Spatial Loss Systems: Exact Simulation and Rare Event Behavior

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ABSTRACT

We consider spatial marked Poisson arrivals in a Polish space. These arrivals are accepted or lost in a general state dependent manner. The accepted arrivals remain in the system for a random amount of time, where the individual sojourn times are i.i.d. For such systems, we develop semi-closed form expressions for the steady state probabilities that can be seen to be insensitive to the sojourn time distribution, and that rely essentially on the static probabilities of marked Poisson objects meeting the state acceptance criteria. The latter observation is then exploited to yield straightforward exact simulation algorithms to sample from the steady state distribution. In addition, for the special case where the arrivals are spheres in a Euclidean space that are lost whenever they overlap with an existing sphere, we develop large deviations asymptotics for the probability of observing a large number of spheres in the system in steady state, under diverse asymptotic regimes. Applications include modeling interference in wireless networks and connectivity in ad-hoc networks.

1. INTRODUCTION

We consider a spatial loss model. It can be used to study the effect of signal interference in wireless networks, or the lack of connectivity in ad-hoc wireless networks (see, e.g., [4], [1]), but it is envisaged that our abstract setting may have diverse other applications as well. Specifically, we assume spatial Poisson marked arrivals in a Polish space. An arrival mark may capture information such as the space or frequency that an arrival uses, etc. Arrivals can be lost from the system depending on the current state of the system as well as arrival's position and mark. (For instance, if a mark corresponds to the space occupied by an arrival, a loss criteria may be to reject an arrival if it leads to intersection of k objects, for some $k \geq 2$.) The accepted arrivals remain in the system for a random amount of time; the individual sojourn times are assumed i.i.d. We model the system as a measure-valued process, and determine a semi-closed form expression for its steady state distribution (see Section 2). Importantly, this expression depends upon the static probabilities that spatially Poisson distributed objects belong to an acceptable state formation. We observe this steady state probability expression allows easy acceptance-rejection based exact simulation algorithms (thus contrasting with more complex exact simulation algorithms for Markov processes, see, e.g., [3], [2]). These static probabilities can also be used to study the effect of lack of connectivity in ad-hoc wireless networks (see, e.g., [4]). Our results are especially applicable to Code Division Multiple Access (CDMA), a channel access method used in wireless communication. Here call arrivals may be modeled as a Poisson process, and a new call may be accepted only if it guarantees that Signal-to-Interference-Noise-Ratio (SINR) at each receiver in the system (a state dependent quantity) exceeds some threshold value (see, e.g., [1]).

To illustrate the utility of the developed steady state probability expressions, we consider in Section 3 a simple setting where spheres of constant radius arrive as a spatial Poisson process in a Euclidean space; if an arriving sphere intersects with an existing sphere, then it is lost, and else it stays (individual sojourn times are i.i.d.). In this framework, we develop large deviation asymptotics for the probability of seeing n or more spheres in steady state, in the asymptotic regime as $n \to \infty$, where the diameter of the spheres reduces at varying rates as n increases.

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2. STATIONARY DISTRIBUTION

In our model, objects arrive as a spatial Poisson process with total rate α, in a complete, separable metric space E, with value lying in $B \subset \mathcal{E}$ assigned probability $\tilde{P}(B) = \int_B \frac{d\alpha(u)}{\alpha}$, where $\mathcal{E}$ denotes the Borel σ-algebra on E. (For example, if arriving objects are spheres in a d dimensional Euclidean space with randomly distributed radii, then an arrival is marked by the location of its center and its radius, and $E = \Re^{d+1}$.) An arrival is lost if the resulting state of the system upon its arrival does not satisfy a specified *acceptance criteria*. Each accepted object stays for an exponentially distributed time with rate β and then leaves the system (in Remark 1, we argue that the analysis remains unchanged if these sojourn times have a general distribution with finite mean).

For any formation $\mathbf{x} = \{x_1, \ldots, x_k\} \in E^k$, $k \in \mathbb{Z}_+$, let $\mu_{\mathbf{x}}$ denote a finite measure on E defined by $\mu_{\mathbf{x}}(C) = \sum_{i=1}^k \delta_{x_i}(C)$, $C \in \mathcal{E}$, where $\delta_x(\cdot) = I(x \in \cdot)$ is the Dirac measure on E that takes unit mass at x. Let $\widehat{M}$ be the set of all such finite measures on E and $\widehat{\mathcal{M}}$ be the Borel σ-algebra on $\widehat{M}$. Let $A \subset \cup_{k=1}^{\infty} E^k$ denote the set of acceptable formations. Let $M \subset \widehat{M}$ be the set of all measures $\mu_{\mathbf{x}}$ for which $\mathbf{x} \in A$. (For instance, in the settings where an arriving object, if it intersects with the existing ones is lost, A may denote all possible formations of non-intersecting objects.) Let σ-algebra $\mathcal{M} = \left\{ M \cap H : H \in \widehat{\mathcal{M}} \right\}$.

Suppose $X = \{X_t : t \geq 0\}$ denotes a measure-valued continuous-time stochastic process that describes the given spatial loss system with the state space $(M, \mathcal{M})$.

Note that if $\mu_{\mathbf{x}} = \sum_{i=1}^{n} \delta_{x_i} \in M$ then $\mu_{\mathbf{x}} - \delta_{x_i} \in M$ for all $i \leq n$ in this loss system. But it is not necessary that $\mu_{\mathbf{x}} + \delta_y \in M$ for all $y \in E$ even though $\mu_{\mathbf{x}} \in M$. Set $\varrho = \alpha/\beta$, and let,

$$\hat{\pi}(H) = \sum_{n=0}^{\infty} \frac{\varrho^n}{n!} \int_{E^n} I(\mu_{\mathbf{x}} \in H \cap M_n) \prod_{i=1}^{n} d\tilde{P}(x_i), \quad H \in \mathcal{M},$$

where $M_n = \Big\{\mu \in M : \mu(E) = n\Big\}$.

We now show that $\hat{\pi}$ is the unique invariant measure of the process X and establish a semi-closed form expression for steady state distribution of number of customers in the system. First note that X is a jump process. Let $\{\lambda(\mu, H), \mu \in M, H \subset M\}$ denote the transition rates of X.

Theorem 1. *The measure $\hat{\pi}$ satisfies the following detailed balance equations:*

$$\hat{\pi}(d\mu)\lambda(\mu, d\eta) = \hat{\pi}(d\eta)\lambda(\eta, d\mu). \tag{1}$$

Furthermore, the measured value process X is positive Harris recurrent with invariant measure $\hat{\pi}$.

Proof. When the system is in state $\mu = \sum_{i=1}^{n} \delta_{x_i}$, an arrival (respectively, departure) takes the system to a state $\eta = \mu + \delta_y$ for some $y \in E$ (respectively, $\eta = \mu - \delta_{x_i}$ for some $i \leq n$). The balance equation (1) is evident if we observe that,

$$\hat{\pi}(d\mu_{\mathbf{x}}) = \varrho^n \frac{1}{n!} I(\mu_{\mathbf{x}} \in M_n) \prod_{i=1}^{n} d\tilde{P}(x_i)$$

and, for $\mu_{\mathbf{x}} \in M_n$,

$$\lambda(\mu_{\mathbf{x}}, d\eta) = \begin{cases} n\beta, & \text{if } \eta = \mu_{\mathbf{x}} - \delta_{x_i}, \\ \alpha I(\eta \in M_{n+1}) d\tilde{P}(y), & \text{if } \eta = \mu_{\mathbf{x}} + \delta_y, \\ 0, & \text{otherwise.} \end{cases}$$

Positive Harris recurrence of X follows, as from every state $\mu \in M$ one can reach the empty state of the system with positive probability and $\hat{\pi}$ is a finite measure [6]. $\square$

From Theorem 1, the invariant probability measure of the process X can be expressed as $\pi(H) = C^{-1}\hat{\pi}(H), H \in \mathcal{M}$, where $C = \sum_{n=0}^{\infty} \varrho^n \frac{P_n}{n!}$ is the normalization constant and

$$P_n = \int_{E^n} I(\mu_{\mathbf{x}} \in M_n) \prod_{i=1}^{n} d\tilde{P}(x_i)$$

is the probability that n independently generated samples in E according to the distribution $\tilde{P}$, satisfy the acceptance criteria (belong to A). Hence, the probability of n objects present in steady state $\pi_n = \pi(M_n) = C^{-1}\varrho^n P_n/n!$.

Remark 1 (Insensitivity). Since the rates $\lambda(\mu, \cdot)$ depend on the sojourn times distribution only through its mean $1/\beta$, one can show that the steady state distribution of X is π even when the i.i.d. sojourn times have a general distribution with mean $1/\beta$. (Proof of [8, Theorem 1] can be easily extended to establish this result.)

Remark 2 (Exact Sampling from π). Algorithm 1 below generates exact samples from the distribution $\pi(\cdot)$;

Algorithm 1.
1. Generate a sample N from Poisson distribution with parameter ϱ.

2. Generate N independent samples in E according to the distribution $\tilde{P}$.

3. If the formation is acceptable then output the formation, otherwise return to Step 1.

The proof of the algorithm is straightforward and omitted. It relies on the observation that in any iteration probability that a formation of size n is accepted is proportional to $\varrho^n \frac{P_n}{n!}$.

Remark 3. Our model fits into the well known germ-grain framework (see, e.g., [2]). Ferrari *et al.* [2] propose exact sampling algorithms for invariant measures of certain spatial birth and death processes that include the loss system that we consider. Algorithm 1 is different and substantially simpler from the algorithms proposed in [2]. In our ongoing work, we extend Algorithm 1 to other cases considered in [2].

Remark 4 (Loss Probability). Loss probability, or the probability that an arrival to the system in steady state is not accepted, is an important performance measure. Since the arrivals are Poisson, it is easy to show that in steady state, arrivals see time averages (PASTA property holds, see, e.g, [7]). Hence, the distribution of number of customers seen by an arrival in the steady state is $\{\pi_n\}$. Furthermore, it is easy to see that the probability of acceptance, $P_{Acc} := 1 - P_{Loss}$, of an arrival in the steady state is

$$P_{Acc} = C^{-1} \sum_{n=0}^{\infty} \varrho^n P_{n+1}/n! = C^{-1} \sum_{n=0}^{\infty} \varrho^n P_n/n! \left(P_{n+1}/P_n\right),$$

where (P_{n+1}/P_n) is the conditional probability that $(n+1)$ object is accepted given that the first n objects are accepted in a static experiment where $n+1$ objects are generated in E using $\tilde{P}$. P_{Loss} is easily estimated via simulation as the average of independent outputs from Algorithm 2.

Algorithm 2.
1. Sample state of the system from $\pi(\cdot)$ using Algorithm 1, and generate another independent sample in E using $\tilde{P}$.

2. If the new formation is acceptable (belongs to A) output 0, *otherwise output* 1.

3. LARGE DEVIATIONS ANALYSIS

Consider the d-dimensional space $S = [0,1]^d$, with Poisson arrivals of total rate α (as in Section 2). Each arrival selects a location in S independently and uniformly; upon arrival it creates a d-dimensional sphere of diameter $1/n^{\eta}$ around the selected location. System accepts the arrival if its sphere is not intersecting with the existing spheres. Each accepted arrival remains in the system for a random amount of time; the corresponding sojourn times are *i.i.d* with mean $1/\beta$. Throughout we assume that $\eta d > 1$, so that the volume consumed by n spheres is, as n grows, asymptotically negligible to the volume of S.

Let $\pi_{i,n}$ be the probability of i spheres present in the system in steady state and N_n be a random variable with distribution $\{\pi_{i,n} : i \geq 0\}$. Suppose that $\mathbb{P}$ is the measure associated with generating spheres independently in S. Let

$P_{i,n}$ be the probability of i spheres non-overlapping when they are generated independently.

Theorem 2 establishes exact asymptotics of steady state tail probabilities $\Gamma_n := P(N_n \geq n)$ by establishing the asymptotics of $P_{n,n}$. (Asymptotics of $P_{n,n}$ for the one-dimensional case, $d = 1$, are reported in [5].)

Note that, with $C_n = \sum_{k=0}^{\infty} \varrho^k P_{k,n}/k!$, we have

$$\Gamma_n = \frac{\varrho^n P_{n,n}}{C_n n!} \left(1 + \sum_{k=n+1}^{\infty} \varrho^{k-n} \frac{P_{k,n}}{P_{n,n}} \frac{n!}{k!}\right). \qquad (2)$$

Since for all $k > n$ it holds that $P_{n,n} \leq P_{k,n}$ and $n!/k! \leq 1/n^{k-n}$, it follows that

$$\Gamma_n = C_n^{-1} \varrho^n \frac{P_{n,n}}{n!} [1 + o(1)].$$

As $P_{k,n} \to 1$ as $n \to \infty$, it follows that $C_n \to e^{\varrho}$. Hence,

$$\Gamma_n \sim e^{-\varrho} \varrho^n P_{n,n}/n!.$$

Theorem 2. *If $\eta d > 3/2$,*

$$\lim_{n\to\infty} \left[P_{n,n} e^{\left(\frac{1}{2}\gamma n^{2-\eta d}\right)}\right] = 1, \qquad (3)$$

and if $\eta d \in \left(\frac{m+3}{m+2}, \frac{m+2}{m+1}\right]$, $m \in \mathbb{N}$,

$$\liminf_{n\to\infty} \left[P_{n,n} e^{\left(\frac{1}{2}\gamma n^{2-\eta d} + \sum_{j=2}^{m+1} r_j \gamma^j n^{(j+1)-j\eta d}\right)}\right] \geq 1, \qquad (4)$$

for some constants $r_2, r_3, \ldots$, and,

$$\limsup_{n\to\infty} \left[P_{n,n} e^{\left(\frac{1}{2}\gamma n^{2-\eta d} + \sum_{j=2}^{m+1} s_j \gamma^j n^{(j+1)-j\eta d}\right)}\right] \leq 1, \qquad (5)$$

for some constants $s_2, s_3, \ldots$.

Remark 5. Let $\tilde{N}$ denote the number of objects in steady state in an associated M/G/∞ system without losses. Then, $\mathbb{P}(\tilde{N} \geq n) = e^{\varrho} \frac{\varrho^n}{n!} [1 + o(1)]$ and $\mathbb{E}[\tilde{N}] = \varrho$. For $\eta d > 1$, $\mathbb{E}[N_n]/\mathbb{E}[\tilde{N}] = C_n^{-1} \sum_{i=0}^{\infty} \varrho^i \frac{P_{i+1,n}}{i!} \to 1$ as $n \to \infty$ while, interestingly, from (2) it follows that $\Gamma_n/\mathbb{P}(\tilde{N} \geq n) \sim P_{n,n}$ and, from Theorem 2, $P_{n,n}$ can be very small for large n, when $\eta d \in (1, 2]$, indicating that while on average loss affects are negligible, they significantly reduce the probability of overcrowding.

Theorem 2 can be easily extended beyond spheres of fixed radius to more general and random shapes. However, the details become lengthy. These would be addressed by authors in a separate work.

Proof of Theorem 2. We start the proof with a couple of general remarks and some notation.

Recall that the volume of a sphere with diameter $1/n^{\eta}$ is $\gamma n^{-\eta d}/2^d$, where $\gamma := \pi^{d/2}/\Gamma(d/2 + 1)$. Clearly, each sphere of diameter $1/n^{\eta}$ 'blocks' at most a set of volume $\gamma/n^{\eta d}$; here 'blocking' means that the center of the next arriving sphere falling in this set would create an overlap.

Let $B(n, i)$ be the volume blocked by the first i spheres with diameter $1/n^{\eta}$, given that there was no overlap yet. Let $\mathbb{Q}$ be the measure associated with generating the i-th sphere uniformly on the non-blocking area, for each $i = 1, \ldots, n$. Evidently, for the first sphere we have $B(n, 1) \leq \gamma/n^{\eta d}$, and in general, $B(n, i) \leq B(n, i - 1) + \frac{\gamma}{n^{\eta d}}$ with equality if the blocking area created by the i-th sphere is a sphere in S and is not intersecting the area blocked by the first $(i - 1)$ spheres. As a consequence of the usual change-of-measure argumentation $\frac{d\mathbb{P}}{d\mathbb{Q}} = \prod_{i=1}^{n-1} (1 - B(n, i))$, and therefore:

$$P_{n,n} = \mathbb{E}_{\mathbb{Q}} \left[\frac{d\mathbb{P}}{d\mathbb{Q}}\right] = \mathbb{E}_{\mathbb{Q}} \left[\prod_{i=1}^{n-1} (1 - B(n, i))\right]. \qquad (6)$$

Below we use this representation to identify asymptotic lower and upper bounds on $P_{n,n}$.

Lower bound: As an immediate consequence of $\eta d > 1$ and $B(n, i) \leq i\gamma/n^{\eta d}$, we have due to (6) that

$$P_{n,n} \geq \prod_{i=1}^{n-1} \left(1 - \frac{i\gamma}{n^{\eta d}}\right)$$

for large values of n. Relying on the Taylor expansion of $\log(1-x)$, the quantity in the right-hand side of the previous display can be expressed as

$$\exp\left(\sum_{i=1}^{n-1} \log\left(1 - \frac{i\gamma}{n^{\eta d}}\right)\right) = \exp\left(-\sum_{j=1}^{\infty} \sum_{i=1}^{n-1} \frac{1}{j} \left(\frac{i\gamma}{n^{\eta d}}\right)^j\right).$$

Using the well known fact that $\sum_{i=1}^{n-1} i^j = \sum_{k=1}^{j+1} c_{j,k} n^k$, for some known constants $c_{j,k}$ (with $c_{j,j+1} = 1/(j+1)$), we thus find that

$$P_{n,n} \geq \exp\left(-\sum_{j=1}^{\infty} \sum_{k=1}^{j+1} \frac{1}{j} \left(\frac{\gamma}{n^{\eta d}}\right)^j c_{j,k} n^k\right).$$

For large n, we can ignore the terms for which $k - j\eta d < 0$, so that we are left with the terms so that $k \geq k_-(j) := \lceil j\eta d \rceil$. At the same time, $k \leq j + 1$. In other words, we have that only the j matter for which $j\eta d \leq j + 1$, or, equivalently, $j \leq j_+ := \lfloor(\eta d - 1)^{-1}\rfloor$. It follows that

$$\liminf_{n\to\infty} P_{n,n} \exp\left(\sum_{j=1}^{j_+} \sum_{k=k_-(j)}^{j+1} \frac{1}{j} \left(\frac{\gamma}{n^{\eta d}}\right)^j c_{j,k} n^k\right) \geq 1.$$

Since $\eta d > 1$, it is immediately seen that $k \geq k_-(j) = \lceil j\eta d \rceil > j$, and hence the summation over the set $k \in \{k_-(j), \ldots, j + 1\}$ reduces to just the term $k = j + 1$. Then,

$$\liminf_{n\to\infty} P_{n,n} \exp\left(\sum_{j=1}^{j^+} \frac{1}{j(j+1)} \left(\frac{\gamma}{n^{\eta d}}\right)^j n^{j+1}\right) \geq 1. \qquad (7)$$

It is instructive to separately consider a few special cases.

- First consider the case $\eta d \geq \frac{3}{2}$. Here, (7) reduces to

  $$\liminf_{n\to\infty} P_{n,n} \exp\left(\frac{\gamma n^{2-\eta d}}{2}\right) \geq 1;$$

  it also entails that $P_{n,n} \to 1$ if $\eta d > 2$ (i.e., the no-overlap probability converges to 1 in this regime), and $\liminf_{n\to\infty} P_{n,n} \geq \exp(-\gamma/2)$ if $\eta d = 2$.

- Also, for $\eta d \in (\frac{4}{3}, \frac{3}{2})$,

  $$\liminf_{n\to\infty} P_{n,n} \exp\left(\frac{\gamma n^{2-\eta d}}{2} + \frac{\gamma^2 n^{3-2\eta d}}{6}\right) \geq 1,$$

  implying that $r_2 = \frac{1}{6}$ in (4).

- It is now easily seen that, for $m \in \{1, 2, \ldots\}$, the above reasoning leads to $m + 1$ terms in the exponent if $\eta d \in$

$\left(\frac{m+3}{m+2}, \frac{m+2}{m+1}\right)$, thus satisfying the structure displayed in (4). If $\eta d = (m+2)/(m+1)$ for some $m \in \mathbb{N}$, then also additional constants appear in the exponent of this asymptotic lower bound.

Upper bound: To prove upper bound, first consider the set

$$\mathscr{N}_n := \left\{i \in \{1, \dots, n-1\} : B(n,i) = B(n,i-1) + \frac{\gamma}{n^{\eta d}}\right\}.$$

Using $1 - x \le e^{-x}$, one obtains:

$$\begin{aligned} P_{n,n} &\le \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\sum_{i=1}^{n-1} B(n,i)\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\sum_{j=1}^{n-1}\sum_{i=j}^{n-1} \bar{B}(n,j)\right)\right], \end{aligned}$$

where $\bar{B}(n,j) := B(n,j) - B(n,j-1)$. Distinguishing between $j \in \mathscr{N}_n$ and $j \notin \mathscr{N}_n$, this expression is majorized by

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\sum_{j\in\mathscr{N}_n} (n-j)\frac{\gamma}{n^{\eta d}}\right)\right].$$

Let K_n be the cardinality of the set $\mathscr{N}_n$ and $\bar{K}_n = n - K_n$. Then

$$\begin{aligned} \sum_{j\in\mathscr{N}_n} (n-j)\frac{\gamma}{n^{\eta d}} &= \sum_{j=1}^{n}(n-j)\frac{\gamma}{n^{\eta d}} - \sum_{j\notin\mathscr{N}_n}(n-j)\frac{\gamma}{n^{\eta d}} \\ &\ge \frac{1}{2}\gamma\frac{n(n-1)}{n^{\beta d}} - \gamma\bar{K}_n n^{1-\eta d}, \end{aligned}$$

and therefore

$$\limsup_{n\to\infty} P_{n,n}\frac{\exp\left(\frac{1}{2}\gamma n^{2-\eta d}\right)}{\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\gamma\bar{K}_n n^{1-\eta d}\right)\right]} \le 1. \tag{8}$$

We are thus left with analyzing $\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\gamma\bar{K}_n n^{1-\eta d}\right)\right]$. The following procedure shows that $\bar{K}_n$, i.e., the number of spheres that cover upon arrival less than the maximal amount $\gamma/n^{\eta d}$ of space, can be stochastically bounded by a binomial random variable. Let $q_{i,n} = \mathbb{Q}\left(B(n,i) < B(n,i-1) + \frac{\gamma}{n^{\eta d}}\right)$. Obeserve that $q_{i,n}$ increases with i, and is therefore bounded above by $\bar{q}_n := q_{n,n}$. In addition,

$$\bar{q}_n = \mathbb{Q}\left(B(n,n) < B(n,n-1) + \frac{\gamma}{n^{\eta d}}\right) \le \frac{2^d\gamma n^{1-\eta d}}{1 - B(n,n-1)}.$$

Since $B(n,n-1) \le (n-1)\gamma/n^{\eta d} \le \gamma n^{1-\eta d}$, for some constants $\varepsilon > 0$ and $L > 0$,

$$\bar{q}_n \le \frac{2^d\gamma n^{1-\eta d}}{1-\gamma n^{1-\eta d}} \le 2^d\gamma n^{1-\eta d}(1+\varepsilon n^{1-\eta d}) \le L n^{1-\eta d} =: q_n. \tag{9}$$

As a consequence, $\bar{K}_n$ is bounded by a binomial random variable with parameters n and q_n, and therefore

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\exp(\bar{K}_n\gamma n^{1-\eta d})\right] &\le \left(q_n e^{\gamma n^{1-\eta d}} + (1-q_n)\right)^n \\ &= \left(1 + L\sum_{j=1}^{\infty}\frac{\gamma^j n^{(j+1)(1-\eta d)}}{j!}\right)^n. \end{aligned}$$

This in turn is majorized by $\exp\left(Ln\sum_{j=2}^{\infty}\frac{\gamma^{j-1}n^{j(1-\eta d)}}{(j-1)!}\right)$. As in the lower bound, some terms in the exponent can be left out without affecting the asymptotic properties; with as before $j_+ = \lfloor(\eta d - 1)^{-1}\rfloor$, the quantity in the previous display is asymptotically equal to

$$\exp\left(L\sum_{j=2}^{j_+}\frac{1}{\gamma(j-1)!}\left(\frac{\gamma}{n^{\eta d}}\right)^j n^{j+1}\right).$$

We thus have, relying on (8), that

$$\limsup_{n\to\infty} P_{n,n}\frac{\exp\left(\frac{\gamma n^{2-\eta d}}{2}\right)}{\exp\left(L\sum_{j=2}^{j_+}\frac{1}{\gamma(j-1)!}\left(\frac{\gamma}{n^{\eta d}}\right)^j n^{j+1}\right)} \le 1.$$

From (9) we conclude that for L we can pick any number larger than $2^d\gamma$, leading to

$$\limsup_{n\to\infty} P_{n,n}\frac{\exp\left(\frac{\gamma n^{2-\eta d}}{2}\right)}{\exp\left(2^d\sum_{j=2}^{j_+}\frac{1}{(j-1)!}\left(\frac{\gamma}{n^{\eta d}}\right)^j n^{j+1}\right)} \le 1. \tag{10}$$

Again consider a few special cases. For $\eta d > 3/2$

$$\limsup_{n\to\infty} P_{n,n}\exp\left(\frac{\gamma n^{2-\eta d}}{2}\right) \le 1,$$

and hence the upper and lower bound match. In particular, we conclude that $\lim_{n\to\infty} P_{n,n} = \exp(-\gamma/2)$ if $\eta d = 2$. For $\eta d \in (\frac{4}{3}, \frac{3}{2}]$, we find

$$\limsup_{n\to\infty} P_{n,n}\exp\left(\frac{\gamma n^{2-\eta d}}{2} - 2^d\gamma^2 n^{3-2\eta d}\right) \le 1.$$

Using the asymptotic upper bound (10) we observe that $m+1$ terms appear in the exponent if $\eta d \in \left(\frac{m+3}{m+2}, \frac{m+2}{m+1}\right)$, just as in the lower bound. □

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