Feedback — Assignment 5

You submitted this quiz on **Sun 24 Mar 2013 4:04 AM PDT -0700**. You got a score of **36.00** out of **36.00**.

Question 1

Which of the options is the correct dual of the linear program below?

$$\begin{aligned} \min\ & c^T x \\ \text{subject to}\ & Ax \le b \\ & x \le 0 \end{aligned}$$

Your Answer		Score	Explanation
○ $\begin{aligned} \max\ & b^T y \\ \text{subject to}\ & A^T y \le c \\ & y \ge 0 \end{aligned}$			
○ $\begin{aligned} \max\ & b^T y \\ \text{subject to}\ & A^T y \ge c \\ & y \ge 0 \end{aligned}$			
○ $\begin{aligned} \max\ & b^T y \\ \text{subject to}\ & A^T y \le c \\ & y \le 0 \end{aligned}$			
◉	✓	3.00	

$$\begin{aligned} &\max\ b^T y \\ \text{subject to}\ \ &A^T y \geq c \\ &y \leq 0 \end{aligned}$$

Total	3.00 / 3.00

Question Explanation

First convert the primal program into standard form

$$\begin{aligned} &\max\ -c^T x \\ \text{subject to}\ \ &\begin{pmatrix} A \\ I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix}. \end{aligned}$$

Then, form the dual of the program

$$\begin{aligned} &\min\ b^T y_1 + 0^T y_2 \\ \text{subject to}\ \ &A^T y_1 + y_2 = -c \\ &y_1, y_2 \geq 0. \end{aligned}$$

Simplify the above dual program by throwing away the "slack" variable y_2

$$\begin{aligned} &\min\ b^T y_1 \\ \text{subject to}\ \ &A^T y_1 \leq -c \\ &y_1 \geq 0. \end{aligned}$$

Then replace $-y_1$ by y to get the result

$$\begin{aligned} &\max\ b^T y \\ \text{subject to}\ \ &A^T y \geq c \\ &y \leq 0. \end{aligned}$$

Question 2

Which of the options is the correct dual of the linear program below?

$$\begin{aligned} &\max\ 2x_1 + 3x_2 + 5x_3 \\ \text{subject to}\ \ &3x_1 + 2x_2 + x_3 \leq 4 \\ &x_2 + 2x_3 \leq 6 \\ &x_3 \leq 5 \\ &x_1 \geq 0 \end{aligned}$$

Your Answer		Score	Explanation
○ $\begin{aligned} \min\ & 4y_1 + 6y_2 + 5y_3 \\ \text{subject to}\ & 3y_1 + y_4 = 2 \\ & 2y_1 + y_2 = 3 \\ & y_1 + 2y_2 + y_4 = 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$.			
○ $\begin{aligned} \min\ & 4y_1 + 6y_2 + 5y_3 \\ \text{subject to}\ & 3y_1 = 2 \\ & 2y_1 + y_3 = 3 \\ & y_1 + 2y_2 + y_3 = 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$.			
○ $\begin{aligned} \min\ & 4y_1 + 6y_2 + 5y_3 \\ \text{subject to}\ & 3y_1 - y_4 = 2 \\ & 2y_1 + y_3 = 3 \\ & y_1 + 2y_2 + y_3 = 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$.			
◉ $\begin{aligned} \min\ & 4y_1 + 6y_2 + 5y_3 \\ \text{subject to}\ & 3y_1 \geq 2 \\ & 2y_1 + y_2 = 3 \\ & y_1 + 2y_2 + y_3 = 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$.	✓	3.00	
Total		3.00 / 3.00	

Question Explanation

After multiplying the last inequality by a -1, we first put the linear program in the standard form where our A matrix reads $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$, b vector reads as $\begin{pmatrix} 4 \\ 6 \\ 5 \\ 0 \end{pmatrix}$, and our c vector reads as $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$. Since we know the dual of our standard primal linear program is

$$\begin{aligned} \min\ & b^T y \\ \text{subject to}\ & A^T y = c \\ & y \geq 0 \end{aligned}$$

we can write the dual linear program as

$$\begin{aligned} \min\ & 4y_1 + 6y_2 + 5y_3 + 0y_4 \\ \text{subject to}\ & 3y_1 - y_4 = 2 \\ & 2y_1 + y_2 = 3 \\ & y_1 + 2y_2 + y_3 = 5 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

We see that we can simplify the above by eliminating the slack variable y_4 that doesn't appear in the objective function to get the final answer

$$\begin{aligned} \min\ & 4y_1 + 6y_2 + 5y_3 \\ \text{subject to}\ & 3y_1 \geq 2 \\ & 2y_1 + y_2 = 3 \\ & y_1 + 2y_2 + y_3 = 5 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

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Question 3

Consider a directed graph $G = (N, A)$ consisting of a set of n nodes V and m arcs A (arcs are directed edges connecting distinct nodes i.e., we denote an arc $a \in A$ with starting node u and ending node v as $a = (u, v)$. Consider the node-arc incidence matrix $M \in \mathbb{R}^{n \times m}$ defined for the graph G as

$$M_{ij} = \begin{cases} 1 & \text{if node } i \text{ is the starting node of arc } j \\ -1 & \text{if node } i \text{ is the ending node of arc } j. \\ 0 & \text{otherwise} \end{cases}$$

Is this node-arc incidence matrix M always totally unimodular for every graph G?

Your Answer		Score	Explanation
Yes	✓	5.00	
No in general, yes only for bipartite graphs.			
Yes in general, except for bipartite graphs.			
Total		5.00 / 5.00	

Question Explanation

The proof that the node-arc incidence matrix M for any graph G is by induction. For $1x1$ sub-matrices the statement is immediate. Consider any $k \times k$sub-matrix L of M. If there is a column that contains at most 1 non-zero element then we expand along this column and use the induction hypothesis to conclude that M is also totally unimodular. If not, then every column has exactly two non-zero entries $+1$ and -1. This means that summing all the rows of the matrix we get the zero vector. Hence the sub-matrix L is singular and has determinant 0.

Question 4

In this question we are now going to see that solving a linear optimization problem is actually no harder than determining the feasibility of a set of linear inequalities. Suppose we had a *feasible* and *bounded* linear program in the standard form in n variables and m constraints:

$$\begin{gathered} \max\ c^T x \\ \text{subject to } \ Ax \leq b. \end{gathered}$$

Of the following systems of inequalities there is exactly one that has the property that the above linear program is feasible and has a bounded optimal solution if and only if the system of inequalities is feasible. Which one is it?

Hint: Think about what you know about feasible and bounded linear programs and their duals.

Your Answer	Score	Explanation
(selected) $\begin{bmatrix} -c^T & b^T \\ c^T & -b^T \\ A & 0_{m\times m} \\ 0_{n\times n} & A^T \\ 0_{n\times n} & -A^T \\ 0_{m\times n} & -I_{m\times m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ b \\ c \\ -c \\ 0_{m\times 1} \end{bmatrix}$	✓ 5.00	
$\begin{bmatrix} -c^T & b^T \\ c^T & -b^T \\ A & 0_{m\times m} \\ 0_{n\times n} & A^T \\ 0_{n\times n} & -A^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ b \\ c \\ -c \end{bmatrix}$		
$\begin{bmatrix} -c^T & b^T \\ c^T & -b^T \\ A & 0_{m\times m} \\ 0_{n\times n} & A^T \\ 0_{n\times n} & -A^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ b \\ c \\ -c \end{bmatrix}$		
$\begin{bmatrix} -c^T & b^T \\ c^T & -b^T \\ A & 0_{m\times m} \\ 0_{n\times n} & A^T \\ 0_{n\times n} & -A^T \\ 0_{m\times n} & -I_{m\times m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ b \\ c \\ -c \\ 0_{m\times 1} \end{bmatrix}$		
Total	5.00 / 5.00	

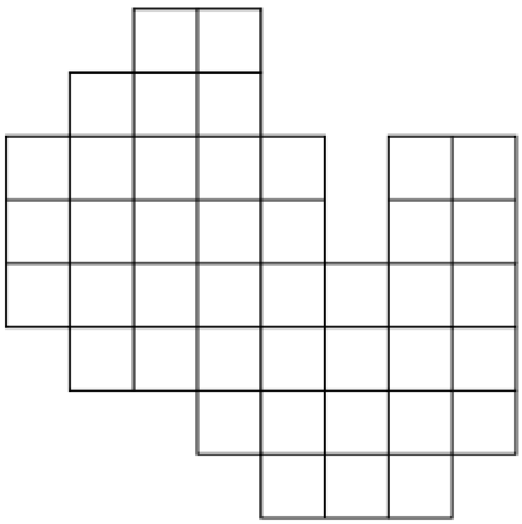
Question Explanation

We know that a feasible and bounded primal linear program also has a feasible and bounded dual linear program whose optimal solutions are equal. So first we write down the dual of the above linear program that maximizes $b^T y$ and has the constraints $A^T y = c$ and $y \geq 0$. By our initial observation it must be true that there exists a primal feasible x and a dual feasible y such that $c^T x = b^T y$. We simply write down

all the inequalities that are implied in the preceding statement to obtain the following system
$$\begin{bmatrix} -c^T & b^T \\ c^T & -b^T \\ A & 0_{m\times m} \\ 0_{n\times n} & A^T \\ 0_{n\times n} & -A^T \\ 0_{m\times n} & -I_{m\times m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ b \\ c \\ -c \\ 0_{m\times 1} \end{bmatrix}$$

Question 5

Can this figure be tiled by dominoes (a domino being two adjacent squares)?

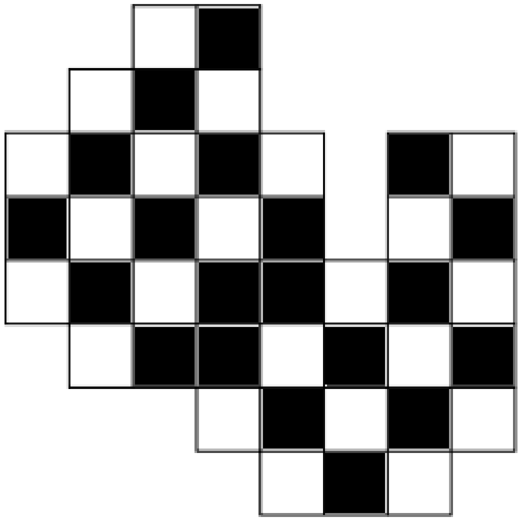


Hint: This is a fun question based on the lecture "Matchings and vertex covers"!

Your Answer		Score	Explanation
No	✓	5.00	
Yes			
Total		5.00 / 5.00	

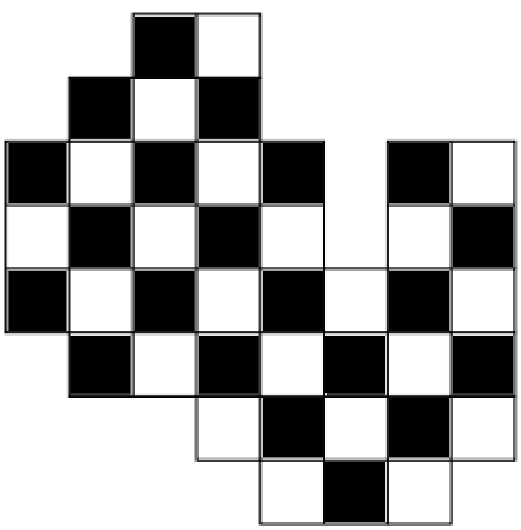
Question Explanation

First transform this figure into a bipartite graph by thinking of each square as a vertex and two vertices being adjacent if and only if the corresponding squares share a side. This graph is bipartite since it has no odd cycles (if you start from a square and move an odd number of steps then you cannot be on the starting square). Now, finding a domino tiling of the figure is exactly the same as a maximum matching of size 21 in the bipartite graph (since there are 42 squares in the figure). But here is a vertex cover of size 20:



From linear programming duality, we know that the size of the maximum matching is always equal to the size of the minimum vertex cover for bipartite graphs. This shows that every matching for this bipartite graph has size at most 20, and hence there is no perfect matching for the graph. So there exists no such domino tiling of the figure in the question.

Note: A first attempt at constructing such a vertex cover will probably lead you to form a vertex cover of size 21 which only implies that every matching is of size at most 21, which is an obvious statement for this graph. To improve it to a vertex cover of lesser size, the main intuition used in the above construction is that it is possible to shift some black squares diagonally and recolor the figure so that we use one black square less due to the nature of the given figure.



Question 6

Assume we run the simplex method on a bounded and feasible linear program . Can we also extract a proof of optimality from our implementation of the simplex method once it terminates after reaching the optimal vertex? Is this proof also short (polynomial in the size of the input)?

Your Answer		Score	Explanation
◉ Yes, the proof is short.	✓	1.00	
○ Yes, but the proof is not short.			
○ No, there is no such proof.			
Total		1.00 / 1.00	

Question Explanation

The set of multipliers λ_B in our simplex algorithm is non-negative after termination since we found an optimal vertex (whose basis is B). But we know that λ is zero outside of the indices in the set B, and that $\lambda_B^T A_B = c^T \implies \lambda^T A = c^T$ for a $\lambda \geq 0$ and thus we have a dual feasible solution λ. Using weak duality, this implies that this vertex is also optimal for the primal program. So our proof of optimality would simply constitute the vector λ.

Question 7

In a bounded and feasible primal maximization linear program, which of the following can **never** happen when a constraint is removed?

Your Answer		Score	Explanation
☐ The dual becomes infeasible	✓	0.75	
☐ The dual optimum stays the same	✓	0.75	
☐ The dual optimum increases	✓	0.75	
☑ The dual optimum decreases	✓	0.75	
Total		3.00 / 3.00	

Question Explanation

By strong duality we know that there exist primal and dual feasible solutions such that their objective function values are equal. When we remove a primal constraint, then either it becomes unbounded or stays bounded (but feasible in both cases). Also, in either case, the primal objective value can only improve (since every feasible solution to the original linear program is still feasible), which means that the dual linear program is either infeasible (if the primal becomes unbounded) or is bounded (if the primal stays bounded) again by strong duality. Hence, it can never be the case that the dual optimum decreases, while it is possible that the dual becomes infeasible.

Question 8

Which of the following matrices are totally unimodular?

Your Answer		Score	Explanation
☑ $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$	✓	0.33	This matrix is the node-edge incidence matrix of a bipartite graph with bipartitions $\{1,2,3\}$ and $\{4,5\}$ and edges between every pair of vertices from different bipartitions. To see this, use the vertex numbering $\{1,4,2,3,5\}$
☐ $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	✓	0.33	The determinant of this matrix is 2. This represents the node-edge incidence matrix of a 3-cycle or a triangle graph.
☑ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	✓	0.33	This matrix is the node-edge incidence matrix of a bipartite graph with bipartitions $\{1\}$ and $\{2,3,4\}$ and edges between $\{1\}$ and each of the vertices in $\{2,3,4\}$.
Total		1.00 / 1.00	

Question 9

Suppose A were a square totally unimodular matrix. Which of the following are also totally unimodular?

Your Answer		Score	Explanation
☑ $\begin{pmatrix} A \\ -A \end{pmatrix}$	✓	1.00	Every submatrix of this matrix is either a submatrix of A or is a submatrix of A with some rows flipped in sign. Since A itself is totally unimodular this submatrix is totally unimodular too.
☐ $\begin{pmatrix} A & A \\ A^T & 0 \end{pmatrix}$	✓	1.00	This contains as a submatrix $\begin{pmatrix} A \\ A^T \end{pmatrix}$ which is not totally unimodular
☑ $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$	✓	1.00	Every submatrix of this matrix will either contain a repeated row or column of A or will be a submatrix of A. In each case the determinant is in the range $\{0, 1, -1\}$.
☑ $\begin{pmatrix} A \\ -I \end{pmatrix}$	✓	1.00	Every submatrix of this matrix is either a submatrix of A or will contain a row from $-I$ which contains at most one non-zero entry along which we can expand the determinant and obtain a smaller matrix of the same type which is totally unimodular by induction.
☑ $\begin{pmatrix} A \\ A \end{pmatrix}$	✓	1.00	Every submatrix of this matrix is either going to be a sub-matrix of A or will contain a repeated row of A. So in every case, the determinant of the resulting submatrix is $0, 1,$ or -1.
☐ $\begin{pmatrix} A \\ A^T \end{pmatrix}$	✓	1.00	Consider for example $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
Total		6.00 / 6.00	

Question 10

This question is based on two systems of inequalities involving a matrix $A \in \mathbb{R}^{m \times n}$.

The first system is $Ax > 0$ while the second system is $\begin{array}{c} A^T y = 0 \\ y \geq 0 \\ y \neq 0 \end{array}$. Of the options below, select the ones that apply to these two systems.

Your Answer		**Score**	**Explanation**
☐ If the second system is feasible then the first is feasible	✓	1.00	
☑ If the second system is feasible then the first is infeasible	✓	1.00	
☐ If the first system is feasible then the second is feasible	✓	1.00	
☑ If the first system is feasible then the second is infeasible	✓	1.00	
Total		4.00 / 4.00	

Question Explanation

Case i) Suppose the first system is feasible for an $x \in \mathbb{R}^n$. Assume towards contradiction that the second system is also feasible for a $y \in \mathbb{R}^m$. Then, $y^T Ax > 0$ since $y \geq 0$. Using $A^T y = 0$, we reach a contradiction. Hence the second system must be infeasible. Case ii) Suppose the second system is feasible for a $y \in \mathbb{R}^m$. Assume towards contradiction that the second system is also feasible for an $x \in \mathbb{R}^n$. Then, $x^T A^T y = 0$. Using $Ax > 0$ and the fact that y is non-negative and has a strictly positive component, we know that $x^T A^T y > 0$ reaching a contradiction. Hence the first system must be infeasible.