Q1) The definition of I^b_{MoI} is as follows, in $\mathcal{F}^b$

$$I^b_{MoI} = \sum_{i=1}^{N} m_i \left((\Delta\vec{r}_i^{\,b})^T (\Delta\vec{r}_i^{\,b}) I_{3\times3} - (\Delta\vec{r}_i^{\,b})(\Delta\vec{r}_i^{\,b})^T \right)$$

where, $\Delta\vec{r}_i^{\,b} = \vec{r}_i - \vec{r}_{CM}$

then let $\mathcal{F}^c$ be the principle axis frame.

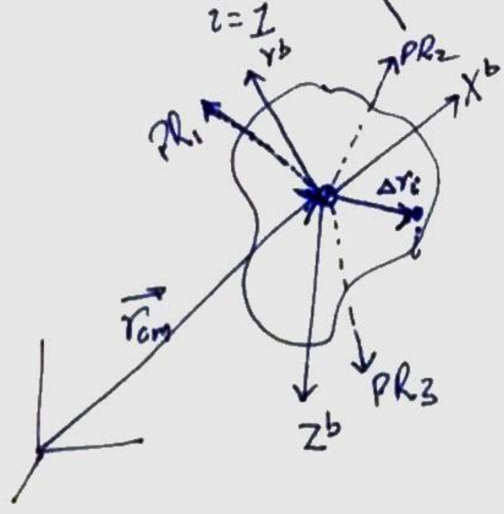
$$I^c_{MoI} = R^T_{pr} I^b_{MoI} R_{pr}$$

where, R^T_{pr} is a rotation matrix from $\mathcal{F}^b$ to $\mathcal{F}^c$.

So, we can write the principle moments of Inertia as follows.

$$I^c_{MoI} = R^T_{pr}\left[\sum_{i=1}^{N} m_i \left((\Delta\vec{r}_i^{\,b})^T (\Delta\vec{r}_i^{\,b}) I_{3\times3} - (\Delta\vec{r}_i^{\,b})(\Delta\vec{r}_i^{\,b})^T \right)\right] R_{pr}$$

$$= \sum_{i=1}^{N} m_i \left((\Delta\vec{r}_i^{\,b})^T (\Delta\vec{r}_i^{\,b}) \underbrace{R^T_{pr} I_{3\times3} R_{pr}}_{I_{3\times3}} - \overbrace{(R^T_{pr}\Delta\vec{r}_i^{\,b})}^{\Delta\vec{r}_i^{\,c}} \underbrace{(\Delta\vec{r}_i^{\,bT} R_{pr})}_{\Delta\vec{r}_i^{\,cT}} \right)$$



because $\mathcal{F}^c$ and $\mathcal{F}^b$ are only rigid rotations, the magnitude of, $|\Delta\vec{r}_i^{\,c}| = |\Delta\vec{r}_i^{\,b}|$

$$\therefore (\Delta\vec{r}_i^{\,b})^T (\Delta\vec{r}_i^{\,b}) = (\Delta\vec{r}_i^{\,c})^T (\Delta\vec{r}_i^{\,c})$$

This fact can also be proved as below

$$\Delta\vec{r}_i^{\,b} = R_{pr}^T \Delta\vec{r}_i^{\,c} \;;\quad \left(\Delta\vec{r}_i^{\,b}\right)^T = \Delta\vec{r}_i^{\,c\,T} R_{pr}$$

$$\left(\Delta\vec{r}_i^{\,b}\right)^T\left(\Delta\vec{r}_i^{\,b}\right) = \left(\Delta r_i^c\right)^T \underbrace{R_{pr} R_{pr}^T}_{I_{3\times 3}} \left(\Delta\vec{r}_i^{\,c}\right)$$

$$= \left(\Delta r_i^c\right)^T\left(\Delta\vec{r}_i^{\,c}\right)$$

So rewriting I_{MoI}^C in its final form in terms $\Delta\vec{r}_i^{\,c}$

$$I_{MoI}^C = \sum_{i=1}^{N} m_i\left[\left(\Delta\vec{r}_i^{\,c}\right)^T\left(\Delta\vec{r}_i^{\,c}\right) I_{3\times 3} - \left(\Delta r_i^c\right)\left(\Delta\vec{r}_i^{\,c}\right)^T\right]$$

All the off-diagonal term are zero because of the property of the Principle axes. So the diagonal term are as follows.

$$I_{pr1} = \sum_{i=1}^{N} m_i\left(\left(\Delta Y_i^c\right)^2 + \left(\Delta Z_i^c\right)^2\right);\quad I_{pr2} = \sum_{i=1}^{N} m_i\left(\left(\Delta X_i^c\right)^2 + \left(\Delta Z_i^c\right)^2\right)$$

$$I_{pr3} = \sum_{i=1}^{N} m_i\left(\left(\Delta X_i^c\right)^2 + \left(\Delta Y_i^c\right)^2\right)$$

$$I_{PR1} + I_{PR2} = \sum_{i=1}^{N} m_i\left(\underbrace{\left(\Delta X_i^c\right)^2 + \left(\Delta Y_i^c\right)^2}_{I_{PR3}} + 2\left(\Delta Z_i^c\right)^2\right)$$

from the last two relations it can be infrered that

$$I_{PR1} + I_{PR2} \geq I_{PR3}$$