Sandipan Dey   Math 685   (5.1–5.3)

18/25.

41. In an $n \times m$ chessboard, there ~~There~~ are $\binom{n}{1}$ ways to choose any row.

Once a row is chosen, there are m places where we can place those two identical rooks, in $\binom{m}{2}$ ways $\Rightarrow$ total # of ways $= \binom{n}{1}\binom{m}{2}$
$= \frac{n m(m-1)}{2}$.

Similarly, there are $\binom{m}{1}$ ways to choose any column and after a column is chosen, ~~the~~ two identical rooks can be placed in this column in $\binom{n}{2}$ ways $\Rightarrow$ total # of ways $= \binom{m}{1}\binom{n}{2}$
$= \frac{m\, n(n-1)}{2}$.

$\therefore$ total # of ways $= \frac{mn}{2}(m+n-2)$

For $m=n=8$, we have, total # of ways $= \frac{8\times 8}{2}(16-2)$
$= 8\times 4\times 14 = 8\times 56 = 448$.

46. Let chessboard size be n. If length of side of a square to be selected is n, # ways of selecting ~~the~~ such squares $= 1$

" " " " " " " $n-1$, # ways " " $= 2\times 2 = 2^2$
$= (n+1-(n-1))^2$

" " " k, " " $= (n+1-k)^2$

$\therefore$ Total # of squares

$= \sum_{k=1}^{n}$ # of squares with length of side k $n=8$?   X

$= \sum_{k=1}^{n}(n+1-k)^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

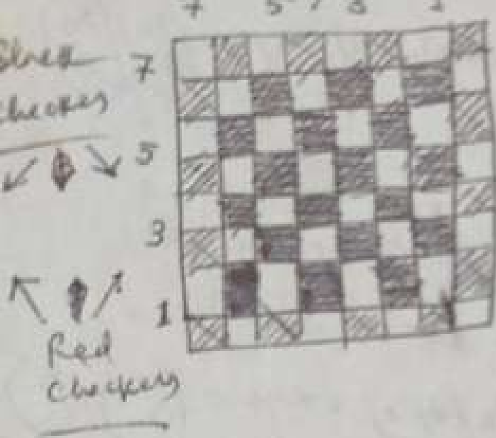
2/5

pick 2 lines on the ~~[illegible]~~ rows of a chess board ($\binom{9}{2}$ ways) ~~[illegible]~~ for the sides of the rectangle, then 2 lines on the columns ~~[illegible]~~ So $\binom{9}{2}\cdot\binom{9}{2}$ ways total.

47. # black squares in a checkerboard $= \frac{64}{2} = 32$

black squares are on // lines to the principal diagonal of the board and there are $2 \cdot 4 = 8$ diagonals lines.

total # black squares $= 1+3+5+7+7+5+3+1$ (as can be



$= \underline{16} + 16 = 32$

A red checker can capture (jump over) a black checker iff the black checker is in any of the blacked out squares and the red checker is on the immediate right square. (if the black check is on any corner square, the red checker can't capture it)

$\therefore$ The total # placements $= \underbrace{32}_{\#\text{ black squares}} - \underbrace{\left(\frac{\overset{8+6+8+6}{28}}{2}\right)}_{\#\text{ black squares in the corner}} = 32 - 14 = 18.$

53.

5.2

Here each square can have exactly one piece on it at any given time. First we have to choose 16 squares out of $8\times8=64$ squares on the chess board, which can be done in $\binom{64}{16}$ ways, but chess board being symmetric from all 4-sides, # different ways will be $\frac{1}{4}\binom{64}{16}$. After we have chosen those 16 locations, now what remains to be done is just permuting the pieces that can be done in $\frac{16!}{8!8!}$ ways.

Hence # ways to place the pieces $= \underbrace{\frac{1}{4}}_{\text{Symmetry}} \underbrace{\binom{64}{16}}_{\text{fix positions}} \underbrace{\left(\frac{16!}{8!8!}\right)}_{\text{permute pieces}}$

~~Alternative way~~

58.

Boys School	Girls School	Coed School
5	8	3
Boys	Girls	
2	3	

Each child goes to a different school. A boy can go to any of $5+3=8$ schools and a girl has $8+3=11$ choices.

~~The 1st boy can go to any of 5+3=8 schools~~

There can be the following disjoint cases:

You're just doing school combinations, not child-school [illegible]

① Both the boys go to boys school: # different ways of selecting schools

$$= \underbrace{2!}_{\text{permute boys}} \binom{5}{2} \underbrace{3!}_{\text{permute girls}} \binom{11}{3} \quad 8+3$$

4/5

② Exactly 1 boy goes to boys school: # different ways

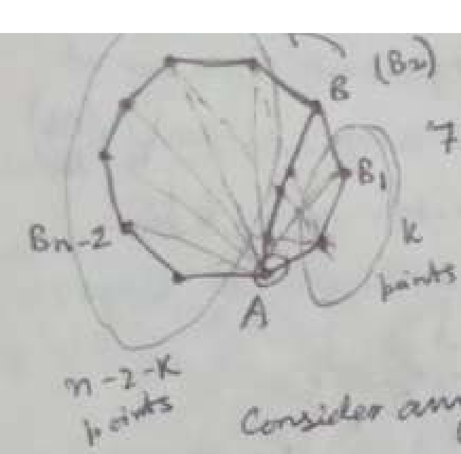
$$= \underbrace{\binom{2}{1}}_{\text{which boy (permute)}} \underbrace{\binom{5}{1}}_{\text{boy going to boys school}} \underbrace{\binom{3}{1}}_{\text{boy going to co-ed school}} \underbrace{3!}_{\text{permute girls}} \binom{8+\overset{\text{remaining co-ed school}}{2}}{3}$$

③ None of the boys go to the boys school: # different ways

$$= \underbrace{2!}_{\text{permute boys}} \underbrace{\binom{3}{2}}_{\text{boys to coed}} \underbrace{3!}_{\text{permute girls}} \binom{8+\overset{\text{remaining co-ed school}}{1}}{3}$$

don't nee[d] perm[ute]

Total # of ways $= 2!\,3!\left[\binom{5}{2}\binom{11}{3} + \binom{5}{1}\binom{3}{1}\binom{10}{3} + \binom{3}{2}\binom{9}{3}\right]$



75. Let's first find # of points of intersection formed by an n gon (convex), assuming no 3 of the chords ~~intersect at same~~ are concurrent.

Consider any chord AB from any ~~vertex~~ corner point A.

Total # of such chords $= n-2$ (all the corner pts except the 2 adjacent ones)

Total # of points of intersection with other chords on AB

$= k \times (n-2-k)$, where $k = \#$ of n-gon corner points on the right side of AB

$\Rightarrow n-2-k = \#$ " " " " left " " "

As B varies from B_1 to B_{n-2} (since $n-2$ chords are there),

K varies from 1 to $n-3$

$\Rightarrow$ Total # points of intersection of all the chords from A with other chords

$$= \sum_{k=1}^{n-3} k(n-2-k) = (n-2)\sum_{k=1}^{n-3} k - \sum_{k=1}^{n-3} k^2$$

$$= (n-2)\cdot\frac{(n-3)(n-2)}{2} - \frac{(n-3)(n-2)(2n-5)}{6}$$

$$= \frac{(n-2)(n-3)}{6}\left[3n-6-2n+5\right] = \frac{(n-1)(n-2)(n-3)}{6} = \binom{n-1}{3}$$

Now, there are n such corner pts of the polygon (one of them is A).

Hence contribution to the # intersection points by all the chords from all these n points $= \frac{n(n-1)(n-2)(n-3)}{6}$.

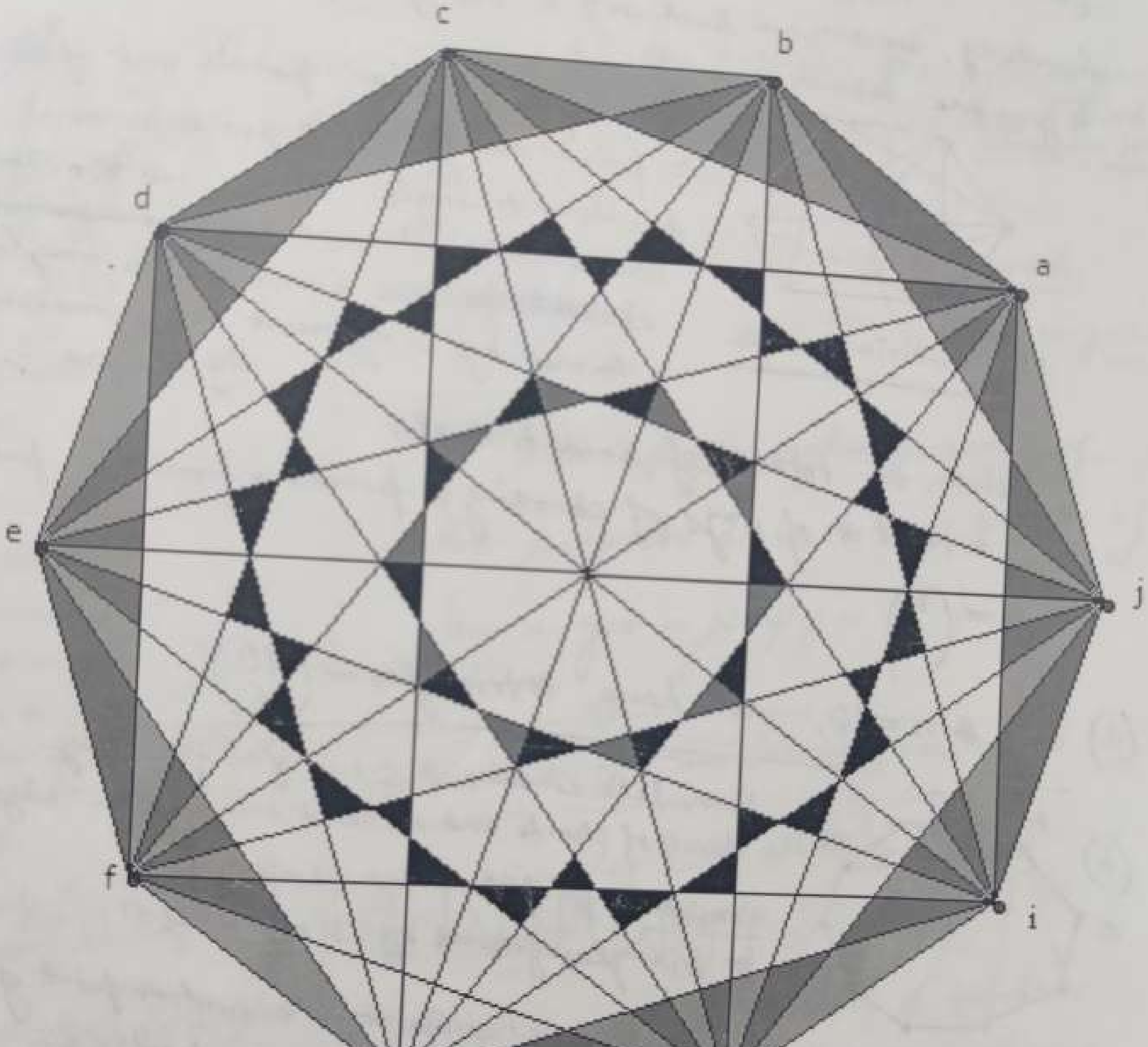
But, as we can see, the same chord is considered twice (by both the endpoints) and a given intersection point is also considered twice.

Hence, getting rid of the repetitions, total # of intersection points

$$= \frac{1}{24}n(n-1)(n-2)(n-3) = \binom{n}{4}$$ (Nice formula!)

All of them are inside points in the n-gon.

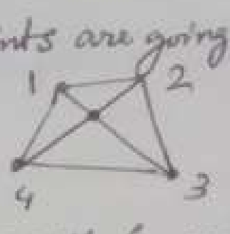
any 4 points are going to give one intersection pt



Some
~~# of~~ inside triangles in 10-gon where some chords are concurrent

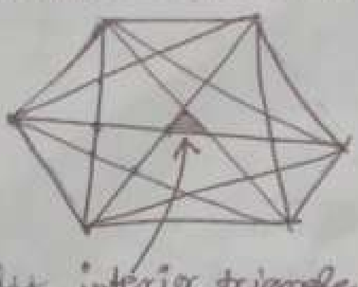
(but our answer will be different as shown in the next page since no 3 concurrent chords are allowed) simplify

We can see that any 4 points are going to give one unique intersection point of the chords. (since no 3 chords are concurrent)



Hence total # intersection points $= \binom{n}{4}$

Similarly, we can see that any 6 out of n points are going to give one interior triangle.



Only interior triangle

Now, every 6 chosen points out of the n points are going to give one new interior triangle, since no two chords are concurrent and ~~all intersection points for two~~ hence the triangle formed by 2 different hexagons can't be the same.

Hence, the total # of such triangles

$=$ total # of ways of choosing 6 points from n points

$= \binom{n}{6}$

(a) For $n=10$, we have, # triangles $= \binom{10}{6}$

(b)

Now, let's consider # of triangles formed by pieces of chords and at least one outside edge.

~~Consider a point A and a chord CD~~

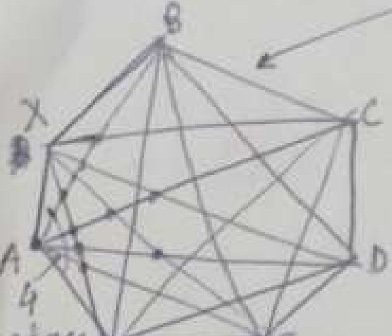
~~# triangles formed by CD with A = 8~~

# of triangles that have exactly one endpoint as a corner point of the n-gon and other two points as the intersection points of the chords $= \left(\binom{n}{4}\atop 2\right) - \binom{n}{6}$ (see for hexagon)

$\binom{n}{6}$ ← # triangles that are formed with 3 intersection points

for each 2 of the interior chord intersection pts, there is exactly one triangle (except the ones that are formed by 3 of the chord intersection points)

Now, # of triangles that have ~~as both the~~ couple of endpoints as the corners of the n-gon and one end point as ~~the~~ an intersection point $= 2n\binom{n-2}{2}$



4 edges incident on A to be considered

$4 = 7-3$ ($7 = n$)

consider AB, AC, AD, AE the 4 chords.
the 1st one (AB) is having 4 triangles with the AX as a side and another pt on AB as an intersection point on AB.
Similarly the 2nd one (AC) will have 3 such triangles, etc.
∴ total # of triangles that chords AB, AC, AD, AE creates with one side as AX is $4+3+2+1$
but there are another 4 chords (XC, XD, XE, XY) incident on Y that will create same # of such triangles again

since as seen from the heptagon, total # triangles any side of the n-gon has with the chords incident on it (with a chord intersection pt as a vertex)

$= 2(1+2+3+\cdots+n-3)$

$= 2\binom{n-2}{2}$ and there are n such sides of the n-gon

~~Hence~~ Also, # of triangles having 3 of the endpoints as the corner points of the n-gon is simply $= \binom{n}{3}$

Hence, we get the following result:

# of triangles with all 3 points as corner points of the n-gon $= \binom{n}{3}$

# of triangles with exactly 2 points as corner points (using a boundary edge of the polygon as its side) and another point as an interior chord intersection point

$= 2n\binom{n-2}{2}$, as argued.

# of triangles with exactly 1 point as corner point of the n-gon and other 2 points as chord inters. pts $= \left(\binom{\binom{n}{4}}{2}\right) - \binom{n}{\cdots}$

$\therefore$ Total # of triangles formed by chords and outside edges

$$= \binom{n}{3} + 2n\binom{n-2}{2} + \binom{\binom{n}{4}}{2} - \binom{n}{6}$$

for any n-gon

$\sim$ 10/10

$$e \begin{array}{|c|c|c|c|c|c|c|} \hline \ & \ & \ & \cdots & n & \cdots & \ \\ \hline \end{array}$$
$$\quad\; 1 \;\; 2 \;\; 3 \qquad\quad i \qquad\quad n$$

$\left(\exists j < i,\ j \geq 1 \;\middle|\; |i-j| = 1,\ \forall i = 2, 3 \cdots n\right)$

26. <u>There are exactly 2^{n-1} ways.</u>

The key observation is to see that when n is at i the position in a permutation, i.e., for all the permutations for which $e[i] = n$, where $2 \leq i \leq n$ $\left(\pi_{e[i]=n}\right)$, we have exactly 2^{i-2} valid permutations. ~~Since~~ Also there is one valid permutation only when $i = 1$, i.e., $e[i] = 1$ in a permutation π (and that is $\boxed{n}\,(n-1)\,(n-2) \cdots 1$ with positions $1, 2, 3, \ldots, n$, where $(n-1)$ can only be since $\exists j < i : |i-j| = 1, \forall i = 2, 3, \ldots, n$). Since i can vary from 1 to n, we have total # of such valid permutations as $= 1 + 2^0 + 2^1 \cdots + 2^{n-2}$, for $i = 1, 2, 3, \ldots, n$ (position of n in π)

$$= 1 + \sum_{i=2}^{n} 2^{i-2} = 1 + \sum_{j=0}^{n-2} 2^j = 1 + 2^{n-1} - 1 = 2^{n-1}$$

Let's see some examples first to understand what is going on.

5

$\boxed{n=2}$ → ✓ $\boxed{2}$ 1 valid; ✓ 1 $\boxed{2}$ valid — total $= 2 = 2^{2-1}$

position of $n=2$ in diff. permutations

$\boxed{n=3}$ →

× $\boxed{3}$ 1 2	# valid = 1	
✓ $\boxed{3}$ 2 1		
× 1 $\boxed{3}$ 2	# valid $= 1 = 2^0$	
✓ 2 $\boxed{3}$ 1		
✓ 1 2 $\boxed{3}$	# valid $= 2 = 2^1$	
✓ 2 1 $\boxed{3}$		

total # valid permutations $= 1 + 2^0 + 2^1 = 4 = 2^{3-1}$

position of $n=3$ in diff. permutations

$\boxed{n=4}$ →

n=4 at i=1	at i=2	at i=3	
× 4 1 2 3	× 1 4 2 3	× 1 2 4 3	✓ 1 2 3 4
× 4 1 3 2	× 1 4 3 2	× 1 3 4 2	× 1 3 2 4
× 4 2 1 3	× 2 4 1 3	× 2 1 4 3	✓ 2 1 3 4
× 4 2 3 1	× 2 4 3 1	✓ 2 3 4 1	✓ 2 3 1 4
× 4 3 1 2	× 3 4 1 2	× 3 1 4 2	× 3 1 2 4
✓ 4 3 2 1	✓ 3 4 2 1	✓ 3 2 4 1	✓ 3 2 1 4
# valid = 1	# valid $= 1 = 2^0$	# valid $= 2 = 2^1$	# valid $= 4 = 2^2$

$\boxed{\Pi}$

total $= 1 + 2^0 + 2^1 + 2^2 = 8 = 2^{4-1}$

Generalizes to $1 + 2^0 + 2^1 + \dots + 2^{n-2} = 2^{n-1}$

Proof that # valid permutations $= 2^{i-1}$ when $e[i] = n$ in Π (Induction)

Now, it's easy to see that when n is at position i of the permutation, the only positions in which the elements can permute are the positions on the left hand side of n and the ~~left half can only~~ all the elements on the right half are fixed, if they are to be valid (i.e. $\exists j < i ; |j - i| = 1 \; \forall i = 2, 3, \dots, n$ holds).

Induct on i (position of n)

Base case

when $i = 1$, the first element is n and the permutation to be valid the next element can't be anything other than $n-1$, " " " " $n-2$, so on " " " to $n-1$ " "

Hence $\exists$ only $\underline{1}$ valid permutation (all elements fixed)

$i = 2$ (n is at pos 2 in Π), the first element can only be $n-1$ and the 3rd must be $n-2$ (because that's the only element left that is within distance 1 from $n-1$ or n), and so on.

Hence $\exists$ only $1 = 2^0$ valid permutation

$i = 3$ (n is at pos 3 in Π), the first two elements can only be $n-1$, ... ⟹ only ... valid ... Π

Hypothesis: Assume when n is at pos i, # valid permutations $= 2^{i-1}$.

Induction step: Consider when n is at pos $i+1$, then the ~~first~~ left i positions can have only $n-1, n-2, \dots, n-i$ in some order but by hypotheses $\exists$ only 2^{i-1} of them and for each of them there are couple of options (first or last), hence $2^{i-1} \times 2 = 2^i$

$\Box$

5.3

27. Given,
$k_1+k_2+k_3=10, \quad k_1, k_2, k_3 \geq 0$

~~The problem can be converted to standard balls & boxes problems where balls~~

$$\sum_{\substack{k_1+k_2+k_3=10\\ k_1,k_2,k_3\geq 0}} P(10; k_1, k_2, k_3) = \sum_{\substack{k_1+k_2+k_3=10\\ k_1,k_2,k_3\geq 0}} \frac{10!}{k_1!\,k_2!\,k_3!}$$

$$= 10! \sum_{k_1=0}^{10}\left(\frac{1}{k_1!}\right)\sum_{k_2=0}^{10-k_1}\left(\frac{1}{k_2!}\right)\cdot\left(\frac{1}{(10-k_1-k_2)!}\right)$$

$$= \sum_{k_1=0}^{10}\frac{1}{k_1!}\left(\frac{10!}{0!\,(10-k_1)!}+\frac{10!}{1!\,(9-k_1)!}+\dots+\frac{10!}{(10-k_1)!\,0!}\right)$$

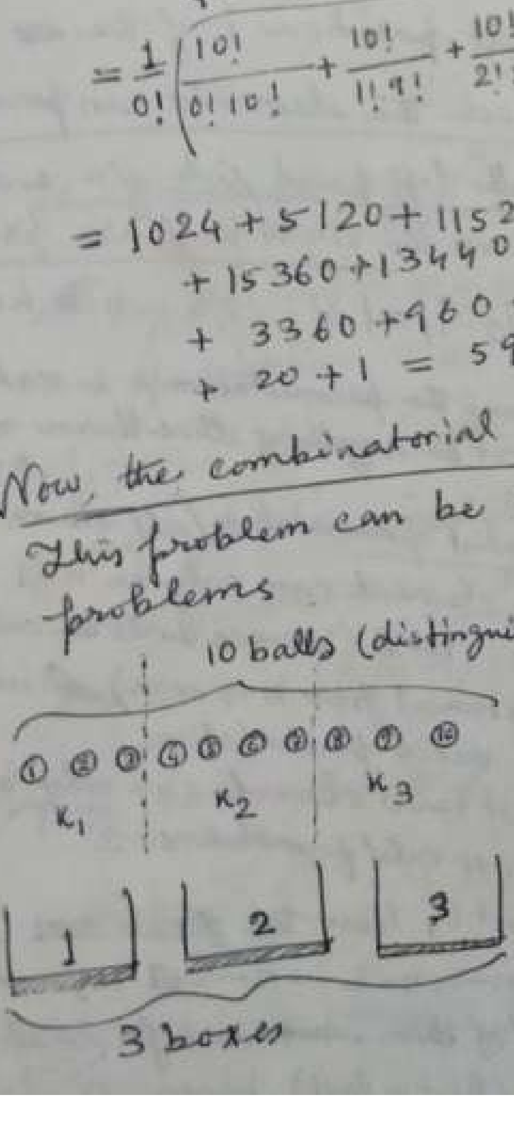
$$= \frac{1}{0!}\left(\overbrace{\frac{10!}{0!\,10!}+\frac{10!}{1!\,9!}+\frac{10!}{2!\,8!}+\dots+\frac{10!}{10!\,0!}}^{\text{10 terms}}\right)+\frac{1}{1!}\left(\overbrace{\frac{10!}{0!\,9!}+\frac{10!}{1!\,8!}+\dots+\frac{10!}{9!\,0!}}^{\text{9 terms}}\right)+\dots$$

$$+\frac{1}{9!}\left(\underbrace{\frac{10!}{0!\,1!}+\frac{10!}{1!\,0!}}_{\text{2 terms}}\right)+\frac{1}{10!}\left(\underbrace{\frac{10!}{0!\,0!}}_{\text{1 term}}\right)$$

$$= 1024+5120+11520+15360+13440+8064+3360+960+180+20+1 = 59049 = 3^{10}$$

Now, the combinatorial argument:

This problem can be thought of as standard balls & boxes problems



# of ways to place 10 distinguisable balls in 3 boxes $= 3^{10}$ (e.g. the firstball can be put in any of 3 boxes in 3 ways, similarly the 2nd ball etc.) s.t. total # ways $= \underbrace{3.3\dots 3}_{\text{10 times for 10 balls}}$

Now, imagine that the balls are ~~partitioned~~ divided into 3 disjoint partitions, s.t. they contain k_1, k_2, k_3 balls respectively, s.t., $k_1+k_2+k_3=10$ and $k_1, k_2, k_3 \geq 0$

Now ~~con let~~ let's consider all possible different values of k_1, k_2, k_3

$k_1 = 10,\ k_2 = 0,\ k_3 = 0$: corresponds to the case when the 1st box is having all 10 balls, 2nd & 3rd 0 balls

$k_1 = 9,\ k_2 = 1,\ k_3 = 0$: " " " " " " " - 9 balls, 2nd one with 1, 3rd with 0 balls.

⋮

$k_1 = 0,\ k_2 = 10,\ k_3 = 0$: " " " 2nd " " all 10 balls, 1st & 3rd with 0 balls

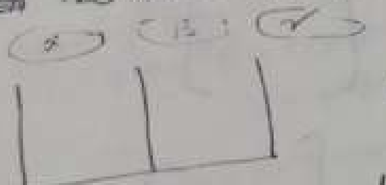
⋮

$k_1 = 0,\ k_2 = 0,\ k_3 = 10$: " 3rd " " all 10 balls, 1st & 2nd with 0 balls

As we can see, by $\sum\limits_{\substack{k_1+k_2+k_3=10 \\ k_1,k_2,k_3 \geq 0}} P(10; k_1, k_2, k_3)$ we consider all the exhaustive set of cases by which 10 (distinguishable) balls can be placed in 3 boxes, ~~two~~ (as shown above), but it's exactly 3^{10}.

$$\Rightarrow \sum_{\substack{k_1+k_2+k_3=10 \\ k_1,k_2,k_3 \geq 0}} P(10; k_1, k_2, k_3) = 3^{10} \quad \text{(Proved)}$$

25. ~~a~~ We have $2n$ α's, $2n$ β's and $2n$ γ's.



~~# elements in the first ha~~ ~~# elements from~~ ~~# ways to divide 3n elements~~ ~~#, ways to~~ the halves and

Fix one of the halves and

Let's consider ~~#~~ the elements that half can contain

	# α's	# β's	# γ's
	~~2n~~	~~n~~	0
n	$2n$	$n+1$	0
	$2n-1$	⋮	⋮
	⋮	$2n-1$	0
	$n+1$	$2n$	0
n	~~n~~		1
	$n-1$	$2n$	⋮
	⋮	$2n$	$n-1$
	1	$2n$	n
$n+1$	0	$2n-1$	$n+1$
	0	⋮	⋮
	⋮		
	0	n	$2n$

Hence, total # of different ways $= 3n+1$ (as shown)

once this half is chosen, the other half becomes fixed. Since the halves are not ~~ord~~ ordered, # different ways $= 3n+1$ ✗ 0/5

See answer in back of book