17.08.05

Prob. Space: tripple $(\Omega, \mathcal{F}, P)$

Ω : a set "sample space", $\omega \in \Omega$ 'sample point'.

$\mathcal{F}$: collection of subsets of Ω called 'event points'.

(i) $\phi, \Omega \in \mathcal{F}$

(ii) closed under complementation & countable unions/ intersection.

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$. $A_i \in \mathcal{F} \Rightarrow$

$\forall i \in \bigcup_i A_i \; \bigcap_i A_i \in \mathcal{F}$

$P : \mathcal{F} \to [0, 1]$

Probability Measure

$\Updownarrow$

(i) $P(\phi) = 0$, $P(\Omega) = 1$

(ii) $P(A^c) = 1 - P(A)$

(iii) $\langle A_i \rangle$ disjoint $\Rightarrow P(\bigcup_i A_i) = \sum_i P(A_i)$

Countable additivity

Additional properties of P:

(i) $A \subset B \Rightarrow P(A) \le P(B)$

(ii) $P(\bigcup_i A_i) \le \sum_i P(A_i)$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Theories of probability
— Terrence Fine

$P(A)$ — Kolmogorov — Measure theoretic formulation

Where do you get P?

(i) usually specified for an algebra $\mathcal{F}_0$ and extended to $\sigma(\mathcal{F}_0)$

$\sigma(\mathcal{F}) =$ smallest σ-field containing $\mathcal{F}$
$=$ intersection of σ-fields containing $\mathcal{F}$

(ii) (a) principle of insufficient reason
Darwin–Fowler

(b) worst-case

$-\sum_i p_i \log p_i$
$i \in \langle 1, \dots x \rangle$

$\sum p_i H_i = c$

$\sum_i p_i = 1, p_i \ge 0$

Mathematical foundation of information theory

$$p_i = \frac{e^{-\beta H_i}}{\sum_j e^{-\beta H_j}}$$ (Gibb's distrⁿ)

Gaussian is actually the worst case.

$$-\int p(x) \log p(x)\, dx$$

$$\int x^2 p(x)\, dx = \sigma^2$$

$$\int x\, p(x)\, dx = 0, \quad p(\cdot) \geq 0, \quad \int p(x)\, dx = 1$$

** (c) physical reasoning

(d) subjective probability.

(e) simplicity. **

Agree/disagree Aumann

** (f) measurement

Random variables

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} (E, \xi)$$

$X^{-1}(A) \triangleq \{\omega : X(\omega) \in A\}$,

$A \in \xi$ (collection of subsets of E)

$\boxed{\triangleq \ \underline{\underline{\text{def}}}}$ $\quad \xi$ is a σ- algebra

Need : $X^{-1}(A) \in \mathcal{F}$ for $A \in \xi$. If this holds, X is said to be an E-valued random variable.

Usually ξ 'obvious'.

'Borel' $\quad \{x : d(y,x) < r\}$, 'completed'.

σ-algebra on open & closed circles are same.

Define prob. measure μ on (E, ξ) by :

$$\mu(A) = P(\{\underset{\omega \in \Omega}{\omega :} X(\omega) \in A\})$$

(i) $\mu(\phi) = 0$, $\mu(E) = 1$.

(ii) $\mu(A^c) = 1 - \mu(A)$.

$$\begin{aligned}
\mu(A^c) &= P(\{\omega : X(\omega) \in A^c\}) \\
&= P(\{\omega : X(\omega) \in A\}^c) \\
&= 1 - P(\{\omega : X(\omega) \in A\}) \\
&= 1 - \mu(A)
\end{aligned}$$

(iii) $A_1, A_2 \ldots$ disjoint in $\mathcal{E}$.

$$\Rightarrow \mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

$$\begin{aligned}
\mu\left(\bigcup_i A_i\right) &= P\left(\left\{\omega : X(\omega) \in \bigcup_i A_i\right\}\right) \\
&= P\left(\bigcup_i \{\omega : X(\omega) \in A_i\}\right) \\
&= \sum_i P(\{\omega : X(\omega) \in A\}) \\
&= \sum_i \mu(A_i)
\end{aligned}$$

μ is called the law of X.

$E = R$ define $F(x) = \mu((-\infty, x])$, $x \in R$.

$F(\cdot)$ carries complete information about μ.

$\mu((a,b)) = F(b) - F(a)$.

1. Claim : $\lim_{x \to -\infty} F(x) = 0$,

2. $\lim_{x \to +\infty} F(x) = 1$

· Countable additivity $\Rightarrow$

$$P\left(\bigcup_i A_i\right) = \lim_{N \uparrow \infty} P\left(\bigcup_{i=1}^{N} A_i\right)$$

$$P\left(\bigcap_i A_i\right) = \lim_{N \uparrow \infty} P\left(\bigcap_{i=1}^{N} A_i\right).$$

$B_1 = A_1$, $B_2 = \underbrace{A_2 \setminus A_1}_{A_2 - A_1}$, .. $B_n = A_n \setminus \left(\bigcup_{j<n} A_j\right)$

$\bigcup_i A_i = \bigcup_i B_i$, $\langle B_i \rangle$ disjoint.

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$$

$$P\left(\bigcup_i A_i\right) = P\left(\bigcup_i B_i\right) = \sum_i P(B_i)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} P(B_i) = \lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$

$\downarrow\downarrow$ strictly decreasing

$$= \lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i \right)$$

$$\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \mu\left((-\infty, x] \right)$$

$$= \lim_{x \to -\infty} \mu\left(\bigcap_{y \geq x} (-\infty, y] \right)$$

~~]]]]~~

$$= \mu\left(\bigcap_{y} (-\infty, y] \right) = \mu(\phi) = 0.$$

~~[[[[~~

3. $\lim_{x \to \infty} F(x) = 1$. $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \mu[x, \infty) = \lim_{x \to \infty} \mu\left(\bigcap_{y \leq x} [y, \infty) \right)$

$$= \mu\left(\bigcap_{y} [y, \infty) \right) = \mu(\phi) = 0$$

$$\lim_{y \downarrow x} F(y) = F(x). \quad \Big(\text{because,}$$

$$\lim_{y \downarrow x} F(y) = \mu\left(\bigcap_{y > x} (-\infty, \cancel{x}] \right)$$

(y decreasing to x)

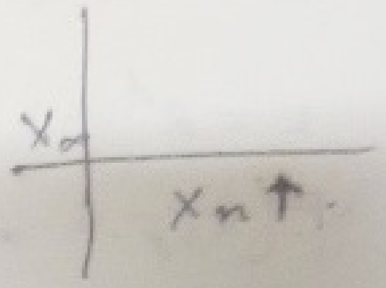
$$= \mu\left((-\infty, x] \right) = F(x)$$

4. $F(\cdot)$ increasing $\left(A \subseteq B \Rightarrow \mu(A) \leq \mu(B) \right)$

~~$= \mu\left((-\infty, x] \right) = F(x)$~~.

5. $\lim\limits_{y \uparrow\uparrow x} F(y)$ exists and is and is $\leq F(x)$

$\left(\because F(\cdot) \text{ nondecreasing and bounded above by } 1 \right)$.



$X_\infty = \text{lub}\, \langle x_n \rangle$

$x_n \to X_N'$

$$\lim_{y \uparrow\uparrow x} F(y) = \lim_{y \uparrow\uparrow x} \mu((-\infty, y])$$

$$= \mu((-\infty, x)) \leq \mu((-\infty, x]) = F(x)$$

if $\lim\limits_{y \uparrow\uparrow x} F(y) < F(x)$.

$$\mu(\langle x \rangle) = \mu((-\infty, x]) - \lim_{y \uparrow\uparrow x} \mu((-\infty, y]).$$

$$= F(x) - \lim_{y \uparrow\uparrow x} F(y).$$

6. $F(\cdot)$ has countably many jumps.

$z_n \triangleq \#$ of jump of size $\geq \frac{1}{n}$ then $z_n \leq n$. let $\tilde{Z}_n = \langle$ jumps of size $\geq \frac{1}{n} \rangle$

$z_n = |\tilde{Z}_n| \leq n$.

$\therefore \bigcup_n Z_n$ is countable.

$E = R^d$.

$F(x_1, \cdots, x_d) = \mu\big((-\infty, x_1] \times (-\infty, x_2) \times \cdots (-\infty, x_d]\big)$.

$$\left.\begin{aligned} &\lim_{x_i \downarrow -\infty} F(x_1 \ldots x_d) = 0. \\ &\lim_{x_1 \uparrow \infty \cdots x_2 \uparrow N} F(x_1 \ldots x_d) = 1. \end{aligned}\right\}$$

Law & Distribution are almost equivalent
↳ more general.

$\lambda(\cdot)$ is a positive measure $(\Omega, \mathcal{F})$,

$\mu(\cdot)$ a signed measure on $(\Omega, \mathcal{F})$.

$l(\cdot)$ Lebesgue measure on R^d, μ is above

Suppose $l(A) = 0 \Rightarrow \mu(A) = 0$.

$\mu << l$.

μ is said to be absolutely continuous w.r.t. l.

Theorem: There exists $p(\cdot) : R^d \to [0,\infty)$

s.t. $\mu(A) = \int_A p(x)\,dx$.

$p(\cdot)$ is called the density of μ.

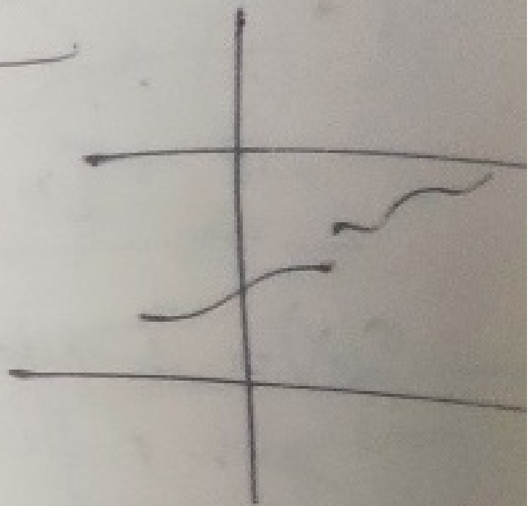
(Thus $F(x) = \int_{-\infty}^{x} p(y)\,dy$).

μ is called the law of X

Define $\delta_{x_0}(A) = 1$ if $x_0 \in A$

$= 0$ if $x_0 \notin A$.

(Dirac function)



(continuous)

Devil's staircase

Not absolutely continuous

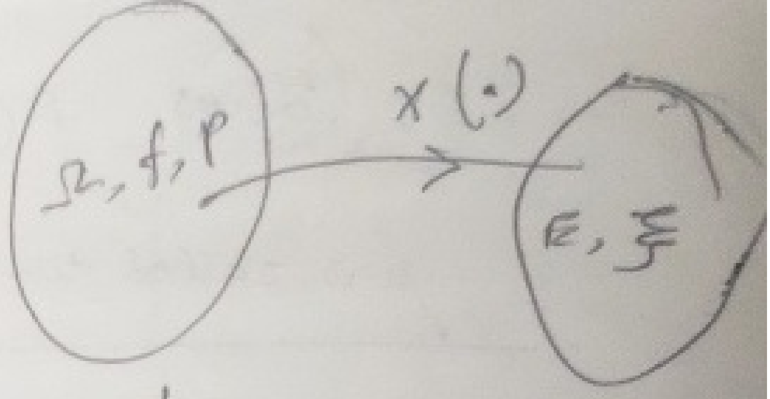
Almost everywhere $\exists$ derivative and 0

where $\nexists$ derivative form cantor set

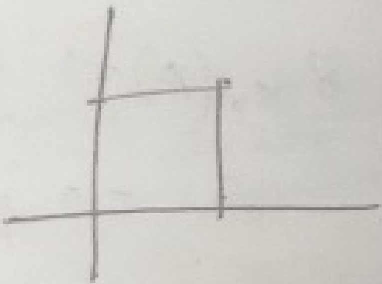
Only absolutely continuous functions satisfy the laws of calculus: fn is integral of its derivative

$$\mu_2 << \mu_1 \Rightarrow \mu_1(A) = 0 \Rightarrow \mu_2(A) = 0$$

$$E_{\mu_2}[X] = E_{\mu_1}[\Lambda X]$$



$\Omega = [0, 1]$



$\sigma(X)$ = smallest sub-σ field of $\mathcal{F}$ which contains $X^{-1}(A)$, $A \in \xi$

(i.e., smallest sub-σ field of $\mathcal{F}$ w.r.t. which X is measurable)

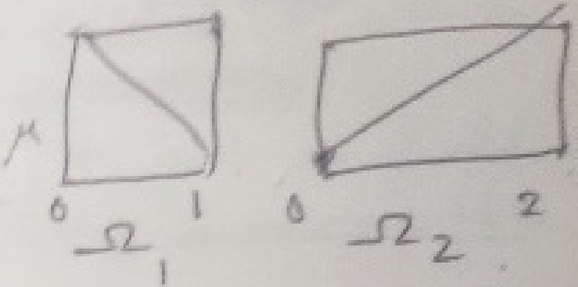
= intersection of all sub-σ-field of $\mathcal{F}$ w.r.t. which X is measurable

$= \langle X^{-1}(A), A \in \mathcal{E} \rangle \Rightarrow$ Needs to check that it's a σ-field

Set opⁿs are preserved under σ | $X^{-1}(A)$

$X_1^{-1}(A) \longleftrightarrow X_2^{-1}(A)$

both uniformly distributed



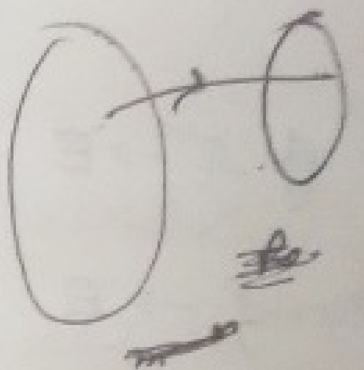
Algebraic picture:

Canonical space:

$\Omega = E$, $\mathcal{F} = \mathcal{E}$,

$P = \mu$

$X(\omega) = \omega$



Quantum $\rightarrow$ Non-commutative algebra

$\sigma(X_\alpha, \alpha \in I)$

= smallest σ-algebra w.r.t. which all X_α are measurable, $\alpha \in I$

$X \quad (X_1 \cdots X_n)$

$(X_1, X_2, \cdots]$

$$\sigma(X_\alpha, \alpha \in I) = \bigcup_{\substack{J \subset I \\ J \text{ countable}}} \sigma(X_\alpha, \alpha \in J)$$

~~this is~~

[only those things are measurable that are countable].

~~Proba~~ Measurement in continuous space

$X : \Omega \to E$

$\mathcal{F} \to \mathcal{E}$

$(\Omega, P) \to (E, \mathcal{E})$

Royden

Real Analysis