Ads/CFT correspondence and lower bound of η/s

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Abstract

We use the Ads/CFT correspondence to calculate the correlation functions of the stress-energy tensor in the gravity regime and use to compute the shear viscosity η of a strongly coupled Yang Mills plasma and use the known entropy density s to calculate η/s. We review the KSS bound on η/s and describe possible ways to violate the lower bound.

1 Introduction

Ads/CFT correspondence is a study of the duality between certain conformal field theories in d dimensions and string theory in Anti De Sitter space in d+1 dimensions. Conformal transformations are basically maps between spaces which preserve angles between directed curves at any given point but not necessarily the distance between points. Quantum field theories which are invariant under conformal transformation are called conformal field theories (CFT). An n dimensional Anti De Sitter space Ads_n is a Lorentzian manifold(with metric signature (1,n-1) and not necessarily positive metric tensor) whose scalar curvature is a constant negative.
In 1997, Juan Maldacena's phenomenal paper proposed a duality betwenn CFTs and Ads in one dimension higher. The basic idea is that the CFT lives on the boundary surface of the Ads which is reminiscent of holography, hence it's also known as holographic duality.
A lot of research has been carried in which this duality is used to study the properties of gauge theories (the CFT side) by studying their gravity dual. One of the most promising fields in which said duality has been utilised is the Quark Gluon Plasma (QGP). The QGP is a strongly correlated and as such it is difficult to obtain microscopic features of it. However, a for a large system size and over long time scales, we study its macroscopic behaviour which involve properties are that are universal or at least belong to a universality class describing a wide range of systems, such as transport coefficients.
This is where Ads/CFT comes in handy (due to universality) because we can use it to study strongly coupled gauge theories in terms of weakly coupled gravitational systems. A large coupling in the gauge theory requires non perturbative

techniques which are analytically difficult and may require brute force numerical computation however on they can be approximated in their lower energy supergravity duals.
The basic idea is that since Ads is described in one dimension more than the CFT, we can describe additional physics in the extra dimension. For example the different masses of different ρ mesons with all other quantum numbers same can be explained by stating that there's a finite 1-D square well in the extra dimensions and the ρ mesons of different masses are different excited states of the same meson in the square well. This is somewhat of an oversimplification but more complicated potentials could help us better approximate the real world masses.
For our purpose we shall first reproduce the work of Son and Starinets to reproduce a gravity dual of the N=4 Super Yang Mills (SYM) plasma which is a close approximation to a real world QGP, and write down the two point correlation function of the stress energy tensor. To deduce the form of the correlators at finite temperature, we shall use the constraints imposed due to hydrodynamics. Then on we shall calculate the shear viscosity coefficient using Kubo formula and the entropy density using black hole thermodynamics. We write down the value of η/s $= 1/(4\pi)$ and discuss the work of KSS on why it could be a possible lower bound. Finally we discuss Cremonini's work on how the bound could be undercome.

2 Some comments on AdS/CFT

2.1 On CFTs

A conformal transformation in a D-dimensional space-time is a change of coordinates that rescales the line element. In differential geometry, it can be thought of as a diffeomorphism.
A conformal transformation is defined by;

$$x_\mu \longrightarrow x'_\mu \qquad g'_{ab}(x) = \Omega^2(x) g_{ab}(x) \tag{1}$$

where $\Omega(x)$ is any arbitrary function of coordinate. taking $\Omega(x) = constant$ we get scale transformation. Conformal transformations rescale lengths but preserve the angles between vectors.
To obtain the formula for conformal transformations at the infinitesimal level, consider $x'_\mu = x_\mu + v_\mu(x)$ and $\Omega(x) = 1 + \frac{\omega(x)}{2}$. we then get,

$$\partial_\mu v_\nu + \partial_\nu v_\mu = w(x) g_{\mu\nu} \tag{2}$$

From taking the trace we get $Dw = 2\partial^\mu v_\mu$ and resubstituting we get,

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{D}(\partial^\tau v_\tau) g_{\mu\nu} \tag{3}$$

A conformal field theory (CFT) is a Quantum Field Theory that is invariant under such transformations. Physically it corresponds to the fact that changing the length scaling should not change the physics of the system. Hence in CFTs we only care about angles and not about distance. Even though CFTs are a subset of QFTs, the machinery needed to describe them is different. Note that scale-invariant QFTs are almost always invariant under the full conformal symmetry.

2.2 On AdS

Ads$_D$ is a D-dimensional solution to the theory of gravitation with a negative cosmological constant Λ and an Einstein hilbert action like action given by

$$S = \frac{1}{16\pi G_{(D)}} \int d^D x \sqrt{-g}(R - 2\Lambda) \tag{4}$$

where g = det(g$_{\mu\nu}$) with g$_{\mu\nu}$ being the metric tensor, R being the Ricci scalar and G$_D$ being Einstein gravitational constant in D dimensions.
In this case the equations of motion are just the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} \tag{5}$$

Taking trace we get,

$$R = g^{\mu\nu} R_{\mu\nu} = 2\frac{D+1}{D-1}\Lambda \implies R_{\mu\nu} = \frac{2}{D-1}_{\mu\nu} \tag{6}$$

Now for the metric, consider a QFT in D dimensions. The most general metric that is Poincare invariant is;

$$ds^2 = \Omega^2(z)(-dt^2 + d\vec{x}^2 + dz^2) \tag{7}$$

where z is the coordinate of the extra dimension, $\vec{x} = (x^1, ..., x^{D-1}$ and $\Omega(z)$ is the function to be determined. If z represents a length scale and the theory is conformal invariant, then ds^2 must be invariant under the transformation

$$x^\mu \longrightarrow \lambda x^\mu, \quad z \longrightarrow \lambda z \tag{8}$$

Then, by imposing the invariance of the metric under the transformation, we obtain that the function $\Omega(z)$ must transform as:

$$\Omega(z) \longrightarrow \lambda^{-1}\Omega(z) \tag{9}$$

Which implies that $\Omega(z)$ would have to be of a from of $\Omega(z) = \frac{L}{z}$ where L is a constant.
Hence the metric is

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \tag{10}$$

From the above metric, direct computation suggest

$$R_{\mu\nu]} = -\frac{D}{L^2}\Lambda g_{\mu\nu} \tag{11}$$

hence we get,

$$\Lambda = -\frac{D(D-1)}{2L^2}, \qquad R = -\frac{D(D+1)}{L^2} \tag{12}$$

For purposes, we will take D = 4 (and unit L) and hence we $R = -20$ and $\Lambda = -6$

AdS space is characterized by its boundary, which is a four-dimensional spacetime that is flat. The principle of AdS/CFT correspondence is that there is an equivalence between a certain type of gravity theory in AdS space and CFTs on the boundary of AdS space. Hence a strongly coupled field theory can be explained by a weakly curved gravitational theory.

3 Correlation functions in holography

In quantum field theory the Lagragian and Hamiltonian are described in terms of field degrees of freedom ϕ(x) at any given space time point x. ϕ describes the state of the system at said point. Dual to ϕ we have is a quantum operator O defined locally at the origin. O is an operator in the CFT, and it is associated with a background field J (like a source current) in Ads space (on whose boundary the CFT lives. Considering our CFT in 4 dimensions we can think of J(x) as the boundary value of a five dimensional background field J(x,z) where z is the coordinate of the extra dimension. The operator/state correspondence states that there is a one-one correspondence between states in the Hilbert space and local operators acting at the origin which can generate these states.

Because of this we can formally write the correspondence using generating functionals as

$$Z_{4D}(J) = e^{iS_{5D}} \tag{13}$$

The LHS is the generating functional in 4D with a source term [1].

$$Z_{4D}(J) = \int [D\phi] exp(i(S_{CFT}(\phi) + \int d^4x \phi \hat{O})) \tag{14}$$

and the RHS being the exponential of the action in 5D at z=0. Using this we can obtain the correlation function as

$$\langle T\, O(x_1)\, O(x_2) \rangle = \frac{\delta^2 S_{5D}(\phi)}{i\delta\phi(x_1)\delta\phi(x_2)}|_{z=0} \tag{15}$$

4 Stress energy tensor

For our purposes we shall be needing the stress energy tensor $T_{\mu\nu}$ to describe the shear viscosity which shall be further explored in the next section. Let's discuss the stress energy tensor for a plasma which is slightly out of equilibrium in which the three three momentum density T_{0i} (i = 1,2,3) is taken to be zero.

$$T_{\mu\nu} = P\delta_{\mu\nu} - \eta(\partial_\mu u_\nu + \partial_\nu u_\mu - \frac{2}{3}\delta_{\mu\nu}\partial_\rho u_\rho) - \zeta\delta_{\mu\nu}\partial_\rho u_\rho \tag{16}$$

Here u_μ is the fluid velocity, P is the pressure, η is the shear viscosity and ζ is the bulk viscosity. For a conformal theory, the stress energy tensor must necessarily be traceless. This implies that the energy density ϵ given by the T_{00} component is 3P and ζ which is proportional to the trace of the tensor would therefore be zero.

5 Shear viscosity

The shear viscosity η of a system gives a measure of its resistance to flow. Flow causes a velocity gradient to be formed in a fluid across surfaces, known as the shear rate and η is the amount of shear stress applied to overcome unit shear rate.
For the QGP, or rather the N=4 SYM plasma, to calculate the shear viscosity from supergravity we shall use the fluctual dissipation theorem of linear response theory (after all the QGP is a many body system). We shall consider our plasma to be in equilibrium and think of η as the response of the system to fluctuations in the stress energy tensor. Ultimately the relationship boils down to η being proportional to the 2-point correlation function of $T_{\mu\nu}$ can be quantified by the Kubo formula [2]

$$\eta = \lim_{x\to 0} \frac{1}{2\omega} \int d^4x e^{i\omega t} \langle |T_{ij}(x), T_{ij}(0)| \rangle \tag{17}$$

where i,j= 1,2,3 as we have only considered the shear components of the stress energy tensor. The 2 point correlator here can be re-written as the imaginary part of the retarded Green's function $G^R_{xy,xy}(\omega, k=0)$. The small frequency ($\omega \to 0$) and zero momentum (k = 0) at equilibrium imply that this is a low energy description for a large system size over a long time. This is where hydrodynamics comes in handy.
Now we shall exploit the mathematical machinery we had developed earlier to find the correlation function and calculate the shear viscosity. On the gravity side, the dual of the stress energy tensor is the metric tensor $g_{\mu\nu}$. So we add shear fluctuation (a perturbation) terms $h_{\mu\nu}$ on $\mathrm{Ads}^5 \times S^5$. Herein we utilise the duality between strongly coupled gauge theory living on the D-brane (generalisation of the boundary or membrane for any given dimension) and supergravity with a background.
While working in Ads_5 we describe the line element as [3]

$$ds^2 = \frac{1}{z^2}[g_{\mu\nu}(x,z)dx^\mu dx^z + dz^2] = \tilde{g}_{MN} dX^M dX^N \tag{18}$$

where $\tilde{g}_{MN}$ is the 5D metric. The effective action that we need to perform said perturbation in $\mathrm{Ads}^5 \times S^5$ is described in the low energy limit of type IIB string theory

$$S_{5D} = \frac{N^2}{8\pi^2} \int d^5x \sqrt{-g}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda) h^{\mu\nu} \tag{19}$$

Here $\mathrm{R}_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant. We can expand $\mathrm{R}_{\mu\nu} = \mathrm{R}^{(0)}_{\mu\nu} + \delta\mathrm{R}_{\mu\nu}$ and $\mathrm{R} = \mathrm{R}^{(0)} + \delta\mathrm{R}$, $\mathrm{g}_{\mu\nu} = \mathrm{g}^{(0)}_{\mu\nu} + \mathrm{h}_{\mu\nu}$. Here the superscript (0) indicates the unperturbed form. Using the Einstein equation

$$R^{(0)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(0)} + g^{(0)}_{\mu\nu}\Lambda = \kappa T_{\mu\nu} \tag{20}$$

with κ being the gravitational constant, we can re write equation (5) as

$$S_{5D} = \frac{N^2}{8\pi^2}\int d^5x\sqrt{-g}(\delta R_{\mu\nu} - \frac{1}{2}g^0_{\mu\nu}\delta R + (\Lambda - \frac{1}{2}R^{(0)})h_{\mu\nu})h^{\mu\nu} \tag{21}$$

Since $\mathrm{T}_{\mu\nu}$ is regarded as an operator in CFT and the metric $\mathrm{g}_{\mu\nu}$ in 4D is considered as its source, we shall consider perturbations in $\mathrm{g}_{\mu\nu}$, $\mathrm{g}^{11}\mathrm{h}_{12} = \phi$ as the bulk field which couples to he operator $\mathrm{T}_{\mu\nu}$ and use (3) to obtain the 2-point correlation function in the Kubo formula.
Since the QGP is a thermal medium, we need to define our correlator at finite temperature T. Following the work of Son and Starinets [4], where they have computed obtain the retarded green's function for a BTZ black hole on the gravity side dual to a 2D CFT, we obtain the thermal green's function as [5]

$$G_{ij,ij}(\omega, k) = \frac{-N^2T^2}{16}(i2\pi T\omega + k^2) \tag{22}$$

where we define the plugging this into the Kubo formula we get

$$\eta = \frac{\pi N^2 T^3}{8} \tag{23}$$

where N is the number of colors. N can be finite (such as N=3, the number of colors of quarks) or it can be infinite. We shall see further use of black holes in the next section.

6 Entropy density

To obtain the entropy we go to the gravity side and consider the entropy of a black hole in Ads_{5D}. Qualitatively we can understand this by the following example. Consider objects falling into the vent horizon of a black hole, adding to its mass. As the radius r of the event horizon is proportional to the black hole mass M (heavier black holes would have more gravitational pull so escape velocity equal to lightspeed would be at a larger distance from the singularity), more mass falling into the black hole should increase it's radius and hence its area. The addition of mass to the black hole would apparently destroy all information about it, however by the second law of thermodynamics it should increase the entropy of the universe. By the work of Hawking and Beckenstein on black hole thermodynamics we find that the entropy is indeed equivalent to its area A.

$$s = \frac{A}{4} = \frac{\pi r^2}{4} \tag{24}$$

and the Hawking temperature of the black hole of mass M_{BH} is proportional to its surface gravity

$$T = \frac{\kappa M_{BH}}{2\pi r^2} \tag{25}$$

κ being the gravitational constant.
In 5 dimensions, using the work of Shi pu [3], we consider the black horizon to be located at z_H in the fifth dimension and describe our Hawking temperature as

$$T = \frac{\sqrt{2}}{\pi Z_H} \tag{26}$$

from which we obtain the entropy density to be

$$s = \frac{\pi^2 N^2 T^3}{2} \tag{27}$$

7 Universality and lower bound of η/s

Using the above computed value of η and s we compute their ratio as

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{28}$$

The above value holds for various gauge groups, at zero and finite chemical potential and even in the presence of external background fields.
Kovtun, Son and Starinets have proposed a conjecture that for a variety of systems, including the ones described by UV complete QFTS, the above value of η/s is indeed a lower bound

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \tag{29}$$

Supporting this conjecture are many pieces of evidence. The first and foremost being the Heisenberg uncertainty principle and kinetic theory. Earlier we described the energy momentum where η appears on the RHS. The energy density $\epsilon = T^{00}$ is therefore proportional to η. Using the relativisitc hyrdrodynamics equation $\partial_\mu T^{\mu\nu} = 0$, we get a linearized equation of motion which in frequency momentum space gives us the the dispersion relation (relation between ω and k). We get 2 transverse modes

$$\omega(k) = -i\frac{\eta k^2}{\epsilon + P} \tag{30}$$

and a sound mode in equilibrium

$$\omega(k) = \sqrt{\frac{dP}{d\epsilon}} - \frac{2\eta k^2}{3(\epsilon + P)} \tag{31}$$

Since the viscosity is proportional to the energy density as well as the mean free time τ between particle collisions while the entropy density is proportional

to the number density n, we have η/s $\sim \epsilon\tau$/n $\sim$ Eτ, with E being the average energy of a particle.By the Uncertainty principle, the E$\tau \geq 1/4\pi$ from which we obtain the bound on η/s.
A slightly more complicated evidence supporting the bound is the work of Buchel, Liu and Starinets, where leading corrections to η/s were obtained in terms of the inverse of powers of the 't Hooft coupling λ

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 + \frac{135\zeta(3)}{8\lambda^3(2N)^{2/3} + ...}) \tag{32}$$

η(z) being the Riemann Zeta function. For further higher order corrections the value can only go up hence $1/4\pi$ is a lower bound. Infact it diverges for $\lambda \to 0$ and becomes equal to the bound at $\lambda \to \infty$.
Besides the above mentioned theoretical evidence, the estimations of η/s obtained in the Relativistic Heavy ion collider (RHIC) based on elliptic flow point towards a value close to 0.08 (KSS bound is around 0.79).

8 Violations of the bound

We shall discuss ways in which the bound could be possibly violated.

8.1 Finite N, finite λ and higher curvature corrections

On the gauge theory side, as we already saw effects due to finite λ lead to corrections that respect the bound however effects due finite N can create corrections that violate it. The bound saturates for N $\to \infty$. We expand η/s in the form

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 - \frac{1}{2N}) + O(\frac{1}{N^2}) \tag{33}$$

The bound saturates for N $\to \infty$.
On the holography side these correspond to higher curvature corrections to the Lagrangian . Curvature arises from the Riemann tensors. The corrections are of the form

$$\alpha' R^2 + \alpha'^2 R^3 + \alpha'^3 R^4 + ... \tag{34}$$

where R^n denote all possible contractions involving n Riemann tensors (after all we can get Ricci tensor and scalr from contractions of the Riemann tensor). These are new terms in the action. Plugging these corrections into the action in (9) and then obtaining η from it, we can see that the bound is undercome.
Kats and Petrov performed curvature corrections starting from the quadratic order at AdS$_5\times$S^5/Z$_2$ (instead of AdS$_5\times$S^5) which described a modified action in 5-D dimesnions as

$$S = \frac{1}{16\pi G_N}\int d^5x\sqrt{-g}(R - 2\Lambda + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \tag{35}$$

Due to which the ratio will take the form

$$\eta/s = \frac{1}{4\pi}(1 + \frac{4}{3}\Lambda\alpha_3) \tag{36}$$

Since cosmological constant Λ is negative for Ads, the bound is violated for any positive value of α_3.

8.2 Entropy mixing in several species

In a non relativistic system, should we have a large number of non-identical, i.e. belonging to different species, degenerate particles then in principle the Gibbs mixing entropy should be higher. Consider a gas with multiple species of molecules. If we employ a logarithmic increase in the mixing entropy of the gas by introducing a large number of different species of molecules, we could potentially reach an entropy value in the denominator of η/s such that it falls below $1/4\pi$. Even though the QGP is not a gas, it might be possible to increase its mixing entropy in a similar or related fashion.
Cohen's work showed that the bound could be violated by increasing the number of species in the fluid even though the dynamics remain independent of the type of particle. However this theory is not UV complete, therefore to violate the bound in this way one might be forced work only in the low energy effective regime.

8.3 Viscosity decrease

Consider the Gauss-Bonnet gravity action

$$S_{GB} = \int d^5x\sqrt{-g}[R - 2\Lambda - \frac{3\lambda_{GB}}{\lambda}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})] \quad (37)$$

which is basically the action we saw earlier in but with αs written in terms of the Gauss-Bonnet 't Hooft coupling and cosmological constant. By the work of Brigante et al [7] using methods similar to ours, we can find that η is proportional to (1-4λ_{GB}).

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 - 4\lambda_{GB}) \quad (38)$$

which violates the bound for any positive η. Brigante, Liu and Myers further showed that to avoid microcausality violations in the dual CFT, we must have

$$\frac{\eta}{s} \geq \frac{16}{100\pi} \quad (39)$$

which leaves $\lambda_{GB} \leq 0.09$, therefore the viscosity never becomes negative.

9 Conclusion

So far we have given a brief understanding of the mathematical quality of Ads and CFT, and exploted it to find the viscosity from the retarded Green's function in the Kubo formula. We used Black hole thermodynamics to obtain the entropy density. We built up on the work of others to arrive at the value of $1/4\pi$ as a lower bound for η/s and explored a few possible ways in which the bound could be undercome.

10 References

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