Comparing the binomial tree model against the Black-Scholes model

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***Abstract*—This report will derive how continuous compounding is obtained, it will explain the differences between call and put options. Moreover, a binomial tree model and the Black-Scholes models will be compared with each other for their option value and hedge parameter. These models matched with each other when European pricing was applied. The binomial model also showed that a call option has a larger value than a put option for American pricing.**

I. Introduction

THE financial market is a seemingly random market at first glance. The price of derivatives is a constant chancing value. This might cause a buyer or seller to misprice a derivative of his. Knowing the underlying characteristics of the derivative gives an advantage on the financial market. This allows the buyer to hedge the risks of the derivative or it would be possible for the buyer or seller to close an arbitrage profit. Furthermore, it allows the buyer and seller to set a price on the option of the derivative.

This report will look into two models to predict the behaviour of a derivative; a binomial tree model and the Black-Scholes model. The binomial tree model is a numeric approximation of the derivative. With the binomial model, it is possible to give an option value to the derivative. The Black-Scholes model on the other hand is an analytical formula, which also gives the option value, but is limited to specific pricing types. Not only are both models able to compute an option value, but they are also able to compute a hedge parameter. Both models will be compared with each other.

II. Theory and method

A. Continuous compounding

The risk-free rate r is the rate at which any money invested in the market or saved in a bank increases its value over time through interest. This interest is added to the original sum with a specific frequency, where the amount of money B after a year is given by

$$B = C * \left(1 + r * \frac{1}{f}\right)^f \tag{1}$$

where C is the originally invested amount of money and f is the frequency at which the interest is added to the sum per year.

When *continuous compounding* is used, the interest does not get added to the money sum in discrete periods per year, but rather continuously through time. Here, there is theoretically an infinite number of periods per year where interest gets added, with f going to ∞. Thus, after a year the formula is given by

$$B = C * \lim_{f\to\infty} \left(1 + r * \frac{1}{f}\right)^f \tag{2}$$

where the limit is defined as the exponential function for r and the formula becomes

$$B = C * e^r. \tag{3}$$

Calculating the amount of money after two years is possible by plugging in the formula for the growth of the original sum C after one year as the starting point in Formula 3 and applying another year of growth:

$$B_{\text{two years}} = B_{\text{one year}} * e^r = C * e^r * e^r = C * e^{2r}. \tag{4}$$

In this way, the growth using continuous compounding after a time of Δt years becomes

$$B = C * e^{r\Delta t}. \tag{5}$$

1) Coupon bond: For example, continuous compounding can be used in a *coupon bond*. Here, one party (the holder) gives another party (the issuer) an initial sum of money. The issuer must pay the holder interest in the form of periodic coupons of a certain constant value. The issuer pays these coupons until the bond has matured at which point it pays the holder the *principal* amount of money. In this way, the coupon bond is like a loan.

For example, a coupon bond could require the issuer to pay coupons of €300 every three months for two years, at which point they pay the holder a principal amount of €50.000. In this market the risk-free rate $r = 1.5\%$. Now, the *fair value* f_0 of the bond can be determined by calculating the initial sum of money the holder would

”loan” the issuer at $t = 0$ for which the issuer would make no profit or loss after the bond has matured.

At time $t = 0$, the issuer would have an amount of money of f_0. In three months, this sum increases its value through the risk-free rate by a factor $e^{0.015*3/12} = e^{0.00375}$ using continuous compounding, but the issuer also has to pay the first coupon. Thus, after three months the issuer has

$$f_0 * e^{0.00375} - €300.$$

This amount of money again increases its value through the risk-free rate in three months, with another coupon being paid at the end of those three months, so after six months the issuer has

$$(f_0 * e^{0.00375} - €300) * e^{0.00375} - €300.$$

This repeats for two years ($2 * 4 = 8$ coupon periods) until the bond matures, at which point the issuer pays the holder the principal amount of €50.000. Thus, after maturity, the issuer has

$$(((f_0 * e^{0.00375} - €300) * e^{0.00375} - €300)...) - €50000$$

which can be expressed as

$$f_0 * e^{0.00375*8} - \sum_{n=0}^{7}[€300 * e^{0.00375*n}] - €50000.$$

The fair value can be determined by setting this formula equal to zero, which would mean that the issuer has made no profit or loss in the end:

$$f_0 * e^{0.00375*8} - \sum_{n=0}^{7}[€300 * e^{0.00375*n}] - €50000 = €0.$$

Rewriting for f_0 gives a fair value of

$$f_0 = \frac{€50000 + \sum_{n=0}^{7} €300 * e^{0.00375*n}}{e^{0.00375*8}} = €50882,20.$$

2) Forward contract: Another example where continuous compounding can be used is a *forward* contract, which is a contract that specifies that one party is going to buy an asset S from another party at time T for the future price K. When the contract begins at $t = 0$, K is set to the *forward price* F. The asset changes in value over time with the risk-free rate r using continuous compounding. The no-arbitrage principle says that neither of these two parties should make a profit or a loss from this deal in the end, assuming only the risk-free rate changes the value. This means that the delivery price K must be equal to the price F that the asset S has at time T - if K is lower, the buying party could sell the asset elsewhere at the true price for a profit, and if K is higher, the buying party paid more for the asset than it is worth. If the asset has a price S_0 at $t = 0$, the forward price F_0 at time 0 can be determined by plugging S_0 into Equation 5 with $\Delta t = T$:

$$F_0 = S_0 * e^{rT}. \tag{6}$$

B. Call and put options

A European option is a contract that gives a party the right to buy or sell an asset S at time T for a strike price K. The contract for buying the asset is called a *call*, with a value of C_t, and the one for selling it a *put*, with a value of P_t. Consider an initial portfolio with a call option to buy a stock S and . After time T, this investment has increased through the continuously compounded risk-free rate to $K * e^{-rT} * e^{rT} = K$, and is thus equal to the strike price. At time T, the party behind this portfolio now has the opportunity to buy the stock S, but will only do so if they can make a profit from the deal. This happens if the value of the stock at that point, S_T, is higher than the strike price. Then, the party gets the stock and loses the strike price, for a payoff of $S_T - K$ - they then have S_T worth of stock. If the stock price is lower than the strike price, the party will simply not exercise their right to buy the stock and the payoff is 0 - they then keep the K amount of money. The portfolio is thus worth $\max(S_T, K)$ at time T. The value of the call option is

$$C_t = \max(S_T - K, 0) \tag{7}$$

which results in a payoff diagram like the one shown in Figure 1.

An example of a put option would be if a party has a portfolio containing a share of stock S and a put option. Here, the opposite is true; if the value S_T of the stock at time T is higher than K, they will not use their option because they could sell the stock for the price S_T elsewhere instead. If S_T is lower than K, they will use their right to sell because they will receive more money than the stock is actually worth. The option value for the put option becomes

$$P_t = \max(K - S_T, 0) \tag{8}$$

resulting in the payoff diagram shown in Figure 2. They either keep the stock or get the money K, so again the portfolio is worth $\max(S_T, K)$ at time T.

If at any point in time the aforementioned call portfolio is worth more than the put portfolio, an arbitrage arises. Then, someone could sell the call portfolio and buy the put portfolio which makes a profit. The opposite would be true if it was worth less. The no-arbitrage

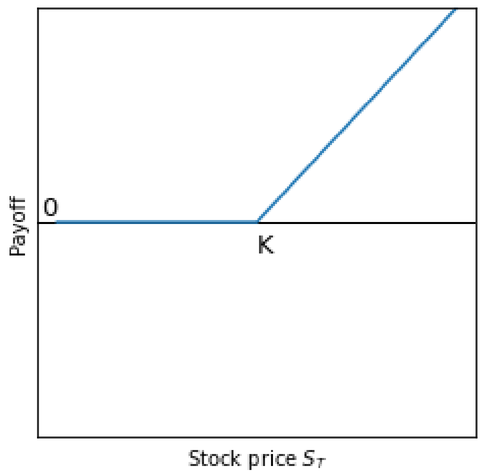


Fig. 1: The payoff diagram for a portfolio with a call option with strike price K and an investment of $K * e^{-rT}$ in the money-market.

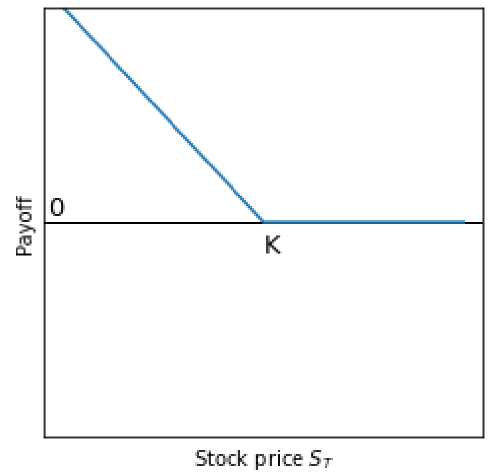


Fig. 2: The payoff diagram for a portfolio with a share of stock S and a put option with strike price K.

principle says that this should not be possible, so both portfolios should be worth the same at all time points t between 0 and T. The call portfolio consists of a call option and an investment of $K * e^{-rT}$ in the money-market at time $t = 0$. At time t, the call option is worth C_t and the investment has grown to $K * e^{-rT} * e^{rt} = K * e^{-r(T-t)}$. The put portfolio consists of the stock S and a put option. At time t these have values of S_t and C_t, respectively. The values of both portfolios should be equal, resulting in the formula

$$C_t + K * e^{-r(T-t)} = S_t + C_t \tag{9}$$

which is called the *put-call parity*.

C. Binomial tree model

The binomial tree model attempts to compute the option price of a stock. To compute the option price, first, the stock price needs to be calculated for each branch in the binomial tree. It does this by discretising time into N intervals, each with a time step of Δt. Each time step the stock can either increase or decrease in value, with a probability p and $1 - p$, respectively. When the stock permutates upwards the value of the stock is multiplied by u and when the stock permutates downwards the stock value is multiplied by d, where u and d are given by

$$u = e^{\sigma\sqrt{\Delta t}} \tag{10}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{11}$$

From this, it is obtained that the stock price i steps into the binomial tree is given by

$$S_{i,j} = S_0 u^j d^{i-j} \tag{12}$$

where j is one of the nodes at timestep i. This allows the entire binomial stock tree to be build.

From the stock price binomial tree, the option value can be calculated. At the ends of the branches, the option value is calculated based on the option type. These values are given by Equation 7 for a call option and by Equation 8 for a put option.

The option values within the tree are then calculated through backwards induction. At each node, the option value is calculated by its two branches. This computation depends on the pricing type. For European pricing the option value becomes

$$f = e^{-r\Delta t}(pf_u + (1-p)f_d) \tag{13}$$

where f_u revers to the option price from the branch above it and f_d refers to the option price from the branch below it. The probability p is given by

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{14}$$

With American pricing, however, the option can be exercised at any time. Therefore, the American option is worth as least as much as the European option with a minimum of zero. Thus the American option value is

$$f = \max\left(e^{-r\Delta t}(pf_u + (1-p)f_d), 0\right) \tag{15}$$

This process continues until the option value is finally calculated at the very start of the tree.

From the binomial tree a hedge parameter Δ can be estimated. The hedge parameter gives the number of shares that need to be bought to create a hedge. This hedge parameter is given by

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \tag{16}$$

D. Black-Scholes model

The Black-Scholes model is a pricing model used to determine the theoretical value for either a call or a put option, based on the Price of the underlying asset S, Strike price of the option K, risk-free rate r, time to expiration t and normal distribution N.

The following derivation is made to replicate why $\Delta_t = N(d_1)$:, given that Δ_t:=$\delta C_t/\delta S_t$

The price of a call option is given as:

$$C = SN(d_1) - e^{-r\tau} KN(d_2)$$

$C = SN(d_1) - e^{-r\tau} KN(d_1 - \sigma\sqrt{\tau})$
$\Delta_t = \frac{\partial C}{\partial S} = \frac{\partial}{\partial S}\left(SN(d_1) - \frac{\partial}{\partial S}\left(e^{-r\tau} KN(d_1 - \sigma\sqrt{\tau}\right)\right)$
$= \left(N(d_1)\frac{\partial}{\partial S}S + S\frac{\partial}{\partial S}N(d_1) - Ke^{-r\tau}\frac{\partial}{\partial S}\left(N(d_1 - \sigma\sqrt{\tau}\right)\right)$
$= \left(N(d_1) + \mathrm{Sn}(d_1)\frac{\partial}{\partial S}(d_1) - Ke^{-r\tau} n(d_1 - \sigma\sqrt{\tau})\frac{\partial}{\partial S}(d_1 - \sigma\sqrt{\tau}\right)$
$= N(d_1) + \mathrm{Sn}(d_1)\frac{\partial(d1)}{\partial S} - Ke^{-r\tau} n(d_1 - \sigma\sqrt{\tau})\frac{\partial(d_1)}{\partial S}$

Hereafter, we express $n(d_1 - \sigma\sqrt{\tau})$ in common terms

$n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$
$n(d_1 - \sigma\sqrt{\tau}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(d_1 - \sigma\sqrt{\tau}\right)^2}$
$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} - \frac{\sigma^2\tau}{2} + d_1\sigma\sqrt{\tau}}$
$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} e^{-\frac{\sigma^2\tau}{2} + d_1\sigma\sqrt{\tau}}$
$= n(d_1)\, e^{-\frac{\sigma^2\tau}{2} + d_1\sigma\sqrt{\tau}}$
$= n(d_1)\, e^{-\frac{\sigma^2\tau}{2} + \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right)\sigma\sqrt{\tau}}$
$= n(d_1)\, e^{-\frac{\sigma^2\tau}{2} + \ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}$
$= n(d_1)\, e^{\ln\left(\frac{S}{K}\right) + r\tau}$
$= n(d_1)\, e^{\ln\left(\frac{S}{K}\right)} e^{r\tau}$
$= n(d_1)\, \frac{S}{K} e^{r\tau}$

After substituting $n(d_1 - \sigma\sqrt{\tau})$ for $n(d_1)\frac{S}{K}e^{r\tau}$:

$\Delta_t = N(d_1) + Sn(d_1)\frac{\partial(d1)}{\partial S} - Ke^{-r\tau} n(d_1 - \sigma\sqrt{\tau})\frac{\partial(d_1)}{\partial S}$
$\Delta_t = N(d_1) + Sn(d_1)\frac{\partial(d1)}{\partial S} - Ke^{-r\tau} n(d_1)\frac{S}{K}e^{r\tau}\frac{\partial(d_1)}{\partial S'}$
$\Delta_t = N(d_1) + Sn(d_1)\frac{\partial(d1)}{\partial S} - Sn(d_1)\frac{\partial(d_1)}{\partial S}$
$\Delta_t = N(d_1)$

Put call theory: Rearranging the terms in the *put-call parity* formula from Equation 9 yields

$$P_t = C_t - S_t + e^{-r(T-t)} K$$

The formula for C_t is defined as:

$$C_t = S_t N(d_1) - e^{-r\tau} KN(d_2)$$

P_t can be rewritten as:

$$P_t = S_t N(d_1) - e^{-r\tau} KN(d_2)) - S_t + e^{-r(T-t)} K$$

Then rearranging this formula gives:

$$P_t = S_t N(d_1) - e^{-r\tau} KN(d_2)) - S_t + e^{-r(T-t)} K$$

$$P_t = S_t(N(d_1) - 1) + e^{-r\tau} K(1 - N(d_2))$$

$$P_t = -S_t N(-d_1) + e^{-r\tau} KN(-d_2))$$

1) Euler method: A hedging simulation is performed using the Euler method. The Euler method is a first-order numerical procedure to solve Stochastic differential equations with one or multiple given values. This is useful to simulate stock price movement, and in turn create a hedging simulation to account for these movements to remain risk-neutral. Ito's differential equation is given as such:

$$\Delta X_t = a(t, X_t)\,\Delta t + b(t, X_t)\,\Delta z_t$$

A property of the Brownian motion is described as follows: $\Delta z_t := z_{t+\Delta t} - z_t \sim \mathcal{N}(0, \Delta t)$. By decreasing the step size of Δt, the simulation will become more accurate, but will also result in a higher computational load as it requires more simulation steps. In this assignment, the Euler method is used to analyse different values of volatility, and hedging adjustments to compute the resulted differences in Hedging strategy for the Black-Scholes and binomial tree model.

III. Results

The binomial model is tested against the Black-Scholes formula. The test is performed with the following parameters: $S = 100$, $\sigma = 0.2$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$, with a call option and a European pricing type. These parameters give a option value of 11.54643 when calculated through the binomial model. The Black-Scholes formula gives a option value of 11.54428. Thus the binomial model overvalues the option value by $2.15 \cdot 10^{-3}$.

By increasing N the binomial tree is better able to converge towards the analytical Black-Scholes value. This is shown in Figure 3. Initially, the binomial tree undervalues the option. In each step the binomial tree overcorrects itself, which results in the oscillating behaviour. These oscillations reduce after more steps are taken.

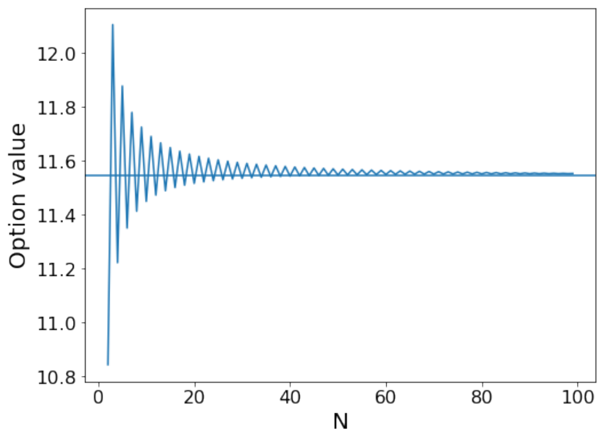


Fig. 3: The convergence of the option value of the binomial model as N increases. The option value converges towards the analytical value of the Black-Scholes model. The binomial model ran with parameters: $S = 100$, $\sigma = 0.2$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$, with a call option and a European pricing type. The horizontal line is the analytical value of the Black-Scholes model.

A. *Comparison binomial and Black-Scholes*

The next step is to compare the binomial model with the Black-Scholes formula for different volatility's. The binomial tree uses the parameters: $S = 100$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$, with a call option and a European pricing type. Figure 4a shows the option value for different volatility's alongside the Black-Scholes value. Figure 4b shows the relative error between the two. This shows that at low volatility's the binomial tree overvalues the option value by $\sim 0.1\%$, while at higher volatility's the binomial tree undervalues the option price by $\sim 0.3\%$.

The hedge parameter for the binomial tree is also compared with its Black-Scholes counterpart. This is performed with the parameters: $S = 100$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$, with a call option and a European pricing type. The results are shown in Figure 5a. The relative error is shown in Figure 5b. For all volatility's the binomial tree gives a lower hedge parameter. For low volatility's this effect is most pronounced.

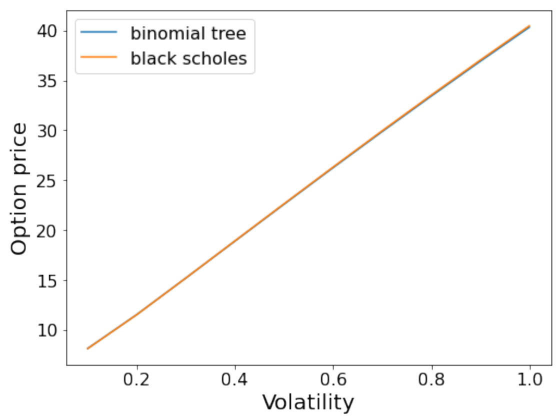
B. *Binomial put and call option comparison*

The put and call option values for an American option pricing are compared with each other in Figure 6. The parameters used were $S = 100$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$. For all volatility values the call option had a larger option value. The option values for both the put option as the call option increase linear with the volatility.

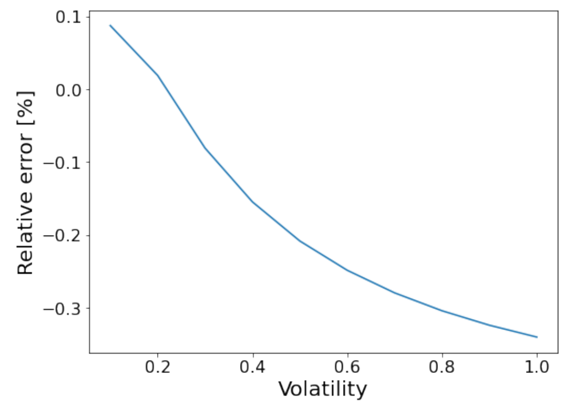
IV. DISCUSSION

From Figure 4 it was shown that the binomial tree and the Black-Scholes model are able to give an option value which are only a few tents of a percentage off from each other when using European pricing. The hedge parameter was also just a few tents of a percentage off, as shown in Figure 3. Thus the binomial model is a good method to compute various attributes of a derivative. Furthermore, the binomial model also allows the pricing of options even when an analytical solution does not exist as is the case for American pricing.

Unfortunately, the Euler method was not able to be implemented due to time constraints.

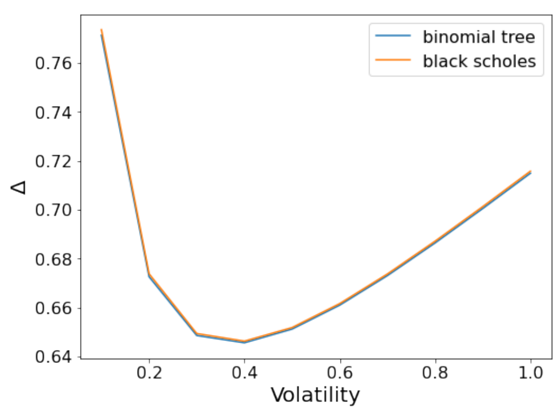


(a) The option price as a function of the volatility for both the binomial model and the Black-Scholes model.

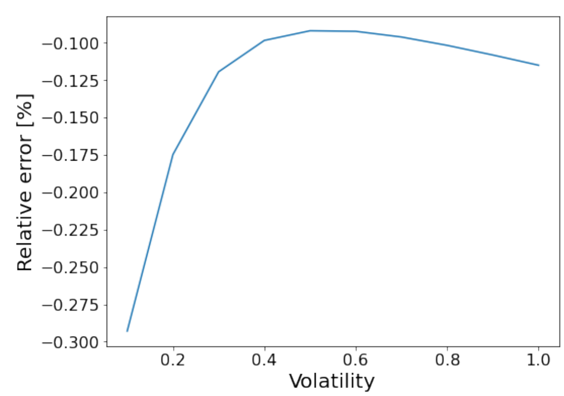


(b) The relative error of the option value computed through the binomial model relative to the analytical Black-Scholes model.

Fig. 4: Comparing the option value between the binomial model and the Black-Scholes model. The parameters used were $S = 100$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$, with a call option and a European pricing type.



(a) The hedge parameter as a function of the volatility for both the binomial model and the Black-Scholes model.



(b) The relative error of the hedge parameter computed through the binomial model relative to the analytical Black-Scholes model.

Fig. 5: Comparing the computed hedge parameter between the binomial model and the Black-Scholes model. The parameters used were $S = 100$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$, with a call option and a European pricing type.

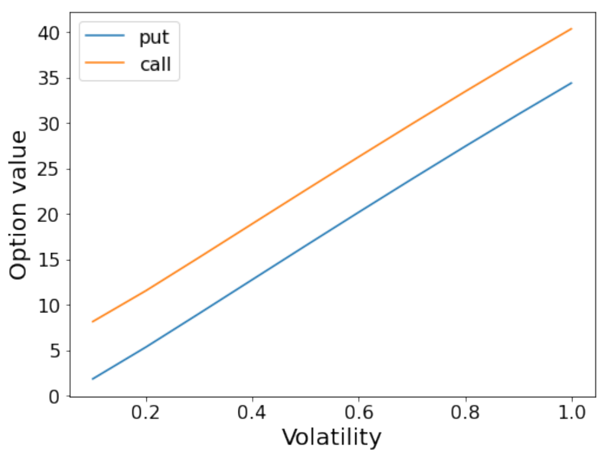


Fig. 6: Comparing the option value for an American put and call option as a function of the volatility. The parameters used were $S = 100$, $T = 1$, $N = 50$, $r = 0.06$, $k = 99$.