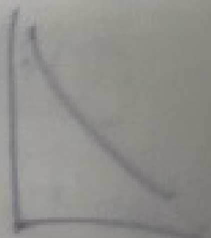
12.09.05

Levy Continuity Theorem

(i) If $\mu_n \to \mu_\infty$, $\varphi_{\mu_n} \to \varphi_{\mu_\infty}$ uniformly on compacts

[Independent of x also, so it is u c]



(ii) If $\varphi_n \to \varphi$ pointwise for some φ continuous at zero, then $\varphi = \varphi_{\mu_\infty}$ for some μ_∞ & $\mu_n \to \mu_\infty$

[Any continuous function on a closed interval is uniformly continuous]

Pf

$A_\delta = \{x = [x_1 \dots x_d],\ |x_i| \le \delta \ \forall i\}$

Volume of $A_\delta = (2\delta)^d$

$$\frac{1}{(2\delta)^d}\int_{A^\delta} \varphi_{\mu_n}(t)\, dt$$

$$= \frac{1}{(2d)}\int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{\sin \delta x_i}{\delta x_i}\, d\mu_n(x)$$

$$\int f I_A = \int_{I_A} f$$

$$\int I_A e^{i\langle t,x\rangle}\, dt = \prod_{j=1}^{d} \int_{-\delta}^{\delta} e^{i\langle t_j x_j\rangle}\, dt_j$$

$$f_d(x) = \frac{1}{(2\delta)^d}\prod \frac{\sin \delta x_i}{\delta x_i}$$

$$f_\delta(\infty) = 0$$

$$\bar{R}^d = R^d \cup \{\infty\}.$$

$$1 = \varphi(0) = \lim_{\delta \to 0} \frac{1}{(2\delta)^d} \int_{A_\delta} \varphi(t)\,dt.$$

$$\frac{1}{(2\delta)^d} \int_{A^\delta} \varphi(t)\,dt = \lim_{k \to \infty} \frac{1}{(2\delta)^d} \int_{A^\delta} \varphi_n(t)\,dt$$

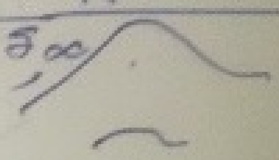
$$= \lim_{k \to \infty} \frac{1}{(2\delta)^m} \int_{\bar{R}^m} f_{\delta,}(x)\,d\mu_n(x).$$

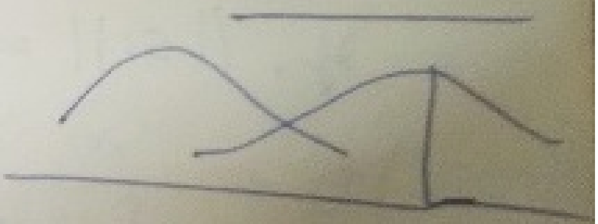
If $\mu_n \to \mu_\infty$, then RHS $= \int_{\bar{R}^m} f_\delta(x)\,d\mu_\infty(x)$

$$= \int_{R^m} f_\delta(x)\,d\mu(x)$$

Given $\langle \mu_n \rangle$ $\quad \mu_n(x) \to a\mu_\infty + (1-a)\delta_\infty$

in general



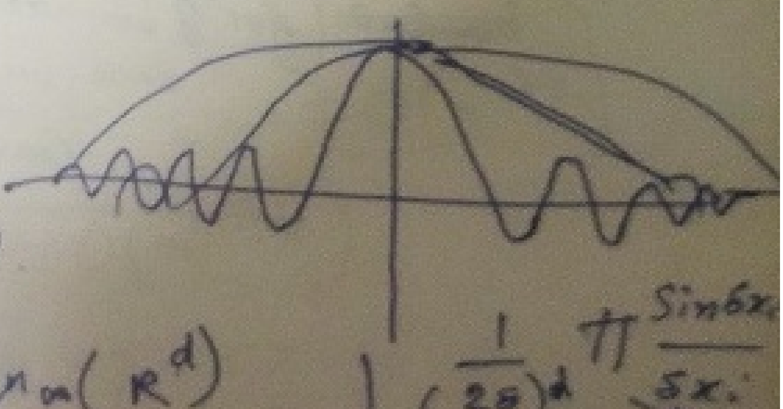


$$\frac{1}{(2\delta)^d} \int_{A^\delta} \varphi(t)\,dt$$

$$= \int_{R^d} f_\delta(x)\,d\mu_\infty(x)$$

$\delta \downarrow 0 \Rightarrow$ RHS $\to \mu_\infty(R^d)$.

LHS $= 1$.



$\frac{1}{(2\delta)^d} \prod \frac{\sin \delta x_i}{\delta x_i} \to 1$ as $\delta \to 0$

$P \cdot \psi \; \mu_\infty = \psi \;\Rightarrow$ limit of μ_n unique.

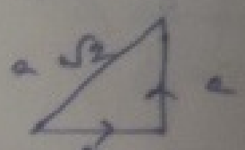
$\liminf \mu_n(G) \geq \mu_\infty(G)$.

Central limit theorem

$\langle X_i \rangle$ i.i.d., $E(X_i) = 0$, $E[X_i^2] = \sigma^2$;

$$S_n = \sum_{i=1}^{n} X_i .$$

$$E[S_i^2] = n\sigma^2 .$$



$$\|X\| = E[X^2]^{1/2} .$$

$$\langle X, Y \rangle = E[XY] .$$

If, $\|S_i\| = \sigma\sqrt{n}$.

SLLN

$$\frac{S_n}{\sqrt{n}\,\sigma} \to N(0,1)$$

~~bounded~~ [for bounded variance case we get Gaussian]

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \to N(0,1)$$

$\sigma^2 = 1$.

$$E\left[e^{i\langle t, \frac{S_n}{\sqrt{n}}\rangle}\right].$$

$$= E\left[\prod_{j=1}^{n} e^{i\langle t, \frac{X}{\sqrt{n}}\rangle}\right].$$

$$= \prod_{j=1}^{n} E\left[e^{i\langle t, \frac{X}{\sqrt{n}}\rangle}\right].$$

$$= E\left[e^{i\langle t, \frac{X_1}{\sqrt{n}}\rangle}\right]^n, \left(\because i.i.d\right)$$

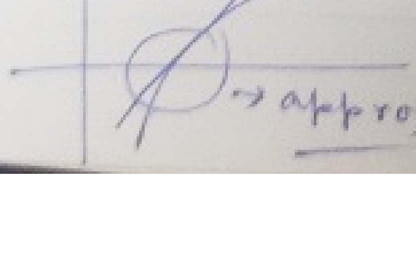
$$\ln E\left[e^{i\langle t, \frac{S_n}{\sqrt{n}}\rangle}\right]$$

$$= n \ln\left[e^{i\left(t, \frac{X}{\sqrt{n}}\right)}\right]$$

$$= n E\left[1 + i\frac{tX_1}{\sqrt{n}} - \frac{1}{2} t^2 \frac{X_1^2}{n} + o\left(n^{-3/2}\right)\right]$$

$$= n \ln\left(1 - \frac{1}{2}\frac{t^2\sigma^2}{n^2} + \underline{O\left(n^{-3/2}\right)}\right).$$

$$\approx -\frac{1}{2}t^2\sigma^2 + O\left(n^{-3/2}\right)$$



approx

$\log x \approx x - 1$.

$\log(1 - x) \approx -x$.

$$\varphi_n(t) \triangleq E\left[e^{i\frac{tS_n}{\sqrt{n}}}\right]$$

$$\varphi_n(t) \longrightarrow e^{-\frac{1}{2}t^2\sigma^2}$$

$\langle X_i \rangle$ indep.

$E[X_i] = 0$,

$E[X_i^2] = \sigma_i^2 < \infty$,

$s_n^2 = \sum_{i=1}^{n} \sigma_i^2$

$$\frac{\sum X_i}{\sqrt{n}} \qquad \frac{\sum X_i}{n} \to 0$$

spread out. SLLN.

Stable distr^n: Gaussian

$$\left(\frac{S_n - a_n}{n^m}\right)$$

Lindeberg condition:

$s_n^2 > 0$ from some n on & $\forall \epsilon > 0$.

$$\sum_{j=1}^{n} \int_{\langle |X_j| > \epsilon s_j \rangle} X_j^2 \, dP = o(s_n^2) \qquad (L)$$

Thm (CLT) $\quad (L) \Leftrightarrow \frac{S_n}{s_n} \to N(0,1)$ in law & $\frac{\sigma_n}{s_n} \to 0$ as $s_n \to \infty$

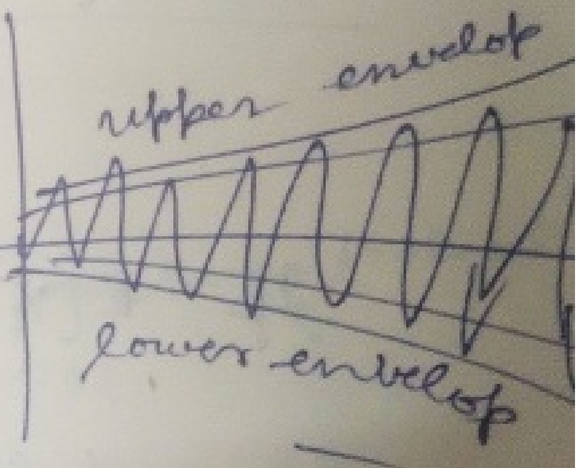
Law of iterated logarithm

Suppose, $\dfrac{\sum_{i=1}^{n} E\left[|X_n|^3\right]}{s_n^3} \le \dfrac{C}{(\ln s_n)^{1+\epsilon}}$

for some $C > 0$, $\epsilon \in (0, 1)$.

Then $\limsup \dfrac{S_n}{\sqrt{2 s_n^2 \ln\ln s_n}} = 1$ a.s.

$\liminf \dfrac{S_n}{\sqrt{2 s_n^2 \ln\ln s_n}} = -1$ a.s.



Cramer's thm.:

$\langle X_i \rangle$ i.i.d.

$E\left[e^{\theta X_1}\right] < \infty$,

$I(x) = \sup_{\theta} \left(\theta x - \ln E\left[e^{\theta X}\right]\right)$

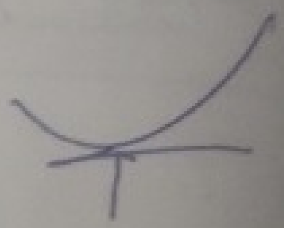
Thm

Suppose $I(\theta) < \infty \ \forall \theta$, then for $B \subset \mathbb{R}$

$$-\inf_{x \in B^{\circ}} I(x) \le \liminf_{n \to \infty} \frac{1}{n} \ln P\left(\frac{S_n}{n} \in B\right)$$

$$\le \limsup_{n \to \infty} \frac{1}{n} \ln P\left(\frac{S_n}{n} \in B\right).$$

$$\le -\inf_{x \in \bar{B}} I(x).$$



$I(E[X_1]) = 0$

$\frac{S_n}{n} \to E[X_1]$.

$I(x) \ge 0$

If $E[X_1] \in B$, all above $= 0$.

$$P\left(\frac{S_n}{n} \in B\right) \simeq e^{-n \inf_B I(x)}$$

$$\frac{1}{n} \log \left(\sum_{n=1}^{N} e^{-\lambda_i n} p_i(n) \right), \quad \lambda_i \ge 0$$

$\lambda_1 < \lambda_2 < \lambda_3 < \dots$

$$\simeq \frac{1}{n} \log \left(e^{-\lambda_1 n} p_1(n) \right)$$

$$\frac{1}{n} \log \left(e^{-\lambda_1 n} \left(p_1(n) + \sum_{i \ge 2} e^{(\lambda_1 - \lambda_i)} p_i \right.\right.$$

$\simeq -\lambda_1 + \frac{\log P_1(n)}{n} \to -\lambda_1$ (slowest exponential)

Large deviations

$$\frac{e^{-\beta \sum V(\sigma_i \sigma_j)} \, \rho(d\sigma)}{\sum e^{-\beta \sum V(\sigma_i \sigma_j)}}$$

Lattice pts.

± 1

Statistical entropy, limits

Tailor expansion of assymptotic function & take approx.

Skorokhod's theorem

$X_n \to X_\infty$ in law, then on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ & r.v.s.

$(\tilde{X}_n \; n = 1, 2, \dots, \infty)$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ s.t.

law of X_n = law of $\tilde{X}_n$ for $n = 1, 2, \dots, \infty$ &

$\tilde{X}_n \to \tilde{X}_\infty$ a.s.

(Careful not taking it twice at a time)

one at a time

$X_n \to X_\infty$ a.s. &

$\langle X_n \rangle$ uniformly integrable $\Rightarrow$

$E[X_n] \to E[X_\infty]$.

one at a time

~~sup~~ $\sup_n \left[|X_n| I \langle |X_n| \geq \alpha \rangle \right] \to 0$

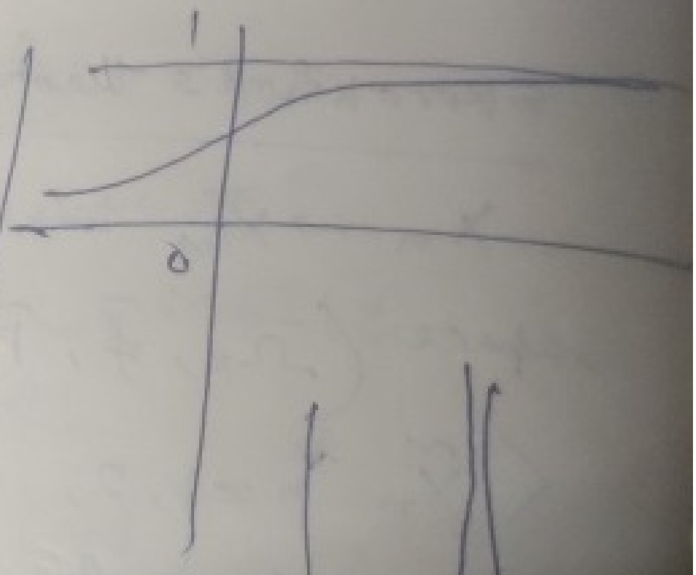
Can't use $E[|X_n - X| > \epsilon] \to 0$

Not one at a time, two a time.

$U \sim$ uniform $[0, 1]$.

$F(X)$ distrⁿ of X, ~~continuity~~

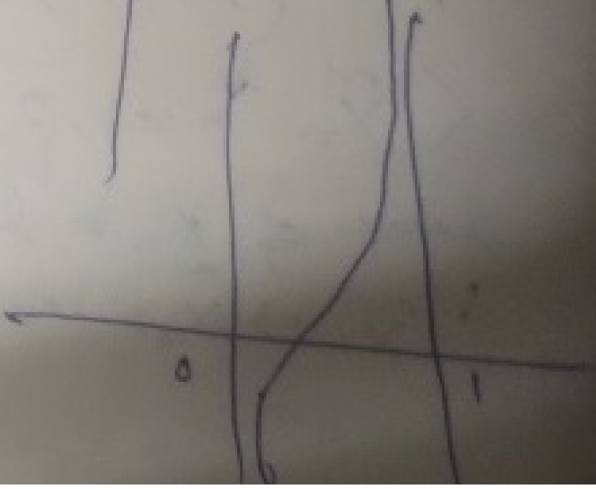
continuous strictly increasing $F^{-1}(U)$ has same law as X.



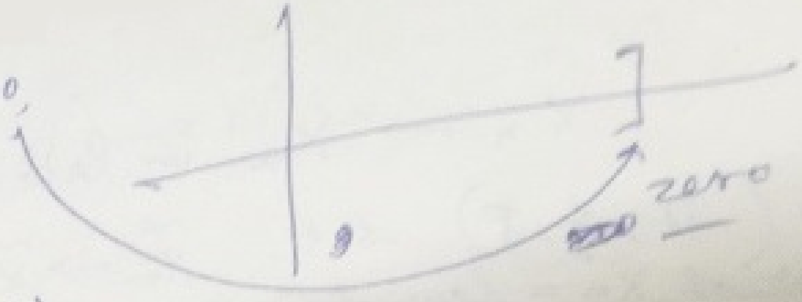
F_n distrⁿ fn of X_n, $n = 1, 2, \dots, \infty$,

F_n continuous strictly increasing.

$X_n \to X_\infty$ in law $\Rightarrow F_n \to F_\infty$



since $P[X=a]=0$,



$F_n^{-1} \to F_\infty^{-1}$

$F_n^{-1}(U) \to F_\infty^{-1}(U)$

Mersenne prime

twister

Chaotic cryptography

Problem books