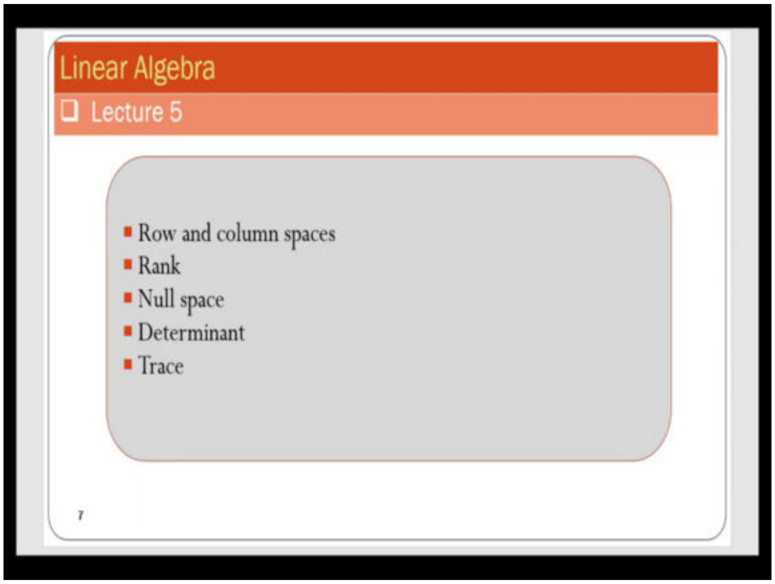
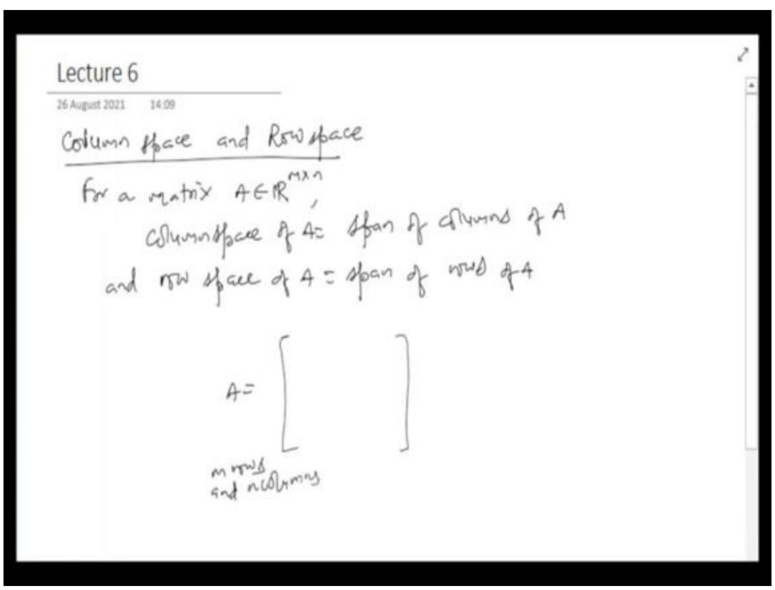
Mathematical Aspects of Biomedical Electronic System Design
Indian Institute of Science, Bangalore
Lecture 16
Linear Algebra – II

(Refer Slide Time: 00:30)



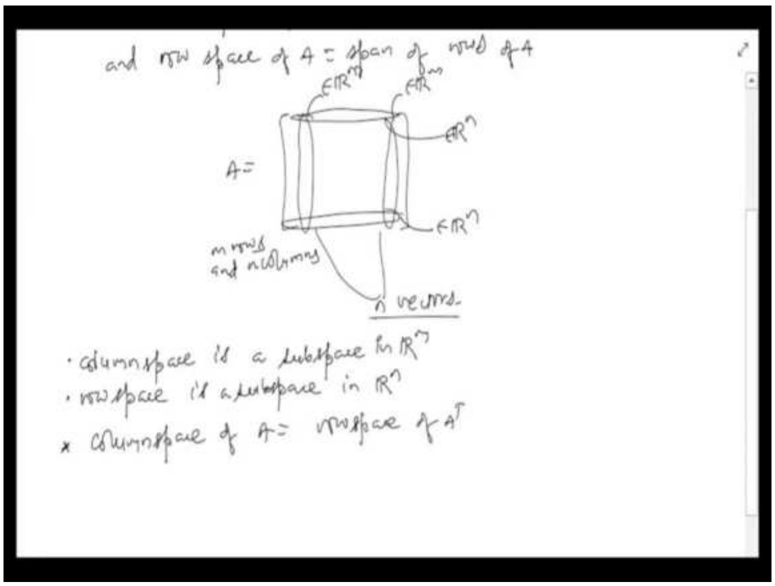
Hello, everyone. Welcome to another lecture of the course Mathematical Aspects of Biomedical Electronic System Design. We will continue with our study of Linear Algebra. In particular, in this lecture we will look at row and column spaces, rank, null space, determinant and trace. So, let us begin the lecture.

(Refer Slide Time: 00:50)



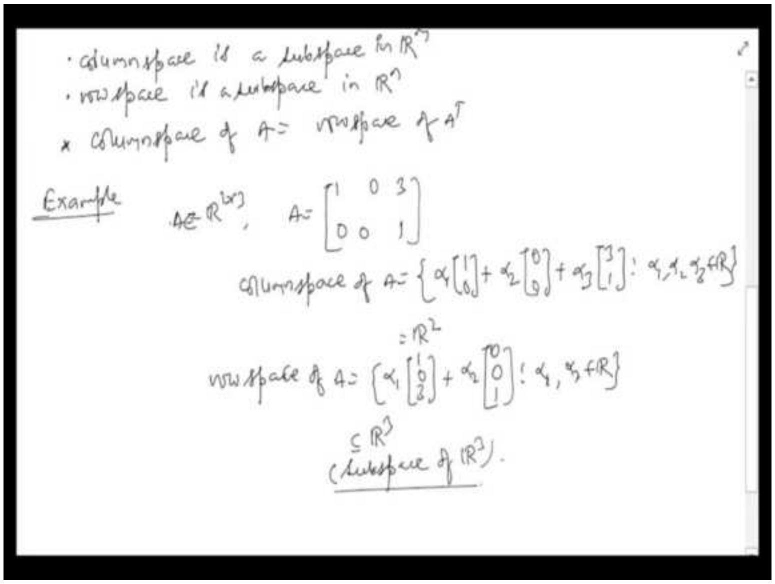
In the previous lecture we learned about spanned of a collection of vectors. We will use that definition to define column and row spaces. Let us start with column space and row space. Given a matrix A, say it is a real value m × n matrix, we define column space of A to be the span of columns of A, span of columns of A. Similarly, we define row space of A to be the span of rows of A. What we mean that if A is an m × n matrix, so it has m rows and n columns, then each of its columns is a vector in R_n. This belongs to R_n and there are, vector R_m. And there are n such vectors.

(Refer Slide Time: 02:17)



So, there are n such vectors. The span of all these n vectors is called column space of A. Similarly, if you look at each of the rows, they are vectors in R_n, vectors in R_n. And there are m such rows. The span of m rows of A is called row space of A. Clearly, each column, being a vector is R_m, the column space is a subspace of R_m, column space is a subspace in R_m. Similarly, row space is a subspace of R_n. Moreover, rows of A are columns of A transpose and columns of A are rows of A transpose. So, column space of A is same as row space of A transpose. Similarly, row space of A is column space of A transpose.

(Refer Slide Time: 03:58)



Let us see an example. Let us consider

$$A \in \mathbb{R}^{2\times 3}, \quad A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, each of the columns of A is an element of $\mathbb{R}^2$. So, columns space of A is span of the three columns that this set of all linear combinations of these three columns and it is, it can be seen that it is entire $\mathbb{R}^2$.

So,

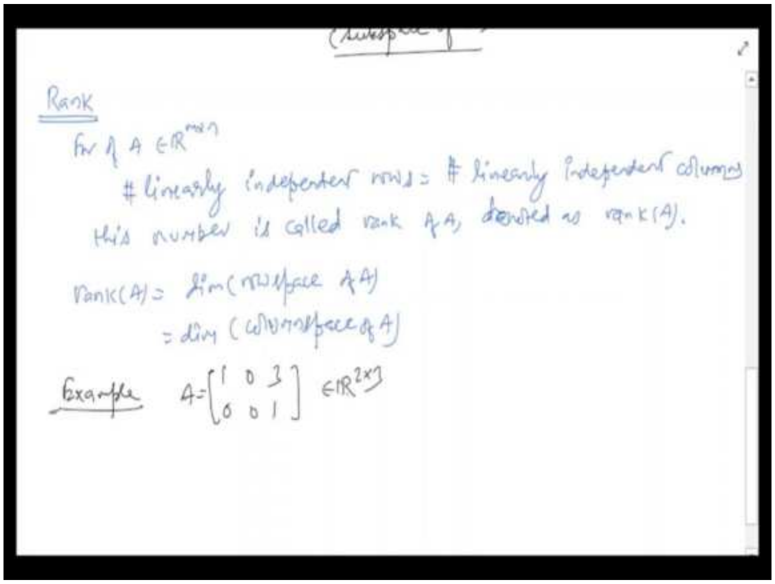
$$column\ space\ of\ A = \left\{\alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\right\}$$

and as I said, it is entire $\mathbb{R}^2$. Similarly, each of the rows of A is an element of $\mathbb{R}^3$. So,

$$row\ space\ of\ A = \left\{\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\right\}$$

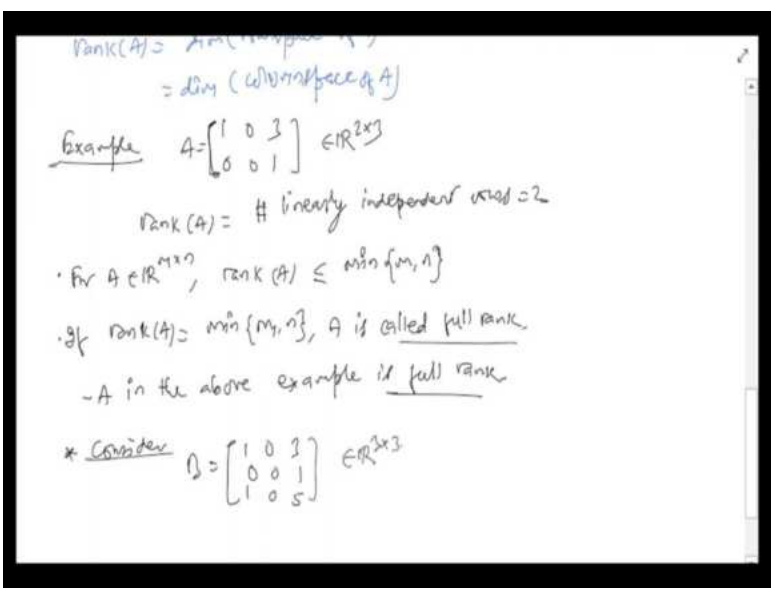
It can be seen that it is a strict subset of $\mathbb{R}^3$. It is a subspace of $\mathbb{R}^3$.

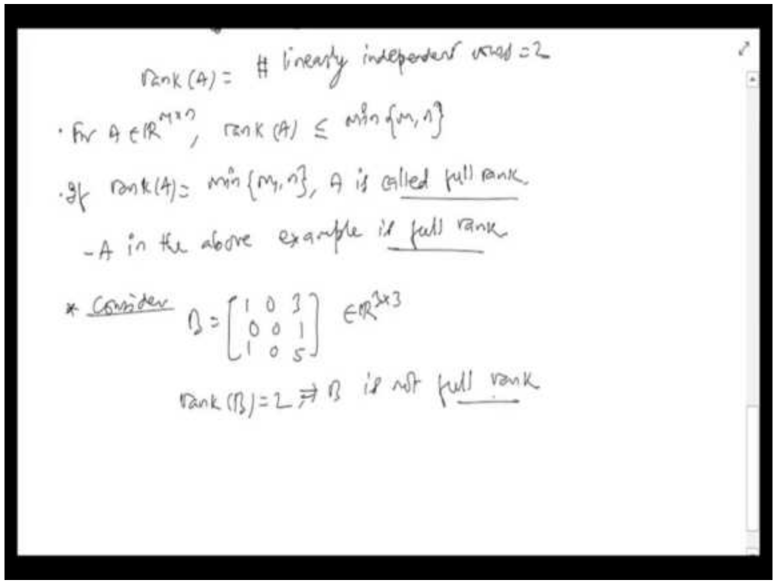
(Refer Slide Time: 05:50)



Next thing that we will see is what we call rank of a matrix. So, for a matrix A, say it is an m × n matrix, it is a fact that its, number of its linearly independent rows is equal to number of linearly independent columns. And this number is called rank of A. Number of linearly independent rows equals to number of linearly independent columns. This number is called rank of A. And it is often denoted as rank, in bracket, A. So, clearly, the rank of A is dimension of row space of A, which in turn is equal to dimension of column space of A. Dimension of row space of A, which in turn equals dimension of column space of A.

(Refer Slide Time: 07:55)





Let us consider the same example that we have seen before. That is,

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

Here, the number of linearly independent rows is 2. There are only 2 rows and the two are linearly independent. And it can be seen that the number of linearly independent columns is also 2. So, rank of A is 2. Number of linearly independent rows, which is 2. It is clear that for

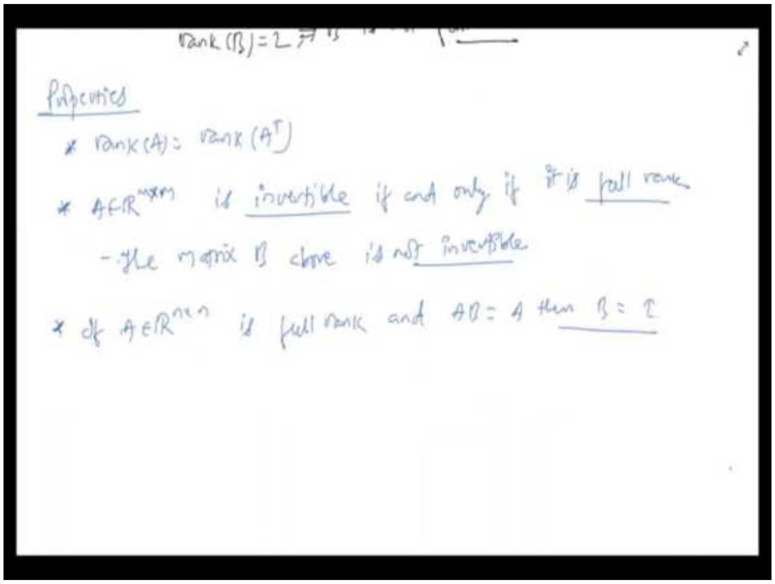
$$A \in \mathbb{R}^{m\times n}, rank(A) \leq \min\{m, n\}.$$

Clearly, the rank of A cannot exceed either m or n. For A, which is m × n, the rank(A) ≤ min{m, n}. For instance, in the above example, we know that rank(A) > min{2,3}, which is 2. If rank(A) = min {m, n}, then A is called a full rank matrix. Equal to min{m, n}, A is called a full rank. So, A in the above example, is full rank. However, if we consider

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} \in \mathbb{R}^{3\times 3} \ ,$$

it can be seen that rank(B) = 2, which is < 3, so B is not full rank. rank(B) = 2, as it has only two linearly independent rows, and two linearly independent column, rank(B) = 2. So, B is not full rank.

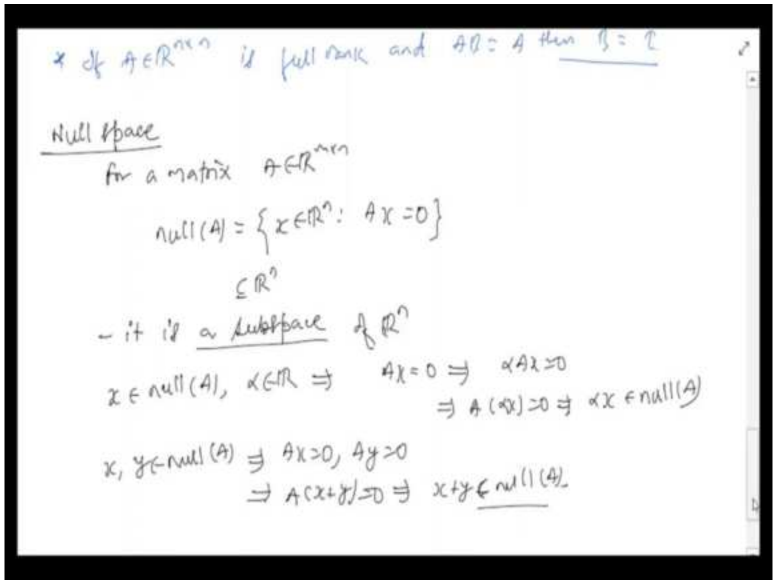
(Refer Slide Time: 10:48)



We will now state a few properties of Rank. The first is easy to see. It says that rank(A) = rank(A^T). It is because the rows of A becomes columns of A transpose and columns of A becomes rows of A transpose. Then, the next property is, a matrix A, square matrix is invertible, if and only if it is full rank. Only if it is full rank. For example, the matrix B in above example is not invertible. Not invertible.

Another important fact about rank is, if another important fact about rank is, if A is a square matrix that is full rank and AB = A, then B has to be an identity matrix. So, let us see what it is. If A, which is a m × n, is full rank and AB = A, then B =I. B cannot be any other matrix.

(Refer Slide Time: 12:58)



The next thing that we will see is what we call null space of a matrix. Null space. Given, $A \in \mathbb{R}^{m \times n}$, its null space is defined as follows.

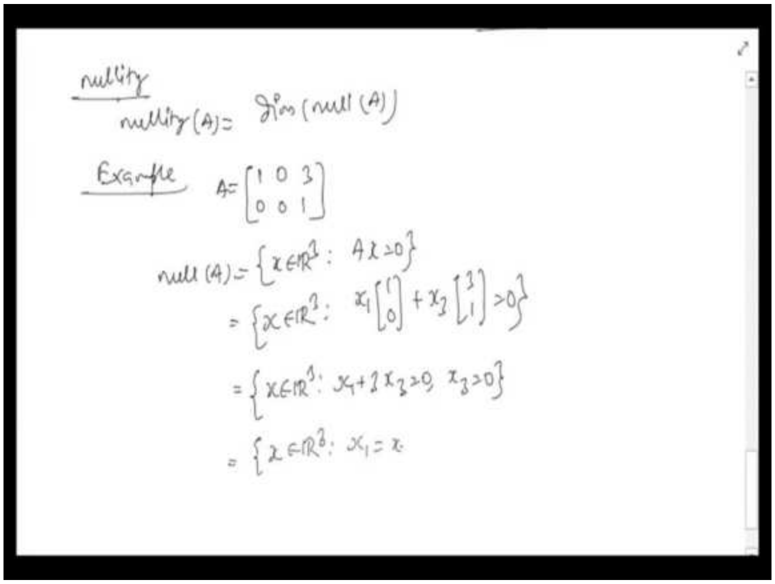
$$null(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

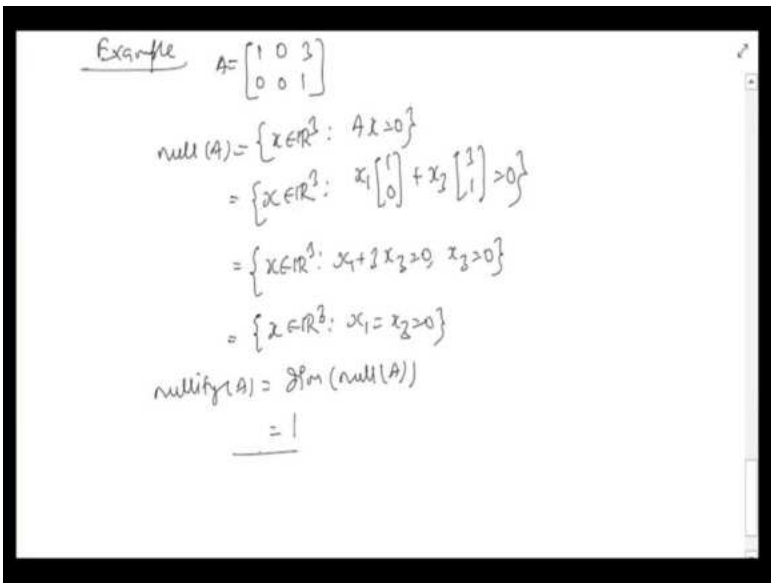
$$\subseteq \mathbb{R}^n$$

and it is can be checked that it is a subspace of R^n. It is subspace of R^n. Recall that to show that a subset of R^n is a subset of R^n, we need to show two properties. One is, if $x \in null(A), \alpha \in \mathbb{R}$, then αx should be in null(A). Let us verify this property. x being in null(A) means $Ax = 0$, which further implies that $\alpha Ax = 0$. And now using associativity we can see that $A(\alpha x) = 0$, which states that $\alpha x \in null(A)$.

Similarly, we have to establish that if $x, y \in null(A)$, then $x + y \in null(A)$. Let us verify this property as well. The fact that $x, y \in null(A)$ implies that $Ax = 0$, and $Ay = 0$. This implies that $A(x + y) = 0$, which in turns means that $x + y \in null(A)$. This way, we can, we have easily verified that $null(A) \subseteq \mathbb{R}^n$.

(Refer Slide Time: 15:18)





The next thing is nullity. $nullity(A) = \dim\,(null(A))$. Nullity of, so we are looking at Nullity. $nullity(A) = \dim\,(null(A))$. Let us see an example. Again, let us consider

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\ .$$

$$null(A) = \{x \in \mathbb{R}^3 : Ax = 0\}$$

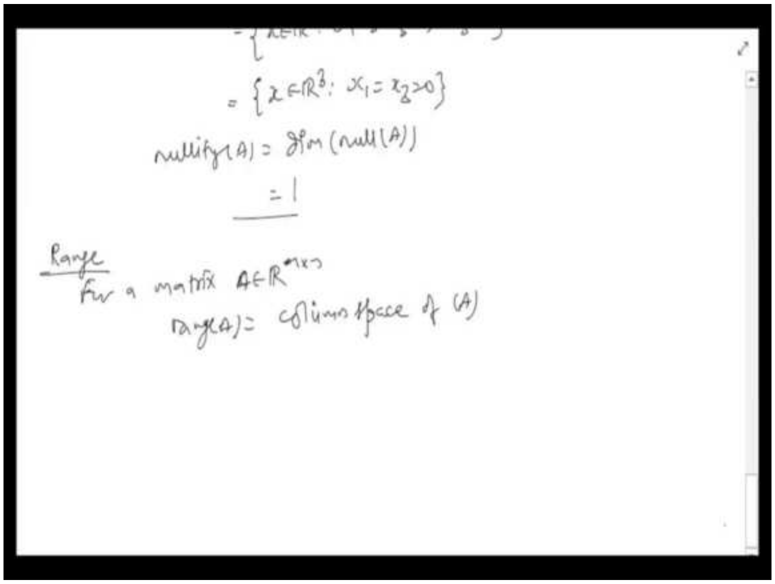
$$= \left\{x \in \mathbb{R}^3 : x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0\right\}$$

$$= \{x \in \mathbb{R}^3 : x_1 + 3x_3 = 0, x_3 = 0\}$$

$$= \{x \in \mathbb{R}^3 : x_1 = x_3 = 0\}$$

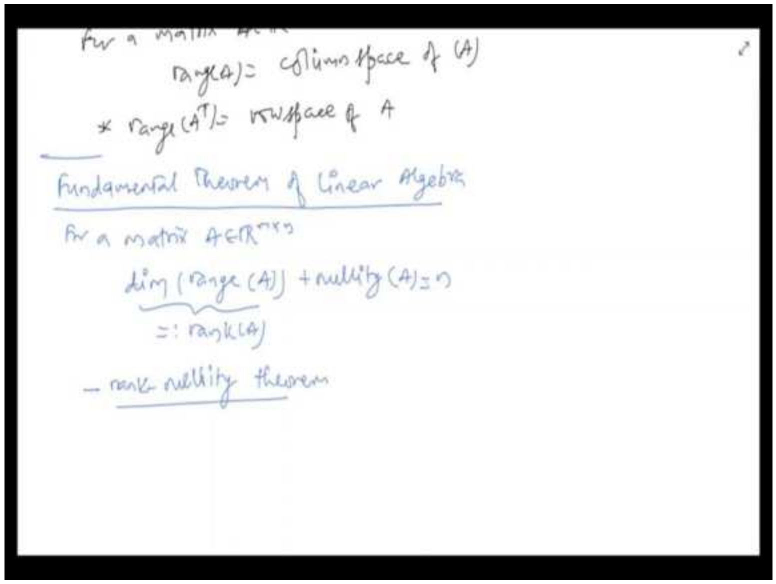
Clearly, $nullity(A) = \dim\,(null(A))$, which is in this case is 1.

(Refer Slide Time: 17:25)



The next notion that we will see is range of A. For a matrix $A \in \mathbb{R}^{m \times n}$, its column space is often referred to as its range. So, the range(A) = column space(A). Clearly, if we are interested in range (A^T), it would be row space(A). Now, we will see a fundamental result in linear algebra. In fact, it is also known as fundamental theorem of linear algebra. It relates dimension of, range of a matrix, its nullity and number of columns in A. So, we will see what we call fundamental theorem of linear algebra.

(Refer Slide Time: 18:50)



It says that for a matrix $A \in \mathbb{R}^{m \times n}$, dimension(range(A)), which is same as

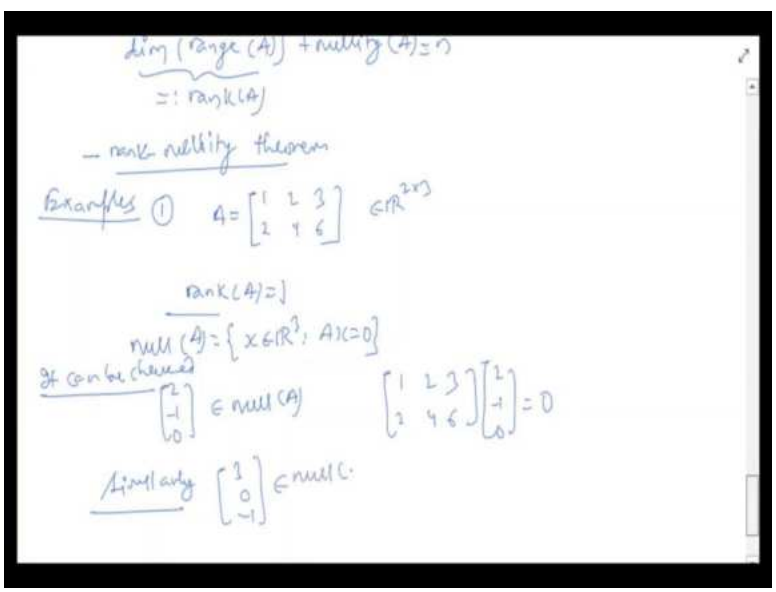
$$dimension\big(columns\ space(A)\big) + \ nullity(A) = \ n.$$

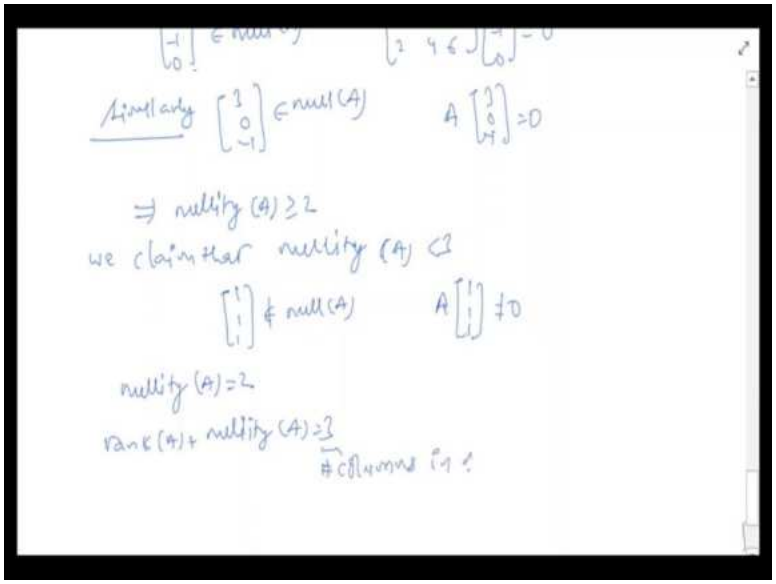
Dimension(range(A)) is often referred to as, is also referred to as rank(A). So, fundamental theorem of linear algebra says that

$$rank(A) + \ nullity(A) = \ n.$$

This result is also referred to as rank nullity theorem, as we see it relates rank and nullity of matrices to their number of columns. Rank nullity theorem.

(Refer Slide Time: 20:17)





Let us see a few examples. Let us take

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

It can be easily seen that rank(A) = 1 because the two rows are linearly dependent. If we multiply the first row by 2, we will get the second row. Clearly, the number of linearly independent rows is just 1. So, the rank(A) = 1.

Now, we will investigate nullity(A). Towards that, let us write what is null(A),

$$null(A) = \{x \in \mathbb{R}^3 : Ax = 0\}.$$

It can be easily checked that vector $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \in null(A)$. Why? We can verify that

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0$. Similarly, it can be checked that, it can be checked, similarly, it can also be checked that the vector $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \in null(A)$. So, A$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 0$. Further, the two vectors that we have written here, are linearly independent. This says that null space of A, which contains at least these two vectors has dimension at least 2. So, what we see is that nullity(A) ≥ 2.

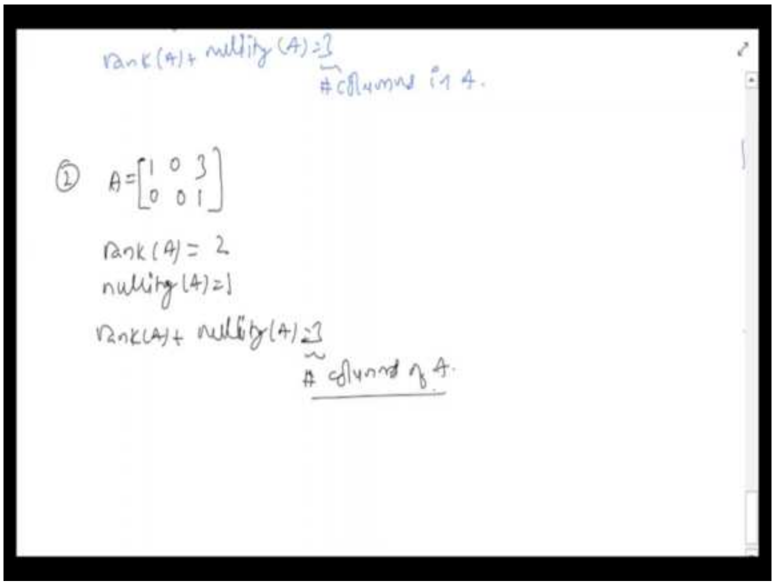
However, we will see that nullity of A cannot be 3. For nullity of A to be 3, all the vectors ∈ R^3 should be null space of A. We argue that, we claim that nullity of A cannot be 3. Why is this true?

Because $\begin{bmatrix}1\\1\\1\end{bmatrix} \notin null(A)$. It can be easily checked that $A\begin{bmatrix}1\\1\\1\end{bmatrix} \neq 0$. This says that nullity(A) = 2. And now, if we see both, rank and nullity, we say that

$$rank(A) + nullity(A) = 3,$$

which is number of columns in A.

(Refer Slide Time: 23:18)

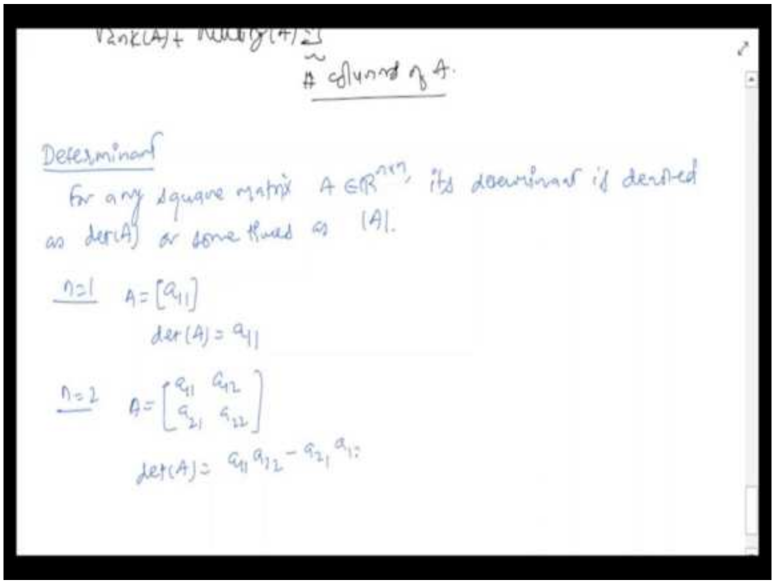


Let us see another example, and this time we will consider

$$A = \begin{bmatrix}1 & 0 & 3\\0 & 0 & 1\end{bmatrix}.$$

So, now, now we will consider $A = \begin{bmatrix}1 & 0 & 3\\0 & 0 & 1\end{bmatrix}$. We have already seen that rank of this matrix, which is equal to dimension of column space of this matrix equals 2, we also saw that nullity of this matrix is 1. So, rank(A) + nullity(A) = 3, which is same as number of columns of A.

(Refer Slide Time: 24:18)

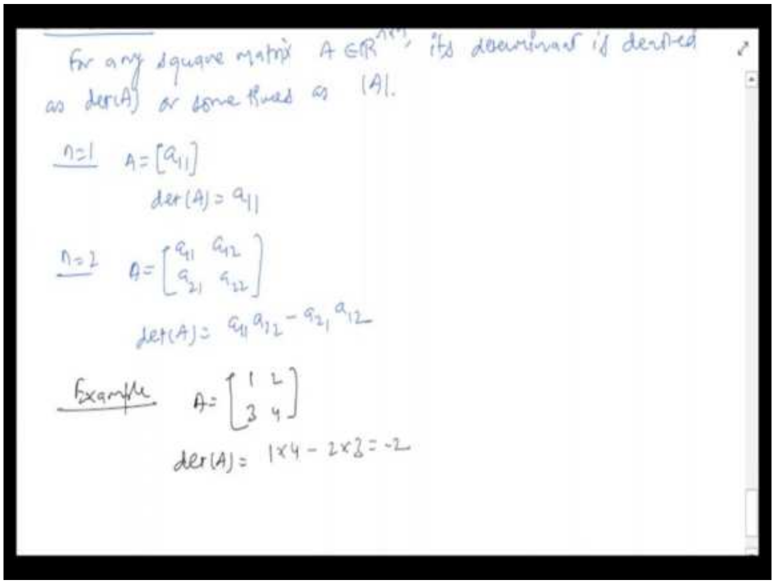


Next, we will look at the notion of determinant of matrix. Determinants are defined for a square matrices. Determinant of any square matrix is a real number. So, $A \in \mathbb{R}^{n\times n}$, its determinant is denoted as det(A). Determinant is denoted as det(A). Or sometimes as this. So, as I already indicated, det(A) is a real number. Let us see how determinants are defined.

When n = 1, that is, we are looking at 1 × 1 matrices, that is A is of the form a_{11}, then det(A) = a_{11}, just the element a_{11}. When n = 2, that is, now we are focusing on 2 × 2 matrices, let us say

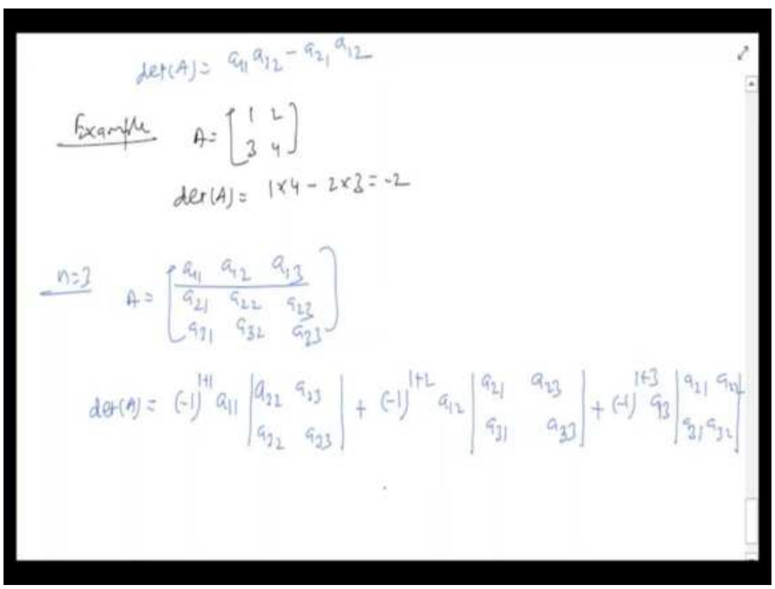
$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $det(A) = a_{11}a_{22} - a_{21}a_{12}$.

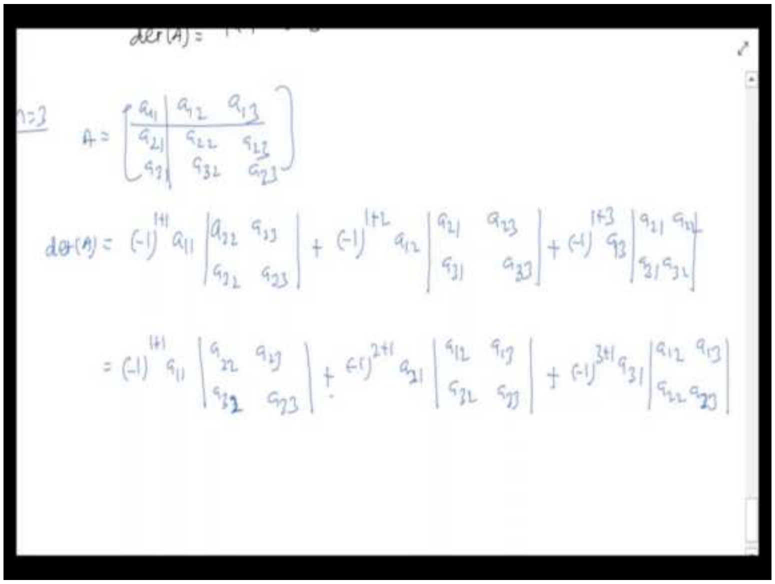
(Refer Slide Time: 26:02)



Let us see an example. Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $det(A) = 1 \times 4 - 2 \times 3 = -2$.

(Refer Slide Time: 26:23)





Now, let us move to n = 3. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Determinant of 3 × 3 matrices can be obtained via expansion along any of the rows or along any of the columns. Let us see what I mean. I will first illustrate what I mean by expansion along first row. So, I will consider first row here for expansion. I will take the first entry of the first row. I will now ignore the row and column corresponding to that entry, that is, I will ignore the first row and first column. And I will write the determinant of the remaining sub matrix. $a_{22}, a_{23}, a_{32}, a_{33}$. So, one thing that I missed to say that the whole thing should be multiplied by $(-1)^{1+1}$, that a_{11}. So, I put here 1 + 1.

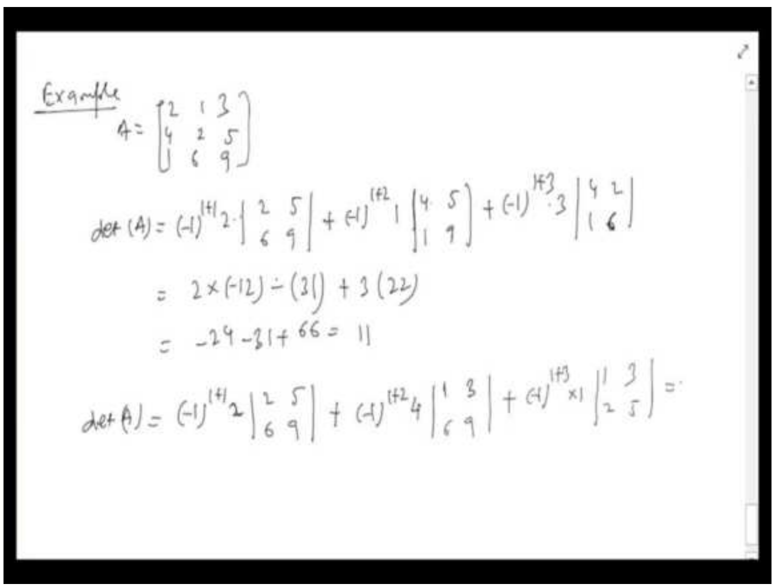
Now, I will move to the next element of the first row, that is a_{12}. So, now I will have $(-1)^{1+2}$, a_{12}. I will again ignore the row and column corresponding to this entry, that is I will ignore the first row and the second column and will consider the determinant of the remaining sub matrix. Which now becomes, is a_{21}, a_{23}, a_{31}, a_{33}.

Finally, I will go to the third entry in this row and I will now have $(-1)^{1+3}a_{13}$ times determinant of the sub matrix a_{21}, a_{22}, a_{31}, a_{32}. This is how determinant of matrix A, which is 3 × 3 matrix, can be written in terms of determinants of 2 × 2 sub matrices. And here, we have expanded along the first row.

We could expand along the first column as well. In that case, we will get the following expression. $(-1)^{1+1}$. So, now I am expanding along the first column. So, $(-1)^{1+1}$, a_{11}, this entries will remain the same a_{32}, a_{23}, a_{32}, this is a_{22}, a_{23}, a_{32}, a_{33}. Now, I will move to the next element of the first column. That is $(-1)^{1+2}$. And this is a_{21}. Sub matrix that now I have considered is a_{12}, a_{13}, a_{32}, a_{33}.

Next, I will go to the third entry of the first column. So, I will have $(-1)^{1+3}$. Now, I will ignore the first column and third row. So, the sub matrix that I will have to consider is a_{12}, a_{13}, a_{22}, a_{23}. So, this is how I will write determinant of A using determinants of sub matrices while expanding along the first column. I could expand along any row, or any column and I will get the same final answer.

(Refer Slide Time: 30:32)



Let us see an example. So, let us consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 5 \\ 1 & 6 & 9 \end{bmatrix}.$$

Let us compute determinant of A via expanding along the first row.

$$\det(A) = (-1)^{1+1}2.\begin{vmatrix} 2 & 5 \\ 6 & 9 \end{vmatrix} + (-1)^{1+2}1.\begin{vmatrix} 4 & 5 \\ 1 & 9 \end{vmatrix} + (-1)^{1+3}3.\begin{vmatrix} 4 & 2 \\ 1 & 6 \end{vmatrix}$$

$$= 2 \times (-12) - (31) + 3(22)$$

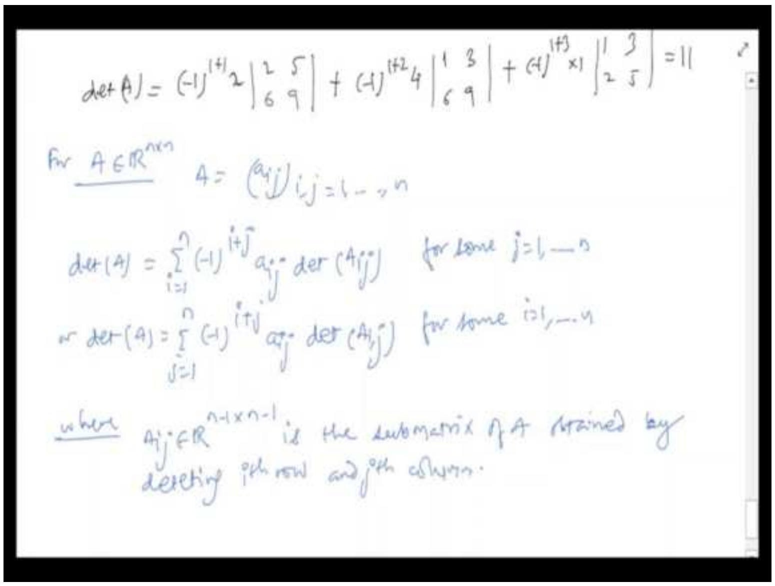
$$= -24 - 31 + 66 = 11$$

Instead, I could have expanded along the first column. Then determinant of A would be computed as

$$\det(A) = (-1)^{1+1}2.\begin{vmatrix} 2 & 5 \\ 6 & 9 \end{vmatrix} + (-1)^{1+2}4.\begin{vmatrix} 1 & 3 \\ 6 & 9 \end{vmatrix} + (-1)^{1+3}1.\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 11.$$

It can be seen that I will get the same final answer as before, that is 11. In fact, as I stated earlier. I could expand along any row or any column. So, this is how we define determinants of 3 × 3 matrices.

(Refer Slide Time: 33:16)



Now, let us take this idea forward and consider n × n matrices. For $A \in \mathbb{R}^{n\times n}$, that is $A = (a_{ij})\ i, j = 1, \dots \dots, n$, I can determine determinant of A, by expanding along any of the rows or any of the columns. So, let us say I am expanding along j$^{\text{th}}$ column. Then determinant of A could be written as

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

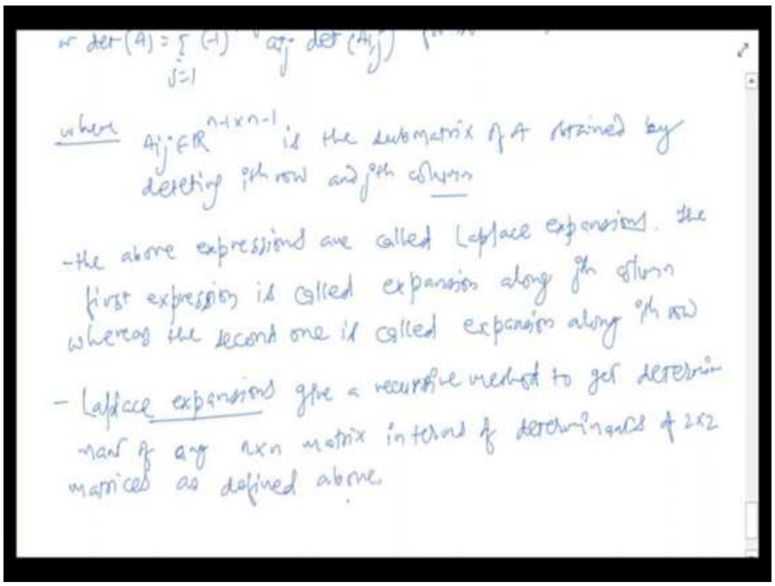
Where a_{ij} is the matrix, sub matrix of A obtained by deleting i$^{\text{th}}$ row and j$^{\text{th}}$ column. So, I could expand along any column. So, for some equal to 1 to n.

Similarly, I could write determinant of A by expanding along say, i$^{\text{th}}$ row. In which case, I will get

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}),$$

where again a_{ij} is the sub matrix that is obtained by deleting i^th row and j^th column. Awesome. So, I could do it by expanding along any of the rows. So, let us see, let me write here, where a_{ij}, is a sub matrix of A, it has dimension n-1 × n-1, this is sub matrix of A obtained by deleting i^th row and j^th column.

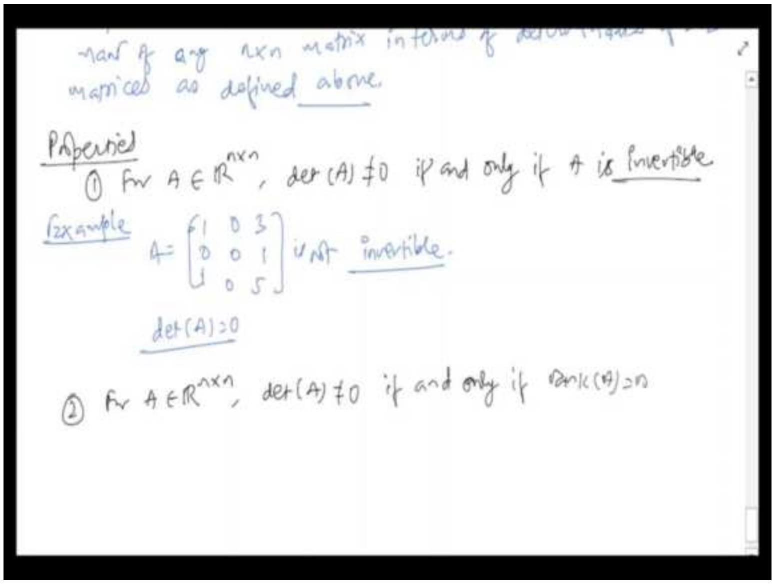
(Refer Slide Time: 35:43)



The above expressions that recursively give determinant of a matrix A in terms of determinants of its sub matrices, they are called Laplace expansions. The above expressions are called Laplace expansions. The first expression is called expansion along j^th column, expansion along j^th column, whereas the second one is called expansion along i^th row. The second one is called expansion along i^th row. As we have already seen Laplace expansion gives a recursive method, expansions and give a recursive method to get determinant of any n × n matrix in terms of determinants of 2 × 2 matrices.

So, how is it achieved? So, you express determinant of n × n matrix in terms of its n-1 × n-1 sub matrices, which in turn could be expressed as determinant of certain n-2 × n-2 matrices, and so on. If we continue, ultimately everything will reduce to determinants of 2 × 2 matrices as defined above.

(Refer Slide Time: 38:33)



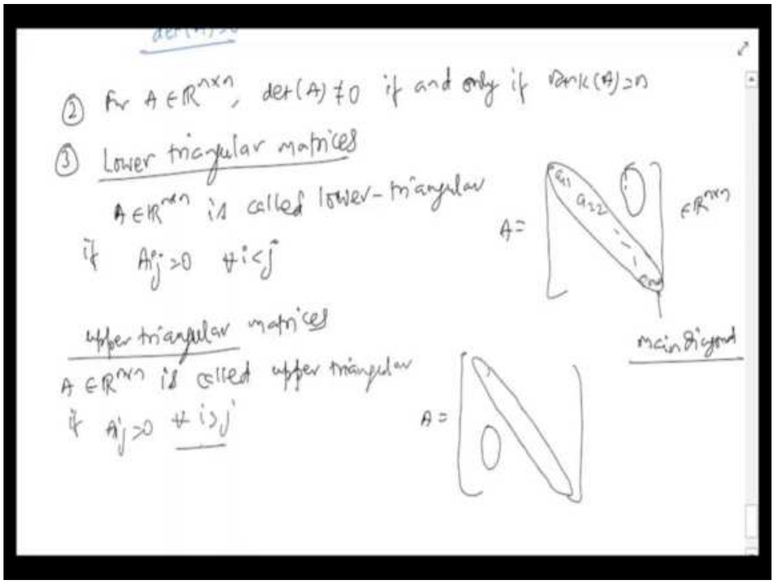
We will now see a few properties of determinants. The first property is, for any matrix, for any square matrix, its determinant is non-zero, if and only if the matrix is invertible. A is invertible. Let us take an example. Recall the following matrix that we had considered earlier also. It was

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}.$$

This matrix is not invertible, is not invertible. It can be easily checked that its determinant is 0. In fact, in this case it is trivial to check if we expand along second column, we see that all the factors, all the terms in the expression of determinant will be 0. So, det(A) = 0.

The second property is also a related one. We have already seen that a square matrix, $A \in \mathbb{R}^{n\times n}$ is invertible if and only if its rank is n. We can combine this statement with the previous property to say that for a matrix A, for a $A \in \mathbb{R}^{n\times n}$, $\det(A) \neq 0$ if and only if rank(A) = n. We will see, now, the third property. But before we see the third property, we need a few definitions. And these are definitions of so-called upper triangular and lower triangular matrices. So, let us see what these are.

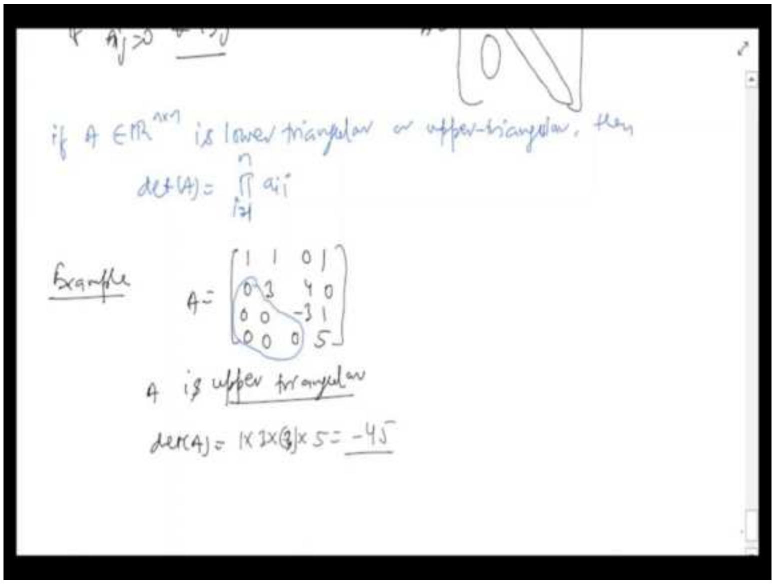
(Refer Slide Time: 40:47)



Lower triangular matrices. Let us consider $A \in \mathbb{R}^{n\times n}$. The entries a_{11}, a_{12}, et cetera up to a_{nn}, these are called entries on the main diagonal. So, this is called main diagonal of A. The matrix is called lower triangular if all the entries above the main diagonal are 0. So, if all these entries are 0, then A is called lower triangular. More precisely, $A \in \mathbb{R}^{n\times n}$ is called lower triangular if $A_{ij} = 0 \ \forall \ i < j$.

We can similarly define upper triangular matrices. As before, let us consider $A \in \mathbb{R}^{n\times n}$, and this is its so-called main diagonal. If all the entries below the main diagonal are 0, then A is called upper triangular matrix. So, is called upper triangular if $A_{ij} = 0 \ \forall \ i > j$.

(Refer Slide Time: 43:06)



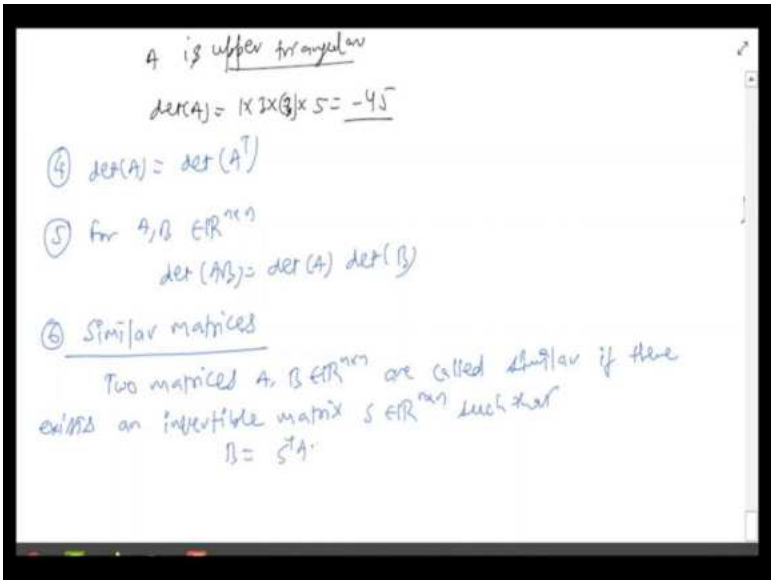
Now, we are ready to state the property of determinants of lower and upper triangular matrices. This property says that if A is upper triangular or lower triangular, then its determinant is just the product of entries along the main diagonal. That is, if A is, $A \in \mathbb{R}^{n\times n}$ is lower triangular or upper triangular, then determinant of A is simply product of entries along the main diagonal. It is product of entries a_{ii}.

Let us see an example. Let us consider a 4 × 4 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Clearly, this is an upper triangular matrix. All entries below the main diagonal are 0. So, det(A) is, in this case product of the diagonal entries. That is, $\det(A) = 1 \times 3 \times (-3) \times 5 = -45$.

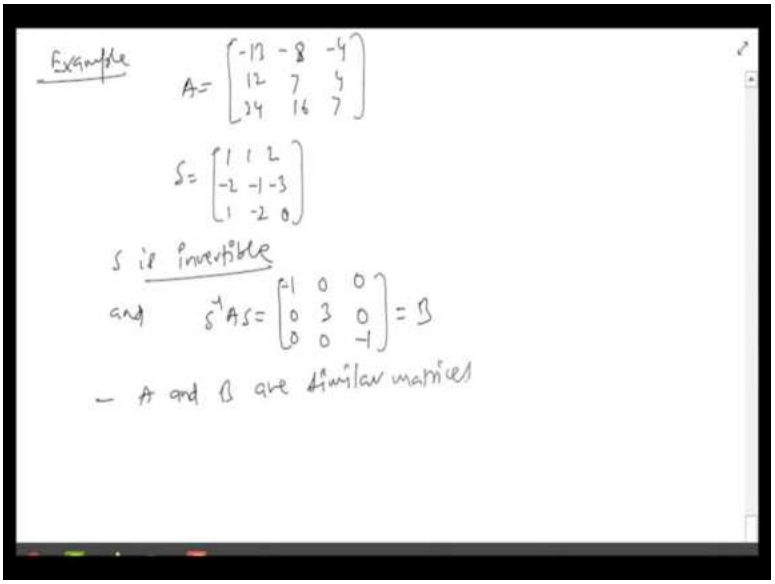
(Refer Slide Time: 45:07)



We now move to the next property of determinants which says that $\det(A) = \det(A^T)$. This is number 4, which says that $\det(A) = \det(A^T)$. Now, we will see yet another property. It says that if $A, B \in \mathbb{R}^{n\times n}$, then $\det(AB) = \det(A)\det(B)$. For $A, B \in \mathbb{R}^{n\times n}$, then $\det(AB) = \det(A)\det(B)$.

Let us see the next property. But again, before I state this property, I will give a couple of definitions. So, the first one is so-called similar matrices. Similar matrices. Two matrices $A, B \in \mathbb{R}^{n\times n}$ are called similar if there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $B = S^{-1}A$.

(Refer Slide Time: 47:07)



Let us see an example to illustrate what I mean. Let us say that A is this matrix, which, 3 × 3 matrix

$$A = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}.$$

Let us say S is

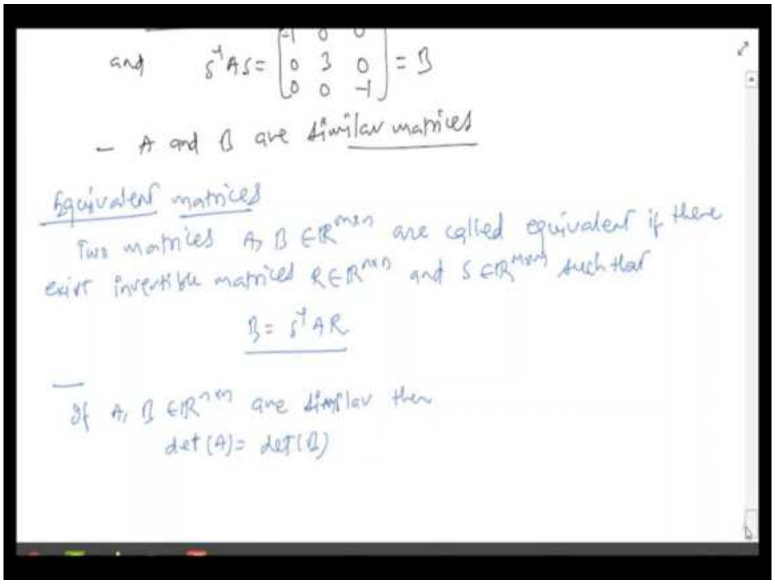
$$S = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix}.$$

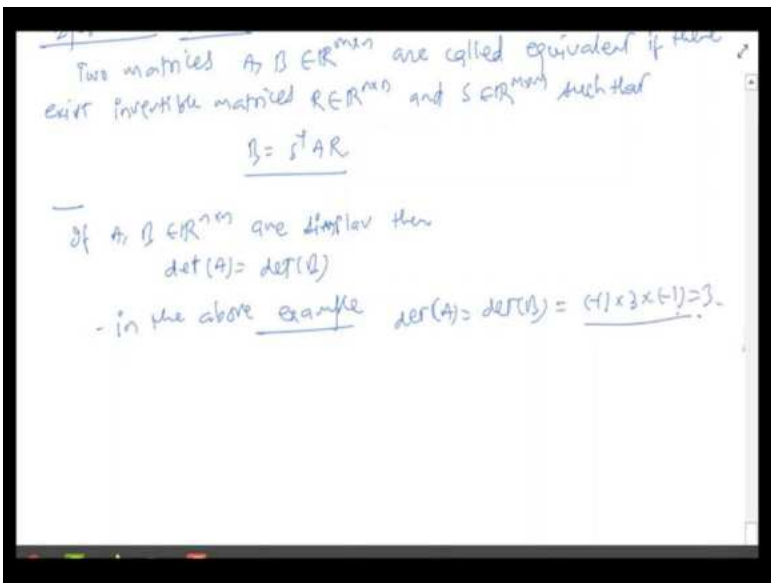
This is, it can easily be checked that S is invertible. And

$$S^{-1}AS = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = B.$$

So, in this case A and B are similar. Similar matrices.

(Refer Slide Time: 48:37)



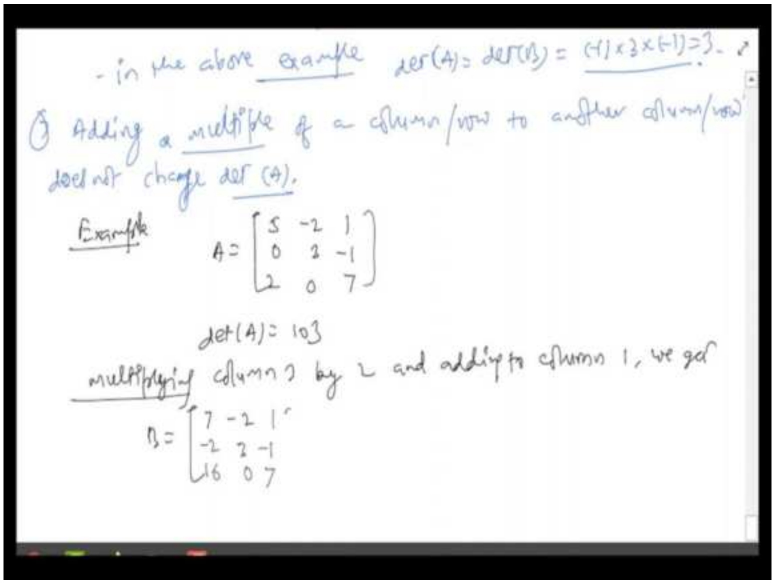


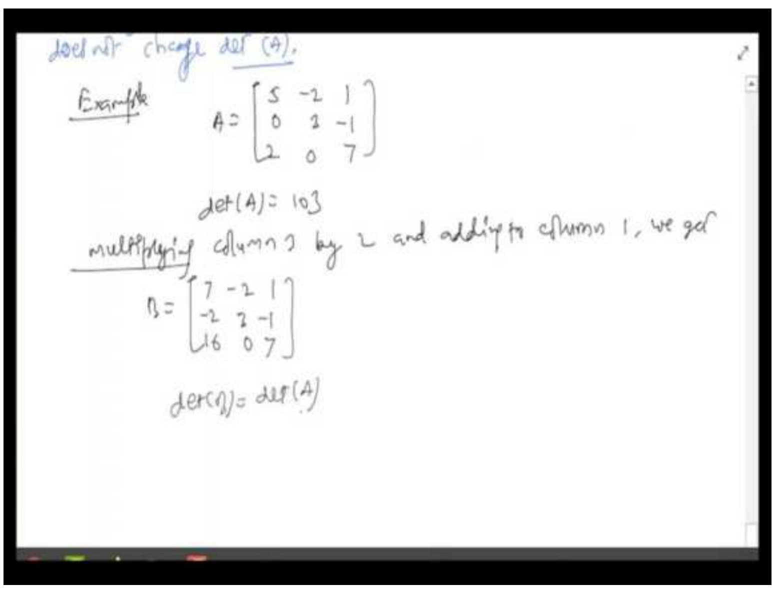
We will also see the notion of equivalent matrices. Two matrices $A, B \in \mathbb{R}^{m\times n}$. They are called equivalent if there exists two other matrices that are invertible, invertible matrices $R \in \mathbb{R}^{n\times n}$, and $S \in \mathbb{R}^{m\times m}$, such that $B = S^{-1}AR$. Having defined this notion of equivalent and similar matrices, now we are ready to state the property that determinants hold for similar matrices. Actually, similar matrices have same determinants.

So, this is the next property. If $A, B \in \mathbb{R}^{n\times n}$, are similar, then det(A) = det(B). For instance, if you look at the above example, where determinant of A is somewhat involved to calculate, but we can readily see that determinant of B which is an diagonal matrix is, and so it is both upper triangular and lower triangular. Determinant of B is product of its diagonal elements. So, $\det(B) =$

$(-1) \times 3 \times (-1) = 3$. So, $det(A) = 3$. In the above example $det(A) = \det(B)$ because A and B are similar, and the latter is $(-1) \times 3 \times (-1) = 3$.

(Refer Slide Time: 51:31)





Now, let us see yet another property. And this will be property number 7. It says that adding a multiple of a column to another column does not change determinant of the matrix. Similarly, adding a multiple of a row to another row does not change determinant of the matrix. Adding a multiple of a column or a row to another row or column, another column or a row, does not change determinant of A. Let us see an illustration of this property. Example, so let us consider

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 7 \end{bmatrix}.$$

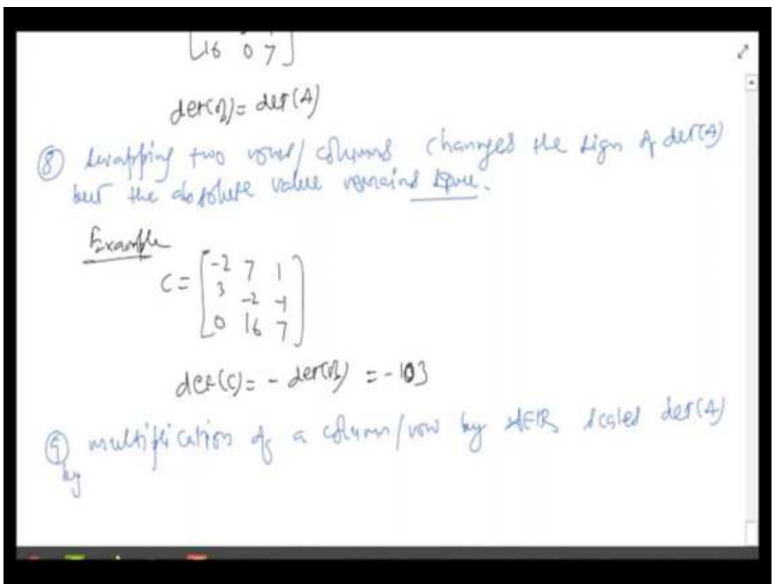
It can be easily checked that det(A) = 103.

Now, if we multiply the last column by 2 and add the resulting (co), vector to the first column, I, we get another matrix. Multiplying column 3 by 2 and adding to column 1, adding to column 1, we get

$$B = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 3 & -1 \\ 16 & 0 & 7 \end{bmatrix}.$$

And the previous property says that det(B) = det(A).

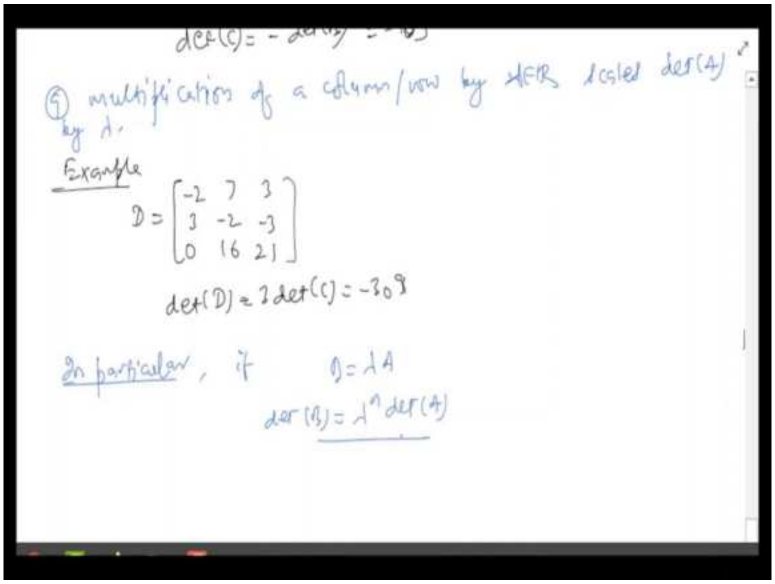
(Refer Slide Time: 54:00)



Let us see another property, that says that swapping two rows or two columns of matrix A merely changes the sign of the determinant. The absolute value remains fixed. So, swapping two rows or columns changes the sign of determinant of A, but the absolute value remains same. For instance, if we swap the two columns of B, say first and second column of B, to get the following matrix C, then $\det(C) = -\det(B) = 103$.

Finally, we will see one more property of determinants that says that multiplication of a column by a constant, scales the determinant by the same constant. Similarly, multiplication of a row by a constant scales the determinant by the same constant. So, it says that multiplication of a column or a row by say, λ, which is a real number, scales determinant of A by λ. So, what it means that if I define another matrix, let us see an example.

(Refer Slide Time: 56:15)

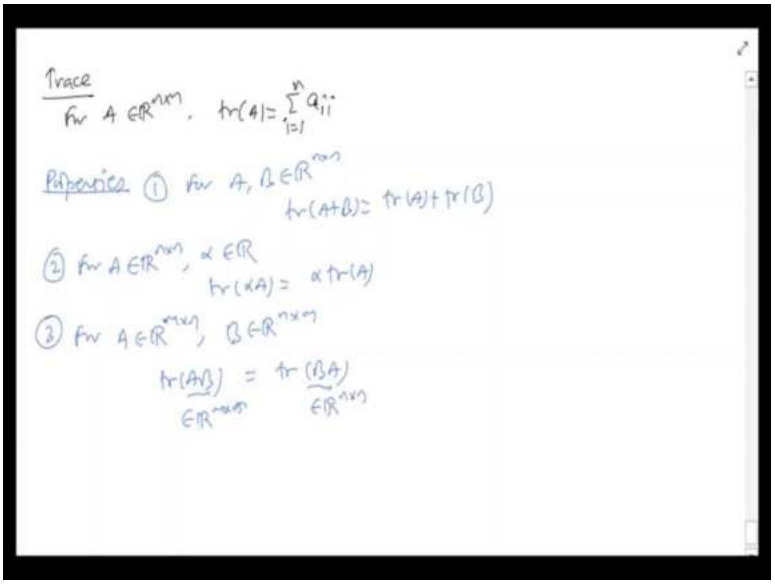


If I define another 3 × 3 matrix D by multiplying the last column of C by 3, that is,

$$D = \begin{bmatrix} -2 & 7 & 3 \\ 3 & -2 & -3 \\ 0 & 16 & 21 \end{bmatrix},$$

then $\det(D) = 3\det(C) = -309$. In particular, this property says that if I multiply all the entries of a matrix by a constant λ, the determinant of the resulting matrix will be λ^n times the original determinant. In particular, if I multiply all the entries of A by λ, that is, if B = λA, then det(B) = λ^ndet(A).

(Refer Slide Time: 57:40)



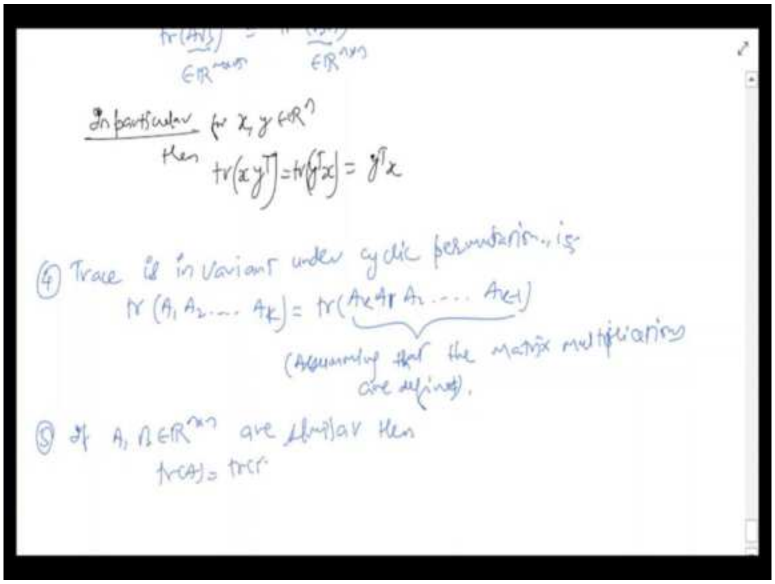
Finally, we look at the notion of trace. Trace is also defined for square matrices. It yields a real number for any square matrix. For $A \in \mathbb{R}^{n\times n}$, its trace is defined as sum of all diagonal entries. Trace is denoted as

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

Let us see a few properties of trace. The first one is an easy one. It says that for matrices $A, B \in \mathbb{R}^{n\times n}$, $tr(A + B) = tr(A) + tr(B)$.

The second property says that if $A \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$, then $tr(\alpha A) = \alpha tr(A)$. This is again, easy to verify. The third property is, says that for matrix $A \in \mathbb{R}^{m\times n}$, and $B \in \mathbb{R}^{n\times m}$, $tr(AB) = tr(BA)$.

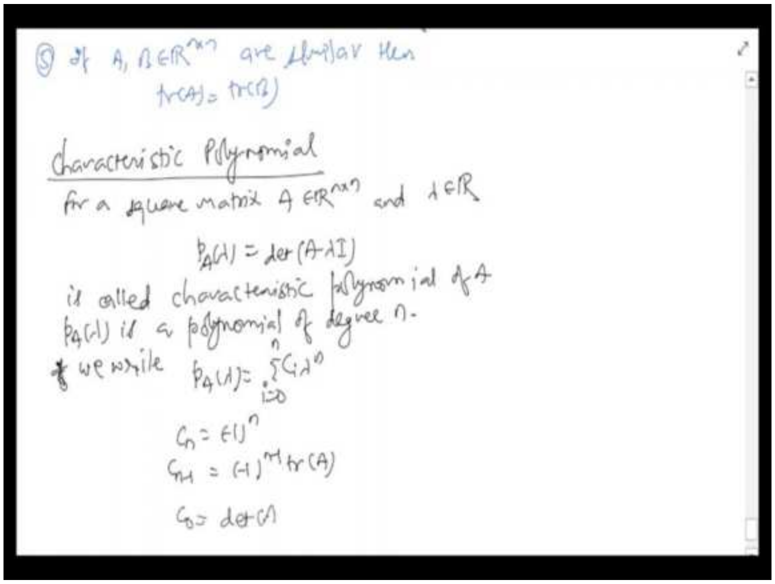
(Refer Slide Time: 59:26)



Let us an illustration of this property. If we consider two vectors x and y, then $xy^T \in \mathbb{R}^{n\times n}$. On the other hand, $y^T x \in \mathbb{R}$. But the above property says that $tr(xy^T) = tr(y^T x) = y^T x$. The fourth property says that trace is invariant under cyclic permutation. Invariant under cyclic permutation. What does it mean? It means that if we consider k matrices A_1, A_2, … A_K, such that they can be multiplied, then trace of their multiplication will be tr(A_K, A1, A_2, … A_{K-1}).

Notice that here we have permuted the matrices in a cyclic order before multiplying them. So, the trace of the left hand side matrix is same as the trace of the right hand side matrix. Of course, here we are assuming that matrix multiplications are defined. Assuming that the matrix multiplications are defined. The next property is in regard to similar matrices. It says that similar matrices have same trace. So, if $AB \in \mathbb{R}^{n\times n}$, then tr(A) = tr(B). This was all about trace. Now, I will quickly introduce a notion that is of characteristic polynomial.

(Refer Slide Time: 62:01)



For a square matrix $A \in \mathbb{R}^{n\times n}$, and a number λ.

$$P_A(\lambda) = \det(A - \lambda I).$$

And this is called characteristic polynomial of A, often denoted as follows $P_A(\lambda)$. So, this is called characteristic polynomial of A. Notice that $P_A(\lambda)$ is a polynomial of degree n. Moreover, if we write

$$P_A(\lambda) = \sum_{i=0}^{n} C_i \lambda^n,$$

then

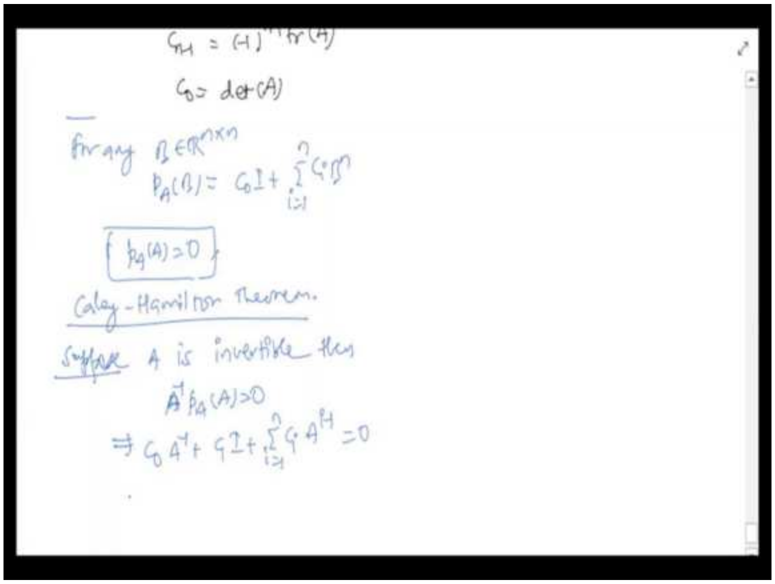
$$C_n = (-1)^n$$

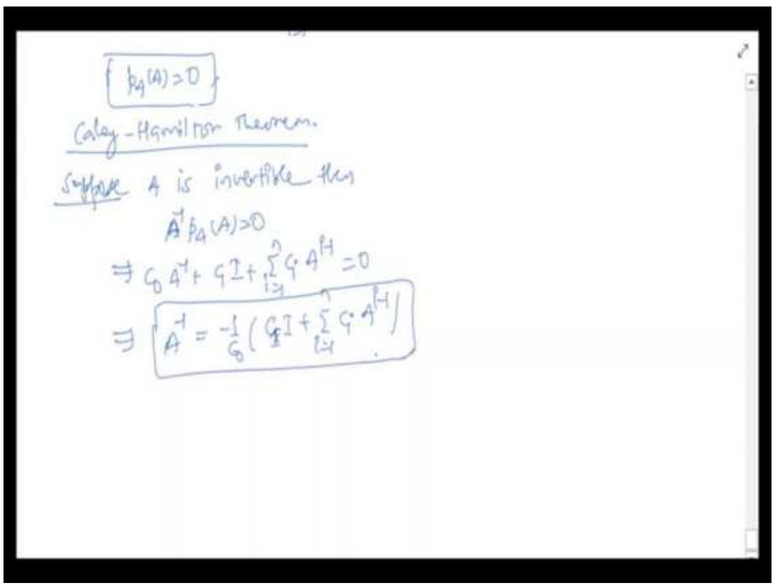
$$C_{n-1} = (-1)^{n-1} tr(A)$$

$$C_0 = \det(A).$$

These properties easily follows from definitions of determinants and trace.

(Refer Slide Time: 63:53)





Further, for any matrix $B \in \mathbb{R}^{n\times n}$, we can define

$$P_A(B) = C_0 I + \sum_{i=1}^{n} C_i B^n.$$

Then, it turns out that

$$P_A(A) = 0.$$

This particular result is referred to as Caley-Hamilton theorem. One of the uses of Caley-Hamilton theorem is in finding the in the inverse of matrices. Suppose A is invertible then from Caley-Hamilton theorem

$$A^{-1}P_A(A) = 0,$$

which implies that

$$C_0A^{-1} + C_1I + \sum_{i=1}^{n} C_iA^{i-1} = 0$$

And on rearranging terms we find that

$$A^{-1} = \frac{-1}{C_0}\left(C_1I + \sum_{i=1}^{n} C_iA^{i-1}\right).$$

So, we see that we can obtain inverse of a matrix A in terms of products of the same matrix. This brings us to the end of this lecture. Thank you.