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(*-----------------------------------------------------------------------------*)
(*---------------- Sandipan Dey, UMBC CSEE ----------------------------------------*)
(*---------------- The Source Code for HW 1 ---------------------------------------*)
(*---------------- Functions -----------------------------------------------------
                 1. ComputeOrthonormalEigenSpaces
                 2. ComputeProjectors
                 3. ComputeProbStates
                 4. ShowOutputTables
                 5. MeasureQuantumSystem
   ---------------------------------------------------------------------------*)

(* ComputeOrthonormalEigenSpaces: Computes the Orthonormal EigenSpaces *)
(* Inputs ⇒ Ω: The Observable *)
(* Output ⇒ EigenValues, EigenVectros and the Dimensions of the EigenSpaces *)
ComputeOrthonormalEigenSpaces[Ω_] := Module[{n, Λ, V, Λo, Vo, i},
  n = Dimensions[Ω][[1]]; (*Ω Square Matrix*)
  {Λ, V} = Eigensystem[Ω]; (*Find EigenValues and Orthogonal EigenVectors*)
  (*Construct Orthonormal EigenKets in the respective EigenSpaces*)
  Clear[Λo]; Do[Λo[Λ[[i]]] = Λ[[i]], {i, n}];
  Clear[Vo]; Do[Vo[Λ[[i]]] = {}, {i, n}]; Do[Vo[Λ[[i]]] = Append[Vo[Λ[[i]]], V[[i]]], {i, n}];
  Do[If[Dimensions[Vo[Λ[[i]]]][[1]] == 1, Vo[Λ[[i]]] = {Normalize[Vo[Λ[[i]]][[1]]]}, Vo[Λ[[i]]] = Orthogonalize[Vo[Λ[[i]]]]], {i, n}];
  Λ = DownValues[Λo][[All, 2]]; V = DownValues[Vo][[All, 2]]; n = Dimensions[Λ][[1]]; (*Dimension of Eigen Space*)
  {Λ, V, n}
 ];


(* ComputeProjectors: Computes the Projectors *)
(* Inputs ⇒ V: EigenSpace buckets containing orthonormal eignevectors, n: Dimension of the EigenSpace *)
(* Output ⇒ n Projectors *)
ComputeProjectors[V_, n_] := Module[{P, pVerify, oVerify, i, j},
  pVerify = oVerify = True;
  P = Table[0, {i, n}, {j, 1}];
  Do[{m, p} = Dimensions[V[[i]]]; P[[i]] = Table[0, {r, p}, {c, p}];
   Do[ket = {V[[i]][[j]]}^T; braw = ket^†; Pr = ket.braw; If [Pr.Pr ≠ Pr, pVerify = False, ] ; P[[i]] = P[[i]] + Pr, {j, m}], {i, n}];
  ZeroMatrix = Table[0, {i, p}, {j, p}]; Do[Do[If[P[[i]].P[[j]] ≠ ZeroMatrix and i ≠ j, oVerify = False,], {i, n}], {j, n}]; (*Verify Kronecker*)
  {P, pVerify}
 ];

(* ComputeProbStates: Computes the Probabilities and the States *)
(* Inputs ⇒ P: n Projectors, Ψ: Quantum System, n: Dimension of the EigenSpace *)
(* Output ⇒ n Probabilities and the States *)
ComputeProbStates[P_, Ψ_, n_] := Module[{prob, state, i},
  prob = Table[Expand[Ψ^†.P[[i]].Ψ][[1]][[1]], {i, n}]; (*Probabilities*)
  state = Table[Map[Simplify, Transpose[{Normalize[ Transpose[P[[i]].Ψ][[1]]]}]], {i, n}]; (*States*)
  {prob, state}
 ];
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(* ShowOutputTables: Shows the Output Tables *)
(* Inputs ⇒ Ω: The Observable, Λ: EigenValues, V: EigenSpace buckets containing orthonormal eignevectors, P: Projectors, Ψ: The Quantum System,
        prob: Probabilities, state: States, n: Dimension of the EigenSpace *)
(* Output ⇒ None *)
ShowOutputTables[Ω_, Ψ_, Λ_, V_, P_, prob_, state_, n_, pVerify_, si_, sv_, so_] := Module[{inputTable, verifyTable, outputTable, i, j, k},
   inputTable  = Table[If[i == 1, Switch[j, 1, "Observable", 2, "QuantumSystem"], Switch[j, 1, MatrixForm[Ω], 2, MatrixForm[Ψ]]], {i, 2}, {j, 2}];
   verifyTable = Table[If[k == 1, Switch[j, 1, "∑P = I", 2, "∑λP = Ω", 3, "Pi.Pi = Pi", 4, "∑p = 1", 5, "Pi.Pj=0, i≠j"],
                       Switch[j, 1, Sum[P[[i]], {i, n}] == IdentityMatrix[p],
                                 2, Sum[Λ[[i]]*P[[i]], {i, n}] == Ω,
                                 3, pVerify,
                                 4, If[Sum[prob[[i]], {i, n}] == 1, True, False],
                                 5, True]],
            {k, 2}, {j, 5}];
   outputTable = Table[If[i == 0, Switch[j, 1, "EigenValue", 2, "EigenSpace", 3, "Projector", 4, "Probability", 5, "State"],
                       Switch[j, 1, Λ[[i]], 2, MatrixForm[V[[i]]ᵀ], 3, MatrixForm[P[[i]]], 4, prob[[i]], 5, MatrixForm[state[[i]]]]],
            {i, 0, n}, {j, 5}];
   (* Show Outputs *)
   Grid[inputTable, Alignment → Center, Spacings → {si, 1}, Frame → All, ItemStyle → "Text", Background → {{None, None}, {Orange, None}}]
   Grid[verifyTable, Alignment → Center, Spacings → {sv, 1}, Frame → All, ItemStyle → "Text", Background → {{None, None}, {None, None}}]
   Grid[outputTable, Alignment → Center, Spacings → {so, 1}, Frame → All, ItemStyle → "Text", Background → {{None, None}, {Green, None}}]
   ];
```

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(* MeasureQuantumSystem: Measures the Quantum System with the Observable *)
(* Inputs ⇒ Ω: The Observable, Ψ: The Quantum System *)
(* Output ⇒ None *)
MeasureQuantumSystem[Ω_, Ψ_, si_:19, sv_:6, so_:4] := Module[{Λ, V, n, P, pVerify, prob, state},
   (* Compute Orthonormal EigenSpaces *)
   {Λ, V, n} = ComputeOrthonormalEigenSpaces[Ω];
   (* Compute Projectors *)
   {P, pVerify} = ComputeProjectors[V, n];
   (* Compute Probabilities and States *)
   {prob, state} = ComputeProbStates[P, Ψ, n];
   (* Shows Output Tables *)
   ShowOutputTables[Ω, Ψ, Λ, V, P, prob, state, n, pVerify, si, sv, so]
  ];

(*----------- Inputs ----------*)
(*Example 1*)
Ω = {{0, 0, 1, -i}, {0, 0, i, -1}, {1, -i, 0, 0}, {i, -1, 0, 0}}; (*Observable*)  Ψ = Transpose[{Normalize[({1, i, 0, -1})]}];(*State*)
MeasureQuantumSystem[Ω, Ψ]
(*Ex 1.1*)
Ω = {{2, 0, 0, i}, {0, 2, 0, 0}, {0, 0, 2, 0}, {-i, 0, 0, 2}}; (*Observable*)  Ψ = Transpose[{Normalize[({1, 1, 1, 1})]}];(*State*)
MeasureQuantumSystem[Ω, Ψ]
(*Ex 1.2*)
Ω = {{5, 0, 0, 3i}, {0, 5, i, 0}, {0, -i, 5, 0}, {-3i, 0, 0, 5}}; (*Observable*)  Ψ = Transpose[{Normalize[({1, 1, 1, 1})]}];(*State*)
MeasureQuantumSystem[Ω, Ψ]
```

(*-------- Output: In the template form ------*)

Observable	QuantumSystem
$\begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$

$\sum P = I$	$\sum \lambda P = \Omega$	$P_i.P_i = P_i$	$\sum p = 1$	$P_i.P_j=0,\ i\neq j$
True	True	True	True	True

EigenValue	EigenSpace	Projector	Probability	State
$-\sqrt{2}$	$\begin{pmatrix} \frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} \\ 0 & \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} \frac{-i+\sqrt{2}}{2\sqrt{3}} \\ -\frac{-2i+\sqrt{2}}{2\sqrt{6}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$
$\sqrt{2}$	$\begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{i}{2} \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} \frac{i+\sqrt{2}}{2\sqrt{3}} \\ \frac{2i+\sqrt{2}}{2\sqrt{6}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$

Observable	QuantumSystem
$\begin{pmatrix} 2 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -i & 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$\sum P = I$	$\sum \lambda P = \Omega$	$P_i.P_i = P_i$	$\sum p = 1$	$P_i.P_j=0,\ i\neq j$
True	True	True	True	True

EigenValue	EigenSpace	Projector	Probability	State
1	$\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{4}$	$\begin{pmatrix} \frac{1}{2} - \frac{i}{2} \\ 0 \\ 0 \\ \frac{1}{2} + \frac{i}{2} \end{pmatrix}$
2	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$
3	$\begin{pmatrix} \frac{i}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{4}$	$\begin{pmatrix} \frac{1}{2} + \frac{i}{2} \\ 0 \\ 0 \\ \frac{1}{2} - \frac{i}{2} \end{pmatrix}$

Observable	QuantumSystem
$\begin{pmatrix} 5 & 0 & 0 & 3i \\ 0 & 5 & i & 0 \\ 0 & -i & 5 & 0 \\ -3i & 0 & 0 & 5 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$\sum P = I$	$\sum \lambda P = \Omega$	$P_i . P_i = P_i$	$\sum p = 1$	$P_i . P_j = 0,\ i \neq j$
True	True	True	True	True

EigenValue	EigenSpace	Projector	Probability	State
2	$\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{4}$	$\begin{pmatrix} \frac{1}{2} - \frac{i}{2} \\ 0 \\ 0 \\ \frac{1}{2} + \frac{i}{2} \end{pmatrix}$
4	$\begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{4}$	$\begin{pmatrix} 0 \\ \frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} \\ 0 \end{pmatrix}$
6	$\begin{pmatrix} 0 \\ \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{4}$	$\begin{pmatrix} 0 \\ \frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} \\ 0 \end{pmatrix}$
8	$\begin{pmatrix} \frac{i}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{4}$	$\begin{pmatrix} \frac{1}{2} + \frac{i}{2} \\ 0 \\ 0 \\ \frac{1}{2} - \frac{i}{2} \end{pmatrix}$