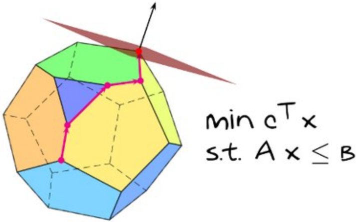
Paths, Cycles and Flows

- Weighted directed graphs
- Shortest paths
- Bellman-Ford Algorithm

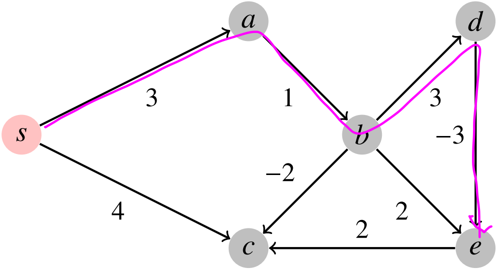


Weighted directed graphs

Let $D = (V, A)$ be a directed graph (without self loops). Let $\ell\colon A \to \mathbb{R}$ be the *lengths* of the arcs. The *length* of a walk $W = v_0, \ldots, v_k$ is the sum of the lengths of its arcs:

$$\ell(W) = \sum_{i=1}^{k} \ell(v_{i-1}, v_i).$$

The *distance* between two nodes s and t is the length of a *shortest path* from s to t.



s, a, b, c length = 2

$d_\ell(s,t)$

$d_\ell(s,e) = \boxed{4}$

Shortest path problem

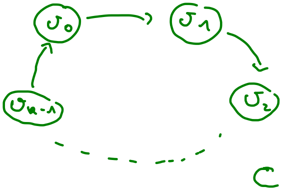
The shortest path problem (single source)

Given a directed graph with edge lengths and a designated node s, compute $d(s, v)$ for each $v \in V$.

- Is NP-complete in general.
- Can be solved in polynomial time, if there are no negative cycles.

A *cycle* is a walk $v_0, v_1, \ldots, v_k$ with $v_0 = v_k$.

$$\ell(C) = \sum_{i=0}^{k-1} \ell(v_i, \underbrace{v_{i+1}}_{\text{mod } k})$$



The Bellman-Ford method

A method to compute minimum length *walks*.

Given: $D = (V, A)$ (no self-loops), $\ell : A \to \mathbb{R}$ and designated node $s \in V$

Goal: Compute shortest path distances from s to all other nodes

Assumption: Each node is reachable from s

The Bellman-Ford method (cont.)

For $k \geqslant 0$ and $t \in V$:

$d_k(t) =$ minimum length of any $s - t$ walk, traversing at most k arcs. (possibly ∞)

$$\underbrace{s \to \circ \to \circ \cdots \cdots \to t}_{\text{traversing at most } k \text{ arcs}}$$

The Bellman-Ford method (cont.)

For $k \geqslant 0$ and $t \in V$:

$d_k(t) =$ minimum length of any $s - t$ walk, traversing at most k arcs. (possibly ∞)

$d_o(s) = 0, \quad d_0(t) = \infty, t \neq s$

The Bellman-Ford method (cont.)

For $k \geqslant 0$ and $t \in V$:

$d_k(t) =$ minimum length of any $s - t$ walk, traversing at most k arcs. (possibly ∞)

Suppose $d_i(t)$ is known for each $i \leqslant k$ and each $t \in V$.

The Bellman-Ford method (cont.)

For $k \geqslant 0$ and $t \in V$:

$d_k(t) =$ minimum length of any $s - t$ walk, traversing at most k arcs. (possibly ∞)

Suppose $d_i(t)$ is known for each $i \leqslant k$ and each $t \in V$.

Now: Compute $d_{k+1}(t)$: for each $t \in V$.

Case 1: ~~The~~ A shortest walk traversing at most $k + 1$ arcs traverses exactly $k + 1$ arcs.

$$\underbrace{s \rightsquigarrow u}_{d_k(u)} \longrightarrow v$$

$d_k(u) + \ell(u, v)$

The Bellman-Ford method (cont.)

For $k \geqslant 0$ and $t \in V$:

$d_k(t) =$ minimum length of any $s - t$ walk, traversing at most k arcs. (possibly ∞)

Suppose $d_i(t)$ is known for each $i \leqslant k$ and each $t \in V$.

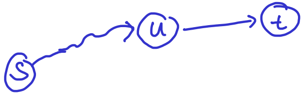
Now: Compute $d_{k+1}(t)$: for each $t \in V$.

Case 2: A ~~The~~ shortest walk traversing at most $k + 1$ arcs traverses at most k arcs.

$$d_{k+1}(t) = d_k(t)$$

The Bellman-Ford method (cont.)

$$d_o(s) = 0, \quad d_0(t) = \infty, t \neq s$$



$$k \geqslant 0,\ t \in V:\ d_{k+1}(t) = \min\{\underbrace{d_k(t),\ \min_{(u,t)\in A}\{d_k(u) + \ell(u,t)\}}\}.$$

- Each term is a valid upper bound on $d_{k+1}(t)$
- At least one of these terms <u>IS</u> $d_{k+1}(t)$

The Bellman-Ford method (cont.)

$$d_o(s) = 0, \quad d_0(t) = \infty, t \neq s$$

$$k \geqslant 0,\ t \in V:\ d_{k+1}(t) = \min\{d_k(t), \min_{(u,t)\in A}\{d_k(u) + \ell(u,t)\}\}.$$

Procedure to compute the values $d_{k+1}(t)$ *assuming* values $d_k(t)$ are pre-computed:

Valid upperbounds for $d_{k+1}(t)$

for each $t \in V$:

$\quad\quad d_{k+1}(t) := \underline{d_k(t)}$

for each $(u, t) \in A$

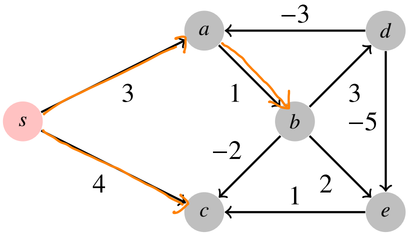
$\quad\quad$ **if**: $d_k(u) + \ell(u,t) < d_{k+1}(t)$

$\quad\quad\quad d_{k+1}(t) := \underline{d_k(u) + \ell(u,t)}$

Correct !

Example

$k = 0$



for each $t \in V$:
$\quad d_{k+1}(t) := d_k(t)$

for each $(u,t) \in A$
$\quad$ **if**: $d_k(u) + \ell(u,t) < d_{k+1}(t)$
$\quad\quad d_{k+1}(t) := d_k(u) + \ell(u,t)$

Quiz: Will (a,b) cause $d_1(b) = \infty$ to be updated?

◯ YES

~~◯~~ NO

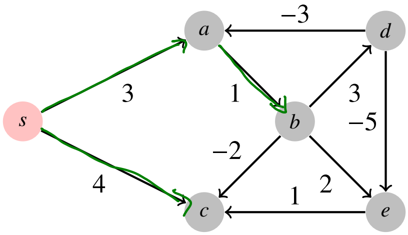
$d_0(s) + \ell(s,a) = 0 + 3 < \infty = d_1(a)$

$d_0(s) + \ell(s,c) = 0 + 4 < \infty = d_1(c)$

$d_0(a) + \ell(a,b) = \infty + 1 = \infty = d_1(b)$

	3	∞	4	∞	∞	d_1
0	∞	∞	∞	∞	∞	d_0
s	a	b	c	d	e	

Example



$k = 1$

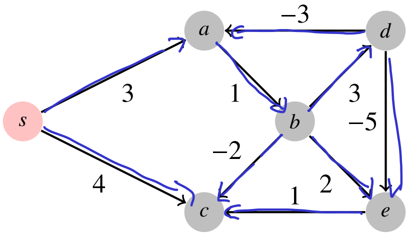
for each $t \in V$:
$$d_{k+1}(t) := d_k(t)$$

for each $(u, t) \in A$
if: $d_k(u) + \ell(u, t) < d_{k+1}(t)$
$$d_{k+1}(t) := d_k(u) + \ell(u, t)$$

	3	$\boxed{4}$	4	∞	∞	d_2
0	3	∞	4	∞	∞	d_1
0	∞	∞	∞	∞	∞	d_0
s	a	b	c	d	e	

Example

$k = 2$



for each $t \in V$:

$d_{k+1}(t) := d_k(t)$

for each $(u, t) \in A$

if: $d_k(u) + \ell(u,t) < d_{k+1}(t)$

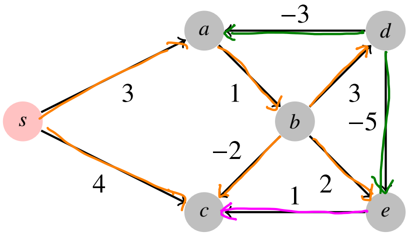
$d_{k+1}(t) := d_k(u) + \ell(u,t)$

$d_2(b) + 3 = 4 + 3 = 7 < \infty = d_3(d)$

$d_2(b) + 2 = 4 + 2 = 6 < \infty = d_3(e)$

	3	4	$\cancel{4}$ 2	7	6	
0	3	4	4	∞	∞	d_2
0	3	∞	4	∞	∞	d_1
0	∞	∞	∞	∞	∞	d_0
s	a	b	c	d	e	

Example



for each $t \in V$:

$\quad d_{k+1}(t) := d_k(t)$

for each $(u, t) \in A$

$\quad$ **if**: $d_k(u) + \ell(u,t) < d_{k+1}(t)$

$\quad\quad d_{k+1}(t) := d_k(u) + \ell(u,t)$

$k = 3$

$d_3(e) + 1$

$= 6+1 = 7 > 2$

Cause update of $d_4(c)$?

YES

NO

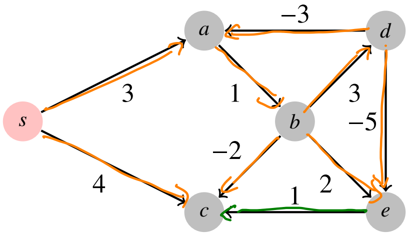
$d_3(d) - 3 = 7 - 3 = 4 \nless d_4(a) = 3$

$d_3(d) - 5 = 7 - 5 = 2 < d_4(e) = 6$

2

0	3	4	2	7	6	d_3
0	3	4	4	∞	∞	d_2
0	3	∞	4	∞	∞	d_1
0	∞	∞	∞	∞	∞	d_0
s	a	b	c	d	e	

Example



$k = 4$

for each $t \in V$:
$\qquad d_{k+1}(t) := d_k(t)$

for each $(u, t) \in A$
$\qquad$**if**: $d_k(u) + \ell(u,t) < d_{k+1}(t)$
$\qquad\quad d_{k+1}(t) := d_k(u) + \ell(u,t)$

$d_4(e) + 1 = 3 > 2 = d_5(c)$

0	3	4	2	7	2	d_6
0	3	4	2	7	2	d_4
0	3	4	2	7	6	d_3
0	3	4	4	∞	∞	d_2
0	3	∞	4	∞	∞	d_1
0	∞	∞	∞	∞	∞	d_0
s	a	b	c	d	e	

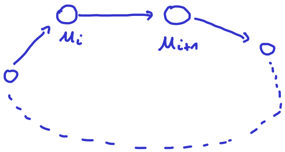
Negative cycles

Theorem

Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, one has $d_n = d_{n-1}$ for $n = |V|$ iff D does not have a cycle of negative length that is reachable from s.

Proof: "⇒" Suppose $u_0, u_1, u_2, \ldots, u_k, u_0$ is a cycle reachable from s

$d_{n-1}(u_i) < \infty \quad \forall i = 0, \ldots, k$



$$\boxed{0} = \sum_{i=0}^{k} \overbrace{d_n(u_{\underbrace{i+1}_{\text{mod}(k+1)}})}^{\le d_{n-1}(u_i) + \ell(u_i, u_{i+1})} - \underbrace{d_n(u_i)}_{= d_{n-1}(u_i)}$$

$$\le \sum_{i=0}^{k} \ell(u_i, u_{i+1}) = \ell(C)$$

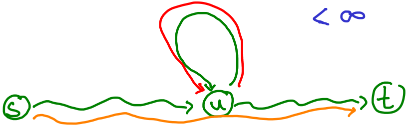
Negative cycles

Theorem

Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, one has $d_n = d_{n-1}$ for $n = |V|$ iff D does not have a cycle of negative length that is reachable from s.

"$\Leftarrow$" Suppose $\underbrace{d_n(t)}_{<\infty} < \underbrace{d_{n-1}(t)}_{<\infty}$ $\Rightarrow$ t is reachable from s

length of shortest s-t walk using exactly n arcs is < length of any s-t walk using n-1 arcs



W_1 $\quad s = w_0, w_1, \dots, \overset{u}{\overset{\shortparallel}{w_i}}, \dots, \overset{u}{\overset{\shortparallel}{w_j}}, \dots, w_n = t$

W_2 $\quad s = w_0, w_1, \dots, w_i, w_{j+1}, \dots w_n = t$

C $\quad w_i, w_{i+1}, \dots, w_j$

Negative cycles

Theorem

Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, one has $d_n = d_{n-1}$ for $n \geq |V|$ iff D does not have a cycle of negative length that is reachable from s.

W_1 $\quad s = w_0, w_1, \dots, \overset{u}{\overset{\shortparallel}{w_i}}, \dots, \overset{u}{\overset{\shortparallel}{w_j}}, \dots, w_n = t$

W_2 $\quad s = w_0, w_1, \dots, w_i, w_{j+1}, \dots, w_n = t$

C $\quad w_i, w_{i+1}, \dots, w_j$

$$\ell(W_1) = \sum_{\mu=0}^{i-1} \ell(w_\mu, w_{\mu+1}) + \overbrace{\sum_{\mu=i}^{j-1} \ell(w_\mu, w_{\mu+1})}^{\ell(C) < 0} + \sum_{\mu=j}^{n-1} \ell(w_\mu, w_{\mu+1})$$

$+$

$\ell(W_2)$

Shortest paths

Theorem

Given $D = (V, A)$, $s \in V$, $\ell : A \to \mathbb{R}$, and suppose that no negative cycle is reachable from s. Then for each $t \in V$ $d_{n-1}(t)$ is the distance between s and t.

Proof: Suppose $d_{n-1}(t) <$ length of shortest path from s to t.

Let W be a shortest walk from s to t using at most $n-1$ arcs and with a minimal number of arcs.

W :

$\ell(W) = \ell(W_2) + \ell(C)$

$\ell(W) < \ell(W_2)$.

$\Rightarrow \ell(C) < 0$ ↯ $\blacksquare$

Computing shortest paths

Compute the values $d_{k+1}(t)$ and the predecessor $\pi_{k+1}(t)$ *assuming* values $d_k(t)$ and $\pi_k(t)$ have been pre-computed:

$s \rightsquigarrow \pi_{k+1}(t) \to t$

for each $t \in V$:

$$d_{k+1}(t) := d_k(t)$$

$$\pi_{k+1}(t) := \pi_k(t)$$

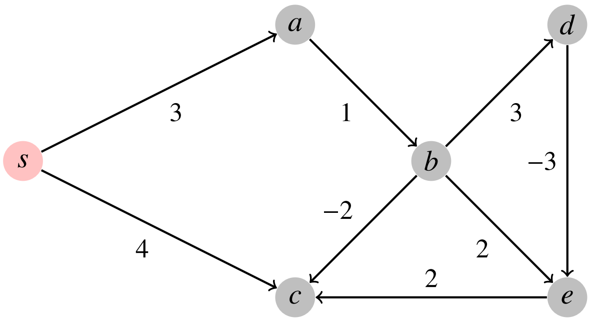
for each $(u, t) \in A$

if: $d_k(u) + \ell(u, t) < d_{k+1}(t)$

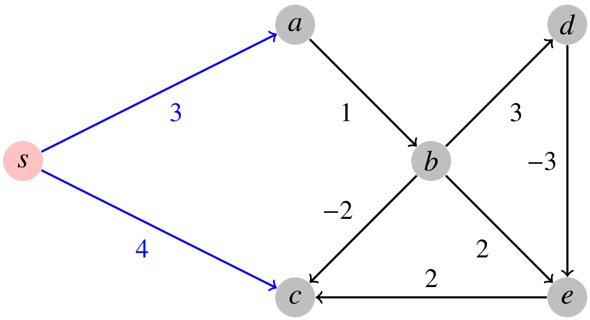
$$d_{k+1}(t) := d_k(u) + \ell(u, t)$$

$$\pi_{k+1}(t) := \boxed{u}$$

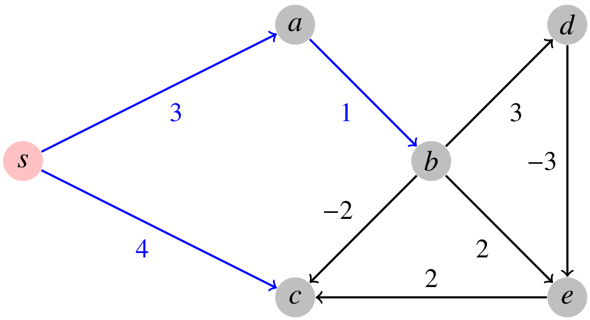
Example



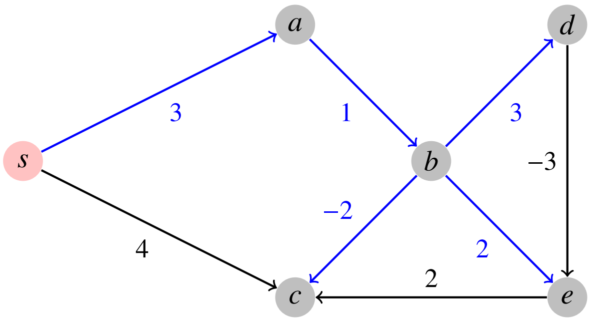
Example



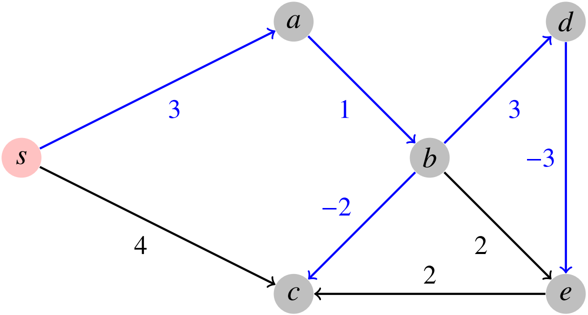
Example



Example



Example



The shortest path tree

No neg. cycles!

Theorem

Let $D = (V, A)$ be a directed graph and suppose that each node is reachable from s. The directed graph $T = (V, A')$ with $A' = \{(\pi_{n-1}(u), u) \colon u \in V \setminus \{s\}\}$ is a directed tree with root s. The unique path from s to any vertex t in T is a shortest path from s to t in D.

Running time of Bellman-Ford

$O(|V| \cdot |A|)$

initialize

$\forall t \in V \setminus \{s\}, d_o(t) = \infty,\ \pi_0(t) = 0$

$d_0(s) = 0$ $\Big\}\ O(|V|)$

$O(|V|(|V|+|A|))$ $\quad |A| = \Omega(|V|)$

for $k = 1$ to n

for each $t \in V$:

$d_{k+1}(t) := d_k(t)$

$\pi_{k+1}(t) := \pi_k(t)$ $\Big\}\ O(|V|)$

for each $(u, t) \in A$

if: $d_k(u) + \ell(u, t) < d_{k+1}(t)$

$d_{k+1}(t) := d_k(u) + \ell(u, t)$

$\pi_{k+1}(t) := u$ $\Big\}\ O(|A|)$

$O(|V|)$

if $\exists t \in V$ with $d_n(t) < d_{n-1}(t)$

D has negative cycle $\Big\}\ O(|V|)$

Running time of Bellman-Ford (cont.)

Theorem

Let $D = (V, A)$ be a directed graph with lengths ℓ and suppose that each node is reachable by $s \in V$, then Bellman-Ford runs in time $O(|V| \cdot |A|)$