Q4) $f_{bk} = f(t_{bk}, x_{bk}) = f\left(t_k + \frac{\Delta t}{2}, x_k + \frac{\Delta t}{2} f_{ak}\right)$

$f_{ak} = f(t_{ak}, x_{ak}) = f(t_k, x_k) \triangleq f_k$

So,

$$f_{bk} = f\left(t_k + \frac{\Delta t}{2}, x_k + \frac{\Delta t}{2} f_k\right)$$

The Taylor Series Expansion gives

$$f_{bk} = f_k + \left(\frac{\partial f}{\partial t}\Big|_{\text{at } x_k, t_k} \frac{\Delta t}{2}\right) + \left(\frac{\partial f}{\partial x}\Big|_{\text{at } x_k, t_k} \frac{\Delta t}{2} f_k\right) + \mathcal{O}(\Delta t^2)$$

$$f_{bk} = f_k + \frac{\Delta t}{2}\underbrace{\left[\frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} f_k\right]_{\text{at } t_k, x_k}}_{\ddot{x}(t_k)} + \mathcal{O}(\Delta t^2)$$

$$f_{bk} = \dot{x}(t_k) + \frac{\Delta t}{2}\ddot{x}(t_k) + \mathcal{O}(\Delta t^2)$$

From the assignment Question

$$x_{k+1} = x_k + \Delta t(b_1 f_{ak} + b_2 f_{bk}) = x_k + \Delta t\, b_1 f_k + \Delta t\, b_2 f_{bk}$$

$$x_{k+1} = x_k + b_1 \Delta t f_k + b_2 \Delta t\left(\dot{x}(t_k) + \frac{\Delta t}{2}\ddot{x}(t_k) + \mathcal{O}(\Delta t^2)\right)$$

$$= x_k + \Delta t\,(b_1 + b_2)\dot{x}(t_k) + b_2 \frac{\Delta t^2}{2}\ddot{x}(t_k) + b_2\mathcal{O}(\Delta t^3)$$

If $b_1 = 0$ & $b_2 = 1$ Ans

$$x_{k+1} = \underbrace{x_k + \Delta t\,\dot{x}(t_k) + \frac{\Delta t^2}{2}\ddot{x}(t_k)}_{\text{first 3 terms of taylor series}} + \mathcal{O}(\Delta t^3)$$

making it a 2nd order method with the Error as $\mathcal{O}(\Delta t^3)$ → polynomial with at least power 3