IB MATH EXPLORATION 2014

What is Euler's Number?

An exploration of e's significance to Mathematics

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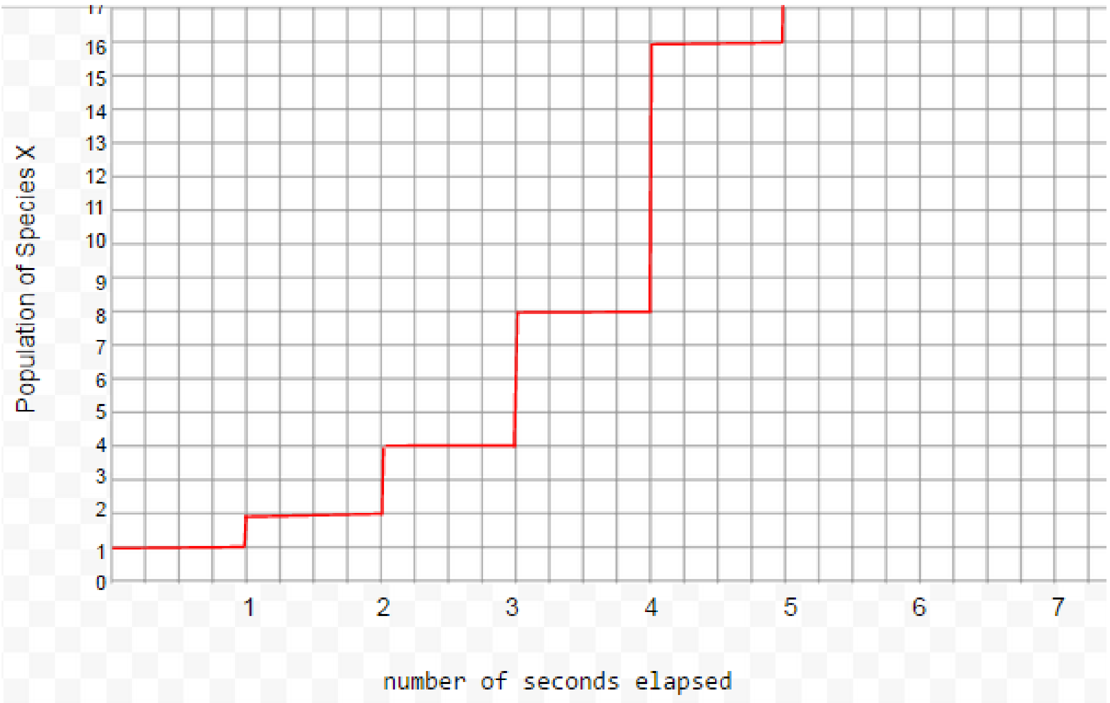
Introduction

Euler's number (e), is one of the most important constants in mathematics. It is named after the Swiss mathematician Leonhard Euler, although its existence was first discovered by John Napier in 1614. As a result, it is also sometimes called Napier's Constant. *e* is an irrational number, and the first few digits of *e* are 2.7128182845.

I have encountered *e* numerous times throughout my study of mathematics and physics. Yet beyond the general high school introduction of compound and continuous interest formulae, I struggled to get an intuitive understanding of *e*. In this exploration, I aim to gain an intuitive and a complete understanding of *e*, and further explore its significance to Mathematics.

Intuition of e

In this section, I will approach *e* from my own reasoning. Consider the following scenario: There is one bacterium of Species X in the pond. The number of bacteria in a pond doubles every second. Therefore, at any second *s*, the population of Species X will be twice the population of the previous second. This can simply be modeled by $p = 2^s$, where p is the population and s is the integer number of seconds elapsed.



*This graph shows the function $\mathrm{p} = 2^{\mathrm{s}}$, when it is compounded every second.

The goal is to solve for p at the end of 1 second, when timeframe of compounds reaches an infinitesimally small length (continuous compound). One can ask the question: if there is a 100% growth every second, what is the growth for half a second, quarter of a second, tenth of a second, until the time frame of compounds approaches zero? In order to answer this, I analyzed the function discussed previously; $\mathrm{p} = 2^{\mathrm{s}}$. This can be derived from the more general formula:

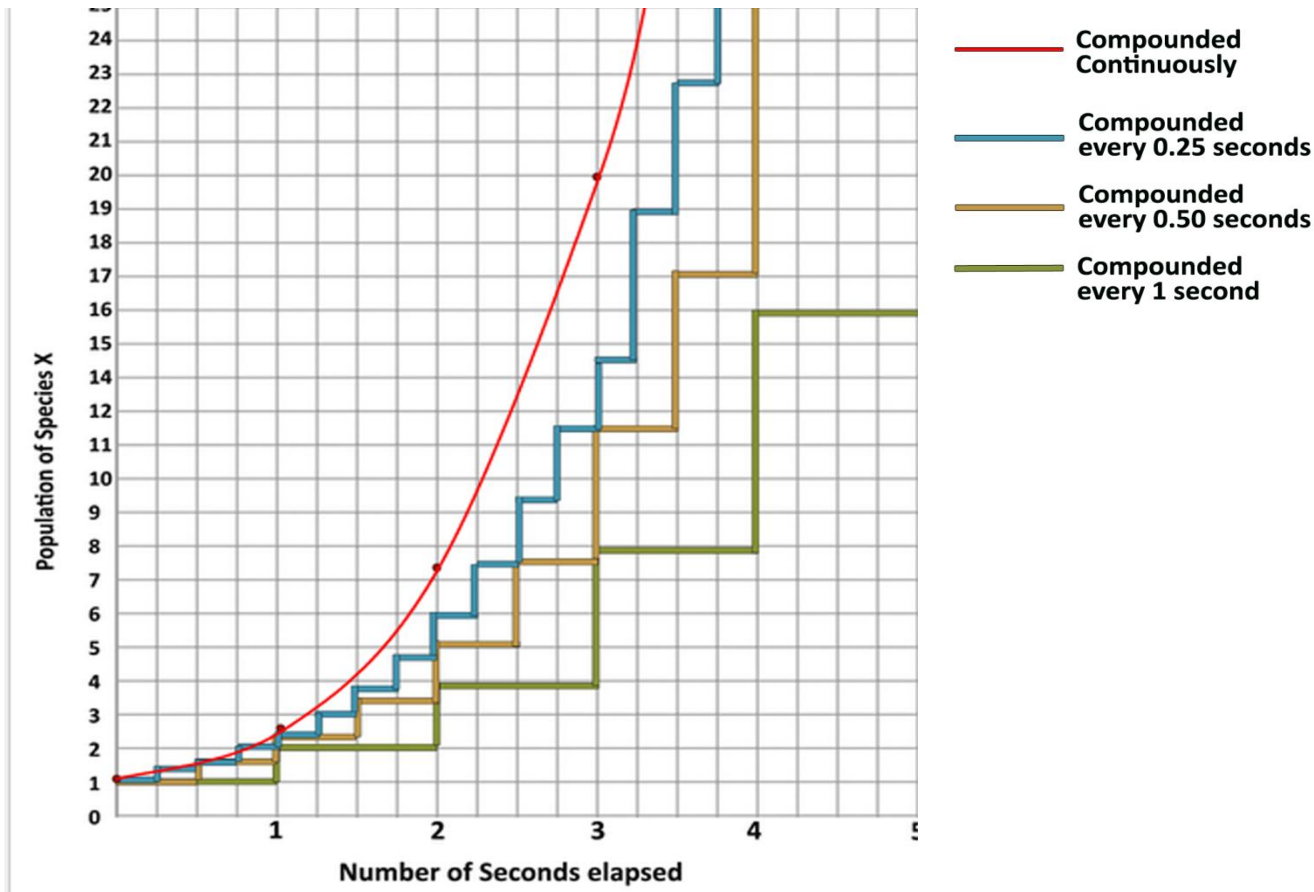
$p = (1 + r)^n$,

where r is the rate of growth in the form $\frac{1}{time}$ and n is the number of compounds.

Since the rate of population growth was a 100% per second, and the number of compounds is equal to the number of seconds elapsed, this yielded $p = (1 + 1)^s = 2^s$

If the rate of growth every second is 100%, then using proportion, the rate of growth every half a second is 50%, rate of growth every quarter of a second is 25%, and so on. Therefore, rate here at a given timeframe of measurement is $r \times s$, or simply s, since $r = \frac{1}{1\,s}$ for this scenario. However, the more we decrease the timeframe, the more frequently we are compounding to get the end result at the end of 1 second. Therefore, the number of compounds is inversely proportional to the length of the timeframe; $n = \frac{1}{s}$. Since rate at any timeframe s is $\times s$; $1 \times s = s$ and $n = \frac{1}{s}$, the general formula $p = (1+r)^n$ becomes $p = (1+s)^{\frac{1}{s}}$

In the following graph, I have made a graph of the population growth for various timeframes s of measurements: Notice how the more frequent the measurements are made, the closer it is to looking like "Compounded Continuously", which is really just the graph of $y = e^x$.



From the graph above, it can be seen that as the timeframe of measurements decreases to zero, *p* approaches a limit which is e. The following shows the formula and calculation to determine this limit e:

$$e = \lim_{s\to 0}(1+s)^{\frac{1}{s}}$$

This limit approaches the value of e, which is 2.71281828459....

s	*e*
5	1.43096908
1	2.00000000
0.1	2.59374246
0.001	2.71692393
0.0000001	2.71828169

Jacob Bernoulli's definition

The conventional definition of *e* is Jacob Bernoulli's definition: $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. [1]This is quite different from the definition that I arrived at on previous chapter, which is:

$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$. While I arrived at $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$ by decreasing the timeframe of measurements so that it approaches zero, one can arrive at Bernoulli's definition by simply increasing the number of compounds (represented by *x*) until it approaches infinity.

Returning to the scenario of Species X in chapter 1, recall that the population starts with 1 bacterium and doubles every second. Therefore, the number of bacteria at the end of the first second will be $(1+1)^1$ if growth is compounded every second. The rate of growth, then simply becomes $\frac{1}{number\ of\ compounds}$ because of proportion. For example, if a growth of 100% per second is compounded twice, once every half a second, it makes sense that the rate is $\frac{1}{2}$, or 50% every half a second. When compounding every half of a second, we can use $\left(1+\frac{1}{2}\right)^2$ to find population at the end of the first second. Notice that we need to compound *twice* (represented by

[1] "Take Wolfram|Alpha Anywhere..." *E*. N.p., n.d. Web. 23 Dec. 2014. <http://www.wolframalpha.com/input/?i=e&a=%2AC.e-_%2ANamedConstant->.

the exponent) because we are now calculating the results for population at the end of 1 second by using compounds every half a second. For compounds every tenth of a second, we would use

$\left(1+\frac{1}{10}\right)^{10}$, and so on... This can give us; $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$, as the number of compounds increases to infinity.

Reconciling $A = Pe^{rt}$ with Bernoulli definition

The formula $A = Pe^{rt}$ is used to calculate the final amount *A*, if the initial amount *P* is compounded continuously at a rate *r* for some time *t*. It seemed to me that both of the definitions of *e* I explored thus far; $e = lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$ and $e = \lim_{x\to 0}(1+n)^{\frac{1}{n}}$, are inconsistent with $A = Pe^{rt}$. This is because in both cases, I arrived at *e*, by initially considering 100% growth per unit time. How would one go about using rates other than 100% per unit time, when using e?

In Bernoulli's definition, $e = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}$, $r = 1$ because of 100% growth. Yet in $A = Pe^{rt}$, e is used, but the rate need not be 100%. Any rate can be used, substituted as r to determine continuous growth for that rate. Because $e^{1} = \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{n}$, in $A = Pe^{rt}$, e^{r}must equal $\lim_{x\to\infty}\left(1+\frac{r}{n}\right)^{n}$ for $A = Pe^{rt}$ to work. In order for this to be true, it must mean that:

$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$. This can be established from the following reasoning:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} = e^{r}$$

Let me define $\frac{r}{n}$ as this: $\frac{r}{n} = \frac{1}{x}$

This gives me: $n = rx$

Since $n \to \infty$ and r is constant, $x \to \infty$.

I will now rewrite the original equation:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} = e^{r}$$

as:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{n} = e^{r}$$

Since $n = rx$, I get:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{xr} = e^{r}$$

This is key to compound interest formula because I've now arrived at e again, since

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$ following Bernoulli's definition. I have now shown that:

$$e^{r} = e^{r}$$

Therefore, $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} = e^{r}$.

This is interesting and extremely useful, because it shows that we are not bound to 100% growth rate when dealing with e. e raised to the power any rate, r can be used to determine continuous growth for that rate.

The derivative of $f(x) = e^{x}$[2]

Here, I will explore a unique property of the derivative of $f(x) = e^{x}$. I will approach this by applying the limit definition of the derivative function, which is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Considering this definition for the function $f(x) = e^{x}$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{e^{\Delta x + x} - e^{x}}{\Delta x}$$

Using law of exponents, we can:

$$f'(x) = \lim_{\Delta x \to 0} \frac{(e^{\Delta x} \times e^{x}) - e^{x}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{e^{x}(e^{\Delta x} - 1)}{\Delta x}$$

$$f'(x) = e^{x} \lim_{\Delta x \to 0} \frac{(e^{\Delta x} - 1)}{\Delta x}$$

$e^{x}$is independent of the limit, so it can be factored out of the limit expression. The limit expression; $\lim_{\Delta x \to 0} \frac{(e^{\Delta x} - 1)}{\Delta x}$, however, approaches 1. See the following table that illustrates this:

Δx	$\frac{(e^{\Delta x} - 1)}{\Delta x}$
5	29.48263…
1	1.718282…

[2] "Take Wolfram|Alpha Anywhere..." *E*. N.p., n.d. Web. 23 Dec. 2014. <http://www.wolframalpha.com/input/?i=e&a=%2AC.e-_%2ANamedConstant->.

0.1	1.051709…
0.0001	1.000050…
0.000001	1.000000…

$$\text{Therefore, } f'(x) = e^x \times 1 = e^x.$$

This shows a remarkable property of the function $f(x) = e^x$. The derivative of $e^x$is e^x. More commonly expressed: $\frac{de^x}{dx} = e^x$. One noteworthy property of this function is that the output of the function at any *x* value, is the rate of change at that point.

Summation Definition

In order to approach the summation definition, I had two choices. I could either approach it using the Binomial Theorem expansion, or I could do it using the Maclaurin Series expansion. I chose the Binomial Theorem because it is easier to explain and illustrate.

e can be defined using this summation series:

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

In this section, I will approach this definition by applying the binomial theorem to $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.[3] The binomial theorem is defined as:

[3] "Binomial Theorem." *Binomial Theorem*. N.p., n.d. Web. 23 Dec. 2014. <http://www.mathsisfun.com/algebra/binomial-theorem.html>.

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r}\ a^{n-r} b^r$$

$$\left(1+\frac{1}{n}\right)^n = \binom{n}{1}\cdot 1^n \cdot \left(\frac{1}{n}\right)^0 + \binom{n}{1}\cdot 1^{n-1}\cdot \left(\frac{1}{n}\right)^1 + \binom{n}{1}\cdot 1^{n-2}\cdot \left(\frac{1}{n}\right)^2 \ldots.$$

Since $\binom{n}{r} = \frac{n!}{r!(n-r)!}$;

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{n!}{1!(n-1)!n} + \frac{n!}{2!(n-2)!n^2} + \frac{n!}{3!(n-3)!n^3} + \frac{n!}{4!(n-4)!n^4} \ldots.$$

It is cool that

$! = (n)(n-1)(n-2)\ldots(1).\ Therefore,\ \frac{n!}{(n-1)!} = \frac{(n)(n-1)(n-2)\ldots(1)}{(n-1)(n-2)\ldots(1)} = n.$

$Likewise, \frac{n!}{(n-2)!} = \frac{(n)(n-1)(n-2)\ldots(1)}{(n-2)\ldots(1)} = n(n-1)$ and $\frac{n!}{(n-3)!} = \frac{(n)(n-1)(n-2)\ldots(1)}{(n-3)\ldots(1)} =$

$n(n-1)(n-3)$

Following this pattern, the series can be written as:

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{n}{n} + \frac{n(n-1)}{2!\,n^2} + \frac{n!(n-1)(n-2)}{3!\,n^3} + \ldots.$$

We can generalize that the k^{th} term (starting with $k = 0$) in this series is equal to:

$$\frac{n(n-1)\ldots(n-k+1)}{k!\,(n^k)}$$

To clarify, the numerator follows the pattern:

$n(n-1)(n-2)(n-3)\ldots(n-x)$, where $x=(1-k)$. *one less than k.

$$\frac{1}{k!}\times\frac{n(n-1)\ldots(n-k+1)}{n^k}$$

In the expression above, $\frac{1}{k!}$ can be separated from the limit by factoring it out.

It seems that for any term k, the expression $\lim_{n\to\infty}\frac{n(n-1)\ldots(n-k+1)}{n^k}$ approaches 1. The following table shows this calculation results from my from GDC (where n is the large number 10^{10}.).

input	K=1	K=2	K=5	K=6	K=7	K=10
output	1	1	1	1	1	1

Since $\lim_{n\to\infty}\frac{n(n-1)\ldots(n-k+1)}{n^k}\approx 1$:

$$\frac{1}{k!}\lim_{n\to\infty}\frac{n(n-1)\ldots(n-k+1)}{n^k}=\frac{1}{k!}$$

Since this is true for all terms in the series, we get the following:

$$\left(1+\frac{1}{n}\right)^n=\sum_{k=0}^{\infty}\frac{1}{k!}$$

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}$$

This was a rather messy way of approaching the summation definition. I particularly dislike the step: $\lim_{n\to\infty}\frac{n(n-1)\ldots(n-k+1)}{n^k}$, where I calculated for nervous values of k to arrive at 1.

Conclusion

I have looked at three major definitions of e:

1) $e = lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

2) $e = \lim_{x\to 0}(1+n)^{\frac{1}{n}}$

3) $e = \sum_{k=0}^{\infty}\frac{1}{k!}$

Additionally, I explored the derivative of $f(x) = e^x$, and analyzed how $A = Pe^{rt}$ works. However, there is a lot more to be learned about e. In further investigations, I wish to analyze the summation series definition of e by using Taylor series expansion. Applying this will allow me to use the derivative of $f(x) = e^x$, which is e^x, to arrive at the summation definition. I think is a more elegant method than arriving at the summation definition than using the Binomial Theorem. After that, I could finally work towards understanding Euler's identity:

$$e^{i\pi} + 1 = 0$$

$$e^{ix} = \cos(x) + i\sin(x).$$