Posting Date: Monday Nov. 18th.
Due Date: Tuesday Dec. 3rd.

1. Prove that the following alternate equilibrium states are, in fact, equilibria of the attitude dynamics model for a rigid-body spacecraft that is orbiting a spherical attracting body in a circular orbit with a mean motion of n, that is subject to gravity-gradient torques, and that has its spacecraft body axes defined to be principal axes:

a) $\underline{q}_{eq} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \vec{\omega}^b_{eq} = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$

b) $\underline{q}_{eq} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \vec{\omega}^b_{eq} = \begin{bmatrix} -n \\ 0 \\ 0 \end{bmatrix}$

c) $\underline{q}_{eq} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\omega}^b_{eq} = \begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix}$

Remember that $\underline{q}$ parameterizes the rotation from local-level orbit-following coordinates to spacecraft body-fixed coordinates and that $\vec{\omega}^b$ is the rotation rate of the spacecraft body-fixed coordinates relative to inertial coordinates and resolved along the body-fixed axes. Assume that the net external torque other than gravity-gradient torque is 0.

Hint: It suffices to show that $\dot{\underline{q}}_{eq} = 0$ and that $\dot{\vec{\omega}}^b_{eq} = 0$ in order to prove that each combination of attitude quaternion and attitude rate constitutes an equilibrium.

2. Develop a method to solve for the straight and level steady-motion of a point-mass model of an aircraft. That is, given the steady-motion (negative) altitude relative to the coordinate system center Z_{eq}, the steady-motion airspeed V_{eq}, and the steady-motion heading angle ψ_{eq}, determine the steady-motion flight-path angle γ_{eq}, the steady-motion thrust T_{eq}, the steady-motion angle of attack α_{eq}, and the steady-motion roll angle ϕ_{eq} by solving the following system of 4 nonlinear equations in these 4 unknowns

$$0 = \dot{Z}_{eq} = -V_{eq} \sin\gamma_{eq}$$

$$0 = m\dot{V}_{eq} = T_{eq}\cos\alpha_{eq} - \tfrac{1}{2}\rho(Z_{eq})V_{eq}^2 S C_D(\alpha_{eq}) - mg\sin\gamma_{eq}$$

$$0 = mV_{eq}\dot{\gamma}_{eq} = [T_{eq}\sin\alpha_{eq} + \tfrac{1}{2}\rho(Z_{eq})V_{eq}^2 S C_L(\alpha_{eq})]\cos\phi_{eq} - mg\cos\gamma_{eq}$$

$$0 = mV_{eq}\cos\gamma_{eq}\,\dot{\psi}_{eq} = [T_{eq}\sin\alpha_{eq} + \tfrac{1}{2}\rho(Z_{eq})V_{eq}^2 S C_L(\alpha_{eq})]\sin\phi_{eq}$$

where

$$C_L(\alpha) = C_{L\alpha}\alpha$$

and

$$C_D(\alpha) = C_{D0} + \frac{[C_L(\alpha)]^2}{\pi ARe}$$

Note that the notation for the atmospheric density function used here, $\rho(Z_{eq})$, differs from the form $\rho(-Z_{eq})$ that has been used in class. The same function of Z_{eq} is implied for by the new formula, as will be seen in Problem 4. The change of notation has been made for the sake of convenience.

Under the reasonable assumption that V_{eq} does not equal zero, the first equation can be solved immediately for γ_{eq} to determine that $\gamma_{eq} = 0$. Similarly, under the reasonable assumption that $[T_{eq}\sin\alpha_{eq} + \frac{1}{2}\rho(Z_{eq})V_{eq}^2 SC_L(\alpha_{eq})] = [T_{eq}\sin\alpha_{eq} + L(Z_{eq}, V_{eq}, \alpha_{eq})]$ does not equal zero, the last equation can be solved immediately for ϕ_{eq} to determine that $\phi_{eq} = 0$. Substitution of these results into the middle two equations yields

$$0 = T_{eq}\cos\alpha_{eq} - \tfrac{1}{2}\rho(Z_{eq})V_{eq}^2 SC_D(\alpha_{eq})$$

$$0 = T_{eq}\sin\alpha_{eq} + \tfrac{1}{2}\rho(Z_{eq})V_{eq}^2 SC_L(\alpha_{eq}) - mg$$

The first of these two equations can be solved to give the unknown T_{eq} in terms of the unknown α_{eq}:

$$T_{eq} = \frac{\frac{1}{2}\rho(Z_{eq})V_{eq}^2 SC_D(\alpha_{eq})}{\cos\alpha_{eq}} = \frac{D(Z_{eq}, V_{eq}, \alpha_{eq})}{\cos\alpha_{eq}}$$

Substitution of this result into the second equation and division of the resulting equation on both sides by $\bar{q}S = \frac{1}{2}\rho(Z_{eq})V_{eq}^2 S$ yields a scalar nonlinear equation in the scalar unknown α_{eq}:

$$0 = f(\alpha_{eq}) = \tan\alpha_{eq} C_D(\alpha_{eq}) + C_L(\alpha_{eq}) - \frac{mg}{\frac{1}{2}\rho(Z_{eq})V_{eq}^2 S}$$

where $f(\alpha_{eq})$ in this case is a nonlinear function whose root needs to be found. It is not a nonlinear dynamics model function in this case. This equation can be solved using Newton's method starting from a first guess of $\alpha_{eqguess} = 0$. After solving for α_{eq}, the result can be used to determine T_{eq}, as given above.

Complete the MATLAB template file `solvesteadystateaircraft01_temp.m` by completing the parts of the code where ???? appears. The result will be the MATLAB function `solvesteadystateaircraft01.m`. This function determines γ_{eq}, T_{eq}, α_{eq}, and ϕ_{eq}. It also computes $\dot{X}_{SM} = V_{eq}\cos\gamma_{eq}\cos\psi_{eq}$ and $\dot{Y}_{SM} = V_{eq}\cos\gamma_{eq}\sin\psi_{eq}$.

Run your code on the input data contained in the files

`steadystateaircraft01_data01.mat` and
`steadystateaircraft01_data02.mat`

Hand in your code and the outputs for γ_{eq}, T_{eq}, α_{eq}, ϕ_{eq}, $\dot{X}_{SM}$, and $\dot{Y}_{SM}$ that you get for the inputs in `steadystateaircraft01_data02.mat`. Express these values to 15 significant digits by using MATLAB's `format long` command. As an aid to your debugging, the following outputs are produced when operating on the data in `steadystateaircraft01_data01.mat`:

`gammaeq` = 0
`Teq` = 3.240358191011175e+03
`alphaeq` = 0.135886048735789
`phieq` = 0
`XdotSM` = -22.231309316168467
`YdotSM` = 61.080020351084045

3. A linearized form can be developed for the attitude dynamics model of a rigid-body spacecraft that is orbiting a spherical attracting body in a circular orbit with a mean motion of n, that is subject to gravity-gradient torques, and that has its spacecraft body axes defined to be principal axes. The nonlinear model's $\underline{f}(\underline{x},\underline{u})$ dynamics function has the following components

$$f_1(\underline{x},\underline{u}) = \tfrac{1}{2}\{x_4[x_5 + 2n(x_1x_2 + x_3x_4)] - x_3[x_6 + n(-x_1^2 + x_2^2 - x_3^2 + x_4^2)] \\ + x_2[x_7 + 2n(x_2x_3 - x_1x_4)]\}$$

$$f_2(\underline{x},\underline{u}) = \tfrac{1}{2}\{x_3[x_5 + 2n(x_1x_2 + x_3x_4)] + x_4[x_6 + n(-x_1^2 + x_2^2 - x_3^2 + x_4^2)] \\ - x_1[x_7 + 2n(x_2x_3 - x_1x_4)]\}$$

$$f_3(\underline{x},\underline{u}) = \tfrac{1}{2}\{-x_2[x_5 + 2n(x_1x_2 + x_3x_4)] + x_1[x_6 + n(-x_1^2 + x_2^2 - x_3^2 + x_4^2)] \\ + x_4[x_7 + 2n(x_2x_3 - x_1x_4)]\}$$

$$f_4(\underline{x},\underline{u}) = \tfrac{1}{2}\{-x_1[x_5 + 2n(x_1x_2 + x_3x_4)] - x_2[x_6 + n(-x_1^2 + x_2^2 - x_3^2 + x_4^2)] \\ - x_3[x_7 + 2n(x_2x_3 - x_1x_4)]\}$$

$$f_5(\underline{x},\underline{u}) = \tfrac{1}{I_{11}^b}[(I_{22}^b - I_{33}^b)x_6x_7 + 6n^2(I_{33}^b - I_{22}^b)(x_2x_3 + x_1x_4)(-x_1^2 - x_2^2 + x_3^2 + x_4^2) + u_1]$$

$$f_6(\underline{x},\underline{u}) = \tfrac{1}{I_{22}^b}[(I_{33}^b - I_{11}^b)x_5x_7 + 6n^2(I_{11}^b - I_{33}^b)(x_1x_3 - x_2x_4)(-x_1^2 - x_2^2 + x_3^2 + x_4^2) + u_2]$$

$$f_7(\underline{x},\underline{u}) = \tfrac{1}{I_{33}^b}[(I_{11}^b - I_{22}^b)x_5x_6 + 12n^2(I_{22}^b - I_{11}^b)(x_1x_3 - x_2x_4)(x_2x_3 + x_1x_4) + u_3]$$

where the body-axes/principal-axes moment-of-inertia matrix takes the form:

$$I_{MoI}^{b} = \begin{bmatrix} I_{11}^{b} & 0 & 0 \\ 0 & I_{22}^{b} & 0 \\ 0 & 0 & I_{33}^{b} \end{bmatrix}$$

The first 4 elements of the $\underline{f}(\underline{x},\underline{u})$ dynamics model function constitute the quaternion kinematics model for the attitude quaternion $\underline{q}$ that parameterizes the rotation from local-level orbit-following coordinates to body-axes coordinates. The last 3 elements of $\underline{f}(\underline{x},\underline{u})$ constitute Euler's rigid-body rotation equations for the inertial spin rate vector of the spacecraft expressed in body-fixed coordinates, $\vec{\omega}^{b}$. Thus, the 7-element state vector for this system takes the form

$$\underline{x} = \begin{bmatrix} \underline{q} \\ \vec{\omega}^{b} \end{bmatrix}$$

with x_1 to x_4 being the quaternion elements and x_5 to x_7 being the angular velocity elements. The control input vector takes the form

$$\underline{u} = \vec{T}^{b}$$

That is, it is the net applied external torque other than the gravity-gradient torque.

Recall from lecture that one of the equilibrium states of this system is $\underline{x}_{eq} = [0,0,0,1,0,-n,0]^T$ when the equilibrium control input is $\underline{u}_{eq} = [0,0,0]^T$. The linearized model about this equilibrium takes the form

$$\Delta\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 0 & n & 0 & \tfrac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & n & 0 & \tfrac{1}{2} & 0 \\ -n & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{6n^2[I_{33}^b - I_{22}^b]}{I_{11}^b} & 0 & 0 & 0 & 0 & 0 & \frac{n[I_{33}^b - I_{22}^b]}{I_{11}^b} \\ 0 & \frac{6n^2[I_{33}^b - I_{11}^b]}{I_{22}^b} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{n[I_{22}^b - I_{11}^b]}{I_{33}^b} & 0 & 0 \end{bmatrix} \Delta\underline{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{I_{11}^b} & 0 & 0 \\ 0 & \frac{1}{I_{22}^b} & 0 \\ 0 & 0 & \frac{1}{I_{33}^b} \end{bmatrix} \Delta\underline{u}(t)$$

This model is in the standard linearized form $\Delta\dot{\underline{x}}(t) = A\,\Delta\underline{x}(t) + B\,\Delta\underline{u}(t)$, with A and B being, respectively, the 7-by-7 and 7-by-3 matrices that appear in the equation given above. Prove that $A_{13} = n$, that $A_{26} = ½$, that $A_{34} = 0$, that $A_{45} = 0$, that $A_{57} = n[I_{33}^b - I_{22}^b]/I_{11}^b$, that $A_{62} = 6n^2[I_{33}^b - I_{11}^b]/I_{22}^b$, that $A_{76} = 0$, and that $B_{62} = 1/I_{22}^b$. Prove these 8 formulas by first determining the appropriate partial derivative of the appropriate element of $\underline{f}(\underline{x},\underline{u})$ to yield a valid general formula for the required partial derivative. Afterwards, substitute the values of the elements of $\underline{x}_{eq}$ and $\underline{u}_{eq}$ into each formula in order to determine the relevant element of A and B. Hand in your derivations and results. Be sure to draw clear boxes around your general formulas for the partial derivatives, which are the ones that apply before substituting in the values in $\underline{x}_{eq}$ and $\underline{u}_{eq}$.

Aside: It makes sense to apply a linearized version of the quaternion unit normalization constraint to this linearized model. It takes the form:

$$
\begin{aligned}
1 &= (\underline{q}_{eq} + \Delta\underline{q})^{\mathrm{T}}(\underline{q}_{eq} + \Delta\underline{q}) = \Delta q_1^2 + \Delta q_2^2 + \Delta q_3^2 + 1 + 2\Delta q_4 + \Delta q_4^2 \\
&= 1 + 2\Delta q_4 + O(\Delta\underline{q}^2) \\
&\cong 1 + 2\Delta q_4
\end{aligned}
$$

which translates into $\Delta q_4 \cong 0$. Therefore, this element can be eliminated from $\Delta\underline{x}$ to define

$$
\Delta\underline{\tilde{x}} = \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \\ \Delta\omega_1^b \\ \Delta\omega_2^b \\ \Delta\omega_3^b \end{bmatrix}
$$

which has the following reduced-order linearized system dynamics model

$$
\Delta\dot{\underline{\tilde{x}}}(t) = \begin{bmatrix}
0 & 0 & n & \tfrac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \tfrac{1}{2} & 0 \\
-n & 0 & 0 & 0 & 0 & \tfrac{1}{2} \\
\frac{6n^2[I_{33}^b - I_{22}^b]}{I_{11}^b} & 0 & 0 & 0 & 0 & \frac{n[I_{33}^b - I_{22}^b]}{I_{11}^b} \\
0 & \frac{6n^2[I_{33}^b - I_{11}^b]}{I_{22}^b} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{n[I_{22}^b - I_{11}^b]}{I_{33}^b} & 0 & 0
\end{bmatrix} \Delta\underline{\tilde{x}}(t) + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{1}{I_{11}^b} & 0 & 0 \\
0 & \frac{1}{I_{22}^b} & 0 \\
0 & 0 & \frac{1}{I_{33}^b}
\end{bmatrix} \Delta\underline{u}(t)
$$

The new A and B matrices in this 6-state model are formed by deleting the 4th rows from the original A and B matrices and the 4th column from the original A matrix. An examination of the 4th rows of the original A and B matrices indicates that $\Delta\dot{q}_4 = 0$. Therefore, its dynamics is not very interesting anyway: It must start at 0 in order to satisfy the unit normalization constraint to first-order, and it must stay constant at this value. It is reasonable to use the 3-parameter attitude representation [Δq_1; Δq_2; Δq_3] in this linearized model because the assumption of small state perturbations assures that no attitude singularities will be encountered when operating near the original equilibrium.

4. The flat-Earth version of the 3D point-mass aircraft dynamics model that has been used throughout the semester can be put in state-space form with the state and control vectors being

$$
\underline{x} = \begin{bmatrix} X \\ Y \\ Z \\ V \\ \gamma \\ \psi \end{bmatrix} \quad \text{and} \quad \underline{u} = \begin{bmatrix} T \\ \alpha \\ \phi \end{bmatrix}
$$

The nonlinear model's $\underline{f}(\underline{x},\underline{u})$ dynamics function has the following components

$$
f_1(\underline{x},\underline{u}) = x_4 \cos(x_5)\cos(x_6)
$$

$$f_2(\underline{x},\underline{u}) = x_4\cos(x_5)\sin(x_6)$$

$$f_3(\underline{x},\underline{u}) = -x_4\sin(x_5)$$

$$f_4(\underline{x},\underline{u}) = \tfrac{1}{m}[u_1\cos(u_2) - \tfrac{1}{2}\rho(x_3)x_4^2 SC_D(u_2)] - g\sin(x_5)$$

$$f_5(\underline{x},\underline{u}) = \tfrac{1}{x_4}\{\tfrac{1}{m}[u_1\sin(u_2) + \tfrac{1}{2}\rho(x_3)x_4^2 SC_L(u_2)]\cos(u_3) - g\cos(x_5)\}$$

$$f_6(\underline{x},\underline{u}) = \tfrac{1}{mx_4\cos(x_5)}[u_1\sin(u_2) + \tfrac{1}{2}\rho(x_3)x_4^2 SC_L(u_2)]\sin(u_3)$$

where the atmospheric density model takes the form:

$$\rho(x_3) = \rho_0 e^{(x_3 - 649.7)/h_s}$$

with ρ_0 being the sea-level atmospheric density and with h_s being the density scale height in meters. The constant 649.7 m is the altitude of the coordinate system's origin, which is the center of the runway of the Blacksburg, VA, airport. The lift and drag models are the same as are given in Problem 2.

One can developed a linearized model about the following system steady motion condition:

$$\underline{x}_{SM}(t) = \begin{bmatrix} \{X(t_0) + x_{4eq}\cos(x_{5eq})\cos(x_{6eq})(t-t_0)\} \\ \{Y(t_0) + x_{4eq}\cos(x_{5eq})\sin(x_{6eq})(t-t_0)\} \\ x_{3eq} \\ x_{4eq} \\ x_{5eq} \\ x_{6eq} \end{bmatrix}$$

The linearized model takes the form.

$$\Delta\dot{\underline{x}}(t) = A\Delta\underline{x}(t) + B\,\Delta\underline{u}(t)$$

with $\Delta\underline{x}(t) = \underline{x}(t) - \underline{x}_{SM}(t)$ and $\Delta\underline{u}(t) = \underline{u}(t) - \underline{u}_{eq}$ and with

$$A = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{(x_{eq},u_{eq})} = \begin{bmatrix} 0 & 0 & 0 & \cos(x_{6eq}) & 0 & -x_{4eq}\sin(x_{6eq}) \\ 0 & 0 & 0 & \sin(x_{6eq}) & 0 & x_{4eq}\cos(x_{6eq}) \\ 0 & 0 & 0 & 0 & -x_{4eq} & 0 \\ 0 & 0 & \frac{-\rho'(x_{3eq})x_{4eq}^2 SC_D(u_{2eq})}{2m} & \frac{-\rho(x_{3eq})x_{4eq}SC_D(u_{2eq})}{m} & -g & 0 \\ 0 & 0 & \frac{\rho'(x_{3eq})x_{4eq}SC_L(u_{2eq})}{2m} & \frac{\rho(x_{3eq})SC_L(u_{2eq})}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \left.\frac{\partial \underline{f}}{\partial \underline{u}}\right|_{(x_{eq},u_{eq})} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\cos(u_{2eq})}{m} & \frac{-[u_{1eq}\sin(u_{2eq})+0.5\rho(x_{3eq})x_{4eq}^2 SC_D^{'}(u_{2eq})]}{m} & 0 \\ \frac{\sin(u_{2eq})}{mx_{4eq}} & \frac{[u_{1eq}\cos(u_{2eq})+0.5\rho(x_{3eq})x_{4eq}^2 SC_L^{'}(u_{2eq})]}{mx_{4eq}} & 0 \\ 0 & 0 & \frac{g}{x_{4eq}} \end{bmatrix}$$

where $\rho'(x_{3eq})$ is the derivative of $\rho(x_{3eq})$ with respect to its input argument, $C_D^{'}(u_{2eq})$ is the derivative of $C_D(u_{2eq})$ with respect to its input argument, and $C_L^{'}(u_{2eq})$ is the derivative of $C_L(u_{2eq})$ with respect to its input argument.

Prove that $A_{14} = \cos(x_{6eq})$, that $A_{25} = 0$, that $A_{35} = -x_{4eq}$, that $A_{43} = -[\rho'(x_{3eq})(x_{4eq})^2 SC_D(u_{2eq})]/(2m)$, that $A_{54} = [\rho(x_{3eq})SC_L(u_{2eq})]/m$, that $A_{64} = 0$, and that $B_{63} = g/x_{4eq}$. Prove these 7 formulas by first determining the appropriate partial derivative of the appropriate element of $\underline{f}(\underline{x},\underline{u})$ and giving a valid general formula for the required partial derivative. Afterwards, substitute the values of the elements of $\underline{x}_{eq}$ and $\underline{u}_{eq}$ into each formula in order to determine the relevant element of A and B. Hand in your derivations and results. Be sure to draw clear boxes around your general formulas for the partial derivatives, which are the ones that apply before substituting in the values in $\underline{x}_{eq}$ and $\underline{u}_{eq}$.

Hints: Remember from Problem 2 that $x_{5eq} = 0$ and that $u_{3eq} = 0$. That is why neither of these quantities appears in the A matrix or the B matrix. Your initial formulas for A_{54} and for B_{63} will probably look much more complicated than the formulas given above. They can be simplified by making substitutions that you can derive from the equation $\dot{x}_{5eq} = 0$.