Discrete: Maths Olympiad

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1. An international society has its members from six different countries. The list of members contains 1978 names, numbered 1, 2,....., 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.
2. The set of all positive integers is the union of two disjoint subsets
$$(f(1), f(2), .., f(n), ..), (g(1), g(2), .., g(n), ..),$$
where
$$f(1) < f(2) < ..... < f(n) < ....,$$
$$g(1) < g(2) < ...... < g(n) < ....,$$
and g(n) = f(f(n)) + 1 for all n $\geq$ 1. Determine f(240).
3. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there.. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$,
$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), n = 1, 2, 3, ...,$$
where x = 2 + $\sqrt{2}$ and y = 2 - $\sqrt{2}$.
Note. A path of n jumps is a sequence of vertices $(P_0, ..., P_n)$ such that
   a) P_0 = A, P_n = E;
   b) for every i, $0 \leq i \leq n-1, P_i$ is distinct from E;
   c) for every i, $0 \leq i \leq n-1, P_i$ and P_{i+1} are adjacent.
4. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set (1, 2,...., n). Each of these subsets has a smallest member. Let F (n, r) denote the arithmetic mean of these smallest numbers; prove that
$$F(n, r) = \frac{n+1}{r+1}$$
5. The function f(x, y) satisfies
   a) f(0, y) = y + 1,
   b) f(x + 1, 0) = f(x, 1),
   c) f(x + 1, y + 1) = f(x, f(x + 1, y)), for all non-negative integers x, y. Determine f(4, 1981).
6. The function f(n) is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n
f(m + n) - f(m) - f(n) = 0 or 1
f(2) = 0, $f(3) > 0$, and f(9999) = 3333.
Determine f(1982).
7. Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, ..., A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for

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every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.

8. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.
9. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:
   a) f(xf(y)) = yf(x) for all positive x, y;
   b) f(x) $\to$ 0 as $x \to \infty$
10. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices ($n > 3$), and let p be its perimeter. Prove that
$$n-3 < \frac{2d}{p} < [\frac{n}{2}][\frac{n+1}{2}] - 2$$
where [x] denotes the greatest integer not exceeding x.
11. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.
12. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by x + y, -y, z + y respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to and end after a finite number of steps.
13. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:
   a) f(xf(y))f(y) = f(x + y) for all x, y $\geq$ 0,
   b) f(2) = 0,
   c) f(x) $\neq$ 0 for $0 \leq x < 2$.
14. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1?
15. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.
16. Prove that there is no function f from the set of non-negative integers into itself such that f(f(n)) = n + 1987 for every n.
17. Let $p_n(k)$ be the number of permutations of the set (1,........n), $n \geq 1$, which have exactly k fixed points. Prove that
$$\sum_{k=0}^{n} k.p_n(k) = n!.$$
18. Show that set of real numbers x which satisfy the inequality
$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$
is a union of disjoint intervals, the sum of whose lengths is 1988.

19. A function f is defined on the positive integers by

$$
\begin{aligned}
f(1) = 1, f(3) = 3,\\
f(2n) = f(n),\\
f(4n+1) = 2f(2n+1) - f(n),\\
f(4n+3) = 3f(2n+1) - 2f(n)
\end{aligned}
$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988, for which f(n) = n.

20. A permutation $(x_1, x_2, ..., x_m)$ of the set (1, 2,........., 2n), where n is a positive integer, is said to have property P if $\left|x_i - x_{i+1}\right|$ = n for at least one i in (1,2,......, 2n - 1). Show that, for each n, there are more permutations with property P than without.

21. Prove that the set (1, 2,........, 1989) can be expressed as the disjoint union of subsets A_i (i = 1, 2,......., 117) such that:
    a) Each A_i contains 17 elements;
    b) The sum of all the elements in each A_i is the same.

22. Let $n \geq 3$ and consider a set E of 2n - 1 distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

23. Let Q^+ be the set of positive rational numbers. Construct a function f : $Q^+ \to Q^+$ such that Q^+ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in Q^+.