UNIVERSITY OF LONDON

Probability and Statistics: To p, or not to p?

Module Leader: Dr James Abdey

6.1 Decision tree analysis

Week 1 introduced the concept of **decision making under uncertainty** whereby decisions are taken in the present with **uncertain future outcomes**.

Decision tree analysis is an interesting modelling technique which allows us to incorporate probabilities in the decision-making process to model and quantify the uncertainty.

Of course, Section 2.1 explained that we could determine probabilities using one of three methods:

- **subjectively**
- **by experimentation (empirically)**
- **theoretically**.

Some examples of managerial decisions under uncertainty include:

- selection of suppliers (which? how many?)
- research and development (R&D) and investment decisions (which project? how many resources?)
- hiring/promotion decisions (to hire, or not to hire?).

In what follows we will be concerned with **decision analysis**, i.e. where there is only one rational decision-maker making non-strategic decisions.

(**Game theory** involves two or more rational decision-makers making strategic decisions. Interested learners are encouraged to study this fascinating field in the future.)

Example

Imagine you are an ice-cream manufacturer, and for simplicity suppose your level of sales can be either high or low. Hence high and low sales are mutually exclusive and collectively exhaustive in this example.[1]

If sales are high you earn a profit of £300,000 (excluding advertising costs), but if sales are low you experience a loss of £100,000 (excluding advertising costs).

You have the choice of whether to advertise your product or not. Advertising costs would be fixed at £100,000.

If you advertise sales are high with a probability of 0.9, but if you do not advertise sales are high with a probability of just 0.6. Note advertising does not guarantee success (not all advertising campaigns are successful!) so here we model advertising as increasing the probability of the 'good' outcome (i.e. high sales). Viewed as conditional probabilities, these are:

$$P(\text{high sales} \,|\, \text{no advertising}) = 0.6$$
$$P(\text{low sales} \,|\, \text{no advertising}) = 0.4$$

and:

$$P(\text{high sales} \,|\, \text{advertising}) = 0.9$$
$$P(\text{low sales} \,|\, \text{advertising}) = 0.1.$$

A standard decision tree consists of the following components:

- **Decision nodes** indicate that the decision-maker has to make a choice, denoted □.
- **Chance nodes** indicate the resolution of uncertainty, denoted ○.
- **Branches** represent the choices available to the decision-maker (if leading from decision nodes) or the possible outcomes if uncertainty is resolved (leading from chance nodes).
- **Probabilities** are written at the branches leading from chance nodes.
- **Payoffs** are written at the end of the final branches.

A decision tree has the following properties:

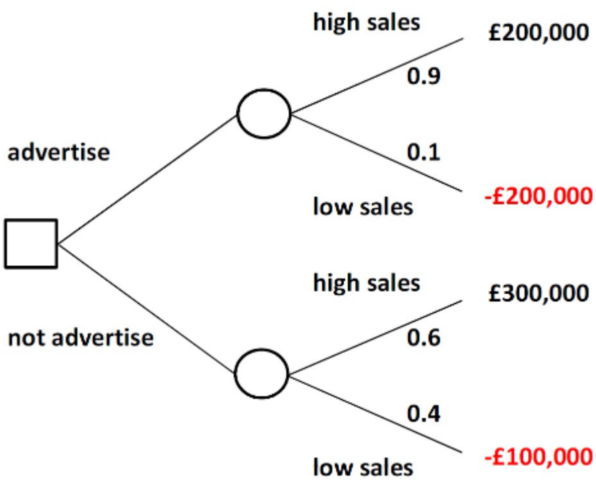
- No loops.
- One initial node.
- At most one branch between any two nodes.

[1]Clearly, in practice a continuum of sales levels could be expected. Applied here would needlessly complicate the analysis – the binary set of outcomes of high and low sales is sufficient to demonstrate the principle and use of decision tree analysis.

- Connected paths.
- At a decision node the decision-maker has information on all preceding events, in particular on the resolution of uncertainty.

We are now in a position to draw the decision tree for the ice-cream manufacturer problem. Note that decision trees are read from left to right, representing the time order of events in a logical manner.

The decision tree is:



On the far left the tree begins with a decision node where we have to decide whether to advertise or not *without knowledge of whether sales turn out to be high or low.* After the decision is made, chance takes over and resolves the realised level of sales according to the respective probability distribution, ultimately resulting in our payoff. Note that the payoffs in the top half of the decision tree (£200,000 and −£200,000) are simply the corresponding payoffs of £300,000 and −£100,000, less the fixed advertising costs of £100,000.

In order to solve the decision tree we calculate the **expected monetary value (EMV)** of each option (advertise and not advertise) and proceed whereby the decision-maker maximises **expected profits**.

The EMV is simply an expected value, and so in this discrete setting we apply our usual probability-weighted average approach. We have:

$$\begin{aligned} \text{E(advertise)} &= 0.9 \times £200{,}000 + 0.1 \times -£200{,}000 \\ &= £160{,}000 \end{aligned}$$

and:

$$\begin{aligned} \text{E(not advertise)} &= 0.6 \times £300{,}000 + 0.4 \times -£100{,}000 \\ &= £140{,}000. \end{aligned}$$

Hence the **optimal (recommended) strategy** is to advertise, since this results in a higher expected payoff (£160,000 > £140,000).

Remember that an expected value should be viewed as a long-run average. Clearly, by deciding to advertise we will *not* make £160,000 – the possible outcomes are either a profit of £200,000 (with probability 0.9) or a loss of £200,000 (with probability 0.1).

However, rather than a 'one-shot' game, imagine the game was played annually over 10 years. By choosing 'advertise' each time, then we would *expect* high sales in 9 years (each time with a profit of £200,000) and low sales in 1 year (with a loss of £200,000), reflecting the probability distribution used. Hence in the long run this would average to be £160,000.

Of course, this example has failed to account for **risk** – the subject of Section 6.2...