Bounded arbitrage and nearly rational behavior

Leandro Nascimento*

July, 2023

Abstract

We establish the equivalence between a principle of almost absence of arbitrage opportunities and nearly rational decision-making. The implications of such principle are considered in the context of the aggregation of probabilistic opinions and of stochastic choice functions. In the former a bounded arbitrage principle and its equivalent form as an approximately Pareto condition are shown to bound the difference between the collective probabilistic assessment of a set of states and a linear aggregation rule on the individual assessments. In the latter we show that our general principle of limited arbitrage opportunities translates into a weakening of the McFadden–Richter axiom of stochastic rationality, and gives an upper bound for the minimum distance of a stochastic choice function to another in the class of random utility maximization models.

Keywords: arbitrage, aggregation of probability measures, stochastic choice.

JEL classification: D70, D81.

^{*}Universidade de Brasília; lgnascimento@unb.br.

1 Introduction

In a series of papers, Nau (1992, 1995a, 1995b, 2015) and Nau and McCardle (1990, 1991) advocated the use of no arbitrage conditions as a unifying principle to characterize rational behavior. They have argued that the absence of arbitrage opportunities is not only a fundamental principle underlying the modern theory of asset pricing, but can also be used to characterize a variety of forms of rational behavior. Indeed, several expressions of rationality in economics, ranging from the axiomatic foundations of models of cardinal preferences to the correlated equilibrium outcomes in finite games, have an equivalent form as a no arbitrage condition.¹

In this paper we show that a weaker principle of bounded arbitrage opportunities characterizes models of approximate rationality for the aggregation of probabilistic opinions and in the context of stochastic choice functions. Our results build on a minimum norm duality connecting the minimum distance between two sets of probability measures to the maximum gain from arbitrage with normalized stakes. Here the possibility of arbitrage denotes a sure gain when betting on a set of states, as in the Coherence Theorem of de Finetti.² A consequence of our duality is that bounding the gains from arbitrage by a multiple of a positive number ϵ gives rise to weaker forms of the postulates of collective rationality in the aggregation of probabilities problem, and to a weaker version of the Axiom of Revealed Stochastic Preference, henceforth ARSP, of McFadden and Richter (1990) in the context of stochastic choice.

In the two cases we obtain the representation of a probability measure P as the sum

$$P = Q_m + e. (1)$$

Here Q_m is a representation (a suitable linear aggregation of other measures) according to the canonical models of rational behavior. The term e stands for an error component whose length does not exceed ϵ . Our characterization of nearly rational decision-making also includes a version of the two models having

$$P = (1 - \epsilon)Q_m + \epsilon R,\tag{2}$$

with Q_m as before, and for some probability measure R. Like equation (1), the expression in (2) carries a similar interpretation as a deviation from the canonical forms of rationality

¹Recently, Beggs (2021) has also drawn a connection between the Nau-McCardle approach and the rationalization of expenditure data \grave{a} la Afriat.

²A weaker version of what is referred to as the Coherence Theorem in this paper was used in Nau and McCardle (1990, 1991) to illustrate the basic idea behind their results. Here we borrow the stronger version of the theorem as found, e.g., in Nielsen (2019, 2021). Details are given at the end of Section 3 below.

and can be viewed as a version of the ϵ -contamination model of Huber (1964). The standard versions of the two models correspond to the case where $\epsilon = 0$.

One such version refers to the aggregation of probabilistic assessments into a single probability measure. This is the linear opinion pool of Stone (1961), where the elements of a set Q are interpreted as the probabilistic assessments of the odds of a set of states by the members of a group, and the probability measure Q_m in equations (1) and (2) is the linear averaging of the probabilities in Q with a weight vector m. The traditional approach to collective rational decision-making links the probabilities in Q to the probability P of a social planner by means of a Pareto unanimity condition. We show in this paper that weaker forms of the Pareto principle characterize the representations with positive ϵ . Particularly, we characterize the version of an imprecise form of linear aggregation of probabilistic opinions as in (2), thereby handling the single-profile case of an aggregation rule due to Genest (1984).

We also investigate in this paper an approximate version of the random utility maximization model of Block and Marschak (1960). The exact version of this model represents a stochastic choice function as a suitably defined linear averaging of deterministic choice functions. Each element of \mathcal{Q} stands for a profile of choice probabilities induced by a single strict preference ordering. In this setting we give conditions ensuring that the answer to the canonical problem is perturbed proportionally to the extent of the deviation from the original condition that makes the choice probabilities in P a linear averaging of the choice probabilities in the $Q \in \mathcal{Q}$. Our condition for an expression resembling (1) is a modified version of ARSP.⁵ We also obtain a similar representation with an expression like equation (2) for a stochastic choice function along those same lines. This last variation of the model was recently proposed by Apesteguia and Ballester (2021), for whom the expression in (2) has the component Q_m as the stochastic choice function in a certain class predicted by the theorist, while R represents unstructured randomness in the choice data that remains unexplained. Our contribution when compared to their paper is to axiomatically characterize their residual behavior representation when the class of stochastic choice functions the theorist wants to fit the data consists of the set of random utility maximization models.

This paper is organized as follows. In Section 2 we introduce the setting in which we work in this paper. Section 3 gives a general result about the proximity of the closed convex hull

³See Mongin (1995) and the references therein.

⁴The original formulation of Genest (1984) uses the multi-profile setting of McConway (1981). For comparison, here we employ techniques that are different from those in Genest's paper to deal with the single-profile case.

⁵The relationship (that we explore in this paper) between the standard version of ARSP and the Coherence Theorem of de Finetti also appears in the work of Clark (1996).

of two compact sets of probabilities in terms of expectations of certain functions, and relate it to the Coherence Theorem and the concept of arbitrage. In Section 4 we specialize our general setting in order to handle the two important applications already mentioned. While Section 5 concludes with some remarks and open problems about the questions addressed in this paper, all proofs are relegated to Appendix A.

2 Setting

The set X is a compact metric space with metric d_X . We denote by $\Delta(X)$ the set of all Borel probability measures on X. That is, each element of $\Delta(X)$ is a countably additive nonnegative measure on the Borel sets of X, and the measure of X is normalized to one.

For notation we write C(X) to represent the vector space of all continuous functions $f: X \to \mathbb{R}$. The vector space C(X) is endowed with the supremum norm $||f||_{\infty} = \sup_{x \in X} |f(x)|$. In view of continuity of the elements of C(X) and compactness of X, the supremum can be replaced by a maximum in the definition of the norm above, as well as in the definition of the oscillation $\omega(f)$ of a function $f \in C(X)$, namely,

$$\omega(f) = \sup_{x_1, x_2 \in X} [f(x_1) - f(x_2)].$$

It is also convenient to define the closed unit ball $B_{\|\cdot\|_{\infty}}(0,1)=\{f\in C(X):\|f\|_{\infty}\leq 1\}.$

The set ca(X) stands throughout for the vector space of all Borel signed measures of bounded variation on X. Note that $\Delta(X) \subseteq ca(X)$. The set ca(X) is endowed with the topology of weak convergence. For the purposes of this paper it suffices to notice that this is the topology where a sequence (R_n) in ca(X) converges to $R \in ca(X)$ if, and only if,

$$\lim_{n \to \infty} \int_X f \, dR_n = \int_X f \, dR \quad \text{ for all } f \in C(X).$$

In the topology of weak convergence, which we refer to as the weak* (or w*, for short) topology, the set of probability measures $\Delta(X)$ is compact.⁶ As a matter of notation, given $\mathcal{R} \subseteq \operatorname{ca}(X)$, the set $\overline{\operatorname{co}} \mathcal{R}$ designates the smallest closed (in the weak* topology) and convex subset of $\operatorname{ca}(X)$ containing \mathcal{R} . This will be referred to as the closed convex hull of \mathcal{R} .

It is also known that the weak* topology on $\Delta(X)$ can be metrized by the so-called Kantorovich-Rubinstein metric d_{KR} . This metric is given by

$$d_{KR}(P,Q) = \sup_{f \in \text{Lip}_1, ||f||_{\infty} \le 1} \left(\int_X f \, dP - \int_X f \, dQ \right), \tag{3}$$

⁶For this and the assertions to follow referencing the space ca(X) and its underlying topology, see Bogachev (2007, 2018).

where

$$Lip_1(X) = \{ f \in \mathbb{R}^X : |f(x_1) - f(x_2)| \le d_X(x_1, x_2) \text{ for all } x_1, x_2 \in X \}.$$

In the more conventional case where X is a finite set, the topological structures of C(X) and $\operatorname{ca}(X)$ coincide with that of a finite-dimensional Euclidean space equipped, respectively, with the maximum norm and the ℓ_1 norm. Otherwise, as is well known, when the set X is not finite there are in particular nonequivalent solutions, which are useful for different purposes, to the problem of giving a metric structure to the set of all probability measures on X. Here the Kantorovich-Rubinstein metric is one of many forms of quantifying the distance between two probability measures. Another important metric on the set $\Delta(X)$ is the one induced by the total variation norm $\|\cdot\|_1$ on $\operatorname{ca}(X)$. Denoting by Σ_X the collection of all Borel sets of X, recall that the total variation of a bounded signed measure $R \in \operatorname{ca}(X)$ is

$$||R||_1 = \sup \left\{ \sum_{n=1}^k |R(E_n)| : (E_n)_{n=1}^k \text{ is a partition of } X, \text{ and } E_n \in \Sigma_X \right\}.$$

This norm induces a metric on $\Delta(X)$ as usual:

$$d_1(P,Q) = \|P - Q\|_1. \tag{4}$$

An alternative expression for the total variation of a signed measure, which reveals a connection between the metrics d_{KR} and d_1 , is based on a Hahn decomposition (X_+, X_-) of X. Namely, X_+ and X_- are disjoint Borel sets of X with $X = X_+ \cup X_-$ and such that $R(X_+) = \sup_{E \in \Sigma_X} R(E)$ and $R(X_-) = \inf_{E \in \Sigma_X} R(E)$. Such a partition can be shown to exist and we have

$$||R||_1 = R(X_+) - R(X_-).$$

Hence we know that

$$d_1(P,Q) = \sup_{f \in \mathcal{F}_X} \left(\int_X f \, dP - \int_X f \, dQ \right), \tag{5}$$

when \mathcal{F}_X is the set of all measurable $f: X \to \mathbb{R}$ with norm at most one. For the purposes of this paper we shall assume that $\mathcal{F}_X = B_{\|\cdot\|_{\infty}}(0,1)$. Both choices of \mathcal{F}_X can be used interchangeably in (5).

We recall that in the special case where X is finite and endowed with the discrete metric, the total variation and the Kantorovich-Rubinstein distance both have the same expression

$$d_1(P,Q) = d_{KR}(P,Q) = \sum_{x \in X} |P(x) - Q(x)|.$$

⁷See Gibbs and Su (2002) for a survey and comparison of the many metrics on the set $\Delta(X)$.

This is also known as the ℓ_1 distance. In general, as long as X is infinite and has at least one accumulation point, the metrics d_1 and d_{KR} induce different topologies on $\Delta(X)$. Without further assumptions on the metric space X, we have

$$d_{KR}(P,Q) \le d_1(P,Q) \tag{6}$$

for all $P, Q \in \Delta(X)$. The inequality in (6) reveals, in particular, that any pair of probability measures that are ϵ -close according to the total variation distance are also ϵ -close in the Kantorovich-Rubinstein metric.⁸

In any case, unless stated otherwise, topological properties of subsets of signed measures of bounded variation on a compact metric space and the convergence of sequences of measures refer to the topology of weak convergence, and not to the notion of distance mentioned in equation (4). Our use of the ℓ_1 distance will be restricted to quantifying the length of the error term in the representations of the form expressed in (1).

3 The general result

To introduce our general result regarding a minimum distance duality for compact sets of probability measures, it is convenient to first provide an expression to the closure of the convex hull of such sets in terms of probability measures on them. In the next proposition we characterize the closed convex hull of a weak*-compact subset of $\Delta(X)$ as the set of all probability measures on X that can be expressed as the expectation of the probabilities in that set according to some second-order probability measure. It is essentially a simple version of Choquet's theorem; see Phelps (2001).

Proposition 1. Suppose that \mathcal{R} is a weak*-compact set of Borel probabilities measures on the compact metric space X. Given any $m \in \Delta(\mathcal{R})$, denote by R_m the element of $\Delta(X)$ defined by

$$R_m(E) = \int_{\mathcal{R}} R(E) \, dm \quad \text{for all Borel sets } E \subseteq X.$$
 (7)

Then $\overline{\operatorname{co}} \mathcal{R} = \{ R_m \in \Delta(X) : m \in \Delta(\mathcal{R}) \}.$

The next theorem is our minimum distance duality for the total variation distance. With the same notation as in (7), the theorem characterizes the value of the problem of minimizing

⁸The converse implication is false as long as the metric space (X, d_X) has a countably infinite subset with an accumulation point for the same reason that in this case the topology induced by the ℓ_1 norm and the topology of weak convergence are not the same. We return to this distinction in Example 1 in the next section.

 $||P_{m_{\mathcal{P}}} - Q_{m_{\mathcal{Q}}}||_1$ over $(m_{\mathcal{P}}, m_{\mathcal{Q}}) \in \Delta(\mathcal{P}) \times \Delta(\mathcal{Q})$, for compact sets \mathcal{P} and \mathcal{Q} of probabilities, with reference to the disagreement about the expected value of normalized continuous functions on X between the $P \in \mathcal{P}$ and the $Q \in \mathcal{Q}$.

Theorem 1. Let \mathcal{P} and \mathcal{Q} denote weak*-compact sets of probability measures on the compact metric space X. Then

$$\min_{P \in \overline{\varpi}\mathcal{P}, Q \in \overline{\varpi}\mathcal{Q}} \|P - Q\|_1 = \sup_{f \in C(X), \|f\|_{\infty} \le 1} \min_{P \in \mathcal{P}, Q \in \mathcal{Q}} \left(\int_X f dP - \int_X f dQ \right). \tag{8}$$

A version of the theorem above appears in Luenberger (1997, p.136) exploring a different but related form of duality in order to bound the distance between two sets of continuous functions under the supremum norm. Such a form is known as the Nirenberg–Luenberger mininum distance duality. In contrast, our Theorem 1 uses the relation between the set of signed measures on the compact metric space X and the continuous linear functionals on C(X) in order to prove a duality result involving two sets of probability measures endowed with the topology of weak convergence. By interpreting the minimum difference $\int_X f \, dP - \int_X f \, dQ$ as the disagreement or discrepancy of the assessment of the expected value of f, Theorem 1 essentially says that the total variation distance between two closed and convex subsets of $\Delta(X)$ is the maximum discrepancy of the assessments of expectations of certain continuous functions with respect to the probabilities in those sets. Geometrically, by viewing

$$\min_{P \in \overline{\operatorname{co}} \mathcal{P}, Q \in \overline{\operatorname{co}} \mathcal{Q}} \left(\int_X f \, dP - \int_X f \, dQ \right) = \min_{P \in \overline{\operatorname{co}} \mathcal{P}} \int_X f \, dP - \max_{Q \in \overline{\operatorname{co}} \mathcal{Q}} \int_X f \, dQ$$

as a scalar multiple of the distance between two parallel hyperplanes supporting, respectively, the sets $\overline{\operatorname{co}} \mathcal{P}$ and $\overline{\operatorname{co}} \mathcal{Q}$, the minimum distance between the two sets corresponds to the maximal separation. This notion is discussed at length in Dax (2006) in a finite-dimensional setting, and our Theorem 1 becomes a special case of Theorem 13 in Dax's paper when X is finite. However, the strategy of proof of that Theorem 13 relies on the compactness of the closed unit ball in finite-dimensional normed spaces, which no longer holds in the context of C(X) with an infinite set X. To derive the expression in (8) we use the characterization of the total variation distance in (5) combined with a version of the minimax theorem due to Kneser (1952).

Regarding the optimization problem in (8), it is not difficult to check that Theorem 1 above implies that for any two compact sets of probabilities, \mathcal{P} and \mathcal{Q} , there are probability measures $m_{\mathcal{P}} \in \Delta(\mathcal{P})$ and $m_{\mathcal{Q}} \in \Delta(\mathcal{Q})$ such that

$$d_1(P_{m_{\mathcal{P}}}, Q_{m_{\mathcal{Q}}}) = \sup_{f \in C(X), \|f\|_{\infty} = 1} \min_{P \in \mathcal{P}, Q \in \mathcal{Q}} \left(\int_X f \, dP - \int_X f \, dQ \right).$$

The identity in (8) is also particularly valuable because it allows us to interchange the roles of maximum and minimum in the original minimization problem. In many concrete cases arising in connection with the dualities presented in Dax and Sreedharan (1997) the expression for the minimum norm can be substantially simplified if we first calculate the minimum of the linear mapping induced by $f \in C(X)$ in (8). In fact, an immediate corollary of Theorem 1 is the following version of an approximate theorem of the alternative.

Corollary 1. Let \mathcal{P} and \mathcal{Q} be as in Theorem 1, and $\epsilon > 0$ be given. Then either

I. $\int_X f \, dP - \int_X f \, dQ > \epsilon$ for all $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ has a solution $f \in C(X)$ with $||f||_{\infty} = 1$, or

II. $d_1(P,Q) \leq \epsilon \text{ has a solution } (P,Q) \in \overline{\operatorname{co}} \mathcal{P} \times \overline{\operatorname{co}} \mathcal{Q},$ but never both.

Corollary 1 can be interpreted as an approximate version of Gordan's Theorem of the Alternative for probabilities. It says that either there is a strong disagreement of the expectations of normalized continuous functions on the set X, or else the closed convex hull of the two sets of probabilities are sufficiently close in the total variation metric of a pair of elements in those sets. For comparison, Gordan's theorem asserts that either there is some disagreement about the expectation of some $f \in C(X)$ according to the measures in \mathcal{P} and \mathcal{Q} in the sense that

$$\min_{P \in \mathcal{P}} \int_{X} f \, dP > \max_{Q \in \mathcal{Q}} \int_{X} f \, dQ,\tag{9}$$

or else the sets $\overline{\operatorname{co}}\mathcal{P}$ and $\overline{\operatorname{co}}\mathcal{Q}$ have a nonempty intersection. This contrast becomes transparent in a finite-dimensional setting after a close inspection of the fourth column of Table 1 in Dax and Sreedharan (1997), which covers the case where X is finite, \mathcal{P} is a singleton, and \mathcal{Q} is the convex hull of a finite set. It should be also noted that the normalization $||f||_{\infty} = 1$ cannot be omitted from system I in Corollary 1 since the inequality in (9) is equivalent to the inequality $\min_{P \in \mathcal{P}} \int_X f \, dP - \max_{Q \in \mathcal{Q}} \int_X f \, dQ > \epsilon$ for all $f \in C(X)$ with arbitrarily large norm

We also mention that the use of the total variation distance in Corollary 1 can be sometimes too restrictive, so the previous corollary to our general result comes at a cost in some circumstances. We illustrate this point with the next example.

Example 1. Consider a collection of approximate systems of the alternative as in Corollary 1, where each system is indexed by n. For

$$X = \{0\} \cup \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}$$

that remains fixed and is viewed as a subset of the real line with the standard topology, in the n-th system we have the sets of degenerate measures δ_x :

$$\mathcal{P}_n = \{\delta_0\}$$
 and $\mathcal{Q}_n = \left\{\delta_{\frac{1}{k}} : k = 1, \dots, n\right\}.$

It is not difficult to verify that in system n we have

$$\min_{P \in \overline{\operatorname{co}} \, \mathcal{P}_n, Q \in \overline{\operatorname{co}} \, \mathcal{Q}_n} \|P - Q\|_1 = 2,$$

even though as n gets larger the degenerate probability $\delta_{\frac{1}{n}}$ intuitively gets closer to $\delta_0 \in \mathcal{Q}_n$ once we take into account the standard metric structure of X. Hence, given any $\epsilon \in (0,2)$, for every n there exists a normalized continuous function f_n such that $f_n(0) > f_n(\frac{1}{k}) + \epsilon$ for all $k = 1, \ldots, n$. Such functions are not difficult to construct. It is also not difficult to notice that for $n \geq \frac{1}{\epsilon}$ they cannot be Lipschitz continuous with the Lipschitz constant equal to 1. In view of the definition of the Kantorovich-Rubinstein metric in (3), obviously either

$$\mathbf{I}_n$$
. $\int_X f dP - \int_X f dQ > \epsilon$ for all $P \in \mathcal{P}_n$ and $Q \in \mathcal{Q}_n$ has a solution $f \in \text{Lip}_1(X)$ with $||f||_{\infty} \leq 1$,

or

$$\mathbf{II}_n$$
. $d_{KR}(P,Q) \leq \epsilon$ has a solution $(P,Q) \in \overline{\operatorname{co}} \mathcal{P}_n \times \overline{\operatorname{co}} \mathcal{Q}_n$,

but never both. Therefore, given any $\epsilon > 0$, for all n sufficiently large, system I_n does not admit a solution and system II_n has a solution, in contrast with the behavior of the related systems under the ℓ_1 metric.

We also provide an equivalent characterization to the existence of a solution to system II in Corollary 1 that will be useful in the sequel.

Corollary 2. Let \mathcal{P} and \mathcal{Q} be as in Theorem 1. There exists $(P,Q) \in \overline{\operatorname{co}} \mathcal{P} \times \overline{\operatorname{co}} \mathcal{Q}$ such that $d_1(P,Q) \leq \epsilon$ if, and only if, for all $f \in C(X)$ with $||f||_{\infty} = 1$,

$$\min_{P \in \mathcal{P}} \int_{X} f \, dP \le \max_{Q \in \mathcal{Q}} \int_{X} f \, dQ + \epsilon. \tag{10}$$

Corollary 2 essentially says that, in order to exist $P \in \overline{\operatorname{co}} \mathcal{P}$ and $Q \in \overline{\operatorname{co}} \mathcal{Q}$ such that P = Q + e for some error term $e \in \operatorname{ca}(X)$ with $\|e\|_1 \leq \epsilon$, the inequality in (10) must hold for any continuous and normalized function f. Geometrically, it says that any continuous linear functional of unit norm on the space $\operatorname{ca}(X)$ cannot separate in excess of ϵ the compact

⁹This is indeed a consequence of the weak convergence of the sequence of degenerate measures $\delta_{\frac{1}{n}}$ to δ_0 .

and convex sets $\overline{\operatorname{co}} \mathcal{P}$ and $\overline{\operatorname{co}} \mathcal{Q}$. By applying a separating hyperplane type of argument it can be also shown that an almost identical inequality (except for a multiplicative term) can be used to characterize, given $\epsilon \in (0,1)$ and a weak* compact and convex set $\mathcal{R} \subseteq \Delta(X)$, the existence of probabilities $P \in \overline{\operatorname{co}} \mathcal{P}$, $Q \in \overline{\operatorname{co}} \mathcal{Q}$ and $R \in \mathcal{R}$ such that P is the convex combination of Q and R with weights $1 - \epsilon$ and ϵ , respectively. This is done in the next proposition.

Proposition 2. Let \mathcal{P} and \mathcal{Q} be as in Theorem 1, and $\epsilon \in (0,1)$. Suppose that \mathcal{R} is a weak* closed and convex subset of $\Delta(X)$. There exist probability measures $P \in \overline{\operatorname{co}} \mathcal{P}$, $Q \in \overline{\operatorname{co}} \mathcal{Q}$ and $R \in \mathcal{R}$ such that

$$P = (1 - \epsilon)Q + \epsilon R \tag{11}$$

if, and only if, for all $f \in C(X)$ with $f(x) \ge 0$ for all $x \in X$ and $||f||_{\infty} = 1$ we have that

$$\min_{P \in \mathcal{P}} \int_{X} f \, dP \le (1 - \epsilon) \max_{Q \in \mathcal{Q}} \int_{X} f \, dQ + \epsilon \max_{R \in \mathcal{R}} \int_{X} f \, dR. \tag{12}$$

Remark 1. The conditions in (10) and (11) have equivalent forms as, respectively,

$$\max_{P \in \mathcal{P}} \int_{X} f \, dP \ge \min_{Q \in \mathcal{Q}} \int_{X} f \, dQ - \epsilon. \tag{13}$$

and

$$\max_{P \in \mathcal{P}} \int_X f \, dP \geq (1 - \epsilon) \min_{Q \in \mathcal{Q}} \int_X f \, dQ + \epsilon \min_{R \in \mathcal{R}} \int_X f \, dR.$$

These are easily verified.

As mentioned in the Introduction, the expressions in (10) and (12) can both be interpreted as bounding the gains from arbitrage with normalized stakes. For concreteness we restrict attention to (10) in its equivalent form as (13), and consider a version of the Coherence Theorem of de Finetti as follows.

Assume that X has M elements and the only element P of \mathcal{P} represents the prices of M bets, and is thus a vector in \mathbb{R}^M . Each $x \in X$ corresponds to a gamble that the individual is willing to bet on. In state i = 1, ..., N, the bet x pays the amount $Q_i(x)$. We therefore write Q_i to represent the profile of payments in state i of each of those M bets. Note that $Q = \{Q_1, ..., Q_N\}$ is a subset of \mathbb{R}^M . If the choice of stakes on each bet x is given by f(x), then the individual receives the amount

$$\sum_{x \in X} f(x)[Q_i(x) - P(x)] \tag{14}$$

in state i. De Finetti's Coherence Theorem can be phrased as saying that either there is a choice of stakes $f \in \mathbb{R}^M$ with $\max_{x \in X} |f(x)| = 1$ giving a sure gain (i.e., an arbitrage opportunity), meaning that the quantity in (14) is positive for all i, or else there exists a probability measure m on the set of states so that the price P(x) of each gamble x corresponds to its expected value $\sum_{i=1}^N m_i Q_i(x)$.

According to our Theorem 1, if instead of restricting the sign of the expression in (14) we assume that

$$\min_{i} \sum_{x \in X} f(x)Q_i(x) - \sum_{x \in X} f(x)P(x) \le \epsilon \tag{15}$$

for all f with $||f||_{\infty} = 1$, then we know that for some m in the simplex of dimension N-1 we have

$$P = \sum_{i=1}^{N} m_i Q_i + e \tag{16}$$

where $||e||_1 \le \epsilon$.¹⁰ The condition in (15) means that (sure) gains from arbitrage obtained with normalized stakes are restricted to the interval $[0, \epsilon]$. Equation (16) in its turn entails that no arbitrage principles like de Finetti's are stable. By stability we mean that when we allow for small gains from arbitrage we obtain a representation that resembles the original one except perhaps for a small error that is proportional to the small gains from arbitrage. The results to follow building on Theorem 1 carry a similar interpretation. Since we use probabilities as primitives in this paper, we interpret conditions such as (15) in terms of expectations of normalized functions (or payoffs) f with respect to P and the probabilities in Q. Hence we read (15) like the inequality in (13): for no normalized function its expected value according to P is less than an ϵ of the minimum expectation evaluated with the probabilities in Q.

4 Applications

4.1 Almost linear aggregation of probabilities

Suppose that \mathcal{P} is a singleton, and \mathcal{Q} is a weak* compact set of probabilities. When the compact metric space X is viewed as a set of states, the only element of \mathcal{P} , which we denote by P, usually has the interpretation of the probability measure expressing the assessment of

¹⁰In the case where \mathcal{P} is a singleton, and X and \mathcal{Q} are finite sets, a version of this result with dual norms follows from Theorem 7.1 in Dax and Sreedharan (1997). Their result does not require that P and Q_i be elements of $\Delta(X)$, thus establishing a more general finite version of the Coherence Theorem.

a decision maker or social planner, or simply of the group, about the likelihoods of events. An element Q of Q is to be interpreted as the probabilistic likelihood assessment of a member of the group of individuals about the events in Σ_X . Depending on the application, these members can be viewed either as members of the society, or as experts or specialists.¹¹

Our first characterization of the case where the probability P is sufficiently close to an element of the closed convex hull of the set Q relies on a condition that resembles the no arbitrage assumption in de Finetti's Coherence Theorem. Two equivalent versions of such condition guarantee that P is nearly represented as the linear average of the elements of Q. This is our next proposition.

Proposition 3. Suppose that $\mathcal{P} = \{P\} \subseteq \Delta(X)$, and let \mathcal{Q} be a nonempty and weak* compact subset of $\Delta(X)$. Let $\epsilon > 0$. There exists a countably additive (Borel) signed measure e on the compact metric space X, and $m \in \Delta(\mathcal{Q})$, such that

$$P = Q_m + e, \quad with \ \|e\|_1 \le \epsilon \tag{17}$$

if, and only if, any of the following two equivalent conditions is satisfied.

(i) For all
$$f \in C(X)$$
 with $||f||_{\infty} = 1$, $\int_X f dP \le \max_{Q \in \mathcal{Q}} \int_X f dQ + \epsilon$.

(ii) For all
$$f \in C(X)$$
 with $||f||_{\infty} = 1$, $\int_X f dP \ge \min_{Q \in \mathcal{Q}} \int_X f dQ - \epsilon$.

Proposition 3 reveals that the conditions ensuring that a probability is close to a linear averaging of probabilities resemble the conditions of bounded arbitrage opportunities in the approximate version of de Finetti's theorem described at the end of Section 3. For interpretation, if the continuous function f in item (i) of Proposition 3 represents the payoffs contingent on the states, and each probability $Q \in \mathcal{Q}$ encodes the opinion of an expert or better informed party about the odds of the states, the assertion in item (i) linking the elements of \mathcal{Q} to P says that, except perhaps for a small error ϵ , the expected value of the normalized payoff vector f according to P cannot exceed the maximum expected payoff according to the opinions of the members of the group. Item (ii) can be interpreted similarly.

Regarding the limiting cases of Proposition 3, note first that the relevant range for the number ϵ is the interval (0,2). For concreteness, for any fixed $Q \in \mathcal{Q}$, we can write P = Q + e, where e = P - Q has norm $||e||_1 \leq 2$. At the same time, the assertion in item (i) in Proposition 3 trivially holds for any $\epsilon \geq 2$ since

$$\int_{X} f \, dP - \int_{X} f \, dQ \le \|f\|_{\infty} \|P - Q\|_{1} \le 2$$

¹¹Compactness of the set Q can be justified when, for instance, the individuals in the group are indexed by $i \in I$, with I denoting a compact metric space such as a finite set or the closed unit interval, and a continuous mapping $i \mapsto Q_i$ associates to each individual in the group her or his assessment of the odds of the states.

for any $Q \in \mathcal{Q}$. And for values of ϵ close to 0, Proposition 3 has as a straightforward corollary the standard condition for linear aggregation of probabilities with reference to de Finneti's notion of coherence. This is the same as the condition in item (i) of Proposition 3 with $\epsilon = 0$, which is certainly equivalent to the mentioned item for all $\epsilon > 0$ arbitrarily small. In fact, set $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$, and find a sequence (m_n) of measures in $\Delta(\mathcal{Q})$ such that $d_1(P, Q_{m_n}) \leq \frac{1}{n}$. By a standard compactness argument involving the weak* topology and the lower semicontinuity of the ℓ_1 norm, we can find $m \in \Delta(\mathcal{Q})$ such that $P = Q_m$. More in general, our condition with $0 < \epsilon < 2$ is compatible with mild violations of coherence.¹²

We also characterize below the almost linear aggregation rule in (17) with a version of the standard Pareto unanimity condition. This is defined next.

Definition 1 (Condition C_{ϵ}). Let $\epsilon \geq 0$. We say that $P \in \Delta(X)$ and $Q \subseteq \Delta(X)$ satisfy the condition C_{ϵ} when for all $f, g \in C(X)$: if $\int_X f dQ \geq \int_X g dQ$ for all $Q \in Q$ then $\int_X f dP \geq \int_X g dP - \frac{\epsilon \cdot \omega(f-g)}{2}$.

Condition \mathbf{C}_{ϵ} reduces to the standard notion of Pareto unanimity about the ranking of the payoff vectors f and g if $\epsilon = 0$. Our notion of approximately Pareto unanimity for a positive ϵ has a correction factor of $\frac{\omega(f-g)}{2}$ in order to account for scaling effects. For instance, if we omit the oscillation of the difference f - g from the last inequality in Definition 1 its assertion would coincide with the standard form of Pareto unanimity.¹³

More important, while the original Pareto condition with $\epsilon=0$ would conclude with the optimality of f when choosing from the set $\{f,g\}$, Definition 1 is less demanding and requires that f be nearly optimal when compared with g. This weaker requirement of approximate optimality is justified, for instance, as long as the decision maker with assessment P of the probability of the events is uncertain about the true opinions of the members of the group, and leaves some room for probabilities that could have been omitted from the set \mathcal{Q} and would weigh in favor of g. For payoff vectors f and g that are sufficiently close, meaning that $||f-g||_{\infty} \leq 1$, a consequence of condition \mathbf{C}_{ϵ} is that for the decision maker f is an ϵ -optimal choice from the set $\{f,g\}$ in the sense of Radner (1980) since in this case we also have $\int_X f \, dP \geq \int_X g \, dP - \epsilon$. Somewhat similar to the justification given in Radner (1980) for such a notion of near optimality, the parameter ϵ could reflect the difficulty the decision maker faces when gauging the preferences of the group, and is thus associated with the costs of discovering the optimal choices based on the opinions of the members of the group.

To illustrate this remark, consider Example 2 in Nielsen (2019), where $X = \{1, 2, 3\}$, $P = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and $Q = \{Q_1, Q_2\}$, with $Q_1 = (\frac{2}{3}, \frac{1}{3}, 0)$ and $Q_2 = (\frac{1}{3}, \frac{2}{3}, 0)$. This example reveals a violation of coherence but, as is easily verified, it is compatible with the relation in (17) when $\epsilon \geq \frac{2}{3}$.

¹³To see this, just multiply each of the functions f and g by $n \in \mathbb{N}$, and take the limit in the resulting inequalities divided by n as $n \to \infty$.

The next proposition characterizes the aggregation by nearly averaging with the weaker notion of Pareto unanimity just described.

Proposition 4. Suppose that \mathcal{P} , \mathcal{Q} and ϵ are as in the statement of Proposition 3. There exist $e \in \operatorname{ca}(X)$ and $m \in \Delta(\mathcal{Q})$ such that the relation in (17) holds if, and only if, P and \mathcal{Q} satisfy condition \mathbf{C}_{ϵ} .

To illustrate the empirical content of Proposition 4 we consider in the next example the aggregation of priors in an Anscombe-Aumann setting with state independent utilities.

Example 2. Each individual i = 1, ..., N in the group has preferences \succeq_i over the Cartesian product \mathcal{C}^M , where \mathcal{C} is a nonempty convex subset of a vector space, and M is a positive integer. The set C represents the set of consequences. We identify C^M with the set of all mappings from the M-element set X to C, and thus interpret X as a set of objective states and \mathcal{C}^M as the set of Anscombe-Aumann acts. Preferences of the individuals, \succcurlyeq_i , and of the decision maker, \geq_0 , satisfy the usual Anscombe-Aumann axioms. We assume further that preferences over consequences are the same among the individuals and the decision maker, so we let them be expressed by a non-constant and affine function $u: \mathcal{C} \to \mathbb{R}$ where $u(c_0) = 0$ for some consequence c_0 . We also view c_0 as the constant act with consequence c_0 in each state. Therefore, there are priors P of the decision maker, and Q_i of the members of the group such that the preference relation \succeq_0 over acts c is represented by $c \mapsto \sum_{x \in X} P(x)u(c(x))$, whereas \succeq_i is represented by $c \mapsto \sum_{x \in X} Q_i(x)u(c(x))$. In this setting, the Pareto condition ensuring that P is nearly the weighted average of the probabilities Q_i says that, if $c \succcurlyeq_i c_0$ for all $i=1,\ldots,N$, then $\frac{2}{2+\epsilon}c+\frac{\epsilon}{2+\epsilon}c^* \succcurlyeq_0 \frac{2}{2+\epsilon}c_0+\frac{\epsilon}{2+\epsilon}c_*$, where c^* is the best consequence in the act c, and c_* represents the worst consequence in c.

In connection with the results in Section 3, we now give a single-profile version of an aggregation rule proposed by Genest (1984).¹⁴ It consists of expressing P as the convex combination of a linear averaging of the probabilities in \mathcal{Q} and an extraneous probability measure $R \in \Delta(X)$. That is,

$$P(E) = (1 - \epsilon)Q_m(E) + \epsilon R(E) \quad \text{for all } E \in \Sigma_X,$$
(18)

for some $m \in \Delta(Q)$, and $0 \le \epsilon \le 1$. The limiting case $\epsilon = 0$ corresponds to the standard linear pool of opinions, whereas the case $\epsilon = 1$ posits no relation between the probability of

¹⁴As mentioned in footnote 4, Genest's (1984) result appeared in the context of extending a result of McConway (1981), who characterized the linear pool of opinions in a multi-profile setting with two properties. The omission of one of these conditions, the so-called zero-probability property, leads to the aggregation rule of Genest in the original setting.

the decision maker and the opinions of the members of the group. In general, the expression in (18) for $\epsilon \in [0, 1]$ is characterized by the following version of the Pareto condition given in Definition 1.

Definition 2 (Condition \mathbf{C}_{ϵ}^*). Let $\epsilon \geq 0$. We say that $P \in \Delta(X)$ and $\mathcal{Q} \subseteq \Delta(X)$ satisfy the condition \mathbf{C}_{ϵ}^* when for all $f, g \in C(X)$: if $\int_X f dQ \geq \int_X g dQ$ for all $Q \in \mathcal{Q}$ then $\int_X f dP \geq \int_X g dP - \epsilon [\omega(f-g) - \max_{x \in X} (f(x) - g(x))]$.

A few remarks are in now order. Like condition \mathbf{C}_{ϵ} , the version of Pareto unanimity given in Definition 2 says that, except perhaps for a term depending on ϵ , the unanimous comparison of expected values of payoff vectors f and g according to the elements of Q is followed by the comparison of those payoffs according to P. But in contrast to the modified Pareto condition in Definition 1, we have a different form for the new term when comparing expected values. Moreover, since we cannot have $\max_{x \in X} (f(x) - g(x)) < 0$ in Definition 2, for otherwise $\int_X f dQ < \int_X g dQ$, we see that condition \mathbf{C}_{ϵ}^* implies condition $\mathbf{C}_{2\epsilon}$. This is not surprising since any probability measure P expressed as in (18) can be also written as the sum $Q_m + e$, where $e = \epsilon(R - Q_m)$ has norm no greater than 2ϵ .

The next proposition characterizes the aggregation rule \grave{a} la Genest with our new version of Pareto unanimity.

Proposition 5. Suppose that \mathcal{P} and \mathcal{Q} are as in the statement of Proposition 3. Let $\epsilon \in [0, 1]$. There exist $R \in \Delta(X)$ such that the relation in (18) holds if, and only if, P and \mathcal{Q} satisfy condition \mathbf{C}_{ϵ}^* .

As a special case of our first aggregation result for probabilities, we also consider P and the elements of Q arising as subjective probabilities in the setting of Savage (1954). Here P and each $Q \in Q$ are nonatomic.¹⁵ We also assume that Q is a finite set of size N. According to a result due to Mongin (1995), a condition that is viewed as a form of consistency between P and the set $\{Q_1, \ldots, Q_N\}$ when comparing the relative likelihood of events is both necessary and sufficient for P to be the linear pooling of the Q_i 's. The probabilistic analogue of our modified Pareto condition can be defined as follows.

Definition 3 (Condition $\mathbf{C}_{\epsilon}^{M}$). Let $\epsilon \geq 0$. We say that $P \in \Delta(X)$ and $\{Q_{1}, \ldots, Q_{N}\} \subseteq \Delta(X)$ satisfy the condition $\mathbf{C}_{\epsilon}^{M}$ when for all $E_{1}, E_{2} \in \Sigma_{X}$: if $Q_{i}(E_{1}) \geq Q_{i}(E_{2})$ for all $i = 1, \ldots, N$, then $P(E_{1}) \geq P(E_{2}) - \epsilon$.

¹⁵Recall that a probability measure $R \in \Delta(X)$ is nonatomic if for all $E_1 \in \Sigma_X$ with $R(E_1) > 0$ there exists $E_2 \in \Sigma_X$ such that $0 < R(E_2) < R(E_1)$.

Our condition $\mathbf{C}_{\epsilon}^{M}$ for the case $\epsilon = 0$ coincides with the consistency condition C_{1} in Mongin (1995). This case is the analogue of Pareto unanimity when comparing the probability of events, or equivalently when accessing the optimality of binary bets. These are the bets on events in Σ_{X} with a payoff of one if the event obtains, and zero otherwise. More precisely, when compared with condition \mathbf{C}_{ϵ} under the assumption that $\epsilon = 0$, condition $\mathbf{C}_{\epsilon}^{M}$ also has an interpretation in terms of bets: replace the set C(X) in Definition 1 with the set of all indicator functions of (Borel) measurable subsets of X. Hence, as long as each probability in \mathcal{Q} ranks a bet on E_{1} above a bet on E_{2} , Definition 3 says that according to P the bet on E_{1} is also ranked above the bet on E_{2} . The novelty in Definition 3 with $\epsilon > 0$ is to allow some room for the bet on E_{1} to be worse than the bet on E_{2} with respect to their expected values. But we require $P(E_{1})$ to be in the closed interval $[P(E_{2}) - \epsilon, P(E_{2})]$, so the bet on E_{1} cannot be too inferior when compared with the other bet.

The next proposition characterizes an almost linear aggregation of probabilities in the case of nonatomic measures.

Proposition 6. Suppose that \mathcal{P} , \mathcal{Q} and ϵ are as in the statement of Proposition 3. Assume further that \mathcal{Q} has size N, and that P as well as the elements of \mathcal{Q} are nonatomic. There exist $e \in \operatorname{ca}(X)$ and $m_i \geq 0$, for all $i = 1, \ldots, N$, such that $P = \sum_{i=1}^{N} m_i Q_i + e$, with $\|e\|_1 \leq \epsilon$, if, and only if, P and \mathcal{Q} satisfy condition \mathbf{C}_{ϵ}^M .

Our Proposition 6 is an extension of part of Proposition 2 in Mongin (1995). It differs from Mongin's result in two aspects. First, we assume from the outset that all the probabilities involved are nonatomic, whereas Proposition 2 in Mongin (1995) only requires that the elements of \mathcal{Q} be nonatomic. Second, whereas Mongin (1995) obtains the exact relation $P = \sum_{i=1}^{N} m_i Q_i$ for a list of nonnegative Pareto weights m_i whose sum is one, the expression for P in our Proposition 6 only entails that $1 - \epsilon \leq \sum_{i=1}^{N} m_i \leq 1 + \epsilon$. In spite of their differences, it is not difficult to see that we can recover Mongin's result as a limiting case when ϵ goes to zero.¹⁷

Finally, we can give a sharper version of Proposition 6 with the more general conditions for aggregation mentioned in Section 3. Our next proposition characterizes a nearly linear aggregation rule for nonatomic measures with normalized Pareto weights using conditions that are similar to those in Proposition 3.

¹⁶ At this time we are also unaware of an example of a representation as in Proposition 6 never holding with $\sum_{i=1}^{N} m_i = 1$.

¹⁷Take $\epsilon = \frac{1}{n}$ and find a sequence of nonnegative and bounded Pareto weights (m_n) in which the representation holds accordingly for each n. In the limit, $P(E) = \lim_{n \to \infty} \sum_{i=1}^{N} m_{n,i} Q_i(E) + e_n(E)$, where $||e_n||_1 \leq \frac{1}{n}$, after passing to a convergent subsequence of (m_n) if needed.

Proposition 7. Suppose that \mathcal{P} , \mathcal{Q} and ϵ are as in the statement of Proposition 6. There exist $e \in \operatorname{ca}(X)$ and $m_i \geq 0$, for all $i = 1, \ldots, N$, with $\sum_{i=1}^{N} m_i = 1$ such that $P = \sum_{i=1}^{N} m_i Q_i + e$, where $\|e\|_1 \leq \epsilon$, if, and only if, any of the following two equivalent conditions is satisfied.

- (i) For all $E \in \Sigma_X$, $P(E) \le \max_i Q_i(E) + \frac{\epsilon}{2}$.
- (ii) For all $E \in \Sigma_X$, $P(E) \ge \min_i Q_i(E) \frac{\epsilon}{2}$.

4.2 Nearly random utility maximization

In the case of a finite set X the results obtained in Section 3 can be given the interpretation of representing a stochastic choice function as if it were nearly generated by the random utility maximization model of Block and Marschak (1960). This is done here by a suitable definition of the set X, and of the elements of \mathcal{P} and \mathcal{Q} .

The setting consists of a grand set \mathcal{Y} of alternatives of size N_0 , and a family \mathcal{D} of subsets of Y. This family represents the choice problems the individual may choose from, and is composed of all nonempty subsets of \mathcal{Y} .¹⁸ A stochastic choice function is a mapping $P_0: \mathcal{D} \to \mathbb{R}$ such that

$$P_0(y,Y) \ge 0$$
, $\sum_{y \in Y} P_0(y,Y) = 1$, for all $Y \in \mathcal{D}$.

Denote by \mathcal{L} the set of all strict preference orderings on \mathcal{Y} , namely, all those relations $\succ \subseteq \mathcal{Y} \times \mathcal{Y}$ that are irreflexive, transitive and satisfy the property that for any two distinct elements y and y' of \mathcal{Y} either $y \succ y'$ or $y' \succ y$. Let $M = N_0 \cdot 2^{N_0 - 1}$, and $N = N_0!$. Define the $M \times N$ matrix A so that each row a(y, Y) has in its column corresponding to $\succ \in \mathcal{L}$ the value

$$a_{\succ}(y,Y) = \begin{cases} 1, & \text{if } \succ \in S(y,Y) \\ 0, & \text{otherwise,} \end{cases}$$

where $S(y,Y) = \{ \succ \in \mathcal{L} : y' \succ y \text{ for no } y' \in Y \}$ is the set of all strict preferences supporting the optimality of y in Y.

The class of all stochastic choice functions that follow the random utility maximization model is the convex hull of the columns of A, which is known as the multiple choice polytope. To describe our approximate version of the model, we rephrase the problem in terms of two

The assumption that $\mathcal{D} = 2^{\mathcal{Y}} \setminus \{\emptyset\}$ is inessential and can be easily relaxed. We keep it for simplicity.

sets of probabilities as in Section 3. Let $X = \{(y, Y) : Y \in \mathcal{D}, y \in Y\}$. The set X has M elements and is thus a compact metric space. It is convenient to have X enumerated as

$$X = \{(y_1, Y_1), (y_2, Y_2), \dots, (y_M, Y_M)\}.$$
(19)

Define $P \in \Delta(X)$ by $P(y,Y) = \frac{P_0(y,Y)}{2^{N_0}-1}$, and for each $\succ \in \mathcal{L}$ the probability measure Q_{\succ} on X by $Q_{\succ}(y,Y) = \frac{a_{\succ}(y,Y)}{2^{N_0}-1}$. The probability P can be viewed as an extended lottery where P(y,Y) represents the joint probability of facing the choice problem Y and choosing the alternative $y \in Y$ from it. A similar interpretation applies to Q_{\succ} .

The stochastic choice function P_0 belongs to the multiple choice polytope if, and only if, the sets $\mathcal{P} = \{P\}$ and the convex hull of the finite set $\mathcal{Q} = \{Q_{\succ} \in \Delta(X) : \succ \in \mathcal{L}\}$ intersect. This the same as saying that, for some $\pi \in \Delta(\mathcal{L})$,

$$P_0 = A\pi. (20)$$

The probability measure π is referred to as a stochastic preference for the obvious reasons.

McFadden and Richter (1990) characterize such an intersection with ARSP. We will show that, more in general, a modification of ARSP characterizes when the system of choice probabilities is ϵ -close to the multiple choice polytope, and instead of the expression in (20), we have

$$P_0 = A\pi + e \tag{21}$$

for some vector $e \in \mathbb{R}^M$ with $\|e\|_1 \leq \epsilon$. With reference to P and Q, the expression in (21) means that $d_1(P, Q_\pi) \leq \frac{\epsilon}{2^{N_0}-1}$ for some $Q_\pi = \sum_{\succ \in \mathcal{L}} \pi(\succ) Q_{\succ} \in \operatorname{co} Q$.

The next example, borrowed from McFadden and Richter (1990), illustrates the representation in (21).

Example 3. Consider $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and a stochastic choice function P_0 where alternatives have an equal probability of being chosen in choice problems of sizes 1, 2 and 4. For choice problems of size 3, the alternative y_{\min} with the smallest index has a slightly better chance of being chosen. Denoting by |Y| the number of elements of Y,

$$P_0(y,Y) = \begin{cases} \frac{1}{|Y|}, & \text{if } |Y| \in \{1,2,4\} \\ \frac{4}{10}, & \text{if } |Y| = 3, y = y_{\min} \\ \frac{3}{10}, & \text{otherwise.} \end{cases}$$

This stochastic choice function cannot be rationalized as random utility maximization (see McFadden and Richter (1990) for the calculations). At the same time, it is not too far from

a stochastic function \tilde{P}_0 where choices are equally likely in any choice problem. In particular,

$$P_0(y,Y) = \begin{cases} \tilde{P}_0(y,Y) - \frac{1}{15}, & if \ y = y_{\min} \\ \tilde{P}_0(y,Y) + \frac{1}{30}, & otherwise \end{cases}$$

for a 3-element set Y. Hence P_0 can be represented as in (21) when $\epsilon = \frac{8}{15}$. It is much less obvious, though, that P_0 can also be expressed as in (21) with $\epsilon = \frac{1}{10}$. This bound is achieved when $\tilde{P}_0(y,Y) = \sum_{\succ \in S(y,Y)} \tilde{\pi}(\succ)$, with $\tilde{\pi}$ suitably chosen.¹⁹

Before proceeding we need a few definitions.

Definition 4. Given the enumeration of X in (19), a tagged trial sequence T is a sequence of finite and fixed length M of the form

$$T = ((y_1, Y_1, t_1), (y_2, Y_2, t_2), \dots, (y_M, Y_M, t_M)),$$

where the t_i are nonnegative integers. The width w_T of a tagged trial sequence is the number $\max_i t_i - \min_i t_i$.

In the original definition of McFadden and Richter (1990) and McFadden (2005), a trial sequence makes reference to a finite list of pairs $(y, Y) \in X$ where repetitions are allowed. To keep track of the greatest and the least number of repetitions in a trial sequence, we use the t_i that are mentioned in our definition of a tagged trial sequence.²⁰ In any case, a tagged trial sequence can be interpreted as a vector of payoffs attaching to each $(y_i, Y_i) \in X$ a nonnegative integer t_i . The expected payoff according to the probability P on X is the value $\sum_{i=1}^{M} P(y_i, Y_i)t_i$. Likewise, each element Q_{\succ} of Q also induces an expected value, which is associated with a column of A. In its original version in McFadden and Richter (1990), ARSP says that the expected payoff according to P of any tagged trial sequence cannot exceed the maximum expected payoff computed with probabilities on X induced by non-stochastic choice functions. That is,

$$\sum_{i=1}^{M} P(y_i, Y_i) t_i \le \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} Q_{\succ}(y_i, Y_i) t_i.$$
(22)

¹⁹A particular choice of a stochastic preference has $\tilde{\pi}(y_i \succ y_j \succ y_k \succ y_l) = \frac{1}{20}$ when $ijkl = 1342, 2341, 3124, 3142, 3241, 4123, 4132, 4231, 4321, <math>\tilde{\pi}(y_i \succ y_j \succ y_k \succ y_l) = \frac{2}{20}$ when ijkl = 1432, 2143, 2431, 3421, and $\tilde{\pi}(y_1 \succ y_2 \succ y_3 \succ y_4) = \frac{3}{20}$. This fact can be verified with the linear program mentioned in the last paragraph of Section 5 below.

²⁰Technically, like in Section 4.1, we need to make reference to the oscillation of certain functions in order to characterize the inexact form of aggregation. This is achieved by keeping track of the maximum and minimum number of repetitions using the tags.

This is the next definition, where we use the original stochastic choice function instead of P, and the matrix A in place of the set Q.²¹

Definition 5 (ARSP). The stochastic choice function P_0 satisfies ARSP if, for every tagged trial sequence T,

$$\sum_{i=1}^{M} P_0(y_i, Y_i) t_i \le \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i.$$
(23)

Our version of ARSP, which we will refer to as ϵ -ARSP, is a weakening of the condition in (23). In terms of the probabilities over pairs (y, Y), it consists of adjusting the right-hand side of (22) in order to accommodate deviations from the original inequality. This is similar to what is done in Corollary 2 in Section 3.

Definition 6 (ϵ -ARSP). Let $\epsilon \geq 0$. The stochastic choice function P_0 satisfies ϵ -ARSP if, for every tagged trial sequence T,

$$\sum_{i=1}^{M} P_0(y_i, Y_i) t_i \le \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i + \frac{w_T \epsilon}{2}. \tag{24}$$

In practice, ϵ -ARSP is a weakening of ARSP that accommodates examples of stochastic functions that are nearly rationalizable by random utility maximization as in Example 3. We note, though, as an illustration of the content of such an axiom, that for $\epsilon < 2$ it has the same implication as ARSP for non-stochastic single-valued choice functions with respect to the Weak Axiom of Revealed Preference. This observation follows from the next example.²²

Example 4. Suppose that a deterministic choice function dictates that y_1 is chosen from Y_1 , and $y_2 \in Y_1$. If the weak axiom fails, for some Y_2 also containing y_1 we have y_2 as the choice from it. Now consider the tagged trial sequence T where, with a slight abuse of notation, $t(y_1, Y_1) = t(y_2, Y_2) = 1$, and t(y, Y) = 0 otherwise. Since each $\succ \in \mathcal{L}$ is transitive and irreflexive, we cannot have $a_{\succ}(y_1, Y_1) = a_{\succ}(y_2, Y_2) = 1$. Hence, using ϵ -ARSP we get that

$$2 = \sum_{(y,Y)\in X} P_0(y,Y)t(y,Y) \le \max_{\succ \in \mathcal{L}} \sum_{(y,Y)\in X} a_{\succ}(y,Y)t(y,Y) + \frac{\epsilon}{2} < 2,$$

which is impossible.

²¹See Border (2007) and the references therein for a discussion of ARSP.

 $^{^{22}}$ The example can be easily extended to the more general case in which the set \mathcal{D} of choice problems is not comprehensive, whereby the choice function satisfies the Strong Axiom of Revealed Preference. By the strong axiom we mean that the revealed (strict) preference relation is acyclic. See McFadden and Richter (1990, pp. 162-165).

We can now state the first characterization of an approximate version of the random utility maximization model.

Proposition 8. The stochastic choice function P_0 satisfies ϵ -ARSP if, and only if, there exist $e: X \to \mathbb{R}$ and $\pi \in \Delta(\mathcal{L})$ such that

$$P_0(y,Y) = \sum_{\succ \in S(y,Y)} \pi(\succ) + e(y,Y),$$
 (25)

where e, viewed as an element of \mathbb{R}^M , has norm $\|e\|_1 \leq \epsilon$.

A second representation of nearly random utility maximization can be given with the notion of residual behavior of Apesteguia and Ballester (2021). It is a weighted combination of two terms. One stands fo the exact representation where the extended vector of choice probabilities is given as in the right-hand side of (20). It is interpreted as the predicted randomness expected to be found in the choice data. The other term is the component $R_0 \in \mathbb{R}^M$, where $R_0(y,Y) \geq 0$ and $\sum_{y \in Y} R_0(y,Y) = 1$ for all $Y \in \mathcal{D}$. When combined with the random utility maximization part, it leads to the expression

$$P_0 = (1 - \epsilon)A\pi + \epsilon R_0. \tag{26}$$

Comparing with (21), the relation in (26) reveals that a fraction of the data is explained by the random utility maximization model through a stochastic preference. This is the $(1-\epsilon)A\pi$ part. The residual component, namely ϵR_0 , is composed of what is referred to in Apesteguia and Ballester (2021) as unstructured residual behavior. In our setting this residual refers to the portion of the data that is not explained by random utility maximization.

The random utility maximization model with the residual behavior just mentioned is characterized by a version of ϵ -ARSP. It is defined next.

Definition 7 (ϵ -ARSP*). Let $\epsilon \in [0,1]$. The stochastic choice function P_0 satisfies ϵ -ARSP* if, for every tagged trial sequence T,

$$\sum_{i=1}^{M} P_0(y_i, Y_i) t_i \le (1 - \epsilon) \max_{r \in \mathcal{L}} \sum_{i=1}^{M} a_r(y_i, Y_i) t_i + (2^{N_0} - 1) \epsilon \max_i t_i.$$
 (27)

Note that any stochastic choice function represented as in (26) can be also expressed as in (21) with $e = \epsilon (R_0 - A\pi)$. Here, $||e||_1 = \epsilon ||R_0 - A\pi||_1 \le (2^{N_0} - 1)\epsilon$. In fact, any stochastic choice function satisfying ϵ -ARSP* also satisfies $\bar{\epsilon}$ -ARSP with $\bar{\epsilon} = (2^{N_0+1} - 2)\epsilon$. More important, as the proposition below shows, ϵ -ARSP* characterizes the representation with residual behavior for the random utility maximization model.

Proposition 9. The stochastic choice function P_0 satisfies ϵ -ARSP* if, and only if, there exist $\pi \in \Delta(\mathcal{L})$, and $R_0 \in \mathbb{R}^M$ with $R_0(y,Y) \geq 0$ and $\sum_{y \in Y} R_0(y,Y) = 1$ for all $Y \in \mathcal{D}$, such that

$$P_0(y,Y) = (1 - \epsilon) \sum_{\succ \in S(y,Y)} \pi(\succ) + \epsilon R_0(y,Y).$$

5 Concluding remarks

Many important results in economics rest on a separating hyperplane theorem. We gave approximate versions to two classical forms of rationality found in the literature. To this end, we provided characterizations of how close a pair of elements of two compact sets of probabilities are with reference to expected values of normalized functions, and also interpreted the conditions in terms of arbitrage opportunities. For the most part we turned standard arguments involving hyperplane separation into the problem of finding conditions under which two closed and convex sets are not too disjoint. We have shown that forms of nearly rational behavior obtain if, and only if, those two specific sets have points not too far from each other. The extent of departure from the classical forms of rational behavior addressed in this paper is directly proportional to the number ϵ mentioned in the text. A natural generalization of the results obtained here would be to have conditions that do not explicitly reference the number ϵ .

Also in relation to the more general question studied in Section 3 of this paper, Hellman (2013) was the first to employ techniques similar to the ones used here to prove a related result about nearly common priors. By relying on the idea of turning the non-existence of a common prior into the possibility of strictly separating two compact and convex sets as in Samet (1998), and using the minimum norm duality theorem given in Dax (2006), Hellman (2013) shows that a condition of absence of bets with expected gains for each player greater than ϵ is equivalent to their priors having a suitably defined distance that does not exceed ϵ . His setting shares a similar mathematical structure with the problem of aggregation of probabilities in the present paper, and we suspect that our results could be adapted in order to expand the framework of a finite state space in Hellman (2013). Such an extension would involve further mathematical complications and is beyond the scope of this paper.²³

The connection between the general results presented in Section 3 above and the Coherence Theorem of de Finetti suggests that other forms of absence of arbitrage opportunities can be weakened in order to allow for approximate versions of linear pricing rules. Indeed, Acciaio et al. (2022) have recently shown that approximately arbitrage-free asset prices are near

²³A recent attempt at such an extension can be found in Hellman and Pintér (2022).

a linear pricing rule. In this respect, the main difference (at least in the finite-dimensional setting) between the technical aspects of their result and the results reported in Section 3 of this paper is that while we obtain an approximate version of Gordan's Theorem of the Alternative, they establish such a version for Stiemke's Alternative. We suspect that the techniques involved in both papers can be also applied to characterize certain nonlinear pricing rules arising in the context of financial markets with frictions, such as the ϵ -contamination pricing rule as described, for instance, in Araujo et al. (2012, 2018) and Chateauneuf and Cornet (2022).

Regarding our solution to the problem of aggregating probabilities through a nearly linear averaging procedure, we characterized the single-profile case of the aggregation rule of Genest (1984) with a weaker form of Pareto unanimity. Genest (1984) originally considered the multi-profile setting, where the aggregation rule is invariant under changes of the probabilities of the members of the group. Namely, if we refer to I as the set of individuals (a compact metric space) in the group, the version of Genest's result translates into the existence of a probability measure m on I, and a probability measure R on X, such that $P_Q(E) = (1-\epsilon) \int_I Q_i(E) dm(i) + \epsilon R(E)$ for all $E \in \Sigma_X$ and continuous profiles of probabilities $i \mapsto Q_i$. In the setting with a finite number of individuals it is not difficult to see that preserving the marginalization property while adding a monotonicity condition and controlling violations of the zero-probability property by $\epsilon > 0$ gives a multi-profile representation of our Proposition $5.^{24}$ We suspect that, by adapting the arguments in Nielsen (2019), we can obtain a multi-profile version of our Proposition 5 under our assumptions about X, the continuity of the profiles $i \mapsto Q_i$, and compactness of I. This would at the same time refine and extend Genest's (1984) characterization of the marginalization property.

Our results about the representation of stochastic choice functions relied on versions of the McFadden-Richter axiom permitting the rationalization of choice probabilities by random utility maximization. Like the McFadden-Richter axiom, the axioms ϵ -ARSP and ϵ -ARSP* introduced in this paper involve infinitely many restrictions. Therefore, in general we are also interested in weaker forms of the (finitely many) conditions involving the Block-Marschak (BM) polynomials used by Falmagne (1978). With the notation of Section 4.2, and especially in connection with our Proposition 8, it consists of employing the double description of the stochastic choice functions P_0 following the random utility maximization model as

$$\{A\pi : \pi \in \Delta(\mathcal{L})\} = \{P_0 \in \mathbb{R}^{N_0} : BP_0 \ge 0\},$$
 (28)

for some square matrix B of size N_0 representing the BM polynomials, to derive the ap-

 $^{^{24}}$ The monotonicity condition is needed because in the original representation of Genest (1984) the weights m(i) are not necessarily nonnegative except perhaps in some particular cases.

proximate form of the representation. By well-known results on approximate solutions to systems of linear inequalities due to Hoffman (1952) and Güler et al. (1995), for a fixed P_0 there exists a constant K > 0, depending only on B, such that for some $\pi \in \Delta(\mathcal{L})$ we have

$$||P_0 - A\pi||_1 \le K ||(BP_0)^-||_1$$

where $(BP_0)_i^- = \max\{-(BP_0)_i, 0\}$. This shows that bounding the violations of the BM inequalities in the definition of the set in the right-hand side of (28) allows us to bound the distance of a stochastic function to the family of random utility maximization models. We suspect that by using a scaled version of the parameter ϵ in the inequalities involving the BM polynomials we can also obtain an error term with $\|e\|_1 \leq \epsilon$ as in Proposition 8. At this point we do not have such a characterization. Yet, like in the random utility maximization model (see McFadden and Richter, 1990, pp. 182-183), we can also turn the problem of deciding whether choice probabilities are nearly generated by a random utility maximization model according to (25) into a linear programming problem. This is done by writing the problem of minimizing $\|P_0 - A\pi\|_1$ subject to $\pi \geq 0$ and $\sum_{\succ \in \mathcal{L}} \pi(\succ) = 1$ in equivalent form as the linear program

$$\min_{z,\pi} \sum_{i=1}^{M} z_i$$
s.t. $-z_i \le P_0(y_i, Y_i) - \sum_{j=1}^{N} \pi_j a_{ij} \le z_i, i = 1, \dots, M$

$$\pi_j \ge 0, j = 1, \dots, N$$

$$\sum_{i=1}^{N} \pi_j = 1.$$

Checking ϵ -ARSP now involves a computational procedure that can be solved with well-known algorithms and the inspection of whether its output has $\sum_{i=1}^{M} z_i \leq \epsilon$.

A Proofs

A.1 Proof of Proposition 1

We first show that R_m is well defined for each $m \in \Delta(\mathcal{R})$. Since for each Borel set E the indicator function $\mathbf{1}_E$ defined on X, as given by

$$\mathbf{1}_{E}(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E, \end{cases}$$

is bounded, the mapping $R \mapsto \int \mathbf{1}_E dR = R(E)$ from \mathcal{R} to \mathbb{R} is Borel measurable in \mathcal{R} since it is the restriction of a Borel measurable mapping on $\Delta(X)$ (Theorem 15.13 in Aliprantis and Border, 2006). Because the mapping $R \mapsto R(E)$ is also bounded, it follows that for any $m \in \Delta(\mathcal{R})$ the value of $R_m(E)$, which is nonnegative, is well defined for any Borel subset E of X. Obviously, $R_m(X) = \int_{\mathcal{R}} \mathbf{1}_X dm = 1$. Now take any countable family $\{E_n : n \in \mathbb{N}\}$ of pairwise disjoint elements of Σ_X . For each k, define the function $f_k \colon \mathcal{R} \to \mathbb{R}$ by $f_k(R) = \sum_{n=1}^k R(E_n)$. By the properties of the Lebesgue integral we obtain that $\int_{\mathcal{R}} f_k dm = \sum_{n=1}^k R_m(E_n)$. Note that, since R is countably additive,

$$\lim_{k \to \infty} f_k(R) = \lim_{k \to \infty} \sum_{n=1}^k R(E_n) = R(\bigcup_{n=1}^\infty E_n)$$

Hence (f_k) is a bounded sequence of measurable functions that converges pointwise to the (measurable) function $f: \mathcal{R} \to \mathbb{R}$ given by $f(R) = R(\bigcup_{n=1}^{\infty} E_n)$. Using the Dominated Convergence Theorem (Theorem 11.21 of Aliprantis and Border, 2006) it follows that

$$R_m(\bigcup_{n=1}^{\infty} E_n) = \int_{\mathcal{R}} f \, dm = \lim_{k \to \infty} \int_{\mathcal{R}} f_k \, dm = \sum_{n=1}^{\infty} R_m(E_n),$$

thus establishing countable additivity of R_m . Therefore, $R_m \in \Delta(X)$.

To complete the proof, define the set of simple probability measures on \mathcal{R} :

$$\Delta_s(\mathcal{R}) = \left\{ m \in \Delta(\mathcal{R}) : m = \sum_{i=1}^k \lambda_i \delta_{R_i}, \exists \{R_1, \dots, R_k\} \subseteq \mathcal{R}, \lambda_i \ge 0, \sum_{i=1}^k \lambda_i = 1 \right\}.$$

Note that $\operatorname{co} \mathcal{R} = \{R_m \in \Delta(X) : m \in \Delta_s(\mathcal{R})\} \subseteq \widetilde{\mathcal{R}}$, where $\widetilde{\mathcal{R}} = \{R_m \in \Delta(X) : m \in \Delta(\mathcal{R})\}$. If $R_m \in \widetilde{\mathcal{R}}$, by the Density Theorem (Theorem 15.10 in Aliprantis and Border, 2006) there exists a sequence (m_n) in $\Delta_s(\mathcal{R})$ converging to m. For any closed set $A \subseteq X$ the indicator function $\mathbf{1}_E$ on X is upper semicontinuous. Hence, the function $R \mapsto R(E) = \int_X \mathbf{1}_E dR$ is also upper semicontinuous, and so is the mapping $m \mapsto R_m(E)$ (see Theorem 15.5 in Aliprantis and Border, 2006). Therefore, for any closed set $E \subseteq X$ we have

$$\lim_{n} \sup_{E} R_{m_n}(E) \le R_m(E), \tag{29}$$

thus showing that $R_{m_n} \stackrel{w*}{\longrightarrow} R_m$. This establishes the inclusion $\widetilde{\mathcal{R}} \subseteq \operatorname{cl}_{w*} \operatorname{co} \mathcal{R}$. It now suffices to show that $\widetilde{\mathcal{R}}$ is closed. If (R_{m_n}) is a sequence in $\widetilde{\mathcal{R}}$ converging to R, by weak* compactness of $\Delta(\mathcal{R})$ we may assume without loss of generality that (m_n) converges to some $m \in \Delta(\mathcal{R})$. By the same argument leading to (29), we have that (R_{m_n}) converges to R_m . By uniqueness of the limit we must have $R_m = R$. We conclude that

$$\overline{\operatorname{co}} \mathcal{R} = \operatorname{cl}_{w*} \operatorname{co} \mathcal{R} \subseteq \widetilde{\mathcal{R}} \subseteq \operatorname{cl}_{w*} \operatorname{co} \mathcal{R},$$

and thus $\overline{\operatorname{co}} \mathcal{R} = \{ R_m \in \Delta(X) : m \in \Delta(\mathcal{R}) \}.$

A.2 Proof of Theorem 1

The total variation norm of the difference of two probability measures P and Q can be written as the composition of the mappings $F \colon \Delta(X) \times \Delta(X) \to \operatorname{ca}(X)$, as given by F(P,Q) = P - Q, and $\|\cdot\|_1$. The latter is viewed as a real-valued function on $\operatorname{ca}(X)$. The Cartesian product $\Delta(X) \times \Delta(X)$ is equipped with the product topology. Hence the mapping F is continuous. Since $\|\cdot\|_1$, as the dual norm of the sup norm on C(X), is a lower semicontinuous function when $\operatorname{ca}(X)$ is endowed with the topology of weak convergence (see Lemma 6.22 in Aliprantis and Border, 2006), we conclude that $\|P - Q\|_1 = \|F(P,Q)\|_1$ defines a lower semicontinuous function on $\Delta(X) \times \Delta(X)$. In particular, the minimum of $\|P - Q\|_1$, with P ranging over $\overline{\operatorname{co}} \mathcal{P}$ and Q ranging over $\overline{\operatorname{co}} \mathcal{Q}$, is attained. This assertion follows from compactness of the set $\overline{\operatorname{co}} \mathcal{P} \times \overline{\operatorname{co}} \mathcal{Q}$, denoted \mathcal{R} , and the lower semicontinuity of $\|F(P,Q)\|_1$. By a similar argument, for each function $f \in C(X)$, the minimum of $\int_X f dP - \int_X f dQ$ with $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ is also attained. Using the definition of the ℓ_1 distance in Section 2, or simply the characterization of the total variation norm in terms of its duality with the sup norm, we obtain that

$$\min_{P \in \overline{\operatorname{co}} \, \mathcal{P}, Q \in \overline{\operatorname{co}} \, \mathcal{Q}} \|P - Q\|_1 = \min_{\boldsymbol{R} \in \mathcal{R}} \sup_{f \in B_{\|\cdot\|_{\infty}}(0,1)} G(f, \boldsymbol{R}),$$

where $G: B_{\|\cdot\|_{\infty}}(0,1) \times \mathcal{R} \to \mathbb{R}$ is the continuous function given by $G(f, \mathbf{R}) = \int_X f dP - \int_X f dQ$ for $\mathbf{R} = (P,Q)$. Note that G is affine in each variable, the closed unit ball $B_{\|\cdot\|_{\infty}}(0,1)$ of C(X) is convex, and \mathcal{R} is a compact and convex subset of $\Delta(X) \times \Delta(X)$. By the minimax theorem (Theorem N' in Kneser (1952), or Corollary 3.3 in Sion (1958)),

$$\min_{\mathbf{R} \in \mathcal{R}} \sup_{f \in B_{\infty}(0,1)} G(f, \mathbf{R}) = \sup_{f \in B_{\|\cdot\|_{\infty}}(0,1)} \min_{\mathbf{R} \in \mathcal{R}} G(f, \mathbf{R})$$

$$= \sup_{f \in B_{\|\cdot\|_{\infty}}(0,1)} \min_{P \in \overline{\infty} \mathcal{P}, Q \in \overline{\infty} \mathcal{Q}} \left(\int_{X} f \, dP - \int_{X} f \, dQ \right)$$

$$= \sup_{f \in B_{\|\cdot\|_{\infty}}(0,1)} \min_{P \in \mathcal{P}, Q \in \mathcal{Q}} \left(\int_{X} f \, dP - \int_{X} f \, dQ \right). \tag{30}$$

Note that the equality in (30) is easily verified since given any two sequences (P_n) and (Q_n) in co \mathcal{P} and co \mathcal{Q} , respectively, we have that

$$\min_{P \in \mathcal{P}, Q \in \mathcal{Q}} \int_{X} f \, dP - \int_{X} f \, dQ \le \int_{X} f \, dP_{n} - \int_{X} f \, dQ_{n}$$

because of linearity of the linear funcional induced by f in the space $\operatorname{ca}(X)$, so by weak convergence, upon letting $P_n \xrightarrow{w*} P$ and $Q_n \xrightarrow{w*} Q$ we obtain that $\min_{P \in \mathcal{P}, Q \in \mathcal{Q}} \int_x f \, dP - \int_x f \, dQ \leq \min_{P \in \overline{\operatorname{co}} \mathcal{P}, Q \in \overline{\operatorname{co}} \mathcal{Q}} \left(\int_x f \, dP - \int_x f \, dQ \right)$.

A.3 Proof of Corollary 1

If system I has \bar{f} as a solution, then for all $(P,Q) \in \mathcal{P} \times \mathcal{Q}$ we have

$$d_1(P,Q) = \sup_{f \in C(X), ||f||_{\infty} \le 1} \left(\int_X f \, dP - \int_X f \, dQ \right) \ge \int_X \bar{f} \, dP - \int_X \bar{f} \, dQ > \epsilon,$$

so system II has none. In view of Theorem 1, when system I has no solution, if we choose $\bar{P} \in \overline{\text{co}} \mathcal{P}$ and $\bar{Q} \in \overline{\text{co}} \mathcal{Q}$ that minimize the distance $d_1(P,Q)$ then we must have $d_1(\bar{P},\bar{Q}) \leq \epsilon$.

A.4 Proof of Corollary 2

Because of Corollary 1, the existence of probability measures $P \in \overline{\operatorname{co}} \mathcal{P}$ and $Q \in \overline{\operatorname{co}} \mathcal{Q}$ with ℓ_1 distance at most ϵ is equivalent to system I in Corollary 1 having no solution. This means that, after employing standard arguments involving continuity and compactness in the weak* topology, the inequality in (10) holds for any continuous function f of norm one.

A.5 Proof of Proposition 2

Suppose that for some $(P,Q) \in \overline{\operatorname{co}} \mathcal{P} \times \overline{\operatorname{co}} \mathcal{Q}$, and $R \in \mathcal{R}$, the expression in (11) holds. Let $f \in C(X)$ be such that $||f||_{\infty} = 1$. Apply a similar reasoning to that leading to (30) in the proof of Theorem 1 to obtain

$$\min_{\widehat{P} \in \overline{\infty} \mathcal{P}} \int_{X} f \, d\widehat{P} = \min_{\widehat{P} \in \mathcal{P}} \int_{X} f \, d\widehat{P}$$

$$\leq \int_{X} f \, dP$$

$$= (1 - \epsilon) \int_{X} f \, dQ + \epsilon \int_{X} f \, dR. \tag{31}$$

The condition in (12) now becomes a consequence of bounding the terms in the expression in (31): $\int_X f dQ \leq \max_{\widehat{Q} \in \overline{co} Q} \int_X f d\widehat{Q}$, and $\int_X f dR \leq \max_{\widehat{R} \in \mathcal{R}} \int_X f d\widehat{R}$.

Define $\widehat{\mathcal{Q}} = \{(1 - \epsilon)Q + \epsilon R : Q \in \overline{\operatorname{co}} \mathcal{Q}, R \in \mathcal{R}\}$, which is compact and convex. If $\overline{\operatorname{co}} \mathcal{P} \cap \widehat{\mathcal{Q}} = \emptyset$, by the Strong Separating Hyperplane Theorem (Theorem 5.79 in Aliprantis and Border, 2006) there exists a nonzero function $f \in C(X)$ such that

$$\min_{P \in \overline{\operatorname{co}}\,\mathcal{P}} \int_X f\,dP > \max_{\widehat{Q} \in \widehat{\mathcal{Q}}} \int_X f\,d\widehat{Q} = (1-\epsilon) \max_{Q \in \overline{\operatorname{co}}\,\mathcal{Q}} \int_X f\,dQ + \epsilon \max_{R \in \mathcal{R}} \int_X f\,dR.$$

Let $\sigma = \min_{x \in X} f(x)$ and $g = f - \sigma \mathbf{1}_X \ge 0$. Note that $||g||_{\infty} = \max_{x \in X} g(x)$. Combining this observation with the last inequality, we obtain that

$$\min_{P \in \overline{\operatorname{co}} \mathcal{P}} \int_X \widehat{g} \, dP > (1 - \epsilon) \max_{Q \in \overline{\operatorname{co}} \mathcal{Q}} \int_X \widehat{g} \, dQ + \epsilon \max_{R \in \mathcal{R}} \int_X \widehat{g} \, dR,$$

upon defining $\widehat{g} = \frac{g}{\|g\|_{\infty}}$, which has norm one.

A.6 Proof of Proposition 3

The following fact proves to be useful.

Fact A.6.1. For all $m \in \Delta(\mathcal{Q})$ and $f \in C(X)$,

$$\int_{X} f \, dQ_m = \int_{\mathcal{O}} \left(\int_{X} f \, dQ \right) dm. \tag{32}$$

Proof. First let $\Delta_s(Q)$ denote the set of simple probability measures on Q (the relevant definition is given in the proof of Proposition 1 above). For $m \in \Delta_s(Q)$ the equality $\int_X f \, dQ_m = \int_Q \left(\int_X f \, dQ \right) \, dm$ is a consequence of the linearity of the linear functional induced on the space of signed measures of bounded variation on X. In fact, in this case $\int_X f \, dQ_m = \sum_{i=1}^k \lambda_i \left(\int_X f \, dQ_i \right)$ when $m = \sum_{i=1}^k \lambda_i \delta_{Q_i}$. For a general $m \in \Delta(Q)$, note that by the Density Theorem (Theorem 15.10 in Aliprantis and Border, 2006) we also know that $\Delta(Q) = \operatorname{cl}_{w*} \Delta_s(Q)$, so that $m_n \stackrel{w*}{\longrightarrow} m$ for some sequence (m_n) in $\Delta_s(Q)$. The mapping $Q \mapsto \int_X f \, dQ$ is weak* continuous (Theorem 15.5 in Aliprantis and Border, 2006). By weak convergence we therefore know that $\lim_{n\to\infty} \int_Q \left(\int_X f \, dQ \right) \, dm_n = \int_Q \left(\int_X f \, dQ \right) \, dm$. At the same time, since for any closed subset E of X the indicator function $\mathbf{1}_E$ on X is upper semicontinuous, the function $Q \mapsto Q(E) = \int_X \mathbf{1}_E \, dQ$ is also upper semicontinuous, and so is the mapping $m \mapsto Q_m(E)$ (see Theorem 15.5 in Aliprantis and Border, 2006). Therefore, for any closed set $E \subseteq X$ we have $\limsup_{n\to\infty} Q_m(E) \le Q_m(E)$. By the characterization of weak convergence we know that $Q_{m_n} \stackrel{w*}{\longrightarrow} Q_m$. Combining the convergence of (Q_{m_n}) and (m_n) just established yields

$$\int_{X} f \, dQ_{m} = \lim_{n \to \infty} \int_{X} f \, dQ_{m_{n}} = \lim_{n \to \infty} \int_{\mathcal{O}} \left(\int_{X} f \, dQ \right) dm_{n} = \int_{\mathcal{O}} \left(\int_{X} f \, dQ \right) dm. \quad \Box$$

Turn to the proof of Proposition 3. Note that for any $f \in C(X)$ with unit norm the function -f also has norm one, and

$$\int_X (-f) dP \le \max_{Q \in \mathcal{Q}} \int (-f) dQ + \epsilon \Leftrightarrow \int_X f dP \ge \min_{Q \in \mathcal{Q}} \int f dQ - \epsilon.$$

Therefore the statements in (i) and (ii) are equivalent. Now assume that P is as in (17). In view of equation (32), simple calculations reveal that

$$\int_{X} f \, dP = \int_{\mathcal{Q}} \left(\int_{X} f \, dQ \right) dm + \int_{X} f \, de$$

$$\leq \max_{Q \in \mathcal{Q}} \int_{X} f \, dQ + \int_{X} f \, de$$

$$\leq \max_{Q \in \mathcal{Q}} \int_{X} f \, dQ + \|f\|_{\infty} \|e\|_{1}. \tag{33}$$

For $||f||_{\infty} = 1$ and $||e||_{1} \leq \epsilon$ the inequality in item (i) is now a consequence of (33). Conversely, suppose that P cannot be expressed as in (17). By Corollary 1 there exists a continuous function $f: X \to \mathbb{R}$ with unit norm such that $\int_{X} f \, dP - \int_{X} f \, dQ > \epsilon$ for all $Q \in \mathcal{Q}$. In particular, because of compactness of \mathcal{Q} and continuity of the mapping $Q \mapsto \int_{X} f \, dQ$ we have that $\int_{X} f \, dP > \max_{Q \in \mathcal{Q}} \int_{X} f \, dQ + \epsilon$, thus showing that the statement in item (i) does not hold. This establishes the converse implication.

A.7 Proof of Proposition 4

Let P be as in (17). Suppose that $f, g \in C(X)$ are such that for all $Q \in \mathcal{Q}$ we have that $\int_X f \, dQ \geq \int_X g \, dQ$. Then $\int_{\mathcal{Q}} \left(\int_X f \, dQ \right) \, dm \geq \int_{\mathcal{Q}} \left(\int_X g \, dQ \right) \, dm$ in view of the monotonicity of the integral with respect to the probability measure $m \in \Delta(\mathcal{Q})$. Let $e_+, e_- \in \operatorname{ca}(X)$ denote nonnegative measures such that $e = e_+ - e_-$ (their existence is assured by the Hahn decomposition theorem). Note that $\|e\|_1 = \|e_+\|_1 + \|e_-\|_1$ and that $0 = e(X) = \|e_+\|_1 - \|e_-\|_1$. Hence

$$\int_{X} f \, dP = \int_{X} f \, dQ_{m} + \int_{X} f \, de_{+} - \int_{X} f \, de_{-}$$

$$\geq \int_{X} g \, dQ_{m} + \int_{X} f \, de_{+} - \int_{X} f \, de_{-}$$

$$= \int_{X} g \, dP + \int_{X} (f - g) \, de_{+} - \int_{X} (f - g) \, de_{-}$$

$$\geq \int_{X} g \, dP + \|e_{+}\|_{1} \min_{x \in X} [f(x) - g(x)] - \|e_{-}\|_{1} \max_{x \in X} [f(x) - g(x)]$$

$$= \int_{X} g \, dP - \frac{\|e\|_{1} \omega (f - g)}{2}$$

$$\geq \int_{X} g \, dP - \frac{\epsilon \cdot \omega (f - g)}{2}.$$

Now assume that P cannot be expressed as in (17). Then system II in Corollary 1 has no solution. Consequently there exists $f \in C(X)$ with $||f||_{\infty} = 1$ and $\int_X f \, dP < \int_X f \, dQ - \epsilon$ for all $Q \in \mathcal{Q}$. This is equivalent to saying that, for $g = (\min_{Q \in \mathcal{Q}} \int_X f \, dQ) \cdot \mathbf{1}_X$, $\int_X f \, dP < \int_X g \, dP - \epsilon$. Since $\omega(f) = \omega(f-g)$ and $\omega(f) \leq 2 ||f||_{\infty}$, we obtain that $\int_X f \, dP < \int_X g \, dP - \frac{\epsilon \cdot \omega(f-g)}{2}$. By construction $\int_X f \, dQ \geq \int_X g \, dQ$ for all $Q \in \mathcal{Q}$. This gives a violation of condition \mathbf{C}_{ϵ} .

A.8 Proof of Proposition 5

Suppose that P can be expressed as in (18). Let $f, g \in C(X)$ be such that $\int_X f dQ \ge \int_X g dQ$ for all $Q \in \mathcal{Q}$. By Fact A.6.1 above and nonnegativity of the mapping $Q \mapsto \int_X (f-g) dQ$, we have $\int_X f dQ_m \ge \int_X g dQ_m$. Hence

$$\int_{X} f \, dP = (1 - \epsilon) \int_{X} f \, dQ_{m} + \epsilon \int_{X} f \, dR$$

$$\geq (1 - \epsilon) \int_{X} g \, dQ_{m} + \epsilon \int_{X} f \, dR$$

$$= \int_{X} g \, dP + \epsilon \int_{X} (f - g) \, dR$$

$$\geq \int_{X} g \, dP + \epsilon \min_{x \in X} [f(x) - g(x)]$$

$$= \int_{X} g \, dP - \epsilon \omega (f - g) + \epsilon \max_{x \in X} [f(x) - g(x)].$$

Now assume that P cannot be expressed as in (18). It follows from Proposition 2 in the case $\mathcal{P} = \{P\}$ and $\mathcal{R} = \Delta(X)$ that for some $f \in C(X)$ we have

$$\int_{X} f \, dP < (1 - \epsilon) \min_{Q \in \mathcal{Q}} \int_{X} f \, dQ + \epsilon \min_{x \in X} f(x). \tag{34}$$

By defining g as the constant function $\left(\min_{Q\in\mathcal{Q}}\int_X f\,dQ\right)\cdot\mathbf{1}_X$ we have by construction $\int_X f\,dQ \geq \int_X g\,dQ$ for al $Q\in\mathcal{Q}$. At the same time, the inequality in (34) gives

$$\int_X f \, dP < \int_X g \, dP + \epsilon \min_{x \in X} [f(x) - g(x)],$$

thus contradicting condition \mathbf{C}_{ϵ}^* .

A.9 Proof of Proposition 6

Suppose that $P = \sum_{i=1}^{N} m_i Q_i + e$, with $m_i \geq 0$ (i = 1, ..., N), $e \in \operatorname{ca}(X)$, and $\|e\|_1 \leq \epsilon$. Let $E_1, E_2 \in \Sigma_X$ be such that $Q_i(E_1) \geq Q_i(E_2)$ for all i = 1, ..., N. Using the Hahn decomposition theorem we know that there are sets $X_-, X_+ \in \Sigma_X$ such that $\|e\|_1 = e(X^+) - e(X^-)$. By the construction of the sets X_+ and X_- in the mentioned theorem we know that $e(E_1) \geq e(X_-)$ and $e(E_2) \leq e(X_+)$. Then

$$P(E_1) = \sum_{i=1}^{N} m_i Q_i(E_1) + e(E_1)$$

$$\geq P(E_2) + e(E_1) - e(E_2)$$

$$\geq P(E_2) + e(X_-) - e(X_+)$$

$$= P(E_2) - ||e||_1$$

$$\geq P(E_2) - \epsilon.$$

Now assume that P and Q satisfy condition $\mathbf{C}_{\epsilon}^{M}$. This part of the proof follows similar steps to those of Proposition 2 in Mongin (1995). Define the set $C \subseteq \mathbb{R}^{N+1}$ so that

$$C = \{z : z_i = Q_i(E_1) - Q_i(E_2) \text{ for } 1 \le i \le N, z_{N+1} = P(E_1) - P(E_2), E_1, E_2 \in \Sigma_X \}.$$

Obviously $C \neq \emptyset$. By the Lyapunov Convexity Theorem (Theorem 13.33 in Aliprantis and Border, 2006) the set C is convex and compact. We also define the set $D \subseteq \mathbb{R}^{N+1}$ so that

$$D = \{z : z_i \ge 0 \text{ for } 1 \le i \le N, z_{N+1} < -\epsilon \}.$$

The set D is also nonempty and convex. Condition $\mathbf{C}_{\epsilon}^{M}$ implies that $C \cap D = \emptyset$. By the Separating Hyperplane Theorem (e.g., Theorem 7.30 in Aliprantis and Border, 2006) there exists a nonzero vector $\lambda \in \mathbb{R}^{N+1}$ such that

$$\sum_{i=1}^{N} \lambda_i [Q_i(E_1) - Q_i(E_2)] + \lambda_{N+1} [P(E_1) - P(E_2)] \ge \sum_{i=1}^{N} \lambda_i z_i + \lambda_{N+1} z_{N+1}$$

for all $E_1, E_2 \in \Sigma_X$, and $z \in D$. Setting $E_1 = E_2$, $z_i = 0$ for all i = 1, ..., N, and $z_{N+1} = -2\epsilon$, we have $0 \ge -2\epsilon \lambda_{N+1}$, and thus $\lambda_{N+1} \ge 0$. If $\lambda_{N+1} = 0$, since $(1, ..., 1, -2\epsilon)$ belongs to the interior of D, we have $0 > \sum_{i=1}^{N} \lambda_i$. At the same time, by setting $E_1 = X$, $E_2 = \emptyset$, and z equal to the zero vector, we obtain that $\sum_{i=1}^{N} \lambda_i \ge 0$, a contradiction. Therefore, we may assume without loss of generality that $\lambda_{N+1} = 1$. Also note that for $E_1 = E_2$, $z_{N+1} = -2\epsilon$, $z_i = n$, and $z_j = 0$ for $j \ne i$ we have that $0 \ge -2\epsilon + \lambda_i n \Leftrightarrow \lambda_i \le \frac{2\epsilon}{n}$. Taking the limit in the last inequality as $n \to \infty$ yields $\lambda_i \le 0$. Define $m_i = -\lambda_i$, for all i = 1, ..., N. For any two sets $E_1, E_2 \in \Sigma_X$, and $z_i = 0$ for all i = 1, ..., N, upon letting $z_{N+1} \uparrow -\epsilon$ we have that

$$P(E_1) - P(E_2) - \sum_{i=1}^{N} m_i [Q_i(E_1) - Q_i(E_2)] \ge -\epsilon.$$

By exchanging the roles of E_1 and E_2 it follows that, for $e = P - \sum_{i=1}^{N} m_i Q_i \in \operatorname{ca}(X)$,

$$|e(E_1) - e(E_2)| \le \epsilon. \tag{35}$$

By the Hahn decomposition theorem, for $X_+, X_- \in \Sigma_X$ such that $||e||_1 = e(X_+) - e(X_-)$, we obtain that $||e||_1 \le \epsilon$ by setting $E_1 = X_+$ and $E_2 = X_-$ in (35).

A.10 Proof of Proposition 7

Assume that $P = \sum_{i=1}^{N} m_i Q_i + e$ with m in the N-1 dimensional simplex and $\|e\|_1 \leq \epsilon$. Equivalence of both conditions follows from taking complements. So we only show that P satisfies (ii). Using the Hahn decomposition theorem we know that there are disjoint sets $X_-, X_+ \in \Sigma_X$ such that $\|e\|_1 = e(X_+) - e(X_-)$. In the decomposition we have $e(X_-) = \inf_{E \in \Sigma_X} e(E)$. At the same time, e(X) = 0, so that $e(X_+) = -e(X_-)$ and thus $e(X_-) \geq -\frac{\epsilon}{2}$. Then

$$P(E) = \sum_{i=1}^{N} m_i Q_i(E) + e(E)$$

$$\geq \min_i Q_i(E) + e(X_-)$$

$$\geq \min_i Q_i(E) - \frac{\epsilon}{2}.$$

Now assume that condition (ii) holds. Let $f \in C(X)$ be non-constant and such that $||f||_{\infty} = 1$. Let $\sigma = \min_{x \in X} f(x)$, and $g = \frac{f - \sigma \mathbf{1}_X}{||f - \sigma \mathbf{1}_X||}$. Clearly g is continuous and $0 \le g \le 1$. By the Lyapunov Convexity Theorem (Theorem 13.33 in Aliprantis and Border, 2006), there exists $E \in \Sigma_X$ such that $P(E) = \int_X g \, dP$, and $Q_i(E) = \int_X g \, dQ_i$, for $i = 1, \ldots, N$. From (ii), $P(E) \ge \min_i Q_i(E) - \frac{\epsilon}{2}$. Since $||f - \sigma \mathbf{1}_X||_{\infty} \le 2$, and thus $-\frac{1}{2} \ge -\frac{1}{||f - \sigma \mathbf{1}_X||_{\infty}}$, we get

$$P(E) \ge \min_{i} Q_{i}(E) - \frac{\epsilon}{\|f - \sigma \mathbf{1}_{X}\|_{\infty}},$$

which is equivalent to

$$\int_{X} f \, dP \ge \min_{i} \int_{X} f \, dQ_{i} - \epsilon \tag{36}$$

after simple algebra. The inequality in (36) is condition (ii) in Proposition 3, and this completes the proof.

A.11 Proof of Proposition 8

Suppose that P_0 is given as in (25). Let e_+ and e_- , both in \mathbb{R}^M , denote the positive and the negative parts of e, respectively. Put $\widehat{N} = 2^{N_0} - 1$. Since

$$\begin{split} \widehat{N} &= \sum_{(y,Y) \in X} P_0(y,Y) \\ &= \sum_{\succ \in \mathcal{L}} \sum_{(y,Y) \in X} a_{\succ}(y,Y) \pi(\succ) + \|e_+\|_1 - \|e_-\|_1 \,, \end{split}$$

and using the fact that, given \succ , for each nonempty set $Y \subseteq \mathcal{Y}$ there is exactly one y for which $a_{\succ}(y,Y) = 1$, we obtain

$$\widehat{N} = \sum_{\succ \in \mathcal{L}} \widehat{N}\pi(\succ) + \|e_+\|_1 - \|e_-\|_1,$$

and thus $||e_+||_1 = ||e_-||_1 = \frac{||e||_1}{2} \leq \frac{\epsilon}{2}$. Let T be a trial sequence of width w_T . Then

$$\sum_{i=1}^{M} P_0(y_i, Y_i) t_i = \sum_{\succ \in \mathcal{L}} \pi(\succ) \left[\sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i \right] + \sum_{i=1}^{M} e_{+}(i) t_i - \sum_{i=1}^{M} e_{-}(i) t_i$$

In view of the inequalities

$$\sum_{\succ \in \mathcal{L}} \pi(\succ) \left[\sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i \right] \leq \max_{\succ \in \mathcal{L}} \left[\sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i \right]$$

and

$$\sum_{i=1}^{M} e_{+}(i)t_{i} - \sum_{i=1}^{M} e_{-}(i)t_{i} \leq \|e_{+}\|_{1} \max_{i} t_{i} - \|e_{-}\|_{1} \min_{i} t_{i}$$

$$\leq \frac{\epsilon}{2} (\max_{i} t_{i} - \min_{i} t_{i})$$

$$= \frac{w_{T}\epsilon}{2}$$

the stochastic choice function satisfies ϵ -ARSP.

Now assume that ϵ -ARSP holds. With respect to the vectors P and Q_{\succ} mentioned in the text this is equivalent to saying that, for every trial sequence T of width w_T ,

$$\sum_{i=1}^{M} P(y_i, Y_i) t_i \le \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} Q_{\succ}(y_i, Y_i) t_i + \frac{w_T \epsilon}{2\widehat{N}}.$$
 (37)

To show that P_0 can be represented as in equation (25), it suffices to show that the set $\mathcal{P} = \{P\}$ and the convex hull of $\mathcal{Q} = \{Q_{\succ} : \succ \in \mathcal{L}\}$ have points whose distance is at most $\frac{\epsilon}{\widehat{N}}$. Let $\bar{\epsilon} = \frac{\epsilon}{\widehat{N}}$. Because of Corollary 1, it suffices to show that for no vector $z \in \mathbb{R}^M$ with $||z||_{\infty} = 1$ we have

$$\sum_{i=1}^{M} P_i z_i > \sum_{i=1}^{M} Q_i z_i + \bar{\epsilon} \quad \text{for all } Q \in \mathcal{Q},$$
(38)

where P_i and Q_i are the *i*-th components of P and Q, respectively.

So suppose that such a vector exists, and denote it by z^* . We will show that this leads to a contradiction. By continuity of the linear functionals on Euclidean spaces of dimension M,

for each $Q \in \mathcal{Q}$ there exists $\gamma_Q > 0$ such that the inequality in (38) holds for all $z \in \mathbb{R}^M$ with $\|z - z^*\|_{\infty} < \gamma_Q$. By letting γ denote the minimum among the γ_Q , for each coordinate i such that $-1 < z_i^* < 1$, choose a rational number z_i sufficiently close to z_i^* . For the coordinates i such that $z_i^* = \pm 1$, put $z_i = z_i^*$. By the denseness of \mathbb{Q} in \mathbb{R} , we can therefore choose points so that the vector z has rational entries, and moreover $\|z - z^*\|_{\infty} < \gamma$. Consequently, it satisfies (38) as well and furthermore $\|z\|_{\infty} = 1$. Now define $\tilde{z} \in \mathbb{Q}^M$ so that $\tilde{z}_i = z_i - \min_j z_j \geq 0$. By construction $\tilde{z}_i = \frac{p_i}{q_i}$ for some $0 \leq p_i \in \mathbb{Z}$ and $0 < q_i \in \mathbb{Z}$, for all $i = 1, \ldots, M$. Note that \tilde{z} also satisfies the inequality in (38), which becomes, after some algebra,

$$\sum_{i=1}^{M} P_i t_i > \max_{Q \in \mathcal{Q}} \sum_{i=1}^{M} Q_i t_i + \bar{\epsilon} \prod_j q_j, \tag{39}$$

where $t_i = p_i \prod_{j \neq i} q_j$. Note that for at least one i we have $p_i = 0$. At the same time, since $\omega(\tilde{z}) = \omega(z) \leq 2 ||z||_{\infty} = 2$, we know that, for all i,

$$p_i \prod_{j \neq i} q_j \le 2 \prod_j q_j. \tag{40}$$

Combining the inequalities in (39) and (40) we obtain that

$$\sum_{i=1}^{M} P_i t_i > \max_{Q \in \mathcal{Q}} \sum_{i=1}^{M} Q_i t_i + \frac{\max_i t_i}{2} \bar{\epsilon}.$$

Since $\min_i t_i = 0$, we know that $w_T = \max_i t_i$ represents the width of the tagged trial sequence constructed from the integers t_i . This is a contradiction with (37).

A.12 Proof of Proposition 9

Assume that P_0 is expressed as in (26). If T is a tagged trial sequence, simple calculations show that

$$\sum_{i=1}^{M} P_0(y_i, Y_i) t_i = (1 - \epsilon) \sum_{\succ \in \mathcal{L}} \pi(\succ) \left[\sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i \right] + \epsilon \sum_{i=1}^{M} R_0(y_i, Y_i) t_i$$

$$\leq (1 - \epsilon) \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i + \epsilon (2^{N_0} - 1) \max_i t_i.$$

This is ϵ -ARSP*.

Conversely, suppose that ϵ -ARSP* holds. Put $\widehat{N} = 2^{N_0} - 1$. Because of Proposition 2 applied to $\mathcal{P} = \{P\}, \ \mathcal{Q} = \{Q_{\succ} : \succ \in \mathcal{L}\}, \ \text{and} \ \mathcal{R} = \{\frac{1}{\widehat{N}}R \in \mathbb{R}^M : R_0(y,Y) \geq 0, \sum_{y \in Y} R(y,Y) = 1\}$

1 for all $Y \subseteq \mathcal{Y}, Y \neq \emptyset$, it suffices to show that for no vector $z \in \mathbb{R}^M$ with nonnegative components and $||z||_{\infty} = 1$ we have

$$\sum_{i=1}^{M} P_{i} z_{i} > (1 - \epsilon) \max_{Q \in \mathcal{Q}} \sum_{i=1}^{M} Q_{i} z_{i} + \epsilon \max_{R \in \mathcal{R}} \sum_{i=1}^{M} R_{i} z_{i}.$$
 (41)

So suppose by way of contradiction that the vector $z^* \in \mathbb{R}^M$ with nonnegative components verifies the inequality in (41). This is equivalent to

$$\sum_{i=1}^{M} P_0(y_i, Y_i) z_i^* > (1 - \epsilon) \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} a_{\succ}(y_i, Y_i) z_i^* + \epsilon \max_{R_0 \in \mathcal{R}_0} \sum_{i=1}^{M} R_0(y_i, Y_i) z_i^*, \tag{42}$$

where $\mathcal{R}_0 = \{\widehat{N}R : R \in \mathcal{R}\}$. For a fixed Y the mappings $y \mapsto P_0(y,Y)$, $y \mapsto a_{\succ}(y,Y)$ and $y \mapsto R_0(y,Y)$ are probability measures on Y. We now identify z(y,Y) with the corresponding coordinate of z for which $(y_i,Y_i)=(y,Y)$. We can now add $1-\max_{y\in Y}z^*(y,Y)$ to both sides of (42) and adjust the vector accordingly so that such an inequality also holds when $\max_{y\in Y}z^*(y,Y)=1$. By repeating this procedure for each nonempty subset Y of \mathcal{Y} , we can assume without any loss of generality that for all $Y\subseteq \mathcal{Y}$ with $Y\neq\emptyset$ we have $z^*(y,Y)=1$ for some $y\in Y$. Now proceed as in the proof of Proposition 8 to adjust again the vector z^* if needed so that $z^*\in\mathbb{Q}^M$, $z_i^*\geq 0$ for all $i=1,\ldots,M$, and the coordinates for which $z_i^*=1$ are preserved. Hence we can write $z_i^*=\frac{p_i}{q_i}$ with $0\leq p_i\in\mathbb{Z}$, and $0< q_i\in\mathbb{Z}$. Let $t_i=p_i\prod_{j\neq i}q_j$. We therefore have

$$\sum_{i=1}^{M} P_0(y_i, Y_i) t_i > (1 - \epsilon) \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i + \epsilon \max_{R_0 \in \mathcal{R}_0} \sum_{i=1}^{M} R_0(y_i, Y_i) t_i^*$$

$$= (1 - \epsilon) \max_{\succ \in \mathcal{L}} \sum_{i=1}^{M} a_{\succ}(y_i, Y_i) t_i + \epsilon (2^{N_0} - 1) \max t_i^*,$$

where the last equality follows from maximizing the second term in the right-hand side of the inequality in (42). This contradicts ϵ -ARSP*.

References

Acciaio, B., Backhoff, J., Gudmund, P. (2022). Quantitative Fundamental Theorem of Asset Pricing. arXiv preprint arXiv:2209.15037.

Aliprantis, C. D., Border, K. C. (2006). *Infinite Dimensional Analysis: a Hitchhiker's Guide*, 3rd ed., New York: Springer.

Apesteguia, J., Ballester, M. A. (2021). Separating predicted randomness from residual behavior. *Journal of the European Economic Association*, 19, 1041-1076.

Araujo, A. Chateauneuf, A., Faro, J. H. (2012). Pricing rules and Arrow–Debreu ambiguous valuation. *Economic Theory*, 49, 1-35.

Araujo, A. Chateauneuf, A., Faro, J. H. (2018). Financial market structures revealed by pricing rules: efficient complete markets are prevalent. *Journal of Economic Theory*, 173, 257-288.

Beggs, A. (2021). Afriat and arbitrage. Economic Theory Bulletin, 9, 167-176.

Block H. D., Marschak, J. (1960). "Random orderings and stochastic theories of responses," in I. Olkin, et al. (eds.), Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling, Stanford University Press, 97-132.

Bogachev, V. I. (2007). Measure Theory, New York: Springer.

Bogachev, V. I. (2018). Weak Convergence of Measures, Providence: American Mathematical Society.

Border, K. C. (2007). *Introductory Notes on Stochastic Rationality*. California Institute of Technology.

Chateauneuf, A. Cornet, B. (2022). Submodular financial markets with frictions. *Economic Theory*, 73, 721-744.

Clark, S. A. (1996). The random utility model with an infinite choice space. *Economic Theory*, 7, 179-189.

Dax, A. (2006). The distance between two convex sets. *Linear Algebra and its Applications*, 416, 184-213.

Dax, A., Sreedharan, V. P. (1997). Theorems of the alternative and duality. *Journal of Optimization Theory and Applications*, 94, 561-590.

Falmagne, J. C. (1978). A representation theorem for finite random scale systems. *Journal of Mathematical Psychology*, 18, 52-72.

Gibbs, A. L., Su, F. E. (2002). On choosing and bounding probability metrics. *International Statistical Review*, 70, 419-435.

Genest, C. (1984). Pooling operators with the marginalization property. *Canadian Journal of Statistics*, 12, 153-163.

Güler, O., Hoffman, A. J., Rothblum, U. G. (1995). Approximations to solutions to systems of linear inequalities. SIAM Journal on Matrix Analysis and Applications, 16, 688-696.

Hellman, Z. (2013). Almost common priors. *International Journal of Game Theory*, 42, 399-410.

Hellman, Z., Pintér, M. (2022). Charges and bets: a general characterisation of common priors. *International Journal of Game Theory*, 51, 567-587.

Hoffman, A. J. (1952). On approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureau of Standards*, 49, 263-265.

Huber, P. J. (1964). Robust estimation of a location parameter. *The Annals of Mathematical Statistics*, 73-101.

Kneser, H. (1952). Sur un théoreme fondamental de la théorie des jeux. $C.\ R.\ Acad.\ Sci.\ Paris,\ 234,\ 2418-2420.$

Luenberger, D. G. (1997). Optimization by Vector Space Methods, New York: John Wiley & Sons.

McConway, K. J. (1981). Marginalization and linear opinion pools. *Journal of the American Statistical Association*, 76, 410-414.

McFadden, D. (2005). Revealed stochastic preference: a synthesis. *Economic Theory*, 26, 245-264.

McFadden, D., Richter, K. (1990) "Stochastic Rationality and Revealed Stochastic Preference," in J. Chipman, D. McFadden, K. Richter (eds.), *Preferences, Uncertainty, and Optimality*, Westview Press, 161-186.

Mongin, P. (1995). Consistent Bayesian aggregation. *Journal of Economic Theory*, 66, 313-351.

Nau, R. F. (1992). Joint coherence in games of incomplete information. *Management Science*, 38, 374-387.

Nau, R. F. (1995a). The incoherence of agreeing to disagree. *Theory and Decision*, 39, 219-239.

Nau, R. F. (1995b). Coherent decision analysis with inseparable probabilities and utilities. Journal of risk and uncertainty, 10, 71-91.

Nau, R. (2015). Risk-neutral equilibria of noncooperative games. *Theory and Decision*, 78, 171-188.

Nau, R. F., McCardle, K. F. (1990). Coherent behavior in noncooperative games. *Journal of Economic Theory*, 50, 424-444

Nau, R. F., McCardle, K. F. (1991). Arbitrage, rationality, and equilibrium. *Theory and Decision*, 31, 199-240.

Nielsen, M. (2019). On linear aggregation of infinitely many finitely additive probability measures. *Theory and Decision*, 86, 421-436.

Nielsen, M. (2021). The strength of de Finetti's coherence theorem. *Synthese*, 198, 11713-11724.

Phelps, R. R. (2001). Lectures on Choquet's Theorem, 2nd ed., New York: Springer.

Radner, R. (1980). Collusive behavior in noncooperative epsilon-equilibria of oligopolists with long but finite lives. *Journal of Economic Theory*, 22, 136-154.

Samet, D. (1998). Common priors and separation of convex sets. *Games and Economic Behavior*, 24, 172-174.

Savage, L. J. (1954). The Foundations of Statistics, New York: John Wiley & Sons.

Sion, M. (1958). On general minimax theorems. Pacific Journal of Mathematics, 8, 171-176.

Stone, M. (1961). The opinion pool. The Annals of Mathematical Statistics, 1339-1342.