Ergodicity Theorem

The foundation of Markov chain theory is the Ergodicity Theorem. It establishes the conditions under which a Markov chain can be analyzed to determine its *steady state* behavior.

A Markov chain can be characterized by the properties of its states. A Markov chain is

- transient if all of its states are transient
- recurrent nonnull if all of its states are recurrent nonnull
- periodic if all of its states are periodic
- aperiodic if all of its states are aperiodic

An irreducibile Markov chain is one in which all states are reachable from all other states (i.e., all states *communicate* - see below).

An irreducible Markov chain is one of the following:

- transient
- recurrent nonnull
- recurrent null

and if it is recurrent, then it is either periodic or aperiodic.

An *ergodic* Markov chain is

- irreducible,
- recurrent nonnull, and
- aperiodic

Most of the systems in which we are interested are modeled with ergodic Markov chains, because this corresponds to a well-defined steady state behavior.

Let $\underline{\pi} = \{\pi_1 \quad \pi_2 \quad \dots \quad \pi_k\}$ be the limiting distribution for the state probabilities (the number of states may be infinite). That is,

$\pi_i = \lim_{n \to \infty} p_i(n)$, where $p_i(n)$ is the probability of being in state i at time n, $n = 0,1,2,\dots$

<u>Ergodicity Theorem</u>

If a Markov chain is ergodic, then a unique steady state distribution $\underline{\pi}$ exists, independent of the initial state $\underline{p}(0) = \{p_1(0) \quad p_2(0) \quad \dots \quad p_k(0)\}$.

Although we will not prove this theorem, ergodicity comes into play roughly as follows:

- aperiodic and recurrent nonnull $\Rightarrow \underline{\pi}$ exists
- irreducible $\Rightarrow \underline{\pi}$ is unique and independent of $\underline{p}(0)$

Some shortcuts exist for helping to determine when a Markov chain is ergodic.

1. A Markov chain with a finite number of states has only transient and recurrent nonnull states (in other words, only a Markov chain with an infinite number of states can be recurrent null).
2. A sufficient test for a state to be aperiodic is that it has a "self-loop" (that is, the probability that the next state is the same as the current state is non-zero) or that it communicates with an aperiodic state. Two states i and j communicate if i is reachable from j and vice versa.
3. In an irreducible, finite state Markov chain, the presence of one aperiodic state guarantees ergodicity.

keywords:

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