Random Graphs and Ramsey Theory

Math 685

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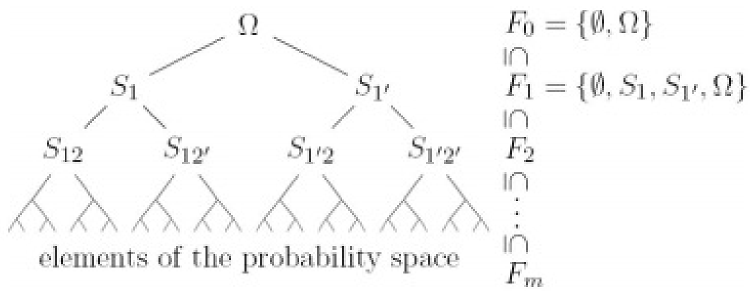
Random Graphs

Definition 1.1

Random Graph:
1) Dynamic: Graph evolves from **empty** (at time 0) to **full**. Choose an edge **randomly**.
2) Static: Given **e**, choose a graph randomly from **all possible** graphs with **e** edges (sample space).
3) Probabilistic (**Erdos-Renyi** model): Given **p**, choose **G(n,p)** s.t. $\Pr[(i,j) \in E[G]] = p$ **,** by **flipping** a **coin** when head occurs, **choose** an **edge.** When $p = e \Big/ \binom{n}{2}$, same as static model.

Random Graph as Martingale

Start with Ω = set of all $2^{\binom{n}{2}}$ graphs with n vertices (**probability space**).
Define $S_1 =$ {all graphs containing edge 1}, $S_{1'} =$ {all graphs not containing edge 1}.
$F_0 = \{0, \Omega\}$, $F_1 = \{0, S_1, S_{1'}, \Omega\}$, $F_2 = \{0, S_1, S_{1'}, S_2, S_{2'}, S_{12}, S_{12'}, S_{1'2}, S_{1'2;}, \Omega\}$ etc.,
a filtration refining Ω by a sequence of σ fields of filtration sequences $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m$



Let X denotes a random variable on our probability space and define

$$X_k = E(X|F_k)$$

Then $\{X_k, F_k\}_{k=0}^{m}$ is a martingale. Note that we have some basic properties of martingale:

$$X_0 = E(X|F_0) = EX,$$

$$X_k = E(X_{k+1}|F_k).$$

Definition 1.2

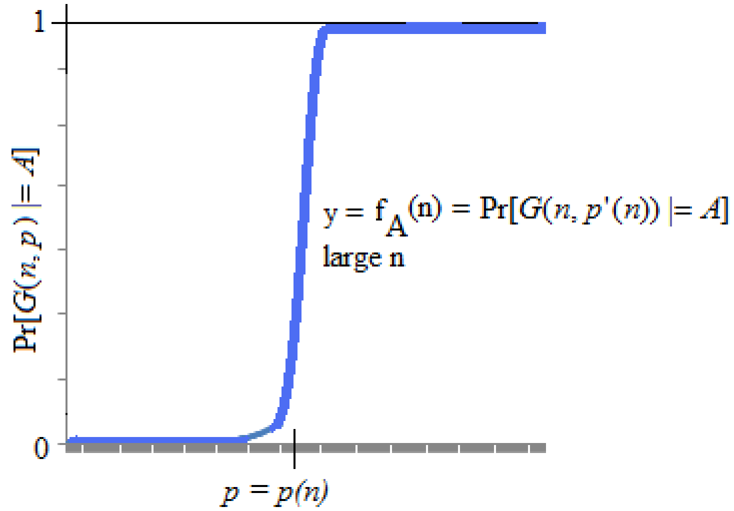
Property A: a monotone property of the graphs $\Pr[G(n.p) \models A]$ as a function of *p* increases from **0** to **1.** Erdos, Renyi discovered that for many natural properties $\Pr[G(n.p) \models A]$ **jumps** from 0 to 1 in a very narrow range.

Definition 1.3

Threshold function: a function $p(n)$ is a threshold function for a property A if

$$p'(n) << p(n) \Rightarrow \Pr[G(n, p'(n)) \models A] \to 0 \ \wedge \ p'(n) >> p(n) \Rightarrow \Pr[G(n, p'(n)) \models A] \to 1$$

(function **jumps** from **0** to **1** around $p'(n) = p(n)$)



Examples

Property A	Threshold
Is connected	$p(n) = \ln n / n$
Contains a k-clique	$p(n) = n^{-2/(k-1)}$
Is planar	$p(n) = 1/n$

Famous **double jump** occurs at $p(n) = 1/n$, when a **giant component** appears.

Theorem 1.1

A be the property that the graph contains a **triangle**. The **threshold function** is $p(n) = 1/n$.

Proof

For each 3-set T, A_T be the event that T is a triangle and X_T be the associated indicator.

$\therefore E[X_T] = P[A_T] = p^3$.

Let X be the number of triangles s. t. $X = \sum X_T$, summation is over all 3-sets, there are $\binom{n}{3}$ of them.

$\therefore E[X] = \sum E[X_T] = \binom{n}{3} p^3 \approx cn^3 p^3$ (by linearity of expectation).

$p = p(n) << \frac{1}{n} \Rightarrow E[X] << 1$ and $\Pr[X > 0] \le E[X] << 1 \Rightarrow$ G does not contain a triangle a.s.

Also, $p = p(n) >> \frac{1}{n} \Rightarrow E[X] >> 1 \Rightarrow P[X = 0] << 1$ (using the second moment method) $\Rightarrow$ G contains a triangle a.s.

First Order Graph Theory: Language with Boolean connectives, existential and universal quantifiers, variables, equality and adjacency I(x, y), with axioms $(\forall x)\neg I(x,x)$ and $(\forall x)(\forall y) I(x,y) \equiv I(y,x)$.

Examples

There exists a path of length 3: $(\exists x)(\exists y)(\exists z)(\exists w) I(x,y) \wedge I(y,z) \wedge I(z,w)$

There are no isolated points: $(\forall x)(\exists y) I(x,y)$

Theorem 1.2 (The Zero-one law)

For every **first order statement** A, $\lim_{n\to\infty} \Pr[G_{n,p} \ \ has \ \ A] = 0 \ \ or \ \ 1$ (holds a.s. or a.n.).

Eigenvalues of graph

Adjacency Matrix A has **real eigenvalues** $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (since **A symmetric)**

Theorem 1.3

If $d_{\max}$ = maximum degree of G, then $\alpha_1 \geq \sqrt{d_{\max}}$

Theorem 1.4 (Perron-Frobenius)

If A is non-negative irreducible then the **spectral radius** $\rho(A)$ is an **eigenvalue** of A and the corresponding eigenvector have all **positive** entries.

Theorem 1.5

If G' is a subgraph of G, then $\alpha_1(G') \leq \alpha_1(G)$.

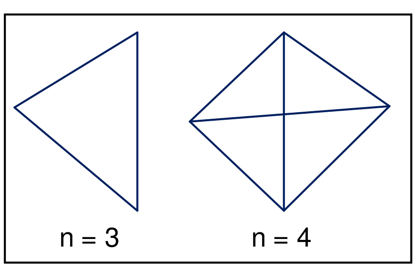
Ramsey Theory

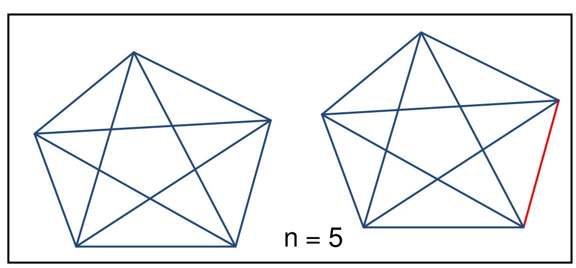
Definition 2.1

Ramsey Number $R(k,l)$: **Smallest** integer ***n*** such that **any 2-coloring** of **edges** of K_n contains either a **monochromatic** (blue) K_k or a (red) K_l.

Examples

$R(5,2) = 5$





$\forall n < 5, \ \exists a \ coloring \ of \ K_n \ \mid$

$\neg\exists \ (a \ blue \ K_5 \ or \ red \ K_2)$

$\Rightarrow R(5,2) \geq 5$

$\forall coloring \ of \ K_5,$

$\exists \ (a \ blue \ K_5 \ or \ red \ K_2)$

$\Rightarrow R(5,2) = 5$

$R(k,2)=k$: Since $\forall n \le k-1,\ \ K_n$ will have a coloring (e.g., color all edges by blue to have a single blue K_{k-1}) which contains neither a blue K_k nor a red K_2, but any (edge) coloring of K_k must contain a blue K_k or a red K_2.

$R(2,k)=k$: Similarly. Hence, we have the following lemma:

<u>Lemma 2.1</u>

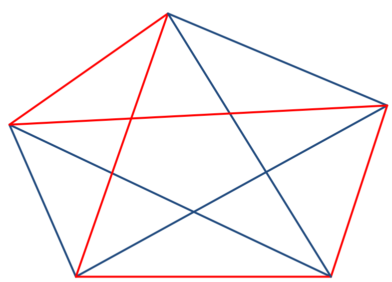
$R(k,2)=R(2,k)=k$.

<u>Symmetry (switch colors)</u>

$R(k,l)=R(l,k)$

<u>More Examples</u>

$R(3,3)=6$:

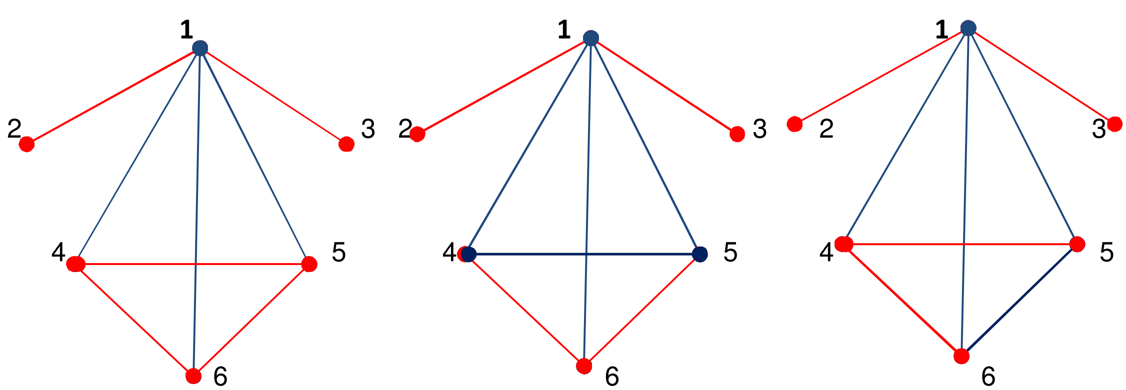


$R(3,3)>5$: $\exists a\ coloring\ of\ K_5 \mid \neg\exists\ (a\ monochromatic\ K_3)$

<u>Theorem 2.1</u>

$R(3,3)=6$: In every 2-coloring (blue or red) of edges of K_6, $\exists$ a monochromatic K_3.

<u>Proof</u>



456: a red K_3 145: a blue K_3 156: a blue K_3

Different ways of (edge) coloring K_6

For n = 6, $\forall coloring \mid \exists\ (a\ blue\ K_5\ or\ red\ K_2) \Rightarrow R(3,3) = 6$

As shown above, fix a vertex (1) of K_6.

At least 3 of **the 5 edges incident** on 1 must be colored by either blue or red in any edge coloring of K_6.

Without loss of generality assume that edges 1-4, 1-5 and 1-6 are colored blue (by any coloring).
Now consider the coloring of the triangle 4-5-6.
No matter how we color the edges of triangle 4-5-6, we must always end up with a **monochromatic** triangle.

Corollary 2.1

In any gathering with 6 people, there always exist 3 people who know each other or 3 people none of whom know each other.

Proof

Represent the relation "**know**" by **blue** coloring and "**does not know**" by **red** coloring of an edge between 2 people: there always exists a **monochromatic** triangle.

Theorem 2.2: Ramsey Theorem

$R(k,l)$ **exists** and is **finite** for any two integers k, l.

Proof (By induction on k, l)

Base case: $R(k,2) = R(2,k) = k$ (by lemma 1).

Induction Hypothesis: Let's assume $R(k-1,l)$ and $R(k,l-1)$ exist for $k,l > 2$.

Induction Step: Let's prove that $R(k,l)$ exists.

Let $r = R(k-1,l) + R(k,l-1)$.

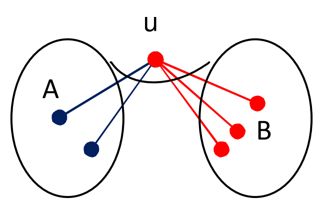
Let's consider **any** edge coloring $C = (E_1, E_2)$ for K_r (colored with **blue** and **red** respectively).

(i.e., $E_1 \cup E_2 = E[K_r]$, $E_1 \cap E_2 = \Phi$, any **2-coloring** of **edges** of K_r forms a **partition** on $E[K_r]$).

Fix $u \in V[K_r]$ and define the following sets:

$A = \{v \in V[K_r] : (u,v) \in E_1\}$

$B = \{v \in V[K_r] : (u,v) \in E_2\}$



$|A| + |B| = d(u) = r - 1 = R(k-1,l) + R(k,l-1) - 1$ [$\because$ every vertex in K_r has **degree** r – 1]

$\Rightarrow |A| \geq R(k-1,l) \vee |B| \geq R(k,l-1),\ w.l.o.g.,\ [if\ not,\ |A| + |B| < r-1]$

Without loss of generality, assume $|A| \geq R(k-1,l)$, the other case is symmetric.

We have, $A \subseteq V[K_r] \Rightarrow r \geq |A| \geq R(k-1,l)$

Also, by **induction hypothesis**, by **any** edge coloring C, K_r must have

1) either a red K_l or

2) a blue K_{k-1}, in which case consider $K_{k-1} \cup \{u\}$, which must be a blue K_k (since u is connected to vertices in set A by blue edges only) .

Hence, K_r either has a blue K_k or a red K_l

$\Rightarrow r = R(k-1,l) + R(k,l-1) \geq R(k.l) \Rightarrow \exists finite\ R(k,l), \ \ \forall k,l \geq 2$

Lemma 2.2 (Upper bound)

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

$$R(k,k) \leq \binom{2k-2}{k-1} \approx O\left(\frac{4^k}{\sqrt{k}}\right)$$

Proof

We have, $R(k.l) \leq R(k-1,l) + R(k,l-1)$ from Ramsey Theorem. Induct on k,l.
Put $k = l$.

Definition 2.2

Property B (E. W. Miller [1937]): A **hypergraph** $\varsigma(V,2^V)$ is said to have the **property B** if $\chi(\varsigma) \leq 2$.

Theorem 3 (P. Erdos [1947], **lower bound**)

$$R(k,l) \geq k2^{\frac{k}{2}}\left(\frac{1}{e\sqrt{2}} - o(1)\right)$$

Proof Sketch

Uses the result $m(n) \geq 2^{n-1}$, where $m(n) =$ minimal $|\varsigma|$ of an n-graph that does not have property B.

For a $\binom{k}{2}$ graph ς the result becomes $\binom{r}{k} = |\varsigma| = m\binom{k}{2} \geq 2^{\binom{k}{2}-1}$.

Theorem 2.4

If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$

Proof

Color K_n randomly with $\Pr[\chi(i,j)=blue]=\Pr[\chi(i,j)=red]=\frac{1}{2}$ (by coin flip).

Let S be a set of k vertices.

Let A_s be the event that S is monochromatic (all the edges in S are colored either in blue or in red).

$\Pr[A_s]=\frac{2}{2^{\binom{k}{2}}}=2^{1-\binom{k}{2}}$, since total possible outcome = #of edges in S = $\binom{k}{2}$, each edge can be colored in 2 ways (i.e., either in blue or in red) and there are only 2 ways (#ways favorable to the event) in which S can be monochromatic (by **classical definition** of probability).

Hence, $\Pr[\bigcup_{\substack{S\subseteq V[K_n]\\|S|=k}} A_S] \le \sum_{\substack{S\subseteq V[K_n]\\|S|=k}} \Pr[A_S]$, by **union bound** of probability

$$=\binom{n}{k}.2^{1-\binom{k}{2}}<1 \qquad \text{(by condition)}$$

$$\Rightarrow \Pr[B]=\Pr\left[\bigcap \overline{A_S}\right]=1-\Pr\left[\bigcup A_S\right]>0 \qquad \text{(\textbf{positive probability})}$$

Hence, B is **not null event** and there is a point in the probability space for which B holds. But a point in probability space here is precisely a coloring of K_n. Hence, there **exists** a **coloring** χ of K_n for which there is **no monochromatic** $K_k \Rightarrow R(k,k)>n$.

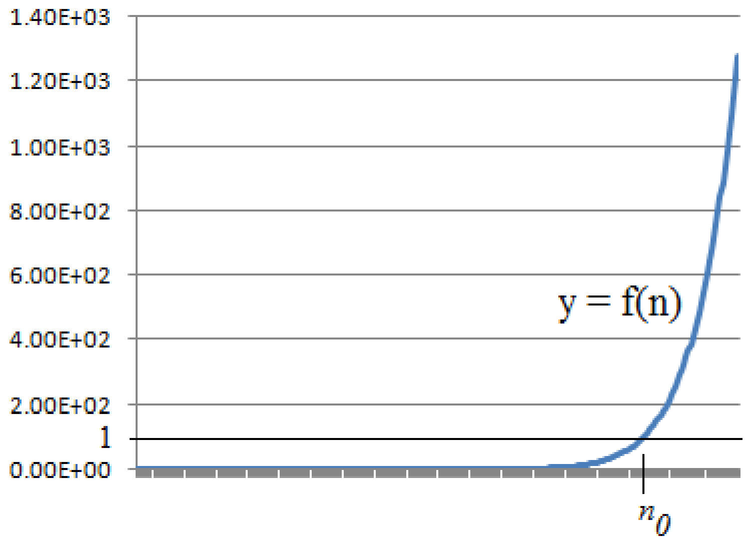
<u>Asymptotics and strong threshold behavior</u>

Maximum n = n(k) for which $\binom{n}{k}2^{1-\binom{k}{2}}<1$

Having $\binom{n}{k}\approx n^k$, $2^{1-\binom{k}{2}}\approx 2^{-\frac{k^2}{2}}$ and applying Stirling's formula, $R(k,k)\approx \frac{k}{e\sqrt{2}}2^{\frac{k}{2}}$

$$Let\ \ f(n)=\binom{n}{k}2^{1-\binom{k}{2}},\ \ n_0\ \ be\ \ s.t.\ \ f(n_0)\approx 1$$

Strong threshold behavior: $n<n_0(1-\varepsilon)\Rightarrow \Pr[\cup A_S]\le f(n) << 1$



Any random coloring has almost surely no monochromatic k-clique.

Easy Algorithm: choose n = n_0 (0.99) and start flipping coins to find such a coloring.

<u>Corollary 2.2 (Generalized version of Theorem 2.4)</u>

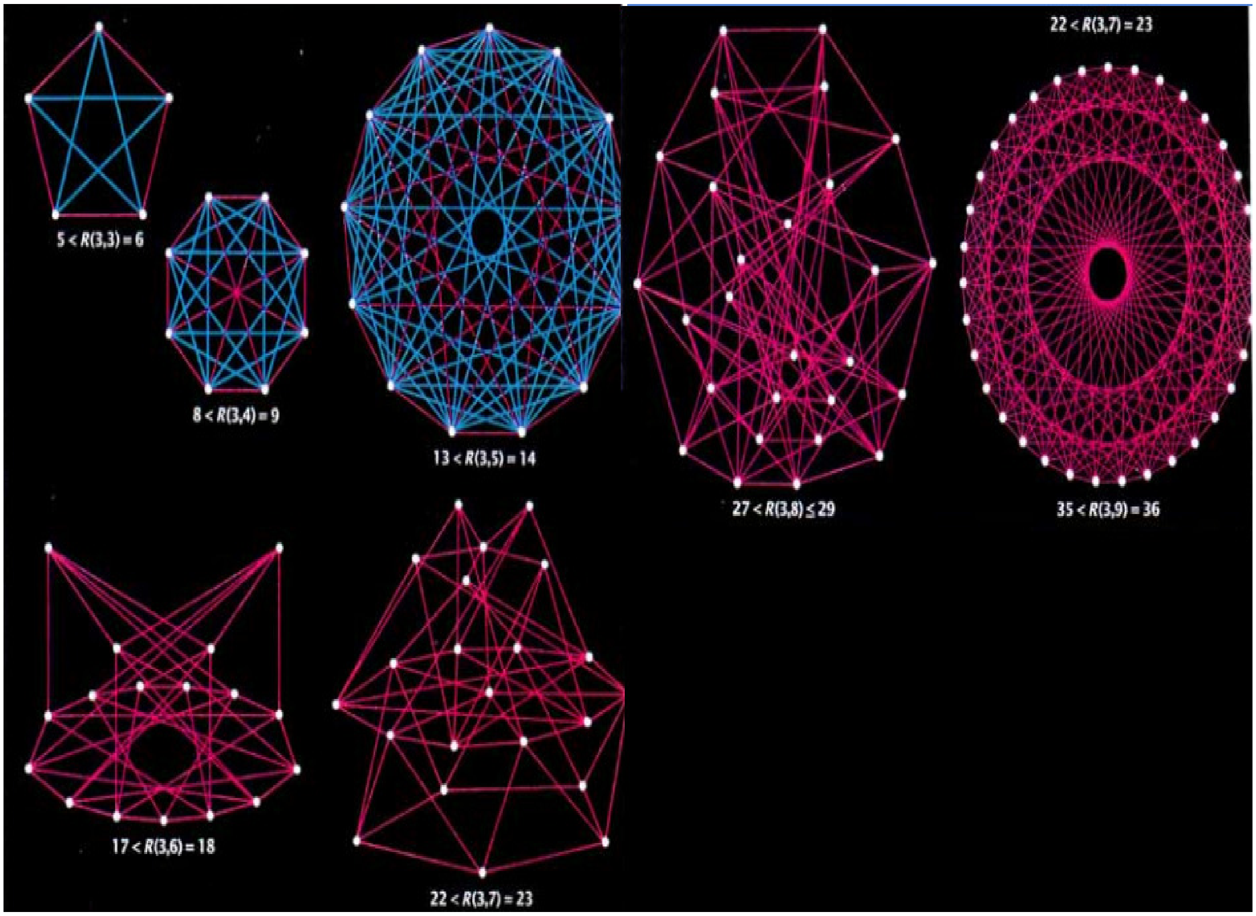
If $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$, for some $p \in [0,1]$, then $R(k,t) > n$.

<u>Hardness of computing R(k,k)</u>

Known: $R(4,4) = 17,\ 44 \le R(5,5) \le 55, R(4,5) = 25$ [known in 1993]

Law of small numbers creates problems.

Erdos asks to imagine an alien force, vastly more powerful than us, leading on Earth and demanding value of R(5,5) or they will destroy our planet. In that case we should marshall all our computers and mathematicians to find the value. But if it's R(6,6), we should destroy the aliens.



References

1. A course in Probability Theory, Kai Lai Chung.
2. The Strange Logic of Random Graphs, J. Spencer.
3. The Probabilistic Method, Noga Alon, J. Spencer, Paul Erdos
4. Random Graph Dynamics, Rick Durrett
5. Probabilistic Methods in Combinatorics, Paul Erdos, Joel Spencer
6. Ten Lectures on the Probabilistic Method, Joel Spencer
7. Ramsey Theory, Ronald L. Graham, Joel H. Spencer, Scientific American (July 1990).
8. Random Graph lecture notes, Linh V. Tran.