# Time-varying feedback stabilization of nonholonomic car-like mobile robots



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# STABILIZATION PAR RETOUR D'ETAT INSTATIONNAIRE DE ROBOTS MOBILES NON-HOLONOMES DE TYPE VOITURE

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## Abstract

Many nonholonomic mechanical systems, such as car-like mobile robots, are controllable but cannot be stabilized to given positions and orientations by using smooth pure-state feedback control. However, as shown in [18], such systems may still be stabilized by using smooth *time-varying* feedbacks, i.e. feedbacks which explicitly depend on the time-variable. This possibility is here applied to the stabilization of a class of nonlinear systems whose equations encompass simple car models. A set of stabilizing smooth time-varying feedbacks is derived and simulation results are given.


**Key words:** mobile robots, nonholonomy, nonlinear systems, controllability, feedback stabilization.


## Résumé

Un grand nombre de systèmes mécaniques non-holonomes, dont les robots mobiles de type voiture, sont commandables mais ne peuvent être stabilisés à une position/orientation désirée en utilisant des commandes par retour d'état continues. Cependant, comme il est montré dans [18], la stabilisation de tels systèmes par retour d'état continu *instationnaire*, c'est à dire dépendant explicitement de la variable temporelle, demeure possible. Cette possibilité est ici appliquée à la stabilisation d'une classe de systèmes non-linéaires dont les équations recouvrent des modèles simples de voiture. Des retours d'état différentiables, instationnaires et stabilisants sont proposés, et l'etude est illustrée par des résultats de simulation.


**Mots clés:** robots mobiles, non-holonomie, systèmes non-linéaires, controlabilité, commande par retour d'état.

# Contents

# 1 Introduction

Path planning for nonholonomic mobile robots is currently a subject of active investigation. Feedback control of mobile robots is another aspect of the general robot control theme. It is also motivating an increasing number of studies and publications in the Robotics and Automatic Control communities [1]-[18]. While the topic may be seen as a logical extension of the much studied case of holonomic robot manipulators, it turns out that this extension is not as straightforward as it may first seem. The basic difference between the two types of systems may be summarized as follows: although mobile robots subject to classical nonholonomic constraints are usually (when the number of actuators is equal to the number of degrees of freedom) completely controllable in their configuration space [1] [4] [19]-[24], they cannot, unlike robot manipulators, be stabilized to a desired configuration by using smooth state feedback controls [1] [4] [16].

The case where stabilizing smooth state-feedbacks do not exist , is a well known problem in nonlinear control theory [25] [28], and it usually steers the conclusion that discontinuous feedback is the alternative. This idea has been exploited in [2], where a discontinuous feedback strategy is proposed to stabilize a knife edge moving in point contact on a plane surface. A different discontinuous feedback may be found in [5]. However, there is another alternative which consists of considering a possible dependency of the feedback control law on the exogeneous time index. This possibility, briefly studied in [18] to stabilize a cart equipped with two independent motorized wheels on the same axis, is here applied to a car-like mobile robot with a steering wheel. As in this previous work, the method used to derive a set of globally stabilizing *time-varying* smooth feedback controls is based on the finding of an adequate time-varying Lyapunov-like function, while the proof of convergence borrows from Lasalle's techniques [26].

As explained in [17], the stabilizability problem evoked above becomes less relevant when the nature of the task to be performed does not require the mobile robot to stop moving in the end. For this reason, the feedback stabilization problem treated in this article should not to be confused with path tracking problems studied in other works ([3] [6] [8] [9] [10] [11] [12] [15]...). It seems that smooth feedback stabilization of car-like mobile robots was first solved theoretically in [30] and that the feedback control laws derived here are original.

The paper is organized as follows.

In Section 2, smooth feedback stabilization of a class of nonlinear systems whose equations encompass simple car models is studied. In particular, it is shown how, for a given set of feedback controls, the controllability property of the system may interact with control time dependency in order to broaden the set of variables that can be stabilized to zero. Sufficient conditions under which asymptotic convergence of the whole state vector to zero is achieved are then given.

Application to car-like mobile robots is discussed in Section 3. The control inputs are assumed to be the car advancement and steering angle velocities. It may be useful to precise that this choice of the control inputs is partly justified by the fact that we are here more concerned with conceptual new possibilities than with realistic modelling and implementation aspects. These will be the subject of other studies. For example, the more realistic case where the control variables are the torques applied to the motorized wheels, has already been considered in [18] for the two-wheel driven cart.

The proposed *time-varying* feedbacks have been tested in simulation, and experimental results are reported and commented upon in Section 4.

Finally, some conclusions are given in Section 5.

# 2 Smooth time-varying feedback stabilization of a class of nonlinear systems

From now on, and for simplicity of exposition, all functions will be taken in the set, denoted as $S$, of analytic matrix valued functions $f(.,t)$ defined on $R^l \times R^+$ ($l \in N$), uniformly bounded with respect to the independent time variable $t$, and with successive partial derivatives also uniformly bounded with respect to $t$.

The following notation will be used also:

$$\frac{\partial^p f}{\partial x_i{}^p}(x_1, ..., x_{l+1}) = f_{x_i}{}^{(p)}(x_1, ..., x_{l+1}) \qquad for\ p \geq 1$$

and the index $p$ will often be omitted when $p = 1$.

Let us then consider a system that satisfies the following equations:

$$\begin{aligned} \dot{x}_1 &= u_1(1 + a(x_1, x_3)x_2) \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= c(x_1, x_3)u_1x_2 \end{aligned} \qquad (1)$$

where $dim(x_1) = dim(x_2) = dim(u_1) = dim(u_2) = 1$, $dim(x_3) = m$, $a(x_1, x_3)$ is a scalar function in $S$, and $c(x_1, x_3)$ is an $m$-dimensional vector function in $S$.

$X = [x_1, x_2, x_3]^T$ is the system's state vector of dimension $(m + 2)$, and $U = [u_1, u_2]^T$ is the two-dimensional system's control vector. Obviously, $(X = 0, U = 0)$ is an equilibrium point of the system. Since the dimension of the control vector is smaller than the dimension of the state vector, it is also clear that when linearizing the system about this equilibrium point, the obtained linearized system is not controllable. However, the nonlinear system 1 may still be controllable, if, as stated in following lemma, the function $c(x_1, x_3)$ satisfies some condition.

**Lemma 2.1**
*The system 1 is completely controllable on $R^{m+2}$ if and only if, for all $X$ in $R^{m+2}$:*

$$rank\ [c, c_{x_1}, ..., c_{x_1}{}^{(p)}, ...](X) \ = \ m \tag{2}$$

The proof of this lemma is given in Appendix A.

Let us now consider the problem of stabilizing the state vector $X$ about zero by using smooth feedback controls.

It is known [1] [4] [16], by application of a Brockett's theorem [25], that any system in the form: $\dot{X} = B(X)U$, with $dim(X) > dim(U)$ (such as system 1), is not stabilizable by using smooth feedbacks that depend on the state vector $X$ only. However, it has been shown in [18] that such a system may still be stabilizable by using smooth *time-varying* feedbacks, i.e. feedbacks which also depend on the independent time variable $t$. The system considered in [18] was the model of a classical mobile cart with two independent actuated wheels mounted on a common axis. This result will here be extended to the class of systems 1 and stabilizing smooth time-varying feedbacks will be derived altogether. The role of controllability in the existence of such controls will also be pointed out. The results will then be applied to a car-like mobile robot equipped with a steering wheel.

**Lemma 2.2**

*Let $k(x_3, t)$ denote a scalar function in $\mathcal{S}$.*
*If the following control:*

$$\begin{aligned} u_1 &= -k_t - g_1(x_1 + k) \\ u_2 &= -u_1[(x_1 + k)(a + c^T k_{x_3}) + c^T x_3] - g_2 x_2 \end{aligned} \tag{3}$$

*(where $g_1(X, t)$ and $g_2(X, t)$ are positive scalar functions in $\mathcal{S}$ bounded from below by some positive number when $X$ remains in a compact set)*
*is applied to the system 1, then:*
*i) The Lyapunov-like function:*

$$V(X, t) \ = \ 1/2((x_1 + k)^2 + x_2{}^2 + ||x_3||^2) \tag{4}$$

*is non-increasing;*
*ii) $(x_1 + k)$, $x_2$, $\dot{x}_3$ and $k_t c^T x_3$ asymptotically converge to zero.*

*Moreover, if the system is controllable, then $k_t{}^{(p)} x_3$ $(p \geq 1)$ tends to zero.*

The proof of this lemma is given in Appendix B.

An interest of this lemma is to point out how the introduction of the function $k(x_3, t)$ may influence the convergence properties of the controlled system. For instance, if $k = 0$, then $x_1$ and $x_2$ converge to zero. However, only convergence of $||x_3||$ to some limit value (not necessarily equal to zero) is obtained in this case. This illustrates the fact that the dimension of the attracting manifold, to which the system's trajectories converge asymptotically, cannot be smaller than $m$ when using smooth time-invariant feedbacks $U(X)$.

In order to have $x_3$ converge to zero, the last part of the lemma suggests of taking advantage of the system's controllability by considering time-dependent functions $k$. The superiority of time-varying feedbacks over time-invariant feedbacks comes from there.

We are now in position of deriving sufficient conditions, bearing upon the choice of the function $k(x_3, t)$, for the convergence of the whole state vector to zero.

**Lemma 2.3**

*Assuming that the system 1 is controllable and that the control 3 is used, if there is a diverging time-sequence $t_i$ ($i \in N$), an integer $p$ and a continuous function $\alpha : R^+ \to R^+$ such that:*

- $$||x_3||^2 \geq l \Rightarrow \sum_{j=1}^{p} (k_t^{(j)}(x_3, t_i))^2 \geq \alpha(l) \quad , \forall i \tag{5}$$

- $$l \neq 0 \Rightarrow \alpha(l) > 0 \tag{6}$$

*then $x_3$ tends to zero.*

*Moreover, if $k(0,t) = 0$ ($\forall t$), then $x_1$ tends to zero.*

The proof of this lemma is given in Appendix C.

A few examples of functions $k(x_3, t)$ that satisfy the properties 5 and 6 of the lemma are given next:

- Any bounded time-function $k(t)$ in S such that $k_t^{(j)}(t)$ does not tend to zero for some $j \geq 1$. Indeed, properties 5 and 6 are in this case satisfied by definition of non-convergence and by taking $\alpha$ equal to a small enough positive number.

  According to the lemma, $x_3$ then tends to zero. However, $x_1$ does not since $k(t)$ cannot tend to zero in this case.

  Notice that any analytic periodic function works, although periodicity is not necessary.

- $k(x_3,t) = sin(t)||x_3||^2$
  In this case we have $k_t(x_3,t) = cos(t)||x_3||^2$ and $k_t(x_3,2i\pi) = ||x_3||^2$. Properties 5 and 6 are thus satisfied by taking $t_i = 2i\pi$ and $\alpha(l) = l^2$. Moreover, $x_1$ tends to zero since $k(0,t) = 0$.

- $k(x_3,t) = \sum_{i=1}^{m} a_i sin(\beta_i t)x_{3,i} \quad with\ a_i \neq 0,\ \beta_i \neq 0.\ and\ |\beta_i| \neq |\beta_j|\ when\ i \neq j$
  $x_{3,i}$ denoting the $i$th component of the vector $x_3$.

  The satisfaction of properties 5 and 6 in this case is proved in Appendix D. Since $k(0,t) = 0$, convergence of $x_1$ to zero is also ensured.

## 3 Application to a car-like mobile robot

A view from above of the considered cart, a three-wheeled vehicle, is depicted in **Fig.1**.

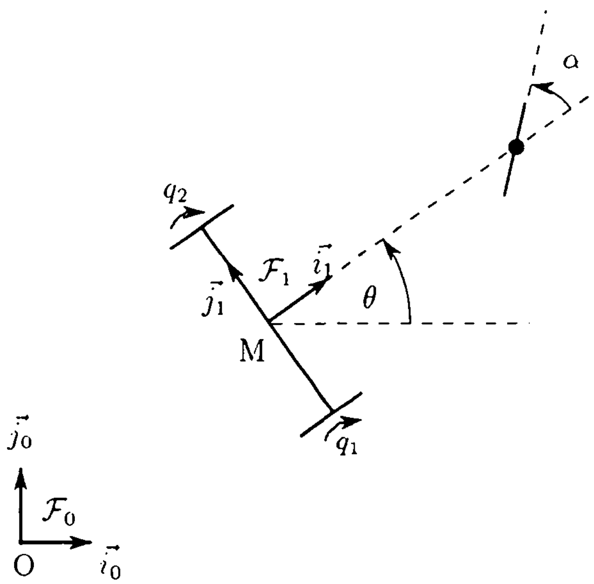


**Fig.1** : Cart's parameters

It is assumed that the cart moves on a horizontal ground. The model equations of the cart's motion are derived under the usual rolling-without-slippage assumption.

They are thus simply obtained by expressing the fact that the point of each wheel in contact with the ground has zero velocity.

The following notations are used:

- $M$: the cart's point located at mid-distance of the two rear wheels
- $r$: radius of the steering wheel
- $d$: distance between $M$ and the steering wheel
- $\mathcal{F}_0 = (O; \vec{i}_0, \vec{j}_0)$: a fixed reference frame such that the point $M$ belongs to the plane $(O; \vec{i}_0, \vec{j}_0)$
- $\mathcal{F}_1 = (M; \vec{i}_1, \vec{j}_1)$: a frame rigidly linked to the cart
- $\alpha$: the steering wheel angle with respect to the cart's body
- $\omega$: the front wheel advancement angular velocity
- $\theta$: the angle $(\vec{i}_0, \vec{i}_1)$ which characterizes the cart's orientation with respect to the fixed frame $\mathcal{F}_0$
- $x$, $y$: the coordinates of the vector $\vec{OM}$ in the basis of the frame $\mathcal{F}_1$
- $v$: the cart's advancement velocity $(\frac{d\vec{OM}}{dt} = v\vec{i}_1)$

Under the rolling-without-slippage assumption, the following equations hold (see [10] [16] [18] for example):

$$\begin{aligned} \dot{x} &= \dot{\theta} y + v \\ \dot{y} &= -\dot{\theta} x \end{aligned} \tag{7}$$

In the case where the rear wheels are motorized and the front wheel is a free rotating wheel, it is possible to take the rear wheels' angular velocities $\dot{q}_i$ $(i = 1,2)$ as the control variables. Moreover, since $(v, \dot{\theta})^T = D(\dot{q}_1, \dot{q}_2)^T$, where $D$ is a known nonsingular matrix that depends on the wheels' radius and the distance between the wheels [17], it is equivalent to take $v$ and $\dot{\theta}$ as the control variables. This cart is *completely controllable* in the variables $x$, $y$ and $\theta$, as it may be shown either by expliciting simple open-loop control strategies [21] or by applying classical techniques of nonlinear control theory [16]. However there is no *stabilizing* smooth (or even continuous) feedback control, depending on these variables, able to make them converge to zero whatever the initials conditions. Nevertheless, stabilizing *non-stationary* smooth feedbacks exist and have been derived in [18].

A difference between this case and the car-like case where the front steering wheel is motorized is that $\dot{\theta}$ no longer corresponds to a physical control variable. For instance, the following equations:

$$\begin{aligned} v &= r\omega \, \cos\alpha \\ \dot{\theta} &= r\omega/d \, \sin\alpha \end{aligned} \tag{8}$$

show that $\dot{\theta}$ is equal to zero when either the front wheel advancement velocity $\omega$ or the steering angle $\alpha$ is equal to zero. As a consequence, $\dot{\theta}$ can no longer be considered as an independent control variable.

From 8, we also have:

$$\tan\alpha = d\dot{\theta}/v \tag{9}$$

and it may be tempting to set the steering angle $\alpha$ as a control variable which can be computed from values determined for $v$ and $\dot{\theta}$. However, this solution is not very satisfactory for at least three reasons: i) the steering angle $\alpha$ is not determined when $v$ is equal to zero, ii) the range of variation of this angle is often mechanically limited (such as on a car), and iii) due to dynamical effects, monitoring of $\alpha$ (which may for example correspond to the position of the rotor of an electric motor) can never be performed instantaneously. For these reasons, the choice of $\dot{\alpha}$, as a control variable, is already more satisfying. Moreover, velocity regulation of an electric motor is a rather standard technique in Robotics since well known *velocity control* schemes have been developped for robot manipulators [31]. As a matter of fact, it would be even better to consider the second derivative of $\alpha$ which is directly related to the torque applied to the motor. But this would involve additional complexities, linked to the derivation and use of a dynamical model of the vehicle, out of the scope of this paper.

By regrouping equations 7-9, we obtain the following model of the car-like robot:

$$\begin{aligned} \dot{x} &= v(1 + (1/d)y \; tan(\alpha) \\ \dot{y} &= -(1/d)vx \; tan(\alpha) \\ \dot{\theta} &= (1/d)v \; tan(\alpha) \\ \frac{d}{dt} \, tan(\alpha) &= (1 + tan(\alpha)^2)\dot{\alpha} \end{aligned} \tag{10}$$

and it is simple to verify that these equations are a particular case of system 1 considered in the previous section, with:

$$
\begin{aligned}
x_1 &= x \\
x_2 &= {g_3}^{1/2}\, tan(\alpha) \\
x_3 &= \begin{bmatrix} {g_4}^{1/2} y \\ {g_5}^{1/2} \theta \end{bmatrix} \\
u_1 &= v \\
u_2 &= {g_3}^{1/2}(1 + tan(\alpha)^2)\dot{\alpha} \\
a(x_1, x_3) &= \frac{1}{d {g_3}^{1/2} {g_4}^{1/2}} x_{3,1} \\
c(x_1, x_3) &= \begin{bmatrix} -\frac{{g_4}^{1/2}}{d {g_3}^{1/2}} x_1 \\ \frac{{g_5}^{1/2}}{d {g_3}^{1/2}} \end{bmatrix}
\end{aligned} \tag{11}
$$

$g_i$ ($i = 3, 4, 5$) are positive numbers introduced to offer extra freedom in the control gains, as it will appear more clearly when expliciting the control expression.

By choosing $tan(\alpha)$, rather than $\alpha$, in the state vector and $v$, rather than $\omega$, in the control vector, it is implicitly assumed that $\alpha$ must be kept in the range $]-\pi/2, +\pi/2[$. However, this is not a theoretical obligation and $\alpha$ and $\omega$ may also be used without $|\alpha|$ having to be smaller than $\pi/2$ (see [30] where this case is treated).

From Lemma 2.1, this system is controllable on $R^4$ since the matrix:

$$
[c, c_{x_1}] = \begin{bmatrix} -\frac{{g_4}^{1/2}}{d {g_3}^{1/2}} x_1 & -\frac{{g_4}^{1/2}}{d {g_3}^{1/2}} \\ \frac{{g_5}^{1/2}}{d {g_3}^{1/2}} & 0 \end{bmatrix} \tag{12}
$$

is always nonsingular.

The associated time-varying feedback control, proposed in Lemma 2.2 and used in the simulation experiments reported in the next section, becomes in this case:

$$
\begin{aligned}
v &= -k_t - g_1(x + k) \\
\dot{\alpha} &= -\frac{1}{1 + tan(\alpha)^2}\left[\frac{v}{d g_3}((x + k)(y - k_y x + k_\theta) + g_4 x y + g_5 \theta) + g_2 tan(\alpha)\right]
\end{aligned} \tag{13}
$$

where $k(y, \theta, t)$ is a function in $\mathcal{S}$, chosen so as to satisfy the conditions of Lemma 2.3 in order to ensure the asymptotic convergence of $x$, $y$, $\theta$ and $\alpha$ to zero.

# 4 Simulations

In the experiments reported here, the vehicle parameters $d$ and $r$ have been set equal to 0.5 and 0.1 respectively.

A real difficulty when trying to implement a nonlinear control such as 13 is to determine the elements in the control that are left free by the stability analysis.

For instance, one has to determine the "control gains" $g_i$ ($i = 1, ..., 5$) and the function $k(y, \theta, t)$. Depending on the choices that are made, the control may either be of practical value or behave poorly, since stability does not necessarily mean good performance. The difficulty of connecting these choices with classical linear control theory techniques is certainly one of the main shortcomings of the control design proposed in this paper. However, simple guidelines may still be followed. For instance, uniform boundedness of the advancement velocity $v$ and of the steering angle $\alpha$ is probably desirable in most practical applications. This can be achieved through the choice of the functions $g_1$, $g_2$ and $k$.

The following functions have been used:

$$k(y, \theta, t) = k_{max} \frac{g_4 y^2 + g_5 \theta^2}{g_4 y^2 + g_5 \theta^2 + 10^{-3}} sin(t) \qquad k_{max} > 0 \tag{14}$$

in order to i) satisfy the conditions of Lemma 2.3, ii) have $|k|$ and $|k_t|$ smaller than $k_{max}$ and limit the car's velocity to $\simeq k_{max}$ when $(x + k)$ is close to zero, and iii) prevent $k$ (and $x$) from getting small too quickly when $y$ and $\theta$ have not yet converged.

$$g_1(x, y, \theta, t) = g_6 / ((x + k)^2 + 1)^{1/2} \qquad g_6 > 0 \tag{15}$$

so as to have an advancement velocity $|v|$ smaller than $(k_{max} + g_6)$. In the simulations, $k_{max}$ and $g_6$ have been set equal to 1 and 2 respectively.

$$g_2(x, y, \theta, \alpha, t) = \frac{1}{d g_3 \; tan(\alpha_{max})} (v^2 + 10^{-4})^{1/2} \left[((x+k)(y - k_y x + k_\theta) - g_4 x y + g_5 \theta)^2 + 1\right]^{1/2} \tag{16}$$

so as to maintain the steering angle $\alpha$ in the range $]-\alpha_{max}, +\alpha_{max}[$. In the simulations reported here, $\alpha_{max}$ has been set equal to the small value 0.1 $rd$ so as to demonstrate some of the control possibilities.

Saturation of the steering angle velocity $\dot{\alpha}$ at $\pm 1$ $rd/s$ has been tried also. This did not affect the control performance significantly. In the present simulations, no saturation has been used.

After a few trials, the constant gains $g_3$, $g_4$ and $g_5$ have been set equal to 5, 1 and 0.1 respectively. At this point of the study, no systematic procedure has been worked out to choose these gains.

Four experiments, corresponding to various initial conditions, are reported.

In the first one, the vehicle's configuration at time $t = 0$ is: $x = 0$, $y = 1$, $\theta = 0$, $\alpha = 0$.

- **Fig.2** shows the motion of the point M in the fixed frame $\mathcal{F}_0$ during a period of 100 seconds. The back and forth motion of the vehicle is not without recalling maneuvers that are performed when parking a car. The motion's amplitude gets continuously smaller when $y$ and $\theta$ get closer to zero.

- **Fig.3** shows the evolution of the variable $x$. Asymptotic convergence to zero is slow.
- **Fig.4** and **Fig.5** show the evolution of $y$ and $\theta$ respectively.
- **Fig.6** shows the evolution of the steering angle $\alpha$. Little steering is performed during the final convergence phase. This explains the slow asymptotic convergence rate and it is certainly one of the control weaknesses. Notice that the imposed bound of 0.1 $rd$ is respected.
- **Fig.7** shows the evolution of the vehicle's advancement velocity $v$, the amplitude of which remains significantly below the imposed limit of 3 $m/s$.
- **Fig.8** shows the evolution of the steering angle velocity $\dot{\alpha}$. The sharpest peaks occur when the advancement velocity $v$ passes through zero, while the other peaks occur when $|v|$ reaches a local maximum.
- The monotonic decreasing of the Lyapunov function $V(X, t)$ used in the control design can be observed in **Fig.9**.

The initial conditions of the three other experiments are respectively:

- $x = \theta = \alpha = 0$, $y = 0.1$ (**Fig.10**);
- $x = \theta = \alpha = 0$, $y = 10$ (**Fig.11**);
- $x = y = \alpha = 0$, $\theta = \pi$ (**Fig.12**).

These simulations illustrate how asymptotic convergence to zero is achieved by the proposed control law. In this respect, they validate the concept of smooth time-varying feedback stabilization applied to a car-like mobile robot.

On the other hand, they point out some of the control shortcomings such as slow convergence near the origin and the difficulty of choosing various control design parameters. Concerning this last point, it should be possible to better use the linearized part of the system which is controllable by connecting the stabilization problem treated here with more classical path tracking problems.

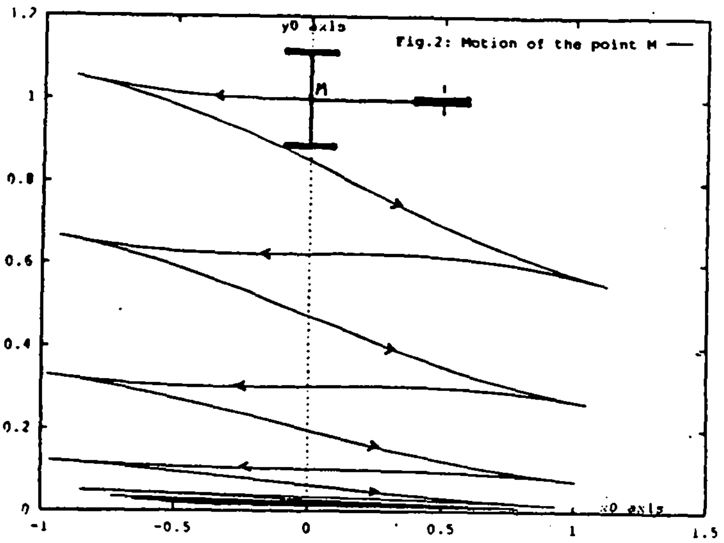
Fig.2: Motion of the point M
y0 axis
x0 axis
M

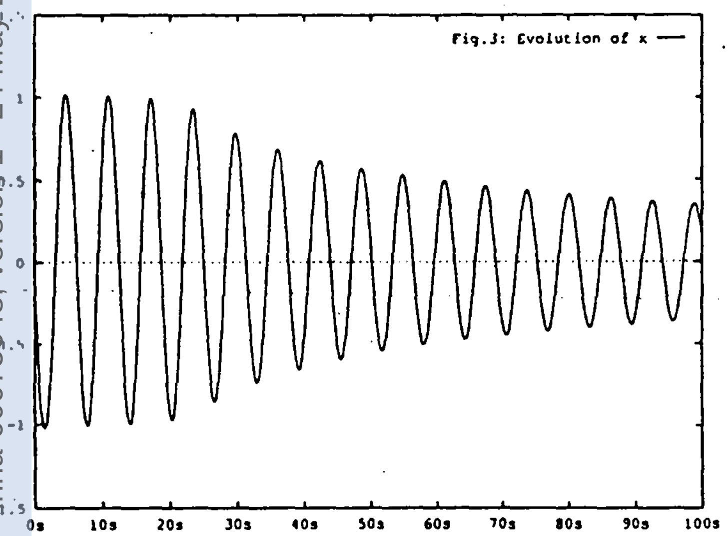
Fig.3: Evolution of x

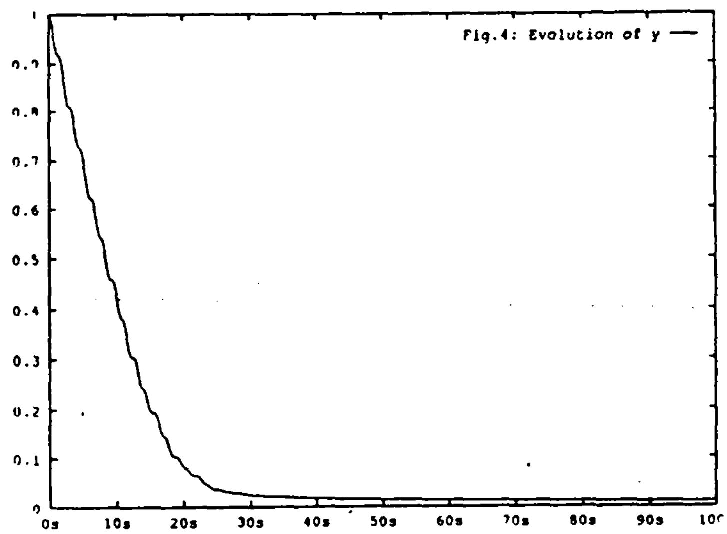
Fig.4: Evolution of y

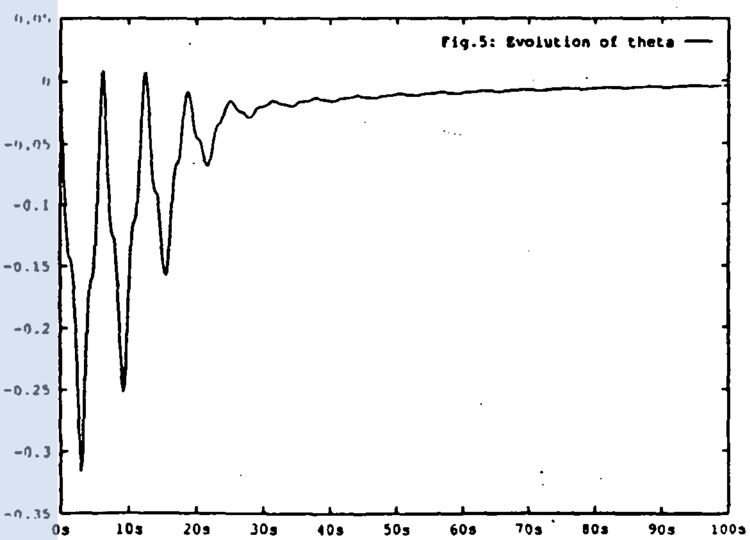
Fig.5: Evolution of theta

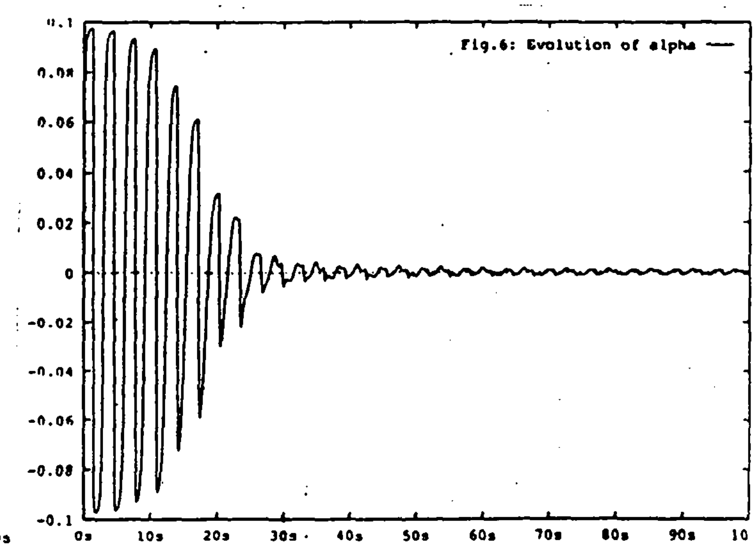
Fig.6: Evolution of alpha

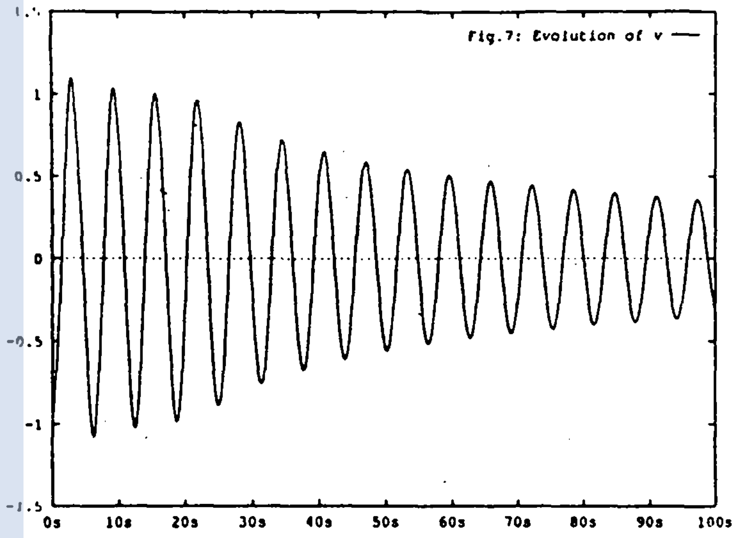
Fig.7: Evolution of v
1
0.5
0
-0.5
-1
0s 10s 20s 30s 40s 50s 60s 70s 80s 90s 100s

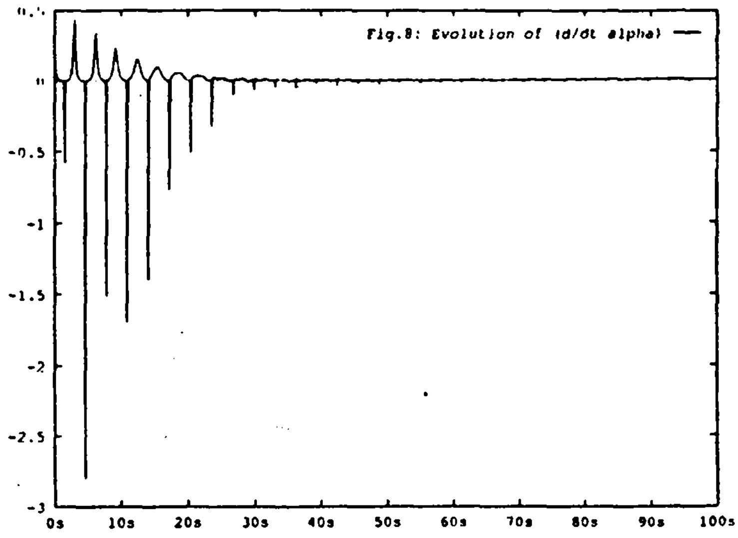
Fig.8: Evolution of (d/dt alpha)
-0.5
-1
-1.5
-2
-2.5
-3
0s 10s 20s 30s 40s 50s 60s 70s 80s 90s 100s

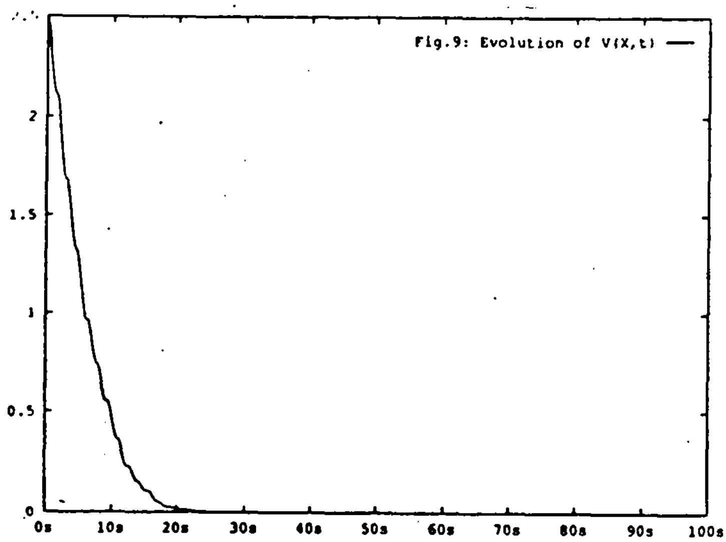
Fig.9: Evolution of V(X,t)
2
1.5
1
0.5
0
0s 10s 20s 30s 40s 50s 60s 70s 80s 90s 100s

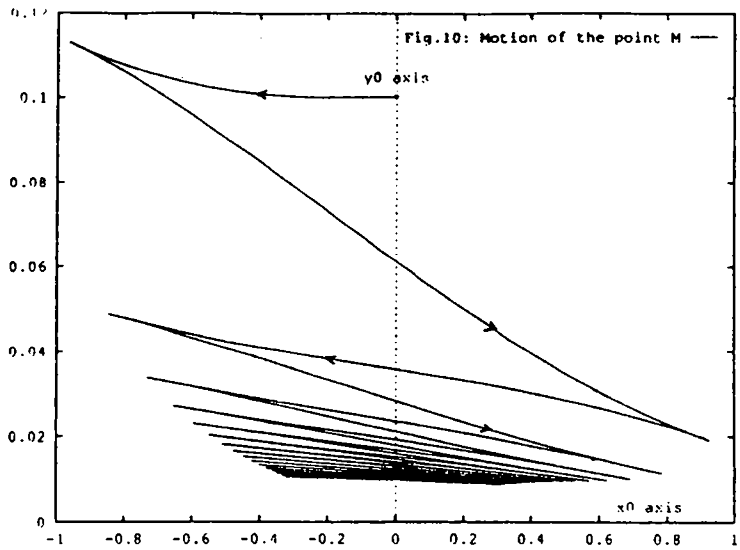
Fig.10: Motion of the point M
y0 axis
x0 axis

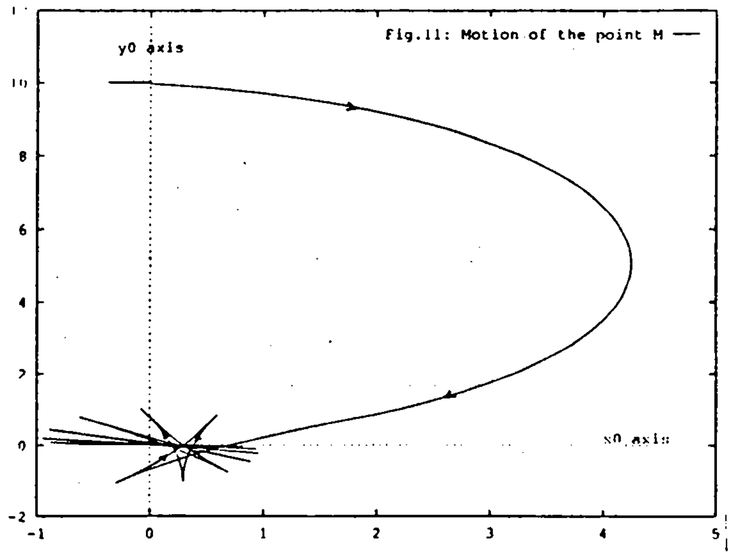
Fig.11: Motion of the point M
y0 axis
x0 axis

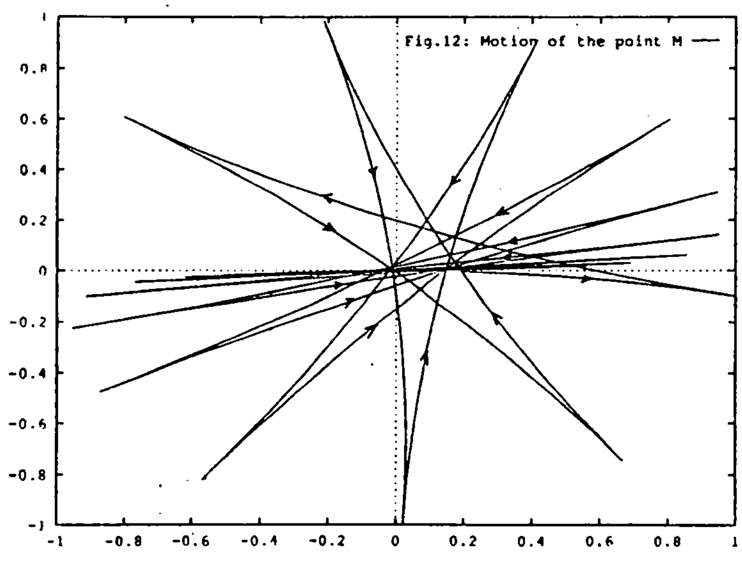
Fig.12: Motion of the point M

## 5 Conclusion

Most mobile robots encountered in practice are, under normal working conditions, controllable without being stabilizable by using smooth pure-state feedback. Existence of smooth *time-varying* feedbacks able to stabilize such systems has been demonstrated in [18] in the case of a cart actuated by two independent wheels on a common axis. This initial result has steered some research effort within the nonlinear control community in order to embed it in a more general framework. Concerning this issue, we are aware of quite interesting results obtained by various authors (Coron, Pomet, Sallet, Sépulchre,...), yet unpublished at the time this article is written. However, more time and work will probably be necessary to explore all aspects and potential applications of this type of feedback.

In this report, a set of smooth time-varying feedbacks for a class of nonlinear systems encompassing simple car models, and sufficient conditions for asymptotic convergence of the state vector to zero have been derived. Although the control design technique proposed here is not general, it applies to different models of nonholonomic mobile robots. How to embed it in a more general theory should prove to be an exciting research topic.

Some simulation results have been given. While showing the applicability of the time-varying feedback concept to the control of a car-like mobile robot, they also point out the importance of improving the method, possibly by better using the controllable linearized part of the system, in order to obtain control schemes of practical value for the engineer.

### Appendix A: proof of Lemma 2.1

The system 1 is completely controllable on $R^{(m+2)}$ if and only if for any $X$ in $R^{(m+2)}$ there are vector fields $f_1, ..., f_p$ in the Lie algebra generated by the system vector fields:

$$b_1(X) = \begin{bmatrix} 1 + a(x_1, x_3)x_2 \\ 0 \\ c(x_1, x_3)x_2 \end{bmatrix} \quad and \quad b_2(X) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{17}$$

such that: $rank[f_1, ..., f_p](X) = m + 2$ [27][29].

Let us then determine vector fields, other than $b_1$ and $b_2$, in the accessibility Lie algebra associated with the system 1.

$$b_3 = [b_2, b_1] = \begin{bmatrix} a \\ 0 \\ c \end{bmatrix}$$

$$b'_1 = b_1 - x_2 b_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$b'_3 = b_3 - ab'_1 = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

$$Ad_{b'_1}{}^1 b'_3 = \begin{bmatrix} 0 \\ 0 \\ c_{x_1} \end{bmatrix}$$

$$Ad_{b'_1}{}^p b'_3 = \begin{bmatrix} 0 \\ 0 \\ c_{x_1}^{(p)} \end{bmatrix} \qquad for\ p \geq 1$$

We also have:

$[Ad_{b'_1}{}^p b'_3, b_2] = 0 \qquad for\ p \geq 0$

Therefore, any vector field generated by $b_1$ and $b_2$ is a linear combination of $b'_1$, $b_2$, $b'_3$, $Ad_{b'_1}{}^1 b'_3$,..., $Ad_{b'_1}{}^p b'_3$,..., and the rank condition of lemma 2.1 follows directly.

## Appendix B: proof of Lemma 2.2

Take the function $V(X,t)$ considered in the lemma, and derive it with respect to time. By using the system's equations 1, one obtain after simple computations:

$$\dot{V} = (u_1 + k_t)(x_1 + k) + x_2[u_1((x_1 + k)(a + c^T k_{x_3}) + c^T x_3) + u_2] \qquad (18)$$

and, by using the control law expression 3:

$$\dot{V} = -g_1(x_1 + k)^2 - g_2 x_2{}^2 \quad (\leq 0) \qquad (19)$$

$V$ is thus non increasing. This implies uniform boundedness of $||X(t)||$ with respect to initial conditions, and existence of the solutions over $R^+$. It also implies boundedness of all the time-derivatives of $X(t)$ and boundedness of the values taken by all functions in $\mathcal{S}$ that depend on the variables $X$ and $t$. In particular, $\ddot{V}$ is bounded and the convergence of $V$ to some limit value, denoted as $V_{lim}$, then implies the convergence of $\dot{V}$ to zero.

Now, since $g_1$ and $g_2$ are bounded from below by some positive number, the convergence of $\dot{V}$ to zero implies, in view of 19, that $(x_1 + k)$ and $x_2$ tend to zero.

As a function in $\mathcal{S}$, the control $U(X,t)$ is bounded along the system's trajectories, and the convergence of $x_2$ to zero thus implies, in view of the last system's equation, the convergence of $\dot{x}_3$ to zero.

Since time-derivatives of any order are bounded, the time-derivative of any variable which tends to a constant converges to zero. In particular $\dot{x}_2$ tends to zero, and so does $u_2$ from the second system's equation.

Since $(x_1 + k)$ tends to zero, $(u_1 + k_t)$ tends to zero according to the expression of $u_1$ in 3. It then results from the expression of $u_2$ in 3, and the convergence of $(x_1 + k)$, $x_2$ and $u_2$ to zero, that $k_t c^T x_3$ tends to zero.

This is as far as one can go without using the controllability of the system. Let us thus assume from now on that the system is controllable, meaning that $c(x_1, x_3, t)$ satisfies the condition 2.

By deriving $k_t c^T x_3$ and using the fact that $\dot{x}_3$ and $(\dot{x}_1 + k_t)$ tend to zero, one obtains that $k_t^{(2)} c^T x_3 - (k_t)^2 c_{x_1}{}^T x_3$ tends to zero. By premultiplying this last function by $k_t$, and substracting the obtained function from $k_t c^T x_3$, itself premultiplied by $k_t^{(2)}$, it comes that $(k_t)^3 c_{x_1}{}^T x_3$ tends to zero. Because of the boundedness of $c_{x_1}{}^T x_3$, this in turn implies that $k_t c_{x_1}{}^T x_3$ tends to zero.

In the same way, it can be shown inductively that $k_t c_{x_1}{}^{(p)T} x_3$ $(p \geq 1)$ tends to zero. Therefore, $k_t x_3{}^T [c, c_{x_1}, ..., c_{x_1}{}^{(p)}, ...]$ tends to zero, and, in view of the controllability condition 2, this implies that $k_t x_3$ tends to zero.

Now, by deriving $k_t x_3$ as many times as necessary and using the convergence of $\dot{x}_3$ to zero, it comes that $k_t{}^{(p)} x_3$ $(p \geq 1)$ tends to zero.

**Appendix C: proof of Lemma 2.3**

From the convergence (previously established) of $(x_1 + k)$ and $x_2$ to zero, and the convergence of $V$ to some limit $V_{lim}$, it results that $||x_3||^2$ converges to $2V_{lim}$. Therefore, there is a time $t_{min}$ such that:

$$t > t_{min} \ \Rightarrow \ ||x_3(t)||^2 \geq V_{lim} \tag{20}$$

and, by using the property 5 in Lemma 2.3:

$$t_i > t_{min} \ \Rightarrow \ \sum_{j=1}^{p}(k_t{}^{(j)}(x_3(t_i), t_i))^2 ||x_3(t_i)||^2 \geq \alpha(V_{lim})V_{lim} \tag{21}$$

On the other hand, from the convergence of $k_t{}^{(j)}x_3$ $(j \geq 1)$ to zero established in Lemma 2.2, we have also:

$$lim_{t \to \infty} \sum_{j=1}^{p}(k_t{}^{(j)}(x_3(t), t))^2 ||x_3(t)||^2 \ = \ 0 \tag{22}$$

Thus:

$$\alpha(V_{lim})V_{lim} \ = \ 0 \tag{23}$$

Assume that $V_{lim} \neq 0$, then $\alpha(V_{lim}) = 0$. But, according to the property 6 in Lemma 2.3, this is possible only if $V_{lim} = 0$. Therefore, 23 is satisfied only if $V_{lim} = 0$, and $x_3$ thus tends to zero.

The conclusion of the second part of the lemma results directly from the uniform continuity, with respect to the variable $t$, of the function $k(x_3, t)$ and from the convergence of $(x_1 + k)$ and $x_3$ to zero.

## Appendix D: properties of the function $k$

The function under consideration is:

$$k(x_3,t) = \sum_{i=1}^{m} a_i sin(\beta_i t) x_{3,i} \qquad with\ a_i \neq 0,\ \beta_i \neq 0,\ and\ |\beta_i| \neq |\beta_j|\ when\ i \neq j \tag{24}$$

We have:

$$[k_t, ..., k_t^{(2i-1)}, ..., k_t^{(2m-1)}] = [a_1 x_{3,1} cos(\beta_1 t), ..., a_m x_{3,m} cos(\beta_m t)] A_1 \tag{25}$$

with:

$$A_1 = \begin{bmatrix} \beta_1 & -\beta_1{}^3 & ... & (-1)^{m+1}\beta_1{}^{2m-1} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \beta_m & -\beta_m{}^3 & ... & (-1)^{m+1}\beta_m{}^{2m-1} \end{bmatrix} \tag{26}$$

By induction, one verifies that:

$$det(A_1) = \begin{cases} \beta_1 \quad if\ m = 1 \\ (\pm 1)(\prod_{i=1}^{m} \beta_i)(\prod_{i>j}(\beta_i{}^2 - \beta_j{}^2)) \quad if\ m \geq 2 \end{cases} \tag{27}$$

$A_1$ is thus nonsingular and 25 implies the existence of a positive number $\gamma_1$ such that:

$$\sum_{i=1}^{m} (k_t{}^{(2i-1)})^2 > \gamma_1 \sum_{i=1}^{m} a_i{}^2 x_{3,i}{}^2 (cos(\beta_i t))^2 \tag{28}$$

In the same way, by considering even-order partial derivatives of $k$, one would prove the existence of a positive number $\gamma_2$ such that:

$$\sum_{i=1}^{m} (k_t{}^{(2i)})^2 > \gamma_2 \sum_{i=1}^{m} a_i{}^2 x_{3,i}{}^2 (sin(\beta_i t))^2 \tag{29}$$

Therefore, by adding the left members of 28 and 29:

$$\sum_{i=1}^{2m} (k^{(i)}{}_t)^2 > inf(\gamma_1, \gamma_2) \inf_{i=1,m} (a_i{}^2) ||x_3||^2 \tag{30}$$

and properties 5 and 6 are satisfied for all $t$ by taking:

$$\alpha(l) = inf(\gamma_1, \gamma_2) \inf_{i=1,m} (a_i{}^2) l \tag{31}$$

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