Feedback — Assignment 2

You submitted this quiz on **Tue 12 Mar 2013 10:22 PM PDT -0700**. You got a score of **36.00** out of **36.00**.

Question 1

Which of the following is true about the linear program

$$\begin{array}{rl} \text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 \leq 0 \\ & x_2 \leq 0 \\ & x_1 + x_2 \geq 1 \end{array}$$

Your Answer		Score	Explanation
☑ Infeasible	✓	0.20	
☑ Bounded	✓	0.20	
☐ Feasible	✓	0.20	
☐ Unbounded	✓	0.20	
☐ Unique optimum	✓	0.20	
Total		1.00 / 1.00	

Question Explanation

The system of inequalities in the linear program is infeasible since adding the first two inequalities yields $x_1 + x_2 \leq 0$which contradicts the third inequality.

Question 2

Which of the following properties apply to the linear program

$$\begin{array}{ll} \text{maximize} & -x_1 - x_2 \\ \text{subject to} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 + x_2 \geq 1 \end{array}$$

Your Answer		Score	Explanation
☑ Infinitely many optima	✓	0.25	
☐ Unbounded	✓	0.25	
☐ Unique optimum	✓	0.25	
☑ Feasible	✓	0.25	
Total		1.00 / 1.00	

Question Explanation

Maximizing $-x_1 - x_2$ is equivalent to minimizing $x_1 + x_2$ which is constrained to be at least 1 by the last inequality. So one example of a feasible solution that is also optimal is $x_1 = 0, x_2 = 1$ But so is $x_1 = 1, x_2 = 0$ In fact, any point on the line $x_1 + x_2 = 1$ from $x_1 = 0$ to $x_1 = 1$ is optimal. Thus the linear program is feasible and has infinitely many optima along the indicated line segment.

Question 3

How many constraints of the polyhedron

$$\left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_3 \leq 0 \\ x_2 + 2x_3 \leq 0 \\ x_1 + x_2 - 3x_3 \geq 1 \end{array} \right\}$$ are active at the point $(1, 2, -1)$?

Your Answer		Score	Explanation
○ 0			
○ 3			
○ 1			
◉ 2	✓	1.00	

Total	1.00 / 1.00

Question Explanation

Substituting $x_1 = 1, x_2 = 2, x_3 = -1$ in the inequalities we find the left hand sides of the three inequalities to be $0, 0$ and 6. So there are two active constraints at the point $(1, 2, -1)$

Question 4

Does the point $(1, 2 - 1)$ define a basic solution for the polyhedron defined in the previous question? Is it also feasible?

Your Answer		Score	Explanation
Not a basic solution but feasible.	✓	1.00	
Basic and feasible solution.			
Neither a basic solution nor feasible.			
Basic solution but not feasible.			
Total		1.00 / 1.00	

Question Explanation

The point $(1, 2, -1)$ is feasible since it satisfies all the constraints. But it does not define a basic solution since only two linearly independent inequalities are active at the given point, while a basic solution requires three.

Question 5

Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ where $A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$, where $m \geq n$. Which of the following is a valid upper bound on the number of basic solutions to $\mathcal{P}$?

Your Answer		Score	Explanation
◉ $\binom{m}{n}$	✓	1.00	
○ $\binom{n}{m}$			
Total		1.00 / 1.00	

Question Explanation

To produce a basic solution we need at least n linearly independent inequalities of the m inequalities that are available. The maximum number of choices that we have is exactly $\binom{m}{n}$.

Question 6

How many basic feasible solutions does the hypercube $\mathcal{H}_n = \{x \in \mathbb{R}^n \mid -1 \leq x_i \leq 1 \; \forall \, i \in \{1, 2, \ldots, n\}\}$ have?

Your Answer		Score	Explanation
○ 2^{n-1}			
○ 2^{2n-1}			
○ 2^{2n}			
◉ 2^n	✓	3.00	
Total		3.00 / 3.00	

Question Explanation

The polyhedron $\mathcal{H}_n$ is the n-dimensional hypercube with the vertex set $\{-1, 1\}^n$. To see that each point from the set is indeed a vertex, we simply note that they give rise to exactly n linearly independent and active constraints thereby forming a basis. Since any such point is also feasible, it is a vertex. To see that there cannot be any other vertex to the hypercube, observe that every basic solution (and hence vertex) has its coordinates set to 1 or -1. Hence there are exactly 2^n vertices (or basic feasible solutions).

Question 7

Consider the linear program described below for answering questions 7 and 8. Select the vertices of the polyhedron described by the LP from the list of points given.

$$\begin{array}{rl} \text{maximize} & 3x + 2y \\ \text{subject to} & 2x + y \leq 18 \\ & 2x + 3y \leq 42 \\ & 3x + y \leq 24 \\ & x \geq 0 \\ & y \geq 0 \end{array}$$

Your Answer		Score	Explanation
☑ $(6, 6)$	✓	0.75	
☐ $(2, 3)$	✓	0.75	
☐ $(1, 1)$	✓	0.75	
☑ $(3, 12)$	✓	0.75	
Total		3.00 / 3.00	

Question Explanation

First note that the objective function here is irrelevant for determination of the vertices. A point (x, y) will correspond to a vertex of the polyhedron above if it is feasible and if the submatrix of A corresponding set of inequalities that are active at (x, y) form a rank 2 matrix. This condition is satisfied only by $(6, 6)$ and $(3, 12)$.

Question 8

Using the multipliers $\frac{5}{4}$ and $\frac{1}{4}$ for the two inequalities $2x + y \leq 18$and $2x + 3y \leq 42$ respectively, which of the following points can you conclude is optimal for the LP described in the previous question?

Your Answer	Score	Explanation

$(6, 6)$		
$(2, -3)$		
$(6, 10)$		
$(3, 12)$	✓	1.00
Total		1.00 / 1.00

Question Explanation

Multiplying the two inequalities by the respective inequalities and adding them we obtain $(2\,\frac{5}{4} + 2\,\frac{1}{4})x + (\frac{5}{4} + 3\,\frac{1}{4})y \leq 18\,\frac{5}{4} + 42\,\frac{1}{4} = 33$. Since we observed in the previous question that $(3, 12)$ is feasible, and moreover one can chek that its value is 33, we conclude that $(3, 12)$ is an optimum solution.

Question 9

Consider the following optimization problem over $(x_1, x_2) \in \mathbb{R}^2$

$$\begin{aligned} \text{minimize} \quad & x_2 \\ \text{subject to} \quad & x_1 x_2 \geq 1 \\ & x_1 \geq 1 \end{aligned}$$

Select the options that apply for the above program.

Your Answer		Score	Explanation
☑ Bounded	✓	0.60	
☑ Feasible	✓	0.60	
☐ Infeasible	✓	0.60	
☐ Unbounded	✓	0.60	
☐ It is a linear program in two variables	✓	0.60	
Total		3.00 / 3.00	

Question Explanation

Consider any $\epsilon > 0$. We can set $x_1 = 1/\epsilon$ which is feasible as long as $\epsilon \le 1$. Then we have $x_2 \ge \frac{1}{x_1} = \epsilon$ which can be satisfied by assigning x_2 to ϵ. Thus every pair $(\frac{1}{\epsilon}, \epsilon)$ is a feasible solution for the above optimization problem with objective function value $\epsilon > 0$. So it has no unique optimum. However, the value of the program is bounded since the two constraints together imply that x_2 is strictly positive. Finally, the above problem is clearly not linear due to the constraint $x_1 x_2 \ge 1$, which is non-linear.

Question 10

Which of the following objective functions makes this linear program unbounded?

$$\begin{aligned} \text{maximize} \quad & c^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{subject to} \quad & x_1 \ge 0 \\ & x_2 \ge 0 \\ & x_1 + x_2 \ge 5 \end{aligned}$$

Your Answer		Score	Explanation
☑ $c = [-1\ 2]^T$	✔	0.75	
☑ $c = [1\ 1]^T$	✔	0.75	
☐ $c = [-1\ \ -1]^T$	✔	0.75	
☑ $c = [5\ \ -1]^T$	✔	0.75	
Total		3.00 / 3.00	

Question Explanation

The simplest way to see this is via a picture. Drawing the feasible region for the given constraints in the plane $\mathbb{R}^2$ we find that the lines $\langle c, x \rangle = \lambda$ corresponding to each of objective functions except for $c = [-1\ \ -1]^T$ intersect the feasible region for all sufficiently large and positive λ. When $c = [-1\ \ -1]^T$, however, λ can be no greater than -5 since the value of the objective function in this case is constrained by the last inequality.

Question 11

Select the linear program that is equivalent to

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Your Answer		Score	Explanation
○ $\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \begin{bmatrix} -A \\ A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \end{array}$			
○ $\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \begin{bmatrix} -A \\ A \\ I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \end{array}$			
○ $\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \begin{bmatrix} A \\ -A \\ I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \end{array}$			
◉ $\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \end{array}$	✓	3.00	
Total		3.00 / 3.00	

Question Explanation

We express the inequality $x \geq 0$ as $-x \leq 0$, and the equality $Ax = b$ by two sets of inequalities $Ax \leq b$ and $-Ax \leq -b$ Using this we see that the correct reformulation of the LP is

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \end{aligned}$$

Question 12

The n dimensional ball with center $y \in \mathbb{R}^n$ and radius $r \in \mathbb{R}$ is defined as the set $\{x \in \mathbb{R}^n \mid \|x - y\|_2 \leq r\}$ Given a y that satisfies the inequality $a^T x \leq b$, what is the radius of the largest ball centered at y and bounded by the given inequality?

Your Answer		Score	Explanation
○ $(a^T y - b)/\|a\|_2$			
◉ $(b - a^T y)/\|a\|_2$	✓	3.00	
○ $(a^T y - b)/\|a\|_2^2$			
○ $(b - a^T y)/\|a\|_2^2$			
Total		3.00 / 3.00	

Question Explanation

Suppose the radius of the largest ball centered at y and contained in the region $a^T x \leq b$ is r. Since a denotes the normal vector of the hyperplane corresponding to the inequality, the unit vector in that direction is $\frac{a}{\|a\|_2}$. As the ball of radius r should be contained in the halfspace $a^T x$, it must contain the vector $y + r\frac{a}{\|a\|_2}$. That is, $a^T\left(y + r\frac{a}{\|a\|_2}\right) \leq b$ Simplifying, we obtain $r \leq (b - a^T y)/\|a\|_2$ Thus the largest possible radius r is $(b - a^T y)/\|a\|_2$.

Question 13

Building on your answer to the previous question, determine which of the following describes a linear program to find the center and radius of the ball with the largest possible radius that is entirely contained in $\{x \in \mathbb{R}^n \mid Ax \leq b\}$. Note that we denote the ith row of A by a_i in the options below.

Your Answer		Score	Explanation
(selected) $$\begin{array}{ll}\text{maximize} & r \\ \text{subject to} & a_i^T y + \|a_i\| r \leq b_i \ \forall\, i \in \{1, 2, \ldots, m\}\end{array}$$	✓	3.00	
$$\begin{array}{ll}\text{maximize} & r \\ \text{subject to} & y + \frac{a_i}{\|a_i\|} r \leq b_i \ \forall\, i \in \{1, 2, \ldots, m\}\end{array}$$			
$$\begin{array}{ll}\text{maximize} & r \\ \text{subject to} & y + a_i \leq r b_i \ \forall\, i \in \{1, 2, \ldots, m\}\end{array}$$			
$$\begin{array}{ll}\text{maximize} & r \\ \text{subject to} & y + r \frac{a_i}{\|a_i\|} \leq \frac{b_i}{\|a_i\|} \ \forall\, i \in \{1, 2, \ldots, m\}\end{array}$$			
Total		3.00 / 3.00	

Question Explanation

The LP we will describe will have $n+1$ variables corresponding to the components of the center y and the radius r of the largest contained ball. It is clear then that our objective function will simply maximize r. Turning to the constraints, as the ball should satisfy each of the inequalities individually, in particular, we have that $a_i^T y + \|a_i\| r \leq b_i \ \forall\, i \in \{1, 2, \ldots, m\}$ using the result from the previous question.

Question 14

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$. Suppose that x^* is a feasible solution with k strictly positive components and the remaining components zero. Furthermore suppose that $\operatorname{rank}(A_1) = k$, where $A_1 x \leq b_1$ is the sub-system of $Ax \leq b$ consisting of the constraints that are active at x^*. Is x^* necessarily a vertex of P?

Your Answer		Score	Explanation
No	✓	3.00	
Yes			
Total		3.00 / 3.00	

Question Explanation

Consider the sub-system $A_1 x \leq b_1$ of $Ax \leq b$ consisting of the constraints that are active at x^*. We are given that the rank of A_1 is k. For x^* to be a vertex we should be able to find $n-k$ other active inequalities whose corresponding sub-sytem has rank $n-k$. However, this may not always be possible. Thus the correct answer is no.

An example where this could happen is as follows: Consider the point $x^* = (1, 0, 0)$ in the following polyhedron:

$$\mathcal{P} = \{x \in \mathbb{R}^3 \mid \begin{bmatrix} 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 0 \end{bmatrix}$$
$$x_1, x_2, x_3 \geq 0\}.$$

Clearly, x^* is a feasible solution and it has one strictly positive component and remaining zeros. Also, the first inequality is tight at x^* so that the rank of the sub-system in this case is also 1. Thus we have satisfied all the conditions of the statement in the question. Now, of the remaining three inequalities only two are active at x^*, namely, $x_2 \geq 0, x_3 \geq 0$. However, together the sub-system corresponding to these three inequalities has rank only 2. Hence x^* is not a vertex of $\mathcal{P}$ in this case.

Question 15

Let $P := \{x \in \mathbb{R}^n \mid Ax \le b\}$ Suppose that at a particular basic feasible solution, there are k active constraints, with $k > n$. Is it true that there exist exactly $\binom{k}{n}$ bases that lead to this basic feasible solution?

Your Answer		Score	Explanation
No	✓	3.00	
Yes			
Total		3.00 / 3.00	

Question Explanation

If we have k active constraints at a basic feasible solution, then we know that we will be able to choose n linearly independent constraints from this set which will define a basis. However, it may be the case that some of the k constraints are redundant (for example one constraint could simply be a positive multiple of another) then it is not the case that every choice of n constraints from the set of k active constraints gives a basis that leads to this basic feasible solution. Thus a choice of n constraints that includes two such redundant constraints is not linearly independent and hence does not lead to a basis.

Question 16

Consider a nonempty polyhedron P and suppose that for each variable x_i we add either the constraint $x_i \ge 0$ or the constraint $x_i \le 0$. Is it true that the new polyhedron has at least one basic feasible solution?

Your Answer		Score	Explanation
No	✓	3.00	
Yes			
Total		3.00 / 3.00	

Question Explanation

Consider a polyhedron P that lies entirely in the positive orthant of $\mathbb{R}^n$. If we add any inequality of the kind $x_i \leq 0$ to this polyhedron, then we have an empty polyhedron which does not have any basic feasible solutions.