Lecture 7: Games With Imperfect Information 1: Bayesian Games

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Review: Normal form games

Normal form game is three things:

- a set of players N
- a set of actions available to each player, $\{A_i\}_{i \in N}$ (with $A = x_{i \in N} A_i$)
- Preferences of each player, depending on the actions of all: $\{u_i\}_{i \in N}$ with $u_i : A \to \mathbb{R}$

Review: Normal form games

And our equilibrium concept was

Nash Equilibrium!

A profile of actions a^* is a (pure strategy) Nash Equilbrium (NE) iff

$$u_i(a^*) \geq u_i(a_i, a^*_{-i})$$

for all a_i, for all $i \in N$

Motivation

Up till now, strategic situations are ones in which everyone knew everything (in equilibrium).

- structure of game
- actions available
- payoffs of others

They were games of complete and perfect information

- complete information: structure of the game, payoffs of all
- perfect information: how all act

Motivation 2

But!! Would like to relax this to study interactions in which actors do not “know everything":

- War: don't know strength of opponent/ don't know opponent's utilities (like to fight?)
- Candidate competition: don't know public preferences/ don't know opponent's “abilities" (war-chest, etc.)

Definition

[Note: departs slightly from Osborne def. 279.1] A Bayesian game in normal form is:

- a set of **players**
- a set of **states** (of nature), $\omega \in \Omega$ (assume countable for now - easily generalized)
- a set of **actions** for each player
- a set of **signals** for each player (also called 'private infromation'), T_i, or "type space"
- von Neumann-Morgernsterm **utility** for each player. i.e. Bernouilli payoffs that depend on others' action and the state of the world: $u_i(\sigma, \omega) \to \mathbb{R}$

and... **beliefs** for each player (almost... more on this later) - if agents don't know about the environment, how are they to act? -what do we assume about how agents form and/ or update beliefs?

Definition, Intuition

Think of a Bayesian game (for now) as:

- Each player has an 'idea' about the world: whatever an agent doesn't **know**, it has **beliefs** about
- Get a signal (private information): something you know that others don't
- Update beliefs: now I know something more than I did at the beginning of the game, namely my own private info. How does that change my beliefs?
- BAYES' RULE
- Take action (perhaps probabilistically): maximize expected utility, given all info and beliefs you have.

Details

- Strategies for normal form games: (new) $\sigma_i : T_i \to \Delta S_i$ now depends on type (private info)
- Payoffs: depend on actions of all, and on state of nature. Bernoulli payoffs $u : S \times \Omega \to \mathbb{R}$.
- Expected utility (depends on type (signal) and actions): $U_i(\sigma, t_i) = \sum_{\omega \in \Omega} Pr[\omega, t_{-i}|t_i] u_i(\sigma_i(t_i), \sigma_{-i}(t_{-i}), \omega)$

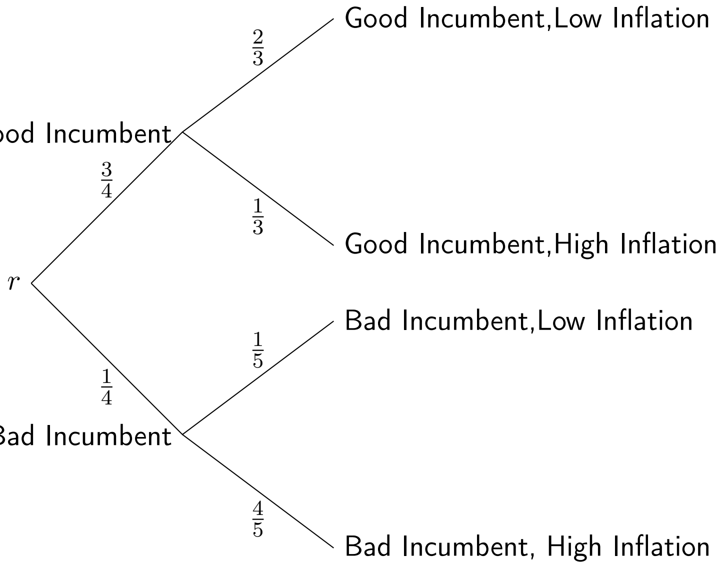
Bayes' Rule

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)} \tag{1}$$

If $C_1, ..., C_N$ are events that partition the whole space i.e., $\sum Pr(C_n) = 1, C_j \bigcap C_k = \emptyset$ and $Pr(C_n) > 0$ for all n, then:

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{\sum_{n=1}^{N} Pr(B|C_n)Pr(C_n)} \tag{2}$$

Learning and Bayes' Rule



Bayesian Reasoning

The likelihood the incumbent is good if we observe low inflation:

1. Agent knows that there is a $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$ probability of reaching the top node.
2. And a $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$ probability of reaching the third node.
3. After observing low inflation its 10 times as likely that the incumbent is good.
4. Let $p(l)$ be probability of good incumbent conditional on low inflation.
5. Because probabilities must sum to 1, $p(l) + \frac{p(l)}{10} = 1$ so that $p(l) = \frac{10}{11}$
6. $10p(l) + p(l) = 10$
7. $p(l)(10 + 1) = 10$
8. $p(l) = \frac{10}{11}$

Bayes' Rule

Let $A_1 ... A_N$ be disjoint events (i.e., no two can occur simultaneously) such that $\sum Pr(A_n) = 1$ and $Pr(A_n) > 0$ for all n. Let B be some other event. Then:

$$Pr(A_j|B) = \frac{Pr(B|A_j)Pr(A_j)}{\sum_{n=1}^{N} Pr(B|A_n)Pr(A_n)} \tag{3}$$

Bayes Incumbent/Inflation Example

Returning to our example, let A_1 be the event that the incumbent is good and A_2 be the event that she is bad. Event B is low inflation. The Bayes formulae are:

$$Pr(A_1|B) = \frac{Pr(B|A_1)Pr(A_1)}{Pr(B|A_l)Pr(A_l) + Pr(B|A_2)Pr(A_2)} \tag{4}$$

$$Pr(A_2|B) = \frac{Pr(B|A_2)Pr(A_2)}{Pr(B|A_l)Pr(A_l) + Pr(B|A_2)Pr(A_2)} \tag{5}$$

Bayes Incumbent/Inflation Example

- $Pr(A_1) = \frac{3}{4}$
- $Pr(A_2) = \frac{1}{4}$
- $Pr(B|A_1) = \frac{2}{3}$
- $Pr(B|A_2) = \frac{1}{5}$

$$Pr(A_1|B) = \frac{\frac{2}{3} \times \frac{3}{4}}{\frac{2}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{1}{4}} \tag{6}$$

and

$$Pr(A_2|B) = \frac{\frac{1}{5} \times \frac{1}{4}}{\frac{2}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{1}{4}} \tag{7}$$

Bayesian Games

- so far, we have assumed that players know each others preferences
- what if players aren't perfectly informed?
- consider a modified version of the Battle of the Sexes

Player 2

		Prob. $\frac{1}{2}$		Prob. $\frac{1}{2}$	
		B	S	B	S
Player 1	B	$2, 1$	$0, 0$	$2, 0$	$0, 2$
	S	$0, 0$	$1, 2$	$0, 1$	$1, 0$

Bayesian Games

Player 2

Player 1

	Prob. $\frac{1}{2}$ B	S		Prob. $\frac{1}{2}$ B	S
B	$2, 1$	$0, 0$		$2, 0$	$0, 2$
S	$0, 0$	$1, 2$		$0, 1$	$1, 0$

- there two 'states' – player 2 'likes' or doesn't 'like' player 1
- there are two 'types' of player 2
- player 2 can calculate expected utilities given strategies of each types

	(B, B)	(B, S)	(S, B)	(S, S)
B	2	1	1	0
S	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Bayesian Games

Player 2

		Prob. $\frac{1}{2}$ B	S	Prob. $\frac{1}{2}$ B	S
Player 1	B	$2, 1$	$0, 0$	$2, 0$	$0, 2$
	S	$0, 0$	$1, 2$	$0, 1$	$1, 0$

- Nash equilibrium
  - each *type of player* chooses optimal action given other types actions
  - player 1 faces uncertainty – expected utility calculation
  - each type of player 2 chooses optimally given player 1's action
  - is $(B, (B, S))$ an equilibrium?

	(B, B)	(B, S)	(S, B)	(S, S)
B	2	1	1	0
S	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Bayesian Games

		Prob. $\frac{1}{2}$ B	S	Prob. $\frac{1}{2}$ B	S
Prob. $\frac{2}{3}$	B	$2,1$	$0,0$	$2,0$	$0,2$
	S	$0,0$	$1,2$	$0,1$	$1,0$

		B	S	B	S
Prob. $\frac{1}{3}$	B	$0,1$	$2,0$	$0,0$	$2,2$
	S	$1,0$	$0,2$	$1,1$	$0,0$

- types, states & signals
  - each player has two types: y, n
  - four states: yy, yn, ny, nn
  - each player receives a signal that reveals his own type
    - P1: $\tau_1(yy) = \tau_1(yn) = y_1$ & $\tau_1(ny) = \tau_1(nn) = n_1$
  - Consider whether each of the four players strategies are optimal. Consider $((B, B), (B, S))$

Can information hurt?

	Prob. $\frac{1}{2}$	:P1:	Prob. $\frac{1}{2}$
	Prob. $\frac{1}{2}$	:P2:	Prob. $\frac{1}{2}$

	L	M	R
T	$1, 2\varepsilon$	$1, 0$	$1, 3\varepsilon$
B	$2, 2$	$0, 0$	$0, 3$

L	M	R
$1, 2\varepsilon$	$1, 3\varepsilon$	$1, 0$
$2, 2$	$0, 3$	$0, 0$

- $0 < \varepsilon < \frac{1}{2}$
  - P2 chooses L: $2\varepsilon > \frac{3}{2}\varepsilon$ and $2 > \frac{3}{2}$
  - P1's best response to L is B
- now, suppose P2 receives signal $\tau(\omega_1) \neq \tau(\omega_2)$
  - now R dominates if in state 1, M if state 2
  - T is best response to M and R

Adverse selection (282.3)

- Firm A taking over firm T
  - A doesn't know value of T: equal probability over each dollar value $\{0, 1, \ldots, 100\}$
  - Value of T 50% greater under A
  - A bids y and true value of T is x
  - A's payoff is $\frac{3}{2}x - y$ and T's payoff is y if offer is accepted and x if rejected
- A's action is a bid y
- T's is a threshold for accepting an offer
  - States: possible values of firm T
  - Actions: Set of possible bids (positive numbers) for A and set of possible thresholds
  - Signals: T gets a different signal for each state, A receives the same signal in each state
  - Beliefs: A assigns equal prob. to each state, T assigns prob. 1 to state indicated by signal

Solution Concepts

Direct application of NE $\rightarrow$ Bayesian Nash Equilibrium

Definition

A strategy profile σ^* is a Bayesian Nash Equilibrium of a Bayesian strategic form game if

$$\sum_{\omega\in\Omega} Pr[\omega, t_{-i}|t_i]u_i(\sigma_i^*(t_i), \sigma_{-i}^*(t_{-i}), \omega)$$

$$\geq \sum_{\omega\in\Omega} Pr[\omega, t_{-i}|t_i]u_i(\sigma_i'(t_i), \sigma_{-i}^*(t_{-i}), \omega)$$

for all i, for all σ_i'

Provision of a Public Good (modified Palfrey-Rosenthal 1988)

- n players
- Actions = contribute or not, $A_i = \{0, 1\}$ for all i

$$u_i(1, a_{-i}) = \begin{cases} 1 - c_i & \text{if } \sum a_i \geq k \\ -c_i & \text{otherwise} \end{cases}$$

$$u_i(0, a_{-i}) = \begin{cases} 1 & \text{if } \sum a_i \geq k \\ 0 & \text{otherwise} \end{cases}$$

- Private information: $c_i \sim U[0, 1]$

Provision of a Public Good, k = 1

Consider aymmetric eq. (all c_i employ same strategy). Now:

- Asymmetric Eq: $a_i = 1, a_{-i} = 0$ is an equilibrium for any i
- Cut off point strategies: $u_i(1, a_{-i}) = 1 - c_i, u_i(0, a_{-i}) = p_i$, so best response function looks like "contribute if $c_i > 1 - p_i$ ". So focus on strategies $\hat{c}_n$ such that "$c_i > \hat{c}_n$" What if $c_i = \hat{c}_n$?
- Others contribtue with prob $\hat{c}_n$. Why?
- $\rightarrow$ Prob. that no one else contributes is $(1 - \hat{c}_n)^{n-1}$
- Contribute if $E[u_i(1, .)] > E[u_i(0, .)]$
- Indifference requires $(1 - \hat{c}_n)^{n-1} = \hat{c}_n$

Notes: $\hat{c}_n$ decresing in n. Why?

Provision of a Public Good, $k > 1$

- Let x_i be realized number of other contributions, $x_i = \sum_{j \neq i} a_j$
- Net utility: $u_i(1, x_{-i}) - u_i(0, x_{-i}) = Pr[x_{-i} = k-1] - c_i$
- Ex ante: $Pr[x_{-i} = k-1] = \binom{n-1}{k-1} \hat{c}_n^{k-1} (1-\hat{c}_n)^{n-k}$. Why?
- Again, indifference implies $\binom{n-1}{k-1} \hat{c}_n^{k-1} (1-\hat{c}_n)^{n-k} = \hat{c}_n$
- Let $\Pi(\hat{c}_n) = \frac{\binom{n-1}{k-1} \hat{c}_n^{k-1} (1-\hat{c}_n)^{n-k}}{\hat{c}_n}$
- Indifference implies $\Pi(\hat{c}_n) = 1$
- ...
- (Approximately) $\hat{c}_n = \frac{k-2}{n-2}$, provided $2 < k < n$

Uncertainty of Candidate Preferences (M&M pg. 164)

Two policy motivated candidates, ideal points (in 1-D) unknown. One median voter. Set up:

- $\theta_1 \in \{0, 1/2\}$, $\theta_2 \in \{1/2, 1\}$
- $u_i(x) = -(\theta_i - x)^2, x =$ implemented policy
- Median voter's ideal point $\sim U[0,1]$
- strategies: $s_1(\theta_1) : \{0, 1/2\} \rightarrow [0, 1/2]$(for simplicity) and vice versa
- Assume 1 uses $s_2(1/2) = a$ and $s_2(1) = b$. What about $\theta_1 = 1/2$?
- $s_1 = 1/2$ dominates any $s_1 < 1/2$. Why?
- So $s_1(1/2) = 1/2$ and $s_2(1/2) = 1/2$

- What about s_1 when $\theta_1 = 0$?
- max

$$-s_1^2\left(\frac{s_1+1/2}{4}+\frac{s_1+b}{4}\right)-\frac{(1/2)^2}{2}\left(1-\frac{s_1+1/2}{2}\right)-\frac{b^2}{2}\left(1-\frac{s_1+b}{4}\right) \tag{8}$$

- (whew)
- Differentiate this and set equal to zero (some more math)
- $b = \frac{11}{7} - \frac{\sqrt{106}}{14} \approx 0.836$
- So: $s_2(1/2) = 1/2$, $s_2(1) \approx 0.836$
- When cand. prefs. uncertain > more divergent platforms than when cand. prefs are known! Why? candidates are policy motivated > would rather lose to a moderate than to an extremist > dampens incentives for extreme candidates to moderate.

Types of Uncertainty

What can agents be uncertain of?

- Payoffs (own or others)
- Actions taken by others

Harsanyi!! -Any game of incomplete information can be transformed into a game of imperfect information (uncertainty about history of play)!
Things to think about...

- Set of actions?
- Number of players?

Homework Questions from Osborne

- Exercises 276.1
- Exercise 290.1
- Exercises 307.1