2. (a) Sandipan Dey   MATH 685   HW-3

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$\forall v \in V(K_n)$, $d(v) = n-1$. $(n \geq 3)$

To have an Euler cycle, $\forall v$, $d(v) \in \mathbb{Z}^+_{even} \Rightarrow n-1 \in \mathbb{Z}^+_{even}$

$\Rightarrow n \in \mathbb{Z}^+_{odd} \Rightarrow n = 3, 5, \ldots$

(b) K_n has an Euler cycle $\forall n \in \mathbb{Z}^+_{odd}$.

For $n \in \mathbb{Z}^+_{even}$ $(n \geq 4)$, K_n does not have an Euler cycle.

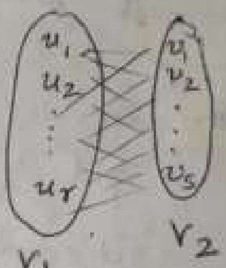
$\forall v$, $d(v) = n-1 \in \mathbb{Z}^+_{odd}$.

But in order to have an Euler trail but not an Euler cycle, it must have ~~at least 2~~ exactly 2 vertices of odd degree. But

$(n \geq 4 \wedge n \in \mathbb{Z}^+_{even}) \Rightarrow$ # vertices with odd degree in $K_n = n \geq 4 \neq 2$ (all vertices)

Hence, there is No such K_n.

(c) K_{rs} is complete bipartite graph



$\Rightarrow \exists V_1, V_2 \subseteq V(K_{rs})$, $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V(K_{rs})$,

$V_1 = \{u_1, u_2, \ldots, u_r\}$, $V_2 = \{v_1, v_2, \ldots, v_s\}$

$\forall i = 1(1)r, \forall j = 1(1)s$, $(u_i, v_j) \in E(K_{rs})$,

$\forall i,j = 1(1)r$, $(u_i, u_j) \notin E(K_{rs})$

$\forall i,j = 1(1)s$, $(v_i, v_j) \notin E(K_{rs})$

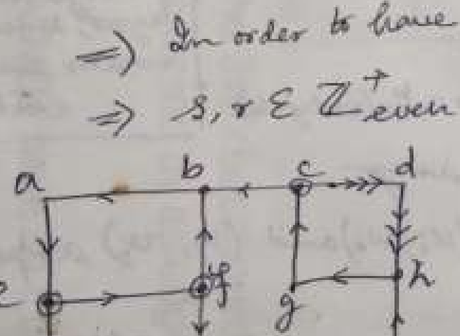
$\Rightarrow$ ~~construct a Euler trail $u_1 - v_1 - u_2 - v_2 - \cdots - u$~~ $\forall v_j \in V_2$, $d(v_j) = r$.

$\forall u_i \in V_1$, $d(u_i) = s$,

$\Rightarrow$ In order to have Euler cycle, $\forall v \in V(K_{rs})$, $d(v) \in \mathbb{Z}^+_{even}$

$\Rightarrow s, r \in \mathbb{Z}^+_{even}$

4.



vertices with odd degrees $= 6$ (degree 3) (b, c, e, f, h, j)

with even degrees $= 5$.

~~Remove any two~~ choose any 4 vertices out of the 6 odd degree vertices (for example ~~Remove~~ e, f, j, b). Remove edges from the graph so that these 4 vertices become degree 2 vertices (for example remove edges ef, fj and fb, at least 3 edges

① f-b-a-e-f-j-k-h-g-c ↑
② e-i-j ← raise pencil
c-d-h

must be removed) The remaining graph has exactly couple of odd degree (degree 3) vertices (namely ~~f and b~~ c, h) Hence, there exists an Euler trail that starts at one and ends at another in the remaining graph (namely ~~f~~ebaeijkhgcdh), hence the trail can be drawn without raising the pencil. Now, there are 3 more edges left to be drawn, namely bf, ef, fj (precisely the ones we removed). Consider this graph now containing vertices b, e, f, j and edges (b, f), (e, f) and (f, j) again this graph has 3 vertices, 3 of them with odd degrees. Again an Euler trail can be drawn (bfj) but still leaves one more edge (namely ef). Hence we have to raise the pencil up at least twice. ~~Hence~~

8. Let u_i $(i=1..k)$ and v_i $(i=1..k)$ be the $2k$ vertices with $d(u_i) \in \mathbb{Z}^+_{odd}$ $\wedge\ d(v_i) \in \mathbb{Z}^+_{odd}$, $\forall i=1..k$.

Let's construct a new graph G' from G in the following manner: add k new vertices w_i $(i=1..k)$ to G and add $2k$ edges (u_i, w_i) and (w_i, v_i) $\forall i=1..k$, s.t. $V(G') = V(G) \cup \{w_1, w_2, \dots, w_k\}$ $\wedge$

$$E(G') = E(G) \cup \{(u_1, w_1), \dots, (u_k, w_k)\} \cup \{(w_1, v_1), (w_2, v_2), \dots, (w_k, v_k)\}$$

Now, $d(u_i) \in \mathbb{Z}^+_{even}$, $d(v_i) \in \mathbb{Z}^+_{even}$ $\forall i=1..k$ in G'.
Also, $d(w_i) = 2 \in \mathbb{Z}^+_{even}$

Rest of the vertices in G' does not undergo change in degree, hence they remain even degree vertices.

Hence, $\forall v \in V(G')$, $d(v) \in \mathbb{Z}^+_{even}$

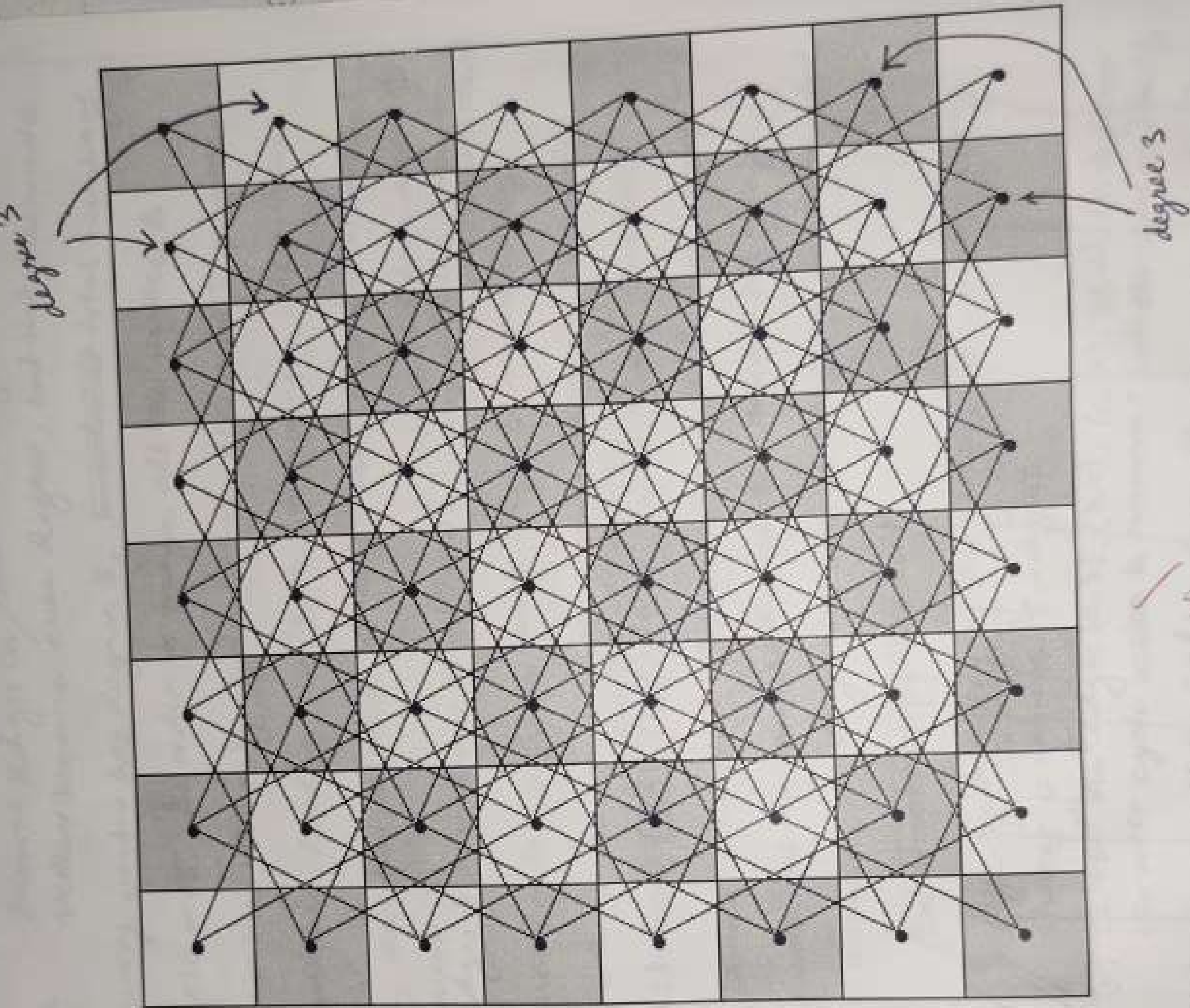
$\Rightarrow \exists$ an Euler trail in G' (which is an Euler cycle)

$\dots u_i w_i v_i \dots$
$\dots v_i w_i u_i \dots$

Also, since $d(w_i) = 2$, $\forall i=1..k$, in G', any Euler trail in G' must have the 2 edges (u_i, w_i) and (w_i, v_i) appeared consecutively, $\forall i=1..k$.

$\Rightarrow$ Removal of w_i $\forall i=1..k$ and $2k$ edges will result in k edge-disjoint open trails of G s.t. each edge in G is present in precisely one of these k trails.

10.

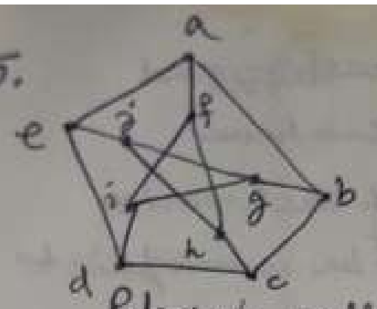


5/5

As can be seen from above, in the graph (with 64 vertices) constructed from the 8x8 chessboard where every square is considered as a vertex and any two vertices (squares) are connected by an edge iff it is a valid move for a knight.

As can be seen, there are more than 2 vertices of odd degree, the graph CAN NOT have an Euler trail. Consequently, it's NOT possible for a night to move around the board so that it makes every possible move exactly once.

15.



Petersen's graph

(a)

~~To find the mini~~ # vertices in the graph = 10

In order to have an Euler Cycle, all vertices must be of even degree. Hence we have to remove edges in such a way that all vertices become even degree, but not disconnected.

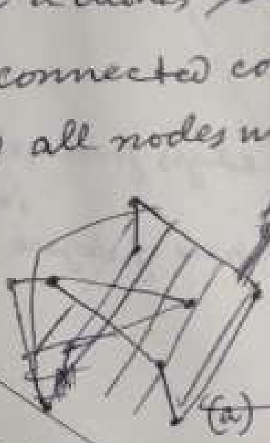
Now, since every vertex has degree 3, ~~in order to~~ total number of edges $= \frac{1}{2} \times 3 \times 10 = 15$. In order to make all the vertices ~~to~~ even degree vertices, the only choice (where we need to remove minimum number of edges) will be: to reduce all vertices to degree 2, then the total # of edges will be $= \frac{1}{2} \times 2 \times 10 = 10$.

Hence, minimum # edges to remove $= 15 - 10 = 5$. (We can't reduce degree of any vertex to 0, since it will become isolated then)

Now, the question is: which ~~edge~~ edges to remove. We must be sure that the remaining graph remains connected still. For instance, {(e,j), (a,f), (g,b), (h,c), (i,d)} can NOT be a choice, since removal of these 5 edges leaves ~~only~~ 2 connected components, ~~each~~ although the remaining graph has all nodes with even degrees, but it becomes disconnected.

NOT a choice

~~But it also appears that~~



By symmetry of the graph, there are following choices to remove 5 edges:

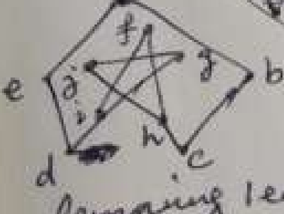
(a) choose all the 5 edges (a, b), (b, c), (c, d), (d, e), (e, a) from the outer cycle ~~to this~~ to remove: but this does not help

(b) choose all the 5 edges from the inner cycle (j, g), (f, h), (i, f), (i, g), (f, g) to remove: this does not help too, since ~~there are~~ some odd degree vertices still remain after removal of these edges.

(c) choose 4 edges from outer cycle and 1 from inner cycle or vice versa; ~~this~~ there still remains odd degree ve

(d) choose 2 edges from outer cycle, 3 edges from inner cy

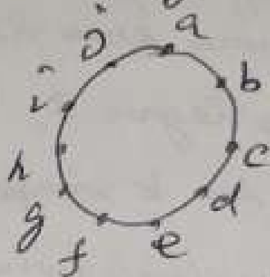
~~but it~~



Remaining 1 edge from external edge group and 3 edges from the middle: still has 2 vertices with degree 3.

G does not have a hamilton cycle

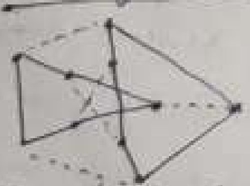
Now, a graph with 10 vertices and 10 edges, with each vertex having degree 2 and having an Euler cycle ~~can~~ must be isomorphic the following graph



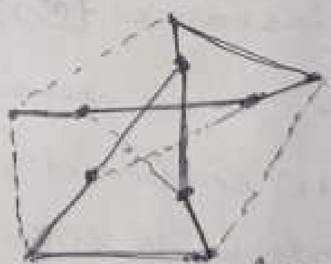
Since ~~Hamilt~~ Petersen's graph does not have a Hamiltonian ~~cycle~~ circuit graph obtained after removing 5 edges from Petersen's graph will not have a hamiltonian ~~cycle~~ circuit. Hence, whatever 5 edges we choose to remove, we can't get the above graph, a contradiction.

~~Hence, edges can't be removed~~

Hence, no matter what 5 edges we remove, we can't get an Euler cycle from Petersen's graph.

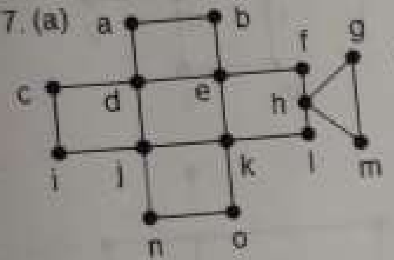


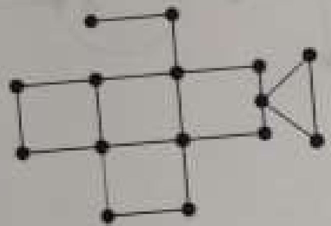
removing 2 edges from external
" 2 " " interal
" 1 edge " middle
~~create~~ disconnects the graph

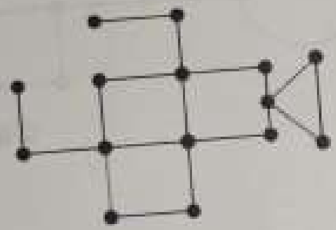


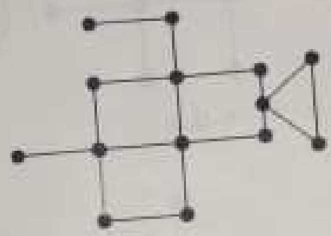
removing 3 edges from external
" 2 " " inernal
Still has vertices of degree 3

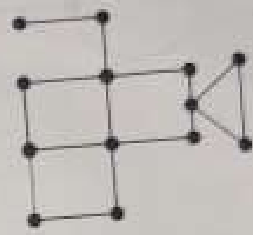
7. (a)

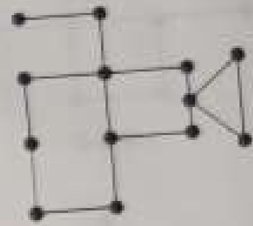


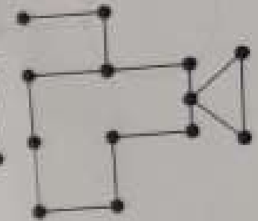

a-d

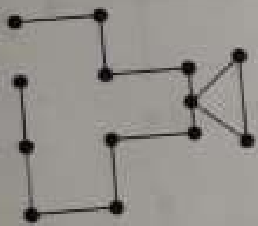

a-d-c


a-d-c-i


a-d-c-i-j


a-d-c-i-j-k


a-d-c-i-j-k-e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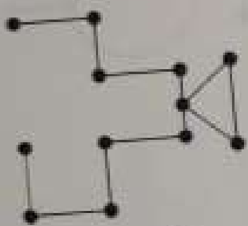
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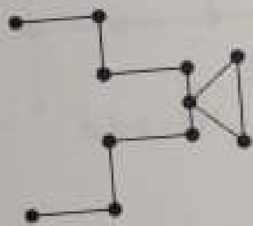
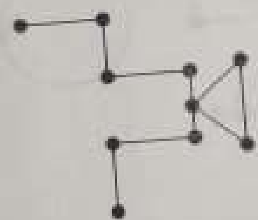
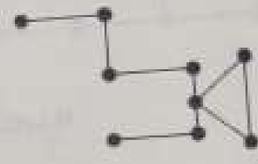
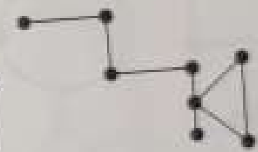
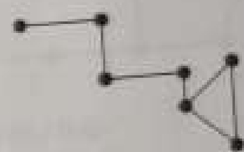
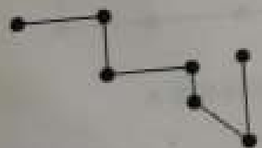

a-d-c-i-j-k-e-d-j-n


a-d-c-i-j-k-e-d-j-n-o


a-d-c-i-j-k-e-d-j-n-o-k


a-d-c-i-j-k-e-d-j-n-o-k-l


a-d-c-i-j-k-e-d-j-n-o-k-l-h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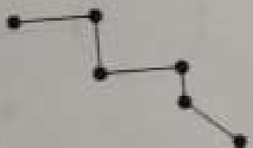

a-d-c-i-j-k-e-d-j-n-o-k-l-h-g-m

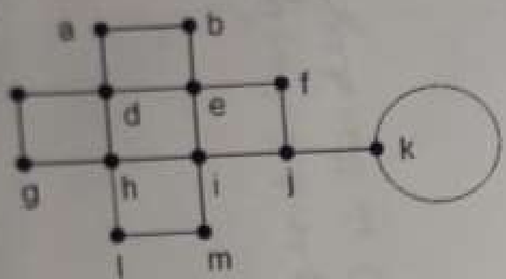

a-d-c-i-j-k-e-d-j-n-o-k-l-h-g-m-h

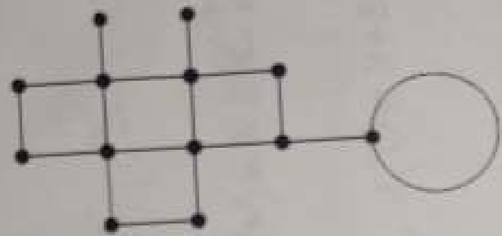

a-d-c-i-j-k-e-d-j-n-o-k-l-h-g-m-h-f


a-d-c-i-j-k-e-d-j-n-o-k-l-h-g-m-h-f-e

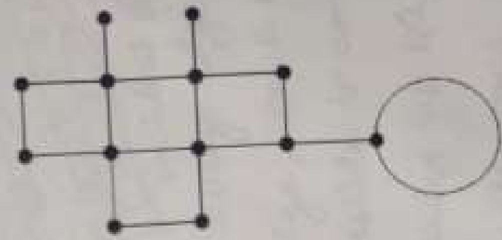

a-d-c-i-j-k-e-d-j-n-o-k-l-h-g-m-h-f-e-b-a

b)

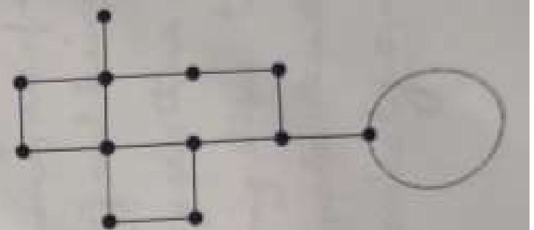




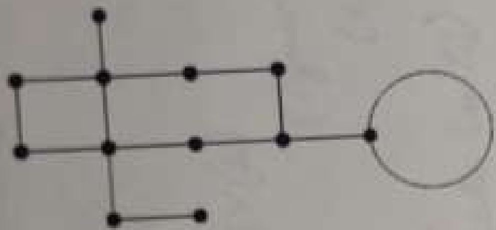
a-b



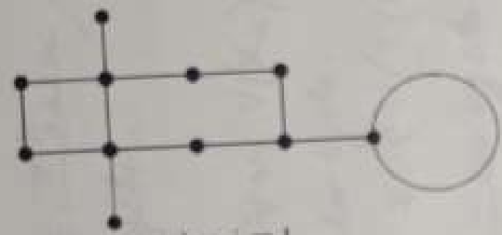
a-b-e



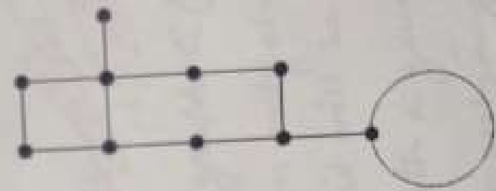
a-b-e-i



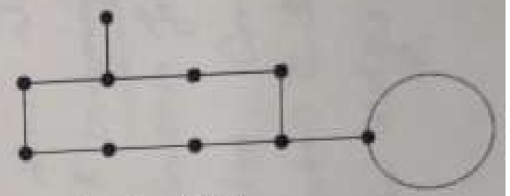
a-b-e-i-m



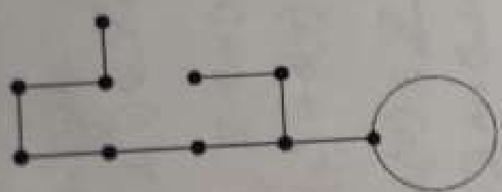
a-b-e-i-m-l



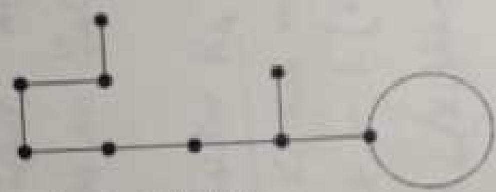
a-b-e-i-m-l-h



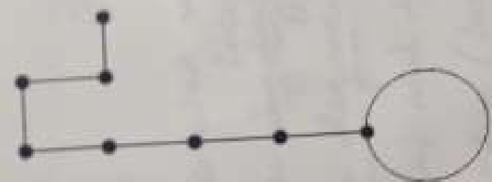
a-b-e-i-m-l-h-d



a-b-e-i-m-l-h-d-e



a-b-e-i-m-l-h-d-e-f



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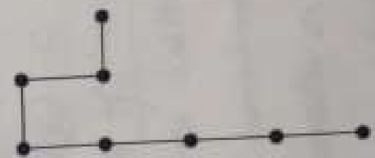

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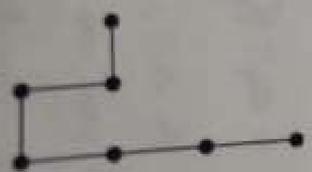

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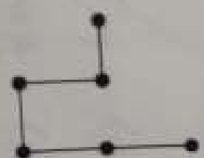

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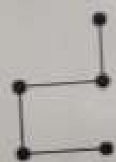
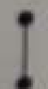


a-b-e-i-m-l-h-d-e-f-j-k-k-j-i-h

a-b-e-i-m-l-h-d-e-f-j-k-k-j-i-h-g

a-b-e-i-m-l-h-d-e-f-j-k-k-j-i-h-g-c



a-b-e-i-m-l-h-d-e-f-j-k-k-j-i-h-g-c-d-a

. (a) L(G) has an Euler cycle if G has an Euler cycle

Proof: G has an Euler Cycle

$\Rightarrow \forall v \in V(G),\ d(v) \in \mathbb{Z}^+_{even}$.

Now, $\forall w \in V(L(G)),\ \exists\, e=(u,v) \in V(G)$

$\wedge\ d(w) = d(u)+d(v)-2$ (ignoring contribution of (u,v) in the degrees)

Now, $d(u), d(v) \in \mathbb{Z}^+_{even}$
(Since $\exists$ an Euler cycle in G)

$\Rightarrow \forall w \in V(L(G)),\ d(w) \in \mathbb{Z}^+_{even}$

$\Rightarrow$ L(G) has an Euler cycle.

5/5

(b)

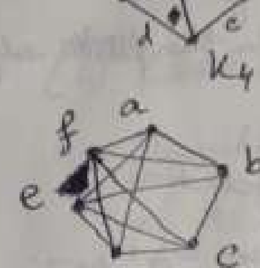
K_4

Consider K_4, $\forall v \in V(K_4),\ d(v)=3 \in \mathbb{Z}^+_{odd}$

$\Rightarrow K_4$ has no Euler cycle.

But in $L(K_4)$, $\forall v \in V(L(K_4)),\ d(v) = 3+3-2 = 4 \in \mathbb{Z}^+_{even}$

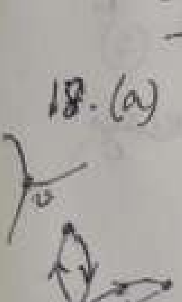
$\Rightarrow L(K_4)$ has an Euler cycle.



$L(K_4)$

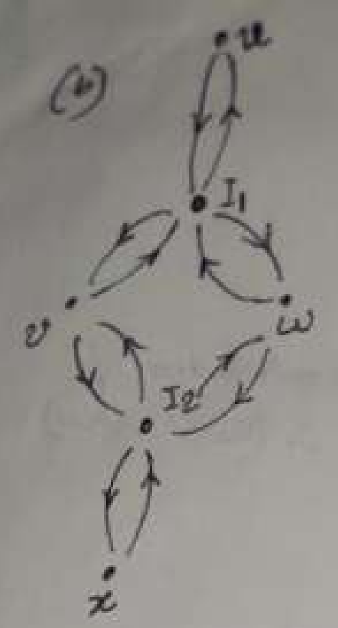
5/5

18. (a)



Since in the undirected network graph G(V,E), every edge incident to a vertex can be thought of as an ~~ingoing~~ incoming and an outgoing edge, each vertex is going have even degree and it will be possible to construct a cycle C that begins and ends with a given vertex $v \in V(G)$ and traversing each edge (incoming or outgoing) exactly once. Again we can consider $V(G)-C$ and since all vertices are of even degree(has equal # of incoming & outgoing edges) and being connected has a common vertex with C, we can find cycles there and fuse it with C at common vertex. Continuing like this, we shall get ~~an euler~~ a directed Euler Cycle that traverses through each of the edges precisely once.

(b)



$u - I_1 - v - I_2 - w - I_1 - w - I_2 - x - v - I_1 - u$

Euler ~~[illegible]~~ cycle

The rule ensures that at any given intersection if some edge is not yet traversed (apart from the return edge to the vertex from which the intersection is arrived at), explore that edge.

Claim: ① No edge can't be uncovered, if the rule is followed: if not, assume to the contrary (proof by contradiction) $\Rightarrow \exists e \in V(h)$ that is uncovered, with endpoints $(v_1, v_2) \in V(h)$.

Case-1 none of (v_1, v_2) is not an intersection:

$\Rightarrow \exists$ exactly a couple of edges $\overrightarrow{v_1v_2}$ and $\overrightarrow{v_2v_1}$ in between v_1, v_2 and that's the only path to reach v_2 from v_1. Hence every walk coming to v_1 must go to $\overrightarrow{v_1v_2}$ before returning via the ~~path~~ edge it came to v_1 (by the rule), a contradiction.

Case-2 (~~[illegible]~~ either v_1 or v_2 is an intersection.

W.l.o.g. assume v_1 is an intersection. Then $\exists$ vertices $w_1, w_2, \ldots$ s.t. $\exists$ edges $\overrightarrow{v_1w_1}, \overrightarrow{v_2w_2}, \ldots$ etc. Any ~~path~~ ~~edge~~ walk that comes to v_1, goes to w_1, i.e., $v_1 \to w_1 \rightsquigarrow w_2 \to v_1$ without visiting v_2, ~~[illegible]~~ goes to the return edge ~~[illegible]~~ at v_1, without traversing ~~[illegible]~~ $\overrightarrow{v_1v_2}$, a contradiction.

Hence all edges must be traversed.

② Each edge will be traversed once: ~~since~~ this is guaranteed by the rule since each vertex has even degree (equal # of incoming & outgoing edges) and ~~the on~~ according to the rule we never ~~return~~ use the same edge twice.

Hence the rule guarantees an Euler cycle.

.2.

515

(a) In order to have Hamilton circuit, G has to have a length-v circuit traversing each of the vertices in $V(G)$ exactly once, where $|V(G)| = v$.

Now, if $|I| = k$, then $e' = \left(\sum_{x \in I} d(x) - 2k\right)$ edges can't be used for Hamilton circuit (since $2k$ edges for each vertex $x \in I$ must be present on Hamilton circuit, remaining edges from $x \in I$ will be unused).

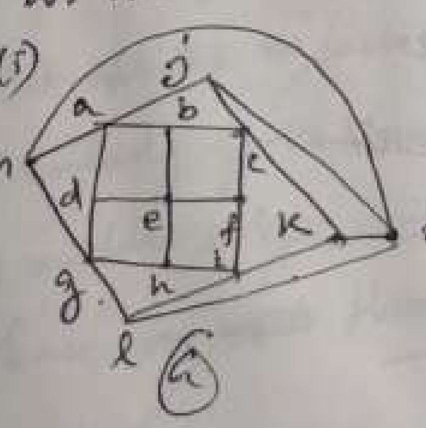
Hence # edges to be used for Hamilton circuit $= e - e'$, when $|E(G)| = e$. But length of Hamilton circuit must be v

$\Rightarrow e - e' \geq v$ for G to have Hamilton circuit

$\Leftrightarrow v < e - e' \Rightarrow$ G can have no Hamilton circuit (proved)

(b) Because if the vertices in I were adjacent, then Hamilton circuit need not use 2 different edges separately for each of $x \in I$, instead it could use the edges in between them as well.

(c) (i)



choosing the independent set $I = \{e, j, k, l, m\}$,

we have $e' = \sum_{x \in I} d(x) - 2 \times |I|$

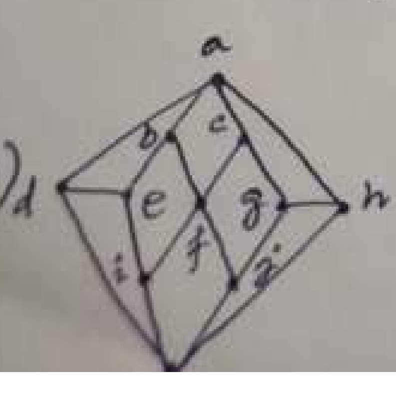
$= 4 + 3 + 3 + 3 + 3 - 2 \times 5$

$= 16 - 10 = 6$

$\Rightarrow e - e' = 24 - 6 = 18$.

Also $v = 14 \Rightarrow v < e - e' \Rightarrow$ G can have no Hamilton circuit

(ii)



choosing the independent set

$I = \{a, e, f, g, k\}$,

we have $e' = \sum_{x \in I} d(x) - 2 \times |I|$

2.2.

515

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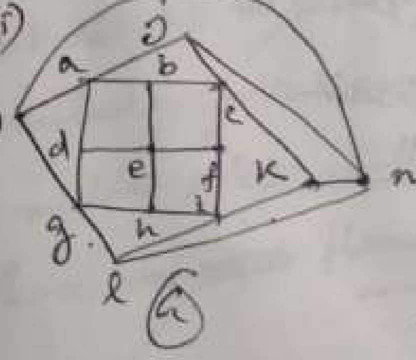

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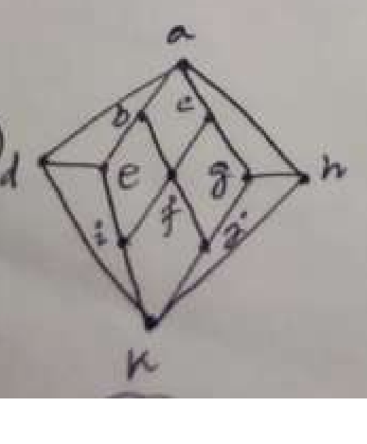


choosing the independent set

$I = \{a, e, f, g, k\}$,

we have $e' = \sum_{x \in I} d(x) - 2 \times |I|$

$= 4 + 4 + 4 + 3 + 3 - 2 \times 5 = 8$

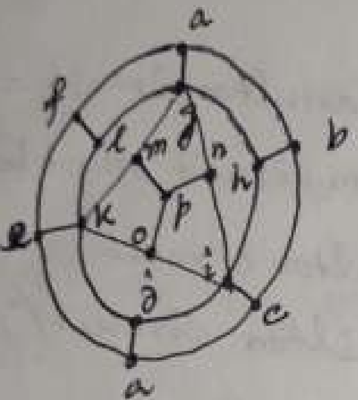
where $e = 18$, $v = 11$.

Hence, $e - e' = 18 - 8 = 10$.

but $v = 11 \Rightarrow e - e' < v$

$\Rightarrow$ G can't have a Hamilton circuit.

(iii)



Let's choose the independent set

$I = \{g, i, k, b, f, d\}$

$\therefore e' = \sum_{x \in I} d(x) - 2|I|$

$= 5+5+5+3+3+3 - 2\times 6$

$= 24 - 12$

$= 12$.

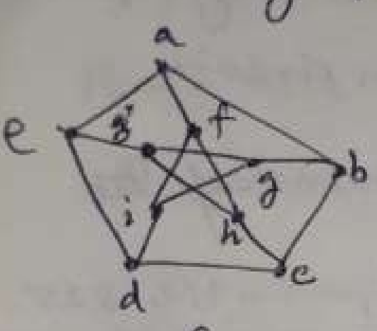
Now, we have, $e = 27$, $v = 16$

$\Rightarrow e - e' = 27 - 12 = 15 < v$

$\Rightarrow$ G can't have a Hamilton circuit.

Petersen's graph is an example of such a graph.

$\forall v \in V(G)$, $d(v) = 3$ (cubic)



G

Proof G has no hamilton circuit. Let's assume to the contrary. (by contradiction)

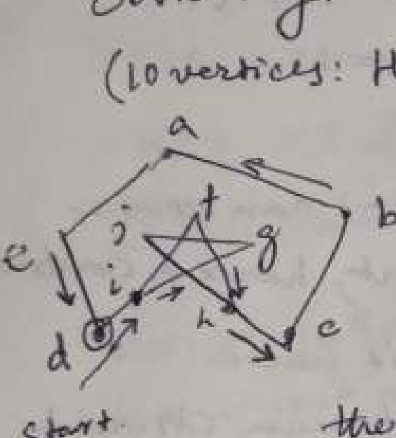
Group edges into external: (e,a), (a,b), (b,c), (c,d), (d,e)

middle: (a,f), (g,b), (h,c), (i,d), (e,j)

internal: (j,g), (f,h), (j,h), (i,g), (i,f).

Obviously Hamilton circuit can't have all the 5 external edges (then can't reach the internal vertices).

(10 vertices: Hamilton circuit should be of length 10).



~~As~~ Let's assume Hamiltonian circuit consists of 4 external edges, doesn't contain edge (c,d)

Obviously then the circuit can't have middle edges (e,j), (a,f), (g,b) can't be on the circuit, since o/w they will meet at one of the vertices a, b or c. On the other hand, we must have the edges (d,i) and (h,c) on the circuit, o/w it will not be a circuit.

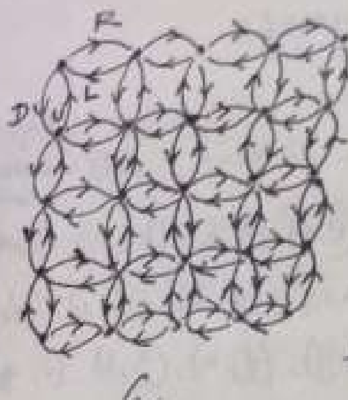
Now, in order that the remaining graph is a ~~cir~~ circuit, there must exist a length-4 path from i to h (since there are 3 vertices in between).

But ~~all the~~ no such path exists! hence such a circuit can't exist. a contradiction

Similarly, for the circuit that contains ~~3 external edges~~ 4 internal or 4 middle edges, will lead to a contradiction.

Hence, no ~~...~~ Hamilton circuit exists.

16.



$|V(G)|=25$
# of Hamilton circuit = length of circuit = 25

It's sufficient to show that the graph G does not have a Hamilton circuit.

Let's assume that the coordinate of the i-th student $\equiv$ ~~...~~ (x_i, y_i), $i = 1, 2, \ldots, 25$.

$\forall i$, x_i must be incremented by 1 or decremented by 1

$\forall i$, y_i " " " " " 1 " " "

$x_i' = x_i + \Delta x_i$

$y_i' = y_i + \Delta y_i$

$\Rightarrow \forall i, \quad \Delta x_i = \pm 1, \quad \Delta y_i = \pm 1$

But, $\left.\begin{array}{l}\sum_{i=1}^{25} \Delta x_i = 0 \\ \sum_{i=1}^{25} \Delta y_i = 0\end{array}\right\}$ which is impossible

Since odd # of +1 and −1 can never cancel out, hence a contradiction.

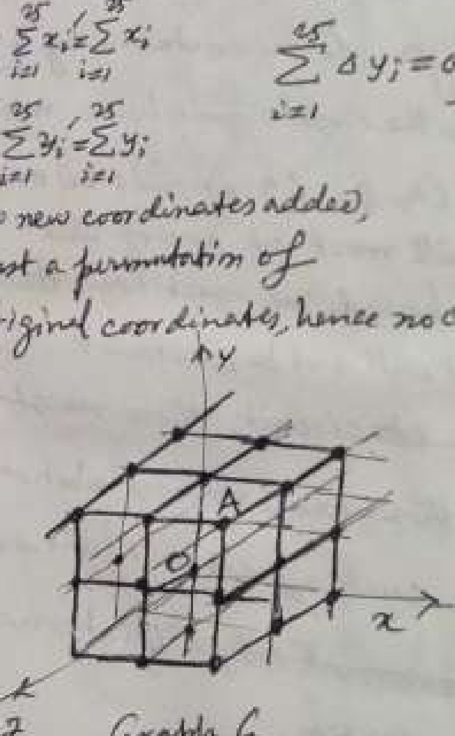
$\left(\because \sum_{i=1}^{25} x_i' = \sum_{i=1}^{25} x_i \right.$

$\sum_{i=1}^{25} y_i' = \sum_{i=1}^{25} y_i$

No new coordinates added, just a permutation of original coordinates, hence no change)

Hence, can't have a Hamiltonian circuit

18.



Graph G

Let's assume the center of the middle cube $O \equiv (0,0,0)$

~~Any order~~ Assuming unit distance between the centers, corner $A \equiv (1, 1, 1)$.

Now, each move along x, y, ~~z~~ or z basically increments ~~by~~. the corresponding coordinate by ±1, the others remain ~~unaffected~~ unchanged.

$\left.\begin{array}{l}\Delta x = \pm 1, 0 \\ \Delta y = \pm 1, 0 \\ \Delta z = \pm 1, 0\end{array}\right\}$ at each move

Now, length of Hamilton ~~circuit~~ path, if one exists must be 26, since there are 27 vertices in the graph G.

$$\Rightarrow \sum_{\substack{26 \text{ moves}}} (\Delta x + \Delta y + \Delta z) = 3 \qquad \left[\begin{array}{l} A \equiv (1,1,1) \\ O \equiv (0,0,0) \end{array}\right]$$

$$\Rightarrow \sum_{\substack{23 \text{ moves}}} \Delta x + \Delta y + \Delta z = 0, \qquad \Delta x, \Delta y, \Delta z \in \{0, \pm 1\} \text{ at each move}$$

but ~~at least one~~ exactly one must be non-zero

But 23 (odd number) +1 and −1 can't cancel each other, hence a contradiction.

No Hamilton path can exist.

✓ {1}

19. (a) # permutations of n vertices in $K_n = n!$

Hence there are $n!$ ~~different~~ Hamilton circuits. But for a given permutation, there are n different cyclic ordering that correspond to the same Hamilton circuit (e.g. $1234\ldots n$, $234\ldots n1$, $34\ldots n12$, .., $n12\ldots n-1$)

Also, from ~~symmetry~~ the same ~~cycle~~ circuit will be consider twice (clockwise & anticlockwise)

Hence # different Hamiltonian circuits in $K_n = \frac{n!}{2n} = (n-1)!/2$

(Since for the same circuit there can be n diff. starting vertex)

(b) Let's prove a more general result,

$\forall n \geq 3$, $n \in \mathbb{Z}^+_{odd}$, K_n can have its edges partitioned into $\frac{1}{2}(n-1)$ Hamilton circuits. (Hence the result holds $\forall n \geq 3$, $n \in$ Prime too)

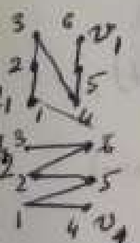
Let's verify the necessary conditions first. Since Hamilton circuit is a circuit with each vertex having degree 2 and K_n has each vertex with degree $n-1 \Rightarrow 2 \mid (n-1) \Rightarrow n-1 \in \mathbb{Z}^+_{even} \Rightarrow n \in \mathbb{Z}^+_{odd}$.

Also, since $|E(K_n)| = \frac{n(n-1)}{2}$, total # edges in $\frac{1}{2}(n-1)$ edge disjoint Hamilton circuits $= \frac{1}{2}(n-1) \times n = |E(K_n)|$, since each one has n edges.

Now, first let's prove the following result:

$\forall m \geq 3, m \in \mathbb{Z}^+_{even}$, K_m can have its edges partioned into $\frac{m}{2}$ Hamilton paths (edge disjoint).

Pf Since $m \in \mathbb{Z}^+_{even}$, $V(K_m)$ can always be partitioned into $\frac{m}{2}$ distinct pairs of vertices, (u_i, v_i), $i=1, .., \frac{m}{2}$, with $u_i \in V(K_m)$, $v_i \in V(K_m)$.



$u_i \neq v_i$. Since K_m is fully connected, one can always have $\frac{m}{2}$ Hamilton paths (P_i), the i th one starting at u_i and ending in v_i, $\forall i = 1..m$. Only thing remains to be proved is that the paths can be constructed in ~~an~~ edge-disjoint manner.

disjoint (K6)

Let's construct the path P_i in the following manner: ~~Choose~~ Start with vertex u_i, choose $m-2$ different vertices in any order then end with vertex v_i, after construction, delete ~~m edges~~ $m-1$ edges $\in P_i$ and construct P_{i+1}. $\forall i = 1, 2, \ldots, \frac{m}{2}-1$

~~But only thing~~ By this construction, it's guranteed that the $\frac{m}{2}$ Hamilton paths ~~are~~ thus constructed are edge-disjoint.

Only thing remains to be proved is that we have sufficient edges for path construction everytime.

Notice that after each path P_i is constructed, the degree of all vertices of the residual graph is decreased. $d(u_i)$ and $d(v_i)$ are decreased by 1, while all the intermediate vertices' degree reduce by 2. $\overset{P_i}{\overrightarrow{u_i\ w_{i_1}\ w_{i_2} \cdots v_i}}$

Initially each ~~vo~~ vertex has degree $m-1$, ~~[illegible]~~ ≥ 2, hence we have enough edges ~~[illegible]~~ to construct P_1 (for $i=1$).
for $i \geq 2$, before constructing P_i, we have $2(i-1)$ vertices each with degree $m-2(i-1)$ and each of rest of the $m-2(i-1)$ vertices have degree $m-2i+1$. (can prove by induction on i easily). Since,
~~$2 \leq i \leq \frac{m}{2}$, there will always be $2(i-1) = 2$ vertices [illegible] (at least) with degree $m-2(\frac{m}{2}-1) = m-2(i-1) \geq m-2(\frac{m}{2}-1) = 2$ and~~
e.g., while choosing the $\frac{m}{2}$ th Hamilton path $P_{\frac{m}{2}}$, we shall still have $2(\frac{m}{2}-1) = m-2$ vertices of degree $m-2(\frac{m}{2}-1) = 2$ and $m-2(\frac{m}{2}-1) = 2$ vertices with degree $m - 2\cdot\frac{m}{2}+1 = 1$, namely $u_{m/2}$, $v_{m/2}$. Hence we can construct $P_{\frac{m}{2}}$. Also, by construction, $P_i \cap P_j = \phi$, $i \neq j$.

Hence, K_m, $m \in \mathbb{Z}^+_{even}$ ~~odd~~ has $\frac{m}{2}$ Hamilton paths (edge-disjoint).
Now, for $n = m+1$, $K_n \in \mathbb{Z}^+_{odd}$, add one more vertex w to K_{n-1} and add all the edges (u_i, w), (v_i, w), $i = 1 \ldots, \frac{m}{2}$ in K_m.
Now, $\frac{m}{2}$ Hamilton paths P_i, $i = 1 \ldots m$ in K_m of the form $u_i \overset{P_i}{\rightsquigarrow} v_i$, with $P_i \cap P_j = \phi$ whenever $i \neq j$, can be extended to $\frac{m}{2} = \frac{n-1}{2}$ disjoint Hamilton circuits $w \to u_i \overset{P_i}{\rightsquigarrow} v_i \to w$ in K_n. (Proved)
(Since $u_i \neq u_j$, $v_i \neq v_j$ for $i \neq j$, (w, u_i), (w, v_i) will be [illegible]

partitioned into $\frac{n-1}{2}$ disjoint Hamilton circuits.

Considering $n=17$ professors and modeling "the sitting next to pair of different professors each day" as # edge disjoint Hamilton cycles, # days the conference ~~show~~ can last

$$= \frac{n-1}{2} = \frac{17-1}{2} = 8.$$

20. (a) $\exists$ an Euler cycle in G

$\Rightarrow \exists$ edge sequence $e_1 e_2 \cdots e_m e_1$, with $e_i \in E(G)$, $|E(G)| = m$ and e_i, e_{i+1} are adjacent (have a common vertex that both are incident upon) and each e_i appears exactly once in the sequence

$\Rightarrow$ In $L(G)$, $v_i = e_i$, $v_i \in V(L(G)) \Leftrightarrow e_i \in E(G)$.

Since, $(v_i, v_j) \in E(L(G)) \Leftrightarrow (e_i, e_j)$ are adjacent in G

$\Rightarrow$ the Euler cycle $e_1 e_2 \cdots e_m e_1$ in G

$\equiv v_1 v_2 \cdots v_m v_1$ in $L(G)$

with each v_i, v_{i+1} adjacent in $L(G)$, since each e_i, e_{i+1} share a common endpoint in G

Also, each e_i appears exactly once in the sequence in G

$\Rightarrow$ each v_i " " " " " " $L(G)$

$\Rightarrow v_1 v_2 \cdots v_m v_1$ is a Hamilitonian cycle in $L(G)$. (proved)

(b) G has a Hamilton circuit

$\Rightarrow \exists$ a sequence of vertices $v_1 v_2 \cdots v_n v_1$, with $v_i \in V(G)$, $|V(G)| = n$ and v_i, v_{i+1} are adjacent, i.e., $(v_i, v_{i+1}) \in E(G)$ and each v_i appears exactly once in the sequence.

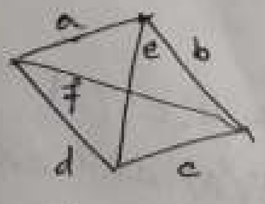
$\Rightarrow$ In $L(G)$, consider the corresponding edges in G. Not all edges might be covered by the Hamilton circuit $v_1 v_2 \cdots v_n v_1$ ~~ep~~, i.e., In $L(G)$, choose the vertices $\underset{e_1}{(v_1, v_2)}, \underset{e_2}{(v_2, v_3)} \cdots, \underset{e_n}{(v_n, v_1)}$ in G. We may be missing some vertices w_i corresponding to e_i in G.

$\boxed{w_i \in V(L(G)) \Leftrightarrow e_i \in E(G)}$

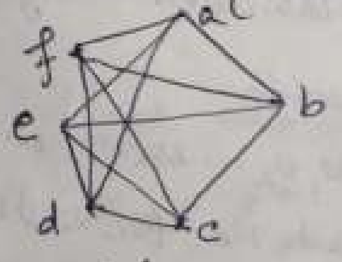
in $L(G)$ corresponding to some edges in G that were not in the Hamilton circuit in G. Construct Hamilton circuit in $L(G)$ in the following manner:

For every consecutive vertices .. $v_{k-1} v_k v_{k+1} \cdots$ in the Hamilton circuit in G, check if there are some other edges incident on v_k other than $e_{k-1} \equiv (v_{k-1}, v_k)$ and $e_k \equiv (v_k, v_{k+1})$. If $\exists$ such edges $e_{i_1}, e_{i_2}, \dots, e_{i_\ell}$ in G that are incident on v_k, but was not included in the Hamilton circuit of G, they must be included in the Hamilton Circuit of $L(G)$. Notice that the corresponding vertices $w_{k-1}, w_k, w_{i_1}, w_{i_2}, \dots, w_{i_\ell}, w_{k+1}$ in $L(G)$ form a complete graph in $L(G)$. Hence, instead of having sequence of vertices $\cdots w_{k-1} w_k w_{k+1} \cdots$ we shall have $w_{k-1} w_k \underbrace{w_{i_1} w_{i_2} \cdots w_{i_\ell}}_{\text{insert unused edges}} w_{k+1}$ to form the circuit in $L(G)$, we shall insert unused edges only iff that edge was not already inserted previously. Hence following the above logic, we shall have a sequence of vertices going through all the vertices in $L(G)$, starting and ending at w_1, hence a Hamilton circuit. (Proved)

(c)



$G = K_4$

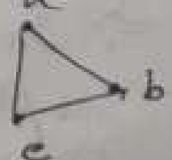


$L(G)$

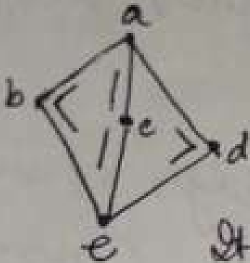
G has vertex with odd degree, hence does not have an Euler Cycle. But $L(G)$ has Hamilton circuit afbcdea

$L(G)$ has Hamilton circuit afbc, but G does not have Hamilton circuit

G



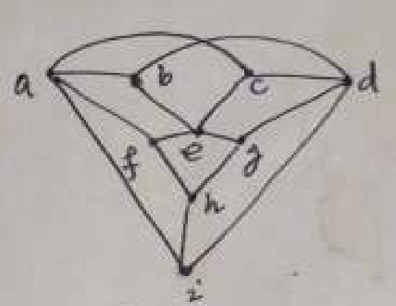
$L(G)$

4.(a) 

~~applying~~
~~By Rule 1, at vertices~~

Let's apply Rule 1 at vertices b, d, e

It violates Rule 2 in the sense that we have proper subcircuits aceb and aced. Hence, can't have Hamilton circuit.

(m) 

G

Using the result proved in problem 10(a),
choose the independent set $I = \{a, e, h, d\}$

~~$I = \{i, f, g, b, d\}$~~

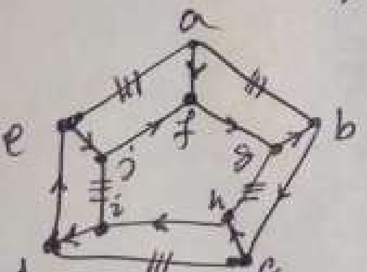
$$\therefore e' = \sum_{x \in I} d(x) - 2|I|$$

$$= (4+4+3+4) - 2 \times 4 = 7$$

Also, $e = |E(G)| = 15$, $v = |V(G)| = 9$

$\Rightarrow e - e' = 15 - 7 = 8 < v$

$\Rightarrow \exists$ no Hamilton circuit in G. (Proved)

6. 

~~Let's~~ Since the edges (a, f) and (c, h) are to be used, at least one edge incident at a can't be used, w.l.o.g. let's assume (a, b) is not used for the Hamilton circuit, delete (a, b) by Rule 3.

5/5

~~Similarly, let's assume (c, d)~~

Now, if (a, b) can't be used, (b, c) must be used for Hamilton circuit, or it will not be a circuit. Also at c, we must get rid of one edge ~~we can't delete e~~ for Hamilton circuit, hence deleting the edge (c, d) that can only be removed. Now, by symmetry we can start from either ~~of~~ from any one of the outer vertices or any one of the inner vertices: there are only 2 choices. Let's start from 'a' (outer vertex). We ~~start~~ go by ...h-c-h-i-d-e-j-f, but by Rule 2 it's a violation.

Similarly if we start from one of the inner vertices, we get to the same ~~con~~ violation of the Rule 2.

Hence, can't have Hamilton circuit using edges af and ch.

(Proved)