09.09.05.

Portmanteau theorem:

The following are equivalent:

(i) $E[f(X_n)] \longrightarrow E[f(X)]$ for bounded continuous f

(ii) $E[f(X_n)] \longrightarrow E[f(X)]$ for bounded uniformly continuous f

(iii) $\liminf_{n\to\infty} P(X_n \in G) \geq P(X \in G)$ for open G.

(iv) $\limsup_{n\to\infty} P(X_n \in F) \leq P(X \in F)$ for closed F.

(v) $\lim_{n\to\infty} P(X_n \in A) = P(X \in A)$ if $P(X \in \partial A) = 0$.

Pf. (i) $\Rightarrow$ (ii) free

(ii) $\Rightarrow$ (iii) follows from the fact that there exists bounded uniformly continuous $\{f_n\}$ s.t. $f_n \uparrow I_G$ (Generally in topology open sets are G, closed sets are F)

$$\therefore \liminf_{n\to\infty} P(X_n \in G) = \liminf_{n\to\infty} E[I_G(X_n)] \geq \lim_{n\to\infty} E[f_m(X_n)] = E[f_m(X)]$$

Let $m \uparrow \infty$, by MCT, RHS

$$E[I_G(X)] = P(X \in G).$$

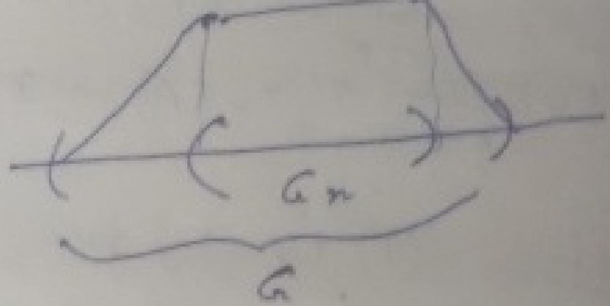
Let $G_n = \langle x \in G : \|x-y\| > \frac{1}{n}$ for $y \in \partial G \rangle$

$$f_n(x) = \frac{d(x, G^c)}{d(x, G_n) + d(x, G^c)}$$

$x \in G_n \Rightarrow d(x, G_n) = 0$

$\therefore f_n(x) = \frac{d(x, G^c)}{d(x, G^c)} = 1$

$x \in G^c \Rightarrow d(x, G^c) = 0 \Rightarrow f_n(x) = 0$



Digression:

Consider $P(R^d)$ = the space of probability measures on R^d.

$\mu_n \to \mu$ in $P(R^d) \iff \int f\, d\mu_n \to \int f\, d\mu$

for bounded continuous f.

(topology → open nbd) open neighbourhoods of

μ, $\{\mu' \in P(R^d) : |\int f_i\, d\mu' - \int f_i\, d\mu| < \epsilon_i, 1 \le i \le n\}$.

$\underbrace{\mu \to \int f\, d\mu}$ the above condition ~~continuous~~ is min to make this f continuous

$x_n \to x$.

$f_\alpha,\ \alpha \in I,$

for any discontinuity it can jump only down

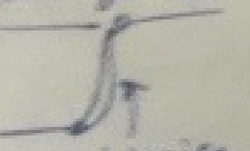
ptwise sup is lower semi-continuous
ptwise inf is upper semi-continuous
we can always approx

$f(x) = \sup_\alpha f_\alpha(x)$

$\therefore \liminf_{n\to\infty} f(x_n) \geq f(x)$.

$\liminf f(x_n) \geq \lim f_\alpha(x_n) = f_\alpha(x)$

$\therefore \liminf f(x_n) \geq f(x)$.



ptwise sup

$\mu \to \int f_n d\mu$

$\mu(E) = \sup_m \int f_m \, dm$.

(iii) $\Leftrightarrow$ (iv) by complementation

(iii), (iv) $\Rightarrow$ (v)

$\max_y f(x,y)$
$\hookrightarrow$ lower semi-continuous
$\min^m$ is lower semi-cont
$\hookrightarrow$ So $\max^m$ will be attended.

$\mathring{A}$

Interior of A
(largest open set in A)

$\mathring{A} = \operatorname{int}(A)$
$=$ largest open set $\subset A$.

$\bar{A}$ = closure of A
= smallest closed set $\supset A$.

boundary $\{ \partial A = \bar{A} - \mathring{A}$

$[1,2) \cup (3,4] \cup \{5\}$

$(1,2) \cup (3,4)$

$$\liminf P(X_n \in A) \geq \liminf P(X_n \in \mathring{A}) \geq P(X \in \mathring{A})$$
$$= P(X \in A)$$

$$\left(\because\ P(X \in \mathring{A}) \leq P(X \in A)\right.$$
$$P(X \in A) \leq P(X \in \mathring{A}) + P(X \in \partial A)$$
$$\left. = P(X \in \mathring{A})\ \right)$$

$$P(X \in A) = P(X \in \mathring{A})$$

$$\begin{cases} P(X \in \mathring{A}) \leq P(X \in A) \\ P(X \in \bar{A}) \leq P(X \in \mathring{A}) \end{cases}$$
$$P(X \in A) = P(X \in \mathring{A})$$

$\mu \to \mu(A)$.

if continuous $\tilde{\mu}(\delta A) = 0$

[things can go wrong only at the boundary, Dirac fn].

only for the limit not in case of X_n

Similarly,

$$\limsup (X_n \in A) \leq \limsup P(X_n \in \bar{A})$$
$$\leq P(X \in \bar{A}) = P(X \in A).$$

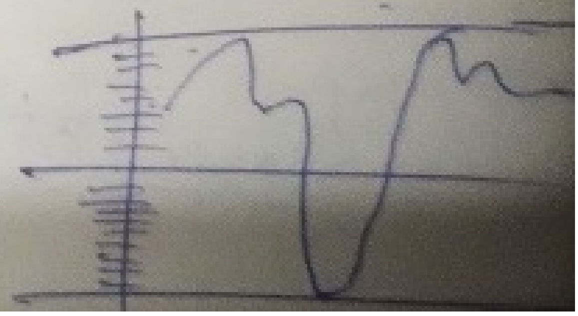
(v) $\Rightarrow$ (i) Let f be bounded continuous,
$$f(x) \in [-M, M]$$

Let $\epsilon > 0$.

Let $-M = a_0 < a_1 < a_2 < \cdots < a_K = M$

s.t. $|a_{i+1} - a_i| < \epsilon$ and

$P(X = a_i) = 0 \ \forall i$



Possible because : $\langle x : P(X = x) > 0 \rangle$ is at most countable

$\because \langle x : P(X \geq x) > \frac{1}{n} \}$ has at most n elements

$\therefore \bigcup_n \langle x : P(X \geq x) > \frac{1}{n} \rangle$ countable.

$f_\epsilon = \sum_{m=0}^{K} a_m \, I \langle a_m \leq f(x) < a_{m+1} \rangle$

Now we don't have to truncate since f_ϵ is a bounded function, unlike the earlier case while approx f_ϵ by simple function

$|f_\epsilon(x) - f(x)| < \epsilon.$

$\therefore |E[f(X_n)] - E[f(X)]| \leq 2\epsilon$

$+ |E[f_\epsilon(X_n)] - E[f_\epsilon(X)]|$

$f(X_n) - f(X)$.

$f_\epsilon(X_n)$ -

$\downarrow$

0 as $n \to \infty$.

ϵ is arbitrary $\Rightarrow E[f(X_n)]$

$\to E[f(X)]$.

$|E[f_\epsilon(X_n)] - E[f_\epsilon(X)]|$

$\leq \sum_{m=0}^{K} |a_m| \, |P(X_n \in \langle x : a_m \leq f(x) < a_{m+1} \rangle)$

$- P(X \in \langle x : a_m \leq f(x) < a_{m+1} \rangle)|$

$\to 0$, by (iv)

Convergence of Prob. Measures by Billingsley

Characteristic function of $\mu \in P(\mathbb{R}^d)$

$$\varphi(t) = \int e^{i\langle t, x\rangle} d\mu$$

$$= \int \cos\langle t, x\rangle d\mu + i\int \sin\langle t, x\rangle d\mu$$

bounded by 1 (well-defined).

Char. fn. of X is $E[e^{i\langle t, X\rangle}]$

$$= E[\cos(\langle t, X\rangle] + i E[\sin\langle t, X\rangle]]$$

In Mathematics, Measure Theory or Topology, Characteristic fn is I_A but in Probability we already have a characteristic fn defined as above, hence I_A is indicator function

Properties:

(i) $\varphi_\mu(0) = 1$, $|\varphi_\mu(t)| \leq 1$, $\overline{\varphi_\mu(t)} = \varphi_\mu(-t)$.

(ii) $|\varphi_\mu(t+h) - \varphi_\mu(t)| \leq \int |e^{i\langle t+h, x\rangle} - e^{i\langle t, x\rangle}| d\mu(x)$

$$\leq \int |e^{i\langle t, x\rangle}| \, |e^{i\langle h, x\rangle} - 1| d\mu(x)$$

$(|e^{i\langle t, x\rangle}| = 1)$

$$= \int |e^{i\langle h, x\rangle} - 1| d\mu(x) \longrightarrow 0$$

as $\|h\| \to 0$

$\therefore \varphi_\mu(\cdot)$ uniformly continuous

[$\because$ no t here].

(iii) If $Y = aX + b$, law of $X \approx \mu$, law of $Y \approx \nu$, then $\varphi_\nu(t) = e^{i\langle t, b\rangle} \varphi_\mu(at)$.

$$E\left[e^{i\langle t, Y\rangle}\right] = e^{i\langle t, b\rangle} E\left[e^{i\langle at, X\rangle}\right]$$

(iv) $\varphi_\mu(\cdot)$ is positive definite: If $x_1, \dots, x_n \in \mathbb{R}^d$, $c_1, \dots, c_n \in \mathbb{C}$,

then $\sum c_j \varphi_\mu(x_j - x_k) \bar{c}_k \geq 0$ $\bar{c}_k \to$ complex conjugate of c_k

$\parallel$

$$\sum c_j \bar{c}_k E\left[e^{i\langle t, X_j - X_k\rangle}\right]$$

$$= E\left[\left|\sum c_k e^{i\langle t, X_k\rangle}\right|^2\right] \geq 0.$$

Thm $\varphi_\mu = \varphi_\nu \Rightarrow \mu = \nu$

Pf

$$\frac{1}{(2\pi)^d}\int e^{i\langle x, y\rangle - \frac{\sigma^2}{2}\|y\|^2} dy$$

$$= \frac{1}{\sqrt{(2\pi)^d \sigma^{2d}}} e^{\frac{-\|x\|^2}{2\sigma^2}}, \quad \sigma > 0,$$

{FT of Gaussian is another Gaussian

$$\therefore \frac{1}{(2\pi)^d} \iint e^{i\langle x-z, y\rangle - \frac{\sigma^2}{2}\|y\|^2} dy\, \mu(dx)$$

replace x by x−z and integrat.

Fubini: When we can change the order of the integrals

(Replace x by (x−z), integrate w.r.t. μ

$$= \frac{1}{\sqrt{(2\pi)^d \sigma^{2d}}} \int e^{\frac{-\|x-z\|^2}{2\sigma^2}} \mu(dx)$$

Interchanging order of integration, since

$$\int e^{i\langle x,y\rangle}\mu(dx) = \varphi_\mu(y)$$

we have

$$\frac{1}{(2\pi)^d}\int \varphi_\mu(y)\, e^{-i\langle z,y\rangle} e^{-\frac{1}{2}\sigma^2\|y\|^2} dy$$

$$= \frac{1}{\sqrt{(2\pi)^d\sigma^{2d}}}\int e^{-\|x-z\|^2/2\sigma^2}\mu(dx).$$

Similarly for ν, since $\varphi_\mu = \varphi_\nu$,

$$\frac{1}{\sqrt{(2\pi)^d\sigma^{2d}}}\int e^{-\|x-z\|^2/2\sigma^2}\mu(dx)$$

$$= \frac{1}{\sqrt{(2\pi)^d\sigma^{2d}}}\int e^{-\|x-z\|^2/2\sigma^2}\nu(dx)$$

$\therefore$ for, bounded continuous f,

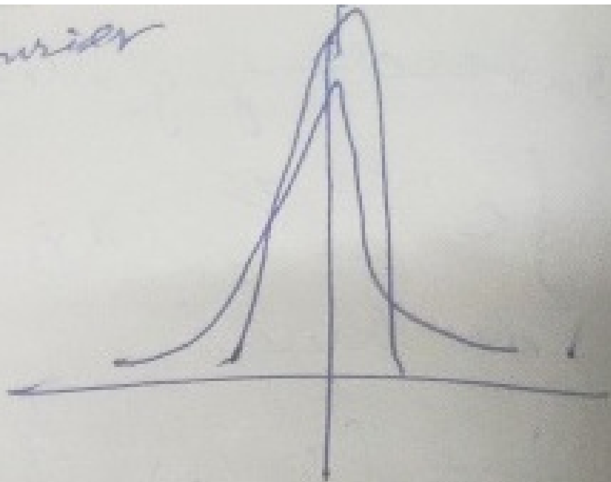
$$\frac{1}{\sqrt{(2\pi)^d\sigma^{2d}}}\iint f(z)\, e^{-\|x-z\|^2/2\sigma^2}\mu(dx)\,dz$$

$=$ 〃 〃 〃 〃 〃 〃 〃 $\nu(dx)\,dz$ (change z by $x-y$)

$$\therefore \frac{1}{\sqrt{(2\pi)^d\sigma^{2d}}}\iint f(x-y)\, e^{-\|y\|^2/2\sigma^2} dy\,\mu(dx)$$

$=$ 〃 〃 〃 〃 〃 $\nu(dx)$

basically it's an inverse fourier transform result.



Let $\sigma \to 0 \Rightarrow$

$$\int f \, d\mu = \int f \, d\nu .$$

$\sigma \to 0$ Gaussian becomes Dirac

Convolution

$$\int f(x-y)\, g(y)\, dy = f * g(x)$$

$$\mu * \nu (B) = \int \mu(B-x)\, \nu(dx) .$$

where $B-x = \langle y-x : y \in B \rangle$

Thm If X, Y indep. with laws μ, ν resp., ~~then~~ law of $X+Y$ is $\mu * \nu$

Pf If ξ is the law of $X+Y$,

$$\varphi_\xi(t) = E\left(e^{i\langle t, X+Y\rangle} \right]$$

$$= E\left[e^{i\langle t, X\rangle} e^{i\langle t, Y\rangle} \right]$$

$$= E\left[e^{i\langle t, X\rangle} \right] E\left[e^{i\langle t, Y\rangle} \right]$$

$$= \varphi_\mu(t)\, \varphi_\nu(t) .$$

[Ft of product of laws is product of Ft of laws?]

$\mathbb{R}$

$$\int e^{i\langle t,x\rangle}\mu * \nu(dx) = \iint e^{\ldots}\mu(\ldots)\nu(dy).$$

$$= \iint e^{i\langle t, x-y\rangle} e^{i\langle t,y\rangle} d\mu(dx-y)\nu(dy)$$

$$= \iint e^{i\langle t,x\rangle} e^{i\langle t,y\rangle}\mu(dx)\nu(dy)$$

$$= \varphi_\mu(t)\,\varphi_\nu(t)$$

$$\therefore \xi = \mu + \nu.$$

Thm (Levy) (i) $\mu_n \to \mu_\infty \Rightarrow \varphi_{\mu_n} \to \varphi_\mu$ uniformly on compacts (for each t pointwise)

Converse is useful (ii) If $\varphi_{\mu_n} \to \varphi$ pointwise for some $\to \varphi$ continuous at zero. Then $\varphi = \varphi_\mu$ for some μ & $\mu_n \to \mu$

Stable distributions (characteristic fn)

Central limit (bounded variance)

$$\sup_{t\in C}|\varphi_{\mu_n}(t) - \varphi_\mu(t)| \to 0 \text{ for } C \text{ bounded.}$$

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time & frequency domain

in time domain what is near 0, becomes sparse in frequency domain