Sandra

$\frac{20}{2}$

1. a) We define the following propositions:

$p \equiv$ I play hockey
$s \equiv$ I am sore (the next day)
$w \equiv$ I use the whirlpool

Hypothesis:

I. $p \to s$
II. $s \to w$
III. $\neg w$

Conclusion:

$\neg p \equiv$ I did not play hockey

Steps:

$s \to w$	(Hypothesis II)	
$\equiv \neg w \to \neg s$	(Contra positive of Hypothesis II)	(IV)
$p \to s$	(Hypothesis I)	
$\equiv \neg s \to \neg p$	(Contra positive of Hypothesis I)	(V)
$\neg w$	(Hypothesis III)	
$\neg w \to \neg s$	(From IV)	
$\therefore \neg s$	(By Modus ponens)	(VI)
$\neg s$	(From VI)	
$\neg s \to \neg p$	(From V)	
$\therefore \neg p$	(Conclusion by Modus ponens)	

b) We define the following propositions:

$P(x) \equiv$ Day x is a partly sunny day
$S(x) \equiv$ Day x is a sunny day
$W(x) \equiv$ I work on a day x

Let the domain of discourse be the set of all Days, i.e.,
$x \in \{\text{Sunday, Monday, ..., Saturday}\}$.

Axiom
$(\forall x)(P(x) \to \neg S(x)) \equiv (\forall x)(\neg P(x) \vee \neg S(x)) \equiv (\forall x)(S(x) \to \neg P(x))$
(The same day can't be both sunny and partially sunny)

Hypothesis:

I. $(\forall x)(W(x) \to (S(x) \vee P(x))$
II. $W(Monday) \vee W(Friday)$
III. $\neg S(Tuesday)$
IV. $\neg P(Friday)$

Conclusion with steps:

$(\forall x)(\neg P(x) \vee \neg S(x))$ (From Axiom)
$\neg P(Friday) \vee \neg S(Friday)$ (Universal instantiation)

$(\forall x)(W(x) \to (S(x) \vee P(x))$ (Hypothesis I)
$W(Friday) \to (S(Friday) \vee P(Friday))$ (Universal instantiation)
$\equiv (\neg W(Friday) \vee S(Friday)) \vee P(Friday)$ ($\vee$ is associative) (V)

$\neg P(Friday)$ (Hypothesis IV)
$(\neg W(Friday) \vee S(Friday)) \vee P(Friday)$ (From V)

$\therefore \neg W(Friday) \vee S(Friday)$ (by Disjunctive syllogism) (VI)
I didn't work on Friday or Friday was sunny.

$W(Friday) \vee W(Monday)$ (Hypothesis I, $\vee$ is commutative)
$\neg W(Friday) \vee S(Friday)$ (From VI)

$W(Monday) \vee S(Friday)$
I worked on Monday or Friday was sunny.

(Note: Nothing without an OR can be concluded)

c) We define the following predicates:

$I(x) \equiv$ x is an insect
$S(x) \equiv$ x is a spider
$D(x) \equiv$ x is a dragon fly
$L(x) \equiv$ x has six legs
$E(x, y) \equiv$ x eats y

Let the domain of discourse be the set of all creatures.

Hypothesis:

I. $(\forall x)(I(x) \to L(x))$
II. $(\forall x)(D(x) \to I(x))$

III. $(\forall x)(S(x) \to \neg L(x))$
IV. $(\forall x)(\forall y)(S(x) \land D(y) \to E(x, y))$

Conclusion:

$(\forall x)(S(x) \to \neg I(x)) \equiv$ Spiders are not insects.

Steps:

$(\forall x)(I(x) \to L(x))$ (Hypothesis I)
$\equiv (\forall x)(\neg L(x) \to \neg I(x))$ (Contra positive of Hypothesis I) (V)

$(\forall x)(S(x) \to \neg L(x))$ (Hypothesis III)
$(\forall x)(\neg L(x) \to \neg I(x))$ (From V)
$\therefore (\forall x)(S(x) \to \neg I(x))$ (Conclusion by Hypothetical syllogism)

d) We define the following predicates:

$I(x) \equiv$ x has an internet account
$S(x) \equiv$ x is a student

Let the universe of discourse be the set of all human beings.

$\therefore I(Homer) \equiv$ Homer has an internet account
$\therefore \neg I(Maggie) \equiv$ Maggie does not have an internet account

Hypothesis:

I. $(\forall x)(S(x) \to I(x))$
II. $I(Homer)$
III. $\neg I(Maggie)$

Conclusion:

$\neg S(Maggie) \equiv$ Maggie is not a student.

Steps:

$(\forall x)(S(x) \to I(x))$ (Hypothesis I)
$\equiv (\forall x)(\neg I(x) \to \neg S(x))$ (Contra positive of Hypothesis I) (IV)

$(\neg I(Maggie) \to \neg S(Maggie))$ (From IV by Universal instantiation)
$\neg I(Maggie)$ (From Hypothesis III)

$\therefore \neg S(Maggie))$ (Conclusion by Modus ponens)

e) We define the following predicates:

$H(x) \equiv$ x is healthy
$G(x) \equiv$ x tastes good
$E(x) \equiv$ I eat x

Let the domain of discourse be the set of all foods.

Hypothesis:

I. $\neg(\forall x)(H(x) \rightarrow G(x))$
II. $H(Tofu)$
III. $(\forall x)(E(x) \rightarrow G(x))$
IV. $\neg E(Tofu)$
V. $\neg H(Cheeseburgers)$

Conclusion:
$\neg(\forall x)(E(x)) \equiv$ I don't eat all foods
(Note: Nothing can be concluded about Tofu or Cheeseburgers)

Steps:

$\neg(\forall x)(H(x) \rightarrow G(x))$ (From Hypothesis I)
$\equiv \neg(\forall x)(\neg H(x) \vee G(x))$
$\equiv (\exists x)(H(x) \wedge \neg G(x))$ (By D' Morgan) (VI)

$(\exists x)(H(x) \wedge \neg G(x))$ (From VI)
$H(f) \wedge \neg G(f)$ (By existential instantiation)
$\therefore \neg G(f)$ (by Simplification) (VII)

$(\forall x)(E(x) \rightarrow G(x))$ (From Hypothesis III)
$E(f) \rightarrow G(f)$ (By Universal instantiation)
$\equiv \neg E(f) \vee G(f)$ (VIII)

$\neg G(f)$ (From VII)
$\neg E(f) \vee G(f)$ (From VIII)
$\therefore \neg E(f)$ (By Disjunctive syllogism)
$(\exists x)(\neg E(x))$ (Existential generalization)
$\therefore \neg(\forall x)(E(x))$

f) <u>We define the following propositions:</u>

$d \equiv$ I am dreaming
$h \equiv$ I am hallucinating
$e \equiv$ I see elephants running down the road

<u>Hypothesis:</u>

I. $d \vee h$
II. $\neg d$
III. $h \to e$

<u>Conclusion:</u>

$e \equiv$ I see elephants running down the road

<u>Steps:</u>

$\neg d$	(Hypothesis II)	
$d \vee h$	(Hypothesis I)	
$\therefore h$	(By Disjunctive syllogism)	(IV)

h	(By IV)
$h \to e$	(Hypothesis III)
$\therefore e$	(Conclusion by Modus ponens)

2. Let's define
   i. $p_1 \equiv n^2$ is odd
   ii. $p_2 \equiv 1 - n$ is even
   iii. $p_3 \equiv n^3$ is odd
   iv. $p_4 \equiv n^2 + 1$ is even

To prove: $p_1 \leftrightarrow p_2 \leftrightarrow p_3 \leftrightarrow p_4$
Sufficient to prove: $p_1 \to p_2 \to p_3 \to p_4 \to p_1$
i.e.
i. $p_1 \to p_2$
ii. $p_2 \to p_3$
iii. $p_3 \to p_4$
iv. $p_4 \to p_1$

i. <u>Indirect proof:</u>

$\neg p_2 \equiv 1-n$ is odd

$\rightarrow 1-n = 2.k+1$, for some $k \in Z$, where Z denotes set of all integers

$\rightarrow n = -2.k$

$\rightarrow n^2 = (-2.k)^2 = 4.k^2 = 2.j$, for some $k^2 = j \in Z$

$\rightarrow n^2$ is even $\equiv \neg p_1$

$\neg p_2 \rightarrow \neg p_1$

$\equiv p_1 \rightarrow p_2$ (By contra positive)

$\therefore p_1 \rightarrow p_2$ (Proved)

ii. Direct proof:

$p_2 \equiv 1-n$ is even

$\rightarrow 1-n = 2.k$, for some $k \in Z$, where Z denotes set of all integers

$\rightarrow n = 1-2.k$

$\rightarrow n^3 = (1-2.k)^3 = 1-6.k+12.k^2-8.k^3$, for some $k^2 = j \in Z$

$\rightarrow n^3 = 2.(6.k^2-3.k-4.k^3)+1$

$\rightarrow n^3$ is odd $\equiv p_3$

$\therefore p_2 \rightarrow p_3$ (Proved)

iii. Proof by contradiction:

We assume $p_3 \wedge \neg p_4$

$p_3 \equiv (n^3 \text{ is odd}) \wedge (n^2+1 \text{ is odd})$

$\rightarrow (\exists k)(\exists l)((n^3 = 2l+1)) \wedge (n^2+1 = 2.k+1))$, $k,l \in Z$

$\rightarrow (\exists k)(\exists l)((n^3 = 2l+1)) \wedge (n^2 = 2.k))$, $k,l \in Z$

$\rightarrow (\exists k)(\exists l)(((n^3)^2 = (2l+1)^2)) \wedge ((n^2)^3 = (2.k)^3))$, $k,l \in Z$

$\rightarrow (\exists k)(\exists l)((n^6 = 4l^2+4l+1)) \wedge (n^6 = 8.k^3))$, $k,l \in Z$

$\rightarrow (\exists k)(\exists l)(4l^2+4l+1 = 8.k^3)$, $k,l \in Z$

$\rightarrow (\exists k)(\exists l)(8.k^3-4l^2-4l = 1)$, $k,l \in Z$

$\rightarrow (\exists k)(\exists l)(2(4.k^3-2l^2-2l) = 1)$, $k,l \in Z$,

which is false, since 2 does not divide 1, a contradiction.

Hence, our assumption was wrong.

$\therefore p_3 \rightarrow p_4$ (Proved)

iv. Direct proof:

$p_4 \equiv n^2+1$ is even

$\rightarrow (\exists k \in Z)(n^2+1 = 2.k)$

$\rightarrow (\exists k \in Z)(n^2 = 2.k-1)$

$\rightarrow (\exists k \in Z)(j = k-1 \wedge n^2 = 2.j+1)$

$\rightarrow (\exists j \in Z)(n^2 = 2.j + 1)$

$\rightarrow n^2$ is odd $\equiv p_1$

$\therefore p_4 \rightarrow p_1$ (Proved)