**Question 1.**

The dual is the following linear program:

$$\max \sum_{i=1}^{n} y_i$$
$$\forall j \in \{1, \ldots, m\}, \sum_{i:e_i \in S_j} y_i \leq w_j$$
$$\forall i \in \{1, \ldots, n\}, y_i \geq 0$$

**Question 2.**

Consider $e_i$ such that $e_i \notin I$. The dual variable $y_i$ corresponding to $e_i$ is increased by the algorithm until it saturates a constraint where $y_i$ appears. However, $y_i$ appears in a constraint iff that constraint corresponds to a subset $S_l$ to which $e_i$ belongs. Since the algorithm then adds $S_l$ to $I$, $e_i$ cannot be considered in a later iteration of the algorithm. It follows that a given dual variable $y_i$ can only be increased once.

**Question 3.**

There is a finite number of dual variables $y_i$, each iteration of the algorithm increases one variable, and each variable can only be increased once, therefore the algorithm has to terminate. Moreover, when the algorithm terminates, there remains no element $e_i$ such that $e_i \notin I$, therefore I is a solution.

**Question 4.**

Let's call $y^*$ the optimal fractional solution for the dual and $x^*$ the optimal fractional solution for the primal. By the strong duality theorem, $val(y^*) = val(x^*)$. Also, the optimal integral solution of the primal necessarily has a higher or equal value to the optimal fractional solution (since the primal is a minimization problem).

Therefore $val(y^*) \leq OPT$.

**Question 5.**

Let's prove by induction that the solution $y$ is feasible. At the initialization step of the algorithm ($y \leftarrow 0$), the current solution is feasible (it satisfies the dual linear program). Then, at each iteration of the algorithm, one variable $y_i$ is increased so as to still keep all constraints satisfied. Therefore the final solution is feasible for the dual.

**Question 6.**

$y$ is feasible and the dual is a maximization problem, therefore $val(y) \leq val(y^*)$. From question 4 it follows that $val(y) \leq OPT$.

**Question 7.**

If a set $S_j$ is in $I$, then by construction of the algorithm the corresponding constraint of the dual is saturated: $\sum_{i:e_i \in S_j} y_i = w_j$.

**Question 8.**

$$\begin{aligned}
\sum_{j \in I} w_j &= \sum_{j \in I} \left( \sum_{i:e_i \in S_j} y_i \right) \\
&= \sum_{i=1}^{n} \left( y_i \cdot \sum_{j:e_i \in S_j} 1 \right) \\
&= \sum_{i=1}^{n} y_i \cdot |\{j : e_i \in S_j\}|
\end{aligned}$$

**Question 9.**

Since $|\{j : e_i \in S_j\}| \leq f$ for all $i$, it follows from question 8 that

$$\sum_{j \in I} w_j \leq f \cdot \sum_{i=1}^{n} y_i = f \cdot val(y)$$

**Question 10.**

By combining questions 6 and 9 we deduce that $\sum_{j \in I} w_j \leq f \cdot OPT$. But $\sum_{j \in I} w_j$ is precisely the value of the set cover output by the algorithm. Therefore the algorithm is a $f$-approximation.