Time Series Problems HT 2009

1. Let $\{X_t\}$ be the ARMA(1, 1) process,

$$X_t - \phi X_{t-1} = \epsilon_t + \theta\epsilon_{t-1}, \qquad \{\epsilon_t\} \sim \mathrm{WN}(0, \sigma^2),$$

where $|\phi| < 1$ and $|\theta| < 1$. Show that the autocorrelation function of $\{X_t\}$ is given by

$$\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+\theta^2+2\phi\theta}, \qquad \rho(h) = \phi^{h-1}\rho(1) \quad \text{for } h \geqslant 1.$$

2. Consider a process consisting of a linear trend plus an additive noise term, that is,

$$X_t = \beta_0 + \beta_1 t + \epsilon_t$$

where β_0 and β_1 are fixed constants, and where the ϵ_t are independent random variables with zero means and variances σ^2. Show that X_t is non-stationary, but that the first difference series $\nabla X_t = X_t - X_{t-1}$ is second-order stationary, and find the acf of ∇X_t.

3. Let $\{S_t, t = 0, 1, 2, \ldots\}$ be the random walk with constant drift μ, defined by $S_0 = 0$ and

$$S_t = \mu + S_{t-1} + \epsilon_t, \qquad t = 1, 2, \ldots,$$

where $\epsilon_1, \epsilon_2, \ldots$ are independent and identically distributed random variables with mean 0 and variance σ^2. Compute the mean of S_t and the autocovariance of the process $\{S_t\}$. Show that $\{\nabla S_t\}$ is stationary and compute its mean and autocovariance function.

4. If

$$X_t = a\cos(\lambda t) + \epsilon_t$$

where $\epsilon_t \sim \mathrm{WN}(0, \sigma^2)$, and where a and λ are constants, show that $\{X_t\}$ is not stationary.

Now consider the process

$$X_t = a\cos(\lambda t + \Theta)$$

where Θ is uniformly distributed on $(0, 2\pi)$, and where a and λ are constants. Is this process stationary? Find the autocorrelations and the spectrum of X_t.

[To find the autocorrelations you may want to use the identity $\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$.]

5. Find the Yule-Walker equations for the AR(2) process

$$X_t = \tfrac{1}{3}X_{t-1} + \tfrac{2}{9}X_{t-2} + \epsilon_t$$

where $\epsilon_t \sim \mathrm{WN}(0, \sigma^2)$. Hence show that this process has autocorrelation function

$$\rho_k = \tfrac{16}{21}\left(\tfrac{2}{3}\right)^{|k|} + \tfrac{5}{21}\left(-\tfrac{1}{3}\right)^{|k|}.$$

[To solve an equation of the form $a\rho_k + b\rho_{k-1} + c\rho_{k-2} = 0$, try $\rho_k = A\lambda^k$ for some constants A and λ: solve the resulting quadratic equation for λ and deduce that ρ_k is of the form $\rho_k = A\lambda_1^k + B\lambda_2^k$ where A and B are constants.]

6. Let $\{Y_t\}$ be a stationary process with mean zero and let a and b be constants.

   (a) If $X_t = a + bt + s_t + Y_t$ where s_t is a seasonal component with period 12, show that $\nabla\nabla_{12}X_t = (1-B)(1-B^{12})X_t$ is stationary.

   (b) If $X_t = (a+bt)s_t + Y_t$ where s_t is again a seasonal component with period 12, show that $\nabla_{12}^2 X_t = (1-B^{12})(1-B^{12})X_t$ is stationary.

7. Consider the univariate state-space model given by state conditions $X_0 = W_0$, $X_t = X_{t-1} + W_t$, and observations $Y_t = X_t + V_t$, $t = 1, 2, \ldots$, where V_t and W_t are independent, Gaussian, white noise processes with var$(V_t) = \sigma_V^2$ and var$(W_t) = \sigma_W^2$. Show that the data follow an ARIMA(0,1,1) model, that is, ∇Y_t follows an MA(1) model. Include in your answer an expression for the autocorrelation function of ∇Y_t in terms of σ_V^2 and σ_W^2.