MAT300: Homework 4

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pp. 62-64: 3.2.6, 3.3.2, Q2P 3

3. Induction
   3.2.6

   Let $x \neq 1$ be a real number. For all $n \in \mathbb{N}$.

   $$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + ... + x^2 + x + 1 \tag{1}$$

   Proof. By induction, let

   $$\begin{aligned} P(n) : \frac{x^n - 1}{x - 1} &= x^{n-1} + x^{n-2} + ... + x^2 + x + 1 \\ &= \sum_{i=0}^{n-1} x^i \end{aligned}$$

   Base Case ($n = 1$):
   When $P(n = 1)$, the left hand side of the equation equals to

   $$\frac{x^1 - 1}{x - 1} = \frac{(x - 1)}{(x - 1)} = 1$$

   while the right hand side equals to $x^0 = 1$. Thus, both sides are equal and $P(n)$ is true for $n = 1$.

Induction Step:
Let k be an arbitrary but fixed element in N and suppose that $P(k)$ is true, i.e. $\frac{x^k-1}{x-1} = \sum_{i=0}^{k-1} x^i$. Then

$$\begin{aligned}
\frac{x^{k+1}-1}{x-1} &= \sum_{i=0}^{k-1} x^i + x^{(k+1)-1} \\
&= (x^{k-1} + x^{k-2} + ... + x^2 + x + 1) + x^k, \textit{ by induction hypothesis} \\
&= x^k + x^{k-1} + x^{k-2} + ... + x^2 + x + 1 \\
&= \sum_{i=0}^{k} x^i \\
&= \sum_{i=0}^{(k+1)-1} x^i
\end{aligned}$$

Thus, $P(n)$ holds for $n = k + 1$. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

3.3.2

Use complete induction to prove that every natural number can be written as the sum of distinct powers of two.

Proof. By complete induction,

$$P(n) : n \geq 1 \textit{ is the sum of distinct power of two}$$

Base Case ($n = 1$):

When $n = 1$, $P(1)$ equals to 2^0. Thus, $P(n)$ holds for $n = 1$.

Induction Step:

Suppose $n \geq 1$ be an arbitrary but fixed element in N and suppose that $P(k)$ is true for all $k \leq n$ by induction hypothesis. Then, **consider** $n+1$,

If $n + 1$ is even, then there exists an integer $k \leq n$ such that $(n + 1) = 2k$. By induction hypothesis, k is the sum of distinct power of 2 and hence so is $(n + 1)$.

If $n + 1$ is odd, then n is even, and by induction hypothesis is a sum of distinct powers of 2, which does not contain a summand of $2^n = 1$. Therefore, $(n + 1) = n + 2^0$ is a sum of distincts powers of 2. Thus, a_n holds for $n = k + 1$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Q2P: 2

Formulate and prove a generalization of the Principle of Mathematical Induction in which the base case is an arbitary $m \in \mathbb{Z}$.

Proof. By complete induction,

$$P(n) : n \geq 1 \; is \; the \; sum \; of \; distinct \; power \; of \; two$$

Base Case ($n = 1$):

When $n = 1$, $P(1)$ equals to 2^0. Thus, $P(n)$ holds for $n = 1$.

Induction Step:

Suppose $n \geq 1$ be an arbitrary but fixed element in N and suppose that $P(k)$ is true for all $k \leq n$ by induction hypothesis. Then, **consider** $n+1$,

If $n + 1$ is even, then there exists an integer $k \leq n$ such that $(n + 1) = 2k$. By induction hypothesis, k is the sum of distinct power of 2 and hence so is $(n + 1)$.

If $n + 1$ is odd, then n is even, and by induction hypothesis is a sum of distinct powers of 2, which does not contain a summand of $2^n = 1$. Therefore, $(n + 1) = n + 2^0$ is a sum of distincts powers of 2. Thus, a_n holds for $n = k + 1$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. ☐