Advanced Signs Switching

Now lets's consider a more challenging version of the problem.

Problem. Is it possible to switch all signs to '$+$' in the following table, where you once again can only flip rows or columns in their entirety?

$-$	$+$	$+$	$-$
$+$	$+$	$+$	$+$
$+$	$+$	$+$	$+$
$-$	$+$	$+$	$-$

The previous argument does not seem to work hereas the number of minuses is even in both the initial and the final configurations.

It turns out that the previous argument (tracking the parity of the number of minuses) can, in fact, be applied here. As usual, we encourage you to try to figure this out before reading further.

The trick is to focus on the top left 2×2 subtable.

$-$	$+$	$+$	$-$
$+$	$+$	$+$	$+$
$+$	$+$	$+$	$+$
$-$	$+$	$+$	$-$

In the original configuration, this subtable contains a single minus sign whereas in the end we want it to contain no minus signs. This is impossible by the previous argument! Indeed, when we switch, say, a row in the 4×4 table, we also switch a row in this 2×2 subtable or do nothing (if the switched row is one of the bottom two rows). Hence, the parity of the number of minuses is preserved in the 2×2 subtable, and this is why the challenge is impossible.

After discussing two versions of this puzzle, it is natural to ask the following general question.

Problem. Call a configuration *solvable*, if it is possible to switch all its signs to pluses. What configurations are solvable?

We've already seen one obstacle: if some 2×2 subtable of a configuration contains an odd number of minuses, then it is unsolvable. It turns out that it is the only obstacle!

Problem. Prove that if every 2×2 subtable of a configuration contains an even number of minuses, then one can switch all signs to pluses.

We will not discuss the solution, but you are encouraged to try to solve it yourself.

To do this, prove two things

1. for any initial configuration, it is possibleto obtain pluses only the in the top row and left column;
2. if every 2×2 subtable of the initial configuration contains an even number of minuses, then, the previous step results in the required (all-pluses) configuration.

Summary.

- Invariants are important tools for proving impossibility, termination, and various bounds.
- Invariants may take many forms: numbers, "parity", equations, inequalities.
- To prove impossibility, one finds a quantity that never changes during a process.
- To prove that a process terminates in a number of steps, one usually finds a quantity that decreases at every step.
- Double counting is a method that uses the sum invariant.