MAT300: Homework 3

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June 1, 2016

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3. Induction
3.2.2

For each natural number n prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

Proof. By induction,
Suppose $P(n)$ is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
Base Case ($n = 1$):
When $P(n = 1)$, the left hand side of the equation equals to 1 while the right hand side equals to $\frac{1(1+1)}{2}$. Thus, both sides are equal to 1 and $P(n)$ is true for $n = 1$.
Induction Step:
Let k be an arbitrary but fixed element in N and suppose that $P(k)$ is true, i.e. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= (k+1) + \sum_{i=1}^{k} i \\
&= (k+1) + \frac{k(k+1)}{2}, \textit{ by induction hypothesis} \\
&= \frac{2(k+1)}{2} + \frac{k(k+1)}{2} \\
&= \frac{2(k+1) + k(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Thus, $P(n)$ holds for $n = k + 1$. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

3.2.3

Let $n \in \mathbb{N}$. Conjecture a formula for

$$a_n = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + ... + \frac{1}{(n)(n+1)}$$

and prove your conjecture.
Conjecture:

$$a_1 = \frac{1}{(1)(1+1)} = \frac{1}{2}$$
$$a_2 = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$
$$a_3 = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$
...
$$a_n = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + ... + \frac{1}{(n)(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + ... + \frac{1}{n^2+n} = \frac{n}{n+1}$$

Proof. By induction,
Base Case ($n = 1$):
When $n = 1$, a_1 equals to $\frac{1}{(1)(1+1)}$ which equals to $\frac{1}{2}$. Thus, a_n holds for $n = 1$.
Induction Step:
Let k be an arbitrary but fixed element in N and suppose that a_k is true, i.e. $a_k = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + ... + \frac{1}{(k)(k+1)} = \frac{k}{k+1}$. Then

$$\begin{aligned}
a_{k+1} &= \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + ... + \frac{1}{(k)(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \textit{ by induction hypothesis} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
&= \frac{(k+1)}{(k+2)}
\end{aligned}$$

Thus, a_n holds for $n = k + 1$. By the principle of mathematical induction, a_n is true for all $n \in \mathbb{N}$. □

3.2.4

Use induction to prove that every positive integer is either even or odd. Then use this result to show that every integer is either even or odd.

Proof. By induction,
Suppose n be an arbitrary but fixed element in N,

$$P(n) = \begin{cases} 2n - 1 \; or \\ 2n \end{cases}$$

By definitions of odd and even integers, there are two cases for $P(n)$ which includes $P(n) = 2n - 1$ when $P(n)$ is odd or $P(n) = 2n$ when $P(n)$ is even.

Case 1: $P(n)$ is odd
Base Case ($n = 1$):
When $P(n = 1)$, $P(n)$ equals to $2(1) - 1 = 1$. Since, by definition of odd integer, 1 is odd. Thus, $P(n)$ holds for $n = 1$.
Induction Step:
Let k be an arbitrary but fixed element in N and suppose that $P(k)$ is true, i.e. $P(k) = 2k - 1$ is odd. Then,

$$\begin{aligned} P(k+1) &= 2(k+1) - 1 \\ &= 2k - 1 + 2, \; \textit{by induction hypothesis} \\ &= P(k) + 2 \end{aligned}$$

Since $P(k)$ is an odd integer, $P(k+1) = P(k) + 2$ is also an odd integer. Thus, $P(n)$ holds for $n = k + 1$. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Case 2: $P(n)$ is odd
Base Case ($n = 1$):
When $P(n = 1)$, $P(n)$ equals to $2(1) = 2$. Since, by definition of even integer, 2 is even. Thus, $P(n)$ holds for $n = 1$.
Induction Step:
Let k be an arbitrary but fixed element in N and suppose that $P(k)$ is true, i.e. $P(k) = 2k$ is even. Then,

$$\begin{aligned} P(k+1) &= 2(k+1) \\ &= 2k + 2, \; \textit{by induction hypothesis} \\ &= P(k) + 2 \end{aligned}$$

Since $P(k)$ is an even integer, $P(k+1) = P(k) + 2$ is also an even integer. Thus, $P(n)$ holds for $n = k + 1$. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

3.2.5

Let m and $n \in \mathbb{N}$. Define what it means to say that m divides n. Now prove that for all $n \in \mathbb{N}$, 6 divides $n^3 - n$.
Suppose m, n be arbitrary but fixed elements in $\mathbb{N}$. If m divides n denoted by $m \mid n$ means that there exist an arbitrary but fixed element $q \in \mathbb{Z}$ such that $mq = n$.

Proof. By induction, suppose that $P(n) = \frac{n^3-n}{6}$.
Base Case ($n = 1$):
When $P(n = 1)$, $P(n)$ is defined by $\frac{1^3-1}{6} = 0$. Since $6 \cdot 0 = 0$. Thus, $P(n)$ holds for $n = 1$.
Induction Step:
Let k be an arbitrary but fixed element in N and suppose that $P(k)$ is true, i.e. $P(k)$ is defined by $6 \mid k^3 - k$. Then

$$
\begin{aligned}
&6 \mid (k+1)^3 - (k+1) \\
&6 \mid k^3 + 3k^2 + 3k + 1 - k - 1 \\
&6 \mid (k^3 - k) + (3k^2 + 3k) + 1 - 1 \\
&6 \mid (k^3 - k) + 3k(k+1)
\end{aligned}
$$

6 divides $(k^3 - k)$ by induction hypothesis. For $3k(k+1)$,
$3 \cdot (k^2 + k) = 3k(k+1)$, given that $k^2 + k \in \mathbb{Z}$. Then 3 divides $3k(k+1)$. And 2 divides $k(k+1)$ since $k \cdot (k+1)$ is equivalent to odd integer times even integer or even integer times odd integer (by 3.2.4). Since 6 is a product of 2 and 3, 6 also divides 3k(k+1). Thus, a_n holds for $n = k+1$. By the principle of mathematical induction, a_n is true for all $n \in \mathbb{N}$. $\square$