Facility location

Primal

$$\min \sum_{\mathbf{i}} \mathbf{f_i y_i} + \sum_{\mathbf{i,j}} \mathbf{c_{ij} x_{ij}} :$$
$$\sum_{\mathbf{i}} \mathbf{x_{ij}} \geq \mathbf{1} \quad \textbf{for all } \mathbf{j}$$
$$\mathbf{x_{ij}} \leq \mathbf{y_i} \quad \textbf{for all } \mathbf{i,j}$$
$$\mathbf{x_{ij}, y_i} \geq \mathbf{0}$$

Dual

$$\max \sum_{\mathbf{j}} \alpha_{\mathbf{j}} :$$
$$\sum_{\mathbf{j}} \beta_{\mathbf{ij}} \leq \mathbf{f_i} \quad \textbf{for all } \mathbf{i}$$
$$\alpha_{\mathbf{j}} \leq \beta_{\mathbf{ij}} + \mathbf{c_{ij}} \quad \textbf{for all } \mathbf{i,j}$$
$$\alpha_{\mathbf{j}}, \beta_{\mathbf{ij}} \geq \mathbf{0}$$

1. Notation

$\mathbf{i}$ **blocked:** $\sum_{\mathbf{j}} \beta_{\mathbf{ij}} = \mathbf{f_i}$

$\mathbf{j}$ **blocked:** $\alpha_{\mathbf{j}} \geq \mathbf{c_{ij}}$ **for some blocked** $\mathbf{i}$

$$\max \sum_{\mathbf{j}} \alpha_{\mathbf{j}} :$$
$$\sum_{\mathbf{j}} \beta_{\mathbf{ij}} \leq \mathbf{f_i}$$
$$\alpha_{\mathbf{j}} \leq \beta_{\mathbf{ij}} + \mathbf{c_{ij}}$$
$$\alpha_{\mathbf{j}}, \beta_{\mathbf{ij}} \geq \mathbf{0}$$

2. A dual solution that grows

$$\max \sum_j \alpha_j :$$
$$\sum_j \beta_{ij} \le f_i$$
$$\alpha_j \le \beta_{ij} + c_{ij}$$
$$\alpha_j, \beta_{ij} \ge 0$$

Initialization: $\alpha, \beta \leftarrow 0$

Repeat

in parallel:

raise every unblocked α_j

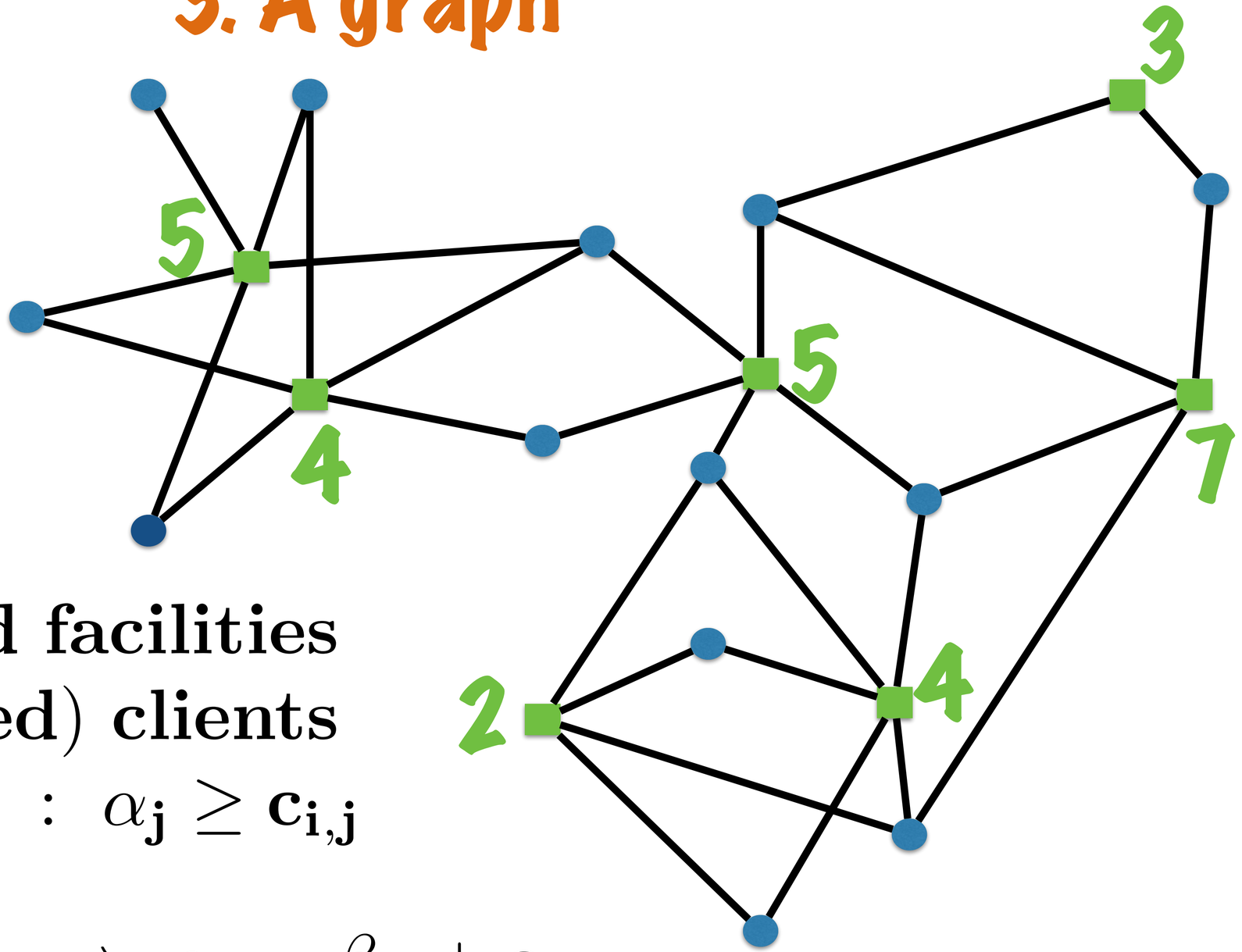
as well as every unblocked β_{ij} **s.t.**

$\alpha_j \ge c_{ij}$ **for some unblocked** α_j

Until every α_j **is blocked**

Fact: $\alpha_j \ge c_{ij} \iff \alpha_j = \beta_{ij} + c_{ij}$

3. A graph



■ : **blocked facilities**

● : **(blocked) clients**

edge $\{\mathbf{i},\mathbf{j}\}$: $\alpha_{\mathbf{j}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}}$

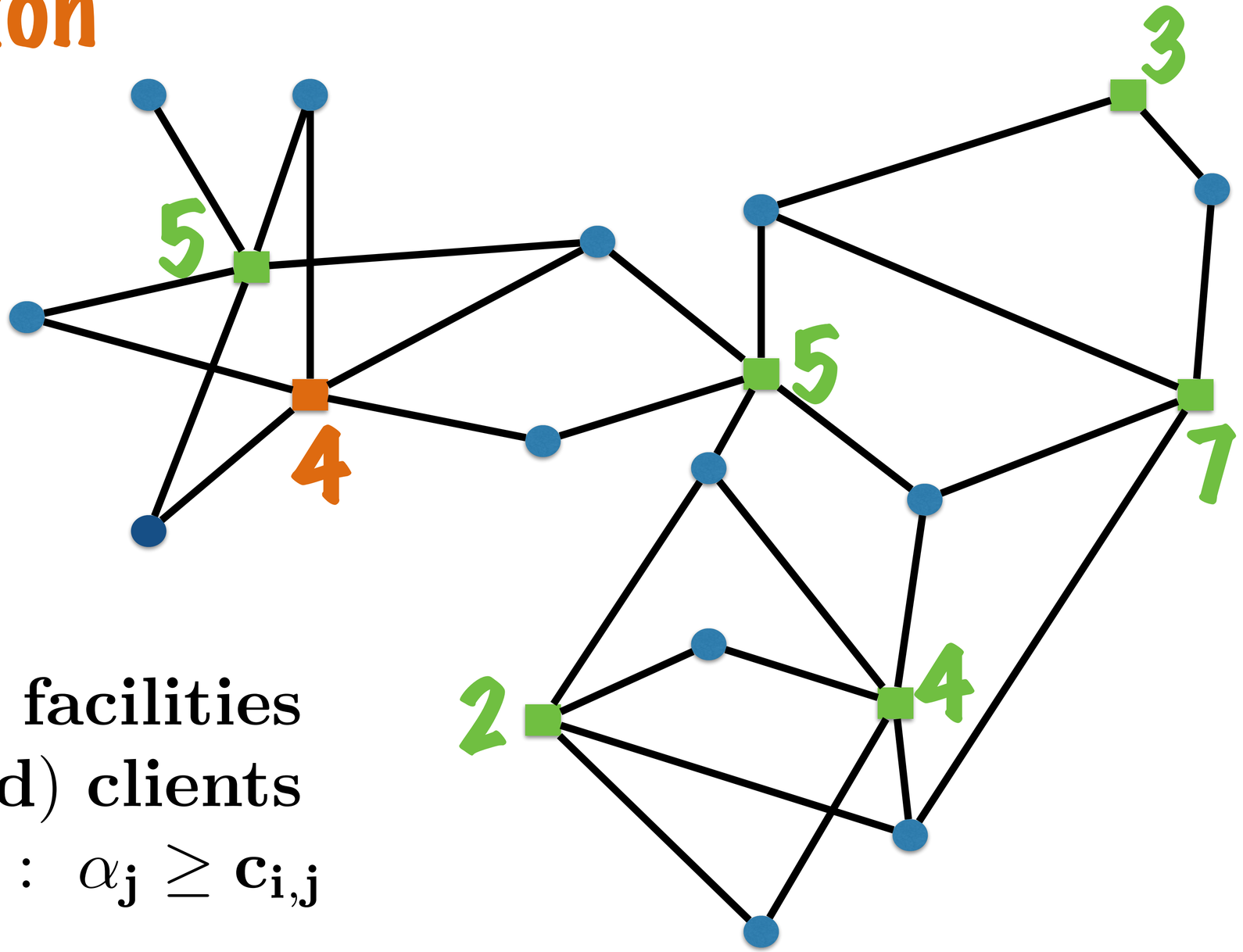
edge $\{\mathbf{i},\mathbf{j}\} \implies \alpha_{\mathbf{j}} = \beta_{\mathbf{ij}} + \mathbf{c}_{\mathbf{ij}}$

4. Constructing a primal solution

Initialization: facilities are pending,
clients are unassigned

While some clients are unassigned:
i_C: pending facility that was blocked first
open i_C
close pending facilities within distance 2
assign to i_C unassigned clients within distance 3

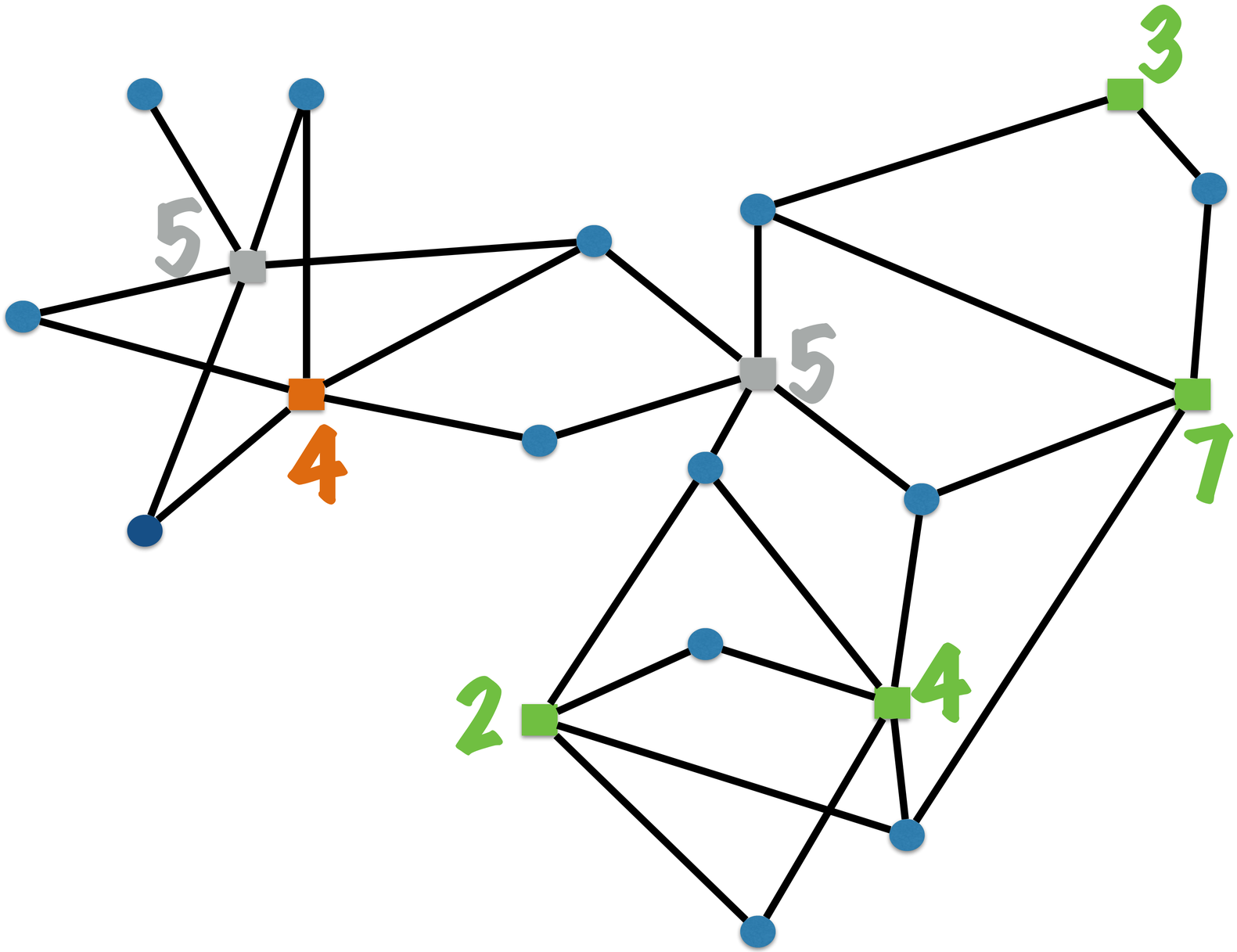
First iteration

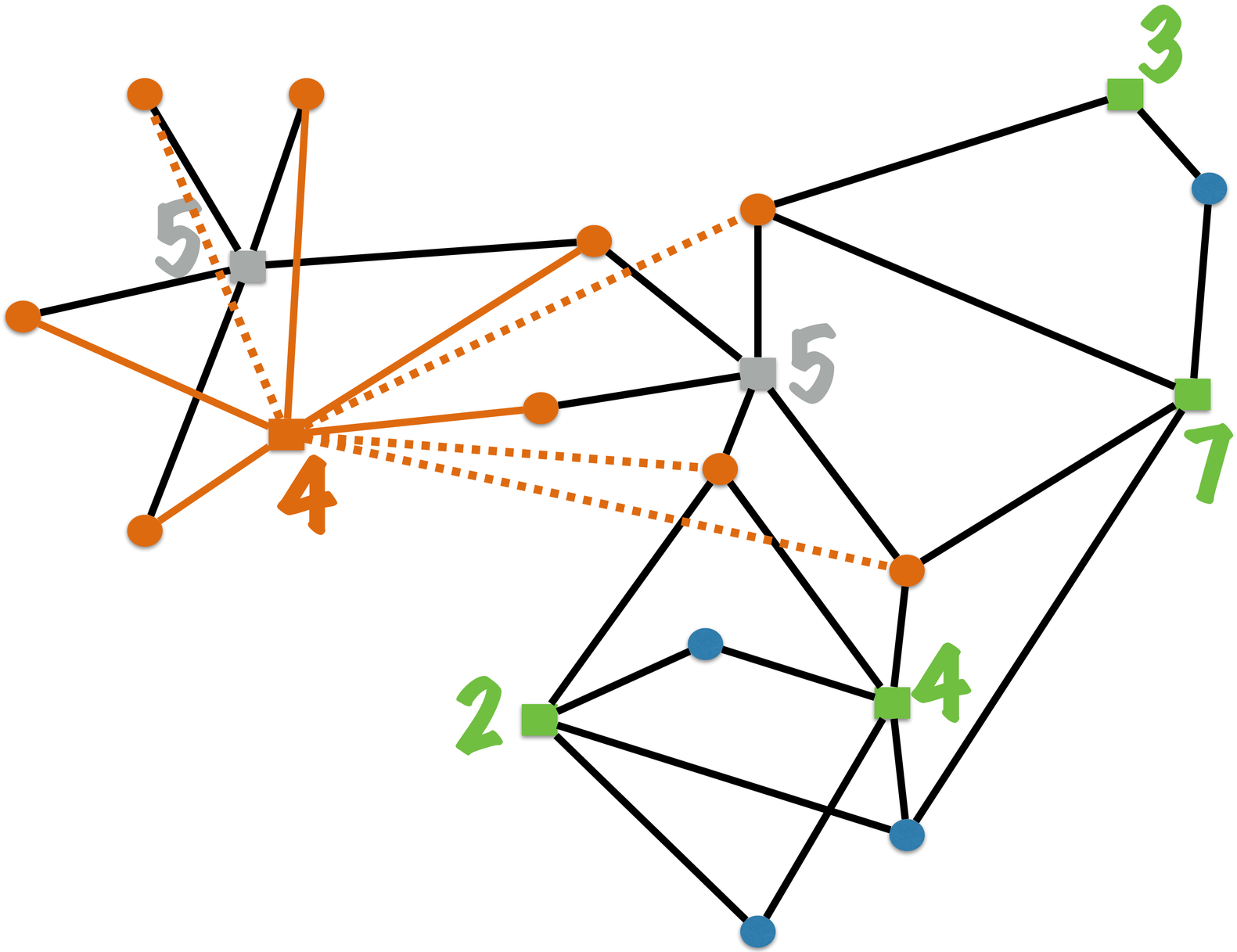


■ : **blocked facilities**

● : **(blocked) clients**

edge $\{\mathbf{i},\mathbf{j}\}$: $\alpha_{\mathbf{j}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}}$


3
5
5
4
7
2
4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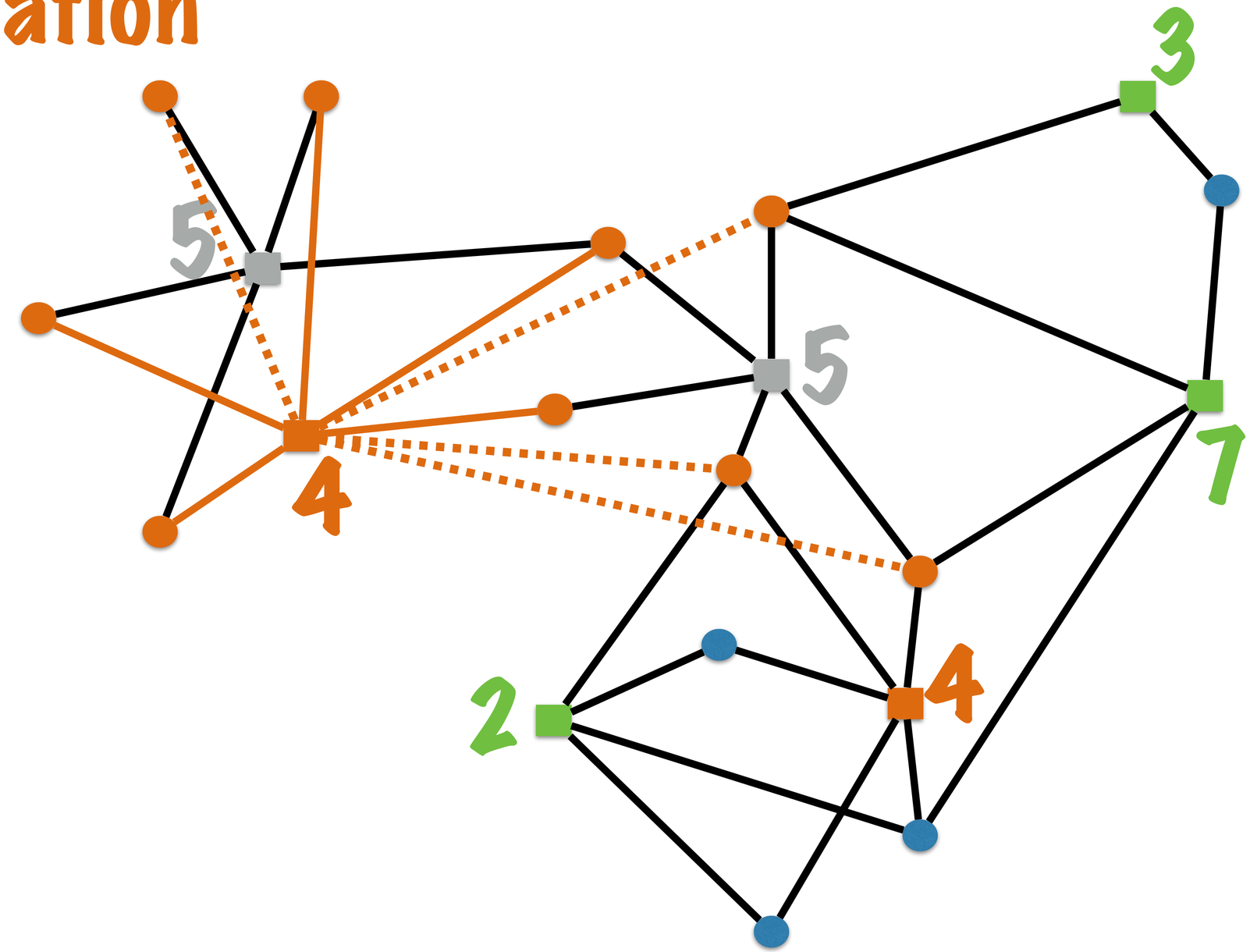
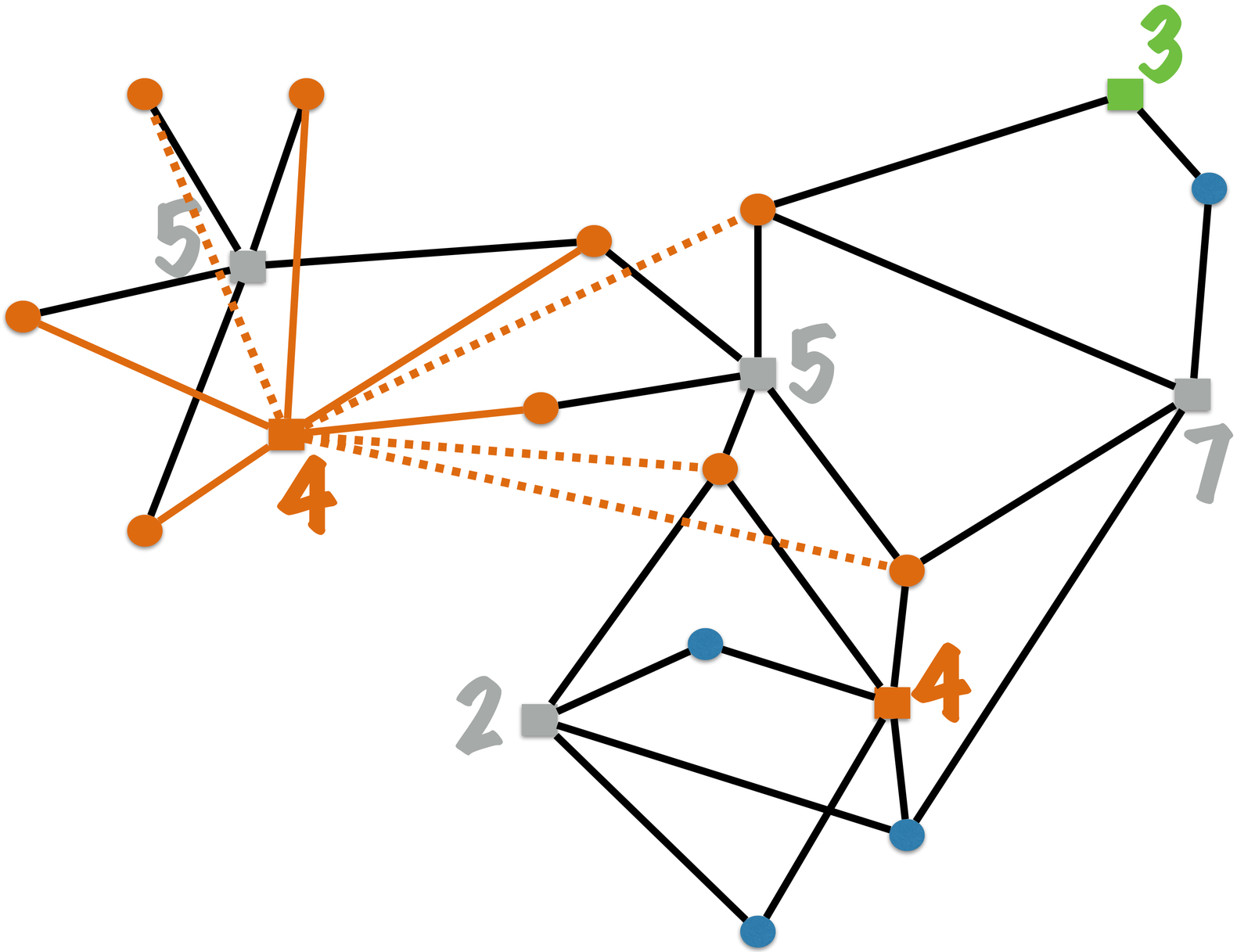
Second iteration

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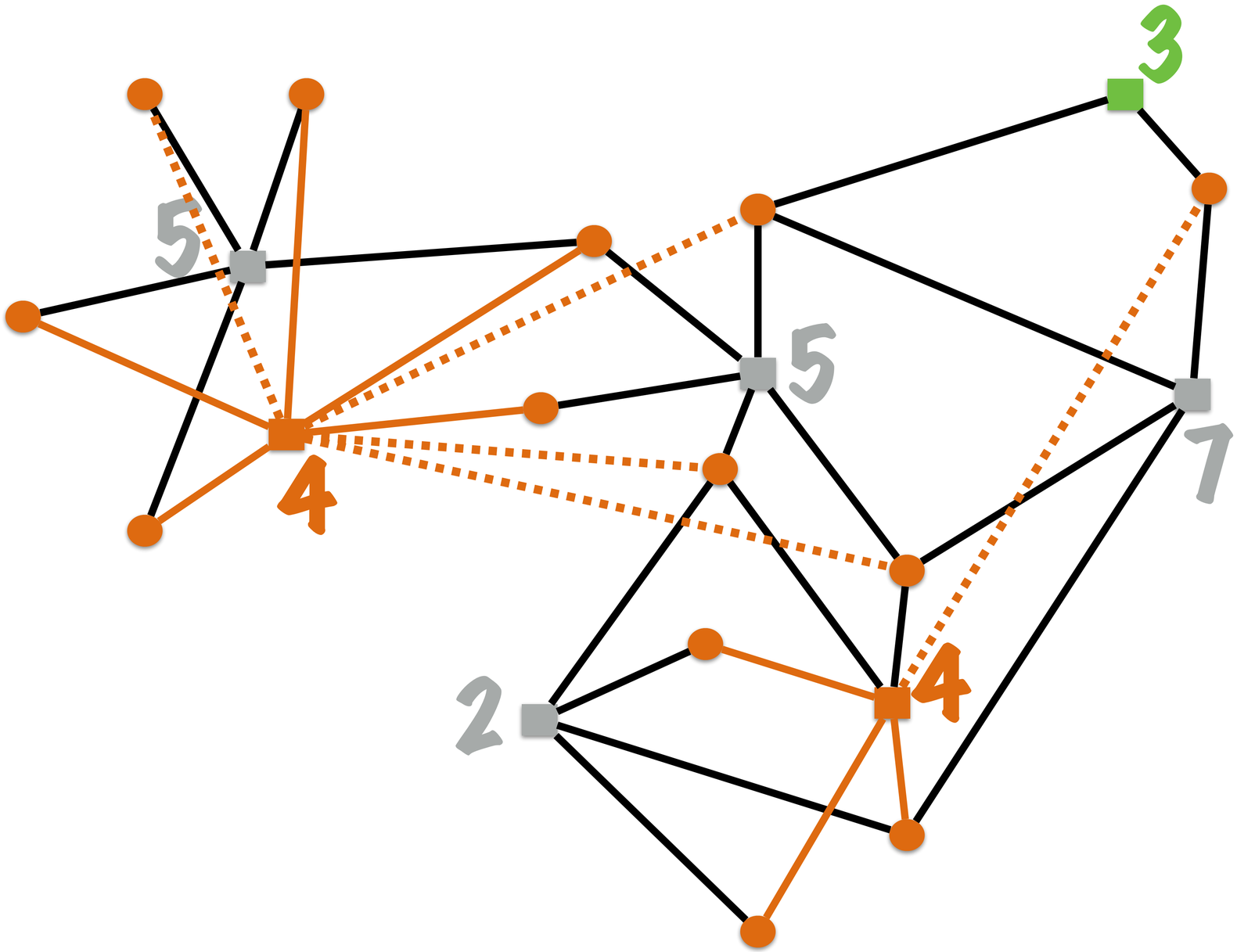
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Facility location