

$$\begin{aligned}
 1. \quad E[y] &= E[a + b'x + e] = E[a] + E[b'x] + E[e] = a + b'x + \mu_e \\
 \text{var}[y] &= E[(y - E[y])^2] = E[(a + x'b + e - (a + x'b + \mu_e))^2] = \\
 &E[(e - \mu_e)^2] = \sigma_e^2
 \end{aligned}$$

$$\begin{aligned}
 2. \quad E[y] &= E[a_1 x_1 - a_2 x_2] = a_1 E[x_1] - a_2 E[x_2] = a_1 \mu_1 - a_2 \mu_2 \\
 \text{var}[y] &= \text{var}[a_1 x_1 - a_2 x_2] = E[(a_1 x_1 - a_2 x_2 - (a_1 \mu_1 - a_2 \mu_2))^2] \\
 &= E[(a_1(x_1 - \mu_1) - a_2(x_2 - \mu_2))^2] = \\
 &E[a_1^2(x_1 - \mu_1)^2 + a_2^2(x_2 - \mu_2)^2 - 2a_1 a_2(x_1 - \mu_1)(x_2 - \mu_2)] = \\
 &a_1^2 E[(x_1 - \mu_1)^2] + a_2^2 E[(x_2 - \mu_2)^2] - 2a_1 a_2 E[(x_1 - \mu_1)(x_2 - \mu_2)] = \\
 &a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 - 2a_1 a_2 \sigma_{12}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad E[y] &= E[a_1 x_1 + (1 - a_1)x_2] = a_1 E[x_1] + (1 - a_1)E[x_2] = a_1 \mu_1 + (1 - a_1)\mu = \mu \\
 \text{var}[y] &= \text{var}[a_1 x_1 + (1 - a_1)x_2] = a_1^2 \sigma^2 + (1 - a_1)^2 \sigma^2 + 2a_1(1 - a_1)\text{cov}[x_1, x_2] \\
 &= (2a_1^2 - 2a_1 + 1)\sigma^2 + (2a_1 - 2a_1^2)e\sigma^2 \\
 &= ((2a_1^2 - 2a_1)(1 - e) + 1)\sigma^2
 \end{aligned}$$

$$\text{cov}[x_1, x_2] = \sigma_1 e \sigma_2 = e \sigma^2$$

$$4. E[y] = E\left[\frac{1}{n} \sum_{i=1}^n x_i\right] = \frac{1}{n} \sum_{i=1}^n E[x_i] = \frac{1}{n} \sum_{i=1}^n \mu = \mu$$

$$a) \text{var}[y] = \text{var}\left[\frac{1}{n} \sum_{i=1}^n x_i\right] = \frac{1}{n^2} \sum_{i=1}^n \text{var}[x_i] + \frac{2}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \text{cov}[x_i, x_j]$$

$$= \frac{1}{n^2} \sum_{i=1}^n \sigma^2 = \frac{1}{n} \sigma^2$$

$$b) \text{cov}[y, x_i] = \text{cov}\left[\frac{1}{n} \sum_{j=1}^n x_j, x_i\right] = E\left[\left(\frac{1}{n} \sum_{j=1}^n x_j - \mu\right)(x_i - \mu)\right]$$

$$= E\left[\left(\frac{1}{n} \sum_{j=1}^n (x_j - \mu)\right)(x_i - \mu)\right] = E\left[\frac{1}{n} \sum_{j=1}^n ((x_j - \mu)(x_i - \mu))\right]$$

$$= \frac{1}{n} \sum_{j=1}^n E[(x_j - \mu)(x_i - \mu)] = \frac{1}{n} E[(x_i - \mu)^2] + \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n E[(x_j - \mu)(x_i - \mu)]$$

$$= \frac{1}{n} \sigma^2$$

$$\text{corr}[y, x_i] = \frac{\text{cov}[y, x_i]}{\sqrt{\text{var}[y] \text{var}[x_i]}}$$

$$= \frac{\sigma^2/n}{\sqrt{\sigma^2/n \cdot \sigma^2}}$$

$$= \frac{\sigma^2/n}{\sigma^2/\sqrt{n}}$$

$$= 1/\sqrt{n}$$

$$5. E[y] = E[h'x] = h'E[x] = \frac{1}{n} h'n' = h'n = M$$

$$\begin{aligned} (a) \text{var}[y] &= E[(y - E[y])(y - E[y])'] = E[(h'x - M)(h'x - M)'] \\ &= E[h'(x - M)(x - M)'h] = h'E[(x - M)(x - M)']h \\ &= \frac{1}{n^2} h' \sigma^2 I_n h = \frac{1}{n} \sigma^2 \end{aligned}$$

$$(b) E[z] = E[Hx] = HE[x] = H \frac{1}{n} h'n = \begin{pmatrix} I_n \\ h'n \end{pmatrix} = M' L_{n+1}$$

$$\begin{aligned} \text{var}[z] &= E[(z - E[z])(z - E[z])'] = E[(H(x - E[x]))(H(x - E[x]))'] \\ &= HE[(x - E[x])(x - E[x])']H' = H \sigma^2 I H' = \sigma^2 H H' \\ &= \sigma^2 \begin{pmatrix} I \\ h' \end{pmatrix} (I' \ h) = \sigma^2 \begin{pmatrix} I I' & I h \\ h' I & h' h \end{pmatrix} \\ &= \sigma^2 \begin{pmatrix} I I' & I h \\ h' I & h' h \end{pmatrix} = \sigma^2 \begin{pmatrix} I & \frac{1}{n} h'n \\ \frac{1}{n} h'n' & \frac{1}{n^2} h'n h' \end{pmatrix} \\ &= \frac{1}{n} \sigma^2 \begin{pmatrix} n I & h'n \\ h'n' & 1 \end{pmatrix} \end{aligned}$$