The contents of this unit:

Line Integrals

Cauchy Integral theorem and Cauchy Integral formula

Power Series Expansions

Taylor Series

Laurent's Series

Types of Singular Points

Cauchy Residue Theorem

Evalution of definite integrals using Cauchy Residue theorem

Complex Integration: let us consider $f(t) = u(t) + iv(t)$, where u and v are real valued functions of t in a closed interval a,b.

We define $\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$

Thus $\int_a^b f(t)dt$ is a complex number in which $\int_a^b u(t)\, dt$ is a real part and $\int_a^b v(t)dt$ is an imaginary part.

Line integral:

Let $f(z)$ be a complex variable defined in a domain D . let C be an arc in a domain D then

$$\int f(z)dz = \int \big(u(x,y) + iv(x,y)\big)(dx + idy)$$

$$= \int (u + iv)(dx + idy)$$

$$= \int (udx - vdy) + i\int (udy + vdx)$$

Problem 1: Evaluate $\int_c (x+y)dx + x^2ydy$ along y=3x between points (0,0), (3,9).

Solution: given points are A (0,0), B(3,9) `

The equation of line passing through A and B is y=3x, then dy=3dx

Now $\int_c (x+y)dx + x^2ydy = \int_0^3 (x+3x)dx + (x^2 3x)3dx$

$$= \int_0^3 (4x)dx + 9x^3dx$$

$= \int_0^3 ((4x) + 9x^3)dx$

$= \left[4\frac{x^2}{2} + 9\frac{x^4}{4}\right]\begin{matrix}3\\0\end{matrix}$

$= 2(3)^2 + \frac{9}{4}(3)^4 - 0\text{-}0$

=18+729/4

=801/4

Problem 2: Evaluate $\int_{1-i}^{2+i}(2x + 1 + iy)dz$ along the line joining $1 - i$ to 2+i.

Solution: given points A(1,-1), B(2,1)

We know that equation of line joining two points $A(x_1, y_1)$, $B(x_2, y_2)$ is

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y + 1 = \frac{1+1}{2-1}(x - 1)$

$y = 2x - 2 - 1$

$y = 2x - 3$, $dy = 2dx$ and x varies from 1 to 2

Now $\int_{1-i}^{2+i}(2x + 1 + iy)dz = \int_1^2 (2x + 1 + i(2x - 3))(dx + i2dx)$

$= \int_1^2 (2x + 1 - 2(2x - 3) + i(2x - 3 + 4x + 2)dx$

$= \int_1^2 \big(-2x + 7 + i(6x - 1)\big)dx$

$= \{-2\frac{x^2}{2} + 7x + i((6)\frac{x^2}{2} - x)\}_1^2$

$= -4 + 14 + i(12 - 2) + 1 - 7 - i(3 - 1)$

=10+10i-6-2i

=4+6i

Exercise:

1.Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the paths i) $y = x^2$ ii) $y = x$.

2. Integrate $f(x) = x^2 + ixy$ from A(1,1) to B(2,8).

Cauchy Integral Theorem:

Statement: let $f(z) = u(x, y) + iv(x, y)$ be analytic on and within a simple closed contour C and $f'(z)$ be continuous then $\int_c f(z)dz = 0$.

Proof:

let $f(z) = u(x, y) + iv(x, y)$ be analytic on and within a simple closed contour C and $f'(z)$ be continuous now we have to prove $\int_c f(z)dz = 0$.

Let $z = x + iy$, $f(z) = u + iv$ then

$$f(z)dz = \big(u(x,y) + iv(x,y)\big)(dx + idy)$$

$$= (u + iv)(dx + idy)$$

$$= (udx - vdy) + i(udy + vdx)$$

$$\int f(z)dz = \int(udx - vdy) + i\int(udy + vdx) \text{..........(1)}$$

Since $f'(z)$ be continuous then the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial x}$ are also continuous in the region enclosed by C

Hence we can Greens theorem i.e

$\oint_C Mdx + Ndy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy$

Therefore equation (1) can be written as

$$\int_c f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

Since $f(z)$ is analytic, by C-R equations $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence

$$\int_c f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dxdy$$

$$=0$$

$$\int_c f(z)dz = 0$$

Problem 1: Evaluate $\int_C \frac{z^2-z+1}{z-1}dz$ where $c: |z| = \frac{1}{2}$ is taken in anticlock wise direction.

Solution : given $\int_C \frac{z^2-z+1}{z-1} dz$

Here $f(z) = \frac{z^2-z+1}{z-1}$ is analytic on and within the circle $|z| = \frac{1}{2}$

Since $z = 1$ lies outside the circle $|z| = \frac{1}{2}$

Now we can apply Cauchy integral theorem , $\int_c f(z)dz = 0$

$\int_C \frac{z^2-z+1}{z-1} dz = 0$

Problem 2: Evaluate $\int_C \frac{e^z}{z-2} dz$ where c: $|z| = 1$.

Solution: given $\int_C \frac{e^z}{z-2} dz$

Let $f(z) = \frac{e^z}{z-2}$ is analytic on and within the circle $|z| = 1$

Since $z = 2$ lies outside the circle $|z| = 1$

Now we can apply Cauchy integral theorem , $\int_c f(z)dz = 0$

$$\int_C \frac{e^z}{z-2} dz = 0$$

Problem 3: Verify Cauchy theorem for the function $f(z) = 3z^2 + iz$, if C is a square with vertices $1\pm i$, $-1\pm i$.

Solution :

Given $f(z) = 3z^2 + iz$, C is a square with vertices $1\pm i$, $-1\pm i$.

i.e A(1,1), B(-1,1), C(-1,-1), D(1,-1)

$$\int_c f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$$

$$f(z) = 3z^2 + iz$$

$$f(z)dz = (3(x+iy)^2 + i(x+iy))(dx+idy)$$

$$f(z)dz = (3x^2 - 3y^2 - y + i(6xy + x))(dx+idy)$$

Along the path C_1 joining A and B, AB

From the points A and B y=1, then dy=0

$f(z)dz = (3x^2 - 3.1^2 - 1 + i(6x.1 + x))(dx + i(0))$

$=(3x^2 - 4 + i(6x + 1)dx$

Now $\int_{C_1} f(z)dz = \int_1^{-1}(3x^2 - 4 + i(6x + x)dx$

$= [3\frac{x^3}{3} - 4x + i\left(\frac{7x^2}{2}\right)]_1^{-1}$

$= -1 + 4 + i\left(\frac{7}{2}\right) - 1 + 4 - i(\frac{7}{2})$

=6

Along the path C_2 joining B(-1,1) and C(-1,-1) , BC

$x = -1, dx = 0$

$$\int_{C_2} f(z)dz = \int_1^{-1} (3 - 3y^2 - y + i(-6y - 1))(idy)$$

$= (i)\int_1^{-1}(3 - 3y^2 - y + i(-6y - 1))dy$

$= i[(3y - 3\frac{y^3}{3} - \frac{y^2}{2} + i(-\frac{6y^2}{2} - y)]_1^{-1}$

$= i((-3 + 1 - \frac{1}{2} + i(-3 + 1) - 3 + 1 + \frac{1}{2} - i(-3 - 1))$

$= i(-4 + 2i)$

$-4i - 2$

Along the path C_3 line joining CD

C(-1,-1) D(1,-1)

From this $y = -1, dy = 0$

$$\int_{C_3} f(z)dz = \int_{-1}^{1} (3x^2 - 3 + 1 + i(-6x + x)dx$$

$= \int_{-1}^{1}(3x^2 - 2 + i(-5x))dx$

$$= [3\frac{x^3}{3} - 2x - 5i\frac{x^2}{2}]_{-1}^{1}$$

$$= 1 - 2 + \frac{5}{2}i + 1 - 2 - \frac{5}{2}i$$

$$= -2$$

Along the path DA i.e C_4

D(1,-1), A(1,1)

From this we conclude $x = 1$ then $dx = 0$

$$\int_{C_4} f(z)dz = \int_{-1}^{1} (3 - 3y^2 - y + i(6y + 1))\,(idy)$$

$$= (i)(3y - 3\frac{y^3}{3} - \frac{y^2}{2} + i(6\frac{y^2}{2} + y))_{-1}^{1}$$

$$= i(3 - 1 - \frac{1}{2} + 4i + 3 - 1 + \frac{1}{2} - 2i)$$

$$= i(4 + 2i)$$

$$= -2 + 4i$$

Now

$$\int_{c} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$$

= 6-4i-2-2-2+4i

=0

Exercise:

1. Show that $\int_C (z + 1)dz = 0$ where C is the boundary of a square whose vertices at points $z = 0, z = 1, z = 1 + i, z = i$.

Cauchy integral formula:

Statement:

Let $f(z)$ be analytic function within and on a closed contour , if $z = a$ is any point within C then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$ where the integral is taken in the positive sense around C.

Proof: let $f(z)$ be analytic function within and on a closed contour. Let $z = a$ is any point within C now we have to prove $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$

Choose a suitably small positive number r_0 and describes a circle C_0 with center at a and radius r_0 so that the circle C_0 entirely within C

Since $f(z)$ is analytic anywhere on C therefore $\frac{f(z)}{z-a}$ is also analytic except $z = a$, thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_0, therefore by generalization of Cauchy theorem we get

$\int_C \frac{f(z)}{z-a}dz = \int_{C_0} \frac{f(z)}{z-a}dz$ on the circle C_0, $z = a + r_0e^{i\theta}, 0 \leq \theta \leq 2\pi$

$$= \int_{\theta=0}^{2\pi} \frac{f(a+r_0e^{i\theta})}{r_0e^{i\theta}} r_0e^{i\theta}.\, id\theta$$

$$= i\int_{\theta=0}^{2\pi} f(a + r_0e^{i\theta})\, d\theta$$

As $r_0 \to 0$ the circle C_0 shrinks to the point a hence allowing $r_0 \to 0$

$$\int_C \frac{f(z)}{z-a}dz = i\int_{\theta=0}^{2\pi} f(a)\, d\theta$$

$$= if(a)\int_{\theta=0}^{2\pi} d\theta$$

$$= if(a)2\pi$$

Hence $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$

Generalization of Cauchy integral formula:

Statement: Let $f(z)$ be analytic function within and on a closed contour , if $z = a$ is any point within C then $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$ where the integral is taken in the positive sense around C.

Proof: from the Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Note that "a" is any point within C and by definition of analytic function

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f'(a) = \lim_{\Delta a \to 0} \frac{f(a+\Delta a) - f(a)}{\Delta a}$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a}\left[\frac{1}{2\pi i}\int_C \frac{f(z)}{z-(a+\Delta a)}dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz\right]$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a}\frac{1}{2\pi i}\int_C f(z)\left[\frac{1}{z-(a+\Delta a)} - \frac{1}{z-a}\right]dz$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a}\frac{1}{2\pi i}\int_C f(z)\left[\frac{\Delta a}{z-(a+\Delta a)(z-a)}\right]dz$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz$$

This is called cauchy's integral formula for the derivative of analytic function

Similarly $f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\, dz$

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}\, dz$$

Therefore $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz.$

Problem 1: Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}dz$. Where C is the circle $|z| = 3$.

Solution: given integral $\int_C \frac{e^{2z}}{(z-1)(z-2)}dz$

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}dz = \int_C e^{2z}\frac{1}{(z-1)(z-2)}dz$$

Let $f(z) = e^{2z}$ which is analytic within and on C, here the circle center at $z = 0$ with radius 3

The integrand has two singular points $z = 1,2$ lies inside the circle

So we use partial fraction to split the integral

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1$ in the above equation, we get

$$1 = -A \text{ implies } A = -1$$

Put $z = 2$ in the above equation, we get

$$1 = B$$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

Now $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C \frac{e^{2z}}{(z-2)} dz - \int_C \frac{e^{2z}}{(z-1)} dz$……………….(1)

Consider $\int_C \frac{e^{2z}}{(z-2)} dz$

Here $z = 2$ is the singular point inside the circle, by Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$\int_C \frac{e^{2z}}{(z-2)} dz = 2\pi i f(2)$$

$$= 2\pi i e^4 \text{…………(2)}$$

Consider $\int_C \frac{e^{2z}}{(z-1)} dz$

Here $z = 1$ is the singular point inside the circle , by by Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$\int_C \frac{e^{2z}}{(z-1)} dz = 2\pi i f(1)$$

$$= 2\pi i e^2 \text{……….(3)}$$

Substituting (2) and (3) in equation (1), we get

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = 2\pi i e^4 - 2\pi i e^2$$

$$= 2\pi i (e^4 - e^2).$$

Problem 2: Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-4)} dz$ where $C: |z| = 2$.

Solution: given integral $\int_C \frac{e^{2z}}{(z-1)(z-4)} dz$

Since $\frac{e^{2z}}{(z-1)(z-4)}$ has two singular points $z = 1, z = 4$ but $z = 4$ lies outside the circle $C: |z| = 2$, so we write the integral as

$$\int_C \frac{e^{2z}}{(z-1)(z-4)} dz = \int_C \frac{\frac{e^{2z}}{(z-4)}}{(z-1)} dz$$

This is of the form $\int_C \frac{f(z)}{z-a} dz$, where $f(z) = \frac{e^{2z}}{(z-4)}$, $a = 1$lies inside the circle , by Cauchy integral formula $\int_C \frac{e^{2z}}{(z-1)(z-4)} dz = 2\pi i f(1)$

$$= 2\pi i \frac{e^{2(1)}}{1-4}$$

$$= -\frac{2}{3}\pi i e^2$$

Problem 3: Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ around i) $C: |z-1| = 3$ ii) $|z| = 2$.

Solution: given integral $\int_C \frac{e^{2z}}{(z+1)^4} dz$

i) $C: |z-1| = 3$

Since $\frac{e^{2z}}{(z+1)^4}$ has singular point $z = -1$, check the point inside the circle or not

$C: |-1-1| = 2 < 3$ so $z = -1$ lies inside the circle

By Cauchy generalized formula, $f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$.

Here $f(z) = e^{2z}$, a=-1, n=3

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1)$$

$$f^1(z) = 2e^{2z}$$

$$f''(z) = 4e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$f'''(-1) = 8e^{2(-1)} = 8e^{-2}$$

Therefore

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{6} 8e^{-2}$$

$$= \frac{8\pi i}{3} e^{-2}$$

ii) $|z| = 2$, $z = -1$ lies inside the circle, so the follow the same procedure as we did in (i)

Problem 4: Evaluate $\int_C \frac{logz}{(z-1)^3} dz$ where $C: |z-1| = \frac{1}{2}$.

Solution : given integral $\int_C \frac{logz}{(z-1)^3} dz$

$\frac{logz}{(z-1)^3}$ has a singular point $z = 1$, we need to check lies inside or not

$$C: |1-1| = 0 < \frac{1}{2}$$

Therefore $z = 1$ lies inside the given circle

By Cauchy generalized formula, $f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$.

Here $f(z) = logz$, a=1, n=2

$$\int_C \frac{logz}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$$

$$f(z) = logz$$

$$f'(z) = \frac{1}{z}$$

$$f''(z) = -\frac{1}{z^2}$$

$$f''(1) = -\frac{1}{1^2} = -1$$

$$\int_C \frac{logz}{(z-1)^3} dz = \frac{2\pi i}{2!}(-1)$$

$$\int_C \frac{logz}{(z-1)^3} dz = -\pi i$$

Exercise: 1. Evaluate $\int_C \frac{3z^2+7z+1}{(z+1)} dz$ where $C: |z+1| = 1$.

2.Evaluate $\int_C \frac{z^3 e^{-z}}{(z-1)^3} dz$ where $C: |z-1| = \frac{1}{2}$.

Entire function: Let $f(z)$ be analytic everywhere in domain D then $f(z)$ is said to entire function.

Liouville's Theorem:

Statement: Every entire bounded function is constant

Maximum modulus theorem:

Statement: let $f: D \to C$ be analytic. If there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$, $\forall z \in D$ then f is constant on D.

Taylor series theorem: let $f(z)$ be analytic function at all points within a circle C_0 with centre at a and radius r_0 then the taylor series expansion of $f(z)$ at a point a is given by

$$f(z) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \ldots \ldots \ldots \ldots$$

Maclaurin's series:

If we take $a = 0$ in taylor series expansion, we get

$$f(z) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \ldots \ldots \ldots \ldots$$

Some Maclaurin series expansion of standard function

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \ldots \ldots \ldots \ldots$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ for } |z| < \infty$$

$$sinz = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \ldots \ldots \ldots \ldots \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \text{ for } |z| < \infty$$

$$cosz = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \text{ for } |z| < \infty$$

$$sinhz = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \ldots \ldots \ldots \ldots \ldots$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$Coshz = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \ldots \ldots \ldots \text{ For } |z| < 1$$

$$= \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \ldots \ldots \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n \text{ for } |z| < 1$$

Note: To obtain Taylor's Series expansion of $f(z)$ about the point $z = a$ put $z - a = w$ then
$f(z) = f(w + a)$

$= \emptyset(w)$

Write the Maclaurin's series expansion of $\emptyset(w)$ either by using standard expansions or direct expansions. Finally substitute $w = z - a$ then we get taylor series expansion of given function .

Problem 1: Expand e^z as Taylor's about $z = 1$.

Solution : Given $f(z) = e^z$

We have to find the Taylor series expansion of $f(z) = e^z$ about $z = 1$

Let $z - 1 = w$

$$f(z) = e^z$$
$$= e^{1+w}$$
$$= e.e^w$$
$$= e(1 + \frac{w}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots \dots \dots \dots \dots) \text{ for all } w$$

Substitute $w = z - 1$

$f(z) = e(1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots \dots \dots \dots \dots)$ for all $z - 1$.

Problem 2: Expand $f(z) = sinz$ in Taylor series about $z = \frac{\pi}{4}$

Solution : given $f(z) = sinz$

The Taylor series expansion of $f(z)$ at a point a is given by

$$f(z) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \dots \dots \dots ....$$

Here $f(z) = sinz,\ a = \frac{\pi}{4}$

$$sinz = f\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)}{1!}f'\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right) + \cdots \dots \dots \dots ....$$

$f(z) = sinz,\ f\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = cosz, f'\left(\frac{\pi}{4}\right) = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(z) = -sinz, f''\left(\frac{\pi}{4}\right) = -sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(z) = -cosz, f'''\left(\frac{\pi}{4}\right) = -cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, and so on

Then

$$sinz = \frac{1}{\sqrt{2}} + \frac{\left(x - \frac{\pi}{4}\right)}{1!}\frac{1}{\sqrt{2}} + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!}(-\frac{1}{\sqrt{2}}) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!}(-\frac{1}{\sqrt{2}}) + \cdots \dots \dots \dots ....$$

Problem 3: Expand $f(z) = \frac{1}{(z^2-z-6)}$ about $i)\ z = -1\ \ ii)\ z = 1.$

Solution : Consider $f(z) = \frac{1}{(z^2-z-6)} = \frac{1}{(z-3)(z+2)}$

Use partial fractions to split $f(z)$

$$\frac{1}{(z-3)(z+2)} = \frac{A}{(z-3)} + \frac{B}{(z+2)}$$

$$1 = A(z+2) + B(z-3)$$

Put $z = -2$ then we get

$$1 = A(0) + B(-5), B = -\frac{1}{5}$$

Put $z = 3$ then we get 1=A(5)+B(0)

$$A = \frac{1}{5}$$

$$\frac{1}{(z-3)(z+2)} = \frac{\frac{1}{5}}{(z-3)} + \frac{-\frac{1}{5}}{(z+2)}$$

$$f(z) = \frac{1}{5}[\frac{1}{(z-3)} - \frac{1}{(z+2)}]$$

i) Now we have to expand $f(z)$ about $z = -1$

Put $z + 1 = w$ implies $z = w - 1$

$$f(z) = \frac{1}{5}[\frac{1}{(w-4)} - \frac{1}{(w+1)}]$$

$$= \frac{1}{5}[\frac{1}{-4(1-\frac{w}{4})} - \frac{1}{(w+1)}]$$

$$= \frac{1}{5}[-\frac{1}{4}(1-\frac{w}{4})^{-1} - (1+w)^{-1}]$$

$$f(z) = \frac{1}{5}[-\frac{1}{4}\left(1 + \frac{w}{4} + \left(\frac{w}{4}\right)^2 + \cdots \ldots..\right) - (1 - w + w^2 + \cdots \ldots)] \quad \text{for } |w| < 1$$

Now substitute $z + 1 = w$ then the required Taylor series expansion is

$$f(z) = \frac{1}{5}[-\frac{1}{4}\left(1 + \frac{z+1}{4} + \left(\frac{z+1}{4}\right)^2 + \cdots \ldots..\right) - (1 - (z+1) + (z+1)^2 + \cdots \ldots)] \text{ for } |z+1| < 1$$

ii) Now we have to expand $f(z)$ about $z = 1$

Put $z-1=w$ implies $z=w+1$

$$f(z)=\frac{1}{5}[\frac{1}{(w-2)}-\frac{1}{(w+3)}]$$

$$=\frac{1}{5}[\frac{1}{-2(1-\frac{w}{2})}-\frac{1}{3(1+\frac{w}{3})}]$$

$$=\frac{1}{5}[-\frac{1}{2}(1-\frac{w}{2})^{-1}-\frac{1}{3}(1+\frac{w}{3})^{-1}]$$

$$f(z)=\frac{1}{5}[-\frac{1}{2}\left(1+\frac{w}{2}+\left(\frac{w}{2}\right)^2+\cdots\ldots..\right)-\frac{1}{3}(1-\frac{w}{3}+(\frac{w}{3})^2+\cdots\ldots)]$$ for $|w|<2$ since the common region between $\left|\frac{w}{2}\right|<1$ *and* $\left|\frac{w}{3}\right|<1$ is $|w|<2$

Now substitute $z-1=w$ then the required Taylor series expansion is

$$f(z)=\frac{1}{5}[-\frac{1}{2}\left(1+\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2+\cdots\ldots..\right)-\frac{1}{3}(1-\frac{z-1}{3}+(\frac{z-1}{3})^2+\cdots\ldots)]$$ for $|z-1|<2$

Exercise: 1. Expand e^z as a Taylor's series about $z=3$.

2. Expand $logz$ by Taylor's series expansion about $z=1$.
3. Obtain the Taylor's series expansion of e^{1+z} in powers of $z-1$.

Laurent's Series:

Statement: let C_1, and C_2 be two circles given by $|z'-a|=r_1$ $|z'-a|=r_2$

respectively, where $r_2<r_1$ and z' is any point on C_1, and C_2. Let $f(z)$ be analytic on C_1, and C_2 throughout the region between the two circles. Let z be any point in the ring shaped region between two circles then

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

Where $a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z')}{(z'-a)^{n+1}}dz'$

$b_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z')}{(z'-a)^{-n+1}}dz'$

Note: if $f(z)$ is analytic on and within C_1 everywhere then the laurent's series expansion of $f(z)$ in power of $z-a$ is just a Taylor's series.

Problem 1: Expand $\frac{1}{z^2-3z+2}$ in the region i) $0 < |z-1| < 1$ ii) $1 < |z| < 2$

Solution: let $f(z) = \frac{1}{z^2-3z+2}$

$= \frac{1}{z^2-2z-z+2}$

$= \frac{1}{(z-1)(z-2)}$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$1 = A(z-2) + B(z-1)$

Put $z = 1$ in the above equation

$1 = -A,\ A = -1$

Put $z = 2$ in the above equation

$1 = B$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

Now $f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

Here the singular points of $f(z)$ are 1,2

i) The function $f(z)$ is analytic in the ring shaped region $0 < |z-1| < 1$

Put $z - 1 = w$ the region is $0 < |w| < 1$

$$z = 1 + w$$

$$f(z) = \frac{1}{(w-1)} - \frac{1}{w}$$

$= \frac{-1}{(1-w)} - \frac{1}{w}$

$= -(1-w)^{-1} - \frac{1}{w}$

$= -(1 + w + w^2 + w^3 + \cdots \ldots) - \frac{1}{w}$ for $0 < |w| < 1$

$$= -(1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots \ldots) - \frac{1}{(z-1)}$$

The above series is valid for $0 < |(z-1)| < 1$

ii) The function $f(z)$ is analytic in the ring shaped region $1 < |z| < 2$

In the given region $1 < |z|$ implies $\left|\frac{1}{z}\right| < 1$ and $|z| < 2$ implies $\left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{-1}{(2-z)} - \frac{1}{(z-1)}$$

$$= \frac{-1}{2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}(1-\frac{1}{z})^{-1}$$

$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots \ldots\right) - \frac{1}{z}(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots \ldots)$$

Valid for $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$

$$= -\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) - \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)$$

$$= -\sum_{n=0}^{\infty}\left(\frac{z^n}{2^{n+1}}\right) - \sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}}\right)$$

Problem 2: Expand $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$ by Laurent's Series.

Solution: $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

Use partial fractions, $\frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{A}{(z-1)} + \frac{B}{(z-3)} + \frac{C}{(z+2)}$

$$z^2 - 6z - 1 = A\big((z-3)(z+2)\big) + B(z-1)(z+2) + C(z-1)(z-3)$$

Put $z = 3$ in the above equation, we get

$$9 - 18 - 1 = A(0) + B(2)(5) + C(0)$$

$$B = -1$$

Put $z = -2$ in the above equation, we get

$$4 + 12 - 1 = A(0) + B(0) + C(-3)(-5)$$

$$C = 1$$

Put $z = 1$ in the above equation, we get

$$1 - 6 - 1 = A(-2)(3) + B(0) + C(0)$$

$$A = 1$$

$$\frac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)} = \frac{1}{(z-1)} - \frac{1}{(z-3)} + \frac{1}{(z+2)}$$

Put $+2 = w$, the region is $3 < |w| < 5$ i.e $3 < |w|$ and $|w| < 5$

$$\left|\frac{3}{w}\right| < 1 \text{ and } \left|\frac{w}{5}\right| < 1$$

$$z = w - 2$$

$$f(z) = \frac{1}{(w-3)} - \frac{1}{(w-5)} + \frac{1}{w}$$

$$= \frac{1}{w(1-\frac{3}{w})} + \frac{1}{5(1-\frac{w}{5})} + \frac{1}{w}$$

$$= \frac{1}{w}(1 - \frac{3}{w})^{-1} + \frac{1}{5}(1 - \frac{w}{5})^{-1} + \frac{1}{w}$$

$$= \frac{1}{w}\left(1 + \frac{3}{w} + \left(\frac{3}{w}\right)^2 + \left(\frac{3}{w}\right)^3 + \cdots \ldots.\right) + \frac{1}{5}\left(1 + \frac{w}{5} + \left(\frac{w}{5}\right)^2 + \left(\frac{w}{5}\right)^3 + \cdots \ldots\right) + \frac{1}{w}$$

$$= \frac{1}{(z+2)}\left(1 + \frac{3}{(z+2)} + \left(\frac{3}{(z+2)}\right)^2 + \left(\frac{3}{(z+2)}\right)^3 + \cdots \ldots.\right) + \frac{1}{5}\left(1 + \frac{(z+2)}{5} + \left(\frac{(z+2)}{5}\right)^2 + \left(\frac{(z+2)}{5}\right)^3 + \cdots \ldots\right) + \frac{1}{(z+2)}$$

$$= \sum_{n=0}^{\infty}\left(\frac{3^n}{(z+2)^{n+1}}\right) + \sum_{n=0}^{\infty}\left(\frac{(z+2)^n}{5^{n+1}}\right) + \frac{1}{(z+2)}$$

Problem 3: Give a 2 Laurent's series expansion in powers of z for $f(z) = \frac{1}{z^2(1-z)}$ and specify the region in which these expansions are valid.

Solution: let $f(z) = \frac{1}{z^2(1-z)}$ the function

The function $f(z)$ has 3 singular points 0,0,1. The Laurent,s series of $f(z)$ can be obtained about $z = 0$ in the region $0 < |z| < 1$

$$\frac{1}{z^2(1-z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{1-z}$$

$$1 = A(z)(1-z) + B(1-z) + C(z^2)$$

Put $z = 0$ in above equation

$$1 = A(0) + B(1-0) + C(0)$$

$$1 = B$$

Put $z = 1$ in the above equation, we get

$$1 = A(0) + B(0) + C(1)$$

$$C = 1$$

Comparing z^2 coefficients in the above equation

$$0 = -A + C$$

$$A = C$$

$$A = 1$$

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{1-z}$$

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + (1-z)^{-1}$$

$f(z) = \frac{1}{z} + \frac{1}{z^2} + 1 + z + (z)^2 + (z)^3 + \cdots \ldots$ for $|z| < 1$ and first two terms are not defined at $z = 0$ so this Laurent's series expansion is valid for $0 < |z| < 1$.

And

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{1-z}$$

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{(-z)(1-\frac{1}{z})}$$

$$f(z) = \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}(1-\frac{1}{z})^{-1}$$

$$f(z) = \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots) \text{ for } \left|\frac{1}{z}\right| < 1 \text{ i.e } |z| > 1$$

Solution : Given $f(z) = \frac{ze^z}{(z-1)^3}$

The poles of $f(z)$ are obtained by equating denominator to zero $(z-1)^3 = 0$

$z = 1,1,1$

$z = 1$ is a pole of order 3

We know that Residue of $f(z)$ at pole $z = a$ of order m is given by

$$Res\,[f(z)]_{z=a} = \frac{1}{m-1!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}((z-a)^m f(z))$$

Here m = 3, a = 1 then

$Res\,[f(z)]_{z=a} = \frac{1}{(3-1)!}\lim_{z\to 1}\frac{d^{3-1}}{dz^{3-1}}((z-1)^3\frac{ze^z}{(z-1)^3})$

$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}(ze^z)$

$= \frac{1}{2}\lim_{z\to 1}(ze^z + 2e^z)$

$= \frac{3e}{2}$

Problem 2: find the residue of $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+1)}$

Solution : for the poles $(z+1)^2(z^2+1) = 0, (z+1)^2 = 0, (z^2+1) = 0$

$z = -1$ is a pole of order 2

$z = i, -i$ are simple poles

Now Residue at simple poles $z = i$

$$Res[f(z): z = i] = \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i}(z-i)\,\frac{z^2-2z}{(z+1)^2(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{z^2-2z}{(z+1)^2(z+i)}$$

$$= \frac{i^2-2i}{(i+1)^2(i+i)}$$

$$= \frac{-1-2i}{(i+1)^2(2i)}$$

$$= \frac{-1-2i}{2i(2i)}$$

$$= \frac{1}{4} + \frac{1}{2}i$$

Now Residue at simple poles $z = -i$

$$Res[f(z): z = -i] = \lim_{z \to -i} (z+i) f(z)$$

$$= \lim_{z \to -i} (z+i) \frac{z^2-2z}{(z+1)^2(z+i)(z-i)}$$

$$= \lim_{z \to -i} \frac{z^2-2z}{(z+1)^2(z-i)}$$

$$= \frac{i^2+2i}{(i+1)^2(-i-i)}$$

$$= \frac{-1+2i}{(-i+1)^2(-2i)}$$

$$= \frac{-1+2i}{4i^2}$$

$$= \frac{1}{4} - \frac{1}{2}i$$

Residue at pole $z = -1$ of order 2 is

$$Res\ [f(z)]_{z=-1} = \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d^{2-1}}{dz^{2-1}} ((z+1)^2 \frac{z^2-2z}{(z+1)^2(z+i)(z-i)})$$

$$= \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} (\frac{z^2-2z}{(z^2+1)})$$

$$= \lim_{z \to -1} \frac{(z^2+1)(2z-2)-(z^2-2z)(2z)}{(z^2+1)^2}$$

$$= \lim_{z \to -1} \frac{2z^3-2z^2+2z-2-2z^3+4z^2}{(z^2+1)^2}$$

$$= \frac{2(1)-2-2}{4} = -\frac{1}{2}$$

Note : if $f(z)$ is of the form $f(z) = \frac{\emptyset(z)}{\varphi(z)}$, $\varphi(a) = 0$, $\emptyset(a) \neq 0$ then Residue of $f(z)$ at $z = a$ is $\frac{\emptyset(a)}{\varphi'(a)}$

Problem 1: find the poles and residue at each pole of $f(z) = \frac{z^2}{z^4-1}$.

We know that Residue of $f(z)$ at pole $z = a$ of order m is given by

$$Res\,[f(z)]_{z=a} = \frac{1}{m-1!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}((z-a)^m f(z))$$

$$Res\,[f(z)]_{z=0} = \frac{1}{6-1!}\lim_{z\to 0}\frac{d^{6-1}}{dz^{6-1}}((z-0)^6\frac{sinz}{z^6})$$

$$= \frac{1}{5!}\lim_{z\to 0}\frac{d^5}{dz^5}(sinz)$$

$$= \frac{1}{120}\lim_{z\to 0}\frac{d^4}{dz^4}(cosz)$$

$$= \frac{1}{120}\lim_{z\to 0}\frac{d^3}{dz^3}(-sinz)$$

$$= \frac{1}{120}\lim_{z\to 0}\frac{d^2}{dz^2}(-cosz)$$

$$= \frac{1}{120}\lim_{z\to 0}\frac{d}{dz}(sinz)$$

$$= \frac{1}{120}\lim_{z\to 0} cosz$$

$$= \frac{1}{120}$$

Evaluation of Integrals of type $\int_0^{2\pi} F(cos\theta, sin\theta)d\theta$ around the unit circle (Contour Integration)

Consider The Evaluation of integrals of type $\int_0^{2\pi} F(cos\theta, sin\theta)d\theta$ where F is real rational function of $sin\theta$, $cos\theta$.

Now we write $z = e^{i\theta}$ so that $dz = ie^{i\theta}\, d\theta$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

We know that $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$

And $sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}(z-\frac{1}{z})$

Also $0 \le \theta \le 2\pi$ where θ travels on the entire unit circle

$$\int_0^{2\pi} F(cos\theta, sin\theta)d\theta = \int_C F(\frac{z+\frac{1}{z}}{2}, \frac{1}{2i}(z-\frac{1}{z}))\frac{dz}{iz}$$

$$= \int_C F(z)\, dz$$

Where C is the unit circle , now by Residue Theorem

$\int_C F(z)\, dz = 2\pi i$(Sum of residues of f(z) at its poles lies inside the circle)

Problem 1: Show that $\int_0^{2\pi} \frac{d\theta}{2+cos\theta} = \frac{2\pi}{\sqrt{3}}$ by Residue Theorem

Solution : Let $I = \int_0^{2\pi} \frac{d\theta}{2+cos\theta}$ …………………………..(1)

Let $z = e^{i\theta}$ so that $dz = ie^{i\theta}\, d\theta$

$$d\theta = \frac{dz}{iz}$$

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

Noe $I = \int_C \frac{1}{2+\frac{z+\frac{1}{z}}{2}} \frac{dz}{iz}$

$= \frac{2}{i} \int_C \frac{1}{4z+z^2+1}\, dz$

$= \frac{2}{i} \int_C f(z) dz$……………………..(2)

Where $f(z) = \frac{1}{4z+z^2+1}$

To get the poles $z^2 + 4z + 1 = 0$

$$z = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$$

$$z = -2 \pm \sqrt{3}$$

$$z = -2 + \sqrt{3}, z = -2 - \sqrt{3}$$

Since $|z| = 1$ the only point $z = -2 + \sqrt{3}$ lies inside the circle

Because $\left|-2 - \sqrt{3}\right| > 1$

Now we have to calculate Residue at $z = -2 + \sqrt{3}$

$$[Res\ f(z): z = -2+\sqrt{3}] = \lim_{z\to -2+\sqrt{3}} (z-(-2+\sqrt{3}))f(z)$$

$$= \lim_{z\to -2+\sqrt{3}} (z-(-2+\sqrt{3}))\frac{1}{4z+z^2+1}$$

$$= \lim_{z\to -2+\sqrt{3}} (z-(-2+\sqrt{3}))\frac{1}{(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))}$$

$$= \lim_{z\to -2+\sqrt{3}} \frac{1}{(z-(-2-\sqrt{3}))}$$

$$= \frac{1}{2\sqrt{3}}$$

Now by Cauchy Residue Theorem

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{2\sqrt{3}}\right)$$

$$= \frac{\pi i}{\sqrt{3}}$$

Now $I = \frac{2}{i}\frac{\pi i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$

$$I = \int_0^{2\pi} \frac{d\theta}{2+cos\theta} = \frac{2\pi}{\sqrt{3}}$$

Problem 2: show that $\int_0^{2\pi}\frac{d\theta}{a+bsin\theta} = \int_0^{2\pi}\frac{d\theta}{a+bcos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, a>b>0

Solution : let $I = \int_0^{2\pi}\frac{d\theta}{a+bsin\theta}$ ………..(1)

Let $z = e^{i\theta}$ so that $dz = ie^{i\theta}\ d\theta$

$$d\theta = \frac{dz}{iz}$$

$$sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$= \frac{1}{2iz}(z^2-1)$$

$$I = \int_C \frac{1}{a+b\frac{1}{2iz}(z^2-1)}\frac{dz}{iz}$$