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UNIT- V

Z - transforms

2ECE
CVT

Def:- The sequence $f(n): \{f(0), f(1), f(2), ----\}$ which is defined for all positive integers 'n' then the z-transforms of $f(n)$ is defined by $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

Where, 'z' is an arbitary complex number.

Note:- Instead of $f(n)$, we can also use $f(k)$ (k is +ve integer)

then the z-transform of $f(k)$ is defined by

$$\boxed{Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k}}$$

Some standard z-transforms:-

1) prove that $Z[1] = \frac{z}{z-1}$

Sol:- By the def

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\therefore Z\{1\} = \sum_{n=0}^{\infty} 1 z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} \qquad [\because (1-x)^{-1} = 1 + x + x^2 + --]$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + ---$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1} = \frac{z}{z-1}$$

2) prove that $Z\{(-1)^n\} = \frac{z}{z+1}$

sol:- $Z\{(-1)^n\} = \sum_{n=0}^{\infty} (-1)^n z^{-1}$

$= 1 - \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ---$

$= \left(1+\frac{1}{z}\right)^{-1} = \left(\frac{z+1}{z}\right)^{-1} = \frac{z}{z+1}$

3) prove that $z\{a^n\} = \frac{z}{z-a}$

sol:- $Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$

$= \sum_{n=0}^{\infty} (a/z)^n$

$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + ---$

$= \left(1 - \frac{a}{z}\right)^{-1}$

$= \left(\frac{z-a}{z}\right)^{-1}$

$= \frac{z}{z-a}$

Note:- $z\{e^{an}\} = z\{(e^a)^n\} = \frac{z}{z-e^a}$

$z\{e^{-an}\} = \frac{z}{z-e^{-a}}$

4) Find $z\left\{\frac{a^n}{n!}\right\}$

sol: $z\left\{\frac{a^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$

$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n = 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + ----$

$= 1 + \frac{(a/z)}{1!} + \frac{(a/z)^2}{2!} + -- = e^{a/z}$

5) Find $z[1/n!]$ -

Sol: Given, $z\left\{\frac{1}{n!}\right\}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= e^{1/z}$$

6) prove that $z\left\{\frac{1}{n+1}\right\} = -z\log(1-1/z) = z\log\left(\frac{z}{z-1}\right)$

Sol: $$z\left\{\frac{1}{n+1}\right\} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right]$$ $\left[\because \log(1-x) = -x - x^2/2 - x^3/3\right.$

$$= z\log\left[1-1/z\right]^{-1}$$ $-\log[1-x] = x + x^2/2 + x^2/3 + \cdots$

$$= z\log\left[\frac{z}{z-1}\right]$$

7) Find $z\{(n)\}$

Sol: $$z\{n\} = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(1 - 1/z\right)^{-2}$$ $\left[\because (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\right.$

$$= 1/z \cdot \frac{z^2}{(z-1)^2} = \frac{z}{(z-1)^2}$$ //

8) Prove that $Z\{n a^n\} = \frac{az}{(z-a)^2}$

Sol:

$$Z\{n.a^n\} = \sum_{n=0}^{\infty} n a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} n.\left(\frac{a}{z}\right)^n$$

$$= \frac{a}{z} + 2\left[\frac{a}{z}\right]^2 + 3\left[\frac{a}{z}\right]^3 + \cdots$$

$$= \frac{a}{z}\left[1 + 2\left[\frac{a}{z}\right] + 3\left[\frac{a}{z}\right]^2 + \cdots\right]$$

$$= \frac{a}{z}\left[1 - a/z\right]^{-2}$$

$$= \frac{az}{(z-a)^2}.$$

9) Prove that $Z\{n a^{n-1}\} = \frac{z}{(z-a)^2}$

Sol:

$$Z\{n a^{n-1}\} = \sum_{n=0}^{\infty} n.a^{n-1} z^{-n}$$

$$= \frac{1}{z} + \frac{2a}{z^2} + \frac{3.a^2}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 - \frac{a}{z}\right]^{-2}$$

$$= \frac{z}{(z-a)^2} \quad //$$

10) Find $Z\left\{\frac{1}{n}\right\} = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n}$

$$= \frac{1}{z} + \frac{1}{2} \cdot \frac{1}{z^2} + \cdots$$

$$= -\log\left(1 - 1/z\right)$$

$$= \log\left(1 - 1/z\right)^{-1}$$

$$= \log\left(\frac{z}{z-1}\right)$$

Linear Property:-

If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$ then

$Z\{a f(n) + b g(n)\} = a F(z) + b G(z)$

Proof:- By def $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z) \rightarrow (1)$

$$\therefore Z\{a f(n) + b g(n)\} = \sum_{n=0}^{\infty} [a f(n) + b g(n)] z^{-n}$$

$$= a \sum f(n) z^{-n} + b \sum g(n) z^{-n}$$

$$\therefore Z\{a f(n) + b g(n)\} = a \cdot F(z) + b G(z) \quad \text{by } (1);$$

Problems 2009 set (2)

(1) find $Z\{2^{2k+3}\}$

Sol: Given $Z\{2^{2k+3}\}$

$$= Z\{2^{2k} \cdot 2^3\}$$

$$= 8 Z\{[(2)^2]^k\}$$

$$= 8 Z\{4^k\} \qquad [\because Z[a^k] = \frac{z}{z-a}]$$

$$= 8 \cdot \left(\frac{z}{z-4}\right) \qquad \text{Here } a = 4$$

(2) find $Z\{\frac{1}{n(n+1)}\}$ [2011 set (3)]

$$= Z\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

Sol: $Z\{\frac{1}{n}\} - Z\{\frac{1}{n+1}\}$

$$= \log\left(\frac{z}{z-1}\right) - z \log\left(\frac{z}{z-1}\right)$$

$$= (1-z) \log\left(\frac{z}{z-1}\right);$$

(3) find $Z\{(-2)^n\}$

Ans:- $\frac{z}{z-(-2)} = \frac{z}{z+2}$

(4) find $Z\{1+(-2)^n\} = Z\{1\} + Z\{(-2)^n\}$

$$= \frac{z}{z-1} + \frac{z}{z+2}$$

5 Find $Z\{x(n)\}$ where $x(n)$ is $n \cdot 2^n$ [2008 set③]

Sol: $Z\{x(n)\} = Z\{n \cdot 2^n\}$

$$[\because Z\{n a^n\} = \frac{a^z}{(z-a)^2}$$

$$= \frac{2 \cdot z}{(z-2)^2}$$

⑥ Find $Z\{(\cos\theta + i\sin\theta)^n\}$ or $Z\{e^{in\theta}\}$

Hence Evaluate ⓐ $Z\{\cos n\theta\}$ ⓑ $Z(\sin n\theta)$

Given

Sol: $Z\{(\cos\theta + i\sin\theta)^n\} = Z\{(e^{i\theta})^n\}$

$$= \frac{z}{z - e^{i\theta}}$$

By Demovirès theorem

$$\Rightarrow Z\{\cos n\theta + i\sin n\theta\} = \frac{z}{z - e^{i\theta}}$$

$$= \frac{z}{z - (\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta} \times \frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}$$

$$= \frac{z(z-\cos\theta) + zi\sin\theta}{(z-\cos\theta)^2 - i^2\sin^2\theta}$$

$$\Rightarrow Z\{\cos n\theta\} + iZ\{\sin n\theta\} = \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

compare real & imaginary parts

We get $Z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

$$Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Note:- $Z\{\cos nt\} = \dfrac{z(z-\cos t)}{z^2 - 2z\cos t + 1}$

Find $Z\{\cos n\pi/2\}$, $Z\{\sin n\pi/2\}$

Sol:

put $\theta = \pi/2$

$$Z\{e^{in\pi/2}\} = Z\{e^{(i\pi/2)n}\}$$

$$= Z\{e^{(i\pi/2)n}\}$$

$$= \frac{z}{z - e^{i\pi/2}}$$

$$= \frac{z}{z - (\cos\pi/2 + i\sin\pi/2)} = \frac{z}{z-i}$$

$$= \frac{z}{z-i} \times \frac{(z+i)}{(z+i)}$$

$$= \frac{z^2 + iz}{z^2 - i^2}$$

$$Z\{\cos n\pi/2 + i\sin n\pi/2\} = \frac{z^2+iz}{z^2+1}$$

Compare real & imaginary parts

real part $Z\{\cos n\pi/2\} = \frac{z^2}{z^2+1}$

Imaginary part $Z\{\sin n\pi/2\} = \frac{z}{z^2+1}$

(8)

Find $Z\left\{\frac{1}{(n+1)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$

Sol:

$$Z\left\{\frac{1}{(n+1)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{1}{z^n}$$

$$= 1 + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= z\left[1+\frac{1}{z}+\frac{1}{2!}(1/z)^2+\frac{1}{3!}(1/z)^3+\cdots\right]$$

$$z\left\{\frac{1}{n+1!}\right\} = z\left[e^{1/z}-1\right] \qquad \left[\because 1+x=\frac{1}{2!}x^2+\frac{1}{3!}x^3=e^x\right]$$

(9) show that $z\left\{\cos\left(\frac{n\pi}{2}+\theta\right)\right\} = \dfrac{z^2\cos\theta - z\sin\theta}{z^2+1}$

Sol:

$$z\left\{\cos\left(n\pi/2+\theta\right)\right\} = z\left\{\cos n\pi/2\cos\theta - \sin n\pi/2\sin\theta\right\}$$

$$= \cos\theta\, z\left\{\cos n\pi/2\right\} - \sin\theta\, z\left\{\sin n\pi/2\right\}$$

$$= \cos\theta\cdot\frac{z^2}{z^2+1} - \sin\theta\cdot\frac{z}{z^2+1}$$

$$= \frac{z^2\cos\theta - z\sin\theta}{z^2+1}$$

(10) show that $z\left\{e^{-an}\cos n\theta\right\} = \dfrac{ze^a(ze^a-\cos\theta)}{z^2e^{2a}-2ze^a\cos\theta+1}$

Sol:

(i) We know that $z\{f(n)\} = z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

By Dumping rule

$$z\{a^{-n}f(n)\} = F(az)$$

$$z\left\{e^{-an}\cos n\theta\right\} = z\left\{(e^a)^{-n}\cos n\theta\right\} \qquad \left[a=e^a,\ a^{-n}=(e^a)^{-n}\right]$$

$$= \left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right] \qquad z\to ze^a$$

$$= \frac{ze^a[ze^a-\cos\theta)}{z^2e^{2a}-2ze^a\cos\theta+1}$$

(ii) $z\left\{e^{-an}\sin n\theta\right\} = \dfrac{ze^a\sin\theta}{z^2e^{2a}-2ze^a\cos\theta+1}$

Similarly for above problem.

2011 set ①.

Dumping rule (or) change of scale property:-

If $Z\{f(n)\} = F\{z\}$ then $Z\{a^{-n} f(n)\} = F(az)$

Proof:- By def $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z) \rightarrow ①$

$$\therefore Z\{a^{-n} f(n)\} = \sum_{n=0}^{\infty} a^{-n} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n) (az)^{-n}$$

$$= F(az) \text{ by } ①$$

Note:-

If $Z\{f(n)\} = F(z)$ then $Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$

$$= F(z) \quad z \rightarrow \frac{z}{a}$$

(or)

$$= Z\{f(n)\} \quad z \rightarrow \frac{z}{a}$$

2013 set-④

If $Z\{U_n\} = \bar{U}(z)$ prove that $Z\{a^n u_n\} = \bar{U}\left(\frac{z}{a}\right)$

Proof:-

① Find $Z\{a^n \cos nt\}$, $Z\{a^n \sin nt\}$ 2012 set ④

Sol: Let $Z\{a^n \cos nt\}$ Let $f(n) = \cos nt$

$$Z\{f(n)\} = Z\{\cos nt\} = \frac{z(z-\cos t)}{z^2 - 2z\cos t + 1} = F(z)$$

By Dumping rule,

$$Z\{a^n f(n)\} = Z\{f(n)\}_{z \rightarrow z/a}$$

$$\therefore Z\{a^n . \cos nt\} = \left.\frac{z(z-\cos t)}{z^2 - 2z\cos t + 1}\right]_{z \rightarrow z/a}$$

$$= \frac{z/a\left[z/a - \cos t\right]}{\frac{z^2}{a^2} - 2\,z/a \cos t + 1} = \frac{z(z - a\cos t)}{z^2 - 2az\cos t + a^2}$$

② Find $Z\{3^n \cos n\pi/2\}$ 2009 set ②

Sol: Let $f(n) = \cos n\pi/2$

$$Z\{f(n)\} = Z\{\cos n\pi/2\}$$

$$= \frac{z^2}{z^2+1}$$

By dumping rule $Z\{a^n f(n)\} = Z\{f(n)\}_{z \to z/a}$

$$\therefore Z\{3^n \cos n\pi/2\} = \left(\frac{z^2}{z^2+1}\right)_{z \to z/3} \qquad \boxed{a=3}$$

$$= \frac{\frac{z^2}{9}}{\frac{z^2}{9}+1} = \frac{z^2}{z^2+9}$$

③ 2012 set ④

find $Z\{a^n \sin nt\}$.

Sol:- Let $f(n) = \sin nt$

$$Z\{f(n)\} = Z\{\sin nt\} = \frac{z \sin t}{z^2 - 2z\cos t + 1}$$

By dumping rule,

$$Z\{a^n f(n)\} = Z\{f(n)\}_{z \to \frac{z}{a}}$$

$$\therefore Z\{a^n \sin nt\} = \frac{z/a \sin t}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos t + 1} = \frac{az \sin t}{z^2 - 2az\cos t + a^2}$$

④ Find $Z\{\cosh nt\}$

Sol:- We know that $Z\{\cos nt\} = \frac{z(z-\cos t)}{z^2 - 2z\cos t + 1}$ $\quad$ ($\because \cos it = \cosh t$)

Replace t by it,

$$Z\{\cos it\} = \frac{z(z-\cos it)}{z^2 - 2z\cos it + 1}$$

$$\therefore \quad z\{\sinh nt\} = \frac{z \sinh t}{z^2 - 2z\cosh t + 1}$$

z- transform of unit step function

A discrete unit step function is defined by

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Hence $z\{u[n]\} = \sum_{n=0}^{\infty} u(n) z^{-n}$ $\quad (-n = k)$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\tfrac{1}{z}\right)^{k}$$

$$= 1 + \tfrac{1}{z} + \tfrac{1}{z^2} + \cdots$$

$$= \left(1 - \tfrac{1}{z}\right)^{-1} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

$\therefore \boxed{z\{u(n)\} = \frac{z}{z-1}}$ where $u[n]$ is a unit step function.

unit Impulse Function:-

unit impulse function $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

z- transform of unit impulse function.

$$z\{\delta_n\} = \sum_{n=0}^{\infty} \delta_n z^{-n} \qquad \therefore z\{\delta[n]\} = 1$$

$$= z^0 = 1.$$

**

1) find $z\{f(k)\}$ where $f(k) = \left(\tfrac{1}{2}\right)^k u(k) \rightarrow$ unit function

Sol: $z\{f(k)\} = \sum^{\infty} f(k) z^{-k}$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k u(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k z^{-k} \qquad \left[\because u=1,\ k \geq 0;\ k=0,\ k<1\right]$$

$$= z\left\{f\left(\frac{1}{2}\right)^k\right\}$$

$$= \frac{z}{z-\frac{1}{2}} = 1+\frac{1}{2z}+\frac{1}{(2z)^2}+\cdots = \left(1-\frac{1}{2z}\right)^{-1} = \frac{2z}{2z-1}$$

(or)

$$= z\left\{\left(\frac{1}{2}\right)^k u(k)\right\}$$

$$= z\{2^{-k} u(k)\}$$

By dumping rule

$$= z\{u(k)\}_{z \to 2z}$$

$$= z\left(\frac{z}{z-1}\right)_{z \to 2z}$$

$$= \frac{2z}{2z-1} \quad //$$

** <u>Second shifting property</u>:-

Statement:-

If $z\{f(n)\} = F(z)$ then $z\{f(n+1)\} = zF(z) - zf(0)$

$$= z\{f(n)\} - z f(0)$$

<u>Proof</u>:-

$$z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1) z^{-n}$$

$$= z \sum_{n=0}^{\infty} f(n+1) z^{-(n+1)} \qquad [\because \text{Divide \& Multiple by } z]$$

Putting $n+1=m$, we get [when $m=0$, m varies from 1 to ∞

$$z\{f(n+1)\} = z \sum_{m=1}^{\infty} f(m) z^{-m}$$

[$f(0) z^0 = f(0)$ add and substitute]

$$= z\left[\sum_{m=0}^{\infty} f(m) z^{-m} - f(0)\right]$$

$z[f(z) - f(0)]$ $\quad \because f(z) = z\{f(n)\}$

similarly $z\{f(n+2)\} = \sum_{n=0}^{\infty} f(n+2) z^{-n}$

$$= z^2 \sum_{n=0}^{\infty} f(n+2) z^{-(n+2)}$$

put $n+2 = m$

$$z\{f(n+2)\} = z^2 \sum_{m=2}^{\infty} f(m) z^{-m}$$

$$= z^2 \left[\sum_{m=0}^{\infty} f(m) z^{-m} - f(0) - f(1) z^{-1} \right]$$

$$= z^2 [f(z) - f(0) - f(1) z^{-1}]$$

$$= z^2 F(z) - z^2 f(0) - f(1) z^{+1}$$

<u>Shifting properties:-</u>

i) shifting $f(n)$ to the Right

statement:- If $z\{f(n)\} = F(z)$ then $z\{f(n-k)\} = z^{-k} F(z)$

Proof:- we have $z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$

$$z\{f(n-k)\} = \sum_{n=0}^{\infty} f(n-k) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n-k) \cdot z^{-n+k-k}$$

$$= z^{-k} \sum_{n=0}^{\infty} f(n-k) \cdot z^{-(n-k)}$$

Let $n-k = m$ then

$$= z^{-k} \sum_{m=-k}^{\infty} f(m) z^{-m}$$

$$= z^{-k} \sum_{m=0}^{\infty} f(m) z^{-m}$$

$$= z^{-k} F(z)$$

[$\because f(m) = 0$ for $m < 0$ sequence defined only for +ve

Note:- $Z\{f(n-1)\} = z^{-1}F(z)$

$Z\{f(n-2)\} = z^{-2}F(z)$

ii) shifting $f(n)$ to the left:-

Statement:- If $Z\{f(n)\} = F(z)$ then $Z\{f(n+K)\} = z^{K}$ $[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} \cdots f(K-1)z^{-(K-1)}]$

proof:- By def

$$Z\{f(n+K)\} = \sum_{n=0}^{\infty} f(n+K) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n+K) \cdot z^{-n-K+K}$$

put $n+K=m$

$$= z^{K} \sum_{n=0}^{\infty} f(n+K) z^{-(n+K)}$$

$$= z^{K} \sum_{m=K}^{\infty} f(m) z^{-m}$$

$$= z^{K} \left[\sum_{m=0}^{\infty} f(m) z^{-m} - \sum_{m=0}^{K-1} f(m) z^{-m} \right]$$

$$= z^{K} \left[Z\{f(n)\} - \sum_{n=0}^{K-1} f(n) z^{-n} \right]$$

$$Z\{f(n+K)\} = z^{K} [F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} \cdots f(K-1)z^{-(K-1)}]$$

((K+1) terms

Note:- i) $Z\{f(n+1)\} = z[F(z) - f(0)]$

ii) $Z\{f(n+2)\} = z^{2}[F(z) - f(0) - f(1)z^{-1}]$

iii) $Z\{f(n+3)\} = z^{3}[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2}] \cdots$

Problems:-

1) Find $Z\left\{\frac{1}{n!}\right\}$ and hence evaluate $Z\left\{\frac{1}{(n+1)!}\right\}$

Sol: We have Let $f(n) = \frac{1}{n!}$

$$Z\left\{\frac{1}{\lfloor n!}\right\} = e^{1/z} = F(z)$$

By shifting property,

$$Z\{f(n+1)\} = Z[F(z) - f(0)]$$

$$\left[f(n) = \frac{1}{n!} \right.$$

$$f(0) = \frac{1}{0!} = 1$$

$$f(1) = 1$$

$$\therefore Z\left\{\frac{1}{(n+1)!}\right\} = Z[e^{1/z} - 1]$$

ii) $Z\{f(n+2)\} = z^2[F(z) - f(0) - f(1)z^{-1}]$

$$\therefore Z\left\{\frac{1}{(n+2)!}\right\} = z^2[e^{1/z} - 1 - z^{-1}]$$

2010 set ③

② Find $Z\left\{\frac{1}{(n+2)(n+1)}\right\}$

Sol:

$$Z\left\{\frac{1}{(n+2)(n+1)}\right\} = Z\left\{\frac{1}{n+1} - \frac{1}{n+2}\right\}$$

$$= Z\left\{\frac{1}{n+1}\right\} - Z\left\{\frac{1}{n+2}\right\} \rightarrow ①$$

Let $f(n) = \frac{1}{n+1}$

We know that

$$Z\left\{\frac{1}{n+1}\right\} = z\log\left[\frac{z}{z-1}\right] = F(z)$$

By shifting $Z\{f(n+1)\} = Z[F(z) - f(0)]$

$$Z\left\{\frac{1}{n+2}\right\} = Z\left[z\log\left(\frac{z}{z-1}\right) - 1\right] \quad \text{sub in ①}$$

$$① \Rightarrow Z\left\{\frac{1}{(n+2)(n+1)}\right\} = z\log\left(\frac{z}{z-1}\right) - z^2\log\left(\frac{z}{z-1}\right) + z$$

$$= z\left[\log\left(\frac{z}{z-1}\right)(1-z) - 1\right]$$

$$\left[\because \text{Let } f(k-1) = a^{k-1}\right.$$

$$f(k) = a^k$$

③ Prove that $z\{a^{k-1}\} = \frac{1}{z-a}$

Sol: Let $f(k) = a^k$

We know that $z\{f(k)\} = z\{a^k\} = \frac{z}{z-a} = F(z)$

By shifting theorem, $z\{f(n-1)\} = z^{-1}F(z)$

$$\therefore z\{a^{k-1}\} = z^{-1}\frac{z}{z-a}$$

$$= \frac{1}{z-a}$$

4) prove that $z\{(k-1)a^{k-1}\} = \frac{a}{(z-a)^2}$

Sol: By shifting theorem, $z\{(k-1)a^{k-1}\} = z^{-1}z\{k \cdot a^k\}$

$$= z^{-1} \cdot \frac{az}{(z-a)^2}$$

$$= \frac{a}{(z-a)^2}$$

5) Find $z\{n+2\} = z\{n\} + 2 \cdot z\{1\}$

Ans:- $\frac{2z^2+z}{(z-1)^2}$

6) find $z\{4 \cdot 3^n + 2(-1)^n\}$

Sol: $4z\{3^n\} + 2z\{(-1)^n\}$

$$= \frac{4 \cdot z}{z-3} + 2 \cdot \frac{z}{z+1}$$

7) If $z\{f(n)\} = \frac{z}{z-1} + \frac{z}{z^2+1}$, find $z\{f(n+2)\}$

Sol: By shifting theorem,

$$z\{f(n+2)\} = z^2[F(z) - f(0) - f(1)z^{-1}]$$

$$= z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 0 - \frac{1}{z}\left[\frac{z}{z-1} + \frac{z}{z^2+1}\right]\right]$$

$$= z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

Find $Z\{2.3^n+5.n\}$ and hence deduce $Z\{2.3^{n+3}+5(n+3)\}$ using shifting theorem.

Sol: $f(n) = 2.3^n+5n$

$$Z\{f(n)\} = Z\{2.3^n+5n\}$$

$$= 2Z\{3^n\} + 5Z\{n\}$$

$$= 2.\frac{z}{z-3} + 5\frac{z}{(z-1)^2} = F(z)$$

Also we have $f(0)=2$, $f(1)=11$, $f(2)=28$

By shifting theorem,

$$Z\{f(n+3)\} = z^3\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2}\right]$$

$$= z^3\left[\frac{2z}{z-3} + \frac{5z-2}{(z-1)^2} - 11z^{-1} - 28z^{-2}\right]$$

(9) find $Z\left\{\frac{1}{(n+2)(n+3)}\right\} = Z\left\{\frac{1}{n+2} - \frac{1}{n+3}\right\}$

Sol: Let $f(n) = \frac{1}{n+1}$ then find $Z\{f(n+1)\}$

$Z\{f(n+2)\}$

Ans: $Z\left[z + z(1-z)\log\frac{z}{z-1} - \frac{1}{z}\right]$

(10) Find z-transform of (i) $\sin(n+1)\theta$, (ii) $\cos(n+1)\theta$

Sol: (i) Let $f(n) = \sin n\theta$ then $f(0)=0$

$$Z\{f(n)\} = Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1} = F(z)$$

By shifting property

$$Z\{f(n+1)\} = Z\left[F(z) - f(0)\right]$$

$$\therefore Z\{\sin(n+1)\theta\} = Z\left[\frac{z\sin\theta}{z^2-2z\cos\theta+1} - 0\right] = \frac{z^2\sin\theta}{z^2-2z\cos\theta+1}$$

ii) similarly let $f(x) = \cos n\theta$.

Ans:- $\dfrac{z^2\cos\theta - z}{z^2 - 2z\sin\theta + 1}$

Multipication by n:-

prove that $z\{n f(n)\} = -z \dfrac{d}{dz}[F(z)]$ where $F(z) = z\{f(n)\}$

proof:- By def $z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z) \rightarrow (1)$

Dif on both sides w.r.t

$(1) \Rightarrow F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$

Differentiate on both sides

$$\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} f(n) \cdot -n z^{-n-1}$$

$$= -z^{-1} \sum_{n=0}^{\infty} n f(n) \cdot z^{-n}$$

$$= -z^{-1} z\{n f(n)\} \text{ by } (2)$$

$$\Rightarrow -z \frac{d}{dz}[F(z)] = z\{n f(n)\}$$

$$\Rightarrow z\{n f(n)\} = -z \frac{d}{dz}[z\{f(n)\}]$$

Note:- $z\{n^p\} = z\{n \cdot n^{p-1}\} = -z \dfrac{d}{dz}[z\{n^{p-1}\}]$

problems:-

1) Find (i) $z\{n^2\}$ (ii) $z\{n^3\}$ (iii) $z\{n^2 a^n\}$

sol: We know that

Let $f(n) = n$

$z\{f(n)\} = z\{n\} = \dfrac{z}{(z-1)^2}$

$$z\{n f(n)\} = -z \frac{d}{dz}[z\{f(n)\}]$$

$$\therefore z\{n \cdot n\} = -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z \left[\frac{(z-1)^2 \cdot 1 - z \cdot 2(z-1)}{(z-1)^4}\right]$$

$$= -z\,\frac{[(z-1)\{z-1-2z\}]}{(z-1)^4}$$

$$= \frac{-z(-z-1)}{(z-1)^3}$$

$$\therefore z\{n^2\} = \frac{z^2+z}{(z-1)^3} \quad //$$

(ii) $z\{n^3\} = z\{n \cdot n^2\}$

$$= -z\,\frac{d}{dz}\left[z(n^2)\right]$$

$$= -z\,\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left[\frac{(z-1)^3(2z+1) - (z^2+z)\cdot 3(z-1)^2}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)(2z+1) - (z^2+z)\cdot 3}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$\therefore z\{n^3\} = \frac{z(z^2+4z+1)}{(z-1)^4} \quad //$$

(iii) $z\{n^2 a^n\}$

Let $f(n) = n^2$

We know that $z\{n^2\} = \dfrac{z^2+z}{(z-1)^3} = F(z)$

By dumping Rule, $z\{a^n f(n)\} = [F(z)]$

$$\therefore z\{a^n n^2\} = \frac{\left(\frac{z}{a}\right)^2 + z}{\left(\frac{z}{a}-1\right)^3} = \frac{\left(\frac{z^2+az}{a^2}\right)}{\frac{(z-a)^3}{a^3}}$$

$$= \frac{a(z^2+az)}{(z-a)^3} \quad //$$

② Find $z\{n\cos n\theta\}$, $z\{n\sin n\theta\}$

Sol:

Here $z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} = \dfrac{z^2-z\cos\theta}{z^2-2z\cos\theta+1} = F(z)$

$$z\{n\cos n\theta\} = -z\cdot\frac{d}{dz}\left[\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right]$$

Differentiate this.

$=$

$=$

2014 set ①

Initial value theorem

If $z\{f(n)\} = F(z)$, then $\underset{z\to\infty}{Lt} F(z) = f(0)$ (or) $\underset{z\to\infty}{Lt} F(z) = \underset{n\to 0}{Lim} f(n)$

Proof:- We know that,

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n)z^{-n} = F(z)$$

$$\therefore F(z) = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

$$= f(0) + f(1)\cdot\tfrac{1}{z} + f(2)\cdot\tfrac{1}{z^2} + \cdots$$

Taking $z\to\infty$ on both sides

we get $\underset{z\to\infty}{Lt} F(z) = f(0)$

$= \underset{n\to 0}{Lt} f(n)$.

Note:- $f(0) = \underset{z\to\infty}{Lt} F(z)$

i) $f(1) = \underset{z\to\infty}{Lt}\, z\,[F(z) - f(0)]$

ii) $f(2) = \underset{z\to\infty}{Lt}\, z^2[F(z) - f(0) - f(1)z^{-1}]$

iii) $f(3) = \underset{z\to\infty}{Lt}\, z^3[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2}]$

final value theorem

If $Z\{f(n)\} = F(z)$ then $\underset{n\to\infty}{Lt} f(n) = \underset{z\to 1}{Lt} (z-1)F(z)$

proof:- we have

$$Z\{f(n+1)-f(n)\} = \sum_{n=0}^{\infty} [f(n+1)-f(n)]z^{-n}$$

$$\Rightarrow Z\{f(n+1)\} - Z\{f(n)\} = \sum_{n=0}^{\infty} [f(n+1)-f(n)]z^{-n}$$

$$\Rightarrow z[F(z)-f(0)] - F(z) = \sum_{n=0}^{\infty} [f(n+1)-f(n)]z^{-n}$$

$$\Rightarrow F(z)(z-1) - z f(0) = \sum_{n=0}^{\infty} [f(n+1)-f(n)]z^{-n}$$

Taking $\underset{z\to 1}{Lim}$ on both sides

We get

$$\underset{z\to 1}{Lt} (z-1)F(z) - f(0) = \sum_{n=0}^{\infty} [f(n+1)-f(n)]\cdot 1$$

$$= \underset{n\to\infty}{Lt} [f(1)-f(0) + f(2)-f(1) + f(3)-f(2) + \cdots f(n+1)-f(n)]$$

$$= \underset{n\to\infty}{Lt} [f(n+1)-f(0)]$$

$$\therefore \underset{z\to 1}{Lt} (z-1)F(z) - f(0) = \underset{n\to\infty}{Lt} f(n+1) - f(0)$$

$$\underset{n\to\infty}{Lt} f(n) = \underset{z\to 1}{Lt} (z-1)F(z)$$

Problems:-

(1) If $F(z) = \dfrac{5z^2+3z+12}{(z-1)^4}$, find value of $f(2)$ & $f(3)$

Sol: Given that $F(z) = \dfrac{5z^2+3z+12}{(z-1)^4}$

$$= \frac{z^2(5+3z^{-1}+12z^{-2})}{z^4(1-z^{-1})^4}$$

$$\therefore f(z) = \frac{1}{z^2}\,\frac{\left(5+\frac{3}{z}+\frac{12}{z^2}\right)}{\left(1-\frac{1}{z}\right)^4}$$

By intial value theorem, $f(0) = \underset{z\to\infty}{Lt}\, f(z)$

$$= \underset{z\to\infty}{Lt}\left[\frac{1}{z^2}\,\frac{5+\frac{3}{z}+\frac{12}{z^2}}{\left(1-\frac{1}{z}\right)^4}\right] = 0$$

$$f(1) = \underset{z\to\infty}{Lt}\, z\left[f(z)-f(0)\right] = \underset{z\to\infty}{Lt}\, z\left[\frac{1}{z^2}\,\frac{\left(5+\frac{3}{z}+\frac{12}{z^2}\right)}{\left(1-\frac{1}{z}\right)^4}\right]$$

$$= 0$$

$$f(2) = \underset{z\to\infty}{Lt}\, z^2\left[f(z)-f(0)-f(1)z^{-1}\right]$$

$$= \underset{z\to\infty}{Lt}\, z^2\left[\frac{1}{z^2}\,\frac{\left(5+\frac{3}{z}+\frac{12}{z^2}\right)}{\left(1-\frac{1}{z}\right)^4}\right] = 5$$

$$f(3) = \underset{z\to\infty}{Lt}\, z^3\left[f(z)-f(0)-f(1)z^{-1}-f(2)z^{-2}\right]$$

$$= \underset{z\to\infty}{Lt}\, z^3\left[f(z)-0-0-5.z^{-2}\right]$$

$$= \underset{z\to\infty}{Lt}\, z^3\left[\frac{5z^2+3z+12}{(z-1)^4} - \frac{5}{z^2}\right]$$ Let L.C.M

$$\therefore f(3) = \underset{z\to\infty}{Lt}\, z^3\left[\frac{5z^4+3z^3+12z^2-5(z-1)^4}{(z-1)^4\, z^2}\right]$$ $\left[\because (x+a)^n = {}^nC_0\, x^n + {}^nC_1 \cdot x^{n-1}a^1\right]$

$$= \underset{z\to\infty}{Lt}\, z^3\left[\frac{5z^4+3z^3+12z^2-5(z^4-4z^3+6z^2-4z+1)}{z^2\left\{z^4\left(1-\frac{1}{z}\right)^4\right\}}\right]$$

$$= \underset{z\to\infty}{Lt}\, z^3\left[\frac{23z^3-18z^2+20z-5}{z^6\left(1-\frac{1}{z}\right)^4}\right]$$

$$= \underset{z\to\infty}{Lt}\, \frac{\cancel{z^3}}{\cancel{z^6}} \times \cancel{z^3}\left[\frac{23-18\left(\frac{1}{z}\right)+\frac{20}{z^2}-\frac{5}{z^3}}{\left(1-\frac{1}{z}\right)^4}\right] = \frac{23-0+0+0}{1}$$

$$= 23$$

(iv) If $f(z) = \dfrac{2z^2+3z+4}{(z-3)^3}$, $|z|>3$, then find the values of $f(1)$, $f(2)$ & $f(3)$.

Sol: Given $f(z) = \dfrac{2z^2+3z+4}{(z-3)^3}$

$$= \frac{z^2(2+3/z+4/z^2)}{z^3(1-3/z)^3}$$

$$= \frac{1}{z}\frac{(2+3/z+4/z^2)}{(1-3/z)^3}$$

By intial value theorem

$$f(0) = \underset{z\to 0}{Lt} f(z) = 0$$

$$f(1) = \underset{z\to 0}{Lt} z\left[f(z) - f(0)\right]$$

$$= \underset{z\to\infty}{Lt} z\left[\frac{1}{z}\frac{[2+3/z+4/z^2)}{(1-3/z)^3}\right] - 0$$

$$= 2$$

$$f(2) = \underset{z\to\infty}{Lt} z^2\left[f(z) - f(0) - f(1)z^{-1}\right]$$

$$= \underset{z\to\infty}{Lt} z^2\left[\frac{2z^2+3z+4}{(z-3)^3} - 0 - \frac{2}{z}\right]$$

$$= \underset{z\to\infty}{Lt} z^2\left[\frac{2z^2+3z^2+4z-2(z-3)^3}{z(z-3)^3}\right]$$

$$= \underset{z\to\infty}{Lt} z^2\left[\frac{2z^3+3z^2+4z-2(z^3-9z^2+27z-27)}{2.z^3(1-3/z)^3}\right]$$

$$= \underset{z\to\infty}{Lt} \frac{z^2}{z^4}\left[\frac{21z^2-50z+54}{(1-3/z)^3}\right]$$

$$= \underset{z\to\infty}{Lt} \frac{z^2}{z^4}\cdot z^2\left[\frac{21-50/z+\frac{54}{z^2}}{(1-3/z)^3}\right]$$

$= 21.$

$$f(3) = \underset{z\to\infty}{Lt}\ z^3\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2}\right]$$

$$= \underset{z\to\infty}{Lt}\ z^3\left[\frac{2z^2+3z+4}{(z-2)^3} - 0 - \frac{2}{z} - \frac{21}{z^2}\right]$$

$$= \underset{z\to\infty}{Lt}\ z^3\left[\frac{2z^4+3z^3+4z^2-2z(z-2)^3-21(z-2)^3}{z^2(z-2)^3}\right]$$

$$= \underset{z\to\infty}{Lt}\ z^3\left[\frac{139z^2-513z+567}{z^2(z-2)^3}\right]$$

$$= \underset{z\to\infty}{Lt}\ z^3 \cdot \frac{1}{z^2} \cdot \frac{z^2\left[139 - \frac{513}{z} + \frac{567}{z^2}\right]}{z^3(1-2/z)^3}$$

$$= 139$$

③ If $z\{f(n)\} = \frac{z}{z-1} + \frac{z}{z^2+1}$ find $z\{f(n+2)\}$

Sol:

Given, $z\{f(n)\} = \frac{z}{z-1} + \frac{z}{z^2+1} \rightarrow ①$

By intial value theorem

$$f(0) = \underset{z\to\infty}{Lt}\ F(z) = \underset{z\to\infty}{Lt}\left[\frac{z}{z-1} + \frac{z}{z^2+1}\right]$$

$$= \underset{z\to\infty}{Lt}\left[\frac{1}{1-1/z} + \frac{1}{z(1+1/z^2)}\right]$$

$$= 1 + 0 = 1$$

$$f(1) = \underset{z\to\infty}{Lt}\ z[F(z) - f(0)] = \underset{z\to\infty}{Lt}\ z\left\{\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right\}$$

$$= \underset{z\to\infty}{Lt}\ z\left[\frac{2z^2-z+1}{(z-1)(z^2+1)}\right]$$

$$= \underset{z\to\infty}{Lt}\ \frac{z \cdot z^2(2-1/z+1/z^2)}{z \cdot z^2(1-1/z)(1+1/z^2)} = 2$$

By shifting theorem

$$z\{f(n+2)\} = z^2[F(z) - f(0) - f(1)z^{-1}] = z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - 2/z\right]$$

If $f(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$ find u_2 and u_3

Sol: Given,

$$u(z) = \frac{2z^2+5z+14}{(z-1)^4}$$

$$u(z) = \frac{z^2(2+5/z+14/z^2)}{z^4(1-1/z)^4} = \frac{1}{z^2}\left[\frac{2+5/z+14/z^2}{(1-1/z)^4}\right]$$

$$u(0) = \underset{z\to\infty}{Lt}\ u(z) = \underset{z\to 0}{Lt}\ \frac{1}{z^2}\frac{[2+5/z+14/z^2)}{(1-1/z)^4} = 0$$

$$u(1) = \underset{z\to\infty}{Lt}\ z\left[u(z)-u(0)\right] = \underset{z\to\infty}{Lt}\ z\cdot\frac{1}{z^2}\left[\frac{2+5/z+14/z^2}{(1-1/z)^4}\right] = 0$$

$$u(2) = \underset{z\to\infty}{Lt}\ z^2\left[u(z)-u(0)-u(1)z^{-1}\right]$$

$$= \underset{z\to\infty}{Lt}\ z^2\cdot\frac{1}{z^2}\left[\frac{2+5/z+14/z^2}{(1-1/z)^4}\right]$$

$$= 2.$$

$$u(3) = \underset{z\to\infty}{Lt}\ z^3\left[u(z)-u(0)-u(1)z^{-1}-u(2)z^{-2}\right]$$

$$= \underset{z\to\infty}{Lt}\ z\ \frac{[2+5/z-14/z^2)}{(1-1/z)^4} - 0 - 0 - \frac{2}{z^2}$$

$$= \underset{z\to\infty}{Lt}\ z\cdot\frac{\left[z^2(2+5/z-14/z^2)-2\right.}{\left(\frac{z-1}{3}\right)^4 z^2}$$

unit-V

Table of some standard z-transforms

$f(n)$ or $f(k)$	$z\{f(n)\}$
1	$\frac{z}{z-1}$
a^n or a^k	$\frac{z}{z-a}$
$(-1)^n$ or $(-1)^k$	$\frac{z}{z+1}$
n (or) k	$\frac{z}{(z-1)^2}$
na^n (or) ka^k	$\frac{az}{(z-a)^2}$
$\frac{1}{n!}$ or $\frac{1}{k!}$	$e^{1/z}$
$\frac{1}{n+1}$ or $\frac{1}{k+1}$	$-z\log(1-1/z)$ or $z\log\left(\frac{z}{z-1}\right)$
$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

The inverse z-transform:-

Def:- If $z\{f(n)\} = F(z)$ then the inverse z-transform is defined by $z^{-1}\{F(z)\} = f(n)$ or $z^{-1}\{F(z)\} = f(k)$.

Some standard Inverse "z" transforms

① $z\{a^n\} = \frac{z}{z-a} \Rightarrow z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$ or a^k

② $z\left\{\frac{1}{n!}\right\} = e^{1/z} \Rightarrow z^{-1}\{e^{1/z}\} = 1/n!$ or $\frac{1}{k!}$

③ $z\{n\} = \frac{z}{(z-1)^2} \Rightarrow z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$ or k

④ $z\{na^n\} = \frac{az}{(z-a)^2} \Rightarrow z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n$ or ka^k

⑤ $z\{a^{k-1}\} = \frac{1}{z-a} \Rightarrow z^{-1}\left[\frac{1}{z-a}\right] = a^{k-1}$ or a^{n-1}

⑥

Problems:-

(1) Find the inverse z-transform of $\frac{z}{(z-1)(z-2)}$

Sol: Let $F(z) = \frac{z}{(z-1)(z-2)}$ $\therefore \frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$

$\therefore \frac{F(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$ $= \frac{A}{z-1} + \frac{B}{z-2}$

$\Rightarrow F(z) = \frac{z}{z-2} - \frac{z}{z-1}$

For A put $z=1$ $\left| \frac{1}{1-2} = -1 \right.$ $\left[\because z^{-1}\left[\frac{z}{z-a}\right] = a^n.\right.$

$F(z) = \frac{z}{z-2} - \frac{z}{z-1}$

For B put $z=2$ $\left| \frac{1}{2-1} = 1 \right.$

Let z^{-1} o.b.s we get $z^{-1}\{F(z)\} = z^{-1}\left[\frac{z}{z-2}\right] - z^{-1}\left[\frac{z}{z-1}\right]$

$= 2^n - 1$ //

*** (2) 2004, 2005, 2007, 2010, 2011, 2012, 2013 set (4)

* Find $z^{-1}\left[\frac{z}{z^2+11z+24}\right]$

Sol: Let $z^{-1}\left[\frac{z}{z^2+11z+24}\right]$

$\Rightarrow \frac{z}{(z+3)(z+8)}$

A put $z=-3$ $\left| \frac{1}{-3+8} = \frac{1}{5} \right.$

$\Rightarrow \frac{F(z)}{z} = \frac{1}{(z+3)(z+8)} = \frac{A}{z+3} + \frac{B}{z+8}$

B $z=-8$ $\left| \frac{1}{-8+3} = \frac{-1}{5} \right.$

$\therefore \frac{F(z)}{z} = \frac{1}{5}\frac{1}{z+3} + \frac{(-1/5)}{z+8}$ $z^{-1}\left[\frac{z}{z-a}\right] = a^n$

$\Rightarrow F(z) = \frac{1}{5}\frac{z}{z+3} - \frac{1}{5}\frac{z}{z+8}$

Let z^{-1} on both sides

$z^{-1}\{F(z)\} = \frac{1}{5} z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} z^{-1}\left[\frac{z}{z+8}\right]$

$= \frac{1}{5}(-3)^n - \frac{1}{5}(-8)^n$ //

Sol: Let $F(z) = \dfrac{z}{z^2+8z+15} = \dfrac{z}{(z+3)(z+5)}$

A put $z=-3$ | $\dfrac{1}{-3+5} = \dfrac{1}{2}$

B put $z=-5$ | $\dfrac{1}{-5+3} = \dfrac{-1}{2}$

$\therefore \dfrac{F(z)}{z} = \dfrac{1}{(z+3)(z+5)} = \dfrac{A}{z+3} + \dfrac{B}{z+5}$

$$\frac{F(z)}{z} = \frac{1/2}{z+3} + \frac{-1/2}{z+5}$$

$$z^{-1}\{F(z)\} = \frac{1}{2} z^{-1}\left(\frac{z}{z+3}\right) - \frac{1}{2} z^{-1}\left(\frac{z}{z+5}\right)$$

$$= \frac{1}{2}\left[(-3)^n - (-5)^n\right]$$

(4) Find inverse z-transform of $z^{-1}\left\{\dfrac{2z^2+38}{(z+2)(z-4)}\right\}$

Sol: Let

$$\Rightarrow \frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$$

$A = 1/6, \quad B = 11/6$

$\therefore F(z) = \dfrac{1}{6} \cdot \dfrac{z}{z+2} + \dfrac{11}{6} \cdot \dfrac{z}{z-4}$ Let

z^{-1} on both sides

$$z^{-1}[F(z)] = \frac{1}{6} z^{-1}\left(\frac{z}{z+2}\right) + \frac{11}{6} z^{-1}\left(\frac{z}{z-4}\right)$$

$$= \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$$

5) Find inverse laplace z transform $z^{-1}\left\{\dfrac{z^2+z}{(z-1)^2}\right\}$

Sol: Let $F(z) = \dfrac{z^2+z}{(z-1)^2}$

$$\frac{F(z)}{z} = \frac{z+1}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2}$$

$A = 1, \quad B = 2$

$$\frac{F(z)}{z} = \frac{1}{z-1} + \frac{2}{(z-1)^2}$$

$$F(z) = \underline{z} + 2\underline{z}$$

$(z-1)^2 \}$

$= 1+2n$ //

⑥ find $z^{-1}\left\{\frac{z^2-z}{(z-1)^2}\right\}$ similarly for above problem.

Ans:- $1-2N$

⑦ find $z^{-1}\left\{\frac{z^2-3z}{(z+2)(z-5)}\right\}$

Sol:

Let $F(z) = \frac{z^2-3z}{(z+2)(z-5)}$

$$\frac{F(z)}{z} = \frac{z-3}{(z+2)(z-5)} = \frac{A}{z+2} + \frac{B}{z-5}$$

$A = 5/7$, $B = 2/7$

$$\frac{F(z)}{z} = \frac{5}{7}\frac{z}{z+2} + \frac{2}{7}\frac{z}{z-5}$$

$$z^{-1}\{F(z)\} = 5/7\, z^{-1}\left(\frac{z}{z+2}\right) + \frac{2}{7} z^{-1}\left(\frac{z}{z-5}\right)$$

$$z^{-1}\{F(z)\} = 5/7 \cdot (-2)^n + 2/7\, (5)^n$$ //

⑧ find $z^{-1}\left\{\frac{z}{z^3-7z^2+14z-8}\right\} = z^{-1}\left\{\frac{z}{(z-1)(z-2)(z-4)}\right\}$

Sol:

$$F(z) = \frac{z}{z^3-7z^2+14z-8}$$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)(z-4)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-4}$$

$\therefore A = 1/3 \quad B = -1/2 \quad C = 1/6$

$$\therefore F(z) = \frac{1}{3}\frac{z}{z-1} - 1/2\,\frac{z}{z-2} + 1/6\,\frac{z}{z-4}$$

z^{-1} on both sides

'3 '2 '6 ''

(9) find inverse z-transform of $\frac{z}{(z-1)(z^2+1)}$

Sol: Let $f(z) = \frac{z}{(z-1)(z^2+1)}$

$$\frac{f(z)}{z} = \frac{1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$1 = A(z^2+1) + Bz+C(z-1)$$

put $z=1 \Rightarrow A = 1/2$

equation z^2 coefficient on b.s we get $A+B=0$ $A=-B$ $\boxed{B=-1/2}$

Equate constant on b.s we get $1 = A-C \Rightarrow$

$C = A-1$

$C = A-1$

$C = 1/2 - 1 = -1/2$

$$\therefore \frac{f(z)}{z} = \frac{1}{2}\cdot\frac{1}{z-1} + \frac{-1/2\, z - 1/2}{z^2+1}$$

$$f(z) = \frac{1}{2}\frac{z}{z-1} - 1/2 \frac{z(z+1)}{z^2+1}$$

$$z^{-1}\{f(z)\} = 1/2\, z^{-1}\left[\frac{z}{z-1}\right] - 1/2\left[z^{-1}\left[\frac{z^2}{z^2+1}\right] + \frac{z}{z^2+1}\right]$$

$$= \frac{1}{2}\left[1 - \cos n\pi/2 - \sin n\pi/2\right]$$

(10) find $z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right]$

Sol: Let $f(z) = \frac{z-4}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\frac{A}{z=-2}$ | $\frac{-2-4}{-2+3} = -6$

$\frac{B}{z=-3}$ | $\frac{-3-4}{-3+2} = 7$

$$= \frac{-6}{z+2} + \frac{7}{z+3}$$

$$z^{-1}\{f(z)\} = 7\, z^{-1}\left[\frac{1}{z+3}\right] - 6\, z^{-1}\left[\frac{1}{z+2}\right]$$

$$\left[\because z^{-1}\left[\frac{1}{z-a}\right] = a^{k-1}\right]$$

$$= 7\cdot(-3)^{n-1} - 6\,(-2)^{n-1}$$

$\{(z-2)(z-1)\}$

Sol:

$$F(z) = \frac{z+1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}$$

$A=3,\ B=-2$

$$z^{-1}\{F(z)\} = 3 \cdot z^{-1}\left\{\frac{1}{z-2}\right\} - 2\, z^{-1}\left\{\frac{1}{z-1}\right\}$$

$$= 3 \cdot (2)^{n-1} - 2(1)$$

12) Find $z^{-1}\left\{\frac{4z^2-2z}{z^3-5z^2+8z-4}\right\} = z^{-1}\left\{\frac{4z^2-2z}{(z-1)(z-2)^2}\right\}$

Sol:

$$\frac{F(z)}{z} = \frac{4z-2}{(z-1)(z-2)^2}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

$(\therefore A=2$
$B=2$
$C=12$

Ans:- $2+2^k+6(k-1)2^{k-1}$

Convolution:-

convolution of two sequences $f(n)$ & $g(n)$ is defined as, $f(n) * g(n) = \sum_{m=0}^{n} f(m)\, g(n-m)$, here $*$ denotes the convolution operator.

convolution theorem:-

If $z^{-1}\{F(z)\} = f(n)$ and $z^{-1}\{G(z)\} = g(n)$ then

$$z^{-1}\{F(z) \cdot G(z)\} = f(n) * g(n) = \sum_{m=0}^{n} f(m) \cdot g(n-m)$$

Proof:- By def of z- transform

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} = F(z)$$

and $z\{g(n)\} = \sum^{\infty} g(n)\, z^{-n} = G(z)$

$$F(z)\cdot G(z) = [f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots + f(n)z^{-n} + \cdots]$$

$$\times [g(0) + g(1)z^{-1} + g(2)z^{-2} + \cdots g(n-1)z^{-(n-1)} + g(n)z^{-n} + \cdots]$$

$$= \sum_{n=0}^{\infty} [f(0)g(n) + f(1)g(n-1) + \cdots + f(n)g(0)]z^{-n}$$

$$= Z\{f(0)g(n) + f(1)g(n-1) + \cdots + f(n)g(0)] \text{ by def z-transform}$$

$$Z^{-1}\{F(z)\cdot G(z)\} = f(0)g(n) + f(1)g(n-1) + \cdots f(n)g(0)$$

$$= \sum_{m=0}^{n} f(m)g(n-m)$$

$$= f(n) * g(n) \quad \text{Hence proved}$$

Problems:-

① Evaluate $Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\}$ by using convolution theorem.

using convolution theorem, evaluate $Z^{-1}\left\{\frac{z^2}{z^2-4z+3}\right\}$

Sol: let $Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\} = Z^{-1}\left\{\frac{z}{z-1}\cdot\frac{z}{z-3}\right\}$

let $F(z) = \frac{z}{z-1}$ and $G(z) = \frac{z}{z-3}$

Z^{-1} on both sides

$$Z^{-1}\{F(z)\} = Z^{-1}\left\{\frac{z}{z-1}\right\} = 1 = f(n),$$

$$Z^{-1}\{G(z)\} = Z^{-1}\left\{\frac{z}{z-3}\right\} = 3^n = g(n)$$

By convolution theorem,

$$Z^{-1}\{F(z)\cdot G(z)\} = \sum_{m=0}^{n} f(m)\cdot g(n-m)$$

$$Z^{-1}\left\{\frac{z}{z-1}\cdot\frac{z}{z-3}\right\} = \sum^{n} 1\cdot 3^{n-m}$$

$= 3 + 3 + 3 + \cdots 3$ is in G.P $\left[\begin{array}{l} a=1 \\ n=3 \end{array}\right.$

$= 1 \cdot \dfrac{3^{n+1}-1}{3-1}$

$= \dfrac{1}{2}(3^{n+1}-1)$ //

(2) JNTU 2012 SCT (4)

* Evaluate $z^{-1}\left\{\dfrac{z^2}{(z-a)(z-b)}\right\}$ by convolution theorem.

Sol:

Let $f(z) = \dfrac{z}{z-a}$ and $G(z) = \dfrac{z}{z-b}$

$z^{-1}\{F(z)\} = z^{-1}\left[\dfrac{z}{z-a}\right] = a^n, \; = f(n)$

$z^{-1}\{G(z)\} = z^{-1}\left[\dfrac{z}{z-b}\right] = b^n, \; = g(n)$

By convolution theorem,

$$z^{-1}\{F(z) \cdot G(z)\} = \sum_{m=0}^{n} f(m) \cdot g(n-m)$$

$$\Rightarrow z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-b}\right\} = \sum_{m=0}^{n} a^m \cdot b^{n-m}$$

$$= a^0 b^n + a^1 b^{n-1} + a^2 b^{n-2} + \cdots + a^n b^0$$

$$= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

is in G.P

$$= b^n \left[\frac{(a/b)^{n+1} - 1}{(a/b) - 1}\right]$$

$$= b^n \left[\frac{\dfrac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\left(\dfrac{a-b}{b}\right)}\right] = b^n \cdot \left[\frac{1}{b^n}\left(\frac{a^{n+1} - b^{n+1}}{a-b}\right)\right]$$

$$z^{-1}\left[\frac{z^2}{\cdots}\right] = \underline{a^{n+1} - b^{n+1}}.$$

$\{(z-4)(z-5)\}$ ()

Sol: $\boxed{\text{2 problem}}$

put $a=4$, $b=5$ in prob ②

We get As:- $\dfrac{4^{n+1}-5^{n+1}}{4-5} = 5^{n+1}-4^{n+1}$;)

④ Evaluate $z^{-1}\left\{\left(\dfrac{z}{z-a}\right)^2\right\}$

Sol: Let $f(z) = \dfrac{z}{z-a}$, $G(z) = \dfrac{z}{z-a}$

$z^{-1}\{F(z)\} = z^{-1}\left\{\dfrac{z}{z-a}\right\} = a^n = f(n)$

$z^{-1}\{G(z)\} = z^{-1}\left\{\dfrac{z}{z-a}\right\} = a^n = g(n)$

By convolution theorem,

$$z^{-1}\{F(z)\cdot G(z)\} = \sum_{m=0}^{n} f(m)\cdot g(n-m)$$

$$\Rightarrow z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = \sum_{m=0}^{n} a^m \cdot a^{n-m}$$

$$= \sum_{m=0}^{n} a^n$$

$$z^{-1}\left(\frac{z^2}{(z-a)^2}\right) = (n+1)a^n$$

(or)

$$= a^n \cdot \sum_{m=0}^{n} (1) \quad (n+1) \text{ times}$$

$$= a^n(n+1);$$

(5) using on the to evaluate $z^{-1}\left\{\left(\dfrac{z}{z-a}\right)^3\right\}$

Sol: We know that $z^{-1}\left\{\left(\dfrac{z}{z-a}\right)^3\right\} = a^n(n+1)$

Let $f(z) = \left(\dfrac{z}{z-a}\right)^2$; $G(z) = \dfrac{z}{z-a}$

$z^{-1}\{F(z)\} = a^n(n+1) = f(n)$, $z^{-1}\{G(z)\} = a^n = g(n)$

by convolution theorem.

$$z^{-1}\left\{\left(\frac{z}{z-a}\right)^2 \cdot \frac{z}{z-a}\right\} = \sum_{m=0}^{\infty} a^m (m+1) a^{n-m}$$

$$= \sum_{m=0}^{n} a^n (m+1)$$

$$= a^n \sum_{m=0}^{n} (m+1)$$

$$= a^n [1+2+3+ -- (n+1)]$$

$$= a^n \frac{(n+1)(n+2)}{2}$$

[∴ sum of 'n' terms

$\sum n = \frac{n(n+1)}{2}$

Sum of $(n+1)$ terms

$\sum n+1 = \frac{(n+1)(n+2)}{2}$

Application of z-transform:-

Difference equation:-

An equation which involves difference functions at one or more values of argument is said to be a difference equation.

Ex|| $y_{n+2} - y_{n+1} + y_n = 2$ $\quad$ [$\Delta y_{n+1} = y_{n+2} - y_{n+1}$

To solve dif equation we using z-transforms

Working rule:-

To solve a given dif equation:-

1) Let $z\{y_n\} = z\{y(n)\} = y(z)$ (or) $y(n) = z^{-1}\{y(z)\}$

2) Take z-transform on b.s of given diff equation

3) use the formulae $z\{y_n\} = y(z)$

$$z\{y_{n+1}\} = z\,[y(z) - y(0)]$$

$$z\{y_{n+2}\} = z^2[y(z) - y(0) - y_1 z^{-1}]$$

4) simplify ...

solution y_n.

Problems:-

① Solve $y_{n+1} - 2y_n = 0$ by using z-transform

Sol: Let $z\{y_n\} = y(z)$

Then, $z\{y_{n+1}\} = z\{y(n+1)\} = z[y(z) - y(0)]$

Apply z-transform to ① on b.s

$① \Rightarrow z\{y_{n+1}\} - 2z\{y_n\} = 0$

$\Rightarrow z[y(z) - y_0] - 2y(z) = 0$

$\Rightarrow (z-2)\, y(z) = zy_0$

$\Rightarrow y(z) = \dfrac{zy_0}{z-2}$

Let z^{-1} on both sides

we get $z^{-1}\{y(z)\} = y_0\, z^{-1}\left[\dfrac{z}{z-2}\right]$

$\Rightarrow y(n) = y_0 \cdot 2^n$ // is sol of ①

② using the z-transform, solve $y_{n+2} - 7y_{n+1} + 12y_n = 0$

Sol: Given that

$y_{n+2} - 7y_{n+1} + 12y_n = 0 \rightarrow ①$

Given, $y_0 = 1 \quad y_1 = 2$

Given eqn ①

Let z-transforms to ① o.b.s

$① \Rightarrow z\{y_{n+2}\} - 7z\{y_{n+1}\} + 12z\{y_n\} = 0$

$\Rightarrow z^2[y(z) - y(0) - y(1)z^{-1}] - 7z[y(z) - y(0)] + 12(y(z)) = 0$

$\Rightarrow y(z)\cdot[z^2-7z+12] = z^2 - 5z$

$$y(z) = \frac{z^2-5z}{z^2-7z+12}$$

$$\Rightarrow \frac{y(z)}{z} = \frac{z-5}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$$

A, put $z=3$

$\frac{-2}{-1} = 2$

$A = 2, \quad B = -1$

$$\therefore \frac{y(z)}{z} = \frac{2}{z-3} - \frac{1}{z-4}$$

$$y(z) = 2\cdot\frac{z}{z-3} - \frac{z}{z-4}$$

Let z^{-1} on b.s

$$z^{-1}\{y(z)\} = 2\cdot z^{-1}\left\{\frac{z}{z-3}\right\} - z^{-1}\left\{\frac{z}{z-4}\right\}$$

$\Rightarrow y(n) = 2\cdot 3^n - 4^n$ is sol of ①

② solve the diff equation $y(n+2)+3(y(n+1)+2y(n)=0$

Sol: Given $y(0)=0$, $y(1)=1$

Let z- transform to ① on b.s

We get $z^2[y(z)-y(0)-y(1)z^{-1}] + 3z[y(z)-y(0)] + 2z[y(n)] = 0$.

$\Rightarrow y(z)[z^2+3z+2] - z = 0$

$$\Rightarrow y(z) = \frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)}$$

A; $z=-1$

$\frac{1}{-1+2} = 1$

$$\therefore \frac{y(z)}{z} = \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \rightarrow ②$$

$$\frac{y(z)}{z} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$y(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

Let z^{-1} on b.s

$$z^{-1}\{y(z)\} = z^{-1}\left\{\frac{z}{z+1}\right\} - z^{-1}\left\{\frac{z}{z+2}\right\}$$

$y(n) = (-1)^n - (-2)^n$ is sol of ①

$u(0)=0,\ u(1)=1$

by z-transforms

Sol: Let z-transforms on both sides to ①

$$① \Rightarrow z\{u_{n+1}\} + \tfrac{1}{4} z[u_n] = z\left\{\left(\tfrac{1}{4}\right)^n\right\}$$

$$\Rightarrow z[u(z) - u(0)] + \tfrac{1}{4} u(z) = \frac{z}{z-\frac{1}{4}}$$

$$u(z)\left(z+\tfrac{1}{4}\right) = \frac{z}{z-\frac{1}{4}}$$

$$u(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)}$$

$$\frac{u(z)}{z} = \frac{1}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z+\frac{1}{4}} \rightarrow ②$$

We get $A=2,\ B=-2$ sub in ②

$$② \Rightarrow \frac{u(z)}{z} = \frac{2}{z-\frac{1}{4}} - \frac{2}{z+\frac{1}{4}}$$

$$\therefore u(z) = 2\cdot\frac{z}{z-\frac{1}{4}} - 2\cdot\frac{z}{z+\frac{1}{4}}$$

Let z^{-1} on both sides

We get $u(n) = 2\left(\frac{1}{4}\right)^n - 2\left(-\frac{1}{4}\right)^n$

2008 set ①

5) Solve $y(k+2) - 5y(k+1) + 6y(k) = 5^k$, given $y(0)=0,\ y(1)=0$

Sol: Let z-transforms on both sides to ①

$$① \Rightarrow z^2[y(z) - y(0) - y(1)z^{-1}] - 5z[y(z) - y(0)] + 6y(z) = \frac{z}{z-5}$$

$$\Rightarrow y(z)[z^2 - 5z + 6] = \frac{z}{z-5}$$

$$\frac{y(z)}{z} = \frac{1}{(z-5)(z^2-5z+6)} = \frac{1}{(z-5)(z-2)(z-3)}$$

② $\Rightarrow \frac{y(z)}{z} = \frac{1}{6}\cdot\frac{1}{z-5} + \frac{1}{3}\frac{1}{z-2} - \frac{1}{2}\cdot\frac{1}{z-3}$

$y(z) = \frac{1}{6}\frac{z}{z-5} + ---$ Let z^{-1} on b.s

We get $y(n) = \frac{1}{6}(5^k) + \frac{1}{3}(2)^k - \frac{1}{2}(3^k)$ is sol of ①

⑥ solve the diff equation $y(k+2) + 2(y(k+1) + y(k) = u(k)$

where $y(0)=0$, $y(1)=0$ and $u(k)=k$ for $k=0,1,2$ - - - - -

Ans:- $\frac{1}{4}n - \frac{1}{4}(1) + \frac{1}{4}(-1)^n - \frac{1}{4}n(-1)^n$

⑦ solve $u_{n+2} - 3u_{n+1} + 2u_n = 0$ given $u_0=0$, $u_1=1$ Ans: 2^n-1

Sol: Given def eqn ①

Let z on both sides to ①

We get $z\{u_{n+2}\} - 3z(u_{n+1}) + 2z[u_n] = 0$

$\Rightarrow z^2[u(z) - u(0) - u(1)z^{-1}] - 3z[u(z) - u(0)] + 2u(z) = 0$

$\Rightarrow z^2[u(z) - 0 - 1.z^{-1}] - 3z[u(z) - 0] + 2u(z) = 0$

$\Rightarrow u(z).(z^2 - 3z + 2) - z = 0$

$\Rightarrow u(z) = \frac{z}{z^2-3z+2}$

$\frac{u(z)}{z} = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow ②$

$1 = A(z-2) + B(z-1)$

put $z=1$

$1 = -A$

$\boxed{A=-1}$

put $z=2$

$1 = B(1)$

$\boxed{B=1}$

sub A, B in ②

② $\Rightarrow \frac{u(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$

$u(z) = -\frac{z}{\quad} + \frac{1}{\quad}$

$[(z+2)(z-5)]$

Sol: Let $f(z) = \dfrac{z^2-3z}{(z+2)(z-5)}$

$$\frac{f(z)}{z} = \frac{z-3}{(z+2)(z-5)}$$

$$= \frac{A}{z+2} + \frac{B}{z-5} \rightarrow (1)$$

$A = 5/7$, $B = 2/7$

$$(1) \Rightarrow \frac{f(z)}{z} = 5/7 \cdot \frac{1}{z+2} + 2/7 \cdot \frac{1}{z-5}$$

$$\therefore f(z) = 5/7 \, \frac{z}{z+2} + 2/7 \, \frac{z}{z-5}$$

Let z^{-1} on b.s

$$z^{-1}\{f(z)\} = 5/7 \, z^{-1}\left[\frac{z}{z+2}\right] + \frac{2}{7} z^{-1}\left[\frac{z}{z-5}\right]$$

$$= 5/7 (-2)^n + 2/7 (5)^n \;;$$