Sandip

18/20

Home work 3: Due 3/17/09

1. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by giving a proof using logical equivalences. (10 points)

Proof:

$$x \in A \cap (B \cup C)$$
$$\Leftrightarrow x \in A \wedge x \in (B \cup C)$$
$$\Leftrightarrow x \in A \wedge (x \in B \vee x \in C)$$
$$\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$$
$$\Leftrightarrow (x \in A \cap B) \vee (x \in A \cap C)$$
$$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$$
$$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

2. Determine if the following functions are 1-1 and/or onto. Show your steps. (10 points)
   a. $f: \mathbf{Z} \rightarrow \mathbf{Z}$, $f(n) = 2n$.
   b. $f: \mathbf{Z} \rightarrow \mathbf{Z}$, $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$
   c. $f: \mathbf{Z} \rightarrow \mathbf{N}$, $f(n) = \begin{cases} -2n & if\ n \le 0 \\ 2n-1 & if\ n > 0 \end{cases}$

Proof:

a. <u>1-1 but not onto</u>

Proof:

1-1: Let $x, y \in Z$

Now, $f(x) = f(y) \rightarrow 2x = 2y \rightarrow x = y$

	...-3 -2 -1 0 1 2 3...	Domain
f	↓ ↓ ↓ ↓ ↓ ↓ ↓	
	...-6 -4 -2 0 2 4 6...	Co-domain

Onto: Let's assume $y \in Z_{odd} \rightarrow y \equiv 1(\bmod 2)$, where
$Z = Z_{odd} \cup Z_{even}$
Now, we can see that $y \notin \mathrm{Im}\,age(f)$,
since $y \in \mathrm{Im}\,age(f) \rightarrow y \equiv 0(\bmod 2)$, a contradiction.
Hence, precisely, we have, $f : Z \rightarrow Z_{even}$, with $f(Z) = Z_{even} \subset Z$,
i.e., $(\exists y \in Z) \mid y \neq f(x),\ x \in Z$

b. <u>Not 1-1 but onto</u>

Proof:

Not 1-1: Let's consider $x = 2n,\ y = 2n+1,\ \forall n \in Z$
Clearly we have, $x \neq y$
But we have, $f(x) = \left\lfloor \frac{x}{2} \right\rfloor = \left\lfloor \frac{2n}{2} \right\rfloor = n$ and

$$f(y) = \left\lfloor \frac{y}{2} \right\rfloor = \left\lfloor \frac{2n+1}{2} \right\rfloor = \left\lfloor n + \frac{1}{2} \right\rfloor = n$$

Hence, we have, $(x \neq y) \wedge (f(x) = f(y)) \rightarrow f$ is not 1-1.

$$\begin{array}{cccccccccc} \ldots -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3\ldots & & \text{Dom} \\ f \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & & \\ \ldots & -2 & & -1 & & 0 & & 1 & \ldots & \text{Co-Dom} \end{array}$$

Onto: Eventually all Z will be exhausted by f since we have all consecutive integers generated by the mapping, without any gap.

c. 1-1 but not onto

Proof: Let's define

$f_1(n) = -2n,\ \ n \le 0$, clearly, $f_1 : Z^- \cup \{0\} \to Z^+_{even} \cup \{0\}$

$f_2(n) = 2n - 1,\ \ n > 0$, clearly, $f_2 : Z^+ \to Z^+_{odd}$

Clearly, $f(n) = \begin{Bmatrix} f_1(n), & n \le 0 \\ f_2(n), & n > 0 \end{Bmatrix}$

1-1: Since $\mathrm{Im}\,age(f_1) \cap \mathrm{Im}\,age(f_2) = \Phi$, to show that *f* is 1-1 it's sufficient to show that *f1* and *f2* both are 1-1.

- $f_1(x) = f_1(y) \to -2x = -2y \to x = y \to f_1$ is 1-1
- $f_2(x) = f_2(y) \to 2x - 1 = 2y - 1 \to x = y \to f_2$ is 1-1

Not onto:

$\mathrm{Im}\,age(f) = \mathrm{Im}\,age(f_1) \cup \mathrm{Im}\,age(f_2) = Z^+_{odd} \cup \{0\} \cup Z^+_{even} = Z^+ \cup \{0\}$

Clearly *f* is not onto, since the negative integers from the co-domain (Z) don't have a pre-image in domain (Z) under the mapping *f*.

$$\begin{array}{ccccccccc} \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \\ f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \ldots & 6 & 4 & 2 & 0 & 1 & 3 & 5 & \ldots \end{array}$$

Domain

Co-domain

$Z \to N$

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