AOE 5234 - Lesson 23
TORQUE-FREE RIGID BODY
(Read Wiesel sections 5.1, 5.2, 5.3)

A common passive means of stabilizing spacecraft attitude is to spin the body about a principal axis. In a principal axis spin with no external torques, Euler's Equations show that the motion should be in equilibrium. That is to say that the angular velocity vector will be constant in the body frame and aligned with the angular momentum vector. The important question from a control perspective is whether the equilibrium point is stable.

Stability can be defined in many ways, but two common definitions are:

Asymptotic Stability - When a system in equilibrium is perturbed slightly, it moves back to the equilibrium condition as time goes to infinity.

Lyapunov Stability - When a system in equilibrium is perturbed slightly, it stays in the neighborhood of the equilibrium point as time goes to infinity.

We will examine the principal axis spin motion for a torque free rigid body, and see that spin about the major and minor axes are in fact Lyapunov stable. This means if we spin the spacecraft about either the largest or smallest principal axis, the spacecraft will remain near this spin state for a long time, even when subject to small disturbing torques. This clearly will reduce the fuel required to maintain vehicle attitude.

To analyze stability, consider a vehicle with no external torques. Euler's equations in a principal axis frame are,

$$\begin{aligned} M_1 &= 0 = A\dot{\omega}_1 + (C - B)\,\omega_2\omega_3 \\ M_2 &= 0 = B\dot{\omega}_2 + (A - C)\,\omega_1\omega_3 \\ M_3 &= 0 = C\dot{\omega}_3 + (B - A)\,\omega_1\omega_2 \end{aligned}$$

Note that for any principle axis spin, we have an equilibrium condition. That is, if we substitute any of the following conditions into the Euler's equations all of the $\dot{\omega}_i$'s are zero, so the initial condition will not change.

$$\begin{aligned} \omega_2 = \omega_3 = 0 \quad \omega_1 = \omega_{1_0} \\ \omega_1 = \omega_3 = 0 \quad \omega_2 = \omega_{2_0} \\ \omega_1 = \omega_2 = 0 \quad \omega_3 = \omega_{3_0} \end{aligned}$$

Now consider (without loss of generality) the first case, which is a perfect spin about axis one. Then $\omega_1 = \omega_{1_0}$ (constant) and $\omega_2 = \omega_3 = 0$, and a perturbed condition could be written as

$$\begin{aligned}
\omega_1 &= \omega_{1_0} + \delta\omega_1 (t) \\
\omega_2 &= \delta\omega_2 (t) \\
\omega_3 &= \delta\omega_3 (t)
\end{aligned}$$

where $\delta\omega_1, \delta\omega_2, \delta\omega_3$ are small perturbations to the equilibrium states. Then Euler's equations give

$$\begin{aligned}
A\delta\dot{\omega}_1 + (C - B)\,\delta\omega_2\delta\omega_3 &= 0 \\
B\delta\dot{\omega}_2 + (A - C)\,(\delta\omega_1 + \omega_{1_0})\,\delta\omega_3 &= 0 \\
C\delta\dot{\omega}_3 + (B - A)\,(\delta\omega_1 + \omega_{1_0})\,\delta\omega_2 &= 0
\end{aligned} \tag{1}$$

If we linearize these equations by dropping all of the higher order terms (i.e., assume δ^2 terms are very, very, small), we have

$$A\delta\dot{\omega}_1 = 0 \tag{2a}$$
$$B\delta\dot{\omega}_2 + (A - C)\,\omega_{1_0}\delta\omega_3 = 0 \tag{2b}$$
$$C\delta\dot{\omega}_3 + (B - A)\,\omega_{1_0}\delta\omega_2 = 0 \tag{2c}$$

The first equation integrates immediately,

$$\delta\omega_1 = \delta\omega_{1_0} = \text{constant} \tag{3}$$

so whatever the initial perturbation of ω_1 is, it stays constant! Solving the next two equations

$$\delta\dot{\omega}_2 = \frac{C - A}{B}\omega_{1_0}\delta\omega_3 \tag{4a}$$
$$\delta\dot{\omega}_3 = \frac{A - B}{C}\omega_{1_0}\delta\omega_2 \tag{4b}$$

Now if we differentiate (2b) and (2c), we have

$$\delta\ddot{\omega}_2 + \frac{A - C}{B}\omega_{1_0}\delta\dot{\omega}_3 = 0 \tag{5a}$$
$$\delta\ddot{\omega}_3 + \frac{B - A}{C}\omega_{1_0}\delta\dot{\omega}_2 = 0 \tag{5b}$$

substituting (4a), (4b) into (5a), (5b)

$$\begin{aligned}
\delta\ddot{\omega}_2 + \frac{(A - C)\,(A - B)}{BC}\omega_{1_0}^2\delta\omega_2 &= 0 \\
\delta\ddot{\omega}_3 + \frac{(B - A)\,(C - A)}{BC}\omega_{1_0}^2\delta\omega_3 &= 0
\end{aligned}$$

Define $\alpha \equiv \frac{(A-C)(A-B)}{BC}\omega_{1_0}^2$ and these become

$$\begin{aligned}
\delta\ddot{\omega}_2 + \alpha\delta\omega_2 &= 0 \\
\delta\ddot{\omega}_3 + \alpha\delta\omega_3 &= 0
\end{aligned}$$

These are uncoupled harmonic oscillators with a frequency $\sqrt{\alpha}$, provided $\alpha > 0$; therefore, the solution for $\delta\omega_2(t), \delta\omega_3(t)$ is a sinusoid with amplitude dependent on $\delta\omega_2(0), \delta\omega_3(0)$ (initial perturbation); The motion is stable in the sense of Lyapunov, i.e. $\delta\omega_2(t), \delta\omega_3(t)$ will stay small if they start small. But this is true only if $\alpha > 0$. Under what condition is this condition satisfied? Careful examination of the definition of α reveals that $\alpha > 0$ if

$$A > B \text{ and } A > C \quad \Rightarrow A \text{ is the major axis}$$

or if

$$A < B \text{ and } A < C \quad \Rightarrow A \text{ is the minor axis}$$

Therefore, if A is the intermediate axis ($B < A < C$ or $C < A < B$), then $\alpha < 0$, and the equilibrium point is unstable.

Dr. Wiesel's text shows this concept in figures containting sets of closed curves called polhodes. Each closed curve describes the motion of the ω_{bi} vector after a disturbance torque is applied (note: in these figures A>B>C)

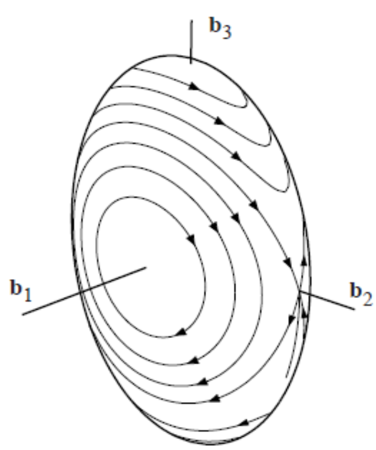


Figure 5.5 The complete set of polhodes. (Wiesel)

Notice that a small disturbance from rotation about the $\widehat{b}_1$ or $\widehat{b}_3$ axis (major and minor axis of inertia respectively) will result in closed path motion of the ω_{bi} vector in the vicinity of the original rotation axis; rigid body rotation about either of these axis is stable. However, a small disturbance from rotation about the $\widehat{b}_2$ axis (the intermediate axis of inertia) will place ω_{bi} on a polhode which departs from the intermediate axis and results in a large displacement; rigid body rotation about this axis is unstable.

It can be shown for bodies with energy dissipation (such as flexible bodies, bodies with sloshing fuel, or bodies with moving parts) the minor axis is also unstable. As is visible in the figure below, deviations in rotation about the $\widehat{b}_3$ axis (minor axis) will slowly move toward rotation about the $\widehat{b}_1$ axis (major axis). This phenomina is called polhode drift and is the direct result of a system seeking the minimum rotational kinetic energy state.

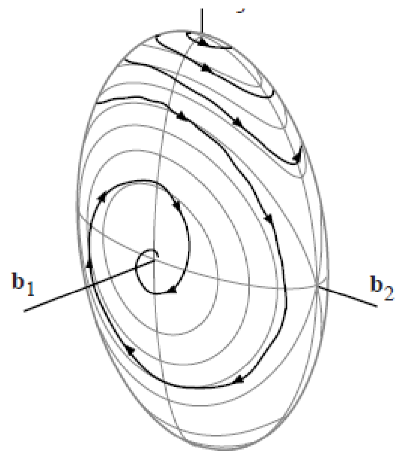
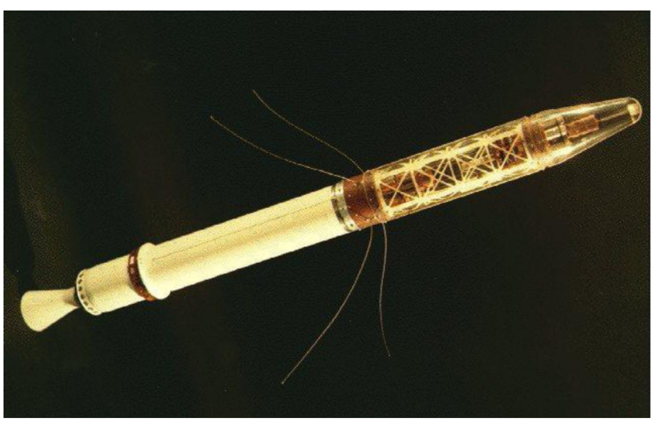


Figure 5.8 Polhode drift for a semi rigid body. (Wiesel)

One of the more famous examples of this phenominon was when the first US satellite Explorer 1 was launched as a minor axis spinner. Unfortunately, its flexible antennas added energy dissipation to the system and Explorer 1 rapidly transitioned to a major axis spin.



Explorer 1(http://en.wikipedia.org/wiki/Explorer_1)

Later satellites were designed as major axis spinners with intentional damping. Then the spacecraft would exibit both *Lyapunov and Asymptotic Stability,* see figures below (note here C>B>A, i.e. $\widehat{b}_3$ is the major axis of inertia; this is called an oblate satellite)

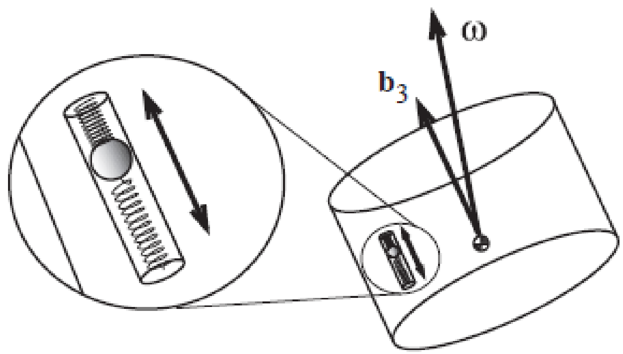


Figure 5.9 An oblate satellite with a nutation damper. (Wiesel)