25.10.05.

Primal.

$\boxed{\begin{array}{l} \min c^T x \\ Ax \geq b \\ x \geq 0 \end{array}}$

Dual

$\max p^T b$

$A^T p \geq c$

$p \geq 0$

If x is primal feasible & y is dual feasible, we have,

$$\left.\begin{array}{l} c^T x \geq p^T y \\ c^T x^T = p^T y^* \end{array}\right\}$$

Sensitivity analysis

Farka's lemma

$$\left.\begin{array}{lcl} Ax = b & \text{or} & p^T A \geq 0 \\ x \geq 0 & & p^T b < 0 \end{array}\right\}$$

x^* primal

$I = \{ \tilde{a}_i^T x = b_i \}$

row of matrix

$\tilde{a}_i^T d \geq 0, \ \forall i \in I.$

must be $c^T d \geq 0$

$\frac{Ax \geq b}{Ax^* \geq b}$ $A(x^* + \epsilon d) \geq b$

$$\left\{\begin{array}{l} \tilde{a}_i^T d \geq 0 \quad i \in I \\ \& \quad c^T d < 0 \end{array}\right\}$$ has no solution

$\tilde{a}_i$ is the i th row

$c^T(x^* + \epsilon d)$
$= c^T x^* + \epsilon c^T d$

Application to asset pricing

Market

— Single period

— n different assets are traded.

— m possible states of nature.

— 1 unit of asset i gives Rs. r_{si} if s state of nature results

$R := \{ r_{si} : s \leq m, i \leq n \}$

$x = (x_1, x_2, \ldots, x_n)^T$ amount of each security.

Rx.

$Rx \geq 0$

$p = (p_1, p_2, \ldots, p_n)^T$

No arbitrage principle

$Rx \geq 0$
then $p^T x \geq 0$

$Rx \geq 0$
$p^T x < 0$ has
no sol^n

$\exists\, q \geq 0$
s.t. $q^T R = p^T$.

or, $p_i = \sum_{j=1}^{m} q_j R_{ji}$. → you give 1 unit of price.

$p_i = E_q R_j$.

Complementary Slackness

min $c^T x$
$Ax \geq b$
$x \geq 0$

dual
max $p^T b$
$A^T p \leq c$
$p \geq 0$

$x^* + p^*$ are primal & dual optimal iff
$p^{*T}(Ax^* - b) = 0$

& $(c - A^T p^*)^T x^* = 0$

$$Ax \geq b \quad \Big| \quad A^T p^* \leq c$$

$$c^T x \geq p^T A x \geq p^T b$$

$c^T x = p^T b$ since optimal.

$$\Rightarrow \left.\begin{aligned} c^T x - p^T A x &= 0 \\ p^T b - p^T A x &= 0 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} &\forall i \\ &p_i(\tilde{a}_i^T x - b_i) = 0 \\ &(c_j - p^T a_j) x_j = 0 \\ &\forall i \end{aligned}}$$

$$\sum_{i=1}^{m} p_i (\tilde{a}_i^T x - b_i) = 0$$

↳ each term is +ve.

$$p_i > 0, \quad \sum_{i=1}^{m} p_i (\hat{a}_i^T x - b_i) = 0$$

$$\tilde{a}_i x > b_i \Rightarrow p_i = 0.$$

constraint is tight & dual variable is zero – this is also possible → degeneracy.

if x_j is +ve constraint in the dual must be ~~also~~ tight.

if not tight x_j is 0.

$$\boxed{\begin{aligned} &\min c^T x \\ &Ax \geq b \\ &x \geq 0 \end{aligned}}$$

$$\boxed{\begin{aligned} &\max p^T b \\ &A^T p \leq c \\ &p \geq 0 \end{aligned}}$$

$$\sum_{i=1}^{m} p_i a_{ij} \leq c_j$$

$$\max p^T b$$

$$\boxed{\tilde{a}_i x - b_i > 0} \ \boxed{p_i = 0}$$

$$x_j (c_j - p^T a_j) = 0$$

cost of the pill

$$x_j > 0 \Rightarrow \boxed{c_j - p^T a_j = 0}$$

$$\boxed{c_j > p^T a_j \Rightarrow x_j = 0}$$

→ Interior point methods

→ Complexity analysis of simplex/ int. pt. methods

→ Primal dual methods.

→ Integer linear Programming.

→ Semi-definite Programming.

~~Sensitivit~~

Sensitivity Analysis

$$\min C^T X$$
$$AX = b$$
$$X \geq 0$$

suppose B is the optimal basis.

$$X_B = B^{-1} b$$

$$\underbrace{C_N^T - c_B B^{-1} N}_{\text{reduced cost}} \geq 0$$

$B^{-1} b > 0$ (each & every element).

$$b \longrightarrow b + \Delta b$$

$$X_B = B^{-1}(b + \Delta b)$$
$$= B^{-1} b + \underbrace{B^{-1} \Delta b}_{\Delta X_B}$$

obj function $= c_B^T X_B$

$$= c_B^T (X_B + B^{-1} \Delta b)$$
$$= c_B^T X_B + \underbrace{c_B^T B^{-1}} \Delta b$$
$$= c_B^T X_B + p^T \Delta b$$

p_j is the marginal price of component b_j

$$\boxed{\frac{\partial z^*}{\partial b_j} = p_j}$$

$B^{-1}(b+\Delta b) \geq 0$

Matrix games

- Finite, two-person, zero sum, one move game.

$$A \quad \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0 & 1 & -2 \\ 2 & -1 & 0 & 1 \\ 3 & 2 & -1 & 0 \end{array}$$

$A = (a_{ij} : i \leq m, j \leq n)$

$r = (r_1, r_2, \ldots, r_m)$

$c = (c_1, c_2, \ldots, c_n)$

$E[r, c] = \sum_{i,j} a_{ij} r_i c_j$

$\max_r \min_c E[r, c]$

$\min_c \max_r E[r, c]$

$$\max_r \left(\min_c E[r,c] \right) \leq \max_r E[r,c]$$

$$\leq E[r,\tilde{c}] \leq \min_c \max_r E[r,\tilde{c}].$$

$$\min E[r,c] = \sum_i \sum_j a_i r_j c_j.$$

$$c = (c_1, c_2, \cdots, c_n)$$

$$c_i \geq 0 \quad \Sigma c_i = 1$$

$$\sum_{i=1}^{m} a_{ij} r_i \qquad \text{(pure strategies)}$$

Result

$\exists$ a strategy $r' \in S_m$ (Simplex in m dimension)

$c' \in S_n$

and a no. v such that

$$\left.\begin{array}{l} E[r', c) \geq v \quad \forall c \\ E[r, c') \leq v \quad \forall r. \end{array}\right\}$$