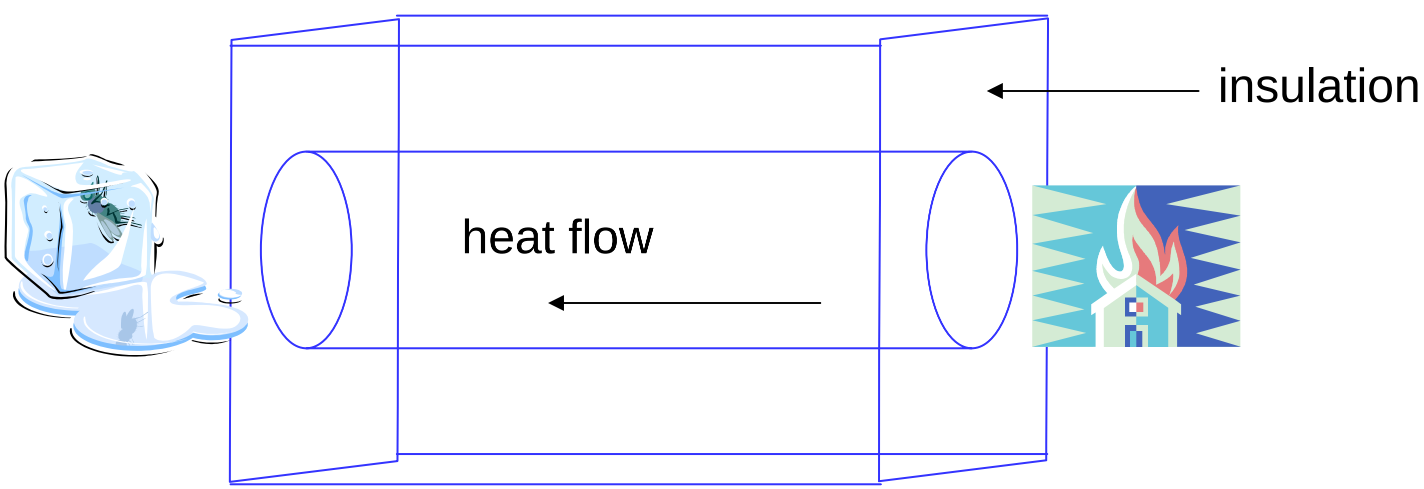
Heat (or Diffusion) equation in 1D*

- Derivation of the 1D heat equation
- Separation of variables (refresher)
- Worked examples

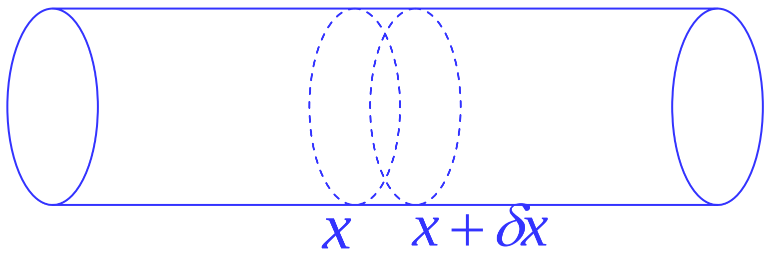
*Kreysig, 8th Edn, Sections 11.4b

Physical assumptions

- We consider temperature in a long thin wire of constant cross section and homogeneous material
- The wire is perfectly insulated laterally, so heat flows only along the wire



Derivation of the heat equation in 1D



Suppose that the thermal conductivity in the wire is K

The specific heat is σ

The density of the material is ρ

Cross sectional area is A

Denote the temperature at point x at time t by $u(x,t)$

Heat flow into bar across face at x : $-KA\left.\dfrac{\partial u}{\partial x}\right|_x$

At the face $x+\delta x$: $KA\left.\dfrac{\partial u}{\partial x}\right|_{x+\delta x}$

So the net flow out is : $KA\dfrac{\partial^2 u}{\partial x^2}\delta x$

Conservation of heat gives :

$$KA\frac{\partial^2 u}{\partial x^2}\delta x = \sigma\rho A\frac{\partial u}{\partial t}\delta x$$

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{K}{\sigma\rho}$$

Boundary and Initial Conditions

As a first example, we will assume that the perfectly insulated rod is of finite length *L* and has its ends maintained at zero temperature.

$$u(0,t) = u(L,t) = 0$$

If the initial temperature distribution in the rod is given by $f(x)$,

we have the initial condition: $u(x,0) = f(x)$

Evidently, from the boundary conditions: $f(0) = f(L) = 0$

Solution by *Separation of Variables*

1. Convert the PDE into two separate ODEs
2. Solve the two (well known) ODEs
3. Compose the solutions to the two ODEs into a solution of the original PDE
   - This again uses Fourier series

Step 1: PDE ➔ 2 ODEs

The first step is to *assume* that the function of two variables has a very special form: the product of two separate functions, each of one variable, that is: Assume that : $u(x,t) = F(x)G(t)$

Differentiating, we find :

$$\frac{\partial u}{\partial t} = F\dot{G}$$

So that $F\dot{G} = c^2 F''G$

$$\frac{\partial^2 u}{\partial x^2} = F''G$$

equivalently $\dfrac{\dot{G}}{c^2 G} = \dfrac{F''}{F} = k$

where $\dot{G} = \dfrac{dG}{dt}$ and $F'' = \dfrac{d^2 F}{dx^2}$

The two (homogeneous) ODEs are:

$$\boxed{\begin{aligned} &F'' - kF = 0 \\ &\dot{G} - kc^2 G = 0 \end{aligned}}$$

Step 2a: solving for F*

We have to solve: $F''-kF=0$

case $k=\mu^2>0$ the general solution is:

$$F(x)=Ae^{\mu x}+Be^{-\mu x}$$

Applying the boundary conditions

$$A+B=0$$

$$Ae^{\mu L}+Be^{-\mu L}=0$$

and so $A=B=0$

case $k=0$:

$$F(x)=ax+b$$

Applying the boundary conditions: $a=b=0$

It follows that the only case of interest is $k=-p^2$

case $k=-p^2<0$: the solution is

$$F(x)=A\cos px+B\sin px$$

Applying the boundary conditions, we find

$$F(0)=A=0$$

$$F(L)=B\sin pL=0, \text{ and so } \sin pL=0$$

$pL=n\pi$ that is:

$$p=n\frac{\pi}{L}$$

so that

$$\boxed{F_n(x)=\sin n\frac{\pi}{L}x}$$

* This analysis is identical to the Wave Equation case we studied earlier

Step 2b: solving for G

$p = n\dfrac{\pi}{L}$, and, as for the wave equation, we denote $\lambda_n = n\dfrac{c\pi}{L}$

We have $\dot{G} + \lambda_n^2 G = 0$

whose general solution is: $G_n(t) = B_n e^{-\lambda_n^2 t}$

Step 2c: combining F&G

The solution to the 1D diffusion equation is:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin n\frac{\pi}{L}x$$

Initial condition: $u(x,0) = \sum_{n=1}^{\infty} B_n \sin n\frac{\pi}{L}x = f(x)$

As for the wave equation, we find: $B_n = \frac{2}{L}\int_0^L f(x)\sin n\frac{\pi}{L}x\,dx$

Analysing the solution

The solution to the 1D diffusion equation can be written as:

$$u(x,t)=\sum_{n=1}^{\infty}B_n u_n(x,t)$$

where

$$u_n(x,t)=e^{-\lambda_n^2 t}\sin n\frac{\pi}{L}x$$

This emphasises that the solution is a weighted sum of functions $u_n(x,t)$

where

(1) The functions u_n are completely determined by the generic problem (that is, the constants L,c) and the boundary conditions $u(0,t)=u(L,t)=0$ in this case; and

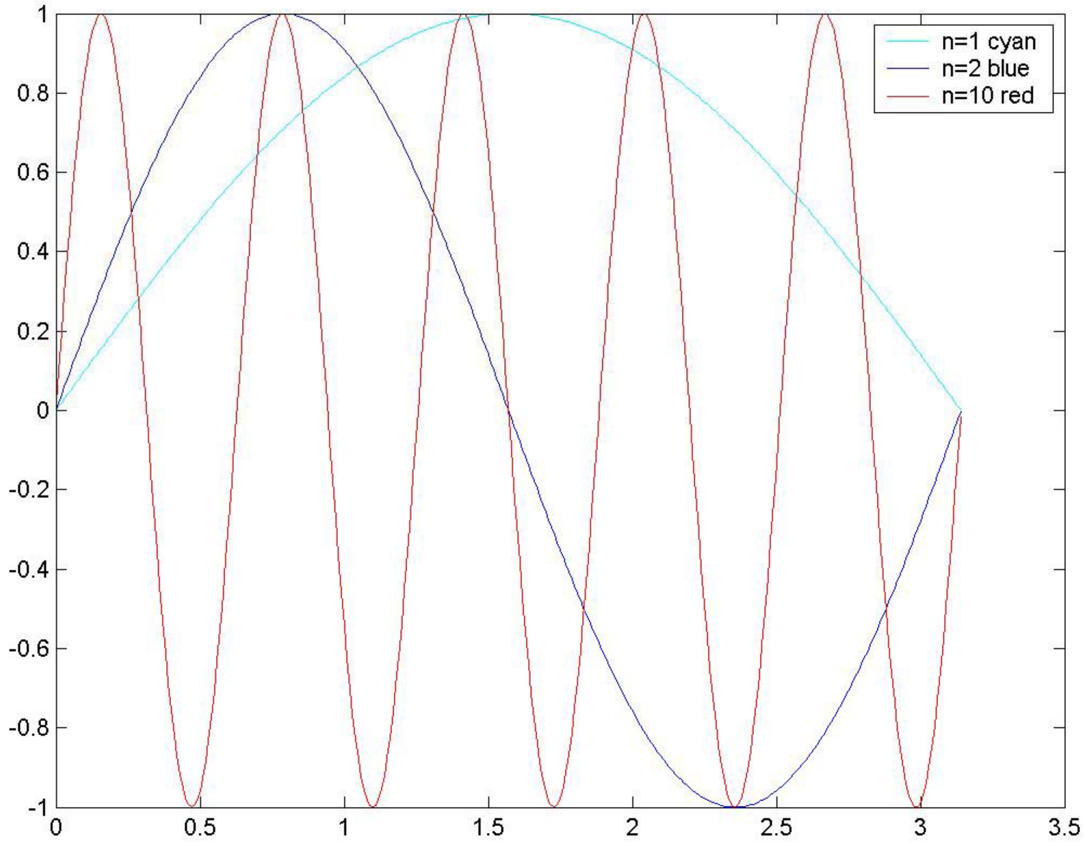
(2) The weights B_n are determined by the initial conditions, since

$$B_n=\frac{2}{L}\int_0^L f(x)\sin n\frac{\pi}{L}x\,dx$$

We now take a closer look at the functions u_n

Sine term

Consider first the sine term: $\sin n\dfrac{\pi}{L}x$



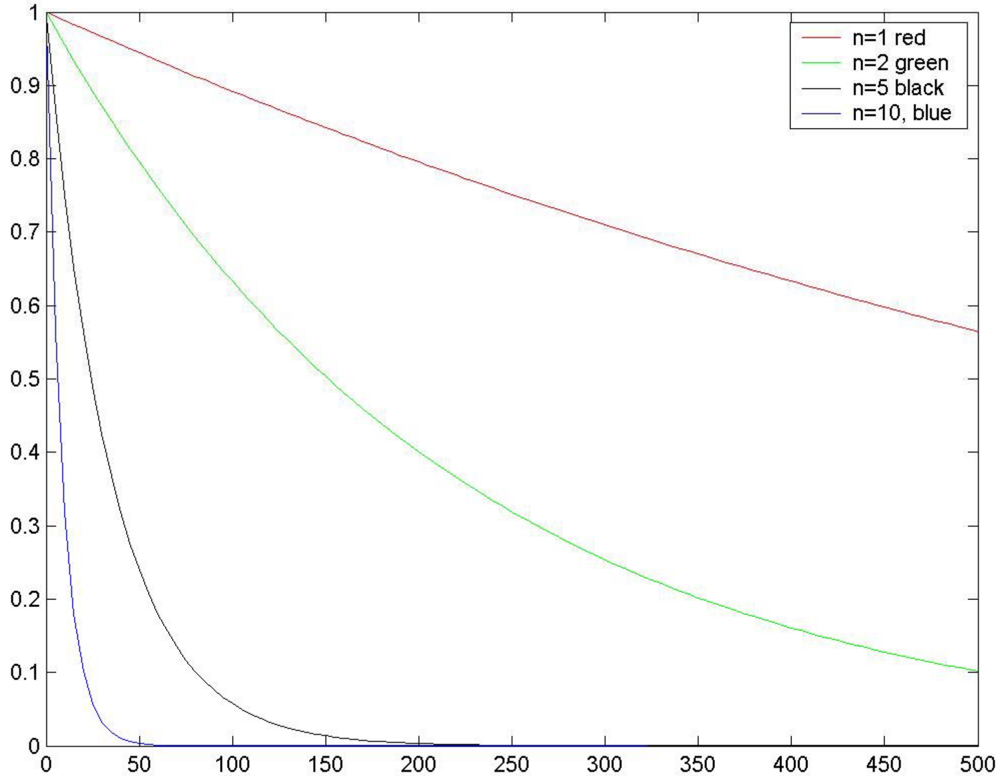
Evidently, the higher the value of *n* the higher the frequency component of *f(x)* that is analysed

Exponential term

Consider next the exponential term: $e^{-\left(\frac{nc\pi}{L}\right)^2 t}$

Suppose that $L = 100$ and (see 2 slides later) $c^2 = 1.158$

Then $\lambda_1^2 = \left(\frac{c\pi}{L}\right)^2 = 0.0011$



Gabor functions

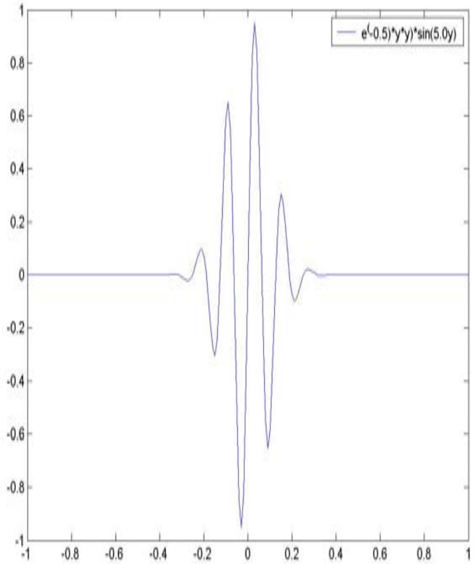
The exponential term is $e^{-\left(\frac{nc\pi}{L}\right)^2 t}$

denote $\sqrt{t} = s$ and $\sigma_n = \left(\frac{L}{nc\pi}\right)$, then the term is

$e^{-\frac{s^2}{\sigma^2}}$ ... a Gaussian

so if we consider (x, s) instead of (x, t) we have to work with a Gaussian multiplied by a sine term.

These are called Gabor functions and are fundamental to signal processing and optics



This appears to be a short burst of sine wave in a Gaussian shaped envelope. This is fundamental to AM/FM communications and to wavelets

Denis Gabor (born Budapest 1900, died London 1979). Left Berlin for London during the 1930s. Eventually professor of electronics at Imperial College. Credited with invention of the hologram in 1947. Awarded the Nobel prize for physics in 1971

Recall the solution …

The solution to the 1D diffusion equation is :

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin n\frac{\pi}{L}x$$

Initial condition : $u(x,0) = \sum_{n=1}^{\infty} B_n \sin n\frac{\pi}{L}x = f(x)$

As for the wave equation, we find : $B_n = \frac{2}{L}\int_0^L f(x) \sin n\frac{\pi}{L}x dx$

Example 1*

Length of the bar :80 cms

Sinusoidal initial conditions : $f(x) = 100\sin\frac{\pi}{80}x$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin n\frac{\pi}{80}x = f(x) = 100\sin\frac{\pi}{80}x$$

$$B_1 = 100, B_2 = B_3 = \ldots = 0$$

Copper : $K = 0.95$ cal/(cm sec ^{o}C)

$\rho = 8.92$ gm/cm^3

$\sigma = 0.092$ cal/(gm ^{o}C)

so

$$c^2 = \frac{K}{\rho\sigma} = \frac{0.95}{0.092 \cdot 8.92} = 1.158 \text{ cm}^2/\sec$$

$$\lambda_1^2 = 1.158\frac{9.870}{6400} = 0.001758$$

$$u(x,t) = 100e^{-0.001785t}\sin\frac{\pi}{80}x$$

time required for the maximum temperature to reduce to 50^{o} :

$$100e^{-0.001785t} = 50 \Rightarrow t = 388s = 6.5\min$$

*Kreysig, 8th Edn, page 603

Example 2

Somewhat more realistically, we assume that the bar is initially entirely at a constant temperature, then at time $t=0$, it is insulated and quenched, that is, the temperatures at its two ends instantly reduce to zero.

Initial condition: $u(x,0) = T_0$

Boundary conditions:

$$u(0,t) = u(L,t) = 0, \text{ for all } t > 0$$

$$u(x,t) \to 0, \text{ as } t \to \infty$$

Solution

Recall that the solution to the 1D diffusion equation is :

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin n\frac{\pi}{L}x$$

Initial condition : $u(x,0) = \sum_{n=1}^{\infty} B_n \sin n\frac{\pi}{L}x = f(x) = T_0$

As for the wave equation, we find : $B_n = \frac{2}{L}\int_0^L f(x)\sin n\frac{\pi}{L}xdx$

$$B_n = \frac{2T_0}{L}\int_0^L \sin n\frac{\pi}{L}xdx$$

$$= \frac{2T_0}{\pi}\int_0^{\pi} \sin n\theta d\theta$$

Solving for B_n

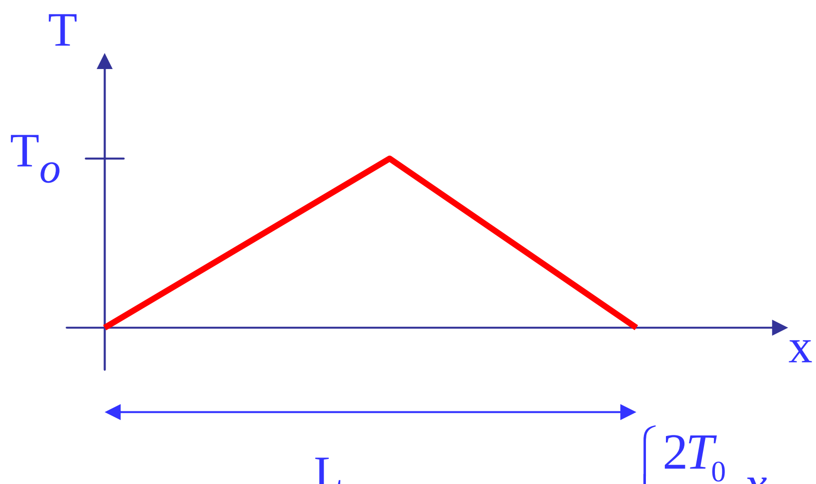
Evidently, $\int_0^{\pi} \sin n\theta d\theta = \left[-\frac{\cos n\theta}{n} \right]_0^{\pi}$

if $n = 2m$ is even, then this is 0

if $n = (2m-1)$ is odd, it equals $\frac{2}{(2m-1)}$

and so, we find $u(x,t) = \sum_{m=1}^{\infty} \frac{2}{(2m-1)} \cdot \frac{2T_0}{\pi} e^{-\left(\frac{(2m-1)c\pi}{L}\right)^2 t} \sin(2m-1)\frac{\pi}{L}x$

Changing the initial condition



Suppose that initially $f(x) = \begin{cases} \dfrac{2T_0}{L}x & \text{if } 0 \le x \le \dfrac{L}{2} \\ \dfrac{2T_0}{L}(L-x) & \text{if } \dfrac{L}{2} \le x \le L \end{cases}$

All the other boundary conditions remain the same:

u(0,t) = 0 for all t > 0

u(L,t) = 0 for all t > 0 (quenched at both ends)

$t \to \infty \Rightarrow u(x,t) \to 0$ (must eventually cool down to zero)

Solution

Recall that the solution to the 1D diffusion equation is:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin n\frac{\pi}{L}x$$

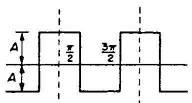
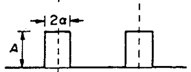
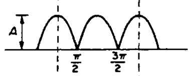
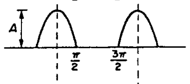
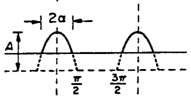
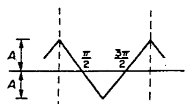
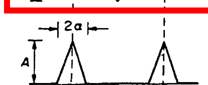
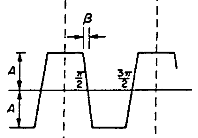
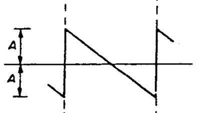
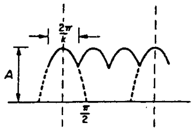
Initial condition: $u(x,0) = \sum_{n=1}^{\infty} B_n \sin n\frac{\pi}{L}x = f(x)$

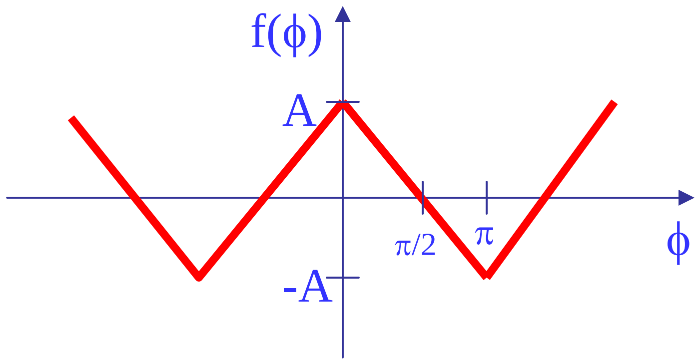
We have to solve for the coefficients using Fourier series.

Instead of orthogonality, we consult HLT

Fourier series for certain waveforms

The series below are expressed in terms of the angular variable θ, the period of each waveform being 2π. Similar waveforms in any variable x with period T can be represented by the same series with the substitution $\theta = 2\pi x/T$. The origin of θ is so chosen as to make the waveforms even functions ($b_n = 0$) wherever possible. α and β are angles, k an integer.

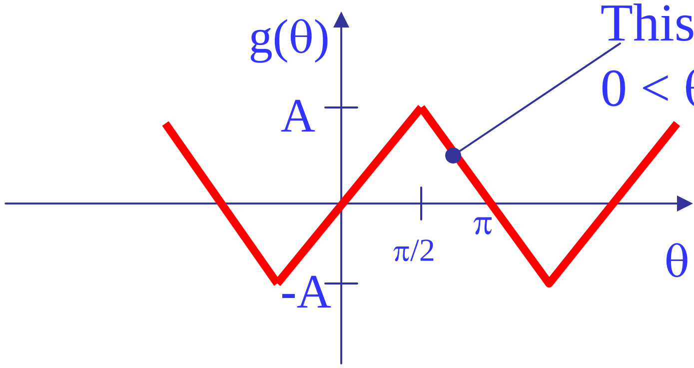
$\theta = 0$ $\theta = 2\pi$	Series	Mean square value
	$\frac{4A}{\pi}\{\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \dots\}$	A^2
	$A\left\{\frac{\alpha}{\pi} + \frac{2}{\pi}(\sin\alpha\cos\theta + \frac{1}{2}\sin 2\alpha\cos 2\theta + \frac{1}{3}\sin 3\alpha\cos 3\theta + \dots)\right\}$	$\frac{A^2\alpha}{\pi}$
	$\frac{2A}{\pi}\{1 + \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta + \frac{2}{35}\cos 6\theta - \dots\}$	$\frac{A^2}{2}$
	$\frac{A}{\pi}\left\{1 + \frac{\pi}{2}\cos\theta + \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta + \dots\right\}$	$\frac{A^2}{4}$
	$\frac{A}{\pi}\{(\sin\alpha - \alpha\cos\alpha) + (\alpha - \frac{1}{2}\sin 2\alpha)\cos\theta + (\sin\alpha + \frac{1}{3}\sin 3\alpha - \cos\alpha\sin 2\alpha)\cos 2\theta + (\frac{1}{2}\sin 2\alpha + \frac{1}{4}\sin 4\alpha - \frac{2}{3}\cos\alpha\sin 3\alpha)\cos 3\theta + \dots\}$	$\frac{A^2}{2\pi}\{\alpha - \frac{3}{2}\sin 2\alpha + 2\alpha\cos^2\alpha$
	$\frac{8A}{\pi^2}\{\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{25}\cos 5\theta + \dots\}$	$\frac{A^2}{3}$
	$\frac{A}{\pi\alpha}\left\{\frac{\alpha^2}{2} + 4\sin^2\frac{\alpha}{2}\cos\theta + \sin^2\alpha\cos 2\theta + \frac{4}{9}\sin^2\frac{3\alpha}{2}\cos 3\theta + \dots\right\}$	$\frac{A^2\alpha}{3\pi}$
	$\frac{4A}{\pi\beta}\{\sin\beta\cos\theta - \frac{1}{9}\sin 3\beta\cos 3\theta + \frac{1}{25}\sin 5\beta\cos 5\theta - \dots\}$	$A^2\left(1 - \frac{4}{3}\frac{\beta}{\pi}\right)$
	$\frac{2A}{\pi}\{\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \dots\}$	$\frac{A^2}{3}$
	$\frac{Ak}{\pi}\sin\frac{\pi}{k}\left\{1 + \frac{2}{k^2 - 1}\cos k\theta - \frac{2}{4k^2 - 1}\cos 2k\theta + \frac{2}{9k^2 - 1}\cos 3k\theta - \dots\right\}$	



which is

$$f(\phi)=\frac{8A}{\pi^2}(\cos\phi+\frac{1}{9}\cos 3\phi+.....)$$

Let's shift this by π/2 in the ϕ direction: θ = ϕ-π/2 to give:



This is now what we need between $0 < \theta < \pi$ with $A = T_o$

$$g(\theta)=f(\theta+\pi/2)=\frac{8A}{\pi^2}(\cos(\theta+\pi/2)+\frac{1}{9}\cos 3(\theta+\pi/2)+.....)$$

$$=\frac{8A}{\pi^2}(\sin\theta+\frac{1}{9}\sin 3\theta+.....)\equiv\sum_{n=1}^{\infty}B_n\sin n\theta$$

So, $$B_n = \frac{8T_0}{\pi^2(2n-1)^2}$$

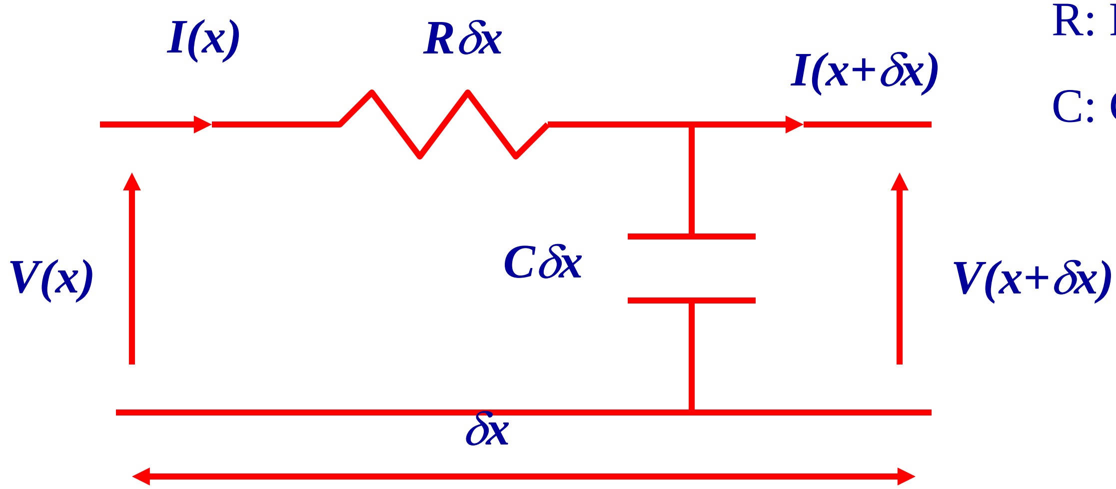
and the final solution for this problem is

$$\boxed{T(x,t) = \sum_{n=1}^{\infty} \frac{8T_0}{(2n-1)^2\pi^2} \sin\frac{(2n-1)\pi x}{L} \exp-\left(\frac{k(2n-1)^2\pi^2 t}{L^2}\right)}$$

Compare this with our previous example of constant initial temperature distribution:

$$\boxed{T(x,t) = \sum_{n=1}^{\infty} \frac{4T_0}{(2n-1)\pi} \sin\frac{(2n-1)\pi x}{L} \exp-\left(\frac{k(2n-1)^2\pi^2 t}{L^2}\right)}$$

Derivation from electrostatics: the 'Telegraph Equation'



R: Resistance per unit length

C: Capacitance per unit length

$$I(x+\delta x)=I(x)+\frac{\partial I}{\partial x}\delta x$$

$$V(x+\delta x)=V(x)+\frac{\partial V}{\partial x}\delta x$$

Ohm's law:

$$V(x+\delta x)-V(x)=-IR\delta x$$

From $I=C\dfrac{\partial V}{\partial t}$, we find:

$$I(x+\delta x)-I(x)=-C\delta x\frac{\partial V}{\partial t}$$

We find

$$-\frac{\partial I}{\partial x}=C\frac{\partial V}{\partial t}$$

$$-\frac{\partial V}{\partial x}=RI$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial x^2}=RC\frac{\partial V}{\partial t}$$

The diffusion equation

Example from electrostatics

Initially, a uniform conductor has zero potential throughout. One end ($x=0$) is then subjected to constant potential V_0 while the other end ($x=L$) is held at zero potential. What is the transient potential distribution?

Initial conditions:

$$u(x,0) = 0, \text{ for } 0 \le x \le L$$

Boundary conditions:

$$u(0,t) = V_0, \text{ all } t > 0$$

$$u(L,t) = 0, \text{ all } t > 0$$

We again use separation of variables; but we need to start from scratch because so far we have assumed that the boundary conditions were

$u(0,t) = u(L,t) = 0$ but this is ***not*** the case here.

We now retrace the steps for the original solution to the heat equation, noting the differences

Step 1: PDE ➔ 2 ODEs

The first step is to *assume* that the function of two variables has a very special form: the product of two separate functions, each of one variable, that is: $\text{Assume that}: u(x,t) = F(x)G(t)$

Differentiating, we find:

$$\frac{\partial u}{\partial t} = F\dot{G}$$

$$\frac{\partial^2 u}{\partial x^2} = F''G$$

where $\dot{G} = \dfrac{dG}{dt}$ and $F'' = \dfrac{d^2F}{dx^2}$

So that $$F\dot{G} = c^2 F''G$$

equivalently $$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k$$

The two (homogeneous) ODEs are:

$$\boxed{\begin{aligned} &F'' - kF = 0 \\ &\dot{G} - kc^2 G = 0 \end{aligned}}$$

No change here!!

Step 2a: solving for F&G .. $k \geq 0$

We have to solve: $F'' - kF = 0$

case $k = \mu^2 > 0$ the general solution is: $F(x) = Ae^{\mu x} + Be^{-\mu x}$

so that $u(x,0) = F(x)G(0) = 0$ from which $A = B = 0$ as before

case $k = 0$:

$$F(x) = ax + b$$

Applying the boundary conditions:

$$F(0) = V_0 = b$$

$$F(L) = 0 = aL + b, \text{ so } a = -\frac{V_0}{L}$$

so that $F(x) = V_0\left(1 - \frac{x}{L}\right)$

Since $k = 0$, we have $\dot{G} = 0$, so that G is constant (ignore)

Step 2b: Solving for F&G …*k*<0

case $k = -p^2 < 0$: the solution is

$$F(x) = A\cos px + B\sin px$$

Applying the boundary conditions, we find

$$F(0) = A = 0$$

$F(L) = B\sin pL = 0$, and so $\sin pL = 0$

$pL = n\pi$ that is:

$$p = n\frac{\pi}{L}$$

so that

$$\boxed{F_n(x) = \sin n\frac{\pi}{L}x}$$

This case is as before

$p = n\dfrac{\pi}{L}$, and, as before, we denote $\lambda_n = n\dfrac{c\pi}{L}$

We have $\dot{G} + \lambda_n^2 G = 0$

whose general solution is: $G_n(t) = B_n e^{-\lambda_n^2 t}$

Step 2c: combining F&G

In this case, the solution to the 1D diffusion equation is:

$$u(x,t) = V_0\left(1-\frac{x}{L}\right) + \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin n\frac{\pi}{L}x$$

Initial condition: $u(x,0) = V_0\left(1-\frac{x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin n\frac{\pi}{L}x = 0$

so that

$$B_n = \frac{2}{L}\int_0^L -V_0\left(1-\frac{x}{L}\right)\sin n\frac{\pi}{L}x\,dx$$

$$= \frac{-2V_0}{\pi}\int_0^{\pi}\left(1-\frac{\theta}{\pi}\right)\sin n\theta\,d\theta$$

which (by parts) $= \frac{-2V_0}{\pi}.\frac{1}{n}$ (or use HLT next to last on earlier list)

Solution is: $u(x,t) = V_0\left(1-\frac{x}{L}\right) - \frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\lambda_n^2 t}\sin n\frac{\pi}{L}x$