A Structured Proof of the QPE Package Theorem `_psi_t_var_formula`

Theorem:

If $\forall_{t\in\mathbb{N}^+}\psi_t = \frac{1}{2^{t/2}}\left((|0\rangle + e^{2\pi i 2^{t-1}\varphi}|1\rangle)\otimes(|0\rangle + e^{2\pi i 2^{t-2}\varphi}|1\rangle)\otimes\cdots\otimes(|0\rangle + e^{2\pi i 2^{0}\varphi}|1\rangle)\right)$,
then $\forall_{t\in\mathbb{N}^+}\psi_t = \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1}\ e^{2\pi i\varphi k}|k\rangle_t$.

Assumptions

1. $\varphi \in [0, 1)$
2. $\forall_{k\in]t[}\ p_k = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 2^k\varphi}|1\rangle)$ (where $]t[= \{0, 1, 2, \ldots, t-1\}$)
3. $\psi_1 = p_0$ and $\forall_{k\in[t-1]}\ \psi_{k+1} = p_k \otimes \psi_k$
   (This produces, *e.g.*, $\psi_t = p_{t-1}\otimes p_{t-2}\otimes\ldots\otimes p_0$.)
4. $\forall_{k\in]t[}\ p'_k = (|0\rangle + e^{2\pi i 2^k\varphi}|1\rangle)$
5. $\psi'_1 = p'_0$, and $\forall_{k\in[t-1]}\ \psi'_{k+1} = p'_k \otimes \psi'_k$
6. $\forall_{t\in\mathbb{N}^+}\forall_{k\in]2^t[}\ \ |0\rangle\otimes|k\rangle_t = |k\rangle_{t+1}$
7. $\forall_{t\in\mathbb{N}^+}\forall_{k\in]2^t[}\ \ |1\rangle\otimes|k\rangle_t = |2^t + k\rangle_{t+1}$
8. $\forall_a\forall_b\forall_J\ \sum_{j\in J} a\otimes b_j = a\otimes\sum_{j\in J} b_j$
9. $\sum_{k=0}^{k=t} a|k\rangle_t = a\sum_{k=0}^{k=t}|k\rangle_t$ (but not yet cited in Induction Step 5 below?)
10. $\psi_{t-1} = \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1}\ e^{2\pi i\varphi k}|k\rangle_t$ iff $\psi'_{t-1} = \sum_{k=0}^{2^t-1}\ e^{2\pi i\varphi k}|k\rangle_t$.
11. Principle of Mathematical Induction:
    If $P(1)$ and $\forall_{t\in\mathbb{N}^+}[P(t)\Rightarrow P(t+1)]$, then $\forall_{t\in\mathbb{N}^+}P(t)$.
12. $\forall_{t\in\mathbb{N}^+}\left[\psi_t = \frac{1}{2^{t/2}}\left((|0\rangle + e^{2\pi i 2^{t-1}\varphi}|1\rangle)\otimes(|0\rangle + e^{2\pi i 2^{t-2}\varphi}|1\rangle)\otimes\cdots\otimes(|0\rangle + e^{2\pi i 2^{0}\varphi}|1\rangle)\right)\right]$.
13. $\forall_{t\in\mathbb{N}^+}\left[f(t) = \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1}\ e^{2\pi i\varphi k}|k\rangle_t\right]$.
14. $\forall_{t\in\mathbb{N}^+}[P(t)\Leftrightarrow \psi_t = f(t)]$
15. $\psi_1 = \frac{1}{2^{1/2}}(|0\rangle + e^{2\pi i\varphi}|1\rangle)$

16. $f(1) = \frac{1}{2^{1/2}} \left(|0\rangle_1 + e^{2\pi i\varphi} |1\rangle_1 \right)$

17. $\forall_{a,b,c\in\mathbb{Z}} \forall_\alpha \ \sum_{k=a}^{c} \alpha_k = \sum_{k=a}^{b} \alpha_k + \sum_{k=b+1}^{c} \alpha_k$

18. $\forall_{k\in\{2^{t'},2^{t'}+1,\ldots,2^{t^*}-1\}} \left[e^{2\pi i\varphi k} |k\rangle_{t^*} = e^{2\pi i\varphi(k-2^{t'}+2^{t'})} \left|k - 2^{t'} + 2^{t'}\right\rangle_{t^*} \right]$

Proof (by induction on t):

Base Case.

1. But $1 \in \mathbb{N}^+$ and $P(1) \Leftrightarrow \psi_1 = f(1)$
and $\psi_1 = \frac{1}{2^{1/2}} \left(\left(|0\rangle + e^{2\pi i\varphi} |1\rangle \right) \right)$
and $f(1) = \frac{1}{2^{1/2}} \left(\left(|0\rangle_1 + e^{2\pi i\varphi} |1\rangle_1 \right) \right)$
and $|0\rangle = |0\rangle_1$ and $|1\rangle = |1\rangle_1$.
Thus $P(1)$.

Inductive Step.

2. But let $t' \in \mathbb{N}^+$ such that $P(t')$
and $\forall_{t\in\mathbb{N}^+} [P(t) \Leftrightarrow \psi_t = f(t)]$.
Thus, $\psi_{t'} = f(t')$.

3. But $t^* = t' + 1$ and $\forall_{a,b,c\in\mathbb{Z}} \forall_\alpha \ \sum_{k=a}^{c} \alpha_k = \sum_{k=a}^{b} \alpha_k + \sum_{k=b+1}^{c} \alpha_k$.
Thus $\sum_{k=0}^{2^{t^*}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} = \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} + \sum_{k=2^{t'}}^{2^{t^*}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*}$

4. But $\forall_{k\in\{2^{t'},2^{t'}+1,\ldots,2^{t^*}-1\}} \left[e^{2\pi i\varphi k} |k\rangle_{t^*} = e^{2\pi i\varphi(k-2^{t'}+2^{t'})} \left|k - 2^{t'} + 2^{t'}\right\rangle_{t^*} \right]$.
Thus $\sum_{k=2^{t'}}^{2^{t^*}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} = \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi(k+2^{t'})} \left|k + 2^{t'}\right\rangle_{t^*}$

5. But $\sum_{k=0}^{2^{t^*}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} = \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} + \sum_{k=2^{t'}}^{2^{t^*}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*}$ and
$\sum_{k=2^{t'}}^{2^{t^*}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} = \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi(k+2^{t'})} \left|k + 2^{t'}\right\rangle_{t^*}$.
Thus $\sum_{k=0}^{2^{t^*}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} = \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} + \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi(k+2^{t'})} \left|k + 2^{t'}\right\rangle_{t^*}$

6. But $\forall_a \forall_b \forall_J \ \sum_{j\in J} a \otimes b_j = a \otimes \sum_{j\in J} b_j$
and $\forall_{t\in\mathbb{N}^+} \forall_{k\in]2^t[} \ |1\rangle \otimes |k\rangle_t = |2^t + k\rangle_{t+1}$.
Thus $\sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi(2^{t'}+k)} \left|2^{t'} + k\right\rangle_{t^*}$
$= e^{2\pi i\varphi(2^{t'})} \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi(k)} \ |1\rangle \otimes |k\rangle_{t'}$
$= |1\rangle \otimes \left(e^{2\pi i\varphi(2^{t'})} \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi(k)} |k\rangle_{t'} \right)$

7. But $\forall_a \forall_b \forall_J \ \sum_{j\in J} a \otimes b_j = a \otimes \sum_{j\in J} b_j$
and $\forall_{t\in\mathbb{N}^+} \forall_{k\in[2^t-1]} \ |0\rangle \otimes |k\rangle_t = |k\rangle_{t+1}$.
Thus $\sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi k} |k\rangle_{t^*} = \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi k} |0\rangle \otimes |k\rangle_{t'} = |0\rangle \otimes \sum_{k=0}^{2^{t'}-1} \ e^{2\pi i\varphi k} |k\rangle_{t'}$

8. But $\sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} + \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi(k+2^{t'})} \left|k+2^{t'}\right\rangle_{t^*}$,
   and $\sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi(2^{t'}+k)} \left|2^{t'}+k\right\rangle_{t^*} = \left|1\right\rangle \otimes e^{2\pi i\varphi(2^{t'})} \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi(k)} \left|k\right\rangle_{t'}$,
   and $\sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = \left|0\right\rangle \otimes \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t'}$.
   Thus $\sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = \left|0\right\rangle \otimes \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t'} + \left|1\right\rangle \otimes e^{2\pi i\varphi(2^{t'})} \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi(k)} \left|k\right\rangle_{t'}$
   $= \left(\left|0\right\rangle + \left|1\right\rangle e^{2\pi i\varphi(2^{t'})}\right) \otimes \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi(k)} \left|k\right\rangle_{t'}$

9. But $\sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = \left(\left|0\right\rangle + \left|1\right\rangle e^{2\pi i\varphi(2^{t'})}\right) \otimes \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi(k)} \left|k\right\rangle_{t'}$
   and $\forall_{k\in]t[}\ p'_k = (\left|0\right\rangle + e^{2\pi i 2^k\varphi} \left|1\right\rangle)$ and $\psi'_{t'} = \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t'}$.
   Thus $\sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = p'_{t'} \otimes \psi'_{t'}$

10. But $\sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = p'_{t'} \otimes \psi'_{t'}$ and $\psi'_1 = p'_0$ and $\forall_{k\in[t-1]}\ \psi_{k+1} = p'_k \otimes \psi'_k$.
    Thus $\sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = \psi'_{t^*}$

11. But $\sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*} = \psi'_{t^*}$ and $[\forall_{a,b,\alpha}$ if $a = b$ then $\alpha a = \alpha b]$
    and $a = \sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*}$ and $b = \psi'_{t^*}$ and $\alpha = \frac{1}{2^{t^*/2}}$.
    Thus $\frac{1}{2^{t^*/2}}\psi'_{t^*} = \frac{1}{2^{t^*/2}} \sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*}$

12. But $\frac{1}{2^{t^*/2}}\psi'_{t^*} = \frac{1}{2^{t^*/2}} \sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*}$ and $\frac{1}{2^{t^*/2}}\psi'_{t^*} = \psi_{t^*}$.
    Thus $\psi_{t^*} = \frac{1}{2^{t^*/2}} \sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*}$.

13. But assuming $\psi_{t'} = \frac{1}{2^{t'/2}} \sum_{k=0}^{2^{t'}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t'}$ and t' was arbitrary,
    we obtain $\psi_{t^*} = \frac{1}{2^{t^*/2}} \sum_{k=0}^{2^{t^*}-1} e^{2\pi i\varphi k} \left|k\right\rangle_{t^*}$.
    Thus, $\forall_{t\in\mathbb{N}^+} [P(t) \Rightarrow P(t+1)]$.

14. But $P(1)$ and $\forall_{t\in\mathbb{N}^+} [P(t) \Rightarrow P(t+1)]$
    and the Principle of Mathematical Induction.
    Thus, $\forall_{t\in\mathbb{N}^+} P(t)$.