3. $K(x)$ is not a computable function

Sandipan Dey

HW-4 CMSC651

Proof (By contradiction) 23/30

Let's assume to the contrary that $K(x)$ is computable

9/10 $\Rightarrow \exists$ TM M that decides $K(x)$ on input x. ($\forall x \in \Sigma^*$, $M(x)$ writes $K(x)$ on its output tape and halts)

Now, let's fix an $n \in \mathbb{N} \cup \{0\}$ and define

$$f_n = \underset{\text{lexicographic}}{\min} \{ x \in \Sigma^* \mid K(x) > n \}$$

Now, f_n is computable since $K(x)$ is, because we can construct TM M_n^* that decides f_n in the following manner: Q is a function not a set. "Compute f_n"?

$M_n(n)$

1. ~~min~~ for all $x \in \Sigma^*$ do /* decidable, M always halts */
   - 1.1. Run $M(x)$ to find $K(x)$
   - 1.2. if $(K(x) > n)$ for the first time
     - 1.2.1. Remember x (cache it) as minx
     - else if $(K(x) > n)$ and $(x \underset{\text{lexicographic}}{<} \text{minx})$
     - 1.2.2. Remember x as mix
2. output minx.

Now, M_n always halts and hence f_n is computable.

Also, K is unbounded $\Rightarrow \forall m \in \mathbb{N}, \exists x \in \Sigma^* \mid K(x) > n$

~~Let's construct the following TM T.~~

~~T(x)~~

~~1. ignore input x.~~

need to argue that $|f_n| \geq n - c - 1$

Hence, M_n with input n halts with x on its tape, where length of the string $\langle M_n, n\rangle = \underbrace{\text{constant}}_{\text{length of encoding of } M_n} + \underbrace{\lceil \lg_2 n \rceil}_{\text{length of } n} < n$ for large n

Hence, $\langle M_n, n\rangle$ is a description of the string x, which is $< n$, where the minimal description $= |d(x)| = K(x) > n$ (by ~~assumption~~ definition of f_n),

hence a contradiction.

2.

7/10

INFINITE $= \{ \langle M \rangle \mid M$ is a TM and $L(M)$ is an infinite set $\}$.

$ALL_{TM} = \{ \langle M \rangle \mid M$ is a TM and $L(M) = \Sigma^* \}$.

<u>To Prove:</u> INFINITE $\leq_m ALL_{TM}$

$f(\langle M \rangle) = \langle M' \rangle$ i.e., $\exists$ computable $f \mid \langle M \rangle \in$ INFINITE $\Leftrightarrow f(\langle M \rangle) \in ALL_{TM}$, $\forall$ TM M.

$\langle M \rangle \xmapsto{f} \langle M' \rangle$.

INFINITE $= \{ \langle M \rangle \mid (\forall n)(\exists y, t) [\, |y| \geq n \wedge M(y)$ accepts in t ~~steps~~ $] \}$

$ALL_{TM} = \{ \langle M \rangle \mid (\forall x \in \Sigma^*)(\exists t) [\, M(x)$ accepts in t ~~steps~~ $] \}$

<u>Proof</u> The following machine F computes a reduction f:

F = "On input $\langle M \rangle$:

~~1. M' on input~~

1. Construct the following machine M':

M' = "on input x:

(nondeterministically) 1. interpret x as $\langle y, n, t \rangle$ triplet.

what are the nondeterministic choices? ?2. if $|y| \geq n$ and $M(y)$ accepts in t steps, M' accepts. else, M' ~~runs forever~~ enters a loop."

F is a function

2. Output $\langle M' \rangle$."

1. <u>f is computable</u> since <u>F is decidable</u> (F just constructs another machine and never halts and outputs M')

2. $\Rightarrow \langle M \rangle \in$ INFINITE $\Rightarrow \forall n \in \mathbb{N}, \exists y, t \mid M(y)$ accepts in t steps and $|y| \geq n \Rightarrow \forall x = \langle y, n, t \rangle$, $M'(x)$ accepts $\Rightarrow \langle M' \rangle \in ALL_{TM}$

iff 3. $\Leftarrow \langle M' \rangle \in ALL_{TM} \Rightarrow \forall x = \langle y, n, t \rangle \in \Sigma^*$, $M'(x)$ accepts $\Rightarrow \forall n \in \mathbb{N}, \exists y, t \mid |y| \geq n$ and $M(y)$ accepts in t ~~steps~~

$\Rightarrow \langle M \rangle \in$ INFINITE (Proved)

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but for fixed y and some choices of n & t M' will loop, so $L(M') \neq \Sigma^*$.

1. MIN-LBA is undecidable

7/10

Proof by Contradiction

If Min-LBA is decidable, $\exists$ TM M that decides Min-LBA.
We show the following: $E_{LBA} \leq_m EQ_{LBA} \leq_m MinLBA$ and prove that if Min-LBA is decidable, then so will be E_{LBA} (by transitivity of $\leq_m$) and therefore a contradiction.

$$\underset{E}{\langle L' \rangle} \xrightarrow{f'} \underset{EQ}{\langle L_1, L_2 \rangle} \xrightarrow{f} \underset{Min}{\langle L \rangle}$$

① Reduction function f for $EQ_{LBA} \leq_m$ Min-LBA works as follows:
F on input $\langle L_1, L_2 \rangle$,
Construct L on input w

1. interpret w as $\langle x, t \rangle$
2. Run $L_1(x)$ & $L_2(x)$ for t steps

2.1. if both accepts, then L accepts and output $min(\langle L_1 \rangle, \langle L_2 \rangle)$

2.2. if $L_1(x)$ accepts in t steps but $L_2(x)$ is still running after t steps, then simulate $L_2(x)$
- if $L_2(x)$ accepts, output $min(\langle L_1 \rangle, \langle L_2 \rangle)$
- else output $\langle L_1 \rangle$

2.3. if $L_2(x)$ accepts in t steps but $L_1(x)$ is still running after t steps, then simulate $L_1(x)$
- if $L_1(x)$ accepts, output $min(\langle L_1 \rangle, \langle L_2 \rangle)$
- else output $\langle L_2 \rangle$

2.4. ow loop.

Why do you believe that the LBA L you constructed is the smallest LBA for that language? There is no reason to. It is possible [illegible] efficient.

$\langle L_1, L_2 \rangle \in EQ_{-LBA} \iff f(L_1, L_2) = L \in$ Min-LBA $\implies EQ_{-LBA} \leq_m$ Min-LBA ①

② Reduction function f' for $E_{LBA} \leq_m EQ_{LBA}$ works as follows.

$$L' \xmapsto{F'} \langle L', L[E_{LBA}] \rangle$$

? Just create a LBA that rejects [illegible] the 1st [illegible]

~~F on input $\langle L' \rangle$,~~
~~Construct L_1, L_2~~
~~L_1 on any input x~~
~~Run $L'(x)$ if accepts, L_1 accept~~

1. output $\langle L', L(E_{LBA}) \rangle$

$\therefore \langle L' \rangle \in E_{LBA} \iff f(L') = \langle L_1, L_2 \rangle = \langle L', L[E_{LBA}] \rangle \in EQ_{LBA}$

$\implies E_{LBA} \leq_m EQ_{LBA}$ ②

Combining ① & ②, we get, $\underset{\text{undecidable}}{E_{LBA}} \leq_m EQ_{LBA} \leq_m \underset{\text{decidable (?)}}{\text{Min-LBA}}$ contradiction

HW-2

3. $E_{TM} \leq_m$ FINITE ?

$E_{TM} = \{ \langle M \rangle \mid \forall x \in \Sigma^* \forall t \in \mathbb{N}$ [M(w) does not accept in $\leq t$ steps] $\}$

FINITE $= \{ \langle M \rangle \mid \exists x \in \Sigma^* \forall y \in \Sigma^*, y_{lex} > x, \exists t \in \mathbb{N}$ [M(y) does not accept in $\leq t$ steps] $\}$

To prove: $\exists$ computable $f \mid \langle M \rangle \in E_{TM} \iff f(\langle M \rangle) \in$ FINITE

$M \xmapsto{f} M'$

Proof: The following machine computes f:

F = "On input $\langle M \rangle$:

1. Construct the following machine M':

   M' = "on input w:

   1. interpret w as triplet $\langle y, x, t \rangle$

   2. if $y \leq_{lex} x$ and M(y) accepts then accept o.w. M' loops."

2. output $\langle M' \rangle$"

Need to argue
$\langle M \rangle \in E_{TM} \Rightarrow \langle M' \rangle \in$ FINITE
$\langle M \rangle \notin E_{TM} \Rightarrow \langle M' \rangle \notin$ FINITE

used where?

HW-3

MIN-LBA is Turing Recognizable

3. Show that MIN-LBA $\leq_m A_{LBA}$.