Problem 1

The goal of this exercise is to show that for the purpose of optimization we can without loss of generality assume that a polyhedron is non-degenerate, *i.e.*, degeneracy does not make optimization harder in theory.

Consider a non-empty polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $\text{rank}(A) = n$. Note that P might be degenerate. We define its perturbed polyhedron

$$P'(\epsilon) = \{x \in R^n : Ax \leq b + \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \epsilon^3 \\ \vdots \\ \epsilon^m \end{pmatrix}\}$$

where $\epsilon > 0$.

Show that there exists a (small enough) ϵ such that all of the following hold.

(a) $P'(\epsilon)$ is non-degenerate,

(b) for any vertex in P there will be a vertex in P' with the same basis, *i.e.*, for any vertex x in P there exists a basis B for v such that B is also a feasible basis in $P'(\epsilon)$,

(c) any infeasible basis B in P is also infeasible in $P'(\epsilon)$,

(d) an optimal feasible basis in $P(\epsilon)$ is an optimal feasible basis in P.

Hint: For (a) and (c) use that a polynomial with finite degree can only have finitely many roots. For (b) show and use that the bases of the perturbed problem are continous functions in ϵ. Conclude (d) from (b) and (c).

Problem 2

Consider a degenerate polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. We will run the Simplex method on P by simulating the Simplex method on $P'(\epsilon)$ for an unknown, but small enough, ϵ. Assume a given feasible basis for P that is also feasible for $P'(\epsilon)$. Starting from this basis, formulate the Simplex algorithm on P that uses the same pivoting as its simulation on $P'(\epsilon)$.

Furthermore, prove that this version of the Simplex method does not cycle.