Unimodality: the shape of the distribution

Unimodality

$$b(k) = b_n(k;p) = \binom{n}{k} p^k q^{n-k} \qquad (k = 0, 1, \ldots, n)$$

Unimodality

$$b(k) = b_n(k; p) = \binom{n}{k} p^k q^{n-k} \qquad (k = 0, 1, \dots, n)$$

$$\frac{b(k)}{b(k-1)} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}}$$

Unimodality

$$b(k) = b_n(k;p) = \binom{n}{k} p^k q^{n-k} \qquad (k = 0, 1, \ldots, n)$$

$$\frac{b(k)}{b(k-1)} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}}$$

$$= \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k!} p^k q^{n-k} \bigg/ \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} p^{k-1} q^{n-k+1}$$

Unimodality

$$b(k) = b_n(k;p) = \binom{n}{k} p^k q^{n-k} \qquad (k = 0, 1, \ldots, n)$$

$$\begin{aligned}
\frac{b(k)}{b(k-1)} &= \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} \\
&= \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k!} p^k q^{n-k} \Big/ \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} p^{k-1} q^{n-k+1} \\
&= \frac{(n-k+1)p}{kq}
\end{aligned}$$

Unimodality

$$b(k) = b_n(k;p) = \binom{n}{k} p^k q^{n-k} \qquad (k = 0, 1, \dots, n)$$

$$\begin{aligned}
\frac{b(k)}{b(k-1)} &= \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} \\
&= \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k!} p^k q^{n-k} \Big/ \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} p^{k-1} q^{n-k+1} \\
&= \frac{(n-k+1)p}{kq}
\end{aligned}$$

What can we conclude?

Unimodality

$$b(k) = b_n(k;p) = \binom{n}{k} p^k q^{n-k} \qquad (k = 0, 1, \ldots, n)$$

$$\begin{aligned}
\frac{b(k)}{b(k-1)} &= \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} \\
&= \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k!} p^k q^{n-k} \Big/ \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} p^{k-1} q^{n-k+1} \\
&= \frac{(n-k+1)p}{kq}
\end{aligned}$$

What can we conclude?

$b(k) > b(k-1)$ if, and only if, $(n - k + 1)p > kq$. Or: $np + p > kp + kq = k(p + q) = k$.

Unimodality

$$b(k) = b_n(k;p) = \binom{n}{k} p^k q^{n-k} \qquad (k = 0, 1, \ldots, n)$$

$$\begin{aligned} \frac{b(k)}{b(k-1)} &= \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} \\ &= \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k!} p^k q^{n-k} \Big/ \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} p^{k-1} q^{n-k+1} \\ &= \frac{(n-k+1)p}{kq} \end{aligned}$$

What can we conclude?

$b(k) > b(k-1)$ if, and only if, $(n-k+1)p > kq$. Or: $np + p > kp + kq = k(p+q) = k$.

Key conclusion:

The binomial probabilities $b(k)$ increase for $k < np + p$ and decrease thereafter.

Expectation: a notion of centre