Assignment 3

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Download all python codes from

```
https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−3/Codes
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and latex-tikz codes from

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https://github.com/satyasm45/Summer−Internship/
    tree/main/Assignment−3
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1 Question No. 2.55

Let $\mathbf{A}$ and $\mathbf{B}$ be the centres of two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at $\mathbf{C}$ and $\mathbf{D}$. Is AB⊥CD?

2 Solution

To perform the given construction let us assume

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.1}$$

Based on the constraints given in the question, $\mathbf{B}$ will lie on the circle with center as $\mathbf{A}$ and radius 3.Without loss of generality,let us assume:

$$\mathbf{B} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.0.2}$$

Then,

$$\|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B}\| = 3 \quad (\because \mathbf{A} = 0) \tag{2.0.3}$$

The centers and radii of the two circles are now given in table 2.1

	Circle 1	Circle 2
Centre	$\mathbf{A}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\mathbf{B}=\begin{pmatrix} 3 \\ 0 \end{pmatrix}$
Radius	r_1=3	r_2=3

TABLE 2.1: Input values

Let us define:

$$\mathbf{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \theta \in [0, 2\pi]. \tag{2.0.4}$$

Then any point $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ on Circle 1 is given by :

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mathbf{A} + r_1\mathbf{u} = 3\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad (\because \mathbf{A} = 0) \tag{2.0.5}$$

This is the locus of Circle 1.
Similarly,locus of Circle 2 is given by:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \mathbf{B} + r_2\mathbf{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 3\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.0.6}$$

Using the locus of the circles Fig. 2.1 was plotted.

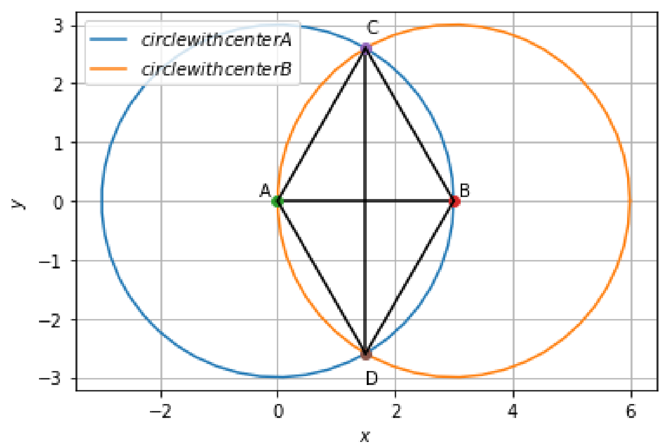

Fig. 2.1: Circles with their points of intersection

We have $\mathbf{C}$ and $\mathbf{D}$ as points of intersection.So,

$$\|\mathbf{D} - \mathbf{A}\| = \|\mathbf{C} - \mathbf{A}\| = r_1 = 3 \tag{2.0.7}$$

$$\|\mathbf{D} - \mathbf{B}\| = \|\mathbf{C} - \mathbf{B}\| = r_2 = 3 \tag{2.0.8}$$

Therefore,in quadrilateral ACBD we have

$$\|\mathbf{D} - \mathbf{B}\| = \|\mathbf{C} - \mathbf{B}\| = \|\mathbf{D} - \mathbf{A}\| = \|\mathbf{C} - \mathbf{A}\| = 3 \tag{2.0.9}$$

So,ACBD is a Rhombus. For a Rhombus we have diagonals bisect each other at right angles.

Therefore it can be concluded that AB⊥CD.

We can also verify the result by using equation 2.0.9 and doing some compuatations:

$$\|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{C}-\mathbf{B}\|^2 \tag{2.0.10}$$

Also, $\|\mathbf{P}\|^2 = \mathbf{P}^T\mathbf{P}$ .So,

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) = (\mathbf{C}-\mathbf{B})^T(\mathbf{C}-\mathbf{B}) \tag{2.0.11}$$

$$\implies \mathbf{A}^T\mathbf{C} - \mathbf{B}^T\mathbf{C} = \mathbf{C}^T\mathbf{B} - \mathbf{C}^T\mathbf{A} + \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{2.0.12}$$

For any two vectors $\mathbf{P}$ and $\mathbf{Q}$ we have
$\mathbf{P}^T\mathbf{Q} = \mathbf{Q}^T\mathbf{P}$.So,

$$2\times\mathbf{A}^T\mathbf{C} - 2\times\mathbf{B}^T\mathbf{C} = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{2.0.13}$$

Similarly,using:

$$\|\mathbf{D}-\mathbf{A}\|^2 = \|\mathbf{D}-\mathbf{B}\|^2 \tag{2.0.14}$$

We get:

$$2\times\mathbf{A}^T\mathbf{D} - 2\times\mathbf{B}^T\mathbf{D} = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{2.0.15}$$

Subtracting equation 2.0.15 from equation 2.0.13:

$$2\times(\mathbf{A}^T - \mathbf{B}^T)(\mathbf{C}-\mathbf{D}) = 0 \tag{2.0.16}$$

$$\implies (\mathbf{A}-\mathbf{B})^T(\mathbf{C}-\mathbf{D}) = 0 \tag{2.0.17}$$

$$\implies AB \perp CD \tag{2.0.18}$$