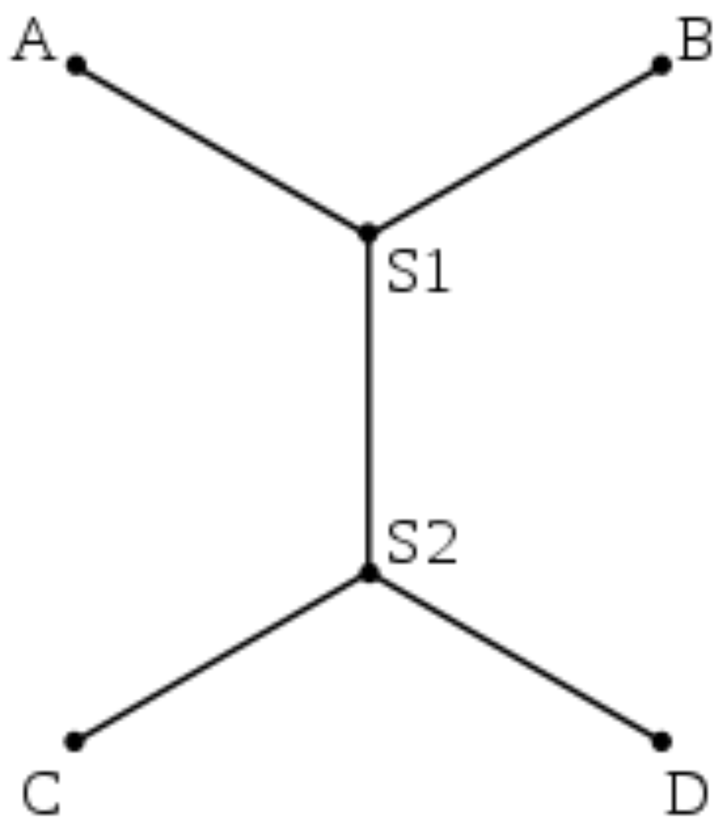
Approximation algorithms, Part II

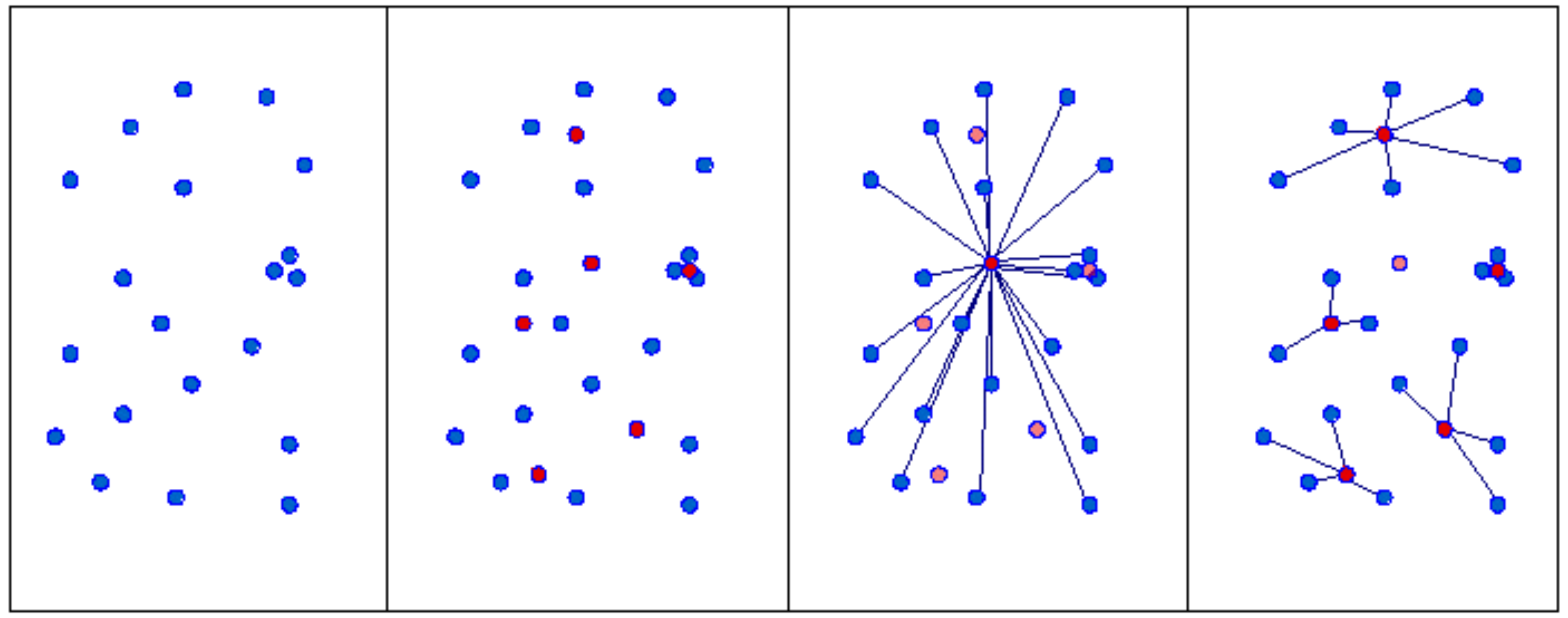
Problems

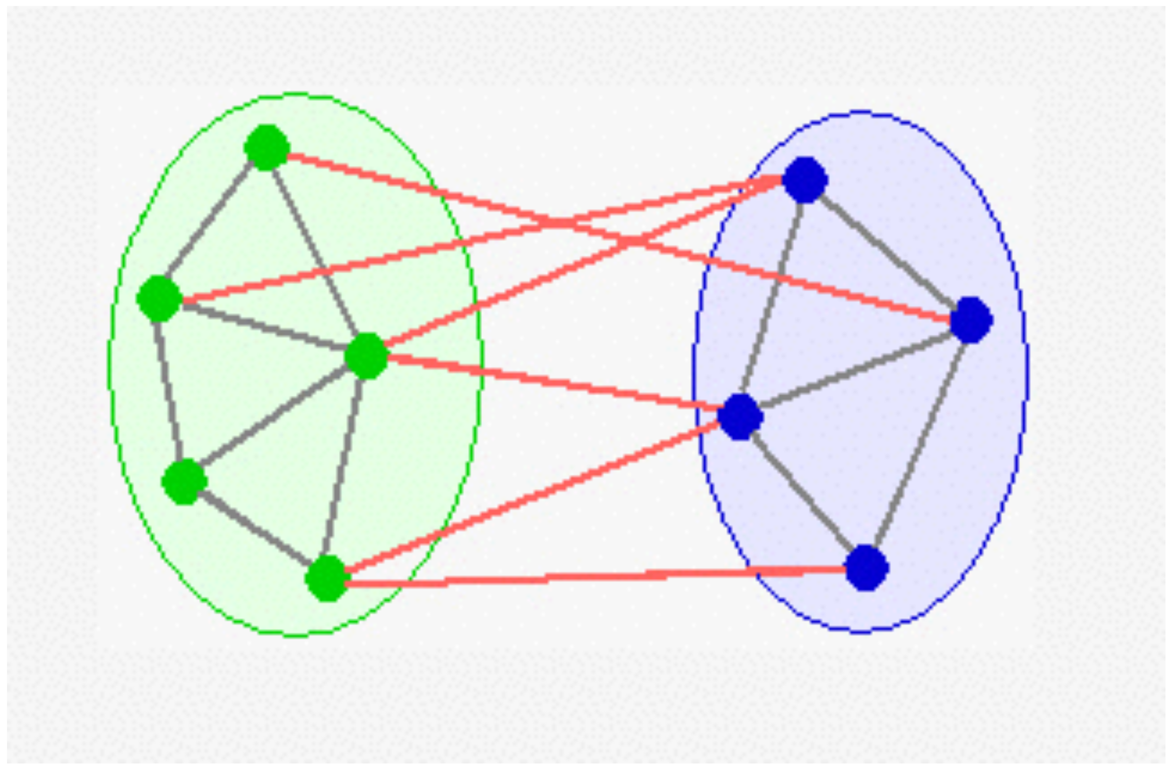
Steiner forest
Facility location
Maximum cut
Sparsest cut...

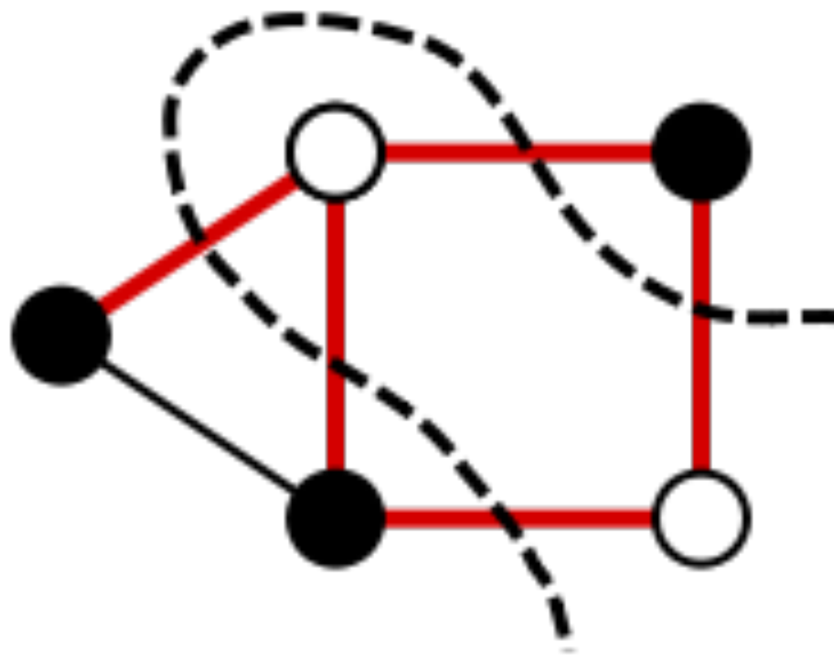
Techniques

Linear programming duality
Semi-definite programming
(More) geometric embeddings









Linear programming duality

Technique 1
Linear programming duality

Bounding the value of an LP

$$\min 7x_1 + x_2 + 5x_3 :$$

$$x_1 - x_2 + 3x_3 \geq 10 \qquad (1)$$

$$5x_1 + 2x_2 - x_3 \geq 6 \qquad (2)$$

$$x_1, x_2, x_3 \geq 0 \qquad (3, 4, 5)$$

How do we certify that OPT is at most 54?

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$$\min 7x_1 + x_2 + 5x_3 :$$

$$x_1 - x_2 + 3x_3 \geq 10 \qquad (1)$$

$$5x_1 + 2x_2 - x_3 \geq 6 \qquad (2)$$

$$x_1, x_2, x_3 \geq 0 \qquad (3,4,5)$$

Try (7,0,1):

- feasible
- objective value = 54

How do we certify an upper bound on OPT?

For minimization:
exhibit a feasible solution
its value is an upper bound

Bounding the value of an LP

$$\min 7\mathbf{x_1} + \mathbf{x_2} + 5\mathbf{x_3} :$$

$$\mathbf{x_1} - \mathbf{x_2} + 3\mathbf{x_3} \geq 10 \qquad (1)$$

$$5\mathbf{x_1} + 2\mathbf{x_2} - \mathbf{x_3} \geq 6 \qquad (2)$$

$$\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3} \geq 0 \qquad (3, 4, 5)$$

How do we certify that OPT is at least 10?

How do we certify that OPT is at least 10?

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$

$$\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad \mathbf{(1)}$$

$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad \mathbf{(2)}$$

$$\mathbf{x_1, x_2, x_3 \geq 0} \quad \mathbf{(3, 4, 5)}$$

$$\mathbf{7x_1 \geq x_1 \text{ and } x_2 \geq -x_2 \text{ and } 5x_3 \geq 3x_3}$$

$$\mathbf{\text{so } 7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10}$$

A better lower bound for OPT

$$
\begin{aligned}
&\min \mathbf{7x_1 + x_2 + 5x_3} : \\
&\mathbf{x_1 - x_2 + 3x_3 \geq 10} && \mathbf{(1)} \\
&\mathbf{5x_1 + 2x_2 - x_3 \geq 6} && \mathbf{(2)} \\
&\mathbf{x_1, x_2, x_3 \geq 0} && \mathbf{(3, 4, 5)}
\end{aligned}
$$

$$\mathbf{2 \times (1) + (2)} \textbf{ implies: } \mathbf{7x_1 + 5x_3 \geq 26.}$$

$$\textbf{so } \mathbf{7x_1 + x_2 + 5x_3 \geq 7x_1 + 5x_3 \geq 26}$$

How do we certify a lower bound on OPT?

For minimization:
exhibit a convex combination of constraints
if each coefficient is less than in objective
then RHS is a lower bound.

What is the best upper bound we can obtain?

Among (x1,x2,x3) such that

$$x_1 - x_2 + 3x_3 \geq 10$$
$$5x_1 + 2x_2 - x_3 \geq 6$$
$$x_1, x_2, x_3 \geq 0$$

Choose the one that minimizes

$$7x_1 + x_2 + 5x_3$$

What is the best lower bound we can obtain?

$$
\begin{aligned}
&\min 7x_1 + x_2 + 5x_3 : \\
&x_1 - x_2 + 3x_3 \geq 10 \quad (1) \\
&5x_1 + 2x_2 - x_3 \geq 6 \quad (2) \\
&x_1, x_2, x_3 \geq 0 \quad (3, 4, 5)
\end{aligned}
$$

Among the convex combinations of constraints

$$y_1 \times (1) + y_2 \times (2)$$

such that

$$7 \geq y_1 + 5y_2 \text{ and } 1 \geq -y_1 + 2y_2 \text{ and } 5 \geq 3y_1 - y_2$$

Choose the one that maximizes

$$10y_1 + 6y_2$$

It's a linear program!

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$
$$\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad (\mathbf{1})$$
$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad (\mathbf{2})$$
$$\mathbf{x_1, x_2, x_3 \geq 0} \qquad (\mathbf{3, 4, 5})$$

Lower bound LP

$$\max \mathbf{10y_1 + 6y_2} :$$
$$\mathbf{y_1 + 5y_2 \leq 7} \quad (\mathbf{1'})$$
$$\mathbf{-y_1 + 2y_2 \leq 1} \quad (\mathbf{2'})$$
$$\mathbf{-3y_1 - y_2 \leq 5} \quad (\mathbf{3'})$$
$$\mathbf{y_1, y_2 \geq 0} \qquad (\mathbf{4', 5'})$$

Primal LP (P)

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$

$$\begin{aligned} &\mathbf{x_1 - x_2 + 3x_3 \geq 10} && (1) \\ &\mathbf{5x_1 + 2x_2 - x_3 \geq 6} && (2) \\ &\mathbf{x_1, x_2, x_3 \geq 0} && (3, 4, 5) \end{aligned}$$

Dual LP (D)

$$\max \mathbf{10y_1 + 6y_2} :$$

$$\begin{aligned} &\mathbf{y_1 + 5y_2 \leq 7} && (1') \\ &\mathbf{- y_1 + 2y_2 \leq 1} && (2') \\ &\mathbf{- 3y_1 - y_2 \leq 5} && (3') \\ &\mathbf{y_1, y_2 \geq 0} && (4', 5') \end{aligned}$$

Linear programming duality