon(\mu) \geq V(\mu)$, which is in the correct direction. Remaining steps in the proof remain unaffected given that Lemmas 31 and 32 hold.

Let us now look at the modification in the proofs of Lemmas 31 and 32.

- Update in equation (72): recall that the stopping statistic is given by $Z(t) = \max_a \min_{b\neq a} Z_{a,b}(t)$, where

$$Z_{a,b}(t) = t \inf_{x\in[-f^{-1}(B), f^{-1}(B)]} \left(\frac{N_a(t)}{t} \mathrm{KL}_{\inf}(\hat{\mu}_a(t), x) + \frac{N_b(t)}{t} \mathrm{KL}_{\inf}(\hat{\mu}_b(t), x) \right).$$

Accordingly, Equation (73) can be updated. Remaining proof for Lemma 31 stays as is.

- In the proof of Lemma 32, inequality in (79) is modified to

$$\mathbb{P}_\mu\left(\hat{\mu}(lm) \notin \mathcal{I}_\epsilon\right) \leq \sum_{i=1}^{K} \mathbb{P}_\mu\left(d_K(\hat{\mu}_i(lm), \mu_i) \geq \zeta\right),$$

  where we change the metric from d_W to d_K.

- The summand above can be bounded using the union bound (over the possible values of $N_a(lm)$) and DKW inequality as below:

$$\begin{aligned}\mathbb{P}_\mu\left(d_K(\hat{\mu}_i(lm), \mu_i) \geq \zeta\right) &= \mathbb{P}_\mu\left(d_K(\hat{\mu}_i(lm), \mu_i) \geq \zeta,\ N_i(lm) \geq \sqrt{lm} - 1\right) \\ &\leq e^{-2\zeta^2(\sqrt{lm}-1)}\left(1 - e^{-2\zeta^2}\right)^{-1}.\end{aligned}$$

- Let $E_1 = e^{2\zeta^2}\left(1 - e^{-2\zeta^2}\right)^{-1}$ and $E_2 = 2\zeta^2$. Using the above bound in Equation (83),

$$\sum_{l=l_0(T)}^{l_2(T)} \mathbb{P}_\mu\left(\hat{\mu}(lm) \notin \mathcal{I}_\epsilon\right) \leq \sum_{l=l_0(T)}^{l_2(T)} E_1 \exp\left(-E_2\sqrt{lm}\right) \leq \frac{E_1 T}{m} \exp\left(-E_2 T^{1/8}\right). \tag{2}$$

- The bound for the other term in the event $\mathcal{G}_T^c(\epsilon)$ remains unaffected.

Next, to fix 2, we construct super-martingales that almost surely dominate the exponentials of $N_a(n)\,\mathrm{KL}_{\inf}(\hat{\mu}_a(n), m(\mu_a))$ (see, [2, Theorem 4.4]), and the proof for δ-correctness follows exactly as in [2].

- Setting

$$\beta(n, \delta) = \ln\frac{K-1}{\delta} + 4\ln(n+1) + 2$$

  in the stopping rule for the algorithm ensures its δ-correctness.

- Update the r.h.s. in (19) using the above mentioned form of β.

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References

[1] S. Agrawal, S. Juneja, and P. Glynn. Optimal δ-correct best-arm selection for heavy-tailed distributions. In *Proceedings of the 31st International Conference on Algorithmic Learning Theory*, volume 117 of *Proceedings of Machine Learning Research*, pages 61–110. PMLR, 08 Feb–11 Feb 2020.

[2] S. Agrawal, W. M. Koolen, and S. Juneja. Optimal best-arm identification methods for tail-risk measures. *CoRR*, abs/2008.07606, 2020.