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$$\mathbf{P}\{X_1 + \dots + X_n = k\} = \binom{k}{k} S_k - \binom{k+1}{k} S_{k+1} + \dots + (-1)^j \binom{k+j}{k} S_{k+j} + \dots + (-1)^{n-k} S_n$$

The selected sample point  $\omega$  will lie in a certain number, say,  $l$  of the sets  $A_j$ :

- \* Identify the sets: suppose  $\omega$  is in  $A_{j_1}, A_{j_2}, \dots, A_{j_l}$ .
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# terms that $\omega$ contributes to on		
	LHS	RHS
$l < k$	0	0



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# A combinatorial identity



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