Sandipan Dey

25/30

Math - 650

Homework - 4

10. (a) The optimization problem for simplex

+10

$$\min f(x) = \min \sum_{i=1}^{n} f_i(x_i), \quad \text{with } f_i(x_i) = \frac{1}{2}(a_i - x_i)^2, \ f(x) = \frac{1}{2}\sum_{i=1}^{n}(a_i - x_i)^2$$

$$\text{s.t. } \sum_{i=1}^{n} x_i = 1,$$

$$x_i \geq 0, \ i = 1, 2, \ldots, n.$$

The variational inequality $\langle \nabla f(x^*), x - x^* \rangle \geq 0$ ~~applied~~ for this problem is

$$\Big[\underbrace{-x_i \leq 0, \ i = 1, \ldots n, \ \sum_{i=1}^{n} x_i \leq 1, \ -\sum_{i=1}^{n} x_i \leq -1}_{(n+2) \text{ constraints}}\Big] \Rightarrow \Big[\underbrace{\langle \nabla f(x^*), x \rangle \leq \langle \nabla f(x^*), x^* \rangle}_{x^* \text{ is a local minimizer of } f}\Big]$$

~~By Farkas's lemma, $\exists$ nonnegative multipliers~~

~~$\underbrace{\{\lambda_i\}_1^n, \{\mu_j\}_1^2}_{n+2}$ s.t. $\sum \lambda_i$~~

i.e., we have

$$\left[\begin{array}{l} \langle \underbrace{-e_i}_{a_i}, \underbrace{x}_{x_i} \rangle \leq 0, \ i = 1, \ldots, n \\ \langle e, x \rangle \leq 1, \\ \langle -e, x \rangle \leq -1 \end{array}\right] \Rightarrow \left[\langle \underbrace{-\nabla f(x^*)}_{c}, x \rangle \leq \underbrace{\langle -\nabla f(x^*), x^* \rangle}_{r} \right]$$

By Farkas's lemma, $\exists \ \{\lambda_i\}_1^n, \{\mu_j\}_1^2 \geq 0$ s.t.

$$\underbrace{-\sum_{i=1}^{n} e_i \lambda_i + \sum_{j=1}^{2} e \mu_j (-1)^{j-1}}_{n+2} = \underbrace{-\nabla f(x^*)}_{c} \text{ and } \sum_{i=1}^{n} \lambda_i \cdot 0 + \sum_{j=1}^{2} \mu_j (-1)^{j-1} \leq \underbrace{\langle -\nabla f(x^*), x^* \rangle}_{r}$$

$$\therefore -\sum_{i=1}^{n} e_i \lambda_i + e \cdot \mu_1 + (-e) \cdot \mu_2 = -\overbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}}^{\lambda} + \overbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}^{e} (\mu_1 - \mu_2)$$

$$= -\lambda + e\mu = -\nabla f(x^*), \quad \text{with } \mu = \mu_1 - \mu_2 \in \mathbb{R}$$

$$\nabla f(x^*) = -(a - x^*) = x^* - a \Rightarrow -\lambda + e\mu = -(x^* - a) \Rightarrow x^* - \lambda = a - e\mu.$$

Also, $\sum_{i=1}^{n} \lambda_i \cdot 0 + \mu_1 \cdot 1 + \mu_2 (-1) = 0 + \mu \leq \langle -\lambda + e\mu, x^* \rangle$ $\quad (\because -\nabla f(x^*) = -\lambda + e\mu)$

$$= -\langle x^*, \lambda \rangle + \mu \langle e, x^* \rangle.$$

$$= -\langle x^*, \lambda \rangle + \mu \underbrace{\left(\sum_{i=1}^{n} x_i^*\right)}_{} = -\langle x^*, \lambda \rangle + \mu$$

$$\Rightarrow \mu \leq \langle -x^*, \lambda \rangle + \mu$$

$$\Rightarrow \langle x^*, \lambda \rangle \leq 0$$

$$\left.\begin{array}{l} \text{but } x_i^* \geq 0, \; i=1,\dots,n \\ \lambda \geq 0 \end{array}\right\} \Rightarrow \langle x^*, \lambda \rangle \geq 0$$

$\therefore \langle x^*, \lambda \rangle = 0$

$\therefore$ we have, $x^* - \lambda = a - \mu e$, $x^* \geq 0$, $\lambda \geq 0$, $\langle x^*, \lambda \rangle = 0$

as an equivalent system.

(b) $\quad x^* - \lambda = a - \mu e, \; x^* \geq 0, \; \lambda \geq 0, \; \langle x^*, \lambda \rangle = 0$

Since we have $x^* \geq \lambda$,

we have $x_i^* = (a_i - \mu)^+ = \max\{0, a_i - \mu\}$.

$$\lambda_i = -(a_i - \mu)^- = \min\{0, a_i - \mu\}$$

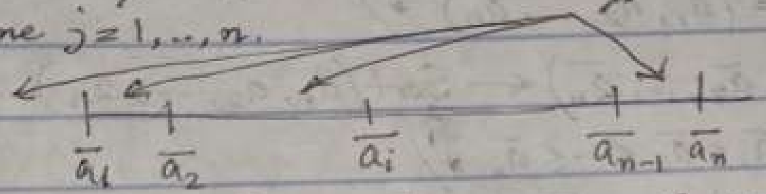
$\forall i = 1, \dots, n$.

(c) $\exists \mu_{\in \mathbb{R}} \Big| \sum_{1}^{n} (a_i - \mu)^+ = \sum_{1}^{n} x_i^* = 1$

(d) we need to find ~~a~~ $\mu_{\in \mathbb{R}}$, s.t. $\sum_{1}^{n} (a_i - \mu)^+ = 1$, given $a \in \mathbb{R}^n$

Can be simplified

Sort the numbers $a_1, a_2, \ldots, a_n$ in increasing order. Let the sorted sequence be $(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$, where $\bar{a}_i = a_j$ for some $j = 1, \ldots, n$.



$(a_1 \le a_2 \le \cdots \le a_{n-1} \le a_n)$

Notice that $\cancel{\bar{a}_1 \le \mu \le \bar{a}_n}$ $\underset{\text{(always true)}}{\mu < \bar{a}_n}$, for if $\mu \ge \bar{a}_n$, we have $(a_i - \mu)^+ = 0 \; \forall i$ $\Rightarrow \sum_{1}^{n} (a_i - \mu)^+ = 0 \ne 1$

Also, when $\mu \le \bar{a}_1$, we have $(a_i - \mu)^+ = a_i - \mu, \; \forall i = 1, \ldots, n$

$\Rightarrow 1 = \sum_{1}^{n} (a_i - \mu)^+ = \sum_{1}^{n} (a_i - \mu) = \sum (\bar{a}_i - \mu) \ge \sum (\bar{a}_i - \bar{a}_1) \ge \bar{a}_n - \bar{a}_1$

$\cancel{\Rightarrow \text{when } \bar{a}_n - \bar{a}_1 < 1}$ $\Rightarrow (\bar{a}_n - \bar{a}_1) \ge 1 \Rightarrow \mu \in [\bar{a}_1, \bar{a}_n)$

Also, $1 = \sum_{1}^{n} (a_i - \mu)^+ \ge n(\bar{a}_1 - \mu) \Rightarrow \frac{1}{n} \ge \bar{a}_1 - \mu \Rightarrow \mu \ge \bar{a}_1 - \frac{1}{n}$

$\Rightarrow$ when $\mu \le \bar{a}_1$, $\mu \ge \bar{a}_1 - \frac{1}{n}$, we need to search in $[\bar{a}_1 - \frac{1}{n}, \bar{a}_1]$

To summerize, we need to search $[\bar{a}_1 - \frac{1}{n}, \bar{a}_1] \cup [\bar{a}_1, \bar{a}_n)$ at max. When $\bar{a}_n - \bar{a}_1 \ge 1$, we need to search only $[\bar{a}_1, \bar{a}_n)$, otherwise search $[\bar{a}_1 - \frac{1}{n}, \bar{a}_n)$. Also, if $\bar{a}_1 = \bar{a}_2 = \cdots = \bar{a}_n$, $\mu = \bar{a}_i - 1/n$.

Also, if $\bar{a}_n - \bar{a}_{n-1} \ge 1$, $\mu = \bar{a}_n - 1 \in [\bar{a}_{n-1}, \bar{a}_n)$

else if $\bar{a}_n - \bar{a}_{n-1} < 1$, $\mu < \bar{a}_{n-1}$,

if $\bar{a}_{n-1} - \bar{a}_{n-2} \ge 1 - (\bar{a}_n - \bar{a}_{n-1})$, $\mu \in [\bar{a}_{n-2}, \bar{a}_{n-1})$

with $\mu = \frac{1}{2} - \frac{\bar{a}_n + \bar{a}_{n-1}}{2}$

$\vdots$

~~in general, if~~ $\cancel{\sum_{j=i+1}^{n} \bar{a}_i - \bar{a}_{i-1} \ge 1 - \sum_{j=i+1}^{n} (\bar{a}_j - \bar{a}_{j-1}), \; \mu \in [}$

in general, if $\sum_{j=i+1}^{n} (\bar{a}_j - \bar{a}_{j-1}) < 1 \;\wedge\; \bar{a}_i - \bar{a}_{i-1} \ge 1 - \sum_{j=i+1}^{n} (\bar{a}_j - \bar{a}_{j-}$

$\mu \in [\bar{a}_{i-1}, \bar{a}_i)$ and can be found by $\mu = \frac{1}{n-i} + \frac{\sum_{j=i}^{n} \bar{a}}{n-}$

$\forall i = 2, \ldots, n$

if $\mu \notin [\bar{a}_1, a_n)$, ...

We can ~~do~~ find μ by binary searching ...

this case.

Hence the following SEARCH routine:

$\triangleright$ PROC SEARCH(a) /* $a \in \mathbb{R}^n$ */ /* assuming n ... $n=1$ is trivial ...

/* $a = (a_1, a_2, \ldots, a_n)$ */

$(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \leftarrow \text{Sort}(a_1, a_2, \ldots, a_n)$

/* $\bar{a}_1 \leq \bar{a}_2 \leq \cdots \leq \bar{a}_n$ */

if $(\bar{a}_1 = \bar{a}_n)$ // all of them equal.

$\mu \leftarrow \bar{a}_1 - \frac{1}{n}$

else /* This block finds μ when $\mu \in [\bar{a}_1, \bar{a}_n)$ */

~~$s \leftarrow 0$~~

~~for $(i \leftarrow 2;\ i <= n;\ i \leftarrow i+1)$~~

~~do /* This loop ... i th iteration finds μ~~

~~if $\mu \in [\bar{a}_{i-1}, \bar{a}_i)$ */~~

~~$s \leftarrow s + d$~~

if $(\bar{a}_n - \bar{a}_{n-1} \geq 1)$ /* $\mu \in [\bar{a}_{n-1}, \bar{a}_n)$ */

$\mu \leftarrow \bar{a}_n - 1$

else sum $\leftarrow \bar{a}_n$

$s \leftarrow \bar{a}_n - \bar{a}_{n-1}$

for $(i \leftarrow n-1;\ i \geq 2;\ i \leftarrow i-1)$

/* i th iteration finds μ if $\mu \in [\bar{a}_{i-1}, \bar{a}_i$ ...

do

if $(s < 1)$ /* $\mu < \bar{a}_i$ */

if $(a_i - a_{i-1} \geq 1 - s)$ /* $\mu \in [\bar{a}_{i-1}$ ...

$\mu \leftarrow \frac{1}{n-i+1}(\text{sum} - 1)$; return μ.

endif

endif

endfor ... $s \leftarrow s + (\bar{a}_i - \bar{a}_{i-1})$ ...

/* if μ is still not found, $\mu \in (\bar{a}_1 - \frac{1}{n}, \bar{a}_1)$ */

/* do binary search to find μ */

epsilon $\leftarrow$ 0.0000001.

$l \leftarrow \bar{a}_1 - \frac{1}{n}$

$u \leftarrow \bar{a}_1$

$\mu \leftarrow \frac{l+u}{2}$.

diff $\leftarrow \sum_{i=1}^{n} (\bar{a}_i - \mu) - 1$

while (|diff| $\geq$ epsilon)

do

  if (diff ≥ 0) /* $\sum_{i=1}^{n} (a_i - \mu) \geq 1$ */

    $l \leftarrow \mu$ /* interval $[\frac{l+u}{2}, u)$ */

  else

    $u \leftarrow \mu$ /* interval $[l, \frac{l+u}{2})$ */

  endif

  $\mu \leftarrow \frac{l+u}{2}$

end while.

return μ.

10. We start with the system

$Ax \leq 0, \; x \in \Delta_{n-1} \qquad (x \in \mathbb{R}^n)$

which is
$$\left.\begin{array}{l} \underset{m\times n}{A}x \leq 0 \\ -e_i^T x \leq 0, \; \forall i=1,\dots,n \\ \underset{n\times 1}{e}^T x = 1 \end{array}\right\}$$

that can be re-written as

$$\left.\begin{array}{l} \underset{(m+n)\times n}{A'}x \leq 0 \\ \underset{(m+n)\times n}{\begin{bmatrix} e^T \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}\right\} \text{①}$$

where $\underset{(m+n)\times n}{0}$ is zero matrix

and

$$\underset{(m+n)\times n}{A'} = \underset{(m+n)\times n}{\begin{bmatrix} A \\ -e_1^T \\ \vdots \\ -e_n^T \end{bmatrix}} \text{ matrix}$$

-4

-2 +6

~~we have~~ By Motzkin transposition theorem, if system ① is inconsistent, ~~then~~ the following system will be consistent:

$$\left.\begin{array}{l} A'^T y' + \begin{bmatrix} e^T \\ 0 \end{bmatrix} w' = 0 \\ \langle 0, y' \rangle + \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, w' \right\rangle + y_0 = 0 \end{array}\right\}$$

each of y', w' are $(n+m)\times 1$ vectors, $y' = \begin{bmatrix} y \\ 0 \end{bmatrix}$, $y \in \mathbb{R}^m$, $0_{n\times 1}$

$$w' = \underset{(n+m)\times 1}{\begin{bmatrix} w_1' \\ w_m' \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

The above system can be re-written as:

$$\underset{n\times(m+n)}{A'^T} y' = \begin{bmatrix} e \\ 0 \end{bmatrix}^T \begin{bmatrix} y_0 \\ 0 \end{bmatrix} \qquad (\because w_1' = -y_0)$$

$$\Rightarrow \underbrace{\begin{bmatrix} \underset{n\times m}{A^T} & -e_1 & -e_2 & \cdots & -e_n \end{bmatrix}}_{n\times(m+n)} y' = \underbrace{\begin{bmatrix} e \\ 0 \end{bmatrix}^T \begin{bmatrix} y_0 \\ 0 \end{bmatrix}}_{n\times 1}$$

$$\Rightarrow \begin{bmatrix} \underset{n\times m}{A^T} & -e_1, -e_2 & \cdots & -e_n \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} e^T \\ 0 \end{bmatrix} \begin{bmatrix} y_0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A^T y - \sum_{i=1}^{m} y_i = y_0 \Rightarrow A^T y = y_0 + \sum_{i=1}^{m} y_i = y_0 + 1 \geq 0$$

why is it equal to 1?

$A^T y > 0, \ y \in \Delta_{m-1} \}$ (2)

hence, system (2) is consistent;

no need!

Similarly, if we start with system (2) as inconsistent, by Motzkin's transposition we can show system (1) to be consistent.

16. (a) $\langle a_i, x\rangle < \alpha_i, \ i=1,\dots,m \Rightarrow \langle c, x\rangle < \gamma$ is consistent

Total 5

$\Leftrightarrow \neg\left(\langle a_i, x\rangle < \alpha_i, \ i=1,\dots,m\right) \vee \langle c, x\rangle < \gamma$ is consistent.

$\Leftrightarrow \neg\left(\neg\left(\langle a_i, x\rangle < \alpha_i, \ i=1,\dots,m\right) \vee \langle c, x\rangle < \gamma\right)$ is inconsistent

$\Leftrightarrow \langle a_i, x\rangle < \alpha_i, \ i=1,\dots,m \wedge \neg\left(\langle c, x\rangle < \gamma\right)$ is inconsistent

$\Leftrightarrow \langle a_i, x\rangle < \alpha_i, \ i=1,\dots,m \wedge -\langle c, x\rangle \le -\gamma$ is inconsistent.

$\Leftrightarrow \exists \lambda_0 \in \mathbb{R}, \ \lambda \in \mathbb{R}^m, \ \mu \in \mathbb{R}$ satisfying

$$\sum_{i\in I} \lambda_i \langle a_i, x\rangle + \mu \langle c, x\rangle = 0$$

$$\sum_{i\in I} \lambda_i \langle a_i, x\rangle + \mu \langle c, x\rangle + \lambda_0 = 0$$

$\mu = 0 \Rightarrow$ L.H.S. is consistent.

$$\sum_{i=1}^{m} \lambda_i \langle a_i, x\rangle - \mu \langle c, x\rangle = 0,$$

$$\sum_{i=1}^{m} \lambda_i \alpha_i - \mu \cancel{\gamma} + \lambda_0 = 0,$$

$$(\lambda_0, \lambda, \mu) \geq 0, \quad (\lambda_0, \lambda) \neq 0$$

(By Motzkin transposition theorem)

-1

But $\mu \neq 0 \overset{\text{set}}{\Rightarrow} \mu = 1$

(b) $\langle a_i, x\rangle \leq \alpha_i,\ i=1,\dots,m \Rightarrow \langle c, x\rangle < \gamma$ is inconsistent

$\Leftrightarrow \langle a_i, x\rangle \leq \alpha_i,\ i=1,\dots,m \wedge -\langle c, x\rangle \leq -\gamma$

~~$\Leftrightarrow \exists \lambda_0 \in \mathbb{R}, \lambda \in \mathbb{R}^{m}, \mu_j \in \mathbb{R}^{m+1}$ satisfying~~

$\Leftrightarrow \exists \lambda_0 \in \mathbb{R}, \mu_j \in \mathbb{R}^m, \mu \in \mathbb{R}$ satisfying

$$\sum_{j=1}^{m} \mu_j \langle a_i, x\rangle - \mu \langle c, x\rangle = 0$$

$$\sum_{j=1}^{m} \mu_j \alpha_i - \mu \gamma + \lambda_0 = 0$$

$$(\lambda_0, \lambda, \mu) \geq 0, \quad (\lambda_0, \lambda) \neq 0$$

-1

(By Motzkin transposition)

(1) $\mu > 0$ else $\langle a_i, x\rangle \leq \alpha_i$, $i=1,\dots,m$ inconsistent

(2) $\lambda_0 > 0$

(c) $\langle a_i, x\rangle < \alpha_i,\ i=1,\dots,m \Rightarrow \langle c, x\rangle \leq \gamma$ is inconsistent

$\Leftrightarrow \langle a_i, x\rangle < \alpha_i,\ i=1,\dots,m \wedge -\langle c, x\rangle < -\gamma$ is inconsistent

$\Rightarrow \exists \lambda_0 \in \mathbb{R}, \lambda \in \mathbb{R}^{m+1}$, satisfying

$$\sum_{i=1}^{m} \lambda_i \langle a_i, \cancel{x}\rangle - \lambda_{m+1} \langle c, \cancel{x}\rangle = 0$$

$$\sum_{i=1}^{m} \lambda_i \alpha_i - \cancel{\lambda_{m+1}} \gamma + \lambda_0 = 0$$

$$(\lambda_0, \lambda, \mu) \geq 0, \quad (\lambda_0, \lambda) \neq 0$$

(By Motzkin)

Again, very incomplete

-1

where's μ?

(d) $\langle a_i, x\rangle = \alpha_i,\ i=1,\dots,m \Rightarrow \langle c,x\rangle < \gamma$ is inconsistent

$\Leftrightarrow \langle a_i, x\rangle = \alpha_i,\ i=1,\dots,m \ \wedge\ -\langle c,x\rangle < -\gamma$ " "

$\Rightarrow \exists\, \lambda_0 \in \mathbb{R},\ \lambda \in \mathbb{R},\ \delta \in \mathbb{R}^m$ satisfying

$$-\lambda\langle c,x\rangle + \sum_{k=1}^{m} \delta_k \langle a_i, x\rangle = 0$$

$$-\lambda\gamma + \sum_{k=1}^{m} \delta_k \alpha_i + \lambda_0 = 0$$

(By Motzkin)

$-1\,1/2$ $\lambda = 1$ crucial $\lambda_0 \geq 0$. $\lambda =$

$(\lambda_0, \lambda) \geq 0 \quad (\lambda_0, \lambda) \neq 0$

(e) $\langle a_i, x\rangle \leq \alpha_i,\ i=1,\dots,m \Rightarrow \langle c,x\rangle = \gamma$ is inconsistent

$\Leftrightarrow \langle a_i, x\rangle \leq \alpha_i,\ i=1,\dots,m \ \wedge\ \langle c,x\rangle < \gamma \ \wedge -\langle c,x\rangle < -\gamma$ is consistent

$\Rightarrow \exists\, \lambda_0 \in \mathbb{R},\ \lambda_1, \lambda_2 \in \mathbb{R},\ \mu \in \mathbb{R}^m$ satisfying

$\cancel{\lambda_1\gamma = \lambda_2\gamma}$ $$(\lambda_1 - \lambda_2)\langle c,x\rangle + \sum_{j=1}^{m} \mu_j \langle a_i, x\rangle = 0$$

$$(\lambda_1 - \lambda_2)\gamma + \sum_{j=1}^{m} \mu_j \alpha_i + \lambda_0 = 0$$

by Motzkin's

-2

$(\lambda_0, \lambda_1, \lambda_2, \mu) \geq 0,\quad (\lambda_0, \lambda) \neq 0$