Institute of Geodesy and Geoinformation für Geodäsie und Geoinformation
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Course: Coordinate Systems

Homework Assignment 1 - Transformations, Quaternions, and Homogeneous Representations

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A. Translation and Rotation

A.1. Task 1

The initial point is $p = [4, 0.9, -0.1]^T$.

(a) Translation $\mathcal{T}_t$ with $t = [1, -1.4, 7.0]^T$

The transformed point p' is given by $p' = p + t$.

$$p' = \begin{bmatrix} 4 \\ 0.9 \\ -0.1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1.4 \\ 7.0 \end{bmatrix} = \begin{bmatrix} 4+1 \\ 0.9-1.4 \\ -0.1+7.0 \end{bmatrix} = \begin{bmatrix} 5 \\ -0.5 \\ 6.9 \end{bmatrix}$$

(b) Rotation $\mathcal{R}_z(\phi)$ with $\phi = 50^\circ$

The rotation matrix $\mathcal{R}_z(\phi)$ is:

$$\mathcal{R}_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With $\phi = 50^\circ$:

$$\mathcal{R}_z(50^\circ) \approx \begin{bmatrix} 0.6428 & -0.7660 & 0 \\ 0.7660 & 0.6428 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformed point p'' is $p'' = \mathcal{R}_z(50^\circ)p$:

$$p'' \approx \begin{bmatrix} 0.6428 & -0.7660 & 0 \\ 0.7660 & 0.6428 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0.9 \\ -0.1 \end{bmatrix} \approx \begin{bmatrix} 2.5712 - 0.6894 \\ 3.0640 + 0.5785 \\ -0.1 \end{bmatrix}$$

$$p'' \approx \begin{bmatrix} 1.8818 \\ 3.6425 \\ -0.1000 \end{bmatrix}$$

(c) Rotation $\mathcal{R}_z(\phi_1)$ followed by $\mathcal{R}_y(\phi_2)$ with $\phi_1 = -30°$ and $\phi_2 = 90°$

The combined rotation $\mathcal{R}$ is $\mathcal{R} = \mathcal{R}_y(\phi_2)\mathcal{R}_z(\phi_1)$.

$$\mathcal{R}_z(-30°) = \begin{bmatrix} \cos(-30°) & -\sin(-30°) & 0 \\ \sin(-30°) & \cos(-30°) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_y(90°) = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The combined matrix is:

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

The transformed point p''' is $p''' = \mathcal{R}p$:

$$p''' = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0.9 \\ -0.1 \end{bmatrix} = \begin{bmatrix} -0.1 \\ -2 + 0.45\sqrt{3} \\ -2\sqrt{3} - 0.45 \end{bmatrix}$$

$$p''' \approx \begin{bmatrix} -0.1000 \\ -1.2206 \\ -3.9142 \end{bmatrix}$$

(d) Translation $\mathcal{T}_t$ followed by a rotation $\mathcal{R}_z(\phi)$

This is calculated as $p'''' = \mathcal{R}_z(\phi)(p + t)$. 1. **Translation:** $p_t = p + t = [5, -0.5, 6.9]^T$ (from part a). 2. **Rotation:** $p'''' = \mathcal{R}_z(50°)p_t$

$$\mathcal{R}_z(50°) \approx \begin{bmatrix} 0.6428 & -0.7660 & 0 \\ 0.7660 & 0.6428 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p'''' \approx \begin{bmatrix} 0.6428 & -0.7660 & 0 \\ 0.7660 & 0.6428 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -0.5 \\ 6.9 \end{bmatrix} \approx \begin{bmatrix} 3.2140 + 0.3830 \\ 3.8300 - 0.3214 \\ 6.9 \end{bmatrix}$$

$$p'''' \approx \begin{bmatrix} 3.5970 \\ 3.5086 \\ 6.9000 \end{bmatrix}$$

A.2. Task 2

The given rotation matrix is:

$$\mathcal{R} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) Euler-Angles with the first rotation around the z-axis, second around y and third around x (Z-Y-X)

The Z-Y-X Euler angle rotation matrix $\mathcal{R}(\alpha, \beta, \gamma)$ is equated to the given matrix $\mathcal{R}$.

$$\mathcal{R} = \mathcal{R}_z(\gamma)\mathcal{R}_y(\beta)\mathcal{R}_z(\alpha) = \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix}$$

i. Finding β

From $R_{33} = \cos\beta$:

$$\cos\beta = \frac{\sqrt{2}}{2}$$
$$\beta = \frac{\pi}{4} \quad (45^\circ)$$

ii. Finding α

From $R_{13} = c_\alpha s_\beta$ and $R_{23} = s_\alpha s_\beta$, using $\sin\beta = \frac{\sqrt{2}}{2}$:

$$R_{13} = 0 = c_\alpha \left(\frac{\sqrt{2}}{2}\right) \implies \cos\alpha = 0$$
$$R_{23} = -\frac{\sqrt{2}}{2} = s_\alpha \left(\frac{\sqrt{2}}{2}\right) \implies \sin\alpha = -1$$

$$\alpha = \arctan 2(-1, 0) = -\frac{\pi}{2} \quad (-90^\circ)$$

iii. Finding γ

From $R_{31} = -s_\beta c_\gamma$ and $R_{32} = s_\beta s_\gamma$:

$$R_{31} = \frac{\sqrt{2}}{2} = -\left(\frac{\sqrt{2}}{2}\right) c_\gamma \implies \cos\gamma = -1$$
$$R_{32} = 0 = \left(\frac{\sqrt{2}}{2}\right) s_\gamma \implies \sin\gamma = 0$$

$$\gamma = \arctan 2(0, -1) = \pi \quad (180^\circ)$$

The Euler-Angles (α, β, γ) are:

$$(-90^\circ, 45^\circ, 180^\circ)$$

(b) Axis-Angle in the minimal form $(\mathbf{k}, \theta)$

The formula for rotation matrix to axis angle is given by : $\theta =$ a tan 2 (—a—, t r R-1) with,

$$a = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = 2sin\theta.r$$

$$tr(R) = r_{11} + r_{12} + r_{13} = 1 + 2cos\theta$$

i. Finding the Angle θ

The rotation angle θ is found from the trace of $\mathcal{R}$: $\text{tr}(\mathcal{R}) = R_{11} + R_{22} + R_{33} = 0 + 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$.

$$\theta = \arccos\left(\frac{\text{tr}(\mathcal{R}) - 1}{2}\right) = \arccos\left(\frac{\frac{\sqrt{2}}{2} - 1}{2}\right) = \arccos\left(\frac{\sqrt{2} - 2}{4}\right)$$

$$\theta \approx 98.421^\circ \quad \text{or } 1.7177 \text{ radians}$$

ii. Finding the Axis $\mathbf{k}$

The unit rotation axis $\mathbf{k} = [k_x, k_y, k_z]^T$ is given by:

$$k_x = \frac{R_{32} - R_{23}}{2\sin\theta}, \quad k_y = \frac{R_{13} - R_{31}}{2\sin\theta}, \quad k_z = \frac{R_{21} - R_{12}}{2\sin\theta}$$

We calculate the numerator terms:

$$R_{32} - R_{23} = 0 - \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$
$$R_{13} - R_{31} = 0 - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$
$$R_{21} - R_{12} = \frac{\sqrt{2}}{2} - (-1) = \frac{\sqrt{2}+2}{2}$$

Using $\sin\theta \approx 0.9892$, the components of $\mathbf{k}$ are:

$$k_x = \frac{\sqrt{2}/2}{2\sin\theta} = \frac{\sqrt{2}}{4\sin\theta} \approx 0.3574$$
$$k_y = \frac{-\sqrt{2}/2}{2\sin\theta} = -\frac{\sqrt{2}}{4\sin\theta} \approx -0.3574$$
$$k_z = \frac{(\sqrt{2}+2)/2}{2\sin\theta} = \frac{\sqrt{2}+2}{4\sin\theta} \approx 0.8624$$

The Axis-Angle representation $(\mathbf{k}, \theta)$ is:

$$\left(\begin{bmatrix} 0.3574 \\ -0.3574 \\ 0.8624 \end{bmatrix}, 98.421^\circ\right)$$

A.3. Task 3

A matrix $\mathcal{M}$ is a true rotation matrix if and only if it satisfies the following two conditions:

1. **Orthogonality:** The matrix inverse is equal to its transpose, $\mathcal{M}^{-1} = \mathcal{M}^T$, which implies $\mathcal{M}^T\mathcal{M} = I$.

2. **Determinant:** The determinant is equal to one, $\det(\mathcal{M}) = 1$.

Task 3. a.
The matrix is:

$$M_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

i. Condition 1: Orthogonality ($\mathcal{M}_1^T\mathcal{M}_1 = I \implies \mathcal{M}_1^T = \mathcal{M}_1^{-1}$)
The transpose of M_1 is:

$$M_1^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

We compute the product $\mathcal{M}_1^T \mathcal{M}_1$:

$$\mathcal{M}_1^T \mathcal{M}_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{16}{16} & 0 & 0 \\ 0 & \frac{16}{16} & 0 \\ 0 & 0 & \frac{4}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $\mathcal{M}_1^T \mathcal{M}_1 = I$, the matrix M_1 is **orthogonal**, satisfying $\mathcal{M}_1^T = \mathcal{M}_1^{-1}$.

ii. Condition 2: Determinant ($\det(\mathcal{M}_1) = 1$)
We compute the determinant using cofactor expansion along the first row:

$$\det(M_1) = \left(\frac{\sqrt{3}}{2}\right) \begin{vmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{1}{2} \end{vmatrix} - \left(-\frac{1}{2}\right) \begin{vmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{vmatrix} + 0$$
$$= \frac{\sqrt{3}}{2}\left[\left(\frac{\sqrt{3}}{8}\right) - \left(-\frac{3\sqrt{3}}{8}\right)\right] + \frac{1}{2}\left[\left(\frac{1}{8}\right) - \left(-\frac{3}{8}\right)\right]$$
$$= \frac{\sqrt{3}}{2}\left[\frac{4\sqrt{3}}{8}\right] + \frac{1}{2}\left[\frac{4}{8}\right]$$
$$= \frac{\sqrt{3}}{2}\left[\frac{\sqrt{3}}{2}\right] + \frac{1}{2}\left[\frac{1}{2}\right]$$
$$= \frac{3}{4} + \frac{1}{4} = 1$$

Since $\det(\mathcal{M}_1) = 1$, the second condition is satisfied.
Conclusion: As $M_1^T = M_1^{-1}$ and $\det(M_1) = 1$, the matrix M_1 **is a true rotation matrix**.

Task 3.b.
i. Condition 1: Orthogonality ($\mathcal{M}_2^T \mathcal{M}_2 = I \implies \mathcal{M}_2^T = \mathcal{M}_2^{-1}$)
The transpose of M_2 is:

$$M_2^T = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We compute the product $\mathcal{M}_2^T \mathcal{M}_2$:

$$\mathcal{M}_2^T \mathcal{M}_2 = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 + \frac{2}{4} + \frac{2}{4} & 0 + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} & 0 + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} \\ 0 + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} & \frac{2}{4} + \frac{1}{4} + \frac{1}{4} & -\frac{2}{4} + \frac{1}{4} + \frac{1}{4} \\ 0 + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} & -\frac{2}{4} + \frac{1}{4} + \frac{1}{4} & \frac{2}{4} + \frac{1}{4} + \frac{1}{4} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $\mathcal{M}_2^T \mathcal{M}_2 = I$, the matrix M_2 is **orthogonal**, satisfying $\mathcal{M}_2^T = \mathcal{M}_2^{-1}$.

ii. Condition 2: Determinant ($\det(\mathcal{M}_2) = 1$)
We compute the determinant using cofactor expansion along the first row:

$$\det(M_2) = 0 \cdot C_{11} - \left(-\frac{\sqrt{2}}{2}\right) \cdot C_{12} + \left(\frac{\sqrt{2}}{2}\right) \cdot C_{13}$$

The minors M_{12} and M_{13} are:

$$M_{12} = \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{vmatrix} = \frac{\sqrt{2}}{4} - \left(-\frac{\sqrt{2}}{4}\right) = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

$$M_{13} = \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{vmatrix} = \frac{\sqrt{2}}{4} - \left(-\frac{\sqrt{2}}{4}\right) = \frac{\sqrt{2}}{2}$$

Substituting into the determinant formula ($\det(\mathcal{M}) = R_{11}C_{11} + R_{12}C_{12} + R_{13}C_{13}$, where $C_{ij} = (-1)^{i+j}M_{ij}$):

$$\begin{aligned}
\det(M_2) &= (0)C_{11} + \left(-\frac{\sqrt{2}}{2}\right)(-\mathbf{M_{12}}) + \left(\frac{\sqrt{2}}{2}\right)(+\mathbf{M_{13}}) \\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{2}{4} + \frac{2}{4} = 1
\end{aligned}$$

Since $\det(\mathcal{M}_2) = 1$, the second condition is satisfied.
Conclusion: As $M_2^T = M_2^{-1}$ and $\det(M_2) = 1$, the matrix M_2 **is a true rotation matrix**.

Task 3. c.
i. Condition 1: Orthogonality ($\mathcal{M}_3^T\mathcal{M}_3 = I \implies \mathcal{M}_3^T = \mathcal{M}_3^{-1}$)
The orthogonality condition requires that the column vectors be orthonormal (unit length and mutually orthogonal). We check the element $(\mathcal{M}_3^T\mathcal{M}_3)_{11}$, which is the squared norm of the first column vector $\mathbf{c}_1$:

$$\mathbf{c}_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$(\mathcal{M}_3^T\mathcal{M}_3)_{11} = \mathbf{c}_1 \cdot \mathbf{c}_1 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2$$

$$(\mathcal{M}_3^T\mathcal{M}_3)_{11} = \frac{3}{4} + \frac{2}{4} + \frac{3}{4} = \frac{8}{4} = 2$$

Since $(\mathcal{M}_3^T\mathcal{M}_3)_{11} = 2 \neq 1$, the matrix is **not orthogonal**.
Conclusion: As M_3 fails the orthogonality condition ($\mathcal{M}_3^T\mathcal{M}_3 \neq I$), M_3 **is not a true rotation matrix**. The determinant check is unnecessary.

B. Quaternions

Task 4
Given the quaternions:

$$q_1 = [0, 1, 2, 1]^T$$

$$q_2 = [3, 2, 1, 2]^T$$

(a) Sum of q_1 and q_2

The sum is computed by adding corresponding components:

$$q_{\text{sum}} = q_1 + q_2 = [0+3, 1+2, 2+1, 1+2]^T$$

$$q_{\text{sum}} = [3, 3, 3, 3]^T$$

(b) Inverse of q_2

The inverse is $q_2^{-1} = \frac{q_2^*}{\|q_2\|^2}$.

- **Squared Norm:**

$$\|q_2\|^2 = 3^2 + 2^2 + 1^2 + 2^2 = 9 + 4 + 1 + 4 = 18$$

- **Conjugate:**

$$q_2^* = [3, -2, -1, -2]^T$$

The inverse is:

$$q_2^{-1} = \frac{1}{18}[3, -2, -1, -2]^T = \left[\frac{1}{6}, -\frac{1}{9}, -\frac{1}{18}, -\frac{1}{9}\right]^T$$

(c) q_1 times the inverse of q_2 ($q_1 q_2^{-1}$)

We compute the product $q_1 q_2^*$ and then divide by $\|q_2\|^2 = 18$.

$$q_1 q_2^* = [0, 1, 2, 1]^T [3, -2, -1, -2]^T$$

- **Scalar Part:** $w_r = 0(3) - ([1, 2, 1] \cdot [-2, -1, -2]) = -(-2 - 2 - 2) = 6$
- **Vector Part:** $\mathbf{v}_r = 0\mathbf{v}_2^* + 3\mathbf{v}_1 + (\mathbf{v}_1 \times \mathbf{v}_2^*)$

$$\mathbf{v}_r = [3, 6, 3]^T + [-3, 0, 3]^T = [0, 6, 6]^T$$

The product $q_1 q_2^*$ is $[6, 0, 6, 6]^T$. Dividing by 18:

$$q_1 q_2^{-1} = \frac{1}{18}[6, 0, 6, 6]^T$$

$$q_1 q_2^{-1} = \left[\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right]^T$$

Task 5 $\mathcal{R}_x(\phi)$ with $\phi = -40°$

(a) Computation of the quaternion q_1 that represents the rotation $\mathcal{R}_x(\phi)$

The rotation $\mathcal{R}_x(\phi)$ is represented by the quaternion $q_1 = [\cos(\phi/2), \sin(\phi/2)\mathbf{i}, 0, 0]^T$, where $\phi = -40°$.

$$\frac{\phi}{2} = -20°$$

$$\cos(-20°) \approx 0.9397$$

$$\sin(-20°) \approx -0.3420$$

$$q_1 = [\cos(-20°), \sin(-20°), 0, 0]^T$$

$$q_1 \approx [0.9397, -0.3420, 0, 0]^T$$

(b) Applying the rotation q_1 to the point x in quaternion form

The point x is represented as a pure quaternion $p = [0, 8, 1.5, 1]^T$. The transformation is $x' = q_1 p q_1^*$, where q_1 is a unit quaternion, which it is by definition.

$$q_1^* \approx [0.9397, 0.3420, 0, 0]^T$$

1. Computing the product $q_1 p$

- **Scalar part** ($w_r = w_1 w_p - \mathbf{v}_1 \cdot \mathbf{v}_p$):

$$w_r = (0.9397)(0) - ([-0.3420, 0, 0] \cdot [8, 1.5, 1]) = -(-2.736) = 2.736$$

- **Vector part** ($\mathbf{v}_r = w_1 \mathbf{v}_p + w_p \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_p$):

$$\mathbf{v}_r = 0.9397[8, 1.5, 1]^T + 0 \cdot \mathbf{v}_1 + [-0.3420, 0, 0]^T \times [8, 1.5, 1]^T$$
$$\mathbf{v}_r \approx [7.5176, 1.4096, 0.9397]^T + [0, 0.3420, -0.513]^T$$
$$\mathbf{v}_r \approx [7.5176, 1.7516, 0.4267]^T$$
$$q_1 p \approx [2.736, 7.5176, 1.7516, 0.4267]^T$$

2. Computing the product $(q_1 p) q_1^*$

$$(q_1 p) q_1^* \approx [2.736, 7.5176, 1.7516, 0.4267]^T [0.9397, 0.3420, 0, 0]^T$$

- **Scalar part** (w_r'): Should be zero (within computational error).

$$w_r' \approx (2.736)(0.9397) - ([7.5176, 1.7516, 0.4267] \cdot [0.3420, 0, 0]) \approx 2.5710 - 2.5710 = 0$$

- **Vector part** ($\mathbf{v}_r'$):

$$\mathbf{v}_r' \approx 2.736[0.3420, 0, 0]^T + 0.9397[7.5176, 1.7516, 0.4267]^T + [7.5176, 1.7516, 0.4267]^T \times [0.3420, 0, 0]^T$$
$$\mathbf{v}_r' \approx [0.9366, 0, 0]^T + [7.0645, 1.6459, 0.4009]^T + [0, 0.1460, -0.5986]^T$$
$$\mathbf{v}_r' \approx [8, 1.7919, -0.1977]^T$$

The Euclidean coordinates of the transformed point are:

$$x' \approx [8, 1.7919, -0.1977]^T$$

(c) Applying to point x the transformation resulting from q_1 followed by q_2

The combined rotation is $q_{\text{total}} = q_2 q_1$. Since q_2 is not a unit quaternion, we must normalize both q_1 and q_2 before combination, or normalize the result. We use the given (unnormalized) $q_2 = [-1, 2, 0, 1]^T$ and the unit q_1 from (a).

i. Computing the total quaternion $q_{\text{total}} = q_2 q_1$

$$q_2 = [-1, 2, 0, 1]^T \quad q_1 \approx [0.9397, -0.3420, 0, 0]^T$$

- **Scalar part** ($w_t = w_2 w_1 - \mathbf{v}_2 \cdot \mathbf{v}_1$):

$$w_t \approx (-1)(0.9397) - ([2, 0, 1] \cdot [-0.3420, 0, 0]) = -0.9397 - (-0.684) = -0.2557$$

- **Vector part** ($\mathbf{v}_t = w_2 \mathbf{v}_1 + w_1 \mathbf{v}_2 + \mathbf{v}_2 \times \mathbf{v}_1$):

$$\mathbf{v}_t \approx (-1)[-0.3420, 0, 0]^T + 0.9397[2, 0, 1]^T + [2, 0, 1]^T \times [-0.3420, 0, 0]^T$$
$$\mathbf{v}_t \approx [0.3420, 0, 0]^T + [1.8794, 0, 0.9397]^T + [0, -0.3420, 0]^T$$
$$\mathbf{v}_t \approx [2.2214, -0.3420, 0.9397]^T$$
$$q_{\text{total}} \approx [-0.2557, 2.2214, -0.3420, 0.9397]^T$$

ii. Computing the final rotation $x'' = \frac{1}{\|q_{\text{total}}\|^2} q_{\text{total}} p q^*_{\text{total}}$

- **Squared Norm** ($\|q_{\text{total}}\|^2$):

  $$\|q_{\text{total}}\|^2 \approx (-0.2557)^2+(2.2214)^2+(-0.3420)^2+(0.9397)^2 \approx 0.0654+4.9346+0.1170+0.8830 = 6.000$$

- **Conjugate:** $q^*_{\text{total}} \approx [-0.2557, -2.2214, 0.3420, -0.9397]^T$

After performing the multiplication $q_{\text{total}} p q^*_{\text{total}}$ and dividing by 6.000, the final rotated point is:

$$x'' \approx [0, 6.784, -4.562, -0.638]^T$$

The Euclidean coordinates are:

$$x'' \approx [6.784, -4.562, -0.638]^T$$

C. Homogeneous Representation

Task 6

(a) Homogeneous representation of l_1, x_1 and x_2

Line l_1

The Euclidean line equation is $y = 2 - x$, which can be rewritten as $1x + 1y - 2 = 0$. The homogeneous representation of a line $l : ax + by + c = 0$ is $l = [a, b, c]^T$.

$$l_1 = [1, 1, -2]^T$$

Points x_1 and x_2

The homogeneous representation of a 2D Euclidean point $x = [x, y]^T$ is $x_H = [x, y, 1]^T$.

$$x_1 = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

(b) Line l_2 passing through x_1 and x_2

In homogeneous coordinates, the line l_2 passing through two points x_1 and x_2 is given by their cross product: $l_2 = x_1 \times x_2$.

$$l_2 = x_1 \times x_2 = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$l_2 = \begin{bmatrix} (-4)(1) - (1)(1) \\ (1)(2) - (4)(1) \\ (4)(1) - (-4)(2) \end{bmatrix} = \begin{bmatrix} -4 - 1 \\ 2 - 4 \\ 4 + 8 \end{bmatrix}$$

$$l_2 = \begin{bmatrix} -5 \\ -2 \\ 12 \end{bmatrix}$$

The Euclidean equation for l_2 is $-5x - 2y + 12 = 0$, or $5x + 2y - 12 = 0$.

(c) The intersection point of l_1 and l_2, if any

The intersection point x_{int} of two lines l_1 and l_2 is given by their cross product: $x_{int} = l_1 \times l_2$.

$$x_{int} = l_1 \times l_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} -5 \\ -2 \\ 12 \end{bmatrix}$$

$$x_{int} = \begin{bmatrix} (1)(12) - (-2)(-2) \\ (-2)(-5) - (1)(12) \\ (1)(-2) - (1)(-5) \end{bmatrix} = \begin{bmatrix} 12 - 4 \\ 10 - 12 \\ -2 + 5 \end{bmatrix}$$

$$x_{int} = \begin{bmatrix} 8 \\ -2 \\ 3 \end{bmatrix}$$

To find the Euclidean coordinates $[x, y]^T$, we normalize by the third component $w = 3$:

$$x_{\text{int, Euclidean}} = \left[\frac{8}{3}, \frac{-2}{3}\right]^T$$

(d) Determining if x_3 with coordinates $x_3 = [4, 2]^T$ lies on l_2

A point x lies on a line l if their dot product is zero: $l^T x = 0$. The homogeneous coordinates for x_3 are $x_3 = [4, 2, 1]^T$. The line l_2 is $l_2 = [-5, -2, 12]^T$.

$$l_2^T x_3 = [-5, -2, 12] \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$l_2^T x_3 = (-5)(4) + (-2)(2) + (12)(1) = -20 - 4 + 12 = -12$$

Since $l_2^T x_3 = -12 \neq 0$, the point x_3 **does not lie on the line** l_2.

Task 7
The given point is $x = [18, 20, -5]^T$. The homogeneous representation is $x_H = [18, 20, -5, 1]^T$.

(a) Translation T_t with $t = [-1, -2, 2]^T$

The translation matrix T_t is:

$$T_t = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed point x' is $x' = T_t x_H$:

$$x' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ 20 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 - 1 \\ 20 - 2 \\ -5 + 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 18 \\ -3 \\ 1 \end{bmatrix}$$

(b) Rotation $\mathcal{R}_z(\phi)$ with $\phi = 56°$

The rotation matrix $\mathcal{R}_z(\phi)$ in homogeneous form is:

$$\mathcal{R}_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using $\phi = 56°$: $\cos(56°) \approx 0.5592$, $\sin(56°) \approx 0.8290$.

$$\mathcal{R}_z(56°) \approx \begin{bmatrix} 0.5592 & -0.8290 & 0 & 0 \\ 0.8290 & 0.5592 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed point x'' is $x'' = \mathcal{R}_z(\phi)x_H$:

$$x'' \approx \begin{bmatrix} 0.5592 & -0.8290 & 0 & 0 \\ 0.8290 & 0.5592 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ 20 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} (0.5592)(18) - (0.8290)(20) \\ (0.8290)(18) + (0.5592)(20) \\ -5 \\ 1 \end{bmatrix}$$

$$x'' \approx \begin{bmatrix} 10.0656 - 16.5800 \\ 14.9220 + 11.1840 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6.5144 \\ 26.1060 \\ -5 \\ 1 \end{bmatrix}$$

(c) Rigid body transformation resulting from (a) followed by (b)

The transformation matrix is $H_{\text{total}} = \mathcal{R}_z(\phi)T_t$ (Rotation applied after Translation).

$$H_{\text{total}} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & -\cos\phi + 2\sin\phi \\ \sin\phi & \cos\phi & 0 & -\sin\phi - 2\cos\phi \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the values $\cos(56°) \approx 0.5592$ and $\sin(56°) \approx 0.8290$:

$$H_{\text{total}} \approx \begin{bmatrix} 0.5592 & -0.8290 & 0 & (-0.5592) + 2(0.8290) \\ 0.8290 & 0.5592 & 0 & (-0.8290) - 2(0.5592) \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0.5592 & -0.8290 & 0 & 1.0988 \\ 0.8290 & 0.5592 & 0 & -1.9476 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed point x''' is $x''' = H_{\text{total}}x_H$:

$$x''' \approx \begin{bmatrix} 0.5592 & -0.8290 & 0 & 1.0988 \\ 0.8290 & 0.5592 & 0 & -1.9476 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ 20 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} (0.5592)(18) - (0.8290)(20) + 1.0988 \\ (0.8290)(18) + (0.5592)(20) - 1.9476 \\ -5 + 2 \\ 1 \end{bmatrix}$$

$$x''' \approx \begin{bmatrix} 10.0656 - 16.5800 + 1.0988 \\ 14.9220 + 11.1840 - 1.9476 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -5.4156 \\ 24.1584 \\ -3 \\ 1 \end{bmatrix}$$

(d) Transformation given by H_1 followed by H_2

The combined transformation is $H_{\text{final}} = H_2 H_1$.

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i. Computing the product $H_{\text{final}} = H_2 H_1$

$$H_{\text{final}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{\text{final}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} & 1 - \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using approximate values ($\sqrt{2} \approx 1.414, \sqrt{3} \approx 1.732, \sqrt{6} \approx 2.449$):

$$H_{\text{final}} \approx \begin{bmatrix} 0.866 & -0.354 & -0.354 & 0.500 \\ 0.500 & 0.966 & 0.259 & 0.134 \\ 0 & -0.707 & 0.707 & 2.000 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii. Applying H_{final} to x_H

The transformed point x'''' is $x'''' = H_{\text{final}} x_H$:

$$x'''' \approx \begin{bmatrix} 0.866 & -0.354 & -0.354 & 0.500 \\ 0.500 & 0.966 & 0.259 & 0.134 \\ 0 & -0.707 & 0.707 & 2.000 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ 20 \\ -5 \\ 1 \end{bmatrix}$$

$$x'''' \approx \begin{bmatrix} (0.866)(18) - (0.354)(20) - (0.354)(-5) + 0.500 \\ (0.500)(18) + (0.966)(20) + (0.259)(-5) + 0.134 \\ 0 - (0.707)(20) + (0.707)(-5) + 2.000 \\ 1 \end{bmatrix}$$

$$x'''' \approx \begin{bmatrix} 15.588 - 7.080 + 1.770 + 0.500 \\ 9.000 + 19.320 - 1.295 + 0.134 \\ 0 - 14.140 - 3.535 + 2.000 \\ 1 \end{bmatrix} = \begin{bmatrix} 10.778 \\ 27.159 \\ -15.675 \\ 1 \end{bmatrix}$$