1st-Apr-20 HW4 SANDEEP K JADA

Prob 1:

The first two Eqs in ω_1 & ω_2 can be written as follow

$$\omega_1 = (\dot\phi \sin\theta)\sin\psi + \dot\theta(\cos\psi)$$
$$\omega_2 = (\dot\phi \sin\theta)\cos\psi + \dot\theta(-\sin\psi)$$

$$\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\psi & \sin\psi \\ -\sin\psi & \cos\psi\end{bmatrix}}_{\text{orthonomal Rot Matrix}}\begin{bmatrix}\dot\theta \\ \dot\phi\sin\theta\end{bmatrix}$$

the iverse in transpose

$$\therefore \begin{bmatrix}\dot\theta \\ \dot\phi\sin\theta\end{bmatrix} = \begin{bmatrix}\cos\psi & -\sin\psi \\ \sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}$$

$$\boxed{\dot\theta = \cos\psi\,\omega_1 - \sin\psi\,\omega_2}$$

$$\dot\phi\sin\theta = \sin\psi\,\omega_1 + \cos\psi\,\omega_2$$

$$\boxed{\dot\phi = \frac{\sin\psi}{\sin\theta}\omega_1 + \frac{\cos\psi}{\sin\theta}\omega_2}$$

Singularity when $\theta = 0 \rightarrow \sin\theta \rightarrow 0$

From the third Eq

$$\dot\psi = \omega_3 - \dot\phi\cos\theta$$

$$\boxed{\dot\psi = \omega_3 - \frac{\sin\psi}{\tan\theta}\omega_1 - \frac{\cos\psi}{\tan\theta}\omega_2}$$

Also affected by singularity at $\theta = 0$

Prob 2: The KE of the Element dm is

$$dT = \frac{1}{2}\,\vec{v}\cdot\vec{v}\,dm$$

$$\int_{body} dT = T = \frac{1}{2}\int_{body}\vec{v}\cdot\vec{v}\,dm \qquad \Big|\ \text{where } \int dm = M$$

$$\vec{v} = \vec{V}_{COM} + \vec{\omega}^{bi}\times\vec{r}$$

$$\vec{v}\cdot\vec{v} = \left(\vec{V}_{COM} + \vec{\omega}^{bi}\times\vec{r}\right)\cdot\left(\vec{V}_{COM} + \vec{\omega}^{bi}\times\vec{r}\right)$$

$$= \vec{V}_{COM}\cdot\vec{V}_{COM} + 2\,\vec{V}_{COM}\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right) + \left(\vec{\omega}^{bi}\times\vec{r}\right)\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right)$$

$$\therefore\quad T = \frac{1}{2}\int \vec{V}_{COM}\cdot\vec{V}_{COM}\,dm + \int \vec{V}_{COM}\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right)dM + \frac{1}{2}\int\left(\vec{\omega}^{bi}\times\vec{r}\right)\cdot\left(\vec{\omega}^{bi}\times\vec{r}\right)dm$$

V_{COM} & $\vec{\omega}^{bi}$ are not functions of positions on Rigid body

$$T = \frac{1}{2}\,\vec{V}_{COM}\cdot\vec{V}_{COM}\int dm + \vec{V}_{COM}\cdot\left(\vec{\omega}^{bi}\times\int\vec{r}\,dm\right) + \text{''}$$

$$T = \frac{1}{2}M\left(\vec{V}_{COM}\cdot\vec{V}_{COM}\right) + \underbrace{\cancelto{0}{\vec{V}_{COM}\cdot\left(\vec{\omega}^{bi}\times\vec{0}\right)}}_{\substack{\text{the Integral Evaluates to the definition}\\ \text{of COM, as } \vec{r} \text{ is dist}\\ \text{from COM, it is simply, } \vec{0}}} + \underbrace{\frac{1}{2}\vec{\omega}^{bi}\cdot I\,\vec{\omega}^{bi}}_{\substack{\text{from lecture notes}\\ \text{def of } \vec{I}}}$$

$$\boxed{T = \frac{1}{2}M\left(\vec{V}_{COM}\cdot\vec{V}_{COM}\right) + \frac{1}{2}\vec{\omega}^{bi}\cdot I\,\vec{\omega}^{bi}}\ \blacksquare$$

Prob 3. a

$$I^{Principal} = \begin{bmatrix} 1.29 & & 0 \\ & 9.68 & \\ 0 & & 10.10 \end{bmatrix} \times 10^6 \, Kg\text{-}m^2 = \begin{bmatrix} A & & \\ 0 & B & 0 \\ & & C \end{bmatrix} \begin{matrix} \rightarrow \text{about } b_1 \\ \rightarrow \text{about } b_2 \\ \rightarrow \text{about } b_3 \end{matrix}$$

The stable configuration finally settles to $C > B > A$ (Already true in our case), so the axes, b_1, b_2, & b_3 will be aligned along.

$b_1 \rightarrow$ (YAW) $\rightarrow$ aligned with radius vector.
$b_2 \rightarrow$ (ROLL) $\rightarrow$ aligned with velocity vector
$b_3 \rightarrow$ (PITCH) $\rightarrow$ in the direction of orbit normal

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Prob 3. b

The Eqs of Motion for a body in orbit under Torque-Free Gravity Gradient are

$$A\ddot{\psi}_1 + (C-B-A)\Omega\dot{\psi}_2 + (C-B)\Omega^2\psi_1 = 0 \quad \text{(Yaw)}$$

$$B\ddot{\psi}_2 + (B+A-C)\Omega\dot{\psi}_1 + 4(C-A)\Omega^2\psi_2 = 0 \quad \text{(Roll)}$$

$$C\ddot{\psi}_3 + 3(B-A)\Omega^2\psi_3 = 0 \quad \text{(Pitch)}$$

where ψ_1, ψ_2, ψ_3 are rotations about axes b_1, b_2, b_3 respectively

Where, $\Omega_L^2 = \frac{\mu}{R^3}$, Can be computed from the time period of the orbit as follows

$$\Omega_L = 2\pi f = \frac{2\pi}{T} \rightarrow T = 90 \text{ min}$$

To find frequencies we need to find the System Poles (or Eigenvalues, or roots of charact-eristic Eq)

1) <u>For Pitch:</u> it is a Spring-Mass System form like $m\ddot{x} + kx = 0$ where

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$\omega_{n,pitch} = \Omega_L \sqrt{\frac{3(B-A)}{C}} = 0.001836 \text{ Rad/sec}$$

2) For Roll & Yaw its a Coupled System of two 2nd order Linear ODE'S, Can be Written is State-Space representation of $\dot{X} = AX$, where

$X^T = [\psi_1 \ \psi_2 \ \dot{\psi}_1 \ \dot{\psi}_2]$, then A becomes

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(C-B)}{A}\Omega_L^2 & 0 & 0 & -\frac{(C-B-A)}{A}\Omega_L \\ 0 & -\frac{4(C-A)}{B}\Omega_L^2 & -\frac{(B+A-C)}{B}\Omega_L & 0 \end{bmatrix}$$

The poles of the Roll & Yaw System are Eigenvalues of A (the Imaginary part of it is the freq)

Eg. $\lambda_{1,2} = -\frac{1}{T} \pm i\,\omega_d$

Real part determines the stability of the System

damped Natural freq

The Eigenvalues of A

$\lambda_{1,2} = \pm i\,6.5794 \times 10^{-4}$ rad/sec

$\lambda_{3,4} = \pm i\,0.002240$ rad/sec

So the Final three frequencies are

$6.5794e^{-4}$ Rad/sec
0.00224023 Rad/sec } Roll & Yaw
0.00183682 Rad/sec → Pitch

The numerical results are Evaluated using MATLAB Script that is attached within the pdf Document.