AOE 6744: Assingment 4

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1 Problem 1

To be begin with the solution of the system, we need to analyze the cost function,

$$V = \lim_{x \to \infty} E \left\{ \left(\frac{x_1}{0.01} \right)^2 + \left(\frac{x_2}{0.1} \right)^2 + \left(\frac{\dot{x}_3}{1} \right)^2 \right\} \tag{1}$$

The variable $\dot{x}_3$ is a function of X, the state and u the control. The expression of $\dot{x}_3$ is as follows. (Note that there is not process noise in $\dot{x}_3$)

$$\dot{x}_3 = R_3 A X + b_3 u \tag{2}$$

Where, R_3 is a row vector [0 0 1 0]. And b_3 is the third element of the B matrix. Not that all the terms in the eq 2 are scalar. Now we define a new control $\bar{u}$ as follows

$$\bar{u} = u + \frac{R_3 A X}{b_3} \tag{3}$$

So we write $\dot{x}_3^2$ and u as

$$\dot{x}_3^2 = b_3^2 \bar{u}^2 \tag{4}$$

$$u = \bar{u} - \frac{R_3 A X}{b_3} \tag{5}$$

We define the matrices; Q,R and L as follows.

$$Q = \begin{bmatrix} 1e4 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R = b_3^2 \tag{6}$$

Substituting eq 4 and eq 6 in the cost in eq 1.

$$V = \lim_{x \to \infty} E \left\{ X^T Q X + R \bar{u}^2 \right\} \tag{7}$$

The given system dynamics in the problem can be re written as,

$$\dot{X} = AX + Bu + \Gamma w \tag{8}$$

We substitute the control definition in terms of $\bar{u}$ in eq 5 in the above state equation

$$\begin{aligned}
\dot{X} &= AX + B\left(\bar{u} - \frac{R_3 AX}{b_3}\right) + \Gamma w \\
&= AX + B\bar{u} - B\left(\frac{R_3 A}{b_3}\right) X + \Gamma w \\
&= AX - B\left(\frac{R_3 A}{b_3}\right) X + B\bar{u} + \Gamma w \\
&= \left(I - \frac{BR_3}{b_3}\right) AX + B\bar{u} + \Gamma w
\end{aligned}$$

$$\dot{X} = \bar{A}X + B\bar{u} + \Gamma w \tag{9}$$

With, $\bar{A} = \left(I - \frac{BR_3}{b_3}\right)$.

1.1 Simulation without noise

A Kalman-Bucy filter was implemented to solve the above problem. This give rise to the dual system, LQG, augmented system as follows.

$$X_{aug} = \begin{bmatrix} X \\ E \end{bmatrix}, A_{aug} = \begin{bmatrix} A - BK & BK \\ 0 & A - GC \end{bmatrix} \tag{10}$$

Where, E is the error in estimation defined as $E = X - \hat{X}$. The dynamics of this augmented system with noise is as follows,

$$\begin{aligned}
\dot{X}_{aug} &= A_{aug} X_{aug} + N \qquad (11) \\
N &= \begin{bmatrix} \Gamma w \\ \Gamma w - Gv \end{bmatrix}
\end{aligned}$$

However, the term N is not considered in this section. The initial conditions are X_0 = [0.1 0 0 0]' and $\hat{X}$ = [0 0 0 0]'. The controller and observer gain are found using the 'LQR' command in MATLAB. Both the systems in eq 8 and 9 are verified to be fully controllable and fully observable. The important issue to take note of here is the problem is ill-conditioned because of,

- The R is small and Q is large, because of the definition in eq 6. The cost function needs to be altered to minimize the control efforts. See fig 3 and 6.
- The sensor noise co-variance V is small. This means a state estimation solely based on Y, is worth considering.

In order to integrate the dynamics in eq 11 the Euler-Cauchy scheme is used with a very small step size.

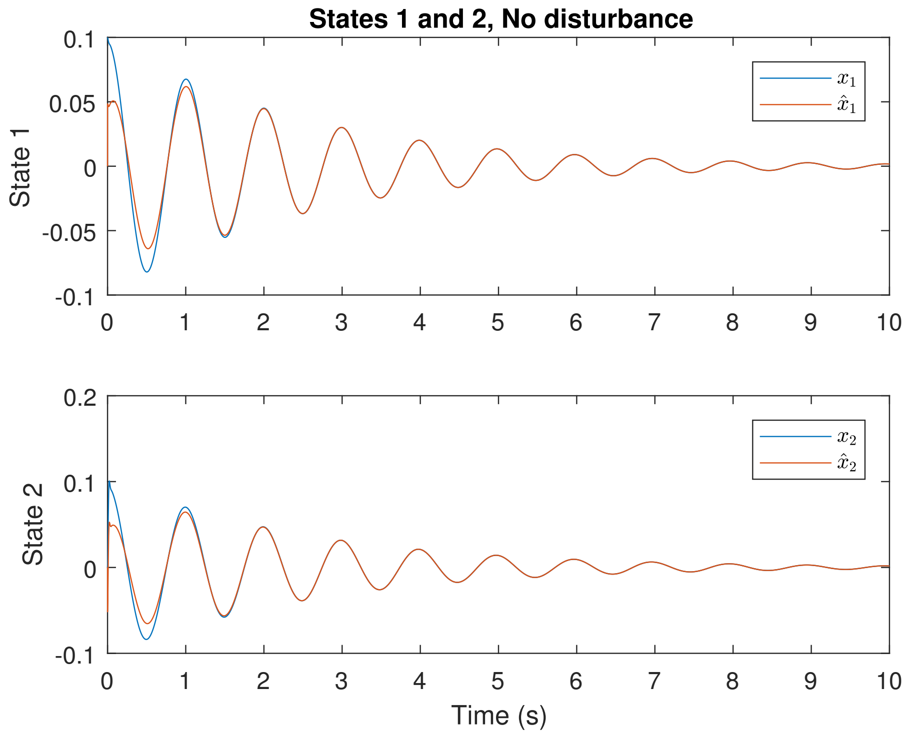

Figure 1: Actual and estimated states 1 and 2 without noise

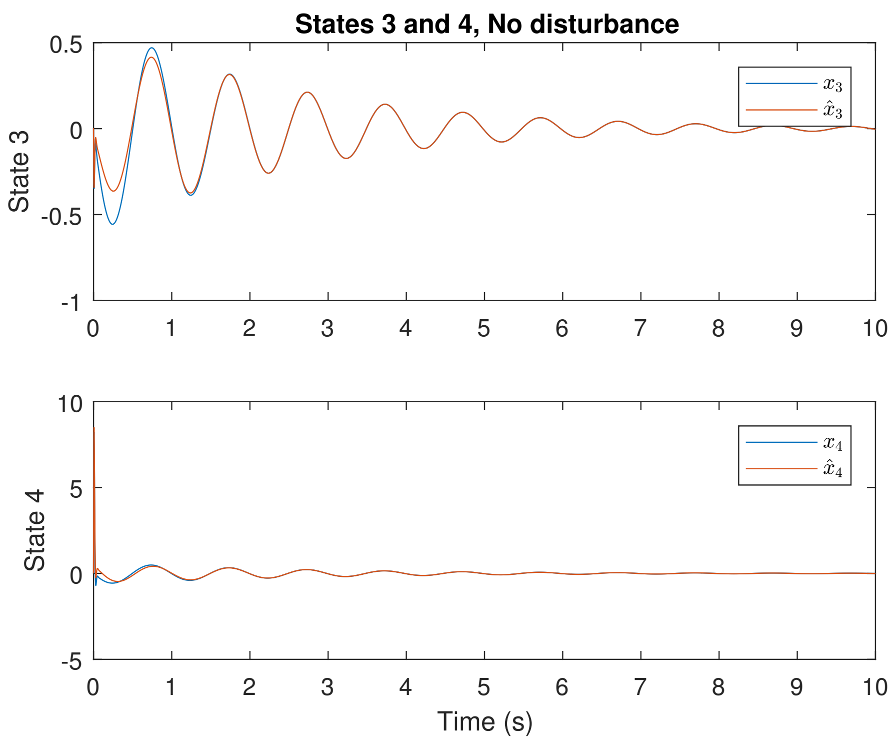

Figure 2: Actual and estimated states 3 and 4 without noise

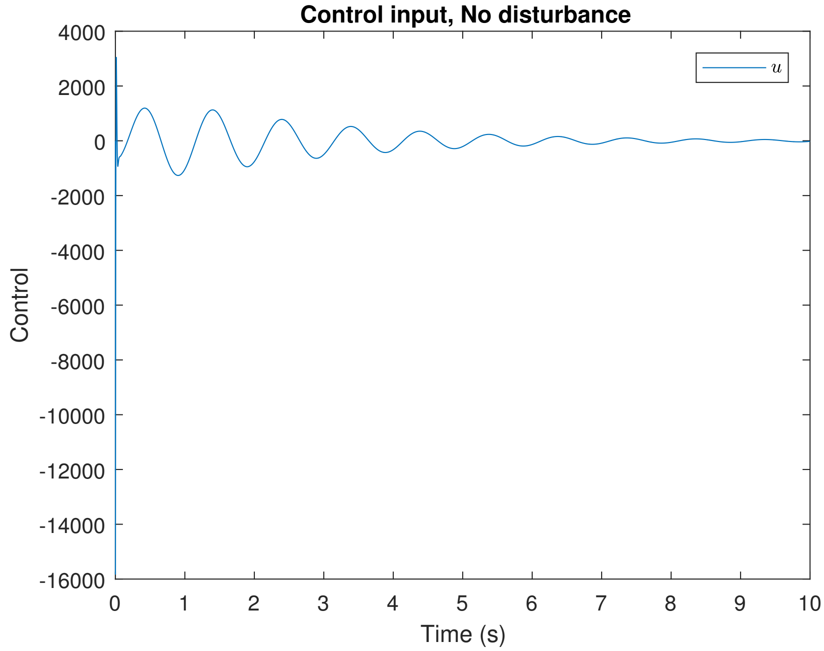

Figure 3: Control effort without noise

1.2 Simulation with Noise

In this section the noise is introduced. Note that the state has small random oscillation around origin because the noise vectors are never truly random. The mean of the noise itself keep randomly changing. Or it might be cause this is a Gauss-Markov random process and how it interacts with noise inputs.

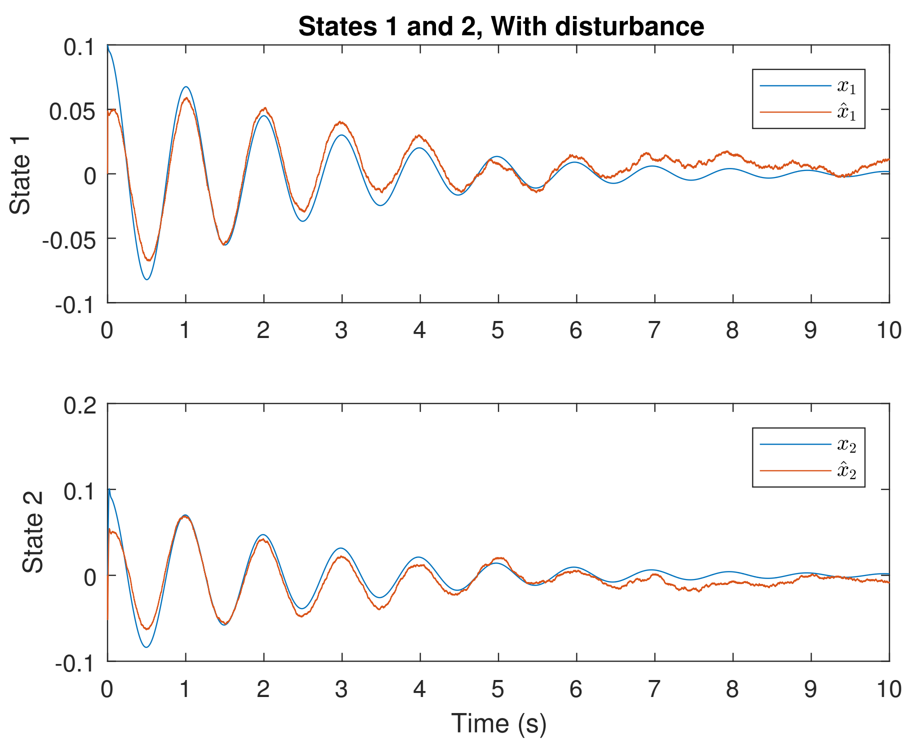

Figure 4: Actual and estimated states 1 and 2 with noise

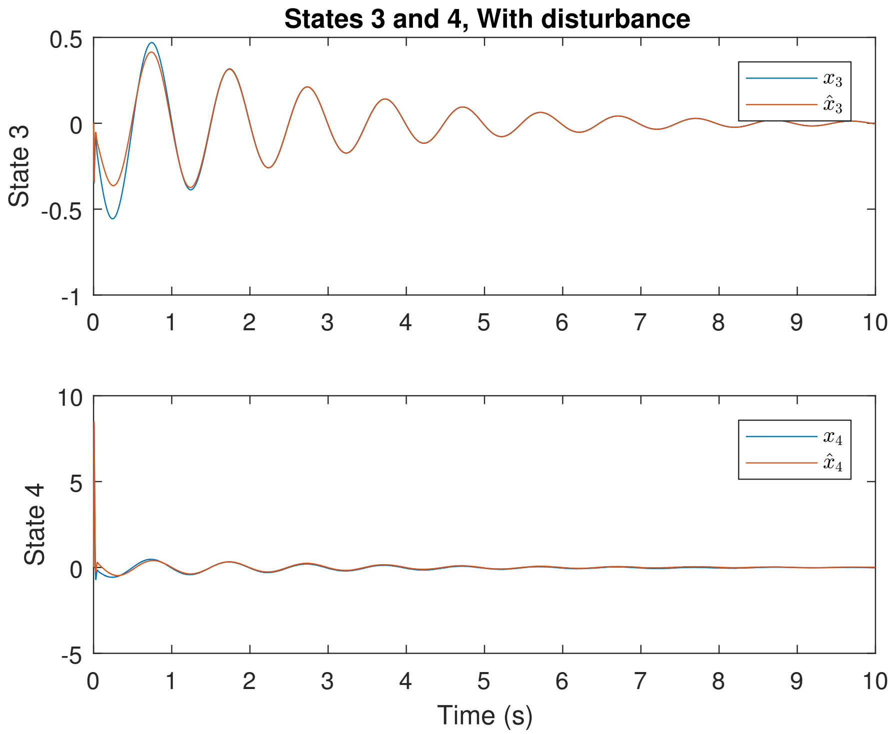

Figure 5: Actual and estimated states 3 and 4 with noise

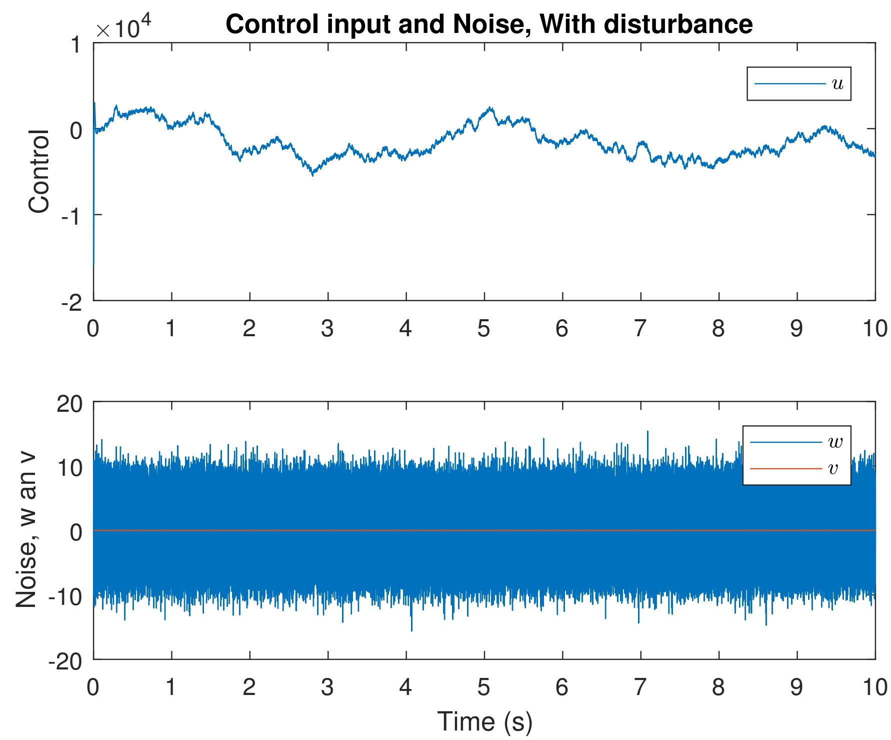

Figure 6: Control effort and noise

1.3 Evaluation of Robustness and Loop Transfer Recovery at Input

For the evaluation of robustness is done through the Bode diagrams. See figures 7 through 11. A summary of the Gain and Phase margins and changes are published in Table 1. The loop transfer recovery in this case is done by modifying W to $W + rBB^T$ and recomputing the observer gain.

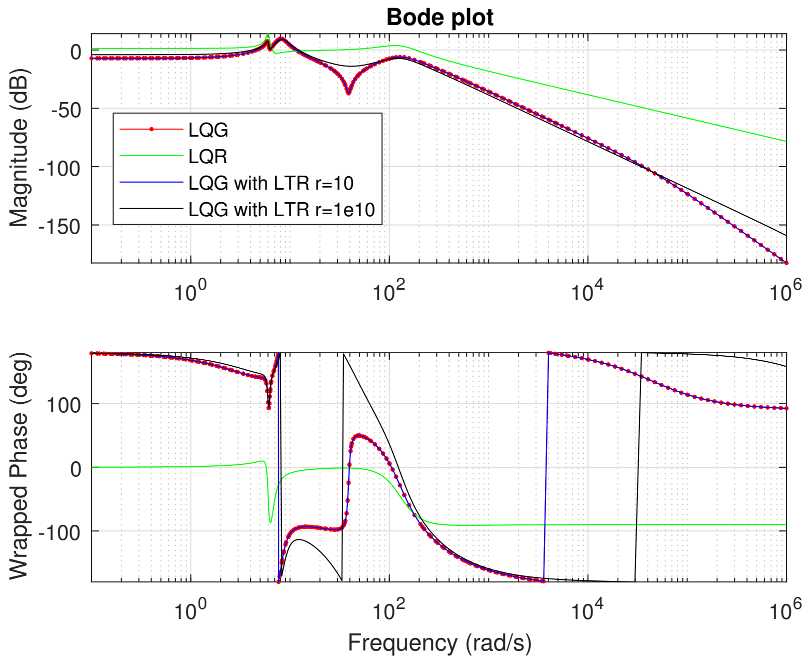

Figure 7: Bode plot comparisons with Phase wrapped -180 to 180

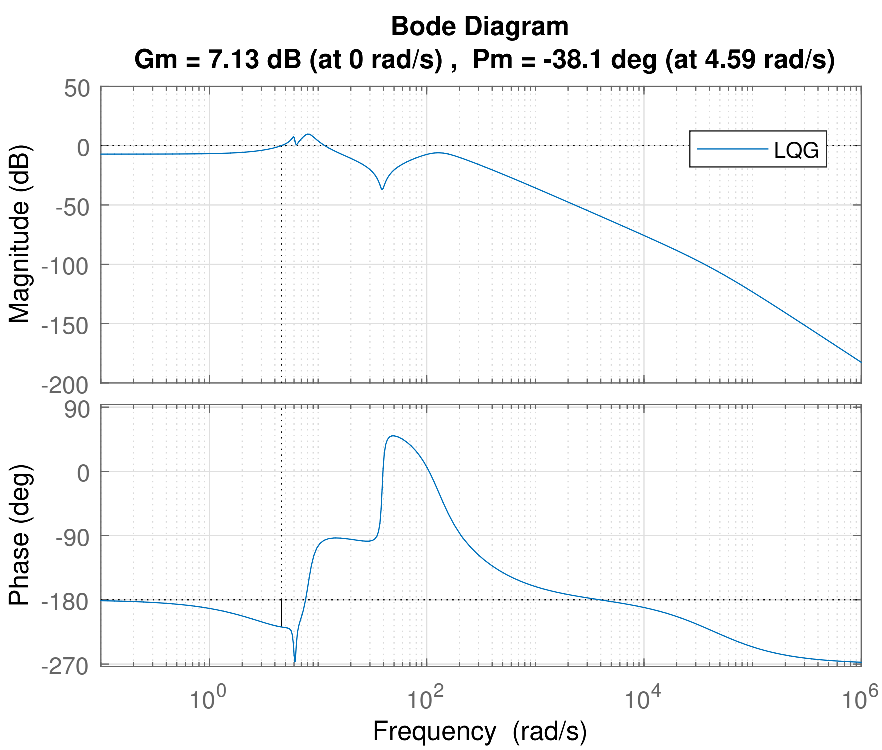

Figure 8: Gain and Phase Margin in with LQG

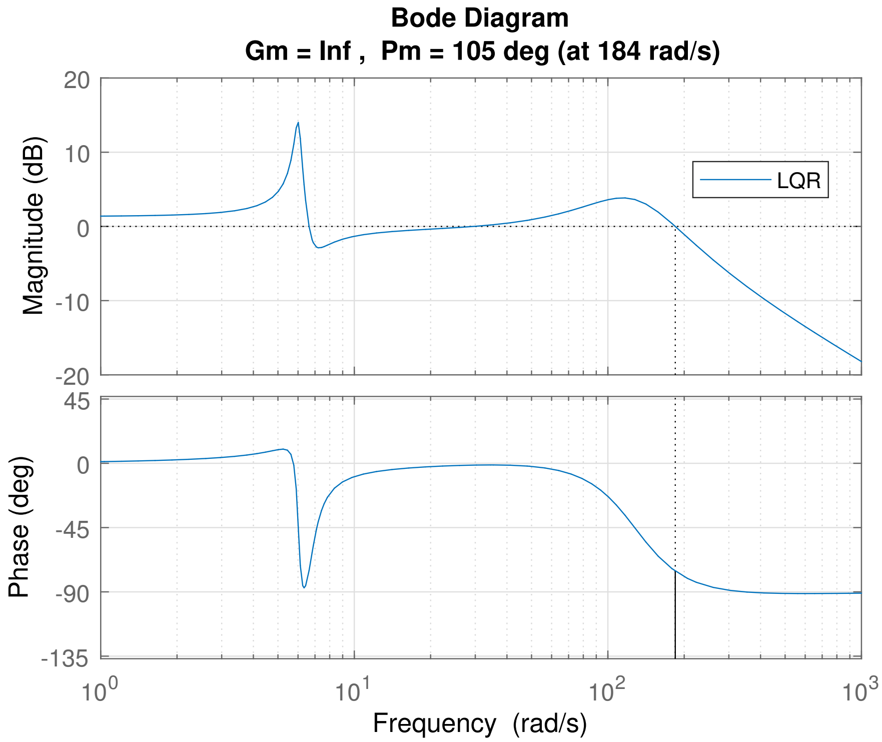

Figure 9: Gain and Phase Margin in with LQR

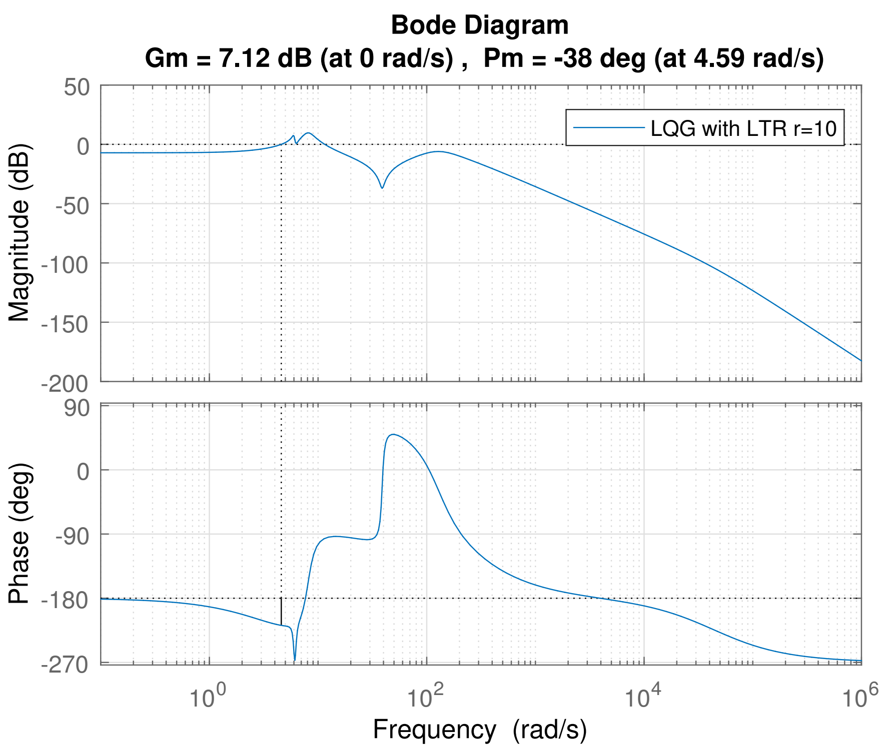

Figure 10: Gain and Phase Margin in with LQG with input LTR, r=10

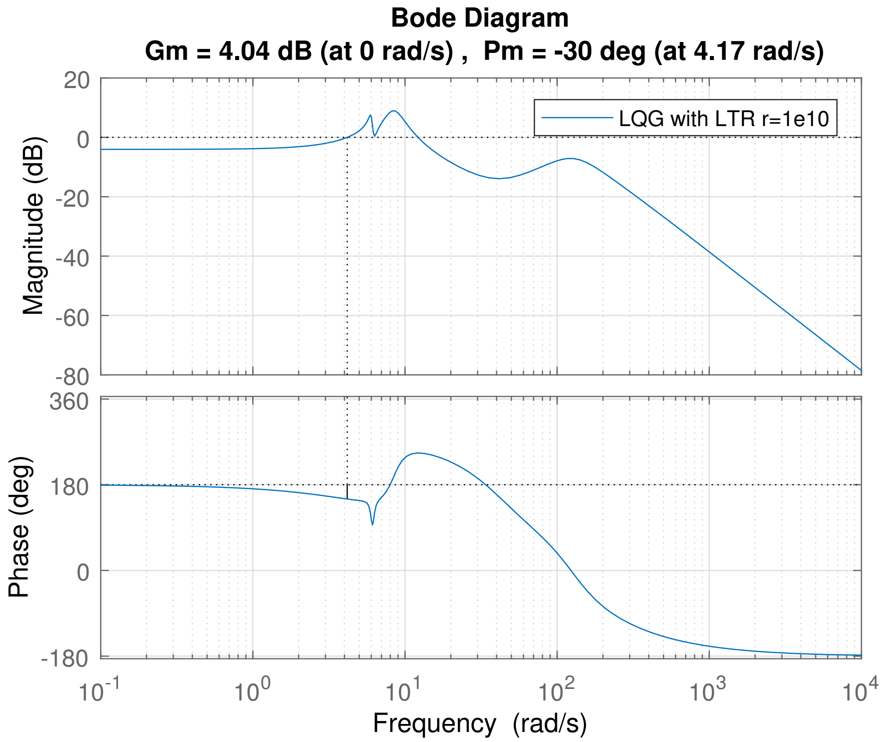

Figure 11: Gain and Phase Margin in with LQG with input LTR, r=1e10

Gain and Phase Margins		
	GM (dB)	PM (deg)
LQG	7.13	-38.1
LQR	inf	105
LQG with LTR, r=10	7.12	-38
LQG with LTR, r=1e10	4.04	-30

Table 1: Summary of Gain and Phase margins

Observations:

- The LQR shows the classic infinite Gain Margin and more than 60 deg Phase margin. In this case the Phase Margin is 105 deg.
- The LTR at input with r=10 does not modify the transfer functions significantly.
- As r is increased the transfer function changes with an improvement at higher frequencies but at zero frequency the Gain Margin reduces.
- The Phase Margin shows improvement with increasing r.
- The point to note is that V is small here, 1e-8, so the observer gain is still large.

2 Problem 2

Consider the nonlinear continuous time model as follows,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{l}(\boldsymbol{x}, \tilde{\boldsymbol{w}}) \tag{12}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{v} \tag{13}$$

Now, lets consider the dynamics of the IMU as follows.

$$\dot{\boldsymbol{\Theta}} = \boldsymbol{H}(\boldsymbol{\Theta})\boldsymbol{\omega} + \boldsymbol{H}(\boldsymbol{\Theta})\tilde{\boldsymbol{w}} \tag{14}$$

$$\text{Where, } \boldsymbol{H} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & \text{-}\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix}$$

$$\boldsymbol{y} = \begin{pmatrix} -\sin\theta \\ \sin\phi \cos\theta \\ \psi \end{pmatrix} + \boldsymbol{v} \tag{15}$$

Base on the assumptions stated in the problems and lecture, the state is $\boldsymbol{\Theta} = [\phi \;\; \theta \;\; \psi]^T$, and the inputs are $\boldsymbol{w} = [\text{p q r}]^T$, which are the body rates. Reconciling the the problem with the general form in eq 12 and eq 13, we write,

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) &= \boldsymbol{H}(\boldsymbol{\Theta})\boldsymbol{\omega} \\ \boldsymbol{h}(\boldsymbol{x}) &= \begin{pmatrix} -\sin\theta \\ \sin\phi \cos\theta \\ \psi \end{pmatrix} \\ \boldsymbol{l}(\boldsymbol{x}, \tilde{\boldsymbol{w}}) &= \boldsymbol{H}(\boldsymbol{\Theta})\tilde{\boldsymbol{w}} \end{aligned} \tag{16}$$

To simulate the EKF, we define the following Jacobian matrices.

$$\begin{aligned} \boldsymbol{A}(t) &= \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right]_{\text{at } \hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)} \\ \boldsymbol{C}(t) &= \left[\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}\right]_{\text{at } \hat{\boldsymbol{x}}(t)} \\ \boldsymbol{L}(t) &= \left[\frac{\partial \boldsymbol{l}}{\partial \tilde{\boldsymbol{w}}}\right]_{\text{at } \hat{\boldsymbol{x}}(t), \tilde{\boldsymbol{w}}(t)} \end{aligned} \tag{17}$$

Evaluating the Jacobians we get the following matrices.

$$\begin{aligned} \boldsymbol{A}(t) &= \begin{bmatrix} \text{q} \cos\phi \tan\theta - \text{r} \sin\phi \tan\theta & \text{q} \sin\phi \sec^2\theta + \text{r} \cos\phi \sec^2\theta & 0 \\ -\text{q} \sin\phi - \text{r} \cos\phi & 0 & 0 \\ \text{q} \cos\phi \sec\theta - \text{r} \sin\phi \sec\theta & \text{q} \sin\phi \sec\theta \tan\theta + \text{r} \cos\phi \sec\theta \tan\theta & 0 \end{bmatrix}_{\text{at } \hat{\boldsymbol{\Theta}}(t), \tilde{\boldsymbol{w}}(t)} \\ \boldsymbol{C}(t) &= \begin{bmatrix} 0 & -\cos\theta & 0 \\ -\text{q} \sin\phi - \text{r} \cos\phi & -\sin\phi \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\text{at } \hat{\boldsymbol{\Theta}}} \\ \boldsymbol{L}(t) &= \boldsymbol{H}(\hat{\boldsymbol{\Theta}}(t)) \end{aligned} \tag{18}$$

2.1 Simulation

The simulation is based on the algorithm given in the lecture 20. The following constants are used in the simulation.

- The process noise standard deviation $\sigma_w = [3\ 3\ 3]^T$ deg/sec
- The process noise Covariance is diag(σ_w^2)
- The sensor noise standard deviation $\sigma_v = [0.2\ 0.2\ 1.5]^T$ (unknown units)
- The sensor noise Covariance is diag(σ_v^2)
- The initial standard deviation of state $\sigma_x = [5\ 5\ 5]^T$ deg
- The initial state Covariance is diag(σ_x^2)
- $\boldsymbol{x_0} = [5\ 10\ 7]^T$ deg and $\boldsymbol{\hat{x}_0} = [0\ 0\ 0]^T$ deg

2.1.1 Without Process Noise

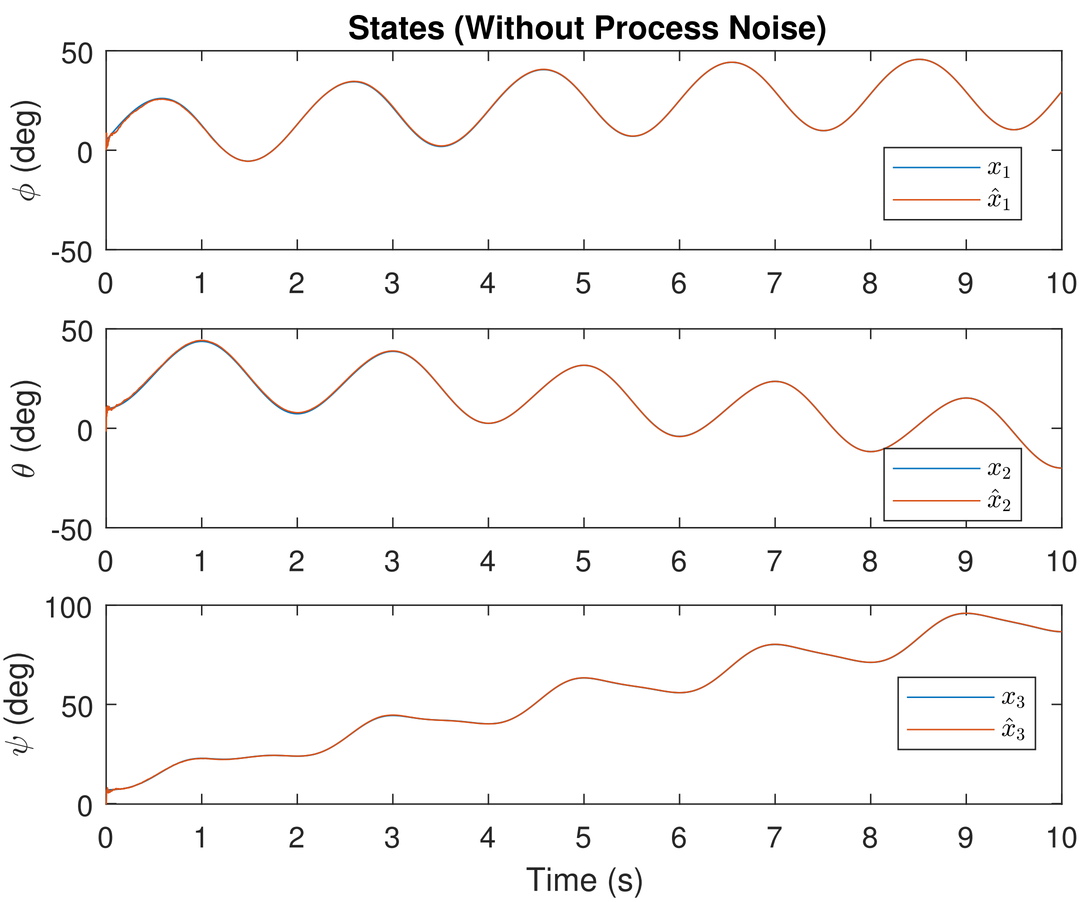

Figure 12: Actual and estimated Euler angles

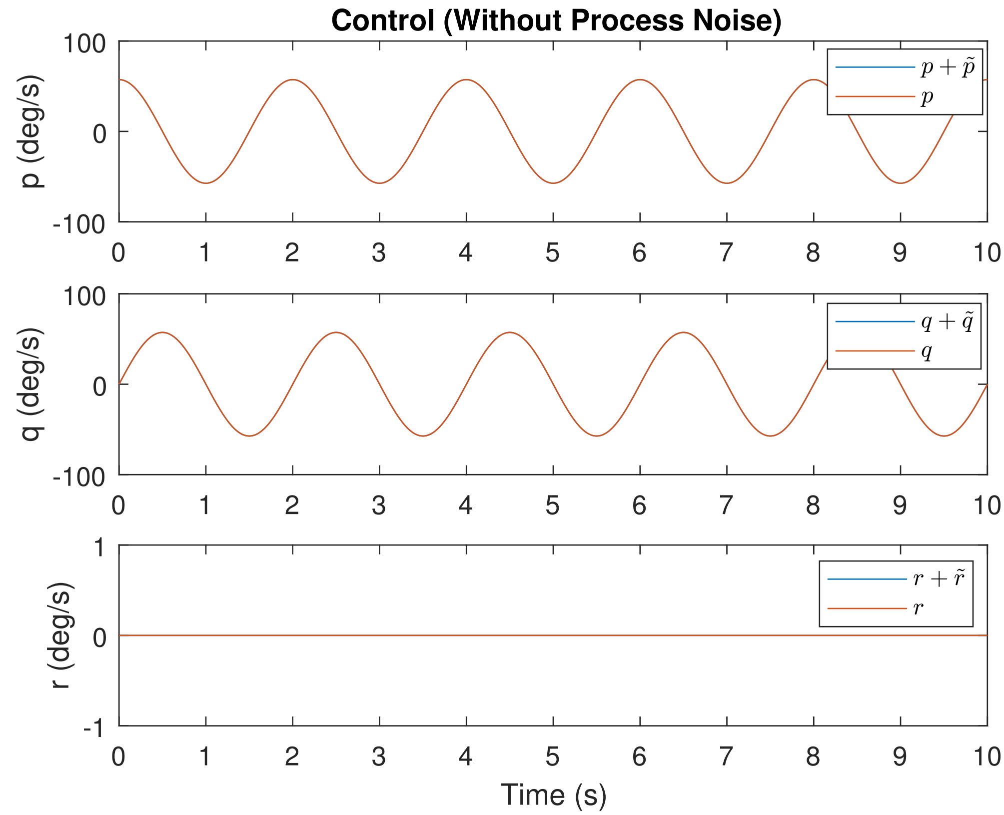

Figure 13: Body rate (without noise)

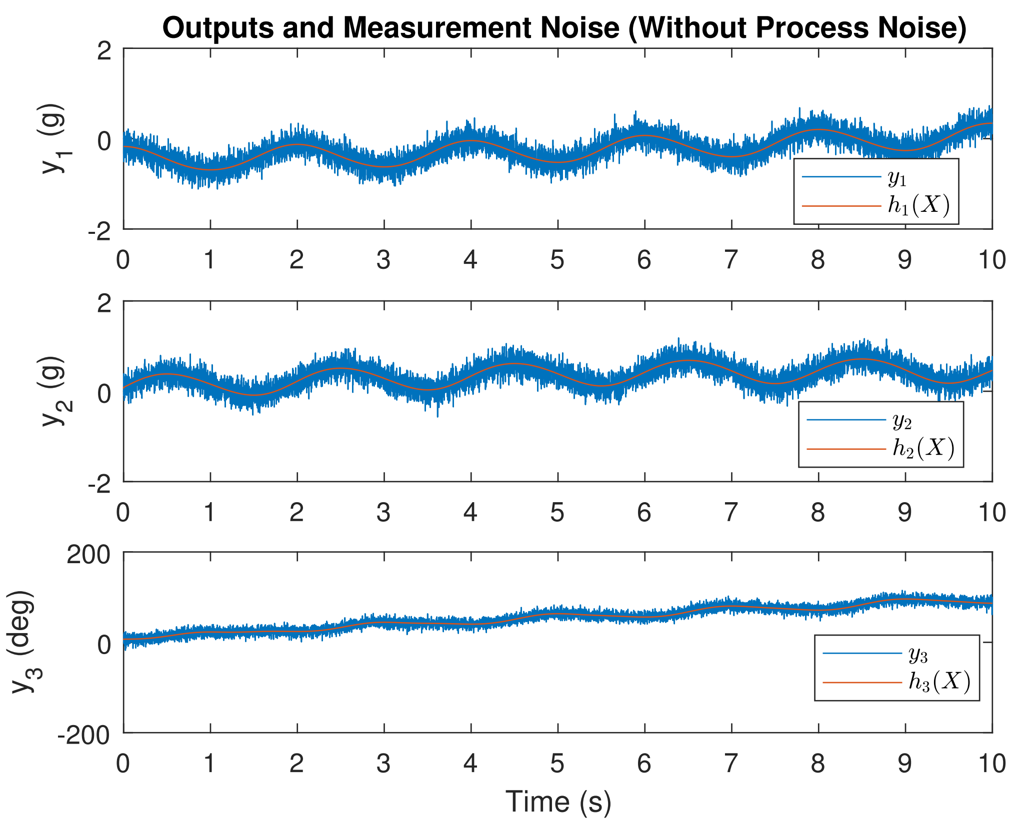

Figure 14: Measurement, with and without noise (without process noise)

2.1.2 With Process Noise

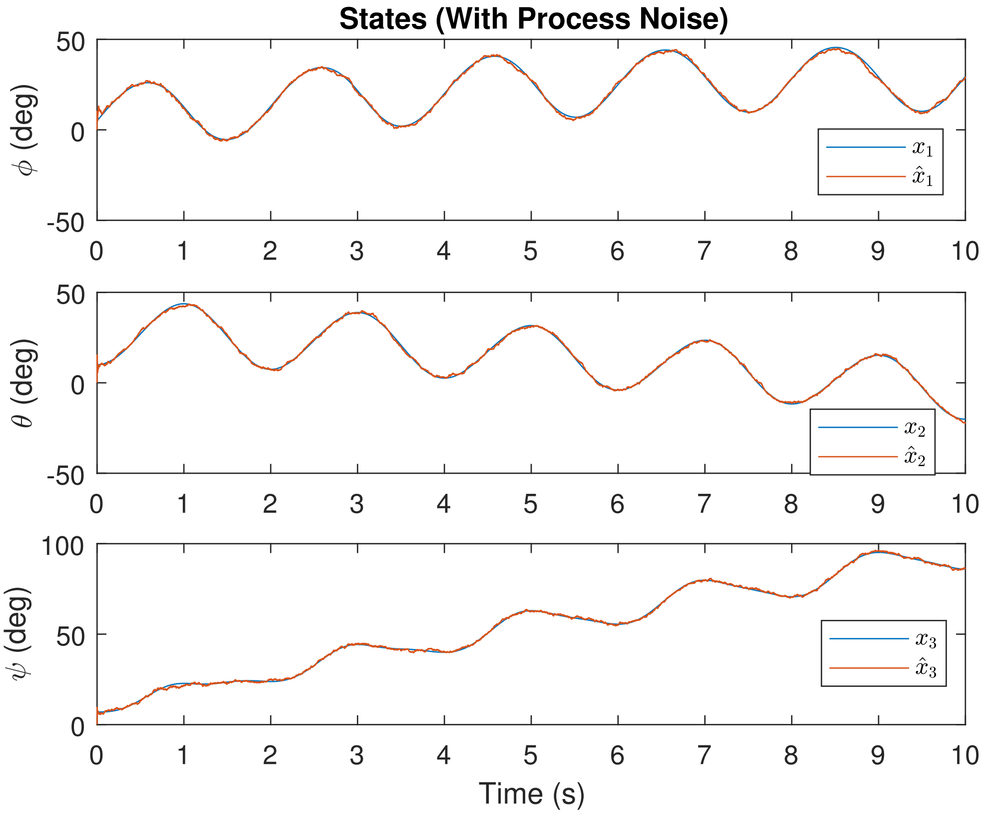

Figure 15: Actual and estimated Euler angles

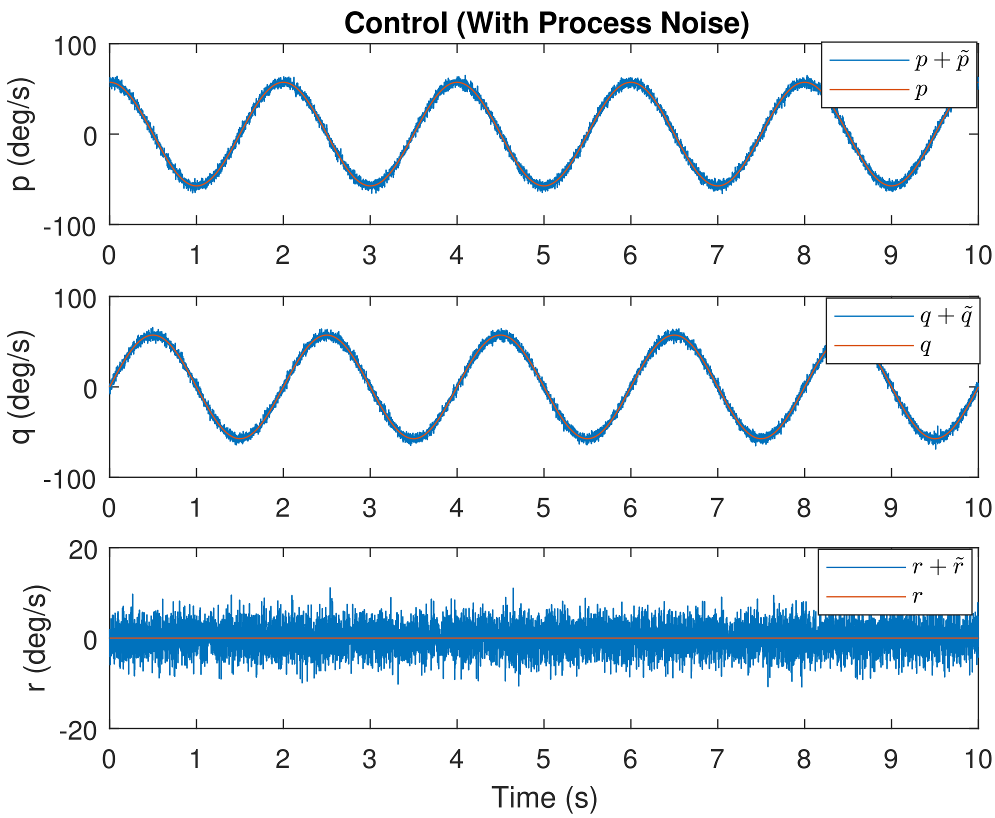

Figure 16: Body rate (with noise)

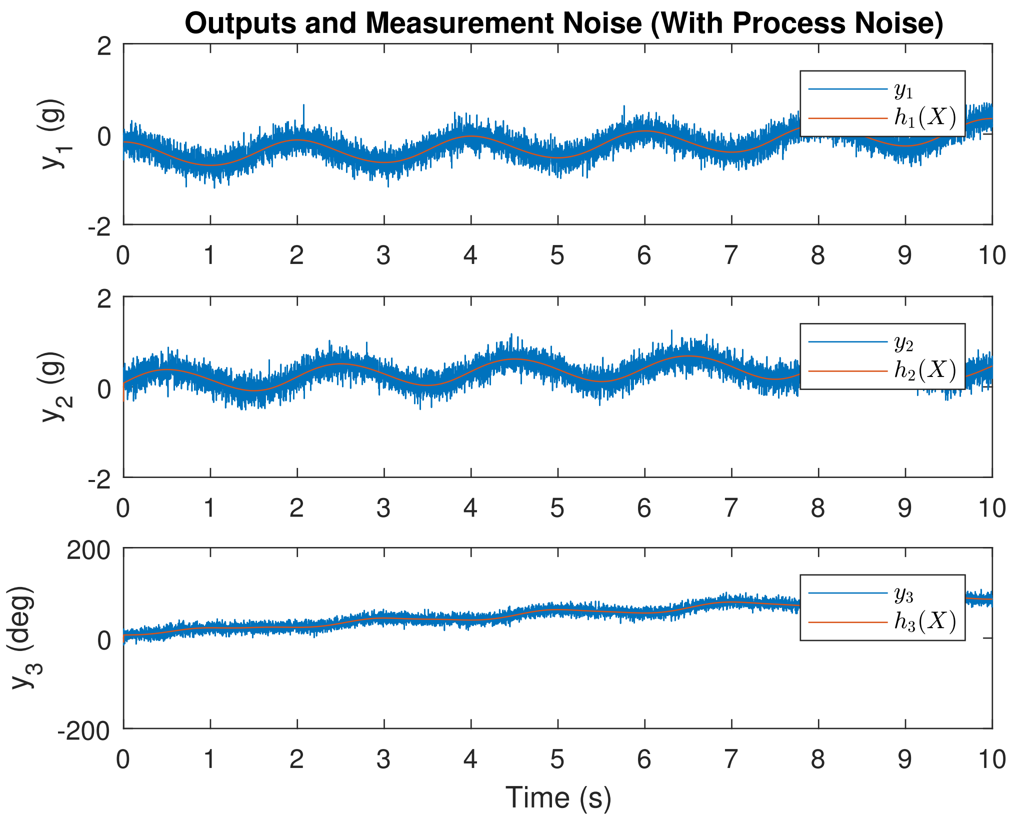

Figure 17: Measurement, with and without noise (with process noise)