Arithmetic Series

The arithmetic series formula computes the sum of all integers from 1 to n as $\frac{n(n+1)}{2}$. We can use induction to prove the arithmetic series formula:

$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

The base case of $n = 1$ is easy to check: $1 = \frac{1 \cdot 2}{2}$. The induction step from n to $n+1$ for every $n \geq 1$ can be proven as follows:

$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1) = (\sum_{i=1}^{n} i) + (n+1) = \frac{n(n+1)}{2} + (n+1)$

where the last equality is due to the induction hypothesis.

Now

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

which finishes the proof by mathematical induction.

While we finished the proof of this beautiful formula, this proof did not reveal a way of arriving at this formula. There is no general recipe for finding such expressions, but the following tricks are often handy.

Arithmetic trick

When looking at the sum $1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$, one may want to pair up the first and the last numbers, then the second and the second to last numbers, and so on. Indeed, each such pair has a sum $(n+1)$, and the number of pairs is $n/2$. A simple visualization of this trick is to write all the numbers from 1 to n in a row, and then write them back one more time in a row below it. We will have n columns each with sum $(n+1)$.

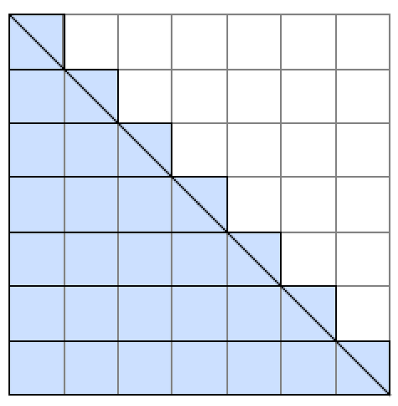
$$\begin{array}{ccccccccc} 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\ \hline (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) \end{array}$$

This trick was allegedly discovered by "the greatest mathematician since antiquity'' Carl Gauss (see more on Wikipedia).

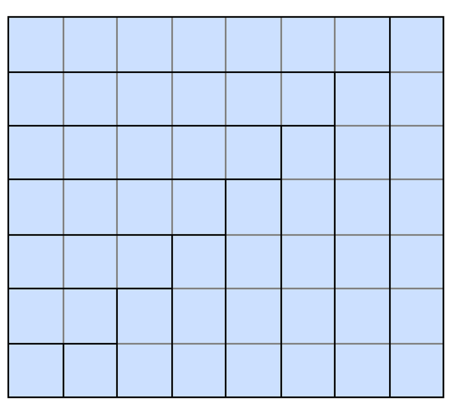
Johann Carl Friedrich Gauss (1777--1855). (Source: Wikipedia.)

Geometric trick

Think of each number i as a strip of length i and width 1. Then the total area of these strips is half the area of an $n \times n$ square plus n little triangles of area $\frac{1}{2}$: total of $\frac{n^2}{2} + n \cdot \frac{1}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$, see the picture below.



Another way to visualize this formula is to take one vertical and one horizontal strip of length i for every i. You can fill in an $(n+1) \times n$ rectangle using these strips:



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