Feedback — Assignment 7

You submitted this quiz on **Fri 19 Apr 2013 7:02 AM PDT -0700**. You got a score of **28.00** out of **28.00**.

Question 1

Consider the following linear program (LP1)

$$\begin{aligned} \text{maximize } & c^T x \\ \text{subject to } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ and its corresponding dual (LP2)

$$\begin{aligned} \text{minimize } & b^T y \\ \text{subject to } & A^T y \geq c \\ & y \geq 0. \end{aligned}$$

If LP1 and LP2 are both feasible, and x^* is an optimal solution to LP1 and y^* is an optimal solution to LP2 then which of the following conditions hold? In order to lighten the notation in the answers, we use the convention that all sums involving index i run from 1 to n and all sums involving index j run from 1 to m.

Your Answer		Score	Explanation
☑ $x_i^* > 0 \implies \sum_j y_j^* A_{ji} = c_i$	✔	0.75	
☐ $x_i^* = 0 \implies \sum_j y_j^* A_{ji} > c_i$	✔	0.75	
☐ $y_j^* = 0 \implies \sum_i A_{ji} x_i^* < b_j$	✔	0.75	
☑ $y_j^* > 0 \implies \sum_i A_{ji} x_i^* = b_j$	✔	0.75	
Total		3.00 / 3.00	

Question Explanation

By weak duality we have the following: $c^T x \leq (A^T y)^T x = y^T Ax \leq y^T b$ for any primal feasible x and dual feasible y. In particular they hold for x^* and y^*. Since x^* and y^* are also the optimal primal and dual solutions respectively, we know by strong duality that $c^T x^* = b^T y^*$. Hence, $(c^T - y^{*T} A)x^* = 0$ and $y^{*T}(Ax^* - b) = 0$ This implies that whenever $x_i^* > 0$ it must be the case that $c_i = \sum_j y_j^* A_{ji}$. Similarly, we can derive $y_j^* > 0 \implies \sum_i A_{ji} x_i^* = b_j$

Question 2

For the following linear program

$$\begin{array}{rl} \text{maximize} & -2x_1 + 3x_2 - 3x_3 \\ \text{subject to} & -x_1 + x_2 - x_3 \leq -1 \\ & -x_1 + 2x_2 + x_3 \leq 0 \\ & -2x_2 - x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$x = [2, 1, 0]^T$ is an optimal solution. Using complementary slackness derive an optimal solution for the dual of the above linear program. Enter your answer (the dual optimal solution is a vector with three components) separated by spaces.

You entered:

1 1 0

Your Answer		Score	Explanation
1 1 0	✓	4.00	
Total		4.00 / 4.00	

Question Explanation

Substituting $x = [2, 1, 0]^T$ in the constraints of the primal we find that the first two inequalities are tight while the third inequality slacks. The dual of the given linear program is

$$\begin{array}{ll} \text{minimize} & -y_1 + 2y_3 \\ \text{subject to} & -y_1 - y_2 \geq -2 \\ & y_1 + 2y_2 - 2y_3 \geq 3 \\ & -y_1 + y_2 - y_3 \geq -3 \\ & y_1, y_2, y_3 \geq 0. \end{array}$$

Let y^* be the dual optimal solution. By complementary slackness our previous observation implies that $y_3^* = 0$. Also, since x_1 and x_2 are strictly positive, by complementary slackness, it must be the case that the first two dual constraints must be tight. Hence, we can solve the resulting 2 variable system of linear equations to obtain the optimal dual solution $y^* = [1, 1, 0]^T$.

Question 3

Over the next three questions, we will solve the shortest s-t path problem using a primal-dual algorithm. First let's start with some definitions. In the shortest s-t path problem, we are given an undirected graph $G = (V, E)$ and non-negative costs $c_e \geq 0$ on all edges $e \in E$ and a pair of distinguished vertices s and t. The objective is to find the minimum-cost path from s to t in G. An s-t cut is a set of vertices that includes s and does not include t. The collection of all s-t cuts is defined as $\mathcal{K} := \{S \subset V \mid s \in S, t \notin S\}$ We can write down the LP relaxation of the shortest s-t path problem as follows

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{e \in E} c_e x_e \\ \text{subject to} & \displaystyle\sum_{e \in \delta(S)} x_e \geq 1, \quad \forall S \in \mathcal{K} \\ & x_e \geq 0 \quad \forall e \in E \end{array}$$

where $\delta(S)$ is the set of all edges that have exactly one endpoint in the vertex set S. The corresponding dual is

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{S \in \mathcal{K}} y_S \\ \text{subject to} & \displaystyle\sum_{S \in \mathcal{K} : e \in \delta(S)} y_S \leq c_e, \quad \forall e \in E \\ & y_S \geq 0, \quad \forall S \in \mathcal{K}. \end{array}$$

The primal-dual algorithm for this problem is as follows: We start with a primal infeasible solution $x = 0$ (corresponding to the set of edges $F = \emptyset$) and dual feasible solution $y = 0$. While there exists no s-t path in (V, F) pick an s-t cut C that is the connected component of (V, F) containing s. Increase y_C until there is an edge $e' \in \delta(C)$ such that $\sum_{S \in \mathcal{K}\,:\, e' \in \delta(S)} y_S = c_{e'}$. Add e' to F (setting $x_{e'}$ to 1) and repeat till an s-t path in (V, F) is found. Finally, output an s-t path P in $G' = (V, F)$.

The first question is: Can G' contain a cycle?

Your Answer		**Score**	**Explanation**
Yes			
No	✓	3.00	
Total		3.00 / 3.00	

Question Explanation

When we add an edge to F it is always an edge e' that belongs to the cut $\delta(C)$ where C is the connected component of (V, F) containing s. By induction (V, F) is a tree before the addition of the edge and since the added edge has exactly one endpoint in C it cannot create a cycle, and hence $(V, F \cup \{e'\})$ is also a tree. Hence, G' does not contain a cycle. Moreover, with the addition of each new edge to F the connected component of (V, F) containing s spans one more new vertex. Hence, G' contains exactly one s-t path P.

Question 4

Suppose that for some s-t cut S, $y_S > 0$. The second question is: What is the value of $|P \cap \delta(S)|$?

Your Answer		**Score**	**Explanation**
1	✓	5.00	
0			
Arbitrary			

- 2

Total	5.00 / 5.00

Question Explanation

We now show that the value of $|P \cap \delta(S)|$is at exactly one for such an s-t cut. Suppose there exists an s-t cut S such that $|P \cap \delta(S)| > 1$, then there must be a subpath P' of P such that P' has only its starting and ending vertices in S and the remaining vertices outside of S. Since $y_S > 0$, we must have increased y_S during some iteration of the primal dual algorithm and at the time C would have been a tree spanning the vertices in S. Thus, $C \cup P'$ contains a cycle. Since the final set of edges contains $C \cup P'$ as a subset, this implies that G' contains a cycle which contradicts the fact that G' is acyclic. Thus, $|P \cup \delta(S)| \leq 1$when $y_S > 0$ for some s-t cut S. However, $|P \cap \delta(S)|$cannot be zero because that would contradict the fact that S is an s-t path, hence it must be exactly 1.

Question 5

Using the previous observations we can carry out the rest of the analysis as follows:

$$c(P) = \sum_{e \in P} c_e = \sum_{e \in P} \sum_{S \in \mathcal{K}\,:\, e \in \delta(S)} y_S = \sum_{S \in \mathcal{K}} |P \cap \delta(S)| y_S.$$

The third question is: Which of the following principles leads to the conclusion that P is optimal?

Your Answer		Score	Explanation
Strong duality			
Weak duality	✓	5.00	
Complementary slackness			
Total		5.00 / 5.00	

Question Explanation

From the previous question, we have $\sum_{S \in \mathcal{K}} |P \cap \delta(S)| y_S = \sum_{S \in \mathcal{K}} y_S$.Together with the analysis presented in the statement of the question, we have

$c(P) = \sum_{S \in \mathcal{K}} y_S$. Since y is a dual feasible solution, weak duality immediately implies that P is optimal (since no primal solution can have an objective function value smaller than the objective function value of any dual feasible solution).

Question 6

For the next two questions, recall some of the definitions from the lecture on the primal-dual algorithm for a minimum weight bipartite perfect matching. We have a complete bipartite graph $G = (A \cup B, E)$ with $|A| = |B| = |V|/2$ where $V = A \cup B$ and real valued weights on the edges $\{w_e\}_{e \in E}$. At an intermediate stage of the execution we have a dual feasible solution y and the corresponding graph $G_y = (V, E_y)$ where E_y is the set of tight edges with respect to the dual solution y. M is a maximum cardinality matching in G_y, and L is the set of nodes reachable in G_y from any exposed node (with the matched edges directed from A to B and unmatched edges directed from B to A).

A vertex cover of a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that for every edge $e \in E$, e has at least one endpoint in S. Is the set $C = (A \cap L) \cup (B \setminus L)$ a minimum cardinality vertex cover for G_y?

Your Answer		Score	Explanation
No			
Yes	✓	5.00	
Total		5.00 / 5.00	

Question Explanation

Yes. We saw in the lecture that no edge can be present between $A \setminus L$ and $B \cap L$. Hence, C is a vertex cover. It follows that $|C| \geq |M|$ since any vertex cover has to include at least one endpoint of each matched edge. Now we will show that $|C| \leq |M|$. First, every vertex of $A \cap L$ is matched (since otherwise if there existed an exposed vertex v in $A \cap L$, we would be able to find an M-augmenting path starting from some exposed node in $B \cap L$ and ending in v contradicting the maximality of M). Second, every vertex of $B \setminus L$ is matched by the definition of L. Third, no matching edge can be between $A \cap L$ and $B \setminus L$ (by the definition of L).

Thus, we have proved that $|C| \leq |M|$. Together, in G_y we now have a matching M and a vertex cover C such that $|M| = |C|$. This immediately implies that C is a minimum cardinality vertex cover for G_y since every vertex cover has to have size at least the size of a matching in G_y.

Question 7

Consider the dual feasible solution y that we maintain throughout the algorithm. Does the objective function value of this dual feasible solution decrease in some iteration?

Your Answer		Score	Explanation
Yes			
No	✓	3.00	
Total		3.00 / 3.00	

Question Explanation

When we update the dual feasible solution y we add $\delta > 0$ to dual variables corresponding to vertices in $A \setminus L$ and subtract δ from all the dual variables corresponding to vertices in $B \setminus L$. The difference in the objective function value caused by this update is exactly $\delta(|A \setminus L| - |B \setminus L|) = \delta(|A| - |A \cap L| - |B \setminus L|) = \delta(|V|/2 - |C|)$ where C is $(A \cap L) \cup (B \setminus L)$ Since we saw in the previous question that $|C| = |M|$, we know that $|C| \leq |V|/2$ since no matching can be of size larger than $|V|/2$. In particular, this shows that whenever we don't have a perfect matching in G_y the dual feasible solution strictly improves in objective value.