Homework 1: Transformations, Quaternions and Homogeneous Representation

A Translation and Rotation

1. **(0)** Consider a point $\boldsymbol{p}$ with coordinates $\boldsymbol{p} = [4, 0.9, -0.1]^T$, provide the coordinates of the transformed points $\boldsymbol{p}'$, $\boldsymbol{p}''$, $\boldsymbol{p}'''$, $\boldsymbol{p}''''$ after applying the following transformation:
   (a) translation $\mathcal{T}_t$ with translation vector $\boldsymbol{t} = [1, -1.4, 7.0]^T$
   (b) rotation $\mathcal{R}_z(\phi)$ with $\phi = 50°$
   (c) rotation $\mathcal{R}_z(\phi_1)$ followed by $\mathcal{R}_y(\phi_2)$ with $\phi_1 = -30°$ and $\phi_2 = 90°$
   (d) translation $\mathcal{T}_t$ as in (a) followed by a rotation $\mathcal{R}_z(\phi)$ as in (b)
2. **(0)** Convert the following rotation

$$\mathcal{R} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

   using the following representation:
   (a) Euler-Angles with the first rotation around the z-axis, second around y and third around x
   (b) Axis-Angle in the minimal form
3. **(0)** Check if the following matrices are true rotation matrices

$$\mathcal{M}_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{2} \end{bmatrix} \qquad \mathcal{M}_2 = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \mathcal{M}_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{4} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

B Quaternions

4. **(0)** Given the quaternions $\mathbf{q}_1 = [0, 1, 2, 1]^T$ and $\mathbf{q}_2 = [3, 2, 1, 2]^T$
   (a) compute the quaternion resulting by the sum of $\mathbf{q}_1$ and $\mathbf{q}_2$
   (b) compute the inverse of $\mathbf{q}_2$
   (c) compute $\mathbf{q}_1$ times the inverse of $\mathbf{q}_2$
5. **(0)** Given a rotation $\mathcal{R}_x(\phi)$ with $\phi = -40°$ in euclidean form
   (a) compute the quaternion $\mathbf{q}_1$ that represent such rotation
   (b) given a point χ with euclidean coordinates $\boldsymbol{x} = [8, 1.5, 1]^T$, apply the rotation $\mathbf{q}_1$ to the point χ in quaternion form
   (c) apply to point χ the transformation resulting from $\mathbf{q}_1$ followed by $\mathbf{q}_2 = [-1, 2, 0, 1]^T$

C Homogeneous Representation

6. **(0)** Given the points χ_1 and χ_2 with their coordinates

$$\boldsymbol{x}_1 = \begin{bmatrix} 4 \\ -4 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   and the line ℓ_1

$$\ell_1 : y = 2 - x.$$

   determine the homogeneous representation of:

(a) ℓ_1, χ_1 and χ_2

(b) line ℓ_2 passing through χ_1 and χ_2

(c) the intersection point of ℓ_1 and ℓ_2, if any

(d) determine if χ_3 with coordinates $\boldsymbol{x}_3 = [4, 2]^T$ lies on ℓ_2

7. **(0)** Given a point χ with coordinates $\boldsymbol{x} = [18, 20, -5]^T$, compute and apply the following transformations in homogeneous representation:

   (a) translation $\mathcal{T}_t$ with translation vector $\boldsymbol{t} = [-1, -2, 2]^T$

   (b) rotation $\mathcal{R}_z(\phi)$ with $\phi = 56°$

   (c) the rigid body transformation resulting from (a) and (b)

   (d) the transformation given by $\mathcal{H}_1$ followed $\mathcal{H}_2$ with:

$$\mathcal{H}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{H}_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$