Institute of Geodesy and Geoinformation für Geodäsie und Geoinformation
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Course: Statistics and Adjustment Theory

Homework Assignment 2 - Variance Propagation

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A Setting up a Covariance Matrix

We observe n 3D points and arrange the measurements in the vector

$$\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n & y_1 & \cdots & y_n & z_1 & \cdots & z_n \end{bmatrix}^T.$$

The standard deviations of the coordinates are

$$\sigma_x = \sigma_y = 1 \text{ cm}, \qquad \sigma_z = 3 \text{ cm}.$$

The correlation between two coordinates of the same type is

$$\rho_{x_i,x_j} = \frac{0.5}{|i-j|}, \qquad i \neq j,$$

and similarly for y_i, y_j and z_i, z_j. All mixed correlations between x, y, and z coordinates are zero. Hence the covariance submatrices are given by

$$\mathrm{Cov}(x_i, x_j) = \begin{cases} \sigma_x^2, & i = j, \\ \sigma_x^2 \dfrac{0.5}{|i-j|}, & i \neq j, \end{cases}$$

$$\mathrm{Cov}(y_i, y_j) = \begin{cases} \sigma_y^2, & i = j, \\ \sigma_y^2 \dfrac{0.5}{|i-j|}, & i \neq j, \end{cases}$$

$$\mathrm{Cov}(z_i, z_j) = \begin{cases} \sigma_z^2, & i = j, \\ \sigma_z^2 \dfrac{0.5}{|i-j|}, & i \neq j. \end{cases}$$

Because the coordinate types are uncorrelated, the full covariance matrix is block diagonal:

$$C = \begin{bmatrix} C_x & 0 & 0 \\ 0 & C_y & 0 \\ 0 & 0 & C_z \end{bmatrix},$$

where each block is an $n \times n$ matrix defined by the expressions above.

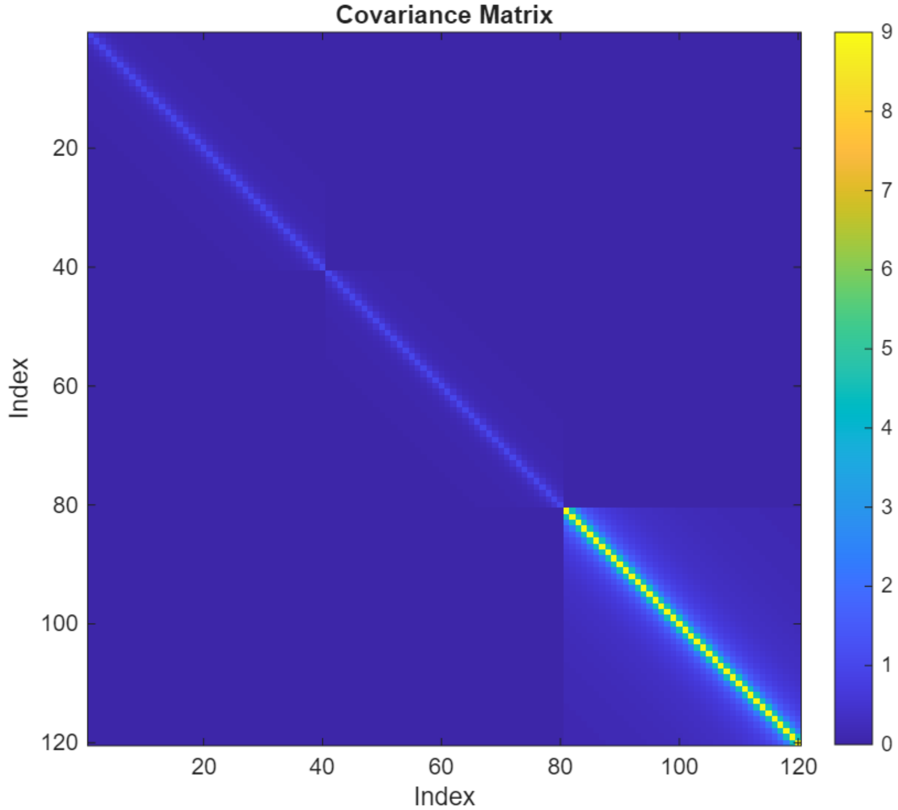

Figure 1: Covariance matrix for n = 40

B Variance Propagation I

We are given n measured points

$$p_i = (x_i, y_i)$$

that lie on a straight line modeled as

$$y = mx + b.$$

The measurement noise is Gaussian with known covariance matrix $C \in \mathbb{R}^{2\times 2}$ for every pair of points.

2 Parameters of the Straight Line

Given two points

$$p_i = (x_i, y_i), \qquad p_j = (x_j, y_j),$$

the slope and intercept are

$$m = \frac{y_j - y_i}{x_j - x_i}, \qquad b = y_i - m\, x_i.$$

We evaluate these formulas for all $\binom{n}{2}$ pairs. From these results we determine:
- the pair for which m is minimal, - the pair for which m is maximal, - the pair for which b is minimal, - the pair for which b is maximal.

3 Variance Propagation

Each point has covariance submatrices

$$C_{ii} = \begin{bmatrix} \sigma_{x_i}^2 & \mathrm{cov}(x_i, y_i) \\ \mathrm{cov}(y_i, x_i) & \sigma_{y_i}^2 \end{bmatrix}, \qquad C_{jj} = \begin{bmatrix} \sigma_{x_j}^2 & \mathrm{cov}(x_j, y_j) \\ \mathrm{cov}(y_j, x_j) & \sigma_{y_j}^2 \end{bmatrix}.$$

Let the measured vector for a point pair be

$$\mathbf{z} = \begin{bmatrix} x_i \\ y_i \\ x_j \\ y_j \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{bmatrix}.$$

The slope is

$$m = f(\mathbf{z}) = \frac{y_j - y_i}{x_j - x_i}.$$

Its gradient is

$$\nabla m = \begin{bmatrix} \frac{\partial m}{\partial x_i} \\ \frac{\partial m}{\partial y_i} \\ \frac{\partial m}{\partial x_j} \\ \frac{\partial m}{\partial y_j} \end{bmatrix} = \frac{1}{(x_j - x_i)^2} \begin{bmatrix} -(y_j - y_i) \\ -(x_j - x_i) \\ +(y_j - y_i) \\ +(x_j - x_i) \end{bmatrix}.$$

Thus the variance of m is

$$\sigma_m^2 = (\nabla m)^T \Sigma \, \nabla m.$$

Similarly, since

$$b = y_i - m x_i,$$

its gradient is computed as

$$\nabla b = \begin{bmatrix} \frac{\partial b}{\partial x_i} \\ \frac{\partial b}{\partial y_i} \\ \frac{\partial b}{\partial x_j} \\ \frac{\partial b}{\partial y_j} \end{bmatrix},$$

and the variance becomes

$$\sigma_b^2 = (\nabla b)^T \Sigma \, \nabla b.$$

We compute (σ_m, σ_b) for all point pairs and report the pairs for which:

RESULTS

- **Minimum slope** m: pair (39,40), $m = -22.542015$
- **Maximum slope** m: pair (28,29), $m = 46.183852$
- **Minimum intercept** b: pair (28,29), $b = -16897.816639$
- **Maximum intercept** b: pair (39,40), $b = 9305.699593$

Variance propagation:

- **Minimum** σ_m: pair (1,39), $\sigma_m = 0.0270513$
- **Maximum** σ_m: pair (39,40), $\sigma_m = 7606.99$
- **Minimum** σ_b: pair (1,39), $\sigma_b = 9.35699$
- **Maximum** σ_b: pair (39,40), $\sigma_b = 3.03585 \times 10^6$

Summary

- The line parameters (m, b) are computed directly from two points.
- Uncertainty propagation is performed using first-order linearization:

$$\sigma_f^2 = (\nabla f)^T \Sigma \, \nabla f.$$

- All results are evaluated for every pair from the dataset.

C Variance Propagation II

4 Distances Between Successive Points

We are given measured points

$$p_i = (x_i, y_i), \qquad i = 1, \dots, n,$$

and the Euclidean distance between consecutive points is

$$d_{i,i+1} = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}.$$

Using the data from `ex02_data.mat`, the following distances were computed:

$$\begin{aligned} d_{1,2} &= 1.881850, \\ d_{11,12} &= 0.432884, \\ d_{21,22} &= 2.458791, \\ d_{31,32} &= 3.358273. \end{aligned}$$

5 Variance Propagation for All Distances

The goal is to compute the covariance matrix of the full distance vector

$$d = (d_1, d_2, \dots, d_{n-1})^T.$$

The covariance matrices of the measured coordinates are given as

$$C_x \in \mathbb{R}^{n \times n}, \qquad C_y \in \mathbb{R}^{n \times n},$$

and they are assumed uncorrelated. Thus, the combined covariance is

$$\Sigma_{xy} = \begin{bmatrix} C_x & 0 \\ 0 & C_y \end{bmatrix}.$$

Jacobian of a Distance

Each distance

$$d_i = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$$

depends only on $(x_i, x_{i+1}, y_i, y_{i+1})$. Let $D_i = d_i$. The partial derivatives are

$$\frac{\partial d_i}{\partial x_i} = \frac{x_i - x_{i+1}}{D_i}, \qquad \frac{\partial d_i}{\partial x_{i+1}} = \frac{x_{i+1} - x_i}{D_i},$$

$$\frac{\partial d_i}{\partial y_i} = \frac{y_i - y_{i+1}}{D_i}, \qquad \frac{\partial d_i}{\partial y_{i+1}} = \frac{y_{i+1} - y_i}{D_i}.$$

These four entries form the nonzero components of the Jacobian row vector

$$J_i = \frac{\partial d_i}{\partial r},$$

where

$$r = (x_1, \ldots, x_n,\, y_1, \ldots, y_n)^T.$$

Variance of a Single Distance

Using first–order error propagation:

$$\boxed{\mathrm{Var}(d_i) = J_i\, \Sigma_{xy}\, J_i^T}$$

and the standard deviation is

$$\sigma_{d_i} = \sqrt{\mathrm{Var}(d_i)}.$$

Covariance Matrix of All Distances

Stacking all Jacobians,

$$J = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_{n-1} \end{bmatrix},$$

the full covariance matrix is

$$\boxed{\Sigma_d = J\, \Sigma_{xy}\, J^T.}$$

Results

From the computed covariance matrix:

$$\sigma_{d_i} = \sqrt{(\Sigma_d)_{ii}}, \qquad i = 1, \ldots, n-1.$$

The resulting minimum and maximum standard deviations are:

$\sigma_{\min} = 1.000000$ at distance index 26,

$\sigma_{\max} = 1.000000$ at distance index 1.

Visualization

The covariance matrix Σ_d was visualized in MATLAB using

$$\texttt{imagesc}(\Sigma_d), \qquad \texttt{colorbar}.$$

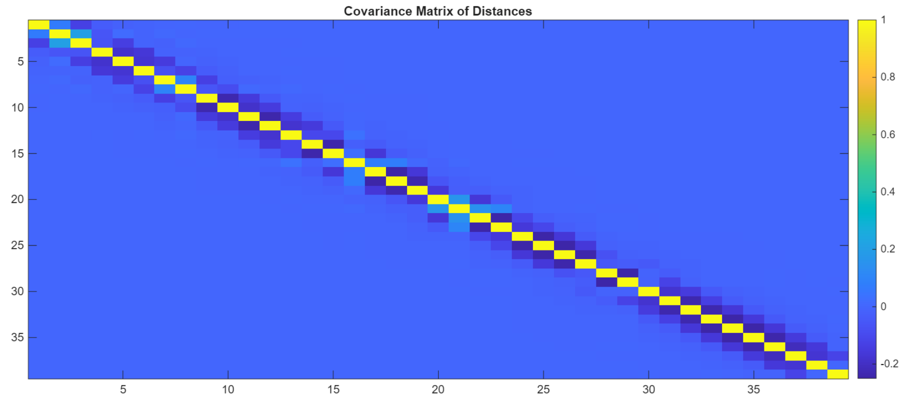

Figure 2: Covariance matrix of Distances

D Variance Propagation III

We are given n measured points

$$p_i = (x_i, y_i), \qquad i = 1, \dots, n,$$

and we want to compute the angle at point i from the x-axis to the segment connecting p_i and p_{i+1}. The angle is defined as

$$r_{i,i+1} = \operatorname{atan2}(y_{i+1} - y_i,\ x_{i+1} - x_i)\,.$$

The measurement uncertainties are given by the standard deviations σ_x and σ_y for all coordinate components. We construct the covariance matrices

$$\Sigma_x = \operatorname{diag}(\sigma_x^2), \qquad \Sigma_y = \operatorname{diag}(\sigma_y^2),$$

and the combined covariance matrix

$$\Sigma_{xy} = \begin{pmatrix} \Sigma_x & 0 \\ 0 & \Sigma_y \end{pmatrix}.$$

6 Computing the angles

The angles r_i for all $i = 1, \dots, n-1$ were computed using the MATLAB code

$$r_i = \operatorname{atan2}(y_{i+1} - y_i,\ x_{i+1} - x_i).$$

The resulting angle values (in radians) are:

$$\begin{aligned} r(i) = \big[&0.5131,\ -1.3373,\ 1.2024,\ -0.1226,\ 0.6258,\ -0.1637,\ 1.0988,\ -0.8792,\ 0.4616,\ 0.1253,\\ &0.8451,\ 0.5871,\ 1.3522,\ 0.2287,\ 0.4791,\ -1.2015,\ 0.6701,\ 0.8262,\ 0.4863,\ -0.4022,\\ &1.8767,\ -0.3373,\ -0.3532,\ 0.5202,\ 0.4752,\ 0.4374,\ 0.1627,\ 1.5491,\ 1.5485,\ 0.0355,\\ &0.8084,\ 0.4916,\ 0.1837,\ 0.3625,\ 0.4836,\ 0.2850,\ 0.8472,\ 0.0455,\ 1.6151,\ 0.5855,\\ &0.6682,\ 0.1948,\ -0.6963,\ 0.4411,\ 0.7883,\ 0.4337,\ 0.5814,\ -0.2335,\ 3.0994,\ 0.3853,\\ &0.3740,\ 0.4255,\ 2.8506,\ 1.4236\big]. \end{aligned}$$

7 Variance propagation of the angles

To propagate the variances, we compute the Jacobian of each angle with respect to all coordinates. For a single angle r_i the partial derivatives are

$$\frac{\partial r_i}{\partial X} = -\frac{Y}{X^2 + Y^2}, \qquad \frac{\partial r_i}{\partial Y} = \frac{X}{X^2 + Y^2},$$

where

$$X = x_{i+1} - x_i, \qquad Y = y_{i+1} - y_i.$$

Applying the chain rule yields the derivatives with respect to the coordinates:

$$\frac{\partial r_i}{\partial x_i} = -\frac{\partial r_i}{\partial X}, \qquad \frac{\partial r_i}{\partial x_{i+1}} = \frac{\partial r_i}{\partial X},$$

$$\frac{\partial r_i}{\partial y_i} = -\frac{\partial r_i}{\partial Y}, \qquad \frac{\partial r_i}{\partial y_{i+1}} = \frac{\partial r_i}{\partial Y}.$$

Stacking all rows gives the full Jacobian matrix J. The covariance matrix of all angles is then computed as

$$\Sigma_r = J\,\Sigma_{xy}\,J^T.$$

The resulting standard deviations of the angles are:

$$\begin{aligned}\mathrm{std}(r_i) = \big[&0.1817, 0.3853, 0.2373, 0.6658, 0.1366, 0.9416, 1.0729, 2.6237, 0.2121, 0.0822,\\ &1.4751, 0.2192, 1.4559, 0.3250, 0.9578, 0.6515, 0.1135, 1.0878, 0.2955, 0.7008,\\ &0.3908, 0.3735, 4.2952, 0.1878, 0.2295, 0.5468, 0.5347, 1.4954, 0.6136, 0.2366,\\ &0.2441, 0.1217, 0.1306, 0.1325, 0.0258, 0.5134, 0.9400, 2.0686, 8.7187, 1.5436,\\ &0.2594, 0.3953, 3.7651, 0.0777, 1.5028, 0.1800, 0.2004, 1.4973, 1.1995, 0.2912,\\ &0.3790, 0.1007, 1.1926, 3.2657\big].\end{aligned}$$

The smallest and largest standard deviations are:

$$\min \mathrm{std}(r_i) = 0.025797 \text{ at angle } i = 35,$$

$$\max \mathrm{std}(r_i) = 8.718669 \text{ at angle } i = 39.$$

Source Code: **MATLAB scripts for Homework 2** (GitHub)