Set, relation and function

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Outline

Set

A set is an unordered collection of distinct objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A, [1].

Note that: A set is a well-defined, unordered collection of similar items where each item is identifiable, and distinct from the other items.

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets. There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. This way of describing a set is known as the **roster method**.

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Set Builder Notation

Another way to describe a set is to use set builder notation. We characterize all those elements in the set by stating the property or properties they must have to be members. The general form of this notation is $\{x \mid x \text{ has property } P\}$ and is read "the set of all x such that x has property P." For instance, the set O of all odd positive integers less than 10 can be written as $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$. Some well known sets are:

$N \quad = \{0, 1, 2, 3, \ldots\}$, the set of all natural numbers
$Z \quad =\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers
$Z^{+} \quad = \{1, 2, 3, \ldots\}$, the set of all positive integers
$Q \quad =\{\frac{p}{q} \mid p \in Z, q \in Z, \ and\ q \neq 0\}$, the set of all rational numbers
R, the set of all real numbers
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Equal Set

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

Subsets

The set A is a subset of B, and B is a superset of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$.

The Size of a Set

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by |S|.

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Power Sets

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Cartesian Products

Ordered n-tuple

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its nth element.

Cartesian Products

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Cartesian Product of n-sets

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,
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Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S. That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

Example

What do the statements $\forall x \in R(x^2 \geq 0)$ and $\exists x \in Z(x^2 = 1)$ mean?

Solution: The statement $\forall x \in R(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as **"The square of every real number is nonnegative."** This is a true statement.

The statement $\exists x \in Z(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as **"There is an integer whose square is 1."** This is also a true statement because $x = 1$ is such an integer (as is -1).

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Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the truth set of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

Example

What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$," $Q(x)$ is "$x^2 = 2$," and $R(x)$ is "$|x| = x$."

Solution: The truth set of P, $\{x \in Z \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

The truth set of Q, $\{x \in Z \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers x for which $x^2 = 2$.

The truth set of R, $\{x \in Z \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is N, the set of nonnegative integers.

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Set Operations

Union

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

An element x belongs to the union of the sets A and B iff x belongs to A or x belongs to B. This tells us that $A \cup B = \{x \mid x \in A \vee x \in B\}$.

Example

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

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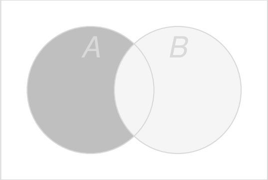

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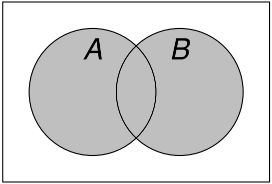

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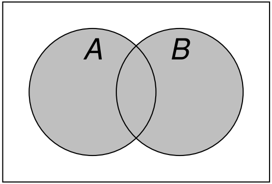

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Intersection

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B. This tells us that $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

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The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.

Disjoint Sets

Two sets are called disjoint if their intersection is the empty set.

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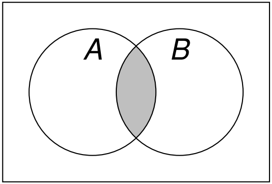

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Two sets are called disjoint if their intersection is the empty set.

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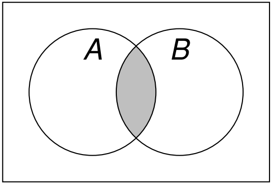

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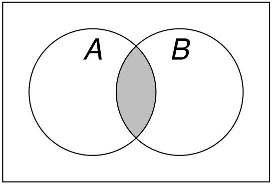

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Set Difference

Let A and B be sets. The difference of A and B, denoted by $A \setminus B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

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The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} \setminus \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$.

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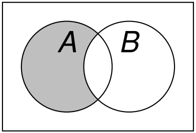

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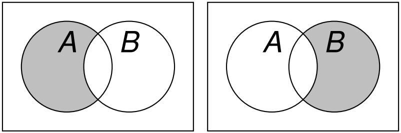

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Complement

Let U be the universal set. The complement of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U \setminus A$.

An element belongs to $\overline{A}$ if and only if $x \notin A$. This tells us that $\overline{A} = \{x \in U \mid x \notin A\}$.

Example

Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

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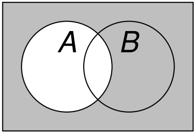

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Set Identities

Category	Identity
Identity Laws	$A \cup \emptyset = A$ $A \cap U = A$
Domination Laws	$A \cap \emptyset = \emptyset$ $A \cup U = U$
Idempotent Laws	$A \cap A = A$ $A \cup A = A$
Complementation Law	$\overline{(\overline{A})} = A$
Commutative Laws	$A \cap B = B \cap A$ $A \cup B = B \cup A$
Associative Laws	$A \cap (B \cap C) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$
Distributive Laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption Laws	$A \cap (A \cup B) = A$ $A \cup (A \cap B) = A$
Complement Laws	$A \cap \overline{A} = \emptyset$ $A \cup \overline{A} = U$

Subset Method

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by subset approach.

Solution: We will prove that the two sets $A \cap B$ and $A \cup B$ are equal by showing that each set is a subset of the other.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. We do this by showing that if x is in $\overline{A \cap B}$, then it must also be in $\overline{A} \cup \overline{B}$. Now suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. Using the definition of intersection, we see that the proposition $\neg((x \in A) \wedge (x \in B))$ is true. By applying De Morgan's law for propositions, we see that $\neg(x \in A)$ or $\neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$. Using the definition of the complement of a set, we see that this implies that $x \in \overline{A}$ or $x \in \overline{B}$. Consequently, by the definition of union, we see that $x \in \overline{A} \cup \overline{B}$. We have now shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

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By De Morgan's law for propositions, we conclude that $\neg((x \in A) \wedge (x \in B))$ is true. By the definition of intersection, it follows that $\neg(x \in A \cap B)$. We now use the definition of complement to conclude that $x \in \overline{A \cap B}$. This shows that $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$.

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Membership Tables

We consider each combination of the atomic sets (that is, the original sets used to produce the sets on each side) that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

Classwork

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

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Generalized Unions and Intersections

As unions and intersections of sets satisfy associative laws, the sets $A \cup B \cup C$ and $A \cap B \cap C$ are well defined; that is, the meaning of this notation is unambiguous when A, B, and C are sets. That is, we do not have to use parentheses to indicate which operation comes first because $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. Note that $A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C, and that $A \cap B \cap C$ contains those elements that are in all of A, B, and C.

Union

The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection. We denote the union of the sets $A_1, A_2, \ldots, A_n$ by

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i.$$

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The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection. We denote the intersection of the sets $A_1, A_2, \ldots, A_n$ by

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Examples

For $i = 1, 2, \dots$, let $A_i = \{i, i+1, i+2, \dots\}$. Then,

1. $$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \dots\} = \{1, 2, 3, \dots\}$$

2. $$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \dots\} = \{n, n+1, n+2, \dots\} = A_n.$$

Note

$$A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i,$$

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Computer Representation of Sets

We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy. Assume that the universal set U is finite. First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$. Represent a subset A of U with the bit string of length n, where the i^{th} bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is 1010101010. Similarly, we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string 0101010101. The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string 1111100000.

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Operation for Computer Representation of Sets

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The bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1, because $x \in A$ if and only if $x \notin A$.

Union

The union of two sets we perform bitwise Boolean operations on the bit strings representing the two sets. The bit in the i^{th} position of the bit string of the union is 1 if either of the bits in the i^{th} position in the two strings is 1 (or both are 1), and is 0 when both bits are 0. Hence, the bit string for the union is the bitwise OR of the bit strings for the two sets.

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Multisets

A multiset (multiple-membership set) is an unordered collection of elements where an element can occur as a member more than once. We can use the same notation for a multiset as we do for a set, but each element is listed the number of times it occurs. So, the multiset denoted by $\{a, a, a, b, b\}$ is the multiset that contains the element a thrice and the element b twice. The set $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ denotes the multiset with element a 1 occurring m_1 times, element a_2 occurring m_2 times, and so on. The numbers $m_i, i = 1, 2, \ldots, r$, are called the multiplicities of the elements $a_i, i = 1, 2, \ldots, r$. (Elements not in a multiset are assigned 0 as their multiplicity in this set.) The cardinality of a multiset is defined to be the sum of the multiplicities of its elements.

Operation on Multisets

Union

Let P and Q be multisets. The union of the multisets P and Q denoted by $P \cup Q$ is the multiset in which the multiplicity of an element is the maximum of its multiplicities in P and Q.

Intersection

The intersection of P and Q denoted by $P \cap Q$ is the multiset in which the multiplicity of an element is the minimum of its multiplicities in P and Q.

Difference

The difference of P and Q denoted by $P - Q$ is the multiset in which the multiplicity of an element is the multiplicity of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is 0.

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Example

Suppose that P and Q are the multisets $\{4 \cdot a, 1 \cdot b, 3 \cdot c\}$ and $\{3 \cdot a, 4 \cdot b, 2 \cdot d\}$, respectively. Find $P \cup Q$, $P \cap Q$, $P - Q$, and $P + Q$.

Solution: We have

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\begin{aligned}
P \cup Q &= \{\max(4,3) \cdot a, \max(1,4) \cdot b, \max(3,0) \cdot c, \max(0,2) \cdot d\} \\
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P \cap Q &= \{\min(4,3) \cdot a, \min(1,4) \cdot b, \min(3,0) \cdot c, \min(0,2) d\} \\
&= \{3 \cdot a, 1 \cdot b, 0 \cdot c, 0 \cdot d\} \\
&= \{3 \cdot a, 1 \cdot b\}, \\
P - Q &= \{\max(4-3,0) \cdot a, \max(1-4,0) \cdot b, \max(3-0,0) \cdot c, \max(0 - \\
&= \{1 \cdot a, 0 \cdot b, 3 \cdot c, 0 \cdot d\} \\
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P + Q &= \{(4+3) \cdot a, (1+4) \cdot b, (3+0) \cdot c, (0+2) \cdot d\} \\
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Function

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Domain/Codomain/Range

If f is a function from A to B, we say that A is the domain of f and B is the codomain of f. If $f(a) = b$, we say that b is the image of a and a is a preimage of b. The range, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.

Equal Function

When we define a function we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain. Two functions are equal when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain. Note that if we change either the domain or the codomain of a function, then we obtain a different function. If we change the mapping of elements, then we also obtain a different function.

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Example

Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set $\{00, 01, 10, 11\}$.

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The domain and codomain of functions are often specified in programming languages. For instance, the Java statement

$$\text{int floor(float real)}\{\ldots\}$$

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Combination of Functions

Let f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to R defined for all $x \in A$ by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, $(f_1 f_2)(x) = f_1(x) f_2(x)$. Note that the functions $f_1 + f_2$ and $f_1 f_2$ have been defined by specifying their values at x in terms of the values of f_1 and f_2 at x.

Image of Set

Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S(t = f(s))\}.$$

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Types of Function

One-to-One (Injective) Function

A function f is said to be one-to-one, or an injection, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be injective if it is one-to-one. We can express that f is one-to-one using quantifiers as $\forall a \forall b(f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$, where the universe of discourse is the domain of the function.

Example

Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Onto (Surjective) Function

A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

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Increasing/Decreasing Functions

A function f whose domain and codomain are subsets of the set of real numbers is called increasing if $f(x) \leq f(y)$, and strictly increasing if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called decreasing if $f(x) \geq f(y)$, and strictly decreasing if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word strictly in this definition indicates a strict inequality.)

Remark

A function f is increasing if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$, strictly increasing if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$, decreasing if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$, and strictly decreasing if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$, where the universe of discourse is the domain of f.

Increasing/Decreasing Functions

A function f whose domain and codomain are subsets of the set of real numbers is called increasing if $f(x) \leq f(y)$, and strictly increasing if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called decreasing if $f(x) \geq f(y)$, and strictly decreasing if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word strictly in this definition indicates a strict inequality.)

Remark

A function f is increasing if $\forall x \forall y(x < y \to f(x) \leq f(y))$, strictly increasing if $\forall x \forall y(x < y \to f(x) < f(y))$, decreasing if $\forall x \forall y(x < y \to f(x) \geq f(y))$, and strictly decreasing if $\forall x \forall y(x < y \to f(x) > f(y))$, where the universe of discourse is the domain of f.

One to One Correspondence/ Bijective

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective.

Note

Suppose that $f : A \to B$.

- To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.
- To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

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Inverse Functions and Compositions of Functions

Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Composition of Functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is the function from A to C defined by $(f \circ g)(a) = f(g(a))$.

Example and matrix representation with diagram in whiteboard.

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The Graphs of Functions

Graphs of Functions

Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{and } f(a) = b\}$.

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Floor and Ceiling Functions

The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. Note that this function has the same value throughout the interval $[n, n+1)$, namely n, and then it jumps up to $n+1$ when $x = n+1$. The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$. Note that this function has the same value throughout the interval $(n, n+1]$, namely $n+1$, and then jumps to $n+2$ when x is a little larger than $n+1$.

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Factorial Function

Another function we will use throughout this text is the factorial function $f : \mathbb{N} \to \mathbb{Z}+$, denoted by $f(n) = n!$. The value of $f(n) = n!$ is the product of the first n positive integers, so

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n$$

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Partial Functions

A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f , of a unique element b in B. The sets A and B are called the domain and codomain of f , respectively. We say that f is undefined for elements in A that are not in the domain of definition of f . When the domain of definition of f equals A, we say that f is a total function.

Example

The function $f : \mathbb{Z} \to \mathbb{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbb{Z}$ to $\mathbb{R}$ where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

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Definition (Relation)

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$. A relation on a set A is a relation from A to A.

Definition (Reflexive Relation)

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Definition (Symmetric Relation)

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

Definition (Transitive Relation)

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Composite Relation

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Boolean Product of the Matrices

Suppose that R is a relation from A to B and S is a relation from B to C. Suppose that A, B, and C have m, n, and p elements, respectively. Let the zero–one matrices for $S \circ R$, R, and S be $M_{S\circ R} = [t_{ij}]$, $M_R = [r_{ij}]$, and $M_S = [s_{ij}]$, respectively (these matrices have sizes $m \times p$, $m \times n$, and $n \times p$, respectively). The ordered pair (a_i, c_j) belongs to $S\ circR$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S. It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k. From the definition of the Boolean product, this means that $M_{S\circ R} = M_R \odot M_S$.

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References

[1] K. H. Rosen. *Discrete mathematics & applications*. McGraw-Hill, 2018.

THANK YOU