Foundations of utilitarianism under risk and variable population

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Abstract

Utilitarianism is the most prominent social welfare function in economics. We present three new axiomatic characterizations of utilitarian (that is, additively-separable) social welfare functions in a setting where there is risk over both population size and individuals' welfares. We first show that, given uncontroversial basic axioms, Blackorby et al.'s (1998) Expected Critical-Level Generalized Utilitarianism is equivalent to a new axiom holding that it is better to allocate higher utility-conditional-on-existence to possible people who have a higher probability of existence. The other two characterizations extend and clarify classic axiomatizations of utilitarianism from settings with either social risk or variable-population, considered alone.

Keywords: Social risk, population ethics, utilitarianism, expected critical-level generalized utilitarianism, prioritarianism.

JEL Classification numbers: D63, D81, J10.

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1 Introduction

Utilitarianism is the most prominent social welfare function in welfare economics and economic policy.[1] Tools to aggregate costs, benefits, and tradeoffs within large and heterogeneous populations are especially important for major public choices such as climate policy. Leading economic analyses of climate policy most often rely on a discounted variant of utilitarianism (Stern, 2006; Nordhaus, 2017). The 2015 Intergovernmental Panel on Climate Change report recognized that evaluating climate policy necessitates the theoretical challenges of both social risk and endogenous population: climate policies and other economic policies impact both the distribution of wellbeing and the chances that future people exist.

So additively separable social evaluation — a key feature of generalized utilitarianism (Blackorby, Bossert and Donaldson, 2005, p. 95) — is at once a core tool of public economics and an enduring focus of debate in welfare economics. In the prior literature, there are two major axiomatic paths to characterizing utilitarianism, each with known criticisms. One, in risky settings, begins with respecting *ex ante* Pareto improvements for individuals (Harsanyi, 1955), but this is criticized by egalitarians and others for ignoring the distribution of utility (*e.g.* Fleurbaey, 2010). Another, in variable-population settings, begins with independence of the existence of unaffected lives (Blackorby and Donaldson, 1984; Blackorby, Bossert and Donaldson, 1995), but this too is criticized in the population literature. We identify a new path to

[1]For example, the widely-used sufficient statistics method of Chetty (2009) and the optimal income tax problem of Mirrlees (1971) both assume a utilitarian social criterion. In some economic experiments, utilitarian allocations receive substantial (although not universal) support Jackson and Yariv (2014).

utilitarianism, by studying risky, variable-population social settings, where the number and lifetime utilities of possible people vary across probabilistic states. We clarify the additional properties of utilitarianism in our more general setting.

We present three new characterizations of Expected Critical-Level Generalized Utilitarianism (ECLGU) for evaluation of variable-population risky social prospects. To do this, we introduce novel axioms about the *combination* of risk and variable population. For example, we propose an axiom that the *probability* of a person existing and that person's *utility*-conditional-on-existence are *complements*, so it is better to allocate higher chances of existence to lives that would have more wellbeing. This axiom allows us to characterize generalized utilitarianism without directly assuming either *ex ante* Pareto or independence of the existence of unaffected lives, which are the classic axioms for additive separability when either risk or population, respectively, are considered in isolation (Blackorby, Bossert and Donaldson, 1998, 2005). Of course, both of these principles are entailed by ECLGU but they need not be the founding principle of utilitarianism. Our other axioms allow other novel characterizations.

The ECLGU family is an important set of social welfare functions that includes leading candidates in the population ethics literature. In particular, classical total utilitarianism is a special case of ECLGU. Because ECLGU allows utilities to be transformed before they are aggregated, it also permits sensitivity to the distribution of utility: a concave transformation yields social evaluation that prefers an equal distribution with the same total utility. We emphasize that such transformation may be, but need not be, individuals' own von Neumann Morgenstern transformations for risky choice. Such generality is a benefit of our avoidance of an explicit axiom respecting individuals' *ex ante* Pareto improvements in expected utility.

Because ECLGU offers an approach to giving priority to lower-utility individuals while retaining additive separability, we contrast ECLGU with a leading alternative approach to inequality: the expected equally-distributed equivalent (EEDE), described by Fleurbaey (2010).[2] Because EEDE, as we make it precise, cares about the *pattern* of inequality across persons, whether to add additional people depends, in part, on their consequences for inequality.

In the next section, we present examples where ECLGU and EEDE disagree or notably agree, in order to motivate our novel axioms. Subsequently, we present our three characterizations. The first theorem uses a new axiom that is sufficient to characterize ECLGU in the presence of a set of uncontroversial basic principles that are used throughout the population economics literature. This axiom requires complementarity between the probability of a person existing and utility-conditional-on-existence. We thus provide an axiomatization of utilitarianism that relies neither on *ex ante* Pareto, nor on independence of the existence of unaffected lives, which is an alternative to the existing literature.

We then explore other axiomatizations of ECLGU in our setting. We show that independence of the existence of unaffected lives and (a variant of) *ex ante* Pareto have to be supplemented with other properties. Our second characterization adds to the basic principles two axioms that make use of

[2]Many of Fleurbaey's results are general enough to include non-egalitarian approaches such as classical utilitarianism and other families; we focus on a particular subset, detailed below, that is separable in same-number, risk-free cases but not otherwise.

both social risk and variable population, namely Existence independence of sure lives and Social risk neutrality in population size. This characterization is important in part because risk-neutrality in the size of perfectly equal populations is an axiom that would be widely accepted among competing approaches to population ethics, including the EEDE approach. Moreover, independence of risk-free individuals has been noted in the prior literature to be attractive in fixed-population evaluations of social risk and inequality, even to egalitarians (*cf.* Fleurbaey, 2018).

Our third theorem is most conceptually similar to prior characterizations, particularly the Harsanyi approach. It uses an "Individual dominance" axiom that respects *social* risk preferences over *single-person* populations. This permits us to characterize ECLGU in a way that generalizes and weakens *ex ante* Pareto beyond expected utility and even beyond individuals' von Neumann-Morgenstern transformations. Again, this axiom has to be supplemented with other properties to obtain ECLGU. With Theorem 2 and 3, we thus improve our understanding of the normative underpinning of the ECLGU family.

We view our contribution both as providing three new avenues for any reader to understand ECLGU and as providing new arguments in favor of ECLGU for readers who find our axioms compelling. This is significant because this family of social welfare functions is of leading practical importance in public economics and contemporary empirical policy analysis. Despite this importance, theorists recognize that prior characterizations have been subject to important criticisms. Theorem 1 provides a new approach that avoids using these controversial axioms. Together with Theorems 2 and 3, these new characterizations offer a better understanding of leading tools for social evaluation that are additively separable, that allow us to evaluate populations of different size, and that allow either utilitarian aggregation or transformed aggregation that is sensitive to inequality in utility.

2 Motivating examples

Harsanyi (1955) and Blackorby, Bossert and Donaldson (1995) have influentially characterized utilitarianism. Here, we provide three more characterizations. These are of course equivalent to one another and to others in the literature. Our further characterizations advance the literature because they clarify what is fundamental to expected utilitarianism; because they highlight new, potentially unintuitive consequences of rejecting expected utilitarianism; and because, in different ways and to different degrees, they sidestep familiar debates in social risk or population ethics.

The examples in this section provide motivating intuition for our new axioms. We focus in this section on the motivation for our Theorems 1 and 2. These examples distinguish expected critical-level generalized utilitarianism (ECLGU), as proposed by Blackorby, Bossert and Donaldson (1998), from expected equally-distributed-equivalent (EEDE) criteria, as proposed by Fleurbaey (2010).

Our paper considers the social evaluation of objective-probability lotteries over states, each of which yields a population welfare outcome, which specifies how many people exist and the lifetime utilities of each. We define ECLGU as the expectation of a value function $\sum_i \left[g(u_i) - g(c)\right]$ where u_i is the utility of individual i existing in a particular population, g is increasing, and c is a critical level of utility. We define EEDE as the expectation of a value function $n^a \times \left[g^{-1}\left(\frac{1}{n}\sum_i g(u_i)\right) - c\right]$ for any fixed $a \in [0,1]$ and $c \in \mathbb{R}$, where n is the number of people who exist.[3] Both ECLGU and EEDE are sensitive to unequal distribution if g is concave, but ECLGU is additively separable in all cases. EEDE is additively separable only in same-population, risk-free cases.[4]

In these examples, rows are individual people (or potential people) and columns are risky social states. Numbers are lifetime utilities. Let $*$ denote a person not existing in a state. We use $\succsim_{\mathrm{E}}$ for EEDE. We use $\succsim_{\mathrm{U}}$ for ECLGU. Example 1, previously discussed by Fleurbaey (2010) and Broome (2015), illustrates why EEDE rejects separability across persons in risky cases.

Example 1 (EEDE rejects separability) *Assume that social states are equally likely. Then:*

$$\overbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}^{u} \succ_E \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{v},$$

but $u \sim_U v$.

This is a fixed-population example. These utilities summarize everything

[3] Although Fleurbaey proposed EEDE in a fixed-population context, for the purposes of comparison with ECLGU we expand it to variable populations with the n^a term, which permits averageist ($a = 0$), totalist ($a = 1$), or variable-value special cases. Parameter c is known in the literature as a critical-level parameter. Blackorby, Bossert and Donaldson (2005) argued in favor of critical-level utilitarianism with positive critical-level. Because all of our Examples that differentiate ECLGU and EEDE use same-population-size pairwise comparisons, they hold for any a and c. For formal definitions of the criteria, see Definitions 2 and 1.

[4] Fleurbaey's (2010) approach is more general and is also compatible with other non-additive social criteria. Fleurbaey also considers non-continuous social orderings.

relevant about lifetime wellbeing, including any experience of inequality. Both people have an equal chance of receiving 1 or 0; so, if individual risk profiles are separable across people (so correlated risks do not matter), then u and v must be equally good. For the top person, the mapping of states to outcomes is the same in v as in u. The bottom person faces the same individual prospect in v as in u. But v will certainly have an unequal outcome, whichever state turns out to realize, and u will certainly have an equal outcome. EEDE favors ex-post egalitarian outcomes and so rejects separability across persons and prefers u over v. ECLGU is indifferent. In the rest of this section, we extend this mechanism to new examples, to further distinguish these two criteria in a risky, variable population setting.

2.1 Leveling down for probabilistic people

Our principal novel axiom holds that, in allocating probabilities of existence and utilities conditional-on-existence among probabilistically possible people, a higher probability of existence and a higher utility-conditional-on-existence are complements. ECLGU holds that it is better if the people who are more likely to exist are also the people who would have better lives, if they existed. In contrast, because EEDE cares about distributional properties, whether EEDE prefers to increase the probability that a person will exist also depend on the chances for inequality among other possible people. Strikingly, EEDE can therefore prefer to "level down" probabilistic people and reject the complementarity of wellbeing and the chances of existence.

Example 2 (Leveling down for probabilistic people) *Assume that all* *three states are equally probable. Assume that $g(u) = -e^{-2u}$. Then:*

$$\overbrace{\begin{pmatrix} 2 & \mathbf{2} & * \\ * & * & 1 \\ 1 & 1 & * \\ * & * & 2 \end{pmatrix}}^{w} \prec_E \overbrace{\begin{pmatrix} * & * & 2 \\ 1 & \mathbf{1} & * \\ 1 & 1 & * \\ * & * & 2 \end{pmatrix}}^{z}, \text{ but } w \succ_U z.$$

In Example 2, both options have a population size of two in every outcome. The person in the top row has higher utility-conditional-on-existence than the person in the second row. Whether the top or second person has a $\frac{2}{3}$ or $\frac{1}{3}$ chance of existence depends on the choice. The people in the bottom two rows are unaffected by the choice.

The decision between w and z allocates the bolded utilities-conditional-on-existence of 2 and 1 between the top two people in the middle state. Under ECLGU, $w \succ_U z$, because the higher utility level is then more likely to exist. But under EEDE, $w \prec_E z$ because *ex post* inequality always happens in w but never happens in z. Choosing z would be a form of leveling down to avoid inequality. In section 4, we characterize ECLGU with the principle behind Example 2.

2.2 Existence independence of sure lives and social risk neutrality in population size

One well-studied path towards ECLGU in risk-free variable population settings is an existence independence property (Blackorby, Bossert and Donaldson, 2005). Such a property holds that social evaluations should not depend on the existence of unaffected people, such as the long dead or those in the distant future. EEDE does not typically satisfy such a property. This is because, in a variable-population setting, adding an extra person has two consequences to an EEDE evaluation: the person's own utility matters and the extra person may change distributional properties, such as inequality. To ECLGU, such distributional effects are irrelevant. But to EEDE, adding a sure person (or learning, for example, that a sure person existed in the past) can reverse a choice. Example 3 shows this by expanding Example 1:

Example 3 (EEDE rejects Existence independence of sure lives) *Assume that social states have equal probability and that $g(u) = -e^{-2u}$. Then:*

$$\overbrace{\begin{pmatrix} 1 & 0 & 0.5 \\ 1 & 0 & 0.5 \end{pmatrix}}^{x} \succ_E \overbrace{\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \end{pmatrix}}^{y}.$$

$$\overbrace{\begin{pmatrix} 1 & 0 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 0 & 0 \end{pmatrix}}^{x'} \prec_E \overbrace{\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0 & 0 & 0 \end{pmatrix}}^{y'}.$$

Notice that, under EEDE, x has the advantage of *ex post* equality, but y has the advantage of 0.9 being a better lifetime utility for one possible person than 0.5. Under some quantitative parameterizations, x is preferred. But adding the sure person in x' changes this advantage, as ex-post inequality appears, so y' is preferred. Such a preference reversal violates existence independence. To ECLGU, $y \succ_U x$ and $y' \succ_U x'$ because only the fact that $0.9 > 0.5$ matters.

In section 5, we present a characterization of ECLGU that builds upon an axiom that rules out choosing the way EEDE does in Example 3. However, we also show that Existence independence is not sufficient for ECLGU. This is because criteria that maximize the expectation of the multiplicative value function $\prod_i g(u_i)$ also satisfy Existence independence.

In section 5, we therefore use an additional property: Social risk neutrality in population size for perfectly equal populations. This principle holds that—in the hypothetical case of no risk or inequality in utility-conditional-on-existence—social evaluation is risk neutral in the number of people who live. Example 4 illustrates this property:

Example 4 (Social risk neutrality in population size) *Assume that social states are equally likely. Then:*

$$\begin{pmatrix} 1 & 1 \\ * & 1 \\ * & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ * & * \end{pmatrix},$$

both for ECLGU and for EEDE with $a = 0$ or $a = 1$.

Such risk neutrality will be an implication of any approach that maximizes the expectation of an additive social welfare function. Importantly, however, social risk neutrality in population size is *also* a property of some EEDE criteria, of expected average utilitarianism, and broadly of other non-separable approaches in the population literature. What singles out ECLGU in the context of our basic axioms, Theorem 2 shows, is the *combination* of Existence independence of sure lives and Social risk neutrality in population size.

To a reader who agrees with EEDE's egalitarian evaluations in Examples 2 and 3, our paper provides new arguments against generalized utilitarianism. Alternatively, to a reader who agrees with generalized utilitarianism's evaluations in these cases, our paper shows that, in the context of some widely accepted basic axioms, these principles are sufficient to characterize ECLGU. Either way, these examples clarify the foundations of utilitarianism for economic policy analysis in the real-world case of both social risk and variable population.

3 Framework and background

3.1 The framework

The set of positive integers is denoted by $\mathbb{N}$. The set of all real numbers is denoted by $\mathbb{R}$. The set of non-negative (resp. positive) real numbers is denoted by $\mathbb{R}_+$ ($\mathbb{R}_{++}$).

The set of possible individuals is $\mathbb{N}$. $\mathcal{N}$ is the set of all possible non-empty finite subsets of $\mathbb{N}$, which are possible populations with typical element N. We consider a welfarist framework where the only information necessary for social decisions is the utility levels of people alive in a certain state of affairs.[5]

[5] Blackorby, Bossert and Donaldson (1998) explicitly state Welfarism as an axiom. Here, to simplify the analysis, we only implicitly make the assumption in order to avoid introducing states of affairs, lotteries over states of affairs, and the corresponding welfarist information. But we could as well use the framework of Blackorby, Bossert and Donaldson (1998). Welfarism itself is often derived from more basic principles, typically Pareto indifference and Anonymity. See Blackorby, Bossert and Donaldson (1999, 2005) for an example in a variable-population framework assuming a multi-profile setting; and Blackorby, Bossert and Donaldson (2006) in a fixed-population framework assuming a single profile of utility. Note that we exclude empty populations where nobody exists for the sake of simplicity: we could account for them by setting the value of empty populations to zero.

A population's welfare information is given by $u = (u_i)_{i \in N} \in \mathbb{R}^N$ where $N \in \mathcal{N}$ is the population, and u_i is for each existing individual i the lifetime utility experienced by i. It is common in the population ethics literature to assume that a life with positive utility is worth living, while a life with negative wellbeing is not. We need not make this interpretation here given that ECLGU can use any critical level.

The set of all possible alternatives is $U = \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$. For each $u \in U$, we denote $N(u)$ the set of individuals alive in u, and $n(u)$ the number of individuals alive in u. For an individual $i \in \mathbb{N}$, we let U_i be the set of alternatives where i exists. Formally, $U_i = \{u \in U | i \in N(u)\}$. For each population $N \in \mathcal{N}$, $U_N = \{u \in U | N(u) = N\}$ is the set of alternatives such that the population is N. For two alternatives u and $v \in U$ such that $N(u) \cap N(v) = \emptyset$, we denote uv the alternative w such that $N(w) = N(u) \cup N(v)$, $w_i = u_i$ for all $i \in N(u)$ and $w_j = u_j$ for all $j \in N(v)$. Similarly, we use uvw when we "merge" three alternatives such that $N(u) \cap N(v) = N(u) \cap N(w) = N(v) \cap N(w) = \emptyset$.

We consider risk over social states of affairs. We do so by introducing lotteries over U. Because U specifies both who exists and their lifetime utilities, such lotteries capture uncertainty over existence, over risk experienced within a lifetime, and over allocation within populations. A lottery p over U is a mapping $p : U \to [0, 1]$ such that there exists a finite set $V \subset U$ such that $\sum_{v \in V} p(v) = 1$, $p(v) > 0$ for all $v \in V$ and $p(u) = 0$ for all $u \in (U \setminus V)$. We denote P the set of all such lotteries. For simplicity, we abuse notation and denote u the degenerate lottery such that alternative $u \in U$ occurs with probability 1 (so that all other alternatives have probability 0).[6] For any $p \in P$, we denote $supp(p) = \{u \in U : p(u) > 0\}$, that is, the support of p.

For any population $N \in \mathcal{N}$, we denote P_N the set of lotteries concerning only population N: that is, $p \in P_N$ if for each $u \in supp(p)$ it is the case that $u \in U_N$. Such same-population lotteries have attracted attention in the discussion about whether we should use an additively separable von Neumann-Morgenstern function in a risky context as suggested by Harsanyi (1955).

Last we introduce two concepts related to the situation individuals face in a lottery. The first concept is that of the probability that an individual i exists with lottery p, that we denote $\pi_i(p)$. Formally, for any $p \in P$ and any individual i, $\pi_i(p) = \sum_{u \in U_i} p(u)$. The second concept is the individual lottery faced by an individual who is sure to exist that we denote p_i. Formally, for any $i \in N$ and any $p \in P$ such that $\pi_i(p) = 1$, p_i is the lottery on $\mathbb{R}$ defined as follows: for any $x \in \mathbb{R}$, $p_i(x) = \sum_{u \in supp(p) | u_i = x} p(u)$. To lighten notation, we further denote p_i the lottery in P such that $p_i(v) = 0$ for all $v \in U$ such that $N(v) \neq \{i\}$, and, for each $u \in U$ with $N(u) = \{i\}$, $p_i(u) = p_i(x)$ where $x = u_i \in \mathbb{R}$. So, p_i is also the social lottery where only individual i exists and faces the individual lottery p_i. So the ranking $p_i \succsim p'_i$ is well-defined and represents a "social assessment" of individual lotteries (see Axiom IDom below).

Note that our framework combines risk in two ways: on who exists and on the utility of the existing persons. Concretely, what we have in mind are situations where people may be born or not (because fertility patterns change

[6] Hence, we implicitly assume the property of Social certainty consistency by Blackorby, Bossert and Donaldson (1998).

for instance) and then face risks *during their lifetimes* that affect their overall lifetime utilities.[7] Note also that we do not use any explicit time structure in our setting. Our risky variable population framework can be applied naturally to intertemporal issues (for instance, our current actions may change who exist in the future). In that case—which is our preferred interpretation—we assume that a population comprises everyone who ever exists in the world, like in Blackorby, Bossert and Donaldson (2005). In this interpretation, time would not play any significant role in welfare evaluation: one of our basic axioms is Anonymity, ruling out discounting or any other discrimination on the basis of time.

We investigate a social welfare ordering $\succsim$ over P (because we assume Social expected-utility throughout, we will assume that $\succsim$ is a complete and transitive ordering). We denote $\succ$ the asymmetric part and $\sim$ the symmetric part of $\succsim$.

3.2 Basic principles

We assume the principles in this section throughout our analysis. Our first three axioms apply only to risk-free distributions with fixed population sizes, meaning only to lotteries where the same population size and allocation occurs for sure. Our fourth axiom introduces risk, assuming that risk is treated by maximizing the expectation of a social objective.[8] Our fifth assumption

[7] Hence, we do not require an interpretation of risky personal identity beyond ordinary usage: a person may or may not be born, and if so may face risk. Individuals here are assumed to be people with well-defined values and preferences. We do not use a more elusive notion of identities or souls that could be incarnate in different places and times.

[8] In contrast with our approach, which preserves Social expected-utility but weakens *ex ante* Pareto in individuals' expected utility, a number of other important contributions to

introduces variable population with a minimal assumption that there is a case in which two populations of different sizes can be compared.

Strong Pareto. For all $N \in \mathcal{N}$, for all u, $v \in U_N$, if $u_i \geq v_i$ for all $i \in N$, then $u \succsim v$; if furthermore $u_j \succ v_j$ for some $j \in N$ then $u \succ v$.

Anonymity. For all u, $v \in U$ such that $n(u) = n(v) > 0$, if there exists a bijection $\varpi : N(u) \to N(v)$ such that $u_i = v_{\varpi(i)}$ for all $i \in N(u)$, then $u \sim v$.

Continuity. For all $N \in \mathcal{N}$, the restriction of $\succsim$ to U_N is continuous in lifetime utilities.

Social expected-utility. There exists a function $V : U \to \mathbb{R}$ such that, for all p, $p' \in P$:

$$p \succsim p' \Longleftrightarrow \sum_{u \in supp(p)} p(u) \times V(u) \geq \sum_{u \in supp(p')} p'(u) \times V(u).$$

Strong Pareto, Anonymity, Continuity, and Social expected-utility all could apply in the same way to a fixed-population setting. The next axiom is our first variable-population axiom. It assumes a minimal degree of comparability across population sizes and rules out an implausible social evaluation that would always regard an additional person as an improvement or as a worsening, irrespective of the utility level.[9]

the recent literature have furthered Harsanyi's theorem by weakening Social expected-utility (Fleurbaey, 2009; Mongin and Pivato, 2015; Zuber, 2016; McCarthy, Mikkola and Thomas, 2020).

[9]In the context of Pareto and continuity in a complete and transitive ordering, Minimal existence of a critical level would follow immediately from the apparently weaker assumption that there are $\bar{c}, \underline{c} \in \mathbb{R}$ such that if $\bar{v}_i = \bar{c}$, $\underline{v}_i = \underline{c}$, and definitions are otherwise as in the axiom, then $\bar{v} \not\prec u$ and $\underline{v} \not\succ u$. This weakening is an application of Broome (2005).

Minimal existence of a critical level. There exist $u \in U$, $c \in \mathbb{R}$, and $i \notin N(u)$, such that if $v \in U$ is defined by $N(v) = N(u) \cup \{i\}$, $v_i = c$, and $v_j = u_j$ for all $j \in N(u)$ then $u \sim v$.

Collectively, we will refer to the five axioms in this section as the **Basic Principles**.

3.3 Variable-population social welfare orderings

For any $N \in \mathcal{N}$, a function $\Xi : \mathbb{R}^N \to \mathbb{R}$ is said to be normalized if $\Xi(u) = a$ whenever $u_i = a$ for all $i \in N$.

Proposition 1 *If the social welfare ordering $\succsim$ on P satisfies the Basic Principles,*[10] *then there exists a function $W : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, increasing and continuous in its second argument, and for each $n \in \mathbb{N}$ a continuous, increasing, symmetric and normalized function Ξ_n such that, for all p, $p' \in P$:*

$$p \succsim p' \iff \sum_{u \in supp(p)} p(u) \times W\Big(n(u), \Xi_{n(u)}(u)\Big) \geq \sum_{u \in supp(p')} p'(u) \times W\Big(n(u), \Xi_{n(u)}(u)\Big).$$

Proof. Given that $\succsim$ satisfies Strong Pareto, Anonymity, and Continuity, we can use Theorem 1 in Blackorby, Bossert and Donaldson (1998) to prove that there exist a function $\bar{W} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ increasing and continuous in its second argument and for each $n \in \mathbb{N}$ a continuous, increasing, symmetric and

[10] Minimal existence of a critical level is actually not needed for this result, but for the sake of simplicity, we assume throughout that all the Basic Principles hold.

normalized function Ξ_n such that, for all u, $v \in U$:[11]

$$u \succsim v \Longleftrightarrow \bar{W}\Big(n(u), \Xi_{n(u)}(u)\Big) \geq \bar{W}\Big(n(v), \Xi_{n(v)}(v)\Big).$$

But the principle of Social expected-utility and Continuity imply that there exists a continuous function $V : U \to \mathbb{R}$ such that, for all u, $v \in U$:

$$u \succsim v \Longleftrightarrow V(u) \geq V(v).$$

Hence there must exist a continuous and increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $V(u) = \Phi \circ \bar{W}\big(n(u), \Xi_{n(u)}(u)\big)$ for all $u \in U$. Denoting $W := \Phi \circ \bar{W}$ and using Social expected-utility, we obtain the result. ■

An example of variable-population social welfare orderings is the class described in Proposition 1 is ECLGU:

Definition 1 *A social welfare ordering $\succsim$ is an ECLGU social welfare ordering if there exist $c \in \mathbb{R}$ and a continuous and increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and for all p, $p' \in P$:*

$$p \succsim p' \Longleftrightarrow \sum_{u \in supp(p)} p(u) \times \sum_{i \in N(u)} \big[g(u_i) - g(c)\big] \geq \sum_{u \in supp(p')} p'(u) \times \sum_{i \in N(u)} \big[g(u_i) - g(c)\big].$$

ECLGU has previously been axiomatized by Blackorby, Bossert and Donaldson (1998). If g is the identity function, then ECLGU is simply critical-level utilitarianism. The case where g is a concave transformation is known as

[11] Note that the proof of Theorem 1 in Blackorby, Bossert and Donaldson (1998) is based on a previous result by Blackorby and Donaldson (1984) that assumes that $\succsim$ restricted to sure alternatives is representable by a function $V : U \to \mathbb{R}$. Because we assume Social expected-utility, we have this representability assumption built into our axiomatization.

prioritarianism, because the social evaluation gives more priority to those who are worse off.[12]

An alternative family, which we contrast with ECLGU, is EEDE by Fleurbaey (2010):

Definition 2 *A social welfare ordering* $\succsim$ *is an EEDE social welfare ordering if there exists a continuous and increasing function* $g : \mathbb{R} \to \mathbb{R}$ *and* $a \in [0, 1]$ *and* $c \in \mathbb{R}$ *such that for all* p*,* $p' \in P$*:*

$$\begin{aligned} p \succsim p' \iff & \sum_{u \in supp(p)} p(u) \times (n(u))^a \times \left[g^{-1} \left(\tfrac{1}{n(u)} \sum_{i \in N(u)} g(u_i) \right) - c \right] \\ \geq & \sum_{u \in supp(p')} p'(u) \times (n(u))^a \times \left[g^{-1} \left(\tfrac{1}{n(u)} \sum_{i \in N(u)} g(u_i) \right) - c \right]. \end{aligned}$$

Although Fleurbaey proposed EEDE in a fixed-population setting, for our purpose of comparison with ECLGU, we extend this family to encompass a range of population-ethical perspectives with the population weighting term n^a and the critical-level parameter c.[13] We discuss EEDE in parallel with ECLGU in order to highlight and isolate points of disagreement between these alternative approaches which may guide the choice of a social ordering.

[12]For a discussion of various prioritarian criteria, see Parfit (1995); Adler and Treich (2017).

[13]Our version of EEDE is in fact a specific case of the *Expected Prioritarian Equally Distributed Equivalent social ordering* axiomatized by Fleurbaey and Zuber (2015).

4 No leveling down for probabilistic lives

Our first representation theorem builds upon the principle in Example 2. We use a novel axiom that combines features of the *ex ante* Pareto axiom that drives the Harsanyi theorem with the existence independence that drives the Blackorby and Donaldson (1984) theorems, without fully assuming either. In a same-number (or, more precisely, a same probabilities of existence) choice with no utility risk conditional on existence, this axiom holds that the *ex ante* probability of existence of an individual is a complement to lifetime utility. In other words, it is better if higher utility-conditional-on-existence is allocated to probabilistic people who are more likely to exist and lower utility-conditional-on-existence is allocated to probabilistic people who are less likely to exist, in any all-else-equal binary choice.[14]

No leveling down for probabilistic lives (NoLD). For all p, $p' \in P$, if there exist two individuals i, $j \in \mathbb{N}$ and two real numbers $a > b$ such that:

- **(i)** For all $k \in \mathbb{N} \setminus \{i, j\}$, $\pi_k = \pi'_k$ and there exists $c_k \in \mathbb{R}$ such that $u_k = c_k$ for all $u \in U_k$ such that $p(u) > 0$ or $p'(u) > 0$;
- **(ii)** $u_i = a$ for all $u \in U_i$ such that $p(u) > 0$ or $p'(u) > 0$;
- **(iii)** $u_j = b$ for all $u \in U_j$ such that $p(u) > 0$ or $p'(u) > 0$;
- **(iv)** $\pi_i(p) + \pi_j(p) = \pi_i(p') + \pi_j(p')$;

then $p \succ p'$ if and only if $\pi_i(p) > \pi_i(p')$.

[14] In our context of Anonymity and Social expected-utility, it is equivalent to assume a version where two utility levels are allocated to fixed probabilities or a version where two probabilities are allocated to fixed utility levels.

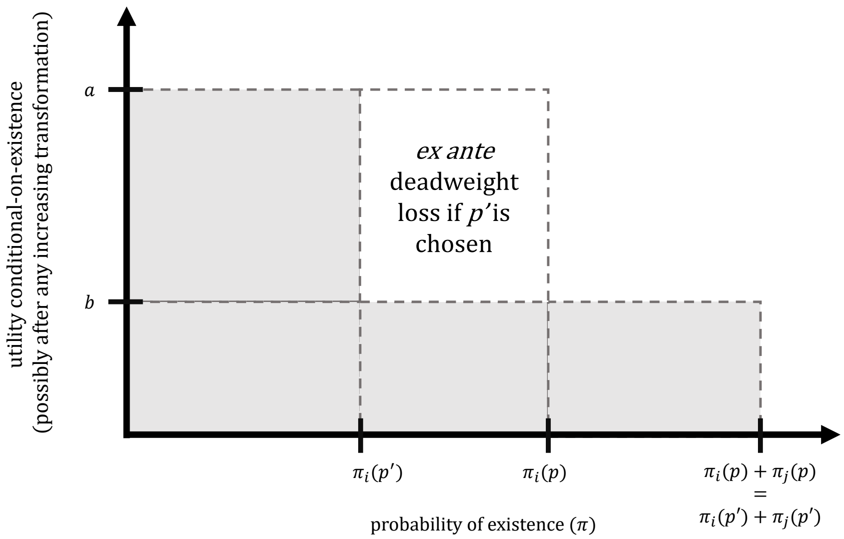

Figure 1: No leveling down for probabilistic lives, illustrated

Figure 1 illustrates the principle, focusing on the two people i and j. Conditional on existence, i has higher utility a than j's utility b. The empty rectangle represents a deadweight loss of *ex ante* wellbeing if p' is chosen over p. The axiom is named in reference to the well-studied "leveling down" objection to those versions of egalitarianism that sometimes prefer making individuals worse off in order to reduce inequality (Parfit, 1995). The axiom has in common with Harsanyi's approach that it pays *some* restricted amount of attention to *ex ante* properties and has in common with Blackorby and Donaldson's approach that it makes *some* variable-population evaluations without regard to the full distribution of welfare.[15] In the context of the basic

[15] This principle is also related to the Probability-adjusted Suppes-Sen principle of Asheim and Zuber (2016), who study social evaluations in a space in which utility-conditional-on-existence is always certain. They construct rank-ordered allocations that are probability-weighted cumulative functions of utility based on individuals' probabilities of existence and (sure) utility levels. Probability-adjusted Suppes-Sen implies that a higher

principles, NoLD is sufficient to characterize ECLGU:

Theorem 1 *A social welfare ordering $\succsim$ on P satisfies the Basic Principles and NoLD if and only if it is an ECLGU social ordering.*

Proof. In the Appendix. ■

5 Existence independence and risk neutrality

5.1 Existence independence

In Example 3, a social ordering was reversed by addition of a person who was certain to exist at a specific, constant utility in any possible state, instead of being certain not to exist. The next axiom rules out such social evaluations:

Existence independence of sure lives (Ind-l). For all p, p', q, $q' \in P$, if there exists an alternative $u \in U$ such that:

- for all $v \in supp(p)$, there exists $w \in U$ such that $N(w) \cap N(u) = \emptyset$, $v = uw$ and $p'(w) = p(v)$;
- for all $v \in supp(q)$, there exists $w \in U$ such that $N(w) \cap N(u) = \emptyset$, $v = uw$ and $q'(w) = q(v)$;

then $p \succsim q$ if and only if $p' \succsim q'$.

In population ethics, Blackorby, Bossert and Donaldson (1998, 2005) have argued that Ind-l is highly plausible because, at a minimum, lifetime utilities

rank-ordered allocation is always better: this happens when some probability has been moved towards higher utility levels.

are sure for people who are dead. Any intuitive attractiveness of existence independence offers one argument against non-separable approaches such as EEDE. EEDE satisfies independence of the sure in fixed-population, risk-free cases but not, as Example 3 shows, in variable-population, risk-free cases. ECLGU satisfies Ind-l, and therefore also satisfies a weaker property, Existence independence for sure alternatives:

Existence independence for sure alternatives (Ind-a). For any u, u', v in U, if $N(u) \cap N(v) = N(u') \cap N(v) = \emptyset$, then $uv \succsim u'v$ if and only if $u \succsim u'$.

Blackorby, Bossert and Donaldson (1998) have characterized social welfare orderings satisfying this principle together with our other Basic Principles.

Proposition 2 (Blackorby, Bossert, Donaldson, 1998) *If $\succsim$ is a social welfare ordering that satisfies the Basic Principles and Ind-a then there exist two continuous and increasing functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = g(0) = 0$, and for all p, $p' \in P$:*

$$p \succsim p' \iff \sum_{u \in supp(p)} p(u) \times \varphi \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] \right) \geq \sum_{u \in supp(p')} p'(u) \times \varphi \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] \right).$$

Proof. This is Theorem 3 of Blackorby, Bossert and Donaldson (1998). Recall that our social welfare ordering is welfarist in the sense of Blackorby, Bossert and Donaldson (1998). We note that Minimal existence of a critical level is assumed by Blackorby, Bossert and Donaldson (1998) but is not listed by them as an axiom. ■

Proposition 2 shows that Ind-a is not enough to characterize ECLGU social orderings in the risky context. Indeed, even the stronger Ind-l is not enough. This is because multiplicative criteria that take the expectation of $\prod_i g(u_i)$ also satisfy the property (and, all our other properties provided, there exists c such that $g(c) = 1$).

5.2 Risk neutrality in population size

Given the result in Proposition 2 we need an additional property to characterize ECLGU. Our next axiom, building upon Example 4, holds that social evaluation is neither risk-loving nor risk-averse in population size, for cases of equal utility-conditional-on-existence:[16]

Social risk neutrality in population size for perfect equality (RiskNeu).

For any alternative $u, v, w \in U$ such that $N(u) \cap N(v) = N(u) \cap N(w) = N(v) \cap N(w) = \emptyset$, and $u_i = v_j = w_k = e$ for all $i \in N(u)$, $j \in N(v)$ and $k \in N(w)$, if $p, p' \in P$ are such that $p(uvw) = 1/2$, $p(w) = 1/2$, $p'(uw) = 1/2$ and $p'(vw) = 1/2$, then $p \sim p'$.

RiskNeu imposes that, for cases of equality and given a pre-existing population w at the equal level, society is risk-neutral with respect to the risk on future population size. Bommier and Zuber (2008) discuss a version of this axiom.[17]

[16] Notice that, in the RiskNeu axiom, utility is equal for all individuals and states of the world. We may want to strengthen the axiom by allowing individuals to have different (sure) utility levels conditional on existence. Such a property would *not* be compatible with EEDE. Assuming that we are indifferent to risk on population given that individuals have sure utility levels conditional on existence and a given probability of existing would imply Property 1 in Appendix A.1: this is sufficient together with our basic axioms to characterize ECLGU without assuming Ind-I.

[17] More precisely, they explore a principle that denies it, which they call Preference for Catastrophe Avoidance. We note that they use Independence of the utilities of the dead, which is a version of Independence for the sure, but do not explicitly cite Independence for

RiskNeu is consistent with a very large variety of social criteria in population ethics, including EEDE with $a = 1$ or $a = 0$.[18] So we interpret RiskNeu to be a weak axiom in the context of our other assumptions for Theorem 1, as it allows most approaches to population ethics that are named in the literature. The next proposition precisely describes the scope of all social orderings that satisfies RiskNeu and our basic principles.

Proposition 3 *If a social welfare ordering $\succsim$ on P satisfies the Basic Principles and RiskNeu, then there exist two continuous functions $\Psi : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$, and for each $n \in \mathbb{N}$ a continuous, increasing, symmetric and normalized function Ξ_n, such that, for all p, $p' \in P$:*

$$p \succsim p' \iff \sum_{u \in supp(p)} p(u) \times \Big[\Phi\left(\Xi_{n(u)}(u)\right) + n(u) \times \Psi\left(\Xi_{n(u)}(u)\right)\Big]$$
$$\geq \sum_{u \in supp(p')} p'(u) \times \Big[\Phi\left(\Xi_{n(u)}(u)\right) + n(u) \times \Psi\left(\Xi_{n(u)}(u)\right)\Big].$$

Proof. In the Appendix. ■

Combining Propositions 2 and 3 we obtain our second characterization of

sure acts. Theirs is the first use of this axiom of which we are aware, although they cite a related principle by Keeney (1980) about risk aversion over quantity of deaths. Although Bommier and Zuber are motivated by the risk of human extinction, we note that this possibility would be represented by a positive population size in our framework (not zero), because some human lives have already been lived.

[18]In particular, it is satisfied by every social criterion that maximizes an expectation of a function of the form $V(u) = \Xi_n(u)$ or $V(u) = n\big(\Xi_n(u) - c\big)$, with Ξ a normalized function like in Proposition 1. This means that it is accepted by average and total versions of utilitarianism and EEDE (each with or without positive critical levels) as well as by maximin. RiskNeu is however rejected by "variable value" approaches to population ethics, which include Rank-Dependent Generalized Utilitarianism (Asheim and Zuber, 2014) and Number-Dampened Generalized Utilitarianism (Ng, 1989), but both of these are already excluded by Ind-a.

ECLGU:

Theorem 2 *A social welfare ordering $\succsim$ on P satisfies the Basic Principles, Ind-l, and RiskNeu if and only if it is an ECLGU social welfare ordering.*

Proof. In the Appendix. ■

EEDE with $a = 1$ or $a = 0$ satisfies RiskNeu. They also satisfy a weak form of separability (weaker than Ind-a) in same-number risk-free cases. Theorem 2 implies that the axiomatic gap between ECLGU and this form of EEDE is that ECLGU satisfies independence in risky cases but EEDE does so only for sure alternatives.

6 *Ex ante* Pareto: a generalization

As noted in the introduction, a leading path to obtaining (critical-level) utilitarianism is to use the *Ex ante* Pareto principle. Expected utilitarianism satisfies *Ex ante* Pareto:

***Ex ante* Pareto.** For all $N \in \mathcal{N}$, for all $p, p' \in P_N$, if $\sum_{u \in U} p_i(u_i) \times u_i \geq \sum_{u \in U} p'_i(u_i) \times u_i$ for all $i \in N$, then $p \succsim p'$. If the inequality is strict for at least one $i \in N$ then $p \succ p'$.

Clearly, EEDE with concave g rejects *Ex ante* Pareto (to see this, add a small amount of utility to each outcome in v in Example 1). But ECLGU also rejects Pareto when g is concave because g plays a role of increasing risk aversion with respect to lifetime utility. We can however provide the following generalization of *Ex ante* Pareto:

Individual dominance (IDom). For all $N \in \mathcal{N}$, for all $p, p' \in P_N$, if $p_i \succ p'_i$ for all $i \in N$, then $p \succ p'$.

IDom only uses social preferences and does not make any reference to individual attitudes under risk and uncertainty.[19] In particular, it allows lifetime utilities to be measured by utility functions different individuals' von Neumann Morgenstern (VNM) utility functions (whereas *Ex ante* Pareto makes sense only if individual preferences satisfy the VNM axioms and lifetime utilities are measured by VNM utilities).[20]

6.1 Number-Weighted and Number-Dampened Utilitarianism

IDom together with our Basic Principles yields the Expected Number-Weighted Generalized Utilitarian (ENWGU) family:

Definition 3 *A social welfare ordering $\succsim$ is an ENWGU social ordering if there exists a continuous and increasing function $g : \mathbb{R} \to \mathbb{R}$, and two*

[19] Related principles in the recent literature are the Anteriority and Reduction to Prospects axioms of McCarthy, Mikkola and Thomas (2020). These axioms hold that risky social distributions can be ranked like individual prospects in specific cases (either when individuals face the same prospects in the two social distributions, or when social distributions are egalitarian and individuals have the same preferences). The Anteriority and Reduction to Prospects axioms are key to their proof that the social preorder is generated by the individual preorder. The main difference in terms of interpretation is that we do not have individual preorders but only a social ordering of individual prospects: that is social distributions where only one individual exists. We do not say anything about individual attitudes towards risk.

[20] See related arguments by McCarthy (2017). In McCarthy's terminology, we permit, but do not require, claim (X): "It is a substantive ethical question what the relation is between the individual preorder and the one-person social preorder" (p. 247). If one agrees with McCarthy that—"(Y) It is a conceptual truth that the individual preorder and the one-person social preorder coincide"—then this flexibility is of no theoretical value.

functions $F : \mathbb{N} \to \mathbb{R}_{++}$ *and* $G : \mathbb{N} \to \mathbb{R}$ *such that, for all* p, $p' \in P$*:*

$$p \succsim p' \iff \sum_{u \in supp(p)} p(u) \times \left(F\big(n(u)\big) \sum_{i \in N(u)} g(u_i) + G\big(n(u)\big) \right)$$
$$\geq \sum_{u \in supp(p')} p'(u) \times \left(F\big(n(u)\big) \sum_{i \in N(u)} g(u_i) + G\big(n(u)\big) \right).$$

Proposition 4 *If a social welfare ordering* $\succsim$ *on* P *satisfies the Basic Principles and IDom, then if it is an ENWGU social ordering.*

Proof. In the Appendix. ■

ENWGU social orderings satisfy IDom as well as Strong Pareto, Anonymity, Continuity and Social expected-utility. For the the statement of Proposition 4 to an if and only statement, we would need to focus on ENWGU social orderings satisfying Minimal existence of a critical level. This corresponds to the following restriction on functions F, G and g: there exists $n \in \mathbb{N}$, and $a, b \in \mathbb{R}$, such that $\frac{F(n)-F(n+1)}{F(n+1)} ng(a) + \frac{G(n)-G(n+1)}{F(n+1)} = g(b)$.

Notice also that both expected total utilitarianism (F and G constant) and average utilitarianism ($F(n) = cn^{-1}$, where $c \in \mathbb{R}_{++}$, and G constant) are special cases of ENWGU. In fact, because F is positive-valued, the subfamily of ENGWU where G is restricted to be constant is named in the literature. Blackorby, et al. (2005, p. 172) define Number-Dampened Generalized Utilitarianism (NDGU), in a risk-free setting, as the social ordering represented by $F\big(n(u)\big) \sum_{i \in N(u)} g(u_i)$, with $F : \mathbb{N} \to \mathbb{R}_{++}$. So, Expected NDGU (ENDGU, defined as the combination of NDGU and Social expected-utility) is the special case of ENWGU with G restricted to be constant. That restriction has been studied in Fleurbaey and Zuber (2015), and is achieved in their paper by the following axiom:

Egalitarian expansion principle. There exists a non-negative lifetime utility level $a \in \mathbb{R}_+$ such that, for all alternatives $u, u' \in U$ such that $N(u) = N(u') \cup \{i\}$ for $i \notin N(u')$, $u_j = u'_j = a$ for all $j \in N(u')$, and $u_i = b$, $u \succsim u'$ whenever $b \geq a$, while $u \prec u'$ whenever $b < a$.

The Egalitarian expansion principle implies that function $F(n)n$ must be non-decreasing and $g(a) = 0$ for the a in the Egalitarian expansion principle. The proof of the following remark is similar to the beginning of the proof of Proposition 2 in Fleurbaey and Zuber (2015).

Remark 1 *If a social welfare ordering $\succsim$ on P satisfies the Basic Principles, IDom, and the Egalitarian expansion principle, then it is an ENDGU social ordering with $F(n)n$ non-decreasing in n and $g(a) = 0$ for some $a \in \mathbb{R}_+$.*

Notice also that ENDGU social orderings also satisfy another interesting property: evaluations of risks on population size n are independent of the individuals' utility level, provided that all existing individuals have the same utility level in all states of the world and in the two lotteries being compared. This property is known as weak additivity in Pollak (1967) and as preferential independence in Keeney (1974). However, this property is also satisfied by ENWGU social orderings with F constant and G non-constant, so it does not characterize ENDGU.

6.2 Characterizing ECLGU using Individual dominance

To obtain our third characterization of ECLGU, we need to add another axiom. As Proposition 3 shows, RiskNeu is not sufficient to obtain ECLGU. Indeed, social criteria that maximize the expectation of a function of the form

$$V(u) = \alpha \tfrac{1}{n(u)} \sum_{i \in N(u)} g(u_i) + (1-\alpha) \sum_{i \in N(u)} \big[g(u_i) - g(c)\big]$$

are ENWGU social orderings and they satisfy RiskNeu.[21] These would be convex combinations of average and critical-level generalized utilitarianism.

To complete the third characterization, our next axiom posits that whether adding risk-free lives is good should be independent of the prospects faced by a fixed set of unconcerned individuals.

Independent addition from the prospects of the unconcerned (IndPUnc). For all p, q, p', $q' \in P$, if there exist $i \in \mathbb{N}$ and $x \in \mathbb{R}$ such that $w \in U$ is defined by $N(w) = \{i\}$ and $w_i = x$ and:

- $supp(p) = supp(p') \subset U \setminus U_i$;
- for all $u \in supp(p)$, $q(uw) = p(u)$ and $q'(uw) = p'(u)$;

then $q \succ p$ if and only if $q' \succ p'$.

This axiom is satisfied by social welfare orderings other than ECLGU, such as a version of maximin that only attends to whether a probability is

[21] Indeed, we can write $V(u) = \alpha g(\Xi_{n(u)}(u)) + n(u)(1-\alpha)\big(g(\Xi_{n(u)}(u)) - g(c)\big)$, where $\Xi_{n(u)}(u) = g^{-1}\left(\frac{1}{n(u)} \sum_{i \in N(u)} g(u_i)\right)$. Hence, it is compatible with the form described in Proposition 3.

zero or positive: $\min_{u \in supp(p)}\{\min_{i \in N(u)} u_i\}$. Adding IndPUnc to IDom will be sufficient to obtain ECLGU social orderings. As the proof of Theorem 3 shows, however, IndPUnc can be replaced with Constant critical level for risk-free distributions, which it implies in the context of our Basic Principles.

Constant critical level for risk-free distributions (CCL-rf). There exists $c \in \mathbb{R}$, such that for all $u \in U$ and for all $i \notin N(u)$, if $v \in U$ is such that $N(v) = N(u) \cup \{i\}$, $v_i = c$ and $v_j = u_j$ for all $j \in N(u)$, then $u \sim v$.

Theorem 3 *A social welfare ordering $\succsim$ on P satisfies the Basic Principles, IDom, and either IndPUnc or CCL-rf if and only if it is an ECLGU social welfare ordering.*

Proof. In the Appendix.[22] ■

7 Conclusion

We characterize generalized utilitarianism in a space that combines variable population and social risk. Because actual policies have uncertain outcomes and may change the size of the population, this setting is the real-world case. Special cases of the ECLGU family include total utilitarianism, critical-level utilitarianism, and versions of these with "prioritarian" transformations that are sensitive to inequality and emphasize changes in the well-being of the worse-off. Because we do not assume *ex ante* Pareto (for example, by using

[22]Note that CCL-rf implies Minimal existence of a critical level, which is one of the basic axioms. So there is redundancy in the statement of the Theorem. We kept this statement for the sake of simplicity.

our IDom axiom), in our representations the transformation g may or may not be interpreted as distinct from individuals' own VNM functions for risk aversion.

Although our characterizations result in a familiar representation—and therefore are logically equivalent to prior characterizations—we have used this rich setting to highlight new axioms. These axioms are designed to avoid begging contested questions in welfare economics. To readers who find our novel axiomatizations compelling, our results can be read as new arguments in favor of separable social evaluation. For example, our axiom against probabilistic leveling down notes a new disadvantage, only apparent in variable-population cases, of non-separable egalitarianism.

To other readers, our results can be read to clarify what is fundamental about utilitarianism, for better or worse. Some may interpret our paper to offer new illustrations of the normative costs of ECLGU. One cost may be that ECLGU (like many other alternative social orderings) prevents social evaluations from having risk aversion over changes in the size of the intertemporal human population. Another may be that ECLGU disregards the fact that adding new people to the population can change its degree of inequality and other distributional properties.

A Appendix: Proofs of the results

A.1 Proof of Theorem 1

Step 1: Proof that ECLGU satisfies all the properties. It can easily be checked that an ECLGU social welfare ordering satisfies the Basic Principles and

NoLD.

Step 2: Proof that $\succsim$ satisfies Property 1. We first show that, if social welfare ordering $\succsim$ on P satisfies the Basic Principles and NoLD then it satisfies the following property:

Property 1. For all p, $p' \in P$, if there exist three alternatives u, v, w in U such that $N(u) \cap N(v) = \emptyset$, $N(w) = \{k\}$ with $k \notin N(v)$, and $p(uv) = 1/3$, $p(w) = 2/3$, $p'(u) = 1/3$, $p'(vw) = 1/3$ and $p'(w) = 1/3$, then $p \sim p'$.

To prove that the property must be satisfied, consider p and $p' \in P$ like those described in Property 1 and assume by contradiction that $p \succ p'$. By Social expected-utility (and the definition of p and p'), it must be the case that

$$\tfrac{1}{3}V(uv) + \tfrac{2}{3}V(w) > \tfrac{1}{3}V(u) + \tfrac{1}{3}V(vw) + \tfrac{1}{3}V(w),$$

where V is the function in the statement of Social expected utility.

Let w' be an alternative such that $N(w') = \{k'\}$, with $k' \notin (N(v) \cup \{k\})$ and $w'_{k'} > w_k$. By Strong Pareto and Anonymity, $V(w') > V(w)$. But we can find $0 < \varepsilon < 1/3$ small enough so that:

$$\tfrac{1}{3}V(uv) + \tfrac{2}{3}V(w) - \tfrac{1}{3}V(uw) - \tfrac{1}{3}V(v) - \tfrac{1}{3}V(w) > \varepsilon\big(V(w') - V(w)\big). \quad \text{(A.1)}$$

Let $\hat{p}$ be the lottery such that $\hat{p}(u) = 1/3$, $\hat{p}(vw) = 1/3$, $\hat{p}(w) = 1/3 - \varepsilon$ and $\hat{p}(w') = \varepsilon$. By Equation (A.1),

$$\tfrac{1}{3}V(uv) + \tfrac{2}{3}V(w) > \tfrac{1}{3}V(u) + \tfrac{1}{3}V(vw) + \left(\tfrac{1}{3} - \varepsilon\right) V(w) + \varepsilon V(w').$$

By Social expected-utility, this implies that $p \succ \hat{p}$. But, by NoLD we should have $p \prec \hat{p}$. Indeed, in $\hat{p}$ we have increased the probability of existence of k' and decreased by the same amount that of k, where k' has higher utility than k, while maintaining the probability of existence of other people.

The contradiction shows that we cannot have $p \succ p'$. We can similarly prove that we cannot have $p \prec p'$ (now taking k' with lower utility than k).

Step 3: Proof that $\succsim$ satisfies Ind-a. Using Property 1, we can show that $\succsim$ satisfies Ind-a.

To see why this is the case, consider any u, u', and v, like described in the axiom Ind-a. Let w be any alternative such that $N(w) = k$, with $k \notin N(v)$. Let us denote p, p', q, $q' \in P$ such that:

- $p(uv) = 1/3$, $p(w) = 2/3$, $p'(u) = 1/3$, $p'(vw) = 1/3$ and $p'(w) = 1/3$;
- $q(u'v) = 1/3$, $q(w) = 2/3$, $q'(u') = 1/3$, $q'(vw) = 1/3$ and $q'(w) = 1/3$.

By Property 1, we know that $p \sim p'$ and $q \sim q'$ so that $p \succsim q$ if and only if $p' \succsim q'$. By Social expected-utility, this means that:

$$\tfrac{1}{3}V(uv)+\tfrac{2}{3}V(w) \geq \tfrac{1}{3}V(u'v)+\tfrac{2}{3}V(w) \iff \tfrac{1}{3}V(u)+\tfrac{1}{3}V(vw)+\tfrac{1}{3}V(w) \geq \tfrac{1}{3}V(u')+\tfrac{1}{3}V(vw)+\tfrac{1}{3}V(w).$$

The equivalence simplifies to $V(uv) \geq V(u'v) \iff V(u) \geq V(u')$. By Social expected-utility, this implies that $uv \succsim u'v$ if and only if $u \succsim u'$.

Step 4: Conclusion. $\succsim$ is a social welfare ordering that satisfies the Basic Principles and Ind-a. By Proposition 2, we thus know that there exist two continuous and increasing functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$\varphi(0) = g(0) = 0$, and for all p, $p' \in P$:

$$p \succsim p' \iff \sum_{u \in supp(p)} p(u) \times \varphi \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] \right) \geq \sum_{u \in supp(p')} p'(u) \times \varphi \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] \right). \tag{A.2}$$

Consider the situation in Property 1, where there are three alternatives u, v, w in U such that $N(u) \cap N(v) = \emptyset$, $N(w) = \{k\}$ with $k \notin N(v)$, and $p(uv) = 1/3$, $p(w) = 2/3$, $p'(u) = 1/3$, $p'(vw) = 1/3$ and $p'(w) = 1/3$. Assume that $w_k = c$, with c the critical-level parameter in Equation (A.2).

Given that $\succsim$ satisfies Property 1, we know that $p \sim p'$. Using the representation in Equation (A.2), this implies:

$$\begin{aligned} & \tfrac{1}{3}\varphi \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] + \sum_{j \in N(v)} \left[g(v_j) - g(c) \right] \right) \\ = \; & \tfrac{1}{3}\varphi \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] \right) + \tfrac{1}{3}\varphi \left(\sum_{j \in N(v)} \left[g(v_j) - g(c) \right] \right) \end{aligned}$$

Denoting a the real number $a = \sum_{i \in N(u)} \left[g(u_i) - g(c) \right]$ and b the real number $b = \sum_{j \in N(v)} \left[g(v_j) - g(c) \right]$, we thus get the equality

$$\varphi(a + b) = \varphi(a) + \varphi(b).$$

We can actually get this equality for any pair of real numbers $(a, b) \in \mathbb{R}^2$. Indeed, any real number can be reached as a sum of transformed utilities minus the transformed critical level.[23] We thus get the Cauchy functional equation $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $(a, b) \in \mathbb{R}^2$. Given that φ is continuous,

[23] We can add the utility of as many people as we want and we can vary their utility level above and below c.

we know that there must exist a real number α such that $\varphi(a) = \alpha a$ for all $a \in \mathbb{R}$ (Aczél, 1966, Chap. 2). Given that φ is increasing, we actually know that $\alpha > 0$.

So we can conclude that there exist a continuous and increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$, and a real number $c \in \mathbb{R}$, such that, for all p, $p' \in P$:

$$p \succsim p' \Longleftrightarrow \sum_{u \in supp(p)} p(u) \times \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] \right) \geq \sum_{u \in supp(p')} p'(u) \times \left(\sum_{i \in N(u)} \left[g(u_i) - g(c) \right] \right).$$

The social welfare ordering $\succsim$ is an ECLGU social welfare ordering.

A.2 Proof of Proposition 3

Given that $\succsim$ on F satisfies the Basic Principles, by Proposition 1, there exist a function $W : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ increasing and continuous in its second argument, and for each $n \in \mathbb{N}$ a continuous, increasing, symmetric and normalized function Ξ_n such that, for all p, $p' \in P$:

$$p \succsim p' \Longleftrightarrow \sum_{u \in supp(p)} p(u) \times W\Big(n(u), \Xi_{n(u)}(u)\Big) \geq \sum_{u \in supp(p')} p'(u) \times W\Big(n(u), \Xi_{n(u)}(u)\Big). \tag{A.3}$$

Consider any $e \in \mathbb{R}$ and any $m \in \mathbb{N}$. Let u, v, w in U be three alternatives such that $n(u) = n(v) = 1$, $n(w) = m$, $N(u) \cap N(v) = N(u) \cap N(w) = N(v) \cap N(w) = \emptyset$, and $u_i = v_j = w_k = e$ for all $i \in N(u)$, $j \in N(v)$ and $k \in N(w)$. By Social risk neutrality in population size for perfect equality, if p and $p' \in P$ are such that $p(uvw) = 1/2$, $p(w) = 1/2$, $p'(uw) = 1/2$ and

$p'(vw) = 1/2$, then $p \sim p'$. By Equation (A.3), this means that:

$$\tfrac{1}{2}W(m+2,e) + \tfrac{1}{2}W(m,e) = \tfrac{1}{2}W(m+1,e) + \tfrac{1}{2}W(m+1,e).$$

Let us denote $\theta_e : \mathbb{N} \to \mathbb{R}$ the function such that $\theta_e(n) = W(n,e)$. The equality implies that for all $m \in \mathbb{N}$,

$$\theta_e(m+2) = 2\theta_e(m+1) - \theta_e(m).$$

Let us prove that $\theta_e(n) = \Phi_e + n\Psi_e$, for all $n \in \mathbb{N}$, where $\Phi_e = 2\theta_e(1) - \theta_e(2)$ and $\Psi_e = \theta_e(2) - \theta_e(1)$. For $n = 1$ and $n = 2$, this is obviously true. Assume that for some $m \in \mathbb{N}$, $\theta_e(m) = \Phi_e + m\Psi_e$ and $\theta_e(m+1) = \Phi_e + (m+1)\Psi_e$: let us show that it is also the case for $m+2$. By the equality above, we obtain

$$\begin{aligned}\theta_e(m+2) &= 2\theta_e(m+1) - \theta_e(m) = 2\big(\Phi_e + (m+1)\Psi_e\big) - \big(\Phi_e + m\Psi_e\big)\\ &= \Phi_e + \big[2(m+1) - m\big]\Psi_e = \Phi_e + (m+2)\Psi_e.\end{aligned}$$

Hence, for any $e \in \mathbb{R}$ and any $n \in \mathbb{N}$, we have that:

$$W(n,e) = \big[2W(e,1) - W(e,2)\big] + n\big[W(e,2) - W(e,1)\big].$$

Denoting $\Phi : \mathbb{R} \to \mathbb{R}$ the continuous function such that $\Phi(e) = 2W(e,1) - W(e,2)$ for all $e \in \mathbb{R}$, and $\Psi : \mathbb{R} \to \mathbb{R}$ the continuous function such that $\Psi(e) = W(e,2) - W(e,1)$ for all $e \in \mathbb{R}$, we obtain the result.

A.3 Proof of Theorem 2

Step 1: Proof that ECLGU satisfies all the properties. It can easily be checked that an ECLGU social welfare ordering satisfies the Basic Principles, Ind-l, and Social risk neutrality in population size for perfect equality.

Step 2: function V can only be specific transformations of an additively separable social welfare function. Given that $\succsim$ on P satisfies the Basic Principles and Ind-l (and therefore Ind-a), we know by Proposition 2 that there exist two continuous and increasing functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = g(0) = 0$, and for all p, $p' \in P$:

$$p \succsim p' \iff \sum_{u \in supp(p)} p(u) \times \varphi\left(\sum_{i \in N(u)} \Big[g(u_i) - g(c)\Big]\right) \geq \sum_{u \in supp(p')} p'(u) \times \varphi\left(\sum_{i \in N(u)} \Big[g(u_i) - g(c)\Big]\right). \tag{A.4}$$

Consider p, p', q, $q' \in P$, such that there exist distinct alternatives u, v, v', w, and w' in U for which:

- $N(u) \cap N(v) = N(u) \cap N(v') = N(u) \cap N(w) = N(u) \cap N(w') = \emptyset$;
- $p(uv) = p(uv') = q(uw) = q(uw') = p'(v) = p'(v') = q'(w) = q'(w') = 1/2$.

By Ind-l, and using Equation (A.4), we obtain that:

$$\begin{aligned}
&\tfrac{1}{2}\varphi\left(\sum_{i \in N(u)} [g(u_i) - g(c)] + \sum_{i \in N(v)} [g(v_i) - g(c)]\right) + \tfrac{1}{2}\varphi\left(\sum_{i \in N(u)} [g(u_i) - g(c)] + \sum_{i \in N(v')} [g(v'_i) - g(c)]\right) \\
&\geq \tfrac{1}{2}\varphi\left(\sum_{i \in N(u)} [g(u_i) - g(c)] + \sum_{i \in N(w)} [g(w_i) - g(c)]\right) + \tfrac{1}{2}\varphi\left(\sum_{i \in N(u)} [g(u_i) - g(c)] + \sum_{i \in N(w')} [g(w'_i) - g(c)]\right) \\
\iff &\tfrac{1}{2}\varphi\left(\sum_{i \in N(v)} [g(v_i) - g(c)]\right) + \tfrac{1}{2}\varphi\left(\sum_{i \in N(v')} [g(v'_i) - g(c)]\right) \\
&\geq \tfrac{1}{2}\varphi\left(\sum_{i \in N(w)} [g(w_i) - g(c)]\right) + \tfrac{1}{2}\varphi\left(\sum_{i \in N(w')} [g(w'_i) - g(c)]\right)
\end{aligned}$$

Denote $a = \sum_{i\in N(u)} \big[g(u_i) - g(c)\big]$, $x = \sum_{i\in N(v)} \big[g(v_i) - g(c)\big]$, $x' = \sum_{i\in N(v')} \big[g(v_i') - g(c)\big]$, $y = \sum_{i\in N(w)} \big[g(w_i) - g(c)\big]$ and $y' = \sum_{i\in N(w')} \big[g(w_i') - g(c)\big]$. We get that, for any real numbers a, x, x', x and y':

$$\varphi(a+x)+\varphi(a+x') \geq \varphi(a+y)+\varphi(a+y') \Longleftrightarrow \varphi(x)+\varphi(x') \geq \varphi(y)+\varphi(y').$$

Let $I = \varphi(\mathbb{R})$, which is an open interval in $\mathbb{R}$ because φ is continuous and increasing. First fix a and denote $\psi_a : I \to \mathbb{R}$ the continuous function such that $\psi_a(x) = \varphi\big(a + \varphi^{-1}(x)\big)$. By letting $z = \varphi(x)$, $z' = \varphi(x')$, $t = \varphi(y)$ and $t' = \varphi(y')$, the above equivalence can be written:

$$\psi_a(z) + \psi_a(z') \geq \psi_a(t) + \psi_a(t') \Longleftrightarrow z + z' \geq t + t',$$

and it holds for all z, z', t and t' in I. So, there must exist an increasing function $\Psi_a : \mathbb{R} \to \mathbb{R}$ such that, for all z, $z' \in I$: $\psi_a(z) + \psi_a(z') = \Psi_a(z + z')$. This is a Pexider functional equation, and it is known that in that case Ψ_a and ψ_a must be affine (Aczél, 1966). Hence there exist $\alpha_a \in \mathbb{R}_{++}$ and $\beta_a \in \mathbb{R}$ such that $\psi_a(z) = \alpha_a z + \beta_a$.

Define the functions $\alpha : \mathbb{R} \to \mathbb{R}_{++}$ and $\beta : \mathbb{R} \to \mathbb{R}$ by $\alpha(a) = \alpha_a$ and $\beta(a) = \beta_a$ for all $a \in \mathbb{R}$. By definition of function ψ_a, we obtain that, for all $x \in \mathbb{R}$, $\psi_a(\varphi(x)) = \varphi\big(a + \varphi^{-1} \circ \varphi(x)\big) = \varphi(a + x)$. But by our result above, it is also the case that $\psi_a(\varphi(x)) = \alpha(a)\varphi(x) + \beta(a)$. We thus end up with the functional equation: $\varphi(a + x) = \alpha(a)\varphi(x) + \beta(a)$ for all $(a, x) \in \mathbb{R}^2$. By Corollary 1 (pp. 150–151) in Aczél (1966), this equation implies that either φ is affine or that it is a positive affine transformation of the function $x \to \alpha e^{\alpha x}$ for some $\alpha \neq 0$.

Step 3: Conclusion. By Step 2, we know that

$$V(u) = \varphi\left(\sum_{i\in N(u)} \big[g(u_i) - g(c)\big]\right),$$

where V is the function in the statement of Social expected utility, and φ is affine or that it is a positive affine transformation of the function $x \to \alpha e^{\alpha x}$ for some $\alpha \neq 0$.

Also given that $\succsim$ satisfies RiskNeu, we know by Prop. 3 that there exist a positive number a, a number b, two continuous functions $\Psi : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$, and for each $n \in \mathbb{N}$ a continuous, increasing, symmetric and normalized function Ξ_n such that, for u such that $n(u) > 0$:

$$V(u) = a\Big[\Phi\left(\Xi_{n(u)}(u)\right) + n(u) \times \Psi\left(\Xi_{n(u)}(u)\right)\Big] + b.$$

By contradiction, suppose that φ is a positive affine transformation of the function $x \to \alpha e^{\alpha x}$ for some $\alpha \neq 0$. Consider any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, and let u be such that $n(u) = n$ and $u_i = x$ for all $i \in N(u)$. We must have:

$$\alpha e^{\alpha n(g(x)-g(c))} = a\Big[\Phi\left(x\right) + n \times \Psi\left(x\right)\Big] + b.$$

If $x \neq c$, then the left-hand side is an exponential function in n, whereas the right-hand side is an affine function in n. But this cannot be true. Hence only the affine form is possible for φ.

A.4 Proof of Proposition 4

Consider any $N \in \mathcal{N}$ and let $n = |N|$. For any $i \in N$ let $\succsim_i$ be the ordering on P_N such that, for any p, $p' \in P_N$, $p \succsim_i p'$ if and only if $p_i \succsim p'_i$. By Social expected-utility, defining $W_i(p) = \sum_{u \in U_i} p_i(u)V(u)$, we have that $p \succsim_i p'$ if and only if $W_i(p) \geq W_i(p')$. Similarly, defining $W_0(p) = \sum_{u \in U_N} p(u)V(u)$, we have that $p \succsim p'$ if and only if $W(p) \geq W(p')$.

Consider $F = (W_0, (W_i)_{i \in N}) : P_N \to \mathbb{R}^{n+1}$. By definition $F(P_N)$ is convex because $W_i(\kappa p + (1-\kappa)q) = \kappa W_i(p) + (1-\kappa)W_i(q)$ for each $i \in N$ or $i = 0$. IDom implies that, if $W_i(p) > W_i(q)$ for all $i \in N$, then $W_0(p) > W_0(q)$. By Proposition 2 in De Meyer and Mongin (1995), there must exist non-negative numbers λ_i and a number γ such that, for each $p \in P_N$:

$$W_0(p) = \sum_{i \in N} \lambda_i W_i(p) + \gamma.$$

Focusing on lotteries yielding sure outcomes, we get that, for all u, $v \in U_N$,

$$u \succsim v \Longleftrightarrow \sum_{i \in N} \lambda_i V(u_i) \geq \sum_{i \in N} \lambda_i V(v_i).$$

By Anonymity and Strong Pareto, it must be the case that all the λ_i must be the same positive number λ. Denoting g the function such that $g(u_i) = V(u_i)$ for each $u_i \in \mathbb{R}$, we obtain that for all p, $q \in P_N$,

$$p \succsim q \Longleftrightarrow \sum_{u \in U_N} p(u) \left[\sum_{i \in N} g(u_i) \right] \geq \sum_{u \in U_N} p(u) \left[\sum_{i \in N} g(u_i) \right].$$

Observe that, by definition and Anonymity, the function g does not depend neither on i nor on N.

By Social expected-utility, and given that a VNM utility function is defined up to an increasing affine transformation, it must be the case that, for each $N \in \mathcal{N}$, for each $u \in U_N$:

$$V(u) = F(N)\left[\sum_{i\in N} g(u_i)\right] + G(N),$$

for some positive $F(N)$ and some number $G(N)$. By Anonymity, $F(N)$ and $G(N)$ only depend on population size.

A.5 Proof of Theorem 3

Step 1: Proof that ECLGU satisfies all the properties. It can easily be checked that an ECLGU social welfare ordering satisfies the Basic Principles, IDom, IndPUnc and CCL-rf.

Step 2: Proof that $\succsim$ has a constant critical level. We show that if $\succsim$ satisfies the Basic Principles and IndPUnc, then it satisfies CCL-rf.

By Minimal existence of a critical level, there exist $u \in U$, $c \in \mathbb{R}$, and $i \notin N(u)$, such that if $v \in U$ is defined by $N(v) = N(u) \cup \{i\}$, $v_i = c$ and $v_j = u_j$ for all $j \in N(u)$, then $v \sim u$. Let ε be some positive number. Define $\tilde{v} \in U$ by $N(\tilde{v}) = N(v)$, $\tilde{v}_i = c + \varepsilon$ and $\tilde{v}_j = u_j$ for all $j \in N(u)$. By Strong Pareto, we must have $\tilde{v} \succ v \sim u$, which by Social expected-utility implies that $V(\tilde{v}) > V(u)$.

Consider any $u' \in U$ and any $k \notin N(u')$, and define $\tilde{v}' \in U$ such that $N(\tilde{v}') = N(u') \cup \{k\}$, $\tilde{v}'_k = c + \varepsilon$ and $\tilde{v}'_l = \tilde{u}_l$ for all $l \in N(u')$. Let p, $p' \in P$ such that $p(u') = p'(\tilde{v}') = q$ and $p(u) = p'(\tilde{v}) = 1 - q$, with $q \in (0, 1)$.

IndPUnc requires that, if $p' \succ p$ for some q, this should be true whatever q is. But $p' \succ p$ means that:

$$qV(\tilde{v}') + (1-q)V(\tilde{v}) > qV(u') + (1-q)V(u)$$

which can be written

$$V(\tilde{v}) - V(u) > \tfrac{q}{1-q}\Big[V(u') - V(\tilde{v}')\Big].$$

Given that $V(\tilde{v}) > V(u)$, this must be true, whatever the values $V(\tilde{v}')$ and $V(u')$, for small enough value of q.

Thus, whatever q is, we have:

$$qV(\tilde{v}') + (1-q)V(\tilde{v}) > qV(u') + (1-q)V(u)$$

which can be written

$$V(\tilde{v}') - V(u') > \tfrac{1-q}{q}\Big[V(\tilde{v}) - V(u)\Big].$$

Given that $\frac{1-q}{q}$ can be as low as we want, we need to have $V(\tilde{v}') - V(u') \geq 0$, and therefore $\tilde{v}' \succsim u'$.

So for every $\varepsilon > 0$, $\tilde{v}' \succsim u'$, where $\tilde{v}' \in U$ is such that $N(\tilde{v}') = N(u') \cup \{k\}$, $\tilde{v}'_k = c + \varepsilon$ and $\tilde{v}'_l = \tilde{u}_l$ for all $l \in N(u')$. By continuity, if $v' \in U$ such that $N(v') = N(u') \cup \{k\}$, $v'_k = c$ and $v'_l = u'_l$ for all $l \in N(\tilde{u})$, then $v' \succsim u'$.

We can prove that $u' \succsim v'$ in the same way by using a negative $\varepsilon < 0$. Hence, $u' \sim v'$.

In conclusion, there exists $c \in \mathbb{R}$, such that for all $u' \in U$ and for all

$k \notin N(u')$, if $v' \in U$ is such that $N(v') = N(u') \cup \{k\}$, $v'_k = c$ and $v'_l = u'_l$ for all $l \in N(u)$, then $u' \sim v'$. The social ordering $\succsim$ satisfies CCL-rf.

Step 3: Conclusion. Given that $\succsim$ satisfies the Basic Principles and IDom, we know by Proposition 4 that it is an ENWGU social ordering. Then there exist a continuous and increasing function $g : \mathbb{R} \to \mathbb{R}$, and two functions $F : \mathbb{N} \to \mathbb{R}$ and $G : \mathbb{N} \to \mathbb{R}$ such that for any $u \in U$ the VNM social utility function is

$$V(u) = F\big(n(u)\big) \sum_{i \in N(u)} g(u_i) + G\big(n(u)\big).$$

By Step 3, we also know that $\succsim$ satisfies CCL-rf. Hence, there exists a $c \in \mathbb{R}$ such that, for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}$, if $u, v \in U$ are such that $n(u) = n$, $N(v) = N(u) \cup \{i\}$ (with $i \notin N(u)$), $u_j = v_j = x$ for all $j \in N(u)$ and $v_i = c$, then $V(u) = V(v)$, so that:

$$F(n)ng(x) + G(n) = F(n+1)\big[ng(x) + g(c)\big] + G(n+1).$$

Given that g is increasing in x and that this equality must be true for all $x \in \mathbb{R}$, we must have $F(n+1) = F(n)$. Hence, we also obtain by the equality that $G(n+1) - G(n) = -g(c)$. Hence, by iterating on n (given that these results are true for all $n \in \mathbb{N}$), we have $F(n) = F(0) = a$ and $G(n) = G(0) - ng(c) = b - ng(c)$, where $a > 0$ and b is a real number. Hence,

$$V(u) = a \sum_{i \in N(u)} \big[g(u_i) - g(c)\big] + b.$$

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