# Measures of physical mixing evaluate the economic mobility of the typical individual

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May 6, 2022

#### Abstract

Measures of economic mobility represent aggregated values for how wealth ranks of individuals change over time. Therefore, in certain circumstances mobility measures may not describe the feasibility of the typical individual to change their wealth ranking. To address this issue, we introduce mixing, a concept from statistical physics, as a relevant phenomenon for quantifying the ability of individuals to move across the whole wealth distribution. We display the relationship between mixing and mobility by studying the relaxation time, a statistical measure for the degree of mixing, in reallocating geometric Brownian motion (RGBM). RGBM is an established model of wealth in a growing and reallocating economy that distinguishes between a mixing and a non-mixing wealth dynamics regime. We show that measures of mixing are inherently connected to the concept of economic mobility: while certain individuals can move across the distribution when wealth is a non-mixing observable, only in the mixing case every individual is able to move across the whole wealth distribution. Then, there is also a direct equivalence between measures of mixing and the magnitude of the standard measures of economic mobility. On the other hand, the opposite is not true. Wealth dynamics, are however, best modeled as non-mixing. Hence, measuring mobility using standard measures in a non-mixing system may lead to misleading conclusions about the extent of mobility across the whole distribution.

Keywords: economic mobility, stochastic processes, wealth inequality

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<sup>&</sup>lt;sup>†</sup>The author thanks the London Mathematical Laboratory fellows, namely Alexander Adamou, Yonatan Berman, Colm Connaughton, and Ole Peters, for helpful discussions and comments on the manuscript.

#### 1 Introduction

Economic mobility describes "dynamic aspects of inequality" (Shorrocks, 1978). It quantifies how wealth (or income<sup>1</sup>) ranks of individuals change over time. Intuitively, when mobility is high, ranks evolve quickly, and the chances of an individual to change their position in the wealth distribution over a given time period are high. When mobility is low, individuals are unlikely to change their rank in the distribution over time, or such changes are slow.

Mobility measures are assumed to be derived from the joint distribution of wealth at two points in time. On this basis, Shorrocks (1978) described several required properties for the statistical measurement of mobility and set the standard for such measures. A particular feature of measures of economic mobility is that they represent an aggregation for the changes in the wealth rankings within the economy. This means that whenever there is a change in the rankings, the measures will suggest the existence of mobility. As a result, in certain circumstances mobility measures may not describe the feasibility of the typical individual to change their wealth rank.

Here we address this issue by introducing *mixing* as a relevant concept when quantifying the feasibility of *every* individual in the economy to change their rank. Mixing is a well-known concept in statistical physics. It describes the property of a dynamical system being strongly intertwined. Translated in economic terms, this means that any measure of mixing will evaluate the extent to which *every* individual in an economy is able to move across the whole steady-state wealth distribution. If wealth dynamics are non-mixing, measures of mixing will be at their lowest value. Otherwise, they will tell us the extent to which the system is mixing. This is significantly different from mobility measures, where any change in the rankings is interpreted as existence of mobility.

We discuss the relationship between mixing and mobility by comparing a measure that accounts for both phenomena to standard measures of mobility in a simple model for wealth dynamics, called Reallocating Geometric Brownian Motion (RGBM) (Marsili et al., 1998; Liu and Serota, 2017; Berman et al., 2020). We call the measure the relaxation time and, interestingly, a variant of it appeared for the first time in the economic mobility literature in the seminal paper by Shorrocks (1978). Formally, the relaxation time is a feature of stochastic processes that evaluates the convergence of the distribution of wealth in an economy towards the steady-state distribution. When wealth is a mixing observable (Peters and Adamou, 2018), then if the wealths of an arbitrary group of individuals is followed over time, the distribution of wealth within this group will gradually become similar to the steady-state wealth distribution. The characteristic time of this convergence process is the relaxation time. Put simply, it is the timescale over which individuals mix into the wealth distribution. When mixing is rapid, i.e. the relaxation time is short relative to the window of observation, we could interpret that as high wealth mixing. Long times is interpreted as low mixing.

In RGBM, individual wealth undergoes random multiplicative growth, modeled as Geometric Brow-

<sup>&</sup>lt;sup>1</sup>We focus on wealth in this paper, but our findings also apply to income.

nian Motion (GBM) (Stojkoski et al., 2020), and is reallocated among individuals by a simple pooling and sharing mechanism (Stojkoski et al., 2019, 2021a). RGBM is a null model of an exponentially-growing economy with social structure. It has three parameters representing common economic growth, random shocks to individual wealth, and economic interaction among agents, quantified by a reallocation rate. This model is known to reproduce several important stylized facts. In particular, when the reallocation is from the rich to the poor, the rescaled wealth distribution converges to a stationary distribution with a Pareto tail. The model has both mixing and non-mixing regimes, characterized by the sign of the reallocation rate parameter (Berman et al., 2020).

We find that in the mixing regime of RGBM, the relaxation time scales with the inverse of the reallocation rate. As the reallocation rate becomes higher, *i.e.* as a larger share of each individual's wealth is pooled and then shared per unit time, the relaxation time becomes shorter proportionally, and mobility increases. As the reallocation rate approaches zero, relaxation times get longer, and mobility lower. In RGBM, decreasing reallocation rates also lead to increasing inequality. Hence, this result is in line with the empirical observation that mobility decreases as inequality increases, and vice versa (Corak, 2013). Furthermore, we show that there is a direct relationship between the relaxation time and standard measures of economic mobility.

A prerequisite for a system to be mixing is to satisfy the ergodic hypothesis, which underlies the assumption that the time average and the expectation value of an observable are the same. In practice, however, many economic systems do not satisfy this hypothesis (Peters, 2019). Hence, they are non-mixing. Indeed, recent studies have shown that the dynamics of wealth do not satisfy the ergodic hypothesis. For instance, Berman et al. (2020) argue that the US economy is best described in RGBM as one in which wealth is systematically reallocated from poorer to richer, *i.e.* the reallocation rate is negative. In this case, even though standard measures of economic mobility might suggest the existence of mobility, there is no mixing. Thus, if we use standard measures of mobility, we may conclude misleadingly that everyone is able to move across the wealth distribution. The thorough study of mobility in RGBM in this regime is outside of the scope of this paper and left for future work.

The paper is organized as follows. In Section 2 we define mixing mathematically and compare its characteristics to those of standard mobility measures. In the same section, we introduce the relaxation time as a measure that evaluates the degree of mixing in an economy. Section 3 relates mobility and mixing in reallocating geometric Brownian motion as a model of wealth. We discuss our findings in Section 4.

## 2 Mixing and mobility

#### 2.1 Definition

The concept of mixing comes from probability theory and statistical physics. It describes the property of a dynamical system being strongly intertwined. In physical terms, this means that, for any set of particles in a dynamical system, the fraction of the particles found within a particular region in the *phase space* (the space of the variable x characterizing the particles, wealth in our case), is proportional to the volume of that region in the phase space. Figuratively, we can think of an economy as a cup of coffee and of some person's wealth as milk poured in the coffee. If the system is mixing, then the milk will spread across the coffee over time and eventually it will be spread equally in the cup.

Mathematically we define mixing as follows. Let  $x_i(t)$  denote the wealth in year t of the i-th individual in a population of size N. Moreover, let  $y_i = x_i/\langle x \rangle_N$ , where  $\langle x \rangle_N = \sum_i x_i/N$  is per capita wealth, be the rescaled wealth of individual i and P(y,t) be the probability density function that describes the distribution of rescaled wealth in the population in the same year, with the initial condition being a Dirac delta function with a mass centered at 1, i.e., P(y,0) = DiracDelta(y-1). If the economy is mixing, then starting from an initial year t=0 in which every individual in the population has identical rescaled wealth  $y_0=1$ , then the distribution P(y,t) will in each subsequent time point resemble more and more a predefined target steady state distribution  $P^*(y)$ . More importantly, it will eventually converge to the target distribution. A standard way for evaluating this property is through the  $\beta$ -mixing coefficient. The coefficient measures the total variational distance between the wealth distribution in year t and the steady state distribution, i.e.,

$$\beta(t) = ||P(y, t) - P^*(y)||,$$

where  $||g(y)|| = \int |g(y)|dy$  is the  $L^1$  norm of g. Formally, the wealth dynamics is said to be " $\beta$ -mixing" if  $\lim_{t\to\infty} \beta(t) = 0$  (Drees, 2000; Stojkoski et al., 2022b).

We hereby emphasize that mixing is strongly related to the concept of ergodicity. However, the latter is a broader concept: an observable x is said to be ergodic if its time average is equal to its ensemble average at any given time. Every mixing system will satisfy this hypothesis because x travels across the phase space in a proportional manner. Nonetheless, the ergodic hypothesis can be satisfied even the system is non-mixing. Hence, every dynamical system that is mixing is also ergodic, but the opposite is not necessarily true.

In economic terms, mixing implies that the system does not discriminate between individuals on the basis of their history: it is possible for everyone to move between any ranks in the distribution over time. Thus, this concept can be seen as a mathematical manifestation of the "American Dream".

Although mixing is naturally linked to economic mobility, as captured by standard measures of mobility, the two are not always the same. As we will see in Section 3.3, there is always a relationship

between measures that incorporate mixing and standard mobility measures, whenever mixing exists in the system. However, the standard measures may still indicate that there is some level of mobility even when mixing does not occur. Yet, the existence of such mobility, a degree of which always exists practically, does not guarantee mixing, as this observation depends on the existence of mobility between *every* quantile in the wealth distribution. This idea was already described by McFarland (1970).

#### 2.2 An example for a mixing measure

Any statistical measure that is derived from the properties of  $\beta(t)$  can be interpreted as a measure for the degree of mixing within an economy. In this work, as an example for a such measure, we consider the reciprocal of the rate of convergence, r, towards the steady state distribution,  $P^*(y)$ , i.e.,

$$r = -\lim_{t \to \infty} \frac{1}{t} \log \beta(t). \tag{2.1}$$

1/r is also known as the relaxation time towards the steady state distribution. It is naturally measured in years and it provides a characteristic timescale over which individuals mix into the wealth distribution. In the coffee analogy, the relaxation time would quantify the time required for the milk to blend with the coffee. This enables the measure to be used for appropriate comparison between different time periods and economies. The relationship between the relaxation time and mobility is intuitive. That is, when the relaxation time is short relative to a relevant window of observation, then there is high wealth mobility and strong mixing. Slow relaxation times are interpreted as indicating low mobility and weak mixing. More importantly, when the mixing time is infinite, then wealth is not a mixing observable, but this does not mean that there is no mobility. A similar measure based on wealth transition matrices, which satisfies normalization properties, was studied in (Shorrocks, 1978).

Quantifying the relaxation time in empirical systems, in which the process governing the wealth dynamics is known, can be done easily by studying the spectral properties of the process (see Section 3.2). The estimation of this measure in situations when the wealth dynamics are unknown, though, is more complicated, but approximations are possible. We show how to construct them in Appendix 1 and Appendix 2.

#### 2.3 Relationship between mixing and measures of mobility

As a means to understand the differences in the information provided by measures of mixing and measures of mobility, here we compare the properties of the relaxation time to those of standard measures of mobility. For this comparison we utilize two standard measures of economic mobility: Spearman's rank correlation and the intragenerational earnings elasticity (IGE)<sup>2</sup>. These measures

<sup>&</sup>lt;sup>2</sup>In fact, the rank correlation and the IGE are both measures of immobility, and to consider them as measures of mobility one has to consider their complement or inverse.

describe attributes of the bivariate joint wealth distribution at two points in time. Such distributions are usually modeled via copulas. Mathematically, a copula can be represented by a simple model in which the wealth transition matrix is parametrized<sup>3</sup>. A widely used model is the Gumbel copula. It is able to reproduce realistic wealth-rank transition matrices, representing higher mobility at the bottom of the distribution than at the top (Jäntti and Jenkins, 2015). The Gumbel copula is uniquely defined by a single parameter  $\theta$ : a larger dependence implies less mobility. Due to its direct relationship with economic mobility, we also include this parameter in the comparative analysis. Technical background for the standard mobility measures is given in Appendix 3.

We consider five properties that a mobility measure may have. First, following Shorrocks (1978) we identify the properties of 1) normalization – the values that the mobility measure may take are bounded in a closed interval; 2) monotonicity – if a new structure is imposed in the wealth dynamics of some individuals then this is reflected in the value of the measure, and 3)  $period\ dependence$  – the mobility predicted by the measure is dependent on the temporal difference  $\delta$  between the periods that are used for its estimation. In addition, we follow Cowell and Flachaire (2018), and add to our analysis the property of 4)  $distribution\ dependence$  – whether the mobility predicted by the measure is dependent on the shape of the empirical wealth distributions that are used for its estimation or not.

Table 1 details the properties that are satisfied by the standard mobility measures which we study. For comparison, the table also gives the properties of the relaxation time, when this measure is used to evaluate the mobility within an economy. We see that the relaxation time does not satisfy the normalization property. Also, the IGE does not satisfy this property. However, we state that normalization is just a standard procedure that can be implemented on any measure, and the relaxation time can be easily adjusted to satisfy this property. For example, the relaxation time can be transformed to an index w valued between 0 and 1, where 1 indicates higher mobility, by defining  $w = e^{-1/r}$ . This measure was actually defined by Shorrocks (1978) in terms of transition matrices. We purposely refrain from the transformation procedure because we want to emphasize whether some value of relaxation time is high or low in terms of the years required for an individual to move across the whole distribution.

Table 1: Properties of mobility measures and relaxation time.

Measure	Property			
	Normalization	Monotonicity	P. dependence	D. dependence
Relaxation time	×	×	×	$\checkmark$
Spearman Correlation	$\checkmark$	$\checkmark$	$\checkmark$	×
IGE	×	$\checkmark$	$\checkmark$	$\checkmark$
Gumbel parameter	✓	$\checkmark$	$\checkmark$	×

<sup>&</sup>lt;sup>3</sup>One might argue that mixing as a concept is equivalent to the notion of *irreducibility* in transition matrices. However, transition matrices already aggregate the wealth dynamics into quantiles and thus may distort the picture of the extent of mobility.

More importantly, the relaxation time (and other measures of mixing) will not satisfy the monotonicity and the period dependence properties. The inability to satisfy the monotonicity property arises because monotonicity implies that, if there is mobility between certain quantiles of the wealth distribution, it will be translated as existence of mobility in the standard measures. The three described measures represent aggregated values of the changes in the wealth rankings of the individuals which constitute the population between two time periods t and  $t + \delta$ . Thus, if the transition matrix is slightly perturbed between the two periods, it will result in a change in the magnitude of mobility predicted by the measure. Measures for mixing, on the other hand, are not monotonic unless there is already mobility between every quantile. In every other case they will imply zero mobility.

Finally, distribution dependence is a phenomenon captured by the relaxation time (and other measures of mixing). From the three standard measures, only the IGE is dependent on the shape of the wealth distributions that are used for its estimation, whereas the other two measures are not. This dependence indicates that measures of mixing will not be interpreted similarly when the underlying wealth distribution remains unchanged, and when it becomes more and more unequal.

### 3 Mixing in a simple model of an economy

#### 3.1 Reallocating geometric Brownian motion

To illustrate the application of mixing in economic systems, we use reallocating geometric Brownian motion (RGBM), a simple model for wealth dynamics (Berman et al., 2020). Under RGBM, wealth is assumed to grow multiplicatively and randomly, in addition to a simple reallocation mechanism. The dynamics of the wealth of person i are specified as

$$dx_i = x_i \left( \mu dt + \sigma dW_i \right) - \tau \left( x_i - \langle x \rangle_N \right) dt, \tag{3.1}$$

with  $\mu > 0$  being the drift term,  $\sigma > 0$  the fluctuations amplitude, and  $dW_i$  is an independent Wiener increment,  $W_i(t) = \int_0^t dW_i$ .  $\tau$  is a parameter that quantifies the rate of reallocation of wealth. It implies that, in every time period dt, everyone in the economy contributes a fraction  $\tau dt$  of their wealth to a central pool. The pool is then shared equally across the population. This parameter encapsulates multiple effects, e.g. collective investment in infrastructure, education, social programs, taxation, rents paid, or private profits.

Under RGBM, the average wealth in a large population grows like  $e^{\mu t}$ . Rescaling by  $e^{\mu t}$ , the dynamic behavior of RGBM is strictly dependent on the relation between  $\tau$  and  $\sigma$ , and the rescaled wealth can be both mixing and non-mixing. When  $\tau > 0$ , rescaled wealth in RGBM is mixing, ergodic and stationary. The model exhibits mean-reversion as each  $x_i$  reverts to the population

average  $\langle x \rangle_N$ . The dynamics of the rescaled wealth  $y_i = x_i/\langle x \rangle_N$  can be described as

$$dy = y\sigma dW - \tau(y-1)dt. \tag{3.2}$$

The stationary (target steady-state) distribution of the model is

$$P^*(y) = \frac{(\zeta - 1)^{\beta}}{\Gamma(\zeta)} \exp\left(-\frac{\zeta - 1}{y}\right) y^{-(1+\zeta)}, \tag{3.3}$$

where  $\zeta = 1 + \frac{2\tau}{\sigma^2}$  and  $\Gamma(\cdot)$  is the Gamma function (see Berman et al. (2020)). The distribution has a power-law tail. The exponent of the power law,  $\zeta$ , is called the Pareto tail parameter, and can be used as a measure of economic equality (Cowell, 2011). More importantly, important stylized facts are recovered: the larger  $\sigma$  (more randomness in the dynamics) and the smaller  $\tau$  (less reallocation), the smaller the tail index and the fatter the tail of the distribution, leading to higher inequality. When  $\tau \leq 0$ , there is no stationary distribution to which rescaled wealth converges.

#### 3.2 Relaxation time in RGBM

The estimation of relaxation time in a stochastic process is usually done by investigating the Fokker-Planck equation which describes the evolution of the probability density function. In RGBM, this equation reads

$$\frac{\partial}{\partial t}P(y,t) = \tau \frac{\partial}{\partial y} \left[ (y-1)P(y,t) \right] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} \left[ y^2 P(y,t) \right]. \tag{3.4}$$

The equation is an appropriate generalization of a transition matrix for a discrete space process to processes with continuum of states. Hence, the analysis follows exactly in the same manner: the critical statistic of the Fokker-Planck equation that governs the convergence of P(y,t) to  $P^*(y)$  is the largest nontrivial eigenvalue of the corresponding Fokker-Planck operator, that is, the second largest eigenvalue. In a mixing system the largest eigenvalue will be zero, whereas the second eigenvalue,  $\lambda_2$ , will be negative, and  $\beta(t) \propto e^{\lambda_2 t}$ . We refer to Gabaix et al. (2016) for a detailed mathematical background on estimating the relaxation time in systems where the governing equation for the wealth dynamics is known.

An extensive study for the eigenvalues of RGBM was done by Liu and Serota (2017). The authors showed that the second largest eigenvalue is simply  $-\tau$  and, therefore, that the relaxation time will exist only when  $\tau > 0$  and it will be equal to  $1/\tau$ . The interpretation behind this result is fairly intuitive – in an economy in which reallocation from the rich to the poor is stronger, mixing is faster. On the other hand, as the reallocation rate approaches zero, relaxation times get longer, and mobility, as defined by the concept of mixing lower. As the model becomes non-stationary for reallocation rates that are equal or less then 0, the possibility for mixing in the economy disappears. Then, the relaxation time is infinite. We point out that while  $\tau < 0$  implies no mixing, that does

not mean that the standard mobility measures will also indicate no mobility. In fact, for any positive fluctuation amplitude  $\sigma$ , there will be randomness in the system. This randomness may drive changes in the observed wealth rankings, and as a result the standard measures may display a certain degree of mobility.

#### 3.3 Relaxation time and standard measures of mobility in RGBM

In RGBM, the standard measures of mobility depend on both  $\tau$  and the fluctuation amplitude  $\sigma$ , unlike the relaxation time. This is a consequence of the randomness playing a significant role in the wealth dynamics when we consider timescales that are shorter than the time required for the system to relax. In what follows, we describe the relationship between relaxation time and the standard measures of mobility in RGBM.

Spearman's rank correlation: Spearman's rank correlation is inversely related to the relaxation time in RGBM. The rank correlation is also dependent on the fluctuation amplitude  $\sigma$  and the temporal difference  $\delta$  between the two periods that are being compared. Larger values for both parameters lead to greater economic mobility. This can be seen in Fig. 1A, where we plot the log of the rank correlation divided by  $\delta$  as a function of  $\tau$  for various noise amplitudes.

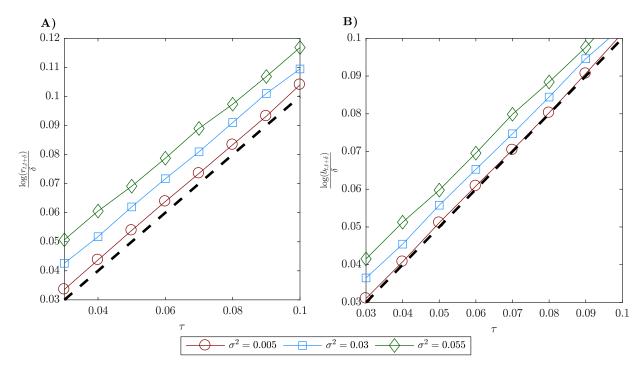


Figure 1: Relaxation time and standard measures of economic mobility. A) Log of Spearman's rank correlation divided by the temporal difference as a function of  $\tau$ . B) Same as A), only on the y axis is the log of the IGE divided by the temporal difference. A-B The dashed black line has a slope 1. The simulations used  $\delta = 20$  years and  $N = 10^4$  people.

**Intragenerational earnings elasticity:** Similarly to the properties of the rank correlation, and as depicted in Fig. 1B, the IGE depends on both the fluctuation amplitude and the reallocation rate.

Transition matrices: As evidenced in Fig. 2A, the transition matrices in RGBM reproduce the asymmetric property of the real world transition matrices and are well-approximated by the Gumbel copula (Fig. 2B). In Fig. 2C we visualize the relationship between Gumbel parameter  $\theta$  and the reallocation parameter  $\tau$  for various fluctuation amplitudes. We find that there is an inverse relationship between  $\theta$  and  $\tau$ , and the Gumbel parameter slope is further determined by the magnitude of  $\sigma$ . As  $\tau$  increases, the value of the  $\theta$  decreases, though disproportionately. We hereby point out that the  $\theta$  parameter and the rank correlation share a direct relationship which cannot be represented analytically. As a way to visualize this relationship in Fig. 2D we plot the rank correlation as a function of  $\tau$ .

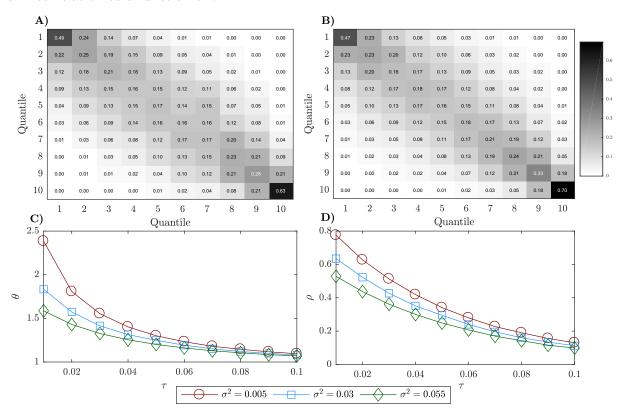


Figure 2: Relaxation time and wealth transition matrices. A) Transition matrix for the stationary regime of RGBM estimated with  $\tau = 0.02$  per year,  $\sigma^2 = 0.01$  per year and  $N = 10^4$  people. B) An example for a transition matrix from data simulated from a Gumbel copula whose parameter  $\theta$  is chosen to be in accordance with the RGBM parameters used in A). C) The relationship between the Gumbel copula parameter  $\theta$  and the reallocation parameter  $\tau$  in the stable state of RGBM. D) The relationship between the Gumbel copula parameter  $\theta$  and the rank correlation  $\rho$  in the stable state of RGBM. The parameters were estimated from a transition matrix in which  $\delta = 20$  years and  $N = 10^4$  people.

#### 3.4 Mobility measures in non-mixing regimes

So far we discussed the relationship between mobility measures and mixing in the stationary regime of RGBM. Studying this relationship in the non-stationary regime, when  $\tau < 0$  is impossible as then the system is not mixing (Stojkoski and Karbevski, 2022). Nonetheless, Spearman's rank correlation, the earnings elasticity and the transition will indicate that there is still some mobility. We visualize this phenomenon numerically in Fig. 3. The results for Spearman's correlation and the IGE suggest that in the negative  $\tau$  regime, the magnitude of the reallocation rate does not impact the extent of mobility. Instead, mobility is dependent on  $\sigma$ , with larger noise amplitudes implying more mobility. Hence, it can be argued that the existence of mobility in the negative regime is solely a result of the randomness present in the system, but not of the reallocation mechanism. More importantly, because of this, individuals are not able to move across each possible rank. The thorough study of mobility in RGBM in this regime is outside of the scope of this paper and left for future work.

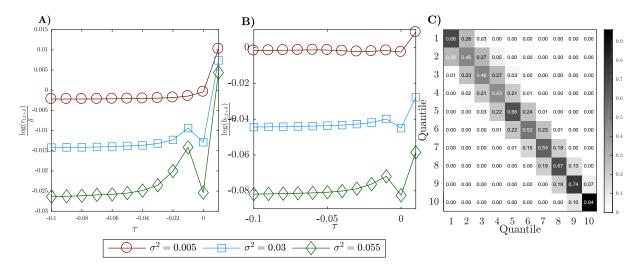


Figure 3: Mobility in the non-mixing regime of RGBM. A) Log of Spearman's rank correlation divided by the temporal difference as a function of  $\tau$ . B) Same as A), only on the y axis is the log of the IGE divided by the temporal difference. C) Transition matrix in RGBM estimated with  $\tau = -0.02$  per year,  $\sigma^2 = 0.01$  per year and  $N = 10^4$  people. A-B The simulations used  $\delta = 20$  years and  $N = 10^4$  people. A-B The results are averaged across 1000 time periods.

#### 4 Discussion and Conclusion

In this paper, we introduced the physical concept of mixing as a relevant phenomenon for quantifying the mobility of an individual between every wealth rank within an economy. We showed that standard mobility measures do not account for this property, whereas measures of mixing do. This is because, as we showed with the relaxation time example, every mobility measure that quantifies the degree of mixing will not be monotonic. This particular characteristic allows measures to quan-

tify whether the system is in a mixing state or not. Whenever the system is not mixing, measures of mixing will suggest no mobility. When the system is mixing, then there is no discrimination between individuals on the basis of their history: it is possible for everyone to move between any ranks in the distribution, and this will happen with certainty in the long run. We used this result to argue that mixing can be seen as a mathematical manifestation of the "American Dream".

Mixing is predicated on the existence of a transformation of wealth which has a steady-state distribution. Studies of wealth inequality often make the hypothesis that the transformation is given by the rescaled wealth. This case was also discussed here. However, a growing body of evidence suggests that in reality rescaled wealth might also be a non-ergodic, and hence a non-mixing observable. For instance, Berman et al. (2020) found that, in the case of RGBM wealth dynamics, negative reallocation ( $\tau < 0$ ) prevails in the US economy. Then, measures such as the relaxation time are undefined and mobility across the whole distribution is non-existent. Nevertheless, another transformation of wealth might exist which is mixing, and measures in terms of it might suggest that there is mixing in the economy. Then mobility between every quantile exists, but can be defined in terms of an another concept. For example, mobility can be defined in terms of growth of wealth or in terms of reduction of the unpredictability of wealth dynamics. Different concepts also require different analytical approaches, as they illuminate the distinct extent to which mobility is socially desirable. In other words, depending on the definition of economic mobility, an increase in economic mobility will not always translate into increased economic welfare. Hence, discovering the relevant wealth transformation is extremely important for policymakers to produce adequate measures for optimizing the mobility within an economy. We refer to Jäntti and Jenkins (2015) for a lengthy discussion on the various concepts of economic mobility and their social implications.

We conclude by noting that in this paper we discussed mixing in terms of wealth mobility, but the same concept can be used for studying income dynamics (Stojkoski et al., 2021b, 2022a). In this context, studying the relationship between mixing and measures of income mobility may represent a fruitful avenue for future research.

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## Appendix 1 Estimation of relaxation time in real systems

Here we present an empirical procedure for estimating the relaxation time. This exact procedure is used in Markov Chains to track the convergence to the steady state and is called the "mixing time" (Aldous and Fill, 2002).

The first step in the procedure is defining a relevant steady state distribution to which the rescaled transformation of wealth converges. We then select a subsample of size K consisting of the individuals that are closest to the mean wealth (i.e. the typical individuals in the economy), track the subsample wealth distribution over time and quantify the difference between the subsample wealth distribution and the steady state distribution with the total variational distance. We stress out that when the relaxation time is estimated using this procedure, one uses an estimate for the wealth distribution based on an empirical histogram. It is widely known that histograms can only resemble the theoretical distribution only to a certain extent because they are a finite sample size approximation. Therefore, in a mixing economy the total variational distance will exhibit two states. First, there will be a relaxation time state during which the two distributions will converge towards each other, and the distance will decrease. After a transitory phase, there will be a stable state. In the stable state the log of the distance reaches a plateau and fluctuates around this plateau. The magnitude of the plateau will be determined by the subsample size: smaller sample sizes will exhibit higher plateau and vice versa. This is a finite sample size effect. In a mixing economy, the total variational distance will be small and exhibit statistical significance in the stable state. The point at which the plateau is reached will, in general, be higher than the relaxation time. Hence, one can use the time point at which the total variational distance becomes significant as an upper bound for the relaxation time and as a sign that the economy is mixing.

Figure A1.1 summarizes the procedure in a fictive example. The blue line is the log of the total variational distance as a function of time. The dashed black line is the slope of the relationship between the statistic and time during the two different states. The inset plots provide snapshots for the wealth distribution of the subsample (i.e., estimates based on a histogram) at different time points (red dashed lines). For comparison, the snapshots also include the form of the selected steady state distribution (black line). Notice that initially, at  $t_1$ , the subsample distribution is very narrow and does not resemble the steady state distribution. The wealths in the subsample evolve, and in  $t_2$  and  $t_3$  the subsample distribution becomes closer to the steady state distribution. Eventually, the subsample distribution resembles the steady state distribution. The upper bound for the relaxation time is the point at which the total variational distance becomes statistically significant. We stress out that in a non-mixing economy, the  $\beta(t)$  will either converge to a fixed value but it will remain statistically insignificant or it will be a divergent quantity. In Appendix Appendix 2 we present an example for the implementation of the procedure in the RGBM model.

We note that the main advantage of the procedure is that it offers a non-parametric approach for estimation of a upper bound for the relaxation time which is independent on the assumption of wealth dynamics. However, this is an expensive procedure to perform in reality as it requires a detailed track for the wealth of a particular set of individuals. This will be even more pronounced if the convergence time to the steady state distribution is slow. We believe that with the rapid development of data gathering methods and the improved understanding of wealth dynamics within a population, some of this shortcoming will be overcome, yielding a more in-depth interpretation of mixing in real economies.

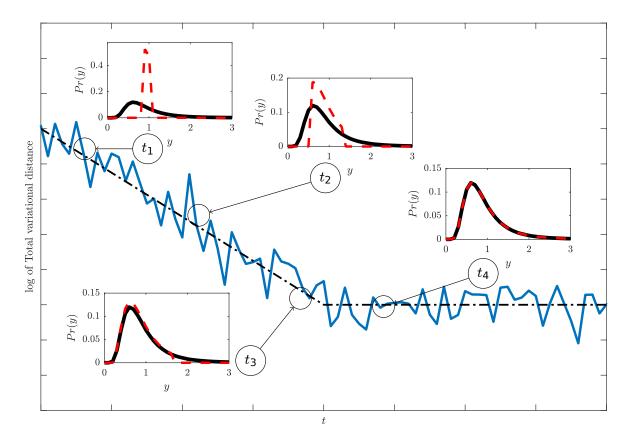


Figure A1.1: Quantifying relaxation times in an empirical system. The blue line shows the log of the total variational distance between the subsample and the target wealth distribution. The black dashed line is the slope of the line which describes the relationship between the log of the distance and time, estimated separately for the relaxation period and the sable state period. The inset plots give snapshots for the empirical form of subsample distribution (red dashed line) and the target distribution at different points in time  $t_1 < t_2 < t_3 < t_4$ .

# Appendix 2 Numerical estimation of Relaxation times in RGBM

We use RGBM to numerically present the procedure described in Section 2. In the concrete example, we focus on the role of the subsample size,  $\tau$  and  $\sigma$  in the duration of the relaxation period and the estimation of the relaxation time measure.

For this purpose, in Figure A2.1A-B we plot the total variational distance (on a log scale) as a function time and vary the subsample size, reallocation rate and the noise amplitude. Intuitively,

the reallocation rate uniquely determines the relaxation time, whereas the noise amplitude has no effect, as argued in Section 3. However, it appears that the subsample size critically determines the behavior of the stable state in the system as it determines the value of the stationary total variational distance  $\beta^*$  (Figure A2.1C). This is because the estimation of the distance relies on the differences between the empirical distribution function (*i.e.* histogram), and the distribution function for the target stationary wealth distribution. Due to the subsample size always being a finite number, in empirical calculations, there will be differences between the empirical distribution and the target distribution, which will be translated in a positive total variational distance. As the subsample size increases, in the limit as the subsample size goes towards infinity, the differences will disappear.

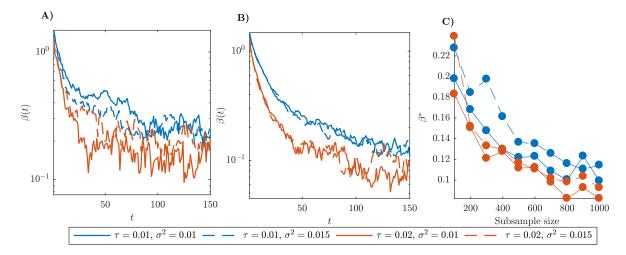


Figure A2.1: **Relaxation time in RGBM.** A) Total variational distance as a function of time for a realization of an RGBM process with a subsample size of  $10^2$  for different noise amplitudes and reallocation parameters. B) Same as A), only with a subsample size of  $10^3$  people. C) The stationary total variational distance as a function of the subsample size for various  $\sigma^2$  and  $\tau$ . In the simulations  $N = 10^5$  people.

# Appendix 3 Definitions of standard mobility measures

**Spearman's rank correlation:** Spearman's rank correlation  $\rho_{t_m,t_n}$  is defined on a joint distribution of wealth at two points in time,  $t_m$  and  $t_n$  ( $t_m < t_n$ ). It is defined as

$$\rho_{t_m,t_n} = 1 - \frac{6\sum_{i} \left[ rg\left(\mathbf{x}_i\left(t_m\right)\right) - rg\left(\mathbf{x}_i\left(t_n\right)\right) \right]^2}{N\left(N^2 - 1\right)},\tag{A3.1}$$

where  $rg(\mathbf{x})$  is the rank transformation of  $\mathbf{x}$ ,  $\mathbf{x}_i(t)$  is the wealth of individual i in period t and N is the population size. This measure is bounded between -1 and 1.  $\rho_{t_m,t_n}=1$  suggests perfect immobility, a state in which there is no change in wealth ranks between the two points in time. Lower values suggest greater economic mobility.

Intragenerational earnings elasticity: The intragenerational earnings elasticity is defined as the slope  $b_{t_m,t_n}$  of the regression

$$\log (x_i(t_n)) = b_0 + b_{t_m,t_n} \log (x_i(t_m)) + u_i,$$
(A3.2)

where  $b_0$  is the intercept and  $u_i$  is the error term. This is a simple linear regression and therefore,

$$b_{t_m,t_n} = \operatorname{corr}\left(\log\left(\mathbf{x}\left(t_n\right)\right), \log\left(\mathbf{x}\left(t_m\right)\right)\right) \frac{\operatorname{var}\left(\log\left(\mathbf{x}\left(t_n\right)\right)\right)}{\operatorname{var}\left(\log\left(\mathbf{x}\left(t_m\right)\right)\right)}, \tag{A3.3}$$

where corr(x, y) is the correlation between the variables x and y and var(x) is the variance of x. As with the rank correlation, lower IGE also indicates greater mobility. However, this measure is unbounded and may take on any real values.

Wealth transition matrix: The wealth transition matrix disaggregates wealth rankings and summarizes economic mobility in a transition matrix  $\mathbf{A}$  in which the elements  $A_{kl}$  quantify the probability that an individual in wealth quantile k in period  $t_m$  is found in wealth quantile l in period  $t_n$ . In a perfectly mobile economy, the entries of the transition matrix are all equal to each other. This would correspond to 0 rank correlation. In an immobile economy, on the other hand, the largest values are concentrated in the diagonal entries. A perfectly immobile case, of rank correlation 1, would correspond to the identity transition matrix.