# Decentralized Finance and Blockchains

Sander Gribling Pieter Kleer Nikolaus Schweizer

# Table of contents

# About

## Course description

Welcome to the course "Decentralized Finance and Blockchains"!

This course approaches decentralized finance and blockchains from three different angles: the computer science or cryptography angle (CS), the game-theoretic or incentives angle (GT), and the financial economics angle (FE).

The three different parts are taught by three different lecturers: Sander Gribling (CS), Pieter Kleer (GT), and Nikolaus Schweizer (FE). Likewise, these lecture notes are also split into three parts. Below, you can find a brief description of each of the three angles, as well as the learning goals of the course.

**Computer Science**: A key principle underlying blockchain technology is maintaining consensus about a distributed ledger, the archive of past transactions. We will study the security aspects related to establishing consensus. We discuss various (im)possibility results in the presence of malicious agents (Byzantine Fault Tolerance) in the general setting of distributed networks. We then discuss the mathematical models behind two famous mechanisms to maintain consensus: Proof of Work and Proof of Stake.

**Game Theory**: Game theory and mechanism design play an important role in the analysis and design of decentralized financial protocols such as those building on blockchain technology. Prominent examples here are cryptocurrencies like Bitcoin. We will be studying such protocols from a game-theoretical perspective by looking at equilibria of their mathematical description, as well as various mechanisms that are used to guide those systems to the desired outcomes.

**Financial Economics**: One of the most intriguing capabilities of the blockchain technology are so-called smart contracts, computer programs that run on the blockchain in a transparent and decentralized manner, thus providing the basis for decentralized finance, the creation of decentralized counterparts to traditional financial institutions. In recent years, a particular success have been decentralized exchanges such as Uniswap which run in the blockchain and replace the traditional market maker and order book of a centralized exchange with an automated market maker. In the course, we will study and compare both types of exchanges using tools from the classical theory of market microstructure and analyze how different aspects of their design affect the functioning of the resulting financial market.

## Learning goals

After successful completion of this course, you are able to:

1. model blockchain technologies from the perspective of distributed computing and prove the (im)possibility of Byzantine Fault Tolerance under various assumptions.

2. explain consensus protocols in distributed networks and compute the probability that malicious agents can change the outcome (e.g. for the Dolev-Strong and longest-chain consensus protocols).
3. model decentralized financial protocols from a game-theoretical perspective and compute their equilibria.
4. explain how mechanism design and game theory can be used to create the desired incentives in decentralized systems and compute the outcome of the relevant mechanisms.
5. analyze both centralized and decentralized financial exchanges using tools from market microstructure theory.
6. explain how the design of a financial exchange affects the properties of the resulting financial market regarding, e.g., liquidity and absence of arbitrage.

## Acknowledgments

The first part of these lecture notes, the computer science perspective, is heavily inspired by a course taught by Tim Roughgarden called "Foundations of Blockchain Protocols". You can find this course here: Course website, Lecture notes.

Note that Roughgarden's course is geared towards computer science students. The expected prior knowledge is thus very different compared to this course. The scope of Roughgarden's course is therefore also much larger than ours: it covers the computer science angle in much more depth than we do here. If you want to learn (much) more about the CS perspective, this is a great starting point.

For full disclosure, the Python code snippets were sometimes/often generated using ChatGPT.

## Contact information

Lecturers:

- Sander Gribling
- Pieter Kleer
- Nikolaus Schweizer (course coordinator)

Note that this is a new course, it is likely that we can still improve the clarity of the exposition. We therefore welcome constructive feedback on these lecture notes either via email, Canvas, or in class.

# Chapter 1

# Introduction

This course is about the science behind blockchain protocols and their applications. The most famous applications being of course cryptocurrencies such as Bitcoin.

Let us first make a disclaimer: in this you will not learn anything about trading in cryptocurrencies. Instead, you will (hopefully) learn the principles behind blockchain protocols and the problems that frequently arise in applications.

At this point, you probably have some rough idea of what a blockchain is and how cryptocurrencies like Bitcoin work, perhaps based on reading some news articles or watching a short video on social media. For the purpose of this course, it is useful to have the following (simplified) picture in mind.

- For a standard currency, for example the euro, there is a central banking agency that can perform tasks such as certifying that a given coin (or entry on a bank account) is indeed a valid euro, or certifying transactions.
- For a cryptocurrency, the guiding principle is often that there is no such central agency. Instead, the current status of "bank accounts" and transactions are maintained in a ***decentralized*** fashion.

The precise meaning of *decentralized* is something that we will explore in this course. For now, imagine that it means that every participant in the (crypto)currency knows the exact amount on every other bank account, as well as a full list of the transactions that have ever taken place.

We often think about the set of participants as a network consisting of *nodes* and *edges*. The nodes represent the participants (we sometimes also referred to participants as parties). The edges represent communication links between two parties. Figure 1.1 below shows an example of a network with three nodes, who can all communicate with each other.

In cryptocurrency applications, it is often convenient (and realistic) to assume that every node can communicate with every other node. This means that all edges are present. This is however not a realistic assumption in every application: if the nodes represent people at a Dutch birthday party for example, then they are likely seated in a circle and can only communicate to their nearest neighbors. While this might seem like a silly example, the resulting network is a common toy model with interesting properties that we will revisit later on.

Every node maintaining a complete list of all bank accounts and transactions might seem like a lot to ask for, and indeed, it is. It is however a useful picture to have in mind. It for example requires us to solve the following problems:

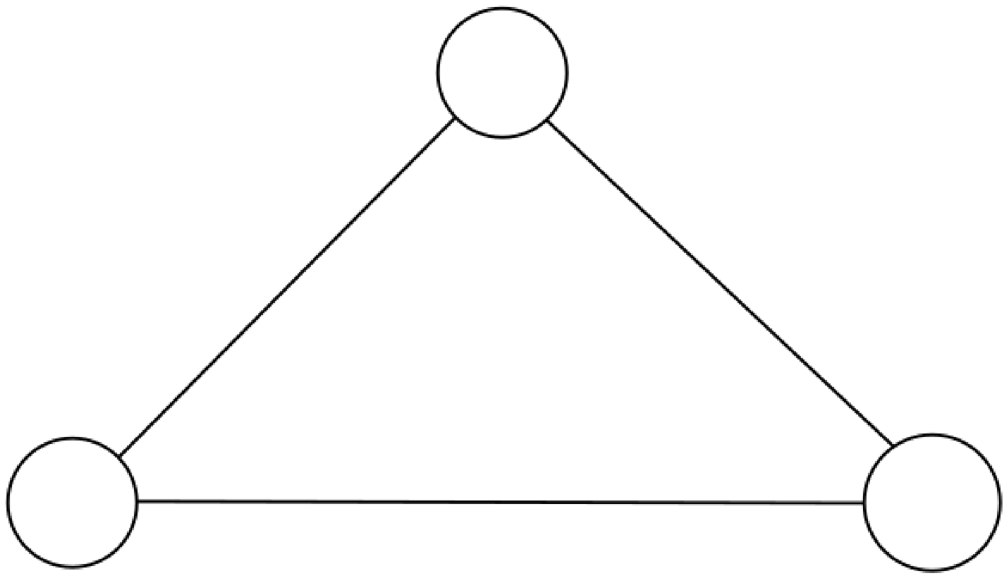


Figure 1.1: A network with three nodes

- If a transaction is to take place, say Alice transfers money to Bob, how do we ensure that everyone updates their records correctly?
- Can we do it in such a way that Alice cannot "double-spend" her money, i.e., also give it to Charlie?

It is not hard to imagine that it is a lot of work to maintain knowledge in a decentralized fashion, indeed, it will take us several chapters to do so! A natural question is therefore, how do we *incentivize* parties to perform this work? If you have ever read a popular science article about Bitcoin, you might have heard the terms "Proof of Work" or "Bitcoin mining" (if not, we will get to this in Chapter 6). These are examples in which parties perform some action (work, e.g., mining a Bitcoin) and are rewarded for this work. Often, the task is too large to perform by a single party and therefore parties form *coalitions*. How do we divide rewards in this case? This is one of the topics covered in Part II.

Once one has settled on the technology and protocols behind a blockchain-based currency, we can turn to applications. This is what Part III is all about.

In the remainder of this introductory chapter, we will give a high-level overview of the three different perspectives on blockchains and their applications that we will discuss in this course. At a first reading, we do not expect you to understand everything in these sections. As the course progresses, you should be able to answer more and more of the questions raised in these sections. (Let us know if we forgot to answer some!)

## 1.1 A computer science perspective

In this part of the course we will focus on the science behind blockchain protocols. Let's start by unpacking the terminology "blockchain protocols" a bit.

The "protocols" here refers to a set of instructions that all participants have to follow. To make it concrete, you think of it as piece of code (a computer program) that all parties have to execute.[1] Such a protocol is designed to perform a certain task. In this course, we will focus on what we want a protocol to do, rather than

[1]For example, look at all information currently available to the participant, do some computation, and report the outcome to all neighboring participants.

how to implement it on a computer. (In other words, there will not be much coding in this course.) In the case of blockchain protocols, the task is to ensure that all parties (eventually) agree on what is written in the blockchain. This last sentence is intentionally a bit vague, we will be more precise later on about what we mean by “eventually” and “agree”.

A “blockchain” simply refers to a chain of blocks. The blocks can be used to store information; again, we will be more precise about which kind of information later on. We want to think of a block all information that is available at a certain moment in time. Logically, we would then like to connect a block to the block that preceded it in time. We can thus think of a chain of blocks as a special kind of network. As opposed to the networks that we described before, a chain requires a *directed* network. What do we mean by *directed*? In our previous description of a network, an edge represented a “communication link” and we implicitly assumed that if, say, Alice can communicate with Bob, then Bob can also communicate with Alice. In a directed network the direction of edges matters. Visually, we will represent this using arrows. If there is an edge between nodes 2 and 1, then we draw an arrow from node 2 to node 1. Here is an example of a chain of three blocks.

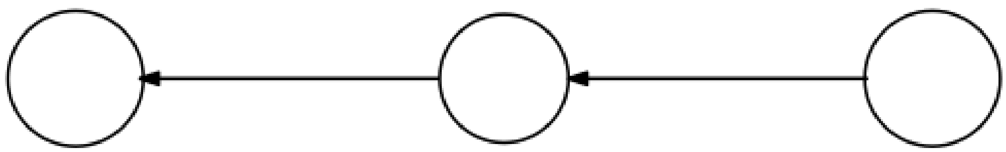


Figure 1.2: A chain with three blocks

We will work towards an understanding of blockchains gradually. We will first get familiar with computation in a decentralized setting. To start, we should decide on a mathematical model for the decentralized setting (or several models). Here one can think for example about questions such as how do we model communication? Is communication instantaneous (the synchronous model), or can there be delays (an asynchronous model)? Do we know all participants in the network in advance or can anyone participate? All of these lead to valid models! We will therefore be explicit about which *assumptions* we are making. When we make an assumption, we should always ask ourselves whether it is a reasonable or realistic assumption. Whether an assumption is realistic or not often depends on the application (instantaneous communication anyone?). Exploring what is and what is not possible under a given set of assumptions is essentially the first part of the course.

To be more specific, we will focus on something we have touched upon before: “agreeing” on information. In the literature we often refer to reaching agreement as “building consensus”. You might see “Proof of Work” and/or “Proof of stake” referred to as consensus-building protocols. The concept of consensus is however separate from blockchains and indeed much older. At this point you should wonder, if everyone can communicate with everyone and we are all following the same protocol, how can we ever disagree? This is a very good question! The answer is that it is often very unreasonable to assume that everyone follows the same protocol! For example, if by deviating from the Bitcoin protocol you could earn a lot of money, then probably someone will decide to deviate. We thus need to design protocols that can resist such dishonest parties, allowing the honest parties to reach consensus. We will call dishonest parties “Byzantine agents” later on. One of the problems that we will discuss is the Byzantine broadcast problem.

The general area to which these questions belong is that of ***distributed computing***. Distributed computing is much broader than what we can cover in this course. Nevertheless, after setting up the framework more formally, we will quickly be able to discuss some foundational results in distributed computing: the state machine replication problem, and the Dolev-Strong protocol for Byzantine broadcast. These are results about the synchronous setting, assuming we know the entire network in advance. Neither of these assumptions is very realistic for blockchains, but is a useful starting point due to its simplicity. It will allow us to quickly get our hands dirty, without getting lost in the details of blockchains.

Having said that, we then move to precisely this: the details of blockchain protocols. More precisely, we will discuss longest chain protocols as a general framework for building consensus in a network that we do not know in advance. We finally discuss in more detail one particular longest chain protocol: the one that is built on the concept of "Proof of work", used for example in Bitcoin where "mining a Bitcoin" is a "proof of work".

## 1.2 A game theory perspective

This part of the course will be concerned with incentives and strategical aspects that may arise in blockchain protocols. We will illustrate these concepts here using using the example of "Bitcoin mining". This is done by solving a complex mathematical puzzle that requires a lot of computation power and is typically done by multiple parties or miners.

Once a Bitcoin is mined, i.e., created, how do we split it fairly between the miners that were involved? A function that decides on the reward that every miner receives is called an allocation rule and can be defined in many ways. Ideally, we want the allocation rule to have some desirable properties that incentivize miners to act faithfully. One such property is sybil-proofness: A miner should not have an incentive to split itself up in multiple parties and receive, in total, more reward from the allocation rule then it would have received when participating as one single party or miner. Reversely, we would also like the allocation rule to be collusion-proof meaning that different miners should not receive more total reward in the allocation rule if they pretend to be one miner, as opposed to the sum of their individual rewards. Our goal here will be to understand and characterize which allocation rules satisfy these, and other, desirable properties by looking at this problem through the lens of cooperative game theory.

One can also take a more competitive view towards Bitcoin mining by considering it to be a so-called Tullock contest. Here every miner invests a certain amount of computing power, but instead of sharing the reward generated by mining a Bitcoin, the Tullock contest gives the whole Bitcoin to the first miner to solve the mathematical puzzle. The probability for a miner to win this contest depends on the amount of computing power that was invested. The miners have a strategic choice to make on how much computing power they want to invest, while optimizing their chance of winning the contest. Our goal will be to analyze this contest through the lens of non-cooperative game theory using stability concepts such as the Nash equilibrium.

## 1.3 A financial economics perspective

From a financial economics perspective, one of the greatest promises and challenges of the blockchain technology is the possibility of organizing counterparts to traditional financial institutions in decentralized ways. An important impulse for this development was the great financial crisis of 2008 when financial institutions that were considered "too big to fail" placed a huge burden on societies and created considerable distrust in traditional "centralized" finance. After the invention of bitcoin, a second important step towards decentralizing financial institutions was the invention of so-called smart contracts, computer programs that can be embedded in later-generation blockchains like Ethereum and that can be used to create automated protocols for financial transactions.

Since the advent of smart contracts, people have looked into various ways of creating decentralized counterparts to traditional financial institutions using smart contracts. Yet there are challenges, some of them as old as financial markets, some of them rather new. Think of decentralized car insurance, implemented as a computer program that automatically sends you an agreed upon amount of cryptocurrency when you upload a photo of your damaged car. What could possibly go wrong? There is more than one answer to this question. Let us just say that, traditionally, most successful insurance markets have operated in environments with a well-functioning legal system in the background.[2]

Automated market makers such as Uniswap are one of the most successful developments in decentralized finance, exhibiting tremendous trading volumes. These are automated exchanges that enable market participants to exchange assets with an automated mechanism that was inspired by the way sports betting markets are organized. Again, traders in these exchanges face some of the same challenges seen in traditional financial exchanges. Yet, there are also new challenges. For instance, due to the full transparency and the block structure of transactions, submitted orders can be seen by others before they are executed – and the order of submissions is not necessarily the order of executions, creating all sorts of potential problems.

The goal in the financial economics part of the course is to get some understanding of the relevant decentralized financial institutions and then take some steps towards understanding them even better by applying models from quantitative finance.

[2]Conversely, in general, the *potential* benefits of decentralized financial institutions may be greatest in environments without well-functioning legal systems, where citizens have to be concerned that those in power confiscate their traditional bank accounts and so forth.

# Chapter 2

# Preliminaries

Here we gather frequently used notation. We also recall some basic concepts that we expect as prior knowledge and we provide some pointers to related literature.

## 2.1 Commonly used symbols

We let $\mathbb{N}$ denote the set of natural numbers, i.e., $\mathbb{N} := \{1, 2, 3, \ldots\}$. For $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$. We use log to denote the natural logarithm.

## 2.2 Graph theory preliminaries

We will use the concepts of undirected and directed graphs, which we recall below.

A *graph* $G$ is defined by a set of vertices $V$ and a set of edges $E$. We write $G = (V, E)$. Here the set of edges is a subset of pairs of vertices. Throughout, we assume an edge consists of a pair of distinct vertices $u, v \in V$ (distinct meaning $u \neq v$). When such an edge is *undirected* we write $\{u, v\}$. When the edge is directed we write $(u, v)$ to denote an edge from vertex $u$ to vertex $v$. We have seen an example of an undirected graph in Figure 1.1: this is the graph with vertex set $V = [3]$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$.

We have also seen an example of a directed graph in Figure 1.2: here the vertex set is again $V = [3]$ and if we label the vertices 1, 2, and 3 from left to right, then the edge set is $E = \{(2, 1), (3, 2)\}$.

## 2.3 Frequently used probability distributions

# Part I

# Computer Science

# Chapter 3

# Introduction to distributed computing

In this chapter we will define some basic concepts and problems from the area of distributed computing.

We start this chapter with an example of a common distributed computing problem: that of making backups of a database and ensuring that these backups are synchronized. This is a problem that you have probably struggled with at some point: how do you properly backup the files on your hard disk? To look ahead, the problem that we aim to solve when we want to base cryptocurrencies on blockchains is very similar: the database would correspond to a complete list of the transactions that have ever occurred.

We then give a formal definition of the underlying problem: the State Machine Replication (SMR) problem. The second half of this chapter is devoted to formalizing and discussing various assumptions about distributed computing networks. The next chapters are dedicated to solutions of the SMR problem under increasingly more realistic assumptions.

## 3.1 Example: maintaining copies of a file

To set the stage, consider the following situation. On my computer, I have an excel file with the grades of all students who take this course. Naturally, I would like to make sure that I don't lose this file. To do so, I can create multiple copies of this file and store them on different computers. That way, if one of the computers breaks, I still have a copy of the file on another computer. Making such a backup once, is easy enough. The problem that we will consider arises when I want to be able to update the file (for example, after the resit) in such a way that the various backups to agree with each other.

We start to see an outline of a distributed computing problem: we can view the different computers as nodes in a graph. If there are $n$ computers, there would be $n$ nodes in the graph. Our task would be to ensure that the $n$ computers each have an up to date copy of the file.

The above example is written from a *centralized* perspective. There is one agent, the "I" persona, that can simply perform the update to each copy of the file. In a *decentralized* setting, we would like the computers to run some sort of protocol that ensures that if I change the file on one of the computers, then the same change is made on all other devices. One quickly realizes that this protocol will require the devices to communicate. This brings us to the second part of the graph: the communication links between the computers determine the edge set of the graph.[1]

[1] In this problem, it is natural to assume that communication works both ways. Therefore the graph would be undirected.

**Connection to blockchains**: If we want to construct a cryptocurrency based on blockchains, then we are essentially interested in solving a similar type of problem. Indeed, we can draw the following analogy. Each participant in the cryptocurrency would correspond to a node in the network. The file that we want each participant to maintain consists of two parts: a list of the current balance of each account, and a full history of all transactions that have ever taken place between participants. In this variant of the problem, there is no natural "I" persona. The decentralized setting is built in: we want each of the nodes to be able to change the file. We do of course want all the nodes to agree on the same file! A change made by one of the nodes should be replicated by all other nodes.

In the next section we will formally define a generalization of the above examples.

## 3.2 State Machine Replication problem

Let us now give a formal definition of a problem that encompasses both examples from the previous section. In the distributed computing literature this problem is known as the State Machine Replication (SMR) problem. To explain the terminology: the state of the machine corresponds to the file that we wanted to maintain in the previous examples.

In the SMR problem we consider the following:

1. There is a set of *nodes* responsible for running a consensus protocol, and a set of *clients* who may submit "transactions" to one or more of the nodes.
2. Each node maintains a local file that we will call its *history*.[2]
3. Nodes can send messages to other nodes, and receive messages from other nodes.

Some remarks are in order. We differentiate here between nodes and clients. The nodes are responsible for maintaining copies of the file. The clients may suggest modifications to the file by sending messages (instructions) to the nodes. For simplicity we assume here that modifications consist of adding information to the file (so no deletions). In a cryptocurrency application this is a reasonable assumption: the file – or history – represents the history of all transactions, which can only grow over time.

Informally, the SMR problem asks to keep all the nodes in sync. Meaning that the local histories of all nodes are the same.

More formally, the SMR problem is to design a protocol that is to be executed by each of the nodes. (Think of a piece of code.) The protocol is allowed to do the following operations:

- maintain or change the local state of the node,
- receive messages from other nodes and from clients,
- send messages to other nodes.

We will see examples of protocol shortly. First, we need to define the properties that we want the protocol to have. In other words, how do we formalize "maintaining a file"? Here we can distinguish two key properties. The first is a *safety* guarantee: we want all nodes to agree on the same file. The second is a *liveness* guarantee: we want to be able to modify the file.

**Goal 1: Consistency.** We say that a protocol satisfies consistency if all the nodes running it always agree on the history. That means that the local history of all the nodes is equal.

In particular, in the case where the local history is supposed to be a list of transactions, all nodes would agree on the order of the transactions.

[2]It represents for example an ordered list of transactions that only grows over time.

**Goal 2: Liveness.** Every "transaction" submitted by a client to at least one node is eventually added to every node's local history.

For the moment, we view "transactions" as simply adding an entry in the file. That is, we ignore the very important financial question of whether the transaction is "valid" – agreeing with the current balance in each of the accounts.

The two goals together are non-trivial to satisfy. It is however not so hard design protocols that reach exactly one of the two goals.[3]

### 3.2.1 Protocol achieving consistency

Here is our first protocol. It will achieve consistency, but not liveness. We will describe the protocol by giving its *pseudocode*. That is, we describe the protocol mostly in words, without committing to a specific programming language.

```
Alg_Consistency:
1.  Initialize: local history H = [ ].
2.  Upon receiving a message m from a client do:
        Nothing.
3.  Upon receiving a message m from a node do:
        Nothing.
```

As you can see, the protocol consists of three lines. The first line describes the initialization that the node performs. In this case, it defines its local history $H$ to be the empty list $H = [\,]$. The second line describes the behavior of the node when it receives a message from a client. The third line does the same for when it receives a message from a node. Here the last two actions are rather trivial: do nothing.

Although this is an extremely naive protocol, we can show that it does satisfy the first goal: consistency.

**Lemma 3.1.** *If all nodes in a distributed network run `Alg_Consistency`, then it satisfies consistency.*

*Proof.* Assume all nodes in a distributed network run `Alg_Consistency`. To argue that we satisfy consistency, we observe that for each node and every moment in time the local history state equals the empty list $[\,]$. In particular, this means that all nodes agree on the same local history, at all times. □

You are recommended to think about the following exercise before proceeding to the next section.

> **Exercise**
>
> Convince yourself that a distributed network whose nodes all run `Alg_Consistency` does not guarantee liveness.

### 3.2.2 Protocol achieving liveness

As a second example, we will give a protocol that achieves liveness, but not necessarily consistence.

[3] Warning: these protocols might seem a bit silly, they are meant as an easy introduction to thinking about protocols.

```
Alg_Liveness:
1.  Initialize: local history H = [ ].
2.  Upon receiving a message m from a client do:
        Append message m to H.
3.  Upon receiving a message m from a node do:
        Append message m to H.
4.  At midnight do:
        Send local history H to all other nodes.
```

A couple of remarks are in order. First, the protocol now actually "does something"! In particular, a node will act upon a message that it receives from either a client or another node. Second, we see that nodes are no longer passive: they occasionally send messages to other nodes as well. Third, the protocol now – implicitly – assumes that a node is aware of the concept of time: it needs to perform a specified action every day at midnight. This is in stark contrast to the protocol `Alg_Consistency` which was purely *event-driven*: a node only had to take action when a message arrived. ("Took action" is maybe a slight exaggeration: the protocol doesn't do anything when a message arrives.) For the purpose of this example, we will assume that all nodes agree on the current time. When an event happens once per day, this is a relatively mild assumption. It also means that messages from clients only get shared once per day, which might not be sufficiently quick depending on the application. At the other extreme, we could imagine the nodes want to share incoming messages every millisecond. In that case however, you might run into all kinds of issues: nodes might be too far apart for a message to pass from node $A$ to node $B$ within that time frame, or nodes might disagree on the current time (we are now measuring milliseconds after all). In that case, agreeing on the time is a much stronger assumption. We will revisit these – and other – assumptions in more detail later on. For now, let us prove that `Alg_Liveness` guarantees the liveness property.

**Lemma 3.2.** *If all nodes in a distributed network run `Alg_Liveness`, then it satisfies liveness.*

*Proof.* Assume all nodes in a distributed network run `Alg_Consistency`. To argue that we satisfy liveness, we need to show that if a client sends a message $m$ to one or more nodes in the network, then it *eventually* gets added to the local history of every node. To that end, assume client $i$ sends a message $m$ to a group of nodes that includes node $A$, on day 1. On day 1 node $A$ receives message $m$ and adds it to its local history $H_A$. Now consider an arbitrary node $B$ in this network that is distinct from $A$. (It might have received message $m$ from client $i$ on day 1 as well, in which case it added $m$ to $H_B$ on day 1 and there is nothing left to show.) At the end of day 1, node $A$ sends their local history $H_A$ to all other nodes. Therefore, on day 2, node $B$ receives $H_A$ and appends it to $H_B$. Since $H_A$ included the message $m$, this means that $H_B$ now contains the message $m$ as well. Thus, the message $m$ is eventually added to the local history of every node.[4] □

It is important to note however that `Alg_Consistency` does not guarantee the consistency property. Can you see why?

**Exercise**

Consider a distributed network containing two nodes $A$ and $B$ where on day 1 client $i$ sends message $m_A$ to $A$ and $j$ sends message $m_B$ to $B$. Describe the local history of each of the nodes on days 1 and 2. What do you observe?

[4] In this case, *eventually* means at most one day after the client sends the message to at least one node in the network.

## 3.3 (Strong) assumptions about the decentralized setting

We will be working with a mathematical model of a real-world situation. We are therefore making some assumptions. In this section we list one possible set of assumptions. Some of the assumptions that we are making here are more restrictive than others. We will call such an assumption relatively *strong*. In later chapters we will replace these strong assumptions with weaker assumptions. While reading the next three assumptions, try to answer the following questions: is it a weak or strong assumption? Do I know an application where it holds / does not hold?

### 3.3.1 Assumption 1: the set of nodes is known

This assumption is also referred to as the *permissioned* setting. It assumes the set of nodes in the network is fixed and known to all nodes, moreover it assumes that each node has a unique identifier that is also known to all other nodes. We will frequently use $n \in \mathbb{N}$ to denote the number of nodes. This allows us to use the numbers between $1$ and $n$ as unique identifiers.

Key advantages:

- It allows *majority voting*.
- One can order the nodes based on their identifier.

### 3.3.2 Assumption 2: signatures exist and cannot be forged

This assumption can be viewed as an extension of Assumption 1. We assume that nodes can add their signature to a message. By this we mean that if node $A$ sends a message $m$ (to an arbitrary node), then they can add their signature to it. All other nodes have a verification procedure that can correctly determine whether node $A$ signed message $m$. No other node can *forge* $A$'s signature: no other node can add a 'signature' that the verification procedure would accept as $A$'s signature. This is an example of a *trusted setup*. This assumption is also referred to as assuming Public Key Infrastructure (PKI).

We will not go into further details about this assumption in this course; we will (happily) assume that it holds and not go into details of how one would implement it in a real-world scenario. (It is a relatively mild assumption however.)

Key advantage:

- It allows us to trust who sent which message.

### 3.3.3 Assumption 3: synchronous model

This is an assumption about the (reliability of the) communication network. Formally, we require the following two sub-assumptions:

1. All nodes have access to a shared clock.
2. Bounded message delays: messages arrive within a predetermined amount of time.

Together, these two assumptions allow us to divide time into smaller intervals in such a way that that messages sent at the start of an interval arrive before the end of the interval. To avoid having to specify the bounded message delay, we will simply number the intervals. Concretely, we thus assume that messages sent at the start of (or simply *in*) interval $t$ arrive at their intended recipient before the start of interval $t + 1$. We will often refer to the intervals as *rounds*.

Key advantage:

- It allows us to define *rounds* (see above).

### 3.3.4 Discussion of the assumptions

Assumption 1 is realistic in some scenarios: if we are using multiple computers to create a backup of a file (e.g. a list of grades), then it reasonable to assume that we know how many computers we are going to use. (There is a central entity that determines the number of nodes.) For our second motivating example however, blockchains, this is a very unrealistic assumption! Not knowing the set of nodes participating in the blockchain protocol is in fact a key feature that we are aiming for. We would like a blockchain protocol to be able to function in a completely decentralized manner, with nodes being able to enter (or leave) the network while the protocol is active.

Assumption 2, as mentioned above, is one that we will simply assume throughout the course.

Assumption 3 is again realistic in some scenarios, but not in others. We have seen an example of a protocol that worked in the synchronous model in Section 3.2.2: the `Alg_Consistency` protocol. The synchronous model makes optimistic assumptions and therefore serves as a good sanity check when designing protocols: the protocol should at least function correctly in the synchronous model. In Chapter 4 we will work with the synchronous model. In the later chapters Chapter 5 and Chapter 6 we will encounter protocols that make milder assumptions.

## 3.4 Honest nodes

The final assumption is about whether we assume nodes to be *'honest'* or *'dishonest'*. We say that a node is *honest* if it executes the intended protocol. Any node that deviates from the intended protocol is called *dishonest* or *faulty*. Note that honesty in this context is a description of the nodes behavior, not its intentions. In the example of creating backups of a list of grades, it is for example perfectly reasonable to assume that all nodes have good intentions, but we would like a protocol to 'work well' even if one of the nodes breaks and is therefore unable to follow the protocol. We therefore prefer the term *faulty* for nodes that are not honest.

The assumption that we will make in this section (and only this section) is a very strong one:

**Assumption 4**: All nodes are honest.

Naturally, we would like to relax this assumption as soon as possible. In the next chapter we will indeed replace it with a much milder assumption: there we assume a bound $f$ on the number of faulty nodes.

### 3.4.1 Solving the SMR problem under assumptions 1,2,3,4

Here we show how to solve the SMR problem under the assumptions 1, 2, 3, and 4. That is, we work in the permissioned, synchronous model, we assume PKI and that all nodes are honest.

As a reminder, we want to design a protocol that guarantees consistency and liveness for the SMR problem. We have already seen two protocols that achieve either consistency or liveness. In particular, in Lemma 3.2 we have shown that `Alg_Liveness` guarantees the liveness property. In the subsequent discussion we have seen that this protocol does not guarantee consistency: it can happen that two nodes disagree on the order of transactions in their local history. The 'issue' here was that every node had the 'right' to append transactions to the local history of other nodes, which could lead to disagreements on the order that transactions are written down. The protocol that we describe here resolves this issue by selecting a leader in each round, who is the only one with the permission to write in that round.

**Coordinating via rotating leaders**: since we are in the permissioned setting, we know the number of nodes participating in the network, say $n$. We can there do the following:

- in round 1, node 1 is called the leader and all other nodes are called followers,
- in round 2, node 2 is called the leader and all other nodes are called followers, $\ldots$
- in round $n$, node $n$ is called the leader and all other nodes are called followers After $n$ rounds, we reset the clock and start again as in round 1. In other words, we are rotating the leaders.

We now describe the protocol for the leader and follower nodes separately. For ease of notation, we always assume that nodes initialize their local history to $H = [\,]$ at the start of the protocol. We additionally allow the nodes to use some local workspace $W$, which they can use to store information (temporarily); it is not part of the local history state.

```
Alg_Leader(t):
1.  Upon receiving a message m from a client do:
        Append message m to the local workspace W.
2.  At the end of the round do:
        Remove from W the messages that are already part of H.
        Append W to H and send W to all other nodes.
        Reset W = [ ].
```

```
Alg_Follower(t):
1.  Upon receiving a message m from a client do:
        Append message m to the local workspace W.
2.  Upon receiving a message m from a node do:
        If m is signed by the leader of the current round t do:
                Append m to H.
        Else do:
                Nothing.
```

Our claim is that if the nodes in a distributed network adhere to the above protocol, then both liveness and consistency are guaranteed.

**Lemma 3.3.** *If all nodes in a distributed network run the 'coordinating via rotating leaders' protocol, then it satisfies liveness and consistency.*

**Exercise**

Prove Lemma 3.3.

# Chapter 4

# The Dolev-Strong protocol

In the previous chapter we have seen a protocol for the SMR problem under the assumptions 1,2,3, and 4. Here we replace the last assumption by a more realistic one: we no longer assume all nodes are honest. Instead, we assume there are at most $f$ faulty nodes in the network, where $f$ is some number between $0$ and $n$.

Our goal in this chapter is to solve the SMR problem under the assumptions 1,2,3 and assuming a bound $f$ on the number of faulty nodes (for some values $f > 0$). The protocol that we will study is due to Dolev and Strong (1983). At a high level, it works similarly to the rotating leaders protocol that we have seen in Section 3.4. If there is even a single faulty node, the rotating leaders protocol no longer satisfies consistency. Can you see why?[1] Informally, the Dolev-Strong protocol gives us a way to detect faulty leaders that try to 'trick' the other nodes into inconsistency. To formalize this, we introduce the *Byzantine Broadcast problem* in Section 4.2. We show how the SMR problem *reduces* to the Byzantine broadcast problem, see Section 4.3. We finally discuss the Dolev-Strong protocol and show that it solves the Byzantine Broadcast problem.

## 4.1 Bounded number of faulty nodes

We recall from the previous chapter that a node is called *honest* if it never deviates from the intended protocol. Any node that is not honest is called *faulty*. Assumption 4 from the previous chapter was that all nodes are honest, or, equivalently, that the number of faulty nodes is equal to $0$. Here we replace this with a more realistic assumption:

**Assumption 4'**: the number of faulty nodes in the network is at most $f$.

We in fact assume the protocol knows the upper bound $f$ on the number of faulty nodes. This means the protocol may depend on $f$. In the previous chapter we have seen a protocol that assumed $f = 0$. Interesting values of $f$ to keep in mind are $f = n/3$ or $f = n/2$, meaning that at most a certain fraction of the nodes is faulty. To start building some intuition, we will consider $f = 1$ and $f = 2$ later on in this chapter.

## 4.2 The Byzantine Broadcast problem

In the *Byzantine Broadcast* problem we consider the following setting:

[1] Suppose there is one faulty node. In some round, it will be elected as the leader. It can then simply choose to violate consistency by sending different messages to different nodes.

1. There are $n$ nodes, one is designated *sender* and the other $n-1$ nodes are *non-sender*. The identity of the sender is known to all non-senders.
2. The sender has a *private input* $v^*$ that belongs to some set $V$. (Private means that only the sender knows $v^*$ at the start of the protocol.)

The set $V$ here represents the set of possible private inputs. In a cryptocurrency application, this might be the set of all valid transactions. For intuition, it suffices to think of only two possible inputs: $V = \{0, 1\}$.

Informally, the goal in the Byzantine broadcast problem is for the sender to send $v^*$ to all other nodes in such a way that all other nodes can be certain that everyone received the same message. Formally, we say that a protocol is a solution to the Byzantine broadcast problem if it guarantees the following three properties:

1. **Termination**: Every honest node $i$ eventually halts with some output $v_i \in V$.
2. **Agreement**: All honest nodes halt with the same output.
3. **Validity**: If the sender is an honest node, then the common output of the honest nodes is the private input $v^*$ of the sender.

Some remarks are in order. First, the requirements are different for honest nodes and faulty nodes. This is by necessity: faulty nodes can deviate in *any* way from the protocol, so we cannot hope to guarantee anything about their output. Second, the condition *agreement* is required to hold both if the sender is honest and if it is faulty. Agreement is comparable to the *consistency* property that we have seen in the SMR problem. Third, note that the *validity* property, necessarily so, is conditioned on the sender being honest. Therefore *validity* is trivially satisfied when the sender is faulty.

Exercise

Design a protocol, for any $0 \leq f \leq n$ specifying both the behavior of the sender and non-sender nodes, that guarantees *termination* and *agreement* whenever assumptions 1,2 and 3 hold.

Solution

Fix some $v_0 \in V$ (e.g. if $V$ is a set of numbers, pick the smallest). Consider the simple protocol in which every node (both sender and non-sender) terminates in round 1 by outputting $v_0$. This protocol clearly satisfies *termination* (every node halts in round 1) and *agreement* (every honest node halts with the same output $v_0$).

## 4.3 SMR reduces to Byzantine Broadcast

Here we show that any protocol that solves the Byzantine Broadcast problem can be used as a black box to solve the State Machine Replication problem. In our original definition of the SMR problem, in particular for liveness and consistency, we assumed all nodes were honest. In the presence of faulty nodes we need modify the guarantees slightly:

**Goal 1: Consistency.** We say that a protocol satisfies consistency if all the *honest* nodes running it always agree on the history.

**Goal 2: Liveness.** Every "transaction" submitted by a client to at least one *honest* node is eventually added to every node's local history.

(The only difference is adding the word *honest* in the right places.)

We now present a protocol that solves SMR, given a protocol for the Byzantine Broadcast problem. Concretely, let us assume we are in the synchronous and permissioned setting (assumptions 1,2,3) and that there are at most $f$ faulty nodes (assumption 4'). Moreover, assume we are given a protocol $\pi$ that solves the Byzantine Broadcast problem under those assumptions. Assume $\pi$ always terminates in at most $T$ rounds. As before, we allow the nodes to use some local workspace $W$ that they can use to store information, but which is not part of the local history state (this is used to implement step 2).

```
Alg_SMR_from_BB(pi,f):

At each round t=0, T, 2T,... that is a multiple of T do:
1.  Define the current leader to be node t/T modulo n.
2.  The leader constructs a list L of transactions it has
    received in the past T rounds, which are not yet part of H.
3.  Use the protocol pi with
    as leader node t/T modulo n and private input L.
4.  At round t+T-1, every node i appends
    its output L_i in the Byzantine Broadcast problem to its local history.
```

**Lemma 4.1.** *Under assumptions 1,2,3,4', assuming $\pi$ solves the Byzantine broadcast problem in at most $T$ rounds, the protocol* `Alg_SMR_from_BB` *solves the SMR problem.*

*Proof.* We need to argue that consistency and liveness are satisfied. For consistency, we can argue in an inductive manner. At the start of the protocol, all honest nodes initialize their local history to $H = [\,]$. Assume all honest nodes agree on the local history at some time that is a multiple of $T$. By the guarantees of $\pi$, when $\pi$ terminates, which happens within $T$ rounds, every honest node agrees on a common output $L$. Therefore all honest nodes append the same message $L$ to their local history state $H$ in round $t + T - 1$, which ensures that the local history states of all honest nodes agree between rounds $t$ and $t + T$.

For liveness, it suffices to observe that every honest node is elected as a leader once very $nT$ rounds. □

## 4.4 The cases $f = 1$ and $f = 2$

In this section we introduce the idea of "cross-checking". We show that this solves the Byzantine broadcast problem when there is at most 1 faulty node, and that it fails when there are 2 faulty nodes. This allows us to build op some intuition about the main idea underlying the Dolev-Strong protocol, without the technicalities that arise when there are multiple faulty nodes. Technically, the Dolev-Strong protocol that we present in the next section is independent from this section, which means that this section is "optional" and can be skipped (at your own risk).

Intuitively, the protocol works by asking the honest nodes to do one simple step of cross-checking: each node verifies whether all other nodes received the same messages from the sender.

Formally, we consider the following protocol.

```
Alg_Cross_Check:

The protocol consists of three rounds:
1.  The sender sends its private value v* to all non-senders (signed).
```

```
2.  Every non-sender i sends the message m_i it
    received from the sender in round 1 to
    all other non-senders with their signature added.
3.  All non-senders choose the most frequently
    received value among the values it received in rounds 1 and 2.
    (Breaking ties in some consistent way.)
    The sender outputs v*.
```

In the following two exercises you are asked to show that `Alg_Cross_Check` solves the Byzantine broadcast problem when there is at most $f = 1$ faulty node and there are at least $4$ nodes in total, but that it breaks when $f = 2$.

> **Exercise**
>
> Under assumptions 1,2,3, $f = 1$, and $n \geq 4$, the protocol `Alg_Cross_Check` solves the Byzantine broadcast problem.

> **Exercise***
>
> Under assumptions 1,2,3, $f = 2$, and $n \geq 4$ even, the protocol `Alg_Cross_Check` does not solve the Byzantine broadcast problem.

Showing that the protocol breaks when $f = 2$ is a bit tricky (hence the * next to the exercise). You are recommended to *try* to solve the exercise on your own, but don't worry if you don't succeed.

A key takeaway from the above two exercises is that one round of cross-checking allows us to deal with one Byzantine node, but not with 2. It thus seems that *more cross-checking* is necessary when there are multiple Byzantine nodes. In a nutshell this is precisely what the Dolev-Strong protocol does: every additional round of cross-checking allows for one more Byzantine node.

## 4.5 The Dolev-Strong protocol

Here we describe a classic protocol due to Dolev and Strong (1983). It solves the Byzantine broadcast problem in the permissioned and synchronous setting, assuming an upper bound $f$ on the number of faulty nodes. Together with the reduction from Section 4.3, this solves the SMR problem under the same assumptions. To describe the protocol, we need one more definition, that of *convincing messages*.

### 4.5.1 Convincing messages

A node $i$ is *convinced of value $v$ in round $t$* if it receives a message prior to round $t$ that satisfies the following three conditions:

1. It contains the value $v$;
2. It is first signed by the sender;
3. It is also signed by at least $t - 1$ other, distinct nodes, none of which are $i$.

### 4.5.2 Protocol description

```
The Dolev-Strong protocol(f):

1.  In round 0 the sender:
    Sends its private value v* to all the non-senders
    Outputs v*
2.  In round t = 1,...,f+1 a non-sender i does:
        If i is convinced of a value v by some message m received
        prior to round t and has not been convinced of v before:
            i adds its signature to m and sends it to all non-senders
3.  At the end of round f+1 a non-sender i does:
        If i is convinced of exactly one value v:
            Output v
        Else:
            Output "failure" (some message not in V)
```

In the above protocol we are outputting "failure" to signal that we have detected a Byzantine sender. The precise message "failure" is of course arbitrary, what matters is that it cannot be confused with a valid private input of the sender. The message "failure" is assumed to be distinguishable from inputs from $V$.

Intuitively, in the Dolev-Strong protocol the $f + 1$ rounds after the first correspond to $f + 1$ rounds of cross-checking. Note that for $f = 1$ the algorithm is not equal to `Alg_Cross_Check`. As we will see in the next section, the list of *distinct* signatures is used to detect Byzantine senders. For now, remark that a node that is newly convinced of a message at the end of round $f + 1$ observes precisely $f + 1$ distinct signatures, at least one more than the number of Byzantine nodes.

### 4.5.3 Proof of correctness

We need to show that under the assumptions 1,2,3 the Dolev-Strong protocol satisfies *termination*, *validity*, and *agreement*. The property *termination* is clear from the description of the protocol. We prove the remaining two properties separately.

**Lemma 4.2.** *Under the assumptions 1,2,3, assuming at most f faulty nodes, the Dolev-Strong protocol satisfies* validity.

*Proof.* Assume that the sender is honest (why are we allowed to do so?) and has private value $v^*$. The sender thus follows the protocol and sends a signed copy of $v^*$ to all non-senders in the first round, it then outputs $v^*$ and terminates. In round 1 all non-senders are therefore convinced of the value $v^*$. Indeed, they have received a message that 1) contains the value $v^*$, 2) is first signed by the sender, and 3) is signed by $1-1=0$ other, distinct nodes, none of which are $i$.

It remains to observe that since the sender is honest, no node is ever convinced of a value other than $v^*$: a *convincing message* needs to contain the signature of the sender. (Here we are using our assumption that signatures cannot be forged.)

At the end of the protocol, all non-senders therefore output $v^*$ as well, establishing *validity*. □

The hard(er) part is to show that the protocol satisfies *agreement*, and this is where we need the multiple rounds.

**Lemma 4.3.** *Under the assumptions 1,2,3, assuming at most $f$ faulty nodes, the Dolev-Strong protocol satisfies* agreement.

*Proof.* Assume that the sender is Byzantine (why are we allowed to do so?). We will show that at the end of the protocol, all honest nodes are convinced of exactly the same set of values. This suffices to establish agreement (can you see why?[2]).

Suppose on an honest node $i$ gets newly convinced of a value $v$ by a message received before the end of time step $t$ for some $t \in \{0, 1, 2, \dots, f + 1\}$. We show that all other honest nodes also get convinced of value $v$ before the end of the protocol. Now we need to distinguish two cases: i) $t \leq f$ or ii) $t = f + 1$.

Case i): if $t \leq f$, then node $i$ still has time to communicate to other honest nodes. In particular, in round $t + 1$ it will add its signature to the message that convinced them of value $v$ and it will send that to all other non-senders. Since node $i$ was convinced of $v$ in round $t$, this newly signed message is a *convincing message* for all other nodes that had not yet been convinced of $v$ (check this!).

Case ii): if $t = f + 1$, then node $i$ no longer has time to communicate. We thus need to show that all other honest nodes had already been convinced of the value $v$ prior to the end of round $t$. To do so, we will crucially use that node $i$ is convinced *for the first time* of value $v$ at the end of round $f + 1$. This means that at the end of round $f + 1$ it receives a message that is signed first by the (Byzantine) sender, and also by $f$ distinct other nodes ($f = t - 1$). Crucially, since the sender is Byzantine, at least one of these $f$ nodes is an *honest* node. Let $j$ be such an honest node. Since $i$ received a message containing the value $v$ and $j$'s signature, the honest node $j$ was newly convinced of the value $v$ in some round $t' \leq f$.[3] We can thus apply case i) to the *honest* node $j$, showing that all non-sender nodes have received a convincing message containing the value $v$. □

[2] Either every honest node is convinced of exactly one value $v$, in which case all honest nodes output $v$. Alternatively, every honest node is convinced of at least two distinct values, in which case all honest nodes output "failure".

[3] In fact, $t' = f$ is the only possibility (since node $i$ is only convinced in round $f + 1$).

# Chapter 5

# Longest-chain protocols

In this chapter we study a second type of consensus protocols, those based on blockchains and in particular the concept of longest chains. This type of consensus protocol lies at the heart of several cryptocurrencies (e.g. Bitcoin). The Bitcoin protocol is one example of a longest-chain protocol, but it is not the only one. In fact, most cryptocurrencies are based on blockchain protocols, see for example the long list on Wikipedia here. All of these protocols follow the same general recipe, but they use different ingredients in certain steps. In this chapter we focus on the general recipe, in the next we will see some of the ingredients that Bitcoin uses (namely proof of work).

To simplify the exposition, we will work in the permissioned setting, assuming PKI, and especially the synchronous model of communication. If we restrict to that setting however, then we already know a good consensus protocol! Indeed, this is precisely what we achieved in the previous chapter. So what is the advantage of doing it again? There are several answers to this question. First, the consensus protocol based on Dolev-Strong is rather slow. It requires many rounds of cross-checking before new information gets added to the local history of each of the nodes. A second answer is that the Dolev-Strong protocol relies on the *permissioned* setting, whereas the protocol that we design in this chapter extends very naturally to the *permissionless* setting. We will go into this in more detail at the end of this chapter.

We start this chapter with a high-level description of a blockchain protocol and the behavior of honest nodes. As usual, we then explicitly state the assumptions that we make in this chapter. We will see that the notion of consensus trivially holds (under the assumptions that we make). Instead, we introduce a new notion – *finality* – that is harder to achieve, but very useful in a cryptocurrency application! In a nutshell, a block is final when it is 'deep enough in the chain'. We will formalize this later on in the chapter, for now the picture to have in mind is that a block that is final can no longer be changed (surprise); if we imagine the block to contain a transaction, we can thus trust that this transaction actually took place. We spend the second half of this chapter to prove that longest-chain protocols achieve finality when, roughly speaking, the majority of the nodes is honest.

## 5.1 Protocol description

The starting point for a blockchain protocol is the concept of a blockchain. As the name suggests, a blockchain is a set of blocks that are connected to form a *chain*. What do we mean by a chain? We say that a set of blocks forms a chain if:

- There is one "genesis" block, forming the start of the chain, we typically denote it with $B_0$,

- Every block that is not $B_0$ points to exactly one other block as its predecessor, in such a way that if we follow the chain of predecessors, we arrive at the genesis block $B_0$.

The above is a somewhat convoluted way of saying that a blockchain looks like Figure 5.1. It is a directed graph where each node has out-degree $1$,[1] except for a single out-degree 0 block $B_0$. It moreover has the property that every block is connected to $B_0$ by a directed path. As a graph, a blockchain is thus a directed tree whose root is the block $B_0$. If you are not familiar with directed graphs, but you are familiar with undirected graphs, then it suffices to think of a blockchain simply as a tree whose root is the block $B_0$. The direction of the arcs is the one that follows the path towards $B_0$.

A *blockchain* is a directed graph $G$ that is an *in-tree* whose root corresponds to $B_0$.[2]

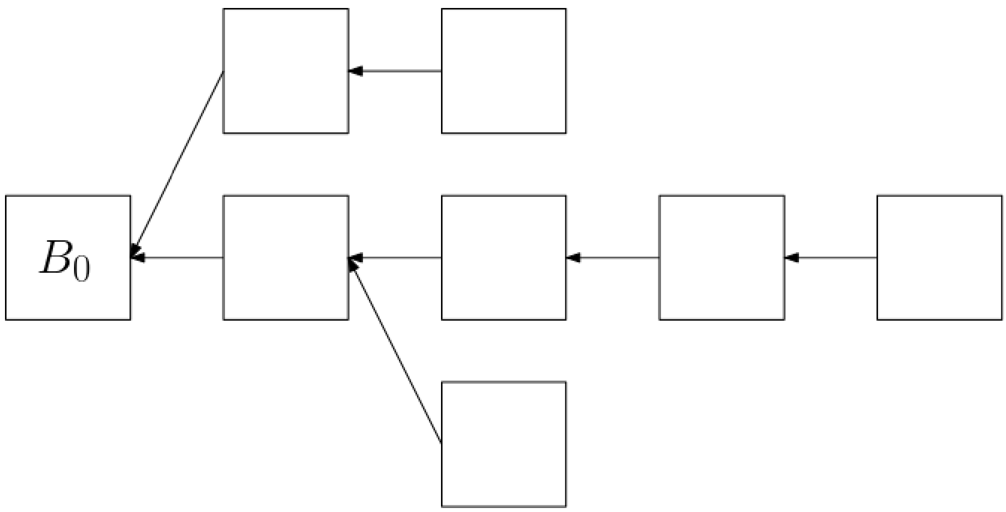



Figure 5.1: A chain of blocks

The purpose of a blockchain is again to contain information. By this we mean that every block represents a chunk of information (say a 1MB file). By storing the information in a chain, we construct a notion of time: the information in a block is created after that of its predecessor. Needless to say, this is a useful property to have when you want to keep track of a list of transactions.

So how does a blockchain fit in the distributed computing framework? We want to think of the blockchain as the information that all nodes have access to. This describes the ideal scenario. If we assume it, then consistency becomes trivial. Such an assumption however essentially amounts to instantaneous communication. We will formalize the assumptions that we work with in the next section.

Now that we have cleared up the concept of a blockchain, we sketch the protocol that we have in mind.

```
A blockchain protocol:

1.  Initialize with a hard-coded genesis block B0
2.  In each round r = 1,2,3,.... do:
    a)  Choose one node i as the leader of round r.
```

[1] Recall that the *out-degree* of a node in a directed graph is the number of arcs leaving the node. (In the picture, the number of arrows coming out of a block.)

[2] Here an in-tree is a directed graph that is defined as follows. If we treat all arcs as undirected edges, then the graph is a simple tree. If we treat $B_0$ as the root node of this tree, then all arcs are directed in such a way that they point towards $B_0$.

```
     b)  Node i proposes a set of blocks,
         each specifying a single predecessor block.
```

This protocol is under-specified: we will fill in the details of step 2 later. In particular, one can imagine several different ways of choosing a leader in step 2a), for example:

1) In the permissioned setting that we used in the previous chapter, we simply select node $i$ as leader in rounds $i, n+i, 2n+i, \ldots$.
2) In a Proof of Work protocol (which we will discuss in the next chapter), the leader in round $r$ is the first node to provide a proof of work after round $r+1$.
3) Proof of Stake is another way to select leaders.

For the moment, either one of these three options is good to have in mind. In the second half of the chapter we prove correctness of the longest-chain protocol under certain assumptions. At that point, you are recommended to revisit the above three options and see whether or not they match the assumptions. We do want to point out that the second option already hints at the fact that *rounds* don't have to correspond to time slots, they rather refer to the periods in between certain events. We will come back to this later.

### 5.1.1 Honest vs. dishonest behavior

Let us now describe the intended behavior of *honest* nodes. In a longest-chain protocol an honest node will do the following when it is elected as the leader of a round $r$:

- it proposes exactly one new block,
- this new block points to exactly one predecessor,
- the predecessor was created in a previous round,

Naturally, the new block may contain some information. For example, a list of newly made transactions or a copy of your favorite recipe for pasta. However, for this chapter we will ignore such information: that information is relevant for applications, but not for the guarantees that we want to achieve here. All that matters for us is that a block contains a pointer to precisely one predecessor that was created in a previous round.

In a longest-chain protocol, there is one additional requirement on the behavior of *honest* nodes:

- the new block extends a *longest chain*.

We should of course define what we mean by a longest chain. A longest chain in a blockchain refers to a sequence of blocks that are on a longest path in the blockchain. Let us revisit the example from Figure 5.1. We will label the blocks for ease of reference; the particular labels that we use are not important. In a blockchain we typically assume that a block is signed by its creator and this signature can be used as a label (it includes sufficient identifying information such as the name of creator, time of creation, predecessor,...).

In this example, we see three distinct (maximal) paths.[3] There is the path $B_0 \leftarrow B_1 \leftarrow B_4$ which contains three blocks. The path $B_0 \leftarrow B_2 \leftarrow B_3 \leftarrow B_6 \leftarrow B_7$ contains five blocks. Finally, there is also the path $B_0 \leftarrow B_2 \leftarrow B_5$ which contains three blocks. In this example, the path containing 5 blocks is the longest. This path would thus be referred to as the longest chain. An honest node in this example would thus create a new block, say $B_8$, that points to $B_7$ as its predecessor.

In the above example, there is a unique longest path, which means the *honest* node does not have to make a choice: its new block has to point to $B_7$. This does not have to be the case however. In the example

[3] A path is *maximal* if it cannot be extended to a longer path, i.e., it is not a subset of a larger path.

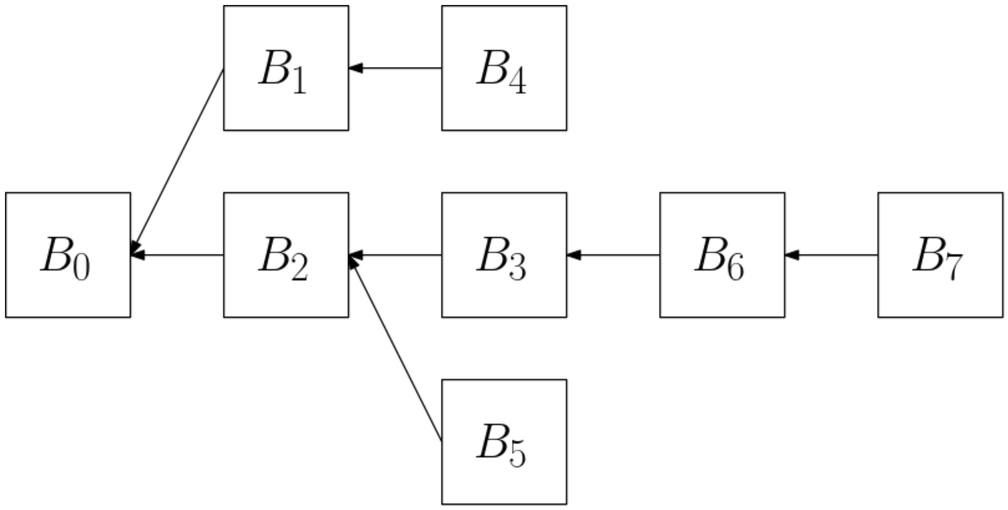



Figure 5.2: A labelled blockchain

below, there are two maximal paths, both containing exactly three blocks (namely $B_0 \leftarrow B_1 \leftarrow B_4$ and $B_0 \leftarrow B_2 \leftarrow B_3$). Either of the two paths would be a longest chain in this example. This is the reason that we ask honest nodes to extend *a* longest chain and not *the* longest chain. When there are several longest chains, honest nodes may break ties in an arbitrary (but fixed) way.

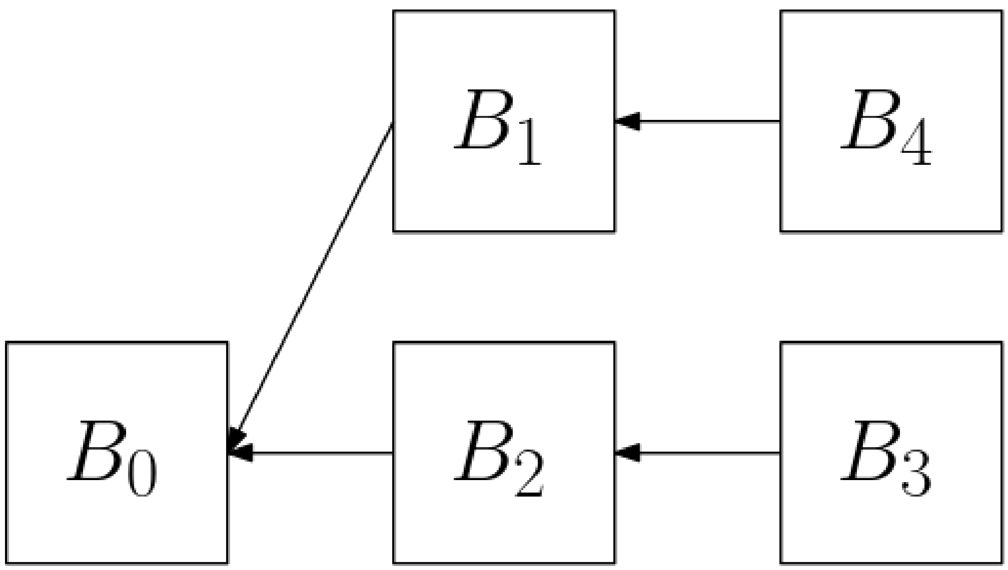



Figure 5.3: A labelled blockchain with 2 longest chains

We have now described the intended behavior of honest nodes. What about nodes that are *dishonest* or *faulty*? As usual, we make no assumptions whatsoever about their behavior. After all, a node is called *faulty* whenever it deviates from the intended protocol (no matter the reason or the manner in which deviates). There is however something we can say. Since the intended behavior is that a new block specifies *exactly* one predecessor, the honest nodes can always disregard any blocks that specify 0 or at least 2 predecessors. We can therefore assume that a block created by a *faulty* node also specifies exactly one predecessor which moreover comes from a previous round.

Looking ahead, let's think of a blockchain as storing a list of transactions. In that case, the honest nodes are working together to maintain a correct history of all the transactions that have taken place. They do so by recording transactions on the longest chain and they regard the longest chain as the one containing the *true* list of transactions.

> **Exercise**
>
> Assume all nodes are honest. What does the blockchain look like after 10 rounds?

If we think of *faulty* nodes as malicious, then their goal could be for example to double-spend some of their money. They could do so by adding (many) blocks to a chain that is currently not the longest. Eventually this would create a new longest chain. If we recall that honest nodes view the longest chain as the one containing the true transactions, then this would allow a faulty node to spend their money twice. Indeed, if we assume they spent some money in the original longest chain in some block $B_i$, then they could extend the path ending at the predecessor of $B_i$ and by doing so, they could spend their money a second time (buying something else).

Figure 5.3 is (or could be) an example of such a situation. Here you can imagine that the honest nodes created blocks $B_2$ and $B_3$ in rounds $1$ and $2$ (recall that the labels are arbitrary). In rounds $3$ and $4$ the leaders happened to be *faulty* and they collaborated to start a new path with the blocks $B_1$ and $B_4$. At this point, the *faulty* nodes have successfully confused the *honest* nodes: there is no way to tell which of the two chains was constructed by *honest* nodes. It might thus be the case that an *honest* node that is elected as leader in round $5$ decides to indicate $B_4$ as its predecessor. By doing so, the top chain would become the unique longest chain. At this point even honest nodes would start extending the top chain. This means the faulty nodes have successfully convinced the honest nodes to abandon their original chain, thus reverting some previously made transactions (blocks).

## 5.2 The assumptions

We now formalize the assumptions that we make about blockchains in a distributed network.

### 5.2.1 Assumption A1: the genesis block is unknown prior to starting the protocol

This first assumption is a trusted setup assumption.

> **A1)** We assume that no node has knowledge of the genesis block prior to the deployment of the protocol.

At this point, it should not be clear why we need this assumption. But if you are already familiar with proof-of-work based protocols, try to answer the following question. (If you are not yet familiar, please revisit this question after we have studied the next chapter.)

> **Exercise**
>
> What could go wrong in a proof of work setting if we don't make this assumption?

Solution

The faulty nodes could cheat by creating valid blocks before the deployment of the protocol! If they manage to create $K$ blocks before the start of the protocol, this gives them the power to create a path of length $K$ “for free”. This means we cannot trust the first $K$ blocks on the blockchain. Another way to phrase assumption 1 is thus that we assume that $K = 0$.

So how do we verify this assumption? Technically we can’t, we just have to take it on faith. There are ways to make the assumption more plausible however. For example, the first block of Bitcoin was created on 3 January 2009. It contained the text “The Times 03/Jan/2009 Chancellor on brink of second bailout for banks”, which is a reference to a headline of that day’s issue of the newspaper The Times. Assuming Nakamoto had no way to influence this headline, it is thus reasonable to assume that nobody knew the genesis block (long) before the deployment of the Bitcoin protocol.

### 5.2.2 Assumptions about leader selection

We need to make two assumptions about how leaders are selected. The first is related to the PKI assumption that we have seen in the BFT protocol.

**A2)** All nodes can efficiently verify whether a given node is the leader of a given round.

**A3)** No node can influence the probability with which it is selected as the leader of a round in step 2a.

In the protocols that we have studied in the previous chapter both were trivially satisfied. Indeed, we were working in the permissioned setting and the synchronous model and the leader-selection protocol simply asserted that in round $t$ node $t$ would be the leader (counting rounds modulo $n$, i.e., node $1$ is the leader in round $n + 1$ as well and so on). Since we assumed signatures exist and cannot be forged, only node $t$ could pretend to be the leader in round $t$. This shows that A2 was satisfied. Since the leader-selection protocol is deterministic, A3 is satisfied trivially. In this chapter, and the next, we want to move away from the permissioned, synchronous setting. Assumptions A2 and A3 clarify the conditions that our leader-selection protocol should satisfy.

In the next chapter we will argue that these assumptions also hold in the proof of work setting.

### 5.2.3 Assumptions about block production

We assume the following about blocks produced in round $r$:

**A4)** Every block produced by the leader in round $r$ must claim as its predecessor a block that belongs to a previous round.

Remember that we assume signatures exist. We can thus assume that when a block is created, the creator includes the round number in the identifier of a block. This assumption implies that if you trace the sequence of predecessors of a single block, you always end up at a/the block that was created in round $0$: the genesis block.

Assumption A4) seems a bit redundant at first: after all how can you create a block that points to a predecessor that doesn’t exist yet? Honest nodes would of course not do this, but faulty nodes might have incentives to do this. This assumption puts some restrictions on their behavior, which we should thus verify for any blockchain protocol. For example, under the assumption A4), faulty nodes are still allowed to create multiple blocks in a single round, but each of these new blocks has to point to a predecessor from a previous round.

The faulty node can thus not propose blocks that point to each other for example. It also prevents faulty nodes from "delaying": if they are correctly selected as leader in round 10, they might want to wait to see what happens in rounds 11 and 12 before announcing their block for example in round 13. This assumption prevents them from using the blocks created in rounds 11 and 12 as predecessors for their block.

Assumption A4) will be crucial in the analysis, but in implementations it is typically not so hard to enforce. For example, in the setting from the previous chapter we would just require the leader of round $r$ to include the round number in the description of the block.

In the proof of work protocol that we will study in the next chapter, we can in fact enforce a stronger version of A4). This stronger version is not needed for the lemmas and theorems in this chapter, but it sometimes simplifies the proofs.

> **A4')** The leader of round $r$ produces exactly one block, and this block claims as its predecessor a block that belongs to a previous round.

We will revisit this assumption in the next chapter. In a nutshell, the proof of work is valid proof only for a single block (the creator has to commit to a block before starting the work).

### 5.2.4 Assumptions about communication

The last assumption that we make is one about our communication model.

> **A5)** At all times, all honest nodes know about the exact same set of blocks.

This is a (very) restrictive assumption; it essentially trivializes the consistency problem that we had to deal with in the previous chapter. We will see how to relax this assumption in the next chapter in the setting of proof of work. So why do we make this assumption? For one, it simplifies our lives (and certainly the exposition) a bit. The main reason however is that the key ideas behind longest-chain protocols are already needed even when we add this restrictive assumption. When we relax the assumption in the next chapter we will see that it still "holds in spirit"; it is therefore a reasonable way to think about longest-chain protocols.[4]

## 5.3 The goals: liveness and *finality*

As in the previous chapter, we will have two goals that we want to achieve in a blockchain protocol. The first will be liveness, as in the previous chapter.

As stated above, assumption A5) trivializes the *consistency* requirement. At least, the way we thought about consistency in the previous chapter. In the previous chapter we thought about consistency as keeping all nodes in sync: their local history states had to be identical at all points in time. This is indeed a key aspect of consistency, but it ignores another very important aspect: consistency over time. By that we mean that there is some notion of consistency between the local history of an honest node at time $t$ and the local history of that same node at some later time $t + t'$. For example, the list of transactions recorded at time $t$ is a prefix[5] of the list of transactions recorded at time $t + t'$. If you take another look at the protocols that we have seen so far, you will realize that they also satisfy this second property. The reason for this is that we only allowed information (transactions) to be added to the local history.

In a blockchain, we think of the (shared) local history state as the information that is stored in the blocks on the longest chain. From our previous discussion, it should be clear that the first aspect of consistency

[4]This is certainly a handwavy statement. We will not make it more precise here.
[5]The first part of…

(consistency across nodes at a given time) is trivial, but the second aspect is not! Indeed, consistency over time is the main goal that we will work towards in this chapter. Concretely, we aim for the following.

> **Goal: Finality** (first version) If an honest node $i$ considers a block $B$ as *finalized* at time $t$, then this block remains finalized at all times after $t$.

Some remarks are in order. First, we have not yet formalized the concept of *finalized*. We will do so in the next section. Second, even without knowing what *finalized* means, we can make sense of the goal *finality*: if we consider the set of finalized blocks as our local history state, then we have achieved consistency *over time*. Indeed, finality precisely ensures that whatever is part of the local history at time $t$ will remain part of the local history at all future times $t + t'$. Third, the observant reader might have noticed the addition "(first version)". This strongly suggests that there will be a second version in the future. There will be one indeed. In the later sections we will end up talking about protocols involving randomness. In such protocols, the above version of finality is too much to ask for. We will need to replace the first version of finality with a slightly weaker version that asks the same probability to hold with high probability. (What do we consider "high probability"? That depends again on the application…)

## 5.4 Finalizing a block

So how do we finalize a block? Recall that we want to think of the longest chain in the blockchain as the one that stores the list of transactions. Intuitively, we would like to argue that if a block $B$ is far enough from the end of this longest chain, then it will (likely) always be a part of a/the longest chain. It would after all require the faulty nodes to extend the chain ending at the predecessor of $B$ to a new chain that is longer than the currently longest chain.

To formalize this intuition, we introduce the parameter $k$.

> The parameter $k$ corresponds to the number of blocks at the end of the longest chain that are still considered not finalized.

Let $G$ be a directed in-tree rooted at a node $B_0$. For an integer $k$ we define

$$\mathcal{B}_k(G) := \text{ the longest chain of } G\text{, with the last } k \text{ blocks removed.}$$

For example, in Figure 5.2 we have

$$\mathcal{B}_0(G) = (B_0 \leftarrow B_2 \leftarrow B_3 \leftarrow B_6 \leftarrow B_7),$$

and

$$\mathcal{B}_1(G) = (B_0 \leftarrow B_2 \leftarrow B_3 \leftarrow B_6).$$

Before reading the rest of this chapter, try to answer the following questions by yourself.

> **Exercise**
>
> In the above example Figure 5.2, what is $\mathcal{B}_2(G)$? What about $\mathcal{B}_1(G)$ and $\mathcal{B}_2(G)$ in the example Figure 5.3? Is there a unique answer?

Solution

For the first question, simply remove $B_6$ from $\mathcal{B}_1(G)$. For the second question, it is not uniquely defined for $\mathcal{B}_1(G)$! In the next sections we will indeed need to argue that under certain assumptions, for $k$ large enough, we can indeed speak of *the* chain $\mathcal{B}_k(G)$ (meaning it's unique). In this example, $\mathcal{B}_2(G)$ is in fact uniquely defined, for both of the longest chains, if you remove the last two blocks, you end up with the chain $(B_0)$.

Exercise*

Assume a network with 10 nodes, at most 1 of which is *faulty*. Suppose we cyclically change the leader in each round. For which value of $k$ is $\mathcal{B}_k(G)$ well-defined?

Coming back to our goal of *finality*, given a parameter $k$, we would like to design a protocol that satisfies *finality* when the honest node $i$ considers the blocks in $\mathcal{B}_k(G_t)$ as finalized. Here we use $G_t$ to denote the graph corresponding to the blockchain in round $t$.

So what should $k$ be? If you back to the blockchain protocol description that we gave in Section 5.1, you will see that there was no mention of a parameter $k$. Indeed, the parameter $k$ is not a parameter of the protocol! Instead, it is a parameter that the user / node / client should decide for themselves, depending on their application!

If we again take cryptocurrencies as motivating example, then it's easy to see that there are competing interests when it comes to the parameter $k$. A smaller value of $k$ means that blocks get finalized more quickly, and thus transactions can be handled more quickly. However, at the same time, a smaller value of $k$ also means that it becomes easier for faulty nodes to roll back the chain to a block that was more than $k$ blocks deep in the chain. Meaning that you might not be able to trust the fact that the transactions in $\mathcal{B}_k(G)$ actually took place.

Continuing the example, a client selling a cup of coffee might be willing to trust all transactions in $\mathcal{B}_1(G)$. Meaning that they will hand you your cup of coffee after waiting for only a single new block to be added to the longest chain. A client selling a house however, would probably want a bit more security before handing you the keys to the house, they might want to wait until the payment is contained in $\mathcal{B}_{100}(G)$.

In the remainder of this chapter we will study various scenarios and prove that

(1) $\mathcal{B}_k(G)$ is well-defined (provided $k$ is large enough),
(2) Blocks contained in $\mathcal{B}_k(G)$ can be considered final.

What do we mean by well-defined? As we have seen in the second-to-last exercise, it can happen that there are several longest chains. In such a case, we say that $\mathcal{B}_0(G)$ is not well-defined because it is not a unique object. The blockchain from Figure 5.3 is an example. Here $\mathcal{B}_0(G)$ is not well-defined, because there are two longest chains. An important observation is that $\mathcal{B}_2(G)$ in that example is well-defined: the two longest chains agree on all blocks, except the last 2. In other words, if you take either of the two longest chains and remove the last two blocks, then you end up with the same chain. In this example, we have $\mathcal{B}_2(G) = (B_0)$.

If you think about it, $\mathcal{B}_k(G)$ is always well-defined "provided $k$ is large enough": if we take $k$ to be one less than the number of blocks in the longest chain, then we always have $\mathcal{B}_k(G) = (B_0)$. Of course, this is not a very satisfying situation for practical applications. We would like to show that $\mathcal{B}_k(G)$ is well-defined even if $k$ is a (small) constant.

A final remark before we continue: in the rest of the chapter we will deal with protocols involving randomness, for example in the decision of who gets to create a block. We therefore cannot guarantee that properties (1) and (2) hold with certainty. The best we can hope for is a statement that (1) and (2) hold *with high probability*. We will make this more precise as we go along.

If we go back to our discussion in Section 5.1.1, we see that honest nodes will always extend the longest chain, whereas faulty nodes might choose to try to extend a shorter chain in the hope of turning it into the longest chain. From the perspective of the honest nodes, the worst case scenario is that the honest nodes keep adding nodes to one chain, while the faulty nodes are collaborating to create an entirely different chain. If the faulty nodes ever manage to extend their chain further than the one of the honest nodes, it would become the longest chain, causing the honest nodes to abandon their original chain. In this worst-case example, no block on the original chain of the honest nodes is finalized. Can we prevent this?

As a motivating example, let us consider the case where two thirds of the nodes are honest and a third of the nodes is faulty. If a new block is proposed by a node chosen uniformly at random[6], then a new block is thus proposed by an honest node with probability $2/3$ and by a faulty node with probability $1/3$. In the long run, we thus expect that the vast majority of the blocks is proposed by honest nodes. In the worst-case situation described above, the chain of the faulty nodes had to become longer than the chain of the honest nodes. In this example, we expect the chain of the honest nodes to be twice as long as the chain of the faulty nodes. It is a nice probability question to turn this "expectation" into a statement that holds with high probability: for a fixed number of blocks $K$, what is the probability that at least $K/2 + 1$ blocks are created by faulty nodes?

This example shows that it is very unlikely that the faulty nodes are able to change the longest chain *entirely*, meaning that the only block in common will be the genesis block $B_0$. Can we say more? Intuitively, I hope you agree that the answer is yes. If two thirds of the nodes are honest, then if the chain becomes long enough, we can probably safely say that the first couple of blocks on the chain are *final with high probability*. Here, by *final with high probability* we mean that with high probability over the randomness in the protocol (i.e., the leader selection) the first couple of blocks will remain part of the longest chain indefinitely.

In the next two sections we make this intuition more precise. We first introduce the concept of *balanced leader sequences*. This allows us to distill the key properties of the "two thirds vs. one third" example. We then show that balanced leader sequences ensure that "the first couple of blocks will remain part of the longest chain indefinitely": we define the common prefix property.

## 5.5 Balanced leader sequences

Eventually, we would like to show that our blockchain protocol satisfies for example *liveness*. For that property, it is important to distinguish between two honest nodes. From the perspective of the blockchain however, there is no need to make this distinction: all honest nodes would act the same. Similarly, we always assume that the faulty nodes are collaborating anyway, so we might as well treat all faulty nodes as identical.

This means that if we think about the sequence of leaders[7] in our protocol, we only need to keep track of who was honest and who was faulty. We will thus represent the leader sequence as a sequence of $H$'s and $F$'s, where $H$ stands for *honest* and $F$ for *faulty*. For example, if the first two leaders are honest, the third is faulty, and the fourth is honest, then this would correspond to the sequence $H, H, F, H$.

We now state the key definition of this section. It depends on a parameter $w$, a positive integer, that stands for window.

[6]This might sound artificial, but it is actually a realistic assumption for many protocols. Proof of work protocols, such as Bitcoin, satisfy this property for example.

[7]Remember, the leader is the one that proposes a new block.

**Definition 5.1** ($w$-balanced leader sequence)**.** We say that a leader sequence $\ell_1, \ell_2, \ell_3, \ldots \in \{H, F\}$ is *$w$-balanced* if, in every window $\ell_i, \ell_{i+1}, \ldots, \ell_{j-1}, \ell_j$ of length at least $w$, the number of $H$'s is strictly larger than the number of $F$'s.

Here are some observations to help you get familiar with the notion of $w$-balanced leader sequences. (The explanation is contained in the footnotes, please take a minute to think about the statements before looking at the footnotes.)

- If a leader sequence contains at least one $F$, then it is not 1- or 2-balanced.[8]
- If a leader sequence is $w$-balanced, then it is also $w'$-balanced for any integer $w'$ that satisfies $w' > w$.[9]

The second observation motivates us to find the smallest value $w$ for which a leader sequence is $w$-balanced.

We will see in the next couple of sections that, as long as the leader sequence is $w$-balanced, then the blocks in $\mathcal{B}_k(G)$ can be considered final, for a large enough $k$. We now investigate several scenarios in which we can prove that a leader sequence is $w$-balanced for some value of $w$.

The first scenario is about the permissioned setting and assumes we elect the leader in a cyclic fashion. Recall that in the permissioned setting, the number of nodes is fixed and denoted by $n$. By electing the leader in a cyclic fashion, we mean that we fix an arbitrary order in which the $n$ nodes will be elected as leader. To obtain our leader sequence, we then repeat this fixed order indefinitely.

Exercise 1

Consider a set of $n$ nodes, with $f < n/3$ faulty nodes. Assume we select the leader in a cyclic fashion. Show that the resulting leader sequence is $n$-balanced.

Exercise 2

Consider a set of $n$ nodes, with $0 < f < n/2$ faulty nodes. Assume we select the leader in a cyclic fashion. Show that there is a worst-case scenario such that the resulting leader sequence is not $n$-balanced.

The second scenario that we consider adds randomness to the process. We still consider the permissioned setting (for now), which means there are $n$ nodes. In each round, we elect the leader uniformly at random from the set of $n$ nodes. That is, each node is equally likely to be chosen as the leader. This is a second natural approach to take in the permissioned setting. More importantly, we will see that this approach extends very naturally to the *permissionless* setting.

### 5.5.1 Random leaders are balanced: the intuition

To build some intuition, let's see how many *faulty* nodes are in a typical leader sequence. Let $\alpha$ denote the fraction of *faulty* nodes in the network, i.e., let $\alpha = f/n$. We assume from now on that $\alpha < 1/2$. This is a pretty harmless assumption: if more than half the nodes are faulty, there is no hope for achieving a $w$-balanced leader sequence for any value of $w$.

[8] Indeed, for $w \in \{1, 2\}$, take a window of length $w$ that contains an $F$. In that window the number of $H$'s is at most 1 (could be zero) and the number of $F$'s at least one.

[9] The key observation here is that the definition of $w$-balanced leader sequences requires the property (strict majority of $H$'s) to hold for every window of length *at least* $w$. If the property holds for every window of length at least $w$, then it in particular holds for every window of length at least $w'$ whenever $w'$ is strictly larger than $w$. Another way to put this is that the definition becomes easier to satisfy the larger $w$ is, since there are less windows to verify.

- If we elect a leader uniformly at random from the set of $n$ nodes, then the probability that the leader is faulty is thus equal to $\alpha$.
- If we elect $K$ leaders in this way, then we thus expect $\alpha K$ of the leaders to be faulty.
- Since $\alpha < 1/2$, we thus expect strictly less than half the leaders to be faulty.

Can we turn this "on expectation" statement into a "with high probability" statement? To gain some intuition, you are invited to experiment with different values of $\alpha$ and $K$, using the Python code snippets below.

To create one such leader sequence, use the following code:

```
import numpy as np

import matplotlib.pyplot as plt

def sample_leaders(K, alpha):
    """
    Samples K Bernoulli random variables, each with probability alpha.

    Parameters:
        K (int): Number of Bernoulli random variables to sample.
        alpha (float): Probability of success (1) for each Bernoulli trial.

    Returns:
        numpy.ndarray: Array of sampled Bernoulli random variables (0s and 1s).
    """
    samples = np.random.binomial(n=1, p=alpha, size=K)
    return samples
```

```
# Example usage:
K = 10      # number of leaders
alpha = 0.45 # probability of leader being faulty
samples = sample_leaders(K, alpha)
print("Sampled leaders:", samples)

print("Fraction of faulty leaders:", np.sum(samples)/K)
```

```
Sampled leaders: [0 0 1 0 1 0 0 0 1 1]
Fraction of faulty leaders: 0.4
```

If you run the above code several times, you will see that the outcome varies a bit. The following code allows you to run the process $T$ times and create a histogram of the fractions of faulty nodes.

```
def sample_T_sequences(T, K, alpha):
    """
    Generates T leader sequences of length K,
    where alpha is the fraction of faulty nodes.

    Returns:
```

```python
        fractions: List of fractions representing the
        fraction of faulty leaders in each leader sequence.
    """
    fractions = []
    for _ in range(T):
        sample = sample_leaders(K, alpha)
        fractions.append(np.sum(sample)/K)
    return fractions

def plot_histogram(fractions, bins=20):
    """
    Plots a histogram of the fractions.

    Parameters:
        fractions (list): List of fraction values.
        bins (int): Number of bins for the histogram.
    """
    plt.hist(fractions, bins=bins, edgecolor='black')
    plt.title("Histogram of the fraction of faulty nodes")
    plt.xlabel("Fraction of faulty nodes")
    plt.ylabel("Frequency")
    plt.grid(True)
    plt.show()
```

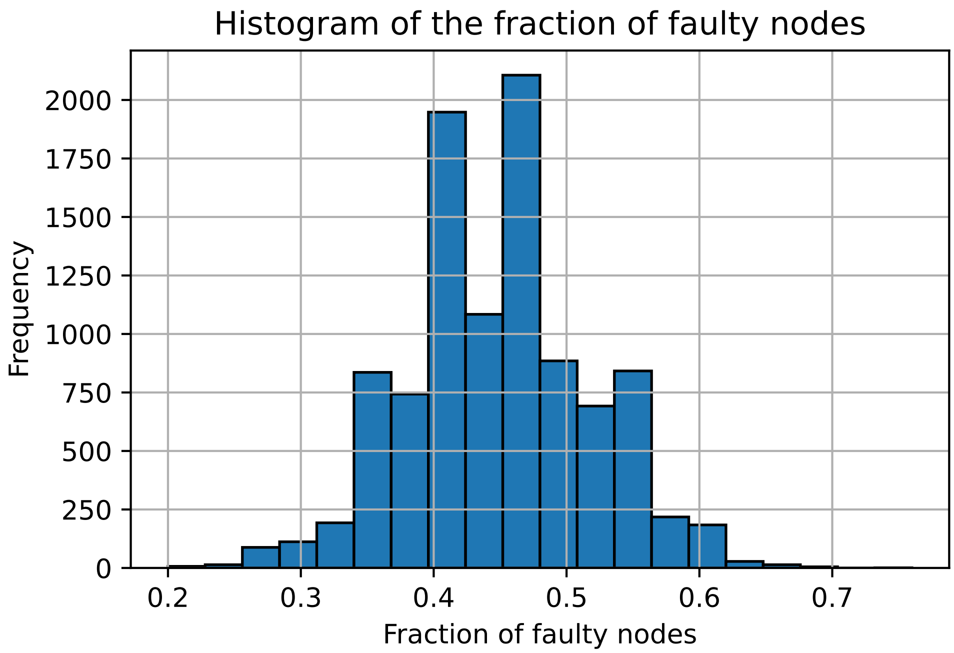
```python
# Example usage:
T = 10000     # Number of leader sequences
K = 50        # Number of leaders per sequence
alpha = 0.45  # Probability of success in each Bernoulli trial

fractions = sample_T_sequences(T,K,alpha)

print("The fraction of leader sequences that are balanced is:",
    sum(1 for f in fractions if f < 0.5)/T)

# Plot the histogram of fractions
plot_histogram(fractions)
```

```
The fraction of leader sequences that are balanced is: 0.7131
```




Try changing the value $K$ in the above snippet to $500$, what do you observe?

If you are lucky[10], you have been able to make the following two observations:

- The longer the leader sequence is, the larger the probability is that strictly less than half the leaders are faulty.
- If you choose $\alpha$ closer to $1/2$, you need to take a longer leader sequence to ensure that a large fraction of the $T$ leader sequences has the property that strictly less than half the leaders are faulty.

### 5.5.2 Random leaders are balanced: the math

To prove that our intuition from above is correct, we will two useful results from probability theory.

The first one is a so-called *concentration inequality*. Roughly speaking, a concentration inequality provides a quantitative statement about the probability that a random variable deviates more than a certain amount from its mean. They typically take on the form[11]

$$\Pr(|Z - \mathbb{E}[Z]| \geq t) \leq \text{ something small.}$$

**Hoeffding's inequality:** The concentration inequality that we use is called Hoeffding's inequality. This inequality is often used in the computer science literature. It applies to a more general setting than what we need here. We first state it in its general form and then we discuss the implications for our setting.

Let $X_1, \ldots, X_K$ be *independent* random variables such that $a_i \leq X_i \leq b_i$. Consider the sum of the random variables:

$$S_K := X_1 + X_2 + \ldots + X_K.$$

[10] In the next section, you will learn that not much luck was needed…

[11] If you want to know (much) more about concentration inequality, I highly recommend Vershynin's book on high-dimensional probability.

As usual, let us use $\Pr(\cdot)$ and $\mathbb{E}[\cdot]$ to denote probability and expectation respectively. Then Hoeffding's inequality states the following.

**Theorem 5.1** (Hoeffding)**.** *Fix a $t > 0$. Then we have*

$$\Pr(|S_k - \mathbb{E}[S_k]| \geq t) \leq 2\exp\left(-\frac{t^2}{\sum_{i=1}^{k}(b_i - a_i)^2}\right).$$

To get a feeling for this bound, let us discuss how we will apply it. In our setting, we want to count the number of faulty nodes in a leader sequence of length $K \geq w$. The variable $X_i$ will take on the values 0 and 1 representing honest and faulty respectively. In this way, $S_k$ indeed counts the number of faulty nodes in the leader sequence of length $K$. As bounds, on $X_i$ we can take $a_i = 0$ and $b_i = 1$. Hoeffding's inequality then shows that

$$\Pr(|S_K - \mathbb{E}[S_K]| \geq t) \leq 2\exp\left(-\frac{t^2}{K}\right).$$

At this point, a natural reaction would be: wait a minute, we have not specified the distribution of the $X_i$'s! Indeed, the distribution of the $X_i$'s is irrelevant for Hoeffding's inequality. All that matters is that the bounds $a_i$ and $b_i$ are known.

So what is the distribution of "our" $X_i$? In our application of counting the number of faulty nodes in a leader sequence, each $X_i$ will be a random variable that takes on the value 0 with probability $1 - \alpha$ and the value 1 with probability $\alpha$.[12] Note that each of the $X_i$'s is identically distributed. They are independent random variables by due to the design of our protocol. So what does Hoeffding's inequality say for our setting? The only thing that remains is to compute $\mathbb{E}[S_K]$. By linearity of the expectation, we have

$$\begin{aligned}\mathbb{E}[S_K] &= \mathbb{E}[X_1 + X_2 + \ldots + X_k] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_K] \\ &= \alpha K.\end{aligned}$$

Hoeffding's inequality thus gives us the following bound:

$$\Pr(|S_K - \alpha K| \geq t) \leq 2\exp\left(-\frac{t^2}{K}\right).$$

Our goal is to show that the sequence $X_1, X_2, \ldots, X_K$ is *balanced*: we want to prove that $S_K < K/2$ *with high probability*. Hoeffding's inequality allows us to quantify this probability. Indeed, we will apply the bound with $t = \left(\frac{1}{2} - \alpha\right)k$ (or a tiny bit smaller if we want the strict inequality). We then obtain

$$\Pr(S_K \geq K/2) \leq 2\exp\left(-\left(\frac{1}{2} - \alpha\right)^2 K\right). \tag{5.1}$$

**The key observation** here is that the right hand side decays exponentially with $K$! A second observation is that the bound deteriorates as $\alpha$ approaches $1/2$, as it should (do you see why?). Both of these observations support the intuition from the previous section: if the fraction of faulty nodes $\alpha$ is strictly less than $1/2$, then if we look at a long enough leader sequence (length $K$), the honest nodes outnumber the faulty nodes with high probability.

[12] As a side remark for those who recently took a course on probability, this means each $X_i$ is drawn according to the Bernoulli distribution with parameter $\alpha$.

Exercise

Revisit the numerical examples from the previous section, does the bound from Equation 5.1 match your observations?

**The union bound:** The above allows us to argue that one particular window is balanced, with high probability. However, to argue that the entire leader sequence is $w$-balanced, we have to consider *all* windows of length at least $w$. To do so, we will need a second fundamental tool from probability theory: the union bound.

We will need the bound in the following form. Let $E_1, \ldots, E_N$ be some events, then we have the inequality

$$\begin{aligned} &\Pr(\text{at least one of the events } E_1, \ldots, E_N \text{ occurs}) \\ &\leq \sum_{i=1}^{N} \Pr(\text{event } E_i \text{ occurs}). \end{aligned}$$

For two events $A$ and $B$, the union bound is a simple consequence of the identity

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Indeed, it suffices to observe that $\Pr(A \cap B) \geq 0$.

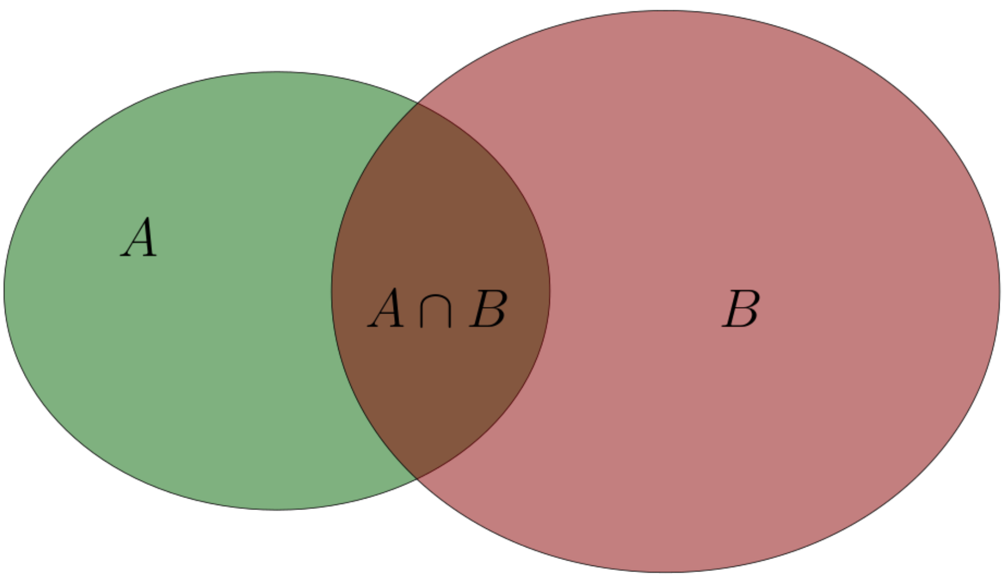



Figure 5.4: Union bound

The union bound for $N$ events can be derived from the one for two events using induction.

Exercise

Prove the above union bound for $N$ events.

How will we apply the union bound? We want to argue that for a given leader sequence, all windows of length at least $w$ have the property that there is a majority of honest leaders. To each window $i$ of length at least $w$ we associate the event $E_i$ that states that at least half the leaders in window $i$ are *faulty*. Notice that we define the event as the *bad event*. In this way, the union bound gives us an upper bound on the probability that at least one bad event occurs. Why do we care about such an upper bound? It precisely gives us a lower bound on the complementary event: the probability that all windows are balanced. In formulas,

$$\begin{aligned}&\Pr(\text{none of the events } E_1, \dots, E_N \text{ occurs})\\ &\geq 1 - \sum_{i=1}^{N} \Pr(\text{event } E_i \text{ occurs}).\end{aligned}$$

**Random leader sequences are balanced:** We now combine the above two key ingredients. Our goal is to show that random leader sequences are $w$-balanced, provided $w$ is large enough. Formally, we will show the following.

**Theorem 5.2** (Random leader sequences are balanced)**.** *A random leader sequence of length $T$ in which the probability that a leader is faulty is $\alpha \in [0, 1/2)$ is $w$-balanced with probability $1 - \delta$ if*

$$w \geq \frac{(1 + 2\log(T) + \log(1/\delta))}{\left(\frac{1}{2} - \alpha\right)^2}.$$

*Proof.* Let $\alpha \in [0, 1/2)$ and consider a leader sequence $\ell_1, \ell_2, \dots, \ell_T$ of length $T$ in which each leader is chosen independently. The probability that a leader is faulty is $\alpha$. Fix some integer $w$. We give a lower bound on the probability that the leader sequence is $w$-balanced.

To do so, we have to show that in each window of length at least $w$, there is a strict majority of honest leaders. Let $N$ denote the number of windows of size at least $w$ in a sequence of length $T$. For future reference, let us observe that $N \leq T^2$: each window is uniquely defined by its start and end point, both of which are integers between 1 and $T$.

For $i \in [N]$, let $E_i$ denote the event that at least half the leaders in the $i$-th window are faulty. Then our leader sequence is $w$-balanced precisely if none of these events occur. The union bound gives us the following lower bound:

$$\begin{aligned}&\Pr(\text{none of the events } E_1, \dots, E_N \text{ occurs})\\ &\geq 1 - \sum_{i=1}^{N} \Pr(\text{event } E_i \text{ occurs}).\end{aligned}$$

We are thus left with providing an upper bound on the quantity

$$\sum_{i=1}^{N} \Pr(\text{event } E_i \text{ occurs}).$$

For that, let us first consider a fixed $i \in [N]$. By assumption, the $i$-th window has length $K_i$ that is at least $w$. Hoeffding's inequality, and in particular Equation 5.1, then gives us the upper bound

$$\begin{aligned}\Pr(\text{event } E_i \text{ occurs}) &\leq 2\exp\left(-\left(\frac{1}{2} - \alpha\right)^2 K_i\right)\\ &\leq 2\exp\left(-\left(\frac{1}{2} - \alpha\right)^2 w\right),\end{aligned}$$

where in the second inequality we have used that $K_i \geq w$. Note that the latter upper bound does not depend on $i$! We therefore have

$$\sum_{i=1}^{N} \Pr(\text{event } E_i \text{ occurs}) \leq 2N \exp\left(-\left(\frac{1}{2}-\alpha\right)^2 w\right)$$
$$\leq 2T^2 \exp\left(-\left(\frac{1}{2}-\alpha\right)^2 w\right).$$

This almost concludes the proof: we have obtained an upper bound on the probability that our leader sequence is not $w$-balanced. By inspection, the bound is decreasing in $w$. For which value of $w$ is this probability at most $\delta$? To determine that, we solve the equation

$$\delta = 2T^2 \exp\left(-\left(\frac{1}{2}-\alpha\right)^2 w\right)$$

for $w$. Taking the natural logarithm on both sides gives

$$\log(\delta) = \log(2) + 2\log(T) - \left(\frac{1}{2}-\alpha\right)^2 w.$$

Rearranging gives the claimed lower bound on $w$ (using that $\log(2) \leq 1$):

$$w \geq \frac{(1 + 2\log(T) + \log(1/\delta))}{\left(\frac{1}{2}-\alpha\right)^2}.$$

□

### 5.5.3 Random leaders are balanced: the takeaway message

Since the proof was a bit lengthy, let us restate what we have just shown, so that we can discuss the main takeaway message.

**Theorem 5.3** (Random leader sequences are balanced)**.** *A random leader sequence of length $T$ in which the probability that a leader is faulty is $\alpha \in [0, 1/2)$ is $w$-balanced with probability $1-\delta$ if*

$$w \geq \frac{(1 + 2\log(T) + \log(1/\delta))}{\left(\frac{1}{2}-\alpha\right)^2}.$$

What do we learn from this theorem? Let us work through an example. Suppose that the fraction of faulty nodes in the network is at most $1/3$, that is, let $\alpha = 1/3$. Then $\left(\frac{1}{2}-\alpha\right)^2 = 1/36$, so the lower bound becomes

$$w \geq 36(1 + 2\log(T) + \log(1/\delta)).$$

Which values of $T$ and $\delta$ are realistic? This of course again depends on your application. The important observation here is that the logarithm function grows *very slowly*; here is a table of the value of the natural logarithm for the first 10 powers of 10.

| Power of 10 | Value (10^k) | Natural Logarithm (log(10^k)) |
|---|---|---|
| 10^0 | 1 | 0.0000 |
| 10^1 | 10 | 2.3026 |
| 10^2 | 100 | 4.6052 |
| 10^3 | 1,000 | 6.9078 |
| 10^4 | 10,000 | 9.2103 |
| 10^5 | 100,000 | 11.5129 |
| 10^6 | 1,000,000 | 13.8155 |
| 10^7 | 10,000,000 | 16.1181 |
| 10^8 | 100,000,000 | 18.4207 |
| 10^9 | 1,000,000,000 | 20.7233 |

For example, if we consider a leader sequence of length $T = 1,000,000,000$, with $\alpha = 1/3$ and $\delta = 1/1,000,000,0000$, then the sequence is $w$-balanced with probability at least $1 - \delta$ whenever $w \geq 26(1 + 3 \cdot 20.7233) \approx 1643$. A blockchain of length $T = 1,000,000,000$ is probably more than we need in any real application, but the point is that $w$ is very small compared to $T$ and $1/\delta$.

**Is the bound tight?**

So far, we have proven *an* upper bound on the probability that a random leader sequence is not $w$-balanced. It is natural to ask whether that bound is *tight*? In other words, can the upper bound be improved? The short answer is yes; the bound can be improved. If you revisit the arguments that we made you will notice that there were essentially three estimates that we made:

1. We used Hoeffding's inquality to upper bound the probability that $S_K$ deviates a lot from its mean.
2. We used the union bound to upper bound the probability of a union of events by the sum of the individual probabilities.
3. We overestimated the number of events in our application of the union bound.

Each of these estimates is likely not tight. Hoeffding's inequality for example only requires the assumption that the random variables are independent and satisfy $a_i \leq X_i \leq b_i$. In our application, we know much more. Indeed, we know the distribution of each of the $X_i$'s: they are Bernoulli random variables. As such, their sum follows the binomial distribution. We could thus replace Hoeffding's inequality with properties of the binomial distribution.[13] The second estimate, the union bound, is perhaps the most loose upper bound. The union bound is an equality only if the all the events are pairwise disjoint. Can you see why? (Hint, consider the case of two events $A$ and $B$, when do you have equality?) In our application, the events correspond to windows not being balanced. Since windows can overlap, it is reasonable to imagine that the events also overlap to some extent. The third estimate, counting the number of possible windows in a leader sequence of length $T$ is in fact pretty close to tight: we used as upper bound $T^2$ and one (you?) can show that the number of windows of length at least $w$ is at least $cT^2$ for some small constant $c$.

From a qualitative perspective though, our analysis "got the job done": the theory supports the intuition that we gathered through numerical examples. To complete the circle, you can try to use numerical methods to investigate how tight the above bound is. The following exercise assumes some familiarity with Python (and hence is marked by a *).

[13] Here it is important that each of the $X_i$'s is independent and *identically* distributed according to the Bernoulli distribution. In the permissionless setting, the random variables might not satisfy the same distribution anymore.

Exercise*

Consider a leader sequence of length $T = 100$, with $\alpha = 1/3$, set $\delta = 0.01$. What is the lower bound on $w$ that follows from Theorem 5.3? Use Python to 1000 such leader sequences. For each of the leader sequences, determine the smallest value of $w$ for which it is $w$-balanced. Make a histogram of the $w$ values. What do you observe?

**Permissioned vs permissionless:** At the start of this section we restricted ourselves to the *permissioned* setting. That is the setting that we are familiar with. It also gives us a clear definition for the probability that a leader that is selected uniformly at random is faulty (namely $\alpha = f/n$). In the remainder of the section, we then only worked with $\alpha$. What about the permissionless setting? The simple definition of $\alpha$ from before no longer works since nodes are allowed to join and leave the network freely. However, if you read the arguments carefully, you will realize that all we needed was *an upper bound* on the probability that a leader is faulty. Later on, in the permissionless setting, we will turn this into an assumption.

## 5.6 The common prefix property

In Section 5.4 we discussed the problem of *finalizing* blocks on a blockchain. Intuitively, we argued that the blocks on the longest chain are final, except for the last few.

More concretely, for a blockchain represented by a directed in-tree $G$ and an integer $k$, we defined the object $\mathcal{B}_k(G)$ as

$$\mathcal{B}_k(G) := \text{ the longest chain of } G\text{, with the last } k \text{ blocks removed.}$$

We have seen examples in which $\mathcal{B}_k(G)$ was not well-defined (in a nutshell: consider a blockchain with two equally long chains and a very small value of $k$). Intuitively, equally long longest chains occur when the number of honest leaders is roughly equal to the number of faulty leaders. In this section we show that this is the only obstruction. For this, we will crucially use the concept of $w$-balanced leader sequences that we introduced and studied in Section 5.5. We show that if the leader sequence is $(2k+2)$-balanced, then $\mathcal{B}_k(G)$ is well-defined. Formally, we prove the following theorem.

**Theorem 5.4** (Common prefix property of longest-chain consensus)**.** *If the leader sequence $\ell_1, \ell_2, \ell_3, \ldots$ is $(2k+2)$-balanced, and assumptions A1, A4', A5 hold, then for every possible resulting in-tree $G$ of blocks known to the honest nodes, $\mathcal{B}_k(G)$ is well-defined.*

Before proving the theorem, let us reflect on the statement for a moment. As usual, we want to assume as little as possible about the behavior of faulty nodes. In this case, that means that we do not place any restrictions on where faulty nodes can add blocks to the blockchain. As a consequence, the leader sequence does not completely specify the resulting in-tree! The statement of the theorem accounts for this: we show that regardless of the behavior of the faulty nodes – "for every possible resulting in-tree $G$" – the object $\mathcal{B}_k(G)$ is well-defined.

We prove the theorem under the assumption A4', which is a slightly stronger version of assumption A4. We do so for two reasons: it simplifies the proof, and it is an assumption that in fact holds when we consider proof of work protocols such as Bitcoin.

### 5.6.1 Proof of Theorem 5.4

The plan is to prove the contrapositive statement: if there is an in-tree $G$ of blocks known to the honest nodes for which $\mathcal{B}_k(G)$ is not well-defined, then the leader sequence is not $(2k+2)$-balanced.

To that end, assume $G$ is an in-tree of blocks known to the honest nodes for which $\mathcal{B}_k(G)$ is not well-defined. Since $\mathcal{B}_k(G)$ is not well-defined, this means that there exist two longest chains in $G$ that do not agree if we remove the last $k$ blocks from each of these two chains. For the remainder of our argument, we will focus on these two chains. It will be useful to visualize these two chains. The following is the picture you should have in mind:

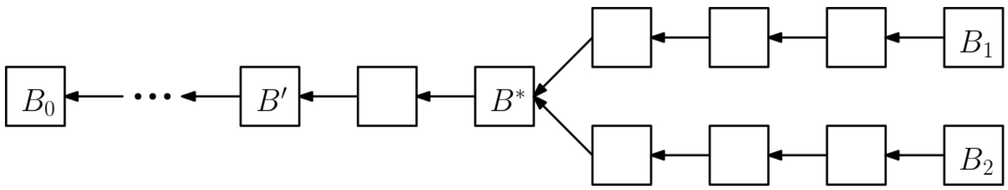



Figure 5.5: Two longest chains

Formally, we assume that there are two longest chains. The first ends at a block labeled $B_1$ (the top chain) and the second ends at a block $B_2$ (the bottom chain). We assume that the block where the two chains meet for the first time is labeled by $B^*$. Since we are assuming that $\mathcal{B}_k(G)$ is not well-defined, the length of the path between $B_1$ and $B^*$ is at least $k + 1$, and the same holds for the path between $B_2$ and $B^*$.

To make it concrete, in Figure 5.5, we have drawn 4 blocks after $B^*$ on either of the chains, so this would be an example in which $\mathcal{B}_1(G)$, $\mathcal{B}_2(G)$, and $\mathcal{B}_3(G)$ are not well-defined. It should be clear how to generalize the figure for larger values of $k$.

We will now use the fact that the length of the paths connecting either $B_1$ or $B_2$ to $B^*$ is at least $k + 1$ to argue that the leader sequence is not $(2k + 2)$-balanced. In other words, we will identify a window in the leader sequence of length at least $2k + 2$ in which there are at least as many faulty leaders as honest leaders.

To do so, it will be useful to introduce the notion of *height* of a block.

**Definition 5.2** (Height)**.** Given an in-tree $G$ with root $B_0$, we define the height of a block $B$ as the length of the path connecting $B$ to $B_0$.

Here are some simple observations:

- The height of $B_1$ is equal to the height of $B_2$. We let $h$ denote the height of $B_1$.
- The height of $B^*$ is at most the height of $B_1$ (or $B_2$) minus $k+1$, that is, its height is at most $h-(k+1)$.

We make one more observation that is slightly less simple and therefore deserves a short proof.

**Lemma 5.1** (One honest block per height)**.** *Under assumption A5, the heights of honestly produced blocks are strictly increasing over time. In particular, there is at most one honestly produced block per height.*

*Proof.* Suppose that $A$ and $B$ are two blocks produced by honest nodes. Since there is only one leader per round, one of the blocks was created first. Suppose that $A$ was created before $B$ (otherwise, swap the roles of $A$ and $B$ in the following argument). Now let $h_A$ denote the height of block $A$. Since $A$ was created by an honest node and we are working under the assumption A5, $A$ was immediately announced to all honest nodes at the time of its creation. This that at the time that $B$ was created, its honest creator was aware of at least one block at height $h_A$. Since an honest node only extends the longest chain that it is aware of, the height of $B$ is at least $h_A + 1$. This proves that the height of honestly created blocks is strictly increasing over time.

The final statement of the lemma is an immediate corollary of the first part. □

We are now ready to finish up the proof of Theorem 5.4. Recall that $B^*$ is the first block that is common on the two longest chains. Let $B'$ denote the first block that is common to both longest chains *and produced by an honest leader*. If $B^*$ is produced by an honest leader, then $B' = B^*$. Otherwise, it will be some block on the path between $B^*$ and $B_0$. (Existence of such a block is ensured by assumption A1: the block $B_0$ is produced by an honest leader.) Let $r'$ be the round in which $B'$ was created.

Since $B'$ is created by an honest node, all blocks that have $B'$ as an ancestor (on the path to $B_0$) are created *after* round $r'$. Let $r_{end}$ denote the round in which the last of $B_1$ and $B_2$ was created. We argue that the leader sequence starting at $r'$ and ending at $r_{end}$ is not $(2k+2)$-balanced.

Indeed, let $h^*$ denote the height of block $B^*$ and similarly $h'$ the height of $B'$. By our assumption, the $h^* - h'$ blocks between $B'$ and $B^*$ are created by faulty nodes. Since $\mathcal{B}_k(G)$ is not well-defined, we know that the $h$, height of $B_1$ (or equivalently $B_2$), is at least $h * +k + 1$. By Lemma 5.1, we know that for each of the heights $h^* + 1, h^* + 2, \dots, h$, there is at most one honestly produced block. The same argument shows that for each of those heights, there was also at least one block produced by a faulty node. This means there are at most $h - h^*$ honest leaders after $r'$ and at least $h - h^*$ faulty leaders. In other words, there was no strict majority of honest leaders in the window from $r'$ to $r_{end}$. It remains to observe that this window has length at least $2k + 2$. Indeed, the two chains ending at $B_1$ and $B_2$ contain at least $k + 1$ blocks that were produced after $B^*$, and therefore after $B'$. Since in each round we create exactly one block (assumption A4'), this means that $r_{end} - r' \geq 2k + 2$. This concludes the proof.

Now that we have established a sufficient condition under which $\mathcal{B}_k(G)$ is well-defined, we can proceed to show that longest-chain protocols achieve the goals of finality and liveness.

## 5.7 Finality of longest-chain consensus

Let us recall the (informal) definition of *finality* that we gave earlier this chapter:

> **Goal: Finality** (first version) If an honest node $i$ considers a block $B$ as *finalized* at time $t$, then this block remains finalized at all times after $t$.

The above goal is a bit vague since it does not specify *when* an honest node should consider a block finalized. Given the results from the previous sections, it should come as no surprise that we will declare the blocks in $\mathcal{B}_k(G)$ to be final. Indeed, we will prove the following theorem.

**Theorem 5.5** (Finality of longest-chain consensus)**.** *Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_T$ be a sequence of in-trees with each in-tree $G_t$ having exactly one more block than $G_{t-1}$. If the common prefix property holds for a given value of k in each of the in-trees $G_1, \dots, G_T$, then we have*

$$\mathcal{B}_k(G_1) \subseteq \mathcal{B}_k(G_2) \subseteq \cdots \subseteq \mathcal{B}_k(G_T).$$

The sequence of in-trees naturally corresponds to the evolution of a blockchain over time, with $G_t$ denoting the in-tree in round $t$. In words, the theorem tells us that once a block is part of $\mathcal{B}_k(G_t)$, it will be part of $\mathcal{B}_k(G_{t'})$ for all subsequent rounds $t' \geq t$. The honest nodes can thus trust the blocks in $\mathcal{B}_k(G)$ to be final.

At this point in the course, you are trained to always question the assumptions. For this particular theorem, try to answer the following question:

> Exercise
>
> Why do we not (need to) refer to any of the assumptions A1-A5 in Theorem 5.5?

We now proceed with the proof of Theorem 5.5.

*Proof.* We assume the common prefix property, with value $k$, holds for each of the in-trees $G_1, \dots, G_T$. In other words, we assume that $\mathcal{B}_k(G_t)$ is well defined for each $t \in [T]$. Let us also assume that there exists a pair of rounds $t, t' \in [T]$ with $t < t'$ and a block $B$ with the property that $B \in \mathcal{B}_k(G_t)$, but also $B \notin \mathcal{B}_k(G_{t'})$. We show that this leads to a contradiction. In particular, we will show that there exists some round $s$ in between rounds $t$ and $t'$ for which $\mathcal{B}_k(G_s)$ is not well defined.

By assumption, $B \in \mathcal{B}_k(G_t)$, the block $B$ belongs to the common prefix of all longest chains in round $t$. Similarly, it does *not* belong to the common prefix of all longest chains in round $t'$. Let $s \in \{t+1, \dots, t'\}$ denote the smallest round number for which $B$ is excluded from at least one longest chain in $G_s$. Let $B_s$ be the block that was added to $G_{s-1}$ to form $G_s$. By minimality of $s$, the block $B$ is contained in the common prefix of all longest chains in $G_{s-1}$. This means that $B_s$ cannot extend a longest chain in $G_{s-1}$, since otherwise $B$ would also be contained in the common prefix of all (in fact *the*) longest chain in $G_s$. The block $B_s$ must thus be added to a chain that was previously not the longest chain, creating a new longest chain. Let $C_1$ denote a longest chain in $G_{s-1}$. Note that $B$ is part of the chain $C_1$. Let $C_2$ denote the newly created longest chain in $G_s$. Let $B^*$ denote the first block that the chains $C_1$ and $C_2$ have in common. Figure 5.6 below illustrates the situation.

We now argue that $\mathcal{B}_k(G_s)$ is not well defined. To do so, observe that $B \in \mathcal{B}_k(G_t)$ implies that $B$ is at least $k$ blocks removed from the end of chain $C_1$. That means that $B^*$ is at least $k+1$ blocks removed from the end of chain $C_1$. Since $C_1$ and $C_2$ have the same length, $B^*$ is also at least $k+1$ blocks removed from the end of $C_2$. Both $C_1$ and $C_2$ are longest chains in $G_s$, and we have just shown that we need to remove at least $k+1$ blocks from the ends of $C_1$ and $C_2$ to reach the first block that they have in common. This proves that $\mathcal{B}_k(G_s)$ is not well defined. □

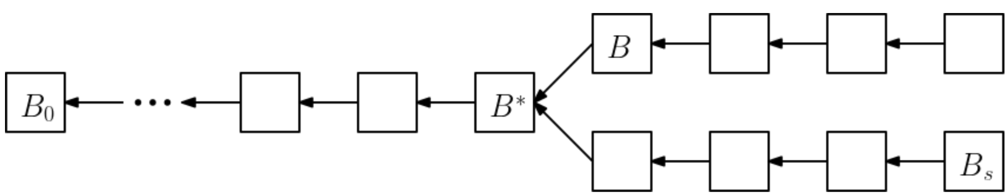



Figure 5.6: Proof of Theorem 5.5.

## 5.8 Liveness of longest-chain consensus

We finally establish liveness of the longest-chain consensus protocol. Recall from the previous chapter:

> **Goal 2: Liveness.** Every "transaction" submitted by a client to at least one *honest* node is eventually added to every node's local history.

We will in fact only be able to show a weaker version of liveness:

> **Goal 2: Liveness.** (weak version) Every "transaction" submitted by a client to all *honest* nodes is eventually added to every node's local history.

The difference between the two lies in whether a transaction is known to at least one honest node or all of them. In a nutshell, we will be able to guarantee that the longest chain will include infinitely many blocks produced by honest nodes, but we cannot guarantee the same for blocks produced by one specific honest node. Formally, we prove the following.

**Theorem 5.6** (Liveness of longest-chain consensus)**.** *Assume the leader sequence $\ell_1, \ell_2, \ldots$ is $(2k+2)$-balanced and assume that the assumptions A1, A4' and A5 hold. Let $G_1, G_2, \ldots$ denote the corresponding sequence of in-trees. For every transaction that is at some point known to all honest nodes, there exists a $t_0$ such that the transaction will be included in $\mathcal{B}_k(G_t)$ for all $t \geq t_0$.*

*Proof.* First note that the assumptions A1, A4', A5 and the fact that the leader sequence is $(2k+2)$-balanced allow us to conclude, via Theorem 5.4, that $\mathcal{B}_k(G_t)$ is well-defined for all $t$. Moreover, we have shown in Theorem 5.5 that finality holds:

$$\mathcal{B}_k(G_1) \subseteq \mathcal{B}_k(G_2) \subseteq \cdots.$$

It thus remains to show that if a transaction is known to all honest nodes at some point, then there is a $t_0$ such that $\mathcal{B}_k(G_{t_0})$ includes that transaction. Since the transaction is known to *all* honest nodes, and honest nodes include all transactions that they are aware of in a new block, it suffices to show that infinitely many finalized blocks are created by honest leaders.

To do so, we will use the property that the leader sequence is $(2k+2)$-balanced. Let us divide the leader sequence into groups of $2k+2$ consecutive leaders. That is, the first group consists of the leaders $\ell_1, \ell_2, \ldots, \ell_{2k+2}$, the second group consists of the leaders $\ell_{(2k+2)+1}, \ldots, \ell_{2(2k+2)}$, and so on. Since the leader sequence is $(2k+2)$-balanced, every group contains at least $k+2$ honest leaders and at most $k$ faulty leaders.

We now argue that the longest chain grows significantly over time: indeed, Lemma 5.1 shows that the height of honestly produced blocks increases strictly over time. Now consider a longest chain at the end of the $T$-th group of leaders. There are at least $(k+2)T$ honest leaders in those groups and at most one honestly produced block per height, therefore the longest chain in $G_{(2k+2)T}$ has length at least $(k+2)T$.

How many honestly produced blocks are there on such a longest chain? We know that there are at most $kT$ faulty leaders in the first $T$ groups. Together, they produced at most $kT$ blocks. Therefore, there are at least $(k+2)T - kT = 2T$ honestly produced blocks on such a chain. How many honestly produced blocks are finalized? We consider blocks finalized when they are part of $\mathcal{B}_k(G)$. For $\mathcal{B}_k(G_{(2k+2)T})$ this means it contains at least $2T - k$ honestly produced blocks, since we are removing the last $k$ blocks and they could happen to be honestly produced. This proves that as $T$ grows, the number of honestly produced blocks that are finalized grows as well. In other words, honestly produced blocks get finalized infinitely often and therefore transactions known to all honest nodes *eventually* get finalized. □

## 5.9 Toward permissionless consensus

In the spirit of "always question the assumptions", for which theorems in this chapter did we rely on the assumption that we are working in the *permissioned* setting?

If you revisit all theorems, you will notice that we used it only to argue that choosing each leader uniformly at random leads to a balanced leader sequence, as long as the fraction of faulty nodes $\alpha = f/n$ is strictly less than $1/2$. Afterwards, all theorems relied on the assumption that the leader sequence is $w$-balanced for a sufficiently large value of $w$.

In other words, if we – somehow – establish that a leader election mechanism in the permissionless setting leads to a $w$-balanced leader sequence, then Theorem 5.5 and Theorem 5.6 immediately show that the longest-chain consensus protocol achieves finality and liveness in that setting as well.

A crucial difference between the permissioned and the permissionless setting is that nodes can join and leave the network freely. In particular, this means that the fraction of faulty nodes in the network might change over time. Let us assume for the moment that we still select a leader uniformly at random from the set of nodes in the network in that round. Can we still use the arguments from Section 5.5 to argue that the resulting leader sequence is sufficiently balanced with high probability? The answer is yes, under mild assumptions. Indeed, if we assume that the fraction of faulty nodes at any point in time is *at most* $\alpha < 1/2$, then we can still apply Hoeffding's inequality in the same way as we did in the proof of Theorem 5.2.

The assumption that we can elect a leader uniformly at random in the permissionless setting is not one that we can make freely. Selecting a uniformly random element from a set of nodes that the protocol does not know is a hard problem. In the permissionless setting we will need to work to design a leader election protocol that has this property. We will do so in the next chapter.

## Chapter 6

# Proof of Work