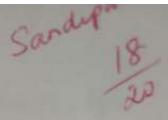
Home work 3: Due 3/17/09



1. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a proof using logical equivalences. (10 points)

Proof:

$$x \in A \cap (B \cup C)$$

$$\Leftrightarrow x \in A \land x \in (B \cup C)$$

$$\Leftrightarrow x \in A \land (x \in B \lor x \in C)$$

$$\Leftrightarrow (x \in A \land x \in B) \lor (x \in A \land x \in C)$$

$$\Leftrightarrow (x \in A \cap B) \lor (x \in A \cap C)$$

$$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$$

$$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Determine if the following functions are 1-1 and/or onto. Show your steps. (10 points)

a.
$$f: \mathbb{Z} \to \mathbb{Z}$$
, $f(n) = 2n$.

b.
$$f: \mathbf{Z} \to \mathbf{Z}$$
, $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$

c.
$$f: \mathbf{Z} \to (\mathbf{N})$$
 $f(n) = \begin{cases} -2n & \text{if } n \le 0 \\ 2n-1 & \text{if } n > 0 \end{cases}$

Proof:

a. 1-1 but not onto

Proof:

1-1: Let
$$x, y \in Z$$

Now,
$$f(x) = f(y) \rightarrow 2x = 2y \rightarrow x = y$$

Domain

Co-domain

Onto: Let's assume
$$y \in Z_{odd} \to y \equiv 1 \pmod{2}$$
, where $Z = Z_{odd} \cup Z_{even}$
Now, we can see that $y \notin \text{Im } age(f)$,

since $y \in \text{Im } age(f) \rightarrow y \equiv 0 \pmod{2}$, a contradiction.

Hence, precisely, we have, $f: Z \to Z_{even}$, with $f(Z) = Z_{even} \subset Z$,

i.e., $(\exists y \in Z) \mid y \neq f(x), x \in Z$

b. Not 1-1 but onto

Proof:

Not 1-1: Let's consider x = 2n, y = 2n + 1, $\forall n \in \mathbb{Z}$ Clearly we have, $x \neq y$

But we have,
$$f(x) = \left\lfloor \frac{x}{2} \right\rfloor = \left\lfloor \frac{2n}{2} \right\rfloor = n$$
 and
$$f(y) = \left\lfloor \frac{y}{2} \right\rfloor = \left\lfloor \frac{2n+1}{2} \right\rfloor = \left\lfloor n + \frac{1}{2} \right\rfloor = n$$

Hence, we have, $(x \neq y) \land (f(x) = f(y)) \rightarrow f$ is not 1-1.



Onto: Eventually all Z will be exhausted by f since we have all consecutive integers generated by the mapping, without any gap.

c. 1-1 but not onto

Proof: Let's define

$$f_1(n) = -2n, \ n \le 0, \text{ clearly}, f_1: Z^- \cup \{0\} \to Z^+_{even} \cup \{0\}$$

$$f_2(n) = 2n-1$$
, $n > 0$, clearly, $f_2: Z^+ \rightarrow Z^+_{odd}$

Clearly,
$$f(n) = \begin{cases} f_1(n), & n \le 0 \\ f_2(n), & n > 0 \end{cases}$$

1-1: Since $\operatorname{Im} age(f_1) \cap \operatorname{Im} age(f_2) = \Phi$, to show that f is 1-1 it's sufficient to show that f1 and f2 both are 1-1.

•
$$f_1(x) = f_1(y) \rightarrow -2x = -2y \rightarrow x = y \rightarrow f_1 \text{ is 1-1}$$

•
$$f_2(x) = f_2(y) \rightarrow 2x - 1 = 2y - 1 \rightarrow x = y \rightarrow f_2 \text{ is } 1-1$$

Not onto:

Im $age(f) = \operatorname{Im} age(f_1) \cup \operatorname{Im} age(f_2) = Z_{odd}^+ \cup \{0\} \cup Z_{even}^+ = Z^+ \cup \{0\}$ Clearly f is not onto, since the negative integers from the codomain (Z) don't have a pre-image in domain (Z) under the mapping f.

$$...-3 - 2 - 1 0 1 2 3...$$
 $f \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$
 $... 6 4 2 0 1 3 5...$

Domain

Co-domain

ZIN