Erasmus
School of
Economics

MOOC Econometrics

Lecture M.1 on Building Blocks: Vectors and Matrices

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Example: Table

Company (abbrev.)	Yearly return (in %)	Size (in billions)	Growth ratio
ABC	15.1	25.5	1.23
DEF	7.9	40.8	1.89
PQR	4.5	30.2	1.55
STV	12.8	4.3	1.18
XYZ	10.5	10.7	1.68

Example: Matrices and vectors

$$A = \begin{pmatrix} 25.5 & 1.23 \\ 40.8 & 1.89 \\ 30.2 & 1.55 \\ 4.3 & 1.18 \\ 10.7 & 1.68 \end{pmatrix} \qquad y = \begin{pmatrix} 15.1 \\ 7.9 \\ 4.5 \\ 12.8 \\ 10.5 \end{pmatrix} \qquad c = \begin{pmatrix} 4.5 & 30.2 & 1.55 \end{pmatrix}$$

$$a_{32} = 1.55 \qquad A_{2\bullet} = \begin{pmatrix} 40.8 & 1.89 \end{pmatrix} \qquad A_{\bullet 2} = \begin{pmatrix} 1.23 \\ 1.89 \\ 1.55 \\ 1.18 \\ 1.68 \end{pmatrix}$$

Scalar multiplication

$$\underset{(p\times q)}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{pmatrix}$$

$$B = c \cdot A = \begin{pmatrix} c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1q} \\ c \cdot a_{21} & c \cdot a_{22} & \cdots & c \cdot a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{p1} & c \cdot a_{p2} & \cdots & c \cdot a_{pq} \end{pmatrix}$$

for all i, j : $b_{ij} = c \cdot a_{ij}$

Matrix addition

$$\underset{(p\times q)}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{pmatrix} \qquad \underset{(p\times q)}{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{pmatrix}$$

$$\underset{(p\times q)}{C} = A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1q}+b_{1q} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2q}+b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}+b_{p1} & a_{p2}+b_{p2} & \cdots & a_{pq}+b_{pq} \end{pmatrix}$$

Question

Test

Does the order of the summation matter?

The order does not matter.

Proof

- Let $C = A + B$.
- For each element $c_{ij} = a_{ij} + b_{ij}$
- And $a_{ij} + b_{ij} = b_{ij} + a_{ij}$
- Since this applies to all elements, $A + B = B + A$

Matrix multiplication

$$\underset{(p\times q)}{A} \cdot \underset{(q\times r)}{B} = \underset{(p\times r)}{C} = AB$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{iq} \\ \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pq} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1r} \\ & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{q1} & \cdots & b_{qj} & \cdots & b_{qr} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1r} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & \cdots & \boxed{c_{ij}} & \cdots & c_{ir} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pj} & \cdots & c_{pr} \end{pmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iq}b_{qj} = \sum_{k=1}^{q} a_{ik}b_{kj} = A_{i\bullet}B_{\bullet j}$$

Question

Test

Let A and B be 2×2 matrices. When does $AB = BA$ hold?.

Answer

Work out the matrix multiplication

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$BA = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{21} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix}$$

Products are generally different, except when $a_{11} = a_{22}$, $a_{12} = a_{21}$, $b_{11} = b_{22}$, and $b_{12} = b_{21}$. Check for example elements $(AB)_{21}$ and $(BA)_{21}$.

Combining multiple additions and multiplications

$$\underset{(q\times r)}{B} + \underset{(q\times r)}{C} + \underset{(q\times r)}{D} = (B+C)+D = (C+D)+B = (B+D)+C$$

$$\underset{(p\times q)}{A} \cdot \underset{(q\times r)}{B} \cdot \underset{(r\times s)}{E} = (A\cdot B)\cdot E = A\cdot(B\cdot E)$$

$$\underset{(p\times q)}{A} \cdot (\underset{(q\times r)}{B} + \underset{(q\times r)}{C}) = AB + AC \neq AB + C$$

Question

Test

Suppose A and B are $p \times p$ matrices. Find an expression without parentheses for $(A+B)^2$.

Answer

$(A+B)^2 = (A+B)\cdot(A+B) = A^2 + AB + BA + B^2$.

Multiplying vectors and matrices

$$\underset{(p\times q)}{A} \cdot \underset{(q\times 1)}{b} = \underset{(p\times 1)}{d}, \qquad d_i = \sum_{k=1}^{q} A_{ik} b_k = A_{i\bullet} \cdot b$$

$$\underset{(1\times p)}{c} \cdot \underset{(p\times q)}{A} = \underset{(1\times q)}{e}, \qquad e_j = \sum_{k=1}^{p} c_k A_{kj} = c \cdot A_{\bullet j}$$

$$\underset{(1\times p)}{u} \cdot \underset{(p\times 1)}{v} = \underset{(1\times 1)}{w}, \qquad w = \sum_{k=1}^{p} u_k v_k$$

$$\underset{(p\times 1)}{v} \cdot \underset{(1\times q)}{x} = \underset{(p\times q)}{Y}, \qquad y_{ij} = v_i x_j$$

Special matrices

- Square matrix: $\underset{(p\times q)}{A}$ with $p = q$, so $\underset{(p\times p)}{A}$
- Diagonal matrix: $\underset{(p\times p)}{A}$ with $a_{ij} = 0$ for $i \neq j$.
- Identity matrix: $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

  $\underset{(p\times q)}{A} \cdot \underset{(q\times q)}{I} = A$ and $\underset{(p\times p)}{I} \cdot \underset{(p\times q)}{A} = A$
- Unit vector: $\iota = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Example: stock returns

$$return_i = b_1 + b_2 \cdot size_i + b_3 \cdot growth_i + e_i$$

$$y = \begin{pmatrix} 15.1 \\ 7.9 \\ 4.5 \\ 12.8 \\ 10.5 \end{pmatrix} \quad X = \begin{pmatrix} 1 & 25.5 & 1.23 \\ 1 & 40.8 & 1.89 \\ 1 & 30.2 & 1.55 \\ 1 & 4.3 & 1.18 \\ 1 & 10.7 & 1.68 \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \; e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}$$

$$y = Xb + e$$

Training Exercise M.1

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).