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Existence and uniqueness of Nash equilibrium in discontinuous Bertrand games: a complete characterization

R. A. Edwards[1] · **R. R. Routledge**[2]

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Abstract

Since (Reny in Econometrica 67:1029–1056, 1999) a substantial body of research has considered what conditions are sufficient for the existence of a pure strategy Nash equilibrium in games with discontinuous payoffs. This work analyzes a general Bertrand game, with convex costs and an arbitrary sharing rule at price ties, in which tied payoffs may be greater than non-tied payoffs when both are positive. On this domain, necessary and sufficient conditions for (i) the existence of equilibrium (ii) the uniqueness of equilibrium are presented. The conditions are intuitively easy to understand and centre around the relationships between intervals of real numbers determined by the primitives of the model.

Keywords Discontinuous payoffs · Existence · Uniqueness · Bertrand competition · Necessary and sufficient conditions

JEL Classification C72 · C62 · D43 · L11

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✉ R. R. Routledge
R.R.Routledge@Liverpool.ac.uk

R. A. Edwards
Robert.Edwards@Nottingham.ac.uk

[1] Business School, University of Nottingham, Nottingham NG8 1BB, UK

[2] Management School, University of Liverpool, Chatham Street, Liverpool L69 7ZH, UK

1 Introduction

The game of Bertrand competition in a homogeneous-good market remains a canonical model of the economic literature because it provides a direct price-making foundation for competitive equilibrium which dispenses with the fiction of the Walrasian auctioneer.[1] The game also remains of theoretical interest because it has discontinuous payoffs, and the existence of equilibrium in such games is often a non-trivial question. In addressing the equilibrium existence question, Reny (1999) used the Bertrand game with linear and symmetric costs to motivate the idea of better-reply security. However, there are many Bertrand games where Reny's pure Nash equilibrium existence result cannot be applied.

The reason for this lack of applicability is that general Bertrand games, with non-linear costs, either fail to be better-reply secure (with convex costs tied payoffs can be higher than non-tied payoffs) or the payoffs cease to be quasiconcave in the seller's own price (which often happens when costs are concave or there is an avoidable fixed cost).[2] Furthermore, the seminal work of Reny (1999), and most of the subsequent extensions, only search for sufficient conditions for the existence of equilibrium in discontinuous games.[3] As a consequence, if a particular condition is violated in these existence theorems, one does not know whether an equilibrium exists or not.

In this work, we study a general Bertrand duopoly with convex costs (not necessarily symmetric) and a tie-breaking rule that permits any deterministic sharing of the market demand at price ties. Furthermore, the price tied at may determine the split in the market demand between the sellers. On this class of games, we try to advance the understanding of the Bertrand game by providing a complete characterization of both the existence and uniqueness of a pure strategy Nash equilibrium. That is, we are able to give necessary, as well as sufficient, conditions for existence and uniqueness of equilibrium. An attractive feature of our conditions is that they are simple to apply: they involve solving for four different prices, which can often be done analytically, and then comparing the intervals determined by these prices.

Following the literature review, Sect. 2 outlines the Bertrand game that we study. Proposition 2.1 presents a characterization of the Bertrand set when it is non-empty. Theorem 2.1 gives a complete characterization of existence and uniqueness of Bertrand equilibrium. In Sect. 3 we explore the intuition of our results by considering three detailed examples. In Sect. 4, we explore the relationship between Bertrand games and the conditions in Reny (1999). In particular, we give a condition for a game to possess a Bertrand equilibrium yet fail to be better-reply secure (Proposition 4.2). Finally, the paper gives some concluding remarks and possible directions for future research.

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[1] Succinct summaries of the literature on Bertrand competition are contained in Vives (1999, Ch. 5) and Baye and Kovenock (2008).

[2] For the case of games where convexity of the cost function results in tied payoffs being higher than non-tied payoffs see, amongst others, Dastidar (1995) and Bagh (2010).

[3] See, for example, Bagh and Jofre (2006) and McLennan et al. (2011). The results in these papers also cannot be applied to the Bertrand game we study because tied payoffs exceeding non-tied payoffs violates their conditions. Carmona (2013) contains a detailed summary of many of the recent contributions to the existence problem in discontinuous games.

1.1 Related literature

This work contributes to the literature on the existence, uniqueness and associated refinements of Nash equilibria in homogeneous-good Bertrand games. The Bertrand price game originated in Joseph Bertrand's 1883 critique of the Cournot model, and has recently received some renewed interest as it is a canonical game with discontinuous payoffs.[4]Dastidar (1995) analyses a Bertrand game in which sellers commit to supplying all the demand forthcoming to them at the prices posted in the market. The sellers have convex cost functions consistent with decreasing returns to scale production functions and the market demand is shared equally between sellers tieing at the minimum price. It is demonstrated that if the sellers are symmetric then the price game possesses a continuum of pure strategy Nash equilibria. If the sellers are not symmetric, then a pure strategy Nash equilibrium still exists, but could be unique. Spulber (1995) considers a price game in which sellers have weakly convex costs determined by a random parameter drawn from an atomless probability distribution. The incomplete information game possesses a symmetric Bayesian Nash equilibrium in which sellers earn positive expected profits, in contrast to zero profits in the classical Bertrand paradox outcome.

Baye and Morgan (1999) study the classical Bertrand game with constant and symmetric marginal costs, but unbounded profits (which could arise from an inelastic market demand). They demonstrate that the price game possesses a continuum of atomless mixed strategy Nash equilibria in which the sellers earn positive profits. Wambach (1999) analyses a Bertrand game in which risk-averse sellers post prices in the market before being fully sure of their marginal cost of production. In this context, sellers price above marginal cost and earn positive expected profits. Hoernig (2002) considers the mixed extension of the (Dastidar 1995) price game with symmetric sellers. It is shown that the game possesses uncountably many mixed equilibria of two different types: (i) sellers could place probability mass upon a finite number of prices (ii) sellers could play a mixed strategy with a connected support, which was atomless on the interior and placed an atom at the upper bound.

Baye and Morgan (2002) study a price game with the "winner-takes-all" sharing rule at minimum price ties. Under this sharing rule, one seller is selected at random, from the sellers tieing at the minimum price, to meet all the market demand forthcoming. It is demonstrated that the left-lower semicontinuity of the monopoly profit function, together with the existence of a monopoly breakeven price, guarantees the existence of a pure strategy Nash equilibrium in which all sellers earned zero profit. Blume (2003) reconsiders the non-existence of pure strategy Nash equilibrium when sellers have asymmetric but constant marginal costs and market demand is split equally at minimum price ties. It is demonstrated that, although there is no pure strategy Nash equilibrium, there exists a continuum of mixed strategy Nash equilibria with the lower cost seller posting the minimum price in the market and the higher cost seller randomizing on an arbitrarily small interval above.

[4] The original article of Bertrand is reprinted in Daugherty (1989). Detailed textbook treatments of the Bertrand game are contained in Mas-Colell et al. (1995, pp. 388–389) and Reny and Jehle (2010, pp. 175–177).

Hoernig (2007) analyses Bertrand games with arbitrary sharing rules at minimum price ties, and profits not necessarily derived from cost and demand primitives. General properties of sharing rules that permitted the application of the (Reny 1999) existence results for pure and mixed strategy equilibria are presented. A pure strategy equilibrium exists whenever the sharing rule was weakly tie-decreasing, coalition monotone, the sum of payoffs was upper semi-continuous and non-tied payoffs were continuous. Tie-decreasing requires that the payoff each tied player receives is non-increasing when an additional player joins the tie, which generates quasiconcave payoffs. Coalition monotone imposes that the sum of the tied players' payoffs increases with the number of tied sellers. With symmetric non-tied payoffs, all players receive zero payoff. With asymmetric non-tied payoffs, one player could earn positive profit.

Chowdhury and Sengupta (2004) apply the Nash equilibrium refinement of coalition-proofness, introduced in Bernheim et al. (1987), to a homogeneous-good price game with convex costs. In their model, the sharing rule at minimum price ties is capacity sharing, which coincides with equal sharing when sellers have symmetric cost functions. They demonstrate the existence of a coalition-proof Nash equilibrium, and its uniqueness is proven when sellers have symmetric cost functions. Saporiti and Coloma (2010) study a price game with a convex variable cost function and a potentially avoidable fixed cost. They prove that the non-subadditivity of the cost function at the tied breakeven price is a necessary and sufficient condition for the existence of a pure strategy Bertrand equilibrium.[5] Bagh (2010) considers a homogeneous-good price game in which sellers have convex cost functions and the market demand is potentially discontinuous. At minimum price ties, an arbitrary and deterministic rule defines how the demand is split between the sellers. Sufficient conditions for the existence of a pure strategy Nash equilibrium, with either symmetric or asymmetric sellers, are provided. These conditions are simply defined by the intersection of relevant intervals of real numbers.

Dastidar (2011a) reexamines the concepts of subadditivity and superadditivity of the cost function in Bertrand games. If sellers have symmetric subadditive cost functions (which generalizes the idea of concave cost functions) then there fails to exist either a pure strategy, or mixed strategy, Nash equilibrium.[6] However, if sellers have symmetric superadditive cost functions then the price games possess a continuum of pure strategy Nash equilibria. Dastidar (2011b) demonstrates that, even if all sellers have subadditive costs, existence of Nash equilibrium is restored by introducing asymmetries in the cost functions. Jann and Schottmuller (2015) study the correlated equilibria of a homogeneous-good game with constant marginal costs and demonstrate that, with symmetric marginal costs, the only correlated equilibrium is the standard Bertrand paradox with both sellers setting prices equal to marginal cost. Amir and Evstigneev (2018) study a standard Bertrand game with sellers having symmetric and constant marginal costs. They provide a necessary and sufficient

[5] Panzar (1989) is an early study of the subadditivity of cost functions in the context of contestable market theory.

[6] A succinct proof of the non-existence of a mixed strategy Nash equilibrium when sellers had concave costs, arising from the presence of an avoidable fixed cost, is contained in Baye and Kovenock (2008).

condition for the uniqueness of pure strategy Nash equilibrium which contains many of the standard models in the literature, and even permits upward-sloping demand functions.

This work is also related to games in which sellers choose both prices and quantities. This type of game is similar to the class of Bertrand–Edgeworth games. It is well-known that Bertrand–Edgeworth games often fail to possess a pure strategy Nash equilibrium. However, when sellers are able to choose the quantities they are willing to supply to the market price-quantity games often possess pure strategy Nash equilibria. Dixon (1992) shows that the competitive equilibrium price is the unique Nash equilibrium if all but one of the sellers can supply the market demand at that price without going bankrupt. More recently, Bos and Vermeulen (2021) give sufficient conditions for the existence of a pure strategy Nash equilibrium when demand may be discontinuous. They identify cases where the outcome under price-quantity competition is the same as the Bertrand price-only game.

The main contribution of this paper is twofold. First, to provide a complete characterization of the Bertrand equilibrium set, and to give necessary and sufficient conditions for the uniqueness of a Bertrand equilibrium. This contrasts with the literature we have just summarized which only provides sufficient conditions for existence/uniqueness of Bertrand equilibrium. Second, we present a number of results relating the conditions of Reny (1999) to Bertrand games. Specifically, we give a sufficient condition for a game to possess a Bertrand equilibrium yet fail to be better-reply secure.

2 The Bertrand game

Consider a set of sellers $N = \{1, 2\}$ who are producing a single perfectly homogeneous good. There is a **market demand** for the good $D : \mathfrak{R}_+ \to \mathfrak{R}_+$. The following standard assumptions are imposed upon the demand function.

Assumption 2.1 The market demand D is such that there exist positive finite real numbers y, z, satisfying $D(0) = y$ and $D(p) = 0$ for all $p \geq z$. The market demand is continuous and strictly decreasing on $[0, z]$.

Each seller is endowed with a **cost function** $C_i : \mathfrak{R}_+ \to \mathfrak{R}_+$, which gives the cost of producing any quantity of output. The following standard assumptions are imposed upon the cost functions.

Assumption 2.2 Each seller's cost function C_i is continuous, convex, strictly increasing and satisfies $C_i(0) = 0$.

In the classical Bertrand price game, the sellers simultaneously and independently post prices in the marketplace with a commitment to meet all the demand forthcoming at the prices posted. If the sellers tie at the same price, then a sharing rule specifies how the market demand is split between the sellers. In this work, an arbitrary sharing rule will be analyzed. A **sharing rule** is a set of functions (g_1, g_2) satisfying $g_i : \mathfrak{R}_+ \to (0, 1)$ and $\sum_{i=1}^{2} g_i(x) = 1$ for all $x \in \mathfrak{R}_+$. This means the functions (g_1, g_2) give the share of the market demand each seller receives, contingent upon the price tied at. The standard equal sharing rule that is used widely in the literature is captured as a special case when we set $g_1(x) = g_2(x) = 1/2$ for all $x \in \mathfrak{R}_+$. However, by specifying the sharing rule in this way we are able to permit uncountably many other possible tie-breaking possibilities.

Given these primitives we can now specify the payoffs of the sellers. Seller $i's$ monopoly profit, as a function of price is:

$$\pi_i(p) = pD(p) - C_i(D(p)).$$

If seller i ties at the same price as seller j then the profit obtained is:

$$\hat{\pi}_i(p) = pg_i(p)D(p) - C_i(g_i(p)D(p)).$$

Given a pair of prices (p_i, p_j) the payoffs of the sellers in the Bertrand game can be succinctly summarized as:

$$u_i(p_i, p_j) = \begin{cases} \pi_i(p_i), & \text{if } p_i < p_j; \\ \hat{\pi}_i(p_i), & \text{if } p_i = p_j; \\ 0 & \text{if } p_i > p_j. \end{cases} \tag{1}$$

A pair of prices (p_1^*, p_2^*) is a **Bertrand equilibrium** if $u_1(p_1^*, p_2^*) \geq u_1(p, p_2^*)$ for every $p \in [0, z]$, and $u_2(p_1^*, p_2^*) \geq u_2(p_1^*, p)$ for every $p \in [0, z]$. The primitives of a **Bertrand game with an arbitrary sharing rule** can be summarized as $G = \{N, D, (C_i, g_i, u_i)_{i \in N}\}$.

The monopoly price of seller i is a price satisfying:

$$p_i^M = \arg\max_{p \in [0,z]} \pi_i(p).$$

As $\pi_i(p)$ is continuous on [0, z], it follows from the Weierstrass theorem that a maximizer is well-defined.

Assumption 2.3 The functions $\hat{\pi}_i(p)$ satisfy $\hat{\pi}_i(p_j^M) > 0$.

This assumption rules out the extreme case where one seller has such a cost advantage that they can post their monopoly price in the market and the other seller cannot profitably tie at that price. The reason for ruling this out is that if it were the case that a seller's monopoly price were lower than the price at which the other seller's shared profit becomes positive then the game possesses a continuum of trivial equilibria in which the low-cost seller chooses their monopoly price and the high-cost seller chooses any higher price.

Define the function $\tilde{\pi}_i(p)$ to be:

$$\tilde{\pi}_i(p) = \pi_i(p) - \hat{\pi}_i(p).$$

When $\tilde{\pi}(p) > 0$, at price p seller i prefers to be the sole monopoly supplier to the market rather than sharing the market demand. When $\tilde{\pi}(p) < 0$, at price p seller i prefers to share the market demand with the competitor rather than serving the market as a monopolist.

Assumption 2.4 The functions $\pi_i(\cdot)$, $\hat{\pi}_i(\cdot)$ and $\tilde{\pi}(\cdot)$ are strictly concave on $(0, z)$ and achieve strictly positive values.

Assumption 2.4 guarantees that the monopoly prices, p_i^M, are unique. The following two prices will play a key role in the results:

$$l_i = \{p \in (0, z) : \hat{\pi}_i(p) = 0\},$$
$$h_i = \{p \in (0, z) : \tilde{\pi}_i(p) = 0\}.$$

l_i defines the breakeven price for seller i when both firms choose the same price and share the market demand. h_i defines the price at which seller i is indifferent between the competitor choosing the same price such that the market demand is shared or a higher price such that seller i is the monopoly supplier. The following four lemmas establish several important properties of these key prices. The proofs are contained in the Appendix.

Lemma 2.1 *The two prices, l_i and h_i, are well-defined and unique. Their relationship on the real line is $l_i \leq h_i < p_i^M$.*

Lemma 2.2 *If (p_1^*, p_2^*) is a Bertrand equilibrium, then $p_1^* = p_2^*$.*

Lemma 2.3 *If $p \in [0, l_i)$ then $\hat{\pi}_i(p) < 0$. If $p \in (l_i, z)$, then $\hat{\pi}_i(p) > 0$.*

Lemma 2.4 *If $p \in [0, h_i)$ then $\tilde{\pi}_i(p) < 0$. If $p \in (h_i, z)$, then $\tilde{\pi}_i(p) > 0$.*

Let the set $P = \cap_{i=1}^2 [l_i, h_i]$.

Proposition 2.1 *Fix a Bertrand game with an arbitrary sharing rule $G = \{N, D, (C_i, g_i, u_i)_{i \in N}\}$. The price vector (p_1^*, p_2^*) is a Bertrand equilibrium if, and only if, $p_1^* = p_2^* \in P$.*

Proof First, the sufficiency part will be demonstrated. Suppose the sellers play prices $p_1^* = p_2^* \in P$. As $p_1^* \geq l_1$ and $p_2^* \geq l_2$ it follows from Lemma 2.3, and the definition of l_i, that $\hat{\pi}_1(p_1^*) \geq 0$ and $\hat{\pi}_2(p_2^*) \geq 0$. If either seller were to deviate to a higher price $p > p_1^* = p_2^*$ they would obtain zero profit. Therefore, this would not be a profitable deviation and no seller can profitably deviate by increasing their price. As $p_1^* \leq h_1$ and $p_2^* \leq h_2$, it follows from Lemma 2.4, and the definition of h_i, that $\tilde{\pi}_1(p_1^*) \leq 0$ and $\tilde{\pi}_2(p_2^*) \leq 0$. Therefore, $\hat{\pi}_1(p_1^*) \geq \pi_1(p_1^*)$ and $\hat{\pi}_2(p_2^*) \geq \pi_2(p_2^*)$. The strict concavity of $\pi_i(\cdot)$, and $h_i < p_i^M$ in Lemma 2.1, imply $\pi_i(p) \leq \hat{\pi}_i(p_i^*)$ for every $p \in [0, p_i^*]$. Consequently, no seller can profitably deviate by reducing their price. Hence, (p_1^*, p_2^*) is a Bertrand equilibrium.

Second, the necessity part will be demonstrated. Let (p_1^*, p_2^*) be a Bertrand equilibrium. It follows from Lemma 2.2 that $p_1^* = p_2^*$. Suppose, without loss of generality, that $P = [l_1, h_2]$. If $p_1^* = p_2^* < l_1$ then $\hat{\pi}_1(p_1^*) < 0$, and seller 1 could profitably deviate by increasing their price and obtaining zero profit. If $p_1^* = p_2^* > h_2$ and $p_1^* = p_2^* \leq p_2^M$, then $\pi_2(p_2^*) > \hat{\pi}_2(p_2^*)$ and seller 2 could profitably deviate to $p_2^* - \epsilon$, $\epsilon > 0$. If $p_1^* = p_2^* > h_2$ and $p_1^* = p_2^* > p_2^M$, then seller 2 could profitably deviate to p_2^M and obtain monopoly profit.

□

The following theorem, which is the main contribution of the paper, is stated without a proof as it is a corollary of the previous proposition.

Theorem 2.1 *Fix a Bertrand game with an arbitrary sharing rule* $G = \{N, D, (C_i, g_i, u_i)_{i \in N}\}$. *Then*:

(i) *there exists a Bertrand equilibrium if, and only if,* $P \neq \emptyset$
(ii) *there exists a unique Bertrand equilibrium if, and only if,* $|P| = 1$.

3 Illustrative examples

In this section we provide further insight regarding the intuition underlying Theorem 2.1 using three examples. Example 1 is a game with linear and symmetric costs, but an asymmetric sharing rule. The existence of a Nash equilibrium follows from Hoernig (2007), Theorem 3, because the sharing rule is norm-tie decreasing, sum upper semicontinuous and coalition monotone. With linear, symmetric costs, this would hold even if the share at price ties varied with price instead of being a constant. Consequently, Reny (1999) conditions can be applied to demonstrate the existence of Bertrand equilibrium. However, Theorem 2.1 also permits us to demonstrate uniqueness, and to easily calculate the equilibrium (often analytically).

The merits of our results are particularly evident in Examples 2 and 3 where the payoffs of the sellers fail to be better-reply secure. Consequently, Reny (1999) conditions and the later generalizations of those results cannot be applied. Under these circumstances, one would normally need to check each set of price strategies and the corresponding payoffs to establish existence and/or uniqueness, which is challenging with discontinuous payoffs. In contrast, using Theorem 2.1 it is a straightforward exercise to answer these questions by calculating and comparing the four key prices.

3.1 Example 1: a game with symmetric linear costs and asymmetric sharing

Suppose the market demand is given by $D(p) = \max\{0, 10 - p\}$. The cost functions of the sellers are $C_1(q) = C_2(q) = 2q$. The sharing rule at price ties is:

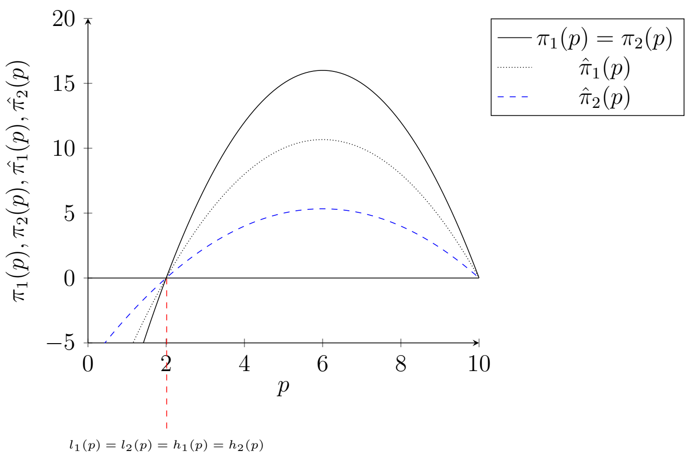

Fig. 1 Symmetric linear costs and asymmetric sharing

$$g_1(p) = \frac{2}{3} \quad \text{and} \quad g_2(p) = \frac{1}{3}.$$

Calculation of the l_i and h_i prices yields:

$$l_1 = h_1 = l_2 = h_2 = 2.$$

It follows from Theorem 2.1 that the games possesses a unique Bertrand equilibrium with $p_1^* = p_2^* = 2$.

Figure 1 illustrates the structure of the payoff functions, where the monopoly profit function is identical for both sellers due to the symmetry of the cost functions. The intuition for the uniqueness of Bertrand equilibrium follows from both sellers having to set the same price in an equilibrium. If they set a price less than 2 both sellers earn negative profit. Either seller could choose to deviate to a higher price to earn a non-negative payoff. Tieing at any price above 2, either seller could profitably deviate to a lower price to earn their higher monopoly profit.

3.2 Example 2: a game with asymmetric non-linear costs and asymmetric sharing

The market demand is given by $D(p) = \max\{0, 10 - p\}$. The cost functions of the sellers are given by $C_1(q) = q^2$ and $C_2(q) = 2q^2$. The sharing rule at price ties is:

$$g_1(p) = \frac{1}{4} \quad \text{and} \quad g_2(p) = \frac{3}{4}.$$

Calculation of the l_i and h_i prices yields:

$$l_1 = 2, \quad h_1 = 5\frac{5}{9} \quad l_2 = 6, \quad h_2 = 7\frac{7}{9}.$$

Therefore

$$P = \cap_{i=1}^{2}[l_i, h_i] = \emptyset$$

and it follows from Theorem 2.1 that the game fails to possess a Bertrand equilibrium. Figure 2 illustrates the structure of the payoff functions, where the cost asymmetry generates asymmetric monopoly payoff functions.

To see the intuition for the non-existence of Bertrand equilibrium, recall that any equilibrium must involve the sellers setting the same price. For any price below l_2, seller 2 earns negative profit and would profitably deviate to a higher price to achieve zero profit. For any price between l_2 and h_2, seller 2 is able to earn a non-negative profit but there exists a profitable deviation for seller 1 who would undercut to serve the entire market as a monopolist. For any price above h_2 where market demand is positive, both sellers would choose to undercut their competitor and become the monopoly supplier. If either seller undercuts, they will also reduce their price to their monopoly price if the original price is above their monopoly price. Hence, there exists no pure strategy Bertrand equilibrium.

3.3 Example 3: a game with asymmetric non-linear costs and symmetric sharing

Suppose the market demand is given by $D(p) = \max\{0, 10 - p\}$. The cost functions of the sellers are given by $C_1(q) = q^2$ and $C_2(q) = 3q^2$. The sharing rule at price ties is:

$$g_1(p) = \frac{1}{2} \quad \text{and} \quad g_2(p) = \frac{1}{2}.$$

Calculation of the l_i and h_i prices yields:

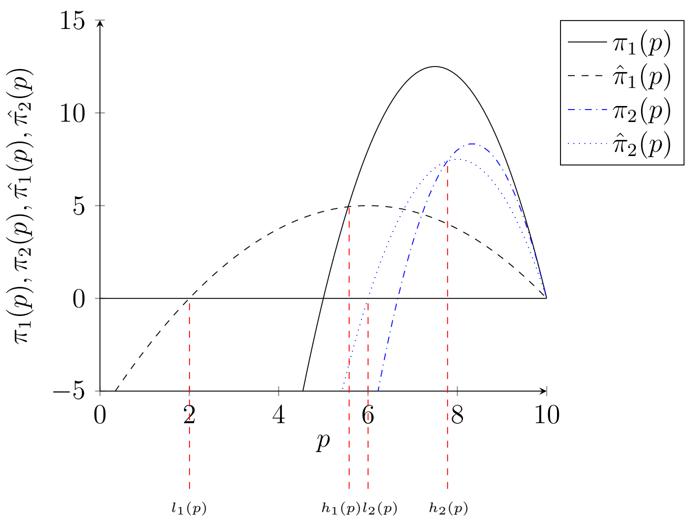

Fig. 2 Asymmetric non-linear costs and asymmetric sharing

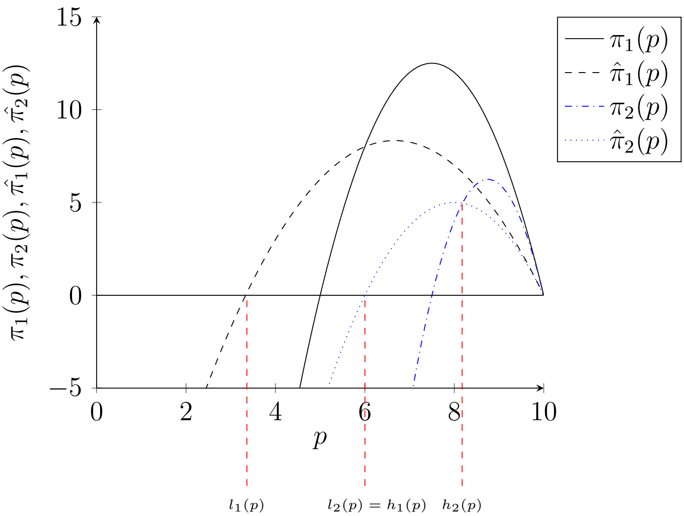

Fig. 3 Asymmetric non-linear costs and symmetric sharing

$$l_1 = 3\frac{1}{3}, \quad h_1 = 6, \quad l_2 = 6, \quad h_2 = 8\frac{2}{11}.$$

Consequently

$$P = \cap_{i=1}^{2}[l_i, h_i] = 6.$$

It follows from Theorem 2.1, and its proof, that this game possesses a unique Bertrand equilibria with $p_1^* = p_2^* = 6$.

Figure 3 illustrates the structure of the payoff functions. It is easily verified by inspection that both sellers posting the same price of 6 is a Bertrand equilibrium.

4 The Reny (1999) conditions and Bertrand games

In this section, we look a little closer into the relationship between Bertrand games and the conditions of Reny (1999). This is of interest in its own right as Reny used the Bertrand game with linear and symmetric costs to motivate the idea of better-reply security. Also, as Example 3 in the previous section demonstrates, a Bertrand game can fail to be better-reply secure and still possess a Nash equilibrium - confirming that better-reply security (along with compact strategy spaces and quasi-concave payoff functions) is only a sufficient condition for the existence of a Nash equilibrium.

A little notation will be helpful. For any Bertrand game with an arbitrary sharing rule $G = \{N, D, (C_i, g_i, u_i)_{i\in N}\}$, let $X = [0, z] \times [0, z]$ denote the joint strategy space.

For any strategy profile $(p_i, p_j) \in X$, let $u(p_i, p_j) = (u_i(p_i, p_j), u_j(p_i, p_j))$ be the vector of payoffs as defined in Eq. (1). The graph of the payoffs, $\Gamma(G)$, is given by:

$$\Gamma(G) = \{((p_i, p_j), u(p_i, p_j)) \in X \times \mathfrak{R}^2\}.$$

Let $\overline{\Gamma(G)}$ be the closure of $\Gamma(G)$. Player i is said to be able to **secure a payoff** of $\alpha \in \mathfrak{R}$ at $(p_i, p_j) \in X$ if there exists a $p_i' \in [0, z]$ such that $u_i(p_i', p_j') \geq \alpha$ for every $p_j' \in B_\epsilon(p_j)$ with $B_\epsilon(p_j)$ being an open ball (interval) of radius ϵ centred on p_j. A game G is **payoff secure** if, for every $((p_i, p_j), u(p_i, p_j)) \in X \times \mathfrak{R}^2$, and any $\epsilon > 0$, player i can secure a payoff of $u_i(p_i, p_j) - \epsilon$ at (p_i, p_j), and player j can secure a payoff of $u_j(p_i, p_j) - \epsilon$ at (p_i, p_j). A game G is **reciprocal upper semicontinuous** if, for any $((p_i, p_j), (u_i, u_j)) \in X \times \mathfrak{R}^2 \in \overline{\Gamma(G)} \setminus \Gamma(G)$ there is a player, say i, such that $u_i(p_i, p_j) > u_i$. The game G is **better-reply secure** if, whenever $((p_i, p_j), (u_i, u_j)) \in X \times \mathfrak{R}^2 \in \overline{\Gamma(G)}$, (p_i, p_j) is not a Nash equilibrium, then one player, say player i, can secure a payoff above u_i at (p_i, p_j).

Lemma 4.1 *Let $b_i \in [0, z]$ be the unique price satisfying $\pi_i(b_i) = 0$. If C_i is strictly convex, then $l_i < b_i < h_i$.*

Proof From the definition of b_i:

$$\pi_i(b_i) = b_i D(b_i) - C_i(D(b_i)) = 0.$$

Rearranging the expression yields:

$$b_i = \frac{C_i(D(b_i))}{D(b_i)}. \tag{2}$$

The shared profit at b_i is:

$$\hat{\pi}_i(b_i) = b_i g_i(b_i) D(b_i) - C_i(g_i(b_i) D(b_i)).$$

From Eq. (2) we have:

$$\hat{\pi}_i(b_i) = \frac{C_i(D(b_i))}{D(b_i)} g_i(b_i) D(b_i) - C_i(g_i(b_i) D(b_i)).$$

To demonstrate $\hat{\pi}_i(b_i) > 0$, we can rearrange the above expression to:

$$\frac{C_i(D(b_i))}{D(b_i)} - \frac{C_i(g_i(b_i) D(b_i))}{g_i(b_i) D(b_i)} > 0. \tag{3}$$

The final inequality following from the strict convexity of C_i. Together $\hat{\pi}_i(b_i) > 0$ and $\pi_i(b_i) = 0$ imply $b_i \in (l_i, h_i)$.

$\square$

Lemma 4.1 demonstrates that the monopoly breakeven price is above the breakeven price when the two sellers tie, but below the price at which the tied payoff and monopoly payoff are equal. As was demonstrated in Reny (1999), establishing that a game is better-reply secure is often a difficult task. It is often easier to show that a game is payoff secure, and reciprocal upper semicontinuous. The two conditions together imply that a game is better-reply secure. The next result shows that under mild conditions Bertrand games often fail to be even payoff secure.

Proposition 4.1 *Fix a Bertrand game with an arbitrary sharing rule* $G = \{N, D, (C_i, g_i, u_i)_{i\in N}\}$. *If (at least) one of the sellers has a strictly convex cost function, then the game is not payoff secure.*

Proof Suppose seller $i's$ cost function is strictly convex. Consider the price profile $p_i = p_j = b_i$. It follows from Lemma 4.1 that $u_i(b_i, b_i) = \hat{\pi}_i(b_i) > 0$. Choose an $\epsilon > 0$ such that $u_i(b_i, b_i) = \hat{\pi}_i(b_i) - \epsilon > 0$. Seller i cannot secure $u_i(b_i, b_i) - \epsilon = \hat{\pi}_i(b_i) - \epsilon$ at (b_i, b_i). To see this, suppose seller i plays $p'_i \leq b_i$ then $u_i(p'_i, p'_j) \leq 0$ for every $p'_j > b_i$ because $\pi_i(p) \leq 0$ for every $p \leq b_i$. If seller i plays $p'_i > b_i$ then $u_i(p'_i, p'_j) = 0$ for every $p'_j < b_i$. It follows that seller i cannot secure payoff $\hat{\pi}_i(b_i) - \epsilon$ at $p_i = p_j = b_i$.

□

Proposition 4.2 *Fix a Bertrand game with an arbitrary sharing rule* $G = \{N, D, (C_i, g_i, u_i)_{i\in N}\}$. *If the set P is non-empty and* $\max P < b_j$, *then the game possesses a Bertrand equilibrium and is not better-reply secure.*

Proof If the set P is non-empty it follows from Theorem 2.1 that the game possesses a Bertrand equilibrium. To see that the game is not better-reply secure, consider any price $p^* \in (\max P, \min\{p_i^M, b_j\})$ and two sequences of prices:

$$p_{i,k} = p^* - \frac{p^*}{k}$$
$$p_{j,k} = p^* + \frac{p^*}{k}.$$

The limit of both sequences is p^*. As seller $i's$ sequence is approaching p^* from below, and seller $j's$ sequence is approaching p^* from above, the limit of the payoffs is converging to:

$$lim_{k\to\infty} u(p_{i,k}, p_{j,k}) = (\pi_i(p^*), 0) \in \overline{\Gamma(G)} \setminus \Gamma(G)$$

The price vector $p_i = p_j = p^*$ is not a Nash equilibrium. The payoffs $(u_i, u_j) = (\pi_i(p^*), 0)$ are in the closure of the graph. However, neither seller can secure a higher payoff. To see this, note that if seller i played $p'_i \leq p^*$ then $u_i(p'_i, p'_j) \leq \pi_i(p^*)$ for all $p'_j > p^*$ as $p^* < p_i^M$. If seller i played $p'_i > p^*$ then $u_i(p'_i, p'_j) = 0$ for all $p'_j < p^*$. If seller j played $p'_j \leq p^*$ then $u_j(p'_i, p'_j) < 0$ for all $p'_i > p^*$ because $p^* < b_j$. If seller j played $p'_j > p^*$ then $u_j(p'_i, p'_j) = 0$ for all $p'_i < p^*$.

Therefore no seller can secure a higher payoff than $(u_i, u_j) = (\pi_i(p^*), 0)$ at $p_i = p_j = p^*$.

□

If one inspects the graphs of the profit functions in Example 3 in the previous section, it is clear the game satisfies the conditions in Proposition 4.2. A natural question to ask is: if the sellers have symmetric costs, and the sharing rule may be asymmetric, is the game better-reply secure? The next result shows that all Bertrand duopolies with symmetric costs are better-reply secure.

Proposition 4.3 *Fix a Bertrand game with an arbitrary sharing rule $G = \{N, D, (C_i, g_i, u_i)_{i \in N}\}$. If the sellers have symmetric cost functions, $C_i = C_j$, then the game is better-reply secure.*

Proof There are three cases to consider. First, suppose $((p_i, p_j), (u_i, u_j)) \in \Gamma(G)$ and $p_i = p_j$. If (p_i, p_j) is not a Nash equilibrium, then one seller, say seller i, has a price p_i' such that $u_i(p_i', p_j') > u_i(p_i, p_j)$ for all $p_j' \in B_\epsilon(p_j)$ with $\epsilon > 0$ being sufficiently small. Second, suppose $((p_i, p_j), (u_i, u_j)) \in \Gamma(G)$, $p_i \neq p_j$, and (p_i, p_j) is not a Nash equilibrium. Without loss of generality, suppose $p_i < p_j$. If $\pi_i(p_i) > 0$, then seller j can secure a higher payoff by choosing $p_j' = p_i - \epsilon$. If $\pi_i(p_i) = 0$, then seller i can secure a higher payoff by choosing $p_i' = p_i + \epsilon$. If $\pi_i(p_i) < 0$, then seller i can secure a higher payoff by choosing $p_i' = p_j + \epsilon$. Finally, suppose $((p_i, p_j), (u_i, u_j)) \in \overline{\Gamma(G)} \setminus \Gamma(G)$, $p_i = p_j$ (as the payoffs are in the closure of the graph) and (p_i, p_j) is not a Nash equilibrium. As the payoffs are in the closure of the graph, but not the graph, it must be that one seller, say seller i, is obtaining his monopoly profit and seller j is obtaining zero: $(u_i, u_j) = (\pi_i(p_i), 0)$. If $\pi_i(p_i) > 0$, seller j can secure a higher payoff by choosing $p_j' = p_i - \epsilon$. If $\pi_i(p_i) < 0$, seller i can secure a higher payoff by choosing $p_i' = p_i + \epsilon$. If $\pi_i(p_i) = 0$, this implies $p_i = p_j = b_i = b_j$, and from Theorem 2.1 and Lemma 4.1, this means $p_i = p_j \in P$ and contradicts (p_i, p_j) not being a Nash equilibrium.

□

However, with more than two sellers, symmetric costs do not guarantee better-reply security in Bertrand games. Throughout the paper we have focussed upon duopolies. If one considers oligopolies, then it is straightforward to find Bertrand games with symmetric costs, and equal sharing at price ties, which fail to be better-reply secure. To see this, consider a Bertrand game with three sellers $N = \{1, 2, 3\}$ each of whom has the same cost function $C_i(q) = q^2$. Suppose the market demand is $D(p) = \max\{0, 10 - p\}$. There is equal sharing at price ties, so $g_i(p) = 1/t$, with $t \in \{2, 3\}$, depending on whether a seller ties with one or two other sellers at the minimum price. The monopoly break-even price is $b_i = 5$ for all sellers (because of their being symmetric). The breakeven price when two sellers tie at the minimum price is $3\frac{1}{3}$, and $2\frac{1}{2}$ when all three sellers tie.

Consider the price vector $(p_1, p_2, p_3) = (3\frac{1}{3}, 3\frac{1}{3}, 4)$. At this price vector, sellers 1 and 2 serve the market, and split the demand equally between them. All three sellers earn zero profit; seller three because they do not post the minimum price in the market. This price vector is *not* a Nash equilibrium because seller three can do better by deviating to $p_3' = 3\frac{1}{3}$ and sharing the market demand with the other two sellers: $u_i(3\frac{1}{3}, 3\frac{1}{3}, 3\frac{1}{3}) = \frac{200}{81}$ for every $i \in N$. But, despite not being a Nash equilibrium, no player can secure a payoff greater than zero at $(p_1, p_2, p_3) = (3\frac{1}{3}, 3\frac{1}{3}, 4)$. This is because seller 3 has to tie precisely at $3\frac{1}{3}$ to obtain a higher payoff. This means seller 3 cannot secure a higher payoff. Consequently, the game is not better-reply secure.

5 Concluding remarks

The aim of this work has been to advance the understanding of homogeneous-good Bertrand games with discontinuous payoffs by providing necessary and sufficient conditions for both the existence and uniqueness of a pure strategy Nash equilibrium. This is in contrast to much of the literature on this canonical game which has tended to look just for sufficient conditions for existence and uniqueness. The model presented here permits tied payoffs to be greater than non-tied payoffs, and consequently, the general results from the literature on discontinuous games, such as the seminal work of Reny (1999), and subsequent extensions, are not always applicable to the model. Furthermore, the model contains, as a special case, the most widely studied case in the literature: constant symmetric marginal costs and equal sharing at minimum price ties.

The results indicate that existence and uniqueness of a Bertrand equilibrium can be characterized by solving for four key prices and then comparing the intervals of real numbers determined by these prices. As the examples in the previous section demonstrate, this can often be accomplished analytically. To conclude, we suggest two possible avenues for further research on this topic.

- Throughout the paper it has been taken as given that the game is one of complete information. Indeed, apart from the works of Spulber (1995) and Wambach (1999), there are few papers which extend the homogeneous-good Bertrand game to permit uncertainty or possible asymmetries of information amongst the sellers. This is probably the consequence of the discontinuous payoffs, and there being few existence results for discontinuous games with incomplete information. However, He and Yannelis (2015) generalized the (Reny 1999) pure strategy Nash equilibrium existence result to games of asymmetric information in which players' strategies must be measurable with respect to their private information. It would be straightforward to model a homogeneous-good Bertrand game with asymmetric information, in which the market primitives (costs and demand) could be state dependent. It would be interesting to know whether, on this richer domain, necessary and sufficient conditions for the existence and uniqueness of a Bayesian Bertrand equilibrium could be found.

- The sharing rule that we have permitted at minimum price ties has been deterministic and modelled by functions mapping the price tied at into the share of the market demand received. This is similar to the approach taken by Bagh (2010). An alternative way of proceeding would be to consider the abstract properties of sharing rules. This was done by Hoernig (2007) in searching for sufficient conditions for the existence of a Bertrand equilibrium.[7] Whether this approach, of not directly specifying the sharing rule but considering abstract properties it may satisfy, could be pursued in searching for necessary and sufficient conditions for existence and/or uniqueness of equilibrium is an interesting open question.

Appendix

This Appendix contains the proofs of the four lemmas that were used to prove Proposition 2.1.

Proof of Lemma 2.1 To see that l_i and h_i exist, are unique, and are well-defined, note that $\hat{\pi}_i(0) < 0$ and $\tilde{\pi}_i(0) < 0$. From Assumption 2.4, the functions $\hat{\pi}_i(\cdot)$ and $\tilde{\pi}_i(\cdot)$ achieve strictly positive values on $(0, z)$. Furthermore, from Assumption 2.4, the strict concavity of the functions $\hat{\pi}_i(\cdot)$ and $\tilde{\pi}_i(\cdot)$ implies there are unique prices l_i and h_i, in $(0, z)$, such that $\hat{\pi}_i(l_i) = 0$ and $\tilde{\pi}_i(h_i) = 0$.

As $\tilde{\pi}_i(p) \leq 0$ for every $p \in [0, h_i]$ and $\tilde{\pi}_i(p) > 0$ for every $p \in (h_i, z)$, to prove that $l_i \leq h_i < p_i^M$, it suffices to show that $\tilde{\pi}_i(l_i) \leq 0$ and $\tilde{\pi}_i(p_i^M) > 0$.

As $\hat{\pi}_i(l_i) = 0$ this implies $\tilde{\pi}_i(l_i) = \pi_i(l_i) - \hat{\pi}_i(l_i) = \pi_i(l_i)$. Therefore:

$$0 = \hat{\pi}_i(l_i) = l_i g_i(l_i)D(l_i) - C_i(g_i(l_i)D(l_i)) \geq l_i g_i(l_i)D(l_i) - g_i(l_i)C_i(D(l_i))$$

where the inequality follows from the convexity of $C_i(\cdot)$. The right-hand side of the inequality is $g_i(l_i)\pi_i(l_i)$. Hence, $\pi_i(l_i) \leq 0$, and consequently, $\tilde{\pi}_i(l_i) \leq 0$.

To see that $\tilde{\pi}_i(p_i^M) > 0$ note that:

$$\hat{\pi}_i(p_i^M) = p_i^M g_i(p_i^M)D(p_i^M) - C_i(g_i(p_i^M)D(p_i^M)) = p_i^M D(p') - C_i(D(p'))$$

with $p' \in (p_i^M, z)$ being the unique price satisfying $D(p') = g_i(p_i^M)D(p_i^M)$. Therefore:

$$\hat{\pi}_i(p_i^M) = p_i^M D(p') - C_i(D(p')) < p'D(p') - C_i(D(p')) \leq \pi_i(p_i^M).$$

Hence, $\tilde{\pi}_i(p_i^M) = \pi_i(p_i^M) - \hat{\pi}_i(p_i^M) > 0$.

□

Proof of Lemma 2.2 We can restrict attention to prices in the interval $[0, z]$. Suppose, a contradiction: that (p_1^*, p_2^*) is a Bertrand equilibrium and $p_1^* \neq p_2^*$. Without loss

[7] The existence of a pure strategy Bertrand equilibrium is often highly sensitive to the sharing rule at minimum price ties. For example, with symmetric concave, or subadditive, costs a pure strategy Nash equilibrium fails to exist under the standard equal sharing rules, but does exist under the winner-takes-all sharing rule. See, for example, Vives (1999, p. 119) and Dastidar (2011a).

of generality, assume $p_1^* < p_2^*$. There are three possible cases to consider. Case 1: $p_1^* < p_2^* \leq p_1^M$. If $\pi_1(p_1^*) < 0$ then 1 could profitably deviate to playing price equal to z. If $\pi_1(p_1^*) \geq 0$, the concavity of $\pi_1(\cdot)$ implies that $\pi_1(p_1^* + \epsilon) > \pi_1(p_1^*)$ and 1 could profitably deviate to $p_1^* + \epsilon < p_2^*$. Case 2: $p_1^* < p_1^M < p_2^*$. Then 1 could profitably deviate to playing p_1^M. Case 3: $p_1^M \leq p_1^* < p_2^*$. By Assumption 2.3, $\hat{\pi}_2(p_1^M) > 0$. It follows from the concavity of $\hat{\pi}_2(\cdot)$, and Lemma 2.3, that $\hat{\pi}_2(p_1^*) > 0$. Therefore 2 could profitably deviate by tieing at p_1^*.

□

Proof of Lemmas 2.3 and 2.4 Note that $\hat{\pi}_i(0) = -C_i(y/2) < 0$. As l_i is the unique price in $(0, z)$ satisfying $\hat{\pi}_i(l_i) = 0$ it follows from the concavity of $\hat{\pi}_i(\cdot)$ that $\hat{\pi}_i(p) < 0$ if $p \in [0, l_i)$, and $\hat{\pi}(p) > 0$ if $p \in [l_i, z)$. Similarly, $\tilde{\pi}_i(0) = C_i(y/2) - C_i(y) < 0$ and h_i is the unique price in $(0, z)$ satisfying $\tilde{\pi}_i(h_i) = 0$. The concavity of $\tilde{\pi}_i(\cdot)$ implies $\tilde{\pi}_i(p) < 0$ if $p \in [0, h_i)$, and $\tilde{\pi}(p) > 0$ if $p \in (h_i, z)$.

□

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