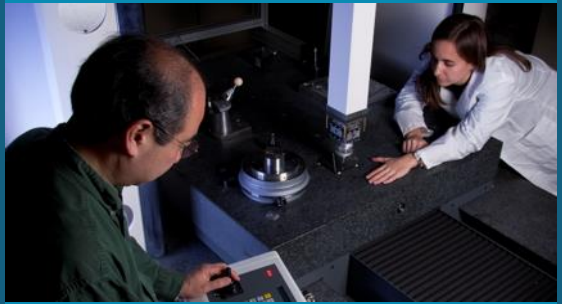
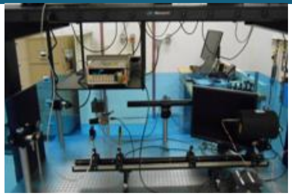
Introduction to Measurement Uncertainty – ENGR224





PRESENTED BY

Primary Standards Lab and Statistical Sciences

V7. SAND2024-07606O

Class Agenda

- ES&H
- Introductions
- Section 1: Measurement Uncertainty and Decision Making
- Section 2: Identifying Sources of Uncertainty
- Section 3: Statistical Basics
- Section 4: Direct Measurements and Simple Measurement Equations
- Section 5: The SI, Traceability, and Calibration
- Section 6: Indirect Measurements
  - Section 6.1: GUM Approach
  - Section 6.2: Monte Carlo Approach

ES&H and Introductions

- ES&H
- Introduce yourself
  - Name
  - Organization
  - Interest in measurement uncertainty

Learning Objectives

Participants will be able to:

- identify sources of uncertainty in a measurement
- define a measurement equation
- perform basic statistical calculations necessary for uncertainty evaluation
- evaluate type A and type B uncertainty contributions to a measurement
- distinguish direct and indirect measurement models
- calculate combined uncertainty of a measurement using the GUM approach
- calculate combined uncertainty of a measurement using the Monte Carlo approach
- have familiarity with the GUM terminology and uncertainty evaluation methods

Relationship ENGR200

Q1. What quantity are you measuring?			Q2: How accurate do the measurements need to be?			Q3: How will you ensure your equipment can make this measurement?			
Quantity Measured	MC	Requirement Number	Value or Range of Values Measured	Specification Limits	Guardbanded Acceptance Limit	Equipment Used (M&TE)	Measurement Uncertainty	TUR (>4 desired)	Calibrate (Yes/No)

Measurement Concerns (criticality, consequences and risk):

1= Performance, 2 = ES&H, 3 = Legal, 4= Corporate reputation, 5 = Personal reputation, 6 = Customer expectations

This course is based on the *Guide to the Expression of Uncertainty in Measurement* (GUM)

(JGCM 100:2008, ANSI/NCSL Z540-2, https://www.bipm.org/en/publications/guides/)

GUM Admonition

Although this guide (class) provides a framework for assessing uncertainty, it cannot substitute for critical thinking, intellectual honesty, and professional skill. The evaluation of uncertainty is neither a routine task nor a purely mathematical one: it depends on detailed knowledge of the nature of the measurand and of the measurement.

GUM (JGCM 100:2008) section 3.4.8

Section 1
Measurement Uncertainty and Decision Making

Section 1 – Measurement Uncertainty and Decision Making

Objective

- Participants will be able to explain the impact that measurement uncertainty has on their decision making.

Content

- Measurement Uncertainty Defined
- Measurement Uncertainty in Decision Making
- Examples of decisions based on measurement

Definition: Measurement Uncertainty

Informal Definition:

The word "uncertainty" means doubt, and thus in its broadest sense "uncertainty of measurement" means doubt about the validity of the result of a measurement.

GUM (JGCM 100:2008) section 2.2.1

Definition: Measurement Uncertainty

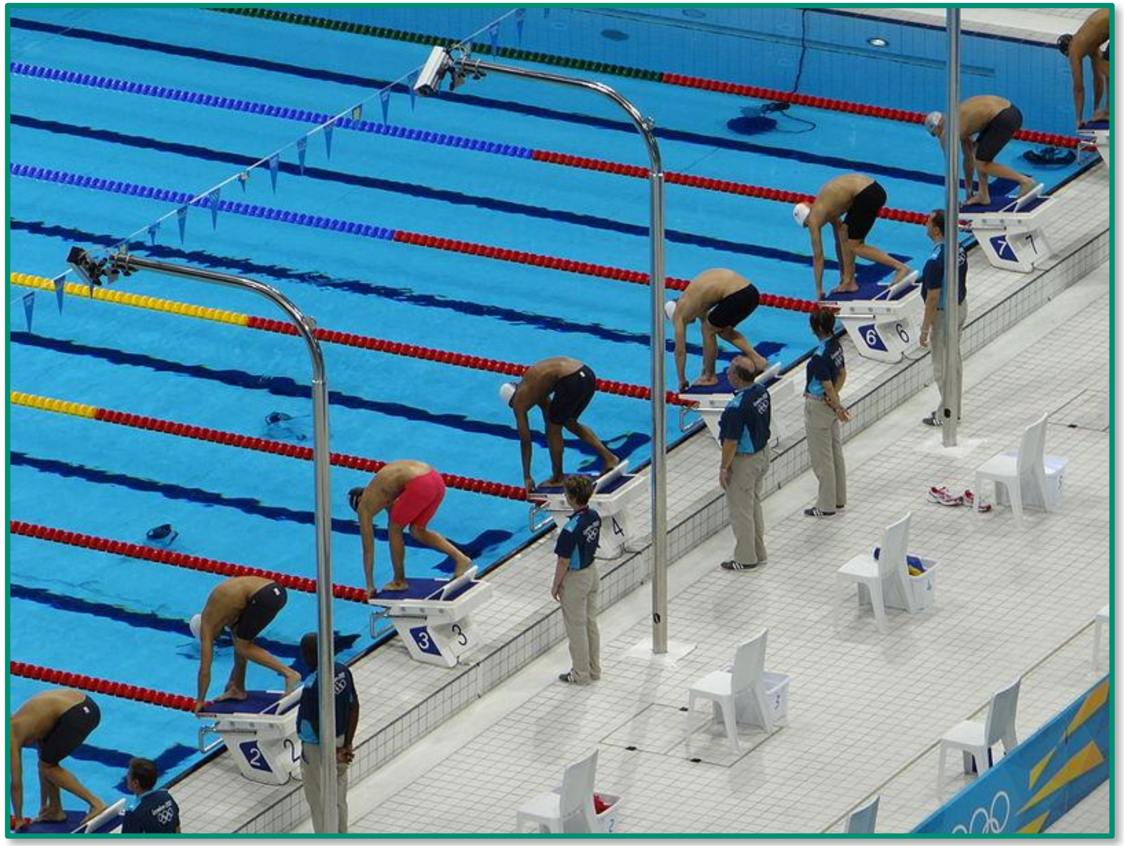
Formal Definition:

uncertainty (of measurement)

parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand.

GUM (JGCM 100:2008) section 2.2.3

Why do we care about uncertainty?



Olympic Swimming Pools

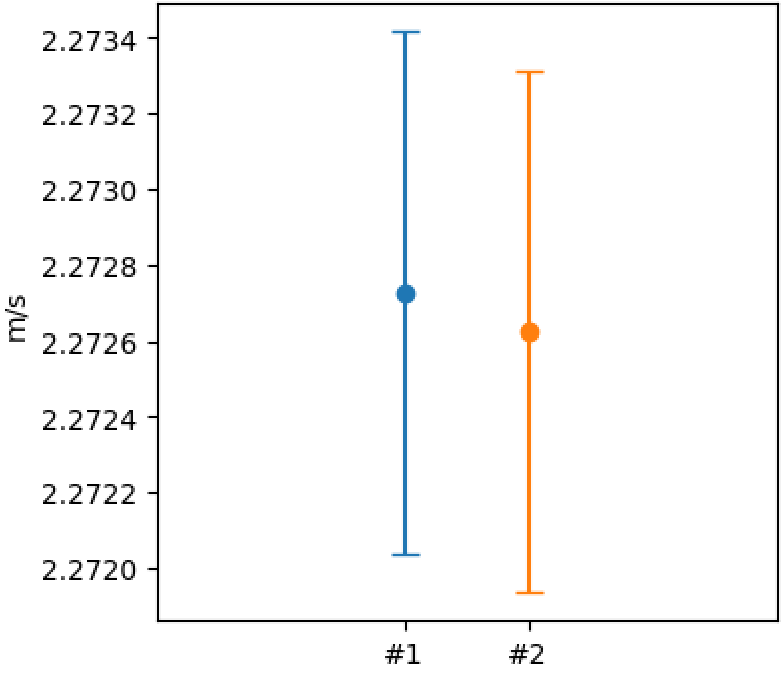
- 50 m long lanes
- ± 1.5 cm tolerance in length between lanes
- At 2.4 m/s → ± 0.006 s uncertainty in time

- 1972 games 400 m Individual Medley event:
  - Tim McKee (USA), 4:31.983
  - Gunnar Larsson (Sweden), 4:31.981

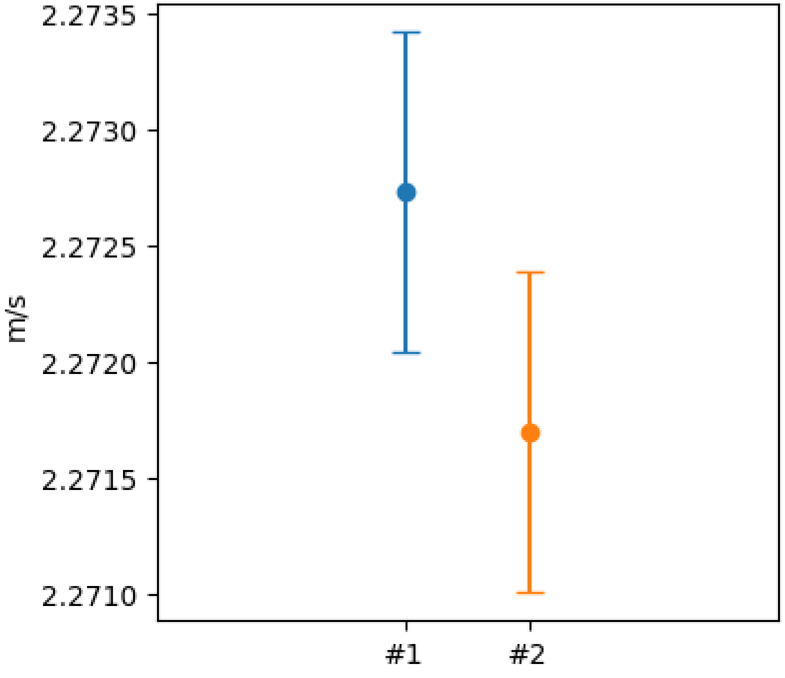
- Resulted in rule change to time swimming events to 0.01 s resolution

Why do we care about uncertainty?

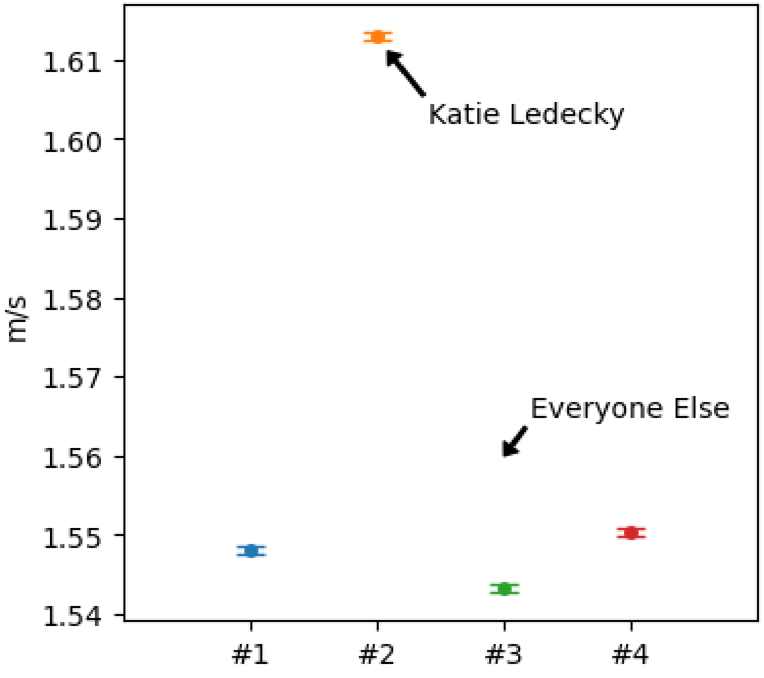
How much confidence do you have in the decision?



0.001 s minimum difference

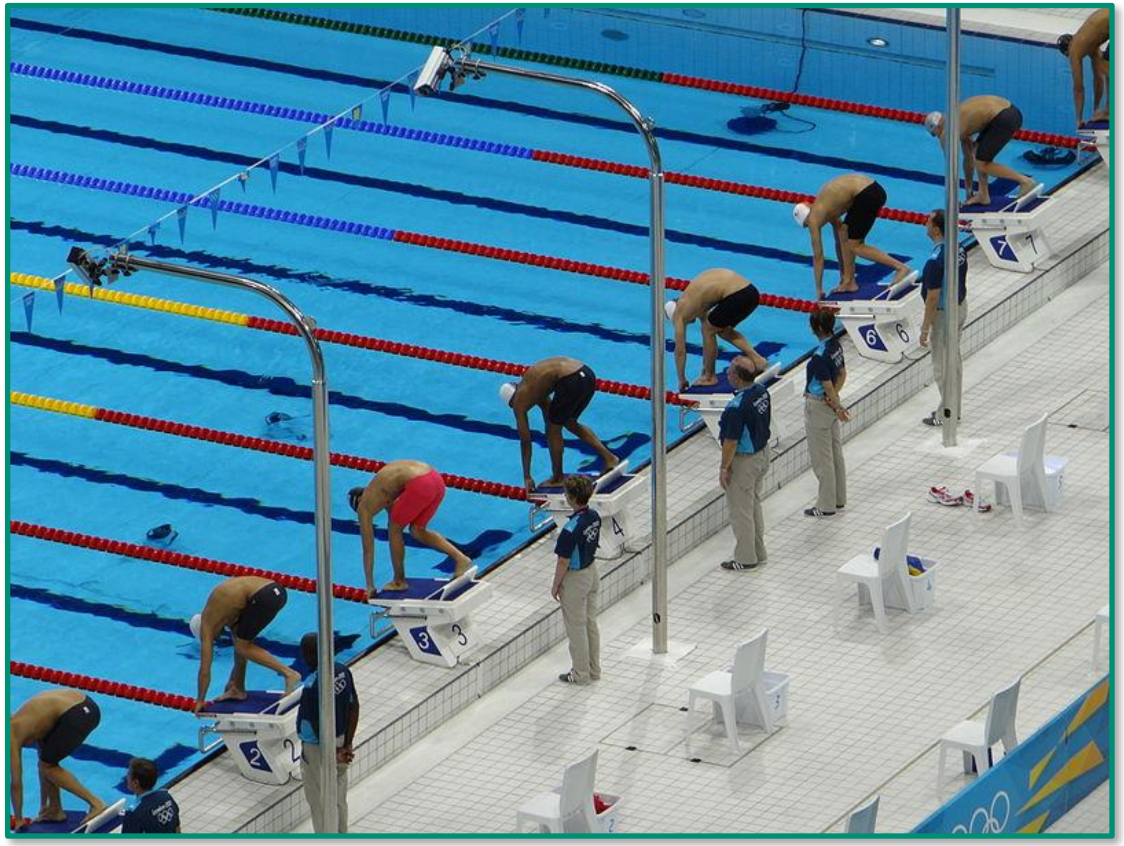


0.01 s minimum difference



2024 Games

Why do we care about uncertainty?



Gary Bembridge/CC-BY-2.0

Swimming facility regulations have tolerances on:

- Dimensions
- Temperature
- Salt content of water
- Water flow/turnover rate
- Illumination
- PA system loudness and frequency response
- Vibrational frequencies of diving boards

Consequences of Bad Measurement Uncertainty – Deflategate

Jeffrey Beall/CC-BY-SA 4.0

Exponent Engineering: The Effect of Various Environmental and Physical Factors on the Measured Internal Pressure of NFL Footballs, 2015.

Use a Measurement Assurance Plan to document measurement adequacy

Step 1 – List product requirements

Step 2 – Evaluate equipment and measurement uncertainty

Step 3 – Calculate the Test Uncertainty Ratio (TUR).

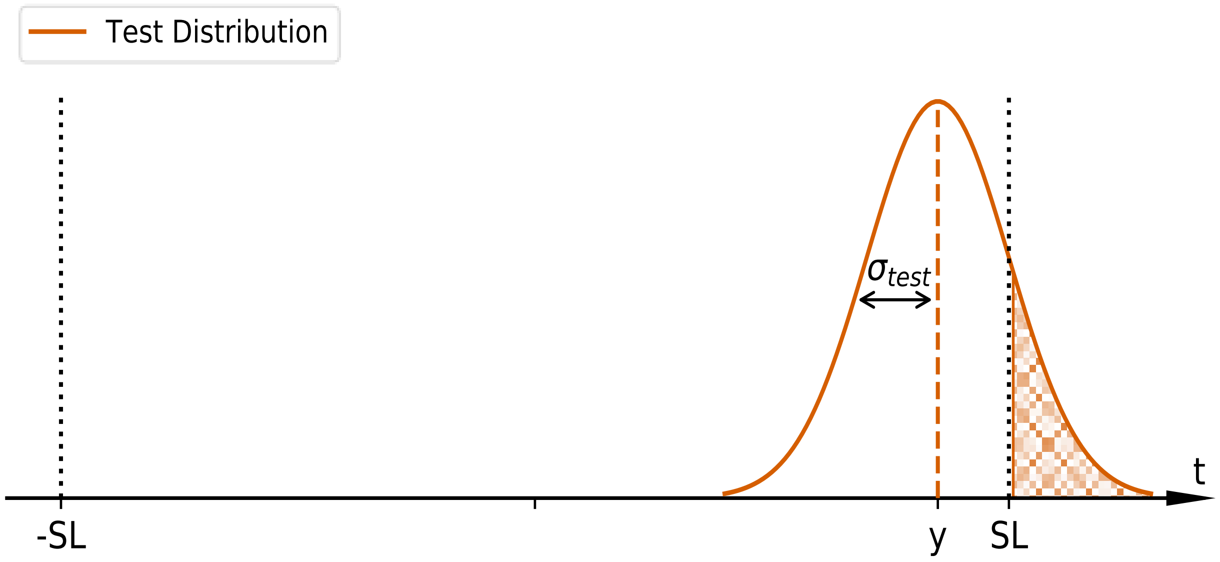
- TUR = (± Tolerance Limits)/(± **Measurement Uncertainty** @ 95%)

Q1. What quantity are you measuring?			Q2: How accurate do the measurements need to be?			Q3: How will you ensure your equipment can make this measurement?			
Quantity Measured	MC	Requirement Number	Value or Range of Values Measured	Tolerance Limits	Guardbanded Acceptance Limit	Equipment Used (M&TE)	Measurement Uncertainty	TUR (>4 desired)	Calibrate (Yes/No)
Football Air Pressure	*****	**Rule 2, Section 1**	**13 "pounds" (psig)**	**± 0.5 "pounds" (psig)**		**Wilson (?) pressure gauge**	**± 1 psig**	**0.5**	**Y**

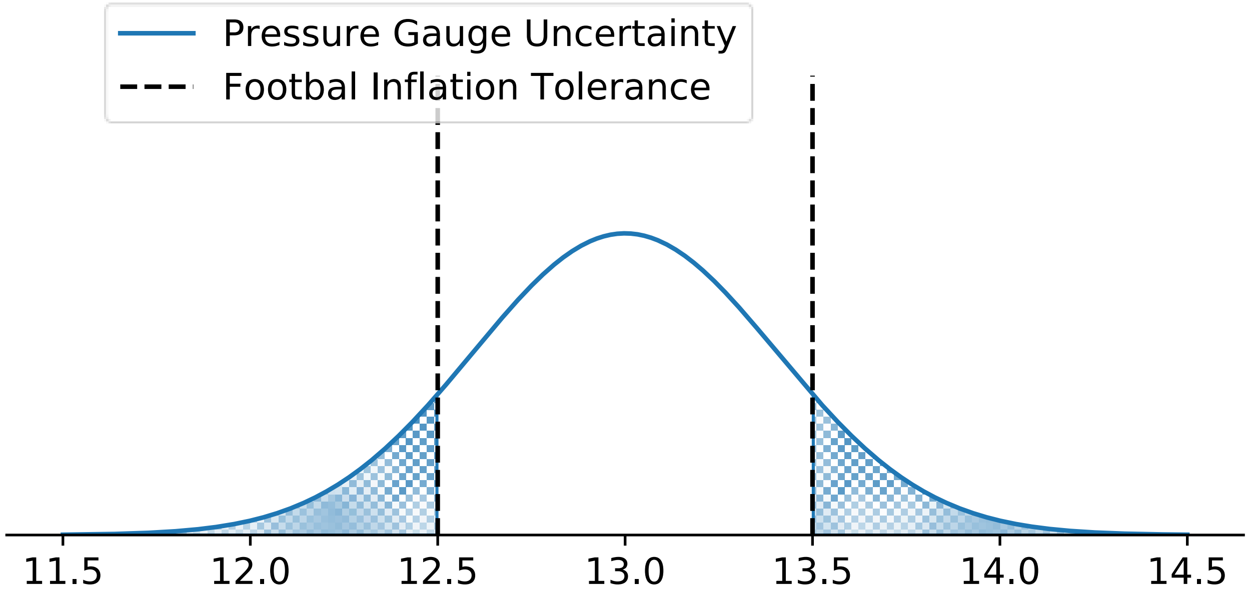
This class: How do you calculate this?

Measurements are used to make decisions

- All measurements have some uncertainty, therefore all measurement decisions have some probability of being incorrect
- An accurate determination of measurement uncertainty is essential for managing this risk
- Test Uncertainty Ratio is a "cheap" way of evaluating the risk



Deflategate Measurement – What happens when your TUR is too low



With a measured value of 13 psig, ~32% probability of a false accept!

Understanding measurement uncertainty is essential for understanding and mitigating risk of making incorrect measurement decisions!

Examples of Measurement Uncertainty in Decision Making

The quality of the measurement may affect the risks associated with decision making

- **Calibration**: Is the power supply operating within tolerance?
- **Manufacturing**: Does my neutron tube component meet specification?
- **Process Monitoring**: Has the metal deposition thickness reached its target value?
- **Surveillance**: Does the stockpile component still meet specification?
- **Research**: Should the grant proposal be funded based on initial measurement data?
- **Application**: Should the aircraft take off based on measured air speed?
- **Commerce**: How much should you pay for that tank of gasoline?
- **Safety**: Is there enough oxygen in this room to breathe?
- **Entertainment**: Can we use this football in the Super Bowl?

Section 2
Identifying Sources of Uncertainty

Section 2 – Identifying Sources of Uncertainty

Objective

- Upon completion of this section, participants will be able to list potential sources of uncertainty and rank sources using engineering judgment

Content

- Determining sources of uncertainty
- Methods for ranking sources of uncertainty

Sources of Uncertainty

- Calibration, reference standards
- Standard procedures
- Environment
- Operator
- Instrumentation
- Imperfect measurement model
- Other random and systematic variation

Identifying Sources of Uncertainty

- Experience
- Subject Matter Experts
- Manufacturer's recommendations
- Published information
- Observation of measurement process

Exercise

Divide into groups of 3 or 4.

- Brainstorm sources of uncertainty in establishing the accuracy of an automobile speedometer using mile markers and a wrist watch. Take 5 minutes.

- Using your best engineering judgment, choose the top 3 sources of uncertainty.

Ranking Sources of Uncertainty

Order of magnitude approximations

Historical data analysis

- Correlation studies
- Variance components

Design of Experiments (DOEx)

- HF101 DOEx class
- See Chapter 9, "Design of Experiments in Metrology," Crowder, et. al (2020).

Section 3
Statistical Basics

Section 3 – Statistical Basics

Objective

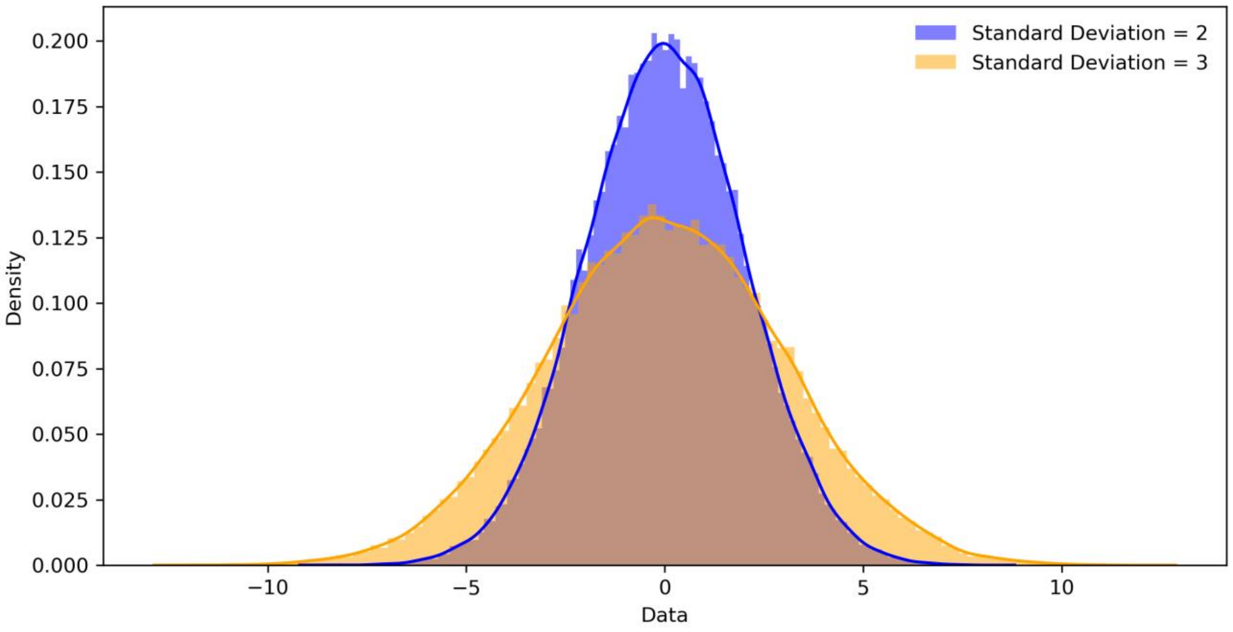
- Upon completion of this section, participants will be able to perform basic statistical analyses common to metrology

Content:

- Mean
- Standard Deviation
- Standard Deviation ("Standard Error") of the Mean
- Variance
- Probability Density Functions (normal, uniform, triangular, t-distribution)

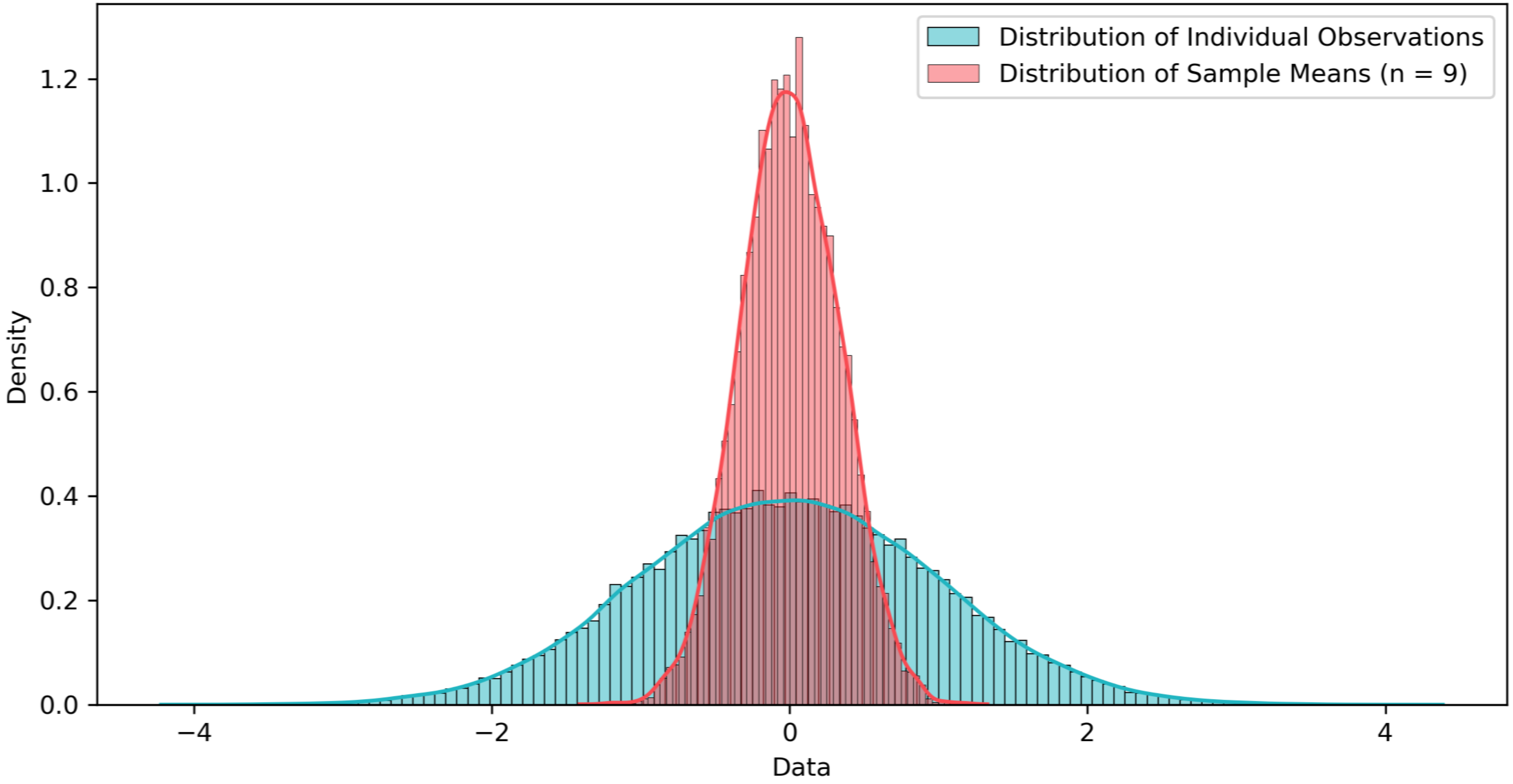
Working Definitions

- Probability density function – a function that characterizes the behavior of a random variable. The PDF is used to compute the probabilities of data falling into specified intervals
- Mean – A measure of the center of a probability distribution
- Standard deviation – A measure of the spread in a probability distribution
- Variance – The square of the standard deviation



Distribution of Sample Mean Illustrated

Standard deviation of the mean ("standard error") provides a measure of the spread in sample means about the true (but unknown) population mean based on n observations.



Standard Error of the Mean:

$$\frac{s}{\sqrt{n}}$$

Standard Error Example

Height (Inches)
65
68
71
64
66
68
73
70
68
67

n = 10
Mean = 68
Standard Deviation = 2.61

n = 100
Mean = 68
Standard Deviation = 2.61

What is the standard error of the mean in each case?
How does sample size affect the standard error of the mean?

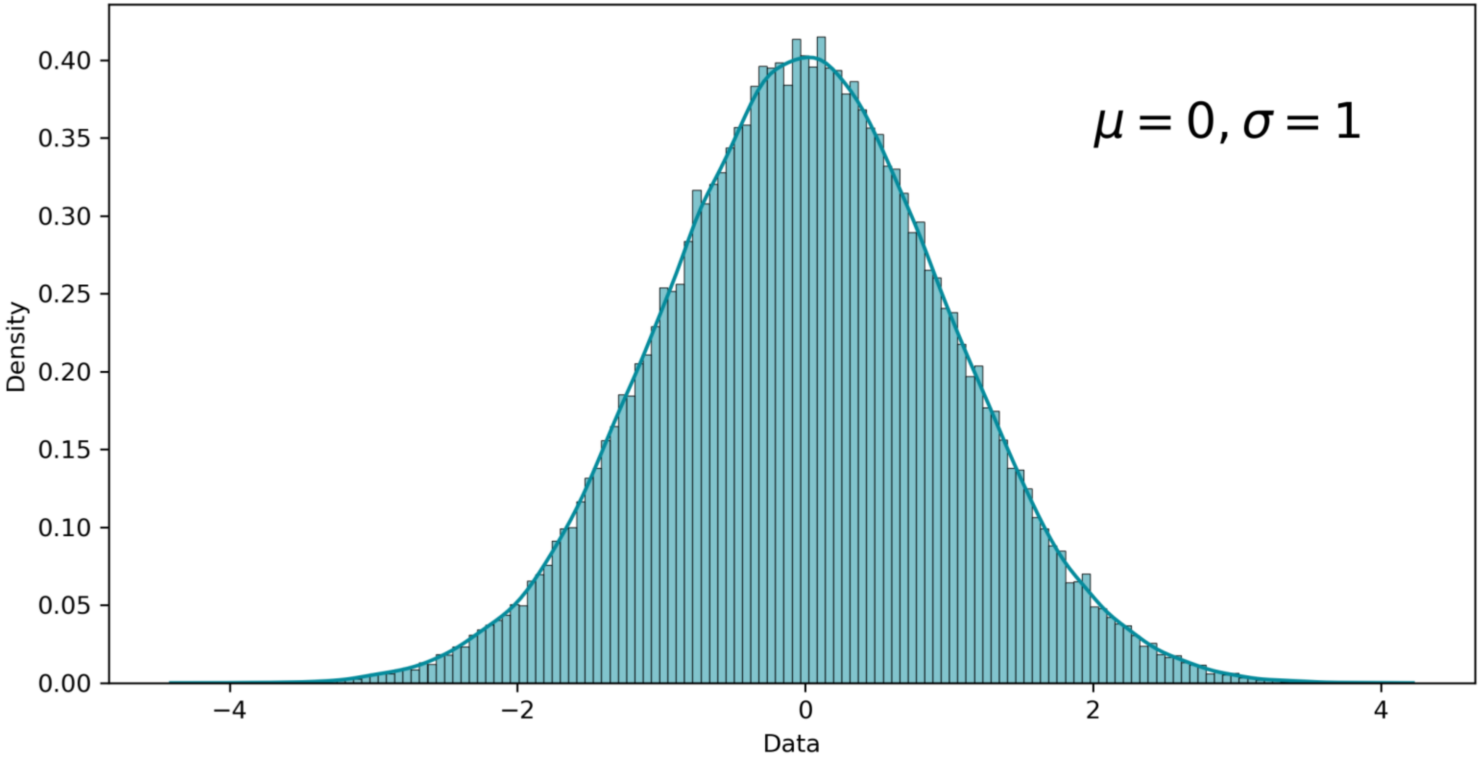
Normal Distribution

Most commonly used probability density function (PDF) in metrology.

$$PDF = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$-\infty < x < \infty$$

$$\text{Mean} = \mu, \text{Standard Deviation} = \sigma$$

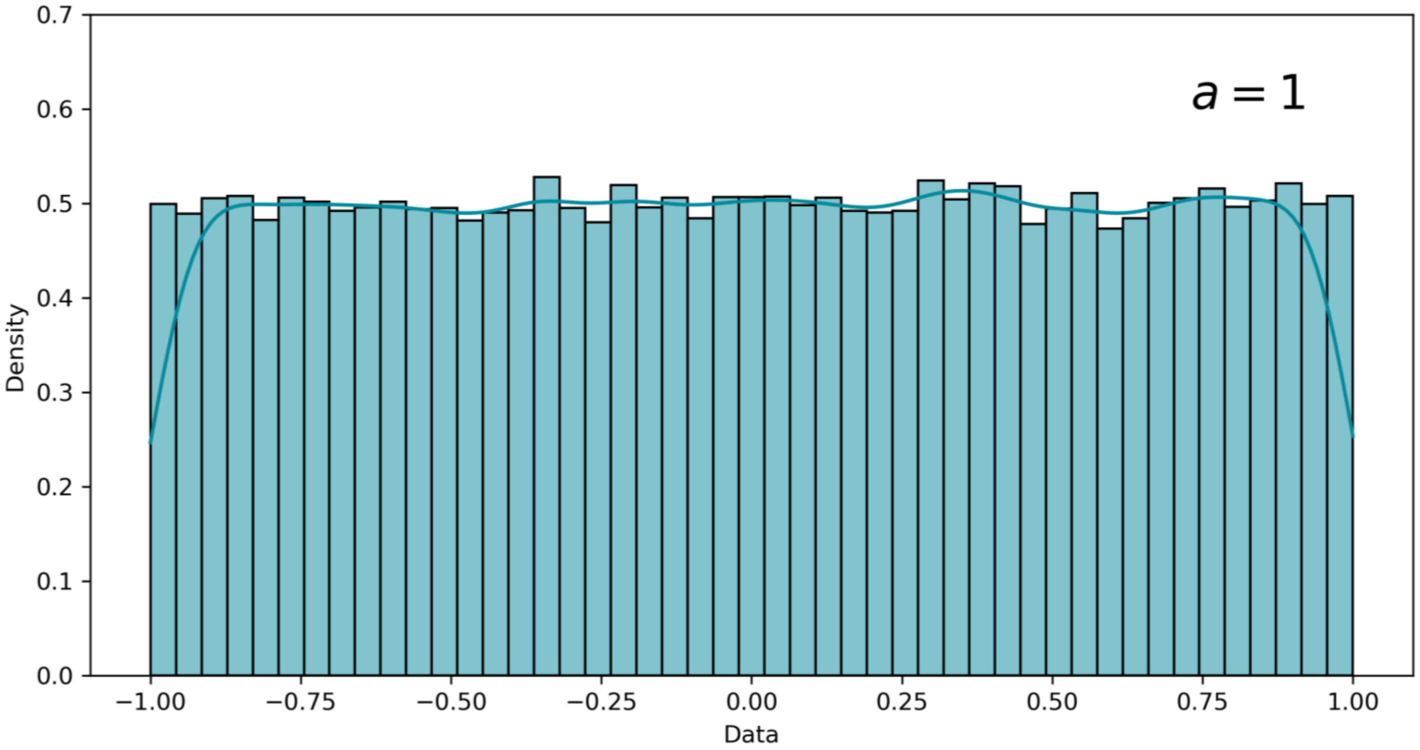


Uniform (Rectangular) Distribution

Used when you have no specific knowledge about the possible values within the interval, i.e., it is equally probable for the value to lie anywhere within the interval.
Requires known lower and upper bounds [-a, a].

$$PDF = \frac{1}{2a}, \qquad where \; -a \le x \le a$$
$$0, \qquad elsewhere$$

$$\text{Mean} = 0, \text{Standard Deviation} = \frac{a}{\sqrt{3}}$$



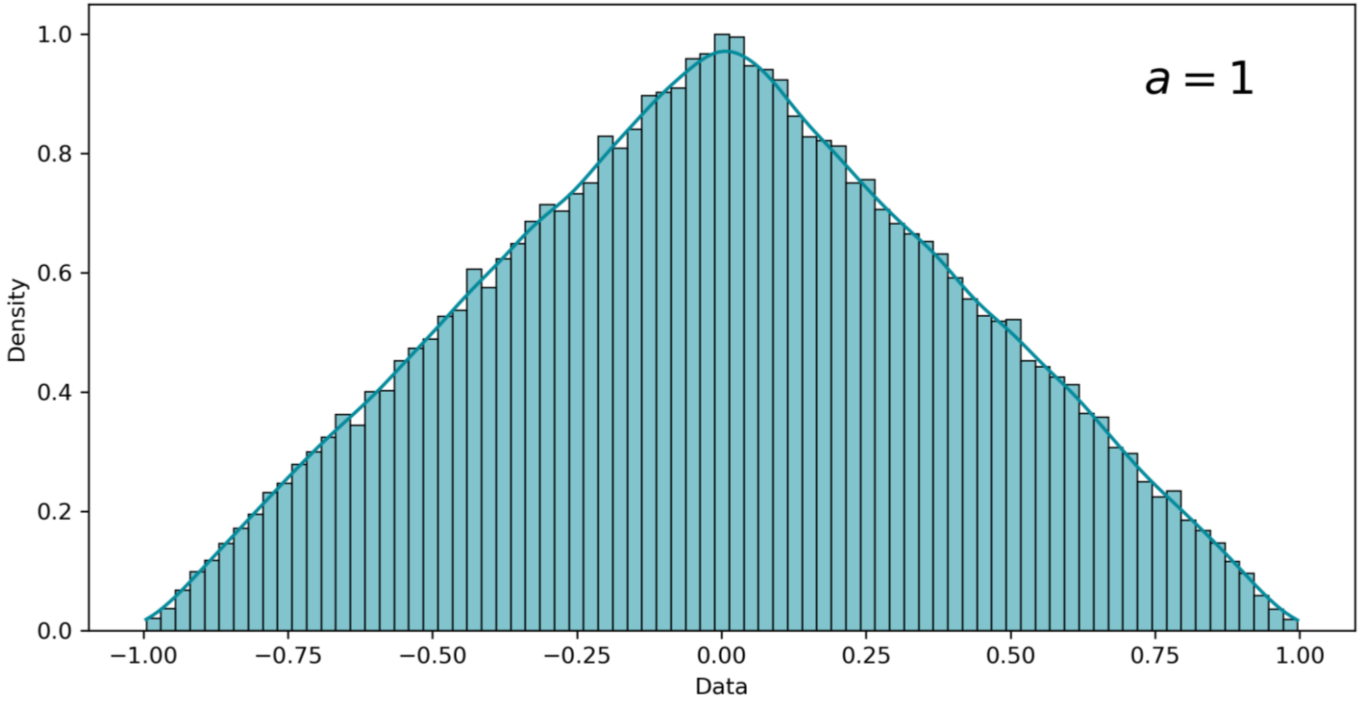
Triangular Distribution

Symmetric distribution used when the values are more likely to fall in the middle of the interval. Requires known lower and upper bounds [-a, a].

$$PDF = \frac{(x+a)}{a^2}, \qquad where \; -a \le x \le 0$$

$$PDF = \frac{(a-x)}{a^2}, \qquad where \quad 0 < x \le a$$

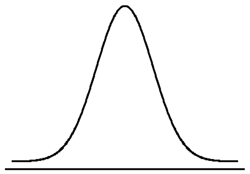
$$\text{Mean} = 0, \text{Standard Deviation} = \frac{a}{\sqrt{6}}$$



Comparison of Distributions

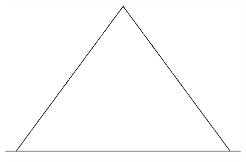
Suppose an uncertainty distribution is defined on the interval [-a, a]= [-1, 1].

- Normal distribution



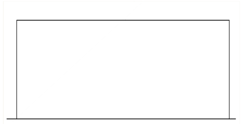
$$\sigma \cong \frac{a}{3} = \frac{1.0}{3} = 0.33$$

- Triangular distribution



$$\sigma \cong \frac{a}{\sqrt{6}} = \frac{1.0}{\sqrt{6}} = 0.41$$

- Uniform distribution



$$\sigma \cong \frac{a}{\sqrt{3}} = \frac{1.0}{\sqrt{3}} = 0.58$$

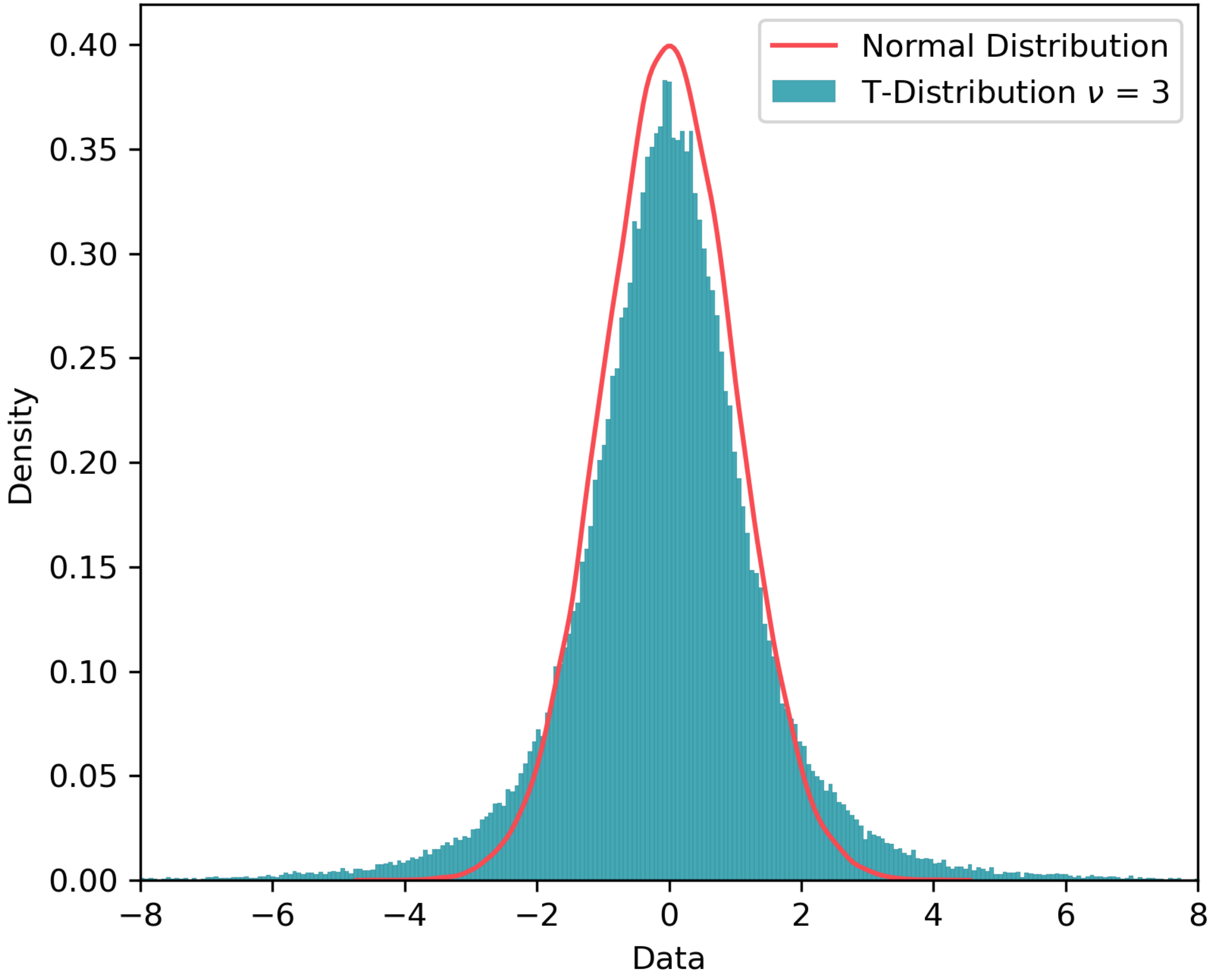
t-Distribution

Used to compute expanded uncertainties.

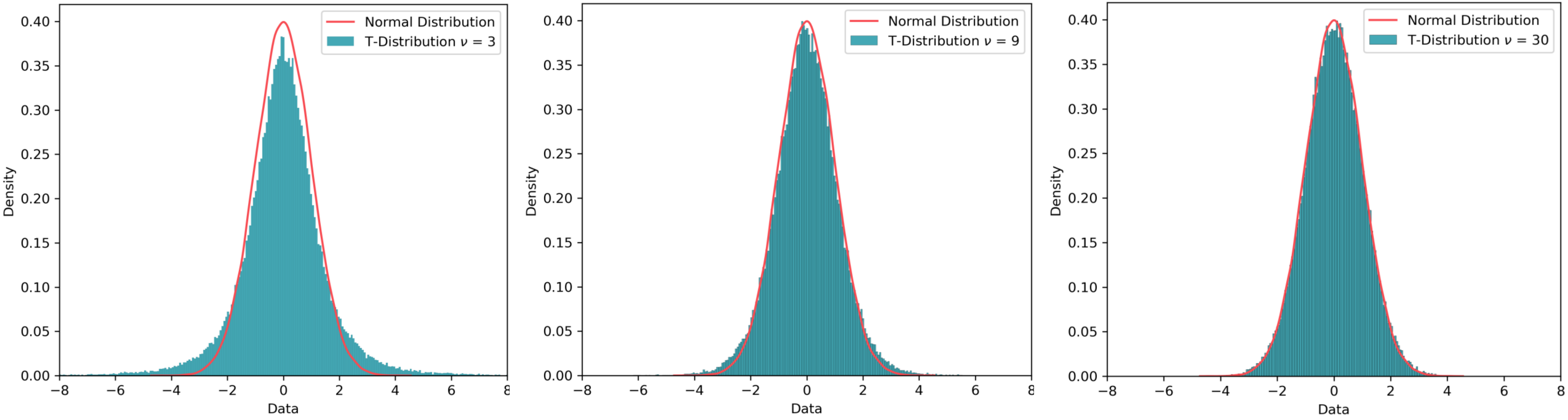
$$PDF = \frac{\Gamma[(\nu+1)/2]}{\Gamma(\nu/2)} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1+\frac{x^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$

$$-\infty < x < \infty$$

$$\text{Mean} = 0, \text{Standard Deviation} = \left(\frac{\nu}{\nu-2}\right)^{\frac{1}{2}}$$



t-Distribution with different degrees of freedom



Distribution Exercise

See SNL Uncertainty Calculator Distribution Tool, Appendix A

Section 4
Direct Measurements

Objective

- Participants will be able to use the basic measurement model and estimate uncertainty associated with direct measurements.

Content

- Basic measurement model
- Estimating uncertainty in direct measurements

Direct Measurement

The measuring instrument is used to measure the object or characteristic of interest, the **measurand**.

The value of the measurand is read directly from the instrument.

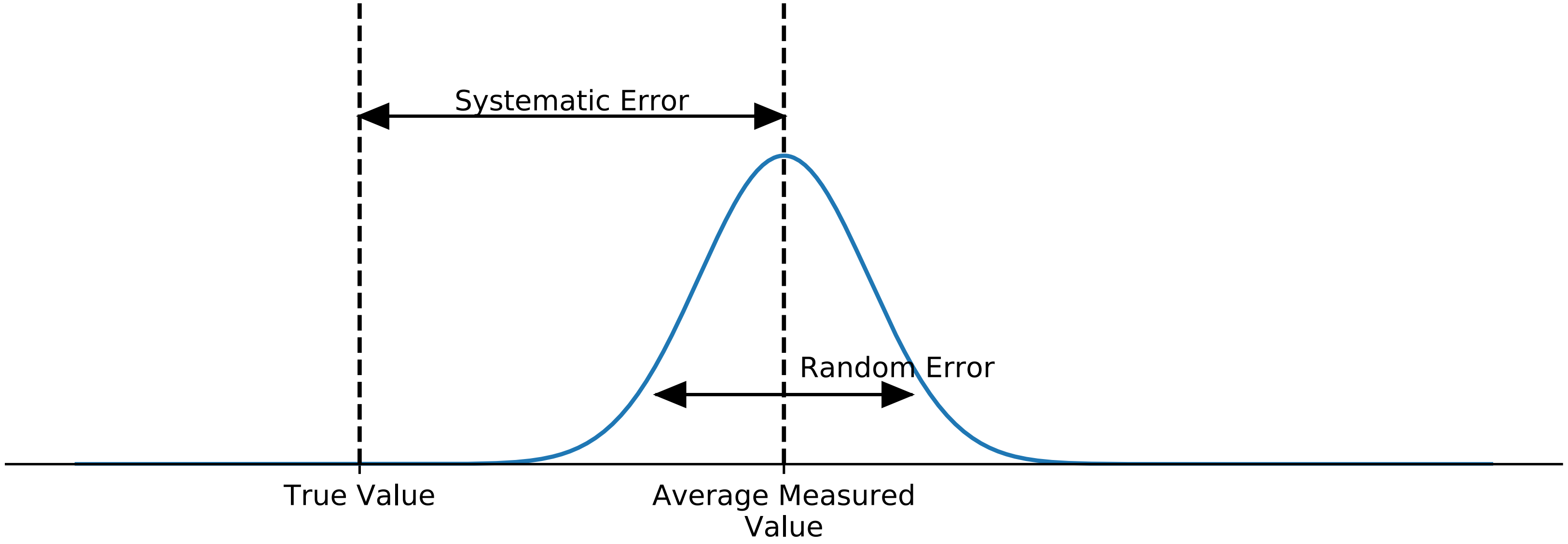
The “Error Approach”

A single measurement of the measurand, y_i, can be expressed as

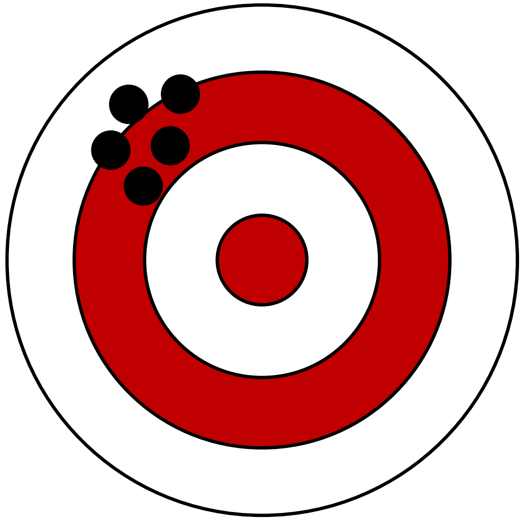
$$y_i = True\ Value + (Measurement\ Error)_i$$

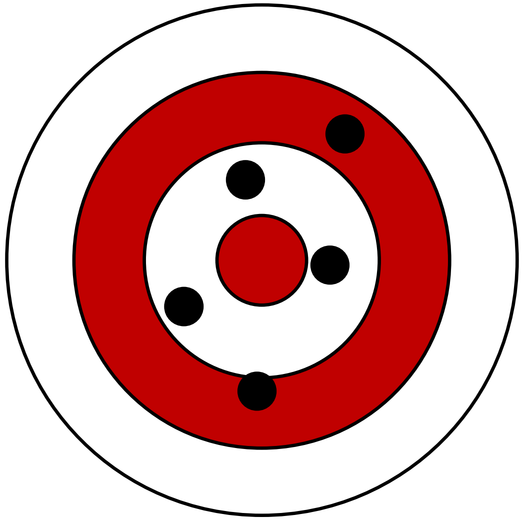
The **error** in one measurement has two components: systematic error and random error:

$$y_i = True\ Value + \varepsilon_{sys} + \varepsilon_{ran,i}$$



Precision vs. Accuracy





Precise but not accurate

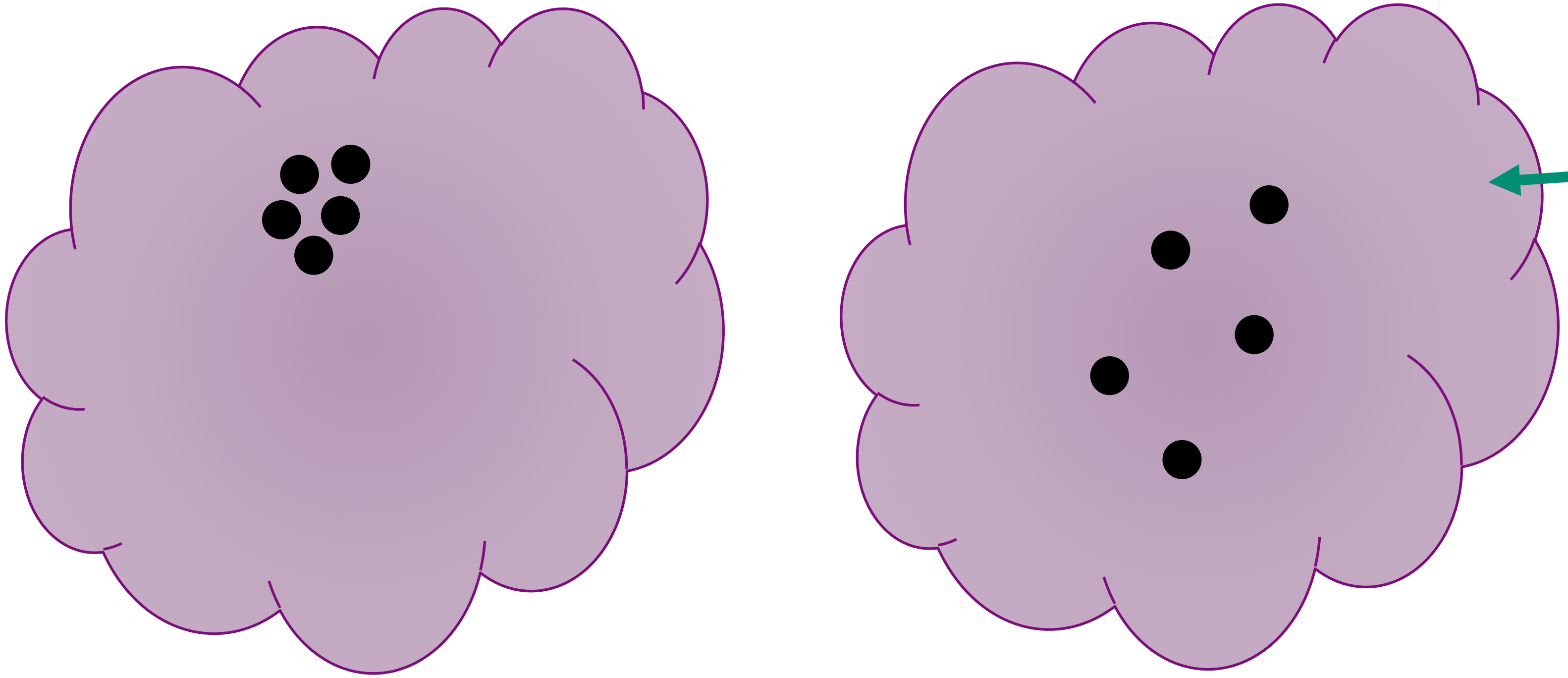
Small **random errors** but
large **systematic error (bias)**

Less precise but more accurate

Larger **random errors** but
smaller **systematic error (bias)**

Precision and Accuracy are useful in a **qualitative** sense.

Reality – We can't know the "True Value" center of the target!



What we know:
- These 5 measured values
- Equipment specs
- Environmental conditions
- Etc. etc.

Uncertainty describes the region where we expect the true value to be based on all available information about a measurement process.

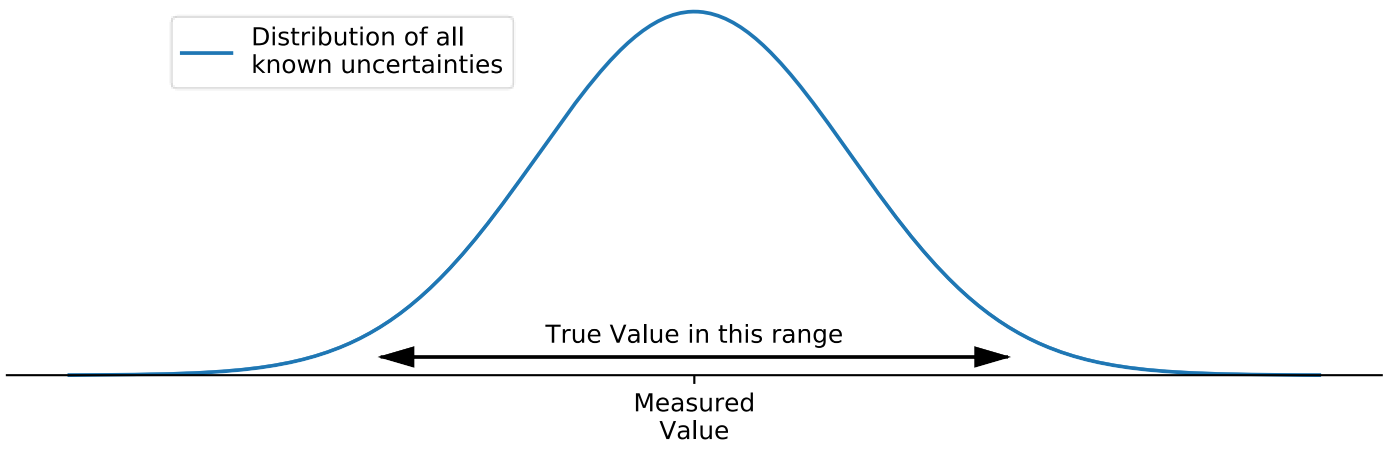
The True Value is unknowable, but we have confidence it is somewhere in that region. Adding information can reduce the size of the cloud.

The "Uncertainty Approach"

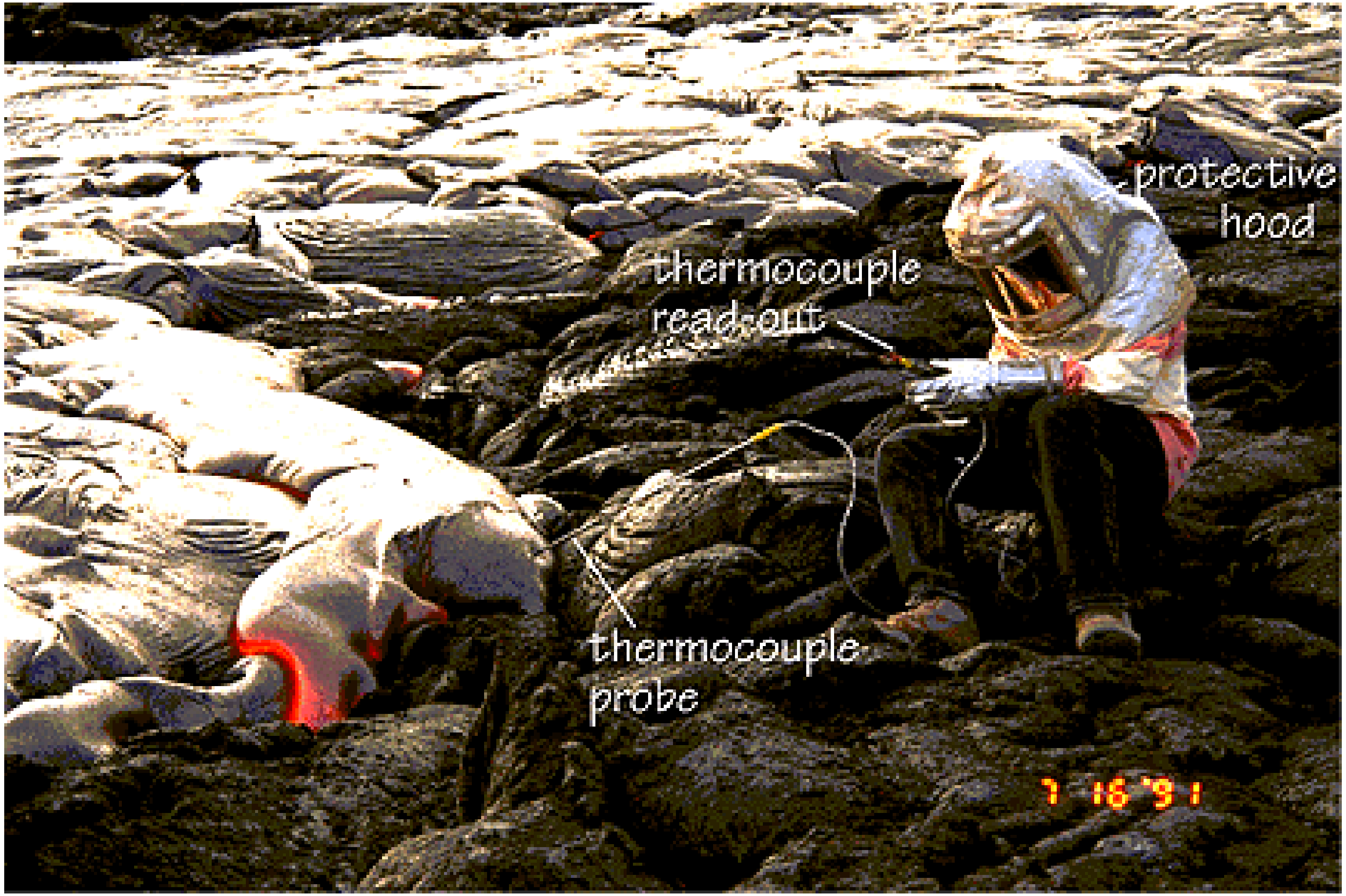
We cannot know the true value, and therefore cannot quantify systematic and random errors. But we can estimate bounds around the measured value where the actual value is believed to lie:

- *Measured Value ± Measurement Uncertainty*

This "Uncertainty Approach" was put forth by the International Committee on Weights and Measures (CIPM) in 1980 to address the problem with the "Error Approach".



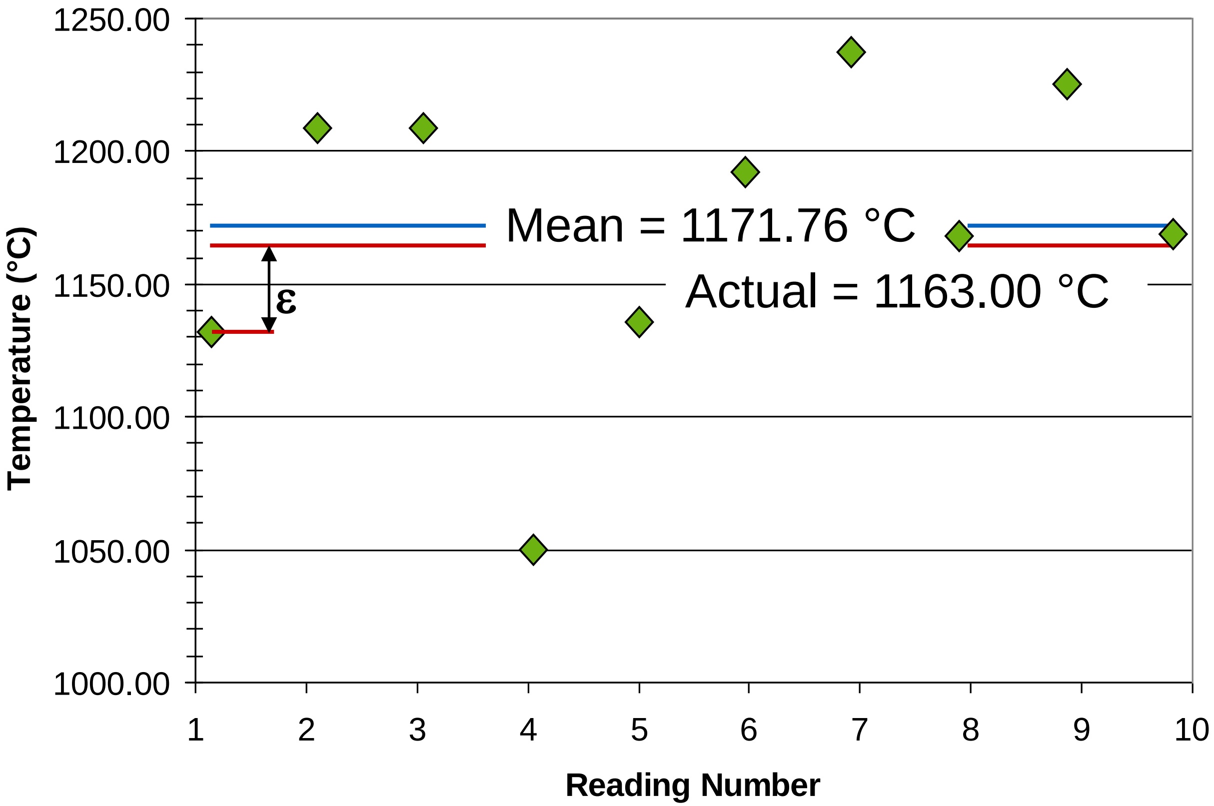
Example: Direct Measurement of Lava Temperature



P. Mouginis-Mark, University of Hawaii at Manoa

Direct Measurement of Lava Temperature

#	Reading
1	1130.60 °C
2	1210.64 °C
3	1211.25 °C
4	1047.01 °C
5	1135.67 °C
6	1189.19 °C
7	1237.59 °C
8	1164.33 °C
9	1225.51 °C
10	1165.79 °C
$\bar{y}$	1171.76 °C



Can we place bounds around the mean value where we can expect the actual value to lie with some level of confidence?

Measurement Uncertainty for Direct Measurements

In the case of multiple measurements, we can express uncertainty as:

$$\bar{y} \pm Measurement\ Uncertainty$$

$$\bar{y} \pm k \cdot (Uncertainty\ Standard\ Deviation)$$

$$\bar{y} \pm t_p(\nu_{eff})\sqrt{u_A^2 + u_B^2}$$

The remainder of this section will provide definitions for, and show how to calculate the terms ν_{eff}, $t_p(\nu_{eff})$, u_A, and u_B.

Definition: Type A and Type B Uncertainty Evaluation

Type A evaluation: method of evaluation of uncertainty by the statistical analysis of series of observations

- Repeated observations.
- Repeatability and Reproducibility (Gauge R & R)
- Precision and accuracy studies (DOEx)
- Historical data analysis, drift
- Data from control charts

Type B evaluation: method of evaluation of uncertainty by means other than the statistical analysis of a series of observations

- Calibration reports
- Correction factors
- Manufacturer's specifications
- Reference data from handbooks

GUM (JGCM 100:2008) section 2.3.2-3 and VIM (JGCM 200:2012) section 2.28-29

Categorization as a Type A or Type B Uncertainty Evaluation

"Type A" and "Type B" are **methods** of determining an uncertainty contribution using available information. They are not related to the physical cause of an error.

These are NOT the same as "random" and "systematic" errors!

There are no "Type A Errors" or "Type B Errors"

Often, the term "Type A Uncertainty" is used to mean "standard uncertainty obtained from a Type A evaluation". Same for "Type B Uncertainty".

Definition: Standard Uncertainty

standard uncertainty

uncertainty of the result of a measurement expressed as a standard deviation

GUM (JGCM 100:2008) section 2.3.1 and VIM (JGCM 200:2012) section 2.30

Type A Evaluation of Standard Uncertainty (Normal Distribution)

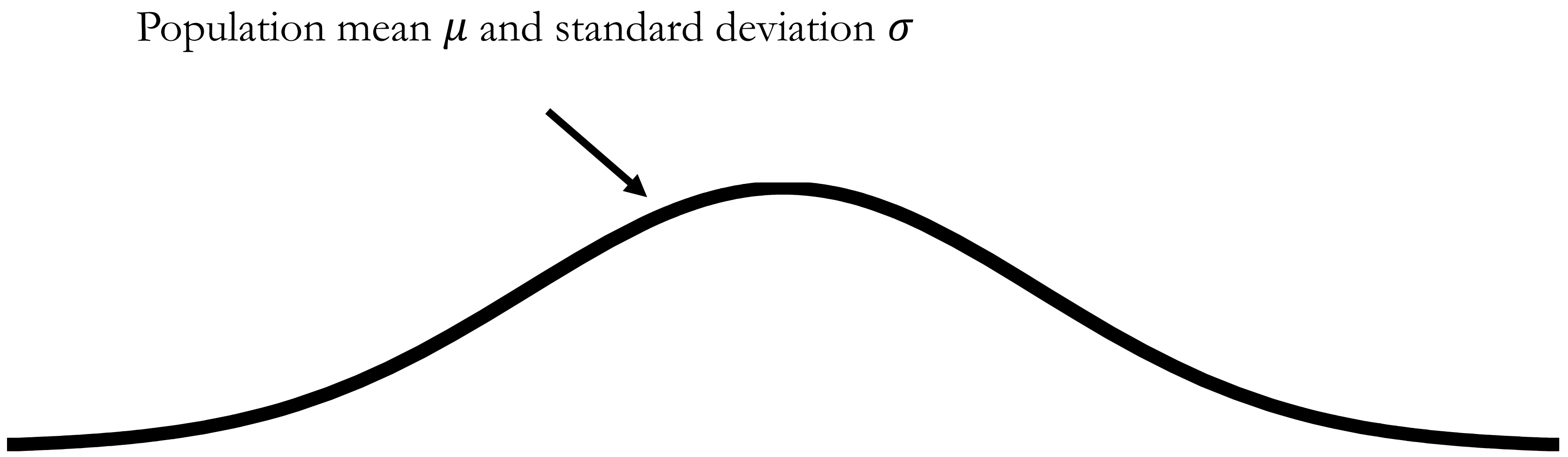
The experimental standard deviation of n measurements of y is:

$$S_Y = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(y_k - \bar{y})^2}$$

The standard uncertainty $u_A(y)$ of the estimated mean, $\bar{y}$, is the standard error of the mean:

$$u_A(y) = S_Y/\sqrt{n}$$

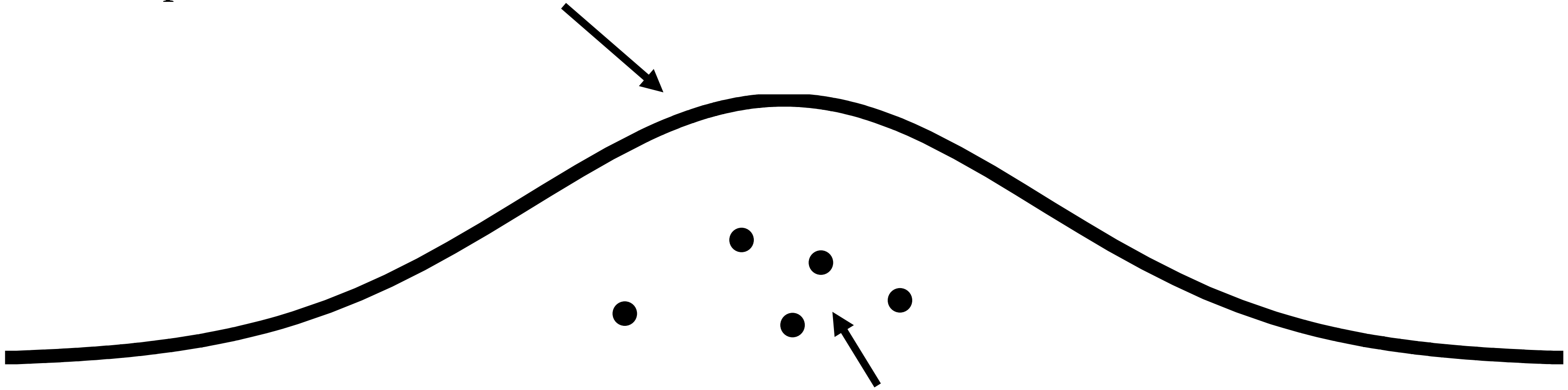
Normal Random Errors



Suppose we could make an infinite number of measurements of some quantity (with random errors that are normally distributed). The resulting distribution of our infinite number of measurements has a mean μ and a standard deviation σ.

Normal Random Errors

Population mean μ and standard deviation σ



Estimate of μ: $\bar{y} = \sum_{i=1}^{5} y_i / 5$

Estimate of σ: $s(y) = \sqrt{\sum_{i=1}^{5} (y_i - \bar{y})^2 / 4}$

Standard Uncertainty: $s(\bar{y}) = s(y)/\sqrt{5}$

In reality we cannot take an infinite number of measurements so we will never exactly know μ or σ. But if we take, say, five *independent* measurements we can estimate μ and σ . We can also use $s(y)$ to get an estimate of the Type A uncertainty in our estimate of μ.

Mini-Exercise: Type-A Evaluation

For our lava temperature measurement example:

- Compute the Type-A standard uncertainty using the lava data
- Hint: Assume the data fit a normal distribution.

Reading
1130.60
1210.64
1211.25
1047.01
1135.67
1189.19
1237.59
1164.33
1225.51
1165.79

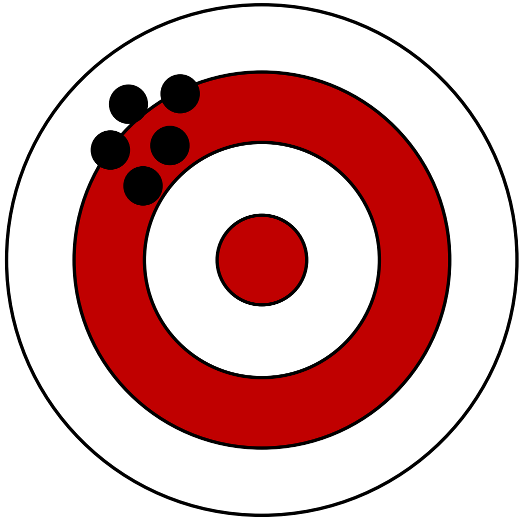
- Mean = 1171.76 °C
- Standard Deviation = 57.01 °C
- Standard Error of the mean = $57.01/\sqrt{10} = 18.03$ °C

Infinite Number of Measurements

$$u_A(y) = s_A(\bar{y}) = s_A(y)/\sqrt{n}$$

If I take billions of measurements, I'll have no uncertainty, right?

WRONG. Type A uncertainty may approach zero, but there's always other Type B uncertainties! Also, the n measurements must be statistically **independent**.



Type B Evaluation of Standard Uncertainty

Type B Evaluations are based on scientific judgment using all available information about the variation in the estimate of the measurand, y

1. Identify the information about the uncertainty
2. Assign a probability distribution depending on what information is available. Common distributions:
   - Normal
   - Triangular
   - Uniform
3. Use the distribution to convert to a **standard uncertainty**

See Table 1 in JCGM101:2008 (GUM-S1) for other fun distributions.

Type B Evaluation of Standard Uncertainty

- Given: uncertainty interval stated as $\pm\, \boldsymbol{U}$ **with coverage factor** (such as k = 2)

  Use a Normal Distribution**:**

  the standard uncertainty is $u_B(y) = U/k$

- Given: uncertainty interval stated as $\pm\, \boldsymbol{U}$ **with level of confidence** (such as 95%)

  Use a Normal Distribution:

  the standard uncertainty is $u_B(y) = U/k$ where $k \;= \Phi^{-1}((1 + conf/2)\,)$

- Example: A quoted uncertainty interval (e.g., from a calibration report) is ±100 V (95% coverage, k = 2), then:

$$u_B(y) = \frac{100V}{2} = 50V$$

Type B Evaluation of Standard Uncertainty

- Given: a range of possible values stated as the **interval [-a, a] (± a)** :

  Use a Uniform (Rectangular) Distribution.

  the standard uncertainty is $u_B(y) = \frac{a}{\sqrt{3}}$

- Given: a range of possible values stated as the **interval [-a, 0, a],** and it is known that it is more likely that the true value is closer to zero than to the endpoints:

  Use a Triangular Distribution.

  the standard uncertainty is $u_B(y) = \frac{a}{\sqrt{6}}$

Mini-Exercise: Type-B Evaluation

For our lava temperature measurement example:

Given the certificate on the next page, calculate the Type B standard uncertainty associated with the calibration of the measurement instrument.

Lava Cal LABORATORY

Report

TEMPERATURE READOUT — File No. 0747

Lava Inc.

Model No. 50B — *LIMITED*

Serial No. 1230744

with THERMOCOUPLE:

Lava Inc.

Model No. Type N

Serial No. 3456

Procedure No. CP – TC (07/22/98)

Lab conditions: Temperature: 23 °C ± 2 °C Humidity: 40% ± 10%

Submitted by: Organization 01234 SNL / NM

Report Date: November 27, 2009

The instrument / thermocouple combination was tested by comparison with a Standard Platinum Resistance Thermometer (SPRT). The (SPRT) is traceable to the National Institute of Standards and Technology (NIST). The type T temperature calibration was performed on the International Temperature Scale of 1990 (ITS-90), using the standard ITS-90 reference functions for thermocouples as defined in NIST Monograph 175. The instrument / thermocouple combination was found to be within the following tolerance:

TC TYPE	RANGE	Tolerance
N	1000°C to 1300 °C	± 75 °C

Limitation: This report is valid only for the instrument / thermocouple combination tested and over the range listed above.

Metrologist: J. Lava, 02542

Approved by: I.M. Outtahere, 02542
Maximo Data Entry Specialist

Copy to: Submitting organization
02542 File

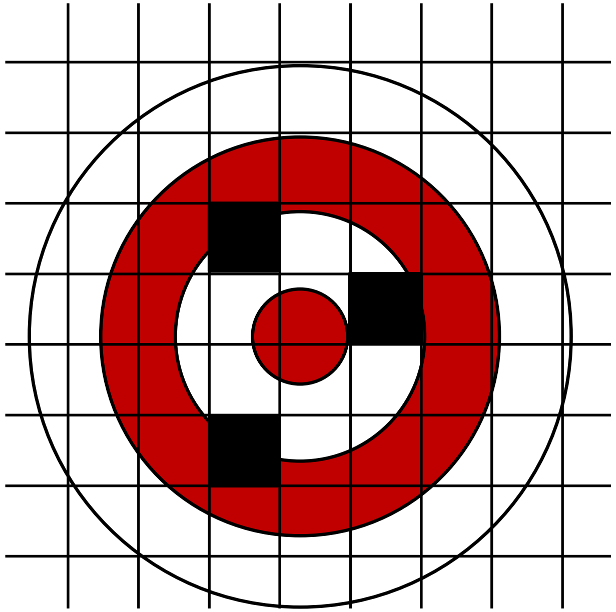
Date received: 11/19/09
Date(s) tested: 11/21-27/09

- Tolerance = 75
- Uniform Distribution
- Standard Uncertainty: $\frac{75}{\sqrt{3}} =$ 43.3 °C

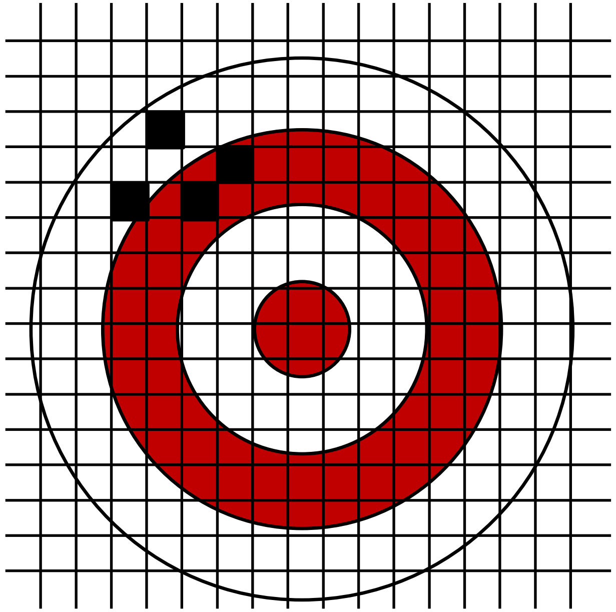
Equipment Resolution

Resolution: Smallest change that can be distinguished by a measurement instrument.

Resolution may contribute to uncertainty, but is not the same thing.



Low resolution



High resolution

Equipment Resolution

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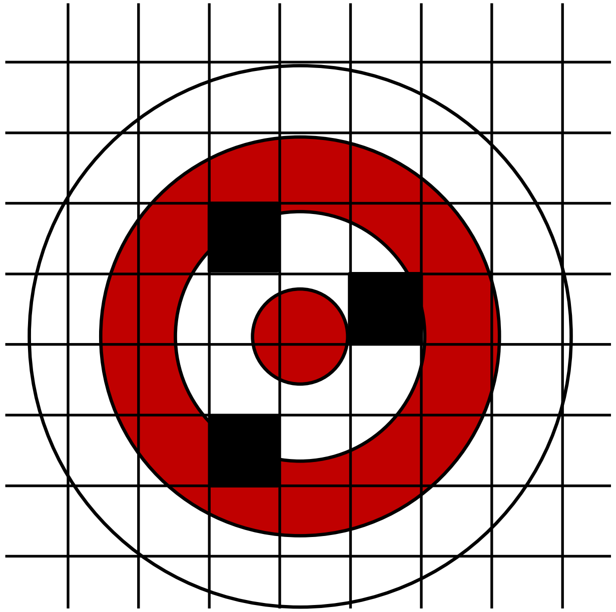

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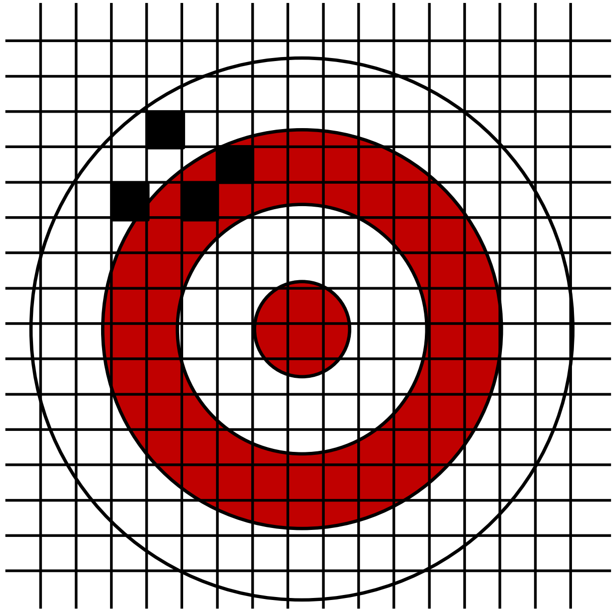
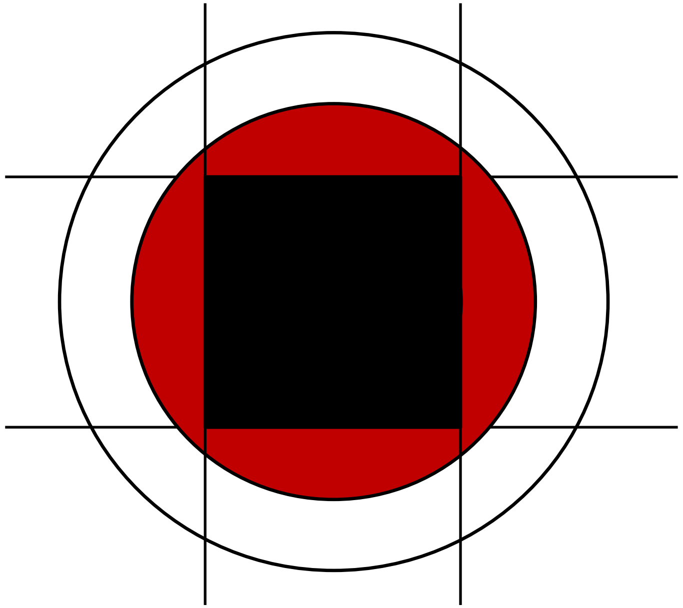


High resolution



Terrible resolution

Mini-Exercise: Type-B Evaluation

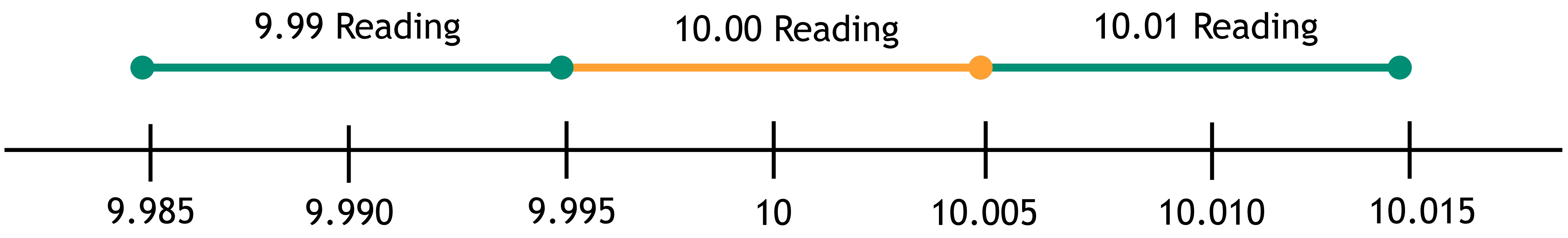
Part 2, for our lava temperature measurement example:

- Compute the Type B standard uncertainty resulting from the limited resolution of the readout (0.01 °C).

Resolution Uncertainty

Equipment resolution is a uniform distribution with $a = r/2$

Resolution, least-significant digit of 0.01:



Combined Standard Uncertainty

Combined Standard Uncertainty – uncertainty obtained by combining the Type A and Type B standard uncertainties

- Computed by taking the positive square root of the sum of the squares (RSS) of the standard uncertainties. Denoted $u_c(y)$:

$$u_c(y) = \sqrt{u_A^2(y) + u_B^2(y)}$$

- In the case of multiple Type A evaluations or multiple Type B evaluations, these terms are also combined as the positive square root of the sum of the squares of the standard uncertainties:

$$u_c(y) = \sqrt{\sum_{i=1}^{N_A} u_{A_i}^2(y) + \sum_{i=1}^{N_B} u_{B_i}^2(y)}$$

Mini Exercise: Lava Temperature

Compute the combined standard uncertainty of the lava measurement, including:

- Type A uncertainty from the 10 repeated measurements (18.03 °C)
- Type B uncertainty from the calibration certificate (43.3 °C)
- Type B uncertainty from the resolution of the readout (0.0029 °C)

Hint: Ignore one of the Type B terms that is insignificant

$$u_c(y) = \sqrt{u_A^2(y) + u_B^2(y)}$$

Combined standard uncertainty

$$u_c(y) = \sqrt{u_A^2 + u_B^2} = \sqrt{18.03^2 + 43.3^2} = 46.90 \ ^\circ\mathrm{C}$$

Definition: Expanded Uncertainty

expanded uncertainty

quantity defining an interval about the result of a measurement that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand

Denoted by U.

GUM (JGCM 100:2008) section 2.3.5 and VIM (JGCM 200:2012) section 2.35

Estimating Expanded Uncertainty

The expanded uncertainty is calculated from the combined standard uncertainty using:

$$U = k \cdot u_c(y)$$

Need to determine:

- Coverage factor k that provides the desired level of confidence

Definition: Coverage Factor

coverage factor

numerical factor used as a multiplier of the combined standard uncertainty in order to obtain an expanded uncertainty

- The coverage factor is often computed from the t-distribution using the effective degrees of freedom ν_{eff} and confidence level p

$$k = t_p(\nu_{eff})$$

- The coverage factor is typically in the range 2 to 3

GUM (JGCM 100:2008) section 2.3.6 and VIM (JGCM 200:2012) section 2.38

Definition: Level of Confidence

level of confidence

probability that the **true value** of a **measurand** is contained within the interval defined by the **expanded uncertainty**

Denoted by p. Frequently, $p = 95\%$.

The above definition paraphrases GUM (JGCM 100:2008) section 6.2.2 and the VIM (JGCM 200:2012) section 2.37. The VIM prefers the phrase "coverage probability" to "level of confidence".

Definition: Degrees of Freedom

Degrees of Freedom

the number of independent pieces of information that go into computing an estimate of the standard deviation

- When estimating a Type A standard uncertainty with n data points, the degrees of freedom is $n - 1$
- Denoted by the symbol ν

Effective Degrees of Freedom

Effective degrees of freedom is computed using the Welch-Satterthwaite approximation when estimating the combined standard uncertainty:

$$\nu_{eff} = \frac{u_c^4(y)}{\frac{u_A^4(y)}{\nu_A} + \frac{u_B^4(y)}{\nu_B}}$$

Effective Degrees of Freedom

Welch-Satterthwaite formula extends to multiple Type A and Type B terms:

$$\nu_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{N_A} \frac{u_{A_i}^4(y)}{\nu_{A_i}} + \sum_{i=1}^{N_B} \frac{u_{B_i}^4(y)}{\nu_{B_i}}}$$

How to determine Degrees of Freedom for Type B uncertainties?

- **Option 1**: Use the value given or calculable from the certificate. Not often available.

- **Option 2**: Let $\nu_B = \infty$. This is a common, but probably not good, assumption that likely overestimates ν_{eff}.

- **Option 3**: Approximate using GUM G.4.2: $\nu_B \cong \frac{1}{2}\left[\frac{\Delta u_B(y)}{u_B(y)}\right]^{-2}$

where the quantity in brackets is the "relative type B uncertainty", a subjective quantity based on estimated confidence in the type B specification.

Example: You believe the manufacturer's spec is good to within 10%, then

$$\nu_B \cong \frac{1}{2}[0.10]^{-2} = 50$$

Mini-Exercise: Effective Degrees of Freedom

Given the values:

- $u_A(y) = 0.57$
- $u_B(y) = 0.25$
- $\nu_A = 9$
- $\nu_B = \infty$

Compute:

$u_c(y)$, the combined standard uncertainty, and

ν_{eff}, the effective degrees of freedom

$$u_c(y) = \sqrt{u_A^2(y) + u_B^2(y)} = \sqrt{0.57^2 + 0.25^2} = 0.622$$

$$\nu_{eff} = \frac{u_c^4(y)}{\frac{u_A^4(y)}{\nu_A} + \frac{u_B^4(y)}{\nu_B}} = \frac{0.622^4}{\frac{0.57^4}{9} + \frac{0.25^4}{\infty}} = 12.8 \rightarrow 12 \text{ (round down)}$$

Degrees of Freedom v	Fraction p in Percent					
	68.27	90	95	95.45	99	99.73
1	1.840	6.310	12.710	13.970	63.660	235.8
2	1.320	2.920	4.300	4.530	9.920	19.21
3	1.200	2.350	3.180	3.310	5.840	9.220
4	1.140	2.130	2.780	2.870	4.600	6.620
5	1.110	2.020	2.570	2.650	4.030	5.510
6	1.090	1.940	2.450	2.520	3.710	4.900
7	1.080	1.890	2.360	2.430	3.500	4.530
8	1.070	1.860	2.310	2.370	3.360	4.280
9	1.060	1.830	2.260	2.320	3.250	4.090
10	1.050	1.810	2.230	2.280	3.170	3.960
11	1.050	1.800	2.200	2.250	3.110	3.850
12	1.040	1.780	2.180	2.230	3.050	3.760
13	1.040	1.770	2.160	2.210	3.010	3.690
14	1.040	1.760	2.140	2.200	2.980	3.640
15	1.030	1.750	2.130	2.180	2.950	3.590
16	1.030	1.750	2.120	2.170	2.920	3.540
17	1.030	1.740	2.110	2.160	2.900	3.510
18	1.030	1.730	2.100	2.150	2.880	3.480
19	1.030	1.730	2.090	2.140	2.860	3.450
20	1.030	1.720	2.090	2.130	2.850	3.420
25	1.020	1.710	2.060	2.110	2.790	3.330
30	1.010	1.700	2.040	2.090	2.750	3.270
35	1.010	1.700	2.030	2.070	2.720	3.230
40	1.010	1.680	2.020	2.060	2.700	3.200
45	1.010	1.680	2.010	2.060	2.690	3.180
50	1.010	1.680	2.010	2.050	2.680	3.160
100	1.005	1.660	1.984	2.025	2.626	3.077
∞	1.000	1.645	1.960	2.000	2.576	3.000

Value of $t_{p,v}$ from the t-distribution for degrees of freedom v

Excel Formula:
=T.INV.2T(1-p, v)

Recap: Calculating Expanded Uncertainty

1. Determine the level of confidence required (95.45% or 99.73% are common choices)
2. Compute the effective degrees of freedom
3. Determine the coverage factor, k, using the t-distribution, the effective degrees of freedom (ν_{eff}), and the selected level of confidence (p)
4. Compute the expanded uncertainty, $U = k \cdot u_c(y)$
5. Report the result by stating $\bar{y} \pm U$ along with the level of confidence

Mini-Exercise: Expanded Uncertainty Calculation

For our Lava Temperature measurement example:

- Compute the effective degrees of freedom using the Welch-Satterthwaite formula
- Compute the coverage factor for a confidence level of 95.45%
- Report the result of the measurement along with the expanded uncertainty

Lava Temperature Example

- $u_c(y) = \sqrt{u_A^2 + u_B^2} = \sqrt{18.03^2 + 43.3^2} = 46.90$ °C

Effective Degrees of Freedom

- $\nu_{eff} = \dfrac{u_c^4(y)}{\frac{u_A^4(y)}{\nu_A} + \frac{u_B^4(y)}{\nu_B}} = \dfrac{46.90^4}{\frac{18.03^4}{9} + \frac{43.3^4}{\infty}} = 412$

Expanded Uncertainty

- $k = 2.01$ (95.45% with $\nu_{eff} = 412$)
- $U = k \cdot u_c(y) = 94$ °C
- Result: $T = (1172 \pm 94)$ ℃ (k=2, 95.45%)

Interlude - Section 5
The SI, Traceability, and Calibration

Section 5 – The SI, Traceability, and Calibration

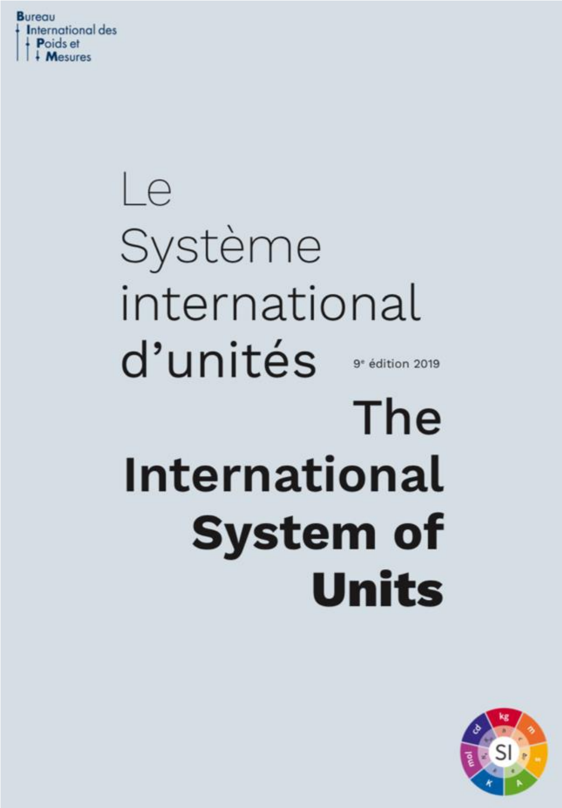
Objective

- Discuss traceability to the SI from the SI base unit definitions to the factory floor.

Content

- SI unit definitions and realizations
- Traceability and Accreditation
- Calibration and Calibration Certificates

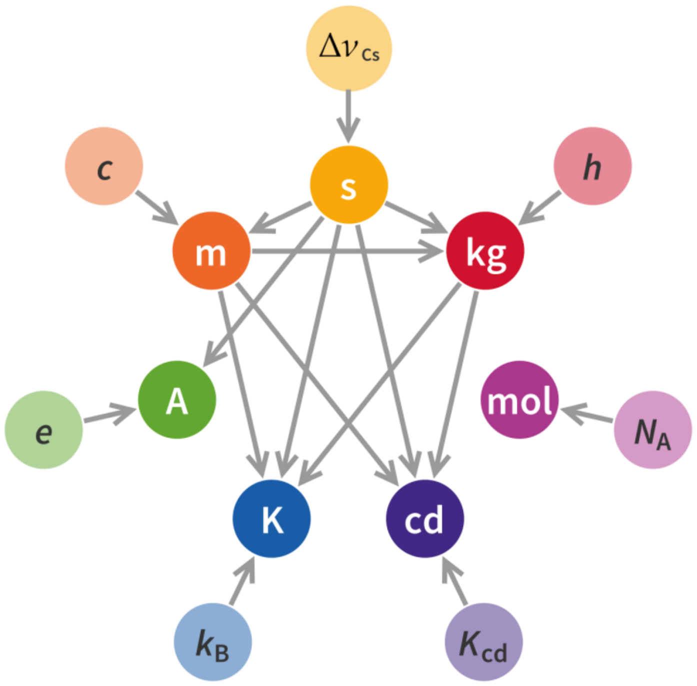
The International System of Units



See the SI Brochure (https://www.bipm.org/en/si/si-brochure/) published by the BIPM.

Think Americans don't use the metric system? Secretly we have all been using it since 1893 (http://www.ngs.noaa.gov/PUBS_LIB/FedRegister/FRdoc59-5442.pdf).

Mechanical Base Units



The **second** is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

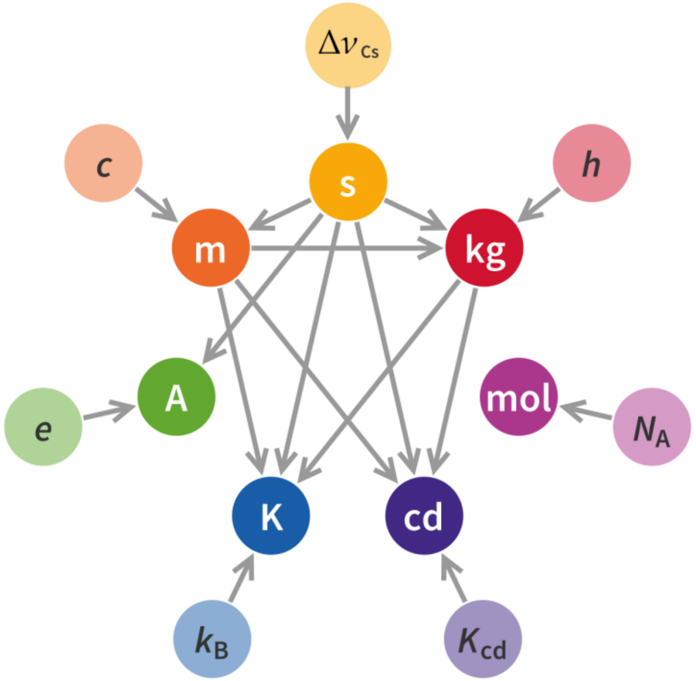
The **meter** is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

The **kilogram** is defined based on an exact value of Planck's constant 6.62607015x10^{-34} kg m^{2}s^{-1} in combination with the definitions of meter and second.

Electrical and Thermodynamic Base Units

The **kelvin** is the change of thermodynamic temperature that results in a change of thermal energy kT by 1.380649×10^{-23} J

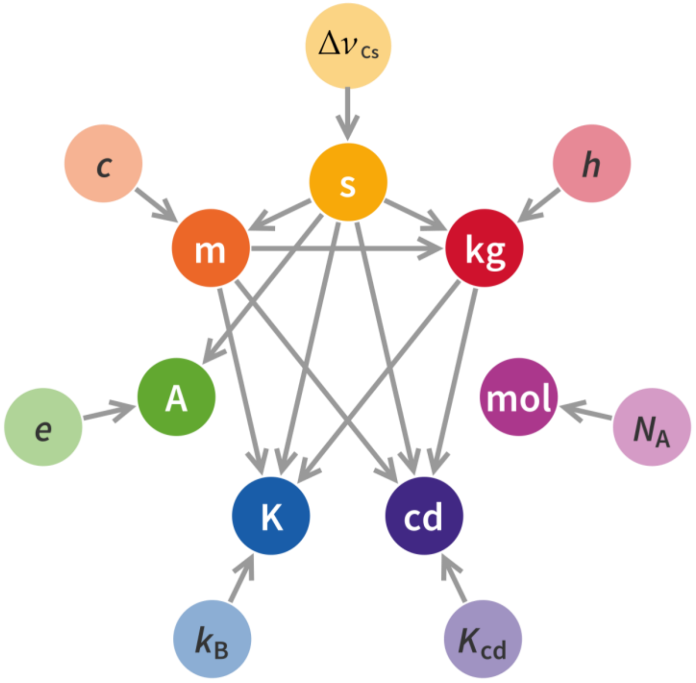
The **ampere** is the current corresponding to the flow of exactly $1/1.602176634 \times 10^{-19}$ elementary charges per second



The Other Two Base Units

The **mole** is the amount of substance of a system which contains exactly $6.02214076 \times 10^{23}$ elementary entities

The **candela** is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.



2019 SI Redefinition

- 1999 – CGPM issues recommendation that kg redefinition is needed
- November 2018– Formal vote passed, approving kilogram based on physical constants
- May 2019 – New definition in effect

1983 - 2019		2019
The **kilogram** is equal to the mass of the international prototype of the kilogram.	→	The **kilogram** is defined based on an exact value of Planck's constant $6.62607015 \times 10^{-34}$ kg $m^2 s^{-1}$ in combination with the definitions of meter and second.
The **ampere** is that constant current which, if maintained in two straight parallel conductors … would produce a force equal to 2×10^{-7} newton per meter of length.	→	The **ampere** is the current corresponding to the flow of exactly $1/1.602176634 \times 10^{-19}$ elementary charges per second
The **mole** is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12.	→	The **mole** is the amount of substance of a system which contains exactly $6.02214076 \times 10^{23}$ elementary entities
The **kelvin** is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.	→	The **kelvin** is the change of thermodynamic temperature that results in a change of thermal energy kT by 1.380649×10^{-23} J

SI History

1795 - French Metric System Established

1875 - Meter Convention (m, kg, s), establishes BIPM and CGPM

1955 - Energy units (A, K, cd) added

1960 - SI formally established

1971 - Mole added as 7th base unit

2019 - The last physical artifact, kg, removed from definitions

There has always been debate and discussion about how to define the SI

See "The Last Artifact", PBS special:

- https://www.youtube.com/watch?v=i2uURgADxGc

Derived Units

All other SI units can be formed by combinations of the base units.

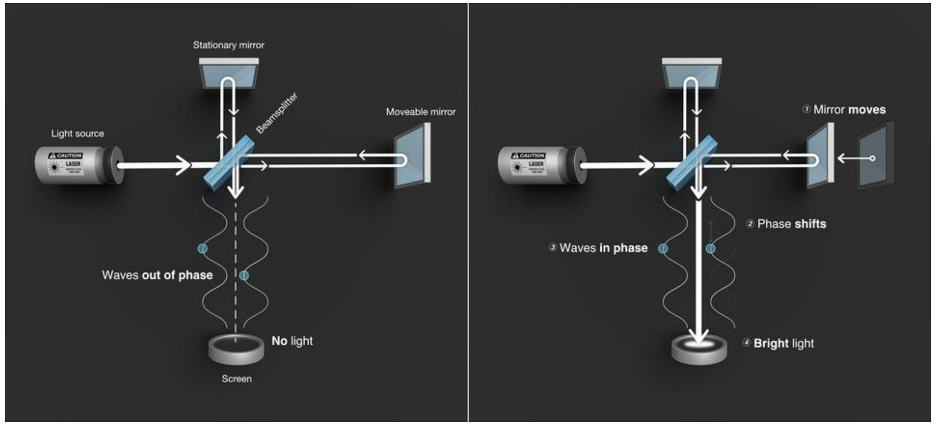
For example:

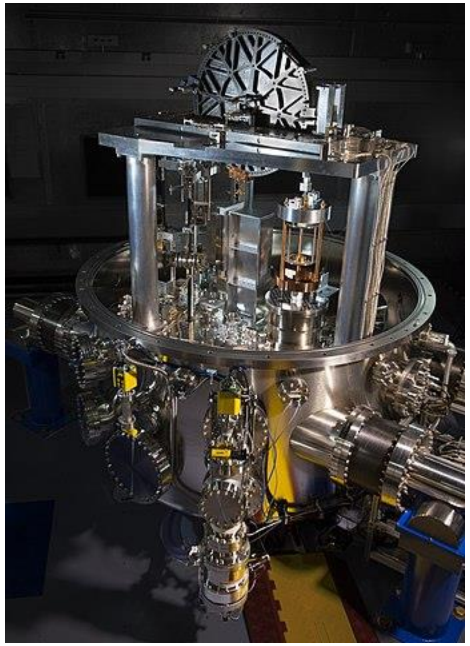
- Frequency, Hz = s^{-1}
- Pressure, Pa = m^{-1} kg s^{-2}
- Energy, J = m^2 kg s^{-2}
- Voltage, V = m^2 kg s^{-3} A^{-1}

Unit Realizations

In practice SI units are usually **realized** without going all the way back to the unit **definition**.

A laser interferometer uses interference patterns between two laser paths to convert a calibrated laser frequency into a distance.





A Kibble Balance is used to realize the unit of kilogram given Planck's constant h.

The Josephson effect can be used to realize the volt given the fixed value of $h/2e$.

All Images: NIST, Public Domain

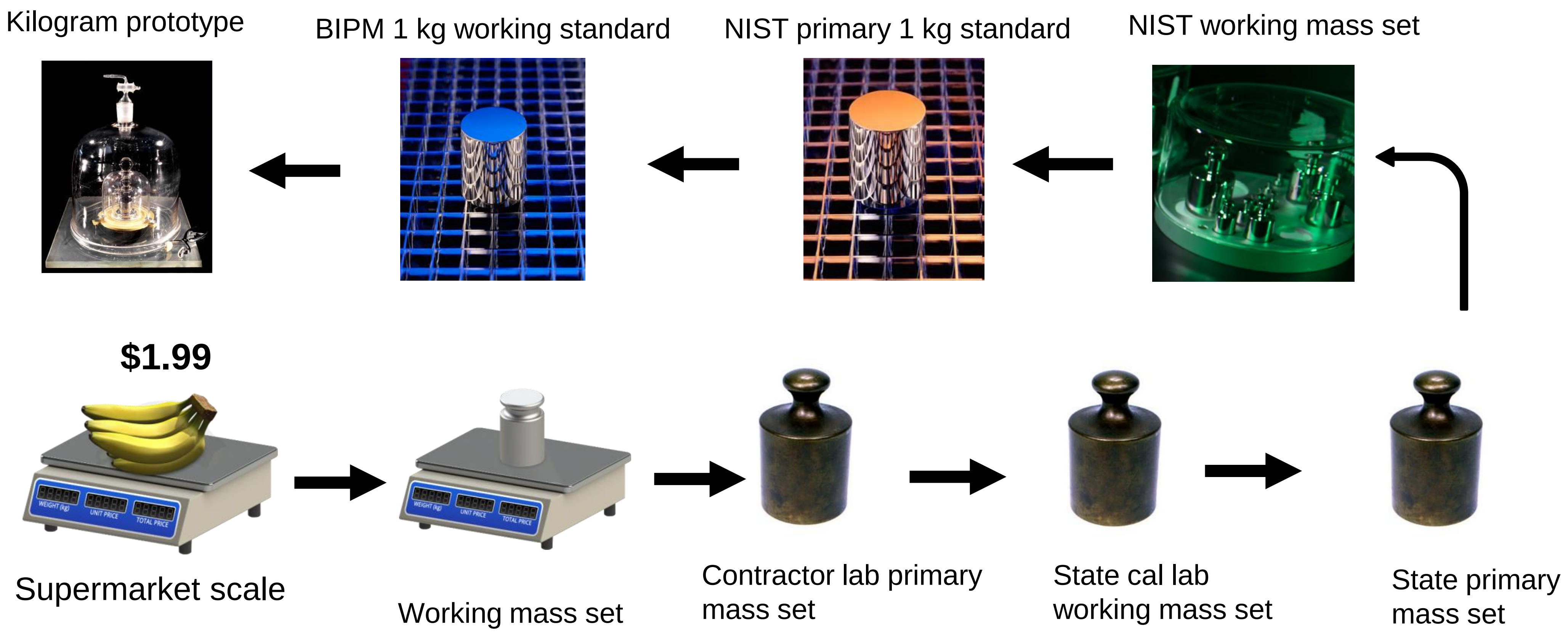
Definition: Traceability

metrological traceability

property of a **measurement result** whereby the result can be related to a reference through a documented unbroken chain of **calibrations**, each contributing to the **measurement uncertainty**

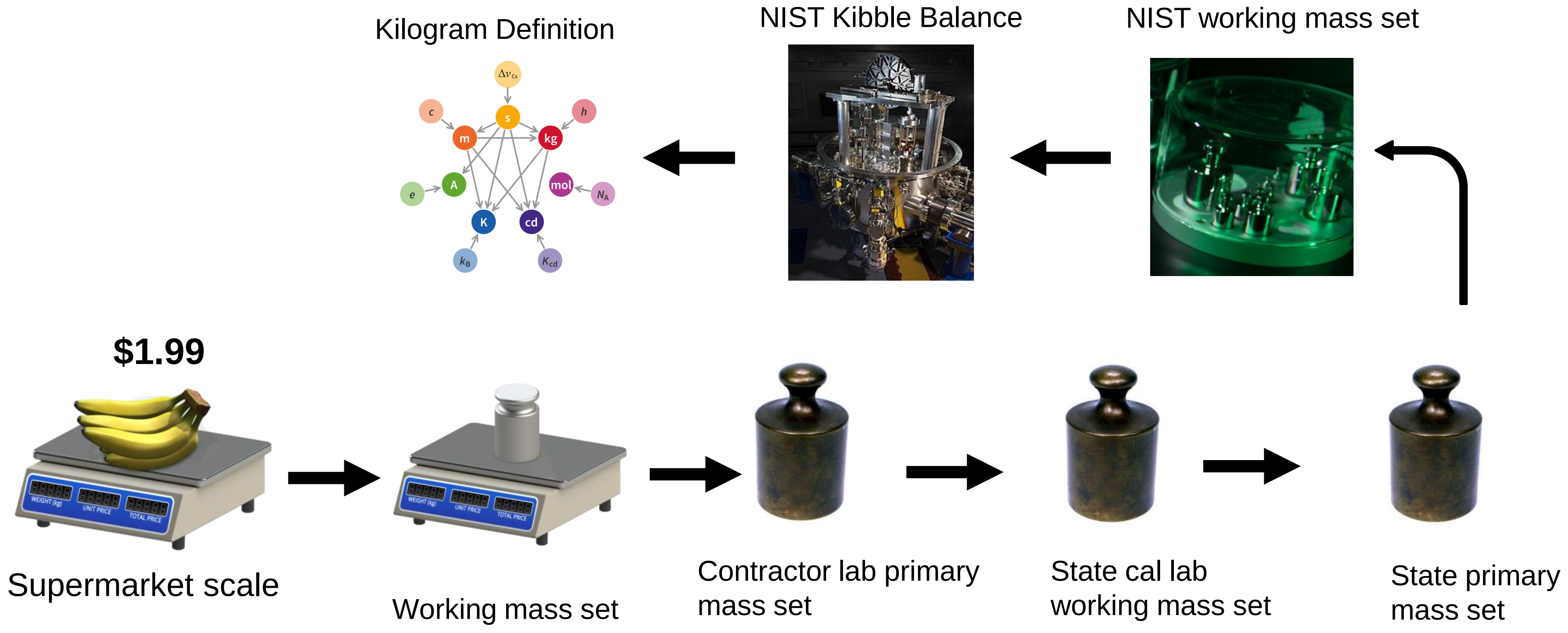
VIM (JGCM 200:2012) section 2.41

A Mass Traceability Chain – Before 2019



Note: the full traceability chain must also include, at each step, a number of other measurements such as temperature, air pressure, and magnetization.

A Mass Traceability Chain – After 2019



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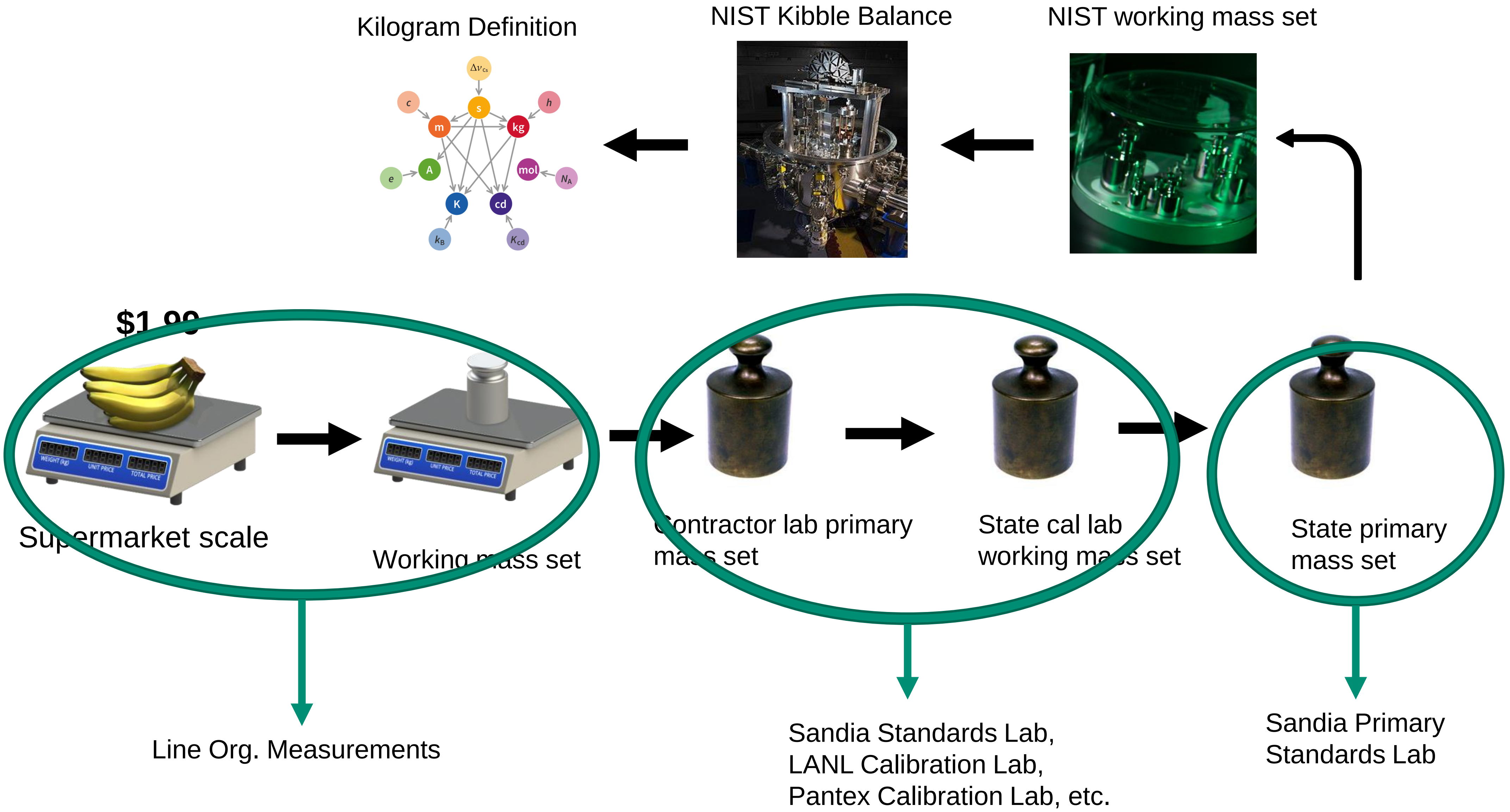
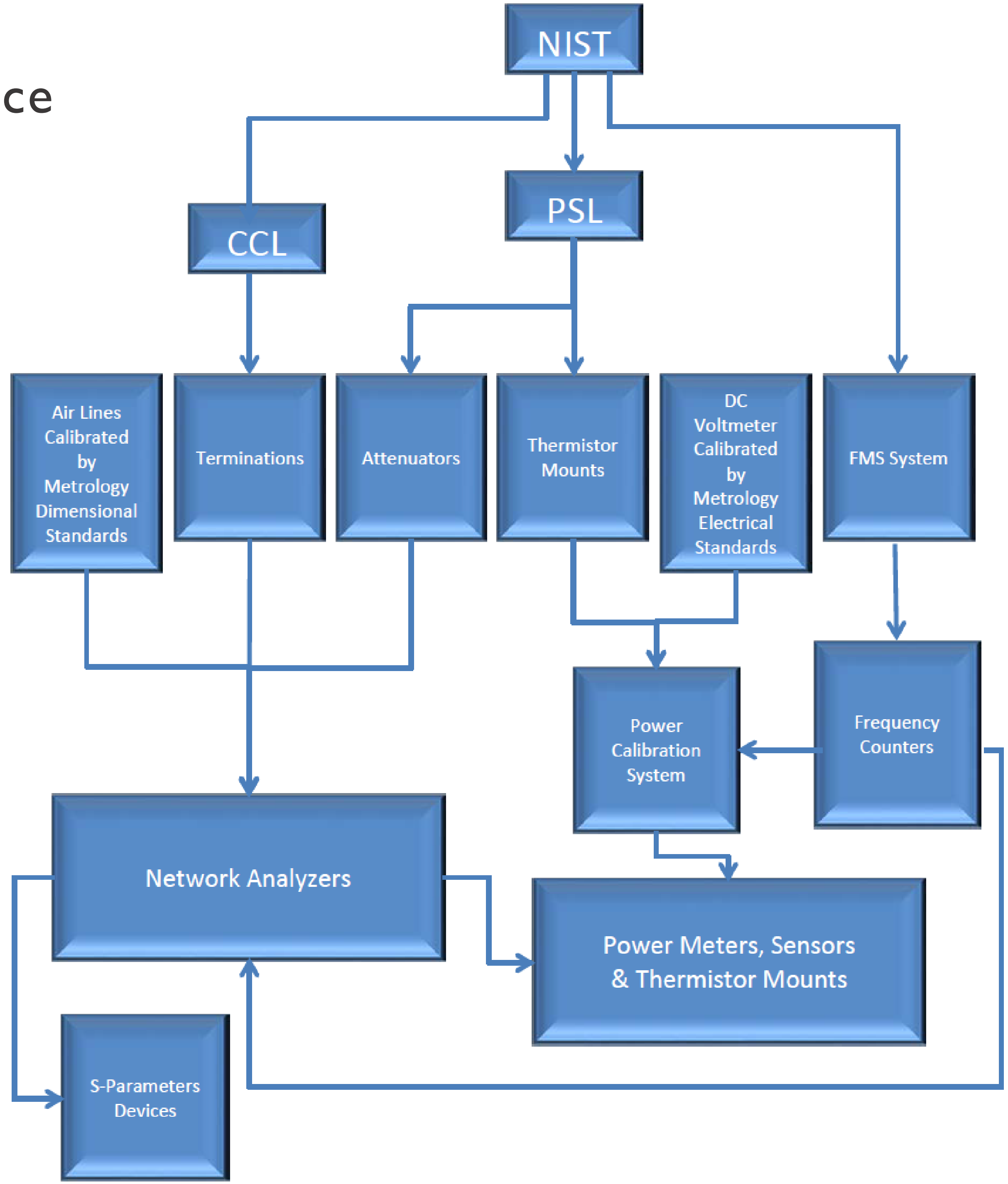
A Mass Traceability Chain – After 2019



Traceability Diagrams in Practice



KSNSC Metrology, Measurement Capabilities NSC-614-3255, 2020

Traceability

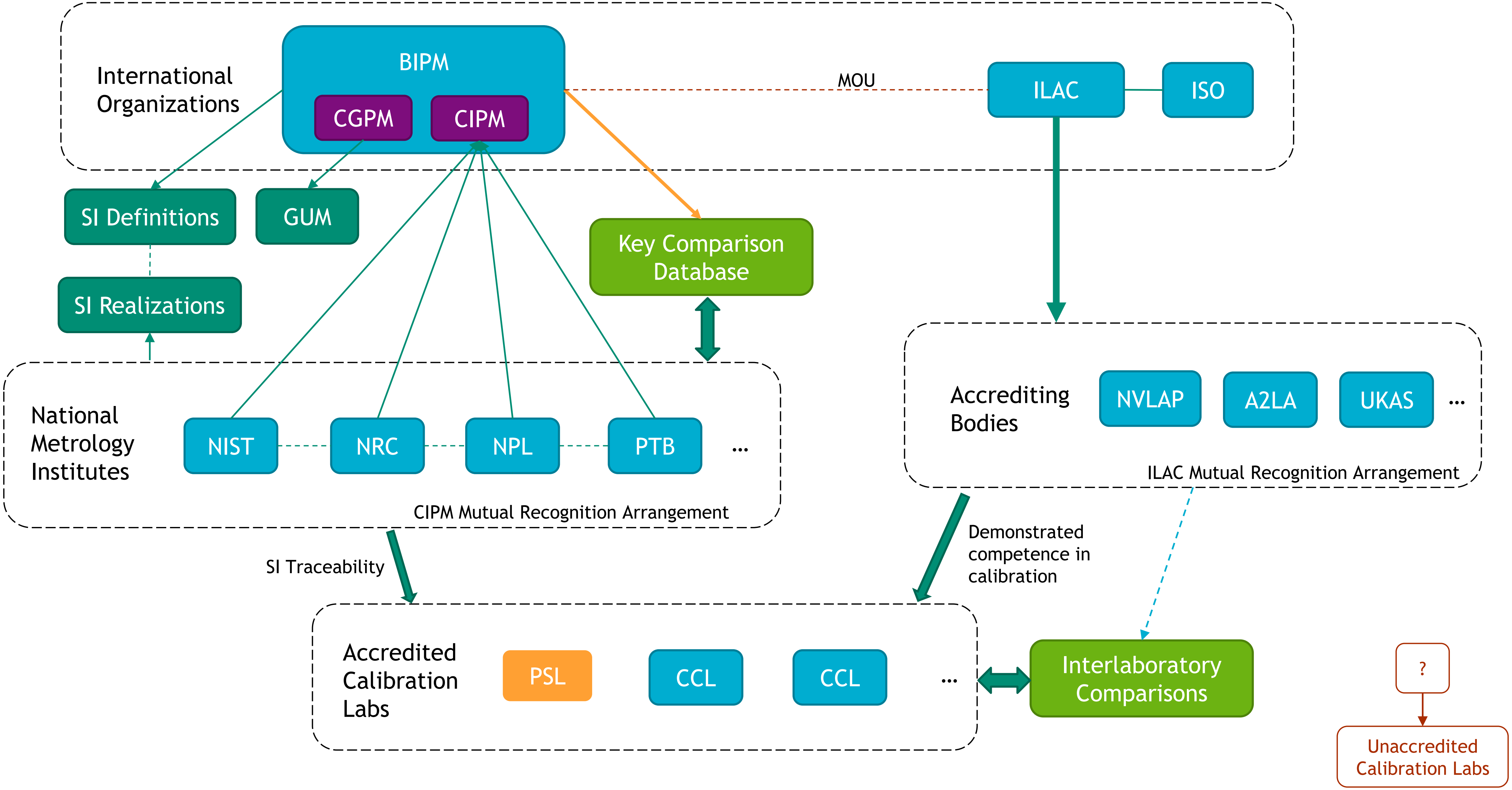
In practice it can be more difficult than anybody admits to properly and completely document traceability. This is because the traceability chain may stretch across many calibration labs and perhaps over many years.

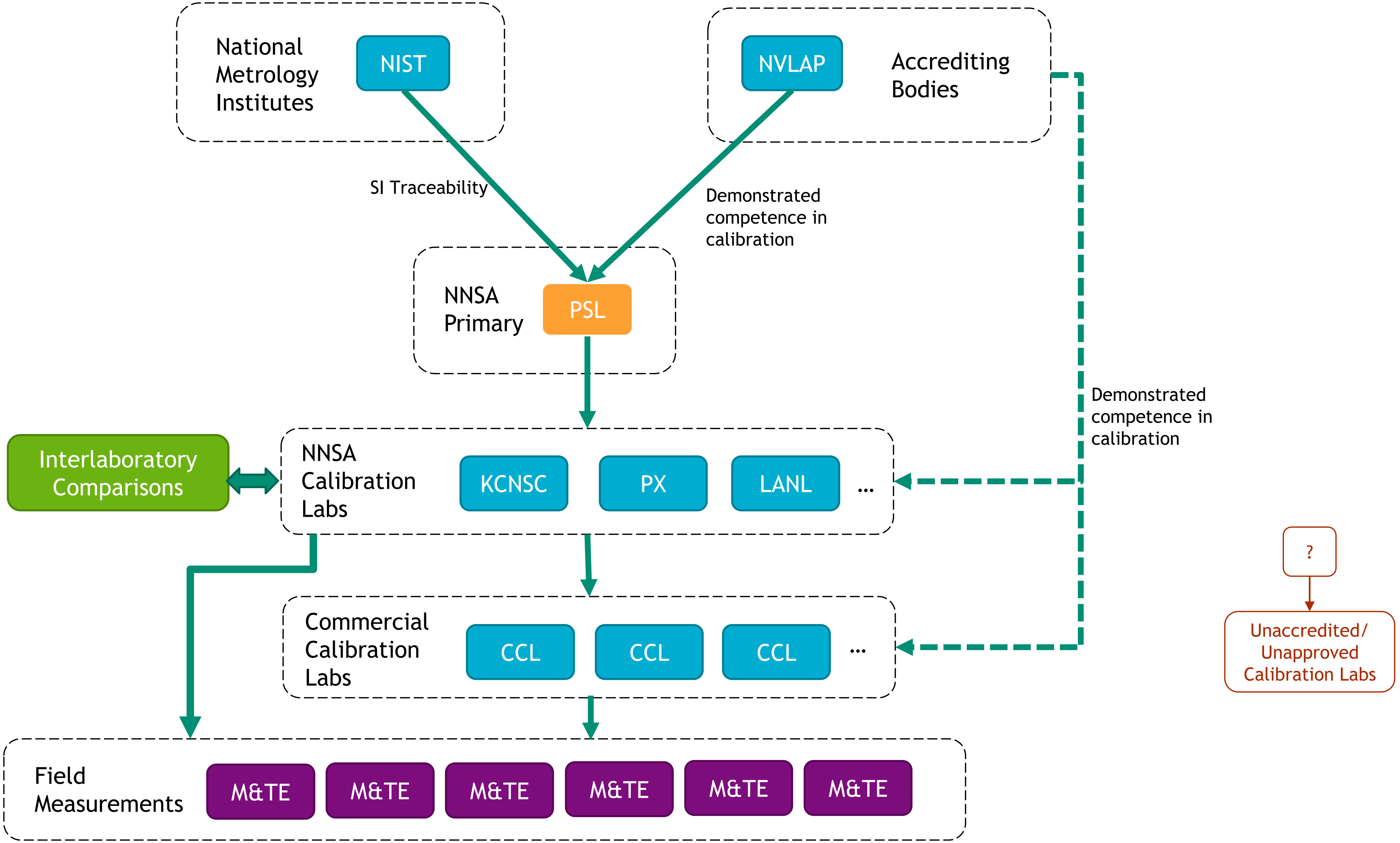
Corollary: If no one ever properly checks traceability then anyone can claim "NIST traceability" even if they have little evidence to back it up.

Accreditation

An accredited calibration lab demonstrates via periodic audits by an independent accreditation body (e.g. NVLAP or A2LA) that they are capable of performing good calibrations as defined by ISO 17025 (or ANSI Z540.3). Note that the accreditation for a particular lab has a well defined scope and an expiration date.

Accreditation not the same as proof of traceability. But we are usually willing to assume that accredited calibrations are traceable since the calibration lab has demonstrated to an independent body that they are capable of making traceable measurements.


International Organizations
BIPM
CGPM
CIPM
MOU
ILAC
ISO
SI Definitions
GUM
SI Realizations
Key Comparison Database
Accrediting Bodies
NVLAP
A2LA
UKAS
...
ILAC Mutual Recognition Arrangement
National Metrology Institutes
NIST
NRC
NPL
PTB
...
CIPM Mutual Recognition Arrangement
SI Traceability
Demonstrated competence in calibration
Accredited Calibration Labs
PSL
CCL
CCL
...
Interlaboratory Comparisons
?
Unaccredited Calibration Labs


National Metrology Institutes
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Demonstrated competence in calibration
NNSA Primary
PSL
Interlaboratory Comparisons
NNSA Calibration Labs
KCNSC
PX
LANL
...
Demonstrated competence in calibration
?
Unaccredited/ Unapproved Calibration Labs
Commercial Calibration Labs
CCL
CCL
CCL
...
Field Measurements
M&TE
M&TE
M&TE
M&TE
M&TE
M&TE

The two terminal resistance of this standard resistor was measured in air at a temperature of 23.0 ± 0.1 °C and a humidity of 40 ± 10%. The resistor is certified over a temperature range of 23 ± 1 °C:

Voltage	Resistance	Interval Uncertainty at *k = 3*
100 V	1.000 59 GΩ	700 ppm

The expanded uncertainty stated above is valid over the certification interval and is given at k(coverage factor) = 3. If this resistor is used at temperatures other than 23 ± 1 °C or at voltages other than 100 V, then the application of appropriate corrections or increase in the uncertainty shall be the responsibility of the user.

This device was measured using an automated high resistance bridge at an appropriate ratio against a selected resistor that is periodically calibrated against a set of Thomas one Ω resistors, whose values are assigned by N.I.S.T.

Note: The uncertainty for this resistor has been increased to 700 ppm to take into account the large drift of this resistor.

DC Voltage

Range	Full Scale	Maximum Resolution	Input Impedance	Temperature Coefficient (ppm of Reading + ppm of Range) / °C	
				Without ACAL[1]	With ACAL[2]
100 mV	120.00000	10 nV	> 10 GΩ	1.2 + 1	0.15 + 1
1 V	1.20000000	10 nV	> 10 GΩ	1.2 + 0.1	0.15 + 0.1
10 V	12.0000000	100 nV	> 10 GΩ	0.5 + 0.01	0.15 + 0.01
100 V	120.000000	1 µV	10 MΩ ± 1%	2 + 0.4	0.15 + 0.1
1000 V	1050.00000	10 µV	10 MΩ ± 1%	2 + 0.04	0.15 + 0.01

Accuracy[3] [ppm of Reading (ppm of Reading for Option 002) + ppm of Range]

Range	24 Hour[4]	90 Day[5]	1 Year[5]	2 Year[5]
100 mV	2.5 + 3	5.0 (3.5) + 3	9 (5) + 3	14 (10) + 3
1 V	1.5 + 0.3	4.6 (3.1) + 0.3	8 (4)+ 0.3	14 (10) + 0.3
10 V	0.5 + 0.05	4.1 (2.6) + 0.05	8 (4) + 0.05	14 (10) + 0.05
100 V	2.5 + 0.3	6.0 (4.5) + 0.3	10 (6)+ 0.3	14 (10) + 0.3
1000 V[6]	2.5 + 0.1	6.0 (4.5) + 0.1	10 (6) + 0.1	14 (10) + 0.1

Transfer Accuracy/Linearity

Range	10 Min, Tref ± 0.5°C (ppm of Reading + ppm of Range)
100 mV	0.5 + 0.5
1 V	0.3 + 0.1
10 V	0.05 + 0.05
100 V	0.5 + 0.1
1000 V	1.5 + 0.05

Conditions

- Following 4 hour warm-up. Full scale to 10% of full scale.
- Measurements on the 1000 V range are within 5% of the initial measurement value and following measurement settling.
- Tref is the starting ambient temperature.
- Measurements are made on a fixed range (> 4 min.) using accepted metrology practices.

1 *Additional error from Tcal or last ACAL ± 1°C .*

2 *Additional error from Tcal ± 5°C.*

3 *Specifications are for PRESET; NPLC 100.*

4 *For fixed range (> 4 min.), MATH NULL and Tcal ± 1°C.*

5 *Specifications for 90 day, 1 year and 2 year are within 24 hours and ± 1° C of last ACAL; Tcal ±5°C; MATH NULL and fixed range.*

ppm of Reading specifications for High Stability (Option 002) are in parentheses.

Without MATH NULL, add 0.15 ppm of Range to 10 V, 0.7 ppm of Range to 1 V, and 7ppm of Range to 0.1V. Without math null and for fixed range less than 4 minutes, add 0.25 ppm of Range to 10 V, 1.7 ppm of Range to 1 V and 17 ppm of Range to 0.1 V.

Add 2 ppm of reading additional error for factory traceability to US NIST. Traceability error is the absolute error relative to National Standards associ-ated with the source of last external calibration.

6 *Add 12 ppm X (Vin / 1000)^2 additional error for inputs > 100 V.*

3.2 Environmental Conditions

Although the instrument has been designed for optimum durability and trouble-free operation, it must be handled with care. The instrument should not be operated in an excessively dusty or dirty environment. Maintenance and cleaning recommendations can be found in the Maintenance Section of this manual.

The instrument operates safely under the following conditions:

- temperature range: 5–40°C (41–104°F)
- ambient relative humidity: 15–50%
- pressure: 75kPa–106kPa
- mains voltage within ±10% of nominal
- vibrations in the calibration environment should be minimized
- altitudes less than 2,000 meters

Section 6
Indirect Measurements

Section 6 – Indirect Measurements

Objective

- Participants will be able to define a measurement model and estimate uncertainty associated with indirect measurements

Content

- Section 6.1: GUM Approach
  - Defining a Measurement Model
  - Calculating Uncertainty
  - Practice Problems

- Section 6.2: Monte Carlo Approach
  - Propagation of Distributions
  - Calculating Uncertainty
  - Practice Problems

Indirect Measurement

A measurement where the value of a measurand is obtained by measuring other quantities functionally related to the measurand.

Defining the Measurement Model

$$Y = f(X_1, X_2, \ldots, X_N)$$

Y represents the measurement of a well-defined physical quantity - the measurand - that can be characterized by an essentially unique value.

f expresses the mathematical relationship between the measurand and the input quantities (The X_i's, the sources of variation).

Defining the Measurement Equation

The function f should contain every quantity (X_i), including all corrections and correction factors, that can contribute a significant amount of uncertainty to the result of the measurement

- $I = V / R$

- $I = f(V, R) = f(X_1, X_2)$

Defining the Measurement Model

If the initial equation f does not model the measurement adequately, additional input quantities (e.g. temperature adjustment) should be included:

- $I = V/R_0\,[1 + a(T - T_0)]$
- $I = f(V, R_0, T, a) = f(X_1, X_2, X_3, X_4)$

Sandia Example

- Film thickness standard $y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$

- Other SNL examples?

Section 6.1: GUM Approach

GUM Approach

1. List the sources of uncertainty
2. Formulate the measurement model
3. Determine the distributions for the input quantities and evaluate the standard uncertainties for each input quantity
4. Determine the combined standard uncertainty
5. Compute the expanded uncertainty using a coverage factor k based on the degrees of freedom and the level of confidence required
6. Estimate the measurand from the measurement equation
7. Report the uncertainty interval

GUM Approach

1. List the sources of uncertainty
   - Brainstorming, cause and effect analysis, engineering judgment may be used.
   - DOEx may also be required

2. Formulate the measurement model
   - The function $Y = f(X_1, X_2, \ldots, X_N)$ should contain every quantity, including all corrections and correction factors, that can contribute a significant component of uncertainty to the result of the measurement

3. Determine the distributions for the input quantities and evaluate the standard uncertainties for each input quantity
   - The distributions are determined on the basis of the statistical analysis of a series of observations (Type A evaluation) or by other means (Type B evaluation).

GUM Approach

4. Determine the combined standard uncertainty of the measurement result y from the standard uncertainties associated with the input quantities (X_i's)

- The combined standard uncertainty, $u_c(y)$, is the positive square root of the combined variance, given by:

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)$$

- Each $u(x_i)$ is a standard uncertainty evaluated as described in Section 4 using either Type A and/or Type B evaluation

GUM Approach

- The partial derivatives are referred to as sensitivity coefficients, which describe how the output estimate y varies with changes in the input quantities (X_i's)

- The partial derivative $\frac{\partial f}{\partial x_i}$ is evaluated at the averages of the input quantity X_i

- If we define the sensitivity coefficients as $c_i = \frac{\partial f}{\partial x_i}$, the uncertainty in y associated with the input quantity X_i is $|c_i|u(x_i)$

Computing Combined Standard Uncertainty (Example)

Suppose the measurement equation is: $P = I^2 R$

Partial derivatives are: $c_1 = \frac{\partial P}{\partial I} = 2IR, \qquad c_2 = \frac{\partial P}{\partial R} = I^2$

Combined standard uncertainty is:

$$u_c(y) = \sqrt{c_1^2 u^2(x_1) + c_2^2 u^2(x_2)}$$
$$= \sqrt{(2IR)^2 u^2(I) + (I^2)^2 u^2(R)}$$

GUM Approach

5. Determine an expanded uncertainty, U, whose purpose is to provide an uncertainty interval $(y - U, y + U)$

   - This interval may be expected to encompass a large fraction of the distribution of values that could be reasonably attributed to the measurand Y

   - U is determined by multiplying the combined standard uncertainty by a coverage factor $k = t_p(\nu_{eff})$ determined from a t-table:

$$U = k\, u_c(y)$$

GUM Approach

6. Estimate the measurand from the measurement equation:

$$Y = f(X_1, X_2, \dots, X_N)$$

- For a linear measurement equation, Y is estimated by:

$$y = f(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n)$$

- For a highly nonlinear measurement equation, Y is estimated by:

$$y = \frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{1}{n}\sum_{k=1}^{n} f(X_{1,k}, X_{2,k}, \dots X_{N,k})$$

  - That is, y is taken as the arithmetic mean or average of n independent determinations Y_k of Y.

Estimating the Measurand (Example)

With measurement equation: $P = I^2 R$

An estimate of the measurand is: $P = \bar{I}^{\,2} \bar{R}$

Or with observations $I_i, R_i,\ i = 1, 2, \ldots, n$

$$P = \frac{1}{n} \sum_{i=1}^{n} I_i^2 R_i$$

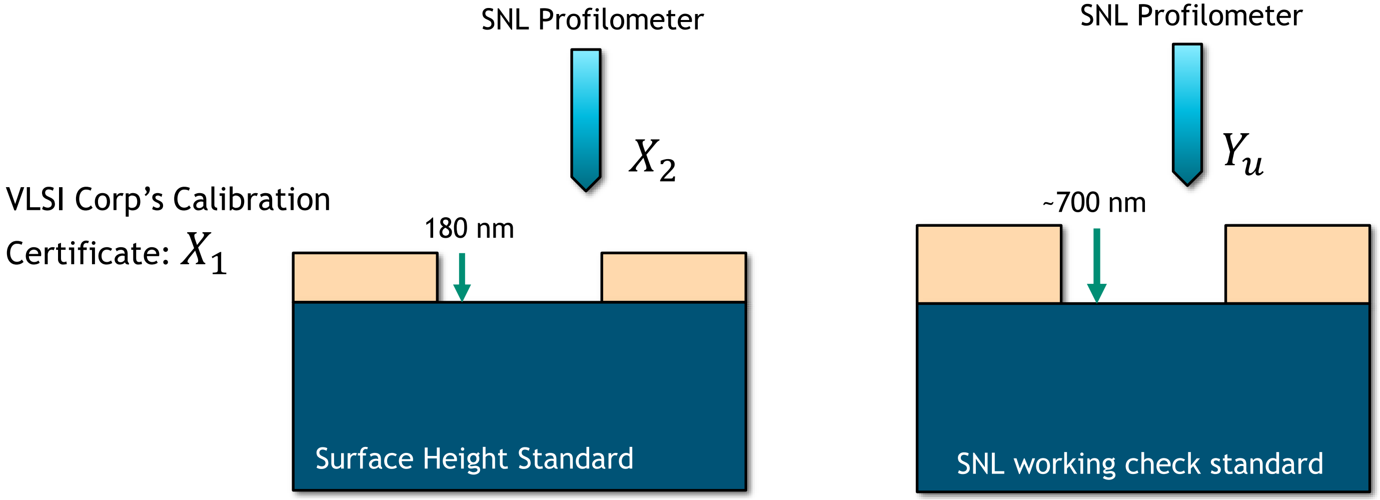
GUM Approach

7. Report the uncertainty interval

   - Express as $y \pm U$
   - Include the level of confidence
   - Include the value of k used to obtain U

Example: Film Thickness Standard

- VLSI Corporation makes Surface Height Standard, provides calibrated height X_1 on a thickness standard
- SNL Measures Surface Height Standard with its own profilometer, obtains height X_2
- Ratio of SNL to VLSI measurements is used to correct profilometer measurements on other samples such as SNL working check standard Y_u
- Thickness standard can then be used as check standard is then used to validate other height measurements



Measurement Equation

$$y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$$

y_{cor} = Corrected measurement of thickness check standard

Y_u = SNL's profilometer measurement of check standard

X_1 = VLSI calibrated measurement of Surface Height Standard (SHS)

X_2 = SNL's profilometer measurement of SHS

Steps to Determine Uncertainty

1. List the sources of uncertainty
2. Formulate the measurement equation, $y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$
3. Determine the distributions for the input quantities, X_1, X_2, Y_u, and evaluate the standard uncertainties for each input quantity. Type A evaluations are used for X_1, X_2, Y_u.
4. Determine the combined standard uncertainty
5. Compute the expanded uncertainty using a coverage factor k based on the degrees of freedom and the level of confidence required
6. Estimate the measurand, y_{cor}, from the measurement equation
7. Report the uncertainty interval

Sources of Uncertainty

- SNL's profilometer measurement of thickness of standard, Y_u
- VLSI corporation's calibrated measurement of Surface Height Standard (SHS), X_1
- SNL's profilometer measurement of Surface Height Standard (SHS), X_2

Distributions for Input Quantities

Input Quantity	Mean (μ)	Standard Uncertainty Type A	Degrees of Freedom	PDF
Y_u	0.6978 μm	0.0026 μm	19	Normal
X_1	0.1820 μm	0.00093 μm	9	Normal
X_2	0.1823 μm	0.00058 μm	19	Normal

Measurement equation is $y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$

Calculation of Type A Combined Standard Uncertainty

$$u_c(y_{cor}) = \left(\left(\frac{Y_u}{X_2}\right)^2 u^2(X_1) + \left(\frac{X_1 Y_u}{X_2^2}\right)^2 u^2(X_2) + \left(\frac{X_1}{X_2}\right)^2 u^2(Y_u)\right)^{1/2}$$

$$= \left(\left(\frac{0.6978}{0.1823}\right)^2 (0.00093)^2 + \left(\frac{(0.1820)(0.6978)}{(0.1823)^2}\right)^2 (0.00058)^2 + \left(\frac{0.1820}{0.1823}\right)^2 (0.0026)^2\right)^{1/2}$$

$$= \left((0.00356)^2 + (0.00222)^2 + (0.00260)^2\right)^{1/2}$$

$$= 0.00494$$

Combined Standard Uncertainty

In this case, we are only provided Type A uncertainties. Typically, we will need to account for both Type A and Type B uncertainties

$$u_c(X_1) = \sqrt{u_A^2(X_1) + u_B^2(X_1)}$$

$$u_c(X_2) = \sqrt{u_A^2(X_2) + u_B^2(X_2)}$$

$$u_c(Y_u) = \sqrt{u_A^2(Y_u) + u_B^2(Y_u)}$$

Calculation of Film Standard Thickness

$$y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$$

$$= \left(\frac{0.1820}{0.1823}\right)(0.6978)$$

$$= 0.6967\ \mu m$$

Determine the Expanded Uncertainty

Obtain the degrees of freedom for the input quantities

Input Quantity	Mean (μ)	Standard Uncertainty Type A	Degrees of Freedom	PDF
Y_u	0.6978 μm	0.0026 μm	19	Normal
X_1	0.1820 μm	0.00093 μm	9	Normal
X_2	0.1823 μm	0.00058 μm	19	Normal

Determine the Expanded Uncertainty

Calculate the effective degrees of freedom using the Welch-Satterthwaite formula*

$$\nu_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{3} \frac{\left(c_i u(x_i)\right)^4}{\nu_i}}$$

$$= \frac{(0.00494)^4}{\left(\frac{0.00356^4}{9} + \frac{0.00222^4}{19} + \frac{0.00260^4}{19}\right)} = 27.7$$

*Note this W-S formula includes the c_i sensitivity coefficients, which were omitted (c_i=1) in Section 4.

Determine the Expanded Uncertainty

Expanded Uncertainty

$$U_{y_{cor}}^{95} = t_{95}(\nu_{eff})u_c(y_{cor})$$

$$= (2.05)(0.00494)$$

$$= 0.0101\ \mu\text{m}$$

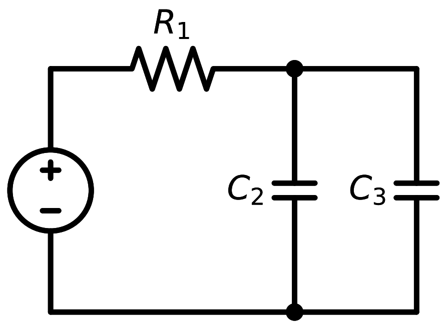
Report the Uncertainty Interval

Uncertainty interval at a 95% level of confidence (k = 2) is:

$$y_{cor} \pm U_{y_{cor}}^{95} = (0.6967 \ \pm 0.0101)\ \mu m$$

$$= (0.6866, 0.7068)\ \mu m$$

Practice: GUM Approach



The time constant of an RC circuit is given by

$$\tau = R_1(C_2 + C_3)$$

Capacitor values are given by the manufacturer:

- **C2 = 0.10 μF ± 0.005 μF**
- **C3 = 0.22 μF ± 0.01 μF**

The resistor value is unknown, so it was determined by a set of measurements obtained using a Fluke 8845A on the 100 kΩ range (right). The multimeter's specification given in a calibration certificate as:

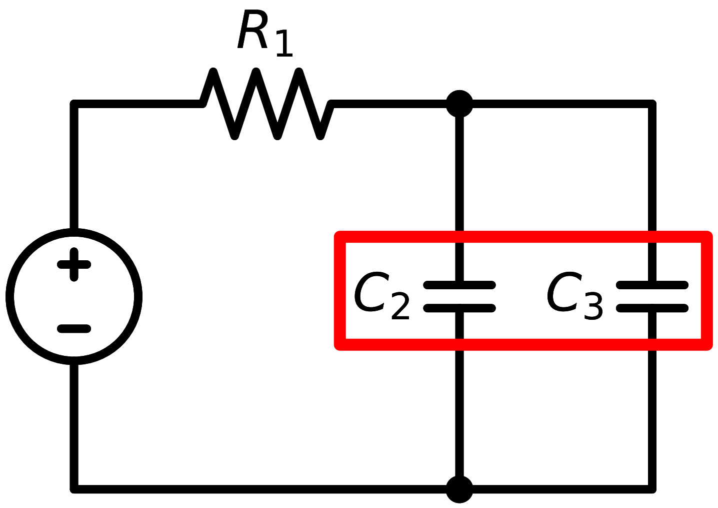
- **±(0.01% reading + 0.001% of range) at 95% level of confidence**

What is the uncertainty in τ?

Measurement	Resistance (kΩ)
1	32.2050
2	32.1878
3	32.2081
4	32.2201
5	32.1807
6	32.1990
7	32.1965
8	32.2120
9	32.1940
10	32.2108

One Step at a Time: Capacitors

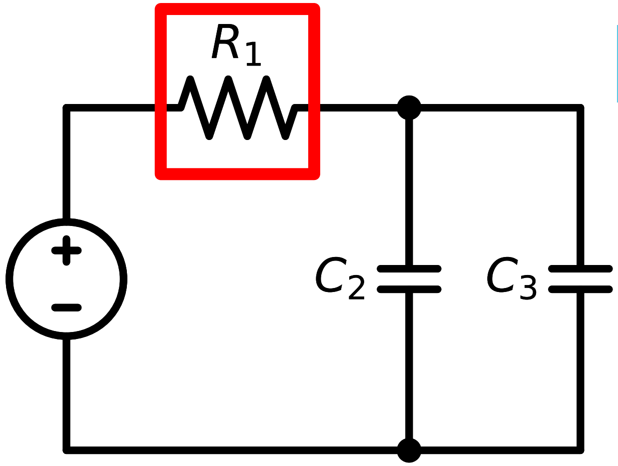
- What do we know about them?
  - **C2 = 0.10 µF ± 0.005 µF**
  - **C3 = 0.22 µF ± 0.01 µF**
- What distribution would be appropriate here?
- What type of uncertainty is described above?
- Is there a type A uncertainty?
- When do we combine uncertainties?



$$u_{C_2} = \frac{.005}{\sqrt{3}} = .0029\mu F$$

$$u_{C_3} = \frac{.01}{\sqrt{3}} = .0058\mu F$$

One Step at a Time: Resistor



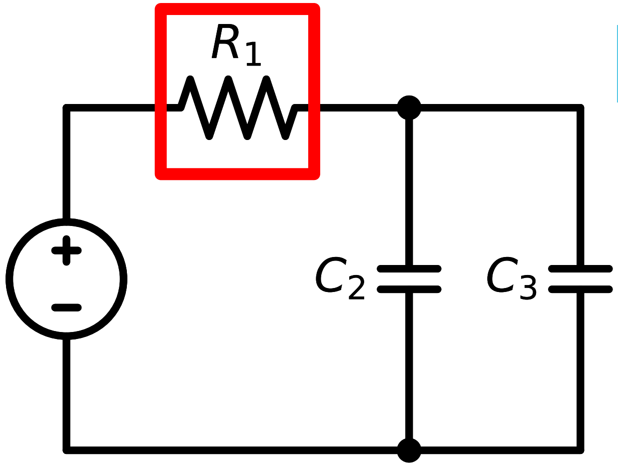
- What do we know about it?
  - **Data collected**
  - **±(0.01% reading + 0.001% of range) at 95% level of confidence**
  - **Tested on 100 kΩ range**
- How do we handle data collected?
  - Distribution?
  - Type?

$$\text{Mean} = 32.2014k\Omega$$
$$\text{Std dev} = .012k\Omega$$

$$u_{R_A} = \frac{.012}{\sqrt{10}} = .0038k\Omega$$

Measurement	Resistance (kΩ)
1	32.2050
2	32.1878
3	32.2081
4	32.2201
5	32.1807
6	32.1990
7	32.1965
8	32.2120
9	32.1940
10	32.2108

One Step at a Time: Resistor Continued



- How do we handle the calibration certificate?
  - Distribution?
  - Type?
    - **±(0.01% reading + 0.001% of range) at 95% level of confidence**
    - **Tested on 100 kΩ range**

$$U_{R_B} = .0001 \times 32.2014 + .00001 \times 100 = .0042k\Omega$$

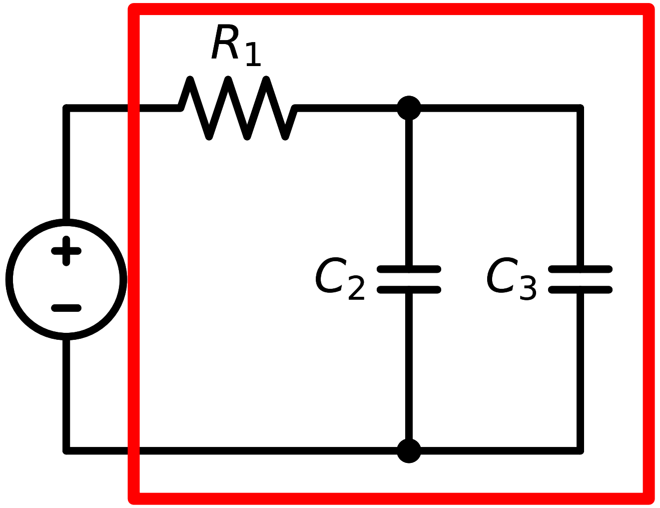
$$u_{R_B} = \frac{.0042}{1.96} = .0021k\Omega$$

- When do we combine uncertainties?

$$u_R = \sqrt{u_{R_A}^2 + u_{R_B}^2} = \sqrt{.0038^2 + .0021^2} = .0042k\Omega$$

One Step at a Time: Combined U

- How do we combine uncertainty of all Xi's?



Sensitivity Coefficients

$$c_R = \frac{\partial \tau}{\partial R} = C_2 + C_3 = 0.10\ \mu F + 0.22\ \mu F = 0.32\ \mu F$$

$$c_{C_2} = \frac{\partial \tau}{\partial C_2} = R_1 = 32.2014\ k\Omega$$

$$c_{C_3} = \frac{\partial \tau}{\partial C_3} = R_1 = 32.2014\ \mathrm{k}\Omega$$

- What is an Ohm*Farad?
  - It is a second??? While this may seem confusing, an electrical engineer is on standby to pretend it makes sense.

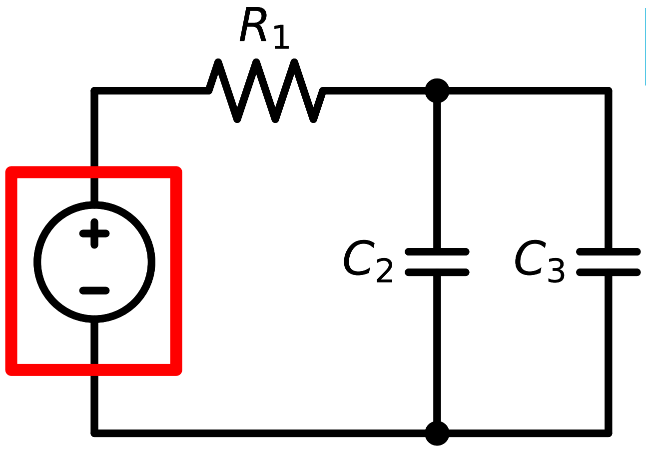
Combined Uncertainty

$$u_c = \sqrt{c_R^2 u_R^2 + c_{C_2}^2 u_{C_2}^2 + c_{C_3}^2 u_{C_3}^2}$$

$$= \sqrt{0.32^2 * 0.00437^2 + 32.2014^2 * 0.00288^2 + 32.2014^2 * 0.00577^2}$$

$$= 0.2078\ ms$$

One Step at a Time: Expanded U



- What do we need to go from u_c to U_c
  - u_c
  - A t-table
    - Our Percent Confidence for our estimate (95%)
    - Our degrees of freedom

Degrees of Freedom

$$v_{eff,R} = \frac{0.00436^4}{\frac{0.0038^4}{9} + \frac{0.0021^4}{\infty}} = 15.6$$

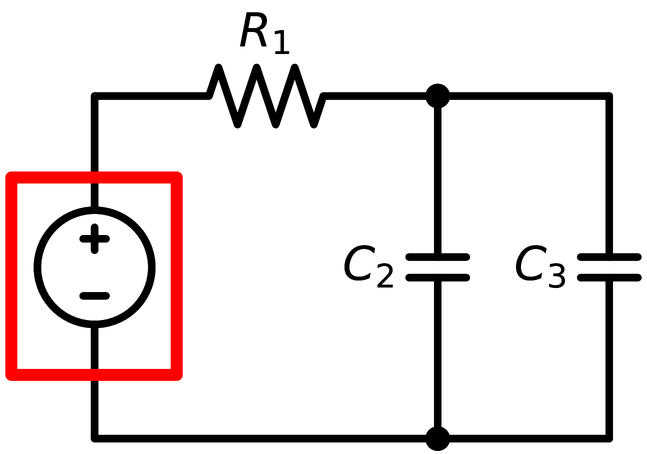
$$v_{eff} = \frac{0.2078^4}{\frac{(3.2 \times 10^{-7} * 0.00437)^4}{15.6} + \frac{(3.22 \times 10^4 * 0.00288^4)}{\infty} + \frac{(3.22 \times 10^4 * 0.00577^4)}{\infty}} = 7.6\ x\ 10^9 = \text{"}big\text{"}$$

t Table

cum. prob	$t_{.90}$	$t_{.95}$	$t_{.975}$
one-tail	0.10	0.05	0.025
two-tails	0.20	0.10	0.05
df			
1	3.078	6.314	12.71
2	1.886	2.920	4.303
100	1.290	1.660	1.984
1000	1.282	1.646	1.962
z	1.282	1.645	1.960
	80%	90%	95%

$$U_c = t_{95}(v_{eff}) \times u_c = 1.96 \times .2078 = .4074\text{ms}$$

One Step at a Time: Tau Estimation



- How do we calculate Tau?

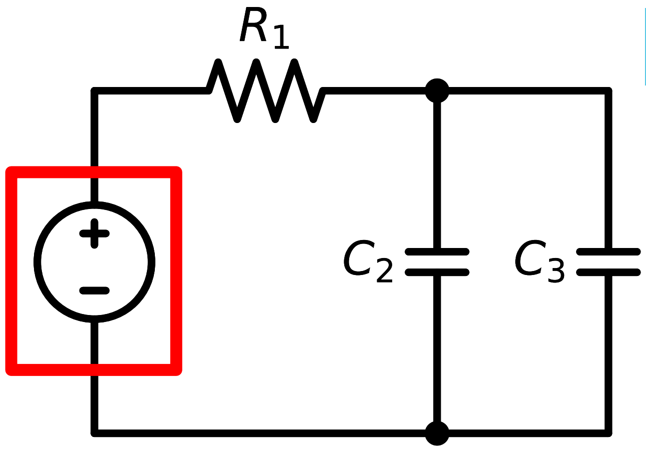
$$\tau = R_1(C_2 + C_3)$$

- What are our best estimates for the Xi's?

$$\tau = \bar{R}_1(\bar{C}_2 + \bar{C}_3)$$

$$= 32.2014\,(.1 + .22) = 10.305ms$$

One Step at a Time: Report Answer



- How do we report?

$$\tau = 10.305\,ms \;\pm 0.4074\,ms$$
$$\tau = 10.30\,ms \;\pm 0.41\,ms\ (k = 1.96, 95\%)$$

Cautions When Applying the GUM

- With correlated input quantities, the combined standard uncertainty must be adjusted
- If the standard uncertainties are large relative to the means, the GUM may underestimate the uncertainty
- If the distribution of the measurand is asymmetric, the t-distribution may not be appropriate
- If the measurement equation is highly non-linear, additional terms may be needed to approximate the combined standard uncertainty
- See Appendix slides and GUM for ways to handle these cases analytically.

Section 6.2: Monte Carlo Approach

Monte Carlo Approach

Used when conditions for applying the GUM methodology may not be met

- Complexity of measurement equation
- Input quantities having large uncertainties
- t-distribution not appropriate
  - Asymmetric PDF
  - Other non-normality

Provides a check on the applicability of the GUM approach

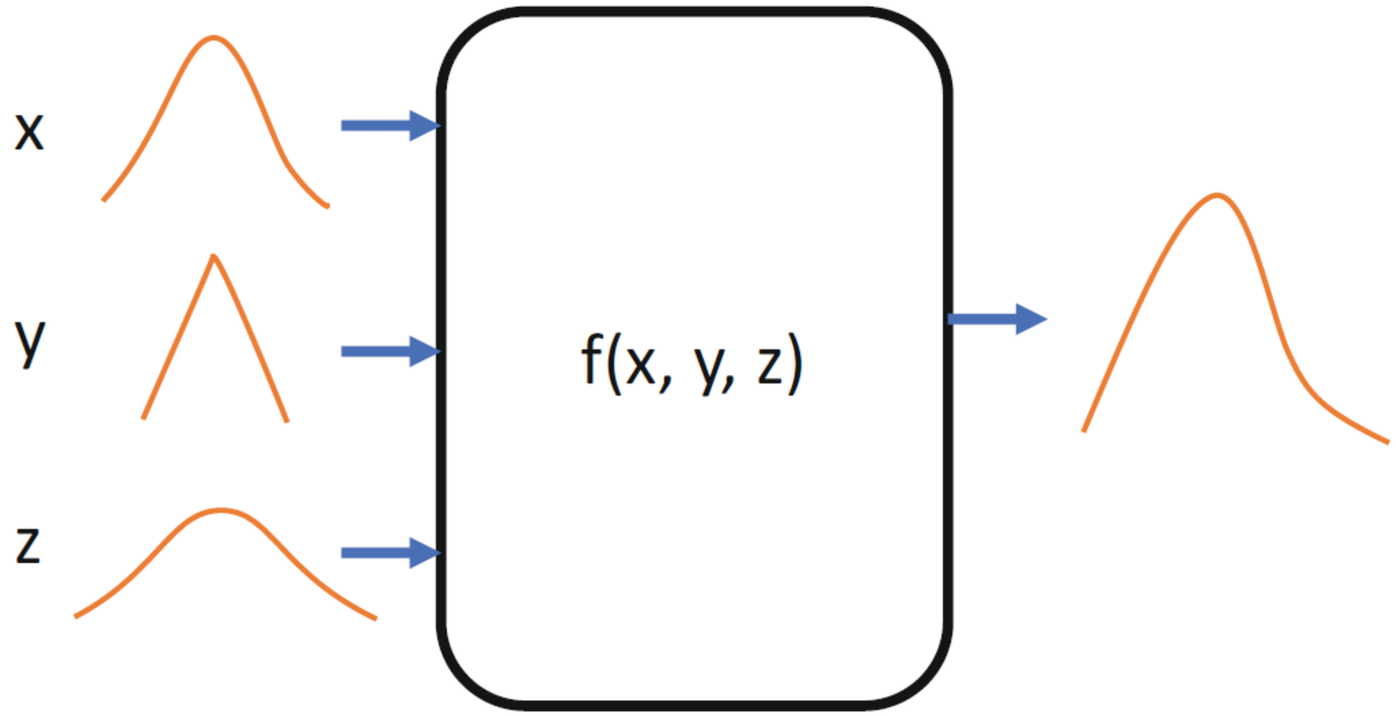
Monte Carlo Approach

Note: Start with a measurement equation and probability density functions (PDFs) for each input quantity (same as with GUM approach)

1. Using Monte Carlo simulation, "propagate" the PDFs for the input quantities through the measurement equation to obtain the PDF of the output quantity (measurand)
2. Compute the mean and standard deviation of the output quantity
3. Obtain an interval (the coverage interval) containing the unknown output quantity value with a specified probability (the coverage probability)

Propagation of Distributions

Illustration with three input quantities:



Monte Carlo Approach

- Select the number M of Monte Carlo trials to be made
  - (M = 10^6 recommended in the GUM Supplement)

- Generate M samples of the input quantities $x_1, x_2, \ldots., x_N$.

- For each sample (i, i = 1 to M), evaluate the measurement equation to give the corresponding output quantity

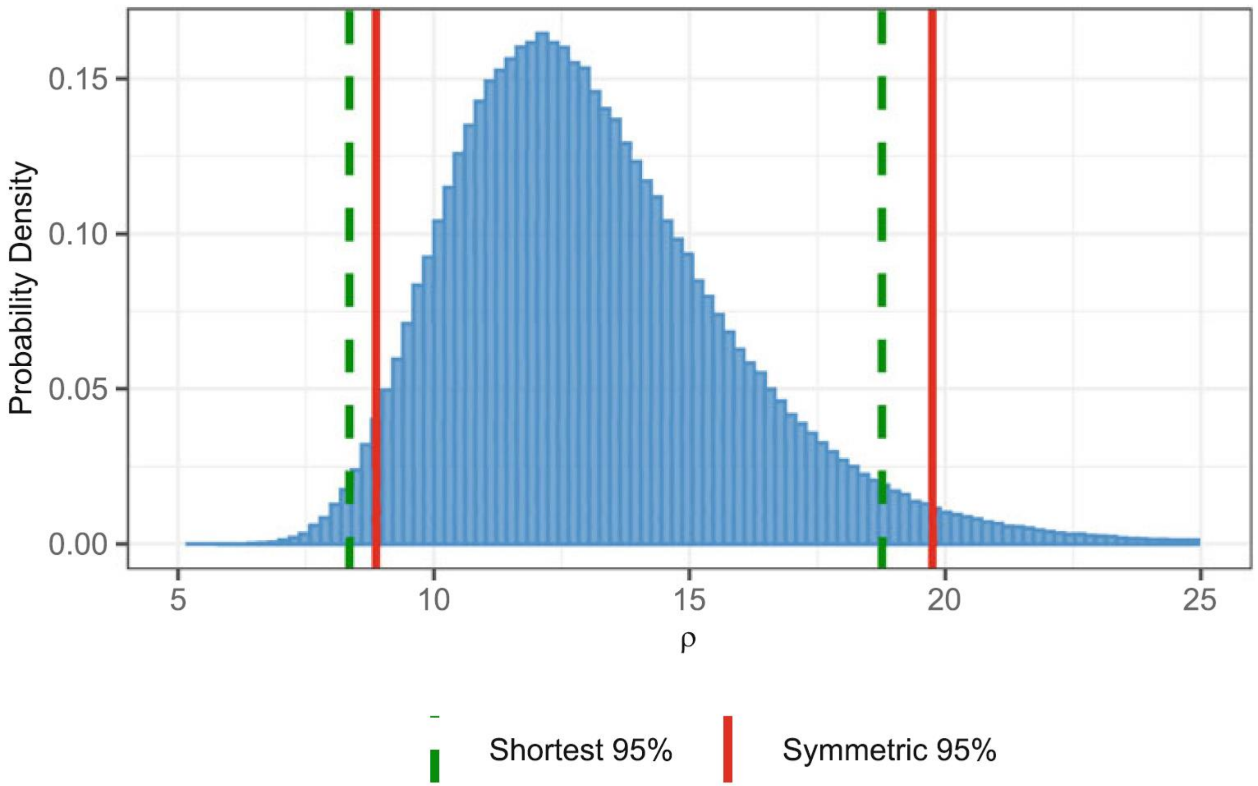
  - $y_i = f\,(x_{1i}, x_{2i}, \ldots., x_{Ni})$

- Plot a histogram of the data

Monte Carlo Approach

Sort the values $(y_1, y_2, \ldots., y_M)$ in increasing order

Form a 95% coverage interval for the output quantity from these sorted values



Example

- x_1 Normal (0,1) distribution
- x_2 Triangular (-1, 0, 1) distribution
- x_3 Uniform (-1,1) distribution

- Measurement equation is: $y = x_1^2 + x_2^2 + x_3^2$

- Compute a 95% coverage interval for y

Question: Suppose the “true” value for y really is 0. Does the Monte-Carlo coverage interval contain the “true” value?

Practice Problem

- x_1 Triangular (-2, 0, 2) distribution
- x_2 Uniform (-2,1) distribution

- Measurement equation is: $y = x_1^2 + x_2^2$

- Compute a 95% coverage interval for y

Steps to Determine Uncertainty

1. List the sources of uncertainty
2. Formulate the measurement equation, $y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$
3. Determine the distributions for the input quantities, X_1, X_2, Y_u, and evaluate the standard uncertainties for each input quantity. Type A evaluations are used for X_1, X_2, Y_u
4. Using Monte Carlo simulation, "propagate" the PDFs for the input quantities through the measurement equation to obtain the PDF of the output quantity (measurand)
5. Compute the mean and standard deviation of the output quantity
6. Obtain an interval (the coverage interval) containing the unknown output quantity value with a specified probability (the coverage probability)
7. Plot a histogram and report the coverage interval

Instructor Demo: Film Thickness Standard

Measurement Equation

$$y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$$

y_{cor} = Corrected measurement of film thickness check standard

Y_u = SNL's profilometer measurement of check standard

X_1 = VLSI calibrated measurement of Surface Height Standard (SHS)

X_2 = SNL's profilometer measurement of SHS

Distributions for Input Quantities

Input Quantity	Mean (μ)	Standard Uncertainty Type A	Degrees of Freedom	PDF
Y_u	0.6978 μm	0.0026 μm	19	Normal
X_1	0.1820 μm	0.00093 μm	9	Normal
X_2	0.1823 μm	0.00058 μm	19	Normal

Measurement equation is $y_{cor} = \left(\frac{X_1}{X_2}\right) Y_u$

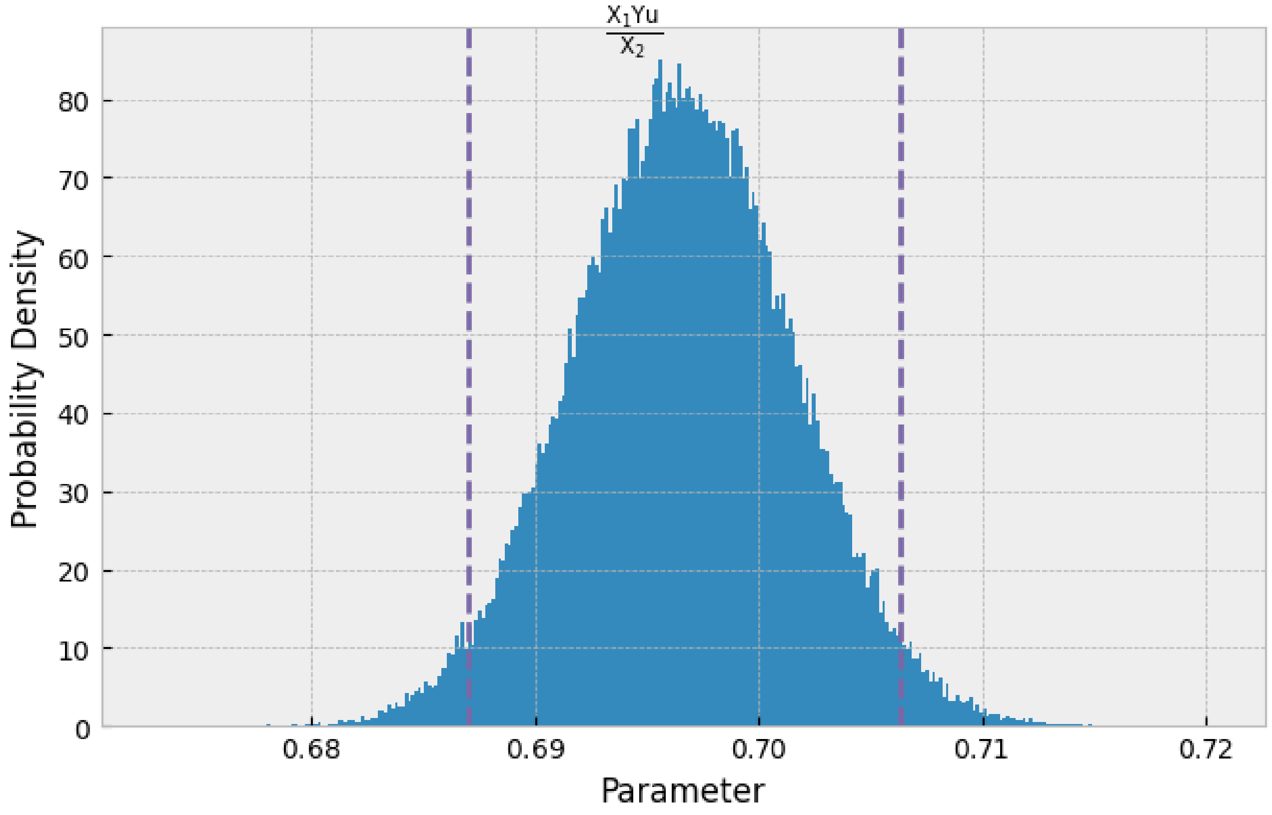
Monte Carlo Approach

- Use M = 10^5 Monte Carlo trials
- Using Distribution Explorer of SNL Uncertainty Calculator:
  - Generate M = 10^5 samples of the input quantities $X_1, X_2,$ and Y_u,
  - For each sample i, i=1 to 10^5, evaluate the measurement equation:

$$y_{cor,i} = \left(\frac{X_{1,i}}{X_{2,i}}\right) Y_{u,i}$$

  - Compute the mean and standard deviation of the output quantity
  - Determine a 95% coverage interval for the output quantity from these sorted values
  - Plot a histogram of the data and report the coverage interval

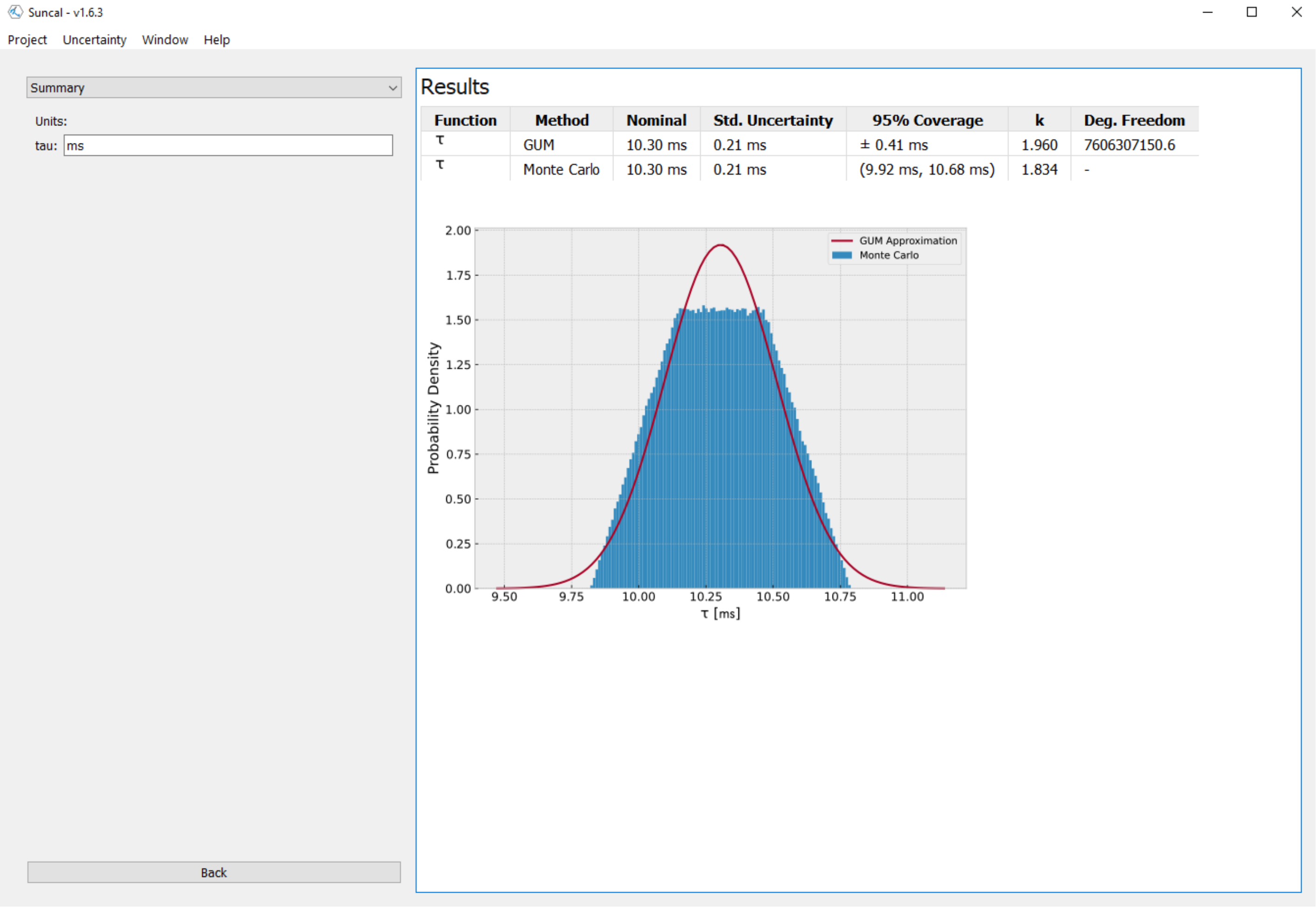
Monte Carlo Solution to Film Thickness Standard Problem



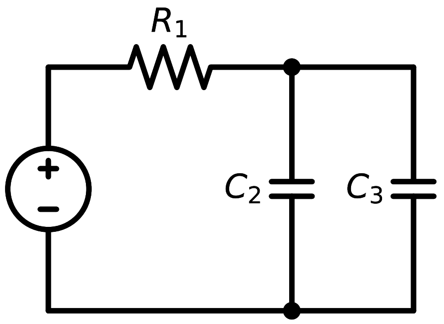
The coverage interval is (0.687, 0.706)

Demo: Sandia's Uncertainty Calculator

See Appendix A



Practice: Monte Carlo Approach



The time constant of an RC circuit is given by

$$\tau = R_1(C_2 + C_3)$$

Capacitor values are given by the manufacturer:

- **$C_2 = 0.10\ \mu F \pm 0.005\ \mu F$**
- **$C_3 = 0.22\ \mu F \pm 0.01\ \mu F$**

The resistor value is unknown, so it was determined by a set of measurements obtained using a Fluke 8845A on the 100 kΩ range (right). The multimeter's specification given in a calibration certificate as:

- **±(0.01% reading + 0.001% of range) at 95% level of confidence**

What is the uncertainty in τ?

Measurement	Resistance (kΩ)
1	32.2050
2	32.1878
3	32.2081
4	32.2201
5	32.1807
6	32.1990
7	32.1965
8	32.2120
9	32.1940
10	32.2108
Average	32.2014
Std. Uncert.	0.0038

GUM Approach versus Monte Carlo Approach

GUM Formula	Monte Carlo Approach
Approximation based on first-order Taylor Series expansion	Approximation based on generating random samples from probability distributions
Assumes the estimate y is normally distributed	No assumption regarding the distribution of y
Uses T-distribution, degrees of freedom, and W-S approximation to account for limited information	Uses original probability distributions and 95% coverage region
Does not perform well for highly non-linear models	Handles non-linear measurement models
Provides a single, reusable expression for uncertainty of a model	Must be re-run for any change in input values
Results in a single plus-or-minus value for uncertainty	Results in a histogram describing the uncertainty distribution
Model must be continuous/differentiable around the input values	Works with complex models that are not readily differentiable symbolically

Summary & Wrap-Up

Participants will be able to:

- identify sources of uncertainty in a measurement
- define a measurement equation
- perform basic statistical calculations necessary for uncertainty evaluation
- evaluate type A and type B uncertainty contributions to a measurement
- distinguish direct and indirect measurement models
- calculate combined uncertainty of a measurement using the GUM approach
- calculate combined uncertainty of a measurement using the Monte Carlo approach
- have familiarity with the GUM terminology and uncertainty evaluation methods

For further assistance, contact the PSL Help Line: (505) 845-8855, pslhelp@sandia.gov

Acknowledgements

Course Developers:

- Steve Crowder
- Tom Wunsch

Contributors:

- Harold Parks (now at NRC)
- Stefan Cular (now at NIST)
- Dan Campbell
- Eric Forrest
- Collin Delker
- Nevin Martin
- Reese Davies

References

NSE, Sandia Stats and PSL:

- Crowder, S., Delker, C., Forrest, E., Martin, N. (2020). Introduction to Statistics in Metrology. Springer.
- ENGR200, Good Measurement Practices Course, TEDS
- Metrology Handbook for the Nuclear Security Enterprise. SAND2020-4666 (OUO).

- NW complex metrology policies are in the D&P manual chapter 13.2 and the Primary Standards Memorandum which is an attachment to chapter 13.2, available at: https://sandialabs.sharepoint.com/sites/EXT_DPBPS/SitePages/Defense%20Programs%20Legacy%20Content.aspx

External:

- GUM, GUM supplements, and VIM, available at: http://www.bipm.org/en/publications/guides/
- NIST Technical Note 1297 (A publication from NIST that gives an overview of the GUM approach), available at: http://physics.nist.gov/Pubs/guidelines/contents.html
- NIST/SEMATECH, Online Engineering Statistics Handbook: https://www.itl.nist.gov/div898/handbook/
- SI unit definitions are in the SI Brochure, available at: http://www.bipm.org/en/si/si_brochure/
- Dieck, R. H., Measurement Uncertainty methods and applications.
- Rabinovich, Semyon G., Measurement Errors and Uncertainties, American Institute of Physics, NY.

Appendix A: Sandia UNcertainty CALculator

- SRN: http://tiny.sandia.gov/uncert
- External: https://sandiapsl.github.io
  - Desktop version (Windows, Mac)
  - Web 'lite' version (online only)

Suncal Features

Given: Measurement model, input values and uncertainties

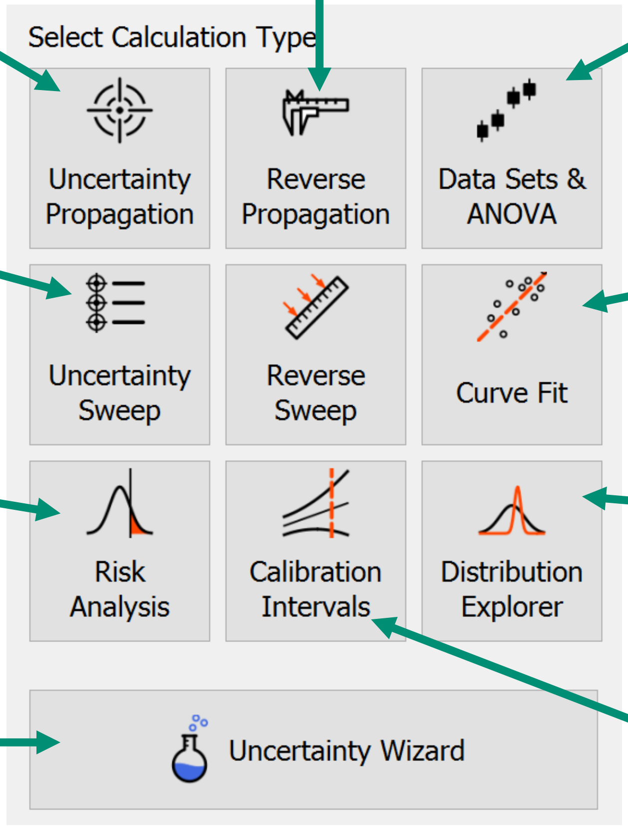
Use: Uncertainty Propagation

Given: Measurement model, output requirement, need input uncertainty

Use: Reverse Propagation

Given: 2-Dimensional data (e.g. N measurements × M days)

Use: Data Sets and ANOVA



Given: Measurement model, inputs could take several values

Use: Sweeps

Given: (x, y) data, need relationship between x and y

Use: Curve Fit

Given: Probability Distributions and tolerance limits

Use: Risk Analysis

Given: Probability Distributions to visualize (mostly for this training)

Use: Distribution Explorer

Guided uncertainty propagation with helpful hints (new in version 1.6)

Given: Historical Calibration Data

Use: Calibration Intervals

Distribution Explorer – Manual Monte Carlo Simulations

Generating random numbers from a PDF

Descriptive Statistics

Histograms

Calculating functions on sampled values

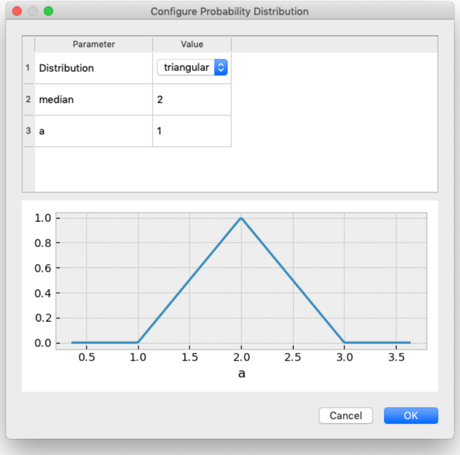
Fitting Distributions to results

Distribution Explorer

Generating Random Numbers

- Enter variable names

- Press “normal…” button to change distribution parameters

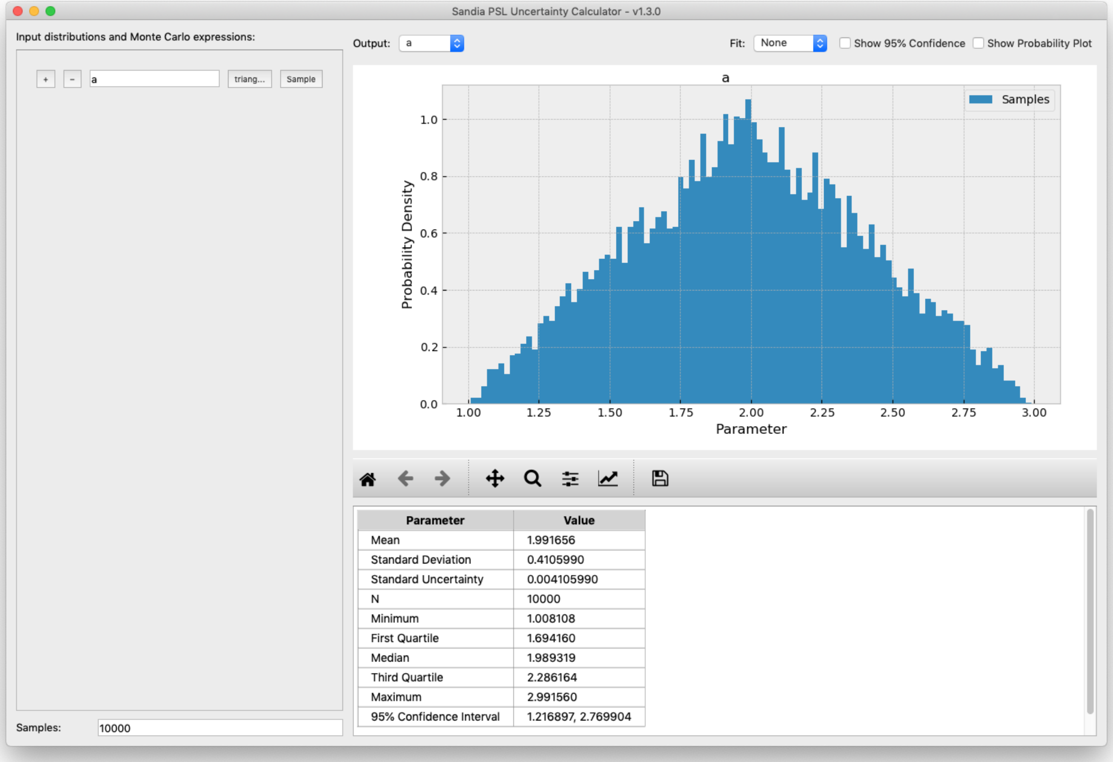


- Press “Sample” to draw random samples from distribution

Distribution Explorer

Histogram and Descriptive Statistics are shown

- Histograms divide sample values into many intervals called bins. Bars represent the number of observations falling within each bin (its frequency).

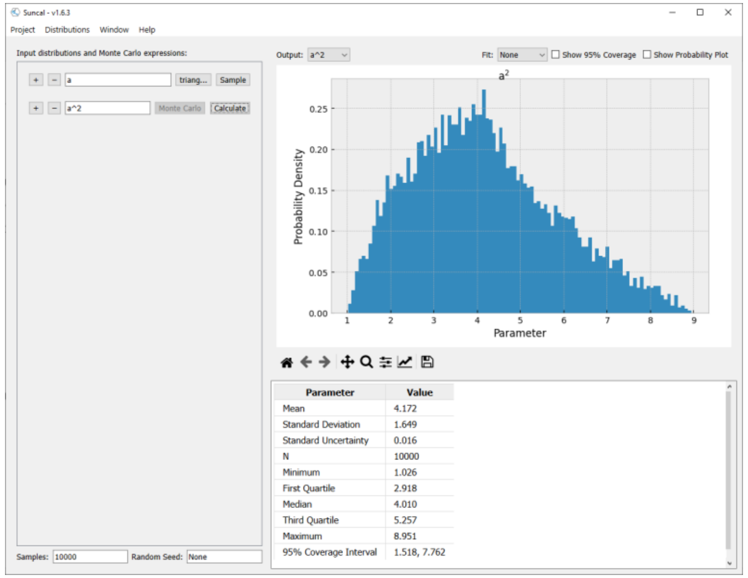


Distribution Explorer

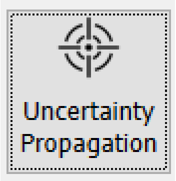
Calculator

- Add additional variables using the **+** button
- Expressions may be entered using previously defined variable names and basic math functions

- Press **calculate** to combine distributions and see the resulting histogram



Uncertainty Calculation

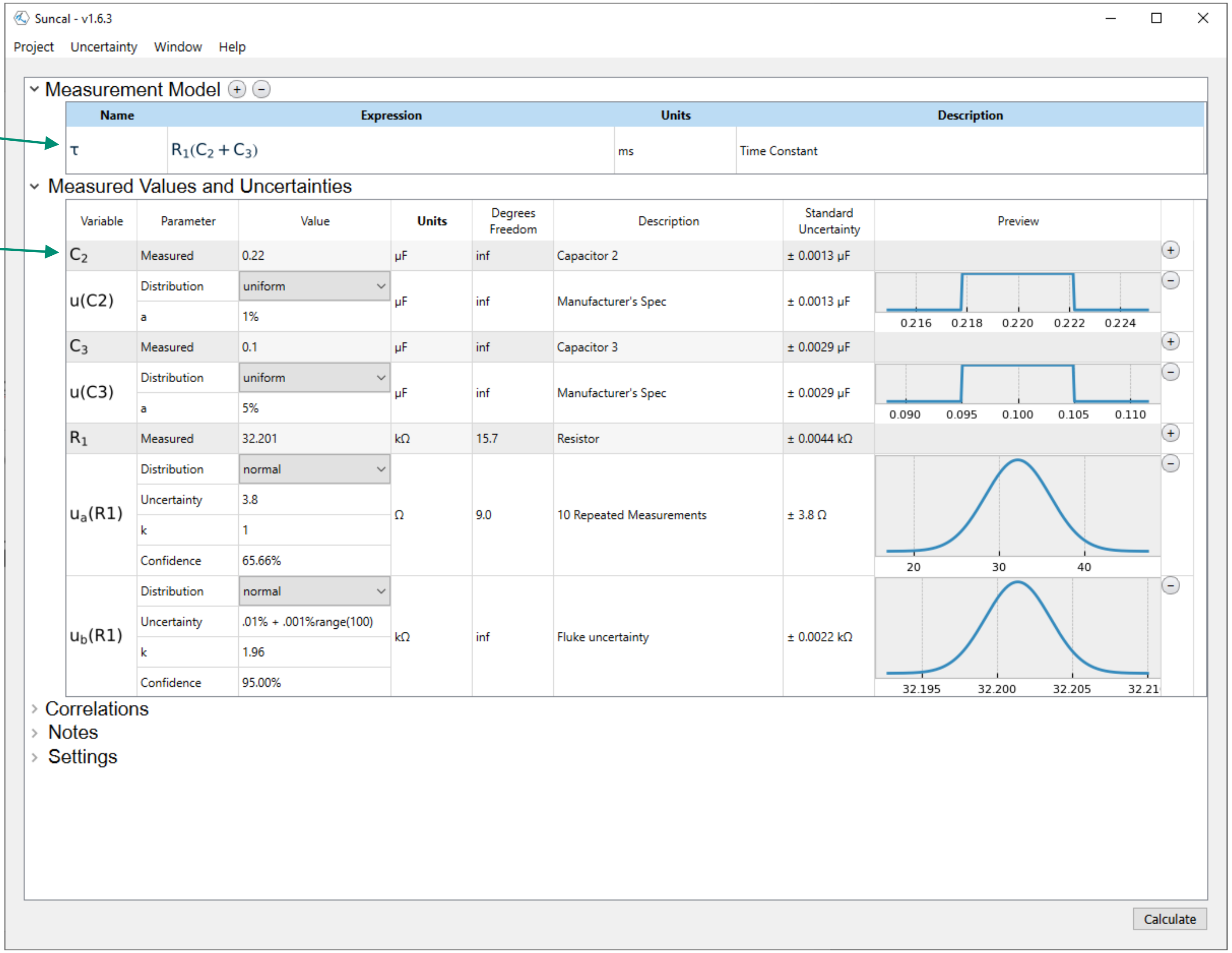


Calculate uncertainty by GUM and Monte Carlo Methods

Enter the Measurement Model

Variables found in the model are filled in for you.

Enter Measured Values and their Uncertainties



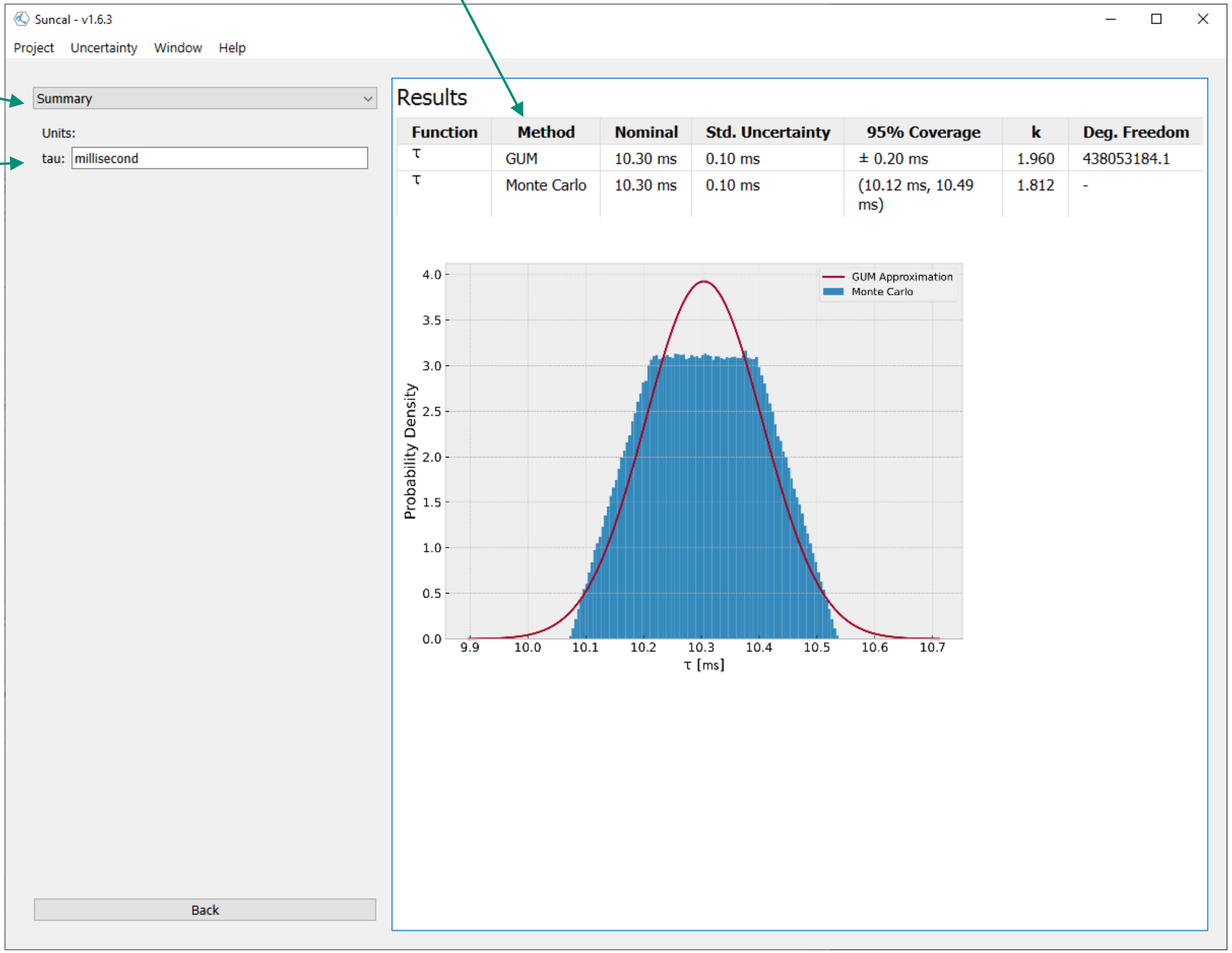
Enter units if desired. Blank units are treated as dimensionless.

Uncertainty Calculation – Results

Comparison of GUM and MC results

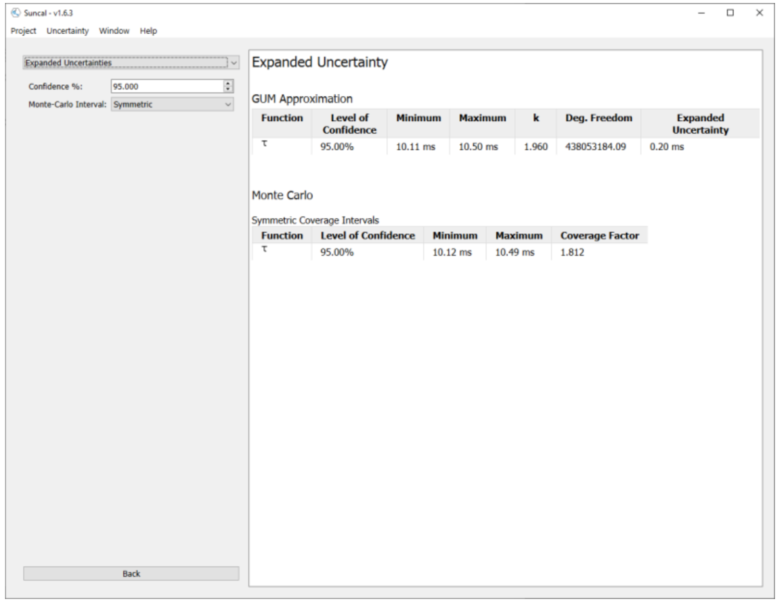
Select detailed reports to view

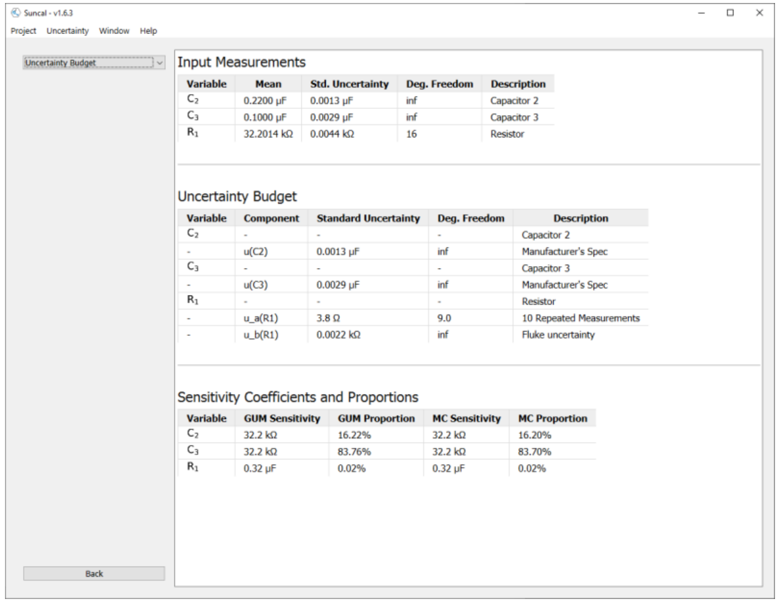
Change units here as desired

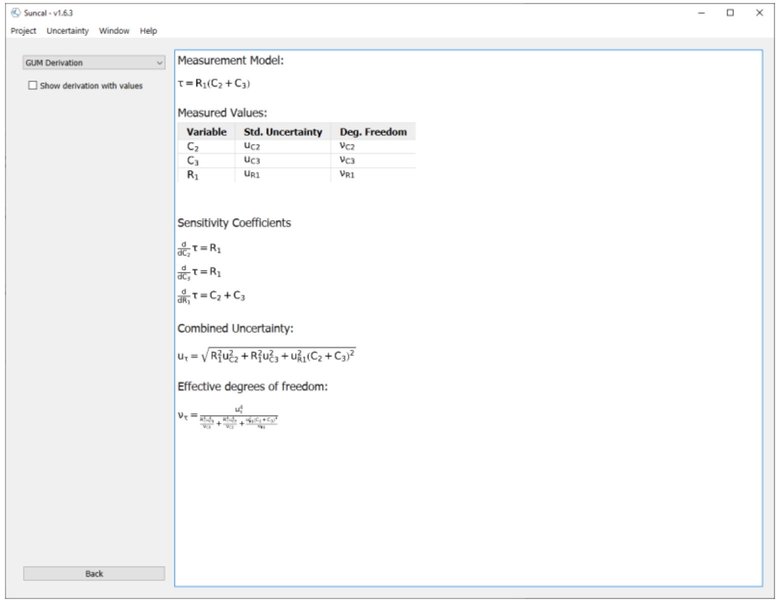


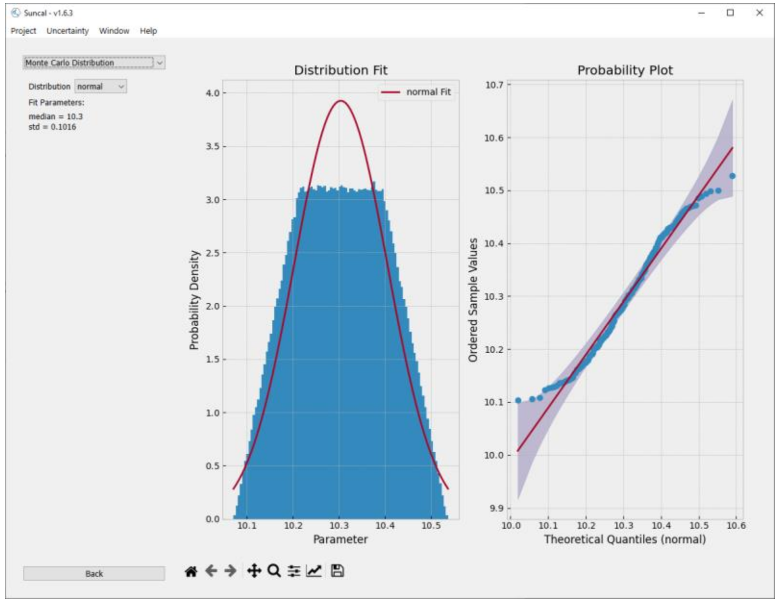
Function	Method	Nominal	Std. Uncertainty	95% Coverage	k	Deg. Freedom
τ	GUM	10.30 ms	0.10 ms	± 0.20 ms	1.960	438053184.1
τ	Monte Carlo	10.30 ms	0.10 ms	(10.12 ms, 10.49 ms)	1.812	-

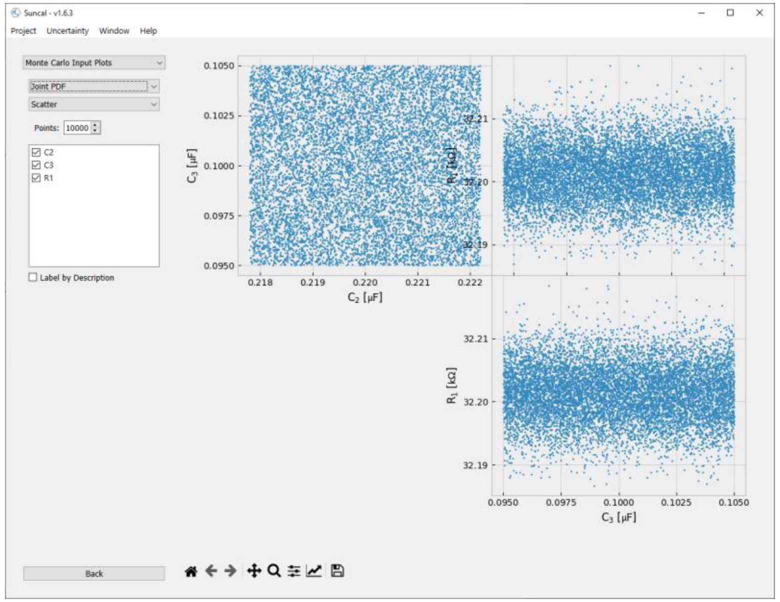
Uncertainty Calculation – Results

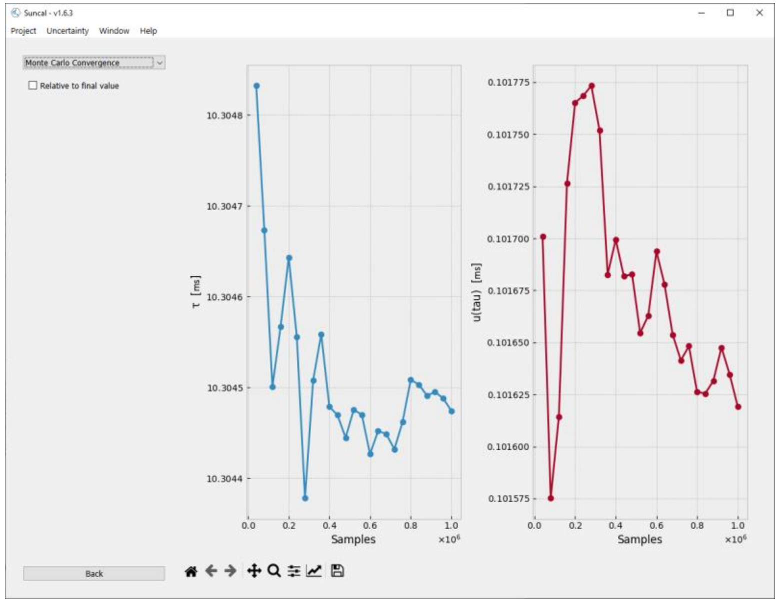












Appendix B: Supplementary Slides

GUM basis

Properties of the Variance Operator (X is random variable, a is constant):

- $Var(X + a) = Var(X)$
- $Var(ax) = a^2 Var(X)$
- $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$

For measurement models involving combinations of the above, i.e. "linear" models in the form:

$$f(X_1, X_2, \ldots X_n) = \sum_{i=1}^{N} a_i X_i + a_k,$$

the GUM equation provides an exact calculation of variance.

In all other cases, the GUM provides an approximation based on a Taylor Series Expansion.

GUM basis – Taylor Series Expansion

First order Taylor Series Expansion of the measurement model:

$$y = f(x_1, x_2, \dots, x_k) \cong f(\mu_1, \mu_2, \dots, \mu_k) + \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(x_i - \mu_i)$$

Assuming independence, the Variance operator gives:

$$u_c^2(y) = Var(y) = \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}\right)^2 Var(x_i - \mu_i)$$

$$= \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)$$

GUM basis – with correlated inputs

When the input quantities are correlated, the variance of the first-order Taylor Series Expansion is

$$u_c^2(y) = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i, x_j)$$

The first sum represents the contribution of the standard uncertainties and the second sum represents the contribution of the covariance terms.

Convert between correlation $r(x_i, x_j)$ and covariance $u(x_i, x_j)$:

$$r(x_i, x_j) = \frac{u(x_i, x_j)}{u(x_i)u(x_j)}$$

GUM basis – Nonlinear models, higher-order terms

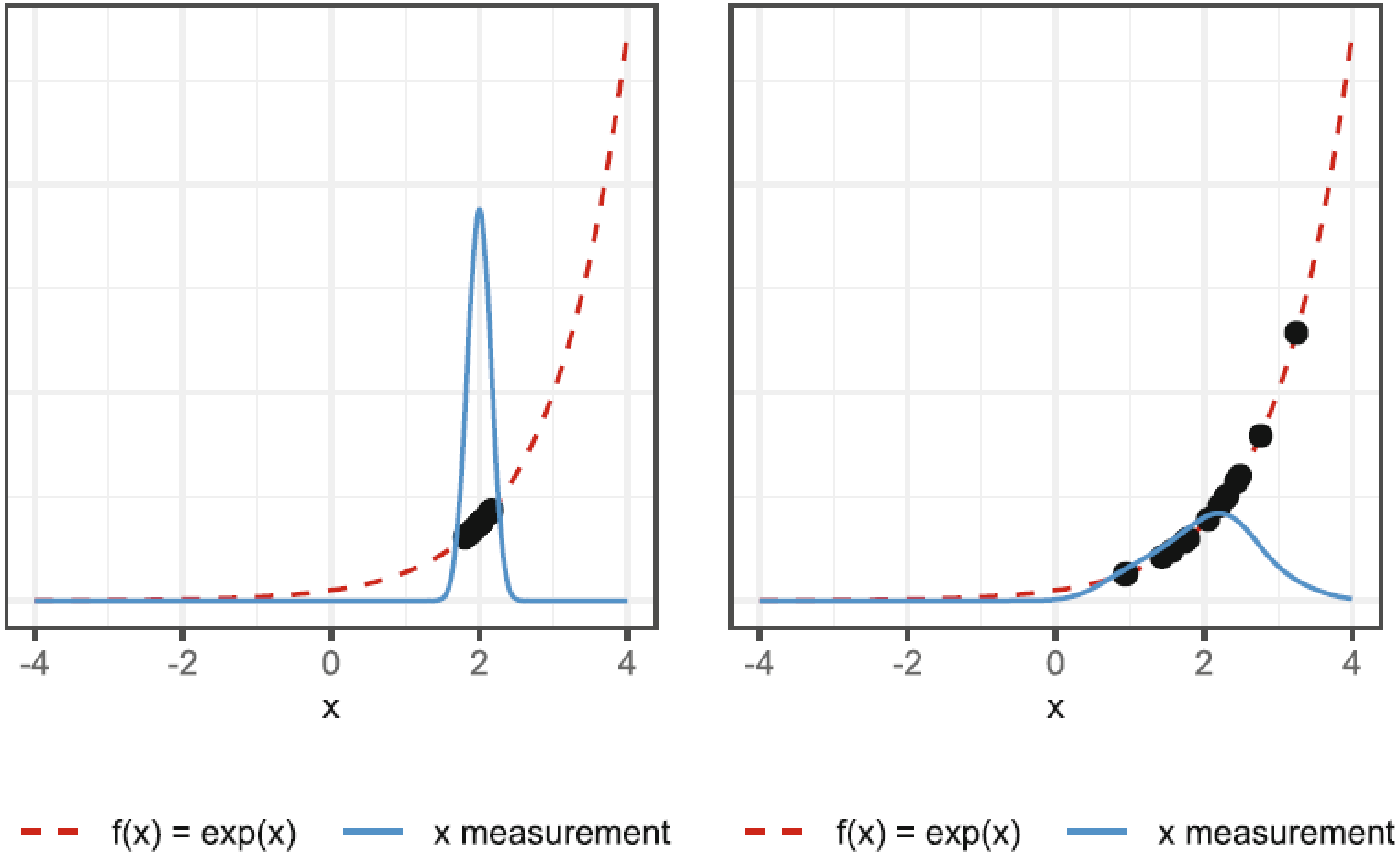
When the nonlinearity of f is significant, higher order terms in the Taylor series expansion must be included in the expression for the combined variance:

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 + \frac{\partial f}{\partial x_i} \frac{\partial^3 f}{\partial x_i \partial^2 x_j}\right) u^2(x_i) u^2(x_j)$$

The first sum represents the contribution of the standard uncertainties and the second sum represents the contribution of the higher order terms.

Correlations between variables is ignored. For non-linear models with correlated inputs, best solution is to use Monte Carlo approach.

GUM basis – Is the Model Linear?



Conditions for application of the GUM for linear models

For linear models, a coverage interval can be determined under the following conditions

- W-S formula is adequate for calculating ν_{eff}
- The X_i are independent
- The PDF for Y can be adequately approximated by a Gaussian or scaled and shifted t-distribution

Conditions for Valid Application of Welch-Satterthwaite

W-S formula may overestimate degrees of freedom when $\nu_B = \infty$ and the type B uncertainty is much larger than the type A uncertainty.

Instead of assuming $\nu_B = \infty$, GUM G.4.2 recommends approximating:

$$\nu_B \cong \frac{1}{2}\left[\frac{\Delta u_B(y)}{u_B(y)}\right]^{-2}$$

where the quantity in brackets is the relative type B uncertainty, a subjective quantity based on estimated confidence in the type B specification.

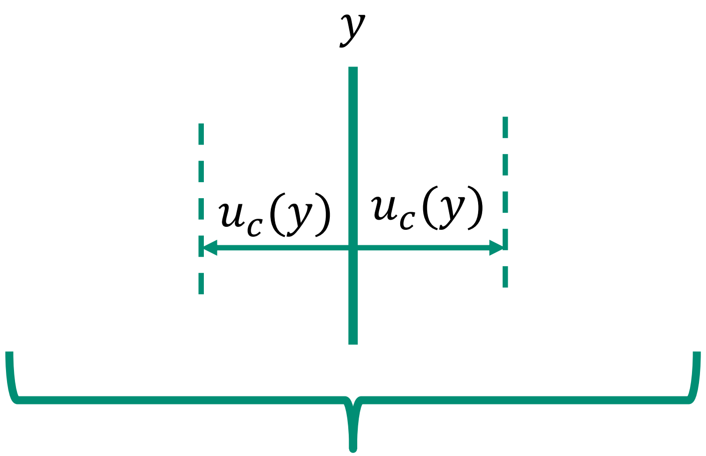
Student t distribution (GUM G.3)

What we know:

- y – best estimate of measurand
- $u_c(y)$ – standard deviation of the estimate

What we don't know:

- Y – the true value of the measurand



Where is Y?

Then the ratio $T = \frac{y-Y}{u_c(y)}$ has a t-distribution with ν_{eff} degrees of freedom.

An expanded uncertainty contains Y with p probability:

$$\Pr\left(-t_p(\nu_{eff}) \leq \frac{y-Y}{u_c(y)} \leq t_p(\nu_{eff})\right) = p$$

or $\Pr\left(y - t_p(\nu_{eff})u_c(y) \leq Y \leq y + t_p(\nu_{eff})u_c(y)\right) = p$

And $\boldsymbol{y \pm t_p(\nu_{eff})u_c(y)}$ is an uncertainty interval containing $\boldsymbol{Y}$ with confidence $\boldsymbol{p}$

Conditions for Valid Application of the GUM for Non-Linear Models

The law of propagation of uncertainty can validly be applied for non-linear models under the following conditions

- The function f is continuously differentiable with respect to the elements X_i
- This applies for all derivatives up to the appropriate order
- The X_i involved in significant higher-order terms of a Taylor series approximation to f are independent

A coverage interval can be determined under the following conditions

- The PDFs assigned to the X_i involved in higher order terms of a Taylor Series (TS) approximation are Gaussian
- Higher-order terms in the TS approximation to f are negligible if the law of propagation of uncertainty based on a first-order TS approximation is used
- W-S formula is adequate for calculating ν_{eff}
- The X_i are independent
- The PDF for y can be adequately approximated by a Gaussian or scaled and shifted t-distribution

Conditions for Valid Application of the Monte Carlo Method

- The function f is continuous with respect to the input quantities (the X_i's) in the neighborhood of the best estimates of the X_i's
- The cumulative distribution function for y is continuous and strictly increasing
- The PDF for y is
  - Continuous over the interval for which this PDF is positive
  - Unimodal (single-peaked) and strictly increasing (or zero) to the left of the mode and strictly decreasing (or zero) to the right of the mode
- $E(y)$ and $Var(y)$ exist
- A sufficiently large value of M is used and samples generated using a suitable pseudo-random number generator

Direct Measurements Using the Monte Carlo Method

Direct Measurement Model is:

- $y_i = Actual + Bias + \varepsilon_{A,i} + \varepsilon_B$
- $y_i = Actual + \varepsilon_{A,i} + \varepsilon_B$ ($Bias$ negligible)
- $i = 1, 2, \ldots, n$

$Actual$ is estimated by: $\bar{y}$ (sample average)

(This estimate will be adjusted if there is an estimate of systematic $Bias$)

Direct Measurements Using the Monte Carlo Method

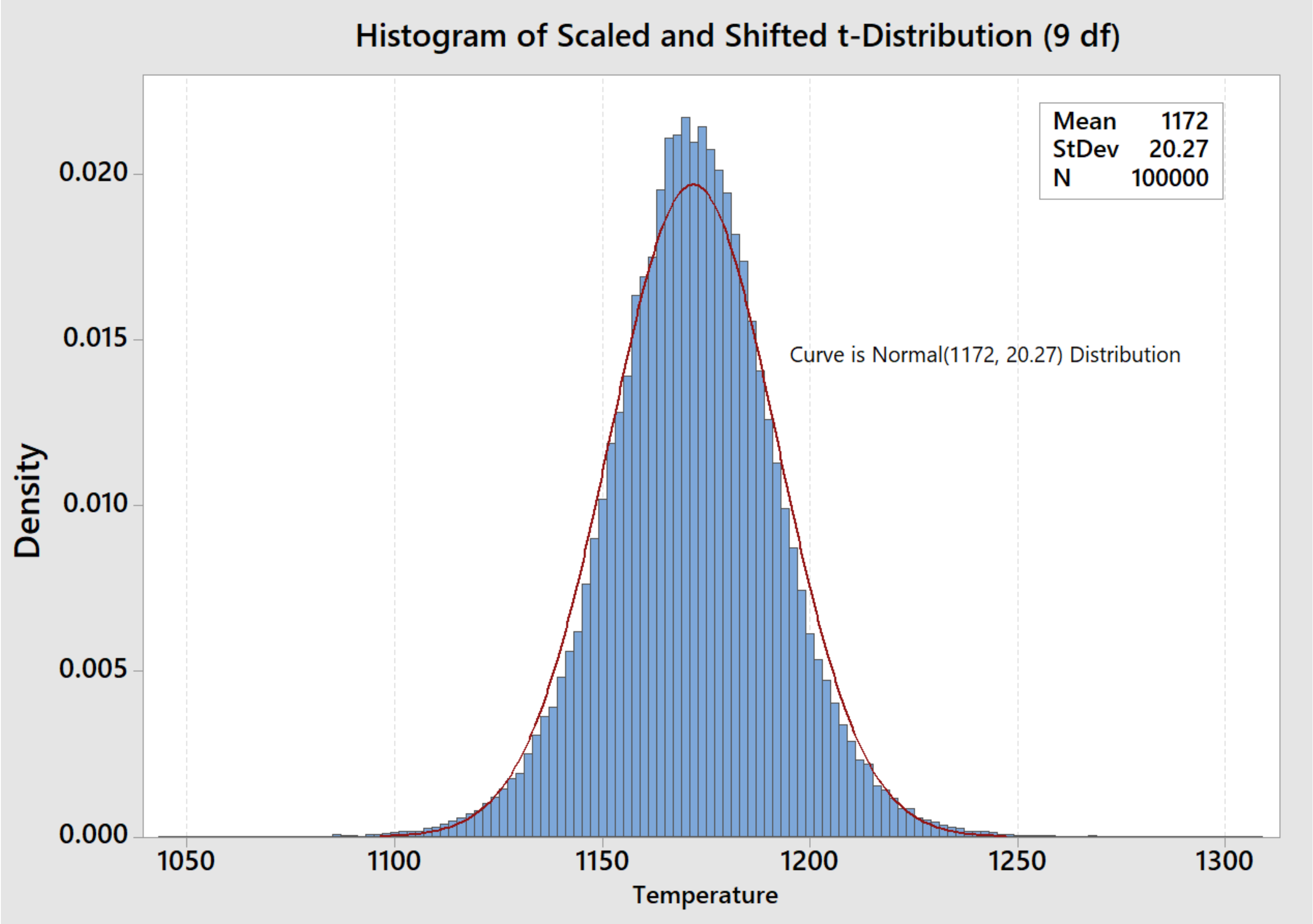
- Use M = 10^5 Monte Carlo trials
- Using Distribution Explorer of SNL Uncertainty Calculator:
  - Generate M = 10^5 samples of the input quantities
    - $Actual_i \sim N(\bar{y}, u_A(y))$ If degrees of freedom < 10, use t-distribution
    - $\varepsilon_{B,i} \sim Uniform(-a, a) \quad i = 1, \ldots 10^5$
  - For each sample i, i=1 to 10^5, evaluate the measurement equation: $y_i = Actual_i + \varepsilon_{B,i}$
  - Compute the mean and standard deviation of the output quantity
  - Determine a 95% coverage interval for the output quantity from these sorted values
  - Plot a histogram of the data and report the coverage interval

Lava Example Using Monte Carlo

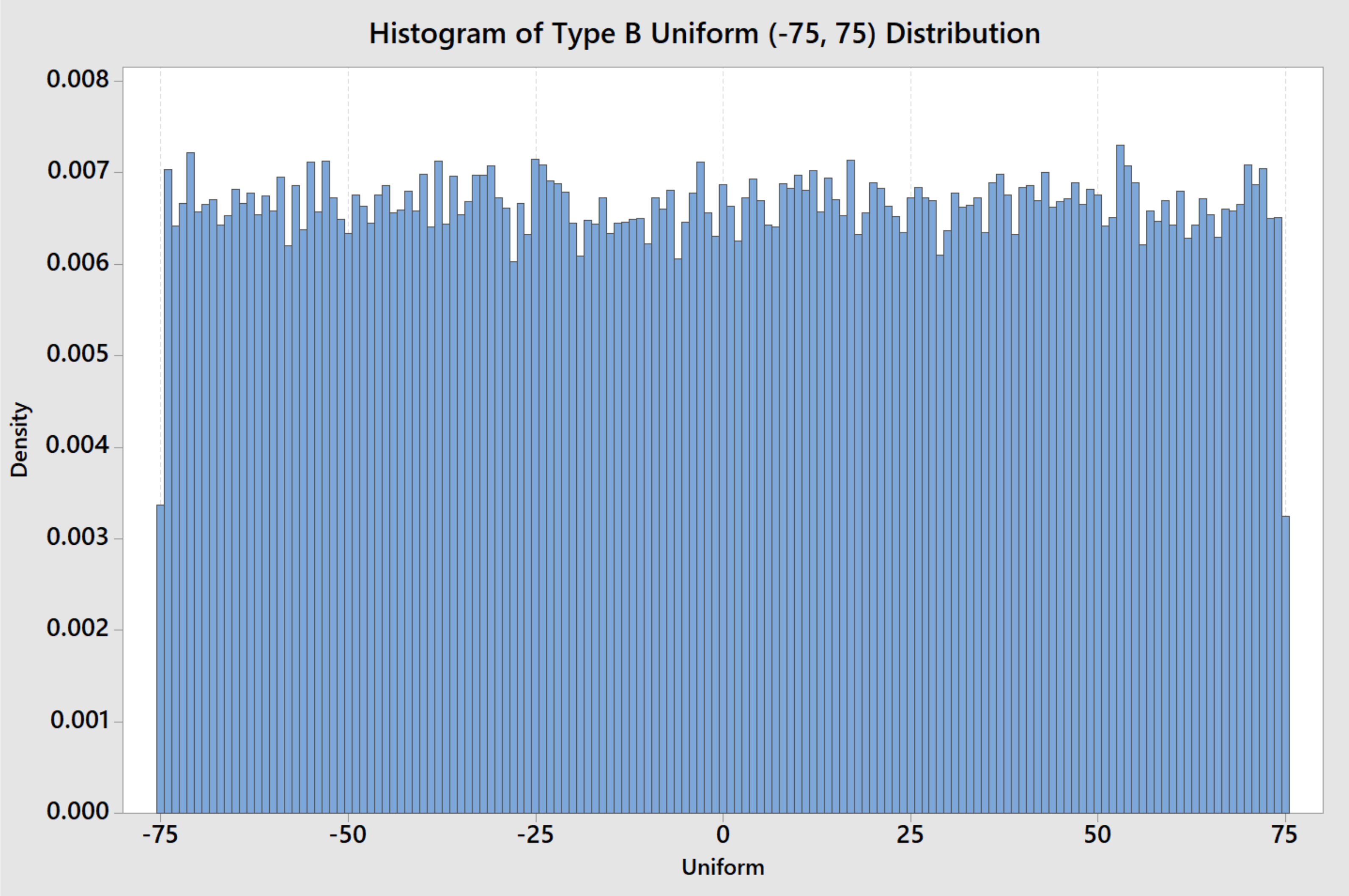
Generate for i=1, …, 10^5:

- $Actual_i \sim t(1171.76, 18.03),\ 9\ df$ (scaled and shifted t-distribution)
- $\varepsilon_{B,i} \sim Uniform(-75, 75)$
- $y_i = Actual_i + \varepsilon_{B,i}$

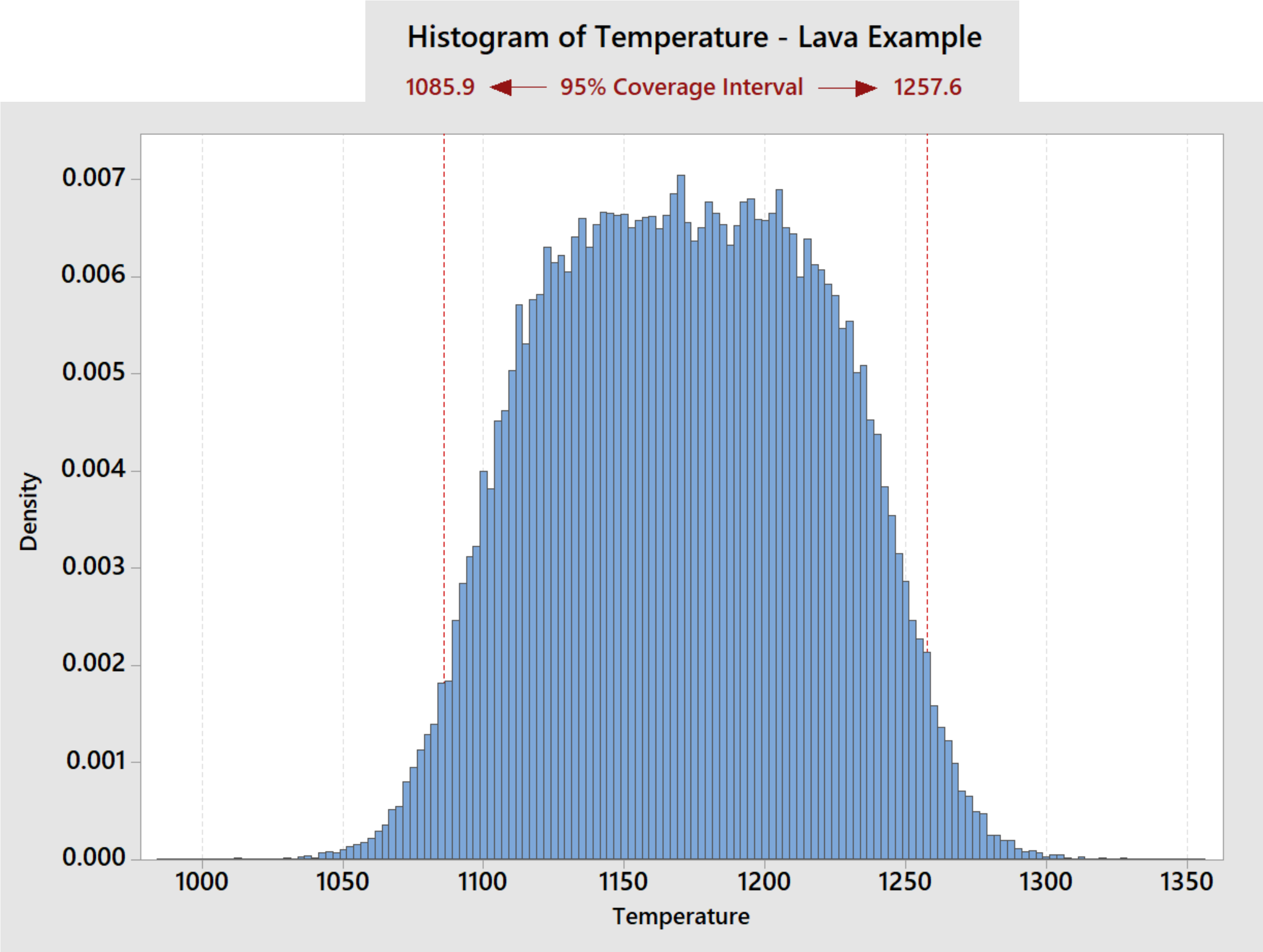
Lava Example Using Monte Carlo: Type A Scaled and Shifted t-Distribution



Lava Example Using Monte Carlo: Type B Uniform Distribution



Lava Example using Monte Carlo



Comparison of GUM and MC Results for the Lava Problem:

- GUM
  - T = (1078, 1266) °C at a 95% level of confidence
- Monte Carlo
  - T = (1086, 1258) °C at a 95% level of confidence

Monte Carlo estimates a 95% coverage interval that is about 10% shorter

Appendix C: Solutions to Example Problems

Lava Temperature Example

Type A (n = 10 measurements)

- Mean = 1171.76
- Standard Deviation = 57.01
- Standard Error of the mean = $57.01/\sqrt{10}$ = 18.03

Type B

- Type N thermocouple with tolerance ±75 °C
- Uniform distribution
- Standard Uncertainty: $\frac{75}{\sqrt{3}} = 43.3$ °C

Type B, instrument resolution

- Readout resolution of 0.01 °C
- Uniform distribution (0.01 is full width)
- Standard uncertainty: $\frac{(0.01/2)}{\sqrt{3}} = 0.0029$ °C

Lava Temperature Example

Combined standard uncertainty

- Neglect insignificant type B component
- $u_c(y) = \sqrt{u_A^2 + u_B^2} = \sqrt{18.03^2 + 43.3^2} = 46.90$ °C

Effective Degrees of Freedom

- $\nu_{eff} = \dfrac{u_c^4(y)}{\frac{u_A^4(y)}{\nu_A} + \frac{u_B^4(y)}{\nu_B}} = \dfrac{46.90^4}{\frac{18.03^4}{9} + \frac{43.3^4}{\infty}} = 412$

Expanded Uncertainty

- $k = 2.01$ (95.45% with $\nu_{eff} = 412$)
- $U = k \cdot u_c(y) = 94$ °C
- Result: $T = (1172 \pm 94)$ °C (k=2, 95.45%)

Welch-Satterthwaite Example

$u_A(y) = 0.57; \quad \nu_A = 9$

$u_B(y) = 0.25; \quad \nu_B = \infty$

$$u_c(y) = \sqrt{u_A^2(y) + u_B^2(y)} = \sqrt{0.57^2 + 0.25^2} = 0.622$$

$$\nu_{eff} = \frac{u_c^4(y)}{\frac{u_A^4(y)}{\nu_A} + \frac{u_B^4(y)}{\nu_B}} = \frac{0.622^4}{\frac{0.57^4}{9} + \frac{0.25^4}{\infty}} = 12.8 \rightarrow 12 \ \ \text{(round down)}$$

Electrical Circuit Problem

Resistance R1

- Type A: n = 10 measurements
  - Mean = 32.2014 kΩ
  - Standard Deviation = 0.0120 kΩ
  - Standard Error of the Mean = 0.0120 / √10 = 0.0038 kΩ
  - Type B: 0.01% reading + 0.001% range at 95% confidence
  - $U_B^{95} = (32.2014)(0.0001) + (100)(0.00001) = 0.00422$ kΩ
  - Standard uncertainty = $\frac{U_B^{95}}{1.96} = 0.00215$ kΩ
  - Combined: $u_R = \sqrt{u_A^2 + u_B^2} = \sqrt{0.0038^2 + 0.00215^2} = 0.00437$ kΩ

Capacitors (Type B)

- Uniform distributions
- $u_{C_2} = \frac{0.005}{\sqrt{3}} = 0.00288$ μF
- $u_{C_3} = \frac{0.01}{\sqrt{3}} = 0.00577$ μF

Electrical Circuit Problem

Sensitivity Coefficients

- $c_R = \frac{\partial \tau}{\partial R} = C_2 + C_3 = 0.32\ \mu\text{F}$
- $c_{C_2} = \frac{\partial \tau}{\partial C_2} = R_1 = 32.2\ \text{k}\Omega$
- $c_{C_3} = \frac{\partial \tau}{\partial C_3} = R_1 = 32.2\ \text{k}\Omega$

Combined Uncertainty

- $u_c = \sqrt{c_R^2 u_R^2 + c_{c_2}^2 u_{c_2}^2 + c_{c_3}^2 u_{c_3}^2} = 0.2078\ \text{ms}$
- $\tau = 10.304\ \text{ms}$

Degrees of Freedom

- Resistor: $\nu_{eff,R} = \frac{0.00436^4}{\frac{0.0038^4}{9} + \frac{0.0021^4}{\infty}}$ =15.31
- Combined: $\nu_{eff} = \frac{0.2078^4}{\frac{(0.32 \times 0.00434)^4}{15.31} + \frac{(32.2 \times 0.00288)^4}{\infty} + \frac{(32.2 \times 0.00577)^4}{\infty}} = 7.5 \times 10^9 =$ “big”
- k = 1.96 (95%)

Result:

- $\tau = 10.304\ \text{ms}\ \pm 0.4074\ \text{ms}$
- $\tau = 10.30\ \text{ms}\ \pm 0.41\ \text{ms}\ (\text{k} = 1.96, 95\%)$

Appendix D: Uncertainty Quick Reference