CMSC 641, Design and Analysis of Algorithms, Spring 2010

Sandipan Dey,
Homework Assignment - 10

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Rough 3-Coloring

Algorithm

Let the graph $G(V, E)$ be with $|V| = n$ vertices $v_1, \ldots v_n \in V$ and $|E| = m$ edges $e_1, \ldots e_m \in E$. Also, we have the 3-color set $C = \{c_1, c_2, c_3\}$.

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for i = 1 to n do
    Randomly pick a color c_j ∈ C and color the vertex v_i with c_j
end for
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Analysis

Let X be the random variable denoting the total number of satisfied edges and X_i be an indicator variable corresponding to the i^{th} edge $e_i \in E$ s.t.
$X_i = \begin{cases} 1 & \text{if } e_i \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}, \forall i \in 1 \ldots m$. Hence, $X = \sum_{i=1}^{m} X_i$.
Now, $P(X_i = 1)$ = probability that the colors picked by the algorithm for two endpoints of e_i are different $= \frac{3 \times 2}{3 \times 3} = \frac{2}{3}$.
Hence, $E[X_i] = 0.P(X_i = 0) + 1.P(X_i = 1) = P(X_i = 1) = \frac{2}{3}$.
By linearity of expectation, we have, $E[X] = E\left[\sum_{i=1}^{m} X_i\right] = \sum_{i=1}^{m} E[X_i] = \frac{2}{3}m \Rightarrow E[X] = \frac{2}{3}c^*$.

Contention Resolution Revisited

Part (a)

Proof: S is conflict free

Let's assume to the contrary $\Rightarrow \exists$ processes $P_i,\ P_j \in S$ s.t. P_j is in conflict with P_i. Also, $X_i = X_j = 1$ by construction. But then P_i must not be selected as an element of S, a contradiction.

Let Z be the random variable denoting the total number of conflict free processes in the set S (i.e., vaule of Z deontes the size of S) and Z_i be an indicator variable, with

$Z_i = \begin{cases} 1 & P_i \in S \\ 0 & \text{otherwise} \end{cases}, \forall i \in 1 \ldots n.$

Hence, $Z = \sum_{i=1}^{n} Z_i$.

Now,

$$\begin{aligned} P(Z_i = 1) = P\left((X_i = 1) \wedge \left(\bigwedge_{X_j \in adj(X_i)} X_j = 0\right)\right) \\ = P(X_i = 1) \prod_{X_j \in adj(X_i)} X_j = 0, \text{ since independent} \\ = \frac{1}{2}.\left(\frac{1}{2}\right)^d, \text{ since } |X_j \in adj(X_i)| = d \\ \Rightarrow E[Z_i] = P(Z_i = 1) = \left(\frac{1}{2}\right)^{d+1} \\ \Rightarrow E[Z] = \sum_{i=1}^{n} E[Z_i] = \frac{n}{2^{d+1}}, \text{ by liniearity of expectation} \end{aligned}$$

Part (b)

As in part (a), we have

$$P(Z_i = 1) = P\left((X_i = 1) \wedge \left(\bigwedge_{X_j \in adj(X_i)} X_j = 0\right)\right)$$
$$= P(X_i = 1) \prod_{X_j \in adj(X_i)} X_j = 0, \text{ since independent}$$
$$= p.\left(1-p\right)^d, \text{ since } |X_j \in adj(X_i)| = d$$
$$\Rightarrow E[Z_i] = P(Z_i = 1) = p.\left(1-p\right)^d$$
$$\Rightarrow E[Z] = \sum_{i=1}^{n} E[Z_i] = np.\left(1-p\right)^d, \text{ by liniearity of expectation}$$

Hence, expected size of $S = f(p) = E[Z] = np.\left(1-p\right)^d$. We want to maximize the size of the independent set S
$\Rightarrow f\prime(p) = (1-p)^d - dp(1-p)^{d-1} = 0 \Rightarrow p = \frac{1}{1+d}$ (we have $f\prime\prime(p) < 0$ at this point).

Hence, maximum expected size of the independent set $= nd\left(1 - \frac{1}{d+1}\right)^{d+1} = \frac{nd^d}{(d+1)^{d+1}}$.

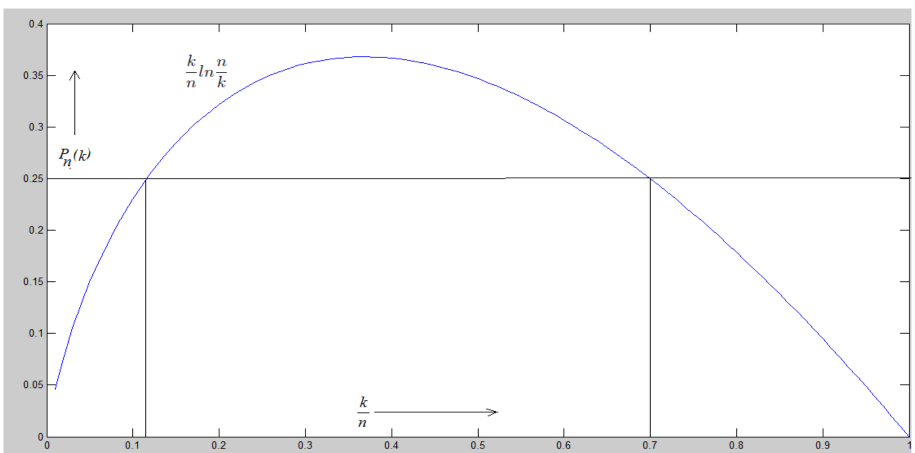
One-Pass Auction

If the seller accepts the first bid, the probability of accepting the highest of the n bids $= \frac{1}{n}$ only. Hence, let's the strategy of the seller be the following: he rejects the first $k-1$ bids ($2 \leq k \leq n$) and accepts the first one which is the highest of all the bids he has seen until that point of time. We have to find k s.t. the seller accepts the highest of the n bids with probability at least $\frac{1}{4}$.

Now probability that he accepts the highest bid using this strategy,

$$P_n(k) = \sum_{i=k}^{n} \text{Probability that } i^{th} \text{ bid is highest and the seller accepts it}$$
$$= \sum_{i=k}^{n} \frac{1}{n}.\frac{k-1}{i-1}, \text{ (since to accept } b_i\text{, the maximum bid from}$$
$$\text{the first } i-1 \text{ bids must be among the first } k-1 \text{ bids)}$$
$$= \frac{k-1}{n} \sum_{i=k}^{n} \frac{1}{n}.\frac{n}{i-1} = \frac{k}{n}\int_{\frac{k}{n}}^{1} \frac{1}{\frac{k}{n}} = \frac{k}{n} ln\frac{n}{k} \text{ for large n, with } n \to \infty$$

Hence, as seen from the graph of $P_n(k)$, if we choose $0.2n \leq k \leq 0.7n$, the seller accepts the highest of the n bids with probability at least $\frac{1}{4} = 0.25$.


$\frac{k}{n} ln \frac{n}{k}$
$P_n(k)$
$\frac{k}{n}$
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1