Le problème des rencontres

de Montmort, 1708

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- What is the chance that no one gets back his own hat?
- What is the chance that exactly k people recover their hats?

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The number of individuals who retrieve their own hats is governed approximately by a Poisson distribution with mean 1.