Impulsive antiperiodic boundary value problem of fractional order

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Abstract

This paper is motivated from some recent papers on impulsive fractional differential equation.In this paper an antiperiodic boundary value problem for an impulsive fractional differential equation of order $q \in (1, 2)$ is studied.We develop an effective way to find solution of such type of problems.A special hybrid singular type Grownwell inequality is established to obtain prior bounds of the solution.The sufficient conditions for existence of the solutions are established by applying fixed point methods under the mixed nonlinear $\mathcal{D}$-contraction condition. an example is given to illustrate the result.

Keywords: Caputo fractional derivative; impulsive fractional differential equation; anti-periodic boundary condition; existence.
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1 Introduction

In recent years the subject of fractional calculus gained much momentum and attracted many researchers and mathematicians.Considerable interest in field of fractional calculus has been developed by the applications to different areas of applied science and engineering like physics, biophysics, aerodynamics,control theory, viscoelasticity, capacitor theory, electrical circuit, description of memory and hereditary properties etc. See [1]-[6] .
Meanwhile many evolution processes are subject to short term perturbations whose duration is negligible in comparison with the duration of processes, that is in form of impulses.A strong motivation for studying impulsive fractional differential equations comes from the fact they have been proved to be valuable tool in a number of fields such as physics,geophysics,regular variation in thermodynamics,electrical circuits etc.For more details one can see the monographs and research papers and references therein, see [7]-[15].we will study the following antiperiodic boundary value problems for impulsive fractional differential equation.

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), \quad t \in J' = J \backslash \{t_1, t_2....t_m\}, \quad J = [0, 1], \\ \Delta u(t_k) = I_k, \quad \Delta u'(t_k) = J_K, \quad k = 1, 2, ...m. \\ 3u(0) = -u(1), \quad 3u'(0) = -u'(1). \end{cases} \tag{1}$$

where ${}^cD_t^q$ denotes the Caputo's fractional derivative of order $q \in (1,2)$ with lower limit zero, $f\colon J \times R \to R$ is jointly continuous, $I_k, J_k \in R$ and t_k satisfy $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = 1$,$\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

2 Mathematical Preliminaries

In this section we introduce notations,definitions and preliminary facts.Throughout this paper, let $C(J,R)$ be the Banach space of all continuous functions from J into R with the norm $\|u\|_c = sup\{|u(t)|\colon t \in J\}$ for $u \in C(J,R)$.We also define $PC(J,R) = \{u : J \to R : u \in (t_k, t_{k+1}], R\}$,$k = 0, 1....m$ and there exist $u(t_k^+)$ and $u(t_k^-)$,$k = 1, 2....m$ with $u(t_k^-) = u(t_k)$ with the norm $\|u\|_{pc} = sup\{|u(t)|\colon t \in J\}$.For measurable functions $l : J \to R$,define the norm $\|l\|_{L^\sigma(J,R)} = (\int_J |l(t)|^\sigma dt)^{\frac{1}{\sigma}}$,$1 \leq \sigma < \infty$.We denote $L^\sigma(J,R)$ the Banach space of all Lebesgue measurable functions l with $\|l\|_{L^\sigma} < \infty$. Let us recall some more definitions of fractional calculus.For more details see [2].

Definition 2.1. *The fractional integral of order γ with the lower limit zero for a function $f : [0,\infty) \to R$ is defined as*

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0,$$

provided that right is point-wise defined on$[0,\infty)$, where $\Gamma(.)$ is the gamma function.

Next we introduce the Caputo fractional derivative.

Definition 2.2. *(See [2]). Caputo fractional derivative of order γ for a function $f : [0,\infty) \to R$ is defined as*

$${}^CD_t^\gamma f(t) =^L D_t^\gamma f(t) \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \quad t > 0, \quad n-1 < \gamma < n.$$

Lemma 2.3. *(See [17]). For $q > 0$ the general solution of fractional differential equation*
${}^cD_t^q u(t) = 0$ is given by

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}.$$

where $c_i \in R$, $\quad i = 0, 1, 2, ..., n-1$, $\quad (n = -[-q])$ and $[q]$ denotes the integer part of real number $q > 0$.

Definition 2.4. *(See [18]). Let X be a infinite dimensional Banach space with the norm $\|.\|$.A mapping $T : X \to X$ is called $\mathcal{D}$-Lipshitzian if there exists a continuous non-decreasing function $\phi_T : R^+ \to R^+$ satisfying*

$$\|T_x - T_y\| \leq \phi_T(\|x - y\|)$$

for all $x, y \in X$ with $\phi_T(0) = 0$.Sometime we call the function ϕ_T a $\mathcal{D}$-function of T on X. If $\phi_T(r) = \alpha r$ for some constant $\alpha > 0$ then T is called a Lipschitzian with a Lipschitz constant α and further if $\alpha < 1$, then T is called a contraction with the contraction constant α. Again if ϕ_T satisfies $\phi_T(r) < r$, $r > 0$ then T is called a nonlinear $\mathcal{D}$-contraction on X.

Now following generalized Gronwell inequality which is introduced by Wang et al.[25] can be used in the fractional differential equation with initial condition.

Lemma 2.5. *(Lemma 2,[16]). Let $y \in C(J, R)$ is satisfying following inequality*

$$y(t) \leq c_1 + c_2 \int_0^t |y(s)|^{\lambda_1} ds + c_3 \int_0^1 |y(s)|^{\lambda_2} ds, \quad t \in J$$

where $\lambda_1 \in [0, 1]$, $\lambda_2 \in [0, 1)$, $c_1, c_2, c_3 \geq 0$ are contants.Then there exists a constant $M^* > 0$ such that

$$|y(t)| \leq M^*|$$

Using Lemma 2.8, we can obtain a new special hybrid singular type Gronwell inequality.

Lemma 2.6. *Let $y \in C(J, R)$ is satisfying following inequality*

$$y(t) \leq c_1 + c_2 \int_0^t (t-s)^{q-1} |y(s)|^{\lambda} ds + c_3 \int_0^1 (1-s)^{q-1} |y(s)|^{\lambda} ds + c_4 \int_0^1 (1-s)^{q-2} |y(s)|^{\lambda} ds \tag{2}$$

where $q \in (1, 2)$, $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{2-q}$, $c_1, c_2, c_3, c_4 \geq 0$ are contants.Then there exists a constant $M^* > 0$ such that

$$|y(t)| \leq M^*$$

Proof. Let

$$x(t) = \begin{cases} 1, & |y(t)| \leq 1, \\ y(t), & |y(t)| > 1. \end{cases}$$

It follows from condition (2.1) and Hölder's inequality that

$$\begin{aligned} |y(t)| \leq |x(t)| &\leq (c_1 + 1) + c_2 \int_0^t (t-s)^{q-1} |x(s)|^{\lambda} ds + +c_3 \int_0^1 (1-s)^{q-1} |x(s)|^{\lambda} ds \\ &+ c_4 \int_0^1 (1-s)^{q-2} |x(s)|^{\lambda} ds \\ &\leq (c_1 + 1) + c_2 \left(\int_0^t (t-s)^{p(q-1)} ds \right)^{\frac{1}{p}} \left(\int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{p}} \end{aligned}$$

$$
\begin{aligned}
&+ c_3 \left(\int_0^1 (1-s)^{p(q-1)} ds \right)^{\frac{1}{p}} \left(\int_0^1 |x(s)|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&+ c_4 \left(\int_0^1 (1-s)^{p(q-2)} ds \right)^{\frac{1}{p}} \left(\int_0^1 |x(s)|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&\leq (c_1+1) + c_2 \left(\frac{1}{p(q-1)+1} \right)^{\frac{1}{p}} \int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds \\
&+ (c_3+c_4) \left(\frac{1}{p(q-2)+1} \right)^{\frac{1}{p}} \int_0^1 |x(s)|^{\frac{\lambda p}{p-1}} ds.
\end{aligned}
$$

By lemma 2.8 one can complete the rest proof immediately. □

Now we collect PC-type Arzela-Ascoli theorem.

Theorem 2.7. *See (Theorem 2.1,[?]). Let X be a Banach space and $\mathcal{W} \subset PC(J, X)$.If the following conditions are satisfied*

1. *$\mathcal{W}$ is uniformly bounded subset of $PC(J, X)$;*
2. *$\mathcal{W}$ is equicontinuous in (t_k, t_{k+1}), $k = 0, 1, 2....m$, where $t_0 = 0$ and $t_{m+1} = 1$;*
3. *$\mathcal{W}(t) = \{u(t) : u \in W, t \in J \setminus \{t_1, t_2, ...t_n\}\}$, $\mathcal{W}(t_k^+) = \{u(t_k^+) : u \in \mathcal{W}\}$ and $\mathcal{W}(t_k^-) = \{u(t_k^-) : u \in \mathcal{W}\}$ are relatively compact sets of X.*

Then $\mathcal{W}$ is a relatively compact subset of $PC(J, X)$.

3 Main results

This section deals with existence and uniqueness of solutions for the problem (1.1).

Definition 3.1. *A function $u \in PC(J, R)$ is said to be a solution of the problem (1.1) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0, t_{k+1}], R)$ satisfies ${}^cD_t^q u_k(t) = f(t, u_k(t))$ a.e. on $(0, t_{k+1})$ with the restriction of $u_{k+1}(t)$ on $[0, t_{k+1})$ is just $u_k(t)$ and the condition $\Delta u(t_k) = J_k$, $\Delta u'(t_k) = I_k$, $k = 1, 2, ...m$ with $3u(0) = -u(1)$ and $3u'(0) = -u'(1)$.*

Before stating and proving the main result we introduce following hypotheses

(H1) : $f : f \times R \to R$ is jointly continuous.

(H2) : f satisfies nonlinear $\mathcal{D}$-contraction on the second variable i.e. there exists a continuous nondecreasing function $\phi : R^+ \to R^+$ such that

$|f(t, u) - f(t, v)| \leq \phi(|u - v|)$ with $\phi(0) = 0$ and $\phi(r) < r$.
Now we are ready to state our result in this paper.

Theorem 3.2. *Assume that (H1) with* $\phi(0) = 0$ *and (H2) hold if*

$$\frac{20+3q}{16\Gamma(q+1)} \leq 1 \tag{3}$$

then the problem (1.1) *has a unique solution on* J.

Proof. Setting $\sup_{t\in f} |f(t,0)| = M$ and $B_r = \{u \in PC(J,R) : \|u\|_{PC} \leq R\}$, where

$$r \geq \frac{\frac{20+3q}{16\Gamma(q+1)}M + \frac{23}{16}\Sigma_{i=1}^m |J_i| + \frac{5}{4}\Sigma_{i=1}^m |I_i|}{1 - \frac{20+3q}{16\Gamma(q+1)}}$$

Define an operator $F : B_r \to PC(J,R)$ by

$$(Fu)(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s))ds - \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s,u(s))ds \\ -\frac{(4t-1)}{16\Gamma(q-1)}\int_0^1 (1-s)^{q-2} f(s,u(s))ds - \frac{1}{16}\sum_{i=1}^m J_i[3+4(t-t_i)] - \frac{1}{4}\sum_{i=1}^m I_i \\ \text{for} \quad t \in [0,t_1). \\ \vdots \\ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s))ds - \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s,u(s))ds \\ -\frac{(4t-1)}{16\Gamma(q-1)}\int_0^1 (1-s)^{q-2} f(s,u(s))ds - \frac{1}{16}\sum_{i=1}^m J_i[3+4(t-t_i)] - \frac{1}{4}\Sigma_{i=1}^m I_i \\ +\sum_{i=1}^k I_i + \sum_{i=1}^k J_i(t-t_i), \\ \text{for} \quad t \in (t_k, t_{k+1}); \\ k = 1,2,....m. \end{cases} \tag{4}$$

Clearly F is well defined. Step 1. We show that $FB_r \subset B_r$, in fact, for $u \in B_r$, $t \in J'$, we have

$$\begin{aligned} |(Fu)(t)| \leq & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u(s))|ds + \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(s,u(s)|)ds \\ & + \left|\frac{(4t-1)}{16\Gamma(q-1)}\right| \int_0^1 (1-s)^{q-2}|f(s,u(s))|ds + \frac{1}{16}\sum_{i=1}^m |J_i||[3+4(t-t_i)|] + \frac{1}{4}\sum_{i=1}^m |I_i| \\ & + \sum_{i=1}^k |I_i| + \sum_{i=1}^k |J_i||(t-t_i)|. \\ & \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u(s) - f(s,0))|ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|ds \\ & + \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(s,u(s) - f(s,0)|ds + \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1} f(s,0)|ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{16\Gamma(q-1)}\int_0^1(1-s)^{q-2}|f(s,u(s))-f(s,0)|ds\\
&+\frac{3}{16\Gamma(q-1)}\int_0^1(1-s)^{q-2}|f(s,0)|ds+\frac{23}{16}\sum_{i=1}^{m}|J_i|+\frac{5}{4}\sum_{i=1}^{m}|I_i|\\
&\le\frac{20+3q}{16\Gamma(q+1)}r+\frac{20+3q}{16\Gamma(q+1)}M+\frac{23}{16}\sum_{i=1}^{m}|J_i|+\frac{5}{4}\sum_{i=1}^{m}|I_i|\\
&\le r.
\end{aligned}
$$

Step 2. We show that F is a contraction mapping.For $u, v \in B_r$ and for each $t \in J'$, we get

$$
\begin{aligned}
&|(Fu)(t)-(Fv)(t)|\\
&\qquad\le\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(s,u(s))-f(s,v(s))|ds\\
&\qquad+\frac{1}{4\Gamma(q)}\int_0^1(1-s)^{q-1}|f(s,u(s))-f(s,v(s))|ds\\
&\qquad+\left|\frac{(4t-1)}{16\Gamma(q-1)}\right|\int_0^1(1-s)^{q-2}|f(s,u(s))-f(s,v(s))|ds\\
&\qquad\le\frac{20+3q}{16\Gamma(q+1)}\|u-v\|_{PC}.
\end{aligned}
$$

Thus F is a contraction mapping on B_r.
By applying the well known Banach's fixed point theorem we know thwt the operator F has a unique fixed point on B_r. therefore the problem (1.1) has a unique solution. □

4 AN EXAMPLE

In this section we give an example to illustrate the above result. Let us consider the impulsive anti-periodic problem.

$$
\begin{cases}
{}^cD_t^{\frac{5}{2}}u(t)=t, & t\in[0,1]\setminus\{\frac{1}{3}\},\\
\triangle u(\frac{1}{3})=\frac{1}{2}, & \triangle u'(\frac{1}{3})=\frac{1}{2},\\
3u(0)=-u(1), & 3u'(0)=-u'(1).
\end{cases}
\tag{5}
$$

Set $f(t,u)=t$, $(t,u)\in[0,1]\times R$. Obviously f is a nonlinear $\mathcal{D}$-contraction on u.One can arrive at following inequality

$$
\frac{20+3q}{16\Gamma(q+1)}=\frac{20+3(\frac{5}{2})}{16\Gamma(\frac{7}{2})}\approx 0.51717379<1.
$$

Thus the assumptions in Theorem (3.2) are satisfied. Hence the problem has at least one solution on $[0,1]$.

References

[1] Diethelm K,The analysis of gractional differential equations,Lecture notes in mathematics,Berlin Heidelberg;Springer-Verlag;2010.

[2] Kilbas AA,Srivastava HM,Trujillo JJ,Theory and applications of fractional differential equations,B.V.;Elevier Science;2006.

[3] Laxmikantham V,Leela S,Vasundhara Devi,J.Theory of fractional dynamic systems,Cambridge Scientific Publishers;2009.

[4] Miller KS,Ross B,An introduction to fractional calculus and differential equations,John Willey;1993.

[5] Pondlubny I,fractional differential equations,Academic press;1993.

[6] Tarasove VE,Fractional dynamics:application of fractional calculus to dynamics of particles,fields and media,Springer,HEP;2011.

[7] Wang G,Ahmad B,Zhang L,Impulsive anti periodic boundary value problem for nonlinear differential equations of fractional order,Nonlinear Anal,TMA 2011;74; 792–804.

[8] Benchohra M,Seba D,Impulsive fractional differential equations in Banach spaces,Electr J Qual Theory Differ Equat Spec ed I 2009;No. 8;1-14.

[9] Ahamad B,Sivasundaram S,Existence of solution for impulsive integral boundary value problems of fractional order,Nonlinear Anal H5 2010;4;134-41.

[10] Ahamad B,Wang G,Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order,Comput Math Appl 2010;59;1341-9.

[11] Cao J,Chen H,Some results on impulsive boundary value problem for fractional differential inclusions,Electr J Qual Theory Differ Equat 2010;No. 111-24.

[12] Wang X,Impulsive boundary value problem for nonlinear differential equation of fractional order,Comput Math Appl 2011;62;2385-91.

[13] Cao J,Chen H,Impulsive fractional differential equations with nonlinear boundary conditions.Math Comput Model 2012;55;303-11.

[14] Li Xi,Chen F,Li Xu,Generalized anti-periodic boundary value problems of impulsive fractioanl differential equations,Commun Nonlinear Sci Numer Simulat 2013;18;28-41.

[15] Myang HK, Hyong CO,Explicit representation of greens function for linear fractional differential operator with variable coefficient, Journal of fractional calculus and applications,2014; 5(1),26-36.

[16] Wang J.,Xiang X,Peng Y,Periodic solutions of semilinear impulsive periodic system of Banach space,Nonlinear Anal 2009;71;e1344-53.

[17] Wang J,Lv L,Zhou Y,Boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces.J Appl Math Comput 2012;38;209-24.

[18] Dhage BC, On so,e nonlinear alternatives of Leray-Shouder type and functional integral equations, Arch Math (Brno) 2008; 42; 11-23.