Mathematical Aspects of Biomedical Electronic System Design
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Lecture 12
Linear Algebra I

Hello everyone, welcome to the next module of the course mathematical aspects of biomedical electronic system design. This module focuses on linear algebra; this module will comprise of three lectures.

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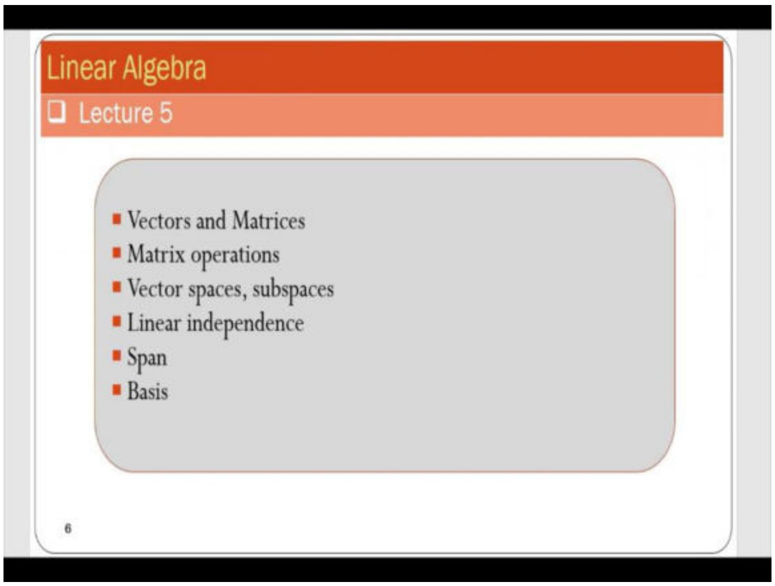

In today's lecture I will introduce vectors, matrices, vector spaces, et-cetera; so let us begin to this lecture.

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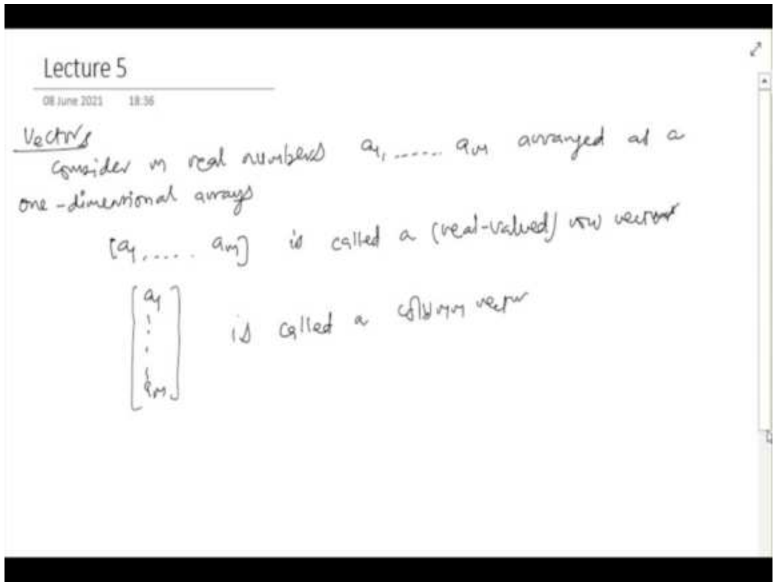

I will start with introducing vectors and matrices. To begin with, we will focus on real valued vectors and matrices. So, let us see, let me introduce vectors. Consider m real numbers, say a_1 up to a_n; arranged as a one-dimensional arrays. One-dimensional array means we could arrange these vectors in two ways; if we arrange them in a row, the resulting object is called row vector. So, this is called real value row vector, real value row vector. Similarly, if we arranged these numbers in a column; the object is called column vector; so this is a column vector.

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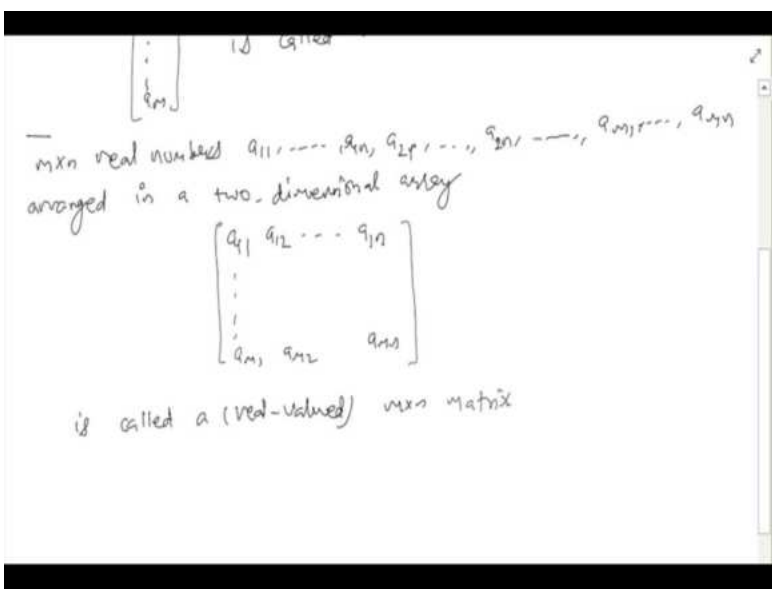

Now, consider $m \times n$ real numbers say a_{11}……a_{1n}, a_{21}…….a_{2n} and so on a_{m1}……a_{mn}. So, we have $m \times n$ real numbers; and if we arrange them in a two-dimensional array, arranged in a two-dimensional array as follows. So, a_{11}, a_{12} ……. a_{1n}, a_{m1}, a_{m2}, a_{mn}; then this object is called the $m \times n$ matrix; so it is called again a real valued, a real valued $m \times n$ matrix.

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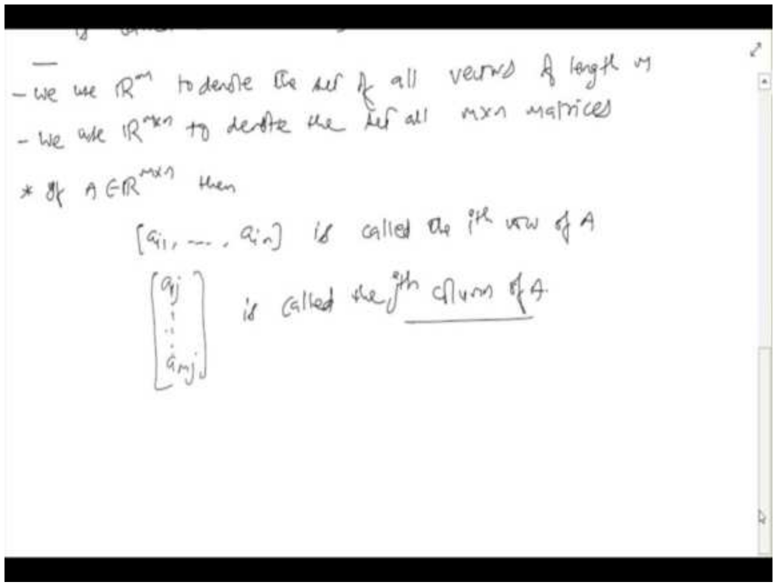

We use R^m to denote the set of all vectors of length m all real valued vectors of length M. And similarly, we use similarly we use $R^{m\times n}$ to denote the set of all real valued m × n matrices. If A is an m × n matrix, then the one-dimensional vector a_{i1} ……. a_{in} is called the i^{th} row of A. And the column vector a_{1j} … a_{mj} is called the j^{th} column of A.

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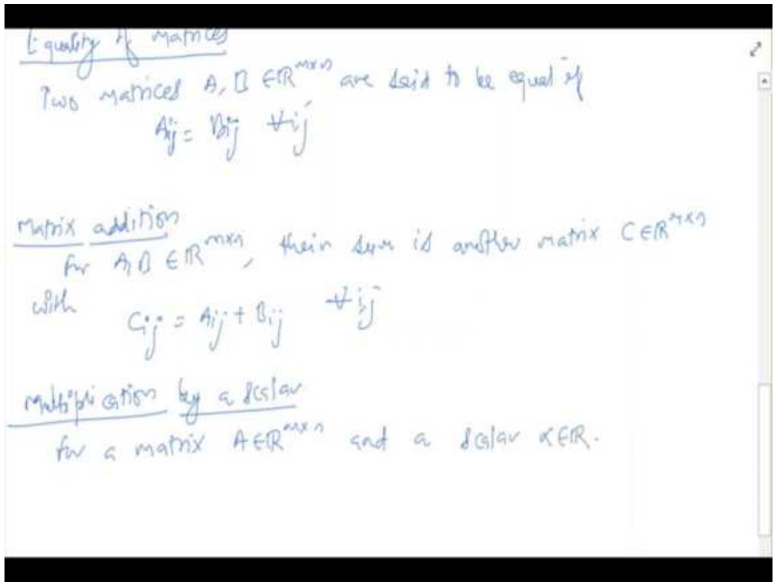

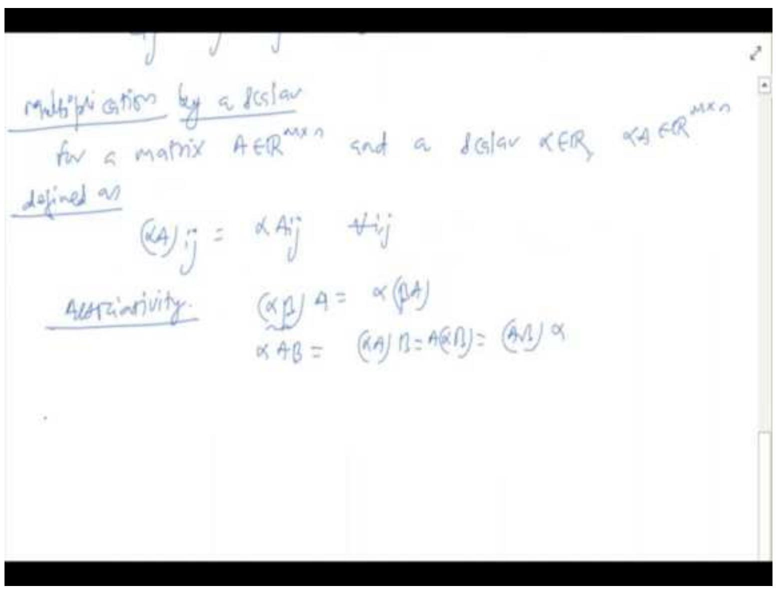

Let us now see a few basic operations on matrices. So, we will begin with equality of matrices. So two $m \times n$ matrices; so, A and B are called equal, if they are equal element wise. So, what I mean is that two matrices A, B, both $m \times n$ are said to be equal if, $A_{ij} = B_{ij}$, $\forall ij$. Here, A_{ij} is the entry in the i^{th} row and j^{th} column of A; and similarly B_{ij} is the entry in the i^{th} row and j^{th} column of B. Next we will see matrix addition. Given two matrices $m \times n$ matrices; their sum is another matrix, another $m \times n$ matrix defined as follows. For A, B $R^{m \times n}$, their sum is another matrix C with C_{ij}; that is

$$C_{ij} = A_{ij} + B_{ij} \quad \forall ij$$

Now, we will see multiplication by a scalar; by scalar here we mean real numbers. So, for a matrix A that is m × n matrix and a scalar α, which as I said is a real number; αA is another m × n matrix. So, this also belongs to $R^{m \times n}$, defined as

$$(\alpha A)_{ij} = \alpha A_{ij} \quad \forall ij$$

The next thing that we will see are properties of associativity and distributivity. So, what is associativity? Associativity properties is that given two scalars α, β, and a matrix A,

$$(\alpha\beta)A = \alpha(\beta A)$$

So, here we are first multiplying α and β giving a real number $\alpha\beta$, and then are multiplying with a matrix A.

Whereas, on the right hand side, I am multiplying α with a matrix βA, which itself is a result of multiplying β with A. Associativity property says that the order of multiplications does not matter. Similarly, given a scalar α and matrices A and B,

$$\alpha AB = (\alpha A)B = A(\alpha B) = (AB)\alpha$$

All these properties come under associativity.

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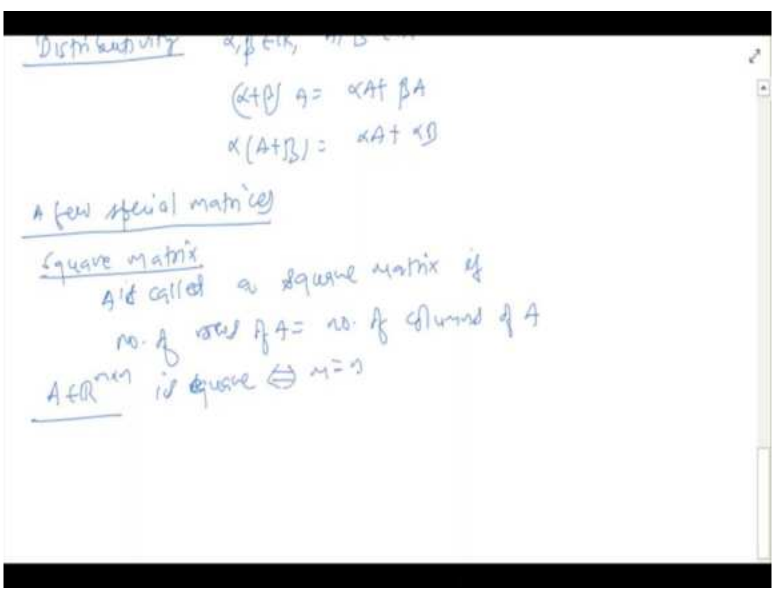

The next property as I mentioned is distributivity. This property says that given real numbers, α, β, and matrices A, B; say both $R^{m \times n}$ matrices.

$$(\alpha + \beta)A = \alpha A + \beta A$$

$$\alpha(A + B) = \alpha A + \alpha B$$

these properties together are called distributivity properties. Now, we will look at a few special matrices; so a few special matrices. First one is class of square matrices; so what is square matrix? A is called a square matrix, if the number of rows of A equals the number of columns in A. So, A is called a square matrix if number of rows of A is same as number of columns of A. In other words, A that is an $m \times n$ matrix is a square, if and only if $m = n$, is square if $m = n$.

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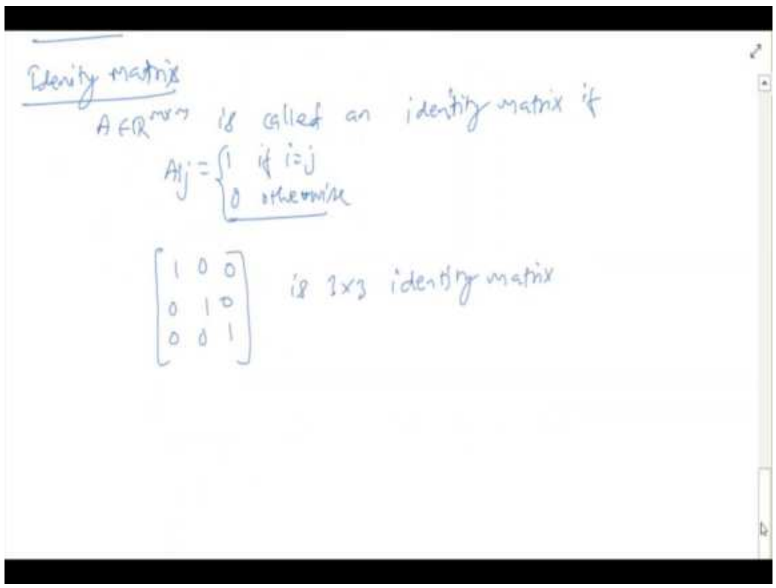

Next we will see identity matrices. So, identity matrix identity matrices are, A which is an $m \times n$ matrix is called an identity matrix, if all its diagonal terms are 1 and off diagonal terms are 0; if $A_{ij} = 1$, if $i = j$ and 0 otherwise. So, for illustration this matrix is 3×3 identity matrix. Next, we will look at what we call diagonal matrices.

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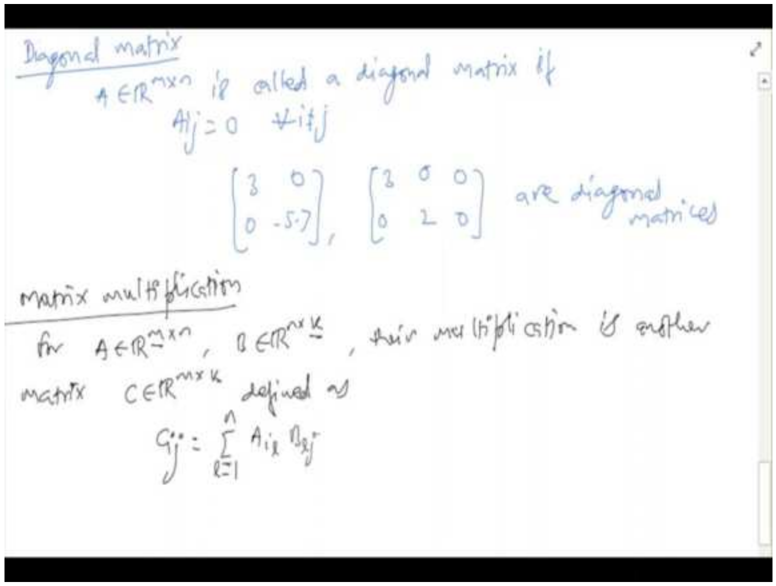

So, a matrix A is called diagonal matrix, if all its off diagonal entries are 0. So, a matrix A which is an m × n matrix is called a diagonal matrix diagonal matrix, if $A_{ij} = 0$, $\forall i \neq j$. Let us see some examples; so, this is a two by two diagonal matrix, let see another example. This is another diagonal matrix, these are diagonal matrices. Notice that in diagonal matrices, diagonal entries could also be 0.

Next we will see what is matrix multiplication. Given two matrices A and B, where A is m × n matrix and B is n × k matrix; we will define their multiplication. So, for A, B their multiplication is another matrix say C, which is an m × k matrix; defined as

$$C_{ij} = \sum_{l=1}^{n} A_{il} B_{lj}$$

This is how we obtain the i-jth element of the product matrix.

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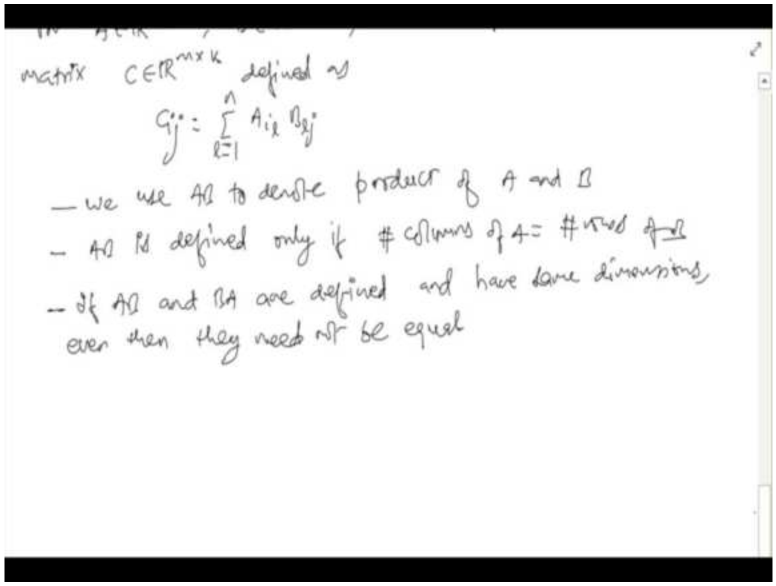

So, from here we see that we cannot multiply arbitrary matrices of arbitrary dimensions; there has to be compatibility within their dimensions. For instance, to multiply A and B, the number of columns in A must be equal the number of rows of B. So, we how do you denote the multiplication? So, we use AB to denote product of A and B. So, as I said earlier AB is defined only if number of columns of A equals number of rows of B.

Next fact is if AB and BA both are defined and have same dimensions; even then they need not be equal. If AB and BA are defined and have same dimensions, even then they need not be equal. Like multiplication of scalars and matrices, matrix multiplications also follow associativity and distributivity properties. Let us see what these properties are.

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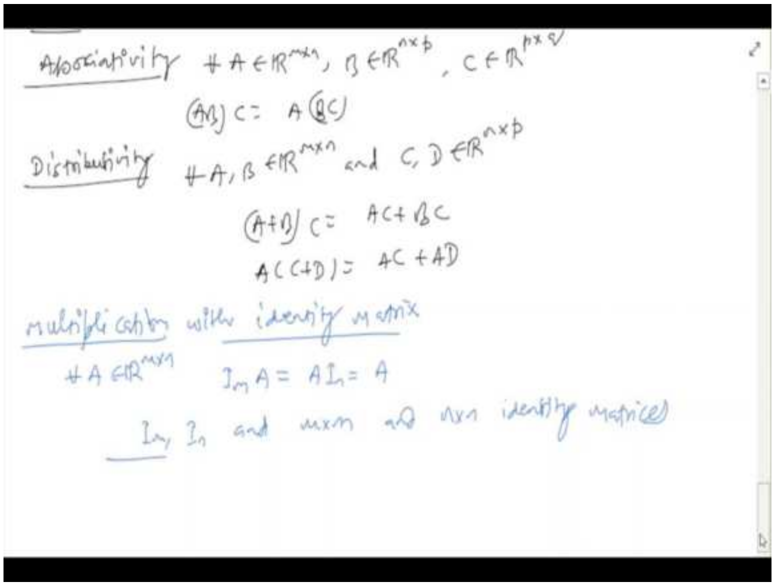

Let us begin with associativity. This says ∀A in $R^{m \times n}$, and B in n × p, and C in p × q.

$$(AB)C = A(BC)$$

Similarly, let us write distributivity property. It says that ∀A, B that are m × n and C, D there are n × p;

$$(A + B)C = AC + BC$$

$$A(C + D) = AC + AD$$

Let us see what happens when we multiply a matrix which is an identity matrix of suitable dimension. So, multiplication with identity matrix; it can be easily verified that ∀A that are m × n matrices

$$I_m A = A I_n = A$$

So, here I_m and I_n are m × n and n × n identity matrices respectively. Next is a very important notion that we will encounter again and again; and this is of transpose.

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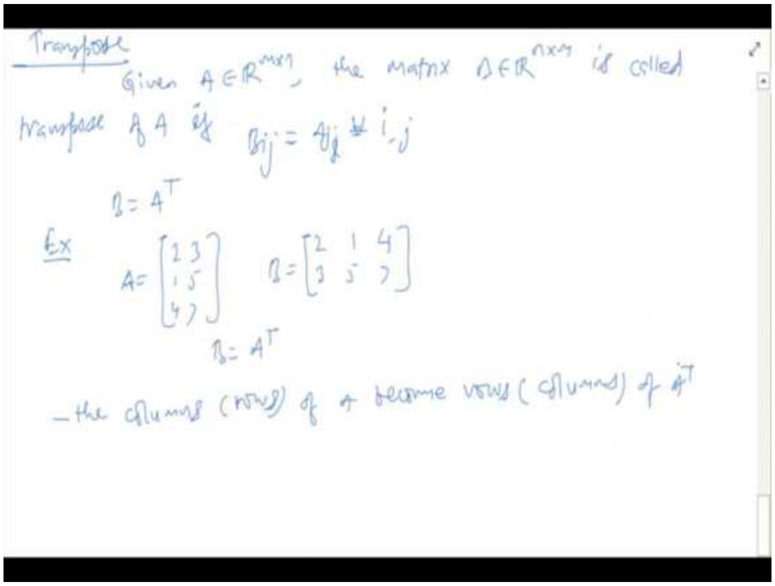

Given a matrix A, which is an m × n matrix; the matrix B is called transpose of A. The matrix B which is actually an n × m matrix is called transpose of A, if $B_{ij} = A_{ji}$, $\forall i, j$. We denote transpose of A B as follows;

$$B = A^T$$

this is transpose signal. Superscript is transpose signal; so, let us see an example.

Let say,

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

so, this is a 3 × 2 matrix. Let say B is 2 × 3 matrix with entries are as follows; then clearly B is transpose of A. So, actually if we see in when we do transpose the columns of A becomes row of A transpose, and rows of A becomes columns of A transpose. So, the columns bracket rows of A become rows columns of A transpose.

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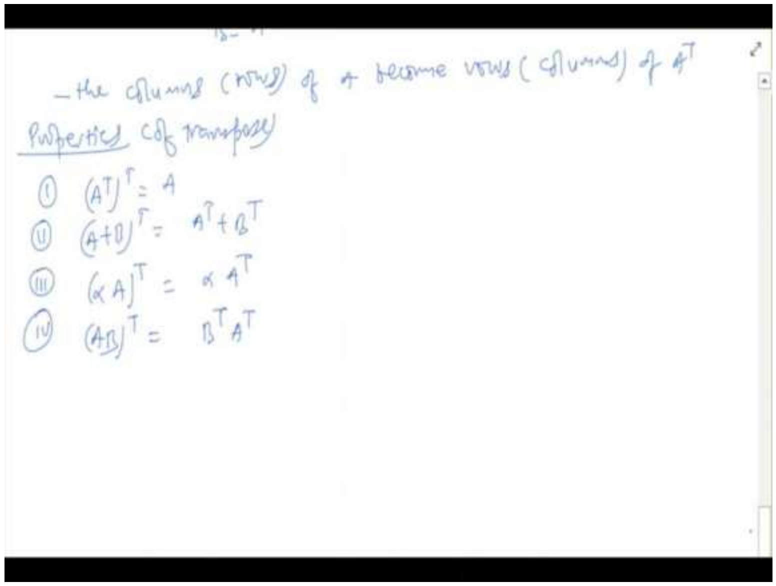

Let here are a few properties of transpose. First property is transpose of transpose is the original matrix itself. Second one says that if I have two matrices A and B, if I add them and take transpose; that is like first taking transpose and then adding. Third property says that if α is a scalar and A is a matrix, then $(\alpha A)^T = \alpha A^T$. Fourth property says that if I have two matrices A and B of suitable dimensions, so that I could multiply those; then transpose of product is product of transposes, but the product is done in reverse order.

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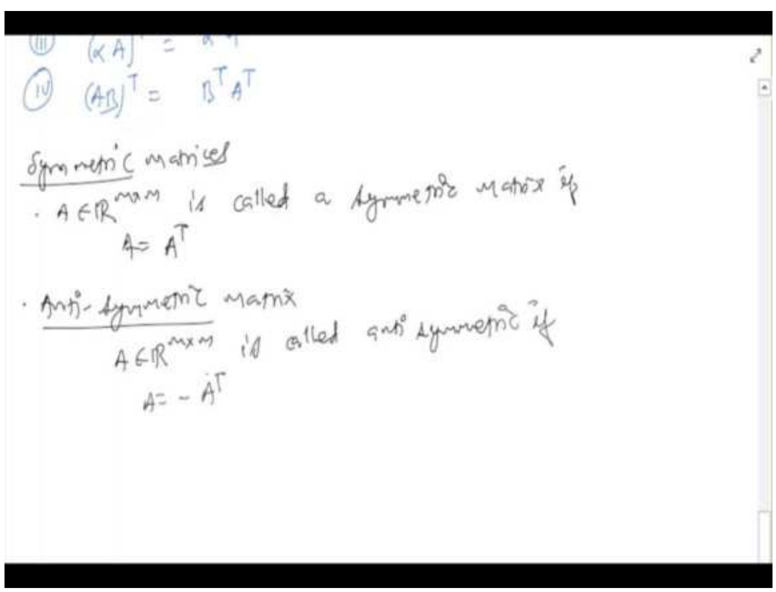

So, let us define another notion and that is of symmetric matrices. So, a matrix A is called symmetric matrix if it is same as its transpose. A and clearly a symmetric matrix has to be a square matrix is called symmetric matrix, if A = A^T. Let us see what are anti-symmetric matrices. So, again A which is an m × m matrix is called anti-symmetric, if A = -A^T; is called anti-symmetric if A = -A^T.

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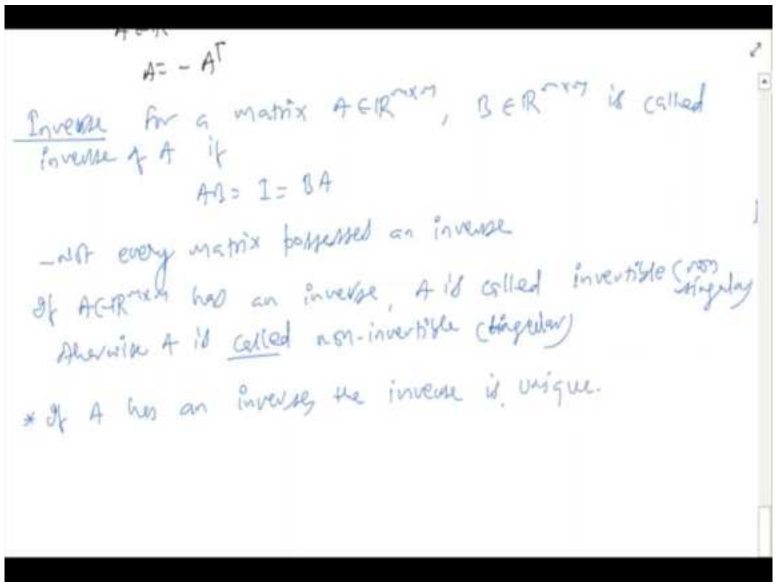

Finally, let us see another very important notion in the context of matrices and that is of inverse of a matrix, inverse of a matrix. So, for a matrix again inverse are defined only for square matrices. So for a matrix A that is an m × m matrix, the matrix B is called inverse, if

$$AB = I = BA$$

B which is again an m × n matrix is called inverse of A if, AB = I. And in this case it is also equal to product of B.

So, if AB = BA = I; I would say that B is inverse of matrix A. Not every matrix possesses an inverse; if a matrix has inverse it is called invertible matrix. Sometimes also called non-singular matrix; otherwise it is called non-invertible or singular matrix; not every matrix possesses an inverse.

If A has an inverse, A is called invertible or as I said non singular; otherwise A is called non-invertible or singular, non-invertible or singular. If a matrix A has inverse, the inverse is unique;

if A has an inverse, the inverse is unique; that is a matrix cannot have two different inverses. So, these are the few facts about inverse of matrices. Until now we have focused on real valued matrices; almost all the concepts notions that we have seen so far, can also be extended to so called complex valued matrices. As the name suggests complex valued matrices are two dimensional arrays of complex numbers.

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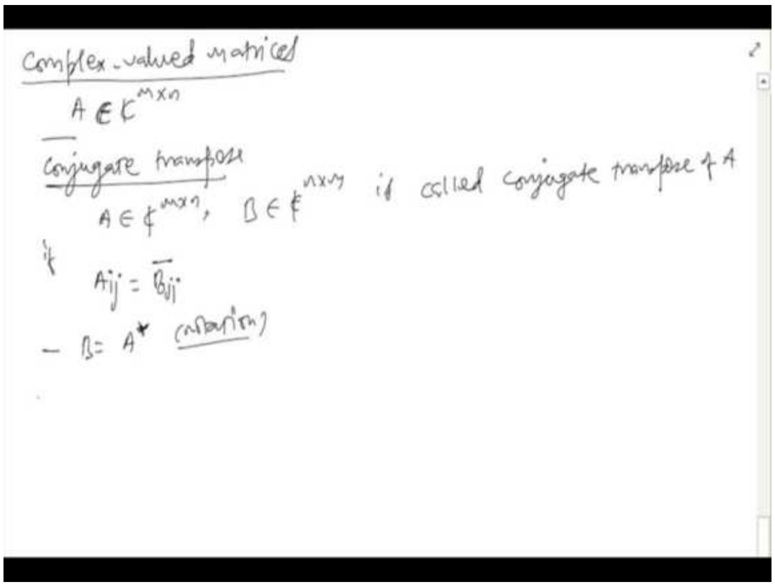

Let us see these also briefly complex-valued matrices. So, C is complex number; we use C to denote complex number. So, complex-valued matrices are as I said are m × n array, but now of complex numbers. Analogous to the notion of transpose in case of complex valued matrices, we have notion of conjugates transpose.

So, given a matrix A, I call another matrix B. So, A is m × n complex valued matrix; I call another matrix B, which is n × m conjugate transpose of A, which called conjugate transpose of A, if

$$A_{ij} = \overline{B_{ij}}$$

Conjugate transpose of A is denoted as A*; so, this is the notation that we use for conjugate transpose. Analogous to the notion of symmetric matrices in the context of complex valued matrices; we have the concept of Hermitian matrices; let us see what these are.

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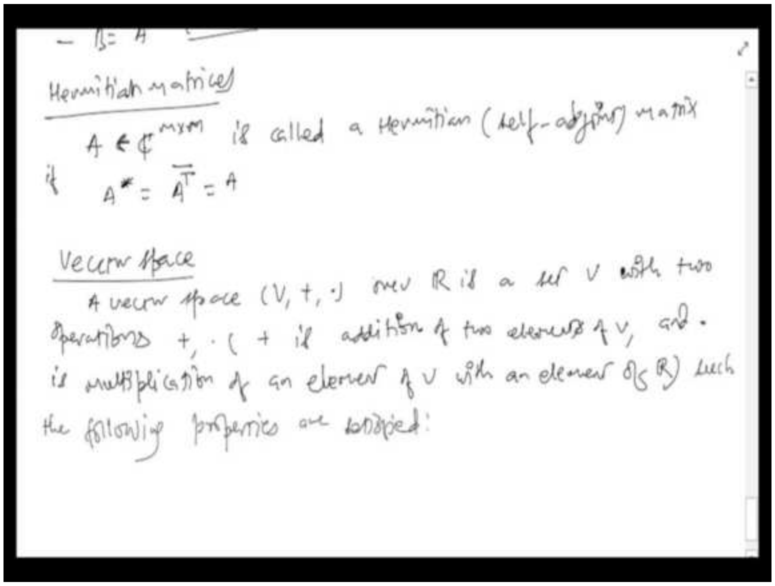

Hermitian matrices, so in A square value is a square complex valued matrix, is called Hermitian, Hermitian matrix or often called self-adjoint matrix. If

$$A^* = \overline{A^T} = A$$

Next we will introduce the notion of vector spaces; we define a vector space over the set of real numbers, vector space.

A vector space V over the set of real numbers are is a set V with two operations. We see those operations are '+' and '.' ; here '+' is addition of two elements of V, and '.' is multiplication of an element of V with an element of R. So, these are two operations such that the following properties are satisfied. So, let us see what are the properties that are needed for vector spaces.

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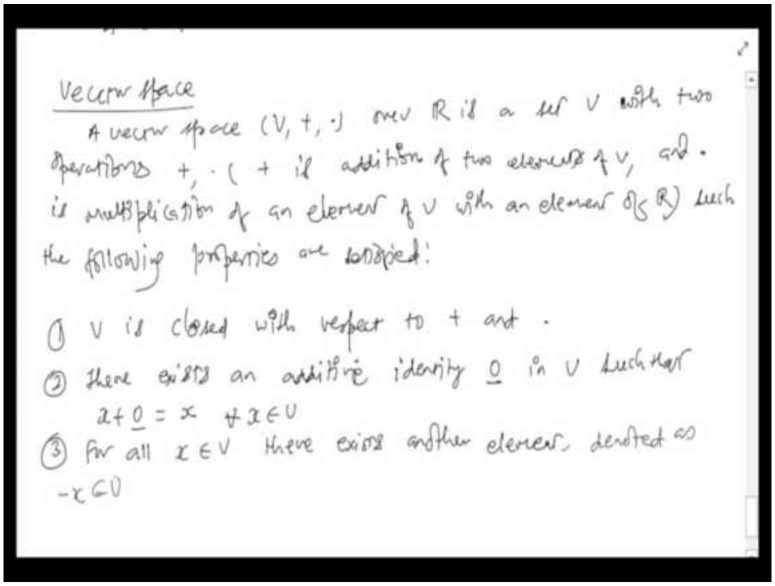

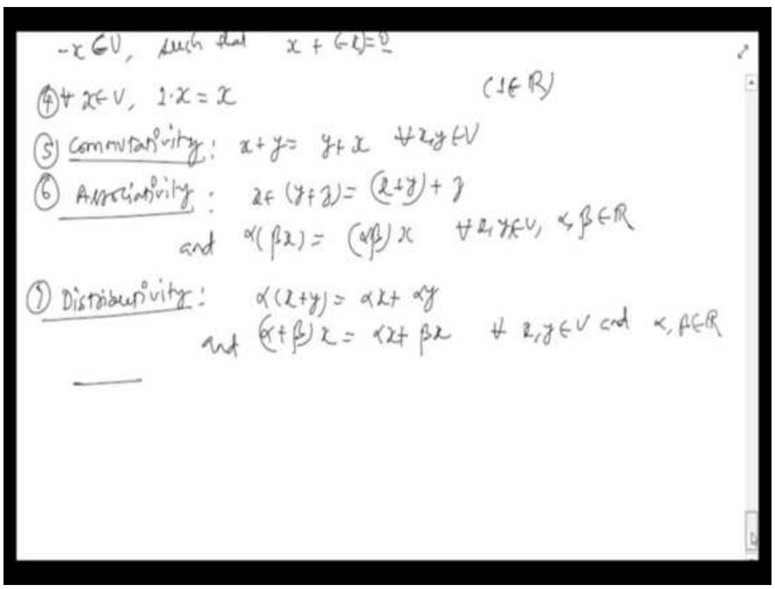

First property is V is closed with respect to these two operations, namely multiplication and addition; closed with respect to '+' and '.'. And what does this mean? This means that if I take two elements from the set V and I perform addition; I get another element of B. Similarly, if I take an element of V and an element of R and multiply the two, again I get an element of V. The second property is that there exists an additive identity, 0 in V. There exists an additive identity is 0 bar in V, such that, if I add this two any element of V, I will get back the same element.

Third property; for each element, so this holds for all x in V. In third property is for all x in V; there exists another element of V, such that if I add these two elements, I will get the 0 additive identity. So, for all x in V, there exists another element denoted as say -x in V, such that

$$x + (-x) = 0$$

Fourth property is, if I multiply any vector in V with 1, I will get the same vector. So, for all x in V, $1.x$, notice that 1 is the multiplicative identity in R; so, $1.x = x$. Fifth property is, what we call commutativity. And this says that

$$x + y = y + x \quad \forall x, y \in V$$

Sixth property is associativity. And this says that,

$$x + (y + z) = (x + y) + z$$

And

$$\alpha(\beta x) = (\alpha\beta)x$$

and this holds $\forall x, y \in V$, $\alpha, \beta \in R$. Seventh property is distributivity. This says that

$$\alpha(x + y) = \alpha x + \alpha y$$

and

$$(\alpha + \beta)x = \alpha x + \beta x \quad \forall x, y \in V \; and \; \alpha, \beta \in R$$

If there is a set V along with two operations, addition and multiplication; such that all these seven properties is satisfied; the state V along with those two operations is called a vector space.

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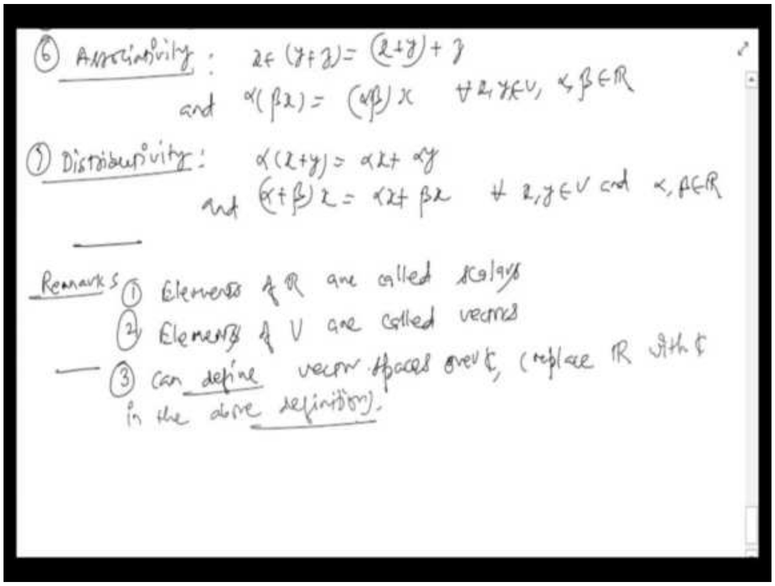

In the context of vector space, the elements of R are called scalars. So, here is a remark, rather I put a couple of remarks. Elements of R are called scalars; and elements of V are called vectors. Just the way we define vector spaces over R, we could similarly define vector spaces over C; that a set of complex numbers. And define vector spaces over C; all we have to do is we have to replace R with C that in above definition; replace R with C in the above definition. But, we will mostly focus on vector spaces over real numbers. Let us see a few examples of vector spaces.

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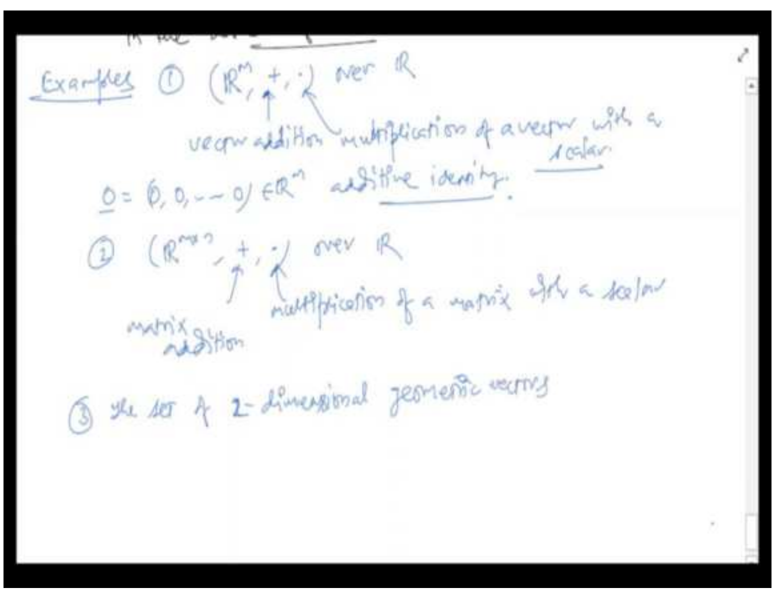

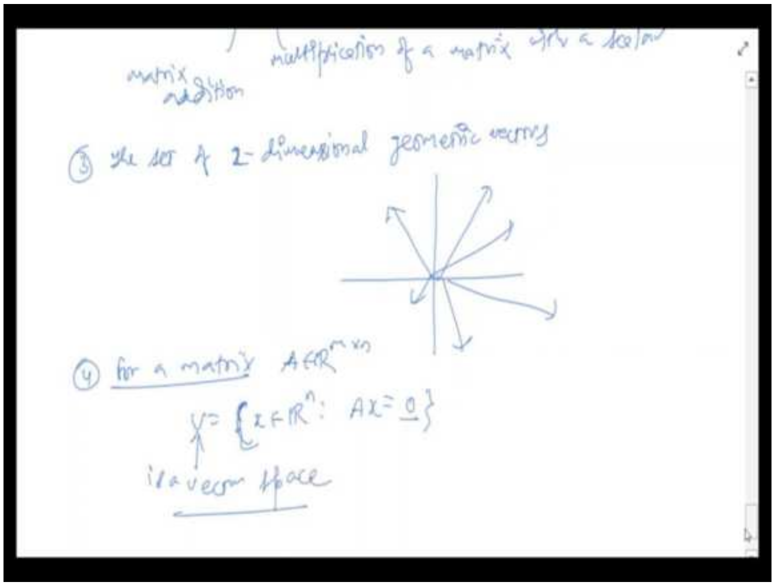

The first example is R^m that is set of vectors of length m, set of real valued vectors of length m, along with these two operations. This is a vector space over R; recall that this is vector addition; and this is multiplication of a vector with a scalar. As we have seen, a while ago, when I introduced vectors and matrices, vector with a scalar; so, this is vector space. What is the additive identity here? It is 0, 0 vector of length m; this is additive identity. Let us see few more examples of vector spaces. The second interesting example this set of all $m \times n$ matrices. Here, this '+' is matrix addition and this '.' is multiplication of a matrix with a scalar; let us see a few more examples.

For instance, it is a familiar example. The set of two dimensional geometric vectors, by this I mean this plane, this two dimensional plane and all the vectors here, these are all vectors here; it is a collection of all such vectors; this is a vector space. Fourth example is let us say a fix a matrix A, say $m \times n$ matrix real valued. Then take all vectors of length n such that $Ax = 0$ vector.

So, this collection V is a vector space. It can be easily shown that this satisfies all the properties all the seven properties that we needed above; namely in particular this V is closed with respect to multiplication with a scalar and vector addition. Having seen these examples, let us make a few observations about the vector spaces; this could also be seen as property of vector spaces.

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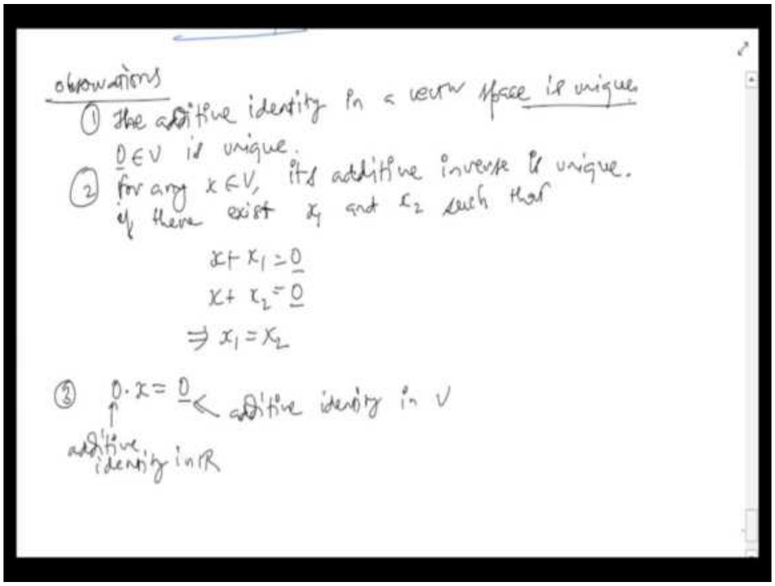

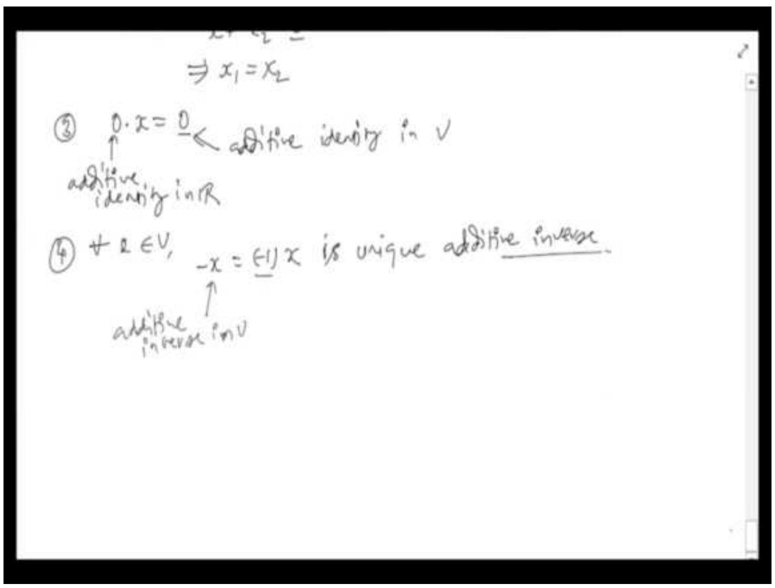

Observations; first observation is, the additive identity is unique, identity in a vector space is unique. Informally, I mean that there can be only one 0 vector in the vector in any vector space; that is 0 that is element of V is unique. The second property is that for any element of the vector space, its additive inverse is unique. So, $\forall x \in V$, its additive inverse is unique.

In other words, if there exist x_1 and x_2, such that $x + x_1 = 0$ vector; and $x + x_2 = 0$ vector. In that case $x_1 = x_2$. What is third property? Third observation about vector space is, it says that $0.x = 0$ vector. This 0 is additive identity in R; that is 0 in 0 real number, additive identity in R; whereas, this 0 is additive identity.

The fourth property says that for all x∈V; -x = (-1)x. -1 is the real number -1; this is the unique additive inverse. Again, this is additive inverse in V, and -1 we know it is a real number -1. So, these were a few properties of vector spaces.

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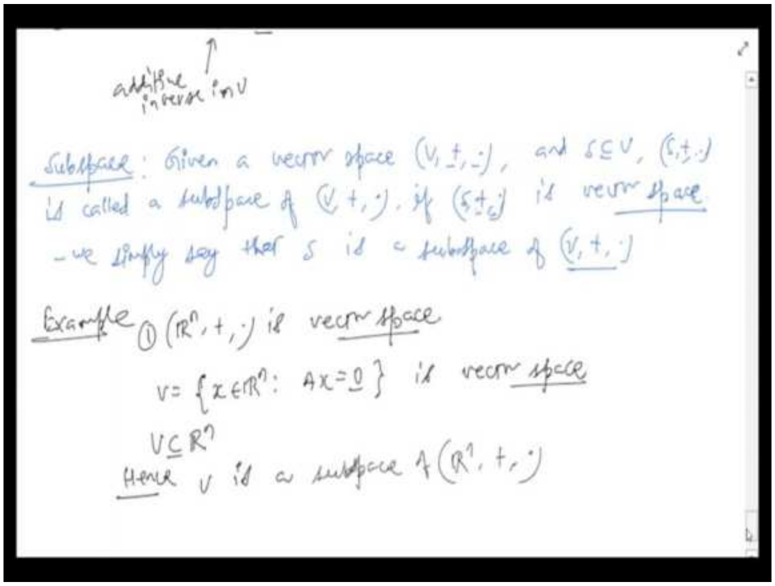

Next, we will see another important notion and that is of subspace. What is a subspace? Given a vector space say (V,+,.) and S ⊆ V. (S,+,.) is called the subspace of (V,+,.), if this itself is a vector space. So, (S,+,.) is called a subspace of (V,+,.); if (S,+,.) is a vector space. That it satisfies all the seven properties that were mentioned above is a vector space.

Here, notice that the addition and multiplication here are same as addition and multiplication in the definition of vector space (V,+,.). So, we often simply say that S is a vector space of (V,+,.); we simply say that S is a subspace. Because we know that the addition and multiplication here would be same as those in the context of the original vector space.

Let us see an example. So, we have already seen that (R^n,+,.) is a vector space. We also saw that V, which is defined as a collection of x ∈ R^n, such that Ax = 0 vector, this is a vector space. Clearly V ⊆ R^n; hence V is a subspace of (Rn, +, .). So, how do I, we identify whether a subset of a vector space is a subspace or not?

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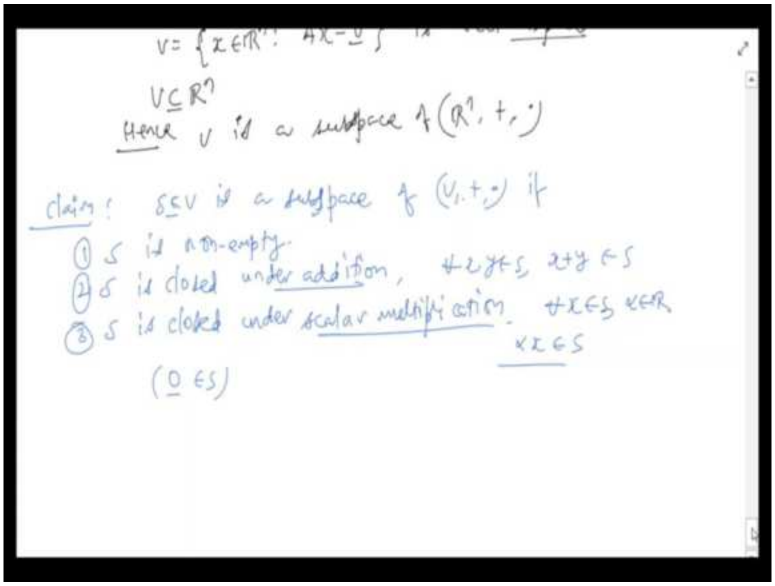

So, here is a claim, an observation. $S \subseteq V$ is a subspace of (V, +, .), if the following three properties are satisfied. First is S has to be non-empty obviously; S cannot be an empty set. The second property is S is closed under addition; that is $\forall x, y \in S$, $x+y \in S$. And the third property is S is closed under scalar multiplication.

So, what we mean is that $\forall x \in S$ and $\alpha \in R$; $\alpha x \in S$. So, we so what I mean that if $S \subseteq V$, where (V, +, .) is a vector space; then S will also be a vector space, in particular, it will be a subspace of V, if these three conditions are satisfied. Third condition in particular implies that $\underline{0} \in S$. If S does not contain the additive identity of V, then S cannot be a subspace of V. Let us see a few more examples.

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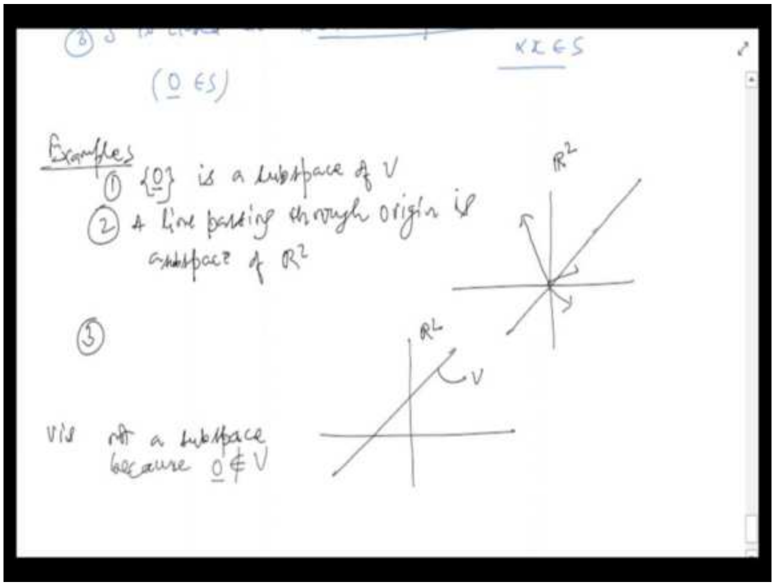

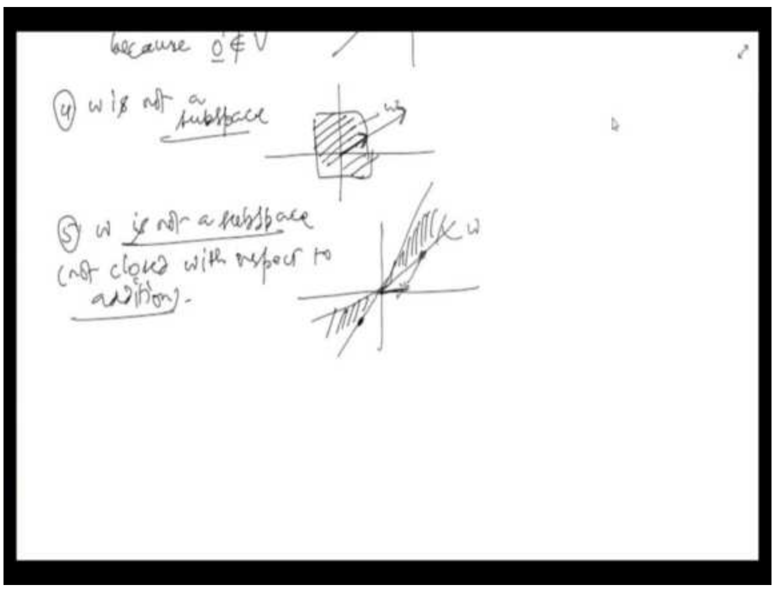

One trivial example is a set containing just the additive identity of V, this set of the singleton itself is a subspace of V. It satisfies all the three conditions that we wrote above. Another example is now consider the set of geometric two dimensional geometric vectors; set of all the vectors. Here if I restrict to the line passing through origin; let us denote the original one as R^2.

Then if I take a line passing through origin, this line is a subspace of R; a line passing through origin is a subspace of R^2. Why it is a subspace? See it is non empty, if I take any two vectors on this line and I add those, I will get another vector on the line. If I take a vector on this line, and I multiply with a scalar; I will still get a vector on the line. So, it meets all the three requirements that I mentioned above; so this line is a subspace.

Now, we will see another example. Say, this line, this does not pass through origin; so, this does not contain the additive identity of R^2. So, this is not a subspace, let us call this line V; then V is not a subspace, because it does not contain the additive identity of R^2. Similarly, consider this set this subset of R^2, let me call this w. w is also not a subspace of R^2, because if I take a vector; let us say I take this vector and then multiply it with 2; then I will get this vector. So, this scalar multiplication of this original vector is not inside w. So, w is not a subspace; let us think of this shape.

Let me call this W, then this W is also not a subspace. What is the reason? The reason is if say I take two vectors, say this and this; and I add the each. Then their summation which will be this vector; this is not in w. So, it is not closed with respect to vector addition not closed with respect to addition. Now, let me define the notion of linear independence of vectors.

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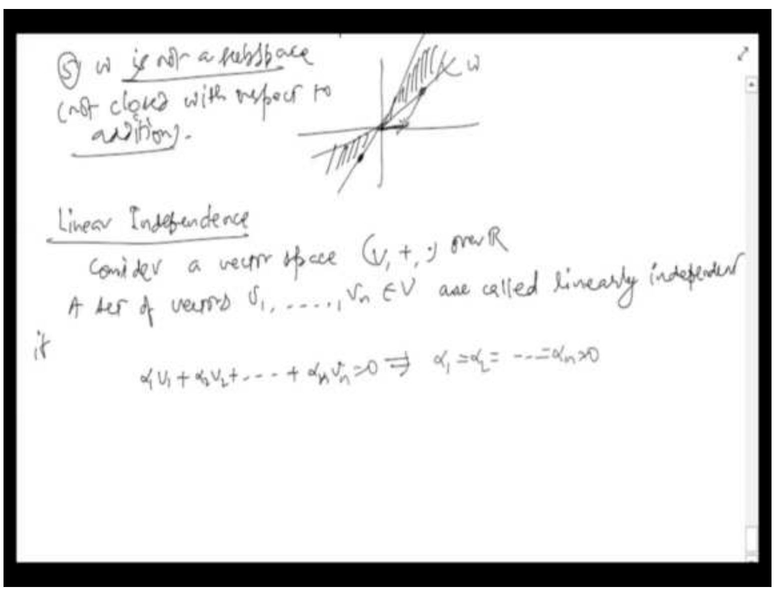

Linear independence; let us consider a vector space (V, +, .) over R; then a set of vectors v_1 …… v_n, ∈ V are called linearly independent, if $\alpha_1 v_1 + \alpha_2 v_2 \dots \dots \dots + \alpha_n v_n = 0$; only if all the alphas are 0. So, this implies that all the alphas are 0. If this happens, we say that the set of vectors v_1 to v_n are linearly independent.

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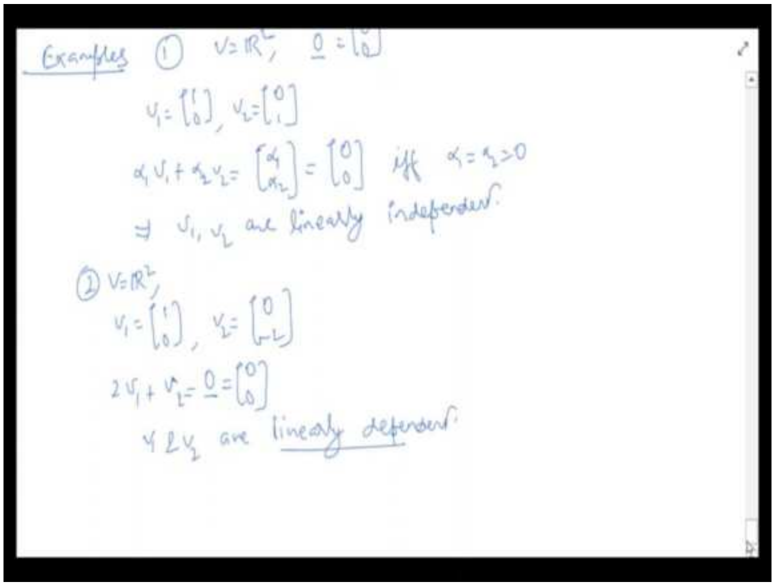

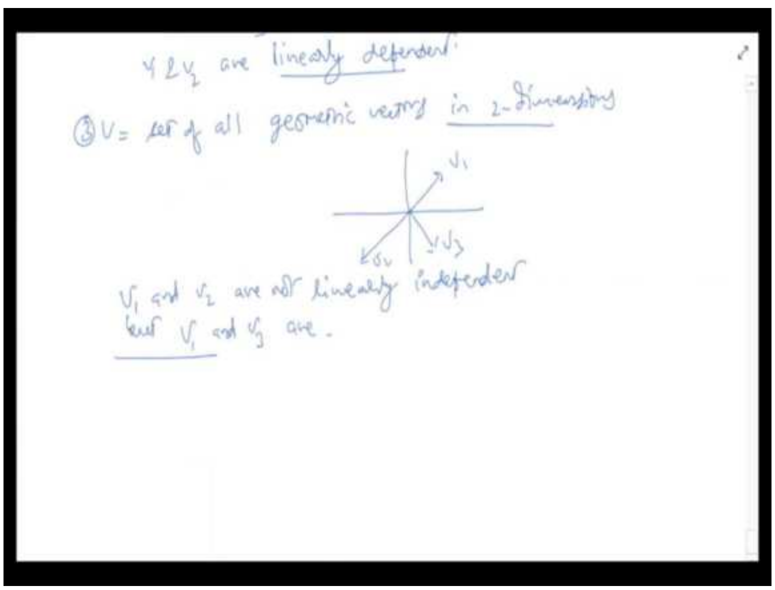

Let us see few examples. Consider V=R^2; in this space the additive identity is 0, 0 vector. Now, consider two vectors $v_1 = \begin{bmatrix}1\\0\end{bmatrix}, v_2 = \begin{bmatrix}0\\1\end{bmatrix}$; then for any scalars $\alpha_1 and\ \alpha_2$,

$$\alpha_1 v_1 + \alpha_2 v_2 = \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad if \alpha_1 = \alpha_2 = 0$$

So, v$_1$, v$_2$ are linearly independent. Let us consider another example. Again, my V=R^2 and now, I am taking $v_1 = \begin{bmatrix}1\\0\end{bmatrix}, v_2 = \begin{bmatrix}0\\-2\end{bmatrix}$. It can be easily checked that

$$2v_1 + v_2 = \underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This means that v_1 and v_2 are not linearly independent; in fact, they are linearly independent. Let us take one more example. And now I take V to be set of all geometric vectors in two dimensions; so, this is my V. In this case, again I take let us say three vectors v_1, v_2, and let say this is v_3. v_1 and v_2 are not linearly independent, independent; but v_1 and v_3 are. Next, I will define the notion of span of vectors.

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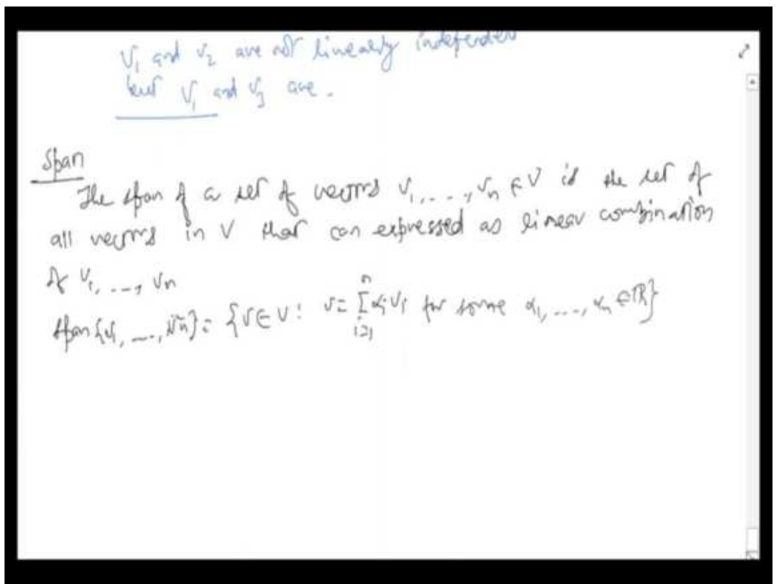

The span of a set of vectors v_1 …… v_n in V, is the set of all linear combinations of these vectors. In other words, this is the set of all vectors in V that can be expressed as linear combination of v_1 …… v_n. In other words, span of v_1 …… v_n is collection of all the $v_i \in V$, such that this V can be written as

$$V = \sum_{i=1}^{n} \alpha_i v_i \qquad for\ \alpha_1, \dots \dots, \alpha_n \in R$$

Let us see examples.

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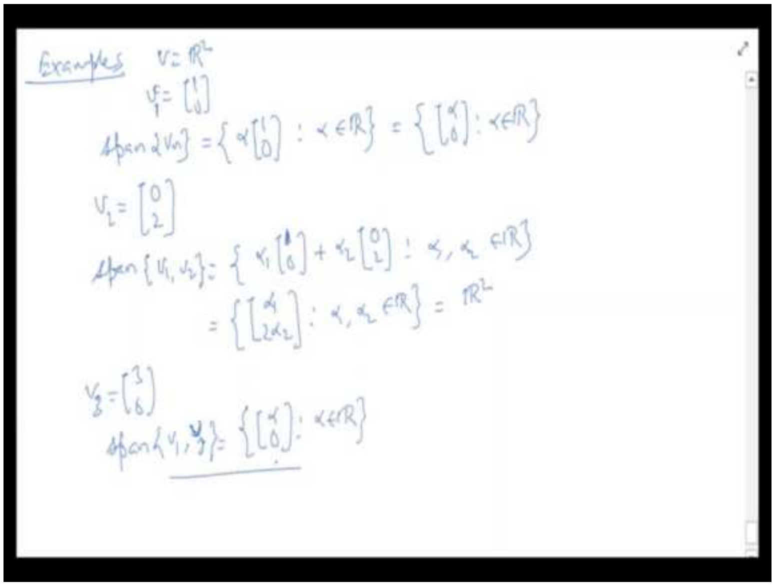

Let us take V=$\mathbb{R}^2$; and I take just one vector that is $\begin{bmatrix}1\\0\end{bmatrix}$, we call it $v_1 = \begin{bmatrix}1\\0\end{bmatrix}$. What is span of v_1?

$$Span\ of\ \{v_n\} = \left\{\alpha\begin{bmatrix}1\\0\end{bmatrix} :\ \alpha \in \mathbb{R}\right\} = \left\{\begin{bmatrix}\alpha\\0\end{bmatrix} :\ \alpha \in \mathbb{R}\right\}$$

Let me take another vector $v_2 = \begin{bmatrix}0\\2\end{bmatrix}$. Now, what is span of v_1, v_2?

$$Span\ of\ \{v_1, v_2\} = \left\{\alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\2\end{bmatrix} :\ \alpha_1, \alpha_2 \in \mathbb{R}\right\} = \left\{\begin{bmatrix}\alpha_1\\2\alpha_2\end{bmatrix} :\ \alpha_1, \alpha_2 \in \mathbb{R}\right\} = \mathbb{R}^2$$

It can be seen that this is whole $\mathbb{R}^2$; so, span of $v_1\ and\ v_2$ which whole of $\mathbb{R}^2$. By the way, if I define $v_3 = \begin{bmatrix}3\\0\end{bmatrix}$; this can easily be checked that span of v_1, v_3 is same as span of v_1. v_1, v_3 is same as span of v_1, that is still it is still

$$Span\ of\ \{v_1, v_3\} = \left\{\begin{bmatrix}\alpha\\0\end{bmatrix} :\ \alpha \in \mathbb{R}\right\}$$

We end this lecture with a discussion on the notion of bases.

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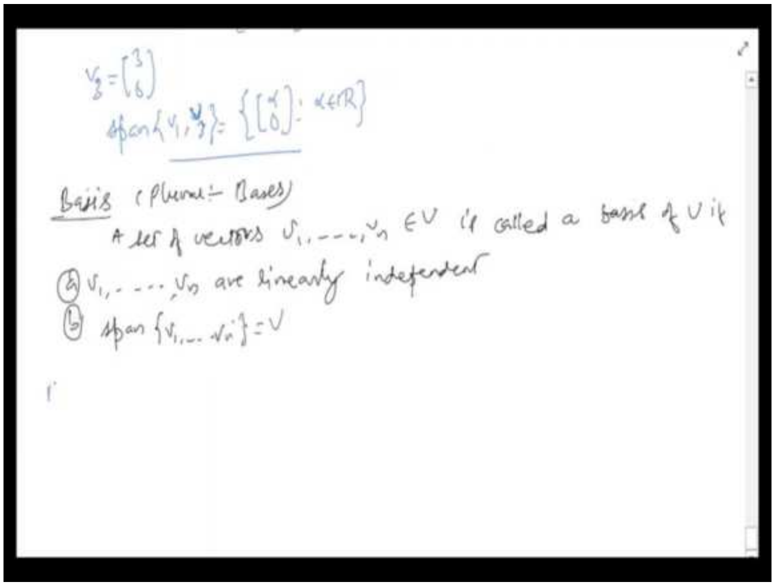

Now, we look at basis; plural is bases, plural is bases. A set of vectors v_1 …… v_n in a vector space V, is called a basis of V, if the following two conditions are satisfied, one is these vectors are linearly independent; and second is, they span the entire space. So,

$$Span\ of\ \{v_1 \dots \dots \dots \dots v_n\} = V$$

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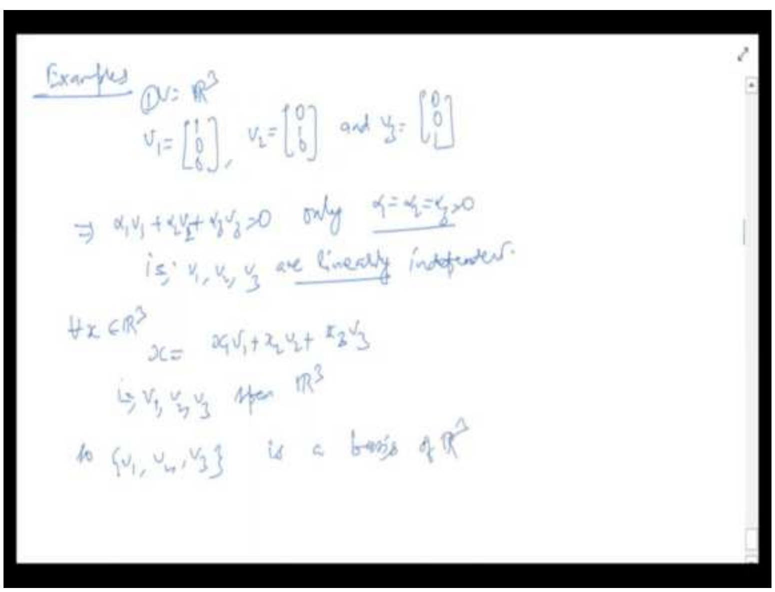

Let us see few examples. Consider V=$\mathbb{R}^3$; then if I consider vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; and \; v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then we can see that,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \quad only\ if\ \alpha_1 = \alpha_2 = \alpha_3 = 0$$

This means that v_1, v_2, v_3 are linearly independent. Moreover, $\forall x \in \mathbb{R}^3$, I can write x as

$$x = x_1 v_1 + x_2 v_2 + x_3 v_3$$

with x_1, x_2, x_3 are three numbers; these three components of x. That means $\forall x \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, v_3. In other words, v_1, v_2, v_3 span the whole of $\mathbb{R}^3$. So, $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. We will now see a few properties of basis.

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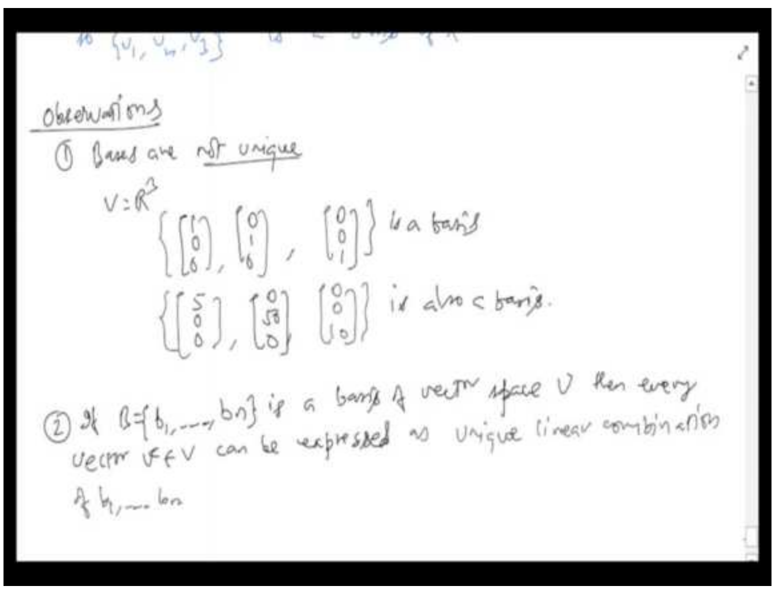

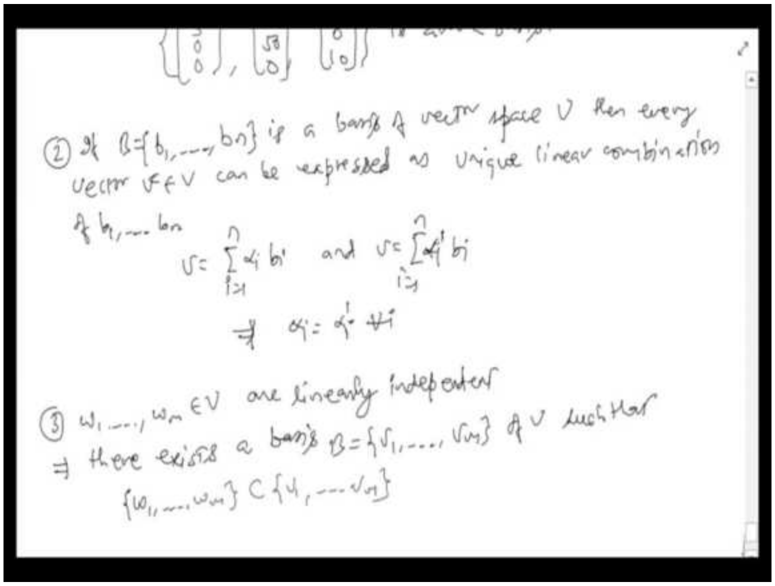

So, observations about basis. First property is bases are not unique. For example, consider V=$\mathbb{R}^3$; we saw a basis just above that was

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

This is a basis; but if I write say

$$\left\{\begin{bmatrix}5\\0\\0\end{bmatrix},\begin{bmatrix}0\\50\\0\end{bmatrix},\begin{bmatrix}0\\0\\10\end{bmatrix}\right\}$$

This is also a basis; so, bases are not unique. Second property is that if $B = \{b_1 \dots \dots \dots \dots b_n\}$ is a basis of the vector space V; then every vector in V can be written as unique linear combination of $b_1 \dots \dots \dots \dots b_n$. Every vector v ∈ V can be expressed as unique linear combination, combination of $b_1 \dots \dots \dots \dots b_n$. What I mean is that if say

$$V = \sum_{i=1}^{n} \alpha_i b_i \quad and \;\; V = \sum_{i=1}^{n} \alpha_i{}' b_i$$

This can happen only if $\alpha_i = \alpha_i{}' \; \forall i$.

So, this was second property. The third observation is that, if $w_1 \dots \dots \dots \dots w_m$; say these are m vectors belonging to V are linearly independent. Then there exists a basis of V that contains these vectors. That means, if $w_1 \dots \dots \dots \dots w_m$ are linearly independent, then there exists a basis B, $B =$

$\{v_1 \dots \dots \dots \dots v_m\}$ of V, such that the collection $\{w_1 \dots \dots \dots \dots w_m\} \subset \{v_1 \dots \dots \dots \dots v_m\}$. Let see what I mean by an example.

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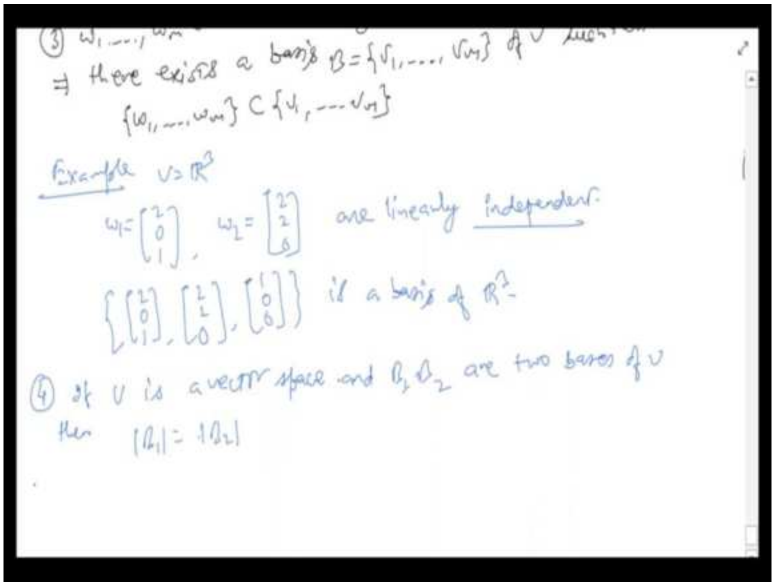

Consider V=$\mathbb{R}^3$. Notice that

$$w_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

These are linearly independent, then there exists a basis that contains these two; so

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$.

Let us look at the fourth property that says that if V is a vector space and v_1, v_2 are two of its basis; then the number of vectors in v_1 will be same as number of vectors in v_2. So, if V is a vector space and B_1, B_2 are two base bases of V, then number of vectors in v_1 often called cardinality of v_1 is same as cardinality of v_2; that is number of vectors in v_2. Let us see one more property.

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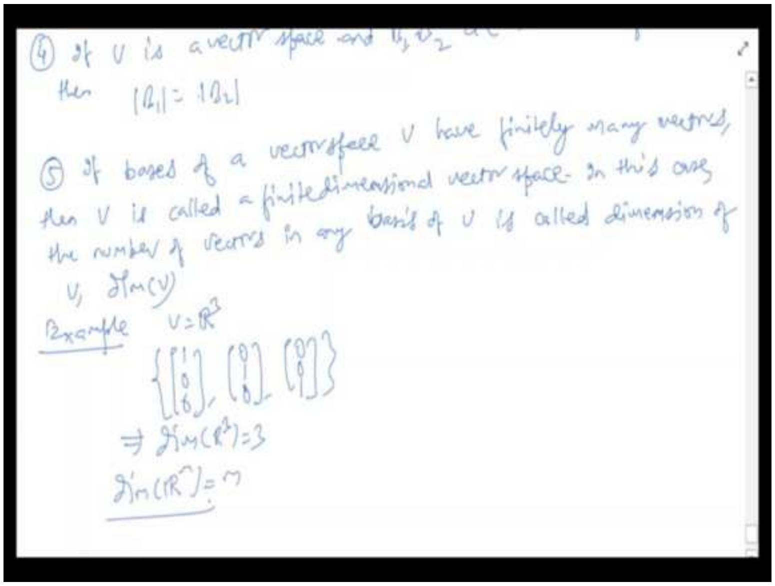

This is more like a definition. It says that if bases of a vector space V have finitely many vectors. That is if the cardinality of basis is finite, that V is called a finite dimensional vector space. If basis of a vector space V have finitely many vectors finitely many vectors; then V is called a finite dimensional vector space, called a finite dimensional vector space. In this case, the number of vectors in any basis is called dimension of V. In this case, the number of vectors in any basis of V is called dimension of V. It is also denoted as dimension v; this is notation. Let us see an example.

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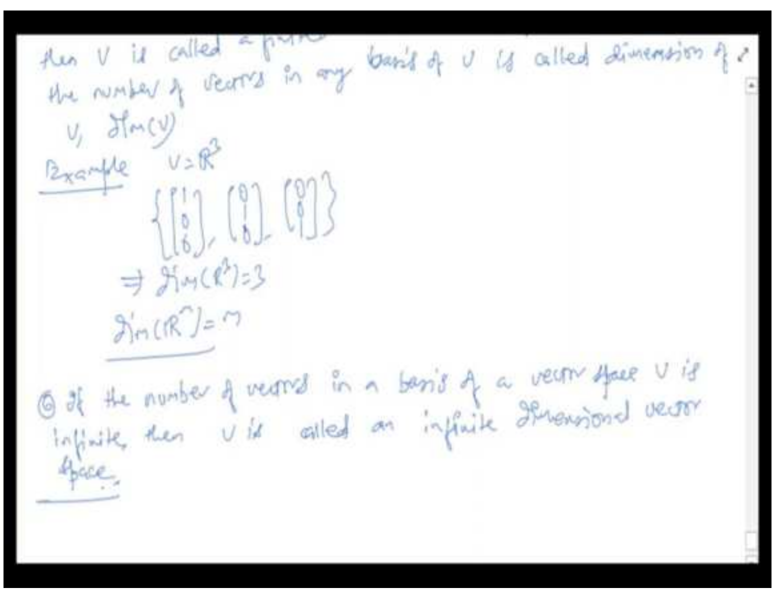

For example, if V=$\mathbb{R}^3$; then we know that one of its basis is

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

So, $\dim(\mathbb{R}^3) = 3$; because this bases has three vectors. Similarly, $\dim(\mathbb{R}^m) = m$. Let us see one final definition. If the number of vectors in a basis of a vector space V infinite; then V is called infinite dimensional vector space. If the number of vectors in a basis of a vector space V is infinite; then V is called and infinite dimensional vector space. So, these were a few properties of bases; and with this we conclude this lecture. Thank you.