Stochastic convergence:

(i) $X_n \to X$ a.s. or with probability 1.

$\Leftrightarrow P(X_n \to X) = 1.$

(ii) $X_n \to X$ in q-th mean ($q \geq 1$)

if $E[|X_n - X|^q] \to 0.$

(iii) $X_n \to X$ in probability if

$P(|X_n - X| \geq \epsilon) \to 0 \;\; \forall \epsilon > 0.$

(iv) $X_n \to X$ in law (in distribution)

if $E[f(X_n)] \to E[f(X)]$ $\forall$ bounded continuous f.

$$\begin{matrix}(i) \\ (ii)\end{matrix} \Rightarrow (iii) \Rightarrow (iv).$$

(ii) $\Rightarrow$ (iii) $\quad P(|X_n - X| \geq \epsilon) \leq \dfrac{E[|X_n - X|^q]}{\epsilon^q} \to 0$

(Chebyshev)

Remarks: (1) (ii) needs q-th abs moments to exist (i.e. $E(|X_n|^q) < \infty$)

(2) (ii), (iii) depend on pair laws of

$\cancel{(iv)}$ (X_n, X), (iv) only on individual laws

$E[|X_n - X|^q]$ X, Y.

$P(|X_n - X| \geq \epsilon)$ $N(0,1)$

$N(0,1)$

(i) $\Rightarrow$ (iii) $\epsilon > 0$.

$$A_N = \{w : |X_n - X| \geq \epsilon \text{ for some } n \geq N\}$$

$$= \bigcup_{n \geq N} \{w : |X_n(w) - X(w)| \geq \epsilon\}$$

$A_{N+1} \subset A_N$, $\quad P(\bigcap_N A_N) = 0$.

$$\bigcap_N A_N = \bigcap_N \bigcup_{n \geq N} \{w : |X_n - X| \geq \epsilon\}$$

$$= \{w : |X_n - X| \geq \epsilon \text{ i.o.}\}$$ [infinetly often]

$$P(\bigcap_N A_N) \leq P(\{X_N \to X\}^c) = 0$$

X_n

X

Let $\delta > 0$, $\lim_{M \uparrow \infty} P(\bigcap_{N=1}^{M} A_N) = 0$

$\Rightarrow \exists N_0$ s.t. $P(\bigcap_{N=1}^{N_0} A_N) = P(A_{N_0}) < \delta$.

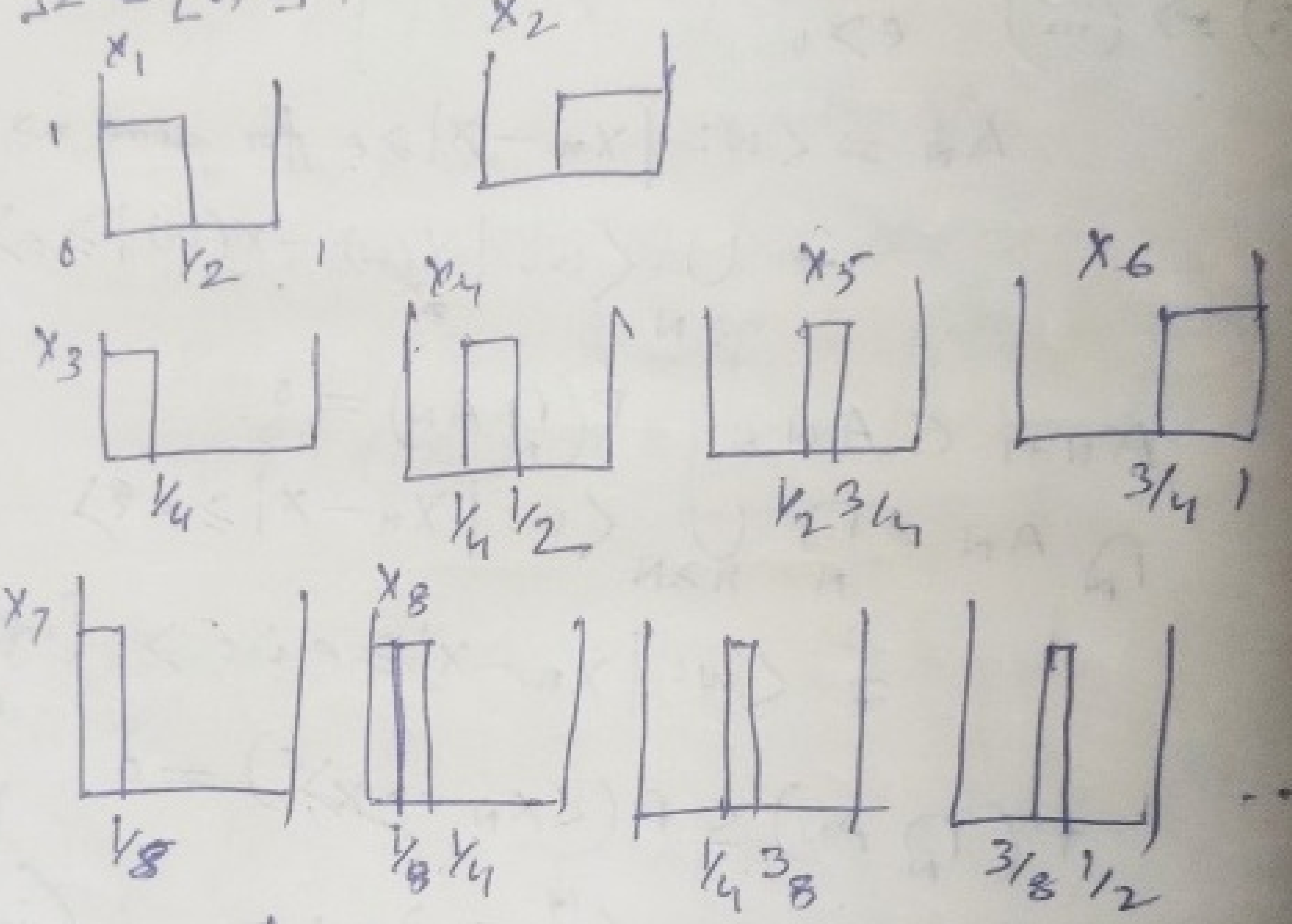
$P(\bigcup_{n \geq N_0} \{|X_n - X| \geq \epsilon\}) < \delta \Rightarrow$

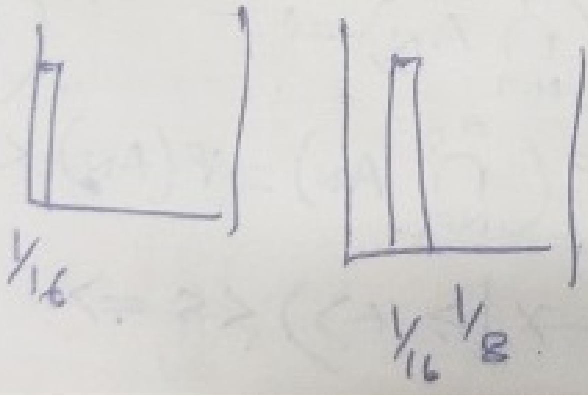
$\Rightarrow P(|X_n - X| \geq \epsilon) < \delta$

$\Rightarrow P(|X_n - X| \geq \epsilon) \to 0$

(ii) $\not\Rightarrow$ (i), (iii) $\not\Rightarrow$ (i), (iii) $\not\Rightarrow$ (ii), (i) $\not\Rightarrow$ (ii)

$\Omega = [0, 1]$.





$P(|X_n| \geq \epsilon) = ?$

$X_n \to 0$ in prob.

$X_n \not\to 0$ a.s.

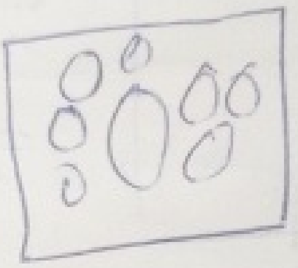
(Shrinking box)

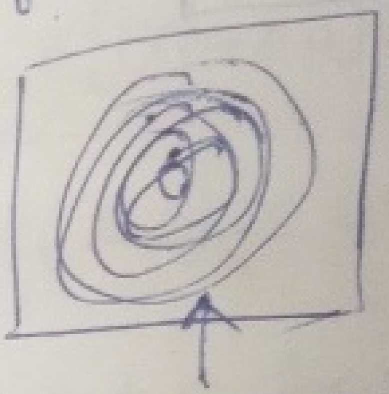
$P(|X_n| \geq \epsilon) = ?$ (take any width ϵ and the prob)

$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \dots$

At any ω, $X_n(\omega) = 1$, i.o.

$\therefore X_n(\omega) \not\to 0$





Convergence.

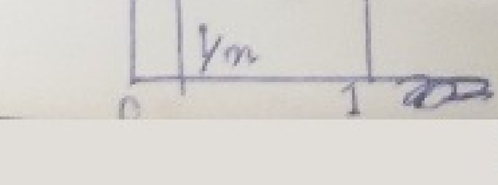
$\therefore$ (iii) $\not\Rightarrow$ (i) [No Relation]

?? (ii) $\not\Rightarrow$ (i) $E[X_n] \to 0$ $\therefore X_n \to 0$ in mean

$E[|X_n|^1] \to 0$.

(i) $\not\Rightarrow$ (ii) $X_n = n\, I_{[0, 1/n]}$

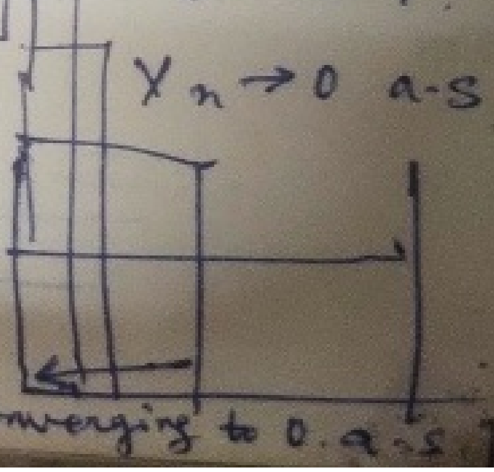
$E[X_n] = 1$



$E[X_n] = 1 \not\to 0$

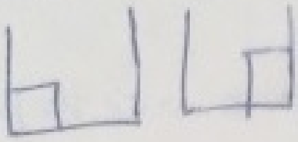
$X_n \to 0$ a.s.

$X_n \to 0$ a.s.

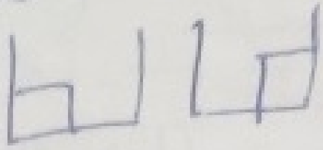


[converging to 0 a.s.]

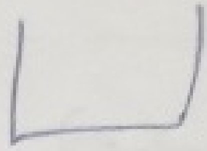
(ii) $\not\Rightarrow$ (i)



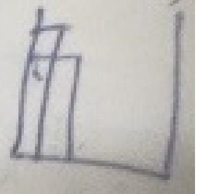
(iii) $\not\Rightarrow$ (i)

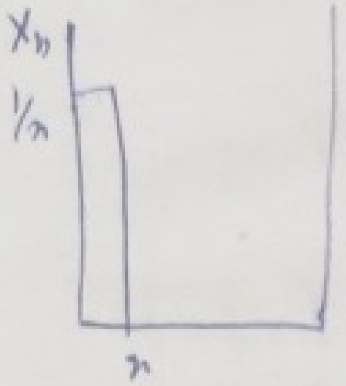


(iii) $\not\Rightarrow$ (ii)



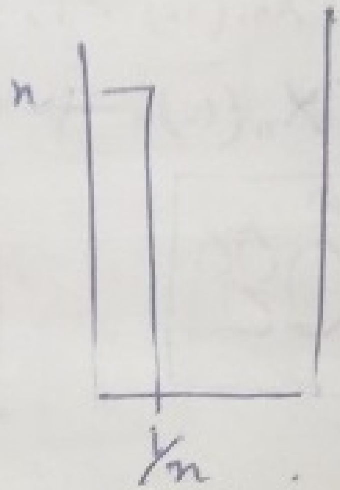
(i) $\not\Rightarrow$ (ii)





$X_n \to 0$ a.s. $\Rightarrow X_n \to 0$ in prob

$X_n \not\to 0$ in mean.



$$\overbrace{\lim_n E[X_n]}^{\to 1} \neq E[\underbrace{\lim_n X_n}_{\to 0}]$$

<u>Bounded Convergence theorem</u>

$X_n \to X$ a.s., $|X_n| \le K$ a.s. $\forall n \Rightarrow$

$E[|X_n - X|] \to 0$.

Note: $E[|X_n - X|] \to 0 \Rightarrow$

$E[X_n] \to E[X]$, because

$E[X_n] - E[X] \le E[|X_n - X|]$

(This is an integral)

$$E[|X_n - X|] = E[|X_n - X| I_{A_N}] + E[|X_n - X| I_{A_N^c}], \quad n \geq N$$

$A_N = \{\omega : |X_n - X| \geq \epsilon$ for some $n \geq N\}$

$$E[|X_n - X| I_{A_N^c}] \leq \epsilon . P(A_n^c) \leq \epsilon$$

$$E[|X_n - X| I_{A_N}] \leq E[|X_n| I_{A_N}] + E[|X| I_{A_N}] \quad \forall n \geq N$$

triangle inequality

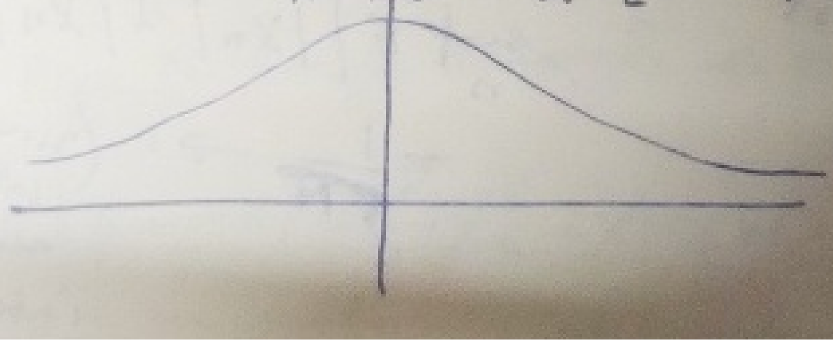
$$\leq K P(A_N) + K P(A_N) \leq 2K\delta .$$

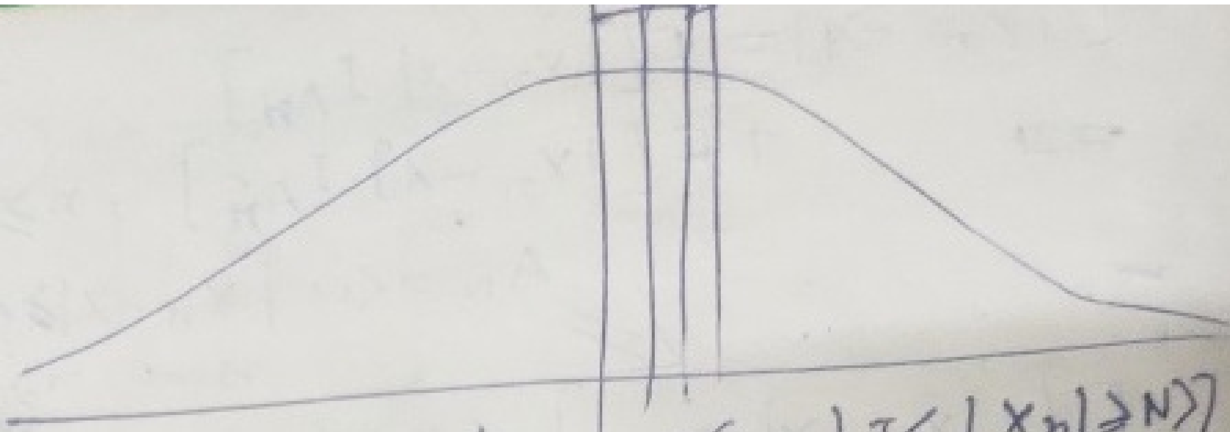
?

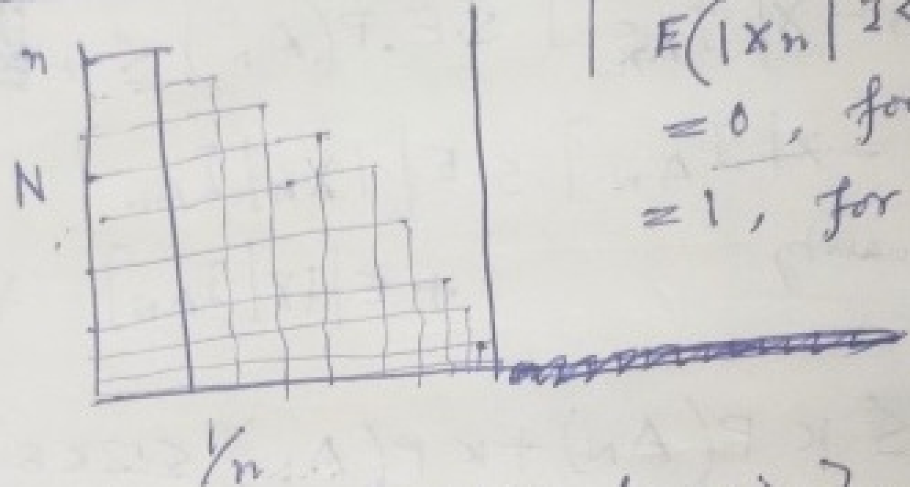
$$\therefore E[|X_n - X|] \leq 2K\delta + \epsilon \text{ for } n \geq N.$$

$$\Rightarrow E[|X_n - X|] \to 0$$

$\{X_\alpha, \alpha \in I\}$ is said to be uniformly integrable if $\lim_{N \to \infty} \sup_\alpha E[|X_\alpha| I\{|X_\alpha| \geq N\}] = 0$





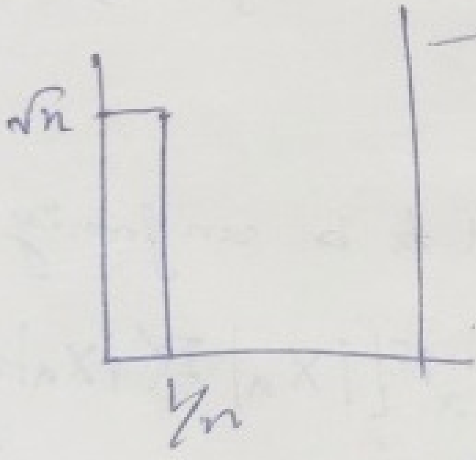


$E(|X_n|\, I\langle |X_n| \geq N\rangle]$

$= 0$, for $n \leq N$

$= 1$, for $n \geq N$??

$\sup_n E(|X_n|\, I\langle |X_n| \geq N\rangle] = 1 \not\to 0$



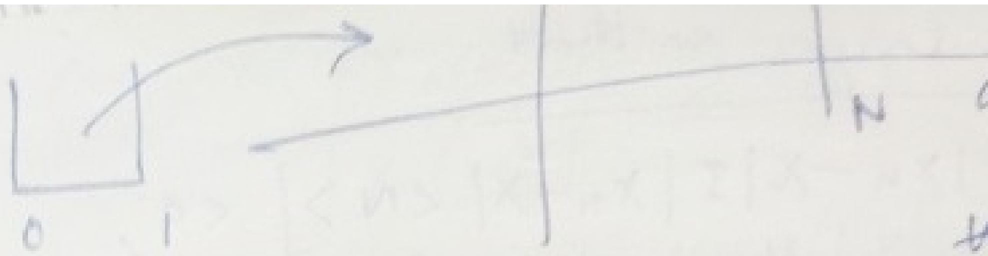
$E[\,|X_n|\, I\langle |X_n| \geq N]$

$= 0, \quad n < N^2$. (?)

$= \frac{1}{\sqrt{n}}, \quad n \geq N^2$.

$\sup_n E[\,|X_n|\, I\langle |X_n| \geq N\rangle]$

$= \frac{1}{N} \to 0$ (unbounded which are uniformly integrable)



after some N all the integrals are uniformly small.

$$\sup_n E\left(|X_n| \, I\langle |X_n| \geq N\rangle\right]$$

do this at the same.

Contribution of ~~to~~ the integrals to the tail part will be uniformly small, as we go farther & farther.

$\therefore X_n \to X$ a.s. $\langle X_n \rangle$ u.i. (uniformly integrable)

$\Rightarrow E\left[|X_n - X|\right) \to 0.$

$$E\left[|X_n - X|\right] = E\left[|X_n - X| \, I\langle |X_n - X| \geq N\rangle\right]$$

$$+ E\left[|X_n - X| \, I\langle |X_n - X| < N\rangle\right]$$

$\epsilon > 0$, Pick N large enough s.t.

$$E\left[|X_n - X| \, I\langle |X_n - X| \geq N\rangle\right] \leq \epsilon.$$

If X_n's are uniformly integrable, so will be $X_n - X$.

Pick n large so that

$E\left[|X_n - X| I\, |X_n - X| < N \right] < \epsilon$,

this is possible because of bounded random variable.

$\therefore E(|X_n - X|) \leq 2\epsilon$

$\therefore E[|X_n - X|] \to 0$

$G : [0, \infty) \longrightarrow [0, \infty)$

$\dfrac{G(t)}{t} \longrightarrow \infty$ as $t \to \infty$

Then $\sup_{\alpha} E[G(|X_\alpha|)] < \infty$

$\Rightarrow \langle X_\alpha \rangle$ u.i.

Proof Let $M = \sup_{\alpha} E[G(|X_\alpha|)]$.

Let $c > 0$ be such that $t \geq c \Rightarrow$

$\dfrac{G(t)}{t} \geq a \triangleq \dfrac{M}{\delta}$ for given $\delta > 0$

on the set $\langle \omega : |X(\omega)| \geq c \rangle$,

$|X(\omega)| \leq \dfrac{G(|X(\omega)|)}{a}$ (from defn)

$$\left(\because \frac{G(|X_\alpha(\omega)|)}{|X_\alpha(\omega)|} \geq a\right).$$

$$\int_{\langle |X_\alpha| \geq c \rangle} |X_\alpha(\omega)| \, dP \leq \int_{\langle |X_\alpha| \geq c \rangle} \frac{G(|X_\alpha(\omega)|)}{a} \, dP$$

$$\leq \frac{1}{a} E\left[G(|X_\alpha|)\right] \leq \frac{M}{a} = \delta.$$

[Since a non.-ve random variable, the integral over the $\langle |X_\alpha| \geq c \rangle$ is $\leq$ the integral over the whole space].

That is,

$$\sup_\alpha E\left[|X_\alpha| I\langle |X_\alpha| \geq c \rangle\right] \leq \delta$$

$$\Rightarrow \lim_{c \to \infty} \sup_\alpha E\left[|X_\alpha| I\langle |X_\alpha| \geq c\right] = 0.$$

$G(t) = t^2$

$t^{1+\epsilon}$

$t \log t$.

All r.v.s whose variances are bounded, they are uniformly integrable

If $X_n \to X$

$\Rightarrow E[X_n] \to E[X]$

$\Rightarrow E[|X_n - X|] \to 0$

if X_n u. i.

1. Bounded convergence theorem
2. Fatou's lemma.
3. Dominated convergence thm.

Bounded convergence thm:

$X_n \to X$ a.s.

$|X_n| \le K \ \forall n$ a.s.

$\Rightarrow E[X_n] \to E[X]$.

$\lim_n E[X_n] = E[\lim_n X_n]$.

Factou's lemma:

$X_n \geq 0$ a.s.

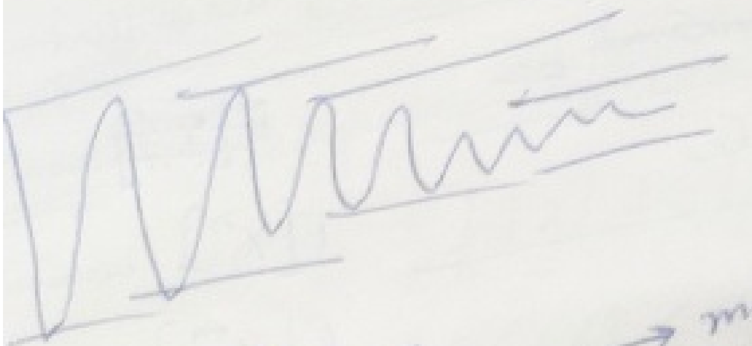
$X_n \to X$ a.s.

$\Rightarrow \liminf_{n\to\infty} E[X_n] \geq E[X]$.

$\liminf_{n\to\infty} X_n = \lim_{n\to\infty}\left(\inf_{m\geq n} X_m\right)$??


asymptotically lower bound
~~upper bound~~

lim



$X_n \to X$ a.s.

$X_n \wedge N \to X \wedge N$ a.s. — min^{m} of the two

Since $E[X_n] \geq E[X_n \wedge N]$

$$\lim_{n\to\infty} \inf E[X_n] \geq \lim_{n\to\infty} \inf E[X_n \wedge N]$$

$$= \lim_{n\to\infty} E[X_n \wedge N] = E[X \wedge N]$$

$$\lim_{n\to\infty} \inf E[X_n] \geq E[X \wedge N], \quad (N \uparrow \infty)$$

$$\Rightarrow \lim_{N\to\infty} \inf E[X_n] \geq E[X]$$

$Y_n = X_n \wedge X$

$$\lim \inf E[X_n] \geq E[X \wedge N]$$

Let $N \uparrow \infty \Rightarrow \lim\inf E[X_n] \geq E[X]$

monotic convergence theorem

??

:-

If higher moment is bounded, then lower ~~moment~~ power will ~~also be bounded~~ be uniformly integrable

$\cancel{E[X_n]}$

$E[X^n]$ bounded

$E[X^m]$ is u.i.

~~$n \geq m$~~

$m \leq n$

~~$n \geq m$~~

if X^2 is bounded

x is u.i.