```
In[204]:= (*----------------------------------------------------------------------------*)
(*---------------- Sandipan Dey, UMBC CSEE ---------------------------------------*)
(*---------------- The Source Code for HW 1.5 ------------------------------------*)
(*---------------- Functions -------------------------------------------------------
                  1. ComputeOrthonormalEigenSpaces
                  2. ComputeProjectors
                  3. ComputeProbStates
                  4. ShowOutputTables
                  5. MeasureQuantumSystem
   ---------------------------------------------------------------------------*)

(* ComputeOrthonormalEigenSpaces: Computes the Orthonormal EigenSpaces *)
(* Inputs ⇒ Ω: The Observrvable *)
(* Output ⇒ EigenValues, EigenVectros and the Dimensions of the EigenSpaces *)
ComputeOrthonormalEigenSpaces[Ω_] := Module[{n, Λ, V, Λo, Vo, i},
   n = Dimensions[Ω][[1]]; (*Ω Square Matrix*)
   {Λ, V} = Eigensystem[Ω]; (*Find EigenValues and Orthogonal EigenVectors*)
   (*Construct Orthonormal EigenKets in the respective EigenSpaces*)
   Clear[Λo]; Do[Λo[Λ[[i]]] = Λ[[i]], {i, n}];
   Clear[Vo]; Do[Vo[Λ[[i]]] = {}, {i, n}];
   Do[Vo[Λ[[i]]] = Append[Vo[Λ[[i]]], V[[i]]], {i, n}];
   Do[If[Dimensions[Vo[Λ[[i]]]][[1]] == 1, Vo[Λ[[i]]] = {Normalize[Vo[Λ[[i]]][[1]]]},
     Vo[Λ[[i]]] = Orthogonalize[Vo[ Λ[[i]]]]], {i, n}];
   Λ = DownValues[Λo][[All, 2]]; V = DownValues[Vo][[All, 2]];
   n = Dimensions[Λ][[1]]; (*Dimension of Eigen Space*)
   {Λ, V, n}
  ];

(* ComputeProjectors: Computes the Projectors *)
(* Inputs ⇒ V: EigenSpace buckets containing orthonormal eignevectors,
n: Dimension of the EigenSpace *)
(* Output ⇒ n Projectors *)
ComputeProjectors[V_, n_] := Module[{P, pVerify, oVerify, i, j},
   pVerify = oVerify = True;
   P = Table[0, {i, n}, {j, 1}];
   Do[{m, p} = Dimensions[V[[i]]]; P[[i]] = Table[0, {r, p}, {c, p}];
     Do[ket = {V[[i]][[j]]}ᵀ; braw = ket†; Pr = ket.braw;
     If [Pr.Pr ≠ Pr, pVerify = False, ] ; P[[i]] = P[[i]] + Pr, {j, m}], {i, n}];
   ZeroMatrix = Table[0, {i, p}, {j, p}];
   Do[Do[If[P[[i]].P[[j]] ≠ ZeroMatrix and i ≠ j, oVerify = False,], {i, n}], {j, n}];
   (*Verify Kronecker*)
   {P, pVerify}
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  ];

(* ComputeProbStates: Computes the Probabilities and the States *)
(* Inputs ⇒ P: n Projectors, ρ: Density Operator, n: Dimension of the EigenSpace *)
(* Output ⇒ n Probabilities and the States *)
ComputeProbStates[P_, ρ_, n_] := Module[{prob, state, i},
   prob = Table[Expand[Tr[P[[i]].ρ]], {i, n}]; (*Probabilities*)
   state = Table[Map[Simplify, Expand[P[[i]].ρ.P[[i]] / Tr[P[[i]].ρ]]], {i, n}];
   (*States*)
   {prob, state}
  ];

(* ShowOutputTables: Shows the Output Tables *)
(* Inputs ⇒ Ω: The Obserrvable, Λ: EigenValues,
V: EigenSpace buckets containing orthonormal eignevectors,
P: Projectors, ρ: The Density Operator,
            prob: Probabilities, state: States, n: Dimension of the EigenSpace *)
(* Output ⇒ None *)
ShowOutputTables[Ω_, ρ_, Λ_, V_, P_, prob_, state_, n_, si_, sv_, so_] :=
  Module[{inputTable, verifyTable, outputTable, i, j, k},
     inputTable = Table[Switch[i, 1,
      Switch[j, 1, "Observable", 2, "Density Operator", 3, "Trace(ρ.ρ)"],
                                2, Switch[j, 1, MatrixForm[Ω],
       2, MatrixForm[ρ], 3, Tr[ρ.ρ]],
                                3, Switch[j, 1, "", 2, "", 3,
       If[Tr[ρ.ρ] ≠ 1, "Mixed ensemble", "Pure ensemble"]]],
              {i, 3}, {j, 3}];
     verifyTable = Table[If[k == 1, Switch[j, 1, "∑P = I",
       2, "∑λP = Ω", 3, "Pᵢ.Pᵢ = Pᵢ", 4, "∑p = 1", 5, "Pᵢ.Pⱼ=0, i≠j" ],
                                 Switch[j, 1, Sum[P[[i]], {i, n}] == IdentityMatrix[p],
                                            2, Sum[Λ[[i]] * P[[i]], {i, n}] == Ω,
                                            3, pVerify,

       4, If[Sum[prob[[i]], {i, n}] == 1, True, False],
                                            5, True]],
                          {k, 2}, {j, 5}];
     outputTable = Table[If[i == 0, Switch[j, 1, "EigenValue",
       2, "EigenSpace", 3, "Projector", 4, "Probability", 5, "State"],
                                  Switch[j, 1, Λ[[i]], 2, MatrixForm[V[[i]]ᵀ],
       3, MatrixForm[P[[i]]], 4, prob[[i]], 5, MatrixForm[state[[i]]]]],
                    {i, 0, n}, {j, 5}];
    (* Show Outputs *)
```

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   Grid[inputTable, Alignment → Center, Spacings → {si, 1},
    Frame → All, ItemStyle → "Text", Background → {{None, None}, {Orange, None}}]
   Grid[verifyTable, Alignment → Center, Spacings → {sv, 1}, Frame → All,
    ItemStyle → "Text", Background → {{None, None}, {None, None}}]
   Grid[outputTable, Alignment → Center, Spacings → {so, 1}, Frame → All,
    ItemStyle → "Text", Background → {{None, None}, {Green, None}}]
   ];

(* MeasureQuantumSystem: Measures the Quantum System with the Observable *)
(* Inputs ⇒ Ω: The Obserrvable, Ψ: The Density Operator *)
(* Output ⇒ None *)
MeasureQuantumSystem[Ω_, ρ_, si_: 18, sv_: 9, so_: 0] :=
  Module[{Λ, V, n, P, prob, state, pVerify, oVerify},
   (* Compute Orthonormal EigenSpaces *)
   {Λ, V, n} = ComputeOrthonormalEigenSpaces[Ω];
   (* Compute Projectors *)
   {P, pVerify} = ComputeProjectors[V, n];
   (* Compute Probabilities and States *)
   {prob, state} = ComputeProbStates[P, ρ, n];
   (* Get Output Tables *)
   ShowOutputTables[Ω, ρ, Λ, V, P, prob, state, n, pVerify, si, sv, so]
  ];

(*----------- Inputs ----------*)
(*Example 1*)
 ρ = {{1 / 4, -ⅈ / 12, 1 / 12, ⅈ / 12}, {ⅈ / 12, 1 / 4, -ⅈ / 12, 1 / 12},
   {1 / 12, ⅈ / 12, 1 / 4, -ⅈ / 12}, {-ⅈ / 12, 1 / 12, ⅈ / 12, 1 / 4}}; (*Density Operator*)
 Ω = {{0, -1, -ⅈ, 0}, {-1, 0, 0, ⅈ}, {ⅈ, 0, 0, 1}, {0, -ⅈ, 1, 0}}; (*Observable*)
MeasureQuantumSystem[Ω, ρ]
(*Ex (a)*)
ρ = {{1 / 4, -ⅈ / 12, 1 / 12, ⅈ / 12}, {ⅈ / 12, 1 / 4, -ⅈ / 12, 1 / 12},
   {1 / 12, ⅈ / 12, 1 / 4, -ⅈ / 12}, {-ⅈ / 12, 1 / 12, ⅈ / 12, 1 / 4}}; (*Density Operator*)
Ω = {{0, 0, 1, -ⅈ}, {0, 0, ⅈ, -1}, {1, -ⅈ, 0, 0}, {ⅈ, -1, 0, 0}}; (*Observable*)
MeasureQuantumSystem[Ω, ρ]
(*Ex (b)*)
 ρ = {{1 / 4, -ⅈ / 12, 1 / 12, ⅈ / 12}, {ⅈ / 12, 1 / 4, -ⅈ / 12, 1 / 12},
   {1 / 12, ⅈ / 12, 1 / 4, -ⅈ / 12}, {-ⅈ / 12, 1 / 12, ⅈ / 12, 1 / 4}}; (*Density Operator*)
 Ω = {{2, 0, 0, ⅈ}, {0, 2, 0, 0}, {0, 0, 2, 0}, {-ⅈ, 0, 0, 2}}; (*Observable*)
MeasureQuantumSystem[Ω, ρ]
(*Ex (c)*)
 ρ = {{1 / 4, -ⅈ / 12, 1 / 12, ⅈ / 12}, {ⅈ / 12, 1 / 4, -ⅈ / 12, 1 / 12},
   {1 / 12, ⅈ / 12, 1 / 4, -ⅈ / 12}, {-ⅈ / 12, 1 / 12, ⅈ / 12, 1 / 4}}; (*Density Operator*)
 Ω = {{5, 0, 0, 3 ⅈ}, {0, 5, ⅈ, 0}, {0, -ⅈ, 5, 0}, {-3 ⅈ, 0, 0, 5}}; (*Observable*)
```

```
MeasureQuantumSystem[Ω, ρ]

(*------------------------------*)
(* Output: In the template form *)
(*------------------------------*)
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Out[209]=

Observable	QuantumSystem
$\begin{pmatrix} 0 & -1 & -i & 0 \\ -1 & 0 & 0 & i \\ i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} \\ -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} & \frac{1}{4} \end{pmatrix}$

Out[210]=

Observable	QuantumSystem
$\begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} \\ -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} & \frac{1}{4} \end{pmatrix}$

EigenValue	EigenSpace	Projector	Probability	Stat
$-\sqrt{2}$	$\begin{pmatrix} \frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} \\ 0 & \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{2}+\frac{1}{6\sqrt{2}}$	$\begin{pmatrix} -\frac{7}{136}\left(-6+\sqrt{2}\right) & \frac{i\left(-1+\sqrt{2}\right)}{4\left(6+\sqrt{2}\right)} \\ -\frac{i\left(-1+\sqrt{2}\right)}{4\left(6+\sqrt{2}\right)} & \frac{1}{136}\left(26+7\sqrt{2}\right) \\ \frac{1}{136}\left(14-25\sqrt{2}\right) & \frac{i\left(1+3\sqrt{2}\right)}{4\left(6+\sqrt{2}\right)} \\ -\frac{i\left(1+3\sqrt{2}\right)}{4\left(6+\sqrt{2}\right)} & \frac{1}{136}\left(14+9\sqrt{2}\right) \end{pmatrix}$
$\sqrt{2}$	$\begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{i}{2} \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2} \end{pmatrix}$	$\frac{1}{2}-\frac{1}{6\sqrt{2}}$	$\begin{pmatrix} \frac{7}{136}\left(6+\sqrt{2}\right) & \frac{i\left(1+\sqrt{2}\right)}{4\left(-6+\sqrt{2}\right)} \\ -\frac{i\left(1+\sqrt{2}\right)}{4\left(-6+\sqrt{2}\right)} & \frac{1}{136}\left(26-7\sqrt{2}\right) \\ \frac{1}{136}\left(14+25\sqrt{2}\right) & \frac{i\left(-1+3\sqrt{2}\right)}{4\left(-6+\sqrt{2}\right)} \\ -\frac{i\left(-1+3\sqrt{2}\right)}{4\left(-6+\sqrt{2}\right)} & \frac{1}{136}\left(14-9\sqrt{2}\right) \end{pmatrix}$

Out[211]=

Observable	QuantumSystem
$\begin{pmatrix} 2 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -i & 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} \\ -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} & \frac{1}{4} \end{pmatrix}$