Geometry: Maths Olympiad

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1. P is a given point inside a given sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V, and W ; Q denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV, and PW. Find the locus of Q for all such triads of rays from P.
2. In triangle ABC, AB = AC. A circle is tangent internally to the circum circle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the in-circle of triangle ABC.
3. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.
4. Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $\frac{(QP+PA)}{QR}$ is a maximum.
5. P is a point inside a given triangle ABC.D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which
$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}$$
is least.
6. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the in-center and the circum center of the triangle and the point O are collinear.
7. The diagonals AC and CE of the regular hexagon ABCDEF are divided by the inner points M and N, respectively, so that
$$\frac{AM}{AC}=\frac{CN}{CE}=r$$
Determine r if B, M, and N are collinear.
8. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all i = 1, 2, 3, M_i is the midpoint of side a_i, and T_i. is the point where the in-circle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1, S_1, M_2S_2, and M_3S_3 are concurrent.
9. Let A be one of the two distinct points of intersection of two unequal co-planar circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches O_1 atP_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1,and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.
10. Let ABC be an equilateral triangle and E the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

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11. Let ABCD be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.
12. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by a(X) the measure of the angle between OA and OX in radians, counter clockwise from OA($0 \leq a(X) < 2\pi$). Let C(X) be the circle with center O and radius of length OX + a(X)/(OX). Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a(Y) > 0$ such that its color appears on the circumference of the circle C(Y).
13. A circle has center on the side AB of the cyclic quadrilateral ABCD.The other three sides are tangent to the circle. Prove that $AD + BC = AB$.
14. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.
15. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all s $\geq$ 4. We construct a set of points P_1, P_2, P_3,......, such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle 120^o clockwise (for k = 0,1,2,.....). Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.
16. Let A, B be adjacent vertices of a regular n-gon (n $\geq$ 5) in the plane having center at O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.
17. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circum circle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral AKNM and the triangle ABC have equal areas.
18. Consider two coplanar circles of radii R and r ($R > r$) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then A = P.)
    a) Find the set of values of $BC^2 + CA^2 + AB^2$.
    b) Find the locus of the midpoint of BC.
19. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that S$\geq 2T$.
20. In an acute-angled triangle ABC the internal bisector of angle A meets the circum circle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:
    a) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.
    b) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.
21. Let ABCD be a convex quadrilateral such that the sides AB, AD, BC satisfy AB = AD + BC. There exists a point P inside the quadrilateral at a distance h from the line CD such that AP = h + AD and BP = h + BC. Show that:
$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$
22. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and

AC at F and G, respectively. If $\frac{AM}{AB}$ = t, find $\frac{EG}{EF}$ interms of t.

23. Prove that there exists a convex 1990-gon with the following two properties:
    a) All angles are equal.
    b) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, ..., 1990^2$ in some order.