CMSC 641, Design and Analysis of Algorithms, Spring 2010

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Homework Assignment - 11

27/30

May 4, 2010

1. PiC (Partition into Cliques) is NP Complete

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1. PiC = $\{\langle G, k\rangle : \exists$ a partition of $V(G)$ into k subsets $V_1, \ldots, V_k$ s.t. each V_i is a clique in $G\}$

2. PiC $\in$ NP : Given $\langle G, k\rangle$ and $\{V_1, \ldots, V_k\}$ (certificate) we have to verify whether it is a valid partition of $V(G)$ into k Cliques. ✓

   (a) Check if the vertex set (all vertices) $V(G)$ is covered by the subsets $V = V_1 \cup V_2 \ldots \cup V_k$: can be done in polynomial time at most $O(|V|^2)$.

   (b) Check if valid partition (all of the k subsets are mutually disjoint) $V_i \cap V_j = \phi$ if $i \neq j$: can be done in polynomial time $O(k^2|V|^2)$.

   (c) Check if each V_i is a clique: this can be done in polynomial time $O(\sum_{i=1}^{k} |V_i|^2) = O(k|V|^2)$.

   Hence the 'Yes' certificate can be verified in polynomial time.

   $\Rightarrow$ PiC $\in$ NP.

3. PiC is NP-hard : We show by reducing the well-known NP-hard problem Graph Coloring to Pic. ✓

   (a) Graph Coloring = $\{\langle G, k\rangle : G$ is k-colorable$\}$.

   (b) To Prove: Graph Coloring $\leq_p^m$ Pic.

   (c) Construction:

      i. Construct the graph $\bar{G}(V, \bar{E})$, the complement graph of $G(V, E)$ (s.t. $(u, v) \in \bar{E} \Leftrightarrow (u, v) \notin E$).

      ii. Claim: G is k colorable iff $\bar{G}$ can be partitioned into k cliques.

(d) Proof:

i. ($\Rightarrow$)

- Suppose G is k colorable.
- $\exists (V_1, V_2, \dots V_k) \subseteq V$ that are colored by colors $1, 2, \dots k$ respectively, with V_i colored by color i.
- $\bigcup_{i=1}^{k} V_i = V$ (since all the vertices of G must be colored).
- $V_i \cap V_j = \phi$ if $i \neq j$ (since every vertex must be colored with exactly one color).
- Each $V_i \subseteq V$ forms an independent set of G (since any two vertices $u, v \in V$ can be colored with the same color only if $(u, v) \notin E$).
- In $\bar{G}$ each of V_i forms a clique $\forall i = 1, \dots, k$ (since for any $u, v \in V_i \Rightarrow (u, v) \notin E \Rightarrow (u, v) \in \bar{E}$), with $\bigcup_{i=1}^{k} V_i = V$ and $V_i \cap V_j = \phi$ if $i \neq j$.

Hence $\bar{G}$ can be partitioned into k cliques.

ii. ($\Leftarrow$)

- Suppose $\bar{G}$ can be partitioned into k cliques.
- $\exists (V_1, V_2, \dots V_k) \subseteq V$, with each V_i being a clique, $\bigcup_{i=1}^{k} V_i = V$, $V_i \cap V_j = \phi$ if $i \neq j$ (mutually disjoint since a partition).
- Each $V_i \subseteq V$ forms an independent set in G (since for any $u, v \in V_i \Rightarrow (u, v) \in \bar{E} \Rightarrow (u, v) \notin E$).
- In G, there are k such independent sets V_i.

Hence G is k colorable.

(e) The Construction is polynomial time: needs $\theta(|E|)$ time to construct the complement graph $\bar{G}$.

3. Parallel Transpose

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Work $= T_1(n) = \theta(n^2)$ (serializing nested for loops on line 2 and 3).
Span $= T_\infty(n) = \theta(lgn) + \theta(1) = \theta(lgn)$.
Parallelism $= \frac{T_1(n)}{T_\infty(n)} = \frac{\theta(n^2)}{\theta(lgn)} = \theta\left(\frac{n^2}{lgn}\right)$.

2. Matrix-Vector Multiplication

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MAT-VEC (A, x)

$n = A.rows$

let y be a new vector of length n

parallel for $i = 1$ to n

$y_i = 0$

~~parallel~~

for $j = 1$ to n

parallel for P_i, $1 \le i \le j$

if $i < j$

then $y_{i+1} = y_i$

endif; $t_{ij} = a_{ij} x_j$;

$y_i = y_i + t_{ij}$;

return y_i;

for $j = 2$ to n

parallel for y_i, $j-1 \le i \le n-1$

$y_{i+1} = y_i$

$y_i = y_i + t_{ij}$;

return y_i;

Hence the parallelism achieved $= \Theta(n^2/\lg n)$, since it's obvious that the span of the algorithm is $\Theta(\lg n)$, when $T_1(n) = n^2$.

$T_\infty(n) = \Theta(\lg n)$

what is this?

how is this different from the original?

?

?

??

return where?

$\max x_1 x_2 \cdots x_n$

s.t. $x_1 + x_2 + \cdots + x_n = N$

$x_i \ge 0$

$f(x) = -x_1 x_2 \cdots x_n$ ~~$+ \lambda(\ldots)$~~

$+ \mu(x_1 + x_2 + \cdots + x_n - N)$

$\nabla_i f(x) = -$ ~~$x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n$~~

$+ \mu$ ~~$= 0$~~

$-x_1 x_2 \cdots x_n$

$\nabla_i f(x) = -\frac{\prod x_i}{x_i}$

$-\lambda = \mu$

$\Rightarrow -\prod x_i = \lambda x_i$

$(-1)^n (\prod x_i)^n$ ~~$= \lambda^n (\prod x_i)$~~

$\max x_1 x_2$

s.t. $x_1 + x_2 = N$

$f = -x_1 x_2 + \mu(x_1 + x_2 - N)$

$-x_2 + \mu = 0$

$-x_1 + \mu = 0$

$\Rightarrow \mu = x_1 = x_2 = \frac{N}{2}$

$\Rightarrow \mu = \frac{x_1 x_2}{x_1} = \frac{x_1 x_2}{x_2}$

$\nabla_i f(x) = -\frac{\prod x_i}{x_i} + \mu = 0$

$\Rightarrow x_i = \frac{P}{\mu}$

$n \cdot \frac{P}{\mu} = N$

$\Rightarrow \mu = \frac{nP}{N}$.