New York University
Department of Economics

ECON-UA 6 C. Wilson
Mathematics for Economists February 21, 2012

Concave Functions of a Single Variable

If $x, y \in \mathbb{R}$ and $\lambda \in (0,1)$, then $\lambda y + (1-\lambda)x$ is a *convex combination of* x and y.

Geometrically, a convex combination of x and y is a point somewhere between x and y.

A set $X \subset \mathbb{R}$ is convex if $x, y \in X$ implies $\lambda y + (1-\lambda)x \in X$ for all $\lambda \in [0,1]$.

The definition of a convex set immediately implies that X is convex if and only if X is either empty, a point, or an interval. Throughout this handout, we suppose that X is a convex subset of $\mathbb{R}$.

Concave Functions

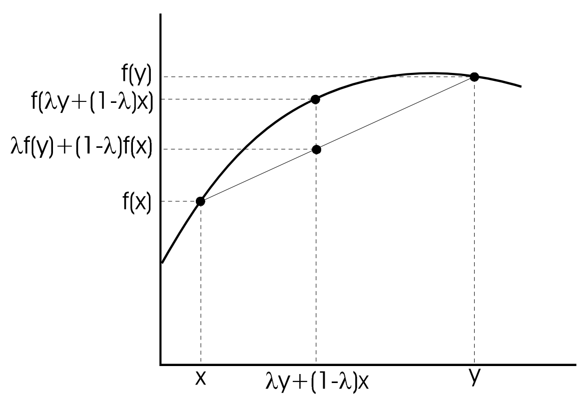
$f : X \to \mathbb{R}$ is *concave* if for any $x, y \in X$, we have, for all $\lambda \in (0,1)$,

$$f(\lambda y + (1-\lambda)x) \geq \lambda f(y) + (1-\lambda) f(x).$$

$f : X \to \mathbb{R}$ is *strictly concave* if for any $x, y \in X$ with $x \neq y$, we have, for all $\lambda \in (0,1)$,

$$f(\lambda y + (1-\lambda)x) > \lambda f(y) + (1-\lambda) f(x).$$

Geometrically, a function f is concave if the cord between any to points on the function lies everywhere on or below the function itself as illustrated in the graph below.



- A constant function is concave. Why?
- A linear function is concave. Why?

Linear Combinations of Concave Functions

Consider a list of functions $f_i : X \to \mathbb{R}$ for $i = 1, ..., n$, and list of numbers $\alpha_1, ..., \alpha_n$. The function $f \equiv \sum_{i=1}^{n} \alpha_i f_i$ is called a *linear combination* of $f_1, ..., f_n$. If each of the weights $\alpha_i \geq 0$, then f is a *nonnegative linear combination* of $f_1, ..., f_n$.

The next proposition establishes that any nonnegative linear combination of concave functions is also a concave function.

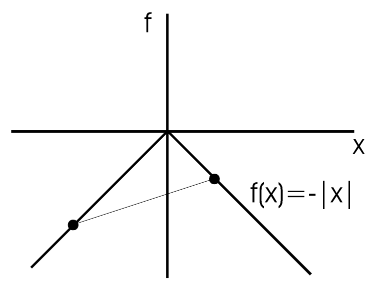
Theorem 1: Suppose $f_1, ..., f_n$ are concave functions and $(\alpha_1, ..., \alpha_n) \geq 0$. Then $f \equiv \sum_{i=1}^{n} \alpha_i f_i$ is also a concave function.
If at least one f_j is also strictly concave and $\alpha_j > 0$, then f is strictly concave.

Proof. Left as an exercise. ■

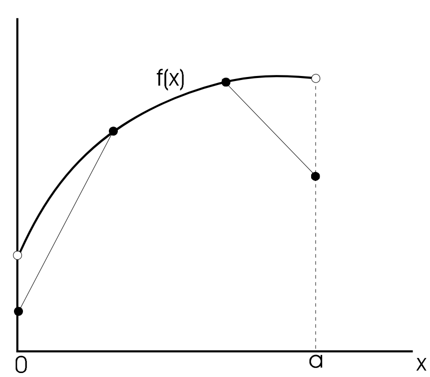
Since a constant function is concave, Theorem 1 implies
- If f is (strictly) concave, then any affine transformation $\alpha f + \beta$ with $\alpha > 0$ is also (strictly) concave.

The Continuity and Differentiability of Concave Functions

A concave function need not be differentiable everywhere. For example $f(x) = -|x|$ is concave function that is not differentiable at 0.



- However, we prove in the Appendix that right and left hand derivatives always exist on the interior of the domain and that $f^{-}(x) \geq f^{+}(x)$. For example, if $f(x) = -|x|$, then $f^{-}(0) = 1$ and $f^{+}(0) = -1$.
- Since both righthand and lefthand deriviates exist on the interior, it follows from our earlier results on differentiable functions that f is both right and left continuous and therefore continuous. However, concave functions need not be continuous at the boundary as illustrated in the example below.



A Characterization of Differentiable Concave Functions

For differentiable functions, the following theorem provides a simple necessary and sufficient conditions for concavity.

Theorem 2: Suppose $f : X \to \mathbb{R}$ is differentiable. (a) f is concave if and only if for each $x, y \in X$, we have

$$f(y) - f(x) \leq f'(x)\,(y - x) \tag{1}$$

(b) f is strictly concave if and only if the inequality is strict for each $x \neq y$.

Proof. (only if) (a) Suppose f is concave. The theorem is trivial if $y = x$. So suppose $y \neq x$.Then for any $\lambda \in (0, 1)$, we have

$$f(\lambda y + (1 - \lambda)\,x) \geq \lambda f(y) + (1 - \lambda)\,f(x)$$

which can be restated as

$$f(\lambda\,(y - x) + x) \geq \lambda\,(f(y) - f(x)) + f(x)$$

which implies

$$\frac{f(\lambda\,(y - x) + x) - f(x)}{\lambda\,(y - x)}\,(y - x) \geq f\,(y) - f(x).$$

But since this relation holds for all $\lambda \in (0, 1)$, it follows that

$$f\,(y) - f(x) \leq \lim_{\lambda \downarrow 0}\left(\frac{f(\lambda\,(y - x) + x) - f(x)}{\lambda\,(y - x)}\right)(y - x) = f'(x)(y - x).$$

(only if) (b) Suppose f is concave, but $f(y) - f(x) = f'(x)(y - x)$ for some $x \neq y$. We will show that f is not strictly concave.

Consider any $z = \lambda y + (1 - \lambda)\,x$, where $0 < \lambda < 1$. Then the definition of concavity implies

$$\begin{aligned}
\lambda f(y) + (1 - \lambda)\,f(x) &= \lambda(f(y) - f(x)) + f(x) \\
&= \lambda f'(x)(y - x) + f(x) \quad \text{(by hypothesis)} \\
&= f'(x)\,(z - x) + f(x) \quad (\text{since } z - x = \lambda\,(y - x)\,) \\
&\geq f(z) \quad (\text{using part (a) and he assumed concavity of } f) \\
&= f(\lambda y + (1 - \lambda)\,x).
\end{aligned}$$

which violates the strict concavity of f.

(if) (a) Suppose relation (1) holds and consider any $x, y \in X$ with $x \neq y$. Next consider any $\lambda \in (0, 1)$ and let $z = \lambda y + (1 - \lambda)\,x = \lambda\,(y - x) + x$. We need to show that relation (1) implies

$$f(z) = f(\lambda y + (1 - \lambda)\,x) \geq \lambda f(y) + (1 - \lambda)\,f(x)$$

Observe first that

$$\begin{aligned}
x - z &= \lambda(x - y) \\
y - z &= (1 - \lambda)\,(y - x)\,.
\end{aligned}$$

Letting z play the role of x in relation (1), we have

$$\begin{aligned} f(x) - f(z) &\leq f'(z)\,(x - z) \\ f(y) - f(z) &\leq f'(z)\,(y - z) \end{aligned}$$

So if we multiply the top relation by $1 - \lambda$ and the bottom equation by λ, and substitute for $x - z$ and $y - z$, we have

$$\begin{aligned} (1 - \lambda)\,(f(x) - f(z)) &\leq \lambda\,(1 - \lambda)\,f'(z)\,(x - y) \\ \lambda\,(f(y) - f(z)) &\leq \lambda\,(1 - \lambda)\,f'(z)\,(y - x) \end{aligned}$$

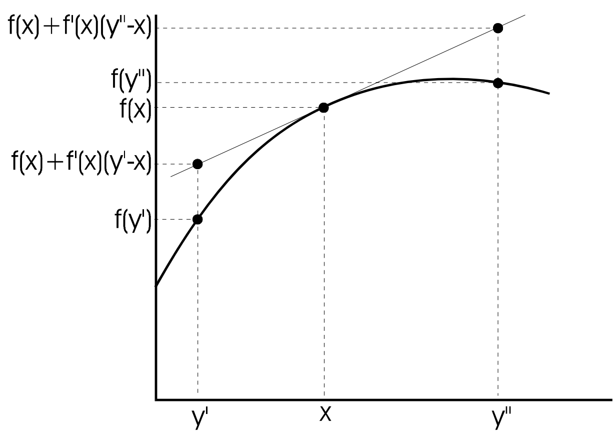
Adding the two relations terms and rearranging terms then yields

$$(1 - \lambda)\,f(x) + \lambda f(y) \leq f(z)$$

which yields the desired inequality.

(if) (b) If the inequality in relation (1) is strict, then all of the inequalities in the proof of (if) (a) are strict and therefore f is strictly concave. ■

Theorem 2 is illustrated below. Notice that the inequality holds both for the case where $y > x$ and $y < x$.



- We show in the Appendix that even if a function is not differentiable everywhere, it is concave if and only if for each $x \in int(X)$, there is an $a \in \mathbb{R}$ such that $f(y) - f(x) \leq a(y - x)$ for all $y \in X$. This is an example of a *supporting hyperplane* (for one dimension) that is an important tool in economic analysis.

Slope of the First Derivative Function

Theorem 2 implies that the first derivative function of a concave function is nonincreasing.

Theorem 3: (a) Suppose $f : X \to \mathbb{R}$ is differentiable. (a) f is concave if and only if f' is nonincreasing. (b) f is strictly concave if and only if f'' is strictly decreasing.

Proof. Consider any $x, y \in X$ such that $x < y$.

(only if) (a) Suppose f is concave. Then by Theorem 2, we have

$$\begin{aligned} f(y) - f(x) &\leq f'(x)(y - x) && (2) \\ f(x) - f(y) &\leq f'(y)(x - y) = -f'(y)\,(y - x) && (3) \end{aligned}$$

Adding these two equations yields

$$0 \leq \big(f'(x) - f'(y)\big)\,(y - x)$$

which implies that $f'(x) \geq f'(y)$

(only if) (b) If f is strictly concave, the the inequalities (2) and (3) are strict. Therefore, $0 < (f'(x) - f'(y))\,(y - x)$ which implies that f'' is strictly decreasing.

(if) (a) Suppose f' is nonincreasing. Then $f'(u) \leq f'(x)$ for all $u \in [x, y]$. Therefore, by the fundamental theorem of calculus, we have

$$f(y) - f(x) = \int_x^y f'(u)du \leq \int_x^y f'(x)du = f'(x)\,(y - x) \tag{4}$$

which implies that f is concave by Theorem 2.

(if) (b) If f' is strictly decreasing, then $f'(u) < f'(x)$ for all $u \in (x, y]$. Therefore, the inequality is relation (4) is strict and it follows that f is strictly concave by Theorem 2. ■

Concave Functions and the Second Derivative

Theorem 4: Suppose $f : X \to \mathbb{R}$ is twice differentiable. (a) f is a concave if and only if $f'' \leq 0$. (b) If $f'' < 0$, then f is strictly concave.

Proof. Let $x, y \in X$ with $x < y$.

(a) (only if) Suppose f is concave. Then Theorem 3 implies $f'(y) \leq f'(x)$. Therefore,

$$f''(x) = \lim_{y \downarrow x} \frac{f'(y) - f'(x)}{y - x} \leq 0.$$

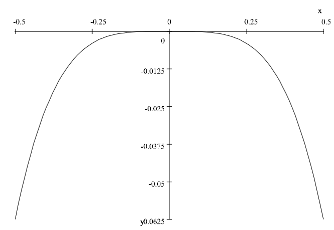
(a) (if) Suppose $f'' \leq 0$. Then the fundammental theorem of calculus implies

$$f'(y) - f'(x) = \int_x^y f''(u)du \leq 0.$$

Therefore f' is nondecreasing and it follows from Theorem 3 that f is concave.

(b) Suppose $f'' < 0$. Then the inequality above is strict. Therefore f is strictly decreasing and Theorem 3 implies that f is strictly concave. ■

Notice that f strictly concave does not imply that $f''(x) < 0$ for **all** x. For instance $f(x) = -x^4$ is a strictly concave function. However $f''(x) = -12x^2$ which implies that $f''(0) = 0$.



Some Important Concave Functions.

Theorem 4 implies that the following functions $f : \mathbb{R}_{++} \to \mathbb{R}$ are strictly concave:

i. $f(x) = \frac{x^\alpha}{\alpha}$ for $\alpha \neq 0$, $\alpha < 1$.

ii. $f(x) = \log x$.

iii. $f(x) = ax - bx^2$ (where $b > 0$)

Convex Functions

$f : X \to \mathbb{R}$ is a *convex* function if for any $x, y \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$f : X \to \mathbb{R}$ is a *strictly convex* function if for any $x, y \in X$, we have, for all $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)$$

NOTE: A convex set and a convex function are two distinct concepts.

The following lemma is an immediate consequence of the definitions of concave and convex functions.

Lemma 3: f is a (strictly) convex function if and only if $-f$ is a (strictly) concave function.

Given Lemma 3, it follows that all of the properties of concave functions carry over to convex functions, perhaps with a change in sign. In particular, we have

- If $f : X \to \mathbb{R}$ is convex and differentiable, then $f(z) - f(x) \geq f'(x)(z - x)$ for all $x, z \in X$.
- If f is twice differentiable, then f is convex if and only if $f'' \geq 0$.
- If $f'' > 0$, then f is strictly convex.
- If f is (strictly) convex then $\alpha f + \beta$ with $\alpha > 0$ is (strictly) convex.

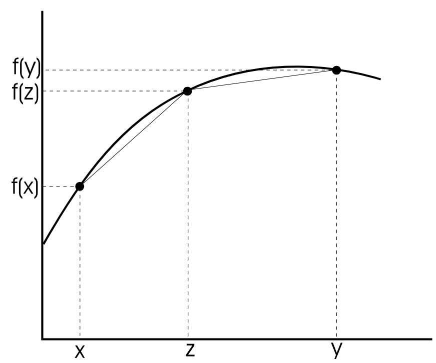
Appendix

For the proof of the results below, we will use the following characterization of a concave function.

Lemma A1: f is concave if and only if for any $x < z < y$, we have

$$\frac{f(y)-f(z)}{y-z} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}.$$

f is strictly concave if and only if the inequalities are strict.



Proof. Choose any $x < z < y$ and define $\lambda = \frac{z-x}{y-x}$. Then $(1-\lambda) = \frac{y-z}{y-x}$ and $z = \lambda y + (1-\lambda)\, x$. Then

$$f(z) \geq \lambda f(y) + (1-\lambda)\, f(x)$$

is equivalent to

$$(1-\lambda)\,(f(z) - f(x)) \leq \lambda\,(f(y) - f(z))$$

Substituting for λ yields and multiplying by $(y-x)$ yields the equivalent statement.

$$(y-z)\,(f(z) - f(x)) \geq (z-x)\,(f(y) - f(z))$$

Adding $(z-x)\,(f(z) - f(x))$ to each side and simplying terms we obtain the equivalent statement

$$(y-x)\,(f(z) - f(x)) \geq (z-x)\,(f(y) - f(x))$$

which is equivalent to

$$\frac{f(z)-f(x)}{z-x} \geq \frac{f(y)-f(x)}{y-x}$$

Similarly, adding $(y-z)\,(f(y) - f(z))$ to each side yields, we obtain the equivalent statement

$$(y-z)\,(f(y) - f(x)) \geq (y-x)\,(f(y) - f(x))$$

and therefore the equivalent statement.

$$\frac{f(y)-f(x)}{y-x} \geq \frac{f(y)-f(z)}{y-z}.$$

The equivalence of each statement also holds if each inequality is strict.

Since f is concave if and only if the first statement is true for all $x < z < y$, the theorem is proved.
■

Lemma A2: Suppose $f : X \to \mathbb{R}$ is concave. Then $x \in int(X)$ implies $f^-(x)$ and $f^+(x)$ exist and $f^-(x) \geq f^+(x)$.

Proof. Note first that Lemma A1 implies that $\frac{f(y)-f(x)}{y-x}$ is nonincreasing in x and y for $x \neq y$. Therefore for all $x < z < y$, we have

$$f^+(z) \equiv \lim_{y \downarrow z} \frac{f(y)-f(z)}{y-z} \leq \frac{f(z)-f(x)}{z-x}.$$

and

$$f^+(z) \leq \lim_{x \uparrow z} \frac{f(z)-f(x)}{y-x} \equiv f^-(z)$$

■

Theorem A1: (a) $f : X \to \mathbb{R}$ is concave if and only if for all $z \in int(X)$ and all $x, y \in X$ with $x < z < y$, we have

$$\begin{aligned} f(z) - f(x) &\geq f^-(z)\,(z-x) \\ f(y) - f(z) &\leq f^+(z)\,(y-z)\,. \end{aligned}$$

(b) f is strictly concave if and only if the inequalities are strict for $x, y \neq z$.

Proof. (only if) Suppose f is concave. The statement is trivial if $y = x$, so suppose that $y \neq x$. Then Lemma A1 implies

$$f^-(y) \equiv \lim_{z \uparrow y} \frac{f(y)-f(z)}{y-z} \leq \frac{f(y)-f(x)}{y-x} \leq \lim_{z \downarrow x} \frac{f(z)-f(x)}{z-x} \equiv f^+(x)$$

Multiplying through by $(y-x)$ then yields the result.

If f is strictly concave, then Lemma A1 implies that the inequalities are strict.

(if) Choose any $x, y \in X$ with $x > y$ and $\lambda \in (0,1)$. Define $z = \lambda x + (1-\lambda)\,y$. By assumption

$$\begin{aligned} f^+(z)\,(y-z) &\geq f(y) - f(z) \\ f^-(z)\,(z-x) &\leq f(z) - f(x) \end{aligned}$$

But $y - z = \lambda(y-x)$ and $z - x = (1-\lambda)\,(y-x)$. Therefore

$$\begin{aligned} \lambda f^+(z)\,(y-x) &\geq f(y) - f(z) \\ (1-\lambda)\,f^-(z)\,(y-x) &\leq f(z) - f(x) \end{aligned}$$

Now if we multiply the first relation by $(1-\lambda)$ and the second relation by λ, and subtract the second from the first, we have

$$0 \geq (1-\lambda)\,\lambda\left(f^+(z) - f^-(z)\right)(y-x) \geq (1-\lambda)\,f(y) + \lambda f(x) - f(z)$$

which, from the definition of z, implies

$$f(\lambda x + (1-\lambda)\,y) \geq \lambda f(x) + (1-\lambda)\,f\,(y)\,.$$

If the initial inequalities are strict, then the concavity is strict. ■

Corollary A1: If $f : X \to \mathbb{R}$ is concave, then for any $x \in int(X)$, there is an $a \in \mathbb{R}$ such that $f(z) - f(x) \leq a(z - x)$ for all $z \in X$.

Proof. Choose a such that $f^{+}(x) \geq f^{-}(x)$. The corollary then follows from Theorem A1. ■

Note that Theorem 2 of the text also follows as an immediate corollary of Theorem A1.