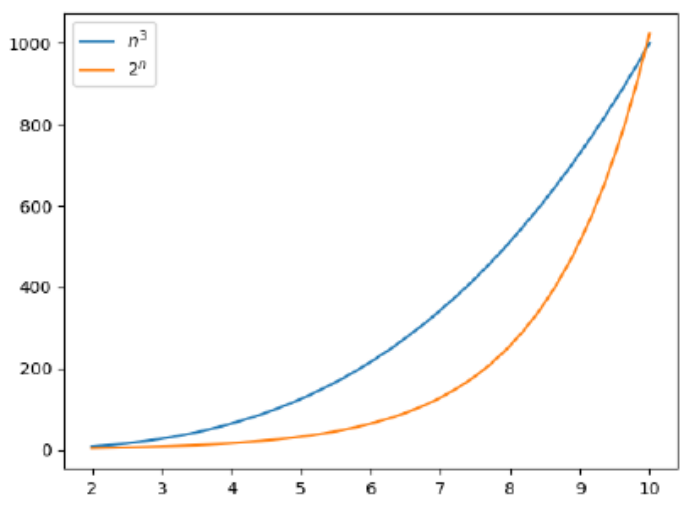
Where to Start Induction?

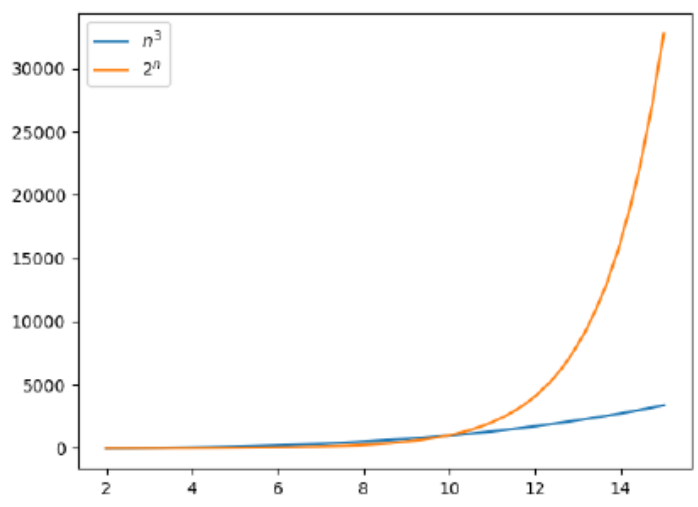
Sometimes we prove statements which do not necessarily hold for all $n \geq 1$, but hold only for $n \geq c$ for some number c. To do this, we can just start our induction base at $n = c$ rather than at $n = 1$.

Problem:
Prove that $2^n \geq n^3$ for all $n \geq 10$.

The following plots show that this statement is not even true for $n < 10$.

```
1   import matplotlib.pyplot as plt
2   import numpy as np
3
4   for m in [10, 15]:
5       plt.clf()
6       n = np.linspace(2, m)
7       plt.plot(n, n ** 3, label='$n^3$')
8       plt.plot(n, 2 ** n, label='$2^n$')
9       plt.legend(loc='upper left')
10      plt.savefig(f'plotn3vs2n{m}.png')
```





We can prove the statement $2^n \geq n^3$ for all $n \geq 10$ using mathematical induction. The base case of $n = 10$ is easy to check: $2^n = 1024 > 1000 = n^3$. For the step from n to $n + 1$, the left-hand side is multiplied by 2, but the right-hand side is multiplied by $\frac{(n+1)^3}{n^3} = \left(1 + \frac{1}{n}\right)^3$. For $n \geq 10$, this expression is bounded from above by $1.1^3 = 1.331 < 2$. Thus, for every $n \geq 10$, we multiply the greater left side by a larger number, and have that $2^n \geq n^3$.