Assignment: Assignment 1

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Assignment 1: A Primal-Dual Algorithm for Set Cover

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A Primal-Dual Algorithm for Set Cover.

In this exercise, we propose to design a primal-dual algorithm for the set cover problem.

The set cover problem is as follows: given a set of elements $E = \{e_1, \ldots, e_n\}$, some subsets of those elements $S_1, S_2, \ldots, S_m \subseteq E$, and a non-negative weight w_j for each subset S_j. The goal is to find a minimum-weight collection of subsets that contains all the elements of E. Namely, we want to find a collection I of subsets that minimizes $\sum_{j \in I} w_j$ and such that subject to $\bigcup_{j \in I} S_j = E$.

Throughout the exercise, we will consider the following linear program LP for the problem.

$\min \sum_{j=1}^{m} x_j \cdot w_j$

subject to,

$\forall i \in \{1, \ldots, n\}, \quad \sum_{j\,:\,e_i \in S_j} x_j \geq 1$

$\forall j \in \{1, \ldots, m\}, \quad x_j \geq 0$

Question 1: What is the dual of this linear program?

We now consider the following primal-dual algorithm.

1. $y \leftarrow 0$
2. $I \leftarrow \emptyset$
3. While I is not a solution (there exists $e_i \notin I$):

- Increase the dual variable y_i of an element e_i that is not covered until there exists an l such that $\sum\limits_{j: e_j \in S_l} y_j = w_l$
- Add the set S_l to I

4. Return I

Correctness.

Question 2: In how many iterations of the while loop can a given dual variable be increased?

Question 3: Using Question 2, argue that the algorithm terminates and so, that the output I is a solution to the problem.

Approximation Ratio. In this section, we assume that each element of the set E appears in at most f sets of $S_1, \ldots, S_m$.

Question 4: Recall a tight lower bound between the value of the optimal fractional solution for the dual val($y*$) and the value of the optimal integral solution for the set cover problem OPT.

Question 5: Argue that the solution y is feasible for the dual.

Question 6: Combining Questions 4 and 5, recall a tight lower bound between the value of the solution y and the value of the optimal integral solution for the set cover problem OPT.

In the following, we want to show

$\sum_{j:S_j \in I} w_j \leq f \cdot \mathrm{val}(y)$.

Question 7: Consider a set $S_j \in I$. What is the relationship between w_j and $\sum_{i:e_i \in S_j} y_i$?

Question 8: Using Question 7, give the relationship between $\sum_{j \in I} w_j$ and the variables y_i.

Question 9: Recall that $|\{j\ :\ e_i \in S_j\}| \leq f$ for all i. Using Question 8, prove that $\sum_{j:S_j \in I} w_j \leq f \cdot \mathrm{val}(y)$.

Question 10: Conclude using Questions 6 and 9.

Answers

1. Dual: max $\sum_{i=1}^{n} y_i$

subject to,

$\forall j \in \{1, \ldots, m\}, \quad \sum_{i:e_i \in S_j} y_i \leq w_j$

$\forall i \in \{1, \ldots, n\}, \quad y_i \geq 0$

2. A given dual variable y_i corresponds to a given element e_i. If e_i is still uncovered, let's consider all the sets $S_k \notin I$ and $e_i \in S_k$ (There must exist at least one such set S_k. If not, then e_i can never be covered by the sets $S_1 \ldots S_m$). Now, in the while loop, when the variable y_i is incremented the value $\sum_{j:e_j \in S_k} y_j$ gets incremented for all those sets S_k (since $y_i \in S_k,\ \forall k$). But each S_k is upper-bounded by a w_k (which is finite, w.l.o.g.), so we can't go on increasing y_i indefinitely while being in the feasible region. Hence, there will be some $l \in k$ for which the constraint $\sum_{i:e_i \in S_k} y_i \leq w_k$ will become tight in the same iteration and it will take just one iteration for a particular element e_i after which the corresponding set S_l will be added to I.

3. Since, as argued in 2, each iteration of the while loop covers an element e_i yet to be covered (and the corresponding set covering the element is added to I), it will take at most n iterations in the worst case to cover all the n elements in E. Hence, the while loop will terminate after at most n iterations with all the n elements covered in the solution .

4. By Weak Duality, we have $val(y*) \leq val(x*)$ where $x*$ is the fractional optimal solution for **Primal** and $y*$ is the fractional optimal solution for the **Dual**. Also, the **OPT** is going to be the optimal integral solution for the **Primal minimization** problem $\Rightarrow val(x*) \leq OPT$. Combining, we have $val(y*) \leq OPT$.

5. Since the while loop guarantees that each of the constraints of the dual are satisfied (all the y_i variables are non-negative and the constraints are at most tight, s.t., $\sum_{i:e_i \in S_j} y_i \leq w_j$ is satisfied $\forall i$, with equality for the sets S_l that are added to I from inside the while loop). Hence the solution y remains feasible.

6.Since y is a feasible solution and $val(y*)$ is the optimal solution for the **Dual maximization** problem, we have $val(y) \leq val(y*)$. Combining with 4 and 5, we have, $val(y) \leq OPT$.

7. For any set $S_j \in I$, we shall have $\sum_{i:e_i \in S_j} y_i = w_j$, as guaranteed by the while loop.

8. Hence, $\sum_{j:S_j \in I} w_j = \ \sum_{j:S_j \in I}\ \sum_{i:e_i \in S_j} y_i = \sum_{i=1}^{n} |j : e_i \in S_j|\ y_i$, since all the elements are there in I, with each of them probably present multiple times in multiple sets. Here $|j : e_i \in S_j|$ present the number of times the element e_i was present in I.

9. Also, given, $|j : e_i \in S_j| \leq f$. Hence, we have,
$\sum_{j:S_j \in I} w_j = \ \sum_{i=1}^{n} |j : e_i \in S_j|\ y_i \leq \ \sum_{i=1}^{n} f.y_i = \ f.\sum_{i=1}^{n} y_i = \ f.val(y) \Rightarrow \ \sum_{j:S_j \in I} w_j \leq \ \ f.val(y)$.

10. Combining $OPT \geq val(y)$ (from 6) and the value of the solution provided by the **Primal-Dual algorithm** $\leq \ \ f.val(y)$ (from 9), we get the **approximation ratio** for this algorithm $= f$.

The answer to this question is the following:

$\max \sum_{i=1}^{n} y_i$

subject to,

$\forall j \in \{1, \ldots, n\}, \sum_{i\,:\,e_i \in S_j} y_i \leq w_j$

$\forall i \in \{1, \ldots, n\}, \quad y_i \geq 0$

1 pt
Yes

0 pts
No

Answer to Question 2 is 1.

1 pt
Yes

0 pts
No

An answer to Question 3 is of the following form:

Whenever a dual variable is increased, the corresponding element is covered at the end of the iteration. Therefore, after at most n iterations of the while loop, all the elements are covered.

2 pts
Yes

0 pts
No

Answer to Question 4 is :

val($y*$) $\leq$ OPT.

2 pts
Yes

0 pts
No

An answer to Question 5 is of the following form:

Remark first that we start with a feasible solution for the dual.

Then, each variable is increased while it does not violate any constraint. Therefore, all the constraints of the LP are satisfied at the end of the while loop.

2 pts
Yes

0 pts
No

An answer to Question 6 has to be of the following form :

val(y) $\leq$ OPT

2 pts
Yes

0 pts
No

An answer to Question 7 has to be of the following form :

$w_j = \sum_{i:e_i \in S_j} y_i$.

- 3 pts
  Yes
- 0 pts
  No

Help Center

An answer to Question 8 is :

$\sum_{j \in I} w_j = \sum_{j \in I} \sum_{i:e_i \in S_j} y_i$

- 1 pt
  Yes
- 0 pts
  No

An answer to Question 9 has to be of the following form :

$\sum_{j \in I} w_j = \sum_{j \in I} \sum_{i:e_i \in S_j} y_i$ implies that $\sum_{j \in I} w_j = \sum_{i=1}^{n} y_i |\{j \,:\, e_i \in S_j \text{ and } j \in I\}|$ and so, $\sum_{j \in I} w_j \leq \sum_{i=1}^{n} y_i f = f \cdot \mathrm{val}(y)$.

- 4 pts
  Yes
- 0 pts
  No

An answer to Question 10 is as follows:

The value of the algorithm is $\sum_{j \in I} w_j$.

We have by Question 9 $\sum_{j \in I} w_j \leq \sum_{i=1}^{n} y_i f = f \cdot \mathrm{val}(y)$.

Therefore, by Question 6, $\sum_{j \in I} w_j \leq f \cdot$ OPT

- 2 pts
  Yes
- 0 pts
  No

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