

Assignment 3

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Download all python codes from

<https://github.com/satyasm45/Summer-Internship/tree/main/Assignment-3/Codes>

and latex-tikz codes from

<https://github.com/satyasm45/Summer-Internship/tree/main/Assignment-3>

1 QUESTION No. 2.55

Let \mathbf{A} and \mathbf{B} be the centres of two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at \mathbf{C} and \mathbf{D} . Is $AB \perp CD$?

2 SOLUTION

To perform the given construction let us assume

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.1)$$

Based on the constraints given in the question, \mathbf{B} will lie on the circle with center as \mathbf{A} and radius 3. Without loss of generality, let us assume:

$$\mathbf{B} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.0.2)$$

Then,

$$\|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B}\| = 3 \quad (\because \mathbf{A} = 0) \quad (2.0.3)$$

The centers and radii of the two circles are now given in table 2.1

	Circle 1	Circle 2
Centre	$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\mathbf{B} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$
Radius	$r_1 = 3$	$r_2 = 3$

TABLE 2.1: Input values

Let us define:

$$\mathbf{u} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \theta \in [0, 2\pi]. \quad (2.0.4)$$

Then any point $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ on Circle 1 is given by :

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mathbf{A} + r_1 \mathbf{u} = 3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (\because \mathbf{A} = 0) \quad (2.0.5)$$

This is the locus of Circle 1.

Similarly, locus of Circle 2 is given by:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \mathbf{B} + r_2 \mathbf{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (2.0.6)$$

Using the locus of the circles Fig. 2.1 was plotted.

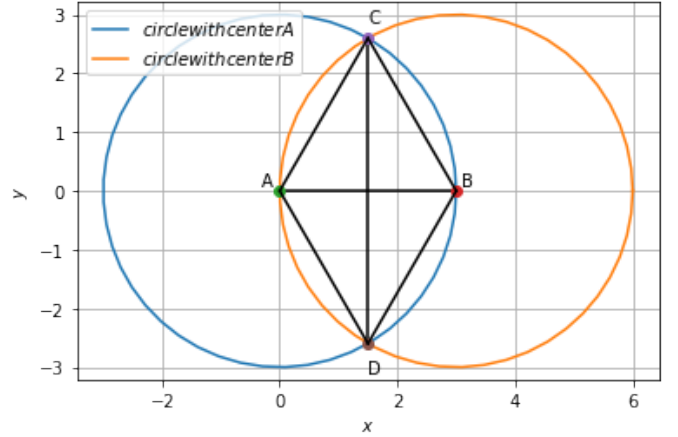


Fig. 2.1: Circles with their points of intersection

We have \mathbf{C} and \mathbf{D} as points of intersection. So,

$$\|\mathbf{D} - \mathbf{A}\| = \|\mathbf{C} - \mathbf{A}\| = r_1 = 3 \quad (2.0.7)$$

$$\|\mathbf{D} - \mathbf{B}\| = \|\mathbf{C} - \mathbf{B}\| = r_2 = 3 \quad (2.0.8)$$

Therefore, in quadrilateral ACBD we have

$$\|\mathbf{D} - \mathbf{B}\| = \|\mathbf{C} - \mathbf{B}\| = \|\mathbf{D} - \mathbf{A}\| = \|\mathbf{C} - \mathbf{A}\| = 3 \quad (2.0.9)$$

So, ACBD is a Rhombus. For a Rhombus we have diagonals bisect each other at right angles.

Therefore it can be concluded that $AB \perp CD$.

We can also verify the result by using equation 2.0.9 and doing some computations:

$$\|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{C} - \mathbf{B}\|^2 \quad (2.0.10)$$

Also, $\|\mathbf{P}\|^2 = \mathbf{P}^T \mathbf{P}$. So,

$$(\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = (\mathbf{C} - \mathbf{B})^T (\mathbf{C} - \mathbf{B}) \quad (2.0.11)$$

$$\Rightarrow \mathbf{A}^T \mathbf{C} - \mathbf{B}^T \mathbf{C} = \mathbf{C}^T \mathbf{B} - \mathbf{C}^T \mathbf{A} + \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \quad (2.0.12)$$

Similarly, we have:

$$\|\mathbf{D} - \mathbf{A}\|^2 = \|\mathbf{D} - \mathbf{B}\|^2 \quad (2.0.13)$$

$$\Rightarrow (\mathbf{D} - \mathbf{A})^T (\mathbf{D} - \mathbf{A}) = (\mathbf{D} - \mathbf{B})^T (\mathbf{D} - \mathbf{B}) \quad (2.0.14)$$

$$\Rightarrow \mathbf{A}^T \mathbf{D} - \mathbf{B}^T \mathbf{D} = \mathbf{D}^T \mathbf{B} - \mathbf{D}^T \mathbf{A} + \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \quad (2.0.15)$$

Subtracting equation 2.0.15 from equation 2.0.12 and using the fact $(\mathbf{PQ})^T = \mathbf{Q}^T \mathbf{P}^T$ we have

$$(\mathbf{A}^T - \mathbf{B}^T)(\mathbf{C} - \mathbf{D}) = (\mathbf{C}^T - \mathbf{D}^T)(\mathbf{B} - \mathbf{A}) \quad (2.0.16)$$

$$= -((\mathbf{A}^T - \mathbf{B}^T)(\mathbf{C} - \mathbf{D}))^T \quad (2.0.17)$$

If we consider $(\mathbf{A}^T - \mathbf{B}^T)(\mathbf{C} - \mathbf{D}) = \mathbf{X}$ then we have

$$\mathbf{X} = -\mathbf{X}^T \quad (2.0.18)$$

Therefore \mathbf{X} is skew-symmetric. Now since $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are 2×1 matrices \mathbf{X} has order 1×1 which indicates:

$$\mathbf{X} = 0 \quad (2.0.19)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^T (\mathbf{C} - \mathbf{D}) = 0 \quad (2.0.20)$$

$$\Rightarrow AB \perp CD \quad (2.0.21)$$