PHYS-UA 210 Computational Physics
Problem Set 05

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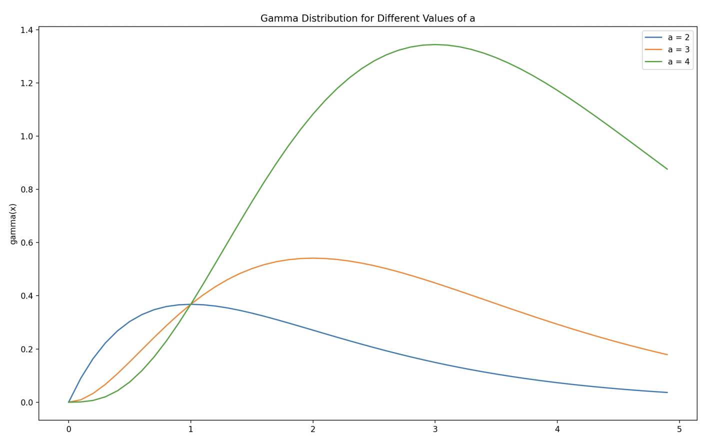
Question 1

Exercise 5.17: The Gamma Function

The gamma function is given by:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, dx$$

Graph of the integrand of the gamma function from $x = [0, 5]$ for $a = 2, 3, 4$:



Finding the maximum at $x = a - 1$:

$$\frac{d}{dx}[x^{a-1}e^{-x}] = (a-1)x^{a-2}e^{-x} - x^{a-1}e^{-x}$$

At max/min, $\frac{d}{dx}[x^{a-1}e^{-x}] = 0$, i.e.,

$$(a-1)x^{a-2}e^{-x} - x^{a-1}e^{-x} = 0$$
$$x = a - 1$$

At max,

$$\frac{d^2}{dx^2}[x^{a-1}e^{-x}] < 0$$
$$(a-1)(a-2)x^{a-3}e^{-x} - (a-1)x^{a-2}e^{-x} - (a-1)x^{a-2}e^{-x} + x^{a-2}e^{-x} < 0$$

At $x = a - 1$,

$$(a-1)(a-2)(a-1)^{a-3} - 2(a-1)(a-1)^{a-2} + (a-1)^{a-2} < 0$$
$$(a-2)(a-1)^{a-3} - 2(a-1)^{a-2} + (a-1)^{a-3} < 0$$
$$(a-1) - 1 - 2(a-1) + 1 < 0$$
$$(a-1) - 2(a-1) < 0$$

Since this is always true, $x = a - 1$ is the maximum.

Using change of variables,

$$z = \frac{x}{c+x}$$

Here, if the peak is expected to be at $z = \frac{1}{2}$, then here, $x = a - 1$ so $c = a - 1$.

Rewriting the integrand using $x^{a-1} = e^{(a-1)ln(x)}$,
Integrand $= e^{(a-1)ln(x)-x}$ This version of the integrand is better for computation since x^{a-1} can grow very large for large values of a, however, the new version does not given the $ln(x$ in the exponent.

Calculating the value of $\Gamma(a)$ for $a = 1.5, 3, 6, 10$:

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Values for gamma functions:
gamma(1.5) =  0.8862269613087213
gamma(3) =  2.0000000000000013
gamma(6) =  119.99999999999999
gamma(10) =  362879.99999999994

Comparing with the factorials:
gamma(3) =  2
gamma(6) =  120
gamma(10) =  362880
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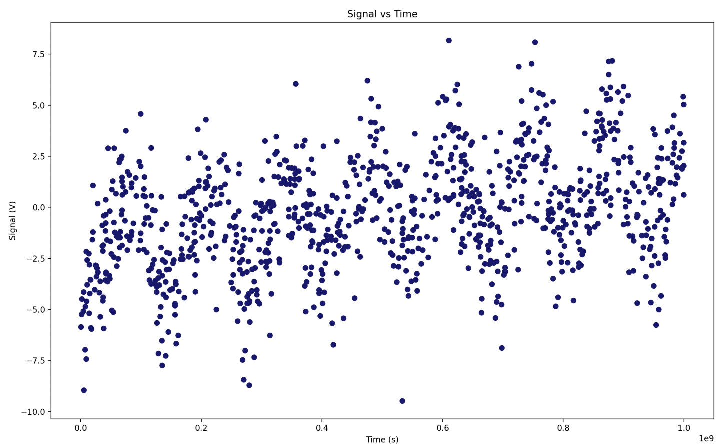
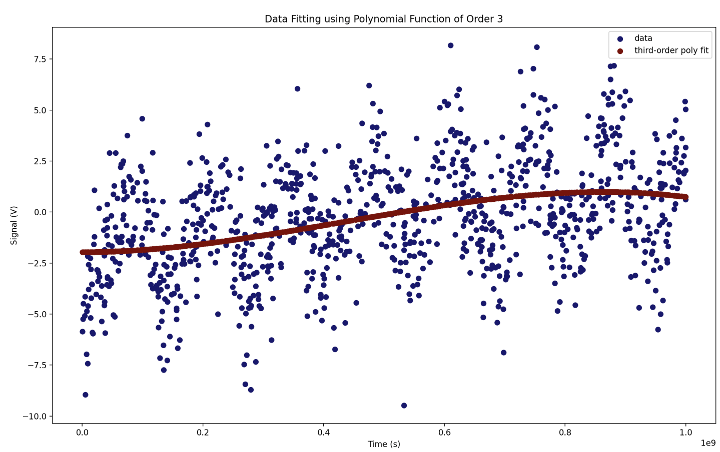
Question 2

Linear Algebra in Signal Analysis

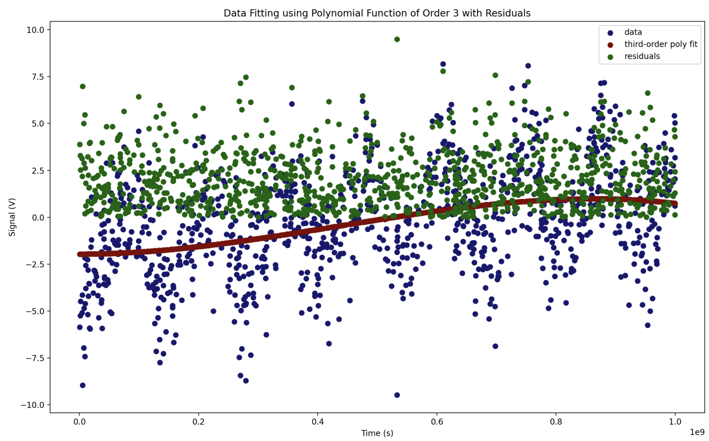
Plotting the data:



Using the SVD technique to find the best third-order polynomial fit in time to the signal:

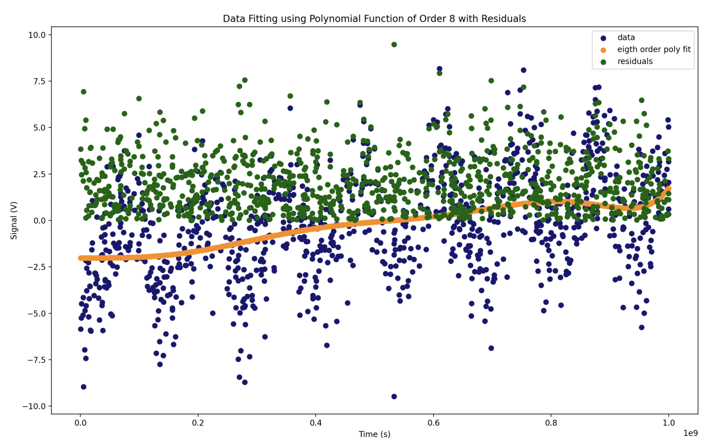


Calculating the residuals of the data with respect to the model:



Here, the residuals are not centered around zero, rather, the fit does little to no justice to the actual data even considering the standard deviation of 2 units since the residuals are higher than 2 units.

Higher order polynomial (order of 8):



Increasing the order of polynomial still does not give a good fit to the actual data given that the residuals are not within the bounds of the standard deviation. No, there is no reasonable degree of polynomial that can fit the data given that even a higher order polynomial does not have a viable condition number:

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Condition number for polynomial fit of order3:  inf
Condition number for polynomial fit of order8:  inf
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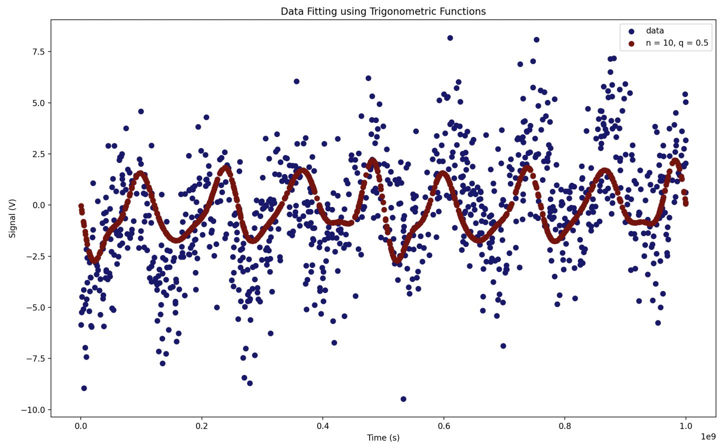
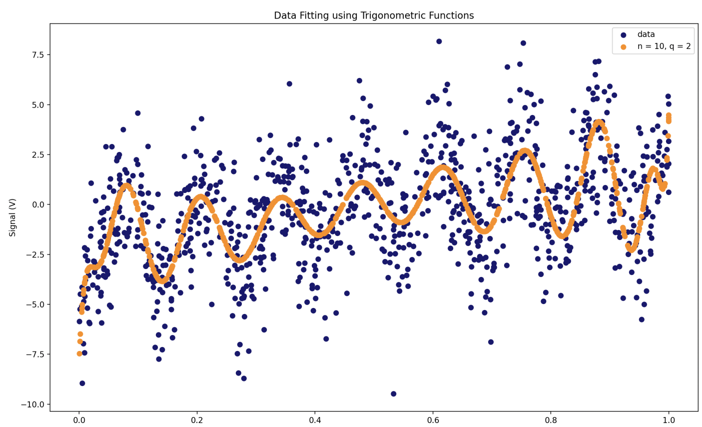
Trigonometric Fits:





Here, the trigonometric functions do a better job at fitting the data than the polynomial fits however, some data points still deviate from the fit beyond their standard deviation of 2.0 units.

The first graph has frequency starting with a period equal to half of the time span covered and in the second graph the frequency is twice the time span covered. Yes, there is a visible periodicity in the data and since the better fit is the one with frequency with period twice the time span covered, this can be a reasonable approximation to the data's periodicity.

Please find my GitHub repository through this: link.