Assignment 3

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Download all python codes from

https://github.com/satyasm45/Summer–Internship/tree/main/Assignment–3/Codes

and latex-tikz codes from

https://github.com/satyasm45/Summer–Internship/tree/main/Assignment–3

1 Question No. 2.55

Let $\mathbf{A}$ and $\mathbf{B}$ be the centres of two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at $\mathbf{C}$ and $\mathbf{D}$. Is AB⊥CD?

2 Solution

The centers and radii of the two circles without any loss of generality are given in table 2.1

	Circle 1	Circle 2
Centre	$\mathbf{A}=\begin{pmatrix}0\\0\end{pmatrix}$	$\mathbf{B}=\begin{pmatrix}3\\0\end{pmatrix}$
Radius	r_1=3	r_2=3

TABLE 2.1: Input values

The choice for $\mathbf{A}$ and $\mathbf{B}$ is valid as:

$$\|\mathbf{B}-\mathbf{A}\| = \|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}\| = 3 \quad (\because \mathbf{A}=0) \tag{2.0.1}$$

Let us define:

$$\mathbf{u} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \theta \in [0, 2\pi]. \tag{2.0.2}$$

Then any point $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ on Circle 1 is given by :

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \mathbf{A} + r_1\mathbf{u} = 3\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad (\because \mathbf{A}=0) \tag{2.0.3}$$

This is the locus of Circle 1.
Similarly,locus of Circle 2 is given by:

$$\begin{pmatrix}x_2\\y_2\end{pmatrix} = \mathbf{B} + r_2\mathbf{u} = \begin{pmatrix}3\\0\end{pmatrix} + 3\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \tag{2.0.4}$$

Using the locus of the circles Fig. 2.1 was plotted.

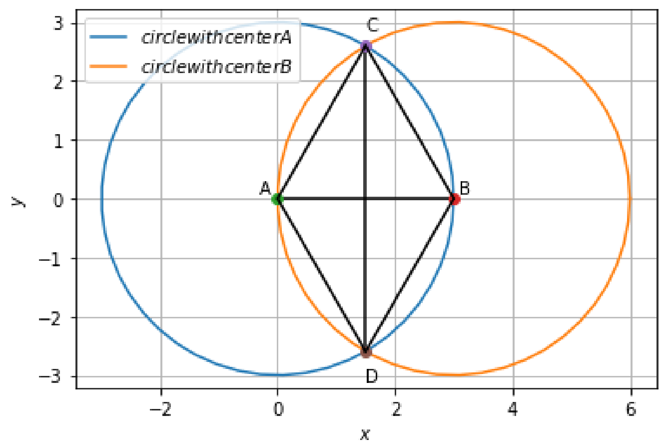

Fig. 2.1: Circles with their points of intersection

We have $\mathbf{C}$ and $\mathbf{D}$ as points of intersection and $r_1 = r_2$.So,

$$\|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{C}-\mathbf{B}\|^2 \tag{2.0.5}$$

$$\implies (\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) = (\mathbf{C}-\mathbf{B})^T(\mathbf{C}-\mathbf{B}) \tag{2.0.6}$$

$$\implies \mathbf{A}^T\mathbf{C} - \mathbf{B}^T\mathbf{C} = \mathbf{C}^T\mathbf{B} - \mathbf{C}^T\mathbf{A} + \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{2.0.7}$$

$$\implies 2\times\mathbf{A}^T\mathbf{C} - 2\times\mathbf{B}^T\mathbf{C} = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{2.0.8}$$

Similarly,using:

$$\|\mathbf{D}-\mathbf{A}\|^2 = \|\mathbf{D}-\mathbf{B}\|^2 \tag{2.0.9}$$

We get:

$$2\times\mathbf{A}^T\mathbf{D} - 2\times\mathbf{B}^T\mathbf{D} = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{2.0.10}$$

Subtracting equation 2.0.10 from equation 2.0.8:

$$2\times(\mathbf{A}^T - \mathbf{B}^T)(\mathbf{C}-\mathbf{D}) = 0 \tag{2.0.11}$$

$$\implies (\mathbf{A}-\mathbf{B})^T(\mathbf{C}-\mathbf{D}) = 0 \tag{2.0.12}$$

$$\implies AB \perp CD \tag{2.0.13}$$

General equation of Circle 1:

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{A}^T\mathbf{x} + \|\mathbf{A}\|^2 - r_1^2 = 0 \tag{2.0.14}$$

Similarly for Circle 2:

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{B}^T\mathbf{x} + \|\mathbf{B}\|^2 - r_2^2 = 0 \tag{2.0.15}$$

Subtracting (2.0.15) from (2.0.14):

$$2\mathbf{B}^T\mathbf{x} = \|\mathbf{B}\|^2 \tag{2.0.16}$$

$$\begin{pmatrix} 3 & 0 \end{pmatrix}\mathbf{x} = 4.5 \tag{2.0.17}$$

which can be written as

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 1.5 \tag{2.0.18}$$

$$\mathbf{x} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{2.0.20}$$

$$\mathbf{q} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \tag{2.0.21}$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.22}$$

Substituting (2.0.20) in (2.0.14)

$$\|\mathbf{x}\|^2 = r_1^2 = 9 \quad (\because \mathbf{A} = 0) \tag{2.0.23}$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 = 9 \tag{2.0.24}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) = 9 \tag{2.0.25}$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) = 9 \tag{2.0.26}$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 = 9 \tag{2.0.27}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 = 9 \tag{2.0.28}$$

$$\|\mathbf{q}\|^2 + 2\lambda\begin{pmatrix} 1.5 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda^2\|\mathbf{m}\|^2 = 9 \tag{2.0.29}$$

$$\|\mathbf{q}\|^2 + 0 + \lambda^2\|\mathbf{m}\|^2 = 9 \tag{2.0.30}$$

$$\lambda^2 = \frac{9 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{2.0.31}$$

$$\lambda^2 = 6.75 \tag{2.0.32}$$

$$\lambda = +\sqrt{6.75}, -\sqrt{6.75} \tag{2.0.33}$$

Substituting the value of λ in (2.0.20)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{2.0.34}$$

$$\mathbf{C} = \begin{pmatrix} 1.5 \\ \sqrt{6.75} \end{pmatrix} \tag{2.0.35}$$

$$\mathbf{D} = \begin{pmatrix} 1.5 \\ -\sqrt{6.75} \end{pmatrix} \tag{2.0.36}$$

Now,

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{C} - \mathbf{D}) = \begin{pmatrix} -3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \times \sqrt{6.75} \end{pmatrix} \tag{2.0.37}$$

$$= 0 \tag{2.0.38}$$

$$\implies AB \perp CD \tag{2.0.39}$$