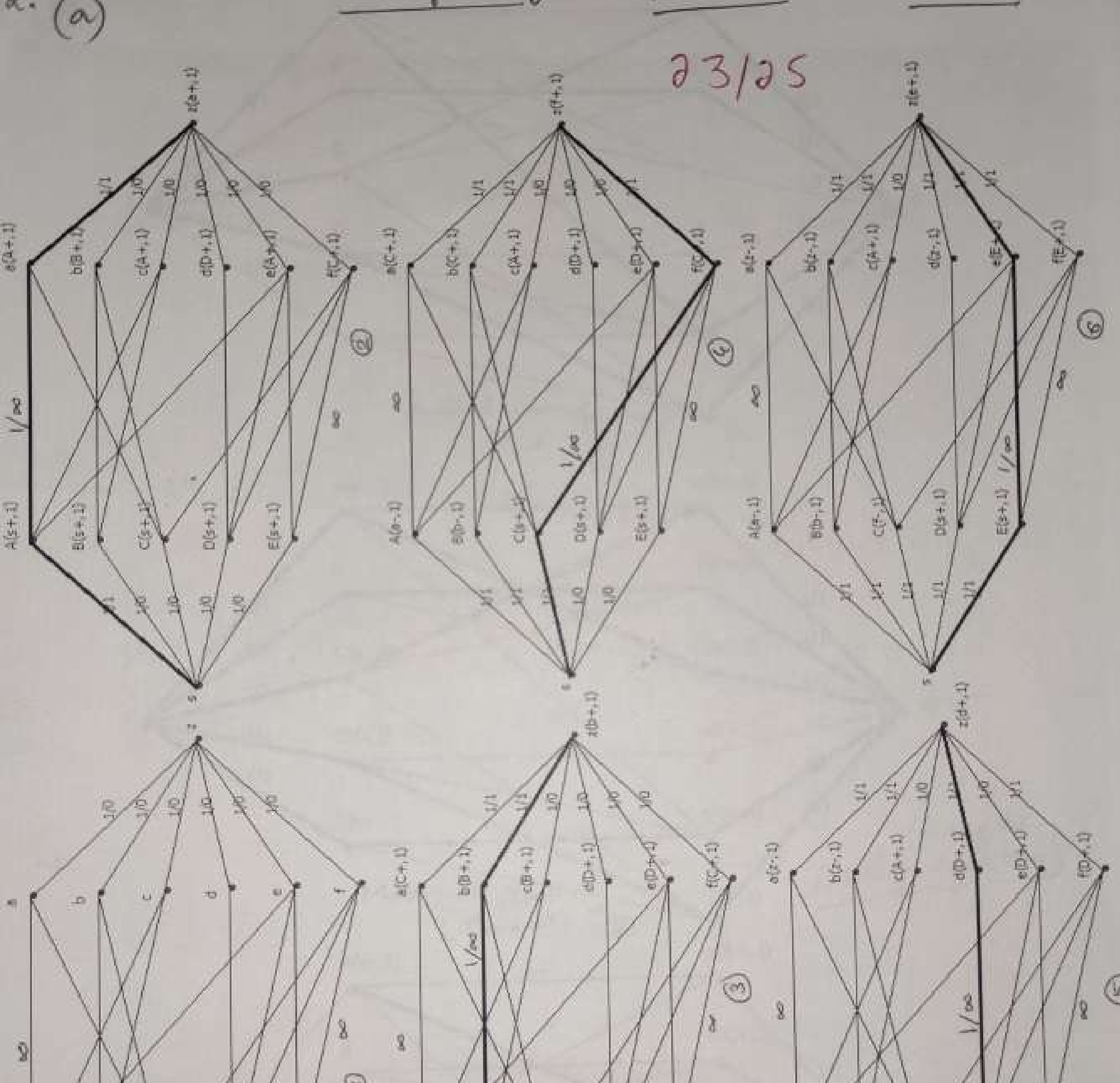
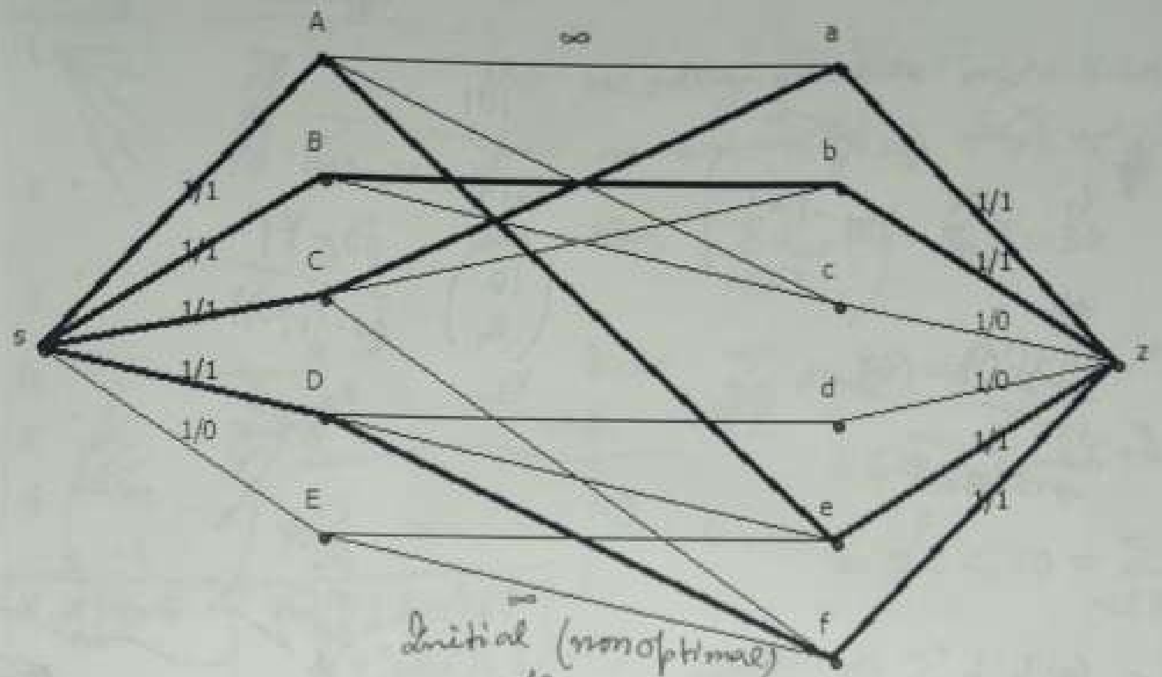
2. (a) Sandipan Dey Math 685 HW 4.4

23/25

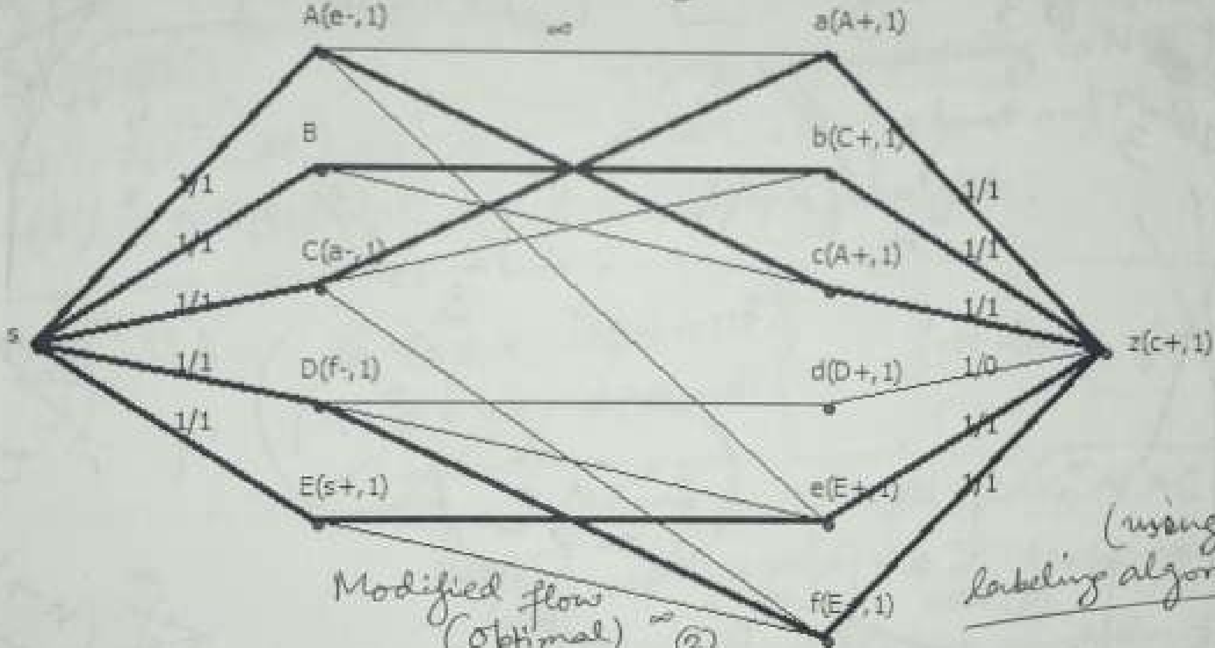


Sandipan Dey Math 685 HW 4.

. (b)

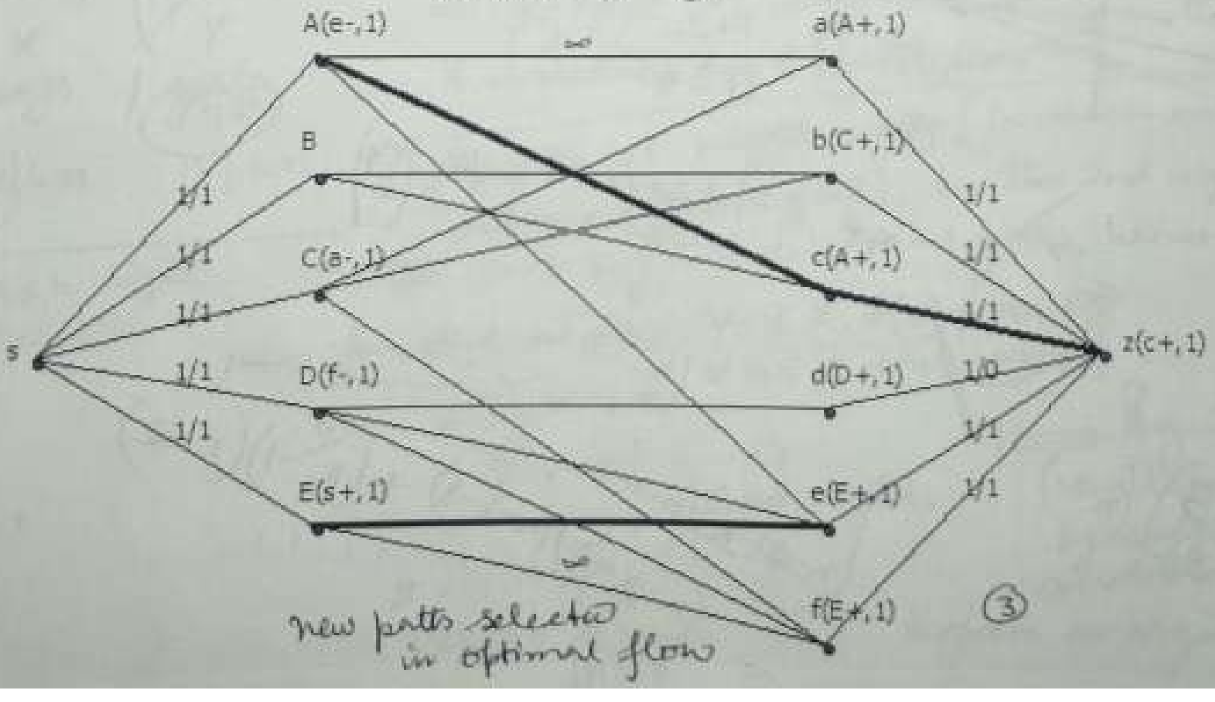


Initial (nonoptimal) flow ①



Modified flow (Optimal) ②

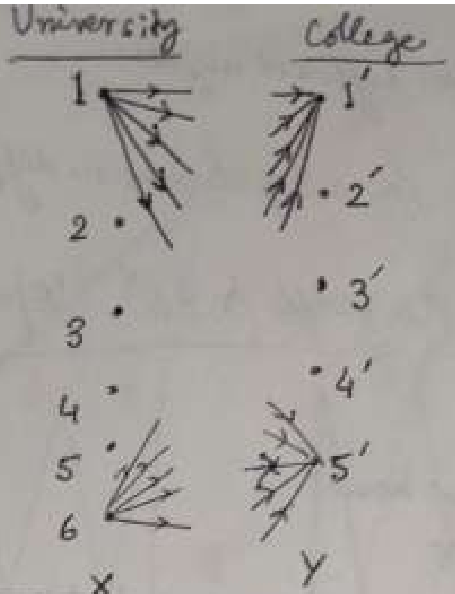
(using the labeling algorithm)



③

new paths selected in optimal flow

5.



Also, no college will hire more than one student from an university $\Rightarrow \forall v \in Y,\ d_{in}(v) \le |X| = 6$ ··· ③ (by Region hole)

~~Now~~, $\sum_{v \in X} d_{out}(v) = 6.5 = 30$

but $\sum_{v \in Y} d_{in}(v) = d(1') + d(2') + d(3') + d(4') + d(5')$

$\le 6+6+6+6+5$ (Combining ① & ③)

$= 29 < 30 = \sum_{u \in X} d_{out}(u)$

~~Hence~~ $\Rightarrow \sum_{u \in X} d_{out}(u) > \sum_{v \in Y} d_{in}(v)$

Given $\forall v \in X, d_{out}(v) = 5$	$d_{in}(1'), d_{in}(2') \le 7$; $d_{in}(3'), d_{in}(4') \le 6$; $d_{in}(5') \le 5$ ··· ①

Combining ① ③ ②,

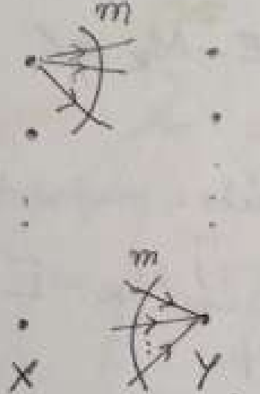
$d(1'), d(2'), d(3'), d(4') \le 6$

$d(5') \le \min(5,6) = 5$

$\Rightarrow$ perfect X matching is NOT possible

$\Rightarrow$ NOT all (at least one) PhD. student will not get a job.

11.



$|X| = m \quad |Y| = n$

1st night

$\left.\begin{array}{l} \forall v \in X,\ d(v) = m \\ \forall u \in Y,\ d(u) = m \end{array}\right\}$ $(m < n)$. for 1st night

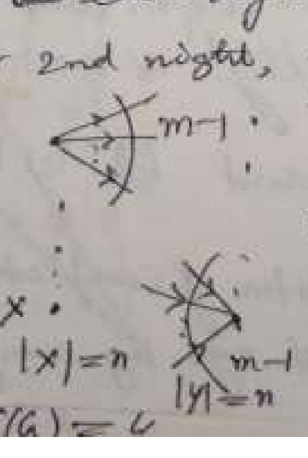
$\exists$ a X matching for 1st night since $\forall A \subseteq X$,

$\sum_{v \in A} d_{out}(v) = m|A| \le \sum_{u \in R(A)} d_{in}(u) = m|R(A)|$

$\Rightarrow |R(A)| \ge |A|$, sufficient condition for X matching due to Hall's theorem.

After the 1st night is over (in which each boy dates a different girl), in the 2nd night ~~the~~ each boy can date $m-1$ girls, hence

for 2nd night, $\left.\begin{array}{l} \forall v \in X,\ d(v) = m-1 \\ \forall u \in Y,\ d(u) = m-1 \end{array}\right\}$.



$|X| = n$ $|Y| = n$

2nd night.

~~Hence~~ Again, $\exists$ an X-matching by sufficient condition of Hall's theorem, as argued earlier.

$X(G) = 4$

Hence we propose the following algorithm for m different nights:

Initialize: $\forall v \in X$ choose m different $u \in Y$ (as a boy chooses m different girls)
$d(v) = m, \ \forall v \in X$
$d(u) = m, \ \forall u \in Y$. /* the graph (X, Y, E), $|E| = mn$ */

for night $k = 1$ to m do
/* at the begining of kth night, we have
$d(v) = m-k+1, \ \forall v \in X$
$d(u) = m-k+1, \ \forall u \in Y$, easy to prove by induction */
begin /* for */
1. Find a (complete) X-matching (containing n pairs) M_k in (X, Y, E)
/* one such matching always exists by Hall's theorem, since as argued, $|R(A)| \geq |A| \ \forall A \subseteq X$ */
2. Delete n pairs from the graph (X, Y, E) to create
current X-matching found.
new graph $(X, Y, E - M_k)$ /* $E \leftarrow E - M_k$ */
end /* for */

Each iteration k (corresponding to kth night) selects a new matching M_k with $|M_k| = n$, $\forall k = 1, .., m$. Also, $\bigcup_{k=1}^{m} M_k = E$

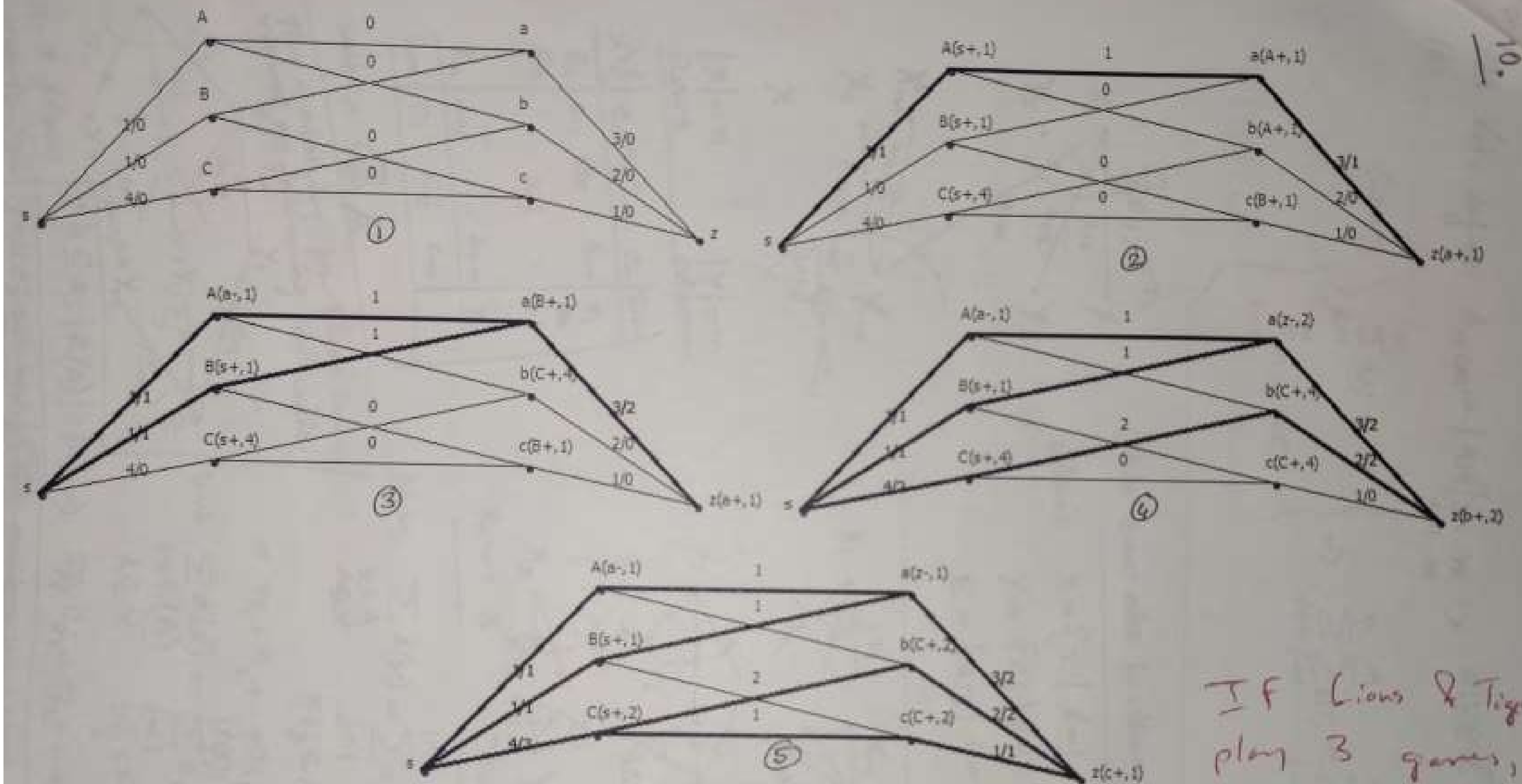
Moreover since M_k is deleted at each step,
$M_i \cap M_j = \phi$, $i \neq j$ (hence a partition)
$i, j = 1, .., m$
(proved)

(b) As follows from the above algorithm, the matchings M_k, even if selected arbitrarily $\forall k = 1, .., m$,

(each iteration deletes n new edges $\Rightarrow$ m iterations delete all the mn edges)

$\bigcup_{k=1}^{m} M_k = E$ and $M_i \cap M_j = \phi$ whenever $i \neq j$, $i, j = 1, .., m$
$\Rightarrow$ a partition of complete matchings M_k can always by completed (Proved)

10.



A : L	a : LT
B : T	b : LV
C : V	c : TV

As can be seen from the figure, there are 3 simultaneous winners (Bear, Lion and Tiger), i.e. there are 2 other co-winners (namely Lion and Tiger) along with bear, since both of them saturate the flow and end up with 26 wins. Vampire, however ends with 22+3=25 wins. There is No solo winner.

If Lions & Tigers play 3 games, one of them must have 2 wins
=) one team has at least 27 wins

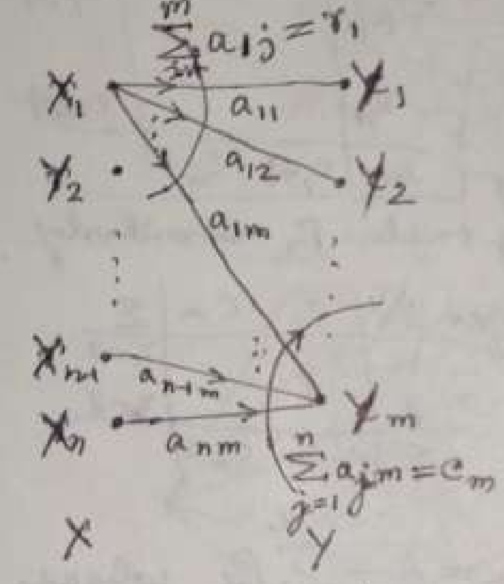
X

3/5

12. Let's define $A_{n\times m}=[a_{ij}]_{\substack{i=1..n\\ j=1..m}}$, $a_{ij}\in \mathbb{Z}^+\cup\{0\}$, $\forall i,j$ (by condition)
(a)

10/10

Given, $\left\{\begin{array}{l}\sum_{j=1}^{m} a_{ij}=r_i\\ \sum_{i=1}^{m} a_{ij}=c_j\end{array}\right\} \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}=\sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}=\sum_{i=1}^{n} r_i=\sum_{j=1}^{m} c_j.$



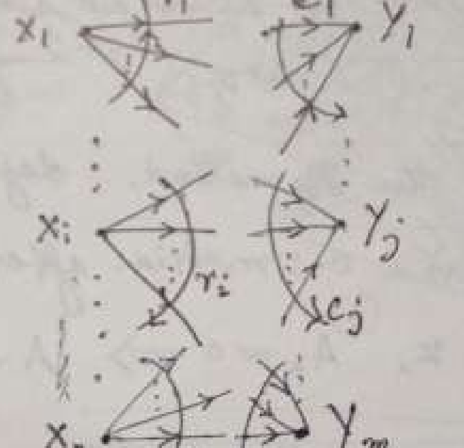
$|X|=n$ Rows $\quad |Y|=m$ columns

This can ~~als~~ be thought as (X, Y, E) matching

where $X=\{r_i \mid i=1,\dots,n\}$
$Y=\{c_j \mid j=1,\dots,m\}$
$E=\{a_{ij} \mid i=1,\dots,n,\ j=1,\dots,m\}$

X \ Y	c_1 c_2 ...	c_m	Sum
R_1	a_{11} ...	a_{1m}	r_1
R_2	a_{ij}		r_i
⋮			⋮
R_n	a_{n1} ...	a_{nm}	r_n
Sum	c_1 ... c_j ...	c_m	

matrix A



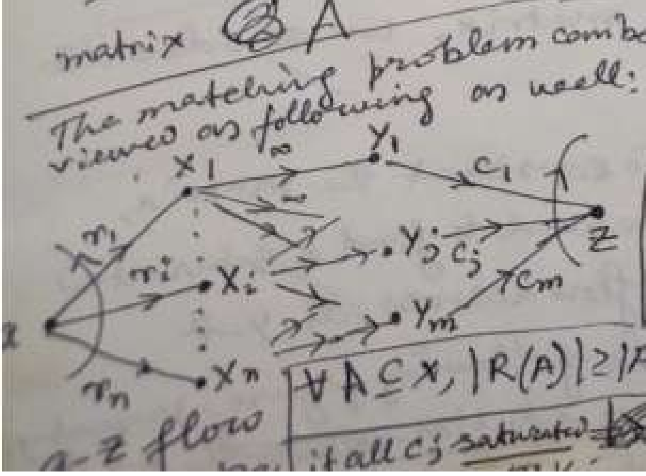
$d(X_i)=r_i, \forall i=1..n$
$d(Y_j)=c_j, \forall j=1..m$

In order that such a matching always exists, ~~the~~ the sufficiency condition by Hall's theorem must always be satisfied $\Rightarrow$
$\forall A\subseteq X,\ |R(A)|\geq|A|$

Now, $\sum_{\substack{x\in A\\ A\subseteq X}} d(x)=\sum_{\substack{j=1\\ x_{ij}\in A}}^{|A|} r_{ij}$
$= r_{i_1}+r_{i_2}+\dots+r_{i_{|A|}}$

but $\sum_{\substack{y\in R(A)\\ A\subseteq X}} d(y)=\sum_{\substack{j=1\\ y_{ij}\in R(A)}}^{|R(A)|} c_{ij}=c_{j_1}+c_{j_2}+\dots+c_{j_{|R(A)|}}$

The matching problem can be viewed as following as well:



a-z flow

$\forall A\subseteq X, |R(A)|\geq|A| \Rightarrow c_{j_1}+c_{j_2}+\dots+c_{j_{|R(A)|}} \geq r_{i_1}+r_{i_2}+\dots+r_{i_{|A|}}$ $\forall A\subseteq X$

if all c_j saturated ~~...~~ and $\sum r_i=\sum c_j$

(15) Now, if $\exists$ an X matching, we shall have the following contigency table

$X \backslash Y$	C_1	C_2	...	C_m	Sum
R_1	0	1	...	0	1
R_2	1	0	...	0	1
$\vdots$					$\vdots$
R_n	0	0	...	1	1
Sum	1	1	...	1	

P_1

which will be a 0-1 matrix with each row, each column having exactly one 1 (corresp. to the matching $r_i \to c_j$) and rest of them

Now, if we ~~assign A to A−P~~ define $A_1 = A - P_1$, we get with row sum $r_i - 1$ and col sum $c_j - 1$.

$X \backslash Y$	C_1 ... C_m	Sum
R_1		$r_1 - 1$
$\vdots$		$\vdots$
R_m		$r_m - 1$
Sum	$c_1 - 1$ ... $c_m - 1$	

If still another matching exists, it can be ~~If we repeat the process~~ expressed by contingency table P_2 similarly.

Now define $A_2 = A_1 - P_2 = A - P_1 - P_2$ to get with row sum $r_i - 2$ and col sum $c_j - 2$ $\forall i, j$.

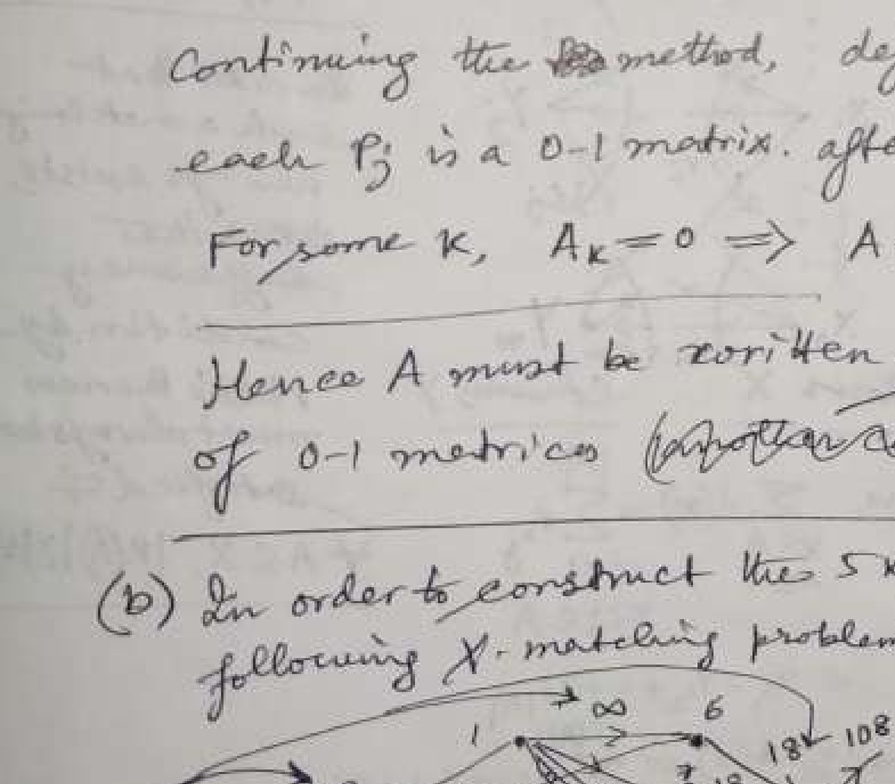
$X \backslash Y$	C_1 ... C_m	Σ
R_1		$r_1 - 2$
R_n		$r_n - 2$
Σ	$c_1 - 2$ ... $c_m - 2$	

Continuing the method, define $A_K = A - \sum_{j=1}^{K} P_j$ where each P_j is a 0-1 matrix. after K-th iteration.

For some K, $A_K = 0 \Rightarrow A = \sum_{j=1}^{K} P_j$ (Similar to the condition obtained for Birkhoff's theorem)

Hence A must be written as sum of 0-1 matrices (~~another constraint~~).

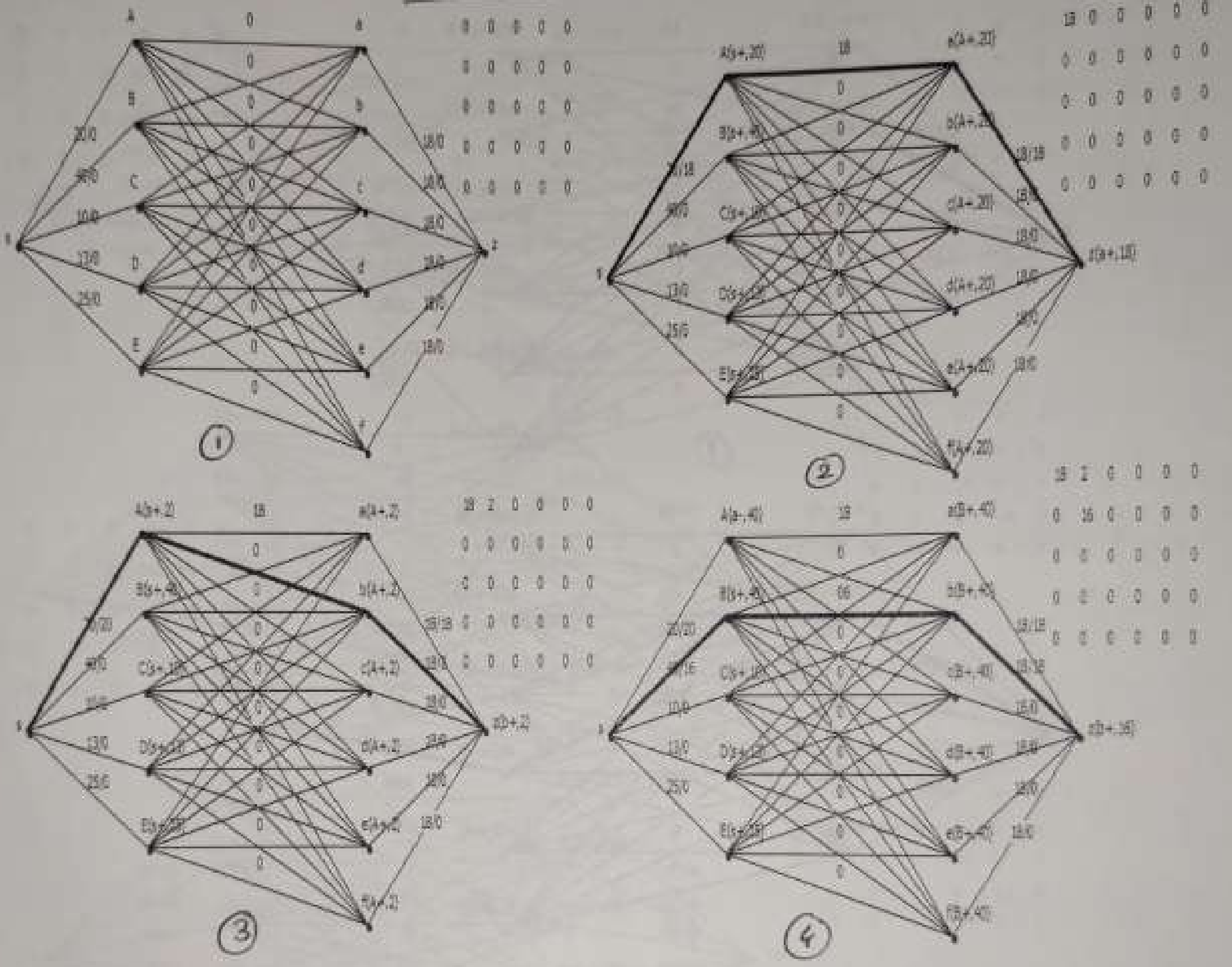
(b) In order to construct the 5×6 matrix, we solve the following X-matching problem:



To construct the matrix, we need to find a saturated flow accross all $Y - Z$

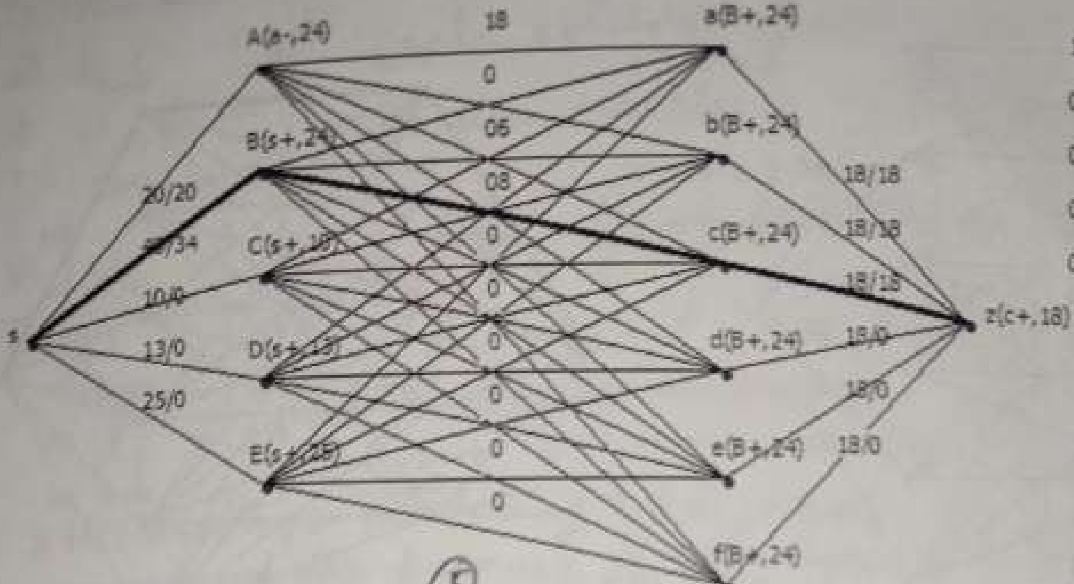
The following figure presents the solution

The 5×6 matrix



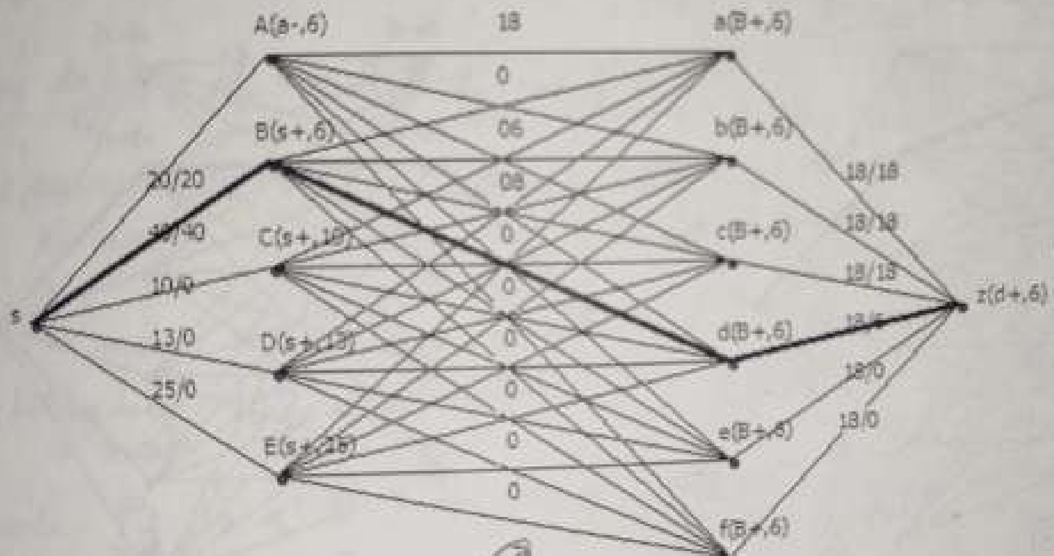
Different steps of the augmenting flow labeling algorithm

The 5×6 matrix



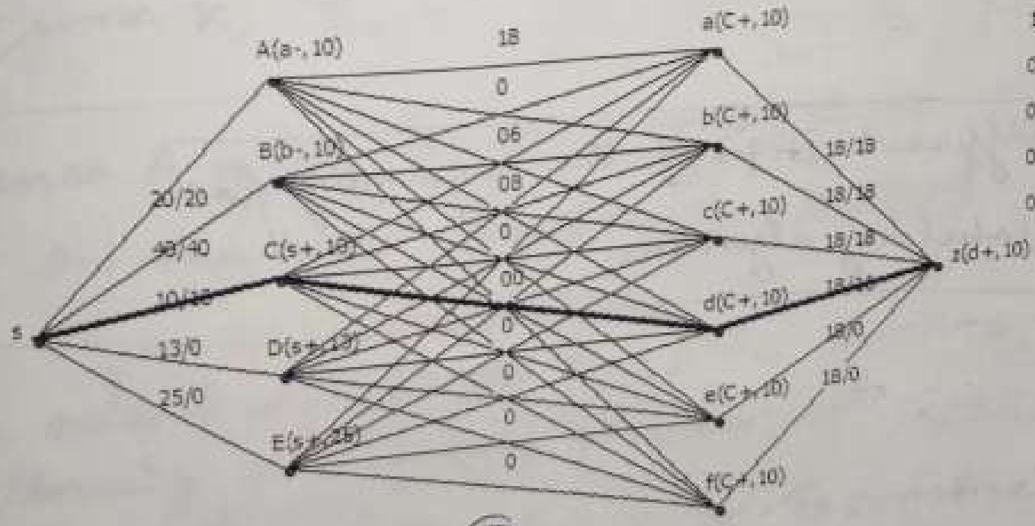
18	2	0	0	0	0
0	16	18	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

5



18	2	0	0	0	0
0	16	18	6	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

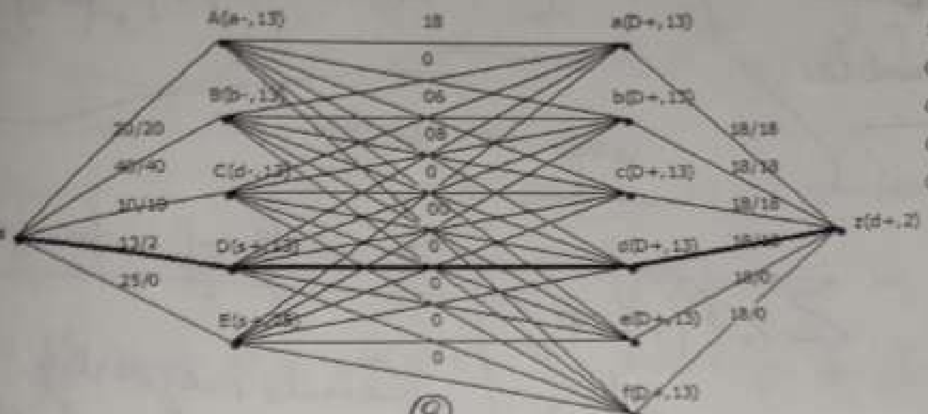
6



18	2	0	0	0	0
0	16	18	6	0	0
0	0	0	10	0	0
0	0	0	0	0	0
0	0	0	0	0	0

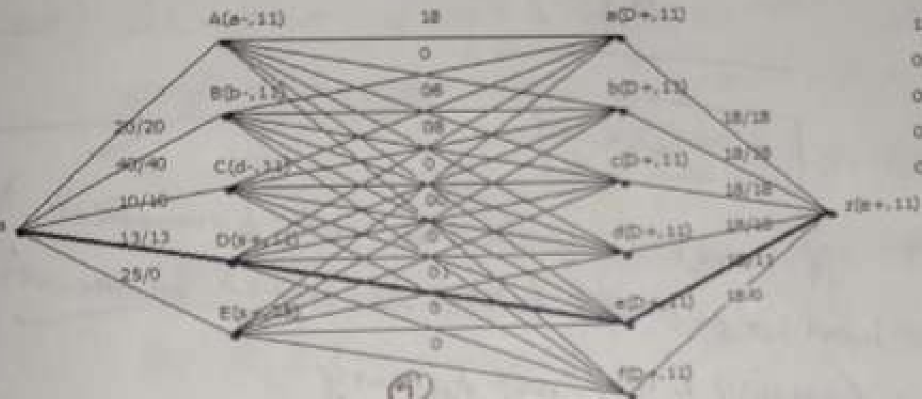
7

The 5×6 matrix



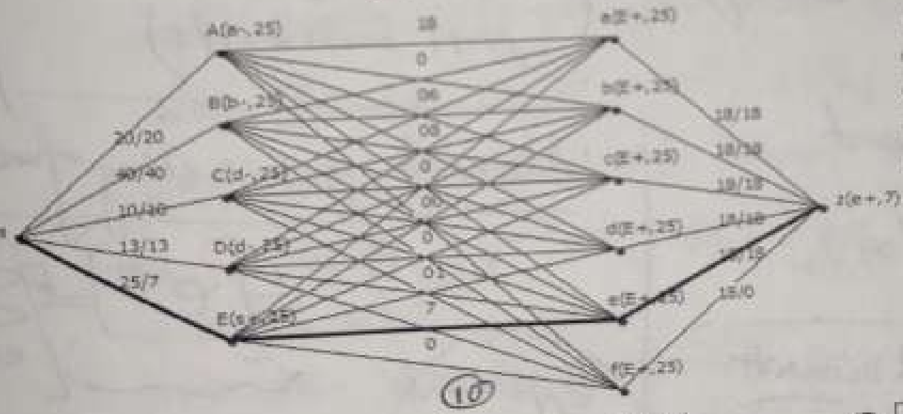
(8)

18	2	0	0	0	0
0	16	18	6	0	0
0	0	0	10	0	0
0	0	0	2	0	0
0	0	0	0	0	0



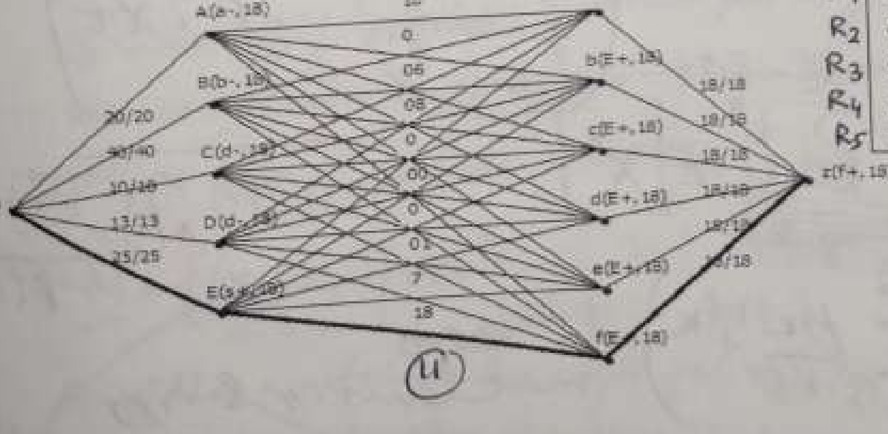
(9)

18	2	0	0	0	0
0	16	18	6	0	0
0	0	0	10	0	0
0	0	0	2	11	0
0	0	0	0	0	0



(10)

18	2	0	0	0	0
0	16	18	6	0	0
0	0	0	10	0	0
0	0	0	2	11	0
0	0	0	0	7	0



(11)

	C_1	C_2	C_3	C_4	C_5	C_6
R_1	18	2	0	0	0	0
R_2	0	16	18	6	0	0
R_3	0	0	0	10	0	0
R_4	0	0	0	2	11	0
R_5	0	0	0	0	7	18

The final 5×6 matrix

$G = (V, E)$, a directed graph. (given)

Let's construct $G' = (X, Y, E')$, a bipartite graph, where

$X = Y = V[G]$, $\forall \overrightarrow{(v_1, v_2)} \in E[G]$, $(x_1, y_2) \in E[G']$

Hence, G has a circuit or a set of vertex-disjoint circuits that pass through each vertex once in the directed graph G iff there is a complete X-matching in G'.

Proof ($\Rightarrow$) Let's assume G has a single directed cycle passing through each of its vertex exactly once: w.l.o.g. let the circuit be $v_{k_1} \to v_{k_2} \to \cdots \to v_{k_{|V|}} \to v_{k_1}$, where $k_1, k_2, \ldots, k_{|V|}$ is just a permutation of $1, 2, \ldots, |V|$. Consider bipartite graph G' now.

For $v_{k_i} \in X$ and $v_{k_{i+1}} \in Y$, we have $\overset{e_i}{(v_{k_i}, v_{k_{i+1}})} \in E[G']$ $\forall i = 1, 2, \ldots, |V|$ and moreover any two edge e_i and e_j don't share any endpoint, since $\nexists$ an edge in between v_{k_i} & v_{k_j} in X $\forall i, j$. (Similarly for Y, since G' is bipartite). Hence, $\forall i \neq j$, $e_i \cap e_j = \emptyset$ and hence we have a matching $M = \{(v_{k_i}, v_{k_{i+1}}) \mid v_{k_i} \in X, v_{k_{i+1}} \in Y, \forall i = 1, 2, \ldots, |V|\}$ with $|M| = |V|$

$\Rightarrow \exists$ an X-matching in G'.

Similarly, if ~~there~~ $\exists$ multiple edge disjoint circuits in G, we can show that $\exists$ a complete X-matching in G'.

($\Leftarrow$) Let G' have a complete X-matching. But $X = V[G]$, which means

$M = \{(v_k, v_{k'}) \mid \forall k = 1, 2, \ldots, |V|, \exists k' \in \{1, 2, \ldots, |V|\}\}$

Since M is a matching the edges $(v_1, v_{1'}), (v_2, v_{2'}), \ldots, (v_{|V|}, v_{|V|'})$ are all disjoint. Also it implies $\exists$ edges $\overrightarrow{(v_1, v_{1'})}, \ldots, \overrightarrow{(v_{|V|}, v_{|V|'})} \in E$

$\{v_{1'}, v_{2'}, \ldots, v_{|V|'}\}$ is a permutation of $\{v_1, v_2, \ldots, v_{|V|}\}$. Hence $v_{1'} = v_i \in V$ and $v_1 \to v_i \in E$. Again from M find the ~~edge starting~~ vertex in Y that pairs with v_2 (which is $v_{2'}^{i} = v_j$ for some j), hence we have a directed path $v_1 \to v_i \to v_j$. In this way we go on building the path and will stop when all the vertices in $\underline{V}$ are present in the path. ~~If some vo G is connected, we get just a single circuit containing all |V| vertices (fo~~

When G is disconnected: $v_1 \to v_i \; v_j \to \cdots \to v_{|V|}$

When G is connected: $v_1 \to v_i \to v_j \to \cdots \to v_{|V|}$

Obviously we shall get a single circuit or a set of circuits (disjoint) containing all the vertices (since we have complete X-matching)

✓ 10/10

Now, by Hall's marriage theorem, the necessary and sufficient condition for complete X-matching in this case is $|R(V')| \geq |V'|$, ~~$\forall |V'|$~~ $\forall V' \subseteq X = V[G]$, i.e., for any arbitrary (non-empty) subset V' of the vertex set $V(G)$ of G, total number of vertices adjacent to any vertex in $V' \subseteq V$ must be at least the size of V'!

22. (a) $G = (X, Y, E)$ is a bipartite graph

By König-Egervary theorem, we have:

size of maximum matching = size of minimum edge cover $= \tau(G)$ (let).

$\delta(G)$ = defficiency of $G = \max_{A \subseteq X} (|A| - |R(A)|)$

It's obvious that the graph G has a perfect X-matching if $\delta(G) = 0$

Clearly, $\underbrace{\delta(G)}_{\text{maximum \# vertices unmatched}} = |X| - \underbrace{\tau(G)}_{\text{size of maximum matching}}$

Also, in any graph G we have, $\alpha(G) + \tau(G) = |V| = |X| + |Y|$ where $\alpha(G)$ = size of maximum independent set of G

Let's prove this first

$\forall$ graph $G=(V,E)$, $\quad \underbrace{\alpha(G)}_{\text{size of max}^m \text{ independent set}} + \underbrace{\tau(G)}_{\text{size of min}^m \text{ edge cover}} = |V|$

Let I be ~~...~~ max independent set of $G=(V,E)$ $\quad (\forall u,v \in I, (u,v) \notin E[G])$

$\Rightarrow V-I$ is a ~~...~~ edge cover (since $\forall (u,v) \in E[G]$, u or v or both $\in V-I$)
with size $|V|-|I|$

But ~~...~~ $\alpha(G)$ is size of max independent set $\Rightarrow |I| = \alpha(G)$

also, $\underbrace{\tau(G)}_{\text{size of min}^m \text{ edge cover}} \leq \underbrace{|V|-|I|}_{\text{size of edge cover } V-I}$, by minimality.

$$\Rightarrow \tau(G) \leq |V| - \alpha(G) \quad \cdots ①$$

Also, V' be min edge cover of $G=(V,E)$ $\quad (\forall (u,v) \in E[G]$ u or v or both $\in V')$

$\Rightarrow V-V'$ is an independent set $(\forall u,v \in V-V', (u,v) \notin E[G]$ ow V' is not an edge cover, not covering $(u,v))$

But $\tau(G)$ is size of min edge cover $\Rightarrow |V'| = \tau(G)$.

also, $\underbrace{\alpha(G)}_{\text{size of max}^m \text{ independent set}} \geq \underbrace{|V|-|V'|}_{\text{size of independent set } V-V'}$, by maximality

$$\Rightarrow \alpha(G) \geq |V| - \tau(G)$$

$$\Rightarrow \tau(G) \geq |V| - \alpha(G) \quad \cdots ②$$

$① \,\&\, ② \Rightarrow \tau(G) = |V| - \alpha(G) \Rightarrow \alpha(G) + \tau(G) = |V|$

$\Rightarrow \alpha(G) + \tau(G) = |X| + |Y| \Rightarrow \alpha(G) = |Y| + (|X| - \tau(G))$

(i) size of min edge cover

= size of max^{m} matching

(by König-Egervary theorem)

$\Rightarrow \delta(G) = |X| - \varepsilon(G)$

|defficiency| |max^{m} matching|

$\Rightarrow \alpha(G) = |Y| + \delta(G)$

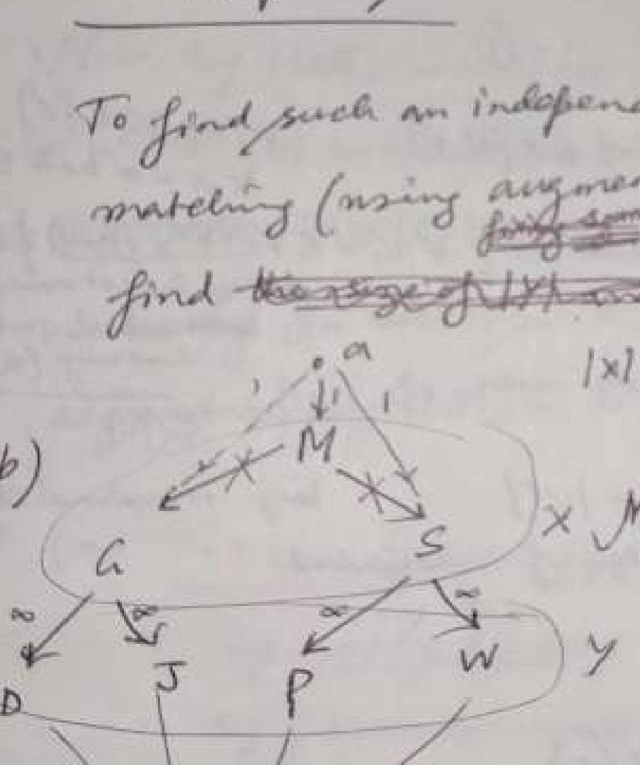
size of maximum independent set — defficiency

To find such an independent set, find $\varepsilon(G)$, ~~a minimum~~ a maximum matching (using augmented labeling flow algorithm) size and ~~find something every other vertex~~

find ~~the size of |Y| and~~ $\delta(G)$. Then add $|Y|$ to it to find $\alpha(G)$

$|X| - '' \varepsilon(G)$

(b)



$M' = \{\{G, D\}, \{S, W\}\}$ can be a maximum matching with size 2.

Hence, $X - \{G, S\} = \{M\}$

Hence $I = Y \cup \{X - \{G, S\}\}$

$= \{D, G, P, W\} \cup \{M\}$

$= \{M, D, G, P, W\}$ is a maximum independent set with $\alpha(G) = 5$

Ignore edges in X, since bipartite

$X = \{M, G, S\}$, $Y = \{D, G, P, W\}$

but $\tau(G)$ = size of min^m edge cover

= size of max^m matching

(by König-Egevary theorem)

$\Rightarrow$ $\delta(G) = |X| - \tau(G)$

|deficiency| |max^m matching|

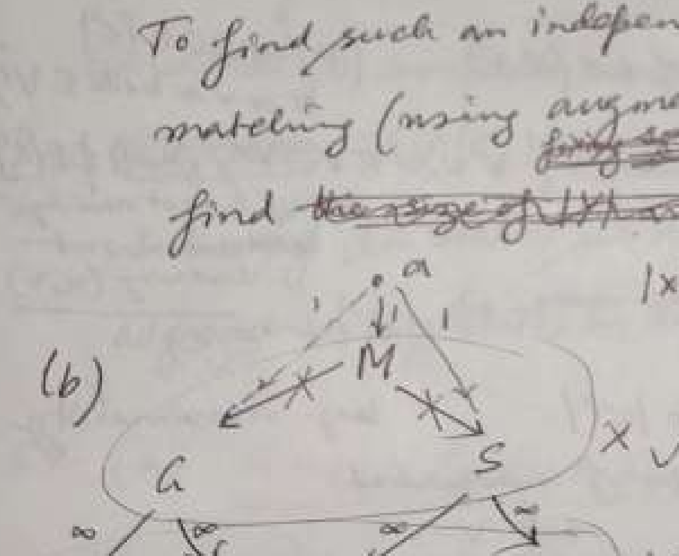
$\Rightarrow$ $\alpha(G) = |Y| + \delta(G)$

size of maximum independent set   deficiency

To find such an independent set, find $\tau(G)$, a maximum size matching (using augmented labeling flow algorithm) and find ~~the size of |Y| and~~ $\delta(G)$. Then add $|Y|$ to it to find $\alpha(G)$

$|X| - \tau(G)$

(b)



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Hence $I = Y \cup \{X - \{G, S\}\}$

$= \{D, G, P, W\} \cup \{M\}$

$= \{M, D, G, P, W\}$ is a maximum independent set with $\alpha(G) = 5$

Ignore edges in X, since bipartite

$X = \{M, G, S\}$, $Y = \{D, G, P, W\}$