FALL 2010

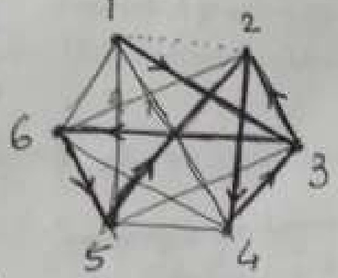
$\frac{18}{25}$

Sandipan Dey

$\frac{81}{105}$

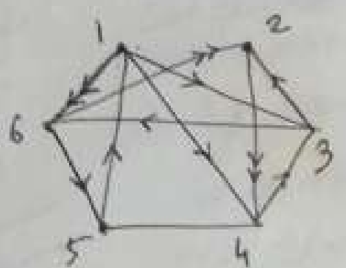
1. **(15 pts)** Let $K'' = (E'', V)$ with vertices $V = \{1, 2, 3, 4, 5, 6\}$ be obtained as a subgraph of K_6 upon removal of the edge 1 – 2.
   (a) Partition E'' into edge-disjoint trails with initial and final vertices of odd degree.
   (b) Can you find two Hamiltonian circuits with disjoint sets of edges?
2. **(10 pts)** Characterize with proof all connected undirected graphs so that each vertex has degree at most 2.

1.



$\frac{10}{25}$

$K'' \equiv K_6 - \{1,2\}$



~~# edges~~ # vertices with odd degree in $K'' = 4$, namely vertices 3 and 4, 5 and 6. But in order to have Euler trail a graph can at most have couple of vertices with degree 1, namely the start and end vertex.

edges in $K'' = \binom{6}{2} - 1 = 15 - 1 = 14$.

Since $14 = 7 \times 2$, we can partition only into 2 edge disjoint trails (since only 2 prime factors) each with 7 edges. One of such partition is:

$T_1 \equiv 1-3-6-5-2-4-3-2$
$T_2 \equiv 1-6-3-5-4-1-3-2$ } with $T_1 \cap T_2$ containing no edges

repeated edge

We can not ~~have~~ Hamilton circuits disjoint ~~as well~~, since we need 7 edges in both the circuits and hence not disjoint

?? 14 : 15 edges to choose from $= \frac{6 \cdot 5}{2} - 1$

Such an

2. $2e = \sum_{v \in V(G)} d(v) \Rightarrow d(v) \leq 2 \Rightarrow 2e \leq 2v$, where $|V(G)| = v$.

$\Rightarrow e \leq v$.

$\frac{8}{10}$

Hence $e \leq v - 1$, ~~in which~~ case it's a tree

Infact a single path

$e = v$, in which case it has exactly one cycle

25/35

16/20

3. (20 pts)

(a) Characterize directed graphs so that each vertex has at most one parent.

(b) What if there is a unique vertex basis?

(c) If $|V| = 1000$ and $|E| = 979$, how many connectivity components are trees?

9/15

4. (15 pts)

(a) A bridge in a connected graph is an edge whose removal disconnects the graph. Show that no connected graph with a bridge admits a Hamiltonian Circuit.

(b) Characterize non-trivial trees with Hamiltonian paths.

can have circuits

3. (a) Each vertex has at most ~~one parent~~ ⇒ ~~if we direct the edge from the parent to the child, we~~ direct the edges from parent to child. It can't have a cycle.
It must be a directed acyclic graph, if it's connected.
Since ~~the~~ each vertex can have at most one parent there can be exactly one path from each vertex to another, hence no cycle.

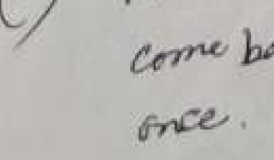
This can't happen

This can't happen

(b) ~~If~~ there is a unique vertex basis, it must be a tree, with root being the vertex basis, since all other children are reachable from the root and the graph is connected, without any cycle.

(c) A tree must have $|E| = |V| - 1$ edges.
If the graph had 999 edges, it would have been minimally connected. Removal of a single edge will create one additional component from this time onwards.
But still 20 edges to remove ⇒ total $1 + 20 = 21$ connectivity components.
initial

4. (a) Hamiltonian circuit ⇒ Need to start from a vertex and come back to it ~~[illegible]~~ after traversing all other vertices exactly once. The graph G has a bridge ⇒ has components G_1, G_2 connected by an edge e.



If the Hamiltonian circuit starts from G_1 (w.l.o.g.) then it eventually it must reach G_2 via the edge e through the vertices x, y. But in order to

complete the circuit, it must come back to G, but
so it must traverse through e, i.e, vertices x, y again once more
⟹ no longer a circuit, contradiction

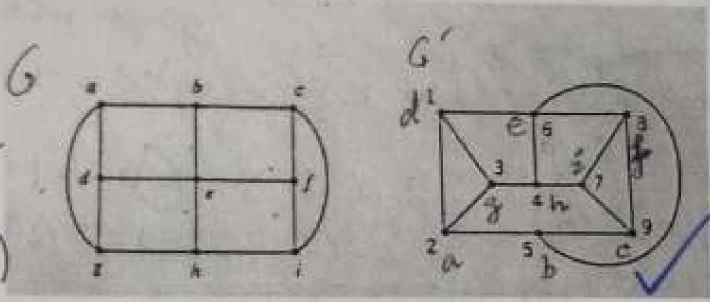
b) missing

5. **(15 points)** Are these two graphs isomorphic? Give the isomorphism or explain why none exists.

15/15

Isomorphic

Since $(u,v) \in E(G) \iff f(u,v) \in E(G')$



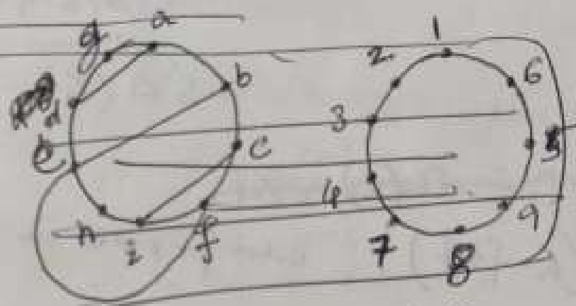
define function
$f : V(G) \to V(G')$

a to 2	e to 6
b to 5	f to 8
c to 9	g to 3
d to 1	h to 4
i to 7	

6. **(MATH 685) (15 pts)** Let $\Delta(G)$ be the maximum degree of a vertex in G. Show that the chromatic number $X(G) \leq \Delta(G) + 1$ (Use induction on $|V|$ and consider removal of vertex of maximum degree from graph)

~~5. Graph G is having~~

23/30



6. Base case $|V| = 1$, $X(G) = 1$, $\Delta(G) = 0$, $\Rightarrow X(G) \leq \Delta(G) + 1$.

8/15

Induction ~~step~~ Hypothesis: Let the assumption be true $\forall\ |V| \leq n$, $n \in \mathbb{N}$

Induction Step: Let's prove for $|V| = n+1$.
If we remove the vertex of maximal degree v from G, ~~G where $\Delta(G$~~.
~~G~~ we get G_v, we have $\Delta(G - \{v\}) \leq \Delta(G)$ by maximality of v

Now, by induction hypothesis, we have,

$X(G - \{v\}) \leq \Delta(G - \{v\}) + 1$ ~~We add the vertex v back along with the edges~~

Let's consider ~~couple of~~ following cases.

① ~~$\Delta(v)$~~ $d(v) = n$, v adjacent to all others ?? then $\chi(G) = \chi(G-\{v\}) + 1$ in the graph with $n+1$ edges

$\Delta(G) = n$?? and $\Delta(G-\{v\}) \leq n-1$ (can have max degree $n-1$. since no self loops or double edges)

$n+1$ ~~edges~~ vertices

$\Delta(G\setminus v) = \Delta(G)$ possible

$\Rightarrow \Delta(G-\{v\}) \leq \Delta(G) - 1$

$\Rightarrow \Delta(G-\{v\}) + 1 \leq \Delta(G)$

$\Rightarrow$ Hence, $\chi(G)$

$= \chi(G-\{v\}) + 1$

$\leq \underline{\Delta(G-\{v\}) + 1} + 1$ by hypothesis

$\leq \Delta(G) + 1$ in the new graph.

~~$d(v) \leq n-1$, $\chi(G) = \chi($~~

$d(v) \geq \Delta(G-\{v\})$, can prove in the similar manner as above.

$d(v) < \Delta(G-\{v\})$, similar manner show that

$\chi(G) \leq \Delta(G) + 1$??

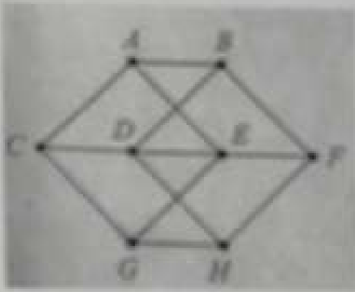
Needs work

7. (a) (10 pts) What is the chromatic polynomial of the edge graph of a cube?

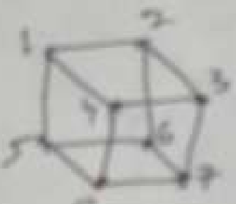
(b) (Math 685)(10 pts) What is the chromatic polynomial of the edge graph of an octahedron?

8. (15 pts) Prove that this graph is non-planar. Begin with a Hamilton cycle to use the circle-chord method.

15/15



7 (a)



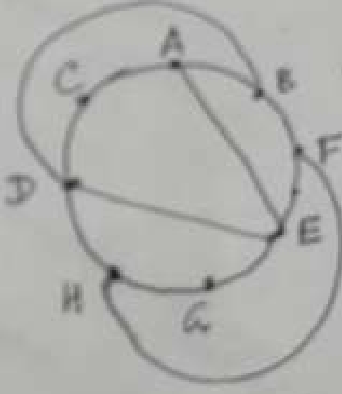
1 can be colored in k ways, after that
2 " " " " k-1 " " "
3 " " " " k-1 " " "
4 " " " " k-2 " " "
5 " " " " k-1 " " "
6 " " " " k-2 " " "
7 " " " " k-2 " " "
8 " " " " k-3 " " "

$\therefore P_k(G)$
$= k(k-1)^3(k-2)^3(k-3)$

(b) Same as cube,

$P_k(G)$
$= k(k-1)^3(k-2)^3(k-3)$

8.



By inside outside symmetry of the circle draw AE inside first. BD, FH must be drawn in the exterior.

Now, ED can be drawn inside.

As we can see, we can no longer draw GC, without intersecting any of the existing edges, hence non planar. (proved)