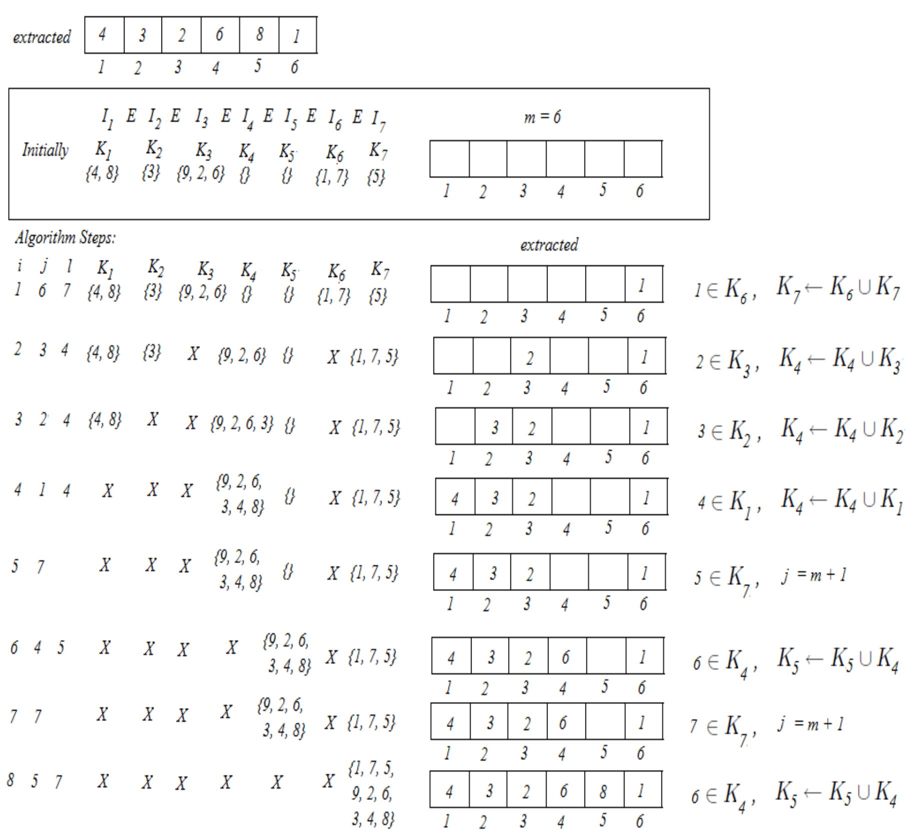
# CMSC 641, Design and Analysis of Algorithms, Spring 2010

Sandipan Dey,
Homework Assignment - 3

March 2, 2010

## Problem 1 (Offline minimum) Solution

### Part (a)



## Part (b)

### Proof

The OFF-LINE-MINIMUM algorithm deals with $n$ INSERT ($I$) operations ($\bigcup_{k=1}^{m} I_k$) and $m$ EXTRACT-MIN ($E$) operations. To prove that the algorithm is correct, we need to prove that $\forall j \in \{1, 2, \ldots m\}$, $extract[j]$ contains the key returned by $j^{th}$ E.

For $i = 1$, the algorithm considers the entire sequence $I_1, E, I_2, E, \ldots I_m, E, I_{m+1}$. It first finds a $j | i \in K_j$. There can be couple of cases:

1. $j = m + 1$, which means that the element 1 is inserted after the last EXTRACT-MIN, in which case it will NOT be part of the *extracted* array, since it will never get a chance to be extracted. The algorithm also does nothing ($j \neq m + 1$ check on line 3 ensures it), simply proceeds to the next larger element. Since the elements $\{1, 2, \ldots n\}$ are considered in the increasing order (ensured by the for loop in line 1), this element will never be considered again. Hence, this behavior is correct.

2. $j \neq m + 1$, which means that some EXRACT-MIN operation has taken place after this INSERT operation $I_j$. 1 being the smallest element in the set $S$, the immediate $E$ operation ($j^{th}$ $E$) must extract this element. The algorithm also correctly assigns $extracted[j] \leftarrow i$ at line 4, where $i = 1$ here.

For the 2nd case, after the INSERT operation of the element 1 and the immediate ($j^{th}$) EXTRACT-MIN is evaluated correctly by the algorithm, the algorithm tries to consider the remaining sequence of operations again, but this time without the particular $I$ and $E$. This is done by the line 6, which performs $K_l \leftarrow K_l \cup K_j$ (since the keys in $K_j$ other than the element $i$ can only be considered for extraction by the following EXTRACT-MINS) and destroys $K_j$, since it already found $extract[j]$, namely the key returned by the $j^{th}$ EXTRACT-MIN.

Therefore, for iterations $i = 2 \ldots n$ it considers only the sequence of operations $I_1, E, I_2, E, \ldots I_{j-1}, E, I_{j+1}, E, \ldots I_m, E, I_{m+1}$, where $l = j + 1$ in this case (it can be $> j + 1$ in other cases when $j + 1$ is already destroyed). Hence after removing the INSERT operation for the element 1 (it's not physically removed, but will never be considered, since $i$ is strictly increasing) and the corresponding $extracted[j]$, the sequence of $n$ INSERT and $m$ EXTRACT-MIN operations get reduced to a different (smaller) sequence of $n-1$ INSERT and $m-1$ EXTRACT-MIN operations, hence a smaller subproblem that is exactly similar and on it the algorithm will work for the iterations $i = 2$ to $n$.

By applying the same logic for the smaller subproblem with $n - 1$ INSERT an $m - 1$ EXTRACT-MIN operations (consdered by the algorithm steps $\forall i =$

$2 \ldots n$), we can divide it into 2 parts again, one for $i = 2$ and the other for still smaller subproblem $i = 3 \ldots n$ and argue that the algorithm works correctly for $i = 2$. Continuing in this manner, $\forall i = k \ldots n$, each time we can divide the current problem into another subproblem with strictly non-increasing size in the sequence of operations (handled by the algorithm in iterations $i = k + 1 \ldots n$) and prove the correctness of the $k^{th}$ iteration. But $i$ is increasing, hence we are done when we have $i = n$.

### Part (c)

### Implementation

- Start with each element as a singleton set in a disjoint set forest, with total $n$ elements.
- In order to form sets $K_j,\ j = 1 \ldots m + 1$ (in the worst case last $n - 1$ of them possibly empty), $n - 1$ UNIONs in the worst case.
- Line 2 basically then reduces to $j \leftarrow FIND-SET(i)$ and we have $n$ such operations.
- Line 5 reduces to $l \leftarrow next(j)$, operation which is executed for $n$ times in the worst case.
- Line 6 reduces to $l \leftarrow LINK(j, l)$ operation which is also executed for $n$ times in the worst case.

Hence, total number of operations $= m\prime = O(n)$

$\Rightarrow$ amortized time $= O(m\prime \log^* n) = O(n \log^*(n))$

or to provide a tighter bound, the amortized time $= O(n\alpha(n))$, where $\alpha$ is the inverse of the Ackerman function.

## Problem 2 Solution

As it can be seen from the figure 1, starting with $2^n + 1$ INSERT operations, followed by an EXTRACT-MIN (with CONSOLIDATE) operations, followed by $2^n - 1$ DELETE operations can create a Fibonacci Heap of height $n$, with $n$ nodes (a chain).

Note that DELETE operation uses DECREASE-KEY + EXTRACT-MIN, where none of the DECREASE-KEY operation here can have cascade-cut, since every non-root node will have its child deleted only once.

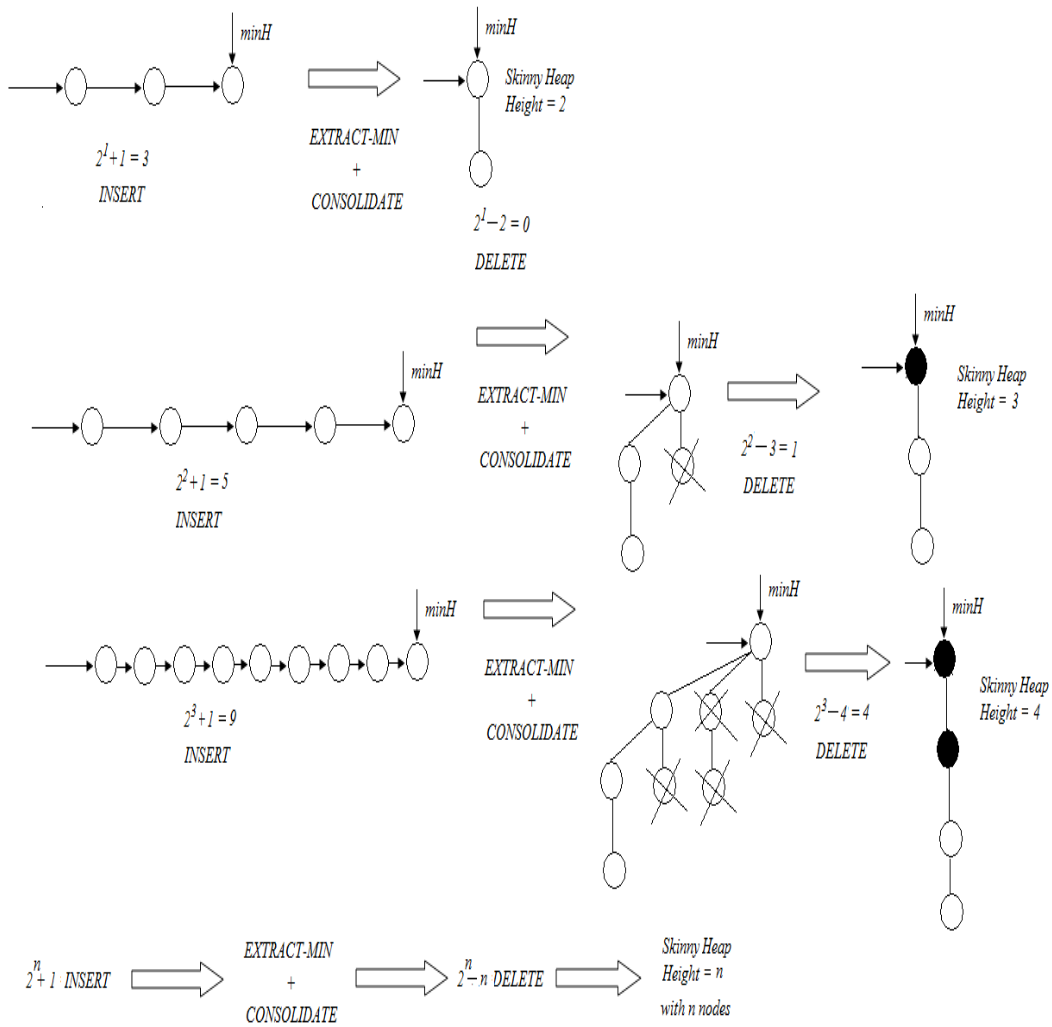



Figure 1: Skinny Fibonacci Heaps with height $O(n)$

## Problem 3 Solution

### Part (a)

**Algorithm 1** Algorithm FIB-HEAP-CHANGE-KEY

$FIB - HEAP - CHANGE - KEY(H, x, k)$

1: **if** $k < key[x]$ **then**
2: call $FIB - HEAP - DECREASE - KEY(H, x, k)$.
3: **else if** $k == key[x]$ **then**
4: return {do nothing}.
5: **else** {increase key}
6: **for** each child $y$ of $x$ **do**
7: call $CUT(H, y, x)$.
8: **end for**
9: $key[x] \leftarrow k$.
10: call $CASCADING - CUT(H, x)$.
11: **end if**

- Lines $1-2$ have an amortized cost of $O(1)$, so have lines $3-4$ (comparison cost).
- Let's analyze the amortized cost for lines $5-10$, i.e., for the increase-key operation.

  By the potential method, potential before increase-key $= t(H) + 2m(H)$.

  Line 7 can increase the number of trees $t(H)$ by at most $D(n)$ (maximum degree of a node in the n-node Fibonacci heap $= O(lg\ n)$).

  Also, if we assume that the number of cascading cut recursive calls line 10 is $c$, then total decrease in number of marked nodes $= O(c)$, where the same call produces $O(c)$ additional trees, where $c$ is a constant. Hence the potential after increase-key $= (t(H)+D(n)+O(c))+2(m(H)-O(c)))$.

  Hence, the change in potential is at most
  $= (t(H) + D(n) + O(c)) + 2(m(H) - O(c))) - (t(H) + 2m(H))$
  $= D(n) - O(c) = O(lg\ n)$.
- Total amortized time for FIB-HEAP-CHANGE-KEY $= O(lg\ n)$.

### Part (b)

Deleting a node $\Rightarrow$ Decrease the corresponding key to $-\infty$, followed by Extract-Min, hence has an amortized cost of $O(1) + O(lg\ n) = O(lg\ n)$.

If we were to delete $min(r, n[H])$ particular nodes the amortized cost would be $= O(min(r, n[H]).lg\ n)$.

But, since we could delete arbitrary nodes, we hope to do better. Deleting singleton trees and leaf nodes is easy. So, by maintaining a pointer to leaf nodes in each tree, the amortized cost of pruning $min(r, n[H])$ nodes is $O(min(r, n[H]))$.