Q4) The Euler's Eq

$$\frac{d\vec{h}}{dt} = \vec{T} \quad \text{where } \vec{h} \rightarrow \text{Angular momentum}$$
$$\vec{T} \rightarrow \text{External torque}$$

for $\vec{h}^b$, Angular momentum Expressed in a body frame Rotating as $\vec{\omega}^b$, the Euler Eq is as follows

$$\frac{d\vec{h}^b}{dt} + \vec{\omega}^b \times \vec{h}^b = \vec{T}^b$$

for a Torque-Free case $\vec{T}^b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Expanding $\vec{h}^b = I^b_{MOI} \vec{\omega}^b$

where I^b_{MOI} is a principle Frame of the body

$$I^b_{MOI} = \begin{bmatrix} I_{tr1} & 0 & 0 \\ 0 & I_{tr2} & 0 \\ 0 & 0 & I_{spin} \end{bmatrix}$$

So, the torque-free rigid body Equations become:

$$\underbrace{\begin{bmatrix} I_{tr1} & 0 & 0 \\ 0 & I_{tr2} & 0 \\ 0 & 0 & I_{spin} \end{bmatrix}}_{I^b_{MOI}} \underbrace{\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}}_{\dot{\vec{\omega}}^b} + \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{\vec{\omega}^b} \times \left(\underbrace{\begin{bmatrix} I_{tr1} & 0 & 0 \\ 0 & I_{tr2} & 0 \\ 0 & 0 & I_{spin} \end{bmatrix}}_{I^b_{MOI}} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{\vec{\omega}^b} \right) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{T}=0}$$

further Simplifying.

$$\begin{bmatrix} I_{Tr1}\dot{\omega}_1 \\ I_{Tr2}\dot{\omega}_2 \\ I_{spin}\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{Tr1}\omega_1 \\ I_{Tr2}\omega_2 \\ I_{spin}\omega_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$I_{Tr1}\dot{\omega}_1 + \omega_2\omega_3(I_{spin} - I_{Tr2}) = 0$$

$$I_{Tr2}\dot{\omega}_2 + \omega_3\omega_1(I_{TR1} - I_{spin}) = 0$$

$$I_{spin}\dot{\omega}_3 + \omega_1\omega_2(I_{TR2} - I_{TR1}) = 0$$

$$\dot{\omega}_1 = \frac{(I_{TR2} - I_{spin})}{I_{Tr1}}\omega_2\omega_3$$

$$\dot{\omega}_2 = \frac{(I_{spin} - I_{TR1})}{I_{TR2}}\omega_1\omega_3$$

$$\dot{\omega}_3 = \frac{(I_{TR1} - I_{TR2})}{I_{spin}}\omega_1\omega_2$$

If ω_1 & ω_2 are very small, then $\omega_1\omega_2 << 0$

$\therefore \dot{\omega}_3 = 0$, so $\omega_3(t) = \text{const} = \omega_{3,avg}$

We substitute $\omega_3(t) = \omega_{3avg}$ in $\dot{\omega}_1$ & $\dot{\omega}_2$ Eq to get

$$\dot{\omega}_1 = \left(\frac{I_{TR2} - I_{spin}}{I_{TR1}}\right) \omega_{3avg}\, \omega_2$$

differentiation once w.r.t to time

$$\ddot{\omega}_1 = \left(\frac{I_{TR2} - I_{spin}}{I_{TR1}}\right) \omega_{3avg}\, \dot{\omega}_2$$

Substituting the Eq for $\dot{\omega}_2 = \left(\frac{I_{TR1} - I_{spin}}{I_{TR2}}\right) \omega_{3avg}\, \omega_1$

we get

$$\ddot{\omega}_1 = \frac{(I_{TR2} - I_{spin})(I_{TR1} - I_{spin})}{I_{TR1}\, I_{TR2}} \omega_{3avg}^2\, \omega_1$$

give the 2nd order ODE as follows.

$$\ddot{\omega}_1 + \underbrace{\frac{(I_{TR2} - I_{spin})(I_{TR1} - I_{spin})}{I_{TR1}\, I_{TR2}} \omega_{3avg}^2}_{\omega_{nut}^2}\, \omega_1 = 0$$

$\omega_{nut}^2 \rightarrow$ the nutation frequency

$$\boxed{\therefore\ \omega_{nut} = \sqrt{\frac{(I_{spin} - I_{TR1})(I_{spin} - I_{TR2})}{I_{TR1}\, I_{TR2}}}\ \omega_{3avg}\ \blacksquare}$$

the assumption here is $I_{spin} > I_{TR1}$ & $I_{spin} > I_{TR2}$

(or) $I_{spin} < I_{TR1}$ & $I_{spin} < I_{TR2}$

The general solutions to the 2^{nd} order PDEs are

$$\omega_1(t) = A \cos(\omega_{nut} t) + B \sin(\omega_{nut} t)$$

$$\omega_2(t) = C \cos(\omega_{nut} t) + D \sin(\omega_{nut} t)$$

$$\dot{\omega}_1(t) = -A\,\omega_{nut} \sin(\omega_{nut} t) + B\,\omega_{nut} \cos(\omega_{nut} t)$$

using the initial dynamics Eq

$$-A\,\omega_{nut} \sin(\omega_{nut} t) + B\,\omega_{nut} \cos(\omega_{nut} t)$$

$$= \frac{(I_{TR2} - I_{spin})}{I_{TR1}} \omega_{3avg}\, \omega_2$$

$$= \left(\frac{I_{TR2} - I_{spin}}{I_{TR1}}\right) \omega_{3avg} \left[C \cos(\omega_{nut} t) + D \sin(\omega_{nut} t)\right]$$

Equating coeff of $\sin(\omega_{nut} t)$ & $\cos(\omega_{nut} t)$, as this relation need to hold for all t, we get

$$B\,\omega_{nut} = C \frac{(I_{TR2} - I_{spin})}{I_{TR1}} \omega_{3avg}$$

$$-A\,\omega_{nut} = D \frac{(I_{TR2} - I_{spin})}{I_{TR1}} \omega_{3avg}$$

$$B\sqrt{\frac{(I_{sp}-I_{TR1})(I_{sp}-I_{TR2})}{I_{TR1}I_{TR2}}} = C\,\mathrm{Sign}(I_{TR2}-I_{spin})\sqrt{\frac{(|I_{TR2}-I_{spin}|)^2}{I_{TR1}^2}}$$

$$-A\sqrt{\frac{(I_{sp}-I_{TR1})(I_{sp}-I_{TR2})}{I_{TR1}I_{TR2}}} = D\,\mathrm{Siagn}(I_{TR2}-I_{spin})\sqrt{\frac{(|I_{TR2}-I_{spin}|)^2}{I_{TR1}^2}}$$

$$B = C\,\mathrm{Sign}(I_{TR2}-I_{spin})\sqrt{\frac{I_{TR2}(I_{spin}-I_{TR2})}{I_{TR1}(I_{spin}-I_{TR1})}}$$

$$D = -A\,\mathrm{Sign}(I_{TR2}-I_{spin})\sqrt{\frac{I_{TR1}(I_{spin}-I_{TR1})}{I_{TR2}(I_{spin}-I_{TR2})}}$$

we healize that when $t=0$

$$\omega_1(0) = A\cos(0) = A \ \&\ \omega_2(0) = C\cos(0) = C$$

$$\therefore\ A = \omega_1(0)\ \&\ C = \omega_2(0)$$

So the final dynamics' Solution look like

$$\boxed{\begin{aligned} \omega_1(t) &= \omega_1(0)\cos(\omega_{nut}t) + \omega_2(0)\,\mathrm{Sign}(I_{TR2}-I_{spin})\sqrt{\frac{I_{TR2}(I_{spin}-I_{TR2})}{I_{TR1}(I_{spin}-I_{TR1})}}\,\sin(\omega_{nut}t) \\ \omega_2(t) &= \omega_2(0)\cos(\omega_{nut}t) - \omega_1(0)\,\mathrm{Sign}(I_{TR2}-I_{spin})\sqrt{\frac{I_{TR1}(I_{spin}-I_{TR1})}{I_{TR2}(I_{spin}-I_{TR2})}}\,\sin(\omega_{nut}t) \end{aligned}}$$