Mathematical Aspects of Biomedical Electronic System Design
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Probability: Important measures and generating functions

Hello everyone, welcome to today's lecture of the course Mathematical Aspects of Biomedical Electronic System Design.

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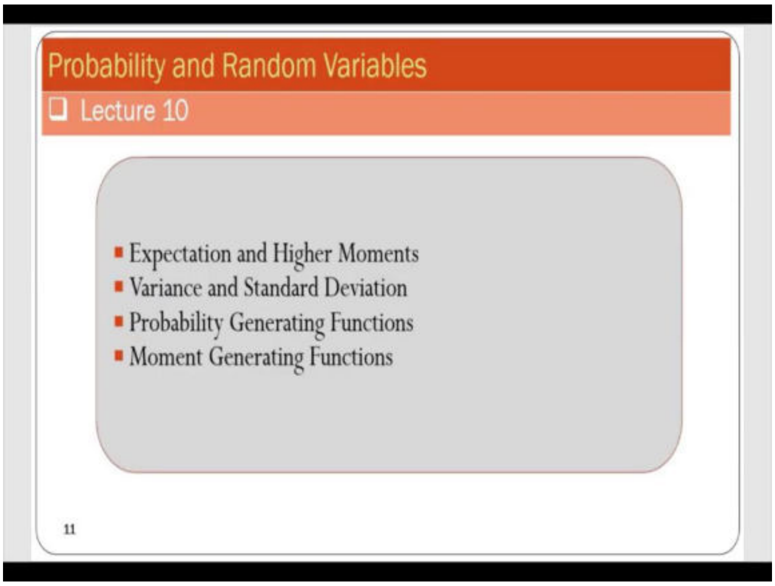

We will continue our discussion on probability and random variables. In today's lecture, we will cover expectation and higher moments, variants and standard deviation. We will also see probability in moment generating functions.

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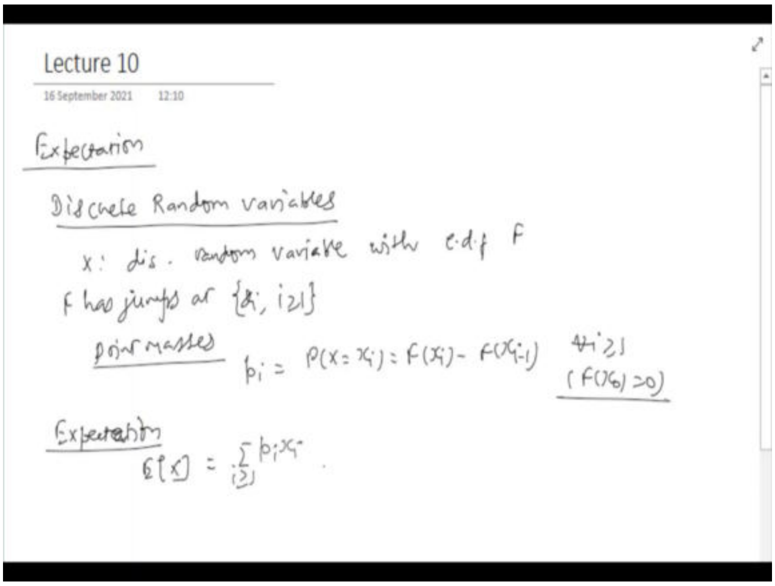

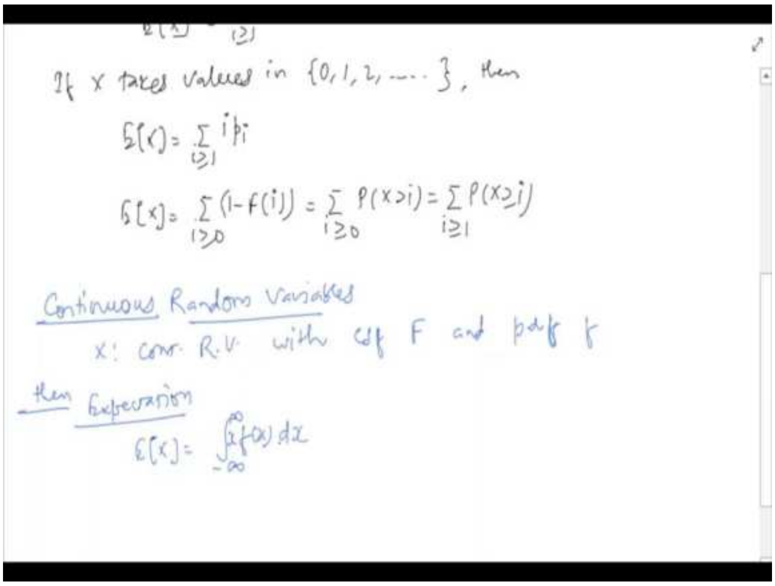

So, let us start today's lecture. We will begin with expectation of random variables. Expectation is a measure of center or average value of a random variable. We will consider the cases of discrete and continuous random variable separately. So, let us begin with discrete random variables.

Suppose X is a discrete random variable with cumulative distribution function F. So, F has jumps and flat portions. Suppose F has jumps at $\{x_i, i \geq 1\}$. This means that the random variable X has point masses p_i at this point x_i, p_i has $P(X = x_i)$. So, point masses p_i which are $P(X = x_i) = F(x_i) - F(x_{i-1})$ $\forall$ $i \geq 1$.

In fact for this definition, we can assume that $F(x_0) = 0$, then expectation is given by the following formula. Expectation denoted as

$$E[X] = \sum_{i \geq 1} p_i x_i.$$

If X takes values in the set of non-negative integers, then we have an alternating formula of expectation we will see that below. So, if X {0, 1, 2, ……}, then as per the previous formula expectation would be given by

$$E[X] = \sum_{i \geq 1} i p_i.$$

As I said we also have an alternative formula that says that expectation will be

$$E[X] = \sum_{i \geq 0} \big(1 - F(i)\big) = \sum_{i \geq 0} P(X > i) = \sum_{i \geq 1} P(X \geq i).$$

So, we have this alternate formula for expectation.

We will see few examples very soon but for now let us turn to continuous random variables. Suppose X is a continuous random variable with cumulative distribution function cdf F and probability density function small f, then expectation x denoted as

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx.$$

If X is a non-negative random variable, then as in discrete case we also have an alternate expression of expectation, which is given as follows. So, if X is a non-negative random variable, then expected value of X is also given by

$$E[X] = \int_{0}^{\infty} (1 - f(x))dx.$$

Let us now see a few examples.

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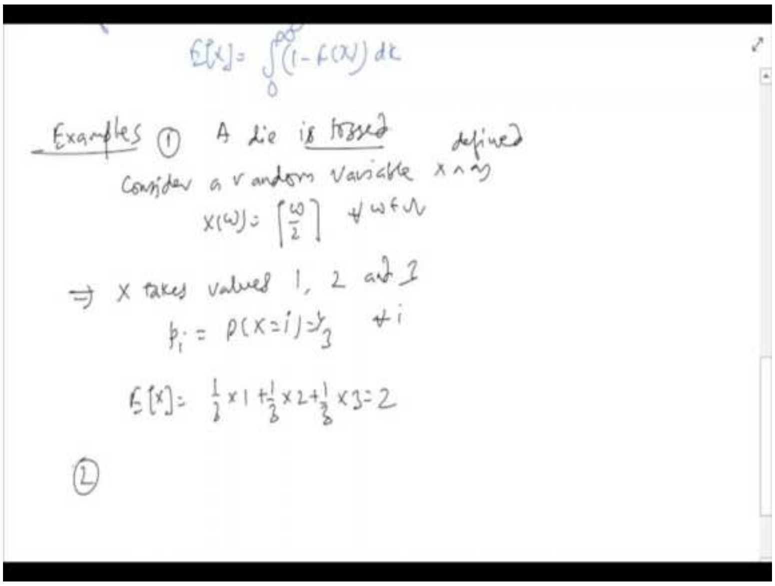

Let us consider a simple example where a die is tossed and we define a random variable X as defined as $X(\omega) = \frac{\omega}{2}$. We have seen this example quite a few times earlier also. Then clearly X takes values 1, 2, and 3 moreover $pi = P(X = i) = \frac{1}{3}$ $\forall i = \{1,2,3\}$. So,

$$E[X] = \frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times 3 = 2$$

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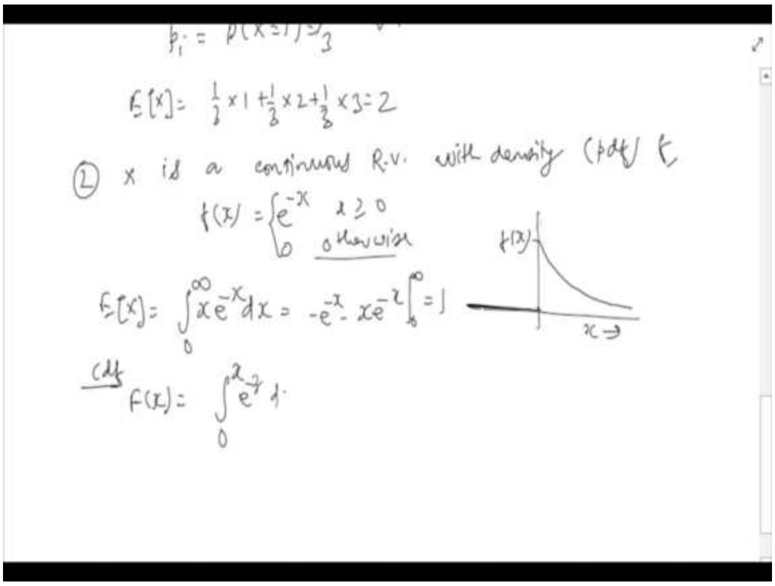

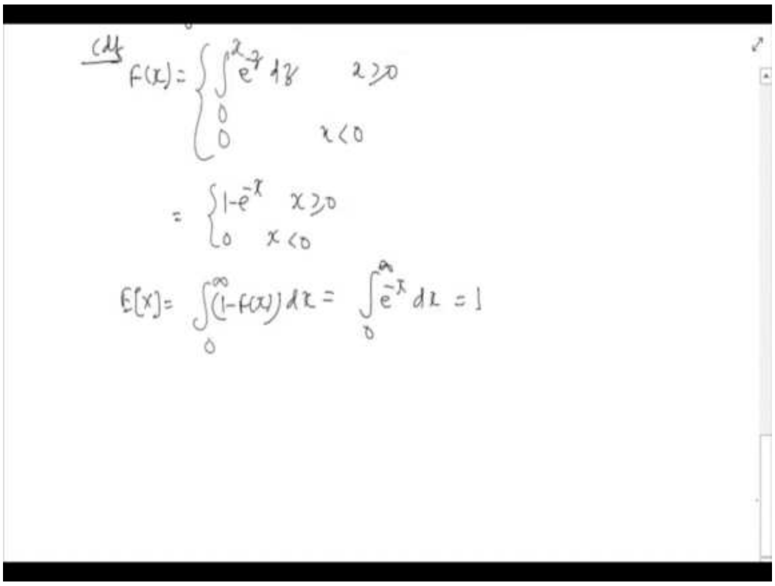

Let us consider another example. Now assume that X is a continuous random variable with density that is probability density function. f given as

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}.$$

So, this density looks as follows. Then, expectation of X can be computed using the formula suggested that is

$$E[X] = \int_0^\infty xe^{-x}dx = -e^{-x} - xe^{-x}\int_0^\infty = 1.$$

I will tell that in this case we can compute the cdf of x as follows, F(x) will be so cdf,

$$F(x) = \begin{cases} \int_0^x e^{-z} dz & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

So, this turns out to be

$$= \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Now we could use even the alternate formula given above to compute the expectation that is

$$E[X] = \int_0^\infty \big(1 - f(x)\big) dx = \int_0^\infty e^{-x} dx = 1.$$

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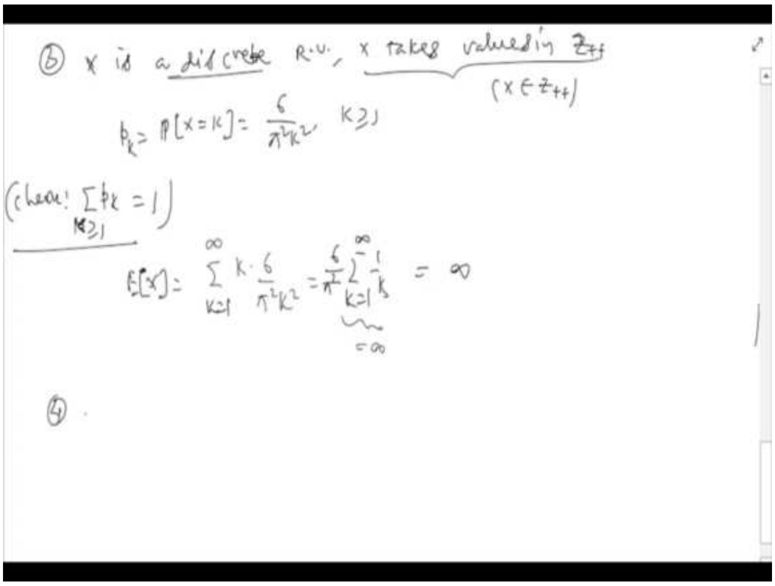

Let us now see another example. Say that X is a discrete random variable, and X takes values in Z_{++}. This fact that X takes values in Z_{++} is often written in short simply as $X \in Z_{++}$. And probability mass function of X is given as follows.

$$p_K = P[x = K] = \frac{6}{\pi^2 K^2}, \forall K \geq 1.$$

It can be checked that $\sum_{K \geq 1} p_K = 1$. So, this p is a valid probability mass function.

Now the

$$E[X] = \sum_{K=1}^{\infty} K \frac{6}{\pi^2 K^2} = \frac{6}{\pi^2} \sum_{K=1}^{\infty} \frac{1}{K} = \infty.$$

So, expectation turns out to be infinity. This tells that expectation of a random variable can be infinity.

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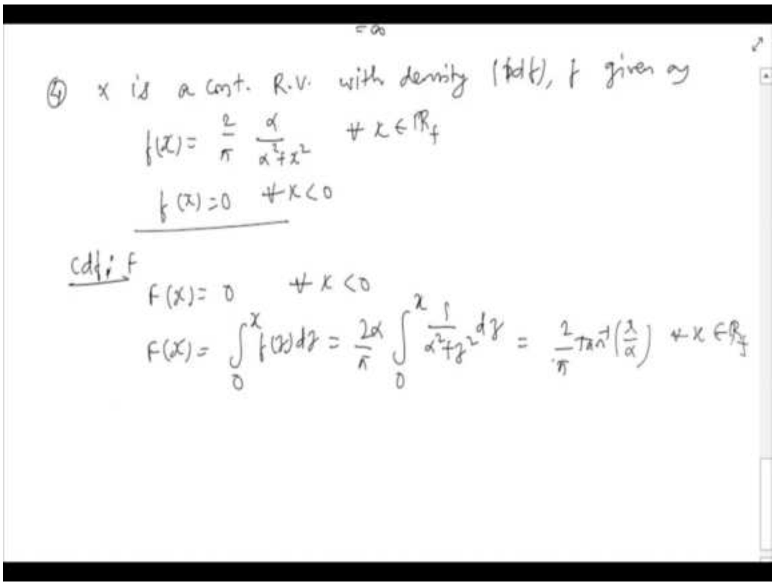

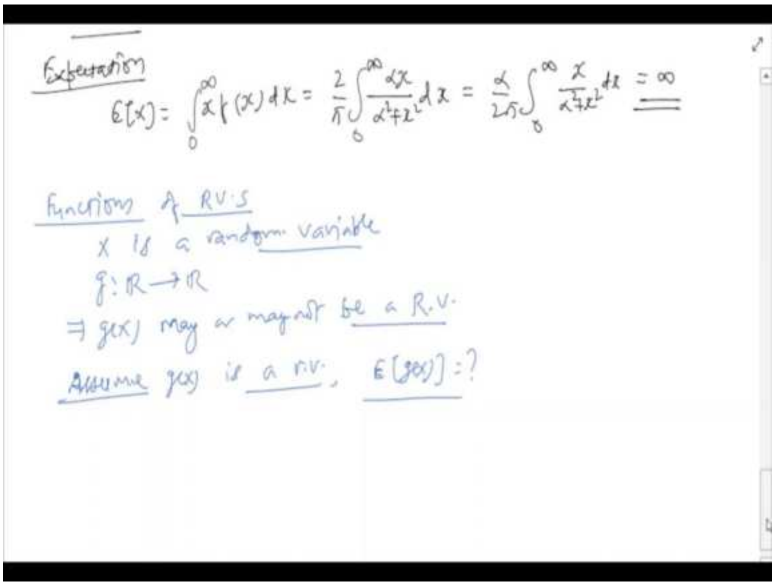

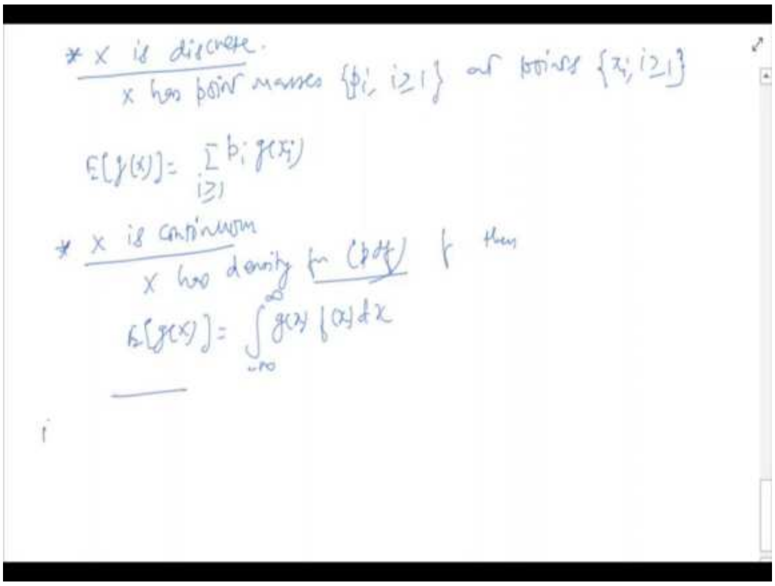

Let us see one more example. This time we will consider a continuous random variable with density that is pdf, f given a

$$f(x) = \frac{2}{\pi}\frac{\alpha}{\alpha^2 + x^2}, \qquad \forall x \in \mathbb{R}_+.$$

f(x) = 0, $\forall x <$ 0 so x is a non-negative random variable. In this case the cdf of x say F is computed as follows.

F(x) = 0, ∀x < 0 because x is a non-negative random variable but for all non-negative values of x,

$$F(x) = \int_0^x f(z)dz = \frac{2\alpha}{\pi}\int_0^x \frac{1}{\alpha^2 + z^2}dz = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{\alpha}\right), \qquad \forall x \in \mathbb{R}_+.$$

Now we can compute the expectation of X using one of the 2 formulae given above.

So, let us use for instance the formula that using – that uses density. So,

$$E[X] = \int_0^\infty xf(x)dx = \frac{2}{\pi}\int_0^\infty \frac{\alpha x}{\alpha^2 + x^2}dx = \frac{\alpha}{2\pi}\int_0^\infty \frac{x}{\alpha^2 + x^2}dx = \infty.$$

So, again we see that expectation of a continuous random variable can be finite or infinite.

There are also random variables which expectations are not defined. However, we will not be interested in those random variables. We will restrict to random variables which expectation is defined. It could be finite or infinite. Having seen the expectation of a random variable, we will now turn to functions of random variables and we will see how their expectations could be computed.

Next we see functions of random variables, suppose X is a random variable and $g: \mathbb{R} \to \mathbb{R}$, then g(X) may or may not be a random variable. However, we are interested in scenarios where g(X) is a random variable and in those scenarios we are interested in expectation of g(X). Assume g(X) is a random variable question is what is expectation of g(X)?

We will again consider the 2 cases of X being discrete and continuous separately, so we will first consider the scenario where X is discrete random variable. Suppose X has point masses $\{p_i, i \geq 1\}$, at points $\{x_i, i \geq 1\}$. In this case, expectation of g(X) can be computed as follows,

$$E[g(X)] = \sum_{i\geq 1} p_i \, g(x_i).$$

On the other hand, if X is a continuous random variable, say X has density function or pdf f, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \, f(x) dx.$$

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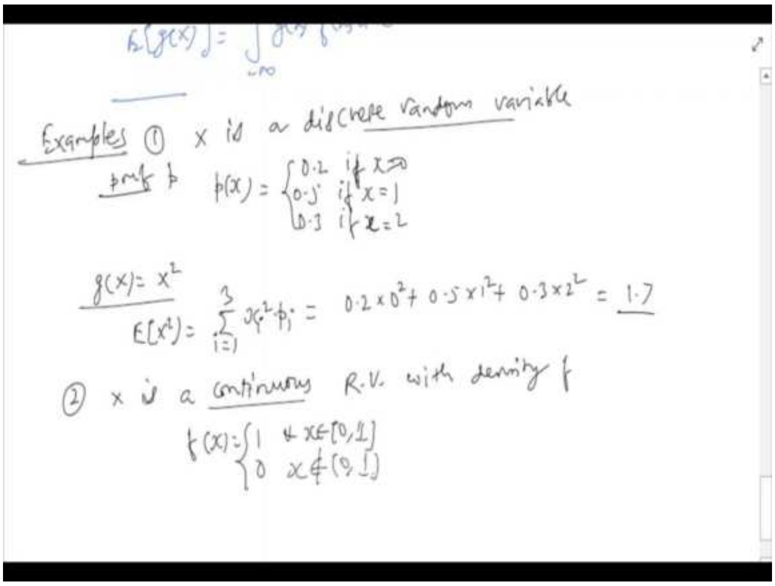
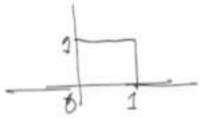


$$E[X^2] = \sum_{i=1}^{3} x_i^2 p_i = 0.2\times 0 + \ldots$$

② X is a continuous R.V. with density f

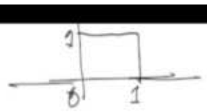
$$f(x) = \begin{cases} 1 & \forall x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$



$g(x) = x^3$

$$E[X^3] = \int_0^1 x^3 \underbrace{f(x)}_{=1}\,dx$$

$$= \int_0^1 x^3\,dx = 1/4$$



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Properties

① $X_1, X_2, \ldots, X_n$ are random variables on $(\Omega, \mathcal{F}, P)$, then

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

(linearity.

② $\alpha, \beta \in \mathbb{R}$ $\quad E[\alpha X + \beta] = \alpha E[X] + \beta$

Example: A die is rolled thrice.

Let X_i be the outcome of the i^{th} roll, $i = 1,2,3$

Also let $X = \sum_{i=1}^{3} X_i$

$$E[X] = \sum_{i=1}^{3} \underbrace{E[X_i]}_{= \sum_{j=1}^{6} j\cdot\frac{1}{6} = \frac{7}{2}} = 3\times\frac{7}{2} = \frac{21}{2}$$

Let us see a few examples. First example is where we assume X to be discrete random variable, so X is a discrete random variable. Suppose the probability mass function of X is as follows. So, p(x) pmf p is as follows,

$$p(x) = \begin{cases} 0.2 \; if \; x = 0 \\ 0.5 \; if \; x = 1. \\ 0.3 \; if \; x = 2 \end{cases}$$

This means that x takes values 0, 1, 2 with probability 0.2, 0.5, 0.3 or x has point masses, 0.5, 0.2, 0.5 and 0.3 at 0, 1 and 2 respectively.

Consider a function of this random variable capital X, $g(X) = X^2$. In this case expected value of X^2 can be computed using the formula given above. So,

$$E[X^2] = \sum_{i=1}^{3} x_i^2 \, p_i = 0.2 \times 0^2 + 0.5 \times 1^2 + 0.3 \times 2^2 = 1.7.$$

Let us consider another example this time of a continuous random variable. X is a continuous random variable with density f given as follows

$$f(x) = \begin{cases} 1 & \forall x \in [0,1] \\ 0 & \forall x \notin (0,1) \end{cases}.$$

Recall that we refer to this random variable as uniformly distributed over 0, 1. This is how the density looks like.

Now consider a function $g(X) = X^3$. Then expected value of X^3 can be computed using the formula given above that suggest that expected value of X^3 will be

$$E[X^3] = \int_0^1 x^3 \, f(x) dx$$

because that is the interval over which f(x) is positive and in fact it is 1 over this interval. So, this becomes

$$= \int_0^1 x^3 \, dx = \frac{1}{4}.$$

Let us know see a few properties of expectation. Suppose x_1, x_2 …. xn are random variables, define on the same probability space, say, (Ω, F, P), then their $\sum_{i=1}^{n} x_i$. This is also a random variable and expectation of this new random variable is sum of expectations of individual random variables. This property is called linearity of expectation. This is quite useful property and in fact we will see one of its uses very soon.

Similarly if α, β are constant, say they are real numbers, then αx + β is also a random variable if x is a random variable. $E[\alpha x + \beta] = \alpha E[x] + \beta$. Let us see an example. Suppose a die is rolled thrice, die is tossed thrice. Let X_i be the outcome of the i^{th} roll, outcome of the i^{th} roll, i toss where i is varying from 1, 2, 3.

Also, let

$$X = \sum_{i=1}^{3} X_i,$$

then expected value of X from linearity of expectation will be

$$E[X] = \sum_{i=1}^{3} E[X_i]$$

and each of these we can compute. So, for instance expected value of X_i would be

$$= \sum_{j=1}^{6} j.\frac{1}{6} = \frac{7}{2} = 3 \times \frac{7}{2} = \frac{21}{2}.$$

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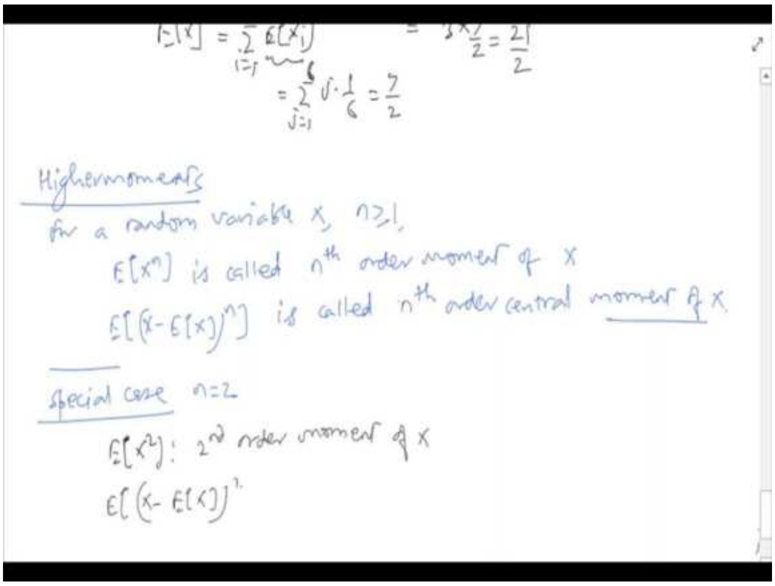

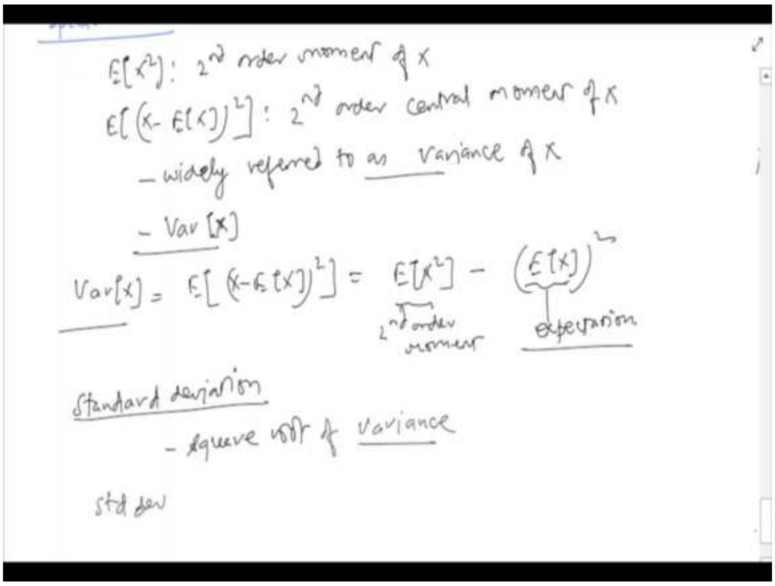

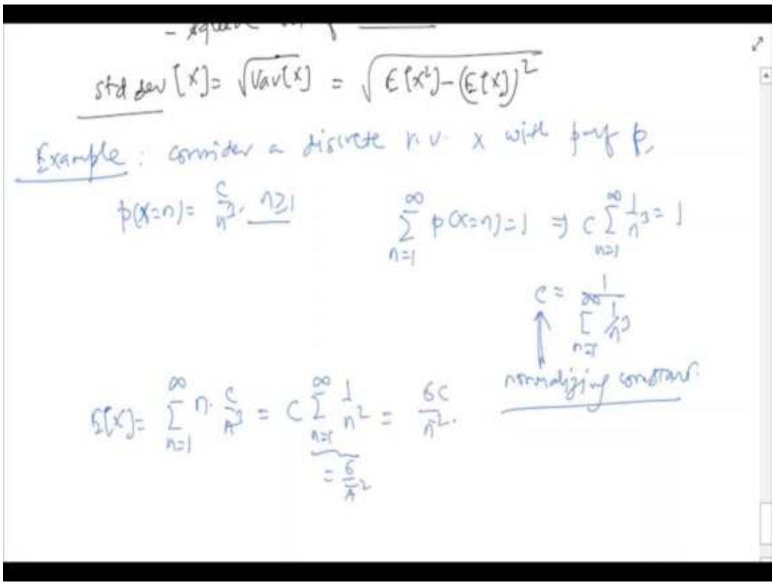

After expectation we now turn our attention to the so called higher moments of random variables, higher moments. For a random variable, X and a non-negative integer rather a positive integer n the quantity $E[X^n]$. This quantity is called the n^{th} order moment of X or sometimes simply n^{th} moment of X, is called n^{th} order moment of X. Related concept is that of central moment. The quantity $E[(X – E[X^n])]$. This is called n^{th} order central moment of X or sometimes simply n^{th} central moment of X, n^{th} order central moment of X.

Of special interest is the scenario when n = 2, special case is when n = 2. In this case, the quantity $E[X^2]$ is clearly second order moment and $E[(X-E[X])^2]$. This is second order central moment and this second order central moment of X is what is widely known as variance of X, widely referred to as variance of X.

In fact it is denoted by Var[X], so $Var[X] = E[(X-E[X])^2] = E[X^2]- (E[X])^2$. So, variance is nothing but second order moment minus square of expectation, square root of variance is known as standard deviation of random variable.

Let us see standard deviation, standard deviation of a random variable is square root of variance. It is often denoted as $std\ dev\ [X] = \sqrt{Var[X]}$. Now we can use any of the above 2 formulae for variance. So for instance it can be written as $= \sqrt{E[X^2] - (E[X])^2}$, so this is standard deviation. So, we have thus defined variance and standard deviation of random variables.

Let us see an example, a random variable X it is discrete random variable so consider discrete random variable X with pmf P given as $P(X = n) = \frac{C}{n^3}$, $n \geq 1$. So, C here is a constant, random variable X is thus taking all positive integer values and these are the point masses at those points. And notice that for P to be a valid probability mass function, we require that

$$\sum_{n=1}^{\infty} P(X = n) = 1.$$

That is we require that

$$C \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.$$

So, C has to be set,

$$C = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^3}}.$$

And so is this constant. For this reason the C is also termed as normalizing constant. So, let us see what will be the expectation and variance of this random variable. Expectation of X can be computed using the formula given above. So, it will be

$$E[X] = \sum_{n=1}^{\infty} n \frac{C}{n^3} = C \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{6C}{\pi^2}.$$

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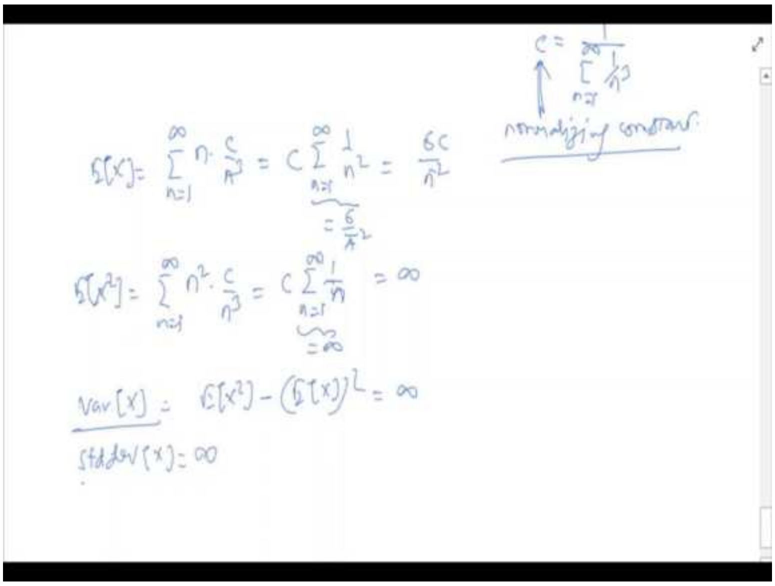

Now let us look at the second point, $E[X^2]$. This from the formula of expectation of functions of random variables can be written as

$$E[X^2] = \sum_{n=1}^{\infty} n^2 \frac{C}{n^3} = C \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

So, the second moment, second order moment of X in this case is ∞.

Now let us turn to variance, Var[x]. As we have seen above, it is difference of second order moment n^2 of expectation, but the second order moment is ∞. Second term is finite but the first is infinite. So, this is also infinite. Clearly, in this case, the std dev[x] =∞.

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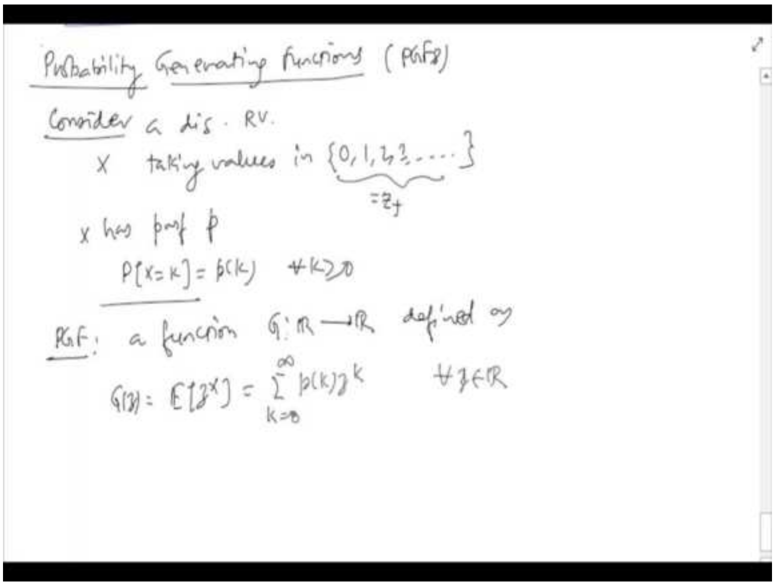

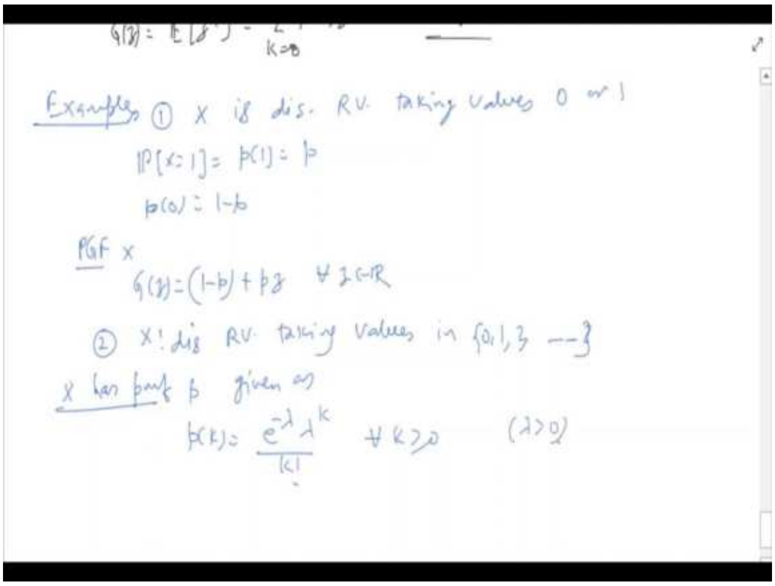

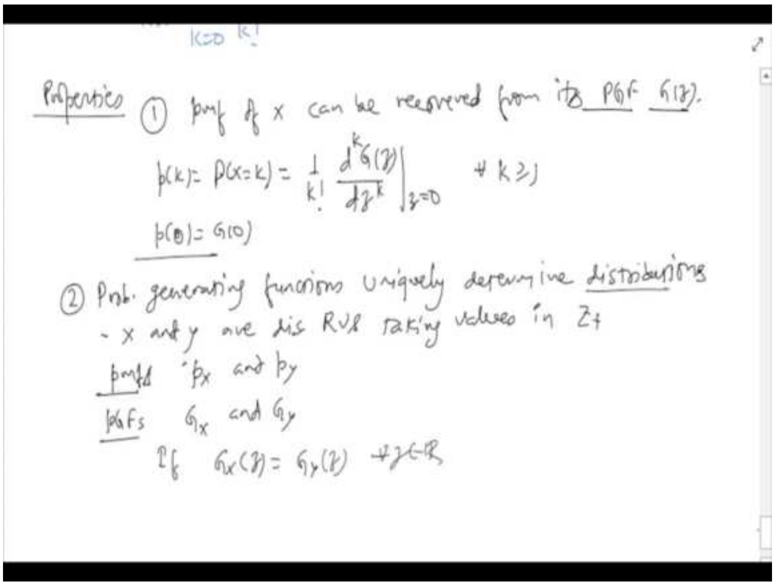

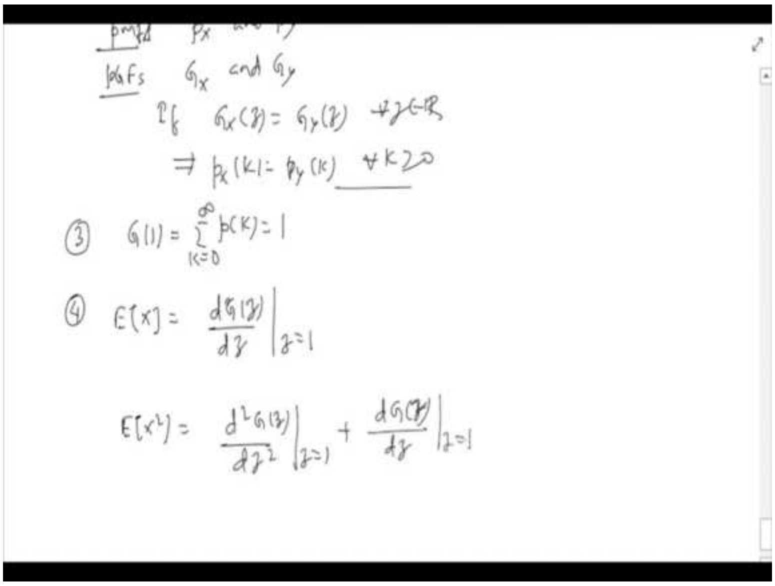

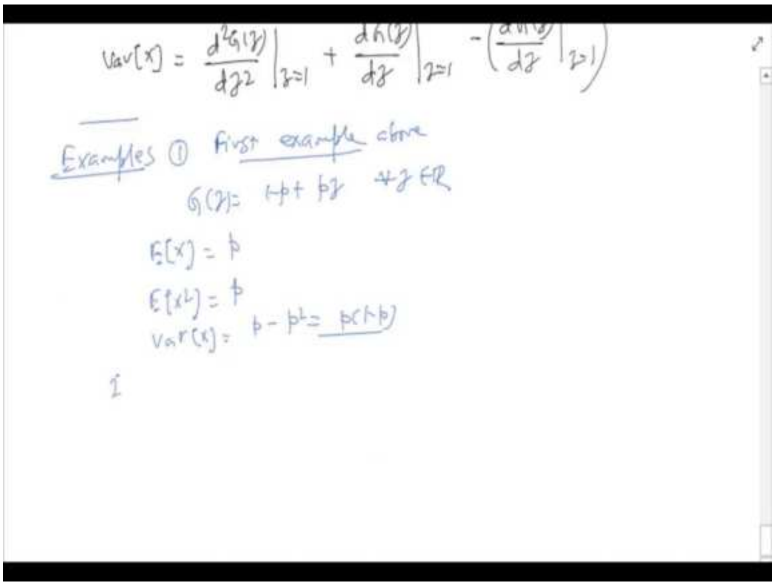

Next let us see the notion of probability generating functions. These are called PGFs in short. Probability generating functions are defined for discrete random variables that is non-negative integer values. Consider a discrete random variable X taking values in {0, 1, 2, 3 …}that is in the set z_+.

Suppose the pmf of X is P. X has pmf P[X=k] = p(k) $\forall k \geq 0$, then the probability generating function of X pgf is a function G: R → R defined as follows the

$$G(z) = E[z^x] = \sum_{k=0}^{\infty} p(k)\, z^x, \quad \forall z \in \mathbb{R}.$$

So, this is how we define probability generating function.

Let us see a couple of examples. First example is suppose is X is a discrete random variable, and it takes value 0 or 1. Moreover, P[x = 1] = p(1) = p clearly p(0) = 1 - p, then pgf of X,

$$G(z) = (1 - p) + pz, \qquad \forall z \in \mathbb{R}.$$

Let us consider another example, suppose again X is a discrete random variable but now it is taking value in z_+ …, suppose X has pmf P that is given as follows given as

$$P(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad \forall k \geq 0, \lambda > 0.$$

Now we can use the formula above to compute the probability generating function of X that is G as

$$G(z) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} z^k = e^{\lambda(z-1)}.$$

Now let us see a few properties of probability generating functions. The first property is the pmf of a random variable can be recovered from its probability generating function. Pmf of X can be recovered from its pgf, G(z). In particular

$$p(k) = P(X = k) = \frac{1}{k!} \frac{d^k G(z)}{dz^k}\bigg|_{z=0}, \quad \forall k \geq 1.$$

In fact, p(0) = G(0). So, this is how we can recover the whole pmf from the pgf of the random variable.

The second property says that probability generating functions uniquely determine the distributions. Probability generating functions uniquely determine distributions, what this means is that if x and y are two discrete random variables taking values in z_+. Suppose their PMFs are p_x and probability. Their pgfs are G_x and G_y. If G_x and G_y are same that is if $G_x(z) = G_y(z)$ $\forall z \in \mathbb{R}$, then their PMFs will also be same that is $p_x(k) = p_y(k)$ $\forall k \geq 0$.

The third property says that

$$G(1) = \sum_{k=0}^{\infty} p(k) = 1$$

given that x is a proper random variable that is an assumption that we are making all through. The fourth property tells that expectation can be computed from PGF of the random variable in particular

$$E[X] = \left.\frac{dG(z)}{dz}\right|_{z=1}$$

In fact moments of all orders of x can be recovered from the pgf can be computed from the pgf. We will not look at moments of all orders but we will be interested in second moment and variance. It turns out that second moment of x that is

$$E[X^2] = \left.\frac{d^2G(z)}{dz^2}\right|_{z=1} + \left.\frac{dG(z)}{dz}\right|_{z=1}.$$

Clearly, we can write even variance of x using the above 2 expressions in terms of pgf of x particular variance of x is given by

$$Var[X] = \left.\frac{d^2G(z)}{dz^2}\right|_{z=1} + \left.\frac{dG(z)}{dz}\right|_{z=1} - \left(\left.\frac{dG(z)}{dz}\right|_{z=1}\right)^2.$$

Let us now illustrate this relations with couple of examples. In fact we will revisit the 2 examples that we saw above example.

First example is above, recall that in the first example above

$$G(z) = (1 - p) + pz, \qquad \forall z \in \mathbb{R}.$$

so mean of the random variable x here can be computed as derivative of pgf that is p evaluated at z = 1 so this is simply p. The second derivative of G(z) at z = 1 plus its first derivative at z = 1, the second derivative of G(z) = 0. First derivative computed at z = 1 is p so that is how we get the second moment and the variance of x can now be computed at $p - p^2 = p(1-p)$.

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$Var(X) =$

① Second example above.

$$G(z) = e^{\lambda(z-1)}$$

$$E[X] = \lambda$$

$$E[X^2] = \lambda^2 e^{\lambda(z-1)}\Big|_{z=1} + \lambda e^{\lambda(z-1)}\Big|_{z=1}$$

$$= \lambda^2 + \lambda$$

$$Var(X) = \lambda$$

Moment Generating functions (MGFs)

Moment generating function of a RV X is a mapping.

Moment generating function of a RV X is a mapping.

$M : \mathbb{R} \to \mathbb{R}$ defined by

$$M(t) = E[e^{Xt}] \quad \forall t \in \mathbb{R}$$

$$* \; e^{tX} = \sum_{n=0}^{\infty} \frac{t^n x^n}{n!}$$

$$\Rightarrow M(t) = \sum_{n=0}^{\infty} \frac{t^n E[X^n]}{n!}$$

Examples ① X: discrete random variable

$$p(1) = P[X=1] = p$$
$$p(0) = 1-p$$

Examples ① X: discrete random variable

$$p(1) = P[X=1] = p$$
$$p(0) = 1-p$$

MGF

$$M(t) = (1-p) + pe^t \quad \forall t \in \mathbb{R}$$

② X: discrete RV.

$$p(k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \forall k \geq 0$$

$$M(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} e^{kt} = \sum_{k=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^k}{k!} = \exp(\lambda(e^t - 1))$$

Let us know see the second example above recall that in this example

$$G(z) = e^{\lambda(z-1)}$$

so expectation of the random variable X can be computed as derivative of G(z) evaluated at z = 1. Its derivative is $\lambda e^{\lambda(z-1)}$ when evaluated at 1 we simply get λ.

$$E[X^2] = \lambda^2 e^{\lambda(z-1)}\Big|_{z=1} + e^{\lambda(z-1)}\Big|_{z=1} = \lambda^2 + \lambda$$

from the above 2 we can see that $Var[X] = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Finally, let us see the notion of moment generating functions. In short, those are called MGFs. The moment generating function of a random variable, X is a mapping, M:ℝ → ℝ defined by $M(t) = E[e^{Xt}], \forall t \in \mathbb{R}$. We assume that this expectation adjusts. Note that

$$e^{Xt} = \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}$$

so

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n E[X^n]}{n!}.$$

Let us see a few examples. The first example let us again consider X to be random variable, discrete random variable, that takes value 0 and 1, p(1) = P[X=1] = p, p(0) = 1 - p. In this case, following the above definition, we can write MGFs,

$$M(t) = (1-p) + pe^t, \qquad \forall t \in \mathbb{R}.$$

Let us see another example. Now we assume that X is discrete and it has PMF. P as follows,

$$p(k) = \frac{e^{-\lambda}\lambda^K}{K!}, \qquad \forall K \geq 0.$$

In this case,

$$M(t) = \sum_{K=0}^{\infty} \frac{e^{-\lambda}\lambda^K}{K!} e^{Kt} = \sum_{K=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^K}{K!} = \exp\left(\lambda(e^t - 1)\right).$$

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$$p(k) = \frac{e^{-\lambda}\lambda^{k}}{k!} \quad \forall k \geq 0$$

$$m(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{k!} e^{kt} = \sum_{k=0}^{\infty} \frac{e^{-\lambda}(\lambda e^{t})^{k}}{k!} = \exp(\lambda(e^{t}-1))$$

③ X: cont R.V.

pdf of X

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$M(t) = \int_{0}^{\infty} \lambda e^{-\lambda x} e^{tx}\, dx .$$

$$M(t) = \int_{0}^{\infty} \lambda e^{-\lambda x} e^{tx}\, dx$$

$$= \frac{\lambda}{\lambda - t} \quad t < \lambda$$

(MGF is defined only for $t < \lambda$)

Properties

① Moment generating functions uniquely determine distributions

X & Y: two random variables

F_X and F_Y: cdf

M_X and M_Y: MGFs

M_X and M_Y: MGFs

If $M_X(t) = M_Y(t) \quad \forall t \in \mathbb{R}$

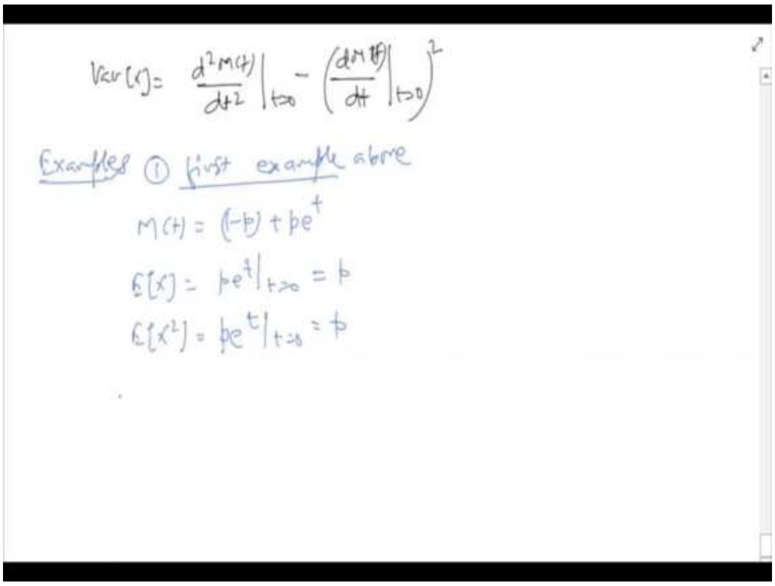
$\Rightarrow F_X(x) = F_Y(x) \quad \forall x \in \mathbb{R}$

② $M(0) = \mathbb{E}[e^{0X}] = 1$

③ $E(X) = \left.\frac{dM(t)}{dt}\right|_{t=0}$

$\forall n \geq 1$

$$E[X^n] = \left.\frac{d^n m(t)}{dt^n}\right|_{t=0}$$



Let us see one more example and this time we take X to be a continuous random variable and probability density function of X is,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Now for this random variable,

$$M(t) = \int_0^{\infty} \lambda e^{-\lambda x} e^{tx} dx$$

and it can be seen that this function is only defined when t is less than the in which case, the function turns out to be

$$= \frac{\lambda}{\lambda - t}$$

so here is $t < \lambda$. MGF is defined only for t less than that.

Let us know see a few properties of moment generating functions. The first property is like probability generating functions, moment generating functions also uniquely determined the distributions. Moment generating functions uniquely determine distributions, that is if we have 2 random variables X and Y, 2 random variables and say these random variables have cumulative distribution functions F_X and F_Y and moment generating functions M_X and M_Y. If $M_X = M_Y$ that is if $M_X(t) = M_Y(t)\ \forall t \in \mathbb{R}$, then $F_X = F_Y$ that is $F_X(x) = F_Y(x)\ \forall x \in \mathbb{R}$, so the distributions of the 2 random variables are also seen.

The second property says that $M(0) = E[e^{0x}] = 1$. The third property says that expectation of the random variable can be recovered from the moment generating function in particular the expected value of X turns out to be

$$E[X] = \left.\frac{dM(t)}{dt}\right|_{t=0}.$$

In fact as in the case of probability generating functions moments of all orders can also be recovered from moment generating functions. In particular ∀ n ≥ 1, the n^{th} order moment of random variable X turns out to be equal to n^{th} derivative of moment generating function evaluated at t = 0. So clearly, we can compute second order moment variance et cetera from the moment generating function in particular

$$Var[X] = \left.\frac{d^2M(t)}{dt^2}\right|_{t=0} - \left(\left.\frac{dM(t)}{dt}\right|_{t=0}\right)^2.$$

Let us see few examples to illustrate this relations. We will revisit the examples that we have seen above. So first example above, I recall that in the first example above the moment generating function was $M(t) = 1 - p + pe^t$ so the expectation of the random variable X can be obtained as derivative of the moment generating function that is $E[X] = pe^t|_{t=0} = p$.

The second moment can be obtained as – the second derivative of moment generating function that is again $E[X^2] = pe^t|_{t=0} = p$ and so before Var[X] = p(1-p).

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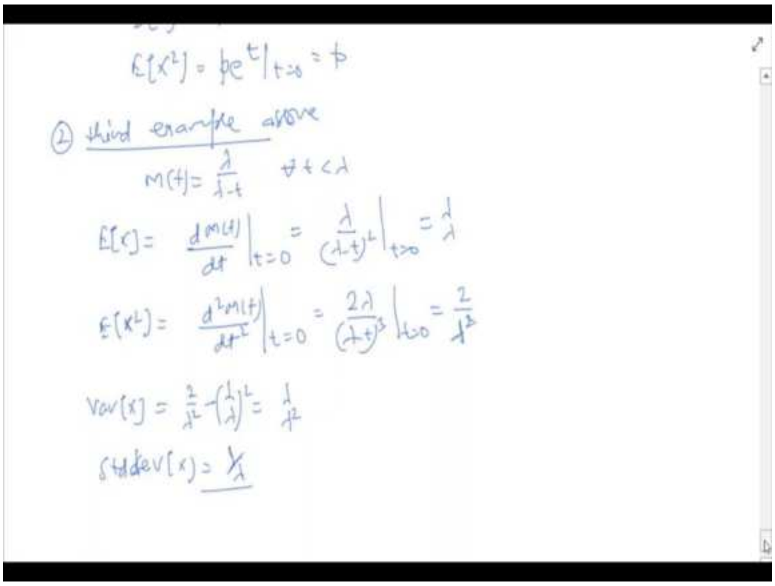

Let us know look at one more example which is actually the third example seen above. Recall that in the third example above, the moment generating function $M(t) = \frac{\lambda}{\lambda - t}, \forall t < \lambda$. So, expectation of x the random variable x can be computed by

$$E[X] = \left.\frac{dM(t)}{dt}\right|_{t=0} = \left.\frac{\lambda}{(\lambda - t)^2}\right|_{t=0} = \frac{1}{\lambda}.$$

Next the second moment of the random variable,

$$E[X^2] = \left.\frac{d^2M(t)}{dt^2}\right|_{t=0} = \left.\frac{2\lambda}{(\lambda - t)^3}\right|_{t=0} = \frac{2}{\lambda^2}.$$

So, variance of x can be computed as

$$Var[X] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

From variance we can also compute the standard deviation of x this is simply $stddev[X] = \frac{1}{\lambda}$. So, this much about moment generating function. This brings us to the end of this lecture. Thank you.