# MAT300: Homework 5

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**pp. 62-64: 3.2.6, 3.3.2, Q2P 3**

3. Induction
   ***3.2.6***

   Let $x \neq 1$ be a real number. For all $n \in \mathbb{N}$.

   $$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + ... + x^2 + x + 1 \tag{1}$$

   *Proof.* By induction, let

   $$\begin{aligned} P(n) : \frac{x^n - 1}{x - 1} &= x^{n-1} + x^{n-2} + ... + x^2 + x + 1 \\ &= \sum_{i=0}^{n-1} x^i \end{aligned}$$

   **Base Case ($n = 1$):**
   When $P(n = 1)$, the left hand side of the equation equals to

   $$\frac{x^1 - 1}{x - 1} = \frac{(x - 1)}{(x - 1)} = 1$$

   while the right hand side equals to $x^0 = 1$. Thus, both sides are equal and $P(n)$ is true for $n = 1$.

**Induction Step:**
Let $k$ be an arbitrary but fixed element in $N$ and suppose that $P(k)$ is true, i.e. $\frac{x^k-1}{x-1} = \sum_{i=0}^{k-1} x^i$. Then

$$\begin{aligned}
\frac{x^{k+1}-1}{x-1} &= \sum_{i=0}^{k-1} x^i + x^{(k+1)-1} \\
&= (x^{k-1} + x^{k-2} + ... + x^2 + x + 1) + x^k, \textit{ by induction hypothesis} \\
&= x^k + x^{k-1} + x^{k-2} + ... + x^2 + x + 1 \\
&= \sum_{i=0}^{k} x^i \\
&= \sum_{i=0}^{(k+1)-1} x^i
\end{aligned}$$

Thus, $P(n)$ holds for $n = k+1$. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

***3.3.2***

Use complete induction to prove that every natural number can be written as the sum of distinct powers of two.

*Proof.* By complete induction,

$$P(n) : n \geq 1 \textit{ is the sum of distinct power of two}$$

**Base Case ($n = 1$):**

When $n = 1$, $P(1)$ equals to $2^0$. Thus, $P(n)$ holds for $n = 1$.

**Induction Step:**

Suppose $n \geq 1$ be an arbitrary but fixed element in $N$ and suppose that $P(k)$ is true for all $k \leq n$ by induction hypothesis. Then, **consider** $n+1$,

If $n+1$ is even, then there exists an integer $k \leq n$ such that $(n+1) = 2k$. By induction hypothesis, $k$ is the sum of distinct power of 2 and hence so is $(n+1)$.

If $n+1$ is odd, then $n$ is even, and by induction hypothesis is a sum of distinct powers of 2, which does not contain a summand of $2^n = 1$. Therefore, $(n+1) = n + 2^0$ is a sum of distincts powers of 2. Thus, $a_n$ holds for $n = k+1$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

***Q2P: 3***

*Proof.* Principle of Mathematical Induction: Suppose $S$ is an arbitary but fixed subset of $\mathbb{Z}^+$ contains 1, and for every $n \in \mathbb{Z}^+$, if $n \in S$ then $(n+1) \in S$ then $S = \mathbb{Z}^+$

By complete induction, suppose $T$ is an arbitary but fixed subset of $\mathbb{Z}^+$ that has no smallest element. Let $S = \mathbb{Z}^+ \setminus T$. We will prove that $S = \mathbb{Z}^+$, which means $T = \emptyset$.

**Base Step:** Since $T$ has no smallest element, 1 cannot be an element of $T$ so $1 \in S$.

**Induction Step:** Suppose $n \in \mathbb{Z}^+$ is an arbitary but fixed and all $i \leq n$ are elements of $S$, i.e. none of them are in $T$.
Then $(n+1)$ cannot be in $T$ either since it would be the smallest element in $T$ if it was. So $(n+1) \in S$. By the principle of mathematical induction, $S \in \mathbb{Z}^+$ *and* $T = \emptyset$. $\square$