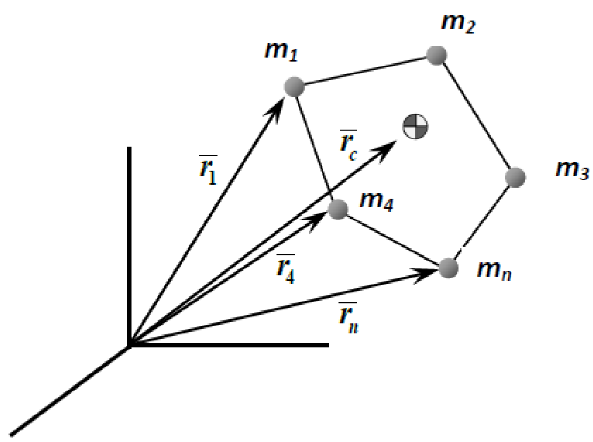
AOE 5234 - Lesson 17
DYNAMICS OF A RIGID BODY
(Read Wiesel sections 4.1 thru 4.3)

Up to this point, spacecraft have been treated as point masses. Consequently, only the motion of a satellite's center of mass was considered as it orbited the Earth. Today we begin adressing the orientation of the vehicle about its center of mass by deriving the rigid body equations of motion.

A rigid body has 6 Degrees Of Freedom (DOF) - 3 translational and 3 rotational. We can think of any rigid body as a system of particles connected in such a way that they remain fixed with respect to each other,



The translational equations of motion can be fround by summing the forces on each particle,

$$\vec{F}_i = \vec{f}_{ie} + \sum_{j \neq i}^{N} \vec{f}_{ij} = m_i \vec{a}_i$$

where $\vec{f}_{ie}$ is the external force on particle i, and $\vec{f}_{ij}$ are the internal forces (forces of other particles) on particle i. Summing over all N particles,

$$\sum_{i=1}^{N} \vec{f}_{ie} + \sum_{i=1}^{N} \sum_{j \neq i}^{N} \vec{f}_{ij} = \sum_{i=1}^{N} m_i \vec{a}_i$$

Newton's third law (every reaction has an equal and opposite reaction) guarantees that for every internal force $\vec{f}_{ij}$ there is another force $-\vec{f}_{ji}$ that exactly cancels it. So we are left with

$$\vec{F}_e = \sum_{i=1}^{N} \vec{f}_{ie} = \sum_{i=1}^{N} m_i \vec{a}_i \tag{1}$$

If we define a position $\vec{r}_c$ as the *center of mass* by ,

$$\vec{r}_c = \frac{1}{M_T}\sum_{i=1}^{N} m_i \vec{r}_i$$

Differentiating twice yields,

$$M_T \frac{d^2\vec{r}_c}{dt^2} = \sum_{i=1}^{N} m_i \frac{d^2}{dt^2}\vec{r}_i = \sum_{i=1}^{N} m_i \vec{a}_i$$

plugging this into equation (1),

$$\vec{F}_e = M_T \frac{d^2\vec{r}_c}{dt^2}$$

In other words, the sum of the external forces equals the total rigid body mass times the acceleration of its center of mass; this is Newtons 2^{nd} law for a fixed mass system. When the external force happens to be the two body gravitational force, we obtain the three translational equations of motion we have been investigating thus far,

$$\begin{aligned} \frac{\vec{F}_e}{M_T} &= -\frac{\mu}{r^3}\vec{r} = \frac{d^2\vec{r}_c}{dt^2} \\ \ddot{\vec{r}} &= -\frac{\mu}{r^3}\vec{r} \end{aligned}$$

The rotational equations of motion can be found with the rotational analagy of Netwon's 2nd law,

$$\vec{M} = \frac{{}^{\hat{\imath}}d}{dt}\vec{H}$$

where $\vec{M}$ is the applied torque and $\vec{H}$ is the angular momentum of the rigid body. However, torques (moments) and angular momentum must be computed with respect to some point. To choose a convenient point, let's begin by taking a completely arbitraty point O, possibly moving in inertial space, as shown in the figure below

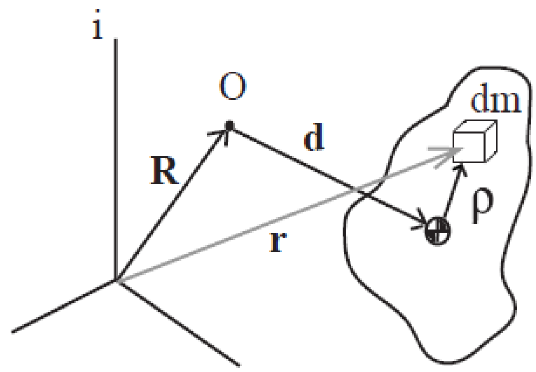


Figure 4.1 Calculation of angular momentum (Wiesel)

We assume each mass element, dm, in a rigid body has an elemental force $d\vec{F}$ applied to it, and that $d\vec{F} = dm\ddot{\vec{r}}$. Taking moments about point O,

$$\vec{M}^o = \int_{body} \left(\vec{d} + \vec{\rho}\right) \times d\vec{F} = \int_{body} \left(\vec{d} + \vec{\rho}\right) \times \ddot{\vec{r}}\, dm \tag{2}$$

Now from the figure, we see

$$\vec{r} = \vec{R} + \vec{d} + \vec{\rho}$$

so

$$\ddot{\vec{r}} = \ddot{\vec{R}} + \ddot{\vec{d}} + \ddot{\vec{\rho}}$$

Then we can expand eq (2) as

$$\vec{M}^o = \int_{body} \left(\vec{d} + \vec{\rho}\right) \times \left(\ddot{\vec{R}} + \ddot{\vec{d}} + \ddot{\vec{\rho}}\right)\ dm$$

$$\vec{M}^o = \int_{body} \left(\vec{d} + \vec{\rho}\right) \times \left(\ddot{\vec{d}} + \ddot{\vec{\rho}}\right) dm + \int_{body} \vec{d} \times \ddot{\vec{R}}\ dm + \int_{body} \vec{\rho} \times \ddot{\vec{R}}\ dm \tag{3}$$

The first term in (3) is the rate of change of angular momentum about O,

$$\begin{aligned}
\frac{d}{dt}\vec{H}^o &= \frac{d}{dt}\left[\int \left(\vec{d} + \vec{\rho}\right) \times \left(\dot{\vec{d}} + \dot{\vec{\rho}}\right)\ dm\right] \\
\frac{d}{dt}\vec{H}^o &= \int \left[\left(\dot{\vec{d}} + \dot{\vec{\rho}}\right) \times \left(\dot{\vec{d}} + \dot{\vec{\rho}}\right) + \left(\vec{d} + \vec{\rho}\right) \times \left(\ddot{\vec{d}} + \ddot{\vec{\rho}}\right)\right]\ dm \\
\frac{d}{dt}\vec{H}^o &= \int \left[\left(\vec{d} + \vec{\rho}\right) \times \left(\ddot{\vec{d}} + \ddot{\vec{\rho}}\right)\right]\ dm
\end{aligned}$$

The third term of eq. (3) is $\int \vec{\rho} dm = \vec{0}$; this is a result of $\vec{\rho}$ being measured from the the center of mass,

$$\int \vec{\rho} \times \ddot{\vec{R}} dm = \int \vec{\rho} dm \times \ddot{\vec{R}} = \vec{0} \times \ddot{\vec{R}} = \vec{0}$$

We can easily integrate the second term of eq. (3) since neither $\vec{d}$ nor $\vec{R}$ varies with the mass element, so

$$\int \vec{d} \times \ddot{\vec{R}}\, dm = \vec{d} \times \ddot{\vec{R}} \int dm = M_T \left(\vec{d} \times \ddot{\vec{R}} \right)$$

Finally we can rewrite eq. (3) as

$$\vec{M}^o = \frac{{}^{\hat{\imath}}d}{dt} \vec{H}^o + M_T \left(\vec{d} \times \ddot{\vec{R}} \right) \tag{4}$$

The last term of (4) can be made zero by correctly choosing the location of point O. For example, O can be fixed in inertial space (i.e. an inertial reference); then $\vec{R}$ is constant and $\ddot{\vec{R}} = \vec{0}$. Similarly, O could be selected coincident with the center of mass, then $\vec{d} = \vec{0}$. Therefore, *we will normally choose an origin that is inertial or located at the center of mass*, and write

$$\vec{M}^o = \frac{{}^{\hat{\imath}}d}{dt} \vec{H}^o$$

If we assume point O is the center of mass and separate the forces into their internal and external components, eq. (2) becomes

$$\vec{M}^o = \int \vec{\rho} \times d\vec{F} = \int \vec{\rho} \times d\vec{f}_{ext} + \int \vec{\rho} \times d\vec{f}_{int}$$

Since all internal forces cancel the second term vanishes, and we are left with

$$\vec{M}^o = \int \vec{\rho} \times d\vec{F}_{ext}$$