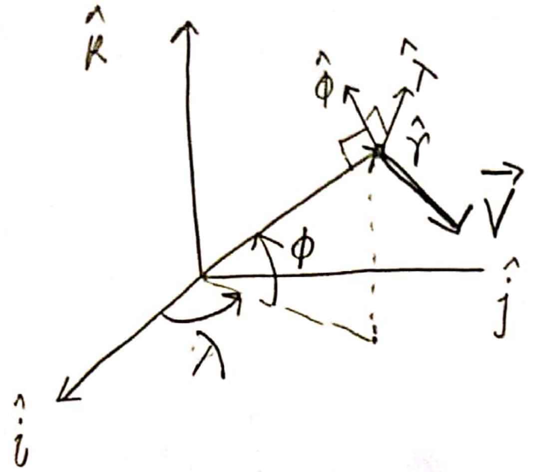
Q1)



the velocity of a particle $\vec{V}$ in $\hat{i}, \hat{j}, \hat{k}$ System Expressed in the Sperical Co-ord $\hat{r}, \hat{\lambda}, \hat{\phi}$ is

$$\vec{V} = \dot{r}\,\hat{r} + r\dot{\lambda}\cos\phi\,\hat{\lambda} + r\dot{\phi}\,\hat{\phi}$$ (from lecture)

the angular momentum, definition (at a fixed pt in inertial Frame)

$$\vec{h} = \sum_{i=1}^{n} m_i(\vec{r}_i - \vec{r}_0) \times \vec{V}_i$$

$n = 1$, in the problem and $\vec{r}_0 = \vec{0}$

$$\vec{h} = m(\vec{r} \times \vec{V})$$

mass of the particle; position of particle in Sperical co-ord; velocity of the particle in Spherical Co-ord

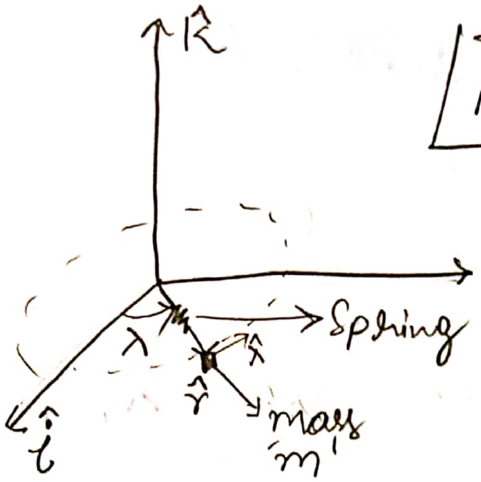
$\vec{r} = r\hat{r}$ → by definition of Sperical Co-ord

$$\vec{h} = m\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -r \\ 0 & r & 0 \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\lambda}\cos\phi \\ r\dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -r^2\dot{\phi} \\ r^2\dot{\lambda}\cos\phi \end{bmatrix}$$

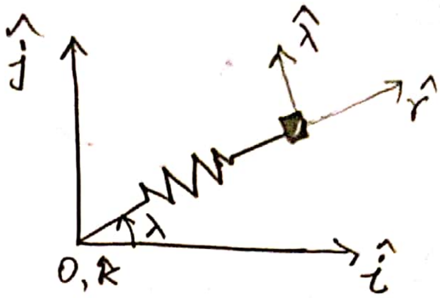
$$\boxed{\vec{h} = -mr^2\dot{\phi}\,\hat{\lambda} + r^2\dot{\lambda}\cos\phi\,\hat{\phi}}$$

Q2)

$\boxed{1.4}$



TOP VIEW

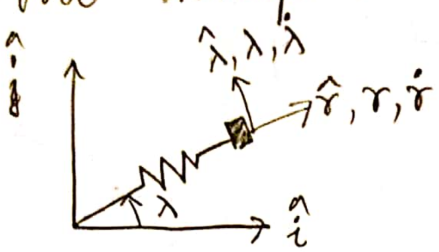


$\boxed{1.1}$ only the mass is the single member of the system

$\boxed{1.2}$ the system can be described by 2-D general planar motion.

$\boxed{1.3}$ the only force producing component is the spring, the slider mechanism produces forces, but are only there to ensure the idealized $\dot{\omega}=0$ & $\phi=\dot{\phi}=0$ assumptions.

2.1 The independent planar co-ord

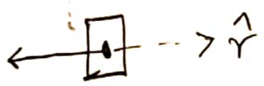


The independent co-ord are $(\lambda, \dot{\lambda}, r, \dot{r})$

as $\phi(t) = 0$, and the F_ϕ ensures this, this dimension is not coupled here, in this analysis.

2.2 The free body diagrams

<u>mass:</u>



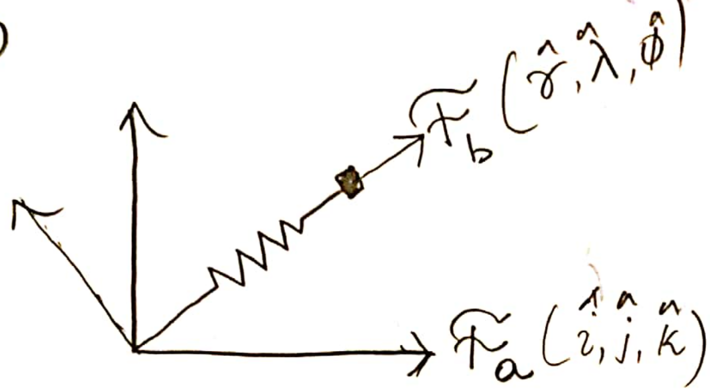
$F_{spring} = -k r$

k is spring stiffness
r is the distance from origin

<u>Spring:</u>

2.3 Newtons laws. (Can only be applied in an inertial frame

The instantaneous acceleration measured in an inertial frame $\mathcal{F}_a$ but transformed to the Rotating frame $\mathcal{F}_b$ is given below



$\mathcal{F}_b$ is rotating at a ang vel ω, with the mass & slider

$$\vec{a}^{\,b}_{inertial} = \vec{a}^{\,b} + \dot{\vec{\omega}}^{\,b} \times \vec{r}^{\,b} + 2\vec{\omega}^{\,b} \times \vec{v}^{\,b} + \vec{\omega}^{\,b} \times (\vec{\omega}^{\,b} \times \vec{r}^{\,b})$$

$$\vec{\omega}^{\,b} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$ because $\phi = 0$ $\hat{k} = \hat{\phi}$

$$\vec{r}^{\,b} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$$ → dist from origin

$$\vec{v}^{\,b} = \begin{bmatrix} \dot{r} \\ 0 \\ 0 \end{bmatrix}$$

$\dot{\vec{\omega}}^{\,b} = \vec{0}$ because the $\lambda(t) = \omega t$, no angular Acceleration.

$$\vec{a}^b_{inertias} = \vec{a}^b + 2\begin{bmatrix}0 & \omega & 0\\ -\omega & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{r}\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0 & \omega & 0\\ -\omega & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\left(\begin{bmatrix}0 & \omega & 0\\ -\omega & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}r\\ 0\\ 0\end{bmatrix}\right)$$

$$= \vec{a}^b + \begin{bmatrix}0\\ -2\omega\dot{r}\\ 0\end{bmatrix} + \begin{bmatrix}-r\omega^2\\ 0\\ 0\end{bmatrix}$$

Now applying Newtons laws

$$\sum \vec{F} = m a$$

$$\begin{bmatrix}F_{spring}\\ F_\lambda\end{bmatrix} = \left(\begin{bmatrix}a_r \hat{r}\\ a_\lambda \hat{\lambda}\end{bmatrix} - \begin{bmatrix}r\omega^2 \hat{r}\\ 2\omega\dot{r}\hat{\lambda}\end{bmatrix}\right) m$$

The Slider needs to apply

$F_\lambda = -2\omega\dot{r}m$ to keep $a_\lambda = 0$

The motion is now solely described by

$$F_{spring} = m a_r - m r\omega^2$$

$a_r = \ddot{r}$ & $F_{spring} = -Kr$

$$\boxed{m\ddot{r} + Kr = m r\omega^2}$$ Eq of motion

Re-writing this as

$$\underbrace{\ddot{r} + \left(\frac{K}{m} - \omega^2\right) r = 0}_{\text{Spring mass System}}$$

$\frac{K}{m} = \omega_n^2 \rightarrow$ the natural freq of Spring-mass

$$\boxed{\ddot{r} + \left(\omega_n^2 - \omega^2\right) r = 0}$$

$\rightarrow$ when $\underline{\omega_n^2 > \omega^2}$ it a normal Spring mass System.

The System exhibits harmonic behavior

$\rightarrow$ when $\omega_n^2 < \omega^2$, the System exhibits unstable behavior.

The System Solution $r(t) = \cosh(\omega^2 - \omega_n^2) r(0)$
$+ \sinh(\omega^2 - \omega_n^2) \dot{r}(0)$

They diverge in a hyperbolic trajectories.

The Energy for this comes from F_λ that Ensures a constant angular velocity ω.

as r grow Exponentially F_λ has to applied to keep a constant ω, which is introducing the growth Energy.

Q3) The Euler angle representation is

$$R = R_2(\alpha) R_1(\beta) R_2(\gamma) \quad , \text{from } \mathcal{F}^a \text{ to } \mathcal{F}^b$$

$R_2(\gamma)$ rotates from $\hat{i}, \hat{j}, \hat{k}$ to $\hat{i}', \hat{j}', \hat{k}'$

$R_1(\beta)$ rotates from $\hat{i}', \hat{j}', \hat{k}'$ to $\hat{i}'', \hat{j}'', \hat{k}''$

$R_2(\alpha)$ rotates from $\hat{i}'', \hat{j}'', \hat{k}''$ to $\hat{i}^b, \hat{j}^b, \hat{k}^b$

$\dot{\gamma}$ is a rotation rate about $\hat{j}^a$ and also $\hat{j}'$

$\dot{\beta}$ " " " " " $\hat{i}'$ " " $\hat{i}''$

$\dot{\alpha}$ " " " " " $\hat{j}''$ " " $\hat{k}^b$

So $\vec{\omega}$ is the sum of

$\begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix}$ in the $\mathcal{F}_a$ but also is $\hat{i}', \hat{j}', \hat{k}'$

$\begin{bmatrix} \dot{\beta} \\ 0 \\ 0 \end{bmatrix}$ is the $\hat{i}', \hat{j}', \hat{k}'$ but also in the $\hat{i}'', \hat{j}'', \hat{k}''$

$\begin{bmatrix} 0 \\ \dot{\alpha} \\ 0 \end{bmatrix}$ is the $\hat{i}'', \hat{j}'', \hat{k}''$ but also in $\mathcal{F}_b$

$$\vec{\omega}^b = \begin{bmatrix} 0 \\ \dot{\alpha} \\ 0 \end{bmatrix} + \left[R_2(\alpha) \right] \begin{bmatrix} \dot{\beta} \\ 0 \\ 0 \end{bmatrix} + \left[R_2(\alpha) R_1(\beta) \right] \begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \dot{\alpha} \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \dot{\beta} \\ 0 \\ 0 \end{bmatrix} + \cdots$$

$$\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \dot{\alpha} \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\alpha\,\dot{\beta} \\ 0 \\ \sin\alpha\,\dot{\beta} \end{bmatrix} + \begin{bmatrix} R_2(\alpha) \end{bmatrix} \begin{bmatrix} 0 \\ \cos\beta\,\dot{\gamma} \\ -\sin\beta\,\dot{\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\,\dot{\beta} \\ \dot{\alpha} \\ \sin\alpha\,\dot{\beta} \end{bmatrix} + \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 0 \\ \cos\beta\,\dot{\gamma} \\ -\sin\beta\,\dot{\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\,\dot{\beta} \\ \dot{\alpha} \\ \sin\alpha\,\dot{\beta} \end{bmatrix} + \begin{bmatrix} \sin\alpha\sin\beta\,\dot{\gamma} \\ \cos\beta\,\dot{\gamma} \\ -\cos\alpha\sin\beta\,\dot{\gamma} \end{bmatrix}$$

$$\boxed{\vec{\omega}^{\,b} = \begin{bmatrix} \cos\alpha\,\dot{\beta} + \sin\alpha\sin\beta\,\dot{\gamma} \\ \dot{\alpha} + \cos\beta\,\dot{\gamma} \\ \sin\alpha\,\dot{\beta} - \cos\alpha\sin\beta\,\dot{\gamma} \end{bmatrix}}$$

Q4)

$$\Xi(q) = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}$$

$$\Xi(q)^T \,\Xi(q) = \ldots$$

$$= \begin{bmatrix} q_4 & q_3 & q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \end{bmatrix} \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}$$

$$= \begin{bmatrix} q_4^2+q_1^2+q_2^2+q_3^2 & \text{Symmetric comp} & \text{Symmetric comp} \\ \cancel{-q_3q_4}+\cancel{q_3q_4}-\cancel{q_1q_2}+\cancel{q_1q_2} & q_4^2+q_1^2+q_2^2+q_3^2 & \text{Symmetric comp} \\ \cancel{q_2q_4}-\cancel{q_1q_3}-\cancel{q_2q_4}+\cancel{q_1q_3} & \cancel{-q_2q_3}-\cancel{q_1q_4}+\cancel{q_2q_3}+\cancel{q_1q_4} & q_4^2+q_1^2+q_2^2+q_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\left(q_4^2+q_1^2+q_2^2+q_3^2\right)}_{\downarrow\ 1\ \text{(from the property of a quaternions)}}$$

$$\boxed{\therefore\ \Xi(q)^T\,\Xi(q) = I_3}$$

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from lecture

$$\dot{q} = \frac{1}{2} \Xi(q) \vec{\omega}^b$$

multiplying both sides by $\Xi^T(q)_{3\times4}$

$$\underbrace{\underset{3\times4}{\Xi^T(\vec{q})} \; \underset{4\times1}{\dot{q}}}_{3\times1} = \frac{1}{2} \underset{3\times4}{\Xi^T(\vec{q})} \; \underset{4\times3}{\Xi(\vec{q})} \; \underset{3\times1}{\vec{\omega}^b}$$

$$\Xi^T(\vec{q})\dot{q} = \frac{1}{2} I_{3\times3} \, \omega^b_{3\times1}$$

$$\boxed{\vec{\omega}^b = 2\,\Xi(\vec{q})^T \dot{q}}$$