AOE 5234 - Lesson 19
DYNAMICS OF A RIGID BODY - EULER EQUATIONS (LSN 19)

(read Wiesel section 4.6)

Although we have discussed the effect of external torque on a rigid body, we haven't yet written explicit equations of motion for rigid body rotation. Such equations are generally divided into two parts. The first part is 3 first order equations that describe the rate of change of angular velocity in the body frame (Eulers's Equations). The second part is 3 first order equations that describe how the ineirital orientation changes based on this angular velocity (we will be discuss this in a later lesson).

Recall that

$$\vec{M} = \frac{{}^i d}{dt}\vec{H} \tag{1}$$

Where the inertial derivative of $\vec{H}$ in the body frame is

$$\frac{{}^i d}{dt}\vec{H} = \frac{{}^b d}{dt}\vec{H} \ + \ \vec{\omega}^{bi} \times \vec{H}_b$$

Also remember that angular momentum, $\vec{H}$, is given by

$$\begin{aligned} \vec{H} &= I\,\vec{\omega}^{bi} \\ \vec{H}_b &= I_b\,\left[\vec{\omega}^{bi}\right]_b \end{aligned}$$

where I is the moment of inertia matrix.

Now equation (1) can be written in the body frame,

$$\vec{M}_b = \frac{{}^b d}{dt}\left\{I_b\ \left[\vec{\omega}^{bi}\right]_b\right\} + \left[\vec{\omega}^{bi}\right]_b \times I_b\ \left[\vec{\omega}^{bi}\right]_b \tag{2}$$

where

$$\left[\vec{\omega}^{bi}\right]_b = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

And, if we use a principal axis frame, I_b is simply

$$
I_b = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}
$$

Noting that I_b is constant, then (2) becomes

$$
\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} A\dot{\omega}_1 \\ B\dot{\omega}_2 \\ C\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} A\omega_1 \\ B\omega_2 \\ C\omega_3 \end{bmatrix}
$$

$$
\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} A\dot{\omega}_1 \\ B\dot{\omega}_2 \\ C\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} (C-B)\,\omega_2\omega_3 \\ (A-C)\,\omega_1\omega_3 \\ (B-A)\,\omega_1\omega_2 \end{bmatrix}
$$

This gives us three scalar equations

$$
\begin{aligned}
M_1 &= A\dot{\omega}_1 + (C-B)\,\omega_2\omega_3 \\
M_2 &= B\dot{\omega}_2 + (A-C)\,\omega_1\omega_3 \\
M_3 &= C\dot{\omega}_3 + (B-A)\,\omega_1\omega_2
\end{aligned}
$$

These three are known as Euler's Equations. They are coupled, nonlinear, first order ODE's. They make up one-half of the rotation equations of motion for a rigid body. We won't try to solve them here, but we will apply them to a specific situation.

Suppose we have an axisymmetric rigid body, therefore $A = B \neq C$, with no applied moment.

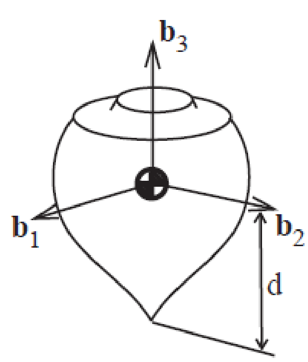


Figure 4.6 Pivot moment of inertia calculation for a top. (Wiesel)

Then Euler's Equations simplify to,

$$\begin{aligned} 0 &= A\dot{\omega}_1 + (C - A)\,\omega_2\omega_3 \qquad (3)\\ 0 &= A\dot{\omega}_2 + (A - C)\,\omega_1\omega_3 \qquad (4)\\ 0 &= C\dot{\omega}_3 \end{aligned}$$

We can integrate the last equation immediately,

$$\omega_3 = \text{constant} = \omega_{3_0}$$

The remaining equations are coupled, linear ODE's and can be solved by various methods. If we define a parameter λ by

$$\lambda \equiv \omega_{3_0}\left(\frac{C - A}{A}\right)$$

then eqs. (3) and (4) become

$$\begin{aligned} 0 &= \dot{\omega}_1 + \lambda\omega_2\\ 0 &= \dot{\omega}_2 - \lambda\omega_1 \end{aligned}$$

Multiply the first by ω_1 and the second by ω_2 and sum to get

$$0 = \omega_1\dot{\omega}_1 + \lambda\omega_1\omega_2 + \omega_2\dot{\omega}_2 - \lambda\omega_1\omega_2$$

$$0 = \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2$$

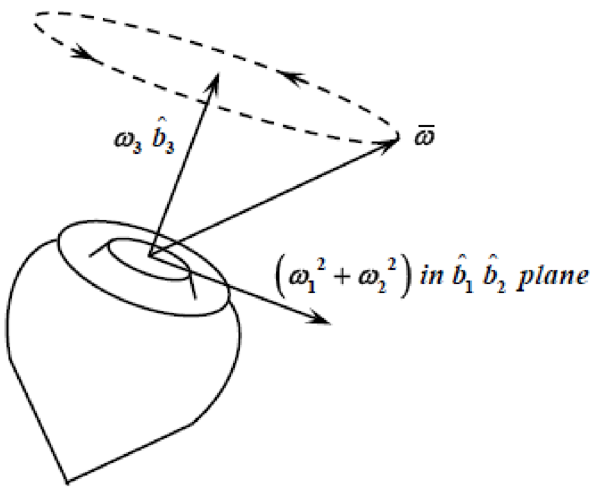
The right-hand side is a perfect derivative which then gives us a constant of the motion

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\left(\omega_1^2 + \omega_2^2\right) &= \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0\\ \frac{1}{2}\left(\omega_1^2 + \omega_2^2\right) &= \text{constant} \equiv \omega_{12} \end{aligned}$$

and since ω_3 is also constant, we see that the magnitude of the angular velocity vector $\vec{\omega}$ must also be constant in the body frame,

$$
\begin{aligned}
\|\vec{\omega}\|^2 &= \omega_1^2 + \omega_2^2 + \omega_3^2 \\
&= \omega_{12}^2 + \omega_{3_0}^2 = \text{constant}
\end{aligned}
$$

Notice that for this axisymmetric body both ω_3 and the sum $\left(\omega_1^2 + \omega_2^2\right)$ are constants. This does not mean that the $\vec{\omega}$ vector must remain fixed in space; it can in fact move as long as these two quantities remain constant. As $\left(\omega_1^2 + \omega_2^2\right)$ = constant describes a circle, the motion of the $\vec{\omega}$ vector looks like,



This in the coning motion that you often see in a spinning symmetric body like a spinning top.