AOE 5234 - Lesson 24
GRAVITY GRADIENT
(Read Wiesel section 5.8)

Spacecraft in orbit are very nearly free of external torques. However, even small torques which create tiny angular accelerations can affect spacecraft orientation over time. The most common environmental torques are: aerodynamic drag, solar radiation pressure, magnetic, and gravity gradient. The relative magnitude of these torques depends on the orbit and geometry of the spacecraft. For low and intermediate altitude satellites, gravity gradient can be significant. This torque can be used as a passive means of attitude control for spacecraft that require nadir (from Arabic meaning "opposite", or in celestial terms directly "below") pointing. The torque is generated by the non-uniform strength of the Earth's gravitational field which varies inversely with the square of the distance from the Earth's center.

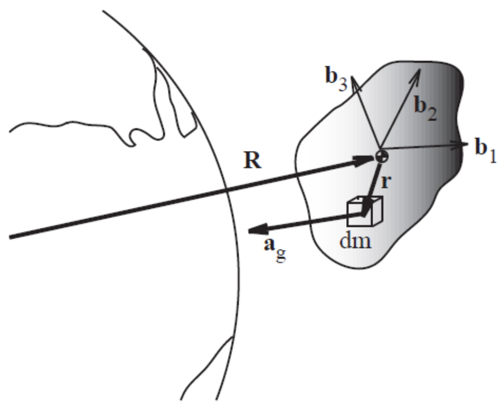

Figure 5.17 Gravity-gradient torques on a close earth satellite. (Wiesel)

The gravitational torque is given by

$$\vec{M} = \int_{body} \vec{r} \times \vec{a}_g dm \tag{1}$$

where the gravitational acceleration $\vec{a}_g$ is given by

$$\vec{a}_g = -Gm_{\oplus} \frac{\vec{R} + \vec{r}}{\left|\vec{R} + \vec{r}\right|^3} \tag{2}$$

Where $\vec{r}$ is position of a mass element dm with respect to the satellite center of mass, $\vec{R}$ is position of the satellite center of mass with respect to the Earth, G is universal gravitational constant, and $m_{\oplus}$ is mass of the Earth.

In equation (1) note that if $\vec{a}_g$ does not vary over the body, then (1) gives

$$\vec{M} = \int_{body} \vec{r} \times \vec{a}_g dm = \int_{body} (\vec{r} dm) \times \vec{a}_g = \vec{0}$$

However, if $\vec{a}_g$ does vary then the integration is somewhat more complicated. First, substitute (2) into (1), giving

$$\vec{M} = \int_{body} \vec{r} \times \vec{a}_g dm = \int_{body} \vec{r} \times \left(-Gm_{\oplus} \frac{\vec{R} + \vec{r}}{\left| \vec{R} + \vec{r} \right|^3} \right) dm$$

$$\vec{M} = -Gm_{\oplus} \int_{body} \vec{r} \times \frac{\vec{R} + \vec{r}}{\left| \vec{R} + \vec{r} \right|^3} dm \tag{3}$$

The numerator can be simplified by noting,

$$\vec{r} \times \left(\vec{R} + \vec{r} \right) = \vec{r} \times \vec{R} + \vec{r} \times \vec{r} = \vec{r} \times \vec{R} \tag{4}$$

If we introduce components in the body frame,

$$\begin{aligned} \vec{r} &= x\hat{b}_1 + y\hat{b}_2 + z\hat{b}_3 \\ \vec{R} &= X\hat{b}_1 + Y\hat{b}_2 + Z\hat{b}_3 \end{aligned}$$

Then we can compute the cross product as

$$\vec{r} \times \vec{R} = (yZ - zY)\,\hat{b}_1 + (zX - xZ)\,\hat{b}_2 + (xY - yX)\,\hat{b}_3 \tag{5}$$

Expanding the denominator of the integrand in (3),

$$\begin{aligned} \left| \vec{R} + \vec{r} \right|^{-3} &= \left(\sqrt{(X+x)^2 + (Y+y)^2 + (Z+z)^2} \right)^{-3} \\ \left| \vec{R} + \vec{r} \right|^{-3} &= \left(X^2 + Y^2 + Z^2 + x^2 + y^2 + z^2 + 2Xx + 2Yy + 2Zz \right)^{-3/2} \\ \left| \vec{R} + \vec{r} \right|^{-3} &= \left(R^2 + r^2 + 2\vec{R} \cdot \vec{r} \right)^{-3/2} \end{aligned}$$

by assuming $\left\| \vec{R} \right\| >> \left\| \vec{r} \right\|$. Then $r^2 \approx 0$,

$$\begin{aligned} \left| \vec{R} + \vec{r} \right|^{-3} &\cong \left(R^2 + 2\vec{R} \cdot \vec{r} \right)^{-3/2} \\ \left| \vec{R} + \vec{r} \right|^{-3} &\cong R^{-3} \left(1 + \frac{2\vec{R} \cdot \vec{r}}{R^2} \right)^{-3/2} \end{aligned}$$

Expressing this as binomial expansion

$$
\begin{aligned}
(1+x)^{-n} &= 1 - nx + \frac{n(n-1)x^2}{2!}... \\
\left|\vec{R}+\vec{r}\right|^{-3} &\cong R^{-3}\left(1 - \frac{3\vec{R}\cdot\vec{r}}{R^2} + \text{higher order terms}\right)
\end{aligned}
$$

Ignoring higher order terms and evaluating the dot product yields,

$$
\left|\vec{R}+\vec{r}\right|^{-3} \cong R^{-3}\left(1 - \frac{3(Xx+Yy+Zz)}{R^2}\right) \tag{6}
$$

Now equations (5) and (6) are substitued into (3); the $\hat{b}_1$ component is,

$$
\vec{M} = \frac{-Gm_\oplus}{R^3}\int_{body}\left[(yZ - zY)\,\hat{b}_1\right]\left(1 - \frac{3(Xx+Yy+Zz)}{R^2}\right)dm
$$

$$
M_1 = -\frac{Gm_\oplus}{R^3}\left\{\begin{array}{c} Z\int ydm - Y\int zdm - 3\frac{XZ}{R^2}\int xydm - 3\frac{YZ}{R^2}\int\left(y^2 - z^2\right)dm \\ -3\frac{Z^2}{R^2}\int yzdm + 3\frac{XY}{R^2}\int xzdm + 3\frac{Y^2}{R^2}\int zydm \end{array}\right\}
$$

There are similar equations for the $\hat{b}_2, \hat{b}_3$ components of $\vec{M}$. Fortunately, several simplifications are possible; since we are using a principal axis frame with origin at the center of mass, $\int ydm = \int zdm = \int xydm = \int yzdm = \int xzdm = 0$. This leaves us with a single integral

$$
\begin{aligned}
M_1 &= \frac{3Gm_\oplus}{R^5}YZ\int\left(y^2 - z^2\right)dm \\
M_1 &= \frac{3Gm_\oplus}{R^5}YZ\int\left(\left(x^2+y^2\right) - \left(x^2+z^2\right)\right)dm \\
M_1 &= \frac{3Gm_\oplus}{R^5}YZ(C-B) \qquad (7)
\end{aligned}
$$

where B, C are the moment of inertia components associated with the $\hat{b}_2, \hat{b}_3$ axes. In the same fashion, we can find

$$
\begin{aligned}
M_2 &= \frac{3Gm_\oplus}{R^5}XZ(A-C) \qquad (8) \\
M_3 &= \frac{3Gm_\oplus}{R^5}XY(B-A) \qquad (9)
\end{aligned}
$$

Now these torques can be used in Euler's equations,

$$\begin{aligned}
\text{(yaw)}\ \frac{3Gm_\oplus}{R^5} YZ\,(C-B) &= A\dot{\omega}_1 + (C-B)\,\omega_2\omega_3 \\
\text{(roll)}\ \frac{3Gm_\oplus}{R^5} XZ\,(A-C) &= B\dot{\omega}_2 + (A-C)\,\omega_1\omega_3 \\
\text{(pitch)}\ \frac{3Gm_\oplus}{R^5} XY\,(B-A) &= C\dot{\omega}_3 + (B-A)\,\omega_1\omega_2
\end{aligned} \tag{10}$$

These equations are not easily solved for $\omega_1(t), \omega_2(t)$ and $\omega_3(t)$, since the forcing functions depend on $X(t), Y(t), Z(t)$, and $R(t)$.

To analyze the dynamics further, we typically assume a circular orbit ($R(t) = R =$ constant) and then write X, Y, Z in terms of orientation angles. Then we consider equilibrium points in which the body spins about principal axis. The orientation of the principal axis with respect to the orbit normal, radial, and tangential directions is shown in the figure below.

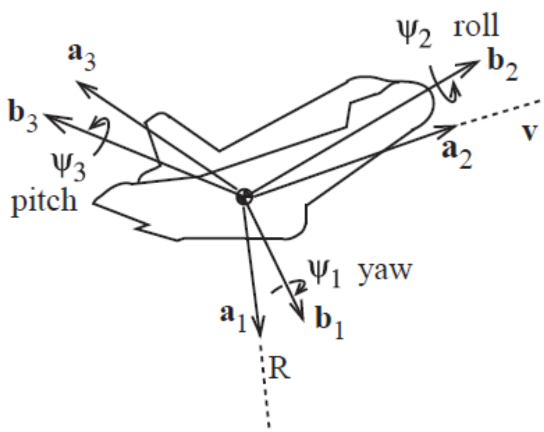

Figure 5.18 Gravity-gradient satellite and orbital reference frame. (Wiesel)

The angular velocity vector $\vec{\omega}^{bi}$ is the sum of the angular velocity vector $\vec{\omega}^{ba}$ and the angular velocity vector $\vec{\omega}^{ai}$. It is straightforward to write the $\vec{\omega}^{ai}$ vector, because it is the same as the rotation of the spacecraft about the Earth in its orbit.

$$\vec{\omega}^{ai} = \Omega\ \hat{a}_3$$

where $\Omega = \sqrt{\frac{\mu}{R^3}}$. The $\vec{\omega}^{ba}$ vector can be written as

$$\vec{\omega}^{ba} = \dot{\psi}_1 \hat{b}_1 + \dot{\psi}_2 \hat{b}_2 + \dot{\psi}_3 \hat{b}_3 \tag{11}$$

The linearized rotation matrix from $\hat{a}$ frame to $\hat{b}$ frame is,

$$
\begin{aligned}
R^{ab} &= R_1(-\psi_1)\ R_2(-\psi_2)\ R_3(-\psi_3) \\
R^{ab} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(-\psi_1) & -s(-\psi_1) \\ 0 & s(-\psi_1) & c(-\psi_1) \end{bmatrix} \begin{bmatrix} c(-\psi_2) & 0 & s(-\psi_2) \\ 0 & 1 & 0 \\ -s(-\psi_2) & 0 & c(-\psi_2) \end{bmatrix} \begin{bmatrix} c(-\psi_3) & -s(-\psi_3) & 0 \\ s(-\psi_3) & c(-\psi_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Assuming small angles and neglecting higher order terms yields,

$$
\begin{aligned}
R^{ab} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \psi_1 \\ 0 & -\psi_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\psi_2 \\ 0 & 1 & 0 \\ \psi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \psi_3 & 0 \\ -\psi_3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
R^{ab} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \psi_1 \\ 0 & -\psi_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \psi_3 & -\psi_2 \\ -\psi_3 & 1 & 0 \\ \psi_2 & 0 & 1 \end{bmatrix} \\
R^{ab} &= \begin{bmatrix} 1 & \psi_3 & -\psi_2 \\ -\psi_3 & 1 & \psi_1 \\ \psi_2 & -\psi_1 & 1 \end{bmatrix}
\end{aligned}
$$

Then $\vec{\omega}^{ai}$ in the $\hat{b}$ frame is,

$$
\vec{\omega}^{ai} = -\Omega\psi_2\hat{b}_1 + \Omega\psi_1\hat{b}_2 + \Omega\hat{b}_3 \tag{12}
$$

and $\vec{\omega}^{bi} = \vec{\omega}^{ba} + \vec{\omega}^{ai}$; written in the body frame this is

$$
\vec{\omega}^{bi} = \left(\dot{\psi}_1 - \Omega\psi_2\right)\hat{b}_1 + \left(\dot{\psi}_2 + \Omega\psi_1\right)\hat{b}_2 + \left(\dot{\psi}_3 + \Omega\right)\hat{b}_3 \tag{13}
$$

Differentiating the components of ω^{bi} in the $\hat{b}$ frame yields,

$$
\begin{aligned}
\dot{\omega}_1 &= \ddot{\psi}_1 - \Omega\dot{\psi}_2 \\
\dot{\omega}_2 &= \ddot{\psi}_2 + \dot{\psi}_1\Omega \\
\dot{\omega}_3 &= \ddot{\psi}_3
\end{aligned} \tag{14}
$$

The position of the Earth with respect to the satellite, in the body frame, is approximated by

$$
\begin{aligned}
X &\cong R \\
Y &\cong -\psi_3 R \\
Z &\cong \psi_2 R
\end{aligned} \tag{15}
$$

We can then substitute values for $X, Y, Z, \omega_1, \omega_2, \omega_3$ and $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$ from equations (13) - (15) into eq (10) to get the complete eqs of motion (recall that

$\Omega^2 = \frac{Gm_\oplus}{R^3} = \frac{\mu}{R^3}$ and that ψ_i and $\dot{\psi}_j$ are small so $\psi_i^2,\ \dot{\psi}_i^2,\ \psi_i\psi_j,\ \psi_i\dot{\psi}_j,\ and\ \dot{\psi}_i\dot{\psi}_j$ are neglected)

$$\begin{aligned}
A\ddot{\psi}_1 + (C - B - A)\,\Omega\dot{\psi}_2 + (C - B)\,\Omega^2\psi_1 &= 0 \quad \text{(yaw)} \\
B\ddot{\psi}_2 + (B + A - C)\,\Omega\dot{\psi}_1 - 4\,(A - C)\,\Omega^2\psi_2 &= 0 \quad \text{(roll)} \\
C\ddot{\psi}_3 + 3\,(B - A)\,\Omega^2\psi_3 &= 0 \quad \text{(pitch)}
\end{aligned}$$

These eqs are linear, constant coefficient ordinary differential equations, so standard methods for linear systems can be used to analyze stability. Wiesel spends some time on this, but the bottom line is that the system is stable when $C > B > A$ (there is a small stability region not covered by this, but it disappears when energy dissipation is considered). This means that the minor ("long") axis of the spacecraft needs to be aligned with the radius vector, the intermediate axis needs to point along the velocity vector, and the major axis needs to be along the orbit normal direction.