PHYS-UA 210 Computational Physics
Final Project Draft2

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GitHub link.

Introduction

This project will create a density field that can tell you information about the gravitational potential. We solve this problem by using a cloud in cell approach and a DFT on the Laplace equation.x

1 Re-scaling the Problem

To rescale the problem, we can think in terms of the equation

$$\frac{d^2r}{dt^2} = \frac{GM}{r^2}$$

.

Our new variables can be written as $r^{'} = \frac{r}{r_0},\ M^{'} = \frac{M}{M_0},\ t^{'} = \frac{t}{t_0},\ G = 1$ so our equation becomes

$$\frac{dr^{'2}}{dt^{'2}} = \frac{M^{'}}{(r^{'})^2}$$

we just want to be sure that the units make sense, so lets take

$$\frac{r^{'}}{(t^{'})^2} = \frac{GM^{'}}{(r^{'})^2}$$

Setting $G = 1$ and the ratio $\frac{M^{'}}{(r^{'})^3} = 1$ we obtain

$$(t^{'})^2 = 1,\ t^{'} = 1$$

With these ratios defined, we have freedom choose r_0.

2 Particle Positions and Density Field

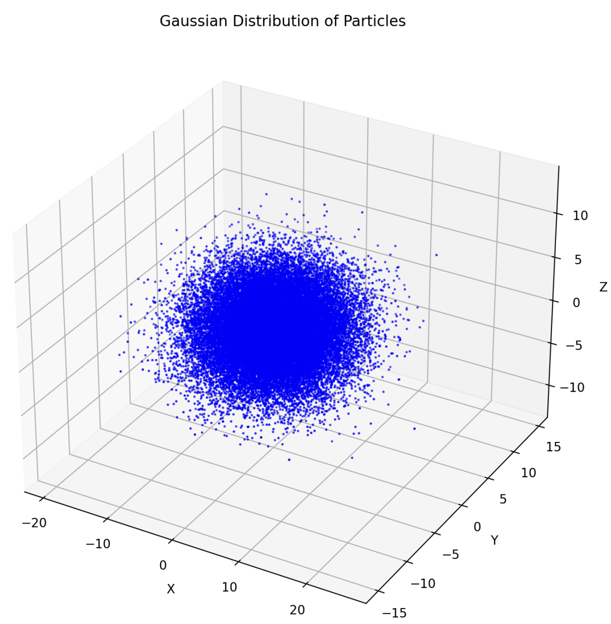
2.1 Particle Distribution

In this simulation, particles are distributed in a 3-dimensional cartesian co-ordinate system following a multivariate Gaussian distribution:

$$\phi(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{p/2} |\Sigma|^{-1/2} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\}$$

Here,
$p = 3$ given our 3-dimensional co-ordinate system.
μ = mean vector
Σ = covariance matrix

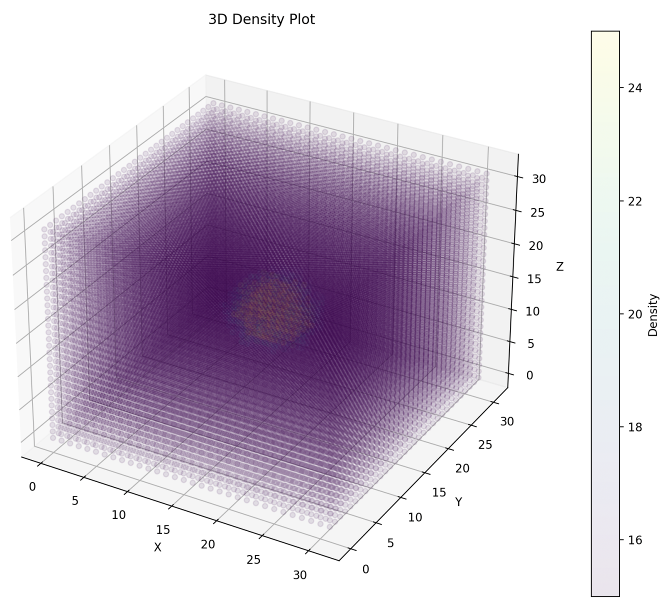
Initially, it produces a particle distribution of $N = 32^3$ number of particles the following way in 32 x 32 x 32 grid ranging from -16 to 16 in all three directions centered at the origin:



2.2 Inferring Density Field

Next, a density field is defined based on the distribution of the particles in the space. Each particle is taken as a cubicle cloud with sides equal to the size of a grid cell. For one cell, its density depends on the fractional overlap of such clouds within its region. For example, if two particles lie on the center of a cell, the relative density becomes 2 units (considering there are no other clouds in the region). The density is a 32 x 32 x 32 grid where each data point refers to the relative density of one cell in the space.

This is a density field for the spatial grid 32 x 32 x 32 for a distribution of 32^3 particles. It makes sense that the density is maximum at the center and falls off as it moves away given the 3D Gaussian distribution of particles.



3 Solving Poisson's Equation

Poisson's equation for gravity is given as:

$$\nabla^2 \phi(\vec{x}) = 4\pi\rho(\vec{x})$$

Poisson's equation is an elliptic partial differential equation whose solution is the potential field given by a mass density distribution. To obtain this equation, we can start from the differential form of Gauss's law for gravity:

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

The gravitational field $\mathbf{g}$ can be expressed in terms of scalar potential ϕ since the gravitational field is conservative and irrotational:

$$\mathbf{g} = -\nabla\phi$$

Then, Gauss's law becomes:

$$\nabla \cdot\ (-\nabla\phi) = -4\pi G\rho$$

which yields the Poisson's equation (taking $G = 1$):

$$\nabla^2\phi(\vec{x}) = 4\pi\rho(\vec{x})$$

Performing Fourier transformation on this formula using the definition of FT in three dimensions, the left hand side yields to:

$$\int_V \nabla^2\phi(\vec{x})e^{-\mathbf{i}\vec{\mathbf{k}}\vec{\mathbf{x}}}\, d^3r$$

Integrating by parts:

$$\int_V d^3r\vec{\nabla}\cdot(e^{-\mathbf{i}\vec{\mathbf{k}}\vec{\mathbf{x}}}\vec{\nabla}\phi) - (-ik\int_V d^3re^{-\mathbf{i}\vec{\mathbf{k}}\vec{\mathbf{x}}}\vec{\nabla}\phi)$$

Looking at the surface term from the integration by parts and employing the divergence theorem:

$$\oint d\vec{S}\cdot(e^{-\mathbf{i}\vec{\mathbf{k}}\vec{\mathbf{x}}}\vec{\nabla}\phi) \to 0$$

when $\vec{\nabla}\phi \to \infty$

$$= ik\int_V d^3r\, e^{-\mathbf{i}\vec{\mathbf{k}}\vec{\mathbf{x}}}\vec{\nabla}\phi$$

Integrating by parts again and getting rid of the surface term again:

$$\begin{aligned} &= ik(-ik)\phi(k) \\ &= k^2\tilde{\phi}(\vec{k}) \end{aligned}$$

Therefore, the Fourier transform of this equation is as follows:

$$k^2\tilde{\phi}(\vec{k}) = 4\pi\tilde{\rho}(\vec{k})$$

3.1 Discrete Fourier Transform of Poisson's Equation

3.1.1 Deriving the discrete version of the Poisson's equation:

$$\frac{\partial\phi}{\partial x}|_{n+\frac{1}{2}} = \frac{\phi_{n+1}-\phi_n}{\Delta}$$

$$\frac{\partial^2\phi}{\partial x^2}|_n = \frac{1}{\Delta}\left[\frac{\partial\phi}{\partial x}|_{n+\frac{1}{2}} - \frac{\partial\phi}{\partial x}|_{n-\frac{1}{2}}\right]$$

$$= \frac{1}{\Delta}\left[\frac{\phi_{n+1}-\phi_n}{\Delta} - \frac{\phi_n-\phi_{n-1}}{\Delta}\right]$$

$$= \frac{1}{\Delta^2}\left[\phi_{n+1} - 2\phi_n + \phi_{n-1}\right]$$

Therefore, the discrete version is given by:

$$\frac{1}{\Delta^2}\left[\phi_{n+1} - 2\phi_n + \phi_{n-1}\right] = 4\pi\rho_n$$

Now, taking Fourier transform on both sides:

$$\phi_n = \Delta \sum_m \tilde{\phi}_m \exp[2\pi imn/N]$$

$$\phi_{n+1} = \Delta \sum_m \tilde{\phi}_m \exp[2\pi im(n+1)/N] = \Delta \sum_m \tilde{\phi}_m \exp[2\pi imn/N]\exp[2\pi im/N]$$

$$\phi_{n-1} = \Delta \sum_m \tilde{\phi}_m \exp[2\pi im(n-1)/N] = \Delta \sum_m \tilde{\phi}_m \exp[2\pi imn/N]\exp[-2\pi im/N]$$

where,

$$FT(\phi_n) = \tilde{\phi}_m$$

$$FT(\phi_{n+1}) = \tilde{\phi}_m \exp[2\pi im/N]$$

$$FT(\phi_{n-1}) = \tilde{\phi}_m \exp[-2\pi im/N]$$

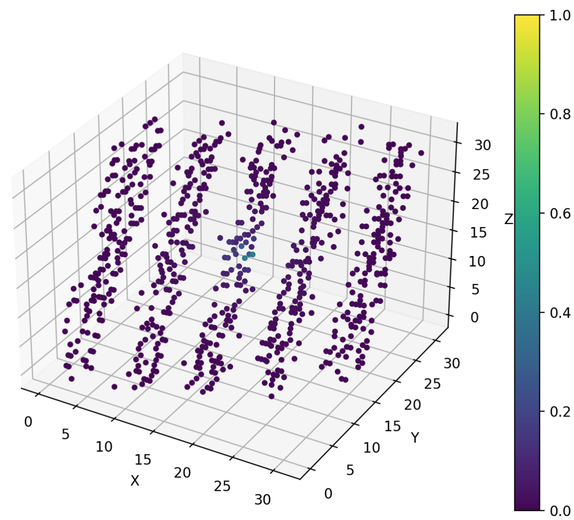
Therefore, the discrete Fourier transform of the Poisson equation is given by:

$$\frac{1}{\Delta^2}[\exp(2\pi im/N) + \exp(-2\pi im/N) - 2]\tilde{\phi}_m = 4\pi\tilde{\phi}_m$$

$$\frac{2}{\Delta^2}[\cos(2\pi m/N) - 1]\tilde{\phi}_m = 4\pi\tilde{\rho}_m$$

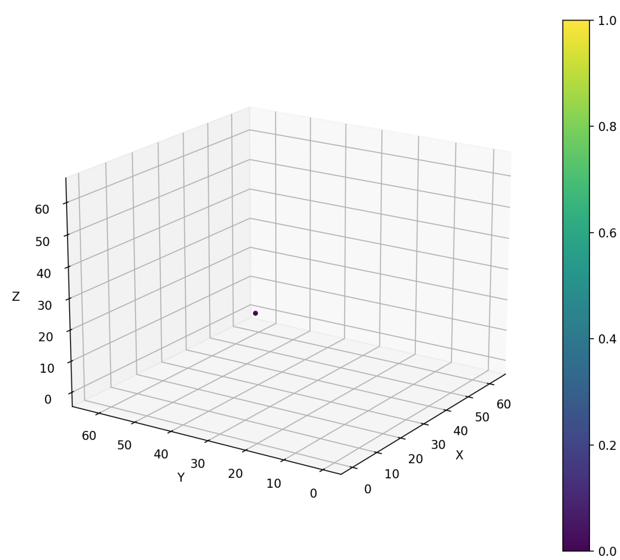
3.1.2 Implementing the DFT derived above

Initially, the DFT implementation resulted in a potential that one would expect from a repeated delta function. This is shown below.



3.2 Isolating the Mass Distribution

After implementing the trick described in the assignment sheet, i.e. isolating the mass and creating the Green's function response, the resulting potential is as shown below.



3.3 Testing a Spherically Symmetric Case for the Potential

3.4 Calculating the Potential with Different Widths and Axis Ratios of the Gaussian