Mathematical Aspects of Biomedical Electronic System Design
Professor Chandramani Singh
Department of Engineering Services Examination
Indian Institute of Science, Bangalore
Lecture 01
Introduction to Signals and Systems

Hello everyone, welcome to the course Mathematical Aspects of Biomedical Electronic System Design. My name is Chandramani Singh. I belong to the Department of ESE at IISc Bangalore. I will take you through the first few modules of the course. Our first module will be on Signals and Systems.

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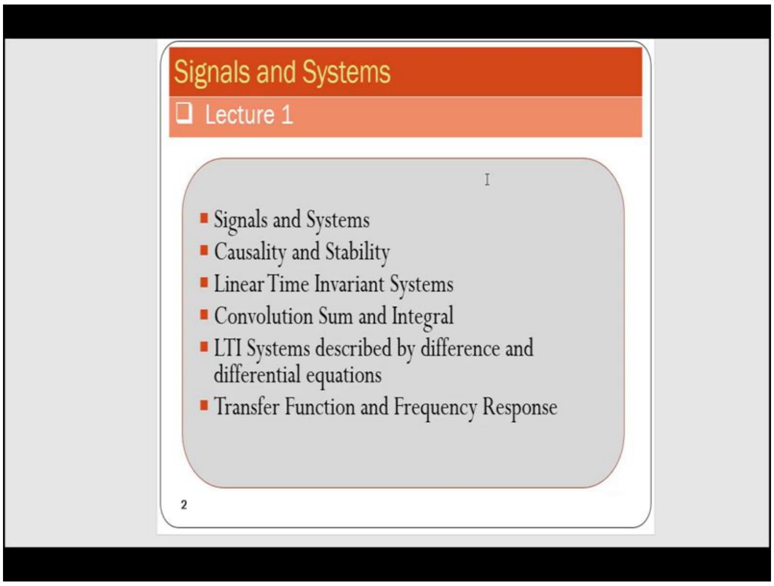

In this module, in particular in this lecture, we will see an introduction to signals and systems, we will see notions of causality instability, linear time invariant systems, we will also see convolution sum and integrals, we will see how LTI systems can be characterized by, are characterized in, when represented by difference and differential equations. Finally, we will see what are transfer function, frequency response.

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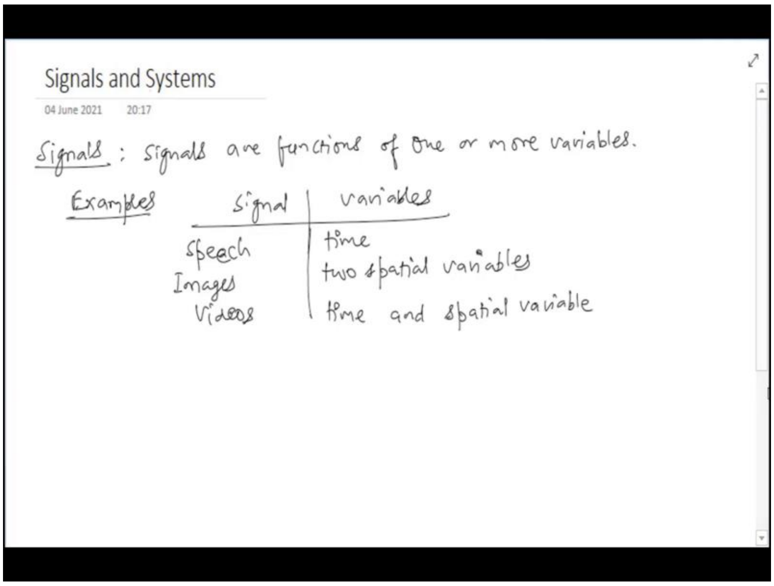

Let just start with signals, signals are functions of one or more variables. Here are a few examples. In speed signal, the variable is time, in images there are two variables, namely two spatial variables in videos we have time as well as spatial variables. In this course, we will restrict our attention to signals with one variable the variable will be usually interpreted as time. Now, there are two types of signals.

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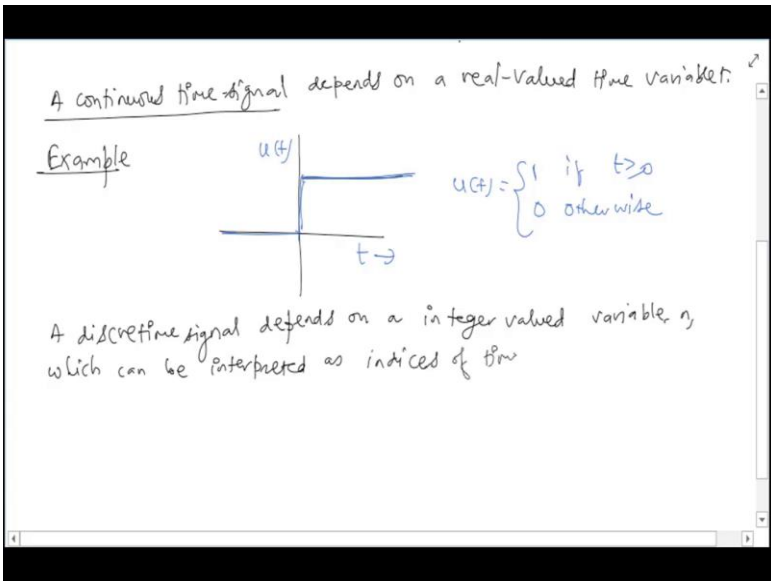

A continuous time signal, it depends on real value time variable, usually denoted as t. Here is an example, might be familiar with this signal, this is called the unit step signal, usually denoted as U(t),

$$U(t) = \begin{cases} 1 \; if \; t \geq 0 \\ 0 \; otherwise \end{cases}$$

On the other hand, a discrete time signal depends on integer value variables, depends on a integer value variables n that indexes instance of time, which can be interpreted as an index of time, as well as I would say indices of time.

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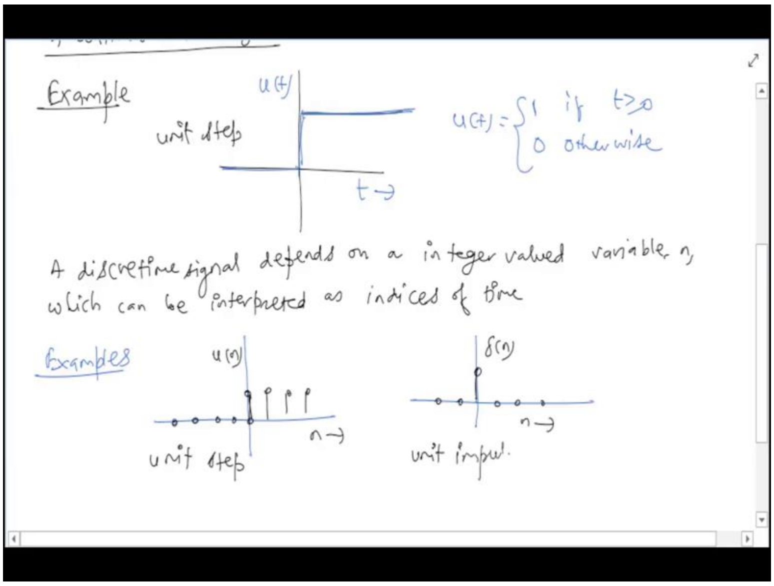

Again, we can have examples, discrete time unit step signal that would look something like this, denoted as U(n) or another quite popular signal often encountered in analysis is Dirac impulse function, do not unit impulse. This is typically denoted as δ(n). So, this is called the unit step, also unit step, continuous time unit step, whereas this function is called impulse or unit impulse function.

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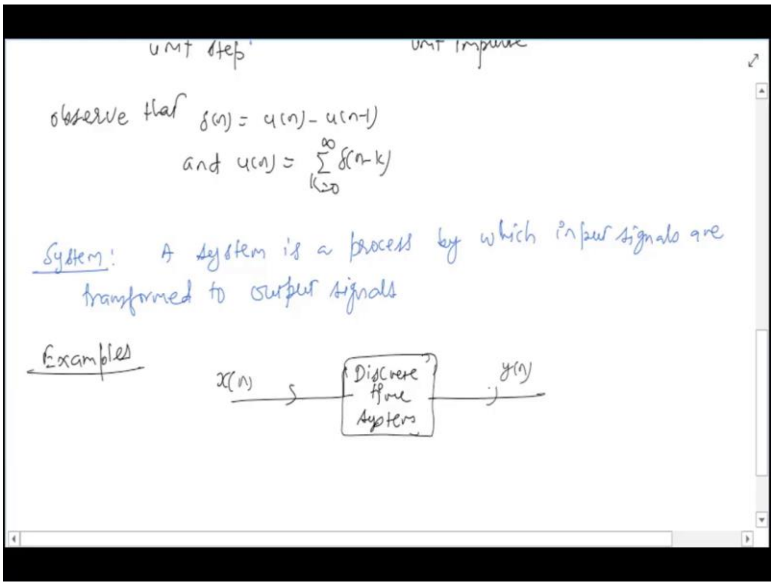

Observe that the two are related, unit step and unit impulse. U(n) is accumulation of infinitely many impulses. Now, let us move to systems. For us a system is a process that transforms input signals to output signals, input signals are transformed to output signals. Here are a few examples. So, we may have a discrete time system, discrete time system that takes a discrete time signal as input and outputs a another discrete time signal, this would be called discrete time system.

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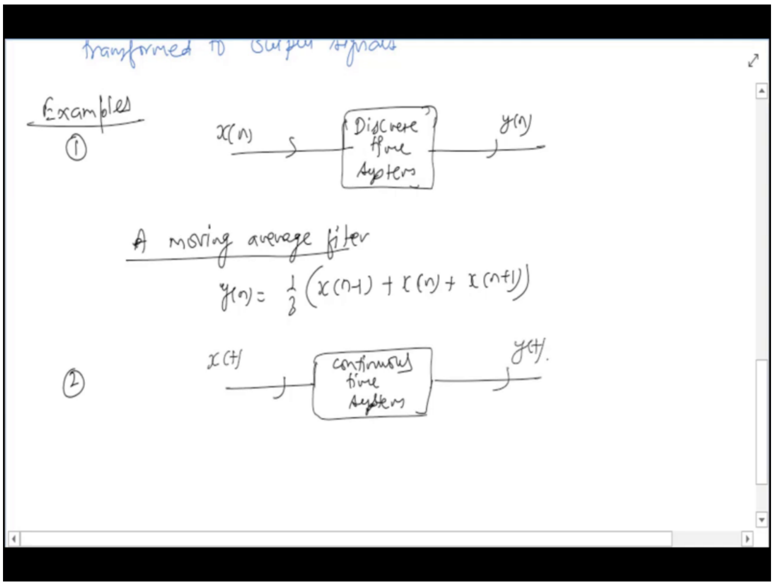

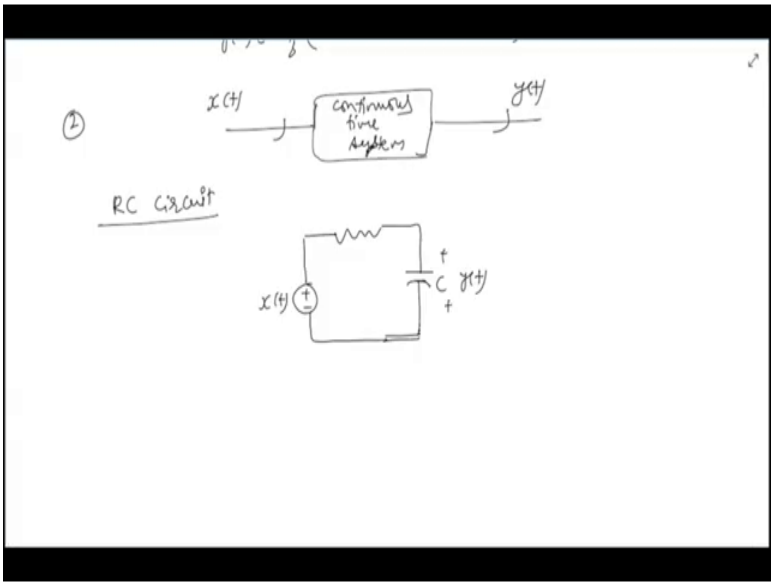

An example of such a system would be, a moving average filter in which y(n) the output at any time index is a moving average of say three of the inputs. So, this is an example of discrete time system. We can also have a continuous time system, where both input as well as output are continuous time signals.

An example of such a system would be an RC circuit. So, here is an RC circuit with an input voltage x(t) and retrieve the voltage, capacitor voltage as the output. We will use this example to illustrate many of the properties of signals and systems as we go along.

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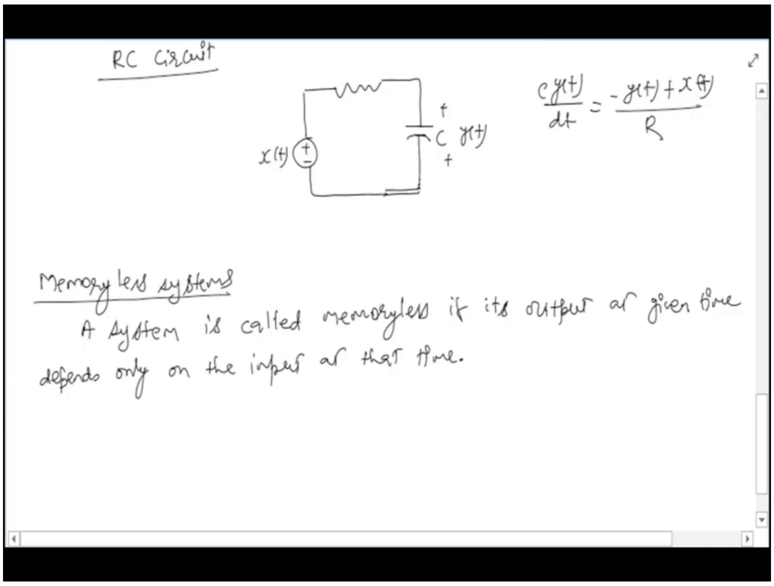

Notice that in this continuous time signal, output and input are related edge for edge. Next is an important class of systems called memoryless system. The system is called memoryless if it’s output at a given time depends on the input only at that time called the memoryless, its output at a given time depends only on the input at that time, input at that time. So here are a couple of examples.

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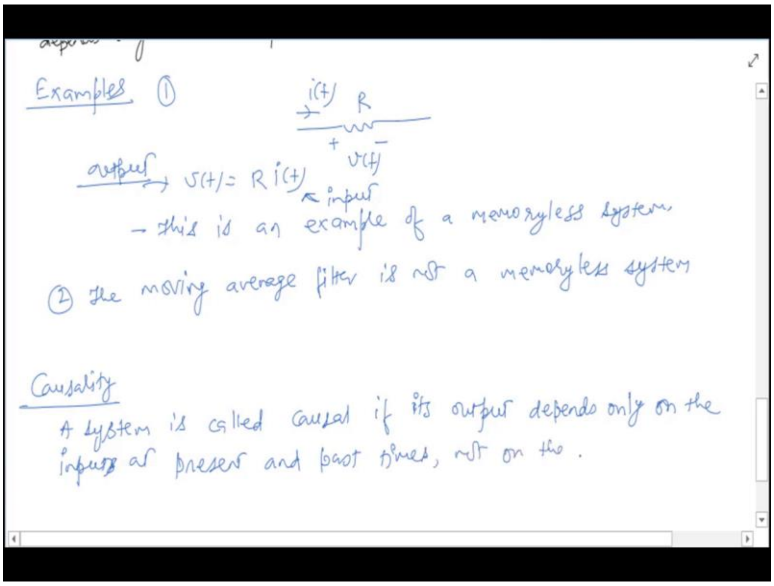

If we consider a simple register and keep the input current i(t), keep the current i(t) as input, voltage across the register as output, then V(t) is Ri(t) and this is an example of a memoryless system. This is, so this is V(t) is the output, i(t) is input, clearly output at any time depends only on input at that time.

So, this is an example of a memoryless system. On the other hand, the moving average filter is not a memoryless system. Next we look at another important property called causality, a system which called causal if its output depends only on the input at present and past times, not on future times. System is called causal if its output at any time depends only on the inputs at present and past times, only on the inputs at present and past times, not on the inputs at a future, inputs at future times.

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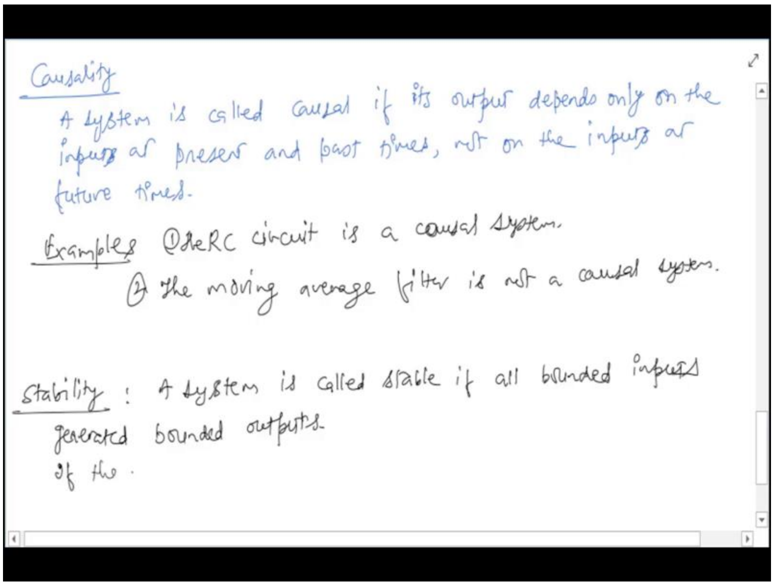

Again example, RC circuit that we described above is a causal system. On the other hand, the moving average filter is not a causal system. Let us now look at another important property of systems namely stability. A system is called stable if all bounded inputs generate bounded outputs.

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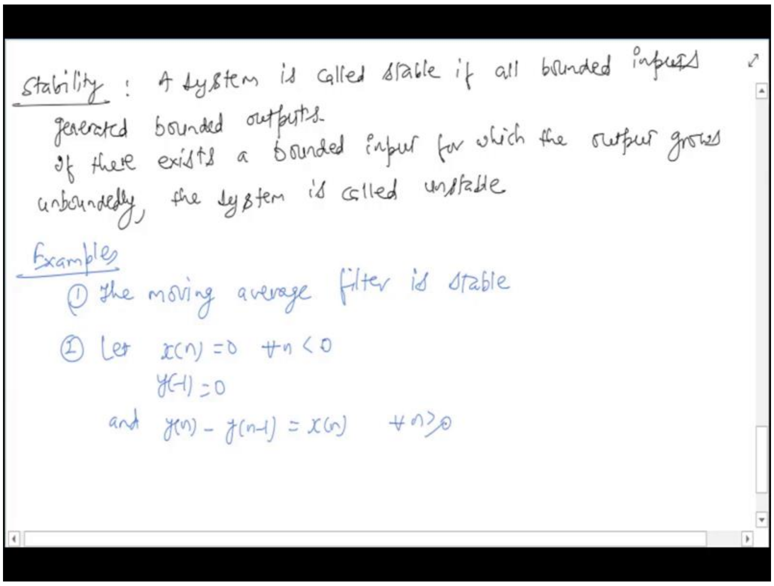

On the other hand, if there exist a bounded input for which the output grows unbounded the system is called unstable, if there exist a bounded input for which the output grows unboundedly, the system is called unstable. Let us look at a couple of examples. Clearly the moving average filter as described above its stable. On the other hand, let us consider the following system, let x(n) be 0 for all $n \leq 0$, $y(-1) = 0$ and y(n) be defined as follows.

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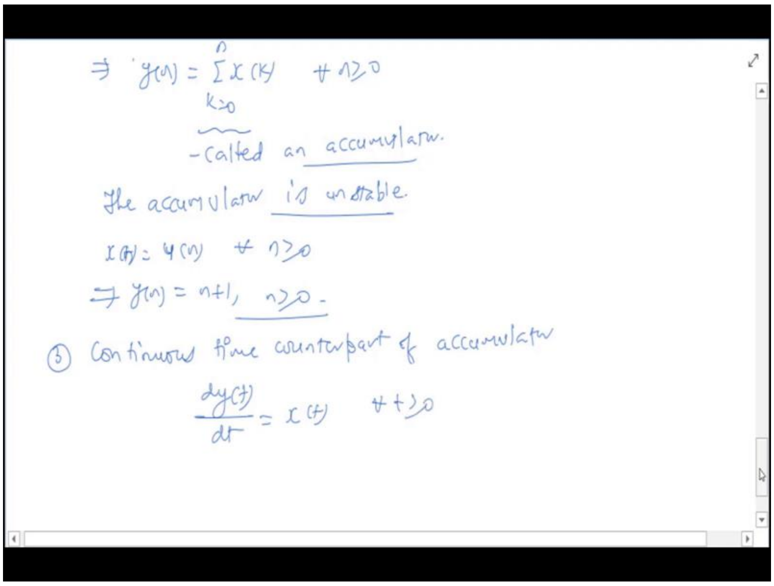

From this, from the above equation, we can see that

$$y(n) = \sum_{k=0}^{n} x\,(k) \qquad \forall\, n \geq 0$$

As we see, as see the feature of this system it's called accumulator. Naturally this is called accumulator.

This, the accumulator is an unstable system, is unstable. For instance, if we give it as input, the unit step signal, so, if

$$x(n) = U(n) \quad \forall\, n \geq 0$$

then we see that y(n) = n+1 which clearly grows unboundedly as we increase n, so while x(n) is bounded y(n) is not. What is the continuous time counterpart of accumulator? It is called integrator, continuous time counterpart or analogue of accumulated it is called integrator and it is described by following differential equation.

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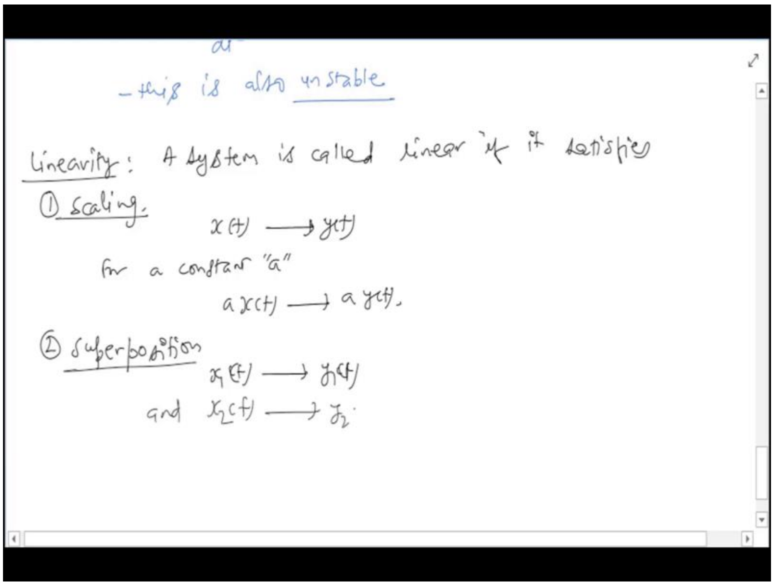

As accumulator this is also unstable. For instance we can give x(t), we can set x(t) to be unit step function and can observe that y(t) grows unbounded. Next very important property that we will see is linearity, a system is called a linear, if it satisfies the following two conditions, what are the two conditions, the first one is scaling, it says that if x(t) results in, input x(t) results in output y(t). Then for a constant, any constant a grid a for any constant a, a x(t) would result in output a of y(t), let me write clearly a of x(t) would result in an output a of y(t) and what is the second condition? The second condition is called superposition. It says that if $x_1(t)$ input $x_1(t)$ results in output $y_1(t)$ and $x_2(t)$ results in output $y_2(t)$.

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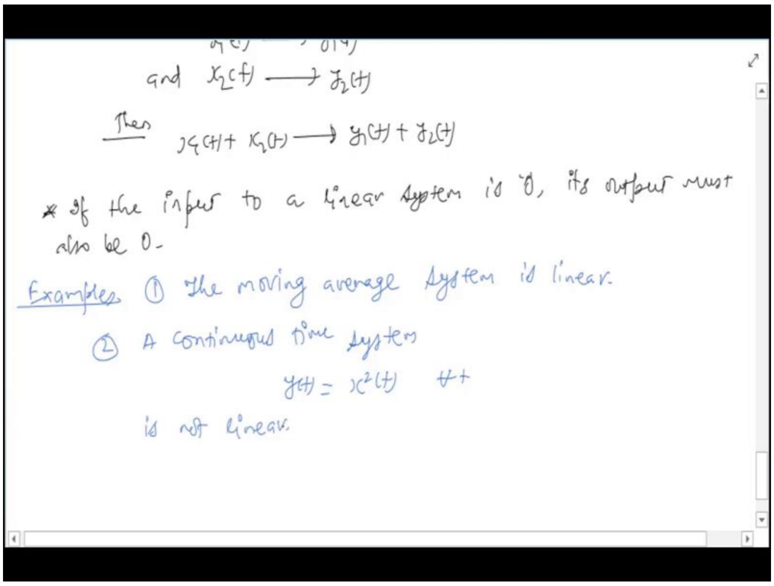

Then $x_1(t) + x_2(t) = y_1(t) + y_2(t)$. Let us see a few examples and non-examples of the linear systems. However, before that I want to mention an important fact that if a system, if the input to a linear system is 0 its output must also be 0, input linear system is 0, its output must also 0. This property can be inferred from the above two conditions.

Now coming to examples, it can be easily verified that the moving average system is linear on the other hand, if I consider a continuous time signal system, where output is square of the input, so $y(t) = x^2t$ for all t is not linear.

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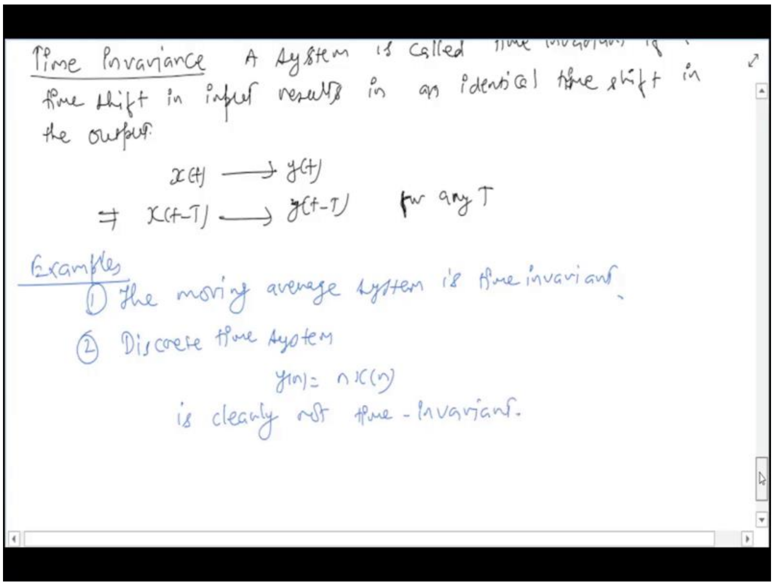

Let us now look at another property called time invariance. The system is called time invariant if a time shift in the input results in an identical time shift in the output, results in an identical time shift in the output. More precisely if input x(t) results in output y(t), then

$x(t-T) = y(t-T) \quad \forall T$

Again examples, the moving average system described above is time invariant, because the rule for generating the output that does not change with time. On the other hand, if you look at the discrete time system y(n) = nx(n), this is clearly not, clearly not time invariant.

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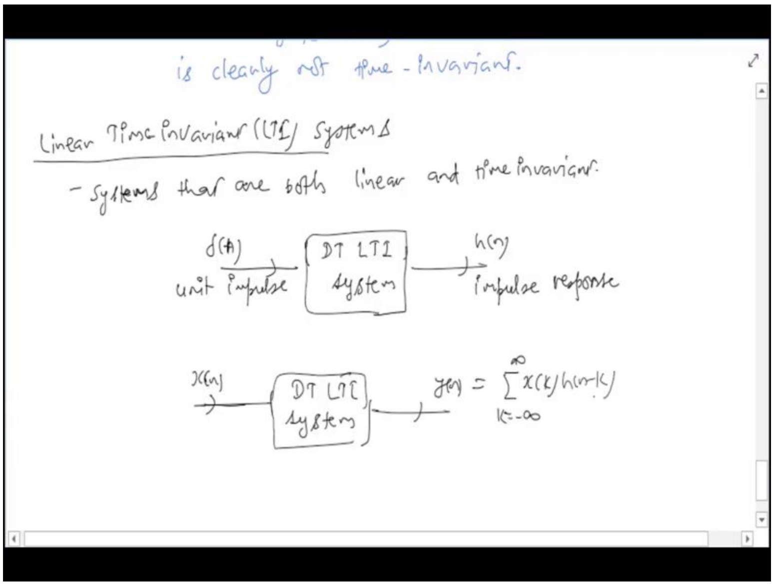

We now arrive at the most important class of systems, namely, linear time invariant systems. As the name suggests, a system that is both linear and time invariant is referred to as a LTI system. The systems in short are called LTI systems. Systems that are both linear and time invariant.

These are most tractable systems in particular, they have the remarkable property that knowing the response to an unit impulse input each enough to compute the response to any other input, let us see what I mean, what I mean that if there is a linear time invariant system, say it is a discrete time, linear time invariant system and we give input δ(n) and the output say h(n) this is unit impulse, this output is called impulse response.

Then for an arbitrary input, the output can be computed using h(n). More precisely, now I give the same system input x(n) the output is

$$y(n) = \sum_{k-\infty}^{\infty} x(k)h(n-k)$$

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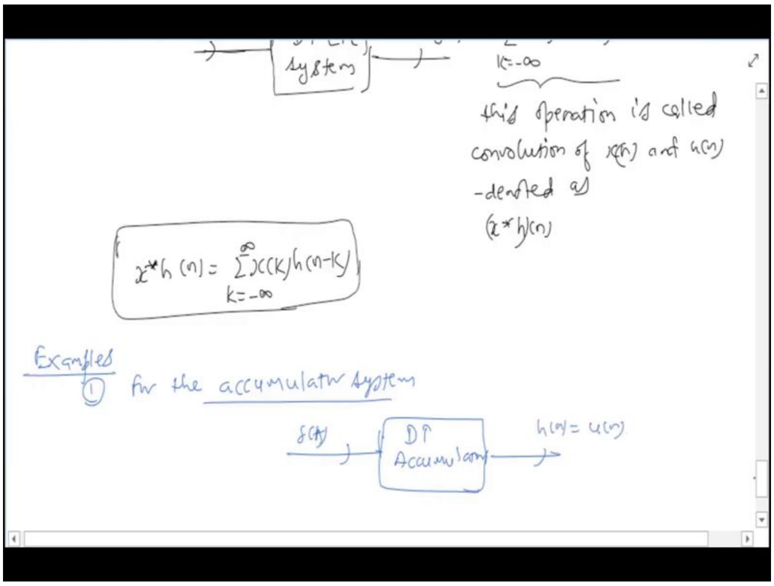

The operation that outputs y(n) from h(n) and x(n) this is called convolution of x(n) and h(n), this operation called convolution of x(n) and h(n) and it is denoted as h(n) x^*h(n), so to be precise

$$x^*h(n) = \sum_{k-\infty}^{\infty} x(k)h(n-k)$$

Let us see a couple of examples.

For the accumulator system that we saw earlier the unit step response turns out to be u(n), for the accumulator system, so this is a discrete time accumulator. If we give it a unit impulse input, the output h(n) turns out to u(n). So, impulse response of discrete time accumulator is u(n).

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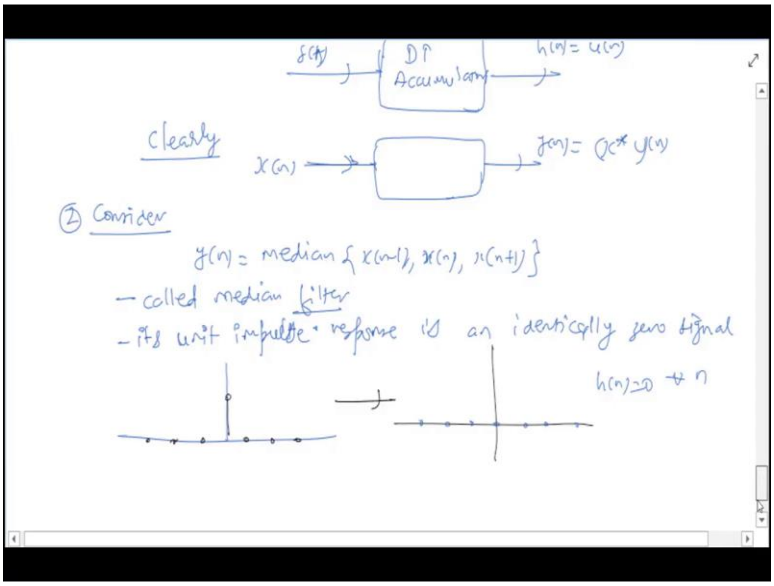

Clearly for any input x(n) to this accumulator, the output y(n) can be obtained as x(n)*, x* y(n). On the other hand, if we consider the following system that is called, let us consider this discrete time system where,

$$y(n) = median\{x(n-1), x(n), x(n+1)\}$$

this is called median filter. Its unit impulse response is, unit impulse response is an all zero signal an identically zero signal. An identically zero signal. To see this, observe that if I input the following signal to this filter, this is unit impulse the output of the filter will be identically 0. So,

$$h(n) = 0 \quad \forall n$$

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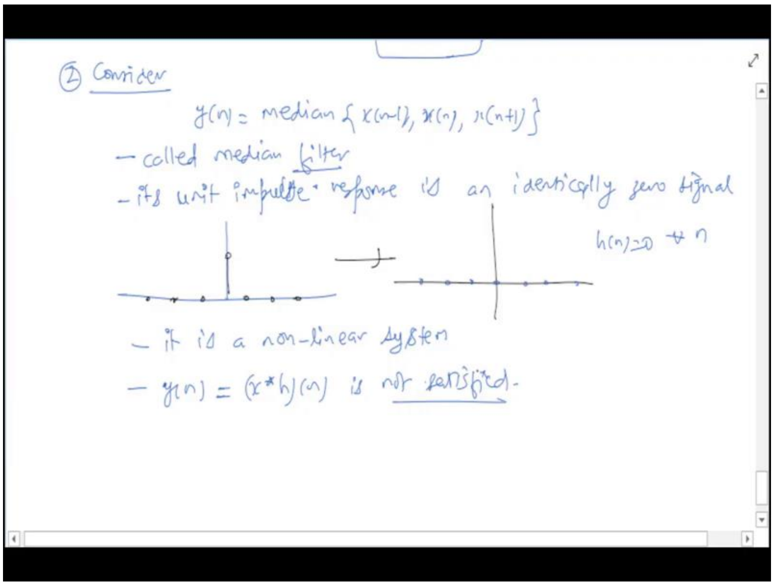

However, in such a system, observe that it is a nonlinear system. So, in such a system, I do not expect to have the following relations satisfied, not satisfied, this is not satisfied because the system is non-linear.

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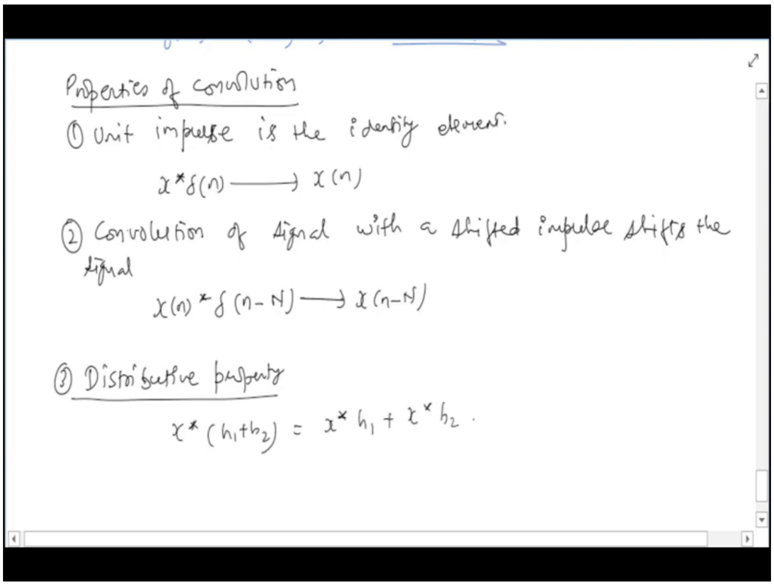

We now see some of the properties of convolution operation and also their implication on LTI systems. So, to see the properties, the first property, properties of convolution. The first property is unit impulse is the identity element of convolution operation. What this means is that if we convolve a signal x with unit impulse, the operation outputs x itself. Second property is convolution of a signal with a shifted impulse, shift signal, shifted impulse shifts the signal.

Mathematically convolution of x(n)*δ(n-N) shifted impulse hx(n-N). The third property is so called distributive property, it says that convolution of $x*(h_1 + h_2) = x*h_1 + x*h_2$.

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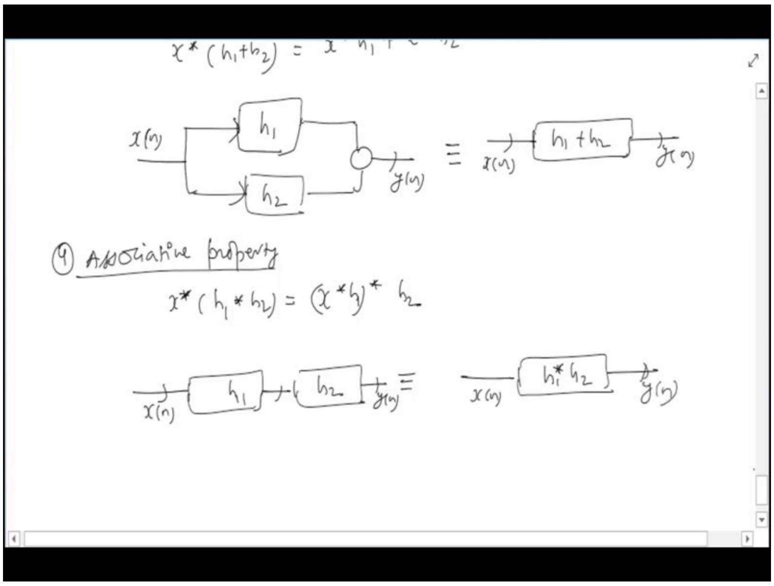

The distributive property implies that the parallel combination of two LTI systems, let us say h_1 and h_2 with impulse can be represented as an equivalent LTI system with impulse response

h_1 plus h_2. So, that is if we input signal x(n) we look at the sum of the outputs call that y(n) this would be same as the output of the following system $(h_1 + h_2)x(n)y(n)$. So, these two systems are equally identical.

The fourth property is associative property, just says that convolution of $x*(h_1*h_2)=(x*h_1)*h_2$. Again, the implication of this property for LTI systems is that we can combine series interconnection of two LTI systems with impulse response such h_1 and h_2 into an equivalent LTI system which impulse response h_1 convolved with h_2. So, that is if we give input x(n) to the system on the left-hand side, then it will produce the same output as if we have given x(n) to the system on right hand side.

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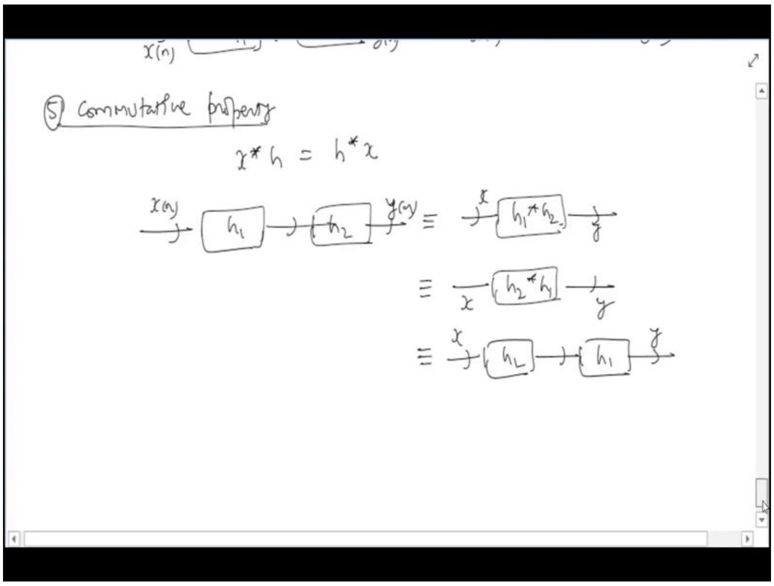

The fifth property is commutative property. This says that x*h = h*x. If we combine this property, it is the associative property, we see that swapping two LTI systems in a series interconnection results in an identical system. What we mean that if we have two LTI systems in series interconnection, then the associative property would imply that the system is equivalent to do this and further using commutative property, commutative property we can conclude that these two are same appealing to associative property once more, we see that the system that we begin with is identical to the system. So, that is swapping the two systems in series interconnection does not have any impact.

Next, we will see, how if whether we can determine causality and stability of linear time invariant systems from their impulse response. We expect this to be the case since impulse response fully characterised LTI systems. Let us see how causality and stability are determined based on unit, based on impulse response, based on impulse responses.

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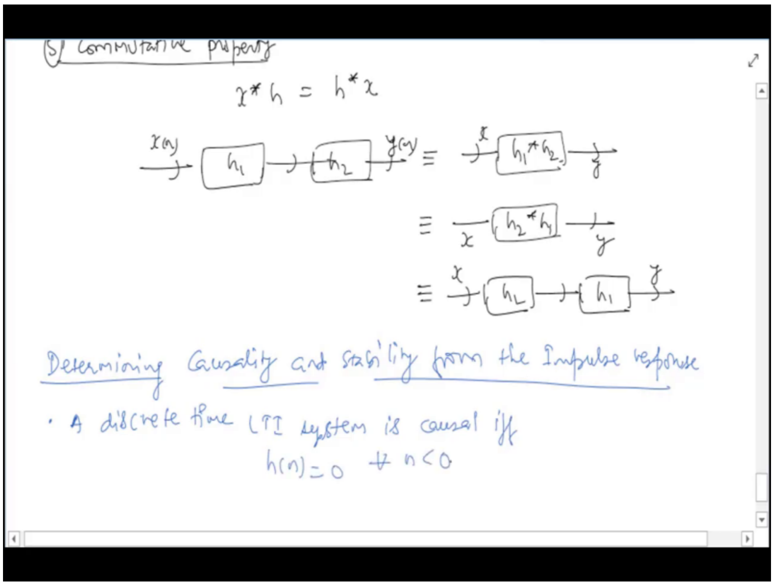

So, determining causality and stability from the impulse response. We find that a discrete time LTI system is causal if and only if impulse response is 0 for all negative values of the argument that is a discrete time LTI system is causal if and only if h(n) = 0 for all n < 0.

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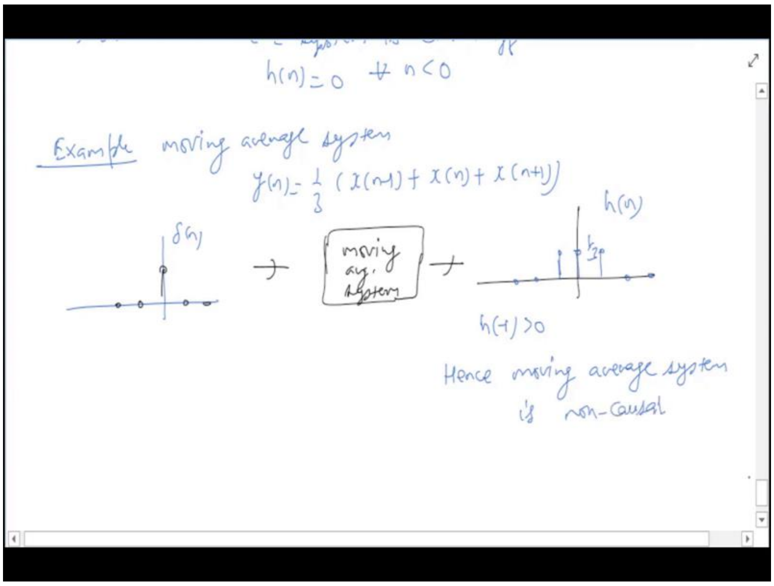

Let us see an example. Recall the moving average system where the output was,

$$y_n = \frac{1}{3}(x(n-1) + x(n) + x(n+1))$$

Let us see what is the impulse response of the system. Here is unit impulse, if we input this to the moving average system the output is the following signal. So, this is h,

$h - 1 > 0$

then from the above fact, we see that moving average system is non-causal, is non-causal.

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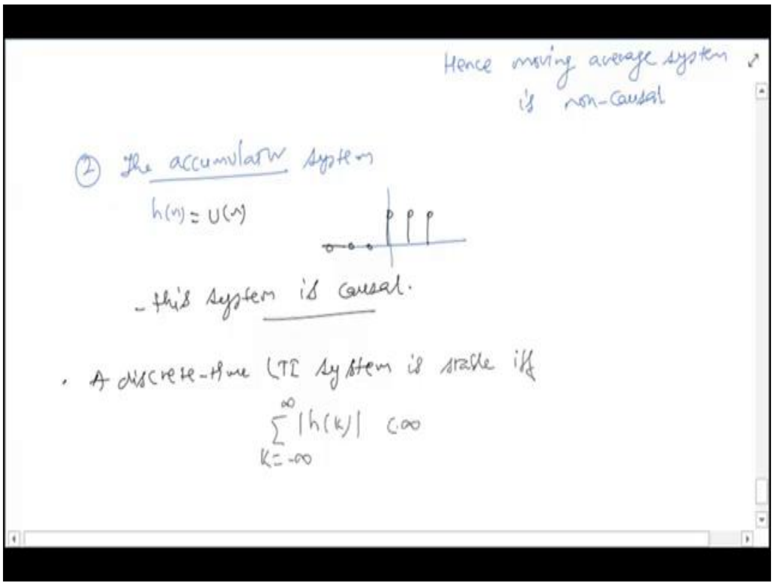

Let us see another example and this is the accumulator system. We know that the impulse response of the accumulator system is u(n). Clearly

$h(n) = 0 \quad \forall n < 0$

So, accumulator system is causal. Next we move to property to see how is property of stability related to impulse response. Discrete time LTI system is stable if and only if the impulse response satisfies the following property. The absolute sum of the terms of impulse response is finite.

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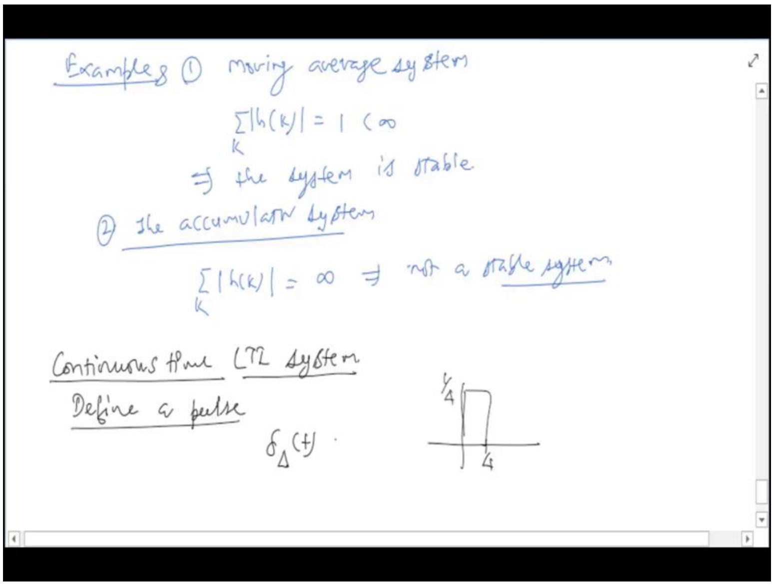

Again, let us see examples. Let us again consider the moving average system. For this system, if we compute the sum as needed to check stability, it can be easily seen that this sum is equal to 1 which is finite. So, moving evidence system is stable. On the other hand, if you look at accumulators system, then this one is clearly infinity and so accumulated is not stable.

Until now, we looked at linear time invariant discrete time systems. Let us now focus on continuous time systems, continuous time LTI systems. As stated before, a continuous time system is linear time invariant if it is both linear and time invariant. Let us focus on the following signal which is called following pulse signal which will help us define convolution integral. So, we define a signal, a pulse, define a pulse delta, capital delta of t as follows, it has width δ and height 1/δ.

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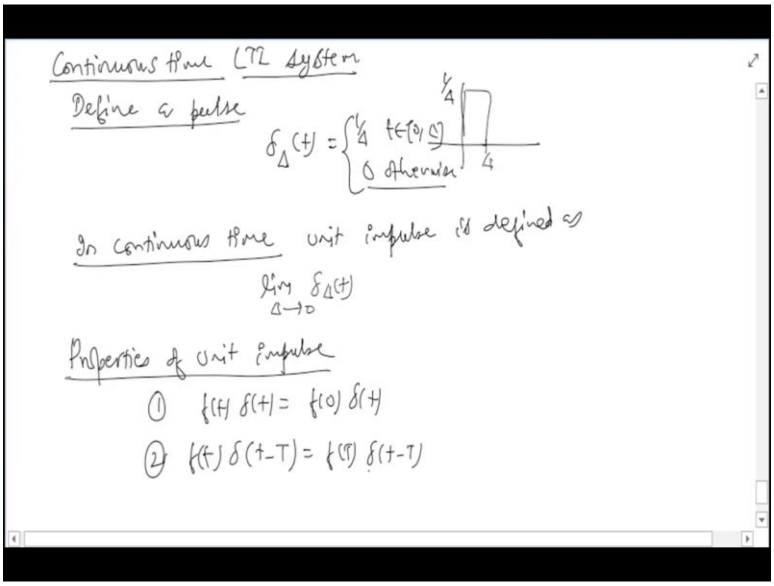

So, this is 1/δ for t in 0 to δ and it is 0 otherwise. Notice that the area of this pulse is unit, in continuous time, the unit impulse is defined as the limit of this function as δ goes to 0, continuous time unit impulse is defined as limit δ tends to 0 pulse signal, notice that the area underneath remains the, underneath remains unit.

Here are a few properties of unit impulse and continuous time. They are analogous to corresponding properties in discrete time. The second property is if we multiply f(t) with time shifted unit impulse, the result is given by this expression.

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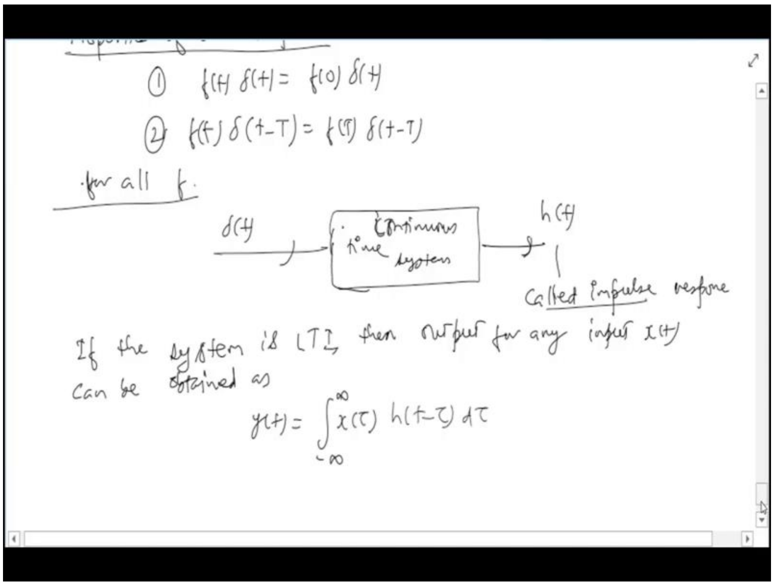

So, these are true for all f, for all f. Now, having defined the impulse unit impulse in continuous time we have the following characterization of, we have the following notion of impulse response, so we have a continuous time system, continuous time system to which we input, give input δ(t), the output of the system with input δ(t) denoted at h(t) is called the impulse response. This is called impulse response.

Further if the system is LTI that is linear time invariant, then output corresponding to output for any input, input x(t) can be obtained as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

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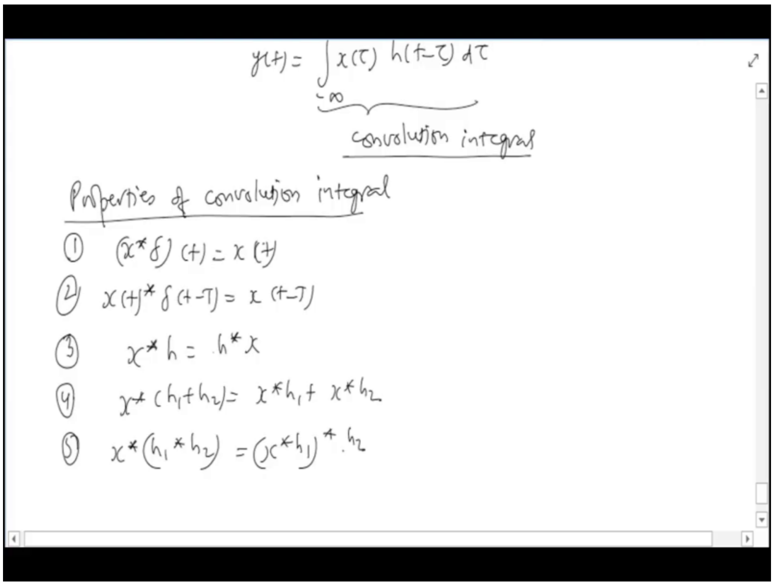

This is the result, this analogous to convolution in discrete time and this integral itself is called convolution integral. Convolution integral possess similar properties to discrete time convolution sum in particular, it has the following properties convolution integral. First one is unit impulse is the identity element that is $(x*\delta)(t) = x(t)$. The second property says that convolution with shifted unit impulse results in shifting the input signal. As for discrete time we have commutative properties that is $x*h = h*x$, distributive properties and associative property.

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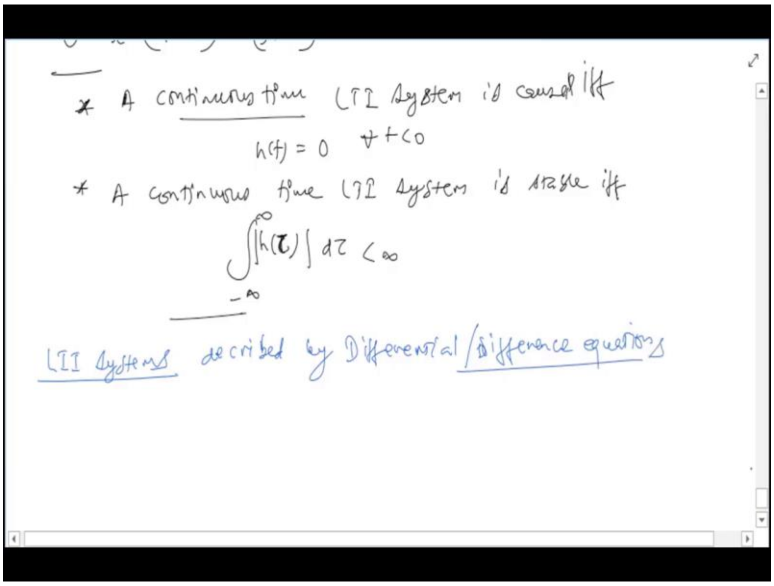

Moreover, we can again write the characterization of causality and stability of linear time invariant systems in terms of impulse response to be precise causality of a continuous time LTI system is equivalent to. So, continuous time LTI system is casual if and only if $h(t) = 0$ for all $t < 0$.

Similarly, a continuous time LTI system is stable, if and only if

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty$$

Having seen these, next we will see the connection between linear time invariance and systems described by differential and difference equations. So, the next topic is LTI systems described by differential equations, differential or a difference equations.

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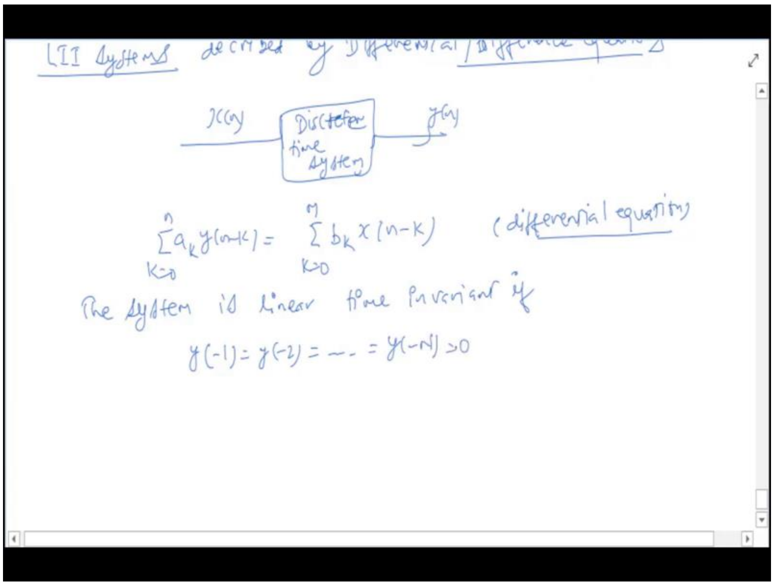

Let us consider a system discrete time system let us say, discrete time system which output is y(n) for input x(n) and moreover, x(n) and y(n) are related as follows

$$\sum_{k=0}^{n} a_k y(n-k) = \sum_{k=0}^{m} b_k x(n-k)$$

this equation is called differential equation. Then the system is linear time invariant, if the following condition is, the system is linear time invariant, if

y(-1) = y(-2) = ……. = y(-N) = 0.

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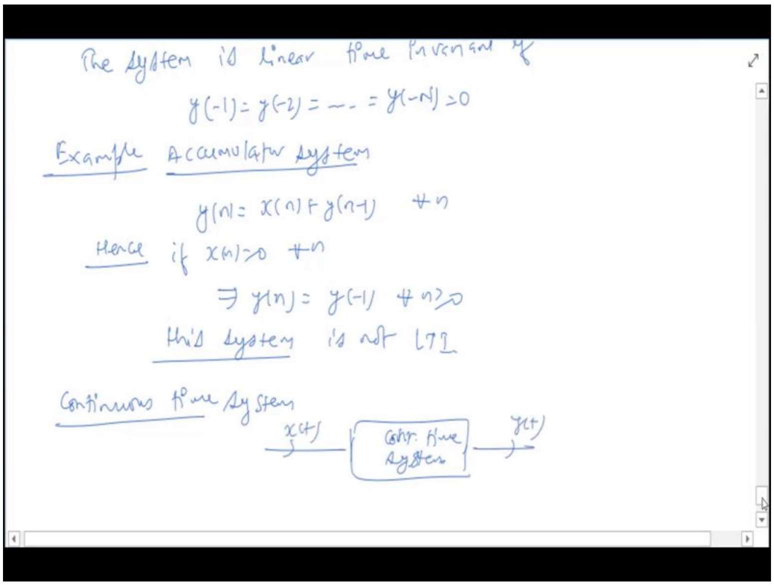

Let us see an example, let us consider the accumulator system. Recall that in the accumulator system, y(n) = x(n) + y(n -1) for all n hence, if x(n) = 0 for all n then we find that y(0) = y(-1), that y(n) = y(-1) for all n ≥ 0. So, this system is not LTI, linear time invariant. Let us now see a continuous time system, so here it is we have continuous time system its input is x(t) and output is y(t).

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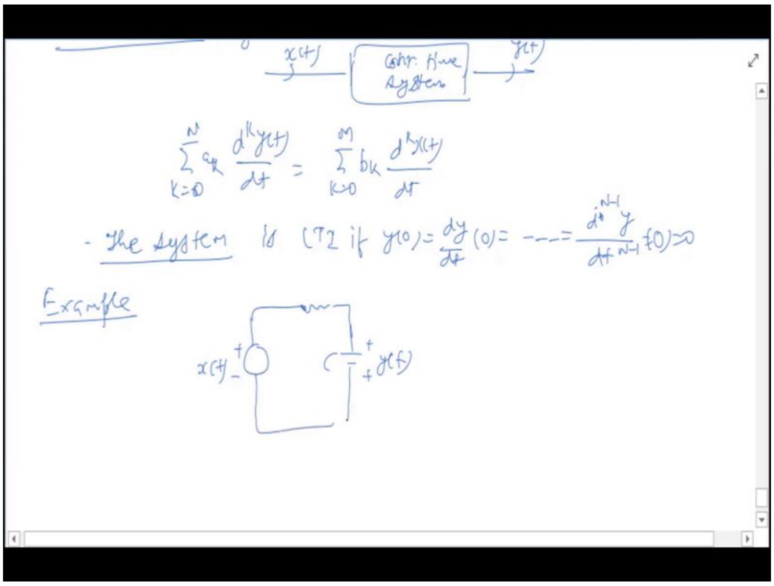

And say y(t) and x(t) are related as follows. So,

$$\sum_{k=0}^{N} a_k \frac{dky(t)}{dt} = \sum_{k=0}^{M} b_k \frac{dkx(t)}{dt}$$

Again it turns out that this system is linear time invariant, if all the initial conditions are 0. If the system is LTI, if

$$y(0) = \frac{dy}{dt}(0) = \cdots \ldots = \frac{d^{n-1}y}{dt^{n-1}}(0) = 0$$

We can use this criteria to see that the RC system that we had seen earlier as an example of continuous time system is not LTI until y(0) = 0. Example let us see this RC system with input x(t) and output voltage across the capacitor.

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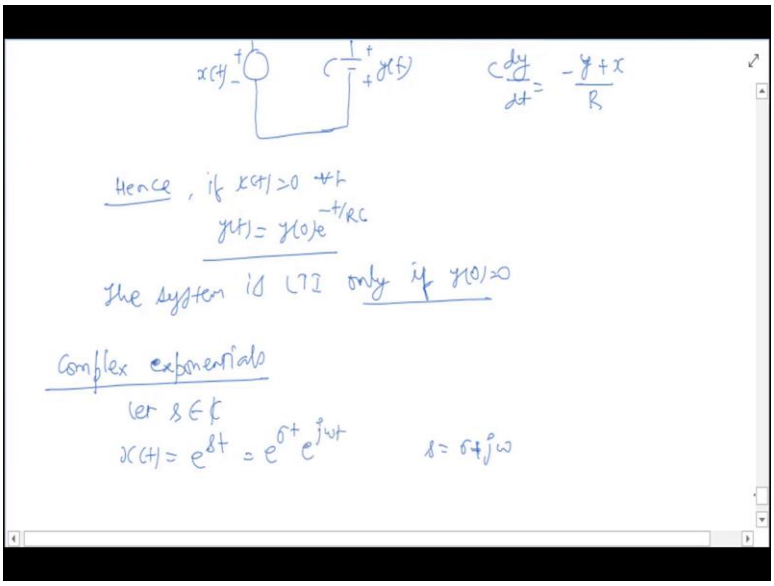

Notice that here input and output are related by

$$C\frac{dy}{dt} = -\frac{y+x}{R}$$

it can be checked that if x(t) = 0 identically y(t) can be written as

$$y(0)e^{-\frac{t}{RC}}$$

The system is linear time invariant only if y(0) = 0. Finally, we look at spatial functions and their responses to linear time invariant systems and this spatial functions are called complex exponentials, let us see what are complex exponential in both discontinuous and discrete domain. Complex exponentials.

Let us S be a complex number signals of the form, e^{St} are called complex exponentials in continuous domain. So, notice that this x(t) can also be written as if

$$S = \sigma + j\omega$$

then this

$$x(t) = e^{\sigma t}e^{j\omega t}$$

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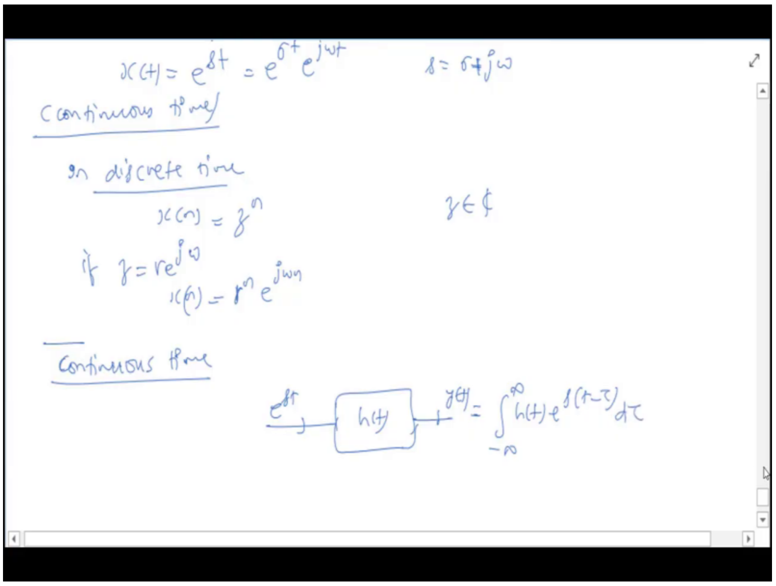

Similarly, if you look in discrete domain. So, this was in continuous time, in discrete time, discrete time signals of the form x(n) = z(n) where z is again complex called discrete time complex exponentials. So, notice that if z = S again our $re^{j\omega}$, if z is

$z = re^{j\omega}$

Then

$x(n) = r^n e^{j\omega n}$

Now, let us see what are the responses of LTI systems to these signals. First, we will consider a continuous time system. So, here is your system with impulse response h(t) we give it a signal, e^{St}, then its output from what we have learned can be written as

$\int_{-\infty}^{\infty} h(t) e^{S(t-\tau)} d\tau$

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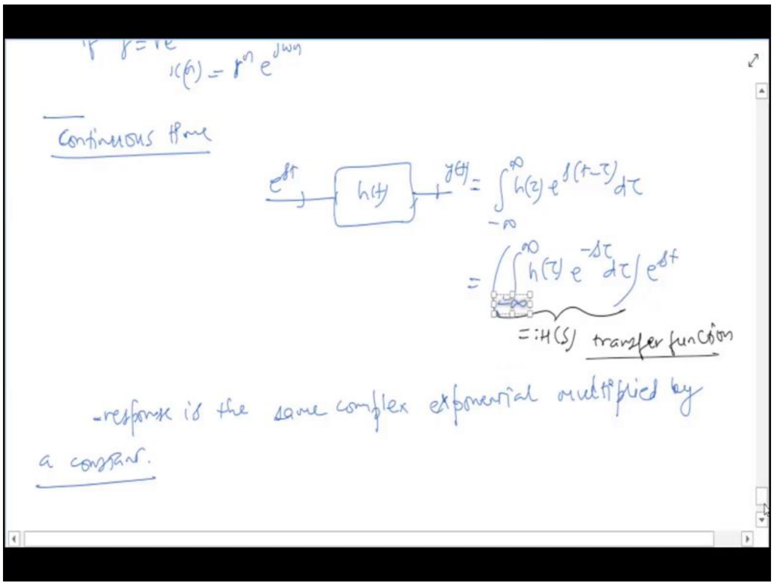

It can be easily seen that this integral equals

$$\left(\int_{-\infty}^{\infty} h(\tau)e^{-S\tau}d\tau\right)e^{St}$$

We see that the response of the LTI system to the complex exponential, its same complex exponential scaled by the constant. Response is the same complex exponential e^{St} multiplied by a constant. The constant here is called H(S), this constant is denoted as H(S) and it is called transfer function. Let us see what happens in discrete time.

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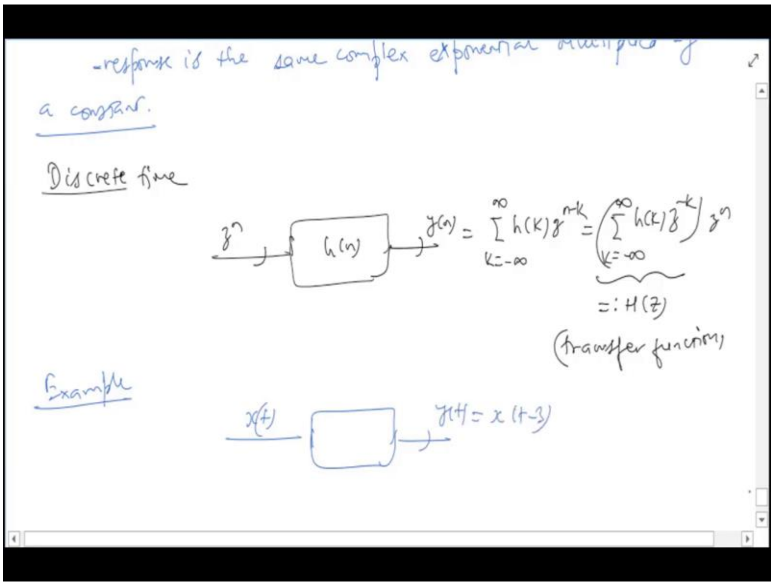

Here again let us consider a system with impulse response h(n), if we input z^n, the output of the system y(n) can be written as,

$$\sum_{k=-\infty}^{\infty} h(k) z^{n-k}$$

which in turn can be written as

$$\left(\sum_{k=-\infty}^{\infty} h(k) z^{-k}\right) z^{n}$$

So, we again see that the output h, constant multiplied by the same constant, same complex exponential this constant, now denoted by H(Z) is again called again call transfer function of the discrete time system. Sometimes these transfer functions are called system functions. Let us see an example. Let us consider a system whose output y(t) to input is x(t-3).

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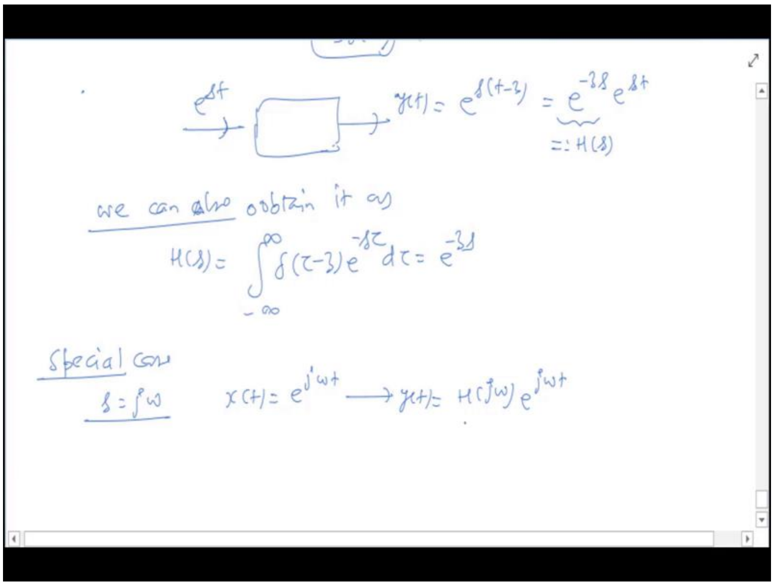

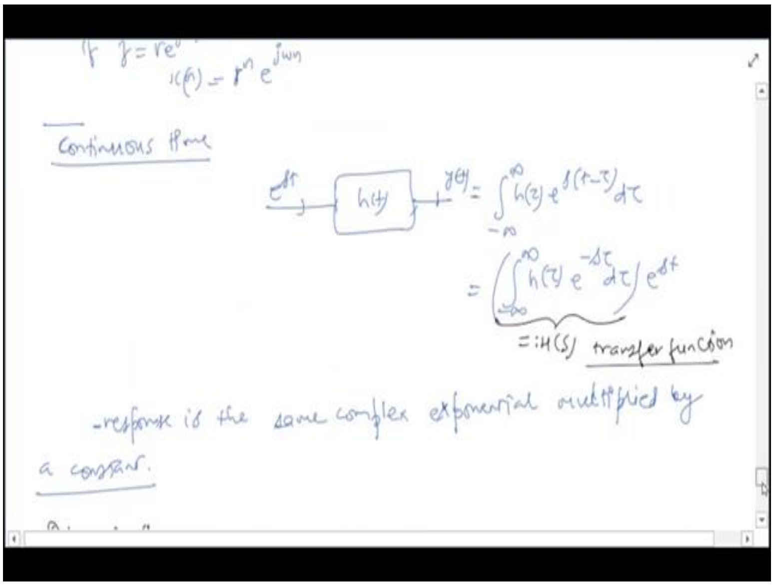

Let us see how do you compute its transfer function. So, we know that transfer function h obtained by inputting to the system, the special complex exponential e^{St}. So, what will the output be, if we input e^{St} the output will be $e^{S(t-3)}$ which we can write as $e^{-3S}e^{St}$ clearly e^{-3S} is the transfer function. The same transfer function can also be obtained using the formula that we have derived above. You can also see, you can also obtain it as H(S) equals to.

Let us see the formula that made

$$\int_{-\infty}^{\infty} \partial(\tau-3)e^{-S\tau}\,d\tau = e^{-3S}$$

which again turns out to be e^{-3S} as expected. Notice that we write $\partial(\tau-3)$ because unit impulse response of the system. This is

$h(\tau) = \partial(\tau-3)$.

Finally, let us look at a special case of this complex exponential application of complex exponential, application of complex exponential.

A special case where S is has only complex part $S = j\omega$, in this case my x(t) which used to be e^{St} would be $e^{j\omega t}$ and the output will be $h(j\omega)$, the constant factor times $e^{j\omega t}$ that is the input signal.

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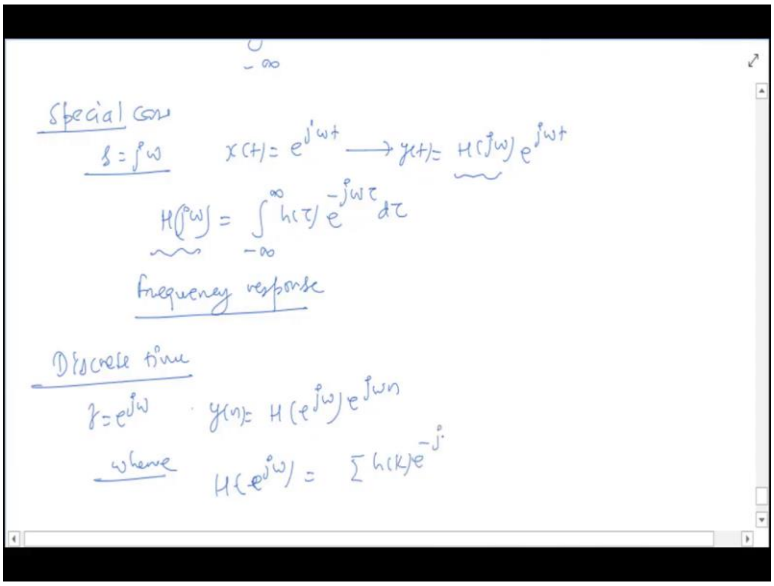

This $h(\omega)$, $h(j\omega)$ can be obtained as follows, can be obtained using the formula that we just derived, it would be $e^{-j\omega\tau}\, d\tau$ and this function $H(j\omega)$ is called a frequency response. Similarly, in discrete time if we said $z = e^{j\omega}$ not having any real part, then again I can write

$$y(n) = H(e^{j\omega})e^{j\omega n}$$

where

$$H\left(e^{j\omega}\right) = \sum h(k)e^{-j\omega}.$$

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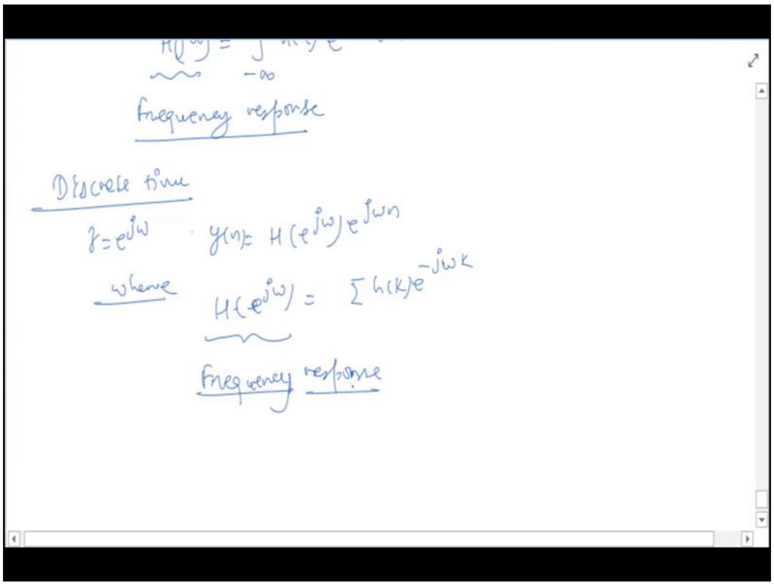

Again this function is called frequency response indiscrete time system. This completes the first module or rather first part of our module on signals and system. Next time when we meet we will see Fourier transforms, Fourier series etc. Thank you.