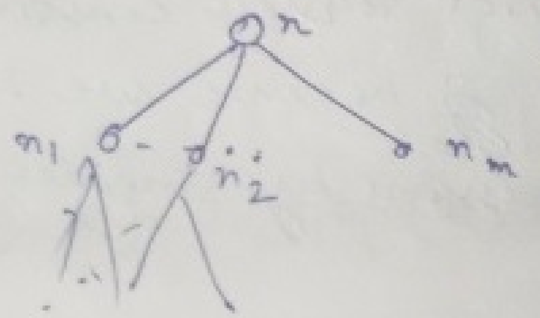
06.09.05.

Konig's Lemma

A tree is infinite if it has infinitely many nodes. If every infinite tree which has finitely many branches it will have an infinite branch (path)

A node is said to be good if it has infinitely many descendants.



Tree will be finite.

$$\neg\left[((P \wedge (Q \Rightarrow (R \vee S))) \Rightarrow (P \vee Q)\right].$$

$\vdash X$ is if there is a closed ~~tableau~~ tableaux for $\neg X$.

$P \wedge (Q \Rightarrow (R \vee S))$

$\neg (P \vee Q)$

P

$Q \Rightarrow (R \vee S)$

Closed

$\neg P$

$\neg Q$

$\times$

We are economical

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Theorem

Soundness: $\vdash_{pt} X$ then $\models X$

Proof

$S \subseteq$ Prop is satisfiable

if for some $v \in VAL$

[for all $x \in S$

$v \models x$

]

∵ Terminology: Tableux branch θ is satisfiable

Tableux T is satisfiable if some branch θ in T is satisfiable.

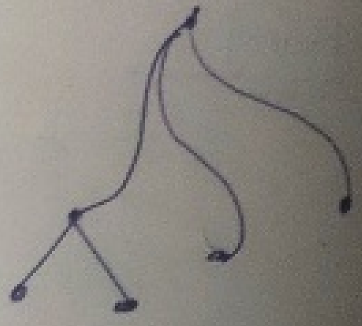
Prosition: If Tab Tree T is satisfiable and T^* is obtained by one application of tab. rule applied to T then T^* is also ~~as~~ satisfiable.

Proof

Let T be satisfiable.

Let Z be a satisfiable branch of T.

Let θ be the branch of T and is expanded by applying some rule ~~to~~ form and X on θ.



Case 1 $\theta \neq Z$ ✓

Case 2 $\theta = Z$

~~Case 3~~ Case analysis on ~~to~~ rules applied to X

Case 2.1 $X = \neg\neg Z$.

~~there exists~~ $\exists\ v$ s.t. $v \vdash \theta$.

$\therefore v \models \neg\neg Z$

$\therefore v \models Z$

$\therefore\ v \models \theta . Z$

Case 2 $\quad X = \alpha .$

$\exists v \qquad v \models \theta$

$\because\ v \models \alpha .$

$\therefore\ v \models \alpha_1$ and $v \models \alpha_2$

$v \models \theta, \alpha_1, \alpha_2$

Remaining cases will be proved similarly.

Observation: Closed branch is not satisfiable.

$$\neg X$$
$$\vdots$$
$$X .$$

Th (Completeness)

$\models X$ then $\vdash_{pt} X .$

Hintika's Lemma

Defn: $H \subseteq PROP$ is called propositional Hintica set, provided,

1. for any $p \in P(S)$, not both p and $\neg p$. i.e. $\{p, \neg p\} \not\subseteq H .$

2. $\bot \notin H, \neg T \notin H$

3. If $\neg\neg X \in H$ then $X \in H$. for all $X \in$ PROP

4. If $\alpha \in H$ then
$\alpha_1, \alpha_2 \in H$.

5. If $\beta \in H$ then either $\beta_1 \in H$ or $\beta_2 \in H$ (or both).

$$\{ (P \wedge (\neg Q \Rightarrow R)), P, (\neg Q \Rightarrow R), \neg\neg Q, Q \}$$

R

Downward ~~[illegible]~~ saturation (add all [illegible] sort of witnesses to the ~~[illegible]~~ truth of the formula)

Hintika's Lemma

H is propositional Hintika set then H is satisfiable.

Proof: Construct v s.t. $v \models H$.

Let V be s.t.

If $p \in H$ then $V(p) = t$.

If $\neg P \in H$ then $V(P) = F$.

If $p \notin H$ & $\neg P \notin H$ then $V(p)$ arbitrary

Claim: If $X \in H$, then $V \models X$.

Induction on structure of X.

Base Case $\quad P, \neg P, \bot, \neg\bot, \top, \neg\top$.

$P \in H$ then $V \models P$ by construction

$\neg P \in H$ then $V \not\models P$, $\therefore V \models \neg P$.

Induction Step. $\quad$ Let $X = \neg\neg Z$.

Case-1 $\quad$ By Ind. hypothesis.

$\therefore Z \in H$

$V \models Z$ by Ind. hypothesis

$X = \neg\neg Z$,
Z simpler.

$\therefore V \models \neg\neg Z$ - semantics

$X = \alpha$,
α_1, α_2, simpler.

Case-2 If $X = \alpha$

$\alpha_1, \alpha_2 \in H$. (by defn of Hintika set)

$V \models \alpha_1$ and $V \models \alpha_2$ by ind. hypothesis

$\Rightarrow V \models \alpha$

Case 3 $\quad$ If $X = B$

Either $\beta_1 \in H$ or $\beta_2 \in H$

$\therefore V \models \beta_1$ or $V \models \beta_2$.

$V \models \beta$

Abstract Completeness

consistency
~~Syn~~ syntactic
↳ visible
contradiction.

unsatisfiable
Valuation.

Complete
Soundness

Abstract Completeness

Abstract Consistency property

$\mathcal{C} \subseteq 2^{PROP}$.

Defn: $\mathcal{C} \subseteq 2^{PROP}$ be collection of sets of prop. formulae.

We call $\mathcal{C}$ has propositional consistency property if

$\forall S \in \mathcal{C}$ the following hold:

1. for $p \in PS$
   $\{P, \neg P\} \not\subseteq S$.
2. $\bot, \neg T \notin S$.
3. $\neg\neg Z \in S$ implies $S \cup \{Z\} \in \mathcal{C}$
4. $\alpha \in S$ implies $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$.
5. $\beta \in S$ implies
   $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$

$\{X\}$

Defn Collection $\mathcal{C}$ has finite character provided $S \in \mathcal{C}$ iff

$\forall$ finite subsets Y of S, $Y \in \mathcal{C}$.

Claim For every prop. consistent $\mathcal{C}$ $\exists$ $\hat{\mathcal{C}} \supseteq \mathcal{C}$ s.t. $\hat{\mathcal{C}}$ is also prop. consistency prop.

Consistency of $\hat{\mathcal{C}}$.

Subset closure of $\mathcal{C}$.

Step (a) $S \in \mathcal{C}$ any $Y \subseteq X$ then

$Y \in \hat{\mathcal{C}}_1$

~~Step~~ **Step** (b) If for any $S \in PROP$,

if $\forall Y \subseteq S$ finite.

$Y \in \mathcal{C}_1$ ~~then~~ let $S \in \hat{\mathcal{C}}$

If $S \in \hat{\mathcal{C}}$ and

$\alpha \in S$ then $S \cup \{\alpha_1, \alpha_2\} \in \hat{C}$.

Step1 Given PCP $\mathcal{C}$;
we construct PCP $\hat{\mathcal{C}}$ s.t. $\mathcal{C} \subseteq \mathcal{C}'$
and $\hat{\mathcal{C}}$ is of finite character.

Step2 If $S_1 \subseteq S_2 \subseteq \cdots \subseteq$ is a chain in $\hat{\mathcal{C}}$
then $\bigcup_i S_i \in \hat{\mathcal{C}}$.

Let $Y \subseteq_{Fin} \bigcup_i S_i$

then $Y = \{z_1, \dots, z_m\}$.

For some k, we have $\{z_1, \dots, z_m\} \subseteq S_k$.

$\{z_1, \dots, z_k\} \in \hat{\mathcal{C}}$.

Claim

Given $S \in \mathcal{C}$.

PS is countable

$\therefore$ PROP is countable. (by lexicographical Ordering)

Let $x_1, x_2, \dots,$ be an enumeration of PROP (all x_i formulae will be in PROP)

Construction of Limit

Given S,

Let $S_1 = S$

$$S_{i+1} = S_i \cup \{x_i\}, \quad \text{if } S_i \cup \{x_i\} \in \mathcal{C}.$$

[if extension is consistent, i.e. within $\mathcal{C}$]

$$= S_i, \quad \text{ow}.$$

$\therefore H = \bigcup_i S_i \in \hat{\mathcal{C}}$

Claim: $\neg\neg z \in H$

Thm: (Model Existence thm).

If $\mathcal{C}$ is PCP and $S \in \mathcal{C}$ then S is satisfiable

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