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# HW4 SANDEEP K JADA

## Prob 1:

The first two Eqs in $\omega_1$ & $\omega_2$ can be written as follow

$$\omega_1 = (\dot{\phi}\sin\theta)\sin\psi + \dot{\theta}(\cos\psi)$$
$$\omega_2 = (\dot{\phi}\sin\theta)\cos\psi + \dot{\theta}(-\sin\psi)$$

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{bmatrix}}_{\text{orthonormal Rot Matrix}}\begin{bmatrix}\dot{\theta}\\ \dot{\phi}\sin\theta\end{bmatrix}$$

orthonormal Rot Matrix
the inverse in transpose

$$\therefore \begin{bmatrix}\dot{\theta}\\ \dot{\phi}\sin\theta\end{bmatrix} = \begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}$$

$$\boxed{\dot{\theta} = \cos\psi\,\omega_1 - \sin\psi\,\omega_2}$$

$$\dot{\phi}\sin\theta = \sin\psi\,\omega_1 + \cos\psi\,\omega_2$$

$$\boxed{\dot{\phi} = \frac{\sin\psi}{\sin\theta}\omega_1 + \frac{\cos\psi}{\sin\theta}\omega_2}$$

Singularity when $\theta = 0 \rightarrow \sin\theta \rightarrow 0$

From the third Eq

$$\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta$$

$$\boxed{\dot{\psi} = \omega_3 - \frac{\sin\psi}{\tan\theta}\omega_1 - \frac{\cos\psi}{\tan\theta}\omega_2}$$

Also affected by singularity at $\theta = 0$

Prob 2: the KE of the Element $dm$ is

$$dT = \frac{1}{2}\vec{v}\cdot\vec{v}\,dm$$

$$\int_{body} dT = T = \frac{1}{2}\int_{body} \vec{v}\cdot\vec{v}\,dm \quad \Big|\ \text{where } \int dm = M$$

$$\vec{v} = \vec{v}_{COM} + \vec{\omega}^{bi}\times\vec{r}$$

$$\vec{v}\cdot\vec{v} = \left(\vec{v}_{COM} + \vec{\omega}^{bi}\times\vec{r}\right)\cdot\left(\vec{v}_{COM} + \vec{\omega}^{bi}\times\vec{r}\right)$$

$$= \vec{v}_{COM}\cdot\vec{v}_{COM} + 2\vec{v}_{COM}\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right) + \left(\vec{\omega}^{bi}\times\vec{r}\right)\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right)$$

$$\therefore\ T = \frac{1}{2}\int \vec{v}_{COM}\cdot\vec{v}_{COM}\,dm + \int \vec{v}_{COM}\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right)dM + \frac{1}{2}\int\left(\vec{\omega}^{bi}\times\vec{r}\right)\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right)dm$$

$V_{COM}$ & $\vec{\omega}^{bi}$ are not fuctions of positions on Rigid body

$$T = \frac{1}{2}\vec{v}_{COM}\cdot\vec{v}_{COM}\int dm + \vec{v}_{COM}\cdot\left(\vec{\omega}^{bi}\times\int\vec{r}\,dm\right) + \text{"}$$

$$T = \frac{1}{2}M\left(\vec{v}_{COM}\cdot\vec{v}_{COM}\right) + \underbrace{\cancel{\vec{v}_{COM}\cdot\left(\vec{\omega}^{bi}\times\vec{0}\right)}}_{\text{the Integral Evaluates to the definiton of COM, as } \vec{r} \text{ is dist from COM, it is simply } \vec{0}}^{\to 0} + \underbrace{\frac{1}{2}\vec{\omega}^{bi}\cdot I\vec{\omega}^{bi}}_{\text{from lecture notes def of } \vec{I}}$$

$$\boxed{T = \frac{1}{2}M\left(\vec{v}_{COM}\cdot\vec{v}_{COM}\right) + \frac{1}{2}\vec{\omega}^{bi}\cdot I\vec{\omega}^{bi}}\ \blacksquare$$

Prob 3.a

$$I^{Principal} = \begin{bmatrix} 1.29 & & 0 \\ & 9.68 & \\ 0 & & 10.10 \end{bmatrix} \times 10^6\ Kg\text{-}m^2 = \begin{bmatrix} A & & 0 \\ & B & \\ 0 & & C \end{bmatrix} \begin{matrix} \rightarrow \text{about } b_1 \\ \rightarrow \text{about } b_2 \\ \rightarrow \text{about } b_3 \end{matrix}$$

The stable configuration finally settles to $C > B > A$ (Already true in our case), so the axes, $b_1$, $b_2$, & $b_3$ will be aligned along.

$b_1 \rightarrow$ (YAW) $\rightarrow$ aligned with radius vector.
$b_2 \rightarrow$ (ROLL) $\rightarrow$ aligned with velocity vector
$b_3 \rightarrow$ (PITCH) $\rightarrow$ in the direction of orbit normal

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Prob 3.b,

The Eqs of Motion for a body in orbit under Torque-Free Gravity Gradient are

$$A\ddot{\psi}_1 + (C-B-A)\Omega\dot{\psi}_2 + (C-B)\Omega^2\psi_1 = 0 \quad \text{(Yaw)}$$

$$B\ddot{\psi}_2 + (B+A-C)\Omega\dot{\psi}_1 + 4(C-A)\Omega^2\psi_2 = 0 \quad \text{(Roll)}$$

$$C\ddot{\psi}_3 + 3(B-A)\Omega^2\psi_3 = 0 \quad \text{(Pitch)}$$

where $\psi_1$, $\psi_2$, $\psi_3$ are rotations about axes $b_1$, $b_2$, $b_3$ respectively

Where, $\Omega_i^2 = \frac{\mu}{R^3}$, can be computed from the time period of the orbit as follows

$$\Omega_i = 2\pi f = \frac{2\pi}{T} \rightarrow T = 90 \text{ min}$$

To find frequencies we need to find the System Poles (or Eigenvalues, or roots of characteristic Eq)

1) For Pitch: it is a Spring-Mass System form like $m\ddot{x} + kx = 0$ where

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$\omega_{n,pitch} = \Omega_i \sqrt{\frac{3(B-A)}{C}} = 0.001836 \text{ Rad/sec}$$

2) For Roll & Yaw its a Coupled System of two 2nd order Linear ODE'S, Can be Written is State-Space representation of $\dot{X} = AX$, where

$X^T = [\psi_1 \; \psi_2 \; \dot{\psi}_1 \; \dot{\psi}_2]$, then A becomes

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(C-B)\Omega_i^2}{A} & 0 & 0 & -\frac{(C-B-A)\Omega_i}{A} \\ 0 & -\frac{4(C-A)\Omega_i^2}{B} & -\frac{(B+A-C)\Omega_i}{B} & 0 \end{bmatrix}$$

The poles of the Roll + Yaw System are <u>Eigenvalues of A</u> (the Imaginary part of it is the freq)

$$Eq. \; \lambda_{1,2} = -\frac{1}{T} \pm i\,\omega_d$$

$-\frac{1}{T}$: Real part determines the Stability of the System

$\omega_d$: damped Natural freq

<u>The Eigenvalues of A</u>

$$\lambda_{1,2} = \pm i\, 6.5794 \times 10^{-4} \text{ rad/sec}$$

$$\lambda_{3,4} = \pm i\, 0.002240 \text{ rad/sec}$$

So the Final three frequencies are

$$\boxed{\begin{array}{ll} 6.5794e^{-4} \text{ Rad/sec} & \multirow{2}{*}{\} Roll & Yaw} \\ 0.00224023 \text{ Rad/sec} & \\ 0.00183682 \text{ Rad/sec} & \rightarrow \text{Pitch} \end{array}}$$

The numerical results are Evaluated using <u>MATLAB Script</u> that is attached within the <u>pdf Document</u>.

# Dynamics of Pitch

```
clc; close all; clear;
format long

A = 1.29*1e6; % Kg m^2
B = 9.68*1e6; % Kg m^2
C = 10.10*1e6; % Kg m^2
T_orb = 90*60; % Sec

Omega = 2*pi*(1/T_orb);

w3 = sqrt(3*Omega^2*(B-A)/C);
```

# Dynamics of Roll and Yaw

X = [psi1 psi2 psi1_dot psi2_dot]

```
Sys_mat = [0 0 1 0;...
           0 0 0 1;...
           -(C-B)*Omega^2/A    0  0  -(C-B-A)*Omega/A;...
           0  -4*(C-A)*Omega^2/B  -(B+A-C)*Omega/B  0];

e = eig(Sys_mat);
w1 = imag(e(1))
w2 = imag(e(3))
w3
```

```
w1 =

     6.579426254356058e-04


w2 =

   0.002240238490951


w3 =

   0.001836821818276
```

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