# AOE 5234 - Lesson 20
# DYNAMICS OF A RIGID BODY - ORIENTATION ANGLES

(Read Wiesel section 4.7)

Last lesson we saw that Euler's Equations are differential equations which describe angular velocity in the body frame. The soluction to these first order ODE's yields the body's angular velocity as a function of time; however, they do not tell us the body's orientation with respect to inertial space. To determine changes in orientation of a body requires a set of differential equations that relates inertial orientation to $\vec{\omega}^{bi}$.

It takes three angles (called Euler angles) to define orientation in three dimensional space. Clearly, there are as many different ways to define these angles as there are problems to solve. For example, all of you are familiar with $\Omega, i, and\ \omega$ which define the orientation of an orbit in inertial space. In this discussion we will use a generic set of Euler angles $\phi$, $\theta$, and $\psi$. In this system, one moves from the $\hat{\imath}$ frame to the $\hat{b}$ frame by rotating about the 3 axis (the $\hat{\imath}_3$ axis) by $\phi$ , then rotating around the new 1 axis (we'll call it the $\hat{n}$ axis) by $\theta$, and finally rotating about the new 3 axis (we'll call it the $\hat{b}_3$) by $\psi$ (these are also called the 3-1-3 Euler angles, see the figure below).

$$R^{ib} = \left(R^{bi}\right)^T = R_3^T\left(\psi\right) R_1^T\left(\theta\right) R_3^T\left(\phi\right)$$

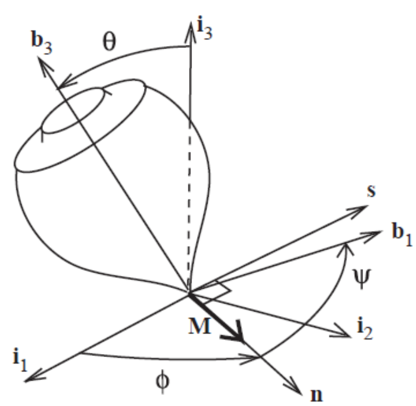


Figure 4.10 The simple top. (Wiesel)

Substituting the transpose of the rotation matricies from lesson 1,

$$R^{ib} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{ib} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi\cos\theta & \cos\phi\cos\theta & \sin\theta \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix}$$

$$R^{ib} = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix}$$

To relate the angular velocity in the body frame to the time derivatives of the Euler angles $(\dot{\phi}, \dot{\theta}, \dot{\psi})$, we must express the angular velocity in terms of the Euler angles.

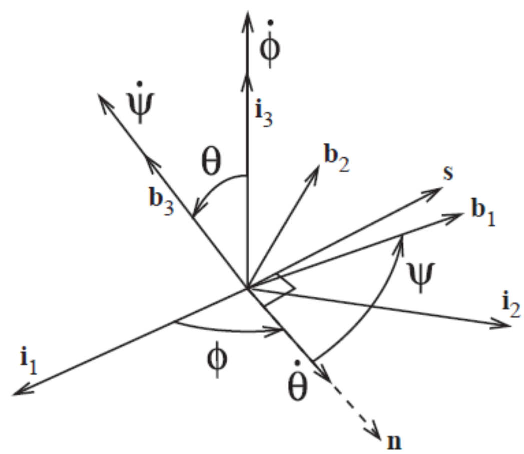


Figure 4.8 The classical Euler angles. (Wiesel)

As we've seen, $\phi$ is a rotation about the $\hat{\imath}_3$ axis, $\theta$ is about the $\hat{n}$ axis, and $\psi$ is about the $\hat{b}_3$ axis. So the angular velocity of the $\hat{b}$ frame with respect to the $\hat{\imath}$ frame can be expressed as,

$$\vec{\omega}^{bi} = \dot{\phi}\hat{\imath}_3 + \dot{\theta}\hat{n} + \dot{\psi}\hat{b}_3$$

Clearly this expression is inadequate for our purposes since only one of the rotations is about a body axis. Therefore, $\hat{n}$ and $\hat{\imath}_3$ must be written in the $\hat{b}$ frame. From fig 4.8 we can see that,

$$\hat{n} = \cos\psi\hat{b}_1 - \sin\psi\hat{b}_2$$

Similarly, we could find $\hat{\imath}_3$ in the $\hat{b}$ frame by looking at figure 4.8. As this is a bit more difficult to visualize, we will simply multiply $\hat{\imath}_3$ by $R^{ib}$ above,

$$[\hat{\imath}_3]_b = R^{ib}\,[\hat{\imath}_3]_{\hat{\imath}} = R^{ib}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{bmatrix}$$

Now we can write the angular velocity of the $\hat{b}$ frame in Euler angles,

$$\vec{\omega}^{bi} = \dot{\phi}\begin{bmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{bmatrix} + \dot{\theta}\begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} + \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or,

$$\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi && (1) \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi && (2) \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi} && (3) \end{aligned}$$

These equations are the *velocity kinematics* of rotational motion and along with the Euler equations complete our rigid body equations of motion. The $\omega_i$ are found using the Euler Equations and then the three coupled nonlinear differential equations above are solved for the Euler angles (note that it is possible to have the torque components ($M_i$'s) be functions of inertial orientation; if this is the case all 6 equations must be solved simultaneously). Note also that these equations depend on how one defines the orientation of the object and are therefore not unique (in this case we used the 3-1-3 Euler angles). Wiesel also developed these expressions for the yaw-pitch-roll system.

Finally, it is important to talk about times when these equations have no clear solution; we call this a singularity. In equations (1)-(3) this singularity occurs at $\theta = 0^0$ *or* $180^0$. For example, if $\theta = 0$ then it is difficult to determine what amount of the rotation about the $\hat{\imath}_3 = \hat{b}_3$ axis is attributed to $\phi$ and what is attributed to $\psi$. Mathematically, the equations for the Euler angles at $\theta = 0$ are

$$\begin{aligned} \omega_1 &= \dot{\theta}\cos\psi \\ \omega_2 &= -\dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi} + \dot{\psi} \end{aligned}$$

Setting the $\dot{\theta}$'s equal to each other in the first two equations yields,

$$\dot{\theta} = \frac{\omega_1}{\cos\psi} = \frac{-\omega_2}{\sin\psi}$$

as $\omega_1$ and $\omega_2$ are independent of each other, this equation may not have a solution.