Quadrilateral Quality

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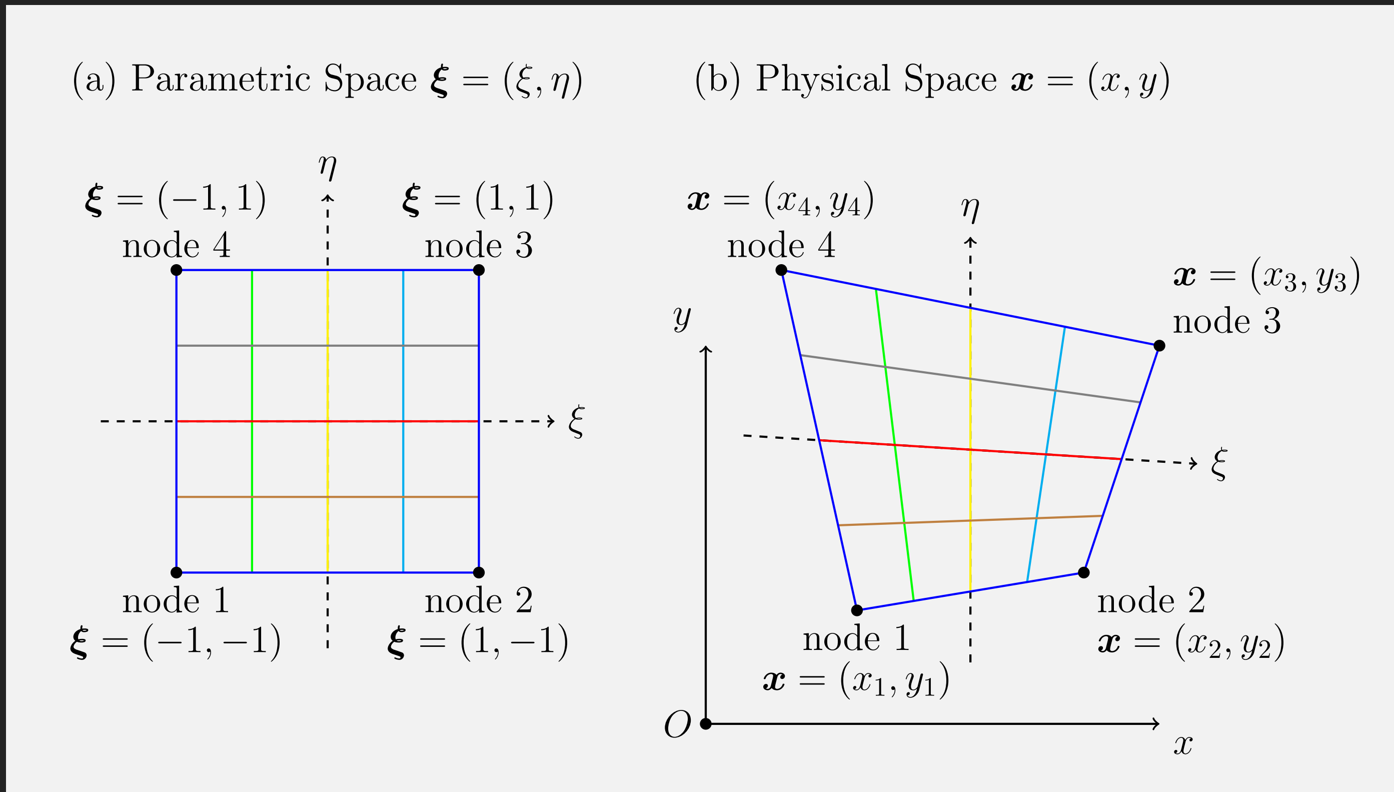

Figure 1: Parametric mapping $\boldsymbol{x} = f(\boldsymbol{\xi})$ from parametric space to physical space.

Chapter 1

Quality

1.1 Isoparametric Mapping

Let the parametric mapping $f : \boldsymbol{\xi} \in [-1, 1] \times [-1, 1] \mapsto \boldsymbol{x} \in \mathbb{R}^2$ be defined as

$$x(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\ x_a, \tag{1.1}$$

$$y(\xi, \eta) = \sum_{a=1}^{4} N_a(\xi, \eta)\ y_a, \tag{1.2}$$

where a nodal **shape function** is defined for each of the four nodes

$$N_1(\xi, \eta) \triangleq \frac{1}{4}(1-\xi)(1-\eta), \tag{1.3}$$

$$N_2(\xi, \eta) \triangleq \frac{1}{4}(1+\xi)(1-\eta), \tag{1.4}$$

$$N_3(\xi, \eta) \triangleq \frac{1}{4}(1+\xi)(1+\eta), \tag{1.5}$$

$$N_4(\xi, \eta) \triangleq \frac{1}{4}(1-\xi)(1+\eta). \tag{1.6}$$

1.2 Jacobian

For the quadrilateral element, the Jacobian $\boldsymbol{J}$ is calculated as the matrix of partial derivatives of $\boldsymbol{x} = (x, y)$ with respect to $\boldsymbol{\xi} = (\xi, \eta)$,

$$\boldsymbol{J}(\xi, \eta) \triangleq \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right] = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}. \tag{1.7}$$

Substituting $x(\xi, \eta)$ and $y(\xi, \eta)$ with shape function equations (1.1)-(1.2) and expanding terms, the Jacobian takes the form

$$\boldsymbol{J}(\xi, \eta) = \frac{1}{4} \begin{bmatrix} -1+\eta & 1-\eta & 1+\eta & -1-\eta \\ -1+\xi & -1-\xi & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \tag{1.8}$$

The determinant of the Jacobian, $\det(\boldsymbol{J})$, can be found to be

$$\det(\boldsymbol{J}\,(\xi, \eta)) = c_0 + c_1 \xi + c_2 \eta, \tag{1.9}$$

where

$$c_0 = \frac{1}{8} \left[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\right], \tag{1.10}$$

$$c_1 = \frac{1}{8} \left[(x_3 - x_4)(y_1 - y_2) - (x_1 - x_2)(y_3 - y_4)\right], \tag{1.11}$$

$$c_2 = \frac{1}{8} \left[(x_2 - x_3)(y_1 - y_4) - (x_1 - x_4)(y_2 - y_3)\right]. \tag{1.12}$$

1.3 Quality

We follow *The Verdict Geometry Quality Library* documentation[1] and implementation[2] for the definitions of quality metrics. The SNL Cubit help manual is also helpful.[3]

1.3.1 Preliminaries

Let the four edge vectors and their respective lengths be defined as

$$\boldsymbol{e}_1 \triangleq \boldsymbol{x}_2 - \boldsymbol{x}_1, \qquad \ell_1 \triangleq \| \ \boldsymbol{e}_1 \ \|, \tag{1.13}$$

$$\boldsymbol{e}_2 \triangleq \boldsymbol{x}_3 - \boldsymbol{x}_2, \qquad \ell_2 \triangleq \| \ \boldsymbol{e}_2 \ \|, \tag{1.14}$$

$$\boldsymbol{e}_3 \triangleq \boldsymbol{x}_4 - \boldsymbol{x}_3, \qquad \ell_3 \triangleq \| \ \boldsymbol{e}_3 \ \|, \tag{1.15}$$

$$\boldsymbol{e}_4 \triangleq \boldsymbol{x}_1 - \boldsymbol{x}_4, \qquad \ell_4 \triangleq \| \ \boldsymbol{e}_4 \ \| \ . \tag{1.16}$$

The two (non-normalized) *principal axes* are the defined though vector addition of the two opposing side lengths

$$\boldsymbol{X} \triangleq \boldsymbol{e}_1 - \boldsymbol{e}_3 = (\boldsymbol{x}_2 - \boldsymbol{x}_1) - (\boldsymbol{x}_4 - \boldsymbol{x}_3), \tag{1.17}$$

$$\boldsymbol{Y} \triangleq \boldsymbol{e}_2 - \boldsymbol{e}_4 = (\boldsymbol{x}_3 - \boldsymbol{x}_2) - (\boldsymbol{x}_1 - \boldsymbol{x}_4). \tag{1.18}$$

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[1] Knupp PM, Ernst CD, Thompson DC, Stimpson CJ, Pebay PP. The verdict geometric quality library. Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA (United States); 2006 Mar 1. OSTI https://www.osti.gov/servlets/purl/901967.

[2] See https://github.com/Kitware/VTK/blob/master/ThirdParty/verdict/vtkverdict/ and in particular, the `quad_scaled_jacobian` function in the `V_QuadMetric.cpp` implementation.

[3] See https://cubit.sandia.gov/files/cubit/16.04/help_manual/WebHelp/cubithelp.htm

At each vertex, there is a normal vector and its respective normalized unit vector

$$\boldsymbol{N}_1 \triangleq \boldsymbol{e}_4 \times \boldsymbol{e}_1, \qquad \hat{\boldsymbol{n}}_1 \triangleq \boldsymbol{N}_1 / \| \boldsymbol{N}_1 \|, \tag{1.19}$$

$$\boldsymbol{N}_2 \triangleq \boldsymbol{e}_1 \times \boldsymbol{e}_2, \qquad \hat{\boldsymbol{n}}_2 \triangleq \boldsymbol{N}_2 / \| \boldsymbol{N}_2 \|, \tag{1.20}$$

$$\boldsymbol{N}_3 \triangleq \boldsymbol{e}_2 \times \boldsymbol{e}_3, \qquad \hat{\boldsymbol{n}}_3 \triangleq \boldsymbol{N}_3 / \| \boldsymbol{N}_3 \|, \tag{1.21}$$

$$\boldsymbol{N}_4 \triangleq \boldsymbol{e}_3 \times \boldsymbol{e}_4, \qquad \hat{\boldsymbol{n}}_4 \triangleq \boldsymbol{N}_4 / \| \boldsymbol{N}_4 \| . \tag{1.22}$$

At the center of the element, there is principal axis normal as well

$$\boldsymbol{N}_c \triangleq \boldsymbol{X} \times \boldsymbol{Y}, \qquad \hat{\boldsymbol{n}}_c \triangleq \boldsymbol{N}_c / \| \boldsymbol{N}_c \| . \tag{1.23}$$

There are four contributions to the quadrilateral area from each of the four nodal areas[4]

$$\alpha_1 \triangleq \boldsymbol{N}_1 \cdot \hat{\boldsymbol{n}}_c, \tag{1.24}$$

$$\alpha_2 \triangleq \boldsymbol{N}_2 \cdot \hat{\boldsymbol{n}}_c, \tag{1.25}$$

$$\alpha_3 \triangleq \boldsymbol{N}_3 \cdot \hat{\boldsymbol{n}}_c, \tag{1.26}$$

$$\alpha_4 \triangleq \boldsymbol{N}_4 \cdot \hat{\boldsymbol{n}}_c. \tag{1.27}$$

[4] It may be tempting to (erroneously) write $\hat{\boldsymbol{n}}_1 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_2 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_3 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_4 \xrightarrow{\text{2D}} \hat{\boldsymbol{n}}_c$ and $\alpha_1 \xrightarrow{\text{2D}} \| \boldsymbol{N}_1 \|, \alpha_2 \xrightarrow{\text{2D}} \| \boldsymbol{N}_2 \|, \alpha_3 \xrightarrow{\text{2D}} \| \boldsymbol{N}_3 \|, \alpha_4 \xrightarrow{\text{2D}} \| \boldsymbol{N}_4 \|$ given that for the 2D quadrilateral case, all unit norms are in the same plane. The problem with such a construction is that it destroys the sign information carried by each normal vector. For the non-degenerate case, the foregoing simplification is true, since all give normals will carry the same sign. However, for degenerate cases, such as when a quadrilateral folds over onto itself, the sign information is no longer homogeneous, and the sign information *must* be retained to accurately calculate Jacobian metrics that go negative.

1.3.2 Signed Area

The **signed area** SA is defined as the average of all nodal area contributions:

$$\mathrm{SA} \triangleq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}. \tag{1.28}$$

The metric dimension is L^2 and the ideal (unit square) value is `1.0`.

1.3.3 Skew

The **skew** is defined as the absolute value of the cosine of the angle between the two principal axes:

$$\mathrm{skew} \triangleq \left| \frac{\boldsymbol{X}}{\parallel \boldsymbol{X} \parallel} \cdot \frac{\boldsymbol{Y}}{\parallel \boldsymbol{Y} \parallel} \right|. \tag{1.29}$$

1.3.4 Aspect Ratio

The **aspect ratio** AR is defined as the maximum edge length ratios taken at the quadrilateral center.[5] This can be expressed in terms of the norms of the principal axes as

$$\mathrm{AR} = \max \left(\frac{\parallel \boldsymbol{X} \parallel}{\parallel \boldsymbol{Y} \parallel}, \frac{\parallel \boldsymbol{Y} \parallel}{\parallel \boldsymbol{X} \parallel}, \right) \tag{1.30}$$

[5]Robinson J. CRE method of element testing and the Jacobian shape parameters. Engineering Computations. 1987 Feb 1.

Alternatively, the perimeter length multiplied by the maximum side length, divided by four times the area to define a triangle aspect ratio that is meaningful for quadrilaterals,[6] with dimension L^0 and acceptable range $[1.0, 1.3]$.

1.3.5 Minimum Jacobian

The **Minimum Jacobian** $J_{\min}$ is defined as the minimum pointwise area of local map at the four corners and center of quadrilateral[7]

$$J_{\min} \triangleq \min\left(\alpha_1, \alpha_2, \alpha_3, \alpha_4\right). \tag{1.31}$$

1.3.6 Minimum Scaled Jacobian

The **Minimum Scaled Jacobian** $\hat{J}_{\min}$ is the minimum nodal area divided by the lengths of the two edge vector connecting that point[8]

$$\hat{J}_{\min} \triangleq \min\left(\frac{\alpha_1}{\ell_4\ell_1}, \frac{\alpha_2}{\ell_1\ell_2}, \frac{\alpha_3}{\ell_2\ell_3}, \frac{\alpha_4}{\ell_3\ell_4}\right), \tag{1.32}$$

We warned previously in Footnote 4 for Jacobians that errors may result if sign information is not properly retained. We note a similar admonishment[9] for Scaled Jacobians, since the

[6]Knupp 2006, *op. cit.* at 38.

[7]Knupp 2006, *op. cit.* at 42.

[8]Knupp 2006, *op. cit.* at 51.

[9]It may (again) be tempting to (erroneously) write for the 2D case $\hat{J}_{\min} \xrightarrow{\text{2D}} \min\left(\sin\theta_1, \sin\theta_2, \sin\theta_3, \sin\theta_4\right)$, where θ_1 is the angle between $\boldsymbol{e}_4$ and $\boldsymbol{e}_1$, θ_2 with $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, θ_3 with $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, and θ_4 with $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$. Such a simplification will only work if $\boldsymbol{\theta}$ is retained as a vector quantity (thus retaining the sign). If θ is considered only as a scalar, errors will result when the Jacobian metric goes negative.

latter is a function of the former.

The dimension is L^0. The full range is $[-1.0, 1.0]$. The acceptable range is typically taken as $[0.3, 1.0]$ in the generous case and $[0.5, 1.0]$ in the more restricted case.

Appendix A

Computational Details

A.1 Mathematica

This section demonstrates the steps used in *Mathematica* to obtain the result in Eq. (1.9).

```
MatrixForm[A = {{-1 + b, 1 - b, 1 + b, -1 - b}, {-1 + a, -1 - a, 1 + a, 1 - a}}]
MatrixForm[B = {{x1, y1}, {x2, y2}, {x3, y3}, {x4, y4}}]
J = 1/4 * (A . B)
Expand[Det[J]]
result = Collect[Det[J] // Expand, {a, b}]
TeXForm[result]
```

produces

$$\det(\boldsymbol{J}(a,b)) = \frac{a}{8}(-x_1y_3 + x_1y_4 + x_2y_3 - x_2y_4 + x_3y_1 - x_3y_2 - x_4y_1 + x_4y_2) \\ + \frac{b}{8}(-x_1y_2 + x_1y_3 + x_2y_1 - x_2y_4 - x_3y_1 + x_3y_4 + x_4y_2 - x_4y_3) \\ + \frac{1}{8}(x_1y_2 - x_1y_4 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 + x_4y_1 - x_4y_3) \tag{A.1}$$