3.1 **HW 5** Saundipan Dey Math 685 30/30

1.(b) $n=5$. Hence number of edges $=4$. $\Rightarrow \sum d(v) = 2\cdot 4 = 8$

2 degree 2 nodes
3 " 1 "
binary
①

1 degree 2 nodes
4 degree 1
4-ary
②

1 degree 3
1 degree 2
3 degree 1 nodes
③

2 degree 1 nodes
3 degree 2 nodes
④

degree 3 nodes ≤ 2 # degree 4 nodes ≤ 1 # degree 2 nodes ≤ 4 since $\sum d(v) = 8$

degree 4 nodes $=1$, all other nodes must be of degree 1. There is exactly one such choice, namely choice ②.

degree 4 nodes $=0$.
 # degree 3 nodes $=2$ no tree corresponds to this choice
 # degree 3 nodes $=1$ # degree 2 nodes $=1$, exactly one choice, namely choice ③.
degree 3 nodes $=0$ # degree 2 nodes $=3$, only one choice, viz. ④
 # degree 2 nodes $=2$, " " " . ①
 # degree 2 nodes $=1$ or 0, no trees possible, since violates connectedness.

2. # edges in the connected graph $= e = 20 = |E(G)|$
vertices " " $= v = |V(G)|$ ✓ 5/5
G is connected $\Rightarrow v-1 \le e \le \binom{v}{2}$
$\Rightarrow v \le e+1 = 20+1 = 21$.
(tree is a minimally (connected) acyclic graph)

3. All trees are bipartite and hence 2 colorable

Proof Start with Root r of the tree T.
Let $\begin{cases} V_1 = \{v \in V[T] \mid d(r,v) \in \mathbb{Z}^+_{odd}\}. \\ V_2 = \{v \in V[T] \mid d(r,v) \in \mathbb{Z}^+_{even}\} \cup \{r\} \end{cases}$
clearly, $V_1 \cup V_2 = V[T]$,
$V_1 \cap V_2 = \phi$ since tree is acyclic
$\forall v_1 \in V_1 \wedge v_2 \in V_2$, $(v_1, v_2) \notin E[T]$.

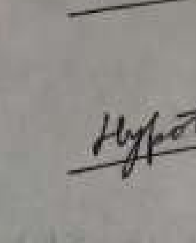
where d denotes the distance, ~~e.g.~~ $d(r,r) = 0$
e.g., if $(r,v) \in E[G]$, $d(r,v) = 1$.
if $(r,u) \in E[G] \wedge (u,v) \in E[G]$ $d(r,w) = 2$ etc.
Since $\forall w \in V$, $\exists$ exactly one path from r to v (being acyclic). $d(r,v)$ is a function

7. Let's prove it by induction on the number of vertices n of the tree.

Basis $n=2$, only possible tree: / which obviously has exactly (hence at least) 2 vertices of degree 1, trivial.

Hypothesis Let $\forall\, n \le m$, any tree with n vertices has at least 2 vertices of degree 1.

Induction Step for $n = m+1$, we need to add one extra vertex to a tree T_m with m vertices, which by hypothesis has at least 2 leaf nodes (vertices). We notice that the new vertex can have an edge with exactly one vertex in T_m, because ~~if~~ had it been connected by edges with more than one vertices in T_m, it would result in cycles, ceasing to be a tree, a contradiction.



T_{m+1}

Hence, for the new vertex $v \in V[T_{m+1}] - V[T_m]$, $\exists\, u \in V[T_m]$ | $(u,v) \in E[T_{m+1}] - E[T_m]$. Now, there can be couple of cases:

case-1 $d(u) \neq 1$ in T_m, then still $\exists$ at least 2 vertices $v_1, v_2 \in V[T_m]$, s.t. $d(v_1)=1$, $d(v_2)=1$. these vertices will be unaltered in T_{m+1} too, hence T_{m+1} will have at least 2 vertices of degree 1.

Case-2 $d(u)=1$ in T_m. but since the new vertex v becomes adjacent to u in T_{m+1}, $d(u)=2$ in T_{m+1}. But by induction hypothesis T_m still have at least one more vertex w of degree 1 (apart from u). Also, in T_{m+1}, $d(v)=1$

$\Rightarrow \exists\, w, v \in V[T_{m+1}]$ s.t. $d(w)=d(v)=1$.

$\Rightarrow T_{m+1}$ at least has 2 vertices of degree 1.

14. Given: the graph G has no circuits and addition of any edge between two existing vertices always creates a circuit.

To prove: G is a tree.

Proof: G is acyclic. ~~To~~ show that G is a tree it's sufficient to show that G is connected.

Direct Proof: Choose any $u, v \in G$. Add (u, v) to $E[G]$ if it ~~did~~ was not there from before. By condition, a circuit is created that contains (u, v) 

$\Rightarrow$ $\exists$ another path from $u \overset{P}{\rightsquigarrow} v$ in G that does not contain the edge (u, v) in it, otherwise $u \overset{P}{\rightsquigarrow} v - u$ could not be a circuit. since u, v were chosen arbitrarily, $\exists$ a path between any two arbitrary vertices in G

$\Rightarrow$ G is connected

/ 5/5

Proof by Contradiction:

Let's assume to the contrary that G is not connected.

$\Rightarrow$ $\exists$ connected components G_1, G_2 in G s.t. $\forall v_1 \in V[G_1] \wedge \forall v_2 \in V[G_2]$ $(v_1, v_2) \notin E[G]$ i.e., $v_1 \in V[G_1]$, ~~&~~ $v_2 \in V[G_2]$, $\nexists$ any path from v_1 to v_2 in G.

Now add the edge (v_1, v_2) in G, which previously did not have this edge. Now, there is exactly one way to reach v_2 from v_1, i.e., by the edge (v_1, v_2), hence $\nexists$ a circuit containing (v_1, v_2)

$\Rightarrow$ adding new edge (v_1, v_2) in G does not create a circuit in G a contradiction. (Proved)

21. It's easy to see that only the leaf nodes of the tree represent ... (since the code is prefix-free, no internal node can encode a letter, o/w it will be ~~...~~ prefix of another letter's code)

Since each vertex of the tree ~~...~~ has two children, an n digit binary sequence (a tree with height n) can have at most $2^{n+1}-1$ leaves, hence maximum $2^{n+1}-1$ letters can be encoded at most

$$2^0+2^1+\dots+2^n=\frac{2^{n+1}-1}{2-1}$$

23. (a) # players entering the tournament $T = 32$
players after the first round $= \frac{32}{2} = 16$
" " " second " $= \frac{16}{2} = 8$
#

$\Rightarrow$ # players eliminated after the 2nd round $= 32-8 = 24 = 16+8$.

~~(b) # players entering the tournament $T' = 16$ (i.e. 16 loosers from 1st round of T)~~
~~# " after the first round $= \frac{16}{2}$~~ (~~...~~ winners from 1st round of T')
~~8~~ (~~looser from 2nd round of T~~)
~~16~~
~~# players after the 2nd round in $T' = \frac{16}{2} = 8$~~
~~# players eliminated after 2nd round =~~

~~6 loosers from 1st round of T~~ $\to$ ~~8 loosers winners from 1st round of T'~~
~~+8 loosers from 2nd round of T~~
~~16 entrants in T'~~ ~~16 players after 1st round of T~~

(b) All tennis tournaments are balanced trees, hence if # players entering tournament $T' = 24$, ~~$2+2^2+2^3+2^4$~~ let the # nodes in the leaf level be l and last but one level is i, then by condition,
$\underbrace{l+i}_{\text{\# nodes in last 2 levels}}=24$, $\underbrace{i}_{\text{\# nodes in last but one level}}=16$ (must be power of 2 and $2^4<16<2^5$) $\Rightarrow l=8$. Hence # players eliminated after the 2nd round $= 4+8=12$

(c) Similarly, for T'', $l+i=12$, $i=8 \Rightarrow l=4$. Hence # players eliminated after 2nd round $= 2+4 = 6$

(d) Next loser's tournament will eliminate $\frac{6}{2} = 3$ people.
Again, " " " " " 1 people in 1st round
1 " " " } hence 2 people
Hence only 1 winner will remain.
It takes total $\lfloor \log_2 24 \rfloor = 4$ loser's tournaments, since each time # people getting half

28. $n = m \cdot i + 1$, $m \equiv 0 \pmod 2 \Rightarrow m \cdot i + 1 \equiv 1 \pmod 2 \Rightarrow n \in \mathbb{Z}^+_{odd}$

Also, $n = i + l \Rightarrow l = i(m-1) + 1$,

Finally, i or l can be even or odd. Let $m = 2$.

$i = 3$, odd, $l = 4$ even

$i = 2$, even, $l = 3$ odd.

✓ 5/5

29. (a) Smallest possible leaf not in sequence is 1, hence must be adjacent to 6th

1 —— 4. remaining sequence (5, 6, 2)

After deleting 1, smallest possible leaf not in sequence is 3

1 —— 4 , remaining sequence (6, 2)

3 —— 5

After deleting 3, smallest possible leaf not in sequence is 4.

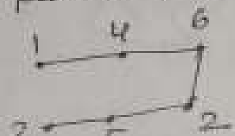
1 —4— 6 , remaing sequence (2)

3 —— 5

After deleting 4, smallest possible leaf not in sequence is 5

After deleting 4, Smallest possible leaf not in sequence is 5.

Hence 2 must be adjacent to 5.



After deleting 5, remaining labels are 2 & 6, they must be adjacent.

is the tree

(b)

Smallest possible leaf not in sequence (& not yet deleted)	remaining sequence	adjacency	deleted vertex
	(8, 8, 3, 5, 4)	(1, 2)	1
1	(8, 3, 5, 4)	(2, 8)	2
2	(3, 5, 4)	(6, 8)	6
6	(5, 4)	(3, 7)	7
7	(4)	(3, 5)	3
3	( )	(4, 5)	5
5			

remaining vertices (4, 8) adjacent.

Hence the tree

c)

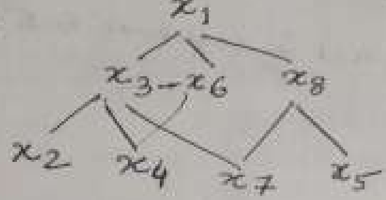
smallest possible leaf not in the sequence & not yet deleted	remaining sequence	adjacency	vertex deleted
1	(3, 3, 3, 3, 3, 3)	(1, 3)	1
2	(3, 3, 3, 3)	(2, 3)	2
4	(3, 3, 3)	(4, 3)	4
5	(3, 3)	(5, 3)	5
6	(3)	(6, 3)	6
7	( )	(7, 3)	7

remaining (3, 8) must be adjacent

The tree:

3.2

4.



Adj(G)

BFS on G

	0-reachable	1-reachable	2-reachable
x_1	x_1	x_3, x_6, x_8	x_2, x_4, x_5, x_7

Since all nodes ~~are~~ (vertices) are reachable from x_1, the ~~adjacency mat~~ graph G is connected.

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6. By induction on # of vertices n of the graph

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Basis: $n=1$, $n=2$ are trivial

Hypothesis: Let's assume $\forall n \le m$, an n-vertex graph with $n-1$ edges ~~with~~ that form no circuit is a spanning tree (tree having all nodes of the graph)

Induction Step: for $n = m+1$, notice that T_{m+1} with $m+1$ vertices and m edges can be obtained by adding a new vertex v and a new edge to T_m, i.e., $V[T_{m+1}] = V[T_m] \cup \{v\}$
$E[T_{m+1}] = E[T_m] \cup \{e\}$, where $e = (v_1, v_2)$, $v_1, v_2 \in V[T_{m+1}]$

If we can show that one of the endpoints of e is the newly added vertex v, then we are done, since ~~it means that we can get a new spanning tree~~ T_{m+1} ~~then~~ essentially becomes a spanning tree covering all the nodes (including the newly added one) in it, when by hypothesis, we have that T_m is a spanning tree.

Now, we argue that it's impossible that for $e=(v_1,v_2)$, both $v_1 \neq v$ and $v_2 \neq v$, ~~where~~ since it then means $v_1 \in V[T_m]$, $v_2 \in V[T_m]$, and if $(v_1,v_2) \notin E[T_m]$, $\exists$ path p from v_1 to v_2 in T_m (since T_m is a spanning tree, must be connected), $v_1 \overset{p}{\rightsquigarrow} v_2$ and adding $e=(v_1,v_2)$ to $E[T_m]$, s.t. $E[T_{m+1}] = E[T_m] \cup \{(v_1,v_2)\}$ will create a circuit in T_{m+1}, a contradiction.

Hence at least one of the end vertices of e must be the newly added vertex $v \Rightarrow T_{m+1}$ is a spanning tree. (Proved)

10. (a) Let's define the distance ~~matrix~~ metric d as a function $d: V[G] \times V[G] \to \mathbb{Z}^+ \cup \{0\}$

as $d(u,u) = 0, \ \forall u \in V[G]$.

$d(u,v) = d(v,u) \geq 0, \ \forall u, v \in V[G]$.

If the root for the spanning trees for both DFS and BFS is a,

we have $d(a,a) = 0$

and $\text{Height}(\underset{\text{rooted at } a}{\text{Spanning Tree}}) = \max\limits_{u \in V[G]} d(a,u)$ in general.

Now, let's pick an arbitrary vertex $v \in V[G]$ and compare d_{DFS} & d_{BFS}:

$$d_{DFS}(a,v) = \begin{cases} 0, & \text{if } a = v \\ 1 + \underset{(a,u) \in E[G],\ u \text{ is the immediate successor of } v}{d_{DFS}(u,v)}, & \text{ow} \end{cases} \quad G \text{ connected, } \cancel{\text{pick any } u}$$

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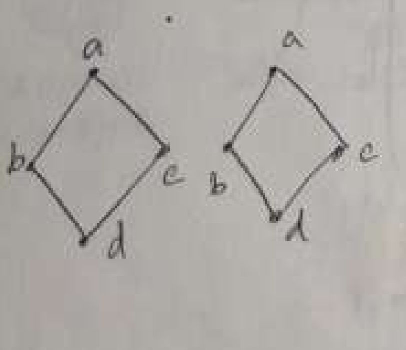
~~$d_{BFS}(a,v) = \min\{\delta(a,v), 1 + d_{BFS}($~~

~~$d_{BFS}(a,v) = 1 + \min$~~



$$d_{BFS}(a,v) = \begin{cases} 0, & \text{if } a = v \\ 1 + \min\limits_{\substack{(a,u) \in E[G] \\ u \text{ already visited}}} d_{BFS}(u,v), & \text{ow} \end{cases} \quad G \ \underline{\text{connected}}$$

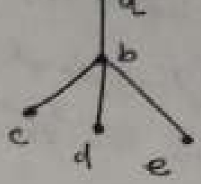
with $d_{DFS}(v,v) = d_{BFS}(v,v) = 0, \ \forall v \in V[G]$.

$\Rightarrow \underline{d_{DFS}(a,v) \geq d_{BFS}(a,v)}$, $\underline{\text{by definition}}$ this is all you really need

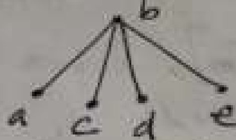
$\Rightarrow \text{Height}(\underset{\text{rooted at } a}{\text{DFS spanning tree}}) = \max\limits_{u \in V[G]^{DFS}} d(a,u) \geq \max\limits_{u \in V[G]^{BFS}} d(a,u)$

$= \text{Height}\left(\underset{\text{rooted at } a}{\text{BFS spanning tree}}\right)$

(b) Consider DFS and BFS of the following graph.

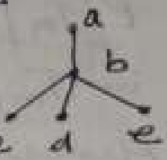


DFS spanning tree when DFS started from ~~with~~ vertex b will look like the following:



Spanning tree with ~~depth~~ height 1.

BFS spanning tree when BFS started from vertex a will look like the following: spanning tree with height 2.



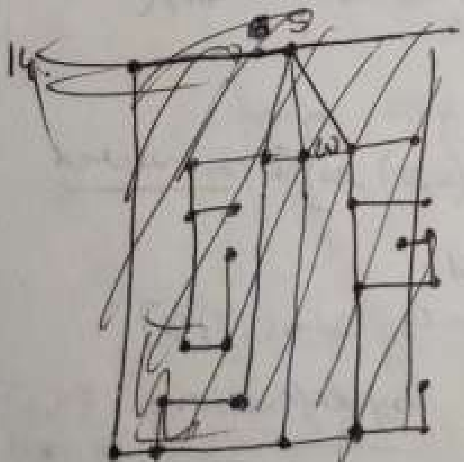
11. Let's assume to the contrary and proof by contradiction.
S be a (nonempty) cut set and T be a spanning tree of the graph G and assume $S \cap E[T] = \phi$, where $S \subseteq E[G]$ and removal of S disconnects G but no proper subset of S disconnects G, also with $V[G] = V[T]$.

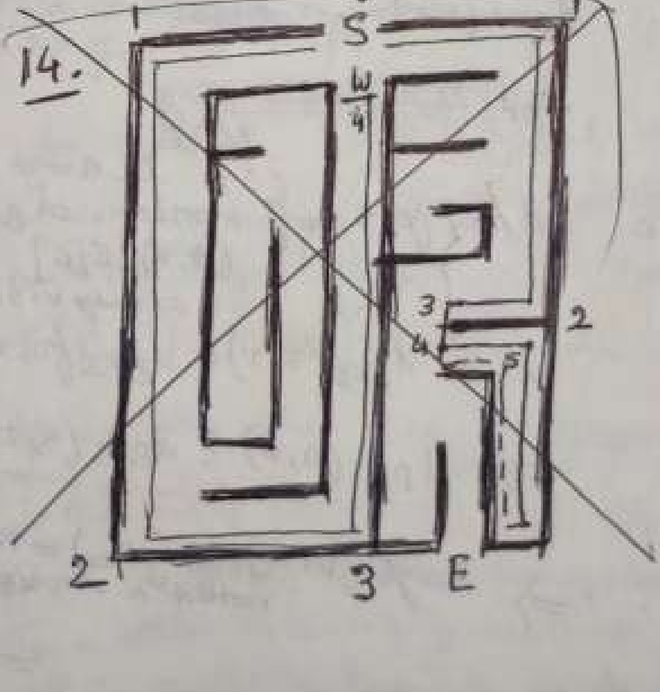
$\Rightarrow E[T] \subseteq E[G] - S$

but removing edges of S from E[G] reduces the graph to at least two connected components that are mutually disconnected from one another, G_1 & G_2, $E[G] - S = E[G_1] \cup E[G_2]$, with $E[G_1] \cap E[G_2] = \phi$.

$\Rightarrow E[T] \subseteq E[G_1] \cup E[G_2]$ with $E[G_1] \cap E[G_2] = \phi$

$\Rightarrow$ T is not a connected graph, a contradiction.





28. Let's assume to the contrary and proof by contradiction.

Let T be the spanning tree and $u_1, v_1 \in V[G]$ and $\underset{e_1}{(u_1, v_1)} \notin E[T]$
Also, $u_2, v_2 \in V[G]$ and $\underset{e_2}{(u_2, v_2)} \notin E[T]$ } with $u_1 \neq u_2, v_2$ or $v_1 \neq v_2, u_2$ (i.e., $e_1 \neq e_2$)

Let's assume ~~both~~ the ~~cycles~~ circuits created by adding (u_1, v_1) and separately (u_2, v_2) to T respectively are not unique, they are the same ~~cycle~~ circuit C. Then C must contain both the edges (u_1, v_1) and (u_2, v_2).

only added (u_1, v_1), (u_2, v_2) was not added to T

① $\Rightarrow E[T] \cup \{(u_1, v_1)\} \equiv C$ and $(u_2, v_2) \in C \Rightarrow (u_2, v_2) \in E[T]$, a contradiction.

② Also, similarly, $E[T] \cup \{(u_2, v_2)\} \equiv C$ and $(u_1, v_1) \in C \Rightarrow (u_1, v_1) \in E[T]$.
only added
since we didn't add (u_1, v_1) it must already be there, again a contradiction.

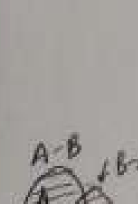
29. Let's induct on m, i.e., number of edges in which the spanning trees T' and T'' differ. Notice that since both of them contain same number of edges, either they will be same or will differ by ~~even~~ some # of edges. $V[T'] = V[T''] = V[G]$.

Basis $m=1$, $T_1 = T_m = T' = T''$, trivial (don't differ)

$m=2$, $E[T'] - E[T''] = \{\underset{e_1}{(u', v')}\}$ where $E[T''] - E[T'] = \{\underset{e_2}{(u'', v'')}\}$

$e_1 \neq e_2$

$T' = T_1$, Now adding e_2 to $E[T']$, creates a unique circuit C, as proved in 28, where $C \equiv E[T'] \cup \{e_2\}$ (differ by one edge)



$A \cup (B-A) - (A-B) = B$

Now, $\underset{E[T_2]}{=} C - \{e_1\} = E[T'] \cup \{e_2\} - \{e_1\} = E[T'] \cup (E[T''] - E[T']) - (E[T'] - E[T''])$

$= E[T'']$

$\Rightarrow T_2 = T''$

$\therefore T_1 = T'$, $T_m = T''$, for $m=2$

Hypothesis Let's assume $\exists$ sequence of (different) spanning trees $T_1, T_2, \dots, T_m$ s.t. $T' = T_1$ and $T'' = T_m$ where T' and T'' differ by $(m-1)$ edges, $\forall m \leq n$, $n \in \mathbb{N}$

Induction Step: Let's prove for $m = n+1$.

Then T' and T'' differ in exactly n edges, i.e. T' has some n edges T'' does not have and vice versa.

Let $E[T'] - E[T''] = \{\overset{e_1'}{(u_1', v_1')}, \ldots, \overset{e_n'}{(u_n', v_n')}\}$

$E[T''] - E[T'] = \{\underset{e_1''}{(u_1'', v_1'')}, \ldots, \underset{e_n''}{(u_n'', v_n'')}\}$

Let's consider the spanning tree $(T'' - \{e_n''\}) \cup \{e_n'\} = T'''$

(spanning since we are not removing vertices, still $V[T'''] = V[T''] = V[G]$, tree, because still connected. Since $C''' \equiv T'' \cup \{e_n'\}$ results in unique circuit C''' and $T''' \equiv C''' - \{e_n''\}$ results in a different spanning tree).

Now, T' and T''' differ in exactly $n-1$ edges and by induction hypotheses $\exists$ sequence of spanning trees

$T_1, T_2, \ldots, T_n$ s.t. $T' = T_1$, $T''' = T_n$. ①

Again T'' and T''' differ by 1 edge,

$$\left.\begin{aligned} E[T''] - E[T'''] &= \{e_n''\} \\ E[T'''] - E[T''] &= \{e_n'\} \end{aligned}\right\}$$

Again by induction hypothesis $\exists$ sequence of spanning trees

T_1', T_2' s.t. $T''' = T_1'$, $T'' = T_2'$. ②

Hence, combining ① & ②, we have:

$\exists$ sequence of spanning trees $T_1, T_2, \ldots, T_{n-1}, T_n, T_2'$

s.t. $T' = T_1$, $T'' = T_2'$, rename T_2' to T_{n+1} $(T_1' = T''' = T_n)$

$\Rightarrow \exists$ a sequence of spanning tree $T_1, T_2, \ldots, T_{n+1}$ s.t.

$T' = T_1$, $T'' = T_{n+1}$, (all trees are different since all the intermediate circuits are unique)

(Proved)

14.

Breadth first search
Starting at S and
ending at E