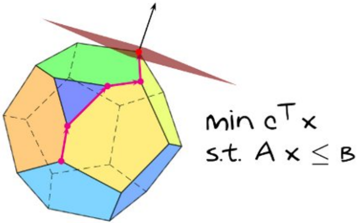
Paths, Cycles and Flows

- Shortest paths and linear programming

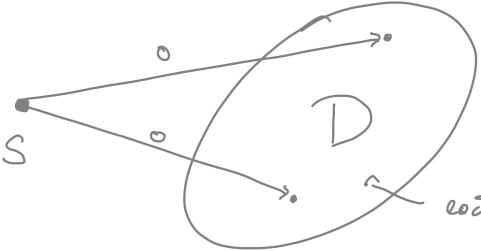


Potentials

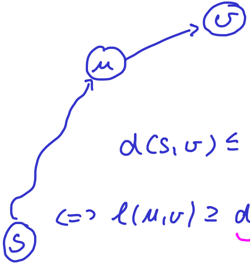
Let $D = (V, A)$ be a directed graph with arc-lengths $\ell : A \to \mathbb{R}$. A function $p : V \to \mathbb{R}$ is a *potential* if

$$\forall a = (u, v) \in A : \quad \ell(a) \geqslant p(v) - p(u).$$

D, ℓ no neg. Cycle $\Rightarrow$ potentials exist



each node in D is reachable by s



$$d(s,v) \leq d(s,u) + \ell(u,v)$$

$$\Leftrightarrow \ell(u,v) \geq \underbrace{d(s,v)}_{p(v)} - \underbrace{d(s,u)}_{p(u)}$$

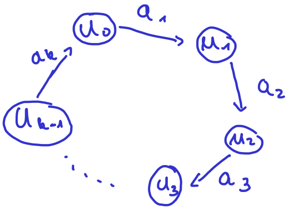
Existence of potentials

Theorem

$D = (V, A)$ with $\ell : A \to \mathbb{R}$ has a potential p if and only if each directed cycle is of non-negative length.

"$\Leftarrow$"

"$\Rightarrow$"



$$\ell(C) = \sum_{i=1}^{k} \underbrace{\ell(a_i)}_{\geq p(u_i) - p(u_{i-1})}$$

mod k

$$\geq 0$$

∎

Computing distances with linear programming

Theorem

Let $D = (V, A)$ be a directed graph with arc-lengths $\ell : A \to \mathbb{R}$, $s \in V$ such that each vertex in V is reachable from s and suppose that each directed cycle is non-negative. Let p be a potential with $p(s) = 0$ and $\sum_{v \in V} p(v)$ maximal. Then

$$\forall t \in V : \ p(t) = \mathrm{dist}_\ell(s, t).$$

proof: Shortest path distances are a potential

$$s \to u_1 \to u_2 \to \cdots \to u_k$$

$p(u_1) \leq \ell(s, u_1)$

$p(u_2) \leq \ell(u_1, u_2) + p(u_1) \leq \ell(u_1, u_2) + \ell(s, u_1)$

$\vdots$

$p(u_k) \leq$ length of PATH.

$p(u) \leq d(s, u)$

$\blacksquare$