Power last frequency

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1 Objective function: Average mechanical power

The states are X_{wec} and F_{pto} which are the WEC position and PTO force time series in the frequency domain, using an unstructured controller for the force. The WEC velocity can be obtained as the derivative of the position.

$$X_{wec} = \begin{bmatrix} x_{dc} \\ x_{a1} \\ x_{b1} \\ x_{a2} \end{bmatrix} \qquad \dot{X}_{wec} = \begin{bmatrix} 0 \\ x_{b1}\omega_1 \\ -x_{a1}\omega_1 \\ 0 \end{bmatrix} \qquad \ddot{X}_{wec} = \begin{bmatrix} 0 \\ -x_{a1}\omega_1^2 \\ -x_{b1}\omega_1^2 \\ -x_{a2}\omega_2^2 \end{bmatrix} \qquad F_{pto} = \begin{bmatrix} f_{dc} \\ f_{a1} \\ f_{b1} \\ f_{a2} \end{bmatrix} \tag{1}$$

Note that the second derivative state can not be obtained by applying the same derivative matrix to the velocity as was applied to the position to obtain velocity. That is, the second derivative operator (matrix) is not the first derivative operator squared; these differ in the treatment of the last frequency component.

In the time domain, these are:

$$\begin{aligned} x_{wec}(t) &= x_{dc} + x_{a1}\cos(\omega_1 t) + x_{b1}\sin(\omega_1 t) + x_{a2}\cos(2\omega_1 t) \\ \dot{x}_{wec}(t) &= x_{b1}\omega_1\cos(\omega_1 t) + -x_{a1}\omega_1\sin(\omega_1 t) \\ f_{pto}(t) &= f_{dc} + f_{a1}\cos(\omega_1 t) + f_{b1}\sin(\omega_1 t) + f_{a2}\cos(2\omega_1 t) \end{aligned} \tag{2}$$

The power in the time domain is the product of velocity and force. This is equivalent to a convolution integral in the frequency domain.

$$\begin{aligned} p(t) =& \dot{x}_{wec}(t) \cdot f_{pto}(t) \\ =& \quad f_{dc}x_{b1}\omega_1\cos(\omega_1 t) \\ & + f_{a1}\cos(\omega_1 t)x_{b1}\omega_1\cos(\omega_1 t) \\ & + f_{b1}\sin(\omega_1 t)x_{b1}\omega_1\cos(\omega_1 t) \\ & + f_{a2}\cos(2\omega_1 t)x_{b1}\omega_1\cos(\omega_1 t) \\ & - f_{dc}x_{a1}\omega_1\sin(\omega_1 t) \\ & - f_{a1}\cos(\omega_1 t)x_{a1}\omega_1\sin(\omega_1 t) \\ & - f_{b1}\sin(\omega_1 t)x_{a1}\omega_1\sin(\omega_1 t) \\ & - f_{a2}\cos(2\omega_1 t)x_{a1}\omega_1\sin(\omega_1 t) \end{aligned} \tag{3}$$

The average power is the integral of power in the time domain, or the DC component of power in the frequency domain. Using orthogonality of sinusoids, many of these terms are zero. Similarly, the integral of a sine or cosine over an integer number of periods is zero.

$$\begin{aligned}
\bar{P}_{mech} =& \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t \\
=& \frac{\omega_1}{2\pi}\int_0^{2\pi/\omega_1} p(t)\,\mathrm{d}t \\
=& \frac{\omega_1}{2\pi}\int_0^{2\pi/\omega_1} \Big[\\
&\quad + (f_{a1}x_{b1}\omega_1)\cos(\omega_1 t)\cos(\omega_1 t) \\
&\quad - (f_{b1}x_{a1}\omega_1)\sin(\omega_1 t)\sin(\omega_1 t) \\
&\Big]\,\mathrm{d}t \\
=& \frac{\omega_1}{2}(f_{a1}x_{b1} - f_{b1}x_{a1})
\end{aligned} \tag{4}$$

If we generalize to n frequencies:

$$\bar{P}_{mech} = \frac{\omega_1}{2}\left(\sum_{k=1}^{n-1} k\left(f_{ak}x_{bk} - f_{bk}x_{ak}\right)\right) \tag{5}$$

Neither the DC component nor the last frequency component of the PTO force or the WEC position have any effect on mechanical power.

2 Dynamics

The linear dynamics can be expressed in the frequency domain, where each frequency component satisfies the equation of motion independently from the other frequency components.

$$(A_i + m)\ddot{X}_i + B_i\dot{X}_i + KX_i = F_{e,i} + F_{pto,i} \tag{6}$$

where $X_i = x_{a,i} + jx_{b,i}$ is the complex amplitude of position, A_i and B_i are the added mass and radiation damping coefficients, and $F_{e,i}$ is the complex amplitude of excitation force for frequency $\omega_i = i * \omega_1$. The hydrostatic stiffness K is constant. The complex amplitude of the PTO force is given as $F_{pto,i} = f_{a,i} + jf_{b,i}$, where j is the imaginary unit.

2.1 DC frequency

For the first (DC) frequency, there is no radiation or excitation.

$$Kx_{dc} = f_{dc} \tag{7}$$

If f_{dc} is arbitrary, as is the case with mechanical power as the objective function and linear dynamics, then the mean (DC) position is related to this arbitrary DC force by the hydrostatic stiffness. In practice the mean position can be driven to zero by adding a max *symmetrical* PTO force constraint, or by considering electric power with losses.

2.2 Nyquist frequency

For the last frequency n, the EOM is

$$\begin{aligned} -x_{a,n}\omega_n^2(A_n+m)+Kx_{a,n} &= f_{ex,a,n}+f_{pto,a,n} \\ x_{a,n} &= \frac{f_{ex,a,n}+f_{pto,a,n}}{K-n^2\omega_1^2(A_n+m)} \end{aligned} \tag{8}$$

If f_{pto} is arbitrary (as above, with objective=average mechanical power), then the last frequency component of the position is related to that arbitrary value by the relation above.

Form observations, the Nyquist component can also be driven to zero by a force constraint (and probably by an electric energy loss).

2.3 Intermediate frequencies

For intermediate frequencies, position, velocity, and acceleration amplitudes are all non-zero. The velocity and acceleration complex amplitudes can be written in terms of the position, as $\dot{X}_i = j\omega_i X_i$ and $\ddot{X}_i = -\omega_i^2 X_i$. The relation between the PTO force and the position is given as

$$X_i = \frac{F_{e,i}+F_{pto,i}}{-\omega_i^2(A_i+m)+j\omega_i B+K} \tag{9}$$

In general all these intermediate frequency components of the PTO force and WEC state affect the objective function and are therefore not arbitrary.

3 Structured Controller

A structured controller changes how the the PTO force is computed but equation 5 still shows no effect of the DC and Nyquist components on the objective function. However, since PTO force is no longer a free variable, the dynamic equations have a specific solution, (even at DC and Nyquist component). For a PI controller, equation 7 is satisfied if either $K_p = K$ (the integral gain equals the hydrostatic stiffness) which is likely suboptimal, or the DC component of position is zero. Equation 8 becomes

$$x_{a,n} = \frac{f_{ex,a,n}}{K-n^2\omega_1^2(A_n+m)-K_p} \tag{10}$$

Similiatly, for a PID equation 7 becomes

$$\begin{aligned} Kx_{dc} &= K_p x_{dc} - K_d n^2 \omega_1^2 x_{dc} \\ &= (K_p - n^2 \omega_1^2 K_d) x_{dc} \end{aligned} \tag{11}$$

which implies either the relation in parenthesis is satisfied (likely suboptimal) or x_{dc} is zero. Equation 8 becomes

$$x_{a,n} = \frac{f_{ex,a,n}}{K - K_p - n^2 \omega_1^2 (A_n + m - K_d)} \tag{12}$$

4 Electrical Power

PTO force and velocity are converted to current and voltage using the impedance matrix. The impedance matrix can capture electrical losses. To compute the current and voltage we use the PTO impedance in transmission matrix form, which contains complex Fourier coefficients,

$$\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \dot{X} \\ F_p \end{bmatrix} \tag{13}$$

Considering that all components in (13) are complex the intermediate components of the real Fourier coefficients become significantly longer and we won't state them right now, since we want to focus on DC and Nyquist components,

$$I = \begin{bmatrix} B_{dc} f_{dc} \\ \cdots \\ \cdots \\ B_{a2} f_{a2} \end{bmatrix} \quad V = \begin{bmatrix} D_{dc} f_{dc} \\ \cdots \\ \cdots \\ D_{a2} f_{a2} \end{bmatrix} \tag{14}$$

However, to re-emphasize, the intermediate components do impact the average electrical power. The key difference in the electrical states (14) vs. the mechanical states (1) is that the DC and Nyquist component of the flow variable (current) are non-zero. We recall, that we manually create a the DC component of the PTO transmission matrix for $\omega = 0$, assuming that the user will define the PTO impedance matrix with the same frequency vector that they used for the BEM code, i.e. no zero component.

The electrical power in the time domain includes extra terms, when compared to the mechanical power (equation 3), that depend on the DC and Nyquist components, because $I_d c$ and $I_{n,a}$ are non-zero and they also do not go to zero when integrated over the full period, including the two below:

$$\begin{aligned} p_e(t) =& i(t) \cdot v(t) \\ =& \quad \cdots \\ &+ I_{dc} V_{DC} + I_{n,a} V_{n,a} \cos^2(\omega_n t) \end{aligned} \tag{15}$$

This results in the average electrical power depending on the DC and Nyquist compononents of both voltage and current. These in turn depend on the DC and Nyquist components of the PTO force. The dynamic equations (7-8) relate the forces to the position. Since the DC and Nyquist components of force are no longer arbitrary neither are the DC and Nyquist components of position.

The DC and Nyquist components of the WEC position and PTO force have an effect on the electrical average power and are therefor not arbitrary.

Note: The current DC and Nyquist components I_dc and $I_{n,a}$ are only non-zero if the PTO transmission matrix has a defined B, this generally requires a gyrating energy transduction within the PTO, such as a generator.

5 Conclusion

For average mechanical power the DC and Nyquist components of WEC position and PTO force are arbitrary (have no effect on the objective function) but ar related to each other through equations 7 and 8 (to satify the dynamics).

For average electrical power using an impedance matrix, the DC and Nyquist components of the WEC position and PTO force have an effect on the objective function and are therefor not arbitrary.

6 Appendix

Electrical states

$$\begin{aligned} A\dot{X} &= (A_a + iA_b)(\dot{X}_a + i\dot{X}_b) \\ &= A_a\dot{X}_a - A_b\dot{X}_b + i(A_a\dot{X}_b + A_b\dot{X}_c) \end{aligned} \tag{16}$$

$$\begin{aligned} BF_p &= (B_a + iB_b)(f_a + if_b) \\ &= B_af_a - B_bf_b + i(B_af_b + B_bf_c) \end{aligned} \tag{17}$$

$$\begin{aligned} C\dot{X} &= (C_a + iC_b)(\dot{X}_a + i\dot{X}_b) \\ &= C_a\dot{X}_a - C_b\dot{X}_b + i(C_a\dot{X}_b + C_b\dot{X}_c) \end{aligned} \tag{18}$$

$$\begin{aligned} DF_p &= (D_a + iD_b)(f_a + if_b) \\ &= D_af_a - D_bf_b + i(D_af_b + D_bf_c) \end{aligned} \tag{19}$$

$$I = \begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} A_a\dot{X}_a - A_b\dot{X}_b + B_af_a - B_bf_b \\ A_a\dot{X}_b + A_b\dot{X}_c + B_af_b + B_bf_c \end{bmatrix} \tag{20}$$

$$V = \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} C_a\dot{X}_a - C_b\dot{X}_b + D_af_a - D_bf_b \\ C_a\dot{X}_b + C_b\dot{X}_c + D_af_a - D_bf_b \end{bmatrix} \tag{21}$$