CMSC 641, Design and Analysis of Algorithms, Spring 2010

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Homework Assignment - 7

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Half Clique is NP-Complete

Construction

We reduce known problem Clique $\in NPC$ to Half-Clique problem. Let $G(V, E)$ be the graph with $|V| = n$ nodes. Also, let (G, k) be an instance of Clique. We transform it to an instance of Half-Clique $(G\prime, k)$ using the following construction:

- If $k \geq \frac{n}{2}$, add $2k - n$ vertices to G, without adding any edges, to obtain graph $G\prime(V\prime, E)$, with $|V\prime| = 2k$.
- If $k < \frac{n}{2}$, add $n - 2k$ vertices and completely connect them to G and each other, to obtain graph $G\prime(V\prime, E\prime)$, with $|V\prime| = 2(n - k)$.
- Run Half-Clique on this altered graph, $G\prime$.

Reduction Proof

We must show that this transformation is a reduction, i.e., we need to show that G has a clique of size k iff $G\prime$ has a Half-Clique.

- G has a clique of size k.
  - When $k \geq \frac{n}{2}$, since the construction of $G\prime$ does not destroy any edges, $G\prime$ still has a k clique. But, $G\prime$ has size $2k \Rightarrow G\prime$ has a Half Clique, by definition.
  - When $k < \frac{n}{2}$ construction creates another clique of size $n-2k$ vertices in $G\prime$ and every vertex of that clique is connected to the old clique from $G \Rightarrow G\prime$ now has a clique of size $k + n - 2k = n - k$. But, $G\prime$ has $2(n - k)$ vertices, by construction $\Rightarrow G\prime$ has a Half Clique, by definition.
- $G\prime$ has a Half-Clique.

- By construction $|V\prime| = 2k$ or $|V\prime| = 2(n-k)$.
- $|V\prime| = 2k \Rightarrow G\prime$ has a clique of at least size k and since the construction of $G\prime$ does not destroy any edges, G must have the same k clique.
- When $|V\prime| = 2(n-k) \Rightarrow G\prime$ has a clique of at least size $n-k$. Going back to G, the construction only removes all the edges from the $n-2k$ vertices in $G\prime$ and hence even in the worst case G is going to have $n-k-(n-2k) = k$. cliques.

- The reduction is a polynomial time reduction ($O(n^2)$).

Dominating Set

Construction

We reduce known problem Vertex Cover $\in NPC$ to Half-Clique problem. Given an undirected graph G and a number k, let's construct a graph $G\prime$ so that G has a vertex cover of size at most k iff $G\prime$ has a dominating set of size at most k.

$G\prime$ will have a vertex for each vertex of G, except for any isolated vertices (vertices of degree 0), and a new vertex v_e for each edge e of G. Each edge $e = (u, v) \in E(G)$ is replaced by a triangle of edges in $G\prime : (u, v), (u, v_e), and (v, v_e)$.

Reduction Proof

We must show that this transformation is a reduction, i.e., G has a vertex cover of size k iff $G\prime$ has a dominating set of size k.

- X is a vertex cover of G.
  - Then X is also a dominating set of $G\prime$ because every edge of G has a vertex in X incident to it, and so every triangle of vertices in $G\prime$ has at least one member of X in it, and so every vertex of $G\prime$ is either in X or adjacent to a vertex in X (Since every vertex of $G\prime$ is in one of the triangles.). So if G has a vertex cover of size k, then $G\prime$ has a dominating set of size k.
- $G\prime$ has a dominating set of size k.
  - Let Y be a dominating set of k nodes of $G\prime$. Note that if any vertex in Y is an edge-vertex v_e rather than a vertex of G, we can replace it by one of the G-vertices for the edge e's endpoints and it will still be a dominating set. (This is because ve only dominated three vertices, those in its triangle, and either of the other two vertices in this triangle also dominate these three vertices.) Once we have

replaced all edge-vertices in Y by G-vertices, the new Y forms a vertex cover of G, because every edge of G must have at least one of its endpoints in Y for Y to be a dominating set of $G\prime$.

- The reduction is a polynomial time reduction ($O(|E|)$).

Clique and Unary Counter

Part (a)

By condition, if $u, v \in V$ and $u \neq v$ then we have,

$$
\begin{aligned}
&(\forall u)(\forall v)\,(x_{uv} \leftrightarrow (u,v) \in E) \\
\equiv &\left(\bigwedge_{(u,v)\in E} x_{uv}\right) \wedge \left(\bigwedge_{(u,v)\notin E} \neg x_{uv}\right)
\end{aligned}
$$

which is conjunction of 2 CNF forms, hence in CNF form.

Hence, we can add clauses $D_1 = \bigwedge_{(u,v)\in E} x_{uv},\ D_2 = \bigwedge_{(u,v)\notin E} \neg x_{uv}$.

Part (b)

By condition, if $u, v \in V$ and $u \neq v$ then we have,

$$
\begin{aligned}
&(\forall u)(\forall v)\,((x_u \wedge x_v) \rightarrow x_{uv}) \\
\equiv &(\forall u)(\forall v)\,((x_u \wedge x_v) \rightarrow x_{uv}) \\
\equiv &(\forall u)(\forall v)\,(\neg x_u \vee \neg x_v \vee x_{uv})
\end{aligned}
$$

which is in CNF form.

Hence, we can add clause $E = \bigwedge_{\substack{u,v\in V\\ u\neq v}} (\neg x_u \vee \neg x_v \vee x_{uv})$.

Part (c)

$n(n+1)$ **variables to represent the value of the n-bit unary counter through the $n+1$ stages**

Let's define the variable

b_i^s = value of the i^{th} bit of the unary counter at stage s.

Now, the unary counter is an n bit counter $\Rightarrow i = 1 \ldots n$ and there are $n+1$ stages of the counter $\Rightarrow s = 0, \ldots n$.

Hence, the total number of variables used to to represent the value of the n-bit unary counter through the $n+1$ stages $= n(n+1)$.

enforce that the counter starts with value 0

By clause $C_1 = \bigwedge_{i=1}^{n} \neg b_i^0$.

enforce that the counter ends with value k

By clause $C_2 = \bigwedge_{i=1}^{k} b_i^n$.

enforce that the counter is incremented (or not) correctly in each stage

We notice that

$(b_i^k \wedge x_i) \to b_{i+1}^k$.

Hence, the corresponding clause will be

$C_3 = \bigwedge_{i=1}^{n} \bigwedge_{k=0}^{n} \left(\neg b_i^k \vee \neg x_i \vee b_{i+1}^k\right)$. (By D'Morgan).

Part (d)

Let's construct our boolean formula $B = C_1 \wedge C_2 \wedge C_3 \wedge D_1 \wedge D_2 \wedge E$.

We notice that B is in CNF form and none of the clauses have more than 3 literals, hence B is in 3-CNF form.

B is satisfiable $\Rightarrow$

Any satisfying assignment τ for $C_1 \wedge C_2 \wedge C_3$ picks a set of k vertices (guaranteed by the unary counter), namely selected those u s.t. $\tau(x_u) = true$.

If τ satisfied $D_1 \wedge D_2 \wedge E$ too, then those k vertices must form a clique (since $x_u \wedge x_v \to x_{uv} \to (u, v) \in E \Rightarrow$ Graph G has a k clique).

Graph G has a k-clique $\Rightarrow$

If $u_1, u_2, \ldots, u_k$ is a k-clique in G, assign $\tau(x_{u_i}) = true$, $\forall i = 1 \ldots k$ and set $\tau(y) = false$ for all other variables $\Rightarrow D_1, D_2, E$ are true (by clique property).

Now, selecting these k variables at the different stages of the unary counter, C_1, C_2, C_3 are also satisfied $\Rightarrow$ this truth assignment satisfies B.