IS 698/800 Spring 2009 Final Exam
Due by May 19th 2009 midnight EST

1. Suppose you want to develop a closed form expression for calculating the following: $\sum_{k=1}^{n} k^3$. **(20 points)**
   a. Show the finite difference table until the differences converge to 0.
   b. Write the system of linear equations to solve the coefficients.
   c. Write the above in matrix form.
   d. Solve for the coefficients (you may use Excel or Matlab).
   e. Write the closed form expression for the above summation.
   f. Prove using mathematical induction that the closed form expression you derived is correct.

Answer:

a.

n	$\sum_{k=1}^{n} k^3$	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
1	1	--	--	--	--	--
2	9	8	--	--	--	--
3	36	27	19	--	--	--
4	100	64	37	18	--	--
5	225	125	61	24	6	--
6	441	216	91	30	6	0
7	784	343	127	36	6	0
8	1296	512	169	42	6	0
9	2025	729	217	48	6	0
10	3025	1000	171	54	6	0

Hence, we have, $\sum_{k=1}^{n} k^3 = a.n^4 + b.n^3 + c.n^2 + d.n + e$ (Since 5th **difference** is vanishing, we have a 4th degree polynomial, with real coefficients *a, b, c, d, e*).

b. System of linear equations:

$$\begin{aligned}
a + b + c + d + e &= 1 \\
16.a + 8.b + 4.c + 2.d + e &= 9 \\
81.a + 27b + 9.c + 3.d + e &= 36 \\
256.a + 64b + 16.c + 4.d + e &= 100 \\
625.a + 125b + 25.c + 5.d + e &= 225
\end{aligned}$$

c. System of equations in matrix form:

$$\underset{5\times 5}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & 64 & 16 & 4 & 1 \\ 625 & 125 & 25 & 5 & 1 \end{bmatrix}} \underset{5\times 1}{\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}} = \underset{5\times 1}{\begin{bmatrix} 1 \\ 9 \\ 36 \\ 100 \\ 225 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & 64 & 16 & 4 & 1 \\ 625 & 125 & 25 & 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 9 \\ 36 \\ 100 \\ 225 \end{bmatrix}$$

d.

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0.041667 & -0.16667 & 0.25 & -0.16667 & 0.041667 \\ -0.58333 & 2.166667 & -3 & 1.833333 & -0.41667 \\ 2.958333 & -9.83333 & 12.25 & -6.83333 & 1.458333 \\ -6.41667 & 17.83333 & -19.5 & 10.16667 & -2.08333 \\ 5 & -10 & 10 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 36 \\ 100 \\ 225 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Solution:} \begin{Bmatrix} a = c = 0.25 = \dfrac{1}{4} \\ b = 0.5 = \dfrac{1}{2} \\ d = e = 0 \end{Bmatrix}$$

e.

$$\sum_{k=1}^{n} k^3 = a.n^4 + b.n^3 + c.n^2 + d.n + e = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{1}{4}n^2(n^2 + 2n + 1)$$

$$= \frac{1}{4}n^2(n+1)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

f. **Basis:**

$$n = 1,\ \sum_{k=1}^{n} k^3 = \sum_{k=1}^{1} k^3 = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

Induction Hypothesis:

Assume that the above identity holds $\forall n \le m,\ m \in Z^+$.

Hence, for $n = m$, we have, $\sum_{k=1}^{m} k^3 = \left(\frac{m(m+1)}{2}\right)^2$

Inductive Step:

For $n = m + 1$, we have,

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{m+1} k^3$$

$$= \sum_{k=1}^{m} k^3 + (m+1)^3$$

$$= \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3 \ (\textit{By Induction Hypothesis})$$

$$= \frac{(m+1)^2}{4}.(m^2 + 4m + 4)$$

$$= \frac{(m+1)^2(m+2)^2}{4}$$

$$= \left(\frac{(m+1)((m+1)+1)}{2}\right)^2$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

Hence, $\forall n \in Z^+$, we have, $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ **(Proved)**

2. Show all work for the following: **(20 points)**

   a. Using symbolic derivations (i.e. not truth tables) show that $(p \to q) \leftrightarrow (\neg q \to \neg p)$

Answer:

$$(p \to q)$$
$$\equiv \neg p \vee q$$
$$\equiv q \vee \neg p \quad (\because \vee \text{ is commutative})$$
$$\equiv \neg(\neg q) \vee \neg p \text{ (by double negation rule)}$$
$$\equiv (\neg q) \to \neg p$$

b. Let $P(m,n)$ be "n is greater than or equal to m" where the universe of discourse is the set of nonnegative integers. What are the truth values of $\exists n \forall m P(m,n)$ and $\forall m \exists n P(m,n)$?

Answer:

Since the set of positive integers is **unbounded from above** (there **does not** exist an **upper bound** to all the members belonging to the set),

$\exists n \forall m P(m,n) \equiv (\exists n)(\forall m)(n \geq m), \ m,n \in Z^{+} \cup \{0\}$ is **False.**

But since any integer is greater than or equal to itself,

$\forall m \exists n P(m,n) \equiv (\forall m)(\exists n)(n \geq m), \ m,n \in Z^{+} \cup \{0\}$ is **True.**

Choose n = m (or n = m+1), for any m, since $\forall m(m \geq m)$ is True.

Note:

The first predicate to be true needs a single element n to upper bound all integers, obviously one such integer does not exist, hence False.

The second predicate to be true does not demand existence of a single n, instead different n 's for different m 's suffice, hence True.

c. Prove or disprove that $(p \to q) \to r$ and $p \to (q \to r)$ are equivalent.

Answer:

P1

$$\equiv (p \to q) \to r$$
$$\equiv \neg(p \to q) \vee r$$
$$\equiv \neg(\neg p \vee q) \vee r$$
$$\equiv (p \wedge \neg q) \vee r \text{ (by D'Morgan)}$$
$$\equiv (p \vee r) \wedge (\neg q \vee r) \ (\because \vee \text{ distributes over } \wedge)$$

P2

$\equiv p \to (q \to r)$

$\equiv p \to (\neg q \vee r)$

$\equiv \neg p \vee (\neg q \vee r)$ $(\because \vee$ *is associative*$)$

Hence, $P1 \equiv (p \vee r) \wedge (\neg q \vee r)$
$P2 \equiv (\neg p) \vee (\neg q \vee r)$

Choose $p = F,\ q = T,\ r = F$,
Under this assignment, P1 is **False** but P2 is **True.**

Since P1 and P2 don't evaluate to same Boolean value for all possible Boolean assignments, they are not equivalent.

d. Prove that all solutions to the equations $x^2 = x + 1$ are irrational.

Answer:

Indirect proof

Let's assume to the contrary, suppose there is a rational root of the equation $x^2 = x + 1$.

$\Rightarrow \exists r \in Q \mid r = \frac{a}{b},\ a \in Z,\ b \in Z^+\ (b \neq 0)$ *and* $\gcd(a,b) = 1$. (*a* may be –ve as well)

Since r is a root of $x^2 = x + 1$, it satisfies the equation, i.e.,

$r^2 = r + 1 \Rightarrow a^2 = ab + b^2$

$\Rightarrow a^2 = b(a+b)...(1)$

Now, $b \in Z^+$, there can be two possibilities,

1. *b>1* (where $r = \frac{a}{b}$ is rational, but not an integer)
2. *b=1* (where *r* is an integer, $r = a$)

Case-1

$b > 1$, in which case *b* must have a prime factor *p*, i.e., *p|b (p divides b)*. From equation (1), since R.H.S. is divisible by *p*, L.H.S. must be divisible by *p* as well.
$\Rightarrow p \mid a^2 \Rightarrow p \mid a$ (since *p* is a prime, *p* does not divide $a \Rightarrow p$ does not divide a^2).

Now, $(p \mid a) \wedge (p \mid b) \Rightarrow p \mid \gcd(a,b) \Rightarrow \gcd(a,b) \geq p > 1$, a contradiction.

Hence, the equation $x^2 = x + 1$ **can't have any rational non-integer solution.**

Case-2

b=1, the equation reduces to $a^2 = a + 1,\ a \in Z$.

- If $|a| < 2$,

  $$\text{i.e., } a = \begin{cases} 0, & \text{L.H.S.} = 0, \text{R.H.S.} = 1 \\ 1, & \text{L.H.S.} = 1, \text{R.H.S.} = 2 \\ -1, & \text{L.H.S.} = 1, \text{R.H.S.} = 0 \end{cases}$$, we have, L.H.S. $\neq$ R.H.S. and

  the equation has no solution in integers in this case.

- If $|a| \geq 2$,

  Writing the equation in the form $a = 1 + \dfrac{1}{a}$, (we can write since $a \neq 0$)

  L.H.S., (a) is an **integer** and R.H.S., $\left(1 + \dfrac{1}{a}\right)$ is a **fraction.**

  (since $0 < \left|\dfrac{1}{a}\right| \leq \dfrac{1}{2} < 1$). Hence, L.H.S. $\neq$ R.H.S. and the equation has no solution in integers in this case as well.

$\therefore \forall a \in Z$, the equation has no solution in integers.

Hence, the equation $x^2 = x + 1$ **can't have any (rational) integer solution.**

Combining **Case-1** and **Case-2**, we have, the equation $x^2 = x + 1$ **can't have a rational solution. (Proved)**

3. Show all work for the following: **(20 points)**

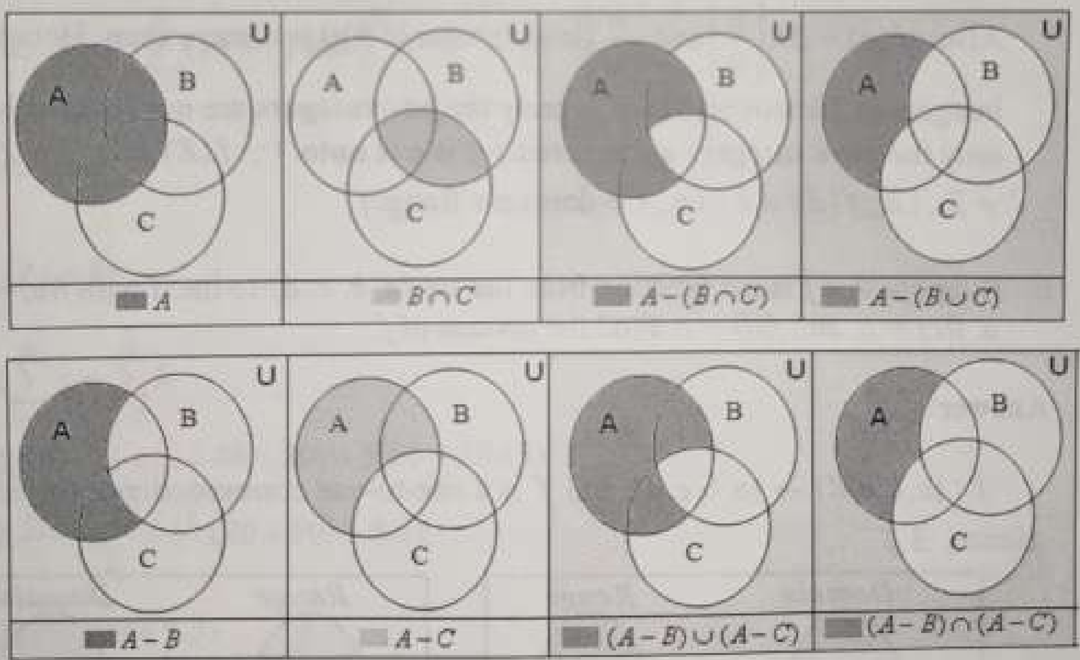
   a. Using either logical equivalences or containment (not membership tables) prove or disprove that $A - (B \cap C) = (A - B) \cap (A - C)$.

Answer:

$x \in A-(B\cap C)$
$\Leftrightarrow x \in A \wedge x \notin B\cap C$
$\Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C)$
$\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$
$\Leftrightarrow (x \in A-B) \vee (x \in A-C)$
$\Leftrightarrow x \in (A-B)\cup(A-C)$
$\therefore A-(B\cap C)=(A-B)\cup(A-C)$

Again,

$x \in (A-B)\cap(A-C)$
$\Leftrightarrow x \in A-B \wedge x \in A-C$
$\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$
$\Leftrightarrow x \in A \wedge (x \notin B \wedge x \notin C)$ ($\wedge$ *is associative and commutative*)
$\Leftrightarrow x \in A \wedge (x \notin B \cup C)$
$\Leftrightarrow x \in A-(B\cup C)$
$\therefore (A-B)\cap(A-C)=A-(B\cup C)$



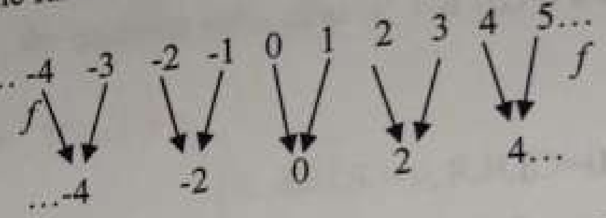
Venn diagram

Hence, it's clear from the above proofs and the Venn diagram that, in general, $A-(B\cap C)\neq(A-B)\cap(A-C)$, since we have the following (in general):

$$A-(B\cap C)=(A-B)\cup(A-C)\neq(A-B)\cap(A-C) \text{ and}$$
$$(A-B)\cap(A-C)=A-(B\cup C)\neq A-(B\cap C)$$

[e.g., $B \cup C = B \cap C$, iff $B=C$, ...

b. Consider the function $f(n) = 2 \times \left\lfloor \frac{n}{2} \right\rfloor$ from **Z** to **Z**. Is the function one-to-one? Is the function onto? Justify your answers.



Z (Domain of *f*)

Z (Co-domain of *f*)

Let's consider *f(2m)* and *f(2m+1)*, $\forall m \in Z$

$$f(2m) = 2 \times \left\lfloor \frac{2m}{2} \right\rfloor = 2 \times \lfloor m \rfloor = 2m$$

$$f(2m+1) = 2 \times \left\lfloor \frac{2m+1}{2} \right\rfloor = 2 \times \left\lfloor m + \frac{1}{2} \right\rfloor = 2m$$

$$\Rightarrow f(2m) = f(2m+1), \ \forall m \in Z$$

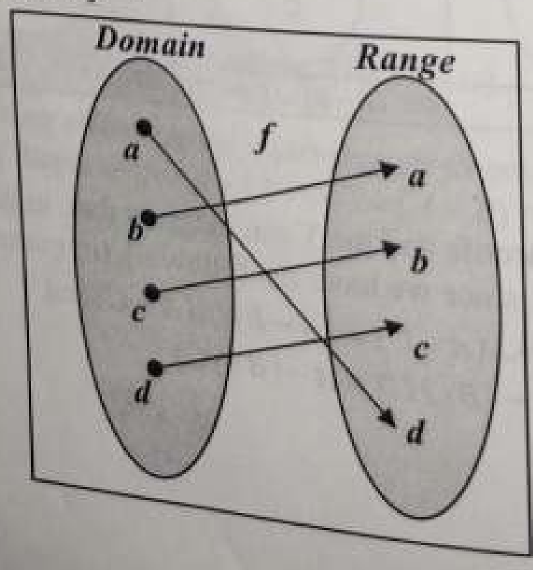
Hence, $(\exists x \in Z)(\exists y \in Z) \mid (f(x) = f(y) \wedge x \neq y) \Rightarrow f$ is **not one-to-one**.

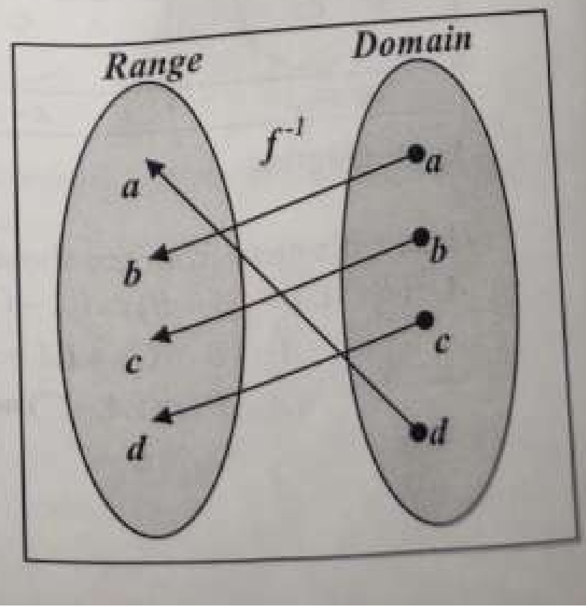
Also, $f(n) = 2 \times \left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lfloor \frac{n}{2} \right\rfloor$ is an integer $\Rightarrow$ *f(n)* is always even. Hence, not all integers in Z is covered by *f*, namely the **odd integers** are not covered by *f* at all, only the **even integers** are covered $\Rightarrow f$ is **not onto**. ($\because f(Z) = Z_{even} = Z - Z_{odd} \neq Z$, i.e., $f(Z) \neq Z$, i.e., Co-domain $\neq$ Range)

c. Suppose that *f* is the function from the set *{a, b, c, d}* to itself with *f(a)=d*, *f(b) = a*, *f(c) = b*, and *f(d) = c*. Find the inverse of *f*.

Answer:

$f : \{a,b,c,d\} \to \{a,b,c,d\}$ and f is a one-to-one correspondence (bijective). Hence, $\exists \ f^{-1}$.





Obviously, $f^{-1}:\{a,b,c,d\}\to\{a,b,c,d\}$ is also a one-to-one correspondence (bijective) and it's defined by the following:

$$
\begin{aligned}
f^{-1}(a) &= b\\
f^{-1}(b) &= c\\
f^{-1}(c) &= d\\
f^{-1}(d) &= a
\end{aligned}
$$

so that

$$
\begin{aligned}
fof^{-1}(a) &= f^{-1}of(a) = I(a) = a\\
fof^{-1}(b) &= f^{-1}of(b) = I(b) = b\\
fof^{-1}(c) &= f^{-1}of(c) = I(c) = c\\
fof^{-1}(d) &= f^{-1}of(d) = I(d) = d
\end{aligned}
$$

d. Find $\sum_{j=20}^{50}(2j+5)$ and $\sum_{j=5}^{100}3^j$.

Answer:

$$
\begin{aligned}
&\sum_{j=20}^{50}(2j+5)\\
&= 2\sum_{j=20}^{50} j + \sum_{j=20}^{50} 5\\
&= 2(20+21+\ldots+49+50)+5(50-20+1)\\
&= 2.\frac{(50-20+1)}{2}.(20+50)+5.31\\
&= 2.\left(\frac{31}{2}\right).70+5.31\\
&= (70+5).31 = 75\times 31 = 2232
\end{aligned}
$$

Sum of A.P series, *20 + 21 + ... + 50*, with

first term = a = 20, common diff. = d = 1, number of terms = n = 50 – 20 + 1=31.

Hence, sum $= \frac{n}{2}.(2a+(n-1)d) = \frac{n}{2}.(1^{st}\ term + last\ term) = \frac{31}{2}(20+50)$

$$\sum_{j=5}^{100} 3^j$$
$$= 3^5 + 3^6 + \dots + 3^{100}$$
$$= 3^5.(\underbrace{1+3+3^2+\dots+3^{95}}_{96\ terms})$$
$$= 3^5.\frac{3^{96}-1}{2}$$
$$= \frac{243}{2}(3^{96}-1)$$
$$\approx 7.7306628109801699655469169464843 \times 10^{47}$$

Sum of G.P series, $1+3+3^2+\dots+3^{95}$, with

first term=a=1, common ratio=r=3, number of terms=n=96,

Hence, $sum = \frac{a(r^n-1)}{(r-1)} = \frac{1(3^{96}-1)}{(3-1)}$

4. Show all work for the following: **(20 points)**

a. Prove or disprove that **AB** = **BA** whenever **A** and **B** are 2 × 2 matrices.

Answer:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$C = AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

$$D = BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{21}b_{12} & a_{12}b_{11}+a_{22}b_{12} \\ a_{11}b_{21}+a_{21}b_{22} & a_{12}b_{21}+a_{22}b_{22} \end{bmatrix}$$

If $AB = BA$, element by element comparisons of matrices C and D, give the following equations:

$$C(1,1) = D(1,1) \Rightarrow a_{11}b_{11}+a_{12}b_{21} = a_{11}b_{11}+a_{21}b_{12} \Rightarrow a_{12}b_{21} = a_{21}b_{12}$$
$$C(1,2) = D(1,2) \Rightarrow a_{11}b_{12}+a_{12}b_{22} = a_{12}b_{11}+a_{22}b_{12} \Rightarrow (a_{11}-a_{22})b_{12} = (b_{11}-b_{22})a_{12}$$
$$C(2,1) = D(2,1) \Rightarrow a_{21}b_{11}+a_{22}b_{21} = a_{11}b_{21}+a_{21}b_{22} \Rightarrow (a_{11}-a_{22})b_{21} = (b_{11}-b_{22})a_{21}$$
$$C(2,2) = D(2,2) \Rightarrow a_{21}b_{12}+a_{22}b_{22} = a_{12}b_{21}+a_{22}b_{22} \Rightarrow a_{12}b_{21} = a_{21}b_{12}$$

So, the following conditions must hold for matrices A and B to commute:

$$\left.\begin{cases} a_{12}b_{21} = a_{21}b_{12} \\ (a_{11} - a_{22})b_{12} = (b_{11} - b_{22})a_{12} \\ (a_{11} - a_{22})b_{21} = (b_{11} - b_{22})a_{21} \end{cases}\right\}$$

For instance, all matrices like $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ commute.

e.g., $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$, $AB = \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix} = BA$

because, the above conditions are satisfied. But, since those conditions are not satisfied for any arbitrary 2X2 matrix, (in general) they are **not commutative**.

e.g., $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$, $BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$, $AB \neq BA$

(***Disproved** with an example*)

b. Find the complexity of: $f(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$. Do not use the formula for the summation.

Answer:

$$\begin{aligned}
& f(n) \\
& = 1^3 + 2^3 + 3^3 + \cdots + n^3 \\
& \leq \underbrace{n^3 + n^3 + n^3 + \ldots + n^3}_{n \text{ times}}, \ \forall n \geq 1 \\
& = n.n^3, \ \forall n \geq 1 \\
& = n^4, \ \forall n \geq 1 \\
& \Rightarrow f(n) \leq n^4, \ \forall n \geq 1 \\
& \Rightarrow f(n) \leq c_2 n^4, \ \forall n \geq k_2, \ c_2 = k_2 = 1 \ (\textit{witnesses}) \\
& \Rightarrow f(n) = O(n^4) \qquad \ldots(1)
\end{aligned}$$

Also,

$$
\begin{aligned}
f(n) &= n^3 + (n-1)^3 + (n-2)^3 + \dots + 3^3 + 2^3 + 1^3 \\
&= n^3 + (n-1)^3 + (n-2)^3 + \dots + \left(\frac{n}{2}+2\right)^3 + \left(\frac{n}{2}+1\right)^3 + \left(\frac{n}{2}\right)^3 + \dots + 3^3 + 2^3 + 1^3 \\
&= n^3 + (n-1)^3 + (n-2)^3 + \dots + \left(\frac{n}{2}+2\right)^3 + \left(\frac{n}{2}+1\right)^3 \\
&\geq \underbrace{\left(\frac{n}{2}+\frac{n}{2}\right)^3 + \left(\frac{n}{2}+\frac{n}{2}-1\right)^3 + \dots + \left(\frac{n}{2}+2\right)^3 + \left(\frac{n}{2}+1\right)^3}_{\text{first } \frac{n}{2} \text{ terms}}, \left(\textit{if } n \textit{ is odd, use } \left\lceil \frac{n}{2} \right\rceil\right) \\
&\geq \underbrace{\left(\frac{n}{2}\right)^3 + \left(\frac{n}{2}\right)^3 + \dots + \left(\frac{n}{2}\right)^3 + \left(\frac{n}{2}\right)^3 + \left(\frac{n}{2}\right)^3}_{\frac{n}{2} \text{ times}}, \ \forall n \geq 1, \left(\text{since each term is } \geq \frac{n}{2}\right)
\end{aligned}
$$

$\Rightarrow f(n) \geq \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^3 = \frac{1}{16}n^4, \ \forall n \geq 1$

$\Rightarrow f(n) \geq c_1 n^4, \ \forall n \geq k_1, \ c_1 = \frac{1}{16}, \ k_1 = 1 \ (\textit{witnesses})$

$\Rightarrow f(n) = \Omega(n^4) \qquad \dots(2)$

Combining (1) and (2), we have,

$c_1 n^4 \leq f(n) \leq c_2 n^4, \ \forall n \geq k, \ c_1 = \frac{1}{16}, \ c_2 = 1, \ k = \max(k_1, k_2) = 1 \ (\textit{witnesses})$

$\Rightarrow f(n) = \theta(n^4)$

c. Let $f(n) = 3n^2 + 8n + 7$. Find the complexity of $f(n)$. Be sure to specify the witnesses.

Answer:

$$
\begin{aligned}
f(n) &= 3n^2 + 8n + 7 \\
&\geq 3n^2, \ \forall n \geq 1
\end{aligned}
$$

$\therefore f(n) \geq 3n^2, \forall n \geq 1$

$\Rightarrow f(n) \geq c_1 n^2, \ \forall n \geq k_1, \ c_1 = 3, \ k_1 = 1 \ (\textit{witnesses}) \qquad \dots(1)$

$\Rightarrow f(n) = \Omega(n^2)$

$$
\begin{aligned}
f(n) &= 3n^2 + 8n + 7 \\
&\leq 3n^2 + 8n^2 + 7n^2, \ \forall n \geq 1
\end{aligned}
$$

$\therefore f(n) \le 18n^2 \ , \forall n \ge 1$

$\Rightarrow f(n) \le c_2 n^2, \ \forall n \ge k_2, \ c_2 = 18, \ k_2 = 1 \ (witnesses) \ \ldots (2)$

$\Rightarrow f(n) = O(n^2)$

Combining (1) and (2), we have,

$c_1 n^2 \le f(n) \le c_2 n^2, \ \forall n \ge k, \ c_1 = 3, \ c_2 = 18, \ k = 1 = \max(k_1, k_2) \ (witnesses)$

$\Rightarrow f(n) = \theta(n^2)$

d. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Find the Boolean product of these two zero-one matrices.

Answer:

$$A \bullet B$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 0) \vee (1 \wedge 0) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

5. **(20 points)** Coke and Pepsi are involved in a competitive struggle to obtain market share from each other. Based on an independent research survey, Coke determined the following tendencies of people each month: 94% of Coke drinkers one week stay with Coke the next week, but 6% of Coke drinkers switch to Pepsi the next week; 9% of Pepsi drinkers switch to Coke the next week, while 91% of Pepsi drinkers stay with Pepsi. Answer the following questions based on the above information (you may use Excel or Matlab for matrix inversion and/or multiplication). Show all your work.

a. Complete the transition matrix for Coke and Pepsi.
b. Suppose that statistics show that last month 72% of people who drink Coke or Pepsi drank Coke. Write a starting matrix to represent this fact.

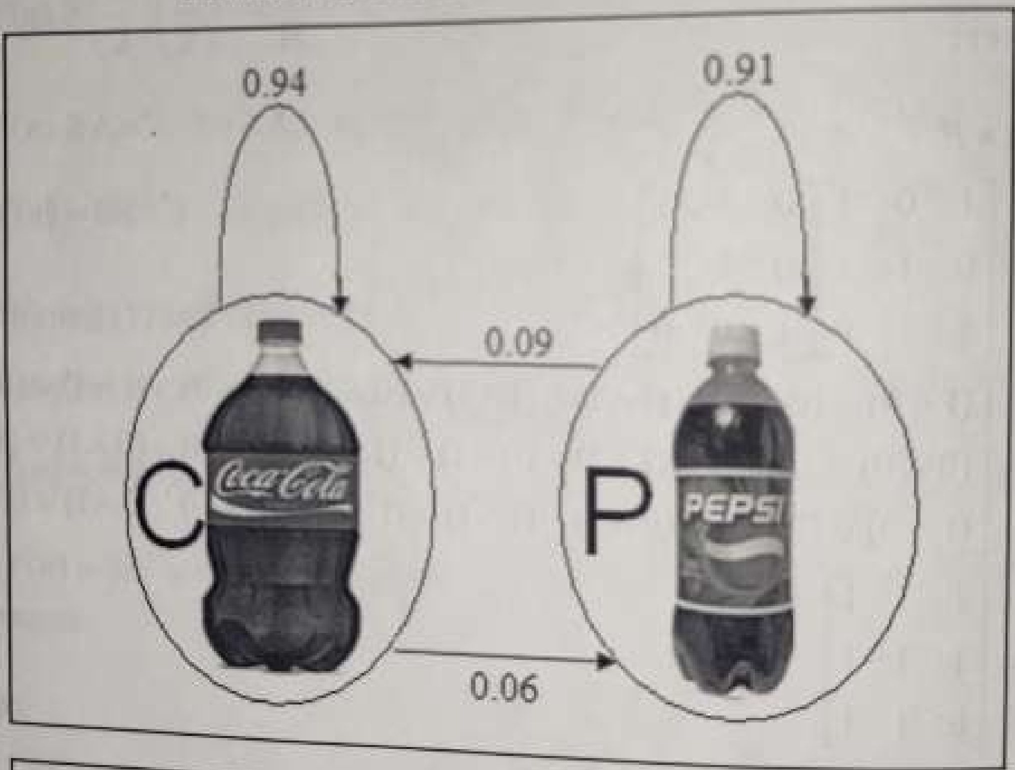
c. Show the predicted sales of Coke and Pepsi for the next six months (starting fr the current month – month 1) if the combined total sale for Coke and Pepsi is $100 million.

d. Pepsi believes that a 30 second advertisement for $2 million will change the Co drinkers so that 12% (instead of the current 6%) of the Coke drinkers will switc to Pepsi by the end of the month. What will be the difference in sales of Pepsi with the advertisement by the end of the month?

Answer:

The problem can be modeled using a Markov chain with two states, namely, **C** (representing **Coke drinkers**) and **P** (representing **Pepsi drinkers**), since the **Markov property** holds here (e.g., probability that a Coke drinker current month will switch to Pepsi next month only depends upon current month probability of the same.

a. **Markov Chain State Transition diagram**



Transition Probability Matrix (T)

$$\begin{array}{c|cc} & C & P \\ \hline C & 0.94 & 0.06 \\ P & 0.09 & 0.91 \end{array}$$

Since we have the following conditional probabilities given:

$P(C|C)$ = *Coke drinkers (current month) will stay with Coke (next month)* = 94%
$P(P|C)$ = *Coke drinkers (current month) will switch to Pepsi (next month)* = 6%
$P(P|P)$ = *Pepsi drinkers (current month) will stay with Pepsi (next month)* = 91%
$P(C|P)$ = *Pepsi drinkers (current month) will switch to Coke (next month)* = 9%

b. **Initial distribution** for the Markov Chain is given by

Starting matrix (S)

$$\begin{array}{cc} C & P \\ \end{array}$$
$$\begin{bmatrix} 0.72 & 0.28 \end{bmatrix}$$

c. Predicted Sales of Coke and Pepsi for different months, given that the combined sale of them remains fixed ($100 million) for each month:

- **Last Month**

Predicted Sale of Coke = $0.72 \times \$100 \ \textit{million} = \$72 \ \textit{million}$

Predicted Sale of Pepsi = $0.28 \times \$100 \ \textit{million} = \$28 \ \textit{million}$

Probabilities for the n^{th} month to be calculated in the following manner:

$P_n = S.\underbrace{T.T.T...T}_{n \ times} = S.T^n$, where $S=P_L$ is the starting matrix (for the last month).

- **Month 1**

$$P_1 = S.T = \begin{bmatrix} 0.72 & 0.28 \end{bmatrix} \begin{bmatrix} 0.94 & 0.06 \\ 0.09 & 0.91 \end{bmatrix} = \begin{bmatrix} 0.7020 & 0.2980 \end{bmatrix}$$

1×2	2×2	1×2
Last month Probabilities	Transition Probability Matrix	Current month (Month 1) Probabilities

Predicted Sale of Coke = $0.7020 \times \$100 \ \textit{million} = \$70.2 \ \textit{million}$

Predicted Sale of Pepsi = $0.2980 \times \$100 \ \textit{million} = \$29.8 \ \textit{million}$

- **Month 2**

$$P_2 = S.T^2 = \begin{bmatrix} 0.72 & 0.28 \end{bmatrix} \begin{bmatrix} 0.94 & 0.06 \\ 0.09 & 0.91 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0.7020 & 0.2980 \end{bmatrix} \begin{bmatrix} 0.94 & 0.06 \\ 0.09 & 0.91 \end{bmatrix} = \begin{bmatrix} 0.6867 & 0.3133 \end{bmatrix}$$

1×2	2×2	1×2
Month 1 Probabilities	Transition Probability Matrix	Month 2 Probabilities

Predicted Sale of Coke = $0.6867 \times \$100\ million = \$68.67\ million$

Predicted Sale of Pepsi = $0.3133 \times \$100\ million = \$31.33\ million$

Calculating in the similar manner,

	Predicted Sale (in million $)	
Month	Coca-Cola	PEPSI
1	70.2	29.8
2	68.67	31.33
3	67.37	32.63
4	66.26	33.74
5	65.32	34.68
6	64.53	35.47

d. After the 30 second advertisement by Pepsi, transition matrix changes to the following:

New Transition Probability Matrix (T)

$$\begin{array}{cc} & \begin{array}{cc} C & P \end{array} \\ \begin{array}{c} C \\ P \end{array} & \begin{bmatrix} 0.88 & 0.12 \\ 0.09 & 0.91 \end{bmatrix} \end{array}$$

Let's calculate the probabilities once again with the new transition matrix:

$$P_1 = S.T = \underset{\substack{1\times 2 \\ \text{Last month} \\ \text{Probabilities}}}{\begin{bmatrix} 0.72 & 0.28 \end{bmatrix}} \underset{\substack{2\times 2 \\ \text{Transition} \\ \text{Probability} \\ \text{Matrix}}}{\begin{bmatrix} 0.88 & 0.12 \\ 0.09 & 0.91 \end{bmatrix}} = \underset{\substack{1\times 2 \\ \text{Current month (Month 1)} \\ \text{Probabilities}}}{\begin{bmatrix} 0.6588 & 0.3412 \end{bmatrix}}$$

Predicted Sale of Coke = $0.6588 \times \$100\ million = \$65.88\ million$

Predicted Sale of Pepsi = $0.3412 \times \$100\ million = \$34.12\ million$

Difference in sales by Pepsi = $\$34.12 - (29.8 + 2.0)\ million = +\$2.32\ million$

Hence, by prediction Pepsi will do $2.32 million more sales.