Algebra: Maths Olympiad

G V V Sharma*

1. m and n are natural numbers with $1 \leq m \leq n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that m + n has its least value.
2. Let a_k(k = 1, 2, 3,....., n,.....) be a sequence of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n}(\frac{a_k}{k^2}) \geq \sum_{k=1}^{n}(\frac{1}{k})$$

3. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5}(kx_k) = a, \sum_{k=1}^{5}(k^3x_k) = a^2, \sum_{k=1}^{5}(k^5x_k) = a^3,$$

4. Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... - \frac{1}{1318} + \frac{1}{1319}.$$

5. a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining n - 1 numbers?
   b) For which values of $n > 2$ is there exactly one set having the stated property?
6. Determine the maximum value of $m^3 + n^3$, where m and n are integers satisfying m, n ∈ (1, 2,...,1981) and

$$(n^2 - mn - m^2)^2 = 1.$$

7. Consider the infinite sequences x_n of positive real numbers with the following properties: $x_0 = 1$, and for all $i \geq 0, x_{i+1} \leq x_i$.
   a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ...\frac{x_{n-1}^2}{x_n} \geq 3.999.$$

   b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ...\frac{x_{n-1}^2}{x_n} < 4$$

8. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n \tag{8.1}$$

has a solution in integers (x, y), then it has at least three such solutions.Show that the equation has no solution in integers when n=2891.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

9. Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that 2abc - ab - bc - ca is the largest integer which cannot be expressed in the form xbc + yca + zab, where x, y and z are non-negative integers.
10. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

    Determine when equality occurs.
11. Prove that $0 \leq yz+zx+xy-2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers for which x + y + z = 1.
12. Find one pair of positive integers a and b such that:
    a) ab(a + b) is not divisible by 7;
    b) $(a+b)^7-a^7-b^7$ is divisible by 7^7.
    Justify your answer.
13. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and ad = bc. Prove that if a + d = 2^k and b + c = 2^m for some integers k and m, then a = 1.
14. Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set M = (1,2,.., $n-1$) is colored either blue or white. It is given that
    a) for each i ∈ M, both i and n - i have the same color;
    b) for each i ∈ M, i ≠ k, both i and $\left|i-k\right|$ have the same color. Prove that all numbers in M must have the same color.
15. For any polynomial $P(x) = a_0 + a_1 + ... + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by w(P). For i = 0, 1,....., let $Q_i(x) = (1+x)^i$. Prove that if $i_1 i_2, ..., i_n$ are integers such that $0 \leq i_1 < i_2 < ..... < i_n$, then $w(Q_{i1} + Q_{i2}, +Q_{in}) \geq w(Q_{i1})$.
16. For every real number x_1, construct the sequence $x_1, x_2, ...$ by setting

$$x_{n+1} = x_n(x_n + \frac{1}{n})$$

    for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n.
17. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set 2, 5, 13, d such that ab - 1 is not a perfect square.
18. Let n be an integer greater than or equal to 2. Prove that if k^2+k+n is prime for all integers k such that $0 \leq k \leq \frac{\sqrt{n}}{3}$, then k^2+k+n is prime for all integers k such that $0 \leq k \leq n-2$.
19. Let n be a positive integer and let $A_1, A_2, ..., A_{2n+1}$ be subsets of a set B. Suppose that
    a) Each A_i has exactly 2n elements,
    b) Each $A_i \cap A_j (1 \leq i < j \leq 2n+1)$ contains exactly one element, and
    c) Every element of B belongs to at least two of the A_i.
    Foe which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has zero assign to exactly n of its elements.
20. Let a and b be positive integers such that ab + 1 divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

    is the square of an integer.
21. Let n and k be positive integers and let S be a set of n points in the plane such that
    a) No three points of S are collinear, and
    b) For any point P of S there are at least k points of S equidistant from P .
    Prove that $k < \frac{1}{2} + \sqrt{2n}$
22. Prove that for each positive integer n there exist n consecutive positive integers none of which is an

integral power of a prime number.

23. Given an initial integer $n_0 > 1$, two players, A and B, choose integers $n_1, n_2, n_3,\ldots\ldots$ alternately according to the following rules: Knowing n_{2k}, A chooses any integer n_{2k+1} such that $n_{2k} \geq n_{2k+1} \geq n_{2k}^2$.Knowing n_{2k+1}, B chooses any integer n_{2k+2} such that $\frac{n_{2k+1}}{n_{2k+2}}$ is a prime raised to a positive integer power.
    Player A wins the game by choosing the number 1990; player B wins by choosing the number 1. For which n_0 does:
    a) A have a winning strategy?
    b) B have a winning strategy?
    c) Neither player have a winning strategy?
24. Determine all integers $n > 1$ such that $\frac{2^n+1}{n^2}$ is an integer.