CMSC 641, Design and Analysis of Algorithms, Spring 2010

Sandipan Dey,
Homework Assignment - 11

May 4, 2010

PiC (Partition into Cliques) is NP Complete

1. PiC = $\{\langle G, k\rangle : \exists$ a partition of $V(G)$ into k subsets $V_1, \ldots, V_k$ s.t. each V_i is a clique in $G\}$

2. PiC $\in$ NP : Given $\langle G, k\rangle$ and $\{V_1, \ldots, V_k\}$ (certificate) we have to verifiy whether it is a valid partition of $V(G)$ into k Cliques.
   (a) Check if the vertex set (all vertices) $V(G)$ is covered by the subsets $V = V_1 \cup V_2 \ldots \cup V_k$: can be done in polynomial time at most $O(|V|^2)$.
   (b) Check if valid partition (all of the k subsets are mutually disjoint) $V_i \cap V_j = \phi$ if $i \neq j$: can be done in polynomial time $O(k^2|V|^2)$.
   (c) Check if each V_i is a clique: this can be done in polynomial time $O(\sum_{i=1}^{k} |V_i|^2) = O(k|V|^2)$.

   Hence the 'Yes' certificate can be verified in polynomial time.

   $\Rightarrow$ PiC $\in$ NP.

3. PiC is NP-hard : We show by reducing the well-known NP-hard problem Graph Coloring to Pic.
   (a) Graph Coloring = $\{\langle G, k\rangle : G$ is k-colorable$\}$.
   (b) To Prove: Graph Coloring $\leq_p^m$ Pic.
   (c) Construction:
      i. Construct the graph $\bar{G}(V, \bar{E})$, the complement graph of $G(V, E)$ (s.t. $(u, v) \in \bar{E} \Leftrightarrow (u, v) \notin E$).
      ii. Claim: G is k colorable iff $\bar{G}$ can be partitioned into k cliques.

(d) Proof:

i. ($\Rightarrow$)

- Suppose G is k colorable.
- $\exists (V_1, V_2, \ldots V_k) \subseteq V$ that are colored by colors $1, 2, \ldots k$ respectively, with V_i colored by color i.
- $\bigcup_{i=1}^{k} V_i = V$ (since all the vertices of G must be colored).
- $V_i \cap V_j = \phi$ if $i \neq j$ (since every vertex must be colored with exactly one color).
- Each $V_i \subseteq V$ forms an independent set of G (since any two vertices $u, v \in V$ can be colored with the same color only if $(u, v) \notin E$).
- In $\bar{G}$ each of V_i forms a clique $\forall i = 1, \ldots, k$ (since for any $u, v \in V_i \Rightarrow (u, v) \notin E \Rightarrow (u, v) \in \bar{E}$), with $\bigcup_{i=1}^{k} V_i = V$ and $V_i \cap V_j = \phi$ if $i \neq j$.

Hence $\bar{G}$ can be partitioned into k cliques.

ii. ($\Leftarrow$)

- Suppose $\bar{G}$ can be partitioned into k cliques.
- $\exists (V_1, V_2, \ldots V_k) \subseteq V$, with each V_i being a clique, $\bigcup_{i=1}^{k} V_i = V$, $V_i \cap V_j = \phi$ if $i \neq j$ (mutually disjoint since a partition).
- Each $V_i \subseteq V$ forms an independent set in G (since for any $u, v \in V_i \Rightarrow (u, v) \in \bar{E} \Rightarrow (u, v) \notin E$).
- In G, there are k such independent sets V_i.

Hence G is k colorable.

(e) The Construction is polynomial time: needs $\theta(|E|)$ time to construct the complement graph $\bar{G}$.

Parallel Transpose

Work $= T_1(n) = \theta(n^2)$ (serializing nested for loops on line 2 and 3).
Span $= T_\infty(n) = \theta(lgn) + \theta(1) = \theta(lgn)$.
Parallelism $= \frac{T_1(n)}{T_\infty(n)} = \frac{\theta(n^2)}{\theta(lgn)} = \theta\left(\frac{n^2}{lgn}\right)$.

Matrix-Vector Multiplication