

FYS4150  
Project 4 - deadline November 15



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University of Oslo, Autumn 2017

### **Abstract**

A c++ program have been developed to calculate various thermodynamic variables of a given lattice of objects using the popular Ising model. Another program in MatLab have also been made to plot these variables against each other and do further calculations such as calculating the variance. The variables has shown to be very sensitive to temperature, rather than the initial configuration of the lattice. When the size of the lattice increases, the overall energy, magnetization, specific heat and susceptibility increase, but the critical temperature, and therefore the phase shift, always happens for  $T = 2.3$ .

# Introduction

This paper will look at the popular Ising model which can determine phase transitions of a given lattice by first knowing a critical temperature. The model is however restricted in such a way that objects in the lattice can only have two values, in a so called binary system. In this paper, these two values represent upwards and downwards spin direction. Even though the model is restricted, it provides a good understanding of how phase transitions work.

A simple 2 by 2 lattice is shown in figure 1, where one can observe how the objects in the lattice interact with each other. The principle of the Ising model can be extended to higher order lattices as well, as will be done later in this paper. Critical temperature and phase transitions will also be looked at near the end of the paper after necessary calculations have been made.

The codes and files used in this project can be found at <https://github.com/sanderwl/Assignment-4>

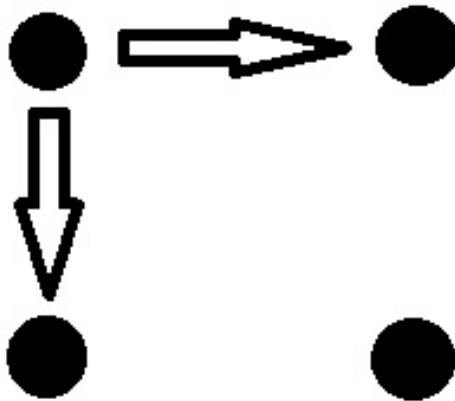


Figure 1: A two by two lattice and how they interact using the Ising model

# Method

To be able to start programming, one first needs to find the expressions for the partition function  $Z$  with its corresponding energy values  $E$ , the mean magnetic moment  $M$ , the specific heat  $C_V$  and the susceptibility  $X$  as functions of the temperature  $T$ . All of this using the periodic boundary condition. The partition function  $Z$  is giving by:

$$Z = \sum_{i=1}^M e^{-\beta * E_i}$$

where  $\beta$  is the inverse temperature given by  $\beta = \frac{1}{kT}$  where  $k$  is the Boltzmann constant and  $T$  is the temperature, so every expression containing  $\beta$  is dependent on the temperature  $T$ .  $E_i$  is the energy for different spin settings given by:

$$E = -J \sum_{\langle kl \rangle}^N s_k s_l$$

where  $J$  is the coupling constant and  $N$  is the total number of spins.  $s_k$  and  $s_l$  are the spins of two neighbouring objects in a lattice. Since we are working 2x2 lattice, the total number of combinations are given by  $2^4 = 16$ , considering we are working with the Ising model where the spins can only be  $-1$  or  $1$ . From figure 1 we see how the Ising model works on a 2x2 lattice and from this the energy is calculated. One can observe that energy is non-zero on lattices where all the objects have the same spin ( $-8J$ ) and where the two diagonals have opposite spin from each other ( $8J$ ). All other settings have zero energy. Knowing the energy values we can calculate the mean energy  $\langle E \rangle$  with our specific partition function:

$$\begin{aligned} \langle E \rangle &= \frac{1}{Z} \sum_{i=1}^M E_i e^{-\beta E_i} \\ &= \frac{1}{2e^{-8} + 2e^8 + 12} \sum_1^{16} E_i e^{-\beta E_i} \\ &= \frac{16e^{-8} - 16e^8}{2e^{-8} + 2e^8 + 12} = -7.983928 \end{aligned}$$

Now we can simply put the same energy values into our partition function:

$$\begin{aligned} Z &= e^{-\beta * -8J} + e^{-\beta * -8J} + e^{-\beta * 8J} + e^{-\beta * 8J} + 12 * e^0 \\ Z &= 2e^{-8J\beta} + 2e^{8J\beta} + 12 \end{aligned}$$

The magnetization is given by:

$$M = \sum_{j=1}^N s_j$$

Unlike in the energy case, the magnetization does not depend the lattice having the same spin or that the diagonals have opposite spin. The only case where the magnetization is zero in a 2x2 lattice is when half of the objects in the lattice opposite spins. Therefore we get the magnetization values:

$$M_i = [4 + 2 + 2 + 2 + 2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + -2 + -2 + -2 + -2 + -4]$$

To calculate the mean magnetic moment or mean magnetization we use the equation below with our calculated magnetic moment and energy values:

$$\begin{aligned} |M| &= \frac{1}{Z} \sum_i^M M_i e^{-\beta E_i} \\ &= \frac{1}{2e^{-8} + 2e^8 + 12} \sum_1^{16} M_i e^{-\beta E_i} \end{aligned}$$

$$= \frac{8e^8 + 16}{2e^{-8} + 2e^8 + 12} = 3.994643$$

To calculate the the specific heat, one only needs to know the total number of spins of the lattice. In the 2x2 lattice case the total number of spins is 4, and remember also that  $\beta J = 1$  and  $k = 1$ . The specific heat is then:

$$\begin{aligned} C_V &= \frac{1}{kT^2} (\langle E^2 \rangle - \langle E \rangle^2) \\ &= \frac{1}{kT^2} \left( \frac{128e^{-8} - 128e^8}{2e^{-8} + 2e^8 + 12} - \left( \frac{16e^{-8} - 16e^8}{2e^{-8} + 2e^8 + 12} \right)^2 \right) \\ &= 0.128329 \end{aligned}$$

The term  $\langle E^2 \rangle - \langle E \rangle^2$  is also called the variance of the energy and is precisely calculated just as shown above. The same variance calculation can be applied to calculating the susceptibility, but the energy terms have to be substituted for mean magnetization. Let's calculate the variance separate from the desired equation this time:

$$\begin{aligned} \sigma_M^2 &= \langle M^2 \rangle - \langle M \rangle^2 = \frac{1}{Z} \sum_{i=1}^M M_i^2 e^{-\beta E_i} - \left( \frac{1}{Z} \sum_{i=1}^M M_i e^{-\beta E_i} \right)^2 \\ &= \frac{32}{2e^{-8} + 2e^8 + 12} (e^8 + 1) - \left( \frac{8e^8 + 16}{2e^{-8} + 2e^8 + 12} \right)^2 = 0.016004 \end{aligned}$$

Remember that the  $\beta = 1$  in the 2x2 lattice case, so the susceptibility would be the same as the variance in that case. We would usually have to calculate the susceptibility with the following equation:

$$X = \frac{1}{k_b T} (\langle M^2 \rangle - \langle M \rangle^2) = \frac{1}{k_b T} \sigma_M^2 = 0.010853$$

To compute the above values, a standard Metropolis algorithm were implemented (kilde). This algorithm takes in a matrix, only ones or negative ones in this case, and then decides to flip or not flip the current value at a random location in the matrix (lattice). The probability of when to flip or not is in this case given by the Boltzmann distribution  $e^{-E_i/T}$ . If the algorithm decides to flip, the computed energy at that position in the lattice is added to the total energy of the system. The same happens for the magnetization. Initial values of the energy and magnetization is calculated beforehand. What initial matrix to use, that is a randomly generated matrix or a matrix with only positive spin, is determined by the user. After the Metropolis algorithm is finished, the cumulative energy, squared cumulative energy, cumulative magnetization and cumulative magnetization squared is calculated which is needed to later calculate the mean energy, mean magnetization, specific heat and susceptibility. This algorithm is based on random events and so one needs to perform the algorithm many times before the results are stable. Therefore the whole Metropolis algorithm is looped inside what is called a Monte Carlo loop. When the number of cycles are increased, the more stable the results will be. However, many cycles require a lot of processing power when the number of cycles get to around  $10^6$ .

To be able to run the program in a time efficient manner, one needs to paralllelize the program, and this was done by using MPI with 8 processors. Using MPI made the processing time much smaller, but it still took a long time for large lattice sizes.

Raw data were produced using c++ while plotting and further calculations were done using MatLab. Further calculations include finding the probability distribution and the thermodynamic limit.

To calculate the critical temperature one have to use the following equation:

$$T_c(L) - T_c(L = \infty) = aL^{-1/v}$$

where  $T_c$  is the critical temperature,  $L$  is the size of the lattice,  $v = 1$  and  $a$  is a constant.  $T_c(L)$  is known and we then get the following equations which can be solved numerically:

$$T_c(L = \infty) = T_c(L) - aL^{-1}$$

Again,  $T_c$  and  $L$  is known. The critical temperature values were graphically found for the specific heat and susceptibility versus temperature plots for each lattice size.  $1/L$  were then plotted against these critical temperatures. The least square rule was then used as the linear regression. By doing this, one would get a graph fitted towards the critical temperature points and where the graph hits the y-axis is the value of  $T_c(L = \infty)$ .

# Results

The specific analytical value found in the method is compared to the numerical value calculated in c++ in the table below:

Table 1: Analytical and numerical solutions

Type	Analytical	Numerical	MC cycles needed
$\langle E \rangle$	-7.983928	-7.98437	10 000
$\langle M \rangle$	3.994643	3.99483	10 000
$C_V$	0.128329	0.124812	100 000
$\langle X \rangle$	0.016004	0.0153613	100 000

To get good results for mean energy and mean magnetization took very few Monte Carlo cycles, only about  $10^4$ . Then the values would be off only by a factor of about  $10^{-3}$ . The susceptibility and the specific heat took more Monte Carlo cycles to get a precise and stable result. Correct results started showing around  $10^5$ . For  $10^4$ , both the specific heat and susceptibility gave unstable results.

The values shown in Table 1 are representing each object in the lattice, but for the rest of the paper the values will be calculated for the entirety of the lattice. To transition from lattice wide values to single object values, one simply have to divide by the size of the lattice, 4 in the above case, and multiply in the reverse case.

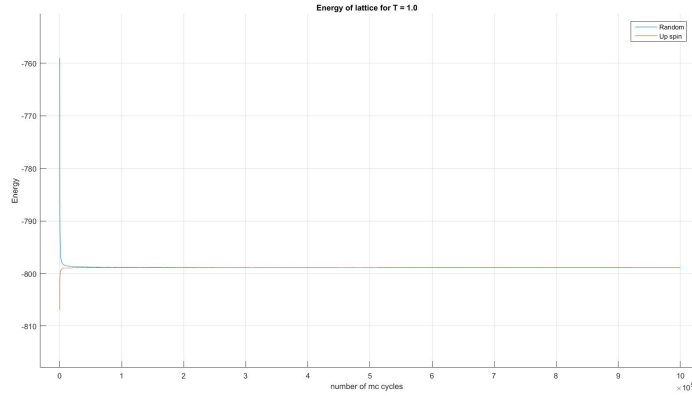


Figure 2: Energy versus Monte Carlo cycles for  $T = 1.0$

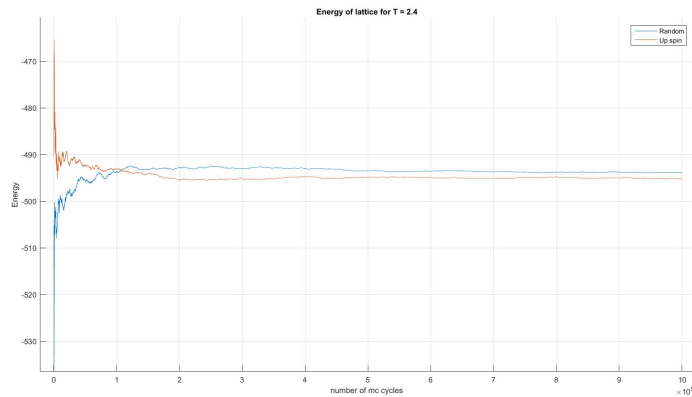


Figure 3: Energy versus Monte Carlo cycles for  $T = 2.4$

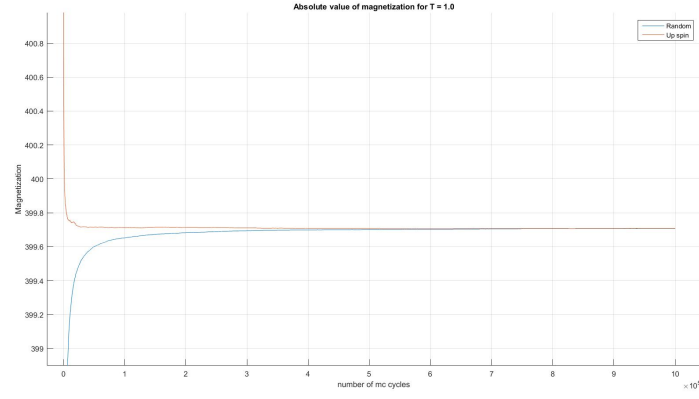


Figure 4: Absolute value of magnetization versus Monte Carlo cycles for  $T = 1.0$

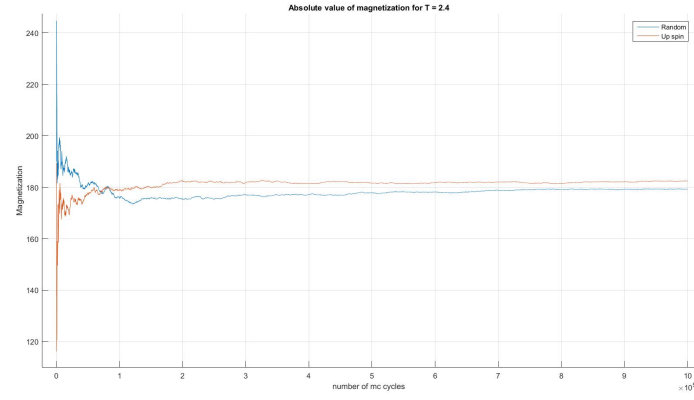


Figure 5: Absolute value of magnetization versus Monte Carlo cycles for  $T = 2.4$

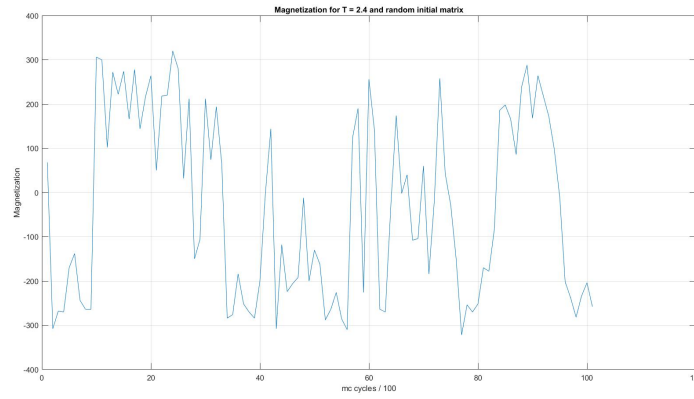


Figure 6: Mean magnetization versus Monte Carlo cycles for  $T = 2.4$

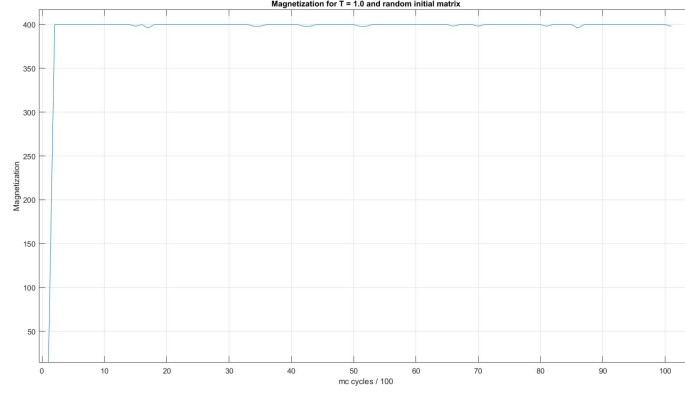


Figure 7: Mean magnetization versus Monte Carlo cycles for  $T = 1.0$

One can see from figure 2, figure 3, figure 4 and figure 5 that the mean energy and magnetization approaches it's equilibrium state. There are fluctuations after the mean energy and magnetization has reached it's equilibrium state, but the value seems to jump back to to the equilibrium state quickly. The values become more unstable when  $T = 2.4$ , and also the value of the equilibrium state is changed. The number of Monte Carlo cycles needed to hit equilibrium is different for different temperatures, but the initial matrix that is used do not seem to affect the time it takes to reach equilibrium. The table below gives an approximation of how many Monte Carlo cycles are needed:

Table 2: EQ values for energy

Orientation	Temperature	EQ state	max MC cycles needed
Random	1.0	-790.881	$2 * 10^4$
Random	2.4	-496.494	$6 * 10^5$
Up spin	1.0	-798.96	$2 * 10^4$
Up spin	2.4	-493.996	$6 * 10^5$

Table 3: EQ values for magnetization

Orientation	Temperature	EQ state	max MC cycles needed
Random	1.0	398.762	$6 * 10^5$
Random	2.4	165.209	$2 * 10^5$
Up spin	1.0	399.732	$10^5$
Up spin	2.4	176.087	$2 * 10^5$

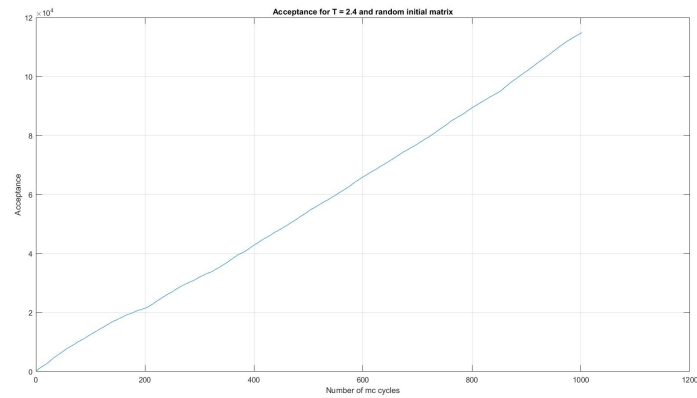
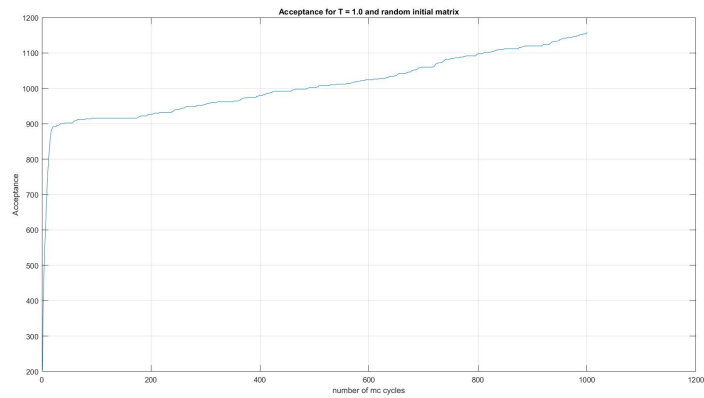


Figure 8: Acceptance versus Monte Carlo cycles for  $T = 1.0$  and  $T = 2.4$  with a random initial matrix

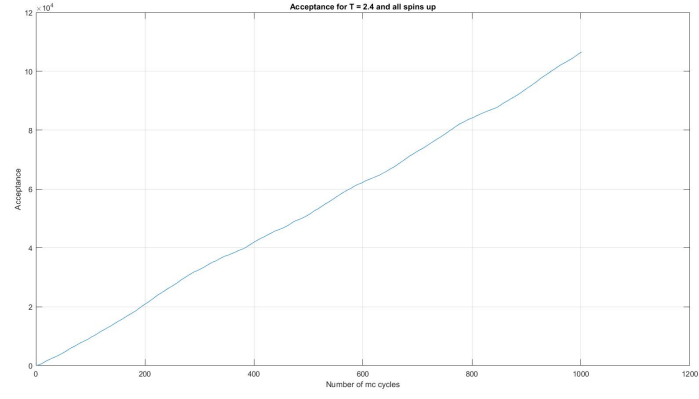
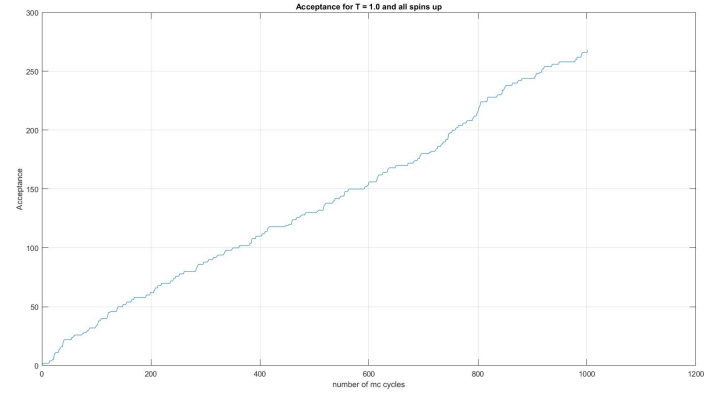


Figure 9: Acceptance versus Monte Carlo cycles for  $T = 1.0$  and  $T = 2.4$  with initial spin upwards

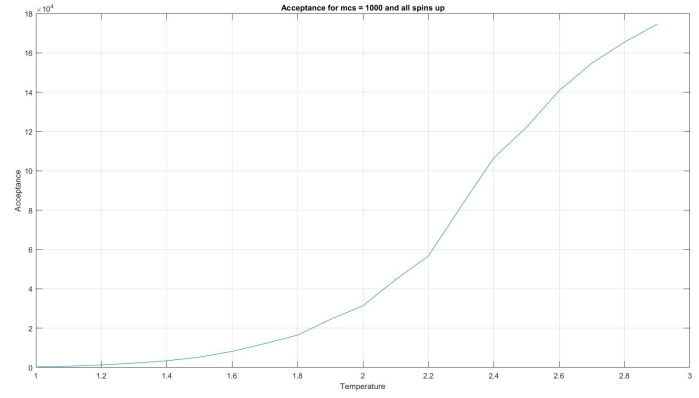
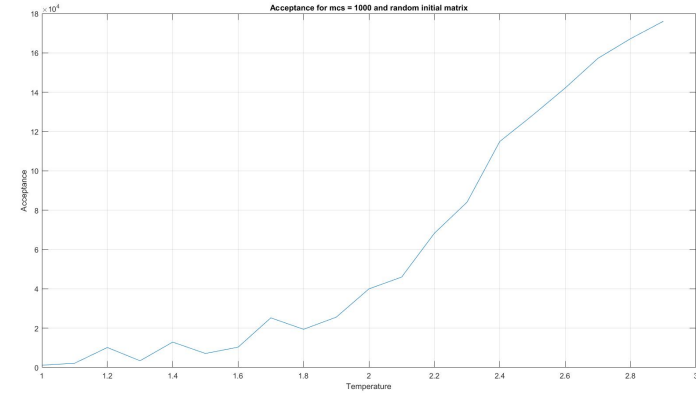


Figure 10: Acceptance versus temperature for a random initial matrix and with initial spin upwards

Acceptance is apparently dependent on both temperature and the initial state of the spins. From figure 8 one can observe that with higher temperature the number of accepted values increase by a factor of 10. For  $T = 1.0$ , one can also observe that the acceptance spikes for few Monte Carlo cycles. If the initial spins are all positive like in figure 9, the acceptance becomes more linear. The line for  $T = 1.0$  is more squiggly than the straighter line for  $T = 2.4$ . The ladder does not spike for few Monte Carlo cycles in this case.

From figure 10, one can observe that the curves are not linear. The curve based on a randomly generated initial matrix is uneven at low temperatures, but smooths out when  $T$  increases. When the initial matrix only consist positive spins, the curve is smooth throughout. These two plots are plotted to a maximum temperature of 3.0, so one would have an idea of how the acceptance would evolve after  $T = 2.4$ .



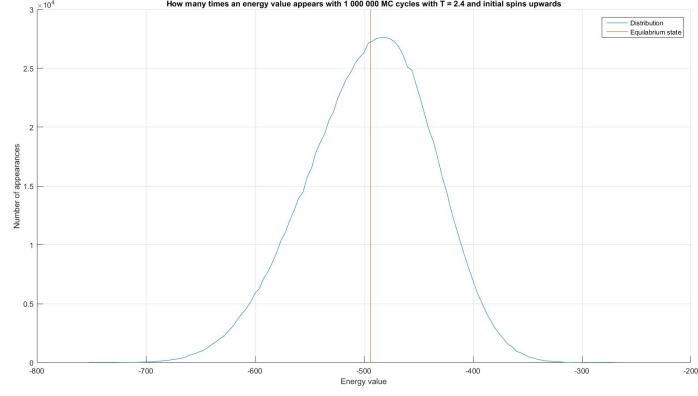
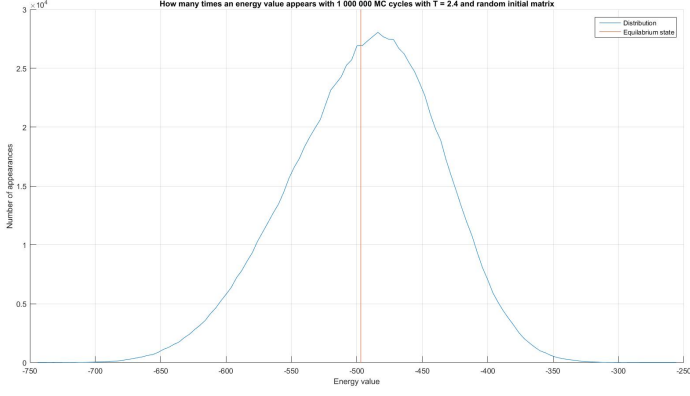


Figure 11: Number of appearances per energy value for a random initial matrix and with initial spin upwards when  $T = 2.4$

With a randomly generated initial matrix, the variance computed in c++ becomes 3226.62. The calculated variance from the energy becomes 3226.9 for  $10^6$  MC cycles (calculated in MatLab script "how-manytimes.m"). This gives great confidence in the variance. For an initial matrix where all the spins point upwards, the calculated variance in c++ becomes 3236.05. The variance calculated from energy values in MatLab becomes 3236.3, again showing the stability of the variance for  $10^6$  MC cycles. With fewer MC cycles, the two variances calculated does not correlate. Variances for other temperatures are shown in the table below:

Table 4: Variance for different temperatures

Temperature	Variance
T=1.0	0.0227441
T=1.25	0.129337
T=1.50	0.444368
T=1.75	1.18547
T=2.0	2.88226
T=2.25	7.89626
T= 2.50	6.18416
T=2.75	4.25249
T=3.0	3.61595

One can observe from figure 11 that the curves are slightly skewed towards the higher energy values. The curve is bell shaped and somewhat centred around the equilibrium state for the lattice at temperature  $T = 2.4$  which is  $-497$  and  $-493$ .

The run time for higher order lattices increased as the lattice size increased. The run times are shown in the table below:

Table 5: Run time for different lattice sizes with  $10^6$  MC cycles and 20 different temperatures ranging from  $T = 2.0$  to  $T = 3.0$

Lattice size	Run time in seconds
60x60	3996.57
80x80	7081.15
100x100	18630.9

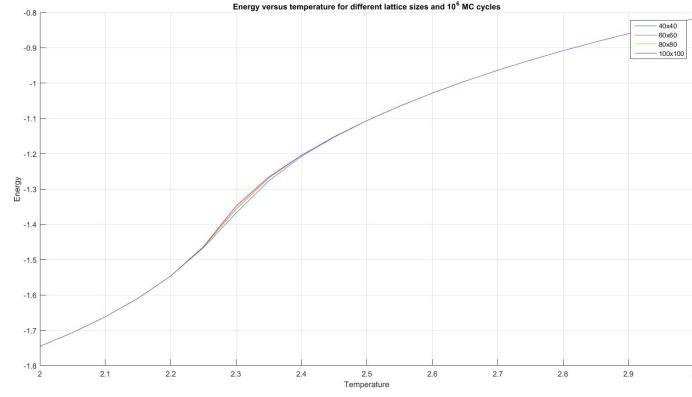


Figure 12: Mean energy versus temperature for  $10^6$  Monte Carlo cycles for single lattice objects

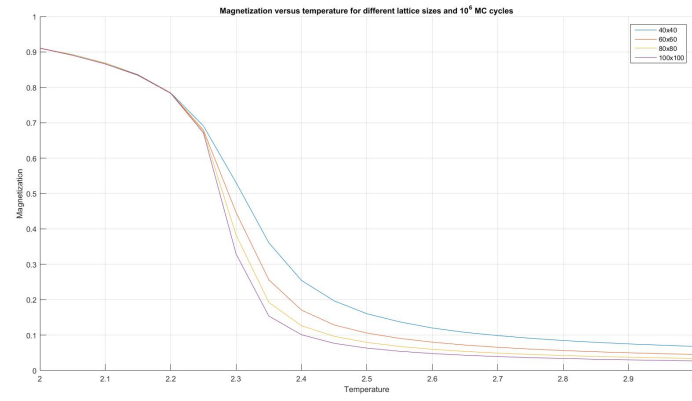


Figure 13: Mean magnetization versus temperature for  $10^6$  Monte Carlo cycles for single lattice objects

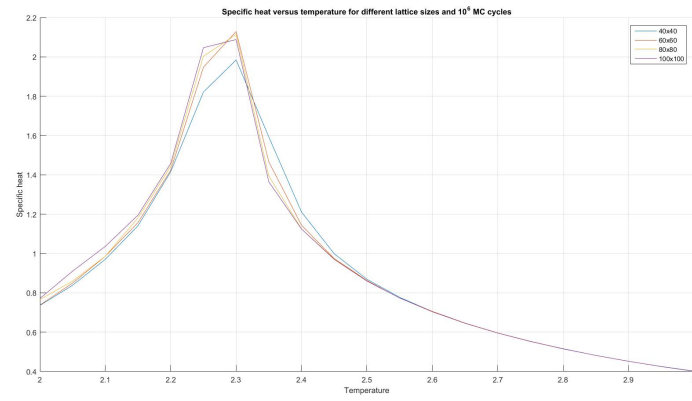


Figure 14: Specific heat versus temperature for  $10^6$  Monte Carlo cycles for single lattice objects

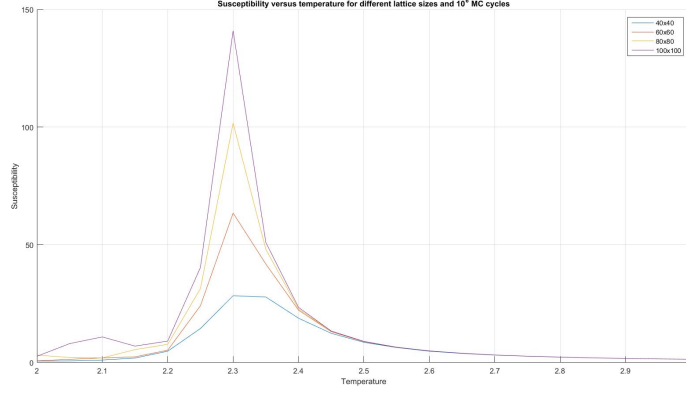


Figure 15: Susceptibility versus temperature for  $10^6$  Monte Carlo cycles for single lattice objects

In the above figures, the respective values are representing an individual object in the lattice, not the entirety of the lattice. This was done because the graphs gave a clearer picture of what was going on. One can observe from figure 12 that the energy values are very similar, except for temperatures ranging from about 2.25 – 4 where they deviate slightly before returning to the aforementioned state.

From figure 13 we observe a similar feature as with the energy. The magnetization values seem to deviate from each other at around  $T = 2.25$ , but in this case, the values does no rejoin at a later temperature. However, the graphs seem to converge when  $T$  approaches 3, so maybe the values rejoin at a temperature larger than 3.

Specific heat values seem to exponentially increase towards  $T = 2.3$  at figure 14, where it hits a peak. After this, the value exponentially decreases towards 0. This is true for all lattice sizes, although the gradient of the different sizes are not equal at all temperatures, but this may be seen with a finer resolution.

In figure 15, one can observe a resembling feature for the susceptibility as in the specific heat case. The susceptibility values increase exponentially towards  $T = 2.3$  followed by a exponential decrease towards 0 when  $T$  gets larger. However, the susceptibility seem to struggle and not increase as much around  $T = 2.1$ . The 40x40 case does not have a distinct peak at  $T = 2.3$ , as the peak seem to extend. A graph with finer resolution may be more informative.

# Discussion

Comparing Table 2 and to figure 2 and figure 3 one can observe that the graph approaches the equilibrium value stated in the table. This is very clear for  $T = 1.0$ , but when  $T = 2.4$ , the energy seems to be oscillate which is natural due to the increased temperature in the system (kilde). Even though the value slightly oscillates after reaching equilibrium, the energy is still said to have reached equilibrium, which value is given in Table 2 after a certain number of Monte Carlo cycles.

The mean magnetization plotted on figure 6 show the magnetization without the absolute value and tells us that the magnetization actually jumps from positive to negative magnetization, and vice versa, rather quickly and frequent after reaching equilibrium. The reason for this is because the energy values are equal for symmetric magnetization values around the equilibrium state. This makes it easy for the magnetization to make these kind of leaps. Looking at figure 4 and figure 5, the absolute value of the magnetization, we lose the above understanding, but we get the correct equilibrium values as calculated in c++.

This is only true when  $T = 2.4$ , because one can observe on figure 7 that when  $T = 1.0$  the magnetization does not fluctuate for positive and negative values. It seems as the magnetization reaches it equilibrium state and then shortly jump to some value, then right back to the equilibrium value. The reason for this is because the temperature in the system is too low for the magnetization to make any great leaps (kilde). From the Boltzmann distribution we can see this mathematically:

$$w_i = e^{-E_i/T}$$

where  $w_i$  is the probability,  $E_i$  is the energy for and  $T$  is the temperature. One can immediately see that when the temperature increase, the probability also increase for positive energy values.

This means that the likelihood of flipping the orientation of the objects in the lattice should be higher for  $T = 2.4$ , since the probability of flipping should be higher than a random number. The magnetization and energy would then fluctuate more when  $T = 2.4$  and this is observed on figure 2, figure 3, figure 4 and figure 5.

Following this train of thought, one would expect the rate of acceptance to be greater for higher temperature values and this can be seen by comparing the y-axis on figure 8 and figure 9. This is confirmed on figure 10 where the acceptance increases when the temperature increases. When the initial matrix is randomized, the results for low temperatures vary, due to the initial starting condition. This does not happen when the initial matrix only has upwards spin since the acceptance does not have to stabilize.

Looking at figure 11, one clearly sees that both graphs are skewed towards higher energy values, when one may have expected the curve to center around it's equilibrium state. The largest area is to the left of the equilibrium, meaning most values is accepted here in the Metropolis algorithm. This makes sense because more values are accepted when the energy is lower according to the Boltzmann distribution.

In figure 12, figure 13, figure 14 and figure 15 something suspicious is going on at  $T = 2.3$ . The energy values split, but rejoins after, the magnetization splits and the specific heat and susceptibility peaks. It so happens that the critical temperature calculated in this paper is  $T_c(L = \infty) = 2.3$ , and we know at the critical temperature, a phase change occurs. The phase change is the reason why the specific heat and susceptibility suddenly decrease after this temperature as well as the split in energy and magnetization. The phenomenon when the variables approach 0 after critical temperature is called a critical phenomenon, as can be seen on figure 14 and figure 15. For an infinitely large lattice size, the specific heat and susceptibility are discontinues at the critical temperature, but this cannot be seen on these graphs (Jensen, 2015, page 431).

Comparing figure 14 and figure 15, one can observe that the susceptibility has a "sharper" peak than the specific heat. Since specific heat is calculated using the energy and susceptibility using magnetization, the "sharpness" can be directly observed on figure 13 by observing that the derivative of the slope. One can also observe from the same figure that with increasing lattice size, the slope and its derivative also increase. One can therefore assume that with an infinite lattice size, the slope would become infinitely steep and approach zero quicker than for smaller lattice sizes. When the magnetization approaches zero after critical temperature, there has been a phase shift(Hjorth-Jensen, 2015).

Splitting of the energy, figure 12, is a sign of the coexistence of two different phases in the split interval (Hjorth-Jensen, 2015, page 429).

From Table 4 one can observe the different variance values which tells us how much the probability  $P(E)$  is spreading. The variance seem to peak around 2.25, which is close the the analytical critical temperature. This means that the likelihood of the energy being in it's equilibrium state is the lowest at critical temperature. One may expect these variance values to increase as the size of the lattice increase.

## Errors

To recognize ones errors is crucial to understanding the results and the limitations when discussing said results. The main errors arising in this paper is related to long run times shown in Table 5, despite having parallellized the code. For instance, figure 12, figure 13, figure 14 and figure 15 does not show the exact critical temperature calculated analytically, due to having too low resolution. By increasing the number of temperature steps in the code, the numerical critical temperature would approach the exact analytical value, 2.269, given in the project when  $dt \rightarrow \infty$ .

## Conclusion

The Ising model has proved to be reliable throughout this project. It gives a good understanding of the thermodynamic variables specific heat and susceptibility as well the energy and magnetization of the system. The Boltzmann probability distribution and its variance derived from this model also seemed solid. From the above results, the critical temperature and a phase change were found, proving the usefulness of the Ising model.

# References

- Hjorth-Jensen, M, (2015), Computational Physics, Lecture Notes Fall 2015, [Internet], Available from: <https://github.com/CompPhysics/ComputationalPhysics/blob/master/doc/Lectures/lectures2015.pdf>, [downloaded 05.11.2017]