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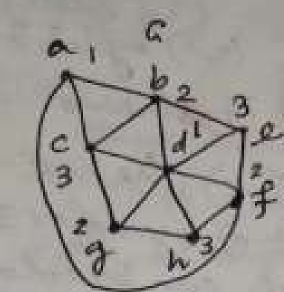
Sandipan Dey Math-685 HW-4

2.3.

1. (a)



abc forms K_3, we need at least 3 colors. (Hence at least 3 colors are needed)

d must have different color from all of b, c, g, h, f, e

f must have different color from a.

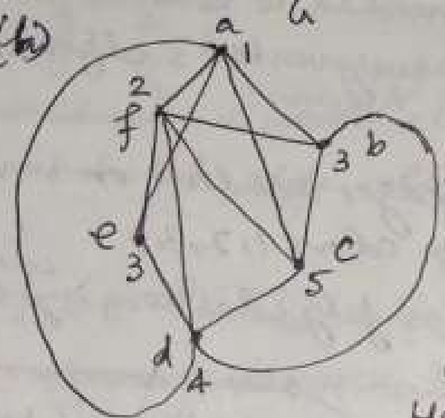
Since b c g h f e is an even-length, # colors required to color (unforced) the cycle = 2. After coloring the cycle we see that the coloring does not lead to any violation if we color d by color 1.

$\Rightarrow \chi(G) = 3$

$\chi(G) \geq 3$ ($\because \exists K_3$ abc)

$\chi(G) \leq 4$ ($\because$ G planar)

(b)



There are triangles (K_3)

aef, efd, acf, bcd, afb.

So we need at least 3 colors to color G.

Let's start coloring G from afb.

a, b must have different color from d.

Hence color of d can't be 1 or 3

$\chi(G) = 5$.

(G is non planar)

~~If color of d is 1, then color of e must be 3~~ can't be 1, 2.

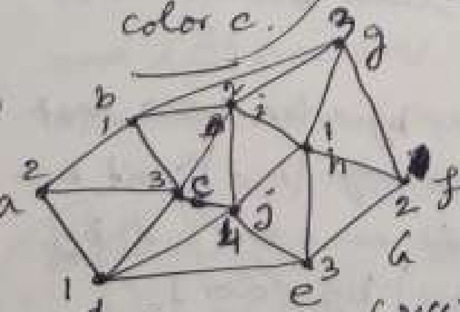
Also color of d can't be 2 since def is a triangle and f is already colored by 2, hence we need 4th color

color of e can't 1, 2, 4, it can be colored with 3.

Now, what remains is to color c. color of c can't be 1, 2, 3, 4 (since a, f, b, d all are adjacent to c. Hence a 5th color is needed to color c.

$\Rightarrow \chi(G) = 5$

(c)



G is planar ($\chi(G) \leq 4$)

G has triangles (e.g. abc, $\chi(G) \geq 3$).

Start with abc by coloring a, b, c by colors 2, 1, 3 respectively.

Now j is adjacent to c, d. can be colored with 2

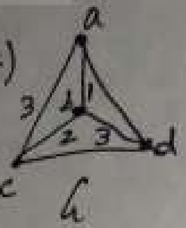
i is adjacent to b, j, c must be colored with 4

g is adjacent to b, i, h can be colored with 3

h is adjacent to {i, g, e, j} can be colored with 1

Now f is adjacent to g, h, e colored previously with colors 3, 1, 3 respectively. Hence we ~~need a third~~ can ~~color to~~ color f by 2. Hence, $\chi(G) = 4$

2. (a)

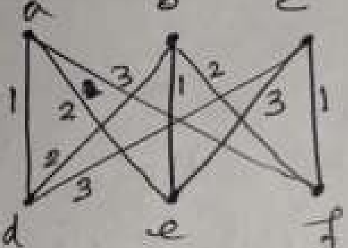


G has K_3's (abc, abd, bcd) ⇒ Needs at least 3 colors to color edges.

Start with abc. Color (a,b), (b,c), (c,a) by colors 1, 2, 3 respectively.

Now, if we consider the edge (b, d) it shares common endpoints both with (a, b) and (b, c), hence can't be colored with color 1 or 2, let's color (b, d) by color 3. The only remaining edge is (a, d) which can be colored by color 1 then. Hence minimal edge coloring is 3.

(b)



$K_{3,3}$

Every vertex in $K_{3,3}$ has degree 3. Hence at least 3 colors are needed to color the edges of $K_{3,3}$ (since at every vertex 3 edges share that vertex).

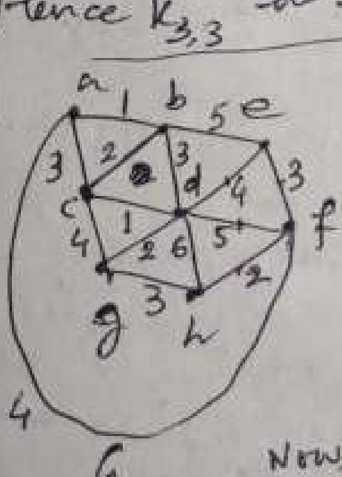
Start with the edges incident on vertex a and color them by colors 1, 2, 3.

Now, consider the edges (d, b), (b, e) and safely color them by colors 2, 3.

Next, consider the edge (b, e) and since it shares common endpoints with both the edges (a, e) and (b, d), it can't be colored by color 2. Color it by color 1 and (b, f) by color 2. Also, we can color (e, c) safely by color 3. Now, the only edge remaining is (c, f) that can be colored by color 1.

Hence $K_{3,3}$ has a minimal edge coloring of 3.

c)



G has vertex 'd' with degree 6, at least 6 colors will be needed to color G.

Color (a, b), (b, c), (c, a) by colors 1, 2, 3 resp.

Now in K_3 bcd, the edge (b, d) can't be colored by color 1, 2, hence color (b, d) by color 3 and color (c, d) by color 1.

Now, d has degree 6. Hence all edges incident to d must be colored with a different color.

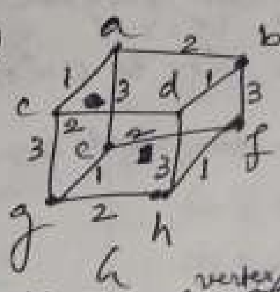
Now, consider other edges and color accordingly to get a minimal edge coloring with 6 colors.

2. (d) 

~~Any~~ Minimal edge coloring must have 4 colors. since $deg(a) = 4$.

coloring edges incident upon a by 4 diff colors 1, 2, 3, 4, color (b, c), (c, d) by colors 1, 4 ~~resp~~.

Then color (b, c), (c, f) by colors 1, 2 resp. and (c, d), (c, f) by colors (4, 2) resp. coloring in this way we obtain a minimal edge coloring with 4 different colors.

(e) 

Any minimal edge coloring of G must ~~have~~ use 3 colors since all vertices are of degree 3.

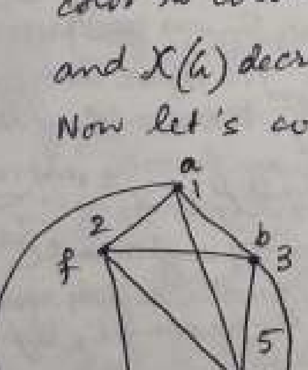
Starting with vertex c and coloring the edges ~~adjacent~~ incident upon it by colors 1, 2, 3 resp., ~~while~~ and considering each vertex & eliminating the choice of colors we get the minimal edge coloring as shown.

3. (a) As shown in 1(a), $\chi(G) = 3$ and deletion of any vertex does not decrease $\chi(G)$. For instance, even after deletion of vertex a, $\chi(G - \{a\}) = 3$. Hence, G is NOT color critical.

(b) As shown in 1(b), $\chi(G) = 5$.

It's easy to see that if any of the vertices a, b, d, f (that are adjacent to c) are removed, we no longer need the 5th color to color the vertex c (since degree of c ~~degree~~ decreases). and $\chi(G)$ decreases.

Now let's consider the case when the vertex e is removed $(G - \{e\})$



$G - \{e\}$

~~Since deg(c) = 4~~ Now color of d in $G - \{e\}$ can't be 1, 2, 3 (adjacent to each of a, b, f). Hence it must be a different 4th color.

Again, c being adjacent to all of a, b, d, f, c must be colored with yet another different color 5. Hence, $\chi(G - \{e\}) = 5$. does not decrease

$\Rightarrow$ G is not color critical.

(n) As shown in 1(n), $\chi(G) = 4$.

It's easy to see that, if we remove the vertex j which is ~~only~~ colored using the 4th color, ~~or exactly one from any one of the adjacent vertices 2, 3~~, or its adjacent vertex i, we no longer need the 4th color.

But what if we remove the vertex e, we can still see that the graph $G - \{e\}$ still requires the 4th color to color the vertex j (since it's still adjacent ~~to the~~ vertices d, c, i, colored with 1, 3, 2 resp.). Hence,

$\chi(G - \{e\}) = 4 \Rightarrow G$ is not color critical.
$= \chi(G)$

8. First let's assume that no two circles are going to touch each other, or they may not be 2-colorable as can easily be seen from the following example:

(3 circles touching each other)

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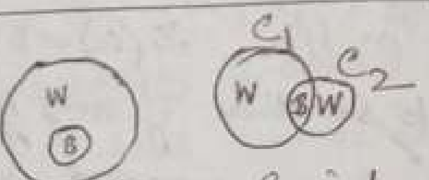
Proof (a) Induction on the # of intersecting circles n

$n = 2$ (Base case).



Ruling out this case since circles don't intersect even then it can be 2 colored: color the smaller circle by B outer part by W.

C_2 intersects C_1 and divides it into 2 regions. Let's say C_1 was previously colored W. After intersecting with C_2, now C_1 has couple of regions: one away from C_2, and not adjacent another is adjacent to C_2. Let's color the region of C_1 adjacent to C_2 by B, the other regions remains W and color the remaining C_2 by W

Induction step

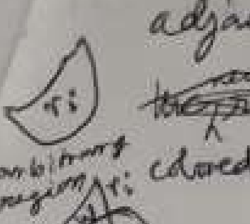
Assume the result be true for $\forall n \leq m$

Let's prove for the case $n = m+1$.

Then the new circle C_{m+1} intersects some of the existing ones $C_1, C_2, .., C_m$. Let the existing regions that are contained by some (or all) of the existing circles be $r_1, r_2, .., r_m$.

When C_{m+1} intersects r_i it again divides r_i into 2 parts: one adjacent to r_i and another will be away from C_{m+1}. ~~We region colors~~

Start with $i=1$.



arbitrary region r_i

after intersecting with C_{m+1}

r_iS: adjacent to C_{m+1}

~~the region adjacent to C_{m+1} by~~ If in the previous step r_i was colored with W, color the region r_iN (away from C_{m+1}, not adjacent to C_{m+1}) remains W still, but color of the intersection region with C_{m+1} (r_iS) is to be colored by B and color remaining region of C_{m+1}, i.e., $(C_{m+1} - r_i)$ by W. Hence, the general algorithm for coloring will be:

$\forall i = 1..m$ // for all regions obtained in coloring with m intersecting circles

1. Choose region r_i
2. If r_i is intersected by C_{m+1} and divided into two parts r_iN and r_iS, don't change the color of r_iN, but color $r_iS = r_i \cap C_{m+1}$ with the other color and then color $C_{m+1} - \bigcup_{j=1}^{i} r_j$ with the same color with which r_iN was colored.

This new coloring scheme ensures that new regions created by C_{m+1} preserves the property that the adjacent regions are colored with the different colors.

(b) Choose the following coloring scheme:
color all the regions that are contained by odd number circles by color W
" " " " " " " even " " " " B

Now let's show that this scheme always gives us a desired 2-coloring:

This can be immediately be proved by observing that any two adjacent regions in the intersecting circles can't have the same parity in terms of # circles containing that region.

proper coloring

e.g. the region contained by 4 circles (even parity) can only be surrounded by the regions contained by 3 circles (odd parity) and this is due to the assumption that none of the circles touch each other.

So moving from outside in color a region by W if # circles containing the region % 2 = 1, and 0 otherwise. This guarantees a proper coloring

[redacted]

10. Let's represent each different animal by different vertex
(e.g., tiger represents a vertex and deer another one). In
general if there are n different elements, $|V(G)| = n$, with
$v_i \in V(G)$ representing ith animal.

Two ~~animal~~ different animals can't live together peacefully
$\Rightarrow$ They must be put in different Areas.

Let's model the edges of graph G: $(v_i, v_j) \in E(G)$ iff $i \neq j$ and
ith animal and jth animal can not live together peacefully.
(e.g., tiger vertex and deer vertex are connected by an edge in between)
Then immediately the problem of assigning different areas
to different animals that can't live together becomes
equivalent to coloring the vertices of the graph with ~~diff~~ the ~~constraint~~
usual constraint that no two adjacent vertices can be
colored with the same color. Here, color corresponds to
assigning an 'area' to an animal. So different animals
can't live together $\Leftrightarrow$ they are adjacent vertices in the graph G
$\Leftrightarrow$ they must be colored with different colors
$\Leftrightarrow$ they must be placed in different areas.

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Proof

levels (L)

. . . . 0

. 1

. . . . . . n

. . . . . . k-1

. . . . k

✓ 10/10

Lets define the sets

$L_n = \{v \in V[G] \wedge level(v) = n\}$

$\forall n = 0, 1, \ldots, k$

consider any level n, where $0 \leq n \leq k$. By definition of level,

$v \in L_n \Rightarrow \forall (u,v) \in E[G], u \in L_m,$ where $m = 0, 1, \ldots, n-1$ i.e., $0 \leq m \leq n-1$

Hence, it's clear that any vertex (team) with level n can't be adjacent to any team (vertex) with level $\geq n$.

Let's prove by induction on maximum level k that such a graph G with k levels can be colored with k colors.

Base case: $k=1$ $\exists$ only 2 distinct levels, viz. level 0 and 1. Color all of the teams (vertices) in level 0 by color 0 and the teams in level 1 by color 1. This is a valid coloring,

color 0 . . . . . L_0

color 1 . . . . . L_1

$G(k=1)$

k=0 corresponds to "no games played"

since no vertices (teams) in L_0 can be adjacent to any other team (vertex), since with outdegree 0. And teams (vertices) in L_1 can only be adjacent to some of the vertices in L_0, but none in L_1. But they are always colored with different colors and hence a valid coloring. Hence level number of each vertex represents a proper coloring for $k=1$.

Induction hypothesis:

Let's assume that the result holds $\forall k \leq n$, $n \in \mathbb{N}$, i.e., $\forall k \leq n$, in graph $G_{(k=n)}$ with maximum level k, the level # of each vertex represents a proper coloring of the vertex.

Induction step:

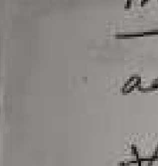
Let's prove for $k=n+1$. The graph $G_{(k=n+1)}$ has an additional level, i.e., the (n+1)th level. Every vertex in this level can only be adjacent to some of the vertices from the levels $0, 1, \ldots, n$, but no other vertex. Hence if we color any $v \in L_{n+1}$ by color $n+1$, it does not violate coloring, since it's adjacent to only the vertices with level (and hence color) $\le n$. Also, by induction hypothesis the graph $G(k=n)$ without this last level added has a proper coloring with n colors. Hence, $\forall k \in \mathbb{N}$, a graph G with maximum level k can be colored properly with each vertex color being same as the level number of the vertex (Proved).

2.4. 9.(c) Let's assume to the contrary that G has a vertex $v \in V[G]$ which is a cut vertex and still G is k-chromatic color critical

$\Rightarrow G - \{v\}$ has at least 2 connected components G_1 and G_2

with $\chi(G - \{v\}) = \max(\chi(G_1), \chi(G_2)) = k-1$

$\Rightarrow \chi(G_1) \le k-1 \wedge \chi(G_2) \le k-1$. but $\chi(G) = k$.



Also, from 9(b) as already proved, $\forall u \in G$, $d(u) \ge k-1$

Let $|V[G_1]| = n_1$, $|V[G_2]| = n_2$, $|V[G]| = n \Rightarrow n = n_1 + n_2 + 1$ (i.e. v)

It's easy to see that any $u \in V[G_1]$ can be connected to at most $n_1 - 1$ vertices in G_1 or to the vertex v, but to none of the vertices in G_2, ~~but $d(u) \ge k-1$~~ hence $n_1 - 1 + 1 \ge d(u)$, but $d(u) \ge k-1$

$\Rightarrow n_1 \ge d(u) \ge k-1 \Rightarrow n_1 \ge k-1$, similarly $n_2 \ge k-1$.

Also ~~at most one if we remove v from~~ if we add v to G, we ~~get~~ must color v with a new color, different from the existing $k-1$ colors. It's easy to see that ~~if we need a~~ we can't have 2 connected components to ensure $d(v) \ge k-1$ $\forall k$, hence a contradiction.

By contradiction

(b) Let's assume to the contrary that $\exists v \in V[G]$ s.t. $d(v) < k-1$.

Now, G is k-chromatic color ~~critic~~ critical $\Rightarrow \chi(G) = k \wedge \chi(G - \{v\}) = k-1$

$\Rightarrow$ ~~Removing v from G,~~ $\Rightarrow$ Adding v along with its edges to $G - \{v\}$ back ~~then~~ must increase the chromatic number from $k-1$ to k, i.e., v must be colored with a new color & can't be colored with any of existing $k-1$ colors.

But, $d(v) < k-1$, when we add v along with $d(v)$ edges back to $G - \{v\}$, # vertices adjacent to v will be $d(v) \leq k-2$, at most $k-2$. ~~Hence~~ Since $\chi(G - \{v\}) = k-1$, ~~then~~ even if all the vertices in $G - \{v\}$ adjacent to v are colored with different colors $\exists$ still ~~a~~ another different color remaining with which v can be safely colored, without increasing the chromatic number, i.e., $\chi(G) = k-1$ still, a contradiction.

~~(c) Please see next page~~

~~(c) G has no vertex~~ Let's assume to the contrary again that G has a vertex $v \in V[G]$ which is a cut vertex and still G is a k-chromatic color critical.

$\Rightarrow$ $G - \{v\}$ has at least 2 connected components G_1 and G_2 and ~~$\exists$~~

$\chi(G - \{v\}) = \max(\chi(G_1), \chi(G_2)) = k-1 \Rightarrow \chi(G_1) \leq k-1 \wedge \chi(G_2) \leq k-1$

~~but $\chi(G) = k$~~

G_1 G_2

$G - \{v\} = G_1 \cup G_2$

Now, by (b), we have, $\forall u \in G$, $d(u) \geq k-1$

~~Case 1~~ Let $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|V(G)| = n = n_1 + n_2 + 1$

Also, $\frac{n_1(n_1+1)}{2} \geq |E(G_1)| = \frac{1}{2}\sum_{G_1} d(u) \geq \frac{n_1(k-1)}{2} \Rightarrow n_1 \geq k-2$ similarly $n_2 \geq k-2$

~~case 1 $n_1 < k$ or $n_2 < k$, in either cases G_1, G_2 can't be disconnected from each other since, for $n_1 < k$, $\forall u \in V(G_1)$, $d(u) \geq k-1 \geq n_1$, but G_1 only has $n_1 - 1$ remaining vertices $\Rightarrow$ it must~~

(Since G_1, G_2 are disconnected from each other) edges from all vertices from G_1 will either have its endpoint in G_1 or end in v, hence $\underbrace{\frac{n_1(n_1-1)}{2}}_{\text{max inside edges}} + n_1 \geq E[G_1]$

10. ($\Rightarrow$) $I_j \subseteq V[L(G)]$,

$L(G)$ can be vertex k-colored

$\Rightarrow \exists$ k independent sets $I_1, I_2, \ldots, I_k$ in $L(G)$.
of vertices color color color

$\Rightarrow$ In any I_j, $w_1, w_2 \in I_j \Rightarrow (w_1, w_2) \notin E[L(G)]$

$\Rightarrow e_{w_1}$ and $e_{w_2} \in E(G)$ does not share a common endpoint

$\Rightarrow$ All edges in G corresponding to all vertices in I_j can be edge colored with the same color

But there are k such independent sets, hence we can use k colors to color the corresponding edges in G, with all edges corresponding to an independent set colored by the same color.

$\Rightarrow$ G can be edge colored with k colors.

($\Leftarrow$) G can be edge colored with k colors.

~~$\Rightarrow \forall v \in V[G], d(v) \le k$~~

~~$\Rightarrow \exists$ k independent~~

$\Rightarrow \exists$ k disjoint edge sets $E_1, E_2, \ldots, E_k$ with $E_j \subseteq E[G]$, $\bigcup_j E_j = E[G]$,
colors colors colors $E_i \cap E_j = \varphi$, $i \neq j$.

Now in any E_j, $e_1, e_2 \in E_j \Rightarrow e_1 \cap e_2 = \varphi$

$\Rightarrow$ corresponding $v_1, v_2 \in V[L(G)]$ will not have an edge in between them, i.e.,

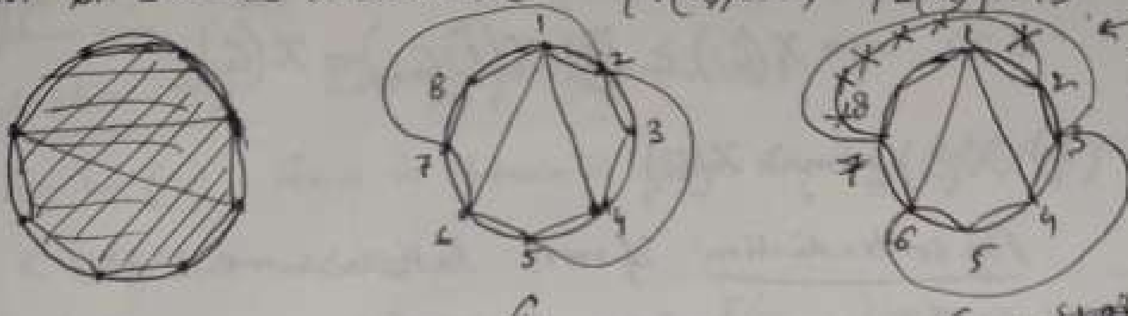
$\forall j$, $e_1, e_2 \in E_j \Rightarrow$ corresponding $(v_1, v_2) \notin E[L(G)]$, with $v_1, v_2 \in V[L(G)]$.

$\Rightarrow$ All vertices corresponding to all edges in E_j can be colored with same color.

but there are k such edge sets

$\Rightarrow L(G)$ can be vertex-colored with k-colors. (Proved)

2. We use circle-chord method to draw the planar graph and show that it can't be 2 colorable. $|V(G)|=8$, $|E(G)|=13$



if we add (7,2) it creates a triangle 723

G_1 $\quad$ G_2 — ~~(1,2)~~

As we can see from above, if we try to obtain a triangle-free planar graph G with maximum # of edges with $|V(G)|=8$. By inside-outside symmetry of the circle, (1,4) and (1,6) edges are drawn inside the circle, still having triangle free planar graph. Now, the next vertex we can choose from 2, 3, 4, 5 (since the graph becomes symmetric, $2\equiv 8$, $3\equiv 7$ and $6\equiv 4$)

If we choose 2 and connect edges preserving triangle free property, we obtain G_1 with 12 edges and observe that adding any further edge will destroy either of triangle-free of planar property.

Similarly, choosing 3 as the next vertex after 1 instead and connecting maximum possible edges we obtain G_2 and observe that adding any further edge to G_2, with $|E(G_2)|=12$ will destroy either planarity of triangle-free property.

By symmetry similar results we can show if we choose 4 or 5 as next vertex after vertex 1 instead.

Hence, with $|V(G)|=8$, $|E(G)|\leq 12$ with G having both triangle-free and planarity.

$\therefore$ Graph G with $|V(G)|=8$ and $|E(G)|=13$ $\wedge$ G planar $\Rightarrow$ G must have a triangle, hence at least 3 colors are required to color G, $\chi(G)\geq 3$, not 2-colorable (Proved)

3. $\chi(G) = \max_i \chi(G_i) = \max\{\chi(G_1), \chi(G_2), \ldots, \chi(G_k)\}$.

<u>Proof</u> (a) $\chi(G) \leq \max_i \chi(G_i)$, since $\max_i \chi(G_i) \geq \chi(G_i)\ \forall i$

$\Rightarrow \max_i \chi(G_i) \geq \chi\left(\bigcup_{i=1}^{k} G_i\right) = \chi(G)$

(b) $\chi(G) \geq \max_i \chi(G_i)$.

<u>By contradiction</u>: if not, let's assume to the contrary, i.e., ~~$\chi(G)$~~ $\chi(G) < \max_i \chi(G_i) = n$ (let)

also, let $j = \arg\max_i \chi(G_i) \Rightarrow \chi(G_j) = n$, $j \in \{1, 2, \ldots, k\}$

Now, G_j is a subgraph of G and $\chi(G) < n \Rightarrow G$ requires less than n colors for a proper coloring, but G_j needs no less than n colors $\Rightarrow$ a contradiction.

$\Rightarrow \chi(G) \geq \max_i \chi(G_i)$

(a) $\wedge$ (b) $\Rightarrow \chi(G) = \max_i \chi(G_i)$, $i = 1, 2, \ldots, k$.

4. Not true in general. Consider $K_{3,3}$ where all vertices are of degree 3 (i.e. ≥ 3). But being bipartite, it's still 2-colorable.

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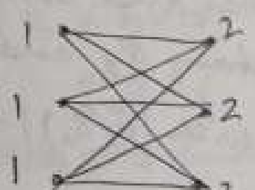


$K_{3,3}$

$\forall v \in V[K_{3,3}]$,

$d(v) = 3 \geq 3$, but it's 2 colorable.

9. (a) Assume to the contrary that G is not connected and still color critical.

$\Rightarrow \exists$ at least 2 connected components in G s.t. $G = G_1 \cup G_2$

Now, $\chi(G) = \max(\chi(G_1), \chi(G_2))$, i.e., $\chi(G) \geq \chi(G_1)$, $\chi(G) \geq \chi(G_2)$

~~Let's remove any vertex from~~ Let's assume w.l.o.g. that $\chi(G_1) \leq \chi(G_2)$. i.e. $\chi(G_1) = \chi(G_2)$. Let's remove a vertex v from G_1. Even if $\chi(G_1 - \{v\}) < \chi(G_1)$, $\chi(G) = \max(\chi(G_1 - \{v\}), \chi(G_2)) = \chi(G_2)$, does not decrease. Hence G is not color critical, a contradiction.

Basis: $n=1$, $G \quad \overline{G}$ $\quad \chi(G)=\chi(\overline{G})=1 \Rightarrow \chi(G)\chi(\overline{G})=1\geq n$

$n=2$, $\quad \chi(G)=2, \chi(\overline{G})=1 \Rightarrow \chi(G)\chi(\overline{G})=2\geq n$.

Hypothesis: Let's assume $\chi(G_n)\chi(\overline{G_n})\geq n \quad \forall n\leq m$.

Induction Step: Let's prove for $n=m+1$.

As in (a), G_{m+1} is obtained from G_m by adding new vertex v_{m+1} + k edges, $0\leq k\leq m$.

$$d(v_{m+1})=k \text{ in } G_{m+1} \Leftrightarrow \overline{d}(v_{m+1})=m-k \text{ in } \overline{G_{m+1}}.$$

Also, if $0\leq k\leq \chi(G_m)-1$, $\left.\begin{array}{l}\chi(G_{m+1})=\chi(G_m)\\ \chi(\overline{G_{m+1}})=\chi(\overline{G_m})+1\end{array}\right\}$ As proved in "(a)

$$\Rightarrow \chi(G_{m+1})\chi(\overline{G_{m+1}})=\chi(G_m)(\chi(\overline{G_m})+1)$$
$$=\chi(G_m)\chi(\overline{G_m})+\chi(G_m)$$
$$\geq \chi(G_m)\chi(\overline{G_m})+1 \qquad \{\because \chi(G_m)\geq 1 \ \forall m\}$$
$$\geq m+1 \qquad \text{(by hypothesis)}$$

if $\chi(G_m)\leq k\leq m$, $\quad \chi(G_{m+1})=\chi(G_m)+1$

$$\chi(\overline{G_{m+1}})=\chi(\overline{G_m})$$
$$\Rightarrow \chi(G_{m+1})\chi(\overline{G_{m+1}})=\chi(G_m)\chi(\overline{G_m})+\chi(\overline{G_m})$$
$$\geq \chi(G_m)\chi(\overline{G_m})+1\geq m+1$$

$\therefore \forall n\in\mathbb{N}$, $\chi(G)\chi(\overline{G})\geq n$. (Proved)

(c) By A.M. $\geq$ G.M. inequality,

we have, $\dfrac{\chi(G)+\chi(\overline{G})}{2}\geq (\chi(G)\chi(\overline{G}))^{1/2}$

As proved in $\left\{\begin{array}{ll}(a), & n+1\geq \chi(G)+\chi(\overline{G})\\ (b), & \chi(G)\chi(\overline{G})\geq n\end{array}\right.$

$$\Rightarrow \frac{\chi(G)+\chi(\overline{G})}{2}\geq (\chi(G)\chi(\overline{G}))^{1/2}\geq n^{1/2} \Rightarrow \chi(G)+\chi(\overline{G})\geq 2\sqrt{n}$$

(Proved)

Let's induct on $n = \#$ vertices in G_n.

Basis: $n=1$ $\quad G \quad \overline{G}$

$\chi(G)=\chi(\overline{G})=1 \Rightarrow \chi(G)+\chi(\overline{G})=2 \leq 1+1=n+1$

$n=2$ $\quad \chi(G)=2, \chi(\overline{G})=1 \Rightarrow \chi(G)+\chi(\overline{G})=3 \leq 2+1=n+1$

Hypothesis: Let's assume the inequality is true $\forall n \leq m$, i.e., $\chi(G_n)+\chi(\overline{G_n}) \leq n+1$ $\forall n \leq m$

Induction Step: Let's prove for $n=m+1$,

G_{m+1} is obtained from G_m by adding extra vertex v_{m+1} and K edges, where $0 \leq K \leq m$, ~~i.e., $\chi(G_m)$~~

G_m

G_{m+1}

~~If n edges~~ Also, it's easy to see that

$$d(v_{m+1}) = K \text{ in } G_{m+1} \Leftrightarrow \overline{d}(v_{m+1}) = m-K \text{ in } \overline{G_{m+1}}$$

(no self ~~edge allowed~~ loop allowed).

~~Now, if $0 \leq K \leq m-1$, $\chi(G_{m+1})$~~

If $0 \leq K \leq \chi(G_m)-1$, v_{m+1} can be safely colored in G_m with the color that is not yet used in the coloring for its neighbors, without using a new color.

$\boxed{\begin{array}{c} d(v_{m+1})=K \\ \text{in } G_{m+1} \\ \overline{d}(v_{m+1})=m-K \\ \text{in } \overline{G_{m+1}} \end{array}}$

$\Rightarrow \chi(G_{m+1}) = \chi(G_m)$ ①

but then $\overline{d}(v_{m+1}) = m-K$

$\Rightarrow m \geq \overline{d}(v_{m+1}) \geq m-\chi(G_m)+1$, in $\overline{G_{m+1}}$

$\Rightarrow$ at least $m+1-\chi(G_m)$ vertices are adjacent to v_{m+1} in $\overline{G_{m+1}}$, but $\chi(G_m)+\chi(\overline{G_m}) \leq m+1$ from hypothesis $\Rightarrow \chi(\overline{G_m}) \leq m+1-\chi(G_m)$

$\Rightarrow m \geq \overline{d}(v_{m+1}) \geq m+1-\chi(G_m) \geq \chi(\overline{G_m})$

$\Rightarrow$ at least $\chi(\overline{G_m})$ vertices are adjacent to v_{m+1} in $\overline{G_{m+1}}$ $\Rightarrow$ we need a new color to color v_{m+1} in $\overline{G_{m+1}}$ $\Rightarrow \chi(\overline{G_{m+1}}) = \chi(\overline{G_m})+1$ ②

① & ② $\Rightarrow \chi(G_{m+1})+\chi(\overline{G_{m+1}}) = \chi(G_m)+\chi(\overline{G_m})+1$
$\leq (m+1)+1$

~~$\Rightarrow \chi(G_n)+\chi(\overline{G_n}) \leq n+1$ holds for $n=m+1$~~

~~$\Rightarrow \chi(G)+\chi(\overline{G}) \leq n+1 \; \forall n \in \mathbb{N}$~~

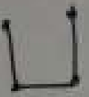
Similarly if $\chi(G_m) \leq K \leq m$, we can show that

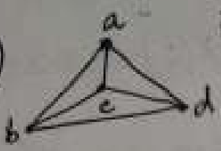
$\chi(G_{m+1}) = \chi(G_m)+1$ but $\chi(\overline{G_{m+1}}) = \chi(\overline{G_m})$

$\Rightarrow \chi(G_{m+1})+\chi(\overline{G_{m+1}}) \leq (m+1)+1$, in this case as well

Hence, $\chi(G)+\chi(\overline{G}) \leq n+1 \; \forall n \in \mathbb{N}$ (Proved)

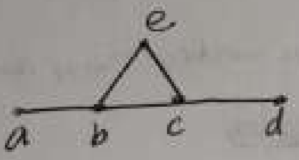
Tree with ~~[illegible]~~ # nodes=4

14. (a) 
$\Rightarrow P_k(G) = k(k-1)^3$ ✓

(b) 

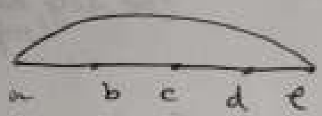
if choose the vertices in the sequence a-c-b-d, we can see that a can be colored in k ways, c in k-1 ways, b in k-2 ways and d in k-3 ways

$\Rightarrow P_k(G) = k(k-1)(k-2)(k-3)$ ✓ (since it's K_4)

(c) 

a can be chosen in k ways.
b " " " " k-1 "
c " " " " k-1 "
e " " " " k-2 "
d " " " " k-1 ways

$\Rightarrow P_k(G) = k(k-2)(k-1)^3$ ✓

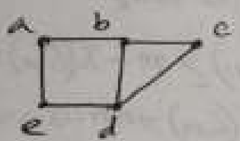
(d) 

K, K-1, K-1, K-1, K-2

R, B, ≠R, ≠B, ≠B ≠R

a can be colored in k ways, after that
b " " " " k-1 " " "
c " " " " k-1 " " "
d " " " " k-1 " " "
e " " " " k-2 " " "

$\therefore P_k(G) = k(k-2)(k-1)^3$ ✗

(e) 

a can be colored in k ways, after that
e " " " " k-1 " " "
d " " " " k-1 " " "
b " " " " k-2 " " "
c " " " " k-2 " " "

$\therefore P_k(G) = k(k-1)^2(k-2)^2$ ✗

use the method from corollary to thm 6

6/10

Let's assume that every planar graph with $|V(G)| < 12$ ~~and~~ has all vertices with degree > 4, to the contrary.

G is planar $\Rightarrow$ $e \leq 3v - 6$,

with $v = |V(G)| < 12$

$$2e = \sum_v d(v) \geq 5v \Rightarrow e > \frac{5}{2}v$$

$\Rightarrow$ ~~$3\cdot 12 - 6 > 3v - 6 \geq e > 4$~~

$$\Rightarrow 3v - 6 \geq e > \frac{5}{2}v \Rightarrow \frac{v}{2} \geq 6 \Rightarrow v \geq 12, \text{ a contradiction}$$

$\Rightarrow$ Every planar graph with $|V(G)| < 12$ has a vertex with degree ≤ 4

Proof by induction on number of vertices n, with $n < 12$

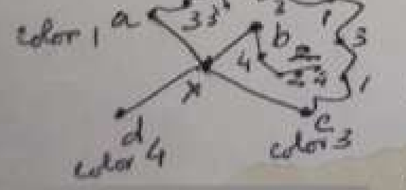
Base case: trivially true for $n \leq 3$, since all of them are 4 colorable.

Hypothesis: Assume true for $\forall n \leq m < 11$, i.e., all planar graphs with $n < m$ are 4 colorable.

Induction Step: Proof for $n = m+1 < 12$: Since $\exists x \in V[G]$ with $d(x) \leq 4$ in G with $m+1$ vertex, We can obtain graph G' with m vertices by removing the vertex x from G, $G' = G - \{x\}$, by induction hypothesis G' is 4 colorable with m vertices.

Now, ~~let's assume~~ if $d(x) < 4$ or in the case when $d(x) = 4$ but two or more of its neighbors are colored with same color, x can be colored with the 4th color and the graph ~~[illegible]~~ G is still 4 colorable.

When $d(x) = 4$ and all its neighbors are colored with 4 different colors, consider a clockwise ordering of its neighbors a, b, c, d for some planar depiction of G. Consider all possible paths emanating



from a and ending at c. If ~~any~~ none of them ~~does not~~ has vertex colored with 1 or 3, change the color of a to 3 on all the paths emanating from a, without affecting color of c. Then color x by color 1

If a-c has an edge or the path from a-c has 1-3 colored vertices, choose b and change it color to 4 and changes vertices on all the paths emanating from b (4 by 2 and 2 by 4). Also notice that due to planarity no path from b can reach to d crossing the path ~~[illegible]~~ a-c.

Now, color x by color 2.

Hence, in either case G is 4 colorable. (Proved).