Section 8.2
Solving a System of Equations Using Matrices (Guassian Elimination)

$$\begin{array}{rcrcrcr} 2x & + & y & + & 3z & = & 1 \\ 3x & - & 2y & + & 4z & = & -1 \\ 2x & - & 4y & + & 2z & = & -2 \end{array}$$

System of Equations

$$\underbrace{\begin{pmatrix} 2 & 1 & 3 \\ 3 & -2 & 4 \\ 2 & -4 & 2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}}=\underbrace{\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}}_{\mathbf{b}}$$

$A\mathbf{x}=\mathbf{b}$

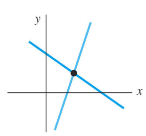
System in matrix form

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 3 & -2 & 4 & -1 \\ 2 & -4 & 2 & -2 \end{array}\right]$$

Augmented Matrix

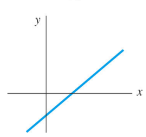
Not every system has a unique solution.

There are three different possible solutions



(a)

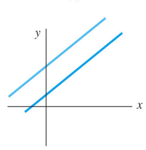
- a unique solution (exactly one solution)



(b)

- infinitely many solutions

$\Big\}$ the system is called **consistent**



(c)

- no solution

$\Big\}$ the system is called **inconsistent**

Starting with an augmented matrix, you have two options:

Use row operations to reduce to:

row-echelon form

- Any row consisting of all zeros must be on the bottom of the matrix
- For all nonzero rows, the first nonzero entry must be a 1. This is called the "leading 1".
- Take any 2 consecutive nonzero rows: The leading 1 for the higher row must be to the left of the leading 1 of the lower row. The leading ones must "staircase down" from left to right.

Reduction to row-echelon form is called :

Gaussian elimination

The solution is then found by back-substitution

reduced row-echelon form

- Row echelon form +
- Find all the leading ones. All other entries in the column containing a leading 1 should be zero (above and below the leading 1).

Reduction to reduced row-echelon form is called :

Gauss-Jordan elimination

The solution is then found by inspection or by a few simple steps

Row-echelon, Reduced row-echelon, or Neither

1.
$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

row-echelon

2.
$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 1 \end{array}\right)$$

neither

3.
$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

reduced row-echelon

4.
$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

neither

5.
$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

reduced row-echelon

6.
$$\left(\begin{array}{ccc|c} 1 & 3 & -4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

row-echelon

Order Matters!

row-echelon – for each column (move left to right), first get the appropriate leading 1, then get 0's underneath it.

reduced row-echelon – for each column (move left to right), first get the appropriate leading 1, then get 0's above and below it.

<u>3 Permitted Row Operations</u> :

(remember: every row represents an equation)

a) Multiply a row by a number

$$\begin{bmatrix} \boxed{3} & -9 & 6 & | & 15 \\ 5 & -2 & 4 & | & -1 \\ 2 & -4 & 2 & | & -2 \end{bmatrix} \tfrac{1}{3} \cdot R_1 \Rightarrow \begin{bmatrix} 1 & -3 & 2 & | & 5 \\ 5 & -2 & 4 & | & -1 \\ 2 & -4 & 2 & | & -2 \end{bmatrix}$$

b) Switch rows

$$\begin{bmatrix} 1 & 7 & 3 & | & 1 \\ 0 & \boxed{-2} & 4 & | & -1 \\ 0 & 1 & 2 & | & -2 \end{bmatrix} \; R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 7 & 3 & | & 1 \\ 0 & 1 & 2 & | & -2 \\ 0 & -2 & 4 & | & -1 \end{bmatrix}$$

c) Add a multiple of $\underbrace{\text{one row}}_{\text{Row that is not changing}}$ to $\underbrace{\text{another row}}_{\text{Row you want to replace}}$

$$\begin{bmatrix} 1 & -2 & 1 & | & -1 \\ \boxed{3} & -2 & 4 & | & -1 \\ 2 & 1 & 3 & | & 1 \end{bmatrix} \; -3R_1 + R_2 = New\ R_2 \quad \Rightarrow \begin{array}{r|rrrcr} -3R_1 & -3 & 6 & -3 & | & 3 \\ + \quad R_2 & 3 & -2 & 4 & | & -1 \\ \hline New R_2 & 0 & 4 & 1 & | & 2 \end{array} \quad \Rightarrow \begin{bmatrix} 1 & -2 & 1 & | & -1 \\ 0 & 4 & 1 & | & 2 \\ 2 & 1 & 3 & | & 1 \end{bmatrix}$$

<u>To get 1's</u> :

a) Switch Rows if there is a 1 in the same column but below the desired spot.

$$\begin{bmatrix} 1 & 7 & 3 & | & 1 \\ 0 & \boxed{-2} & 4 & | & -1 \\ 0 & 1 & 2 & | & -2 \end{bmatrix} \; R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 7 & 3 & | & 1 \\ 0 & 1 & 2 & | & -2 \\ 0 & -2 & 4 & | & -1 \end{bmatrix}$$

b) If k is the entry in the desired spot, multiply the row by $\dfrac{1}{k}$ if every other entry in the row is divisible by k.

$$\begin{bmatrix} \boxed{3} & -9 & 6 & | & 15 \\ 5 & -2 & 4 & | & -1 \\ 2 & -4 & 2 & | & -2 \end{bmatrix} \tfrac{1}{3} \cdot R_1 \Rightarrow \begin{bmatrix} 1 & -3 & 2 & | & 5 \\ 5 & -2 & 4 & | & -1 \\ 2 & -4 & 2 & | & -2 \end{bmatrix}$$

c) Do step b) followed by step a) if there is another row where every entry is divisible by k.

$$\begin{bmatrix} 2 & 1 & 3 & | & 1 \\ 3 & -2 & 4 & | & -1 \\ \boxed{2} & -4 & 2 & | & -2 \end{bmatrix} \tfrac{1}{2} \cdot R_3 \Rightarrow \begin{bmatrix} 2 & 1 & 3 & | & 1 \\ 3 & -2 & 4 & | & -1 \\ 1 & -2 & 1 & | & -1 \end{bmatrix} R_3 \leftrightarrow R_1 \begin{bmatrix} 1 & -2 & 1 & | & -1 \\ 3 & -2 & 4 & | & -1 \\ 2 & 1 & 3 & | & 1 \end{bmatrix}$$

These are the "easier" ways to get a 1

To get 1's : (continued)

d) Use "elimination" step – add one row to a multiple of another row

$$\left[\begin{array}{ccc|c} \boxed{3} & -2 & 4 & -1 \\ 2 & 1 & 3 & 1 \\ 2 & -4 & 7 & -2 \end{array}\right] -R_2 + R_1 = New\ R_1 \quad \Rightarrow \quad \begin{array}{r|cccc} -R_2 & -2 & -1 & -3 & \mid\ -1 \\ +\ R_1 & 3 & -2 & 4 & \mid\ -1 \\ \hline NewR_1 & 1 & -3 & 1 & \mid\ -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & -2 \\ 2 & 1 & 3 & 1 \\ 2 & -4 & 7 & -2 \end{array}\right]$$

e) Last Resort – Introduce fractions by multiplying by $\frac{1}{k}$

$$\left[\begin{array}{ccc|c} \boxed{2} & -5 & 7 & 11 \\ 4 & -9 & 4 & -1 \\ 6 & -4 & 2 & -2 \end{array}\right] \frac{1}{2}\cdot R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{-5}{2} & \frac{7}{2} & \frac{11}{2} \\ 4 & -9 & 4 & -1 \\ 6 & -4 & 2 & -2 \end{array}\right]$$

These are the "harder" ways to get a 1

To get 0's :

Use "elimination" step : add a multiple of $\underbrace{\text{one row}}_{\text{Row with the leading 1 in it}}$ to $\underbrace{\text{another row}}_{\text{Row you want to replace}}$

The leading 1 is always obtained before getting the zero(s)

Multiply the row with the "leading" 1 by the same # but opposite sign of the number you would like to be zero.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ \boxed{3} & -2 & 4 & -1 \\ 2 & 1 & 3 & 1 \end{array}\right] -3R_1 + R_2 = New\ R_2$$

$$\Rightarrow \begin{array}{r|cccc} -3R_1 & -3 & 6 & -3 & \mid\ 3 \\ +\ R_2 & 3 & -2 & 4 & \mid\ -1 \\ \hline NewR_2 & 0 & 4 & 1 & \mid\ 2 \end{array} \quad \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 4 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{array}\right]$$

$$\begin{array}{rcrcrcr} 2x & + & y & + & 3z & = & 1 \\ 3x & - & 2y & + & 4z & = & -1 \\ 2x & - & 4y & + & 2z & = & -2 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 2 & 1 & 3 & 1 \\ 3 & -2 & 4 & -1 \\ 2 & -4 & 2 & -2 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 2 & 1 & 3 & 1 \\ 3 & -2 & 4 & -1 \\ \boxed{2} & -4 & 2 & -2 \end{array}\right] \tfrac{1}{2}\cdot R_3 \Rightarrow \left[\begin{array}{rrr|r} 2 & 1 & 3 & 1 \\ 3 & -2 & 4 & -1 \\ 1 & -2 & 1 & -1 \end{array}\right] R_3 \leftrightarrow R_1 \left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 3 & -2 & 4 & -1 \\ 2 & 1 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ \boxed{3} & -2 & 4 & -1 \\ 2 & 1 & 3 & 1 \end{array}\right] -3R_1 + R_2 = New\ R_2 \Rightarrow \begin{array}{r|rrr|r} -3R_1 & -3 & 6 & -3 & 3 \\ +\ R_2 & 3 & -2 & 4 & -1 \\ \hline NewR_2 & 0 & 4 & 1 & 2 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 0 & 4 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 0 & 4 & 1 & 2 \\ \boxed{2} & 1 & 3 & 1 \end{array}\right] -2R_1 + R_3 = New\ R_3 \Rightarrow \begin{array}{r|rrr|r} -2R_1 & -2 & 4 & -2 & 2 \\ +\ R_3 & 2 & 1 & 3 & 1 \\ \hline NewR_3 & 0 & 5 & 1 & 3 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & 5 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 0 & \boxed{4} & 1 & 2 \\ 0 & 5 & 1 & 3 \end{array}\right] -R_3 + R_2 = New\ R_2 \Rightarrow \begin{array}{r|rrr|r} -R_3 & 0 & -5 & -1 & -3 \\ +\ R_2 & 0 & 4 & 1 & 2 \\ \hline NewR_2 & 0 & -1 & 0 & -1 \end{array}$$

then
$\times(-1)$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & \boxed{5} & 1 & 3 \end{array}\right] -5R_2 + R_3 = New\ R_3 \quad \Rightarrow \begin{array}{r|rrr|r} -5R_2 & 0 & -5 & 0 & -5 \\ +\ R_3 & 0 & 5 & 1 & 3 \\ \hline NewR_3 & 0 & 0 & 1 & -2 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \Rightarrow x - 2y + z = -1 \;\Rightarrow x - 2(1) + (-2) = -1 \;\Rightarrow x - 4 = -1 \Rightarrow x = 3 \\ \Rightarrow y = 1 \\ \Rightarrow z = -2 \end{array}$$

Solution : $\boxed{(3,1,-2)}$

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & -6 \\ 2x & - & 3y & & & = & -7 \\ -x & + & 3y & - & 3z & = & 11 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 1 & -6 \\ 2 & -3 & 0 & -7 \\ -1 & 3 & -3 & 11 \end{array}\right) \begin{array}{l} \\ -2R_1+R_2 \\ R_1+R_3 \end{array} \quad \left(\begin{array}{rrr|r} 1 & -2 & 1 & -6 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & -2 & 5 \end{array}\right) \begin{array}{l} \\ \\ -R_2+R_3 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & -2 & 1 & -6 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{c} x-2y+z=-6 \\ y-2z=5 \\ z \text{ is free} \end{array} \quad \begin{array}{c} x=-6+2y-z \\ y=5+2t \\ \text{let } z=t \end{array} \quad x=-6+10+4t-t$$

$$\boxed{\begin{array}{c} x=4+3t \\ y=5+2t \\ z=t \end{array}}$$

Infinitely many solutions

$$\begin{array}{rcrcrcr} 3x & + & 6y & + & 6z & = & 5 \\ 3x & - & 6y & - & 3z & = & 2 \\ 3x & - & 2y & & & = & 1 \end{array}$$

$$\left(\begin{array}{rrr|r} 3 & 6 & 6 & 5 \\ 3 & -6 & -3 & 2 \\ 3 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \tfrac{1}{3}R_1 \\ \\ \\ \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 2 & \tfrac{5}{3} \\ 3 & -6 & -3 & 2 \\ 3 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -3R_1+R_2 \\ -3R_1+R_3 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 2 & \tfrac{5}{3} \\ 0 & -12 & -9 & -3 \\ 0 & -8 & -6 & -4 \end{array}\right) \begin{array}{l} \\ -\tfrac{1}{12}R_2 \\ \\ \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 2 & \tfrac{5}{3} \\ 0 & 1 & \tfrac{3}{4} & \tfrac{1}{4} \\ 0 & -8 & -6 & -4 \end{array}\right) \begin{array}{l} \\ 8R_2+R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 2 & \tfrac{5}{3} \\ 0 & 1 & \tfrac{3}{4} & \tfrac{1}{4} \\ 0 & 0 & 0 & -2 \end{array}\right) \quad \begin{array}{c} 0x+0y+0z=-2 \\ \Rightarrow 0=-2 \\ \textit{FALSE} \\ \text{Inconsistent System} \end{array}$$

No solution

A system of linear equations is said to be **homogeneous** is each of its equations has a constant term of 0.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & = & 0 \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & - & 0 \end{array}$$

$x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is always a solution of a homogeneous system.

This solution is called the **trivial solution**.

This means that a homogeneous system is always consistent.

For a homogeneous system, there are only two different possible solutions :

- a unique solution (the trivial solution)
- infinitely many solutions

What is more interesting is when there is a solution that has one or more of the variables not zero. A solution of this type is called a **non-trivial solution**.

$$\begin{array}{ccccccc} x & + & 2y & - & z & = & 0 \\ & & 2y & + & 3z & = & 0 \\ x & + & 4y & + & 2z & = & 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 4 & 2 & 0 \end{array}\right) -R_1 + R_3 \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right)$$

repeated row

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{c} x + 2y - z = 0 \\ 2y + 3z = 0 \\ z \text{ is free} \end{array} \quad \begin{array}{l} x = z - 2y \Rightarrow x = 4t \\ y = {}^{-3}\!/_{2}\, t \\ \text{let } z = t \end{array}$$

There are infinitely many solutions, one in particular is (8, -3,2). This is a nontrivial solution found by letting *t* be 2.

In a homogeneous system of equations, if you have more variables than equations, you are guaranteed to have nontrivial solutions

$$\begin{array}{ccccccccc} x & + & 3y & - & 2z & = & 0 \\ x & + & 3y & + & 4z & = & 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right) -R_1 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right) \begin{array}{l} x+3y-2z=0 \quad \Rightarrow x=-3y \\ 6z=0 \Rightarrow z=0 \end{array}$$

y is free
let $y=t$

$$\boxed{\begin{array}{c} x=-3t \\ y=t \\ z=0 \end{array}}$$

With fewer equations than variables you will always have at least one free parameter, this leads to infinitely many nontrivial solutions