1.3

Sandipan Dey

Homework - 2

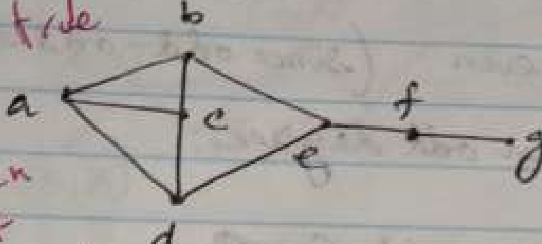
28/30

2.(a) In undirected graph G, let each vertex v denote a person.
Degree of v = $d(v)$ = # children the person has.
nodes in G with odd degree must be even

this is not necessarily true → ⇒ # people having odd # children is even

⇒ # people " " # brothers & sisters is even ✓

(b)



$N(a) = \{b, c, d\}$ $N(c) = \{a, b, d\}$

$N(b) = \{a, c, f\}$ $N(d) = \{a, c, e\}$

$N(e) = \{b, d, f\}$ $N(f) = \{e, g\}$

$N(g) = \{f\}$

Since # children are not their parents' children (so the graph is directed)

$s(a) = |N(b) \cup N(c) \cup N(d)| = |\{a, b, c, d, e, f\}| = 5$

$s(b) = |N(a) \cup N(c) \cup N(e)| = |\{a, b, c, d, f\}| = 5$

$s(c) = |N(a) \cup N(b) \cup N(d)| = |\{a, b, c, d, e\}| = 5$

$s(d) = |N(a) \cup N(c) \cup N(e)| = |\{a, b, c, d, f\}| = 5$

$s(e) = |N(b) \cup N(d) \cup N(f)| = |\{a, c, e, g\}| = 4$

$s(f) = |N(e) \cup N(g)| = |\{b, d, f\}| = 3$

$s(g) = |N(f)| = |\{e, g\}| = 2$

∴ # x with odd $s(x)$ is 5 not even.

In general $s(x)$ represents number of vertices reachable by length 0 or length 2 path from x.

Not necessarily.

5 odd $s(x)$ vertices in the graph in (b) is enough to prove it.

x with odd $s(x)$ = 0, even

x with odd $s(x)$ = 1, odd

5/5

4. $|V(G)| = n$.

odd degree vertices of G must be even.

Since G has $n-1$ odd degree vertices, $n-1 \in \mathbb{Z}^+_{even} \Rightarrow n \in \mathbb{Z}^+_{odd}$

In $\bar{G}$, $\forall v \in V(\bar{G})$, $d_{\bar{G}}(v) = n - d_G(v)$

Now, for $n-1$ vertices $\underline{\text{in } G}$, d_G is odd,

$\Rightarrow$ " " vertices in $\bar{G}$, $d_{\bar{G}}$ is even (since odd − odd = even)

$\Rightarrow$ Exactly one vertex in $\bar{G}$ has an odd degree.

8. We get the following scheduling graph G.

G_1 G_2

$|G_1| = 13$ $|G_2| = 13$

Conf 1 Conf 2.

Let's remove the connecting edges in between G_1 & G_2 (or all cross conference games) 13

Now consider G_1 or G_2. Each of the vertices in G_1 (or G_2) has degree 11.

$\Rightarrow$ total degree of vertices in G_1 (or G_2) is $= 13 \times 11$ which is odd, hence can't be satisfied.

5/5

9. It is obvious that total number of incoming edges to all the vertices must equal the total number of outgoing edges from all vertices (since each incoming edge to a vertex must come from an outgoing edge from an another vertex)

$\Rightarrow$ Sum of the in-degrees of vertices

= Sum of the out-degrees " " in the directed graph

But total number of edges (undirected)

= total number of directed incoming edges to all the vertices

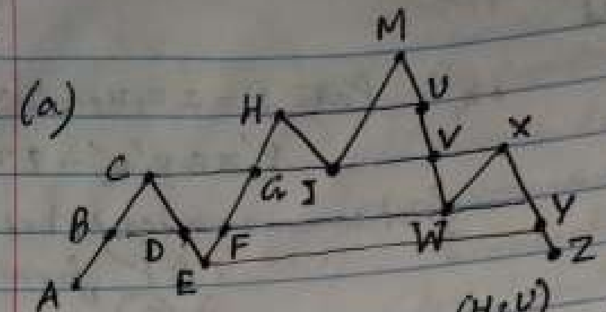
= " " " directed outgoing " from " " "

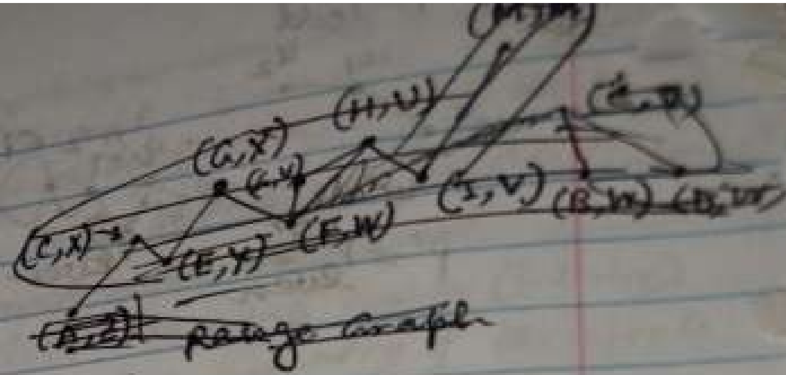
= sum of in-degrees of vertices

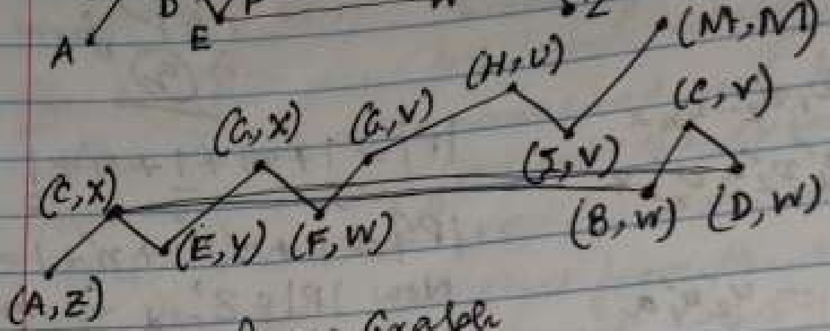
= sum of out-degrees " " (Proved)

1.3

10. (a)



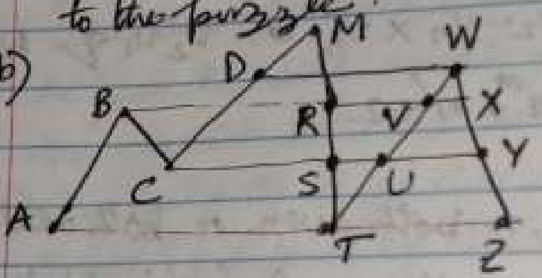


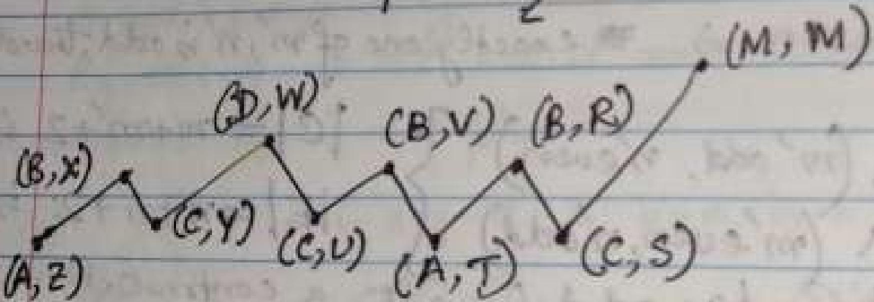


Range Graph

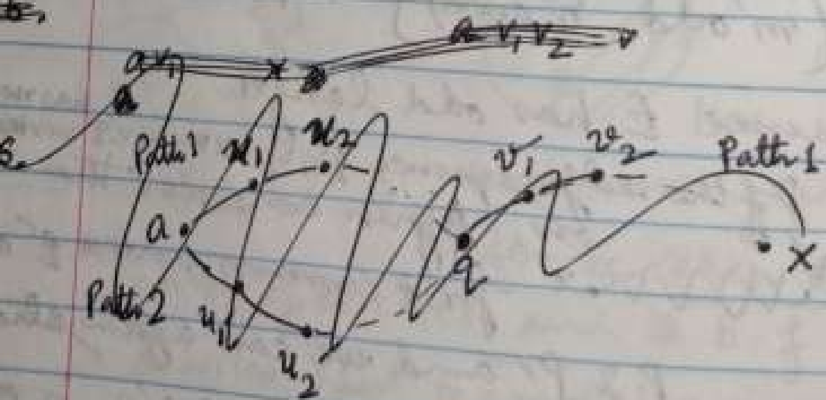
There is a path from (A,Z) to (M,M) which is the solution to the puzzle

(b)





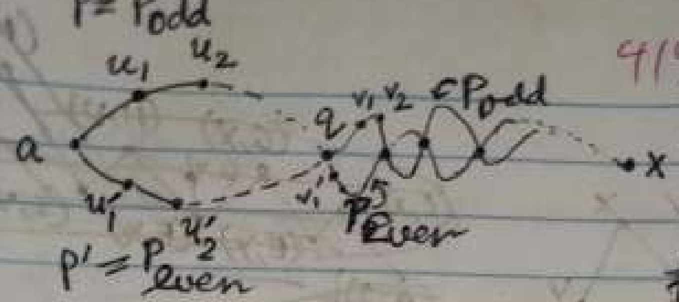
$\exists$ a path from (A,Z) to (M,M): Solution of the puzzle



1.3. $P = P_{odd}$

415

16.



Path $P \equiv a\, u_1 u_2 \cdots q\, v_1 v_2 \cdots$

$P' \equiv a\, u_1' u_2' \cdots q\, v_1' v_2' \cdots$

first common vertex in between P & P' from a is q

$P \equiv a\, u_1 u_2 \cdots u_m\, q\, v_1 v_2 \cdots v_n\, x$ $\qquad |P| = \underset{a}{1} + m + \underset{q}{1} + n + \underset{x}{1} = m+n+3$

$P' \equiv a\, u_1' u_2' \cdots u_{m'}'\, q\, v_1' v_2' \cdots v_{n'}'\, x$ $\qquad |P'| = 1 + m' + 1 + n' + 1 = m'+n'+3$

Now $|P| \in \mathbb{Z}^+_{odd} \Rightarrow m+n \in \mathbb{Z}^+_{even}$

$|P'| \in \mathbb{Z}^+_{even} \Rightarrow m'+n' \in \mathbb{Z}^+_{odd}$

$\therefore C \equiv a\, u_1 u_2 \cdots u_m\, q\, u_{m'}' \cdots u_2' u_1'\, a$

is a circuit with length $\underset{a}{1} + m + m' + \underset{x}{1} = m + m' + 2$

Also the edge sequence $E \equiv q\, v_1 v_2 \cdots v_n\, x\, v_{n'}' \cdots v_2' v_1'\, q$ has

length $= \underset{q}{1} + n + n' + \underset{x}{1} = n + n' + 2$

Now, $m+n$ even $\Rightarrow$ m, n both even or both odd.

$m'+n'$ even $\Rightarrow$ exactly one of m', n' is odd, the other is even

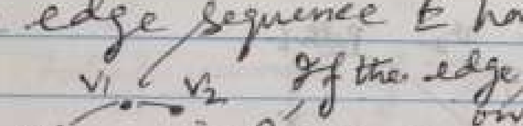
Case-1 $(m, n \text{ even}) \wedge (m' \text{ odd}, n' \text{ even})$ OR $(m, n \text{ odd}) \wedge (m' \text{ even}, n' \text{ odd})$ $\Big\}$ $|C| = m+m'+2$ is odd, $|E| = n+n'+2$ is even

$\Rightarrow$ The circuit C has odd length, a contradiction.

Case-2 $(m, n \text{ even}) \wedge (m' \text{ even}, n' \text{ odd})$ OR $(m, n \text{ odd}) \wedge (m' \text{ odd}, n' \text{ even})$ $\Big\}$ $|C| = m+m'+2$ is even, $|E| = n+n'+2$ is odd

$\Rightarrow$ The edge sequence E has odd length.

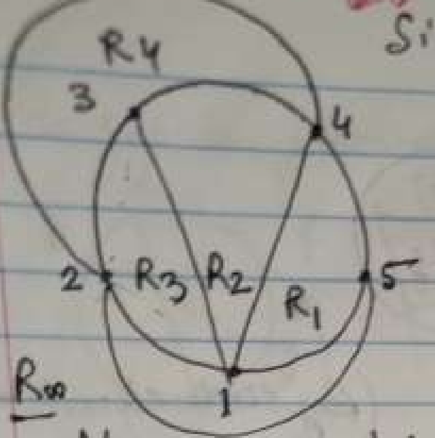
For case-2



If the edge sequence $q v_1 v_2 \cdots x \cdots v_2' v_1' q$ has no repeated vertex, we are done. Ow, Again consider the first common vertex q' on P & P' and we can again show by the similar argument as earlier that either $\underbrace{q q'}_{P} \underbrace{q}_{P'}$ is an odd length circuit or edge sequence $q' \xrightarrow{P} x \xrightarrow{P'} q'$ has odd length. If the sequence has no repeated vertex, it is an odd length circuit and we are done ow continue like before.

Why must it [illegible]

1.4.
2.



Since in K_5 all vertices are connected to each other choose any circuit 1-2-3-4-5-1.
By inside-outside symmetry of the circle let's draw chord 1-4 inside (interior).
~~Now 3-5 must be connected and~~
Similarly let's draw 1-3 inside (interior) by inside-outside symmetry of the circle.

Now, 2-4 must be connected and to avoid crossing we must connect from ~~out~~ the exterior (R_∞) of the circle.
Now, to connect 3-5, there are ~~couple of~~ 3 choices.

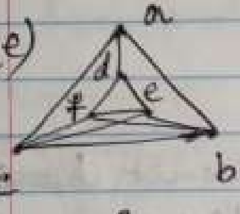
(1) The first choice can start from the exterior of the region R_1 ~~define~~ defined by the circuit 1-4-5-1 and go to the interior of the circuit, in which case it must cross the circuit.

(2) The second choice similarly has to start from interior of region R_4 and must go to exterior of region R_4 defined by the circuit 2-3-4-2 and hence must cross the circuit.

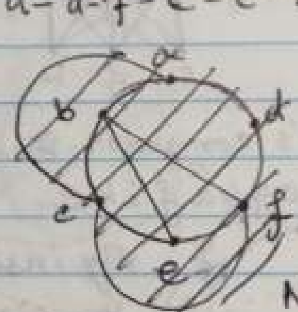
(3) Similarly the third choice ~~should~~ must start from inside region R_3 defined by circuit 1-2-3-1 and go to the exterior of R_3 to connect to 5 and it must also cross the circuit.

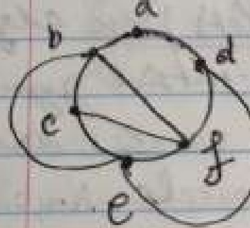
Hence, K_5 is not planar.

3. (e)



a-d-f-e-c-b-a is a circuit

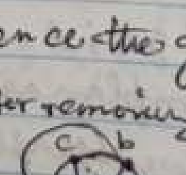




Let's pick b-f to be the first edge to be drawn by inside-outside symmetry of circle let's draw it inside.

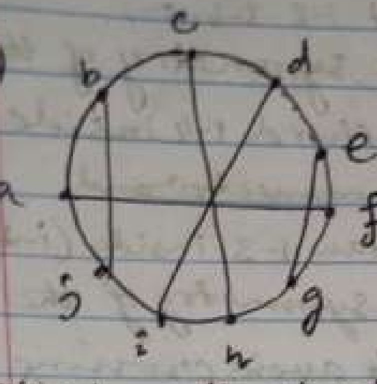
Now, we have to connect d-e. To avoid crossing edges, d-e must be connected through exterior. Then c-f must be connected through interior of the circle and b-f must be connected again through exterior. But to connect a-c we need to cross either the circuit bceb or bcfb. Hence the graph is non-planar.

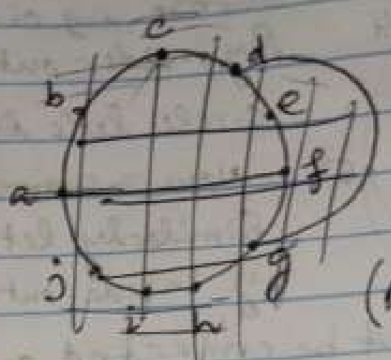
Now this configuration (after removing degree-2 vertex a) looks exactly like K5 (as in prob 2)



$f \equiv 1$ $b \equiv 4$
$e \equiv 2$ $d \equiv 5$
$c \equiv 3$ } (Prob 2)

3.(f)

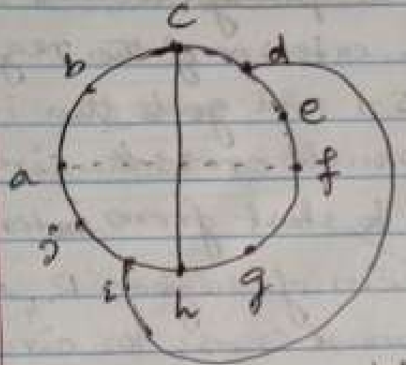




(By circle-chord method)

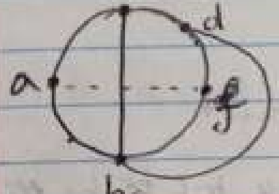
Let's choose the circuit abcdefghija

Let's pick the edge ch and with inside-outside symmetry for a circle let's draw the edge in the interior. Now the edge d-i must be drawn in the exterior of the circle. Now in order to draw the edge a-f one must either cross the circuit chgfedc or the circuit defghid.



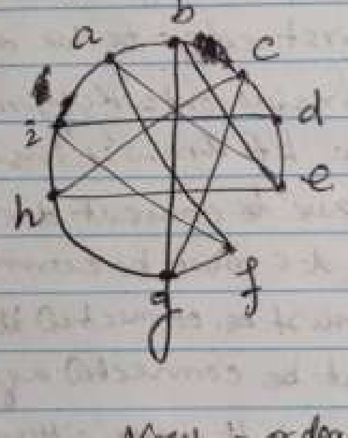
Hence the graph is nonplanar.

Now let's consider the subgraph above and remove vertices with degree 2. we get

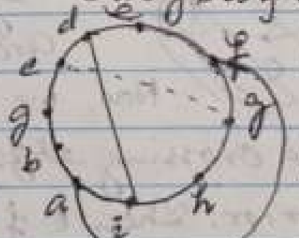


which is a K3-3 configuration.

(g)

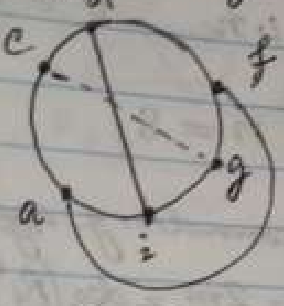


Let's consider the circuit ~~[illegible]~~ edcgbafihe. Let's pick edge di first. By inside-outside symmetry of the circle, we can draw di inside the circle. Now, in order to draw a-f, it must be drawn in the exterior of the ~~[illegible]~~ circle.



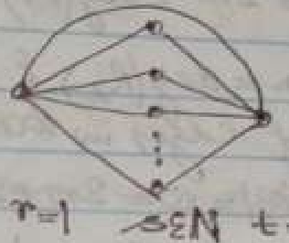
Now, in order to draw the edge cg, one must either cross the circuit defghid or the circuit aihgfa. Hence the graph is nonplanar.

Now lets consider the subgraph above and remove vertices with degree 2 to get the following graph:



which has a $K_{3,3}$ configuration.

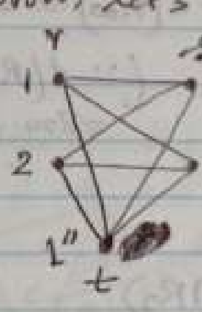
6.

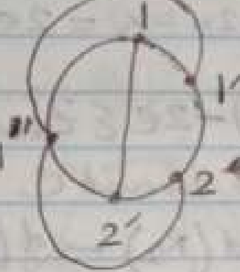


r=1 $s \in N$ t=1

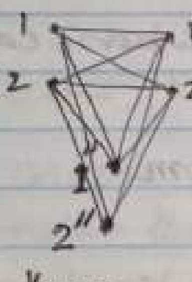
As can be seen. $K_{r,s,t}$ is planar for $r=t=1, \forall s \in \mathbb{N}$ similarly $r=s=1, \forall t \in \mathbb{N}$ and $s=t=1, \forall t \in \mathbb{N}$ by symmetry, i.e., K_{n11}, K_{1n1} & K_{11n} are planar $\forall n \in \mathbb{N}$
$n = 1, 2, \dots$

Now, let's consider $K_{2,2,1}$ which has a circuit $1\text{-}1'\text{-}2\text{-}2'\text{-}1''\text{-}1$.

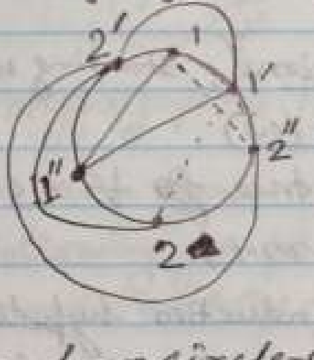


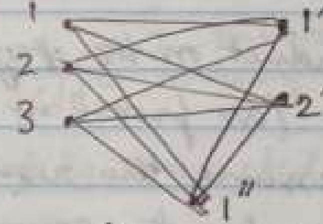


by circle chord let's redraw the graph. By inside-outside symmetry of the circle, let's pick the edge 1-2' first and draw it inside the circle. Hence we see that $K_{2,2,1}$ and by symmetry $K_{1,2,2}$ and $K_{2,1,2}$ are planar.

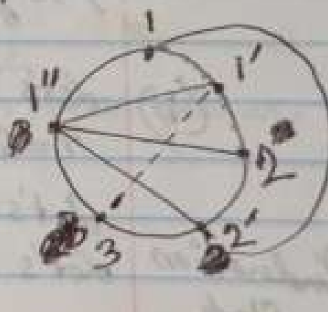


$K_{2,2,2}$





$K_{3,2,1}$



As can be seen from circle-chord method, $K_{2,2,2}$ is not planar

As can be seen, $K_{3,2,1}$ and equivalently $K_{1,2,3}$ and $K_{3,1,2}$ are not planar.

Hence only planar tripartite graphs are

$$\left\{ \begin{array}{l} K_{11n}, K_{1n1}, K_{n11}, \quad n = 1, 2, 3, \dots \\ K_{122}, K_{212}, K_{221} \end{array} \right.$$

8.

Using Euler's formula, $r = e - v + 2$,
given $r = 10$, $2e = \sum_v d(v) = 4n$, $v = n$

11. (a)

$\Rightarrow 10 = 2n - n + 2 \Rightarrow n = 8$

Let's assume the # of boundary ~~ages~~ edges in region $R_1 = 2m = deg(R_1) \in \mathbb{Z}^+_{even}$

boundary edges in $R_2 = 2n = deg(R_2) \in \mathbb{Z}^+_{even}$

Let's additionally assume that # of edges in the common boundary $= K$. Hence $|E(G)| = e = \cancel{2m+2n-K}$

~~Now, sum of de~~ Now, $\sum_i d(R_i) = 2e = 2|E(G)|$

$\cancel{\Rightarrow 2m + 2n} \Rightarrow d(R_1) + d(R_2) + d(R_\infty) = 2e$

$\Rightarrow 2m + 2n + 2m + 2n - K = 2e$ $\quad (\because d(R_\infty) = 2m + 2n - K)$

$\Rightarrow K = 4(m+n) - 2e \in \mathbb{Z}^+_{even}$.

~~Now,~~ Hence, # edges on circuit C

$= d(R_1) + d(R_2) - d(R_1 \cap R_2)$

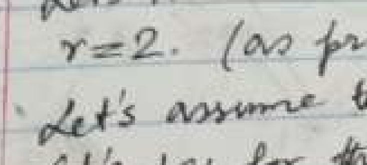
$\Rightarrow$ length of $C = 2m + 2n - K \in \mathbb{Z}^+_{even}$ (proved).

(b) Let's induct on the # of regions r, starting with base case $r = 2$. (as proved in part (a)).

Let's assume the result is true for $\forall\, r \leq m$

Induction Step

Let's try for the case $r = m+1$.



Now, by induction hypothesis, we know that the cycle C is an even-length cycle.

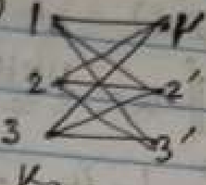
Form R_1' by removing all the interior edges from $R_1 \cup R_2 \cup \dots \cup R_m$

Let's remove all the ~~edges~~ vertices from the graph that are interior to all of those m regions to get graph G', ~~thus, the,~~ ~~additional region~~ with the combined region R_1' and another additional new region R_2'. By hypothesis both R_1', R_2' have even # of edges on boundaries and hence again by hypothesis the circuit C will be even length cycle (Prov

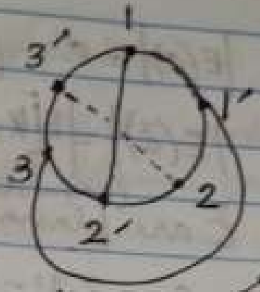
Circle-chord

13. (a)

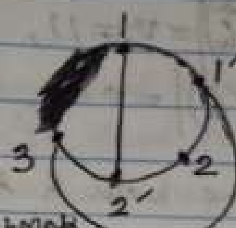
(i)



$K_{3,3}$



$K_{3,3}$

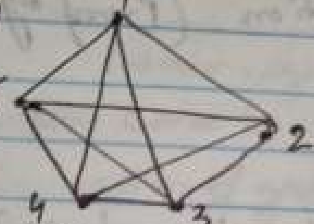


subgraph of $K_{3,3}$ after removing vertex 3′

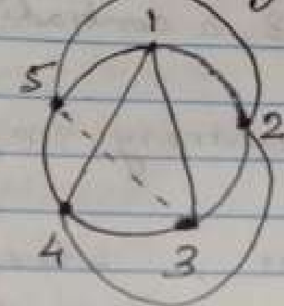
As can be seen, just removing a single vertex (e.g 3′ or equivalently 2) we can get a planar ~~planar graph~~ subgraph of $K_{3,3}$

Hence, $K_{3,3}$ is critically planar graph

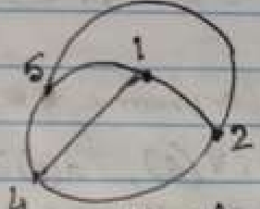
(ii)



K_5



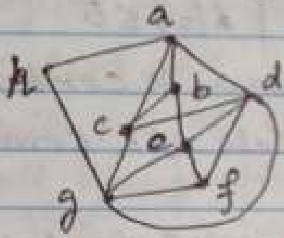
K_5

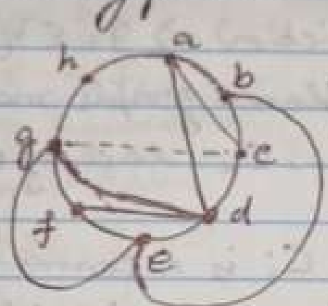


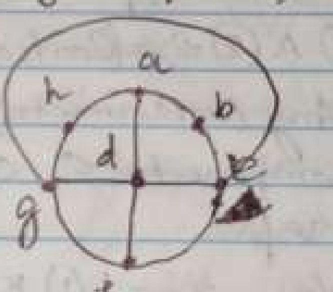
subgraph of K_5 after removing vertex 3 we get a planar subgraph of K_5

Hence, K_5 is critically planar.

(iii)







subgraph after removing vertex c. planar

Hence, the original graph is critically planar.

13. (b) A Critical non planar graph <u>may not be connected.</u>

Consider the following graph consisting of 2 components C_1 & C_2

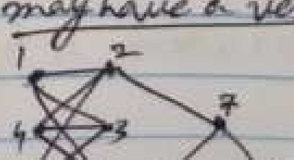
$K_{3,3}$ C_1 K_3 C_2

$C_1 \equiv K_{3,3}$, $C_2 \equiv K_3$

This graph is still ~~cri~~ critically nonplanar, since removal of a vertex from C_1 makes it planar.

A critical nonplanar graph <u>may have a vertex whose removal</u> disconnects the graph.

Consider the following graph



$K_{3,3}$ K_3

As we can see it's critically disconnected, still removal of vertex 7 <u>disconnects</u> the graph

✓ ~~[illegible]~~ SIS.

1.4

16. For G, $|V(G)| = v = 11$, $|E(G)| = e$ (let)

$\Rightarrow$ $|V(\bar{G})| = v = 11$, $|E(\bar{G})| = |E[K_{11}]| - e = \frac{11 \times 10}{2} - e = 55 - e$

Let's assume both G & $\bar{G}$ are (connected) planar, to the contrary.

$$\Rightarrow \begin{cases} \text{for } G, & e \le 3v - 6 = 3 \cdot 11 - 6 = 27 \Rightarrow e \le 27 \\ \text{for } \bar{G}, & 55 - e \le 3v - 6 = 27 \Rightarrow e \ge 55 - 27 = 28 \end{cases}$$

$\Rightarrow$ $e \le 27 \wedge e \ge 28$, a contradiction (Proof by contradiction)

18. (a) Let's assume to the contrary, i.e.,

$\forall v \in V(G)$, $d(v) \ge 6$

$\Rightarrow 2e = \sum_v d(v) \ge 6v$, where $|E(G)| = e$, $|V(G)| = v$.

$\Rightarrow e \ge 3v$ ①

Also, G is connected planar $\Rightarrow e \le 3v - 6$ ②

① $\wedge$ ② $\Rightarrow$ $\times$ (Contradiction) $\Rightarrow \exists v \in V(G) \mid d(v) \le 5$

(b) Any (unconnected) planar graph may have k connected components $C_1, C_2, \ldots, C_k$, each of which is a connected graph.

Now, $\forall i = 1(1)k$, C_i is a connected graph and from (a) we know, $\exists v_i \in V(C_i) \mid d(v_i) \le 5$, $\forall i$

$\Rightarrow \exists v \in V(G) \mid d(v) \le 5$,

where $V(G) = \bigcup_{i=1}^{k} V(C_i)$ and choose $v = v_i$ for some $i = 1, \ldots, k$

(Proved)

14. (a) Proof by contradiction

Let's assume to the contrary that there exist a region in the maximal planar graph that has degree > 3 (not triangular). That particular region may look like the following

interior regions $\geq$ degree 4

In this case in any arbitrary configuration of graph there must exist couple of points that can be connected by still having the planarity.

consider the degree 4 ~~graphs~~ regions as graphs and by Euler's eq^n, $r = e - v + 2$, with $r = 2 \Rightarrow e = v$ hence all these graphs must have a single cycle, with $v \geq 4$. If all the vertices are on the cycle, we can join any two nonadjacent to increase edges to $e+1$ and regions to $r+1$, still maintaining the planarity.

ow, $\exists$ a vertex with degree 1, connect it with another vertex (nonadjacent) ($\exists$ such a vertex) without disturbing planarity $r+1 = e+1-v+2$

The regions with degree > 4 can also be similarly proved.

exterior regions $\geq$ degree 4

In this case, any arbitrary configuration of the graph must contain more than 3 edges on its boundary, s.t. the exterior region has degree at least 4.

In this case also, arguing as before, we must have at least two vertices on the boundary that can be safely joined to maintain planarity.

Since

Hence, the graph is not a maximal planar graph a contradiction.

1.4.

4/5

14.(b) A graph is planar $\Rightarrow e \leq 3v-6$

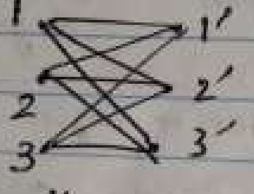
Given $v=n \Rightarrow e \leq 3n-6$

By Euler's eqn, $r = e-v+2 \leq 3n-6-n+2 = 2n-4$

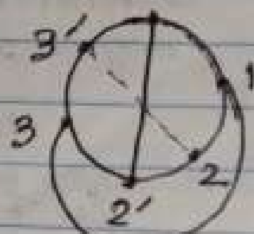
Hence a maximal planar graph with n vertices has $3n-6$ vertices and $2n-4$ regions.

Why does r achieve its minimum? Need prove.

12. (a) Crossing edges

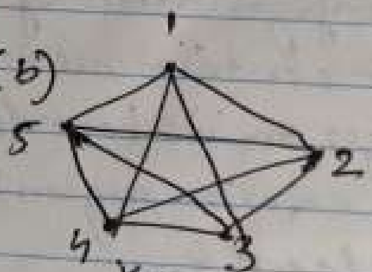


$K_{3,3}$

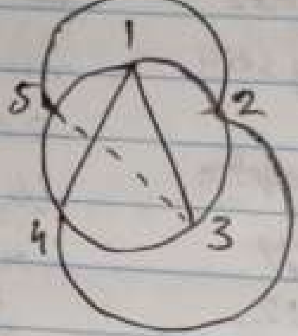


Hence, crossing number $c(K_{3,3}) = 1$ (only one pair of crossing edges, e.g. $(1,2')$ and $(2,3')$)

(b)



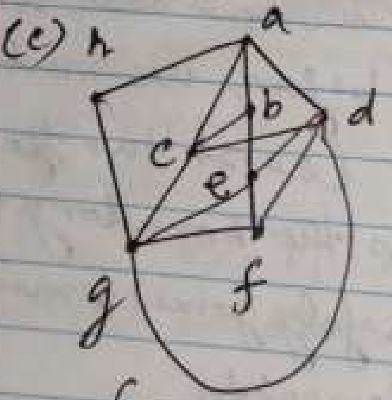
K_5



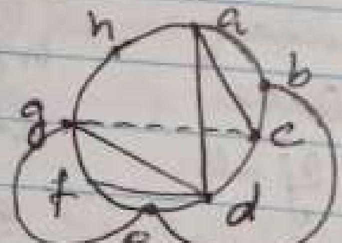
$c(K_5) = 1$

crossing edge pair: $(1,4)$, $(3,5)$.

(c)



G



Circle-chord

G

Only one pair of edges namely ad and gc are intersecting,

$c(G) = 1$.

$e \leq 3v-6$ does not mean e can achieve $3v-6$ for arbitrary v