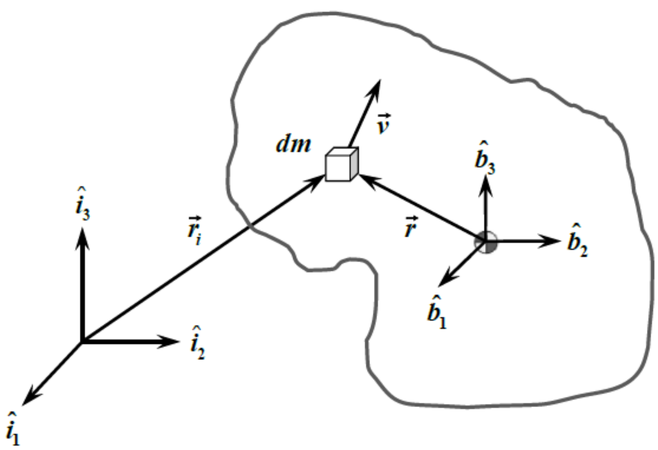
AOE 5234 - Lesson 18
DYNAMICS OF A RIGID BODY - ANGULAR MOMENTUM & ENERGY

(Read Wiesel sections 4.3, 4.4, 4.5)

Last lesson we showed that the rate of change of angular momentum was equal to the external moment applied. Furthermore, we proved that these equations are most easily evaluated when the momentum is evaluated about either an inertial origin or the center of mass of the body. Now we will develop equations for the angular momentum and kinetic energy of a rigid body. Remember that angular momentum of a point mass is given by the cross product of inertial position and linear momentum,

$$\vec{H} = m\left(\vec{r} \times \vec{v}\right)$$

A rigid body can be considered as a sum of many small point masses, dm; therefore, total momentum can be calculated by integrating over the body,



$$\vec{H}^o_{\hat{i}} = \int_{body} \vec{r}_i \times \vec{v}_i dm$$

While this statement is true, it can be very inconvenient to aply in a moving body; this is because the position of each element, dm, changes with time. If instead we use a frame attached to the body with origin at the center of mass, then the inertial velocity (caused by rotation) relative to the origin becomes $\vec{v} = \vec{\omega}^{bi} \times \vec{r}$. The angular momentum relative to the center of mass can be written as,

$$\vec{H} = \int \vec{r} \times \left(\vec{\omega}^{bi} \times \vec{r}\right) dm$$

Note that $\vec{\omega}^{bi}$ is the same for all points on the rigid body (however, it is not easy to factor $\vec{\omega}^{bi}$ out of the integral).

Writing these vectors in the $\hat{b}$ frame,

$$\begin{aligned} \vec{r} &= x\hat{b}_1 + y\hat{b}_2 + z\hat{b}_3 \\ \vec{\omega} &= \omega_1\hat{b}_1 + \omega_2\hat{b}_2 + \omega_3\hat{b}_3 \end{aligned}$$

and computing the cross products yields

$$\vec{H} = \int \left\{ \begin{array}{c} \left(\omega_1\left(y^2+z^2\right) - \omega_2 yx - \omega_3 zx\right)\hat{b}_1 \\ \left(-\omega_1 xy + \omega_2\left(x^2+z^2\right) - \omega_3 yz\right)\hat{b}_2 \\ \left(-\omega_1 xz - \omega_2 yz + \omega_3\left(x^2+y^2\right)\right)\hat{b}_3 \end{array} \right\} dm$$

This result can be rewritten in matrix-vector form

$$\vec{H}_b = I_b\vec{\omega}_b$$

where

$$I_b = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm$$

$$\vec{\omega}_b = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

The matrix I_b is the *moment of inertia matrix.* The components of this moment of inertia matrix are frame-dependent in much the same way the components of a vector are dependent on the frame used to describe the components. It is a second-order tensor (a vector is a first-order tensor & a scalar is a zero order tensor) and constant for a rigid body if computed in the body frame.

Rotational kinetic energy can also be useful in the analyzing rigid body motion. Again, it can be computed by integrating motion of discrete masses over the body, as

$$T = \int \frac{1}{2}\vec{v}\cdot\vec{v}\ dm$$

When computed in the body frame, this becomes

$$T = \frac{1}{2}\int \left(\vec{\omega}^{bi}\times\vec{r}\right)\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right)\ dm$$

Using our earlier component definitions, the cross products can be computed as

$$\begin{aligned}
T &= \frac{1}{2}\int \left\{ (\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2 \right\} dm \\
&= \frac{1}{2}\int \left\{ \begin{array}{c} \omega_2^2 z^2 - 2\omega_2\omega_3 yz + \omega_3^2 y^2 + \omega_3^2 x^2 \\ -2\omega_1\omega_3 xz + \omega_1 z^2 + \omega_1^2 y^2 - 2\omega_1\omega_2 xy + \omega_2^2 x^2 \end{array} \right\} dm \\
&= \frac{1}{2}\vec{\omega}^{bi} \cdot I\vec{\omega}^{bi}
\end{aligned}$$

Principle body axis frame

We mentioned that the inertia matrix should be computed in a body-fixed frame to ensure that it is constant. Note that there is a preferred orientation of the body frame that makes the inertia matrix even simpler. Recall that angular momentum is given by

$$\vec{H} = I_b \vec{\omega}^{bi}$$

In general, $\vec{H}$ and $\vec{\omega}^{bi}$ (angular momentum and angular velocity) do not have to be aligned. However, let's consider a case in which they are aligned; then $\vec{H}$ and $\vec{\omega}$ differ only by a scalar multiple, λ

$$\vec{H} = I_b \vec{\omega} = \lambda \vec{\omega}$$

Rearranging,

$$[I_b - \lambda I]\,\vec{\omega} = 0 \tag{1}$$

where I is a 3x3 identity matrix. This is an eigenvalue-eigenvector problem. It can be shown that for $\vec{\omega} \neq 0$, solutions only exist when

$$\det\,[I_b - \lambda I] = 0$$

This determinant yields a cubic polynomial in λ, and the three roots of the polynomial are known as the *eigenvalues of the matrix I*. By substituting one of these eigenvalues, λ_i, into eq. (1), we can solve for the $\vec{\omega}$ which solves the three simultaneous equations. The resulting direction is the called the *eigenvector related to λ_i*. By writing a unit vector in this direction and repeating for each λ_i, we can use the eigenvectors as the basis for a new body frame known as the *principal axis frame* (note that there must be three eigenvectors because I_b is positive definite). In the principal axis frame the moment of inertia matrix is diagonal with the eigenvalues on the diagonal

$$I_b = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

When the body has angular velocity about one of the principal directions (eigenvectors), the angular momentum will be aligned with the angular velocity. For example if $\vec{\omega} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, then,

$$\begin{aligned} \vec{H} &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda_3 \end{bmatrix} \\ &= \lambda_3 \vec{\omega} \end{aligned}$$

Parallel axis theorem

When calculating the moment of inertia matrix for a spacecraft with many components, it is often useful to translate the effect of a component's inertia matrix from the component center of mass to the overall center of mass. This can be done using the parallel axis theorem. As this theorem is derived in the same manner as the Inertia matrix above, it will simply be stated here

$$I_b = I_{b'} + m \begin{bmatrix} \Delta y^2 + \Delta z^2 & -\Delta x \Delta y & -\Delta z \Delta x \\ -\Delta x \Delta y & \Delta x^2 + \Delta z^2 & -\Delta y \Delta z \\ -\Delta z \Delta x & -\Delta y \Delta z & \Delta x^2 + \Delta y^2 \end{bmatrix}$$

where I_b is the MOI about the component's center of mass in a reference frame parallel to the body frame, m is the component mass, and $\Delta x, \Delta y, \Delta z$ are the distances between the component's center of mass and the body frame axes.

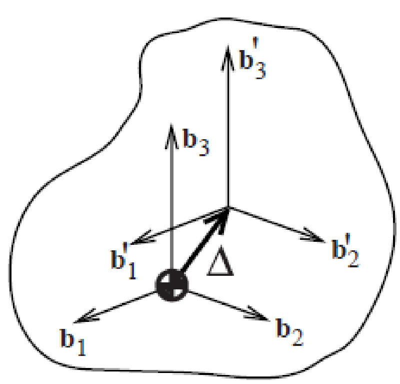


Figure 4.6 Parallel traslation of body frame axes (Wiesel)