Prob-3

HCW Eq

$$\ddot{x} = 3n^2x + 2n\dot{y}$$

$$\ddot{y} = -2n\dot{x}$$

$$\ddot{z} = -n^2 z$$

First solving for $x(t)$ & $y(t)$ because they are coupled.

$\ddot{y} = -2n\dot{x}$ → this is substituted in

$$\dddot{x} = 3n^2\dot{x} + 2n(-2n\dot{x})$$

$$\dddot{x} = 3n^2\dot{x} - 4n^2\dot{x} = -n^2\dot{x}$$

$$\dddot{x} = -n^2\dot{x}$$

$(\dddot{x} + n^2\dot{x}) = 0$ → dE to be solved

at $t=0$

$$\ddot{x}_0 = 3n^2x_0 + 2n\dot{y}_0$$

this $\ddot{x}_0$ will be used and will be substituted later in terms of x_0 & $\dot{y}_0$

Taking Laplace transform.

$$\mathcal{L}(\dddot{x}(t) + n^2 \dot{x}(t)) = 0$$

$$\mathcal{L}(\dddot{x}(t)) + n^2 \mathcal{L}(\dot{x}(t)) = 0$$

$$s^3 X(s) - s^2 x_0 - s\dot{x}_0 - \ddot{x}_0 + n^2 s X(s) - n^2 x_0 = 0$$

$$(s^3 + sn^2) X(s) = s^2 x_0 + n^2 x_0 + s\dot{x}_0 + \ddot{x}_0$$

$$X(s) = \frac{(s^2+n^2)x_0 + s\dot{x}_0 + \ddot{x}_0}{s(s^2+n^2)}$$

$$X(s) = \frac{x_0}{s} + \frac{\dot{x}_0}{s^2+n^2} + \frac{\ddot{x}_0}{s(s^2+n^2)}$$

The term $\frac{\ddot{x}_0}{s(s^2+n^2)}$ need to be split by partial fractions.

$$\frac{\ddot{x}_0}{s(s^2+n^2)} = \frac{a}{s} + \frac{bs+c}{s^2+n^2}$$

$$\ddot{x}_0 = as^2 + an^2 + bs^2 + cs$$

$$\ddot{x}_0 = (a+b)^2 + cs + an^2$$

$a+b=0$ & $an^2 = \ddot{x}_0$ $\Rightarrow$ $\boxed{a = \frac{\ddot{x}_0}{n^2} \quad b = -\frac{\ddot{x}_0}{n^2}}$

$\boxed{c=0}$

$$X(s) = \frac{x_0}{s} + \frac{\dot{x}_0}{s^2+n^2} + \frac{\ddot{x}_0}{n^2}\frac{1}{s} - \frac{\ddot{x}_0}{n^2}\frac{s}{s^2+n^2}$$

$$\mathcal{L}^{-1}(X(s)) = x(t) = x_0 + \frac{\dot{x}_0}{n}\sin(nt) + \frac{\ddot{x}_0}{n^2} - \frac{\ddot{x}_0}{n^2}\cos(nt)$$

$$x(t) = x_0 + \frac{\ddot{x}_0}{n^2} + \frac{\dot{x}_0}{n}\sin(nt) - \frac{\ddot{x}_0}{n^2}\cos(nt)$$

$$\ddot{x}_0 = 3n^2x_0 + 2n\dot{y}_0$$

$$x(t) = x_0 + \frac{(3n^2x_0 + 2n\dot{y}_0)}{n^2} + \frac{\dot{x}_0}{n}\sin(nt) - \frac{(3n^2x_0+2n\dot{y}_0)}{n^2}\cos(nt)$$

$$= x_0 + 3x_0 + \frac{2}{n}\dot{y}_0 + \frac{\dot{x}_0}{n}\sin(nt) - \left(3x_0 + \frac{2}{n}\dot{y}_0\right)\cos(nt)$$

$$\boxed{x(t) = 4x_0 + \frac{2}{n}\dot{y}_0 + \frac{\dot{x}_0}{n}\sin(nt) - 3x_0\cos(nt) - \frac{2}{n}\dot{y}_0\cos(nt)}$$

The Second Eq.

$$\ddot{y}(t) = -2n\dot{x}(t) \rightarrow \text{the Second eq}$$

$$\int \ddot{y}(t)\,dt = -2n\int \dot{x}(t)\,dt + C_1$$

$$\dot{y}(t) = -2n\, x(t) + C_1$$

at $t=0$

$$x(t=0) = 4x_0 + \frac{2}{n}\dot{y}_0 - 3x_0 - \frac{2}{n}\dot{y}_0$$

$$= x_0$$

$$\dot{y}_0 = -2n\, x_0 + C_1$$

$$\boxed{C_1 = \dot{y}_0 + 2n\, x_0}$$

$$\int \dot{y}(t)\, dt = -2n \int x(t) + C_1 t + C_2$$

$$\int x(t) = 4x_0 t + \frac{2}{n}\dot{y}_0 t - \frac{\dot{x}_0}{n^2}\cos(nt) - \frac{3x_0}{n}\sin(nt) - \frac{2}{n^2}\dot{y}_0 \sin(nt)$$

$$y(t) = \left(4x_0 t + \frac{2}{n}\dot{y}_0 t - \frac{\dot{x}_0}{n^2}\cos(nt) - \frac{3x_0}{n}\sin(nt) - \frac{2}{n^2}\dot{y}_0 \sin(nt)\right)(-2n) + C_1 t + C_2$$

$$y(t) = -8n x_0 t - 4\dot{y}_0 t + \frac{2}{n}\dot{x}_0 \cos(nt) + 6x_0 \sin(nt) + \frac{4}{n}\dot{y}_0 \sin(nt) + C_1 t + C_2$$

$$y(t) = -6n x_0 t - 3\dot{y}_0 t + \frac{2}{n}\dot{x}_0 \cos(nt) + 6x_0 \sin(nt)$$
$$+ \frac{4}{n}\dot{y}_0 \sin(nt) + C_2$$

at $t = 0$

$$y_0 = \frac{2}{n}\dot{x}_0 + C_2$$

$$C_2 = y_0 - \frac{2}{n}\dot{x}_0$$

The $x(t)$ & $y(t)$ are:

$$\boxed{\begin{aligned} \underline{\underline{x(t)}} &= 4x_0 + \frac{2}{n}\dot{y}_0 + \frac{\dot{x}_0}{n}\sin(nt) - 3x_0\cos(nt) - \frac{2}{n}\dot{y}_0\cos(nt) \\ y(t) &= y_0 - \frac{2}{n}\dot{x}_0 - 6n x_0 t - 3\dot{y}_0 t + \frac{2}{n}\dot{x}_0\cos(nt) + 6x_0\sin(nt) + \frac{4}{n}\dot{y}_0\sin(nt) \end{aligned}}$$

for $z(t)$

$$\ddot{z} + n^2 z = 0$$

Taking laplace transform

$$s^2 Z(s) - s z_0 - \dot{z}_0 + n^2 Z = 0$$

$$(s^2+n^2) Z(s) = s z_0 + \dot{z}_0$$

$$Z(s) = z_0 \frac{s}{s^2+n^2} + \frac{\dot{z}_0}{n} \frac{n}{s^2+n^2}$$

$$\mathcal{L}^{-1}(Z(s)) = z_0 \cos(nt) + \frac{\dot{z}_0}{n} \sin(nt)$$

$$\boxed{z(t) = z_0 \cos(nt) + \frac{\dot{z}_0}{n} \sin(nt)}$$

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