CS 535: Notes I

1 Probability

Each possible outcome of an experiment is called a *sample point*. The set of all sample points is called *sample space*, denoted by Ω. For example, in the experiment of 100 coin tosses, the sample space is $\{0,1\}^{100}$. An *event* E is a subset of Ω. For example in the above experiment, the set of coins tosses with even number of heads is an event. Two events E_1 and E_2 are disjoint if $E_1 \cap E_2 = \emptyset$. A collection of events $E_1, E_2 \cdots$ are mutually disjoint if for every $i \neq j$, $E_i \cap E_j = \emptyset$. A function P from Ω to $[0,1]$ is called a probability function if the following hold: 1) $\sum_{x\in\Omega} P(x) = 1$, 2) For every event E, $0 \leq P(E) \leq 1$, for every collection of mutually disjoint events $\{E_i\}_i$ (possibly countably infinite), $P(\cup_i E_i) = \sum_i P(E_i)$. Here $P(E)$ denotes $\sum_{x\in E} P(x)$. Probability of an event E is $\sum_{x\in E} P(x)$. Given an event E, $\overline{E} = \Omega - E$. Observe that $P(\overline{E}) = 1 - P(E)$. We call a probability function *uniform* if $P(x) = \frac{1}{|\Omega|}$ for every $x \in \Omega$. Note that if P is uniform probability function, then $P(E) = \frac{|E|}{|\Omega|}$.

1.1 Examples

- Experiment: Toss a fair coin n times. Sample space: $\{0,1\}^n$ (View Head as 1 and Tail as 0). Probability function: $P(x) = 1/2^n$ or every $x \in \{0,1\}^n$. Consider the event E: We see exactly 5 heads. This event corresponds to the following set: Set of all n-bit sequences that have exactly 5 heads. Since P is uniform probability function, we have that $P(E) = \frac{|E|}{|\Omega|}$. Thus to compute $P(E)$, it suffices to compute the size of E. Note that size of E is exactly $\binom{n}{5}$. Thus $P(E)$ equals $\binom{n}{5} \times \frac{1}{2^n}$.

- Experiment: Roll a dice twice. Sample space $\{1,\cdots,6\} \times \{1\cdots,6\}$. Probability function: $P(\langle i,j\rangle) = 1/36$.

- Toss a biased coin (with probability of Head 3/4) twice. Sample space $\{11, 10, 01, 00\}$. Probability Function: $P(11) = 9/16, P(10) = 3/8, P(01) = 3/8, P(00) = 1/16$.

- Toss a biased coin (with probability of Head 3/4) n times. Sample space: $\{0,1\}^n$. Let x be a sample point with ℓ 1's and $n-\ell$ zeros. Now $P(x) = \binom{n}{\ell} \times (\frac{3}{4})^\ell \times (\frac{1}{4})^{n-\ell}$. Note that P is not uniform. Let E be an event that denotes: We see exactly 5 heads.
  $P(E) = \sum_{x \text{ has exactly 5 ones}} P(x)$, which equals $\binom{n}{5} \times (\frac{3}{4})^5 \times (\frac{1}{4})^{n-5}$

- Uniformly at random pick an integer x from $\{1,\cdots,n\}$. Pick another integer at random from $\{1,\cdots,n\}$. Sample Space: $\{\langle i,j\rangle \mid i,j \in \{1,\cdots,b\}\}$. Probability function: $P(\langle i,j\rangle) = 1/n^2$, when $i,j \in \{1,\cdots,n\}$. What is the probability of the event E: At least one of the numbers we see is 5. We can calculate this probability, by decomposing E into smaller disjoint events. Let

$$E_1 = \{\langle 5, j\rangle \mid j \in \{1, \cdots, n\} - \{5\}\}.$$
$$E_2 = \{\langle i, 5\rangle \mid i \in \{1, \cdots, n\} - \{5\}\}.$$
$$E_3 = \{\langle 5, 5\rangle\}.$$

Note that E is union of E_1, E_2, and E_3. Moreover, E_1, E_2, and E_3 are mutually disjoint. Thus $P(E) = P(E_1) + P(E_2) + P(E_3)$. Since P is uniform probability measure, $P(E_1)$ is $\frac{|E_1|}{|\Omega|}$ which is $\frac{n-1}{n^2}$. Similary, we have $P(E_2) = \frac{n-1}{n^2}$, finally $P(E_3) = \frac{1}{n^2}$. Thus $P(E) = \frac{2n-1}{n^2}$.

- Toss a fair coin till you see a head. Sample space: $\{0, 01, 001, 0001, \cdots\}$. $P(0^i 1) = 1/2^{i+1}$.

Union Bound. If $E_1, E_2, \cdots E_m$ are events, then

$$P(\cup_i E_i) \leq \sum_i P(E_i).$$

Suppose we toss a fair coin n times, what is the probability that we see $2\log n$ successive heads? Let us denote this event with E. Let us consider the following events $E_1, E_2, \cdots E_{n-2\log n+1}$, where E_i is the event that ith, $(i+1)^{st}, \cdots, (i + 2\log n - 1)$ tosses are all heads. Observe that for each E_i, $P(E_i) = (1/2)^{2\log n} = 1/n^2$. Note that $E = \cup E_i$. By Union bound

$$\begin{aligned} P(E) &\leq P(E_1) + P(E_2) + \cdots + P(E_{n-2\log n+1}) \\ &\leq \frac{n - 2\log n + 1}{n^2} \leq \frac{1}{n} \end{aligned}$$

Thus the probability the we see $2\log n$ consecutive heads is very small.