CMSC 641
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Design and Analysis of Algorithms
Quiz 5 Approximation Algorithms

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Bin Packing by First-Fit Decreasing

Instructions: Below is a proof that the First-Fit Decreasing (FFD) heuristic uses no more than $3/2 \cdot c^*$ bins, where c^* is the number bins used in the optimum packing. At various points in the proof, you will be asked to provide an explanation or analysis. You do not have to answer these questions in order.

Recall that in the Bin Packing problem we have n items with sizes $s_1, s_2, \ldots, s_n$. We are asked to place the items into bins so that the sum of the sizes of the items in each bin is ≤ 1. The objective is to do so with as few bins as possible.

In the First-Fit Decreasing (FFD) heuristic, we sort the items by size. So, we can assume that

$s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n$.

We take the items in order o decreasing size and place the current item in the first bin with enough remaining space to hold the current item. If none of the bins have enough space, then we start a new bin. The bins are considered in the order they were started — oldest bin first.

Let c^* be the number of bins used in the optimum bin packing. Let i_0 be the index of the first item that FFD places in bin number $c^* + 1$.

Question: what if no such item exists?

If no such item exists, then the solution obtained by FFD is optimal. Since the number of bins must be c^* (OPT). [can't be less].

Claim 1: $s_{i_0} \leq 1/2$.

Proof by contradiction: suppose $s_{i_0} > 1/2$. Then $s_1, s_2, \ldots, s_{i_0}$ are all $> 1/2$. So, there are $c^* + 1$ items that are $> 1/2$ which means the optimum bin packing must use at least $c^* + 1$ bins, a contradiction. QED.

Elaborate: How do we know there are $c^* + 1$ items $> 1/2$? Why must the optimum bin packing use at least $c^* + 1$ bins?

$\text{Since } \left(\underset{1st}{s_1} > \underset{2nd}{s_2} > \cdots > \underset{(c^*+1)th}{s_{i_0}}\right) \wedge \left(s_{i_0} > 1/2\right) \Rightarrow$ all (c^*+1) items are $> 1/2$

why are there c^*+1 ?

~~Since the OPT bin packing must use~~ has to ~~pack~~ use

If $s_{i_0} > \frac{1}{2}$, then all possible bin packings must have $s_1, s_2, \ldots, s_0 > \frac{1}{2}$
OPT bin packing must be one of them, hence must use at least c^*+1 bins.

because 2 items w/ size > 1/2 can't fit in a bin.

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Let c be the total number of bins used by FFD. We call the bins numbered $c^* + 1$ through c the "extra bins." Let t be the number of items that FFD placed in the extra bins and let z_1, z_2, …, z_t be their sizes.

Question: there are $n - i_0$ items that come after item i_0, the first item to be placed in an extra bin. Why isn't t just equal to $n - i_0 + 1$?

$\underbrace{s_1\ s_2 \cdots}_{c^*}\ s_{i_0}\ s_{i_1} \cdots s_{i_c}$

$c^*{+}1 \quad c^*{+}2 \quad c$

Since ~~$s_{i_0} > s_{i_1} > \cdots > s_{i_c}$~~ ~~and~~ bins are open, some of the items from $n - i_0 + 1$ can be placed in older bins i.e., from $1, 2, \ldots, c^*$

Claim 2: $t \le c^* - 1$.

Proof by Contradiction: Suppose $t \ge c^*$. For $1 \le j \le c^*$, let b_j be the sum of the sizes of the items that FFD placed in bin number j. Then, for $1 \le j \le c^*$, $b_j + z_j > 1$. Thus,

$$\sum_{j=1}^{c^*}(b_j + z_j) > \sum_{j=1}^{c^*} 1 = c^*$$

Since the sum of the sizes of all of the items cannot exceed c^*, this is a contradiction. QED.

Elaborate: Why is $b_j + z_j > 1$? Why can't the sum of all the sizes exceed c^*?

~~$b_j + z_j$~~ ~~b_j~~ Assume to the contrary.
If $b_j + z_j \le 1$, z_j would have been placed ~~to~~ in bins from $1, \ldots, c^*$ (since open),
a contradiction.

since c^* is OPT, (feasible) sol^n ~~sum of bs~~ all the bin sizes till c^* must be ~~will be~~ enough to hold all the item.

Finally, since each item that FFD placed in an extra bin has size $\le 1/2$ and since there are no more than $c^* - 1$ of them, FFD uses no more than $\lceil (c^* - 1)/2 \rceil \le c^*/2$ extra bins. Thus, FFD uses no more than $3/2 \cdot c^*$ bins in total.

Elaborate: How do we know that FFD uses no more than $\lceil (c^* - 1)/2 \rceil$ extra bins? Why is the total number of bins used by FFD less than or equal to $3/2 \cdot c^*$?

extra bin size $\le \frac{1}{2}$?

need to argue that each bin will hold at least 2 items.

since sum of the sizes can't exceed c^*,

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$(c^* - 1)\frac{1}{2} + \underbrace{e}_{\#\text{ extra bins}} \le c^* \Rightarrow \underbrace{e}_{\#\text{ extra bins}} \le c^*/2$

$\therefore$ Total # bins used by FFD $\le c^* + c^*/2 = 3c^*/2$.

$\therefore$ Approx factor $= \frac{A\ FFD}{OPT} = \frac{3c^*/2}{c^*} = \frac{3}{2}$