# Dynamic Portfolio Credit Risk and Large Deviations

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**Abstract** We consider a multi-time period portfolio credit risk model. The default probabilities of each obligor in each time period depend upon common as well as firm specific factors. The time movement of these factors is modelled as a vector autoregressive process. The conditional default probabilities are modelled using a general representation that subsumes popular default intensity models, logit-based models as well as threshold based Gaussian copula models. We develop an asymptotic regime where the portfolio size increases to infinity. In this regime, we conduct large deviations analysis of the portfolio losses. Specifically, we observe that the associated large deviations rate function is a solution to a quadratic program with linear constraints. Importantly, this rate function is independent of the specific modelling structure of conditional default probabilities. This rate function may be useful in identifying and controlling the underlying factors that contribute to large losses, as well as in designing fast simulation techniques for efficiently measuring portfolio tail risk.

## 1 Introduction

Financial institutions such as banks have portfolio of assets comprising thousands of loans, defaultable bonds, credit sensitive instruments and other forms of credit exposures. Calculating portfolio loss distribution at a fixed time in future as well as its evolution as a function of time, is crucial to risk management: Of particular interest are computations of unexpected loss or tail risk in the portfolio. These values are important inputs to the amount of capital an institution may be required to hold for

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regulatory purposes. There is also interest in how this capital requirement evolves over time.

In this short note, we develop a discrete time dynamic model that captures the stochastic evolution of default probabilities of different firms in the portfolio as a function of time. We then develop an asymptotic regime to facilitate analysis of tail distribution of losses. We conduct large deviations analysis of losses in this regime identifying the large deviations rate function associated with large losses. This tail analysis provides great deal of insight into how large losses evolve over time in a credit portfolio.

There is a vast literature on modelling credit risk and on modelling a portfolio of credit risk (see, e.g., [9, 15, 10]). [11, 4, 12, 2, 17] are some of the works that conduct large deviations analysis for large portfolio losses in a static single period setting.

*Our contributions:* As mentioned earlier, we model the evolution of the credit portfolio in discrete time. The conditional probabilities of default of surviving firms in any time period is modelled as a function of a linear combination of stochastic covariates. This subsumes logit function models for conditional probabilities, default intensity models (in discrete time) as well as threshold based Gaussian and related copula models (see [8, 7, 6, 3, 16] as examples where similar dependence on stochastic covariates is considered). We model the stochastic evolution of the stochastic covariates as a vector AR process, although the essential features of our analysis are valid more broadly.

As is a common modelling practice, we assume that these stochastic variates are multivariate Gaussian distributed, and can be classified as:

- *Systemic common covariates:* These capture macroeconomic features such as GDP growth rates, unemployment rates, inflation, etc.
- *Class specific covariates:* All loans in our portfolio belong to one of a fixed number of classes. These capture the common exposure to risk to obligors in the same industry, geographic region, etc.
- *Idiosyncratic variates:* This captures the idiosyncratic risk corresponding to each obligor.

We embed the portfolio risk problem in a sequence of problems indexed by the portfolio size $n$. We develop an asymptotic regime where the conditional default probabilities decrease as $n$ increases. In this regime we identify the large deviations rate function of the probability of large losses at any given time in future. Our key contribution is to show that the key component to ascertaining this rate function is a solution to a quadratic program with linear constraints. Further we observe in specialized settings that the resultant quadratic program can be explicitly solved to give a simple expression for the large deviations rate function. Our other contribution is to highlight that in a fairly general framework, the underlying structure of how portfolio losses build up is independent of the specific model for conditional default probabilities - thus whether we use default intensity model, logit based model or a

Gaussian Copula based model for default probabilities, to the first order (that is, on the large deviations scaling), the portfolio tail risk measurement is unaffected.

Our large deviations analysis may be useful in identifying and controlling parameters that govern the probability of large losses. It is also critical to development of fast simulation techniques for the associated rare large loss probabilities. Development of such techniques is part of our ongoing research and not pursued here. In this paper, we assume that each class has a single class specific covariate and these are independent of all other covariates. This is a reasonable assumption in practice and makes the analysis substantially simpler. As we discuss later in Section 3, relaxing this and many other assumptions, is part of our ongoing research that will appear separately.

*Roadmap:* In Section 2, we develop the mathematical framework including the asymptotic regime for our analysis. We end with a small conclusion and a discussion of our ongoing work in Section 3. Some of the technical details are kept in the appendix in Section 4.

## 2 Mathematical Model

Consider a portfolio credit risk model comprising $n$ obligors. These are divided into $K$ classes $\{1,2,\ldots,K\}$, $\mathscr{C}_j$ denotes the obligors in class $j$. As mentioned in the introduction, we model conditional default probabilities using structures that subsume discrete default intensity models considered in Duffie, Saita, Wang [8] as well as Duan et. al. [7], popular logit models (see, e.g., [3], [16]), as well as threshold based Gaussian and related copula models (see, e.g., [11, 12, 2]).

First consider the discrete default intensity model and suppose that time horizon of our analysis is a positive integer $\tau$. We restrict ourselves to discrete default intensities taking the proportional-hazards form as in [8, 7]. Specifically, suppose that one period conditional default probability for a firm $i$ in $\mathscr{C}_j$, at period $t \leq \tau$, is given by

$$p_{i,j,t} = 1 - \exp[-\exp(P_{i,j,t})],$$

where,

$$P_{i,j,t} = -\alpha_j + \beta^T \mathbf{F}_t + \gamma_j G_{j,t} + \varepsilon_{i,t},$$

where the above variables have the following structure:

- For $j \leq K$, $\alpha_j > 0$ and for for $d \geq 1$, $\beta \in \Re^d$. $(\gamma_j, j \in K)$ are w.l.o.g. non-negative constants.
- Random variables $\varepsilon = (\varepsilon_{i,t} : i \leq n, t \leq \tau)$ are assumed to be i.i.d. (independent, identically distributed) with standard Gaussian distribution with mean zero and variance one.
- $(\mathbf{F}_t \in \Re^d : t = 0,\ldots,\tau)$ denote the common factors that affect default probabilities of each obligor. To keep the analysis simple we assume that $(\mathbf{F}_t : t = 0,\ldots,\tau)$ follows the following VAR(1) process.

$$\mathbf{F_{t+1}} = \mathbf{A}\mathbf{F}_t + \tilde{\mathbf{E}}_{t+1}$$

where $\mathbf{A} \in \Re^{d \times d}$ and $\tilde{\mathbf{E}} = (\tilde{\mathbf{E}}_t : t = 1, \ldots, \tau)$ is a sequence of i.i.d. random vectors, assumed to be Multi-variate Gaussian with mean 0 and positive definite variance covariance matrix $\Sigma$. Further let $\mathbf{B}$ be a matrix such that $\mathbf{B}\mathbf{B}^T = \Sigma$. Then, we can model

$$\mathbf{F_{t+1}} = \mathbf{A}\mathbf{F}_t + \mathbf{B}\mathbf{E}_{t+1}$$

where, each $\mathbf{E}_t = (E_{t,j} : j \leq d)$ is a vector of independent mean zero, variance one, Gaussian random variables. Then, it follows that for $t \geq 1$,

$$\mathbf{F_t} = \mathbf{A}^t \mathbf{F}_0 + \sum_{i=1}^{t} \mathbf{A}^{t-i} \mathbf{B}\mathbf{E}_i.$$

- The random variables $(G_{j,t} : j \leq K, t \leq \tau)$ capture the residual class risk (once the risk due to the common factors is accounted for by $\tilde{\mathbf{E}}$). These are assumed to be independent of $\tilde{\mathbf{E}}$ as well as $\varepsilon$. Further, we assume that they follow a simple autoregressive structure

$$G_{j,t} = \eta_j G_{j,t-1} + \Lambda_{j,t}$$

  where $(\Lambda_{j,t} : j \leq K, t \leq \tau)$ are assumed to be i.i.d., mean zero, variance one, standard Gaussian distributed.
- It follows that

$$G_{j,t} = \eta_j^t G_{j,0} + \sum_{i=1}^{t} \eta_j^{t-i} \Lambda_{j,i}. \tag{1}$$

To keep the analysis notationally simple, we assume that exposure $e_i$ of each obligor $i \in \mathscr{C}_j$ equals $ex_j$. This denotes the amount lost if an obligor in $\mathscr{C}_j$ defaults net of recoveries made on the loan.

An analogous logit structure for conditional probabilities corresponds to setting

$$p_{i,j,t} = \frac{\exp(P_{i,j,t})}{1 + \exp(P_{i,j,t})}.$$

In the remainder of the paper, we assume that

$$p_{i,j,t} = F(P_{i,j,t})$$

where $F : \Re \to [0,1]$ is a distribution function that we assume is strictly increasing. Thus, $F(-\infty) = 0$ and $F(\infty) = 1$. In the setting of Logit function

$$F(x) = \frac{e^{\theta x}}{1 + e^{\theta x}} \tag{2}$$

and for default intensity function

$$F(x) = 1 - \exp(-e^{\theta x}) \tag{3}$$

for $\theta > 0$.

Another interesting setting to consider is the J. P. Morgan's threshold based Gaussian Copula models extensively studied in literature, see, e.g., [11] and [12]. Adapting this approach to our setting, an obligor $i$ in class $j$ that has survived till time $t-1$, defaults at time $t$ if

$$\beta^T \mathbf{F}_t + \gamma_j G_{j,t} + \varepsilon_{i,t} > \alpha_j$$

for large $\alpha_j$. These models are studied in literature for a single time period, but can be generalized for multiple time periods by having a model for time evolution of common and class specific factors, as we consider in this paper.

One way to concretely fit this to our outlined framework, express

$$\varepsilon_{i,t} = \frac{\varepsilon_{i,t}(1) + \varepsilon_{i,t}(2)}{\sqrt{2}}$$

where $\varepsilon_{i,t}(1)$ and $\varepsilon_{i,t}(2)$ are independent Gaussian mean zero, variance one random variables. Then, set

$$P_{i,j,t} = -\alpha_j + \beta^T \mathbf{F}_t + \gamma_j G_{j,t} + \frac{1}{\sqrt{2}}\varepsilon_{i,t}(1)$$

to get

$$p_{i,j,t} = F(P_{i,j,t}) = \bar{\Phi}(-P_{i,j,t}) \tag{4}$$

where $\bar{\Phi}(\cdot)$ denotes the tail distribution function of a mean zero, variance half, Gaussian random variable (here $F(x) = \bar{\Phi}(-x)$).

### 2.1 Probability of large losses

In this note, our interest is in developing large deviations asymptotic for the probability of large losses in the portfolio by any specified time $\tau$. It may be useful to spell out a Monte Carlo algorithm to estimate the probability that portfolio losses $L$ by time $\tau$ exceed a large threshold $u$.

*Monte Carlo Algorithm:* Suppose that the current time is zero and our interest is in generating via simulation independent samples of portfolio losses by time $\tau$. We assume that $\mathbf{F}_0$ and $(G_j(0) : j \leq K)$ are available to us.

In the algorithm below, let $\mathscr{S}_t$ denote the surviving, non-defaulted obligors at (just after) time $t$ and $\mathscr{L}_t$ denote the losses incurred at time $t$. $\mathscr{S}_0$ denotes all the obligors. The algorithm then proceeds as follows

1. Set time $t = 1$.
2. While $t \leq \tau$,

a. Generate independent samples of $(\varepsilon_{i,t} : i \in \mathscr{S}_{t-1})$, $\mathbf{E}_t$ and $(\Lambda_{j,t} : j \leq K)$ and compute $p_{i,j,t}$ for each $(i \in \mathscr{S}_{t-1}, j \leq K)$.
b. Generate independent uniform numbers $(U_{i,t} : i \in \mathscr{S}_{t-1})$. Obligor $i \in \mathscr{S}_{t-1}$ defaults at time $t$ if $U_{i,t} \leq p_{i,j,t}$. Recall that obligor $i \in \mathscr{C}_j$ causes loss $e_j$ if it defaults. Compute $\mathscr{S}_t$ as well as $\mathscr{L}_t$.

3. A sample of total loss by time $T$ is obtained as $L = \sum_{t=1}^{\tau} \mathscr{L}_t$.
4. Set $I(L > u)$ to one if the loss $L$ exceeds $u$ and zero otherwise. Sample average of independent samples of $I(L > u)$ then provides an unbiased and consistent estimator of $P(L > u)$.

As mentioned in the introduction, we analyze the probability of large losses in an asymptotic regime that we develop in Section 2.2.

## 2.2 Asymptotic regime

Let $(\mathscr{P}_n : n \geq 1)$ denote a sequence of portfolios. $\mathscr{P}_n$ denotes a portfolio with $n$ obligors. As before the size of class $\mathscr{C}_k$ in $\mathscr{P}_n$ equals $c_k n$ so that $\sum_{k=1}^{K} c_k = 1$. To avoid unnecessary notational clutter we assume that $c_k n$ is an integer for each $k$ and $n$.

In $\mathscr{P}_n$, for each $n$, the conditional probability of default $p_{i,j,t}(n)$ at time $t$ for obligor $i \in \mathscr{C}_j$ that has not defaulted by time $t-1$ is denoted by $F(P_{i,j,t}(n))$, where

$$P_{i,j,t}(n) = -\alpha_j m_n + \tilde{m}_n \beta^T \mathbf{F}_t + \tilde{m}_n \gamma_j G_{j,t} + \tilde{m}_n \varepsilon_{i,t}$$

for each $n, i$ and $t$. Here, $m_n$ and $\tilde{m}_n$ are positive sequences increasing with $n$. The sequence of random vectors $(\mathbf{F}_t : t \leq \tau)$ and $(G_{j,t} : j \leq K, t \leq \tau)$ evolve as specified in the previous section and notations $(\mathbf{E}_t : t \leq \tau)$ and $(\Lambda_{j,t} : j \leq K, t \leq \tau)$ remain unchanged. For notations $(\mathscr{S}_t, \mathscr{L}_t : t \leq \tau)$, $(U_{i,t} : i \leq n, t \leq \tau)$ we simply suppress dependence on $n$ for presentation simplicity. Here, $(U_{i,t} : i \leq n, t \leq \tau)$ are used to facilitate Monte Carlo interpretation of defaults.

The following assumption is needed.

**Assumption 1**

$$\limsup_{n \to \infty} r_n = \frac{m_n}{\tilde{m}_n} = \infty. \tag{5}$$

*Remark 1.* There is a great deal of flexibility in selecting $\{m_n\}$ and $\{\tilde{m}_n\}$ allowing us to model various regimes of default probabilities. When $r_n$ increases to infinity at a fast rate, the portfolio comprises obligors with small default probabilities. When it goes to infinity at a slow rate, the portfolio comprises obligors with relatively higher default probabilities.

Let $\tilde{A}_{i,t}$ denote the event that obligor $i$ defaults at time $t$ in $\mathscr{P}_n$, i.e., $i \in \mathscr{S}_{t-1}$ and $U_{i,t} \leq p_{i,j,t}(n)$. Then,

$$A_{i,t} = \cup_{s=1}^{t} \tilde{A}_{i,s}$$

denotes the event that obligor $i$ defaults by time $t$.

The aim of this short note is to develop the large deviations asymptotics for the probabilities

$$P(\sum_{i=1}^{n} e_i I(A_{i,\tau}) > na)$$

as $n \to \infty$.

Note that obligor $i \in \mathscr{S}_{t-1} \cap \mathscr{C}_j$ defaults at time $t$ if

$$U_{i,t} \leq F(P_{i,j,t}(n)).$$

Equivalently, if

$$P_{i,j,t}(n) \geq F^{-1}(U_{i,t}).$$

This in turn corresponds to

$$\begin{aligned} -m_n\alpha_j + \tilde{m}_n\beta^T(\mathbf{A}^t\mathbf{F}_0 + \textstyle\sum_{i=1}^{t}\mathbf{A}^{t-i}\mathbf{B}\mathbf{E}_i) + \\ \tilde{m}_n(\eta^t\gamma_j G_{j,0} + \gamma_j \textstyle\sum_{i=1}^{t}\eta^{t-i}\Lambda_{j,i}) + \tilde{m}_n\varepsilon_{i,t} \geq F^{-1}(U_{i,t}). \end{aligned} \tag{6}$$

Let $H_t = \beta^T(\sum_{j=1}^{t}\mathbf{A}^{t-j}\mathbf{B}\mathbf{E}_j)$. For each $i, j$, let $\mathbf{h}_j = (h_{j,k} : 1 \leq k \leq d)$ be defined by

$$\mathbf{h}_j = \beta^T\mathbf{A}^j\mathbf{B}.$$

Recall that $\mathbf{E}_t = (E_{t,k} : k \leq d)$ is a vector of independent mean zero variance 1, Gaussian random variables. Thus, we may re-express

$$H_t = \sum_{j=1}^{t}\sum_{k=1}^{d} h_{t-j,k}E_{j,k}.$$

Then $H_t$ is a mean zero Gaussian random variable with variance

$$v(H_t) = \beta^T\left(\sum_{j=1}^{t}\mathbf{A}^{t-j}\Sigma(\mathbf{A}^{t-j})^T\right)\beta.$$

Let $Y_{j,t} = \gamma_j\sum_{k=1}^{t}\eta^{t-k}\Lambda_{j,k}$ and for $i \in \mathscr{C}_j$,

$$Z_{i,t}(n) = \varepsilon_{i,t} - \tilde{m}_n^{-1}F^{-1}(U_{i,t}) + \beta^T\mathbf{A}^t\mathbf{F}_0 + \eta^t\gamma_j G_{j,0}.$$

Then, $\tilde{A}_{i,t}$ occurs if $i \in \mathscr{S}_{t-1} \cap \mathscr{C}_j$ and

$$H_t + Y_{j,t} \geq r_n\alpha_j - Z_{i,t}(n).$$

Below we put a mild restriction on $m_n$, $\tilde{m}_n$, tail distribution of each $\varepsilon_{i,t}$, and the functional form of $F$:

**Assumption 2** *There exists a non-negative, non-decreasing function $g$ such that $g(x) \to \infty$ as $x \to \infty$, and*

$$\limsup_n \sup_{t\le\tau, j\le K, i\in\mathscr{C}_j} P(Z_{i,t}(n)\ge x) \le e^{-g(x)}.$$

*Further, there exists a* $\delta\in(0,1)$ *such that*

$$\liminf_{n\to\infty} \frac{g(r_n^\delta)n}{r_n^2} = +\infty. \tag{7}$$

*Remark 2.* Since, for fixed $\mathbf{F}_0$ and $G_{j,0}$, the term $\beta^T\mathbf{A}^t\mathbf{F}_0+\eta^t\gamma_j G_{j,0}$ can be uniformly bounded by a constant, call it $c$, and

$$P(\varepsilon_{i,t}-\tilde{m}_n^{-1}F^{-1}(U_{i,t})\ge x-c)\le P(\varepsilon_{i,t}\ge (x-c)/2)+P(-\tilde{m}_n^{-1}F^{-1}(U_{i,t})\ge (x-c)/2),$$

in Assumption 2, the key restriction is imposed by the tail distribution of $-\tilde{m}_n^{-1}F^{-1}(U_{i,t})$ and we look for a function $g$ and $\delta\in(0,1)$ such that

$$P(-\tilde{m}_n^{-1}F^{-1}(U_{i,t})\ge x)\le e^{-g(x)} \tag{8}$$

for all sufficiently large $n$, and (7) holds. Equation (8) is equivalent to finding $g$ so that

$$\log\left(\frac{1}{F(-\tilde{m}_n x)}\right)\ge g(x), \tag{9}$$

for all sufficiently large $n$. Consider first the case of $F$ in (2) as well as (3). In that case, the LHS is similar to

$$\theta\tilde{m}_n x$$

for large $\tilde{m}_n x$, and condition (7) holds if $\tilde{m}_n, r_n$ and $\delta\in(0,1)$ are selected so that

$$\frac{\tilde{m}_n r_n^\delta n}{r_n^2}\to\infty.$$

This is achieved, for instance, if for $\kappa\in(0,1)$, $r_n=n^\kappa$, and

$$2-1/\kappa<\delta<1,$$

for arbitrarily increasing $\{\tilde{m}_n\}$.

Now consider $F$ in (4) where $F(x)=\bar{\Phi}(-x)$. Then, LHS of (9) is similar to

$$\tilde{m}_n^2 x^2$$

for large $\tilde{m}_n x$, and condition (7) holds if $\tilde{m}_n, r_n$ and $\delta$ are selected so that

$$\frac{\tilde{m}_n r_n^\delta n}{r_n^2}\to\infty.$$

This is achieved, for instance, if for $\kappa>0$, $r_n=n^\kappa$, and

$$1-1/(2\kappa)<\delta<1,$$

for arbitrarily increasing $\{\tilde{m}_n\}$.

Let

$$N_j(t) = \sum_{i \in \mathscr{C}_j} I(\tilde{A}_{i,t})$$

denote the number of defaults for class $j$ at time $t$ for each $j \leq K$ and $t \leq \tau$.

Let

$$\mathscr{N} = \{\frac{N_1(\tau)}{n} \geq a_\tau\},$$

where $a_\tau \in (0, c_1)$.

In Theorem 1 below we argue that on the large deviations scaling, the probability of default of any fraction of total customers in a single class at a particular time equals the probability that the complete class defaults at that time. It also highlights the fact that in the single class setting, in our regime, large losses are much more likely to occur later rather than earlier. This then provides clean insights into how large losses happen in the proposed regime.

**Theorem 1.** *Under Assumptions 1 and 2,*

$$\lim_{n \to \infty} \frac{1}{r_n^2} \log P(\mathscr{N}) = -q^*(\tau),$$

*where $q^*(t)$ is the optimal value of the quadratic program*

$$\min \sum_{k=1}^{t} \sum_{p=1}^{d} e_{k,p}^2 + \sum_{k=1}^{t} l_k^2,$$

*subject to,*

$$\sum_{k=1}^{t} \sum_{p=1}^{d} h_{t-k,p} e_{k,p} + \gamma_j \sum_{k=1}^{t} \eta_1^{t-k} l_k \geq \alpha_1,$$

*and, for $1 \leq \tilde{t} \leq t-1$,*

$$\sum_{k=1}^{\tilde{t}} \sum_{p=1}^{d} h_{\tilde{t}-k,p} e_{k,p} + \gamma_j \sum_{k=1}^{\tilde{t}} \eta_1^{\tilde{t}-k} l_k \leq \alpha_1.$$

*Further, $q^*(t)$ equals*

$$\frac{\alpha_1^2}{\sum_{k=1}^{t} \sum_{p=1}^{d} h_{t-k,p}^2 + \gamma_j^2 \sum_{k=1}^{t} \eta_1^{2(t-k)}}. \tag{10}$$

*Note that it strictly reduces with $t$.*

*Remark 3. In Theorem 1, its important to note that $q^*(\tau)$ is independent of the values $a_\tau \in (0, c_1)$.*

Some notation, and Lemma 1 are needed for proving Theorem 1. For each $j \leq K$, let

$$\mathscr{H}_{j,t} = \{H_t + Y_{j,t} \geq r_n \alpha_j + r_n^{\delta}\}$$

and

$$\tilde{\mathscr{H}}_{j,t} = \{H_t + Y_{j,t} \leq r_n \alpha_j - r_n^{\delta}\}.$$

Let,

$$\mathscr{H}_j^t = \left(\mathscr{H}_{j,t} \cap (\cap_{\tilde{t}=1}^{t-1} \tilde{\mathscr{H}}_{j,\tilde{t}})\right).$$

**Lemma 1.**

$$\lim_{n\to\infty} \frac{1}{r_n^2} \log P(\mathscr{H}_1^t) = -q^*(t),$$

*where $q^*(t)$ is defined in (10).*

## *2.3 Key result*

Recall that our interest is in developing large deviations asymptotics for $P(\sum_{i=1}^n e_i I(A_{i,\tau}) > na)$.

Let $\mathbf{b}$ denote a sub-collection of indices of $\{1,2,\ldots,K\}$ such that $\sum_{i\in\mathbf{b}} ex_j c_j > a$, while this is not true for any subset of $\mathbf{b}$. We call such a set $\mathbf{b}$ a minimal set, and we let $\mathscr{B}$ denote a collection of all such minimal sets (similar definitions arise in [12]). Consider the question that losses from the portfolio exceed $na$ when we count losses only from classes indexed by $\mathbf{b}$. Our analysis from Theorem 1 can be repeated with minor adjustments to conclude that the large deviations rate for this is the smallest of all solutions to the quadratic programs of the form described below.

For $\mathbf{t} = (t_j, j \in \mathbf{b})$ such that each $t_j \leq \tau$. Set $t_{\max} = \max_{j\in\mathbf{b}} t_j$ and let $q^*(\mathbf{t},\mathbf{b})$ be the solution to quadratic program below (call it **O2**),

$$\min \sum_{k=1}^{t_{\max}} \sum_{p=1}^{d} e_{k,p}^2 + \sum_{j\in\mathbf{b}} \sum_{k=1}^{t_j} l_{j,k}^2$$

subject to, for all $j \in \mathbf{b}$,

$$\sum_{k=1}^{t_j} \sum_{p=1}^{d} h_{t_j-k,p} e_{k,p} + \gamma_j \sum_{k=1}^{t_j} \eta_j^{t_j-k} l_{j,k} \geq \alpha_j,$$

and, for $1 \leq \tilde{t} \leq t_j - 1$,

$$\sum_{k=1}^{\tilde{t}} \sum_{p=1}^{d} h_{\tilde{t}-k,p} e_{k,p} + \gamma_j \sum_{k=1}^{\tilde{t}} \eta_j^{\tilde{t}-k} l_k \leq \alpha_j.$$

Set

$$\tilde{q}(\tau,\mathbf{b}) = \min_{\mathbf{t}: j\in\mathbf{b}, t_j \le \tau} q^*(\mathbf{t},\mathbf{b}).$$

It is easy to see that there exists an optimal $\mathbf{t}^*$ such that $\tilde{q}(\tau,\mathbf{b}) = q^*(\mathbf{t}^*,\mathbf{b})$ with the property that the respective constraints for each $\tilde{t} < t_j^*$ are not tight. Whenever, there exists $\tilde{t} < t_j$ such that constraint corresponding to $\tilde{t}$ is tight, a better rate function value is achieved by setting such a $t_j = \tilde{t}$. This then helps complete the proof of the large deviations result. It is also then easy to see that the most likely way for $\{\sum_{i=1}^n e_i I(A_{i,t}) > na\}$ to happen is that all obligors belonging to class $\mathbf{b}$ default by time $t$, where $\mathbf{b}$ is selected as the most likely amongst all the classes in $\mathscr{B}$. In other words,

$$\lim_{n\to\infty} \frac{1}{r_n^2} \log P(\sum_{i=1}^n e_i I(A_{i,\tau}) > na) = -\min_{\mathbf{b}\in\mathscr{B}} \tilde{q}(\tau,\mathbf{b}).$$

The proof again is straightforward and relies on the following simple lemma (see, e.g., [5], Lemma 1.2.15).

**Lemma 2.** *Let $N$ be a fixed integer. Then, for every $a_\varepsilon^i \ge 0$,*

$$\limsup_{\varepsilon\to\infty} \varepsilon \log\left(\sum_{i=1}^N a_\varepsilon^i\right) = \max_{i\le N} \limsup_{\varepsilon\to\infty} \varepsilon \log a_\varepsilon^i.$$

*In particular, the* $\limsup$ *above can be replaced by* $\lim$ *above if* $\max_{i\le N} \lim_{\varepsilon\to\infty} \varepsilon \log a_\varepsilon^i$ *exists.*

### *2.4 Single period setting*

In practice, one is often interested in solving for portfolio credit risk in a single period, that is, $\tau = 1$. In that case, it is easy to arrive at a simple algorithm to determine $q^*(1,\mathbf{b})$ and the associated values of the variables.

Note that the optimization problem **O2** reduces to

$$\min \sum_{p=1}^d e_{1,p}^2 + \sum_{j\in\mathbf{b}} l_{j,1}^2,$$

subject to, for all $j \in \mathbf{b}$,

$$\sum_{p=1}^d h_{0,p} e_{1,p} + \gamma_j l_{j,1} \ge \alpha_j.$$

Call this problem **O3**. The following remark is useful in solving **O3**.

*Remark 4. It is easy to see that for the optimization problem - minimize $\sum_{k=1}^n x_k^2$ subject to*

$$\sum_{k=1}^n a_k x_k \ge b, \tag{11}$$

*the solution for each k is*

$$x_k^* = b\frac{a_k}{\sum_{j=1}^n a_j^2}$$

*and (11) is tight. The optimal objective function value is*

$$\frac{b^2}{\sum_k a_k^2}.$$

To simplify the notation, suppose that $\mathbf{b} = \{1,2,\ldots,k\}$ and that $\alpha_1 \geq \alpha_2 \geq \alpha_k > 0$. In view of Remark 4, solving **O3** can be reduced to solving the quadratic program

$$\min \quad cx^2 + \sum_{j \leq k} c_j y_j^2$$

subject to

$$x + y_j \geq \alpha_j \tag{12}$$

for all $j \leq k$, where $c = 1/(\sum_{p=1}^d h_{0,p}^2)$ and $c_j = 1/\gamma_j^2$ for each $j$.

It is easily seen using the first order condition that there exists a $1 \leq j^* \leq k$ such that under the unique optimal solution, constraints (12) hold as equalities for $1 \leq j \leq j^*$, that optimal $x^*$ equals

$$\frac{\sum_{j \leq j^*} c_j \alpha_j}{c + \sum_{j \leq j^*} c_j}$$

and this is $\leq \alpha_{j^*}$. Then, $y_j = \alpha_j - x^*$ for $j \leq j^*$, and $y_j = 0$ otherwise.

Further, the optimal objective function equals,

$$(c + \sum_{j \leq j^*} c_j)\left(\frac{\sum_{j \leq j^*} c_j \alpha_j^2}{c + \sum_{j \leq j^*} c_j} - \left(\frac{\sum_{j \leq j^*} c_j \alpha_j}{c + \sum_{j \leq j^*} c_j}\right)^2\right).$$

The algorithm below to ascertain $j^*$ is straightforward.

**Algorithm**

1. If
$$\frac{\sum_{j \leq k} c_j \alpha_j}{c + \sum_{j \leq k} c_j} \leq \alpha_k$$
then $j^* = k$. Else, it is easy to check that
$$\frac{\sum_{j \leq k-1} c_j \alpha_j}{c + \sum_{j \leq k-1} c_j} > \alpha_k$$
2. As an inductive hypothesis, suppose that

$$\frac{\sum_{j\le r} c_j \alpha_j}{c + \sum_{j\le r} c_j} > \alpha_{r+1}.$$

If the LHS is less than or equal to $\alpha_r$, set $j^* = r$, and STOP. Else, set $r = r-1$ and repeat induction.

It is easy to see that this algorithm will stop as,

$$\frac{c_1 \alpha_1}{c + c_1} < \alpha_1.$$

## 3 Conclusion and ongoing work

In this paper we modelled portfolio credit risk as an evolving function of time. The default probability of any obligor at any time depended on common systemic covariates, class dependent covariate and idiosyncratic random variables - we allowed a fairly general representation of conditional default probabilities that subsumes popular logit, default intensity based representations, as well as threshold based Gaussian and related copula models for defaults. The evolution of systemic covariates was modelled as a VAR (1) process. The evolution of class dependent covariates was modelled as an independent AR process (independent of systemic and other class co-variates). We further assumed that these random variables had a Gaussian distribution. In this framework we analyzed occurrence of large losses as a function of time. In particular, we characterized the large deviations rate function of large losses. We also observed that this rate function is independent of the representation selected for conditional default probabilities.

This was a short note meant to highlight some of the essential issues. In our ongoing effort we build in more realistic and practically relevant features including:

1. We conduct large deviations analysis
    a. when the class and the systemic covariates are dependent with additional relaxations including allowing exposures and recoveries to be random. Further, as in [8], we also model firms exiting due to other reasons besides default, e.g., due to merger and acquisitions. We also allow defaults at any time to explicitly depend upon the level of defaults occurring in previous time periods (see, e.g., [16]).
    b. when the covariates are allowed to have more general fatter-tailed distributions.
    c. When the portfolio composition is time varying.
2. Portfolio large loss probabilities tend to be small requiring massive computational effort in estimation when estimation is conducted using naive Monte Carlo. Fast simulation techniques are developed that exploit the large deviations structure of large losses (see, e.g., [14, 1] for introduction to rare event simulation).

## Appendix: Some proofs

Let $||x||^2 = \sum_{i=1}^n x_i^2$. Consider the optimization problem

$$\min ||x||^2 \tag{13}$$

$$s.t. \sum_{j=1}^n a_{i,j} x_j \geq b_i \quad i = 1, \ldots, m, \tag{14}$$

and let $x^*$ denote the unique optimal solution of this optimization problem. It is easy to see from first order conditions that if $(b_i : i \leq m, b_i > 0)$, is replaced by $(\alpha b_i : i \leq m)$, $\alpha > 0$, then the solution changes to $\alpha x^*$.

Let $(X_i : i \leq n)$ denote i.i.d. Gaussian mean zero variance 1 random variables and let $d(n)$ denote any increasing function of $n$ such that $d(n) \to \infty$ as $n \to \infty$.

The following lemma is well known and stated without proof (see, e.g., [12]).

**Lemma 3.** *The following holds:*

$$\lim_{n\to\infty} \frac{1}{d(n)} \log P\left(\sum_{j=1}^n a_{i,j} X_j \geq b_i d(n) + o(d(n)) \ i = 1, \ldots, m\right) = -||x^*||^2.$$

**Proof of Lemma 1:** Recall that we need to show that

$$\lim_{n\to\infty} \frac{1}{r_n^2} \log P(\mathscr{H}) = -q^*(t). \tag{15}$$

where $P(\mathscr{H})$ denotes the probability of the event that

$$\sum_{j=1}^t \sum_{k=1}^d h_{t-j,k} E_{j,k} + \sum_{k=1}^t \eta^{t-k} \Lambda_{1,k} \geq r_n \alpha_1 + r_n^\delta$$

and

$$\sum_{j=1}^{\tilde{t}} \sum_{k=1}^d h_{\tilde{t}-j,k} E_{j,k} + \sum_{k=1}^{\tilde{t}} \eta^{\tilde{t}-k} \Lambda_{1,k} \leq r_n \alpha_1 - r_n^\delta$$

for $1 \leq \tilde{t} \leq t-1$.

From Lemma 3, to evaluate (15), it suffices to consider the optimization problem (call it **O1**),

$$\min \sum_{k=1}^t \sum_{p=1}^d e_{k,p}^2 + \sum_{k=1}^t l_k^2 \tag{16}$$

$$\text{s. t.} \ \sum_{k=1}^t \sum_{p=1}^d h_{t-k,p} e_{k,p} + \sum_{k=1}^t \eta_1^{t-k} l_k \geq \alpha_1, \tag{17}$$

$$\text{and } \sum_{k=1}^{\tilde{t}} \sum_{p=1}^{d} h_{\tilde{t}-k,p} e_{k,p} + \sum_{k=1}^{\tilde{t}} \eta_1^{\tilde{t}-k} l_k \le \alpha_1. \tag{18}$$

for $1 \le \tilde{t} \le t-1$.

We first argue that in **O1**, under the optimal solution, the constraints (18) hold as strict inequalities.

This is easily seen through a contradiction. Suppose there exists an optimal solution $(\hat{e}_{k,p}, \hat{l}_k, k \le t, p \le d)$ such that for $\hat{t} < t$,

$$\sum_{k=1}^{\hat{t}} \sum_{p=1}^{d} h_{\hat{t}-k,p} \hat{e}_{k,p} + \sum_{k=1}^{\hat{t}} \eta_1^{\hat{t}-k} \hat{l}_k = \alpha_1$$

and if $\hat{t} > 1$, then for all $\tilde{t} < \hat{t}$ (18) are always strict. We can construct a new feasible solution with objective function at least as small with the property that constraints (18) are always strict.

This is done as follows: Let $s = t - \hat{t}$. Set $\bar{e}_{k+s,p} = \hat{e}_{k,p}$ for all $k \le \hat{t}$ and $p \le d$. Similarly, set $\bar{l}_{k+s} = \hat{l}_k$ for all $k \le \hat{t}$. Set the remaining variables to zero.

Also, since the variables $(\bar{e}_{k,p}, \bar{l}_k k \le t, p \le d)$ satisfy constraint (18) with variables $(\bar{e}_{k,p}, \bar{l}_k k \le s, p \le d)$ set to zero, the objective function can be further improved by allowing these to be positive. This provides the desired contradiction. The specific form of $q^*(t)$ follows from the straightforward observation in Remark 4. $\Box$.

**Proof of Theorem 1:**

Now,

$$P(\mathscr{N}) \ge P(\mathscr{N} | \mathscr{H}_1^{\tau}) P(\mathscr{H}_1^{\tau}).$$

We argue that $P(\mathscr{N} | \mathscr{H}_1^{\tau})$ converges to 1 as $n \to \infty$. This term equals

$$P(\frac{N_1(\tau)}{n} \ge a_\tau, \frac{\sum_{t=1}^{\tau-1} N_1(t)}{n} \le c_1 - a_\tau | \mathscr{H}_1^{\tau}).$$

This may be further decomposed as

$$P(\frac{\sum_{t=1}^{\tau-1} N_1(t)}{n} \le c_1 - a_\tau | (\cap_{t=1}^{\tau-1} \tilde{\mathscr{H}}_{1,t})) \tag{19}$$

times

$$P(\frac{N_1(\tau)}{n} \ge a_\tau | \frac{\sum_{t=1}^{\tau-1} N_1(t)}{n} \le c_1 - a_\tau, \mathscr{H}_{1,\tau}). \tag{20}$$

To see that (19) converges to 1 as $n \to \infty$, note that it is lower bounded by

$$1 - \sum_{t=1}^{\tau-1} P(\frac{N_1(t)}{n} \ge \varepsilon | (\cap_{t=1}^{\tau-1} \tilde{\mathscr{H}}_{1,t}))$$

for $\varepsilon = (c_1 - a_\tau)/(\tau - 1)$. Consider now,

$$P(\frac{N_1(1)}{n} \geq \varepsilon | \tilde{\mathscr{H}}_{1,1})$$

This is bounded from above by

$$2^{c_1 n} P(Z_{1,1}(n) \geq r_n^{\delta})^{\varepsilon n}$$

where $2^{c_1 n}$ is a bound on number of ways at least $\varepsilon n$ obligors of Class 1 can be selected from $c_1 n$ obligors. Equation 19 now easily follows.

To see (19), observe that this is bounded from above by

$$2^{c_1 n} P(Z_{i,\tau}(n) \leq -r_n^{\delta})^{(c_1 - a_\tau)n}$$

Since this decays to zero as $n \to \infty$, (19) follows.

In view of Lemma 1, we then have that

$$\lim_{n\to\infty} \frac{1}{r_n^2} \log P(\mathscr{N} \cap \mathscr{H}_1^{\tau}) = -q^*(\tau),$$

and thus large deviations lower bound follows. To achieve the upper bound, we need to show that

$$\limsup_{n\to\infty} \frac{1}{r_n^2} \log P(\mathscr{N}) \leq -q^*(\tau). \tag{21}$$

Observe that

$$P(\mathscr{N}) \leq P(H_\tau + Y_{1,\tau} \geq r_n \alpha_1 - r_n^{\delta}) + P(\frac{N_1(\tau)}{n} \geq a_\tau, H_\tau + Y_{1,\tau} \leq r_n \alpha_1 - r_n^{\delta}).$$

Now, from Lemma 3 and proof of Lemma 1,

$$\lim_{n\to\infty} \frac{1}{r_n^2} \log P(H_\tau + Y_{1,\tau} \geq r_n \alpha_1 - r_n^{\delta}) = -q^*(\tau).$$

Now,

$$P(\frac{N_1(\tau)}{n} \geq a_\tau, H_\tau + Y_{1,\tau} \leq r_n \alpha_1 - r_n^{\delta})$$

is bounded from above by

$$2^n P(Z_{i,\tau} > r_n^{\delta})^{n a_\tau}$$

so that due to Assumption 2,

$$\limsup_{n\to\infty} \frac{1}{r_n^2} \log P(\frac{N_1(\tau)}{n} \geq a_\tau, H_\tau + Y_{1,\tau} \leq r_n \alpha_1 - r_n^{\delta}) = -\infty,$$

and (21) follows. $\Box$

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