ENPM667 - Control of Robotic Systems

Technical Report of Final Project

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Chapter 1

INTRODUCTION

The final project consists of obtaining state space representation for a crane with two loads suspended from cables and actuated by an external force F and designing an LQR controller and LQG controller for the system. There are two components to the project.

- The first step in the first component is to calculate the equations of motion for the system using the Lagrangian method and write them in a nonlinear state-space representation.
- The next step is to linearize the system around the equilibrium points, rewrite the equations of motion in linearized state-space representation, and obtain conditions for which the linearized system will be controllable.
- The final step in the first component is to check whether the system is controllable for the given conditions. If the system is controllable, an LQR controller is designed and the results are simulated at the initial condition and are applied to both linearized system and nonlinear system. Finally, the stability of the system is certified using Lyapunov's indirect method.

The next steps involved in the second component are -

- After designing an LQR controller, the system observability is checked for the given output vectors in the first step.
- In the next step, a Luenberger observer is obtained for every given output vector for which the system is observable. Then, the response to initial conditions and unit step input is simulated for both linearized system and nonlinear system.
- The final step in the second component is to design an output feedback controller for the smallest output vector using the LQG method and apply the resulting output feedback controller to the original nonlinear system. The system's performance with the LQG controller is depicted by simulating it.

Chapter 2

EQUATIONS OF MOTION FOR THE SYSTEM

The given system is a crane that moves along one direction with mass M and input force F. Two loads with mass m_1 and m_2 are suspended using cables with lengths l_1 and l_2 and are attached to the crane. θ_1 is the angle made by the cable 1 and θ_2 is the angle made by the cable 2.

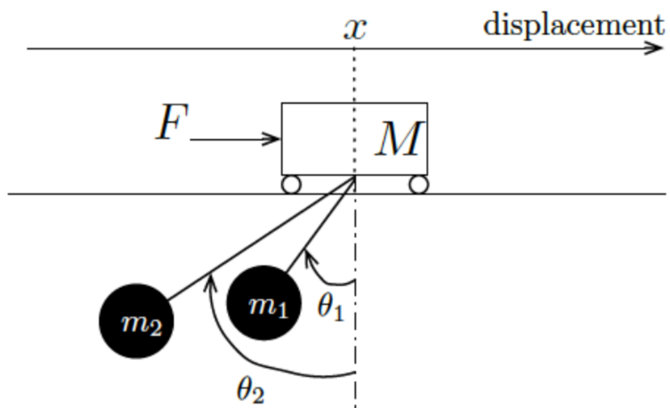

Figure 2.1: Crane with two loads attached

- Obtaining the equation for the position of the cables with mass m_1 and mass m_2 as a function of θ_1 and θ_2 respectively-
  Cable 1 position along x axis = $x - l_1 \sin\theta_1$
  along y axis = $-l_1 \cos\theta_1$
  Cable 2 position along x axis = $x - l_2 \sin\theta_2$
  along y axis = $-l_2 \cos\theta_2$

$$x_1 = (x - l_1 \sin(\theta_1))\,\hat{i} + (-l_1 \cos(\theta_1))\,\hat{j} \tag{2.1}$$

$$x_2 = (x - l_2 \sin(\theta_2))\,\hat{i} + (-l_2 \cos(\theta_2))\,\hat{j} \tag{2.2}$$

- Velocity is the time derivative of the position function and the equation of velocity for both the cables is obtained as below -

$$v_1 = \left(\dot{x} - l_1 \cos(\theta_1)\,\dot{\theta_1}\right)\hat{i} + \left(l_1 \sin(\theta_1)\,\dot{\theta_1}\right)\hat{j} \tag{2.3}$$

$$v_2 = \left(\dot{x} - l_2 \cos(\theta_2)\,\dot{\theta_2}\right)\hat{i} + \left(l_2 \sin(\theta_2)\,\dot{\theta_2}\right)\hat{j} \tag{2.4}$$

- Potential energy (P.E) and Kinetic energy (K.E) of the system are formulated using the above derived position equations and velocity equations respectively -

$$\begin{aligned}\text{K.E} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\left(\dot{x} - l_1\dot{\theta}_1\cos(\theta_1)\right)^2 + \frac{1}{2}m_1\left(l_1\dot{\theta}_1(\sin(\theta_1))\right)^2 + \\ \frac{1}{2}m_2(\dot{x} - \dot{\theta}_2 l_2\cos(\theta_2))^2 + \frac{1}{2}m_2\left(l_2\dot{\theta}_2(\sin(\theta_2))\right)^2\end{aligned} \tag{2.5}$$

$$P.E = -m_1 g l_1\cos(\theta_1) - m_2 g l_2\cos(\theta_2) \tag{2.6}$$

- The total energy of the system is calculated below

$$L = K \cdot E - P \cdot E \tag{2.7}$$

$$\begin{aligned}\text{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta_1}^2\cos^2(\theta_1) - m_1 l_1\dot{\theta}_1\dot{x}\cos(\theta_1) + \frac{1}{2}m_1 l_1^2\dot{\theta_1}^2\sin^2(\theta_1) \\ +\frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2\cos^2(\theta_2) - m_2 l_2\dot{\theta}_2\dot{x}\cos(\theta_2) + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2\sin^2(\theta_1) \\ +g\left[m_1 l_1\cos(\theta_1) + m_2 l_2\cos(\theta_2)\right]\end{aligned} \tag{2.8}$$

- The above equation is simplified and written below -

$$\begin{aligned}\text{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2\dot{\theta_2}^2 - \\ \dot{x}(m_1 l_1\dot{\theta}_1\cos(\theta_1) + m_2 l_2\dot{\theta}_2\cos(\theta_2)) + g[m_1 l_1\cos(\theta_1) + m_2 l_2\cos(\theta_2)]\end{aligned} \tag{2.9}$$

- Lagrange Equations is obtained by computing the derivative of L w.r.t $\dot{x}, \dot{\theta}_1, \dot{\theta}_2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F \tag{2.10}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \tag{2.11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \tag{2.12}$$

- Computing the first Lagrangian Equation -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F \tag{2.13}$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + (m_1 + m_2)\dot{x} - m_1 l_1\dot{\theta}_1\cos(\theta_1) - m_2 l_2\dot{\theta}_2\cos(\theta_2) \tag{2.14}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = M\ddot{x} + (m_1 + m_2)\ddot{x} - [m_1 l_1 \ddot{\theta}_1 \cos(\theta_1) \\ -m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1)] - [m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)] \tag{2.15}$$

Since L is independent of x,

$$\frac{\partial L}{\partial x} = 0 \tag{2.16}$$

- Final Lagrangian equation for 2.10

$$[M + m_1 + m_2]\,\ddot{x} - m_1 l_1 \ddot{\theta}_1 \cos(\theta_1) + m_1 l_1 \dot{\theta_1}^2 \sin(\theta 1) - m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) \\ + m_2 l_2 \dot{\theta_2}^2 \sin(\theta_2) = F \tag{2.17}$$

- The double derivative equation for the state variable x is obtained as below by using the above equation -

$$\ddot{x} = \tfrac{1}{M+m_1+m_2}\left[m_1 l_1 \ddot{\theta}_1 \cos\theta_1 + m_2 l_2 \ddot{\theta}_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F\right] \tag{2.18}$$

- Computing the second Lagrangian Equation -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \tag{2.19}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 \dot{x} l_1 \cos(\theta_1) \tag{2.20}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta_1} - \left[m_1 l_1 \ddot{x} \cos(\theta_1) - m_1 \dot{x} l_1 \dot{\theta}_1 \sin(\theta_1)\right] \tag{2.21}$$

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1 \dot{\theta}_1 \dot{x} \sin(\theta_1) - m_1 l_1 g \sin(\theta_1) \tag{2.22}$$

- Final Lagrangian equation for 2.11 -

$$m_1 l_1^2 \ddot{\theta_1} - m_1 \ddot{x} l_1 \cos(\theta_1) + m_1 \dot{\theta}_1 \dot{x} l_1 \sin(\theta_1) - m_1 \dot{\theta}_1 \dot{x} l_1 \sin(\theta_1) + m_1 l_1 g \sin(\theta_1) = 0 \tag{2.23}$$

and the simplified equation is

$$m_1 l_1^2 \ddot{\theta_1} - m_1 \ddot{x} l_1 \cos(\theta_1) + m_1 l_1 g \sin(\theta_1) = 0 \tag{2.24}$$

- The double derivative equation for the state variable θ_1 is obtained as below by using the above equation -

$$\ddot{\theta}_1 = \frac{\ddot{x}\cos\theta_1}{l_1} - \frac{g \sin\theta_1}{l_1} \tag{2.25}$$

- Computing the third Lagrangian Equation -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \tag{2.26}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 \dot{x} l_2 \cos(\theta_2) \tag{2.27}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 - \left[m_2 \ddot{x} l_2 \cos(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2)\right] \tag{2.28}$$

$$\left(\frac{\partial L}{\partial \theta_2}\right) = m_2 \dot{x} l_2 \dot{\theta}_2 \sin(\theta_2) - m_2 l_2 g \sin(\theta_2) \tag{2.29}$$

- Final Lagrangian equation for 2.12 -

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} \cos(\theta_2) + m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2) + m_2 g l_2 \sin(\theta_2) \tag{2.30}$$

  and the simplified equation is

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} \cos(\theta_2) + m_2 g l_2 \sin(\theta_2) = 0 \tag{2.31}$$

- The double derivative equation for the state variable θ_2 is obtained as below by using the above equation -

$$\ddot{\theta}_2 = \frac{\ddot{x}\cos\theta_2}{l_2} - \frac{g\sin\theta_2}{l_2} \tag{2.32}$$

State-space representation of the Non-linear system

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2} \\ \dot{\theta}_1 \\ \frac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{\left(M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2\right) l_1} - \frac{g\sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{\left(M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2\right) l_2} - \frac{g\sin\theta_2}{l_2} \end{bmatrix} \tag{2.33}$$

Chapter 3

LINEAR SYSTEM

3.1 Linearizing the system around an equilibrium point

- The state space representation for the crane system is non-linear and the system is linearized around the equilibrium point which is given in the problem as $x = 0$, $\theta_1 = 0$ and $\theta_2 = 0$. Assuming the below limiting condition at equilibrium,

$$\sin\theta_1 \approx \theta_1$$
$$\sin\theta_2 \approx \theta_2$$
$$\cos\theta_1 \approx 1$$
$$\cos\theta_2 \approx 1$$
$$\dot{\theta}_1^2 = \dot{\theta}_2^2 \approx 0$$

- Using the above assumptions and taking $x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2$ and $\dot{\theta}_2$ as state variables the linearized system is obtained as below -

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{-1}{M}\left[m_1 g\theta_1 + m_2 g\theta_2 - F\right] \\ \dot{\theta}_1 \\ \frac{1}{l_1}\left(\ddot{x} - g\theta_1\right) \\ \dot{\theta}_2 \\ \frac{1}{l_2}\left(\ddot{x} - g\theta_2\right) \end{bmatrix} \tag{3.1}$$

- The above state space form can be expanded by taking Force F as input 'U'.

$$\dot{X} = AX + BU$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M\,l_1} \\ 0 \\ \frac{1}{M\,l_2} \end{pmatrix}$$

$$Y = CX + DU$$

Since we are taking output for all the state variables, taking C as the identity matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = 0$$

The controllability matrix is found out by:

$$C = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}$$

$$\text{C} = \begin{pmatrix} 0 & \frac{1}{M} & 0 & \sigma_2 & 0 & \sigma_1 \\ \frac{1}{M} & 0 & \sigma_2 & 0 & \sigma_1 & 0 \\ 0 & \frac{1}{M\,l_1} & 0 & \sigma_6 & 0 & \sigma_4 \\ \frac{1}{M\,l_1} & 0 & \sigma_6 & 0 & \sigma_4 & 0 \\ 0 & \frac{1}{M\,l_2} & 0 & \sigma_5 & 0 & \sigma_3 \\ \frac{1}{M\,l_2} & 0 & \sigma_5 & 0 & \sigma_3 & 0 \end{pmatrix}$$

where

$$\sigma_1 = \frac{\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1} + \frac{g^2\,m_1\,m_2}{M^2\,l_2}}{M\,l_1} + \frac{\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1}}{M\,l_2}$$

$$\sigma_2 = -\frac{g\,m_1}{M^2\,l_1} - \frac{g\,m_2}{M^2\,l_2}$$

$$\sigma_3 = \frac{\frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2} + \frac{g^2\,m_1\,(M+m_1)}{\sigma_7}}{M\,l_1} + \frac{\frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2} + \frac{g^2\,m_1\,m_2}{\sigma_7}}{M\,l_2}$$

$$\sigma_4 = \frac{\frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2} + \frac{g^2\,m_2\,(M+m_2)}{\sigma_7}}{M\,l_2} + \frac{\frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2} + \frac{g^2\,m_1\,m_2}{\sigma_7}}{M\,l_1}$$

$$\sigma_5 = -\frac{g\,(M+m_2)}{M^2\,{l_2}^2} - \frac{g\,m_1}{\sigma_7}$$

$$\sigma_6 = -\frac{g\,(M+m_1)}{M^2\,{l_1}^2} - \frac{g\,m_2}{\sigma_7}$$

$$\sigma_7 = M^2\,l_1\,l_2$$

The above controllability matrix is found out using Matlab and the output is shown. The rank of the matrix is also found using Matlab and it is determined as full rank (n_r = 6).

Chapter 4

CONTROLLABILITY CONDITIONS FOR LINEARIZED SYSTEM

- If the linear time-invariant system is controllable, the Grammian controllability matrix will be invertible. If the rank of the controllability matrix is full, the Grammian controllability matrix is invertible. By checking the rank of the controllability matrix, the system's controllability can be determined.

$$C = \begin{pmatrix} 0 & \frac{1}{M} & 0 & \sigma_1 & 0 & \sigma_4 \\ \frac{1}{M} & 0 & \sigma_1 & 0 & \sigma_4 & 0 \\ 0 & \frac{1}{M\,l_2} & 0 & \sigma_6 & 0 & \sigma_2 \\ \frac{1}{M\,l_2} & 0 & \sigma_6 & 0 & \sigma_2 & 0 \\ 0 & \frac{1}{M\,l_2} & 0 & \sigma_5 & 0 & \sigma_3 \\ \frac{1}{M\,l_2} & 0 & \sigma_5 & 0 & \sigma_3 & 0 \end{pmatrix}$$

where

$$\sigma_1 = -\frac{g\,m_1}{M^2\,l_2} - \frac{g\,m_2}{M^2\,l_2}$$

$$\sigma_2 = \frac{\frac{g^2\,m_2\,(M+m_1)}{\sigma_8} + \frac{g^2\,m_2\,(M+m_2)}{\sigma_8}}{M\,l_2} + \frac{\frac{g^2\,(M+m_1)^2}{\sigma_8} + \frac{g^2\,m_1\,m_2}{\sigma_8}}{M\,l_2}$$

$$\sigma_3 = \frac{\frac{g^2\,m_1\,(M+m_1)}{\sigma_8} + \frac{g^2\,m_1\,(M+m_2)}{\sigma_8}}{M\,l_2} + \frac{\frac{g^2\,(M+m_2)^2}{\sigma_8} + \frac{g^2\,m_1\,m_2}{\sigma_8}}{M\,l_2}$$

$$\sigma_4 = \frac{\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_2} + \sigma_7}{M\,l_2} + \frac{\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \sigma_7}{M\,l_2}$$

$$\sigma_5 = -\frac{g\,(M+m_2)}{\sigma_8} - \frac{g\,m_1}{\sigma_8}$$

$$\sigma_6 = -\frac{g\,(M+m_1)}{\sigma_8} - \frac{g\,m_2}{\sigma_8}$$

$$\sigma_7 = \frac{g^2\,m_1\,m_2}{M^2\,l_2}$$

$$\sigma_8 = M^2\,{l_2}^2$$

- The above controllability matrix is found when $l_1 = l_2$. The rank of the controllability matrix when $l_1 = l_2$ is 4. Hence the system is not controllable.
- To obtain the conditions for the linearized system to be controllable, $l_1 = l_2$, $l_1 = 0$ or $l_2 = 0$ conditions is applied and controllability matrix rank is found. In both the conditions, the rank is not full and therefore in both the scenarios, the system is uncontrollable.
- The conditions for which the linearized system is controllable are -
  $l_1 \neq l_2$, and $l_1, l_2 \neq 0$

Chapter 5

LQR CONTROLLER AND LYAPUNOV STABILITY

- In the next task, the mass, and length of the system values are entered and an LQR controller is designed for the system. The parameters of the LQR cost are adjusted to find the minimal weight. The resulting response to initial conditions is simulated using Matlab after the controller is applied to the system. Finally using Lyapunov's indirect method the stability (locally or globally) of the closed-loop system is certified.

- For the LQR Controller, if A,B_k is stabilizable, then we can look for k that minimizes the following cost:

  $$J(k, \vec{X}(0)) = \int_0^\infty \vec{X}^T(t)Q\vec{X}(t) + \vec{U}_k^T(t)R\vec{U}_k(t)dt$$

  where Q and R are positive definite matrices.
  The optimal solution is given as $K = -R^{-1}B_K^T P$,
  where P is a positive symmetric solution of the below Ricatti equation:

  $$A^T P + PA - PB - R^{-1}B^T P = -Q$$

- Q is selected based on how good the performance error should be and the R is selected based on the how expensive actuator is. For saving the input used, penalizing the actuator by increasing the value of R, and for increasing the performance, penalizing the error by increasing the value of Q.

- These Q and R values are found by trial and error method by taking an initial Q, R and then obtaining the gain matrix, and then changing the values of Q and R again based on the output response by the gain matrix

- The values of Q and R matrices can be validated with the different output graphs plotted

The output of the question 1-D is below:

```
C_M = 6×6
10^-3  ×
         0    1.0000         0   -0.1472         0    0.1419
    1.0000         0   -0.1472         0    0.1419         0
         0    0.0500         0   -0.0319         0    0.0227
    0.0500         0   -0.0319         0    0.0227         0
         0    0.1000         0   -0.1128         0    0.1249
    0.1000         0   -0.1128         0    0.1249         0

Rank of the controllability matrix =
     6
Since the rank of the controllability matrix is full, the system is the controllable
Q = 6×6
    50     0     0     0     0     0
     0   100     0     0     0     0
     0     0   150     0     0     0
     0     0     0   200     0     0
     0     0     0     0   250     0
     0     0     0     0     0   300

C = 6×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
```

Figure 5.1: 1 - D Controllability Matrix and the Q matrix

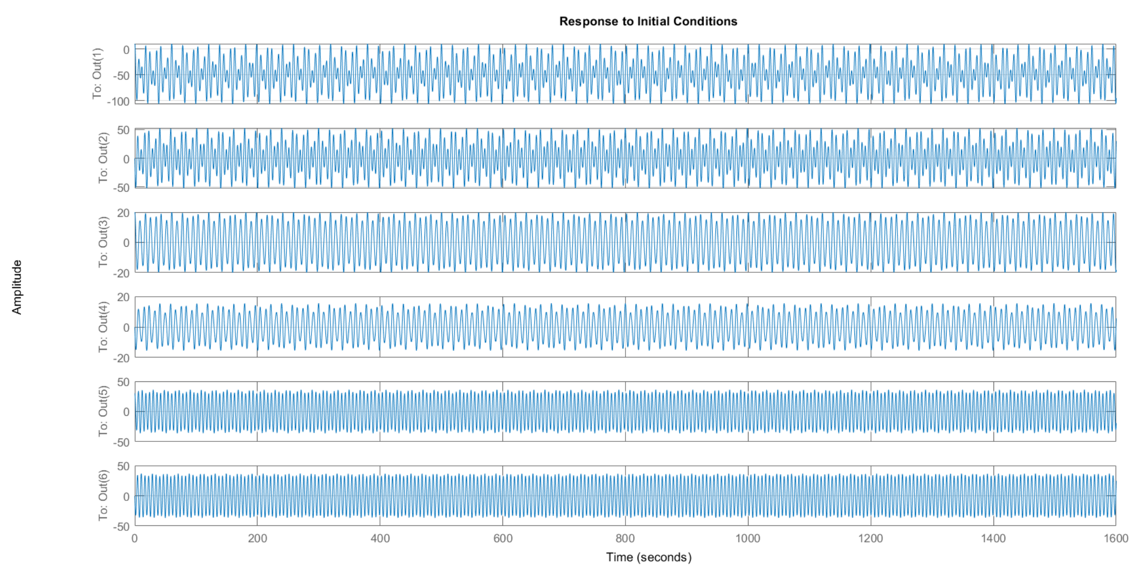

Figure 5.2: LQR response for open loop system

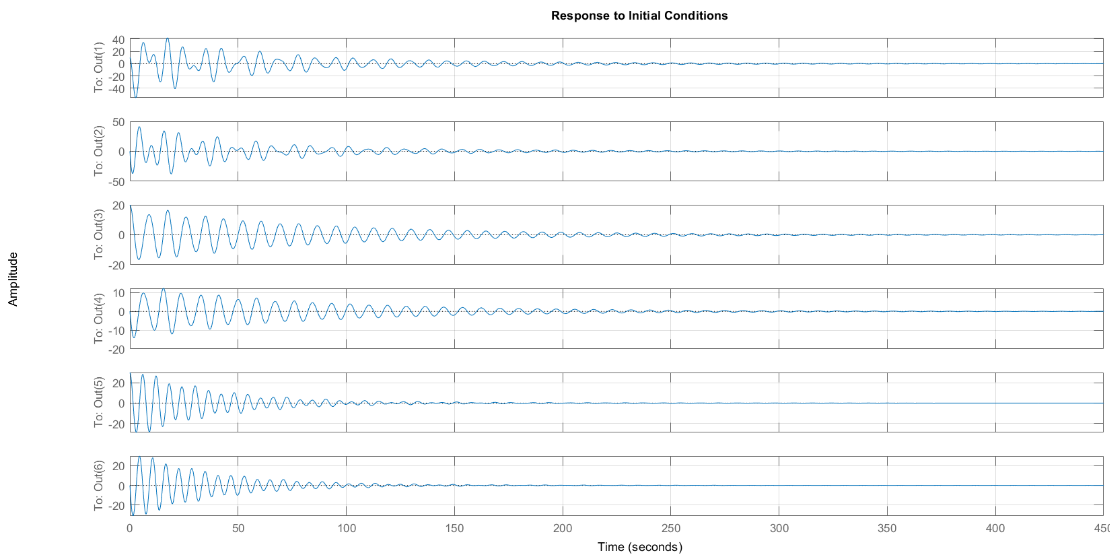

Figure 5.3: LQR response for closed loop system - with linear state feedback

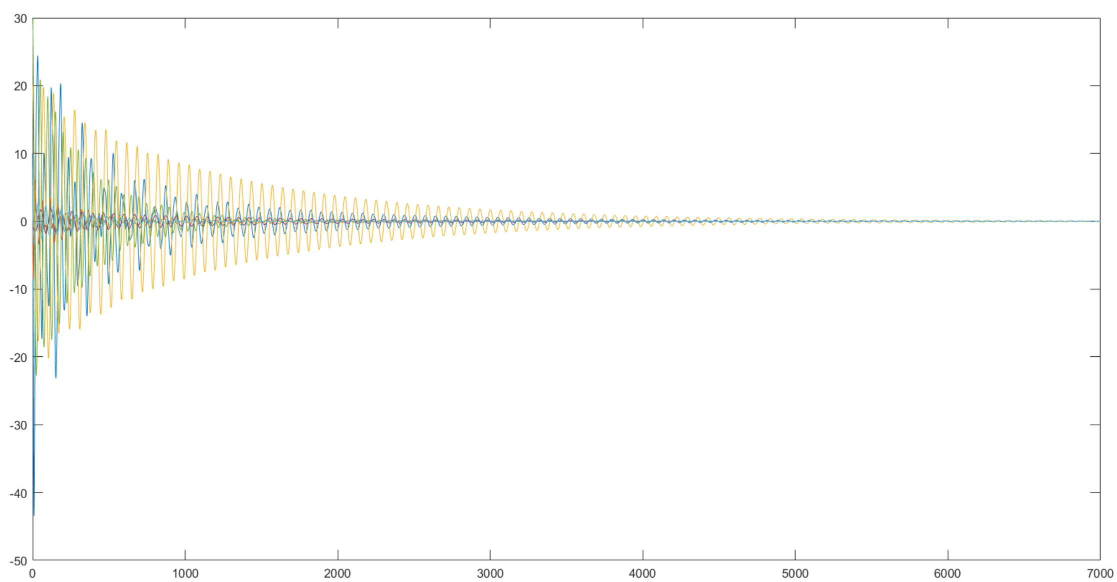

Figure 5.4: Plot for Non-linear system using LQR Controller

Lyapunov's indirect method stability certification

- As per Lyapunov's indirect method, the eigen values of pair (A, B_k) should be on the left half of the plane. Eigen values for the system with LQR controller is displayed below -

  Eigen value 1 = - 0.0118 + 0.7262i
  Eigen value 2 = - 0.0118 - 0.7262i
  Eigen value 3 = - 0.0271 + 1.0403i
  Eigen value 4 = - 0.0271 - 1.0403i
  Eigen value 5 = - 0.2548 + 0.2328i
  Eigen value 6 = - 0.2548 - 0.2328i

- It can be inferred from the above values that all the eigen values lie on the left half of the plane. So according to Lyapunov's indirect stability criterion, the system is stable.

Chapter 6

OBSERVABILITY OF THE SYSTEM

- The linear state equation is called observable on $[t_0, t_f]$ if any initial state x(t_0) = x_0 is uniquely determined by the corresponding response y_t, t $\in [t_0, t_f]$
- If the pair (A^T, C^T) is stabilizable then we say that (A, C) is detectable
- If the pair (A^T, C^T) is controllable then we say that (A, C) is observable
- The rank of the observability matrix can be found out using

$$\begin{bmatrix} C^T & A^T C^T & (A^2)^T C^T & (A^3)^T C^T & (A^4)^T C^T & (A^5)^T C^T \end{bmatrix}$$

- For the question 1(E), given four output vectors and we have to find among those output vectors where the linearized system is observable. This can be done by finding the rank of the observability matrix
- When the rank of the observability matrix is full the system is observable

The output for the question 1 - (E) is shown below for all the given output vectors:

–> For the output vector - x(t)
Rank of the observability matrix is 6. So, the system is observable

–> For the output vector - (θ_1(t), θ_2(t))
Rank of the observability matrix is 4. So, the system is not observable

–> For the output vector - (x(t), θ_2(t))
Rank of the observability matrix is 6. So, the system is observable

–> For the output vector - (x(t), θ_1(t), θ_2,(t))
Rank of the observability matrix is 6. So, the system is observable

The output for the question 1 - (F) is shown below:

```
L1 = 6×1
10^5 ×
      0.0004
      0.0070
     -1.1134
     -0.6959
      1.0541
      0.4426

L3 = 6×2
     25.4112    4.1945
    214.9357   69.9650
   -702.2536 -340.2790
   -626.8389 -379.6182
      1.3312   16.5888
     18.6937   67.4882

L4 = 6×3
     17.4440   -1.8833    0.0000
     72.9264  -17.8593   -0.9809
     -1.9594   18.5560   -0.0000
    -17.6676   82.5333   -0.0493
      0.0000   -0.0000    6.0000
      0.0000   -0.0981    6.9209
```

Figure 6.1: Observer Gain Matrices (L1, L3, L4)

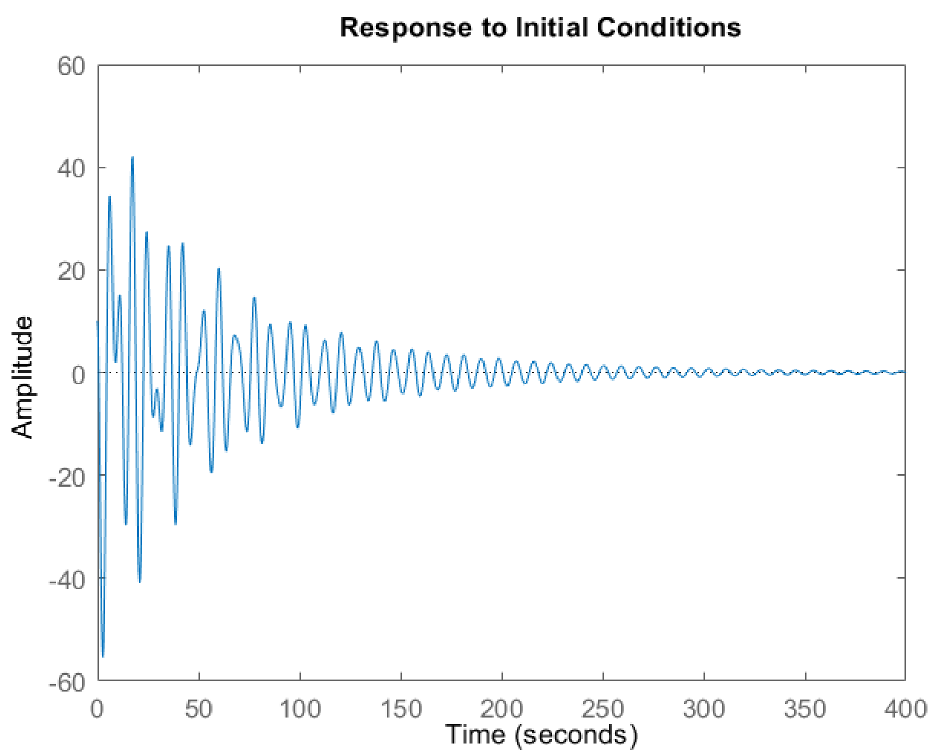

Figure 6.2: Luenberger Observer's response for initial condition for x(t)

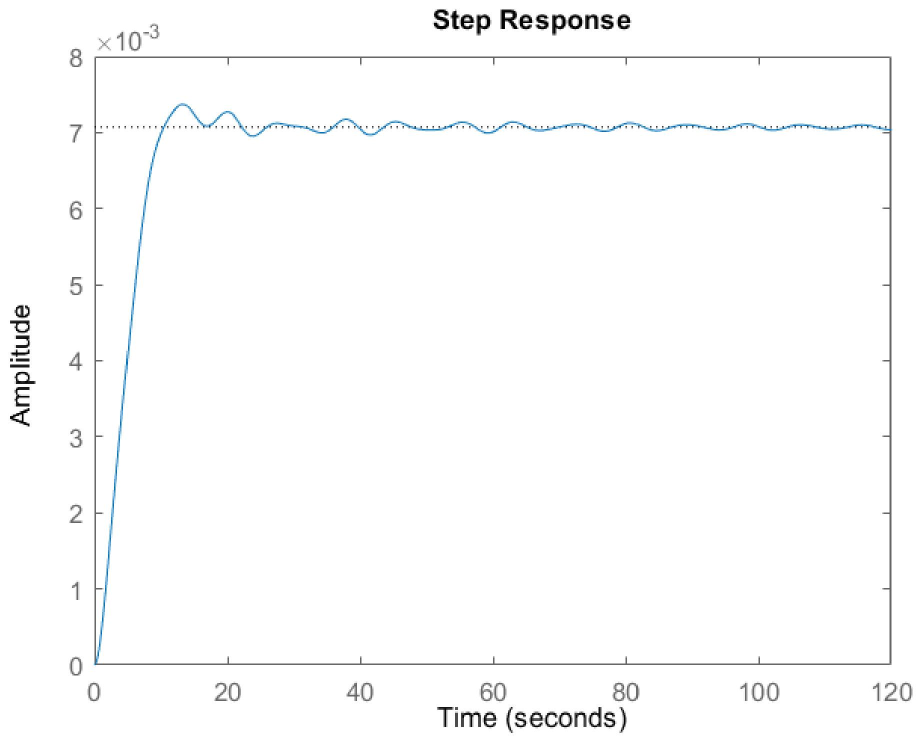

Figure 6.3: Luenberger Observer’s response for unit step input for x(t)

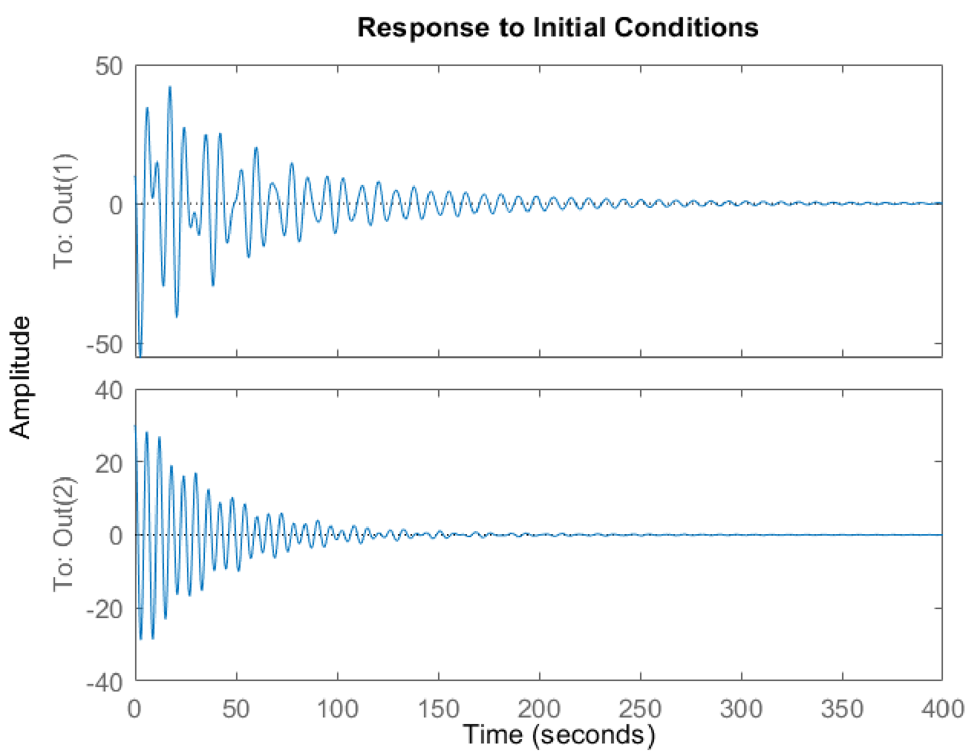

Figure 6.4: Luenberger Observer’s response for initial condition for x(t), θ_2

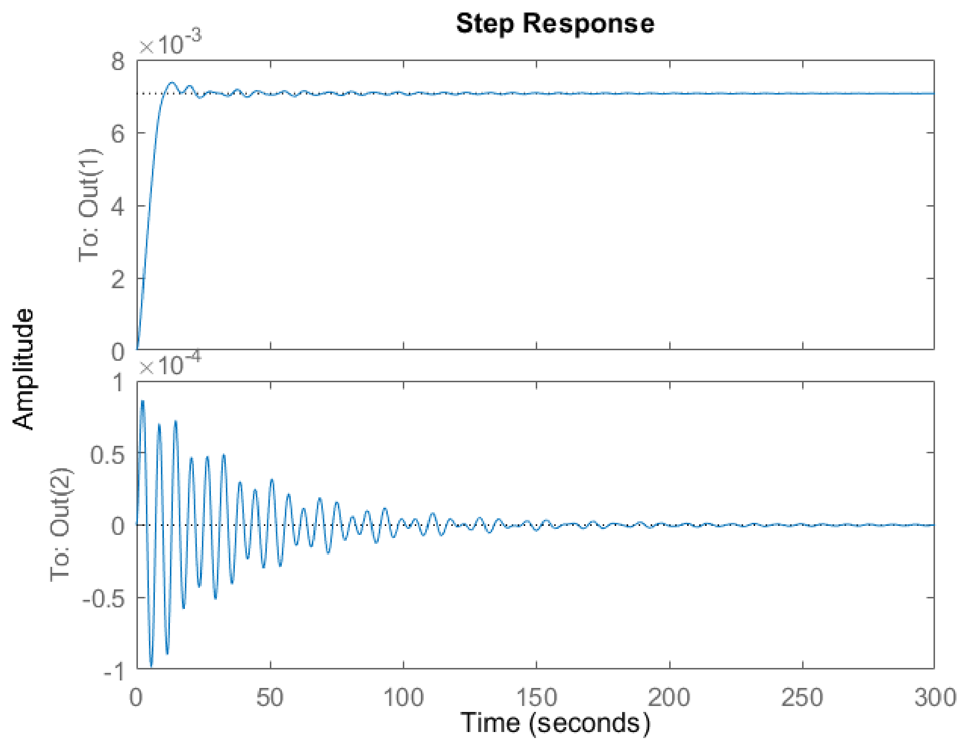

Figure 6.5: Luenberger Observer's response for unit step input for x(t), θ_2

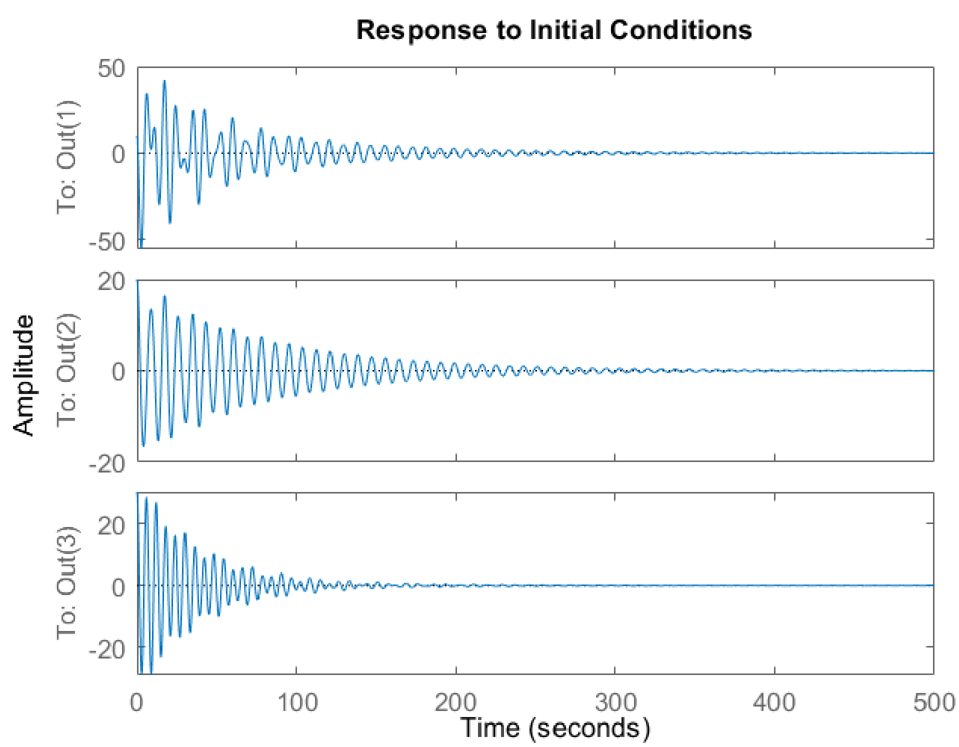

Figure 6.6: Luenberger Observer's response for initial condition for x(t), θ_1,θ_2

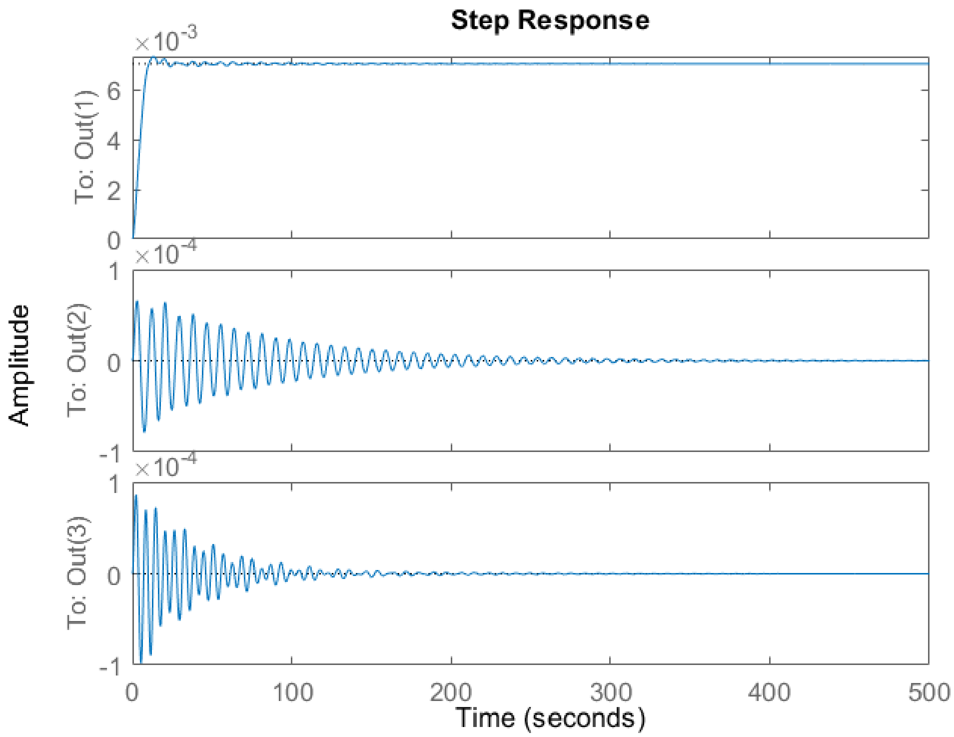

Figure 6.7: Luenberger Observer's response for unit step input for x(t), θ_1, θ_2

```
------Non-Linear System-------
L1 = 6×1
10^5 ×
       0.0004
       0.0070
      -1.1134
      -0.6959
       1.0541
       0.4426

L3 = 6×2
      25.4112    4.1945
     214.9357   69.9650
    -702.2536 -340.2790
    -626.8389 -379.6182
       1.3312   16.5888
      18.6937   67.4882

L4 = 6×3
      17.4440   -1.8833    0.0000
      72.9264  -17.8593   -0.9809
      -1.9594   18.5560   -0.0000
     -17.6676   82.5333   -0.0493
       0.0000   -0.0000    6.0000
       0.0000   -0.0981    6.9209
```

Figure 6.8: Non-Linear System - Observer Gain Matrices (L1, L3, L4)

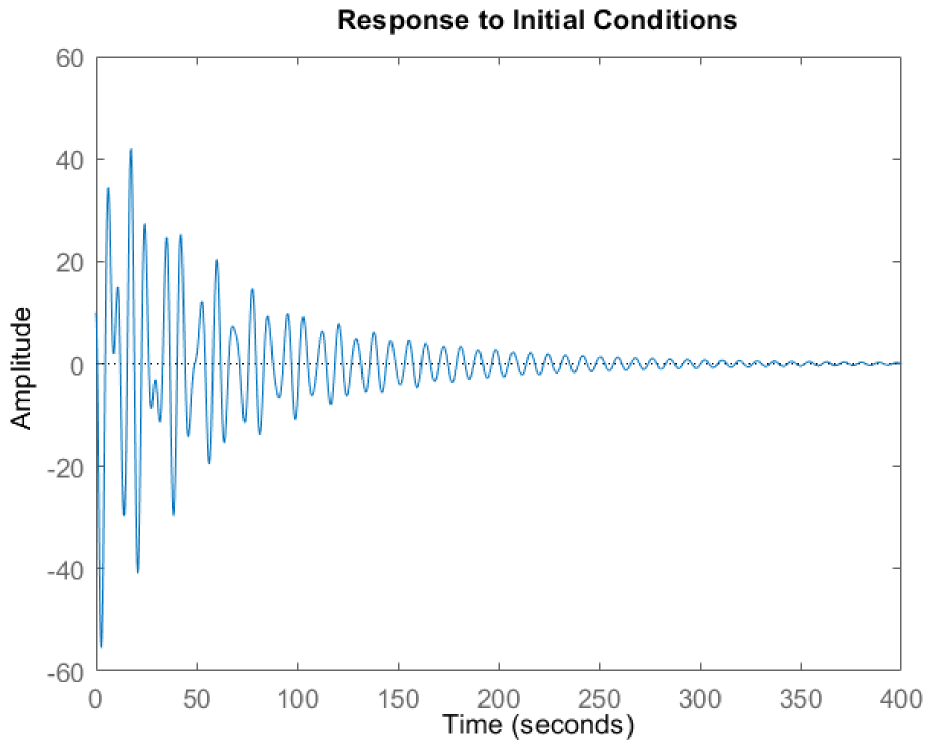

Figure 6.9: NL - Luenberger Observer's initial response condition for x(t)

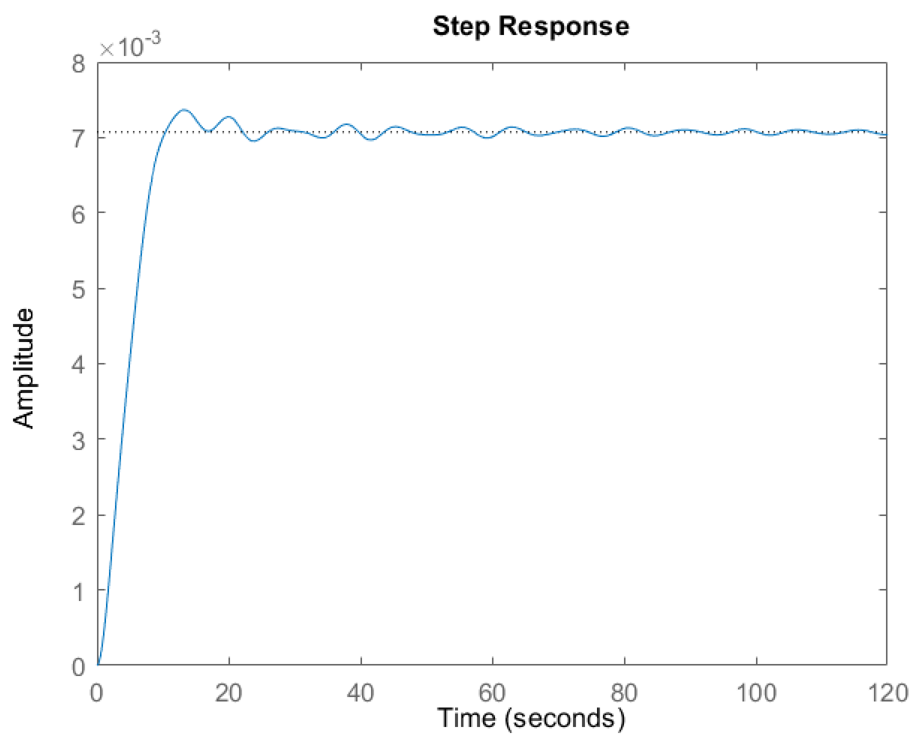

Figure 6.10: NL - Luenberger Observer's response for unit step input for x(t)

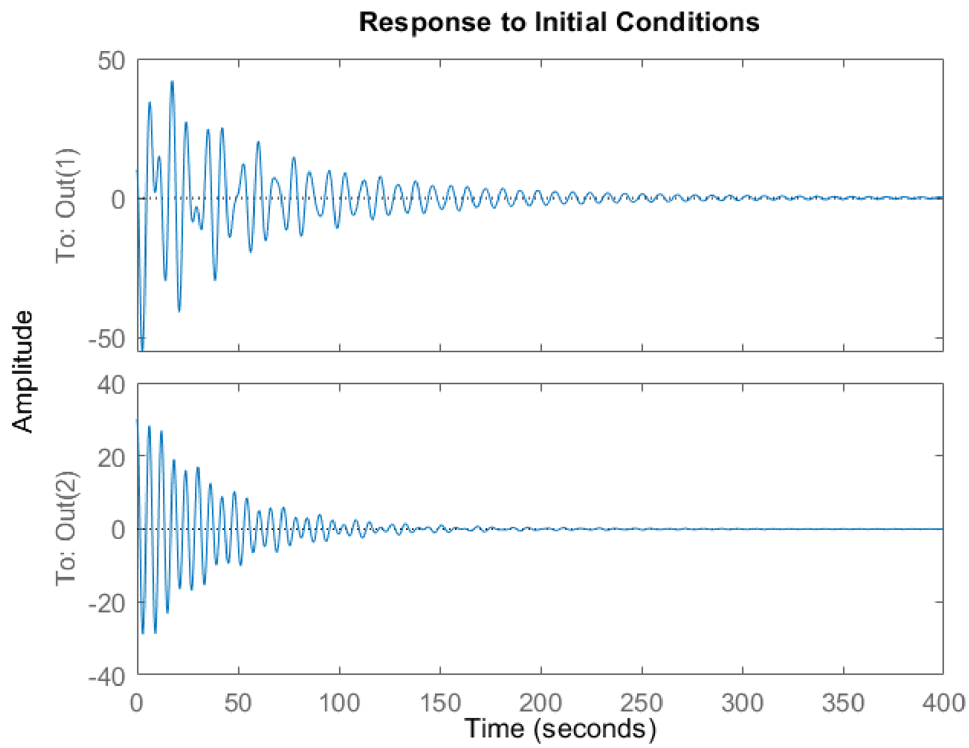

Figure 6.11: NL - Luenberger Observer's initial response condition for x(t), θ_2

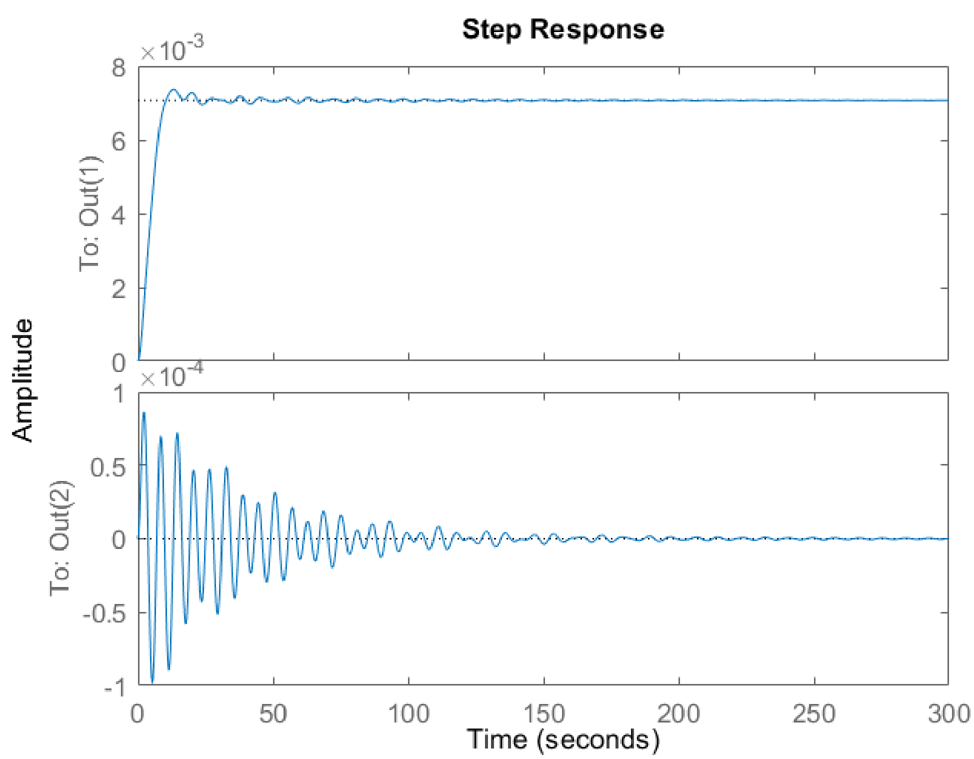

Figure 6.12: NL - Luenberger Observer's response for unit step input for x(t), θ_2

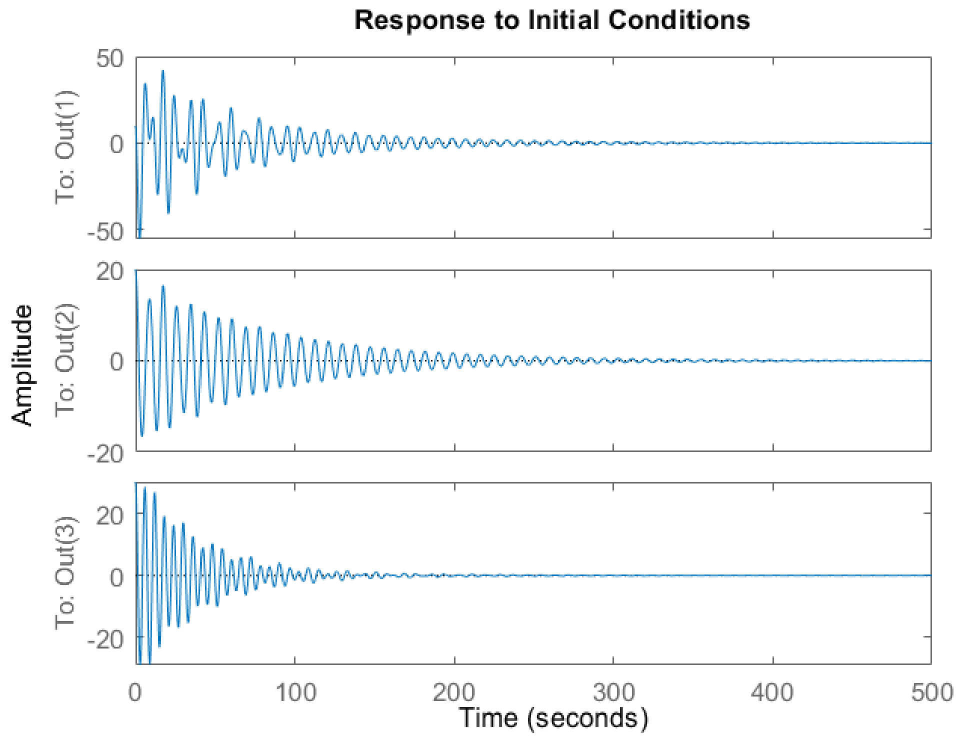

Figure 6.13: NL - Luenberger Observer's initial response condition for x(t), θ_1, θ_2

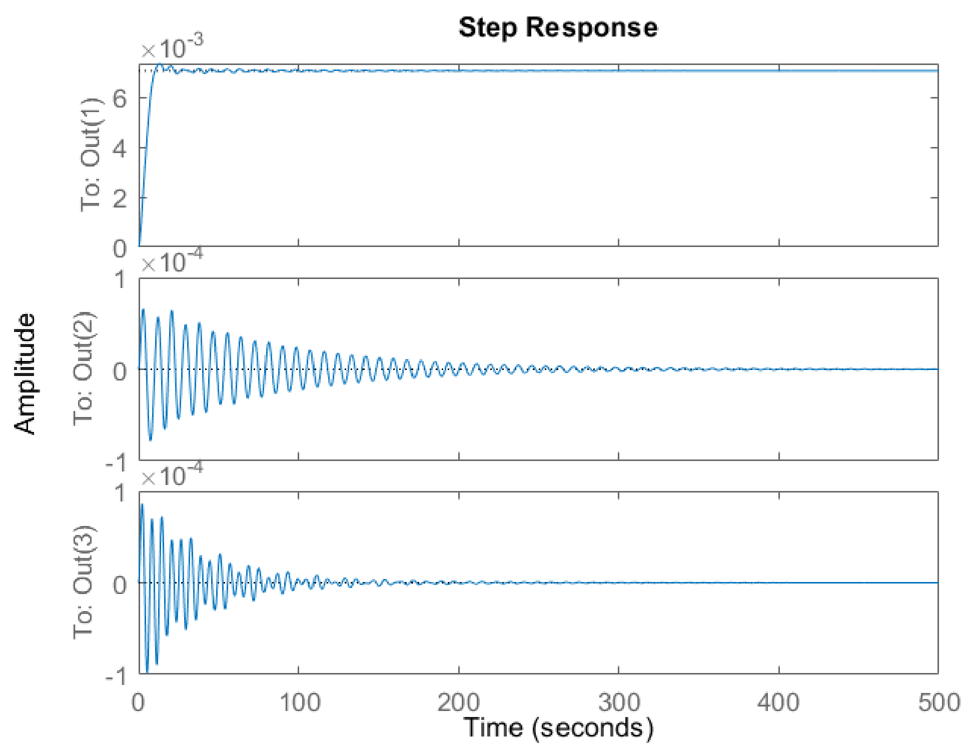

Figure 6.14: NL - Luenberger Observer's response for unit step input for x(t), θ_1, θ_2

Chapter 7

LQG CONTROLLER

- The LQG controller is nothing but the combination of LQR controller and the Kalman filter. The below responses and performance of system with LQG controller are simulated as below using Matlab.
- According to the problem statement, an output feedback controller of the smallest output vector x(t) is designed and LQG method is applied to the original nonlinear system.
- The simulation shows the LQG controller's performance response for initial condition and response to step input and for the non-linear system

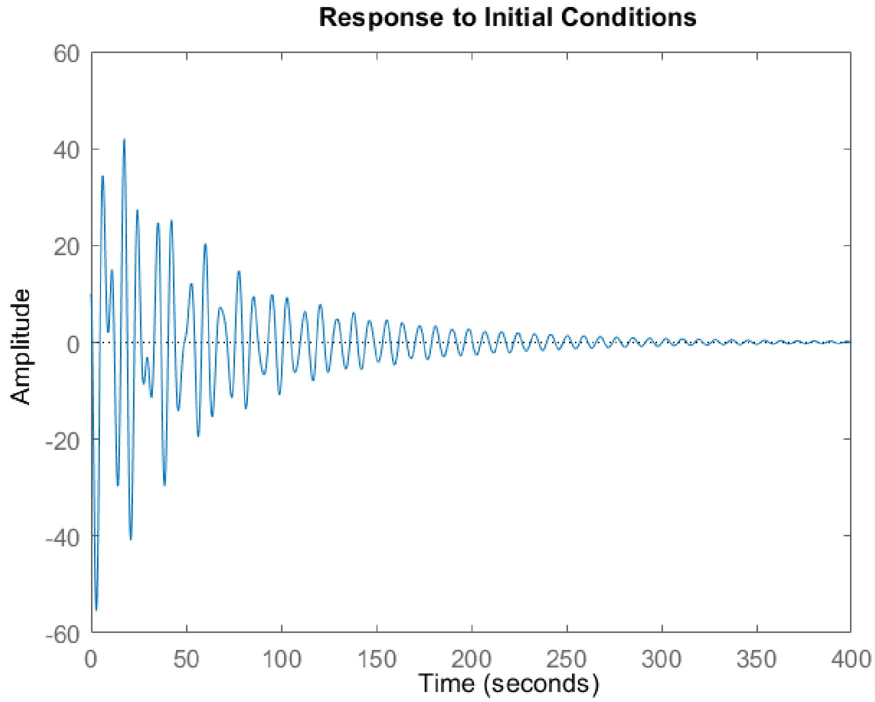

Figure 7.1: LQG controller's performance response for initial condition - System 1

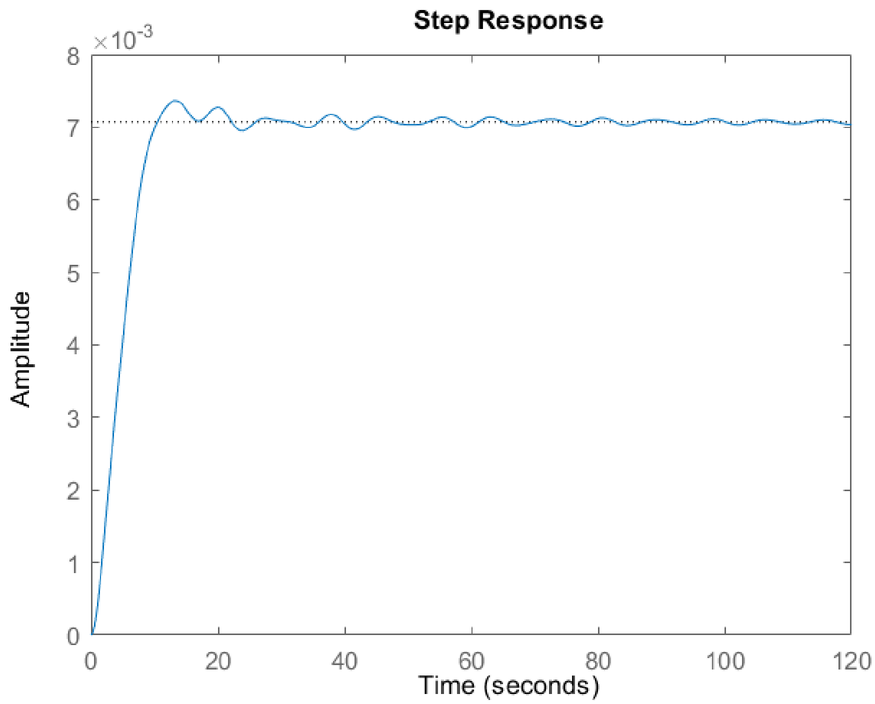

Figure 7.2: LQG controller's performance response for unit step input - System 1

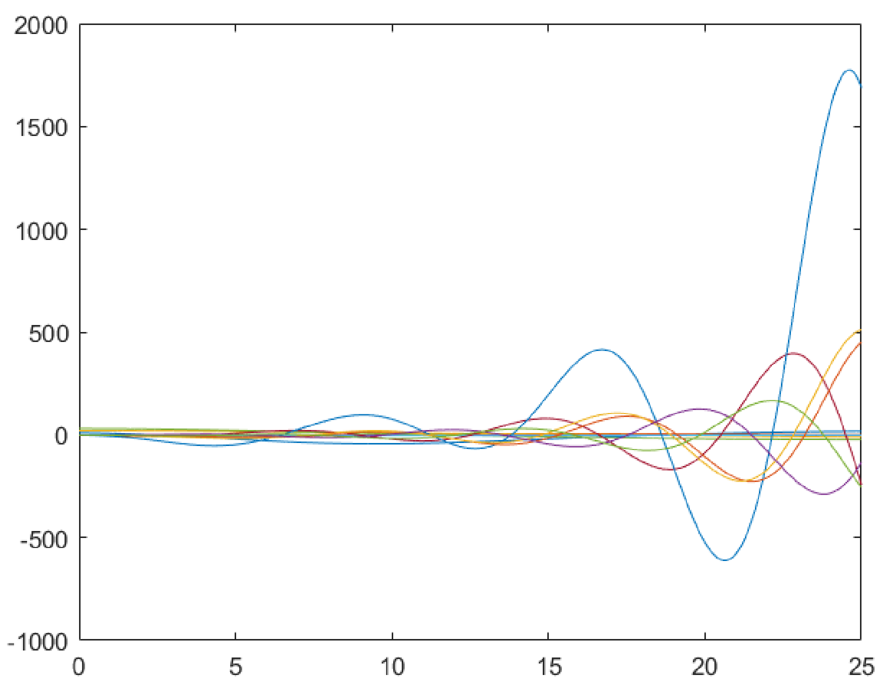

Figure 7.3: Plot for Non-linear system using LQG Controller

- Re-configuring Controller to asymptotically track a constant reference on x - For the most optimal Reference Tracking, main aim is to minimize the following cost function:

  $$\int_0^\infty (X(t) - X(d))^T(t)Q(X(t) - X(d)) + (Uk - U\infty)^T R(Uk - U\infty)dt$$

  If there is U_∞ such that $AX_d + B_k U_\infty$, then the optimal solution is given by $U(t) = K(X(t) - X_d$, where $K = -R^{-1}B_k^T P$ and P is the symmetric positive definite solution of the Ricatti equation.

  The controller is modified to minimize the optimal reference tracking cost function.

- The controller design can reject constant force disturbances applied on the cart. These disturbances are accounted by using the Gaussian process in the LQG controller and minimize the error.

  To run the Matlab code check the readme.md file attached in the file.