More Coffee

Problem. There are two equally sized cups: cup 1 contains coffee and cup 2 contains milk. Both cups are half full (we are optimists). Your favorite drink is $1/3$ coffee and $2/3$ milk. Can you get such a drink in cup 1 by transferring (any amount of) liquid between the two cups? Any amount of your favorite drink would work --- the right proportion is what matters.

We are going to show that, at any point of time, *at least half of the drink in cup 1 is coffee* (yes, this is our invariant!). Since the total amounts of coffee and milk are equal (and will remain equal after any number of pourings), this is the same as saying that *at least half of the drink in cup 2 is milk*. And this implies that we cannot achieve our goal: no matter what we do, cup 1 will always contain at least as much coffee as milk.

To show that this property holds, we start by noting that it clearly holds in the very beginning: cup 1 contains coffee only whereas cup 2 contains milk only. Now, consider the two allowed operations. When we pour drink from cup 1 to cup 2, the proportion of milk and coffee in cup 1 stays the same. Hence, cup 1 still contains at least as much coffee as milk. And when we pour drink from cup 2 to cup 1, the proportion in cup 2 does not change. This means that cup 2 still contains at least as much milk as coffee and this, in turn, means that cup 1 contains at least as much coffee as milk.