Investigating different Monte Carlo techniques in Finance

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***Abstract*—Monte Carlo methods can be used to price more advanced derivatives for which no deterministic methods are available. This report investigates the accuracy of Monte Carlo methods for calculating option values, hedge parameter δ using the bump-and-revalue, pathwise, and likelihood ratio method, and finally applies the control variate technique to reduce the variance in the price of an Asian call option. The Monte Carlo method for option valuation converges to the deterministic solution for a European put option as the number of iterations increases. The bump-and-revalue method works to calculate δ for options with a mostly smooth payoff function, but if that is not the case the pathwise and likelihood ratio methods are necessary to improve the accuracy, with the likelihood ratio performing the best. Future research is suggested to improve the reliability of the pathwise method. The control variate technique reduces the variance in the price of an Asian call option significantly compared to the standard Monte Carlo method.**

I. Introduction

Simple derivatives can be priced through deterministic methods. These can be analytical solutions or might be computed through a binomial model [1]. More advanced derivatives, however, do not have this luxury. Monte Carlo methods are used to price the more advanced derivatives. Understanding the advantages and disadvantages of these Monte Carlo methods can give an edge on the financial market.

This report will set up the basis to simulate stocks with a Monte Carlo method. Several options will be described; these include a European put, European call, digital, and an Asian call option for both arithmetic and geometric means. Furthermore, this report will look into three methods to obtain the hedge parameter δ using Monte Carlo simulation. These methods are the bump-and-revalue, pathwise, and likelihood ratio method. At last, this report will also look at a variance reduction method called the control variates technique. This technique will be applied to an Asian call option with arithmetic averages.

II. Theory and method

A. Different payoffs

The type of option on a stock determines what the payoff of the derivative is. This report will describe several different types of options; European put, European call, digital, and an Asian option.

1) European put: The European put gives the seller the option to sell the derivative for the strike price K once maturity is reached. When the stock price at maturity S_T is worth less than the strike price the seller exercises his right to sell the derivative. When S_T however is larger than K the seller can choose to not use his option to sell the derivative. Therefore the payoff of the derivative is

$$\text{payoff} = \max\left(K - S_T, 0\right). \tag{1}$$

2) European call: The European call gives the buyer the option to buy the derivative for the strike price K once maturity is reached. When the stock price at maturity S_T is worth more than the strike price the buyer is able to exercise his right to buy the derivative. When S_T however is smaller than K the buyer can choose to not use his option to buy the derivative. Therefore the payoff of the derivative is

$$\text{payoff} = \max\left(S_T - K, 0\right). \tag{2}$$

3) Digital: A digital option only has two possible payoffs, either 0 or a predetermined amount of money P_D. Like the previously discussed option types, the payoff depends on the strike price K and the stock price at maturity S_T. A digital call option only has payoff P_D if $S_T > K$ while a digital put option only pays the sum when $S_T < K$. This report looks at digital calls, whose payoff function is then

$$\text{payoff} = P_D * I_{S_T > K}. \tag{3}$$

4) Asian call: The Asian call option has a payoff that is related to the average stock price during a specified time window. In this time window N samples are taken from the stock at an interval of

$$\Delta T = \frac{T - \tau}{N - 1}, \tag{4}$$

where T is the time of maturity and τ time at which the time window starts. Thus the time values at which the stock will be sampled are

$$t_i = \tau + i \cdot \Delta T, \tag{5}$$

for $i = 1$, ..., N. For this report the time-window will span the entire duration of the stock. Therefore τ will be set to 0 throughout this report.

From the sampled stock values a mean value μ is calculated. The payoff then becomes

$$\text{payoff} = \max\left(\mu - K, 0\right). \tag{6}$$

There are two methods to compute the mean value. This can be either the arithmetic mean or the geometric mean. The arithmetic mean is given by

$$\mu = \frac{\sum_{i=1}^{N} \mu_i}{N}, \tag{7}$$

where N is the number of values and μ_i are the data values.

The geometric mean on the other hand is given by

$$\mu = \left(\prod_{i=0}^{N} \mu_i\right)^{\frac{1}{N}}. \tag{8}$$

For the geometric Asian call option, an exact solution of its price exists, which is

$$C(S,T) = e^{-rT}\left(Se^{\widetilde{r}T}\phi(\widehat{d}_1) - K\phi(\widehat{d}_2)\right), \tag{9}$$

where ϕ is the standard normal cumulative distribution function and

$$\widehat{d}_1 = \frac{\log\frac{S}{K} + \left(\widetilde{r} + 0.5\widetilde{\sigma}^2\right)T}{\sqrt{T}\widetilde{\sigma}} \tag{10}$$

$$\widehat{d}_2 = \frac{\log\frac{S}{K} + \left(\widetilde{r} - 0.5\widetilde{\sigma}^2\right)T}{\sqrt{T}\widetilde{\sigma}} \tag{11}$$

with

$$\widetilde{\sigma} = \sigma\sqrt{\frac{2N+1}{6(N+1)}} \tag{12}$$

$$\widetilde{r} = \frac{r - 0.5\sigma^2 + \widetilde{\sigma}^2}{2} \tag{13}$$

B. Monte Carlo option valuation

The stock price S can be simulated through a Monte Carlo method. Each time step dt the stock changes in value by an amount dS, where

$$dS = rSdt + \sigma SdZ, \tag{14}$$

where r is the risk free rate, σ the volatility and $dZ = \phi\sqrt{dt}$, where ϕ is a random variable generated from a normal distribution with a mean of 0 and a standard deviation of 1. The next value of the stock is then

$$S_{i+1} = S_i + dS = S_i(1 + r\Delta t + \sigma\phi\sqrt{\Delta t}), \tag{15}$$

where Δt is the chosen time step.

There is no need to simulate the stock price in time increments when r and σ are fixed. When this is the case the stock price at maturity can directly be calculated through

$$S(T) = S_0 e^{(r-0.5\sigma^2)T + \sigma Z\sqrt{T}}, \tag{16}$$

where S_0 is the price at time $t = 0$, T is the time of maturity and $Z = \phi$.

By performing many runs the option value V can be computed by the arithmetic average of the payoffs of the stocks

$$V(S_0, t) = \frac{\sum_{m=1}^{M} \text{payoff}(S(t))}{M} \tag{17}$$

where M is the number of distinct stock evaluations. To compute the option value at time $t = 0$ the payoff is calculated by taking the discounted value, which becomes

$$V(S_0, t=0) = e^{-rT}\frac{\sum_{m=1}^{M} \text{payoff}(S(T))}{M} \tag{18}$$

C. Hedge parameter δ

The hedge parameter δ is an expression of how sensitive an option's value (at $t = 0$) can be to a change in the stock price S_0, and is defined here as

$$\delta = \frac{\partial V}{\partial S_0}. \tag{19}$$

1) Bump-and-revalue method: The value of δ can be approximated using the forward Euler method by changing S_0 with a very small "bump" ϵ, and dividing the resulting change in V by the change in S. This results in the approximated value

$$\delta \approx \frac{V(S_0 + \epsilon) - V(S_0)}{(S_0 + \epsilon) - S_0} = \frac{V(S_0 + \epsilon) - V(S_0)}{\epsilon}. \tag{20}$$

This formula is most accurate to the true value of δ when $\epsilon \to 0$, but a very small ϵ could result in machine precision inaccuracies when calculating it computationally. Thus, a suitable value should be chosen for ϵ.

If the two values of V that are subtracted are both computed from Equation 16 using separate random numbers Z, this could result in problems. The Monte Carlo simulation would need to run enough times that the variance in V is much smaller than the difference between $V(S_0 + \epsilon)$ and $V(S_0)$ in Equation 20. Since the goal is to have ϵ be as small as possible, this would likely require a very large amount of runs. A solution would be to use the same *seed* to generate the values of Z for both calculations, so that the same sequence of random numbers is used. Any error from the Monte Carlo simulation should hopefully be the same for both calculations, with only the difference remaining. This report investigates the bump-and-revalue method for both a fixed seed and a different seed.

2) Pathwise method: The bump-and-revalue method presents problems when it is applied to a digital option. In that case, even a small change ϵ to S_0 can move S_T beyond the "threshold" and instantly change the payoff by a large amount of money P_D. Since the small number ϵ is in the denominator of Equation 20, this big difference can get even bigger and result in a very inaccurate δ.

One way to get a usable value for δ is using the pathwise method, which can be applied on the stocks used in this report. Indicating the payoff function at time T as $f(S_T)$ and recalling that V is the average of the discounted payoff over many runs, the pathwise method works by rewriting the formula of δ to

$$\begin{aligned} \delta &= \frac{\partial[\text{mean}(e^{-rT} * f(S_T))]}{\partial S_0} \\ &= e^{-rT} * \text{mean}\Big(\frac{\partial f(S_T)}{\partial S_0}\Big) \end{aligned} \tag{21}$$

where instead of differentiating over the average, now δ is calculated by averaging the derivative. This trick only works on the condition that the payoff function is mostly smooth. Of course, this is not true for a digital option, so the payoff function must be smoothed first. In this report, the digital payoff function f is smoothed by replacing it with the cumulative distribution function of a normal distribution with a mean of K (multiplied by the payoff amount P_D). The amount of smoothing can be adjusted by changing the standard deviation of the normal distribution.

The derivative of the payoff function in Equation 21 can now be rewritten to

$$\frac{\partial f(S_T)}{\partial S_0} = \frac{\partial f(S_T)}{\partial S_T}\frac{\partial S_T}{\partial S_0} \tag{22}$$

where the first derivative (of the payoff function) is now simply the *probability* density function of the normal distribution. From Equation 16, the derivative of S_T with respect to S_0 is simply $\frac{S_T}{S_0}$. Multiplying these quantities and calculating the average value then yields an approximation of δ.

3) Likelihood ratio: The likelihood ratio technique can be a helpful technique because it does not differentiate the payoff function. It works by first writing V as

$$V(S_0) = e^{-rT} * \int f(S_T)g(S_T, S_0)dS_T \tag{23}$$

where $g(S_T, S_0)$ is the probability density function of a payoff S_T occurring given an initial S_0. Then, assuming

$$\begin{aligned} \delta = \frac{dV}{dS_0} &= e^{-rT} * \int f(S_T)\frac{\partial g}{\partial S_0}dS_T \\ &= e^{-rT} * \int f(S_T)\dot{g}dS_T \\ &= e^{-rT} * \int f(S_T)\frac{\dot{g}}{g}gdS_T \\ &= e^{-rT} * \text{mean}\Big(f(S_T) * \frac{\dot{g}}{g}\Big) \end{aligned} \tag{24}$$

gives an expression for δ which only differentiates g instead of f. Now, when Monte Carlo valuation is used, this can be written using Black-Scholes Theory as

$$\delta = e^{-rT} * \text{mean}\Big(f(S_T)\frac{Z}{\sigma S_0\sqrt{T}}\Big) \tag{25}$$

D. Control variate technique

Monte Carlo simulations rely heavily on generating random numbers. These random numbers can bring a lot of variance into the results. To combat this, several methods can be used to reduce the variance over the results. One of these methods is the control variate technique.

The control variate technique estimates the mean of a random variable by looking at a simpler random variable. Suppose there is a derivative A which has no exact solution. A Monte Carlo method can give an estimate of derivative A, however, the Monte Carlo method will not be the true solution due to the random nature of the Monte Carlo method. If it is possible to know by how much the Monte Carlo estimate is wrong then the true solution can be better approximated.

A simpler, but hopefully closely related, derivative B is used to find out by how much the Monte Carlo estimate is wrong. The simpler derivative B samples the same sequence of random numbers for its Monte Carlo estimator as A - the idea is that any error from the Monte Carlo simulation should be similar for both. B needs to have a known true value for this method to work, like an analytical solution. By looking at the difference between the Monte Carlo estimate of B and the true value of B the error for the Monte Carlo method is deduced. The error of the Monte Carlo method is then subtracted from the Monte Carlo estimate of A to find a better estimate of A. In a formula this becomes

$$\widetilde{C}_A = \widehat{C}_A - \beta(\widehat{C}_B - C_B), \tag{26}$$

where $\widetilde{C}_A$ is the control variate estimate of A, $\widehat{C}_A$ the Monte Carlo estimate of A, $\widehat{C}_B$ the Monte Carlo estimate of B, C_B the true value of B, and β a coefficient which aims to minimise the standard error of $\widetilde{C}_A$.

The standard error of $\widetilde{C}_A$ is equal to

$$\sigma_{\widetilde{C}_A} = \sigma_A^2 + \beta^2\sigma_B^2 - 2\rho\beta\sigma_A\sigma_B, \tag{27}$$

where ρ is the correlation between A and B and is given by

$$\rho = \frac{\mathrm{Cov}(A, B)}{\sigma_A\sigma_B}. \tag{28}$$

Thus, the method performs best when there is a strong correlation between A and B. The standard error is then minimised when

$$\frac{\partial\sigma_{\widetilde{C}_A}}{\partial\beta} = 2\beta\sigma_B^2 - 2\rho\sigma_A\sigma_B = 0. \tag{29}$$

Solving for β gives the coefficient β^* for which the standard error is minimised:

$$\beta^* = \frac{\rho\sigma_A}{\sigma_B}. \tag{30}$$

This technique can be applied to the price of an Asian call option. The analytical solution of the price using arithmetic averages is not known, so the option price using geometric averages is used as B since its analytical solution is given. First, the prices for both methods are calculated using the same random sampling seed. From the resulting values, the correlation and standard deviations can be computed to choose the optimal β^* and finally use Equation 26.

III. Results

THIS section will show the results for the behaviour of the Monte Carlo method, the performance of the different methods for estimating δ, and the accuracy of the control variate technique.

A. Basic option valuation

To determine the price of an option M stock prices need to be calculated. With Equation 18 the option price is then calculated. Figure 1 shows that the option price converges when M increases. The calculations were performed with parameters $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ for a European put option and was repeated 100 times. These results are compared with a binomial tree model [1] with 1000 binomial tree steps. The binomial tree model equated an option value of 4.77937. The Monte Carlo method converges towards this result. The standard error also decreases at a higher M and thus the accuracy of the results increase when M becomes larger.

The impact of K was tested for the Monte Carlo method. Figure 2 shows how the option price changes as a function of K. This was performed with parameters $M = 10^3$, $T = 1$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ for a European put option and was repeated 10 times. When K is sufficiently low ($\sim$ 0 to $\sim$ 80) the option value goes to 0. This is because all of the stocks prices are larger than the strike price. Therefore the put option is not exercised and has no value. When K approaches S ($\sim$ 80 to $\sim$ 120) some stocks will have a price worth less than K. At this point the put option starts to gain value. For even higher values ($\sim$ 120 or greater) almost all stock prices will be less than K. At this point, the option values start scaling one-to-one with K.

The impact of σ was also tested for the Monte Carlo method. Figure 3 shows how the option price changes as a function of σ. This was performed with parameters $M = 10^3$, $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, for a European put option and was repeated 10 times. At a volatility of 0 the option does not have any value, since all stocks will be worth more than K. As the volatility increases more stocks will start to get a value lower than K due to the increased randomness. At this point, the option starts to gain value. The increased volatility also causes more variance in the results, since the stock prices will now have a larger spread of results.

B. δ computation

The hedge parameter δ was computed with the bump-and-revalue method, the pathwise method, and the likelihood ratio method.

1) Bump-and-revalue method: The hedge parameter δ was computed with the bump-and-revalue method for a system with parameters $M = 10^3$, $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ for a European call option and was repeated 10 times. The bump-and-revalue method used $\epsilon = 0.01$. This was performed when using different seeds and fixed seeds. The different seed method resulted in $\delta = 17.92397 \pm 59.88642$. The fixed seed method resulted in $\delta = 0.67429 \pm 0.02099$. The Black Scholes formula gives a δ of 0.67374. Thus the bump-and-revalue method matches well with the analytical Black Scholes formula when using a fixed seed. The bump-and-revalue method does not give precise values when using different seeds.

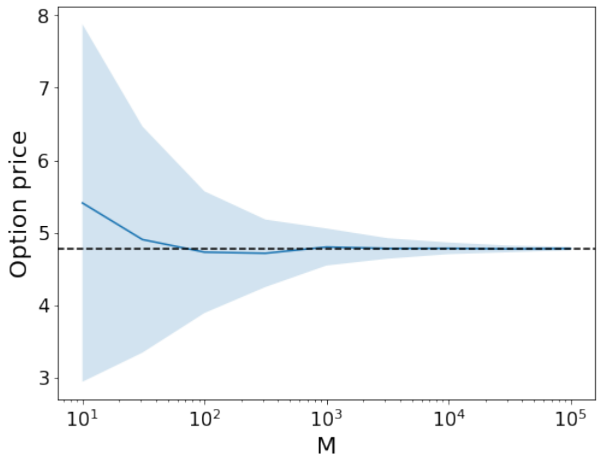

Fig. 1: The option price as a function of M for the Monte Carlo method. The dashed line shows the expected option price calculated with a binomial model with $N = 1000$. These calculations were performed with $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and an European option. As M increases the results converge towards the binomial tree value and the confidence interval becomes smaller. The shaded area shows the 95% confidence interval.

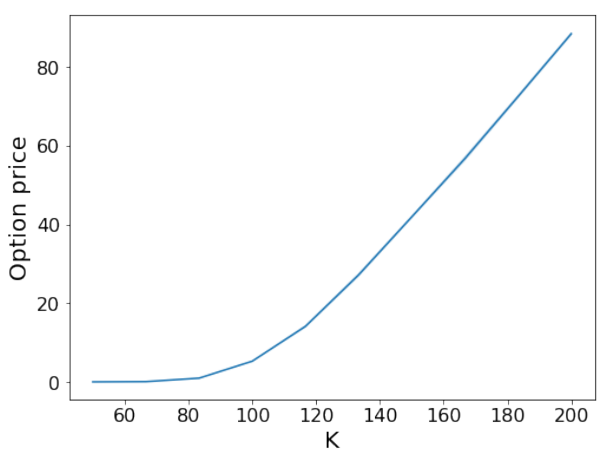

Fig. 2: The option price as a function of K for the Monte Carlo method. This was calculated with $M = 10^3$, $T = 1$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ for an European option. The calculations were repeated 10 times. The, barely visible, shaded area shows the 95% confidence interval.

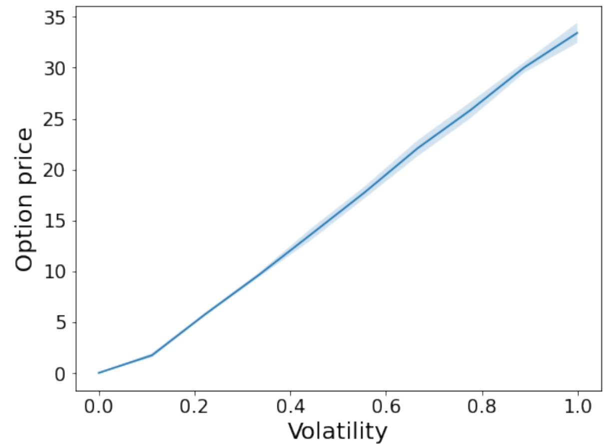

Fig. 3: The option price as a function of σ for the Monte Carlo method. This was calculated with $M = 10^3$, $T = 1$, $K = 99$, $r = 0.06$, $S = 100$ for an European option. The calculations were repeated 10 times. The shaded area shows the 95% confidence interval.

TABLE I: The calculated δ and standard deviation for a digital option ($P_D = 1$) using three different methods. Each method used $M = 10^3$, $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and was averaged across 10^4 runs.

	Mean δ	σ
Bump-and-revalue	0.012347	2.059804
Pathwise method	0.018089	0.006831
Likelihood ratio	0.018205	0.000882

2) Digital option: The hedge parameter δ of a digital option was investigated with $P_D = 1$ and parameters $M = 10^3$, $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, $\sigma = 0.2$. The bump-and-revalue method used a fixed seed, and the pathwise method used a smoothing distribution with $\sigma_{\text{smoothing}} = 0.1$. The results are shown in Table I

As expected, the bump-and-revalue method is very inaccurate with a sizeable standard deviation. The other two methods perform much better, although the likelihood ratio method has a standard deviation that is 7.5 times as low as the pathwise method. The pathwise method also results in a slightly different average δ, which might be fixed by lowering $\sigma_{\text{smoothing}}$ - although a value too low quickly results in machine precision errors.

C. Control variate technique

The control variate technique is applied to get an estimate for the option price for an Asian call option with an arithmetic mean. The control derivative is the option price for an Asian call option with a geometric mean. The option value, with geometric mean, was calculated with the parameters $M = 10^4$, $T = 1$, $K = 99$, $r = 0.06$, $S = 100$ and $\sigma = 0.2$ and was repeated 10 times. An option value was found of 6.30485 ± 0.05530. The analytical solution of Equation 9 found a value of 6.32288. The analytical solution is within the confidence interval of the Monte Carlo method.

The impact of M on an Asian call option with an arithmetic mean with $N = 100$ is shown in Figure 4 for both the raw Monte Carlo method and the control variate technique. This was performed with parameters $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and repeated 10 times. The control variate technique has a significantly smaller confidence interval. Furthermore, the average of the control variate technique is impacted very little by M. The control variate technique, therefore, does not require a lot of individual stock simulations.

Figure 5 shows the impact of the strike price on the Asian call option with an arithmetic mean and $N = 100$. The calculations were performed with $M = 10^4$, $T = 1$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and repeated 10 times. Very little variance is observed for both the raw Monte Carlo method as the control variate technique, as the option price changes between 0 and 50 which is much larger than the errors.

The impact of N on the price of an Asian call option with an arithmetic mean is also studied and shown in Figure 6. This is performed with $M = 10^4$, $T = 1$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and repeated 10 times. The control variate technique has again a much smaller confidence interval compared to the raw Monte Carlo method. The control variate technique also does not have any fluctuations in its average, which is however observed in the raw Monte Carlo method.

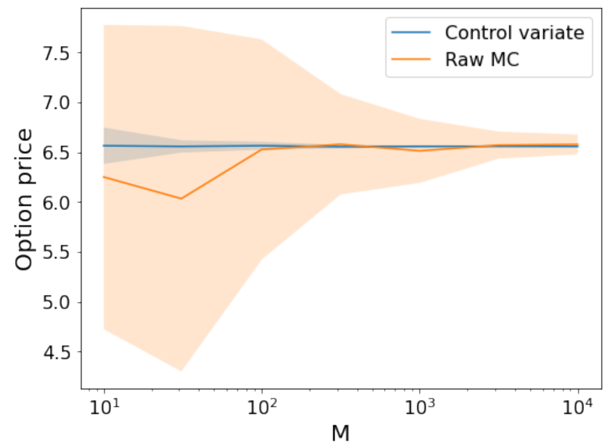

Fig. 4: The option price as a function of M for both the raw Monte Carlo method and the control variate technique. The calculations were performed with $T = 1$, $K = 99$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and an Asian call option evaluated with an arithmetic mean and $N = 100$. The shaded areas show the 95% confidence interval. The control variate technique has a significantly smaller confidence interval.

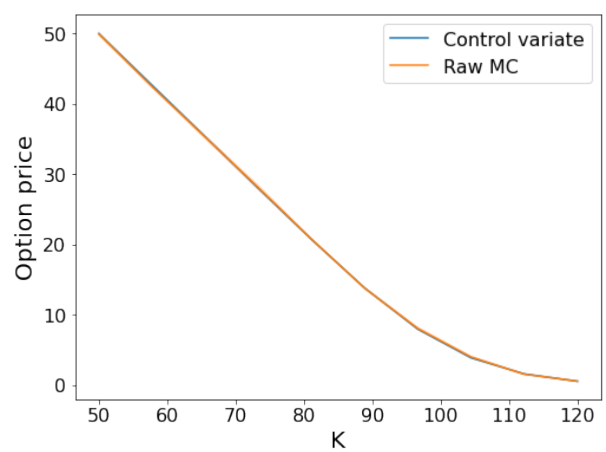

Fig. 5: The option price as a function of K for both the raw Monte Carlo method and the control variate technique. The calculations were performed with $M = 10^3$, $T = 1$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and an Asian call option evaluated with an arithmetic mean and $N = 100$. The calculations were repeated 10 times. The, barely visible, shaded areas show the 95% confidence interval.

IV. Discussion & conclusion

This report shows that Monte Carlo methods approximate option values well as they converge for large numbers of simulations. The control variate technique is effective at reducing the variance in the calculation of an Asian call option price.

The bump-and-revalue method works well to approximate δ when an option's payoff function is mostly smooth but breaks down for a digital option that has a discontinuous payoff function. Here, the pathwise method and likelihood ratio methods improve accuracy significantly, although the pathwise method has an added point of inaccuracy in its smoothing parameter $\sigma_{\text{smoothing}}$. Future research could be directed at how to mitigate this problem.

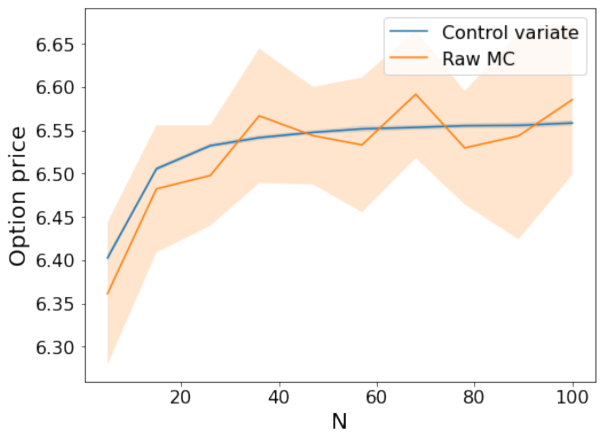

Fig. 6: The option price as a function of N for both the raw Monte Carlo method and the control variate technique. The calculations were performed with $M = 10^3$, $T = 1$, $r = 0.06$, $S = 100$, $\sigma = 0.2$ and an Asian call option evaluated with an arithmetic mean. The calculations were repeated 10 times. The shaded areas show the 95% confidence interval.

REFERENCES

[1] S. Broos and N. van Santen, "Comparing the binomial tree model against the black-scholes model," February 2022.