MAT300: Homework 7

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June 10, 2016

pp.76-82: 4.2.18, 4.2.22, 4.3.8, 4.3.11, 4.3.16, 4.3.21

4.2.18

In a totally ordered set, immediate successors and immediate pedecessors (when they exists) are unique

Immediate Successor

Proof. Suppose A is a totally ordered set. Let x be an arbitrary but fixed element of A. Assume that x has two immediate successors y_1 and y_2. Since y_1 is an immediate successor of x, by definition of immediate successor, there does not exist an element $z \in A$ such that $x < z < y_1$. Analogously, since y_2 is an immediate successor of x, there does not exist an element $z \in A$ such that $x < z < y_2$. Thus, $y_1 = y_2$, since every totally ordered set is antisymmetric. As a result, immediate successors are unique if they exist. □

Immediate Predecessor

Proof. Suppose A is a totally ordered set. Let x be an arbitrary but fixed element of A. Assume that x has two immediate predecessors w_1 and w_2. Since w_1 is an immediate predecessor of x, by definition of immediate predecessor, there does not exist an element $u \in A$ such that $w_1 < u < x$. Analogously, since w_2 is an immediate predecessor of x, there does not exist an element $u \in A$ such that $w_2 < u < x$. Thus, $w_1 = w_2$, since every totally ordered set is antisymmetric. As a result, immediate predecessors are unique if they exist. □

4.2.22

Let A be a partially ordered set. Let K be a nonempty subset of A. If K has a least upper bound, it is unique.

Proof. Let A be a partially ordered set. Let K be a nonempty subset of A. Suppose there exist elements u_1 and u_2 of K which are least upper bounds. Since u_1 is a least upper bound and u_2 is also a least upper bound, $u_1 \leq u_2$. Analogously, since u_2 is a least upper bound and u_1 is also a least upper bound, $u_2 \leq u_1$. Thus, $u_1 = u_2$ since every partially ordered set is antisymmetric. As a result, if $K \subseteq A$ has a least upper bound then the least upper bound is unique. □

4.3.8

Let A be a set and let Ω be a subset of $\mathcal{P}(A)$. Then the relation $\sim_\Omega$ associated with Ω is symmetric.

Side Notes: We want to prove that $\sim_\Omega$ is symmetric i.e. we need to show that $\forall x, y \in A(x \sim_\Omega y) \implies (y \sim_\Omega x)$

Proof. Let A be an arbitrary but fixed set and let Ω be an arbitrary but fixed subset of $\mathcal{P}(A)$. Suppose there exist arbitrary but fixed elements a and b of A in which a is related to b. By definition of relation (4.3.6), a and b are elements of the same set S where S is an element of Ω. Since a and b are in the same set S, b is also related to a. Thus, $a \sim_\Omega b$ implies that $b \sim_\Omega a$. As a result, the relation $\sim_\Omega$ associated with Ω is symmetric. □

4.3.11

Let $A = \{1, 2, 3, 4, 5, 6\}$

1. Consider the following subset of $\mathcal{P}(A)$:

$$\Omega = \{\{1, 2, 3, 4\}, \{5, 6\}.\}$$

   Find $\sim_\Omega$.

$$\begin{aligned} \sim_\Omega = \{&(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),\\ &(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4),\\ &(5,5),(5,6),(6,5),(6,6)\} \end{aligned}$$

2. Consider the following relation on A:

$$\sim = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,4),(2,1),(2,4),(4,1),(4,2),(3,6),(6,3)\}.$$

   Find $\Omega_\sim$

$$\Omega_\sim = \{\{1, 2, 4\}, \{3, 6\}, \{5\}\}$$

4.3.16

Let A be a set and let Ω be a subset of $\mathcal{P}(A)$. Suppose that the elements of Ω are pairwise disjoint. Then the relation $\sim_\Omega$ associated with Ω is transitive.

Proof. Let A be an arbitrary but fixed set and let Ω be an arbitrary but fixed subset of $\mathcal{P}(A)$. Suppose that the elements of Ω are pairwise disjoint which means that if S_1, S_2 are arbitrary but fixed elements of Ω, then $S_1 = S_2$ or $S_1 \cap S_2 = \emptyset$. Let x, y, z be arbitrary but fixed elements of A and $x \sim_\Omega y$ and $y \sim_\Omega z$. By definition of relation, this means that there exists an arbitrary but fixed set R which is an element of Ω such that $x \in R$ and $y \in R$. Analogously, since $y \sim_\Omega z$, this means that there exists an arbitrary but fixed set S which is an element of Ω such that $y \in S$ and $z \in S$. Since y is an element of R and S, by the hypothesis which assume pairwise disjoint, $R = S$. Thus $x \in S$, $z \in S$, and $x \sim_\Omega z$. As a result, the relation $\sim_\Omega z$ associated with Ω is transitive. $\square$

4.3.21

Let $\sim$ be an equivalence relation on a set S. Then $\Omega_\sim$ forms a partition of S. That is,

- $\bigcup\limits_{x \in S} T_x = S$, and
- for x and y in S, either $T_x = T_y$, or $T_x \cap T_y = \emptyset$.

Proof. Suppose A is an arbitrary but fixed set and $\sim$ is an equivalence relation on S. Suppose a, b are arbitrary but fixed elements of A and $T_a \cap T_b \neq \emptyset$. Let x be an arbitrary but fixed element of T_a. Since $x \in T_a$, $a \sim x$. Since $T_a \cap T_b \neq \emptyset$, there exists an element c of $T_a \cap T_b$ such that $a \sim c$ and $b \sim c$. Since an equivalence relation is symmetric and transitive, b is related to x. Therefore $x \in T_b$ which means that $x \subseteq T_b$. $\square$