31.08.05.

Probability

Given: X integrable $(E[|X|]<\infty)$

Then $E[|X| I<|X| \geq N>] \xrightarrow{N \uparrow \infty} 0$

$E[|X| I<|X| \leq N>] \xrightarrow{N\uparrow\infty} E(|X|)$

Roydon: $X \geq 0$, X bdd, ≥ 0 $\int X dP = \sup_{Y \leq X} \int Y dP$

Y simple

~~Given~~ General $X \geq 0$.

$$\int X\, dP = \sup_{\substack{Y \leq X \\ Y \text{ bdd}}} \int Y dP$$

$$\int X dP \geq \sup_{\substack{Y \leq X \\ Y \text{ bdd}}} \int Y dP \geq \sup_{\substack{Y \leq X \\ Y \text{ simple}}} \int Y dP$$

$$= \int X dP$$

Bdd. conv. thm

$X_n \to X$ a.s., $|X_n| \leq K$ a.s.

$\Rightarrow E[|X_n - X|] \to 0$, $E[X_n] \to E[X]$.

$$A_N = \bigcup_{n \geq N} \langle |X_n - X| \geq \epsilon \rangle$$

$$\Rightarrow \lim_N P(A_N) = 0.$$

$$E[|X_n - X|] = E[|X_n - X| I_{A_N}] + E[|X_n - X| I_{A_N^c}]$$

$$\leq 2K P(A_N) + \epsilon \text{ for } n \geq N.$$

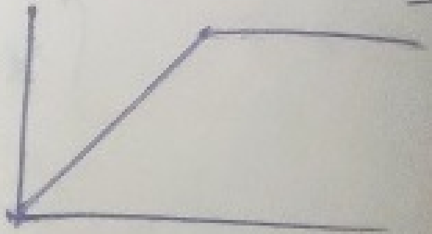
[$\because$ for A_N^c, each is bounded above by ϵ]

Fatou's lemma:

$X_n \geq 0$, $X_n \to X$ a.s. $\Rightarrow \underline{\lim}\, E[X_n] \geq E[X]$ ($\underline{\lim}$: liminf)

$X_n \to X$ a.s. $\Rightarrow X_n \wedge N \to X \wedge N$ a.s.

($\because x \to x \wedge N$ continuous)



$\therefore E[X_n \wedge N] \to E[X \wedge N]$

$E[X_n] \geq E[X_n \wedge N] \;\forall n$

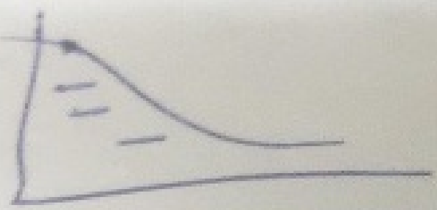
$\therefore \liminf_{n\to\infty} E[X_n] \geq E[X \wedge N]$, $N \uparrow \infty$

$E[X \wedge N] \to E[X]$

for any Y simple, $0 \leq Y \leq X$,

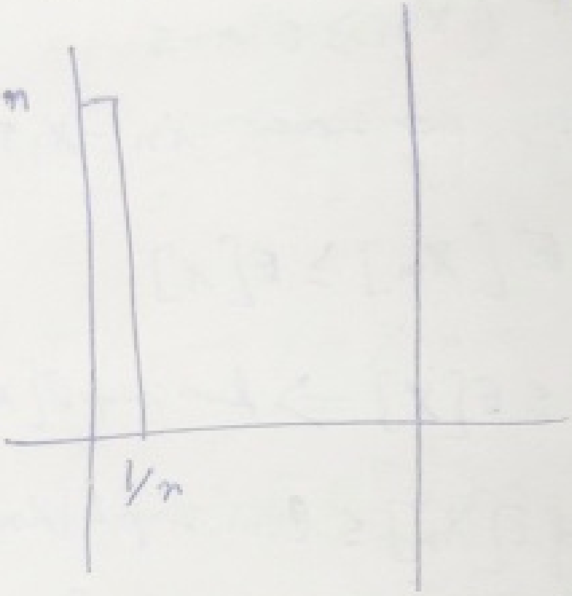
$\exists N$ s.t. $E[Y] \leq E[X \wedge N] \leq E[X]$

Since Y is simple, Y has finitely many values,
hence supremum of Y exists
for $E[Y] \le E[X \wedge N]$ we should take N larger than Y's.



$\therefore E[X]$

$$= \sup_{\substack{Y \le X \\ \text{simple}}} E[Y] \le \lim_{N \uparrow \infty} E[X \wedge N] \le E[X].$$



$E(X_n) = 1$, $X_n \to 0$ a.s.

Monotone convergence theorem:

$X_n \uparrow X$ a.s., $\langle X_n \rangle$ integrable (two arrows for strictly increasing)

$E[X_n] \uparrow E[X]$ (can be $+\infty$)

$X_{n+1} \ge X_n$ a.s.

Similarly for $X_n \downarrow X$ ($E[X]$ can be $-\infty$)

Remark: $X_n \uparrow X \Rightarrow X \geq X_n \Rightarrow E[X] \geq E[X_n]$

[$\because$ ~~X can't be~~ the integration of X can't be undefined, since integration of X_n is ~~not undefined~~ defined, hence bounded from below, can be $+\infty$]

suppose $E[X] < \infty$ (wo loss of generality)

w.l.o.g., let $(X_n \geq 0$ a.s.

(ow, consider $X_n - X_1 \uparrow X - X_1$)

Fatou $\Rightarrow \liminf E[X_n] \geq E[X]$

But $E[X_n] \leq E[X] \Rightarrow \limsup E[X_n] \leq E[X]$

$E[X] \leq \liminf E[X_n] \leq \limsup E[X_n] \leq E[X]$

$\therefore E[X_n] \to E[X]$

$E[X] = \infty$, $X_n \wedge N \uparrow E[X \wedge N]$

$\therefore E[X_n \wedge N] \to E[X \wedge N]$

$\therefore \liminf E[X_n] \geq \lim E[X_n \wedge N] = E[X \wedge N]$

let $N \uparrow \infty \Rightarrow E[X \wedge N] \uparrow \infty \Rightarrow \lim E[X_n] = \infty$

Dominated convergence theorem

$X_n \to X$ a.s. [very frequently used]

$|X_n| \leq Y$ a.s.

$E[Y] < \infty \Rightarrow E[X_n] \to E[X]$

Pf $Y - X_n \to Y - X$ a.s., $Y - X_n \geq 0$ a.s

$\therefore \liminf E[Y - X_n] \geq E[Y - X]$

$\therefore E[Y] - \limsup E[X_n] \geq E[Y] - E[X]$.

$\therefore \limsup E[X_n] \leq E[X]$ [$\because \liminf(-X_n) = -\limsup(X_n)$]

$Y + X_n \to Y + X$ a.s.

$\therefore \liminf E[Y + X_n] \geq E[Y + X]$

$\Rightarrow \liminf E[X_n] \geq E[X]$.

~~$\Rightarrow$ lim inf~~ $\Rightarrow E[X_n] \to E[X]$

F.N.

Generalizations: (1) $X_n \geq 0$ a.s.

$\liminf E[X_n] \geq E[\liminf X_n]$

(2) $|X_n| \leq Y_n$, $X_n \to X$ a.s. & $Y_n \to Y$ a.s. &

$E[Y_n] \to E[Y] \Rightarrow E[X_n] \to E[X]$

Uniform Integrability:

$\langle X_\alpha, \alpha \in I \rangle$ u.i.

if $\lim\limits_{c \uparrow \infty} \sup\limits_{\alpha} E[|X_\alpha| I\langle |X_\alpha| > c\rangle) = 0$.

Claim: I: finite $\Rightarrow$ u.i.

$$E[|X| I\langle |X| > c\rangle] \xrightarrow{c \uparrow \infty} 0 \text{ by DCT}$$

$$\sup_\alpha E[|X_\alpha| I\langle |X_\alpha| \geq c\rangle] < \epsilon,$$

for c sufficiently large

[take max of all c's]

$X_n \to X$ a.s., $\langle X_n\rangle$ u.i $\Rightarrow E[|X_n - X|] \to 0$

$E[|X_n - X|] = E[|X_n - X| I\langle |X_n - X| \leq c\rangle]$

$\downarrow$ 0 (By Bounded Convergence Theorem)

$+ E[|X_n - X| > c\rangle]$

After fixing C in the R.H.S second term, we can ~~always~~ always choose an $n \geq N$ for the first term, so that the first term is made less than ϵ, for any C we can always choose n, so that it becomes independent of n

$$E[|X_n| I\langle |X_n - X| > C\rangle]$$

$$E[|X_n| I\langle |X_n| > B\rangle I\langle |X_n - X| > C\rangle]$$

$$= E[|X_n| I\langle |X_n| > B\rangle I\langle |X_n - X| > C\rangle]$$

$$+ E[|X_n| I\langle |X_n| \leq B\rangle I\langle |X_n - X| > C\rangle]$$

$$\leq E[|X_n| I\langle |X_n| > B\rangle] + B P(|X_n - X| > C)$$

Choose B large $\Rightarrow E[|X_n| I\langle |X_n| > B\rangle] < \epsilon/2$ ($\because \langle X_n\rangle$ u.i)

Chosen B, pick C large having n so that $P(|X_n - X| > C) < \frac{\epsilon}{2B}$.

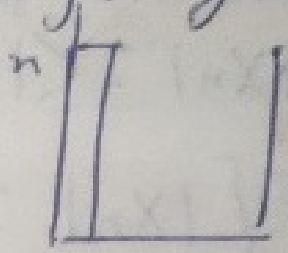
To see that this is possible, note that $E[|X_n|] = E[|X_n| I\langle |X_n| \geq a\rangle] + E[|X_n| I\langle |X_n| < a\rangle]$

Choose a. s.t. 1st term $< \epsilon$ ($\because$ u.i.)

$\therefore E[|X_n|] \leq a + \epsilon$, $\therefore \sup_n E[X_n] < \infty$.

If u.i. then expectations are uniformly bounded

not the converse. e.g. $\longrightarrow$



here u.b. but not u.i.

$$P(|X_n - X| \geq c) \leq \frac{E[|X_n - X|]}{c}$$

$$\leq \frac{\sup_n E[|X_n|] + E[|X_n|]}{c}$$

$$< \frac{\epsilon}{2} \text{ for } C \text{ large.}$$

$$f(x_n, y) \to 0$$

$$f(x_n, y_n) \not\to 0$$

Test for u.i: $\frac{G(t)}{t} \uparrow \infty \quad G: R^+ \to R^+$.

$\sup_\alpha E\left[G\left(|X_\alpha|\right)\right] < \infty \Leftrightarrow \langle X_\alpha, \alpha \in I \rangle$ u.i.

[It must be faster than the linear and also expectation is bounded $\Rightarrow$ u.i.]

e.g., $\sup_\alpha E\left[|X_\alpha|^2\right] < \infty$. $\left[\begin{matrix}\text{e.g.} \\ t\log t\end{matrix}\right]$

$\Rightarrow \langle X_\alpha \rangle$ u.i.

Independence

Events A, B are indep. if $P(AB) = P(A)P(B)$

random variables X, Y are independant if

$P(X \in A, Y \in B) = P(X \in A) P(Y \in B) \quad \forall A, B$.

$A_1, \ldots, A_n$ indep. if $P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$

$X_1, \ldots X_n$ indep. if $P(X_i \in A_i, 1 \le i \le n)$

$= \prod_{i=1}^{n} P(X_i \in A_i)$

for arbitrary families $\begin{cases} A_\alpha, \alpha \in I \text{ indep if every finite subfamily} \\ X_\alpha, \alpha \in I \quad " \quad " \quad " \quad " \end{cases}$

(0) A, B indep $\Rightarrow A, B^c$ indep, A^c, B indep, $A^c B^c$ indep

Remarks: (1) $A_1, \cdots A_n$ indep $\Leftrightarrow I_{A_1}, \cdots I_{A_n}$ indep.

$$P(AB) = P(A)P(B) \Rightarrow$$

$$P(A^c B) + P(AB) = P(B)$$

$$\Rightarrow P(A^c B) = P(B) - P(AB)$$

$$= P(B) - P(A)P(B)$$

$$= P(B)(1 - P(A))$$

$$= P(A^c)P(B)$$

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(2) X, Y indep $\Leftrightarrow E[f(X)g(Y)] = E[f(X)] E[g(Y)]$.

whenever expectations are defined

$$P(X \in A, Y \in B) = E[I_{\langle X \in A \rangle} I_{\langle Y \in B \rangle}].$$

$$\therefore E[f(X)g(Y)] = E[f(X)] \; E[g(Y)]$$

For $f = I_A, \; g = I_B$

[Monotone Class Theorem to prove extension of ~~sub-class to~~ a whole class.]

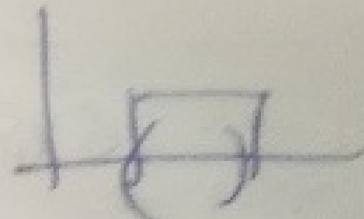
Measure theoretically,

$X_1, \dots, X_n$ indep $\iff$ law of $(X_1, \dots, X_n)$ is a product measure

$(S_i, \mathcal{S}_i)$, $i = 1, \dots, n$.

Product space: $S_1 \times S_2 \times \dots \times S_n = \{(x_1, \dots, x_n) : x_i \in S_i\}$

nbds of product spaces is defined as product of nbd spaces



$\mu(dx_1 \dots dx_n) = \mu(dx_1)\,\mu(dx_2) \dots \mu_n(dx_n)$

$f(x, y)\,dx\,dy$ will define a probability measure

$$\mu(A_1 \times A_2 \times \dots \times A_n) = \prod_{i=1}^{n} \mu_i(A_i)$$

$f(x,y)\,dx\,dy$

If $f(x,y) \neq f_1(x) f_2(y)$ it will not be a product measure

Kolmogorov's 0-1 law: $X_1, X_2, \dots$ independent.

$\bigcap_n \sigma(X_i, i \geq n) = \mathcal{F}_T$ = tail σ-field [distant features]

$A \in \mathcal{F}_T \Rightarrow P(A) = 0$ or 1.

$A \in \mathcal{F}_\tau \subseteq \sigma(X_i, i \geq n), \forall n,$

$A \in \sigma(X_{n+1}, X_{n+2}, \dots)$

σ-field $\mathcal{G}_i \subset \mathcal{F}$ indep $\Leftrightarrow$

$(A_i \in \mathcal{G}_i \Rightarrow \langle A_i \rangle$ indep $)$

$\langle X_i \rangle$ indepn $\Leftrightarrow \langle \sigma(X_i) \rangle$ indep

$\therefore$ A indep of $\langle X_1, \dots, X_n \rangle$

$\therefore$ A indep of $\sigma(X_m, m \geq 1)$

$\therefore$ A indep of $\mathcal{F}_\tau$

$\therefore$ A indep of itself.

$\therefore P(A) = P(AA) = P(A)^2 \Rightarrow P(A) = 0$ or 1

tail event will be true or false

$\limsup \frac{1}{n} \sum_{m=1}^{n} X_i > 100$ whenever we drop the first few terms limit will be same

Ω, ϕ is independent of itself | Since tail limit

Borel - Cantelli lemma

(i) $\sum_n P(A_n) < \infty \Rightarrow P(A_i \text{ i.o.}) = 0$.

(ii) $\langle A_i \rangle$ indep, $\sum_n P(A_n) = \infty \Rightarrow P(A_n \text{ i.o}) = 1$.

Proof (i) $\sum_n P(A_n) = \sum_n E[I_{A_n}] = E[\sum_n I_{A_n}]$ by MCT (monotone convergence Theorem)

$E[\sum_n I_{A_n}] < \infty$

$\Rightarrow \sum_n I_{A_n} < \infty$ a.s.

$\Rightarrow P(A_n \text{ i.o.}) = 0$ [from some pt. onwards must be 0]