Math 650, Foundations of Optimization, Spring 2010

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Homework Assignment - 4

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Problem 3 Solution

The problem $max\{x^2 + (y+1)^2 : -x^2 + y \geq 0, x + y \leq 2\}$ can be converted to the following minimization problem P:

$$
\begin{aligned}
min - \frac{1}{2}x^2 - \frac{1}{2}(y+1)^2 \\
s.t.\ \ x^2 - y \leq 0 \\
x + y - 2 \leq 0
\end{aligned}
$$

(a) We have the objective function $f(x,y) = -\frac{1}{2}x^2 - \frac{1}{2}(y+1)^2$ (maximize radius of the circle at centered at $(0,-1)$ satisfying the following constraints) $g_1(x,y) = x^2 - y \leq 0,\ g_2(x,y) = x + y - 2$ and $h(x,y) = 0$.

By the FJ condition, if a point (x^*, y^*) is a local minimizer of P, then there exist multipliers $(\lambda_0, \lambda_1, \lambda_2)$, not all zero, $(\lambda_0, \lambda_1, \lambda_2) \geq 0$, s.t.,

$$
\begin{aligned}
\lambda_0 \nabla f(x^*, y^*) + \lambda_1 \nabla g_1(x^*, y^*) + \lambda_2 \nabla g_2(x^*, y^*) = 0,\ \lambda_1, \lambda_2 \geq 0 \\
g_1(x^*, y^*) \leq 0,\ g_2(x^*, y^*) \leq 0,\ \lambda_1 g_1(x^*, y^*) = 0,\ \lambda_2 g_2(x^*, y^*) = 0
\end{aligned}
$$

Now, $\nabla f(x,y) = \begin{bmatrix} -x \\ -y-1 \end{bmatrix}$, $\nabla g_1(x,y) = \begin{bmatrix} 2x \\ -1 \end{bmatrix}$, $\nabla g_2(x,y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Equivalently we could also form the weak Lagrangian
$L(x,y,\lambda) = \lambda_0.\left(-\frac{1}{2}x^2 - \frac{1}{2}(y+1)^2\right) + \lambda_1(x^2 - y) + \lambda_2(x + y - 2)$ and have the above FJ conditions $\Rightarrow$

$$\frac{\partial L}{\partial x} = (-\lambda_0 + 2\lambda_1)x^* + \lambda_2 = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} = -\lambda_0 y^* - \lambda_0 - \lambda_1 + \lambda_2 = 0 \tag{2}$$

$$\lambda_1 \geq 0,\ x^{*2} - y^* \leq 0,\ \lambda_1(x^{*2} - y^*) = 0 \tag{3}$$

$$\lambda_2 \geq 0,\ x^* + y^* - 2 \leq 0,\ \lambda_2(x^* + y^* - 2) = 0 \tag{4}$$

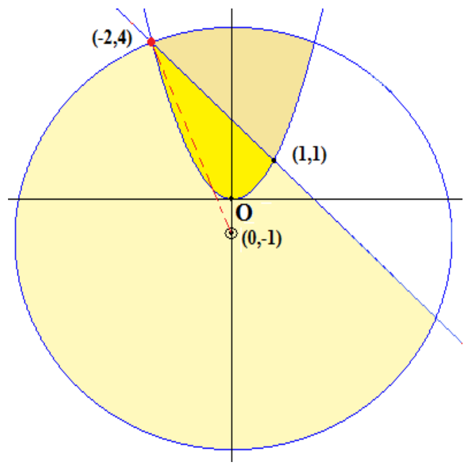
$$(\lambda_0, \lambda_1, \lambda_2) \neq 0 \tag{5}$$

Let's assume to the contrary $\lambda_0 = 0$

Now $\lambda_0 = 0 \Rightarrow \lambda_1 = \lambda_2$ (from (2)) $\Rightarrow \lambda_1(2x^* + 1) = 0$.
But λ_1 can't be zero, since it implies $(\lambda_0, \lambda_1, \lambda_2) = 0$, which can't be, by (5).
Hence, $x^* = \frac{1}{2} \Rightarrow y^* = \frac{1}{4}$, from (3), since $\lambda_1 \neq 0$.
Also, $x^* = \frac{1}{2} \Rightarrow y^* = 2 - x^* = \frac{3}{2}$, from (4), since $\lambda_2 \neq 0$, a contradiction.

Since, $\lambda_0 \neq 0$, the KKT condition holds.

(b) As seen from the graph, the optimal solution point is $(-2, 4)$. The opti-



mal value of the objective function is shown.

(c) Since KKT condition holds, scaling λ_0 to 1, we have the following from (1) and (2),

$$x^* = \frac{\lambda_2}{1 - 2\lambda_1} \tag{6}$$

$$y^* = \lambda_2 - \lambda_1 \tag{7}$$

Considering the sign of the multipliers λ_1, λ_2 (combinatorial game!),

1. $\lambda_1 > 0,\ \lambda_2 > 0$, then from (3) and (4) we have, $x^{*2} = y^*$ and $x^* + y^* = 2 \Rightarrow x^* + x - 2 = 0 \Rightarrow x^* = -2, 1$. Hence the two points are $(x^*, y^*) = (-2, 4)$ and $(x^*, y^*) = (1, 1)$. The point $(-2, 4)$ is a valid KKT point since the multipliers $(\lambda_1, \lambda_2) = (2, 6)$ at this point (both positive). But for the point $(1, 1)$, we have $\lambda_1 = 0$, which is impossible.

2. $\lambda_1 = 0,\ \lambda_2 > 0$, then from (1), (2) and (4) we have, $x^* = \lambda_2$ and $y^* = \lambda_2 - 1$ and $x^* + y^* = 2$ respectively $\Rightarrow \lambda_2 = \frac{3}{2} \Rightarrow (x^*, y^*) = (\frac{3}{2}, \frac{1}{2})$, which is feasible as well, hence another KKT point.

3. $\lambda_1 > 0,\ \lambda_2 = 0$, then from (1), (2) and (3) we have, $x^*(-1+2\lambda_1) = 0$ and $y^* = -\lambda_1 - 1$ and $y^* = x^{*2}$ respectively. Now, $x^* = 0 \Rightarrow y^* = 0 \Rightarrow \lambda_1 = -1$ hence imossible and $\lambda_1 = \frac{1}{2} \Rightarrow y^* = -\frac{3}{2} = x^2$ again impossible, hence none of the points are KKT points in this case.

4. $\lambda_1 = 0,\ \lambda_2 = 0$, then from (1) and (2) we have, $x^* = 0$ and $y^* = -1$ is a possible KKT point, but this point is not feasible, hence not a KKT point.

To summerize, we have 2 KKT points, $(-4, 2)$ and $(\frac{3}{2}, \frac{1}{2})$. Hence, the global maximizer is the point $(-2, 4)$.