MATH-650

Homework-1

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1. Let's define the polynomial, g_n: $(\forall n \in \mathbb{N} \cup \{0\})$

$$g_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

(we can always define g_n, $\forall x \in I = (c,d)$, since f is n-times differentiable in I, $\exists f^{(k)}(x)$, $\forall k = 1, 2, \ldots, n$)

start with the assumption of the top!

~~Lemma~~ Claim:

$$g_n^{(k)}(a) = \begin{cases} f^{(k)}(a), & \forall (k \in \mathbb{N}) \wedge (0 \le k \le n) \\ 0, & \forall (k \in \mathbb{N}) \wedge (k > n) \end{cases} \quad (1)$$

this is obvious!

Proof

First let's prove that

$$g_n^{(k)}(x) = f^{(k)}(a) + f^{(k+1)}(a)(x-a) + \frac{f^{(k+2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-k)!}(x-a)^{n-k},$$
$$\forall\ 0 \le k \le n,$$

by induction on k

no need to be pedantic!

Basis: $k=0$, $g_n^{(0)}(x) = g_n(x) = \sum_{i=0}^{n} f^{(i)}(a)\frac{(x-a)^i}{i!}$, by definition of g_n.

$k=1$, $g_n^{(1)}(x) = \sum_{i=0}^{n} f^{(i)}(a) \cdot \frac{d}{dx}\left(\frac{(x-a)^i}{i!}\right)$

$$= \sum_{i=1}^{n} f^{(i)}(a)\frac{(x-a)^{i-1}}{(i-1)!}$$

Induction Hypothesis $\forall k \le m$, let's assume $g_n^{(k)}(x) = \sum_{i=k}^{n} f^{(i)}(a)\frac{(x-a)^{i-k}}{(i-k)!}$

Induction Step: for $k = m+1$, $g_n^{(m+1)}(x) = \frac{d}{dx}\left(g_n^{(m)}(x)\right)$

$$= \frac{d}{dx}\left(\sum_{i=m}^{n} f^{(i)}(a) \cdot \frac{(x-a)^{i-m}}{(i-m)!}\right), \text{ by induction hypothesis}$$

$$= \sum_{i=m}^{n} f^{(i)}(a) \cdot \frac{d}{dx}\left(\frac{(x-a)^{i-m}}{(i-m)!}\right)$$

$$= \sum_{i=m+1}^{n} f^{(i)}(a) \cdot \frac{(x-a)^{i-(m+1)}}{(i-(m+1))!} \quad \text{(proved)}$$

Hence we have, $g_n^{(k)}(x) = \sum_{i=k}^{n} f^{(i)}(a) \dfrac{(x-a)^{i-k}}{(i-k)!}, \ \forall\, k \in \mathbb{N} \wedge 0 \le k \le n$

$$\Rightarrow g_n^{(k)}(x) = f^{(k)}(a) + f^{(k+1)}(a)(x-a) + \frac{f^{(k+2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-k)!}(x-a)^{n-k}$$

$$\Rightarrow g_n^{(k)}(a) = f^{(k)}(a) + f^{(k+1)}(a).(a-a) + \frac{f^{(k+2)}(a)}{2!}(a-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-k)!}(a-a)^{n-k}$$

$$= f^{(k)}(a) + 0 + 0 + \cdots + 0 \quad \text{(all higher order terms vanish)}$$

$$= f^{(k)}(a), \quad \forall\, 0 \le k \le n.$$

Now, we have to prove that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o\left((x-a)^n\right)$$

$$\Leftrightarrow f(x) = g_n(x) + o\left((x-a)^n\right)$$

$$\Leftrightarrow \lim_{x\to a} \frac{f(x) - g_n(x)}{(x-a)^n} = 0.$$

Proof (By induction on n). define $L_n = \lim_{x\to a} \dfrac{f(x) - g_n(x)}{(x-a)^n}$

Basis: for $n=0$, $L_n = \lim_{x\to a} \dfrac{f(x) - g_0(x)}{(x-a)^0} = \lim_{x\to a} \dfrac{f(x) - f(a)}{1} = 0$

you shouldn't consider this case

$n=1$, $L_n = \lim_{x\to a} \dfrac{f(x) - g_1(x)}{(x-a)}$

Start with this

$$= \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a}$$

$\left[\left(\frac{0}{0}\right)\right.$ form, by L'Hospital's rule, $\lim_{x \to h} \frac{p(x)}{q(x)} = \lim_{x \to h} \frac{p'(x)}{q'(x)}$, if $p(h) = q(h) = 0$ (or ∞) $\exists\ p', q'$]

$$= \lim_{x \to a} \frac{\frac{d}{dx}\left[f(x) - f(a) - f'(a)(x-a)\right]}{\frac{d}{dx}\left[x-a\right]} \quad \text{(L'Hospital)}$$

$$= \lim_{x \to a} \frac{f'(x) - f'(a)}{1} = f'(a) - f'(a) = 0.$$

this assumes the continuity of f', an assumption you shouldn't make!

Induction Hypothesis:

Let's assume $L_n = 0$, $\forall\ n \leq m$.

Induction Step:

for $n = m+1$, $\quad L_{m+1} = \lim_{x \to a} \frac{f(x) - g_{m+1}(x)}{(x-a)^{m+1}}$

$$= \lim_{x \to a} \frac{f(x) - g_m(x) - \frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^{m+1}}{(x-a)^{m+1}}$$

$$= \lim_{x \to a} \frac{f(x) - g_m(x)}{(x-a)^{m+1}} - \frac{f^{(m+1)}(a)}{(m+1)!}$$

$$= \lim_{x \to a} \frac{f^{(1)}(x) - g_m^{(1)}(x)}{(m+1)(x-a)^m} - \frac{f^{(m+1)}(a)}{(m+1)!} \quad \left[\because g_m(a) = f(a). \ \tfrac{0}{0} \text{ form, apply L'Hospital rule}\right]$$

$$= \lim_{x \to a} \frac{f^{(m)}(x) - g_m^{(m)}(x)}{(m+1)!(x-a)} - \frac{f^{(m+1)}(a)}{(m+1)!} \quad \left[\because g_m^{(k)}(a) = f^{(k)}(a) \text{ by (1), applying L'Hospital's rule for } m \text{ times}\right]$$

$$= \lim_{x \to a} \frac{f^{(m)}(x) - f^{(m)}(a)}{(m+1)!(x-a)} - \frac{f^{(m+1)}(a)}{(m+1)!}$$

$$\because g_m^{(m)}(x) = \underbrace{0 + 0 + \cdots + 0}_{m \text{ times}} + \frac{d^m}{dx^m} \frac{f^m(a)}{m!}($$

$$= \frac{f^m(a)}{m!} \frac{d^m}{dx^m}(x-$$

$$= \frac{f^m(a)}{m!} \cdot m! = f$$

$$\cancel{= \lim_{x \to a} f^{(m+1)}}$$

$$= \lim_{x \to a} \frac{\frac{d}{dx}\left(f^{(m)}(x) - f^{(m)}(a)\right)}{(m+1)! \frac{d}{dx}(x-a)} - \frac{f^{(m+1)}(a)}{(m+1)!}$$

(by L'Hospital again, since $\frac{0}{0}$ form)

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$$= \lim_{x \to a} \frac{f^{(m+1)}(x)}{(m+1)!.1} - \frac{f^{(m+1)}(a)}{(m+1)!}$$

again, you're using the cont. of $f^{(m+1)}(x)$; an assumption that shouldn't be made!

$$= \frac{f^{(m+1)}(a)}{(m+1)!} - \frac{f^{(m+1)}(a)}{(m+1)!}$$

$$= 0.$$

Hence, $L_n = \lim_{x \to a} \frac{f(x) - g_n(x)}{(x-a)^n} = 0, \quad \forall n \in \mathbb{N} \cup$

$$\Rightarrow f(x) = g_n(x) + o\left((x-a)^n\right).$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o\left((x-a)^n\right)$$

(Proved)

This problem shouldn't take four pages!

Given $\|f(x)\| \leq \|x\|^2$

$$\left.\begin{array}{l} \Rightarrow \|f(0)\| \leq 0 \quad \Rightarrow \|f(0)\| = 0 \\ \Rightarrow f(0) = 0 \end{array}\right\}$$ (by positive definiteness of norm)

$(f : \mathbb{R}^n \to \mathbb{R})$

Now, $$\frac{\partial f(0)}{\partial x_i} = \lim_{h \to 0} \frac{f(\overset{x_1}{0}, \overset{x_2}{0}, \overset{x_3}{0}, \ldots, \overset{x_i}{0+h}, \ldots, \overset{x_n}{0}) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(0,0,\ldots,h,\ldots,0)}{h}$$ (h is a scalar here)

$$\cancel{\frac{\partial f(0+0)}{\partial x_i} = \lim_{h \downarrow 0} \frac{f(0,0,\ldots,h,\ldots,0)}{h} \Rightarrow \left\| \frac{\partial f(0+0)}{\partial x_i} \right\| = \lim_{h \to 0}}$$

$$= \lim_{h \to 0} \frac{\|f(0,0,\ldots,h,\ldots,0)\|}{h}$$ $(\because \|f(0,0,\ldots,h,\ldots)\| = f(0,0,\ldots,h,\ldots)$

$$\leq \lim_{h \to 0} \frac{\|(0,0,\ldots,h,\ldots,0)\|^2}{h}$$ $(\because \|f(x)\| \leq \|x\|^2, \ \forall x \in \mathbb{R})$

$$= \lim_{h \to 0} \left(\frac{h^2}{h}\right) = 0$$

As we notice, $$\frac{\partial f(0+0)}{\partial x_i} \leq \lim_{h \downarrow 0} \frac{h^2}{h} = 0$$

$$\frac{\partial f(0-0)}{\partial x_i} \leq \lim_{h \uparrow 0} \frac{h^2}{h} = \lim_{K \downarrow 0} \frac{(-K)^2}{-K} = \lim_{K \downarrow 0} -\left(\frac{K^2}{K}\right) = 0$$

good!!

In order that $\frac{\partial f(0)}{\partial x_i}$ exists, $$\frac{\partial f(0+0)}{\partial x_i} = \frac{\partial f(0-0)}{\partial x_i}$$

$$\Rightarrow \frac{\partial f(0)}{\partial x_i} = 0, \quad \forall x_i$$

$$\therefore \nabla f(0) = 0 \Rightarrow \langle \nabla f(0), h \rangle = 0, \quad \forall h \in \mathbb{R}^n$$

$$\therefore \lim_{h \to 0} \left\| \frac{f(0+h) - f(0) - \langle \nabla f(0), h \rangle}{\|h\|} \right\| = \lim_{h \to 0} \frac{\|f(h) - 0 - 0\|}{\|h\|} = \lim_{h \to 0} \frac{\|f(h)\|}{\|h\|}$$

$$\leq \lim_{h \to 0} \frac{\|h\|^2}{\|h\|} = \lim_{h \to 0} \|h\| = 0$$ (h is a vector here)

By positive definiteness of norm agains we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0) - \langle \nabla f(0), h \rangle}{\|h\|} = 0 \Rightarrow f(0+h) = f(0) + \langle \nabla f(0), h \rangle + o($$

$\Rightarrow f$ is Fréchet differentiable at

18. Let's define $\varphi(t) = F(x+t(y-x))$, $t \in [0,1]$

$\Rightarrow \varphi(1) = F(y)$, $\varphi(0) = F(x)$

F has Lipschitz derivative $\Rightarrow \varphi$ is differentiable.

$\Rightarrow \varphi'(t) = DF(x+t(y-x)) \cdot (y-x)$ (By chain rule)

Now, by FTC, we have $[\varphi'(0) = DF(x)(y-x)]$

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\,dt$$

$$\Rightarrow \varphi(1) - \varphi(0) - \varphi'(0) = \int_0^1 (\varphi'(t) - \varphi'(0))\,dt$$

$$\Rightarrow \|\varphi(1) - \varphi(0) - \varphi'(0)\| \le \left\| \int_0^1 (\varphi'(t) - \varphi'(0))\,dt \right\|$$ (taking norm from both sides)

$$\le \int_0^1 \|\varphi'(t) - \varphi'(0)\|\,dt$$ (by triange inequality)

$$= \int_0^1 \|(DF(x+t(y-x)) - DF(x))(y-x)\|\,dt$$

$$\le \int_0^1 \|DF(x+t(y-x)) - DF(x)\|\,\|y-x\|\,dt$$

(by property of matrix norm)

$\boxed{(\because \|Ax\| \le \|A\|\|x\|) \text{ by definition of norm}}$

$$= \int_0^1 L\|x+t(y-x)-x\|\,\|y-x\|\,dt$$

($\because$ F is Lipschitz differentiable, $\exists L \ge 0$ $\|DF(y) - DF(x)\| \le L\|y-x\|$)

$$= L\|y-x\|^2 \int_0^1 \|t\|\,dt$$

$$= L\|y-x\|^2 \int_0^1 t\,dt \quad (\because \|t\| = t,\ t \ge 0)$$

$$= L\|y-x\|^2 [t^2/2]_0^1 = \frac{L}{2}\|y-x\|^2$$

$$\Rightarrow \|F(y) - F(x) - DF(x)(y-x)\| \le \frac{L}{2}\|y-x\|^2$$ (Proved)