UMBC, CSEE Department
CMSC 651 Automata Theory and Formal Languages

November 9, 2010
Exam 2

Instructions

1. This is a closed-book, closed-notes exam.
2. You have 75 minutes for the exam.

Problem 1 (20 pts). What, if any, is the relationship between each of the following pairs of complexity classes? Make sure that you distinguish between $C_1 \subseteq C_2$ and $C_1 \subsetneq C_2$. Briefly justify your answer using the Time and Space Hierarchy Theorems.

a. DSPACE$[n^2]$ and DSPACE$[f(n)]$, where $f(n) = n$ for odd n and $f(n) = n^3$ for even n.

b. DTIME$[2^n]$ and DTIME$[3^n]$.

c. NSPACE$[2^n]$ and DSPACE$[5^n]$.

d. DSPACE$[n]$ and DTIME$[(\log n)^n]$.

Problem 2 (20 pts). Consider 1-tape Turing machines that are "ink bounded" — that is, Turing machines that are limited by the number of times that they can change a symbol on the tape. We think of each change (including erasing a symbol) as taking 1 unit of "ink". Let INK$^1[n]$, TIME$^1[n]$ and SPACE$^1[n]$ be the class of languages recognized by 1-tape Turing machines that on inputs of length n use at most n units of ink, take at most n steps and use at most n tape cells, respectively.

Discuss how INK$^1[n]$, TIME$^1[n]$ and SPACE$^1[n]$ are related. Justify your remarks as fully as you can.

Problem 3 (20 pts). Consider the PATH language defined in the textbook:

$$\text{PATH} = \{ (G, s, t) \mid G \text{ is a directed graph that has a path from } s \text{ to } t \}.$$

For this problem, we will work with a variation:

$$\text{DAG\_PATH} = \{ (G, s, t) \mid G \text{ is a directed acyclic graph and } G \text{ has a path from } s \text{ to } t \}.$$

We want to show that DAG_PATH is complete for NLOG (nondeterministic logspace) using logspace transducers. For both PATH and DAG_PATH, the graph G is given as an adjacency matrix.

a. Show that DAG_PATH is in NLOG by giving a high-level description of a nondeterministic logspace Turing machine for DAG_PATH. Note that your machine must *reject* when G contains a cycle.

b. Describe a logspace transducer that reduces PATH to DAG_PATH. Give a high-level description, then argue that your machine uses no more than logarithmic space on the work tape.

Short Exam 2 / Math 650 / Spring 2010 / Dr. Güler
(Thursday, March 25, 2010)

Name Sandipan Dey

1. **[10 pts.]** Let C be a convex set in $\mathbb{R}^n$, and let f be a Gateaux differentiable function on an open set containing C.

(a) If $x^* \in C$ is a local minimizer of f on C, then

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C. \tag{1}$$

(b) If f is convex and (1) is satisfied at $x^* \in C$, then x^* is a global minimizer of f on C.

2. **[10 pts.]** Prove one the following:
a. Prove that

$$\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{12} + \frac{w}{12}\right)^4 \leq \frac{1}{2}x^4 + \frac{1}{3}y^4 + \frac{1}{12}z^4 + \frac{1}{12}w^4$$

with equality holding if and only if $x = y = z = w$.
b. Consider the optimization problem

$$\min\{||x - d||^2 : ||x||^2 \leq 1\},$$

where $||d|| > 1$. Use the variational inequality to find the optimal solution solution.

3. **(Extra credit)** **[4 pts.]** Let C be a convex set in $\mathbb{R}^n$, and let f be a Gateaux differentiable function on an open set containing C. Prove that f is convex on C if and only if the tangent plane at any point $x \in C$ lies below the graph of f, that is,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \text{ for all } x, y \in C.$$

Short Exam 2 / Math 650 / Spring 2010 / Dr. Güler
(Thursday, March 25, 2010)

Name Sandipan Dey

1. **[10 pts.]** Let C be a convex set in $\mathbb{R}^n$, and let f be a Gateaux differentiable function on an open set containing C.

(a) If $x^* \in C$ is a local minimizer of f on C, then

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in C. \tag{1}$$

(b) If f is convex and (1) is satisfied at $x^* \in C$, then x^* is a global minimizer of f on C.

2. **[10 pts.]** Prove one the following:

a. Prove that

$$\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{12} + \frac{w}{12}\right)^4 \leq \frac{1}{2}x^4 + \frac{1}{3}y^4 + \frac{1}{12}z^4 + \frac{1}{12}w^4$$

with equality holding if and only if $x = y = z = w$.

b. Consider the optimization problem

$$\min\{||x - d||^2 : ||x||^2 \leq 1\},$$

where $||d|| > 1$. Use the variational inequality to find the optimal solution solution.

3. **(Extra credit)** **[4 pts.]** Let C be a convex set in $\mathbb{R}^n$, and let f be a Gateaux differentiable function on an open set containing C. Prove that f is convex on C if and only if the tangent plane at any point $x \in C$ lies below the graph of f, that is,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad \text{for all} \quad x, y \in C.$$

Sandipan Dey

18/20

(a) ~~Now~~ First order necessary condition

f is a convex function and $x^* \in C$ is a local minimizer. ~~Since~~ Choose any $x \in C$. Since x is convex all the points in the interval $x - x^* \in C$.

Now, let's consider $f(x^* + t(x - x^*))$, where $t \in \mathbb{R}$.

For arbitrary small t $(t \to 0)$, since x^* is a local minima, we have, $f(x^* + t(x - x^*)) \geq f(x^*)$. ..①

Also, f is Gateaux differentiable, ~~and~~

~~$f(x^*, x - x^*) = \lim_{t \to 0} \frac{f(x^* + t(x - x^*))}{t}$~~ $\therefore f(x^*, x - x^*) = \lim_{t \downarrow 0} \frac{f(x^* + t(x - x^*))}{t}$ exists.

not needed here!

Also, f is convex $\Rightarrow$ $f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle$, $\forall x, y \in C$.

Putting $y = x^* + t(x - x^*)$ and $x = x^*$, we have, $f(x^* + t(x - x^*)) - f(x^*) \geq t \langle \nabla f(x^*), x - x^* \rangle$. ·②

from ① & ②, we have,

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0,$$

but x was arbitrarily chosen

$$\Rightarrow \langle \nabla f(x^*), x - x^* \rangle \geq 0 \ \forall x \in C.$$

1. (b) ~~x^* is a global~~ f is convex and we have

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C.$$

Also, $f(x) - f(x^*) \geq \langle \nabla f(x^*), x - x^* \rangle \geq 0$, since f is convex $\forall x \in C$.

$\Rightarrow \ f(x) \geq f(x^*), \ \forall x \in C.$

$\Rightarrow x^*$ is a global minimizer.

~~2. a)~~ choose ~~$f(x,y,z,w) = \frac{x}{2} + \frac{y}{3} + \frac{z}{4}$~~

~~Let $f(x,y,z,w) = \frac{x}{2} + \frac{y}{3}$~~

2. a) Let $f(x) = x^4$, now, $f''(x) = 12x^2 \geq 0, \ \forall x \in \mathbb{R}$,

$\Rightarrow f(x)$ is convex.

By Jensen's inequality, we have,

$$f\left(\sum_i \lambda_i x_i\right) \leq \sum_i \lambda_i f(x_i),$$

with $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$.

Let, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, $\lambda_3 = \lambda_4 = \frac{1}{12}$, as positive.

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{5+1}{6} = 1$, sums to one.

$\therefore$ By Jensen's inequality on $f(x_i) = x_i^4$, we have,

$$\left(\sum_{i=1}^{4} \lambda_i x_i\right)^4 \leq \sum_{i=1}^{4} \lambda_i (x_i)^4$$

$$\Rightarrow \left(\frac{x_1}{2} + \frac{y}{3} + \frac{z}{12} + \frac{w}{12}\right)^4 \leq \frac{1}{2}x^4 + \frac{1}{3}y^4 + \frac{1}{12}z^4 + \frac{1}{12}w^4$$

(Proved)

equality case? -4

3. ~~f(y) - f(x)~~ Choose $x, y \in C$

By Taylor's Theorem,

$$f(x + t(y-x)) = f(x) + t\langle \nabla f(x), y-x\rangle + o(t).$$

$$\leq (1-t) f(x) + t f(y) \quad (\text{since } f \text{ is convex})$$

$$\Rightarrow t(f(y) - f(x)) \geq t\langle \nabla f(x), y-x\rangle + o(t).$$

$$\Rightarrow f(y) - f(x) \geq \langle \nabla f(x), y-x\rangle + \frac{o(t)}{t}$$

At. $t \to 0$, $\frac{o(t)}{t} \to 0$

$$\Rightarrow f(y) - f(x) \geq \langle \nabla f(x), y-x\rangle \quad (\text{Proved})$$

$\forall x, y \in C$.

+2

$\Leftarrow$?