Problem 3: Given image has dimensions of 2000x1968 with RGB channel ranks of 1888 (R), 1820 (G), and 1888 (B) if we consider intensity 0 - 1.

a) The top 200 singular values are sufficient to approximate the image so that it looks indistinguishable. This can be observed in fig1 as the error in Frobenius norm and 2-norm becomes shallow (fig 2).

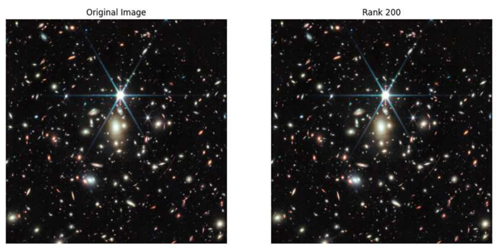

Fig1. This figure shows the comparison between the original image (left) and the approximate image reconstructed using the top 200 singular values (right). The approximate image retains the major features of the original, demonstrating the effectiveness of singular value decomposition in image compression

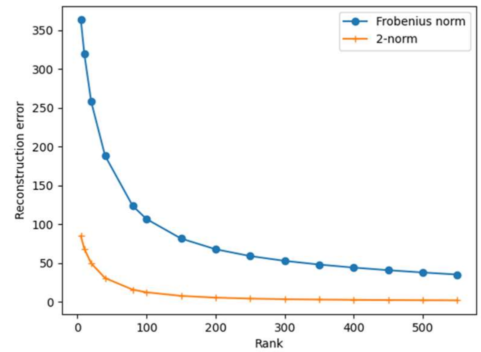

Fig2. This figure depicts the reconstruction error between the original image and the approximated image as a function of the rank (number of singular values used). The Frobenius norm (blue curve) and 2-norm (orange curve) of the reconstruction error are plotted.

Total Error plotted in fig2 for frobenius norm and 2-norm are calculated as follows:

$$\text{Total Error in frobenius} = \sqrt{||A_0 - A_{v0}||_F^2 + ||A_1 - A_{v1}||_F^2 + ||A_2 - A_{v2}||_F^2} \qquad (1)$$

$$\text{Total Error in 2-norm} = \sqrt{||A_0 - A_{v0}||_2^2 + ||A_1 - A_{v1}||_2^2 + ||A_2 - A_{v2|}|_2^2} \qquad (2)$$

Remember the following information:

- A_0, A_1, A_2 represent the original image channels (e.g., Red, Green, Blue).

- A_{v0}, A_{v1}, A_{v2} stand for the reconstructed image channels using singular values.

- The overall error is computed as the square root of the sum of the squared norms for each channel.

b) The approximated image has rank 200 the number of entries in the right singular matrix, diagonal matrix, and left singular matrix are 2000*200, 200*200, and 200 *1968. Every channel (RGB) will have a similar number of entries.

Total numbers transmitted for approx. image (SVD) = 3*(2000*200 + 200*200 + 200 *1968) =2500800

Total numbers transmitted for original image = 3*(2000*1968) = 11808000

c) the 2-Norm and Frobenius-Norm errors between the matrix representation of the original image and the approximate image obtained for different numbers of singular values (rank) are given in Table 1(a) and Table 1(b) respectively.

From the table 1(a) and table 1(b) we can say that for the matrix A of rank r, with singular values σ_1, σ_2, $\sigma 3$, ... σ_r, A_v is the v-rank approximation of A.$\left(\boldsymbol{A_v} = \sum_{i=1}^{v} \sigma_i \boldsymbol{u_i} \boldsymbol{v_i^T}\right)$ such that 1< v < r, then: $|A - A_v|_F = \sqrt{\sigma_{v+1}^2 + \sigma_{v+2}^2 + \cdots + \sigma_r^2}$, $\quad |A - A_v|_2 = \sigma_{v+1}$

Table 1 (a):

Rank	Frobenius RGB Theo. - R	Frobenius RGB Exp. - R	Frobenius RGB Theo. - G	Frobenius RGB Exp. - G	Frobenius RGB Theo. - B	Frobenius RGB Exp. - B
1	223.48386	223.48235	205.92361	205.92531	199.39362	199.39218
2	196.11081	196.10936	180.3734	180.37463	176.7866	176.78549
3	157.8561	157.85509	144.2481	144.24927	144.21748	144.21648
4	114.99139	114.99069	103.42394	103.424644	107.64662	107.64597
5	76.43782	76.4374	64.87497	64.87529	72.43908	72.43868
6	66.48712	66.48681	54.66095	54.661232	63.113873	63.113617
7	51.805946	51.805683	39.37038	39.370605	49.2266	49.226395
8	43.689293	43.689106	30.979784	30.979925	41.517673	41.517506
9	38.421467	38.421295	25.824308	25.824436	36.53164	36.531475
10	34.667255	34.667114	22.358469	22.358557	32.936394	32.936275
11	31.750214	31.750107	19.80308	19.803167	30.091057	30.090961
12	29.302792	29.30267	17.814804	17.814884	27.717272	27.71715
13	27.174013	27.173899	16.219667	16.219734	25.671028	25.670923
14	25.267317	25.26721	14.87354	14.873595	23.84932	23.849228
15	23.52887	23.52877	13.701034	13.701091	22.198404	22.198324

Table 2 (a):

Rank	2-norm RGB Theo. - R	2-norm RGB Exp. - R	2-norm RGB Theo. - G	2-norm RGB Exp. - G	2-norm RGB Theo. - B	2-norm RGB Exp. - B
1	51.663624	51.663624	49.161522	49.161522	45.847248	45.847248
2	42.79276	42.79276	38.84556	38.84556	36.64282	36.64282
3	30.317627	30.317629	28.432049	28.432049	26.869957	26.869957
4	18.663061	18.663061	17.24715	17.24715	17.131653	17.131653
5	9.47665	9.47665	8.786462	8.786462	8.887383	8.887383
6	7.420379	7.420379	6.7944946	6.7944946	7.0215335	7.0215335
7	4.655424	4.655424	4.1621523	4.1621523	4.3907413	4.3907413
8	3.331354	3.3313537	2.8033657	2.8033657	3.1444588	3.1444588
9	2.5744588	2.5744588	2.0705616	2.0705616	2.4573026	2.4573026
10	2.1231124	2.1231124	1.6130053	1.6130054	2.0350926	2.0350926
11	1.8313282	1.8313282	1.32303	1.32303	1.7609655	1.7609652
12	1.6294286	1.6294284	1.1178546	1.1178545	1.553491	1.5534909
13	1.4767994	1.4767995	0.9693855	0.96938556	1.4025065	1.4025066
14	1.3519466	1.3519466	0.8632491	0.863249	1.2849782	1.2849783
15	1.2518247	1.2518247	0.77693635	0.77693635	1.184997	1.184997

Problem 3 :

(a) $A \in \mathbb{R}^{m\times m} \rightarrow$ symmetric, therefore eigenvalues decomposition of A exist

$$\underset{m\times m}{A} = \underset{m\times m}{Q}\ \underset{m\times m}{\Lambda}\ \underset{m\times m}{Q^T}$$

— Q is a orthogonal matrix,
Λ is a diagonal matrix having eigenvalues of A as diagonal elements.

$A = U \Sigma V^T \leftarrow$ SVD of A — Σ is a singular matrix having singular values as diagonal elements

U: eigenvector of AA^T

$$AA^T = Q\Lambda Q^T \ Q \Lambda Q^T$$
$$= Q\Lambda^2 Q^T \quad \text{— (1)}$$
$$AA^T = U\Sigma V^T V \Sigma U^T$$
$$= U \Sigma^2 U^T \quad \text{— (2)}$$

comparing (1) & (2)

$$\Sigma^2 = \Lambda^2$$

Hence, $U = Q$

$$\Sigma = |\Lambda|$$

diagonal values of Σ will be square root of diagonal values of Λ^2, as diagonal values of Λ^2 are square of eigenvalues of A, then we say

$$\sigma_i = \sqrt{(\lambda_i)^2} = |\lambda_i|$$

σ_i: singular value of A, or diagonal element of Σ

(b) Show that $|x^T A x| \le \|A_2\|$ for any non-zero unit vector $x \in \mathbb{R}^m$.

$$|x^T A x| \le \|x^T\|_2 \, \|Ax\|_2 \leftarrow \text{Cauchy Schwarz inequality}$$

$\|x\| = 1$ or $\|x^T\| = 1$

$$|x^T A x| \le \|Ax\|_2 \le \|A\|_2 \, \|x\|_2$$

$$\boxed{|x^T A x| \le \|A\|_2}$$

(c) $u \in \mathbb{R}^m$

$Au = \lambda u$ ← eigenvector of A

← eigenvalue of A,

$$A \xrightarrow{\text{changed to}} \tilde{A}$$

$$u \longrightarrow \tilde{u}$$

$$\lambda \longrightarrow \lambda + \delta\lambda$$

$$\frac{\|\delta A\|}{\|A\|} = O(\epsilon_{mach})$$

$$\tilde{A} = A + \delta A$$

$$\tilde{u} = u + \delta u$$

$$\tilde{\lambda} = \lambda + \delta\lambda$$

$|\delta\lambda| \le \|\delta A\|_2$ ← need to prove.

$$Au = \lambda u$$

$$(A + \delta A)(u + \delta u) = (\lambda + \delta\lambda)(u + \delta u)$$

$$\cancel{Au} + \delta A u + A\delta u + \delta A \delta u = \cancel{\lambda u} + \lambda \delta u + \delta\lambda\, u + \delta\lambda\, \delta u$$

ignoring 2nd order terms.

$$\delta A u + A\delta u = \lambda \delta u + \delta\lambda\, u$$

multiplying both sides with u^T

$$u^T \delta A u + u^T A \delta u = \lambda u^T \delta u + \delta\lambda\, u^T u$$

$\because u^T A = u^T A^T = (Au)^T = (\lambda u)^T = \lambda u^T$; as A is symmetric

$$u^T \delta A u + \lambda u^T \delta u = \lambda u^T \delta u + \delta\lambda\, u^T u$$

$$u^T \delta A u = \delta\lambda\, u^T u$$

$$|\delta\lambda| = \frac{|u^T \delta A u|}{|u^T u|} = \frac{|u^T \delta A u|}{\|u\|_2^2}$$

$$|\delta\lambda| = \frac{|\underset{\sim}{u}^T \delta A \underset{\sim}{u}|}{\|\underset{\sim}{u}\|_2^2} \leq \frac{\|\underset{\sim}{u}^T\|_2 \|\delta A\|_2 \|u\|_2}{\|\underset{\sim}{u}\|_2^2}$$

$$\boxed{|\delta\lambda| \leq \|\delta A\|_2}$$

(d) $A \in \mathbb{R}^{m\times m}$ $\qquad Au = \lambda u$

input : A

output : λ

$$\kappa^R = \max_{\delta A} \frac{\frac{|\delta\lambda|}{|\lambda|}}{\frac{\|\delta \underset{\sim}{A}\|_2}{\|\underset{\sim}{A}\|_2}} = \max_{\delta A} \frac{|\delta\lambda|}{|\lambda|} \times \frac{\|A\|_2}{\|\delta A\|_2}$$

$\frac{|\delta\lambda|}{\|\delta A\|_2} \leq 1$ using (c)

$\max_{\delta A} \frac{|\delta A|}{\|\delta A\|_2} = 1$

$$\boxed{\kappa^R = \frac{\|A\|_2}{|\lambda|}}$$

(e) $M = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$; $a \neq 0$ eigenvalues of $M =$ a, a

$\kappa^R = \frac{\|M\|_2}{|\lambda|}$ $\qquad \|M\|_2 = a$

$|\lambda| = a$

$\kappa^R = \frac{|a|}{|a|} = 1$

$$\boxed{\kappa^R = 1}$$

(f) Algorithm S, S is backward stable $\tilde{f}(x) = f(\tilde{x})$

relative forward error $= \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$

$f(M) : \lambda \to a$ $\qquad f(M)$ gives exact eigenvalue

$f(\tilde{M}) : \lambda \to \tilde{a}$ $\qquad f(\tilde{M})$ give Approximate eigenvalue

Relative forward error $= \dfrac{\|a(1+\varepsilon_M) - a\|}{\|a\|} = O(\varepsilon_M)$

(g) $\underset{\sim}{M} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $a \in \mathbb{F}$

characteristice equation of $\det(\underset{\sim}{M} - \lambda \underset{\sim}{I}) = 0$

$$(\lambda - a)^2 = 0$$

$$\lambda^2 + a^2 - 2\lambda a = 0 \quad \text{—(1)}$$

error will occur, while evaluating coefficients of characteristics equ

$$fl(a^2) = a \otimes a = a^2(1+\varepsilon_M)$$

$$fl(2a) = a \oplus a = 2a(1+\varepsilon_M)$$

$\varepsilon_M \rightarrow$ Machine epsilon

After putting errors in coefficients in equ (1), equ(1) will look like

$$\lambda^2 + a^2(1+\varepsilon_M) - 2a(1+\varepsilon_M)\lambda = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{2a(1+\varepsilon_M)\lambda \oplus \left[\sqrt{2a(1+\varepsilon_M) \otimes 2a(1+\varepsilon_M) \ominus 4 \otimes a^2(1+\varepsilon_M)}\right](1+\varepsilon_M)}{2}(1+\varepsilon_M)$$

$$= \frac{2a(1+\varepsilon_M)\lambda \oplus \left[\sqrt{(4a^2(1+\varepsilon_M)^3 - 4a^2(1+\varepsilon_M)^2)(1+\varepsilon_M)}\right](1+\varepsilon_M)}{2}(1+\varepsilon_M)$$

$$= \frac{2a(1+\varepsilon_M)\lambda \oplus \left[\sqrt{4a^2(1+\varepsilon_M)^4 - 4a^2(1+\varepsilon_M)^3}\right](1+\varepsilon_M)}{2}(1+\varepsilon_M)$$

$$= a(1+\varepsilon_M)^3 \pm \sqrt{a^2\{(1+\varepsilon_M)^4 - (1+\varepsilon_M)^3\}}\,(1+\varepsilon_M)^3$$

Made with Goodnotes

$$(1+\varepsilon_M)^4 = (1+\varepsilon_M^2+2\varepsilon_M)^2 = (1+\varepsilon_M^2+2\varepsilon_M)(1+\varepsilon_M^2+2\varepsilon_M)$$
$$= 1+\varepsilon_M^2+2\varepsilon_M+\varepsilon_M^2+\varepsilon_M^4+2\varepsilon_M^3+2\varepsilon_M+2\varepsilon_M^3+4\varepsilon_M^2$$

$$(1+\varepsilon_M)^3 = (1+\varepsilon_M^3+3\varepsilon_M+3\varepsilon_M^2)$$

$$(1+\varepsilon_0)^4-(1+\varepsilon_M)^3 = \varepsilon_M+O(\varepsilon_M^2)$$

$$\lambda = a(1+\varepsilon_M) \pm a\sqrt{\varepsilon_M}\,(1+\varepsilon_M)$$

$$\lambda_1 = a(1+O(\varepsilon_M)) + O\sqrt{\varepsilon_M}) = a[1+O(\sqrt{\varepsilon_M})]$$

$$\lambda_2 = a(1+O(\varepsilon_M)) - O\sqrt{\varepsilon_M}) = a(1+O(\varepsilon_M)-O(\sqrt{\varepsilon_M})]$$
$$= a[1-O(\sqrt{\varepsilon_M})]$$

(h)

Forward relative error $\Rightarrow$?

$$\text{Forward relative error} = \frac{\|\tilde{f}(x)-f(x)\|}{\|f(x)\|} = \frac{|a(1+\varepsilon_M\pm\sqrt{\varepsilon_M})-a|}{|a|}$$

$$= \frac{|\varepsilon_M \pm \sqrt{\varepsilon_M}|}{|a|} = O(\sqrt{\varepsilon_M})$$

as the relative forward error is in the order of $O(\sqrt{\varepsilon_M})$ it is not backward stable (forward relative error $\leq K\,O(\varepsilon_M)$)

check for stability

$$\frac{|\tilde{f}(x) - f(\tilde{x})|}{f(\tilde{x})} = \frac{|a(1+\varepsilon_m \pm \sqrt{\varepsilon_m}) - a(1+\varepsilon_m)|}{a(1+\varepsilon_m)}$$

$$= \frac{\sqrt{\varepsilon_m}}{1+\varepsilon_m} = \frac{\sqrt{\varepsilon_m}}{1} \qquad \varepsilon_m \lll 1$$

not stable it should be $O(\varepsilon_m)$ to become stable.