Time Series Solutions HT 2008

1. Let $\{X_t\}$ be the ARMA(1, 1) process,

$$X_t - \phi X_{t-1} = \epsilon_t + \theta\epsilon_{t-1}, \qquad \{\epsilon_t\} \sim \mathrm{WN}(0, \sigma^2),$$

where $|\phi| < 1$ and $|\theta| < 1$. Show that the autocorrelation function of $\{X_t\}$ is given by

$$\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+\theta^2+2\phi\theta}, \qquad \rho(h) = \phi^{h-1}\rho(1) \quad \text{for } h \geqslant 1.$$

Solution. Taking expectations $E(X_t) = \phi E(X_{t-1})$, and using $\phi < 1$ and stationarity we get $E(X_t) = E(X_{t-1}) = 0$.

For $k \geqslant 2$: multiplying

$$X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1}$$

by X_{t-k} and taking expectations we get $\gamma_k = \phi\gamma_{k-1}$, and hence $\gamma_k = \phi^{k-1}\gamma_1$ for $k \geqslant 2$.

Multiplying the same equation by X_t and taking expectations we get

$$\gamma_0 = \phi\gamma_1 + E[X_t(\epsilon_t + \theta\epsilon_{t-1})]$$

and

$$\begin{aligned} X_t &= \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1} \\ &= \phi[\phi X_{t-2} + \epsilon_{t-1} + \theta\epsilon_{t-2}] + \epsilon_t + \theta\epsilon_{t-1} \\ &= \phi^2 X_{t-2} + \phi\epsilon_{t-1} + \phi\theta\epsilon_{t-2} + \epsilon_t + \theta\epsilon_{t-1} \end{aligned}$$

so

$$
\begin{aligned}
\gamma_0 &= \phi\gamma_1 + E[(\phi^2 X_{t-2} + \phi\epsilon_{t-1} + \phi\theta\epsilon_{t-2} + \epsilon_t + \theta\epsilon_{t-1})(\epsilon_t + \theta\epsilon_{t-1})] \\
&= \phi\gamma_1 + \sigma^2[\phi\theta + 1 + \theta^2].
\end{aligned}
$$

Also

$$
\begin{aligned}
\gamma_1 &= E(X_t X_{t+1}) \\
&= E[X_t(\phi X_t + \epsilon_{t+1} + \theta\epsilon_t)] \\
&= \phi\gamma_0 + E[(\phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t+1} + \theta\epsilon_t)] \\
&= \phi\gamma_0 + \theta\sigma^2.
\end{aligned}
$$

We can now solve the two equations involving γ_0, γ_1, and then find γ_k, and hence ρ_k, as required.

2. Consider a process consisting of a linear trend plus an additive noise term, that is,

$$X_t = \beta_0 + \beta_1 t + \epsilon_t$$

where β_0 and β_1 are fixed constants, and where the ϵ_t are independent random variables with zero means and variances σ^2. Show that X_t is non-stationary, but that the first difference series $\nabla X_t = X_t - X_{t-1}$ is second-order stationary, and find the acf of ∇X_t.

Solution. $E(X_t) = E(\beta_0 + \beta_1 t + \epsilon_t) = \beta_0 + \beta_1 t$ which depends on t, hence X_t is non-stationary.

Let $Y_t = \nabla X_t = X_t - X_{t-1}$. Then

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 t + \epsilon_t - \{\beta_0 + \beta_1(t-1) + \epsilon_{t-1}\} \\ &= \beta_1 + \epsilon_t - \epsilon_{t-1}. \end{aligned}$$

So

$$\begin{aligned} \text{cov}(Y_t, Y_{t+k}) &= \text{cov}(\epsilon_t - \epsilon_{t-1}, \epsilon_{t+k} - \epsilon_{t+k-1}) \\ &= E(\epsilon_t \epsilon_{t+k} - \epsilon_{t-1}\epsilon_{t+k} - \epsilon_t \epsilon_{t+k-1} + \epsilon_{t-1}\epsilon_{t+k-1}) \\ &= \begin{cases} 2\sigma^2 & k = 0 \\ -\sigma^2 & k = 1 \\ 0 & k \geqslant 2. \end{cases} \end{aligned}$$

Hence Y_t is stationary and its acf is

$$\rho_k = = \begin{cases} 1 & k = 0 \\ -\frac{1}{2} & k = 1 \\ 0 & k \geqslant 2. \end{cases}$$

3. Let $\{S_t, t = 0, 1, 2, \ldots\}$ be the random walk with constant drift μ, defined by $S_0 = 0$ and

$$S_t = \mu + S_{t-1} + \epsilon_t, \qquad t = 1, 2, \ldots,$$

where $\epsilon_1, \epsilon_2, \ldots$ are independent and identically distributed random variables with mean 0 and variance σ^2. Compute the mean of S_t and the autocovariance of the process $\{S_t\}$. Show that $\{\nabla S_t\}$ is stationary and compute its mean and autocovariance function.

Solution.

$$
\begin{aligned}
S_t &= \epsilon_t + \mu + S_{t-1} \\
&= \epsilon_t + \mu + \epsilon_{t-1} + \mu + S_{t-2} \\
&= \epsilon_t + \epsilon_{t-1} + 2\mu + S_{t-2} \\
&= \dots \\
&= \sum_{j=0}^{t-1} \epsilon_{t-j} + t\mu + S_0
\end{aligned}
$$

So $E(S_t) = 0 + t\mu + 0 = t\mu$.

For the autocovariance of S_t, the autocovariance at lag k is

$$
\begin{aligned}
E[\{S_t - t\mu\}\{S_{t+k} - (t+k)\mu\}] &= E(\sum_{j=0}^{t-1} \epsilon_{t-j} \sum_{i=0}^{t+k-1} \epsilon_{t+k-i}) \\
&= \sum_{j=0}^{t-1} E(\epsilon_{t-j}\epsilon_{t-j}) \\
&= t\sigma^2
\end{aligned}
$$

since, when moving from the first line to the second line of the above display, $E(\epsilon_{t-j}\epsilon_{t+k-i}) = 0$ unless $i = j + k$.

$Y_t = \nabla S_t = S_t - S_{t-1} = \mu + \epsilon_t$, which is clearly stationary.

$E(Y_t) = \mu$.

For the autocovariance of Y_t, note $Y_t - \mu = \epsilon_t$, and similarly $Y_{t'} - \mu = \epsilon_{t'}$, and so for $t \neq t'$ each Y_t depends on a different ϵ_t, and therefore cov$(Y_t, Y_{t'}) = 0$ for all $t \neq t'$. So the autocovariance function is σ^2 at lag 0, and is zero at all other lags.

4. If

$$X_t = a\cos(\lambda t) + \epsilon_t$$

where $\epsilon_t \sim \mathrm{WN}(0, \sigma^2)$, and where a and λ are constants, show that $\{X_t\}$ is not stationary.

Now consider the process

$$X_t = a\cos(\lambda t + \Theta)$$

where Θ is uniformly distributed on $(0, 2\pi)$, and where a and λ are constants. Is this process stationary? Find the autocorrelations and the spectrum of X_t.

[To find the autocorrelations you may want to use the identity $\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$.]

Solution. $E(X_t) = E(a\cos(\lambda t) + \epsilon_t) = a\cos(\lambda t)$, which depends on t, so X_t is not stationary.

Now for $X_t = a\cos(\lambda t + \Theta)$ we need to consider the joint distributions of $(X(t_1), \ldots, X(t_k))$ and of $(X(t_1 + \tau), \ldots, X(t_k + \tau))$. Since shifting time by t is equivalent to shifting Θ by λt, and since Θ is uniform on $(0, 2\pi)$, these two joint distributions are the same, and so X_t is stationary.

$$
\begin{aligned}
E(X_t) &= aE(\cos(\lambda t + \Theta)) \\
&= \frac{a}{2\pi}\int_0^{2\pi} \cos(\lambda t + \theta)\, d\theta \\
&= \frac{a}{2\pi}[\sin(\lambda t + \theta)]_0^{2\pi} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma_t = E(X_t X_0) &= a^2 E(\cos(\Theta)\cos(\lambda t + \Theta)) \\
&= a^2 E\left[\frac{1}{2}\{\cos(\lambda t + 2\Theta) + \cos(\lambda t)\}\right] \\
&= \frac{a^2}{2}[\frac{1}{2\pi}\int_0^{2\pi} \cos(\lambda t + 2\theta) + \cos(\lambda t)\, d\theta] \\
&= \frac{a^2}{2}\cos(\lambda t)
\end{aligned}
$$

So $\rho_t = \cos(\lambda t)$.

The spectrum is F where $\gamma_t = \int_{-\pi}^{\pi} e^{it\omega}\, dF(\omega)$. Try the discrete distribution for F, $F(\lambda) = F(-\lambda) = c$, a constant, $F(\omega) = 0$ otherwise.

Then

$$
\begin{aligned}
\gamma_t &= e^{it\lambda}c + e^{-it\lambda}c \\
&= c[\cos(t\lambda) + i\sin(t\lambda) + \cos(t\lambda) - i\sin(t\lambda)] \\
&= 2c\cos(\lambda t).
\end{aligned}
$$

So we want $2c = a^2/2$, or $c = a^2/4$. So $F(\lambda) = F(-\lambda) = a^2/4$.

5. Find the Yule-Walker equations for the AR(2) process

$$X_t = \tfrac{1}{3}X_{t-1} + \tfrac{2}{9}X_{t-2} + \epsilon_t$$

where $\epsilon_t \sim \mathrm{WN}(0, \sigma^2)$. Hence show that this process has autocorrelation function

$$\rho_k = \tfrac{16}{21}\left(\tfrac{2}{3}\right)^{|k|} + \tfrac{5}{21}\left(-\tfrac{1}{3}\right)^{|k|}.$$

[To solve an equation of the form $a\rho_k + b\rho_{k-1} + c\rho_{k-2} = 0$, try $\rho_k = A\lambda^k$ for some constants A and λ: solve the resulting quadratic equation for λ and deduce that ρ_k is of the form $\rho_k = A\lambda_1^k + B\lambda_2^k$ where A and B are constants.]

Solution. The Yule-Walker equations are

$$\rho_k = \tfrac{1}{3}\rho_{k-1} + \tfrac{2}{9}\rho_{k-2}.$$

So as in the hint, to solve

$$\rho_k - \tfrac{1}{3}\rho_{k-1} - \tfrac{2}{9}\rho_{k-2} = 0$$

try $\rho_k = A\lambda^k$. Substituting this into the above equation, and cancelling a factor of λ^{k-2}, we get

$$\lambda^2 - \tfrac{1}{3}\lambda - \tfrac{2}{9} = 0$$

which has roots $\lambda = \frac{2}{3}$ and $\lambda = -\frac{1}{3}$, so $\rho_k = A(\frac{2}{3})^k + B(-\frac{1}{3})^k$. We also require $\rho_0 = 1$ and $\rho_1 = \frac{1}{3} + \frac{2}{9}\rho_1$. Hence we can solve for A and B: $A = \frac{16}{21}$ and $B = \frac{5}{21}$. So

$$\rho_k = \tfrac{16}{21}(\tfrac{2}{3})^k + \tfrac{5}{21}(-\tfrac{1}{3})^k.$$

6. Let $\{Y_t\}$ be a stationary process with mean zero and let a and b be constants.

   (a) If $X_t = a + bt + s_t + Y_t$ where s_t is a seasonal component with period 12, show that $\nabla\nabla_{12}X_t = (1-B)(1-B^{12})X_t$ is stationary.

   (b) If $X_t = (a + bt)s_t + Y_t$ where s_t is again a seasonal component with period 12, show that $\nabla_{12}^2 X_t = (1-B^{12})(1-B^{12})X_t$ is stationary.

Solution.

(a)

$$\begin{aligned}\nabla X_t &= a + bt + s_t + Y_t - [a + b(t-1) + s_{t-1} + Y_{t-1}] \\ &= b + s_t - s_{t-1} + Y_t - Y_{t-1}\end{aligned}$$

$$\begin{aligned}\nabla\nabla_{12} X_t &= b + s_t - s_{t-1} + Y_t - Y_{t-1} \\ &\quad - [b + s_{t-12} - s_{t-13} + Y_{t-12} - Y_{t-13}] \\ &= Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}\end{aligned}$$

and this is a stationary process since Y_t is stationary. (We have used the fact that $s_t = s_{t-12}$ for all t.)

(b)

$$\begin{aligned}\nabla_{12} X_t &= (a+bt)s_t + Y_t - [(a+b(t-12))s_{t-12} + Y_{t-12}] \\ &= Y_t + 12bs_{t-12} - Y_{t-12}\end{aligned}$$

$$\begin{aligned}\nabla_{12}^2 X_t &= Y_t + 12bs_{t-12} - Y_{t-12} \\ &\quad - [Y_{t-12} + 12bs_{t-24} - Y_{t-24}] \\ &= Y_t - 2Y_{t-12} + Y_{t-24}\end{aligned}$$

and this is stationary since Y_t is stationary (again using $s_t = s_{t-12}$ for all t.)

7. Consider the univariate state-space model given by state conditions $X_0 = W_0$, $X_t = X_{t-1} + W_t$, and observations $Y_t = X_t + V_t$, $t = 1, 2, \ldots$, where V_t and W_t are independent, Gaussian, white noise processes with $\text{var}(V_t) = \sigma_V^2$ and $\text{var}(W_t) = \sigma_W^2$. Show that the data follow an ARIMA(0,1,1) model, that is, ∇Y_t follows an MA(1) model. Include in your answer an expression for the autocorrelation function of ∇Y_t in terms of σ_V^2 and σ_W^2.

Solution.

$$\begin{aligned}
\nabla Y_t &= Y_t - Y_{t-1} \\
&= (X_t + V_t) - (X_{t-1} + V_{t-1}) \\
&= X_t - X_{t-1} + V_t - V_{t-1} \\
&= W_t + V_t - V_{t-1}
\end{aligned}$$

and so ∇Y_t is an MA(1).

As V_t, V_{t-1} and W_t are independent,

$$\begin{aligned}
\gamma_0 &= Var(\nabla Y_t) \\
&= \sigma_W^2 + 2\sigma_V^2 .
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\gamma_1 &= Cov(\nabla Y_t, \nabla Y_{t+1}) \\
&= Cov(W_t + V_t - V_{t-1}, W_{t+1} + V_{t+1} - V_t) \\
&= -\sigma_V^2,
\end{aligned}$$

and, from the independence, $\gamma_k = 0$ for $|k| \geq 2$. Hence the acf is $\rho_0 = 1$,

$$\rho_1 = -\frac{\sigma_V^2}{\sigma_W^2 + 2\sigma_V^2},$$

and $\rho_k = 0$ for $|k| \geq 2$.