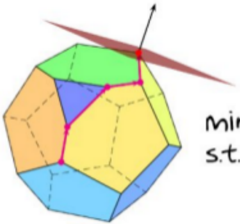
Weak duality via LP duality

- Integer programming
- LP-relaxations
- Node-edge incidence matrix

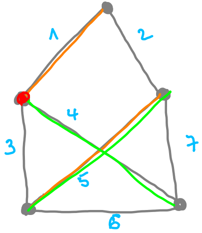


$$\min c^T x \quad \text{s.t. } Ax \leq b$$

Towards a second proof of weak duality via LP-duality

Idea

Describe *characteristic* vectors χ^M of matchings by linear constraints and the *integrality* constraint.



Enumerate edges

Matching can be described as 0/1-vector

$$\chi^M = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\chi^M = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Quizz:

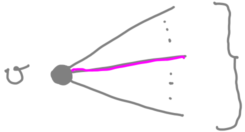
IS $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ a characteristic vector of a matching

YES NO

The description

For $v \in V$ we denote the set of edges *incident* to v by

$$\delta(v) = \{e \in E \colon v \in e\}.$$

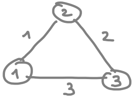


$\delta(v)$

The set $\{\chi^M : M$ matching of $G\}$ is the set of *feasible solutions of*

$$\begin{array}{ll} v \in V: & \sum_{e \in \delta(v)} x_e \leqslant 1 \\ e \in E: & x_e \in \{0, 1\}. \end{array}$$

Quiz:



$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$x \in \{0,1\}^3$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$x \in \{0,1\}^3$

Integer programming

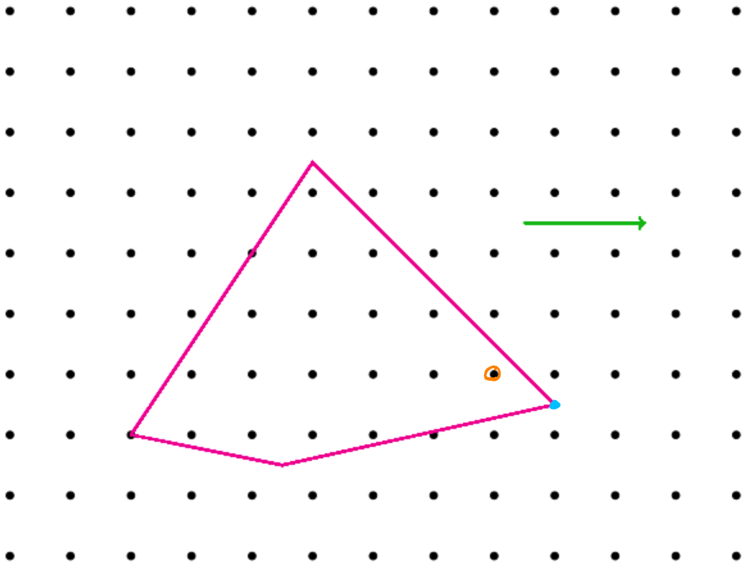
Integer program

An optimization problem of the form

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leqslant b \\ & \boxed{x \in \mathbb{Z}^{\mathbf{n}}} \end{aligned} \qquad x \in \mathbb{R}^n$$

is an *integer linear program*.

Integer programming



Integer programming

Integer program

An optimization problem of the form

$$\begin{aligned} \max \quad & c^T x \\ & Ax \leqslant b \\ & \boxed{x \in \mathbb{Z}^{\mathbf{n}}} \end{aligned}$$

is an *integer linear program*.

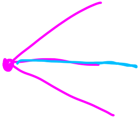
- The integer programming problem is NP-hard. Unless $P = NP$, it cannot be solved in polynomial time.
- Many researchers believe that $P \neq NP$.
- The $P \neq NP$ question is one of the *millennium prize* problems of the *Clay Mathematics Institute*. (1 million $ prize)

An integer programming formulation of max-weight matching

$$\max \sum_{e \in E} w_e \cdot x_e$$

$$\begin{array}{ll} v \in V: & \sum_{e \in \delta(v)} x_e \leqslant 1 \\ e \in E: & x_e \geqslant 0 \\ & \mathbf{x} \in \mathbb{Z}^{|\mathbf{E}|}. \end{array}$$

$$x_e = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$$



feasible solutions are characteristic vectors of MATCHINGS

An integer programming formulation of max-weight matching

$$\max \textstyle\sum_{e\in E} w_e \cdot x_e$$

$$v \in V: \quad \textstyle\sum_{e\in\delta(v)} x_e \leqslant 1$$
$$e \in E: \quad x_e \geqslant 0$$
$$\mathbf{x} \in \mathbb{Z}^{|\mathbf{E}|}.$$

Integer program

$\geq$

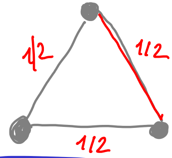
$\leq$

$$\max \textstyle\sum_{e\in E} w_e \cdot x_e$$

$$v \in V: \quad \textstyle\sum_{e\in\delta(v)} x_e \leqslant 1$$
$$e \in E: \quad x_e \geqslant 0$$
$$\mathbf{x} \in \mathbb{R}^{|\mathbf{E}|}.$$

LP relaxation

Quiz



$$\max \sum_{e \in E} w_e \cdot x_e$$

$$\forall v \in V: \sum_{e \in \delta(v)} x_e \leq 1$$

$$x \geq 0$$

ALL weights $= 1$

Max. cardinality of matching : $\boxed{1}$

Max. value of LP-relax : $\boxed{3/2}$

$$2 \cdot x_1 + 2 \cdot x_2 + 2 \cdot x_3 \leq 3$$

$$x_1 + x_2 + x_3 \leq 3/2$$

An IP formulation of min. w-vertex cover

$$\min \textstyle\sum_{v \in V} y_v$$

$$\begin{aligned} uv \in E: &\quad y_u + y_v \geqslant w_{uv} \\ v \in V: &\quad y_v \geqslant 0 \end{aligned}$$

$$\mathbf{y} \in \mathbb{Z}^{|\mathbf{V}|}.$$

$y_u + y_v \geq w_{uv}$

Quiz

$\leq$

$$\min \textstyle\sum_{v\in V} y_v$$

$$\begin{array}{rl} uv \in E: & y_u + y_v \geqslant w_{uv} \\ v \in V: & y_v \geqslant 0 \end{array}$$

$$\mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}.$$

$\geq$

$$\min \textstyle\sum_{v\in V} y_v$$

$$\begin{array}{rl} uv \in E: & y_u + y_v \geqslant w_{uv} \\ v \in V: & y_v \geqslant 0 \end{array}$$

$$\mathbf{y} \in \mathbb{Z}^{|\mathbf{V}|}.$$

LP- Relaxation

Min. value w- vertex cover problem

IP - FORMULATION

Proving weak duality via LP duality

Theorem

The max. weight of a matching is at most the min. value of a w-vertex cover.

$$
\begin{array}{rl}
\multicolumn{2}{c}{\max \sum_{e \in E} w_e \cdot x_e} \\
v \in V: & \sum_{e \in \delta(v)} x_e \leqslant 1 \\
e \in E: & x_e \geqslant 0 \\
\multicolumn{2}{c}{\mathbf{x} \in \mathbb{Z}^{|\mathbf{E}|}.}
\end{array}
\quad \leq \quad
\begin{array}{rl}
\multicolumn{2}{c}{\max \sum_{e \in E} w_e \cdot x_e} \\
v \in V: & \sum_{e \in \delta(v)} x_e \leqslant 1 \\
e \in E: & x_e \geqslant 0 \\
\multicolumn{2}{c}{\mathbf{x} \in \mathbb{R}^{|\mathbf{E}|}.}
\end{array}
\quad = \quad
\begin{array}{rl}
\multicolumn{2}{c}{\min \sum_{v \in V} y_v} \\
uv \in E: & y_u + y_v \geqslant w_{uv} \\
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\multicolumn{2}{c}{\mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}.}
\end{array}
\quad \leq \quad
\begin{array}{rl}
\multicolumn{2}{c}{\min \sum_{v \in V} y_v} \\
uv \in E: & y_u + y_v \geqslant w_{uv} \\
v \in V: & y_v \geqslant 0 \\
\multicolumn{2}{c}{\mathbf{y} \in \mathbb{Z}^{|\mathbf{V}|}.}
\end{array}
$$

MAX MATCHING

MIN. VAL. W-VERTEX COVER

$\text{MAX MATCHING} \leq \text{MIN. VAL. W-VERTEX COVER}$

Proving weak duality via LP duality (cont.)

$$\max \sum_{e \in E} w_e \cdot x_e$$

$$\begin{aligned} v \in V: &\quad \sum_{e \in \delta(v)} x_e \leqslant 1 \\ e \in E: &\quad x_e \geqslant 0 \end{aligned}$$

$$\mathbf{x} \in \mathbb{R}^{|\mathbf{E}|}.$$

$$\min \sum_{v \in V} y_v$$

$$\begin{aligned} uv \in E: &\quad y_u + y_v \geqslant w_{uv} \\ v \in V: &\quad y_v \geqslant 0 \end{aligned}$$

$$\mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}.$$

$$\max w^T x \qquad \qquad \min \mathbf{1}^T y$$

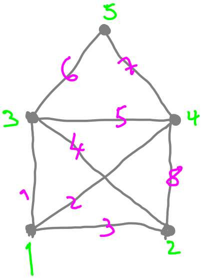
$$\begin{aligned} Ax &\leqslant \mathbf{1} \\ x &\geqslant 0 \end{aligned} \qquad = \qquad \begin{aligned} A^T y &\geqslant w \\ y &\geqslant 0 \end{aligned}$$

Back to matchings: The node-edge incidence matrix

Let $G = (V, E)$ be a graph and suppose the nodes and edges are ordered as $v_1, \dots, v_n$ and $e_1, \dots, e_m$. The matrix $A^G \in \{0, 1\}^{n \times m}$ with

$$A^G_{i,j} = \begin{cases} 1 & \text{if } v_i \in e_i, \\ 0 & \text{otherwise} \end{cases}$$

is the *node-edge incidence* matrix of G.



$A_G =$

	1	2	3	4	5	6	7	8
1	1	1	1	0	0	0	0	0
2	0	0	1	1	0	0	0	1
3	1	0	0	1	1	1	0	0
4	0	1	0	0	1	0	1	1
5	0	0	0	0	0	1	1	0

Proving weak duality via LP duality (cont.)

$$\max \sum_{e \in E} w_e \cdot x_e$$

$$v \in V: \quad \sum_{e \in \delta(v)} x_e \leqslant 1$$

$$e \in E: \quad x_e \geqslant 0$$

$$\mathbf{x} \in \mathbb{R}^{|\mathbf{E}|}.$$

$$\min \sum_{v \in V} y_v$$

$$uv \in E: \quad y_u + y_v \geqslant w_{uv}$$

$$v \in V: \quad y_v \geqslant 0$$

$$\mathbf{y} \in \mathbb{R}^{|\mathbf{V}|}.$$

$$\max w^T x \qquad = \qquad \min \mathbf{1}^T y$$

$$A^G x \leqslant \mathbf{1} \qquad\qquad (A^G)^T y \geqslant w$$

$$x \geqslant 0 \qquad\qquad y \geqslant 0$$

Weak duality via LP duality

Lemma (Weak duality)

Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{N}_0$ be edge-weights. If M is a matching of G and if y is a w-vertex cover of G, then

$$w(M) \leqslant \sum_{v \in V} y_v.$$