Math 194, problem set #7, SOLUTIONS

For discussion Tuesday November 27

1. Prove for $n \geq 1$ that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}. \qquad \text{(Engel)}$$

   Solution: For $n = 1$, the inequality holds. Assume the inequality holds for $n < k$, and consider
$$\begin{aligned}
\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2k-3}{2k-2} \cdot \frac{2k-1}{2k} &\leq \frac{1}{\sqrt{3(k-1)+1}} \cdot \frac{2k-1}{2k} \cdot \frac{\sqrt{3k+1}}{\sqrt{3k+1}} \\
&= \frac{1}{\sqrt{3k+1}} \cdot \frac{2k-1}{2k} \cdot \frac{\sqrt{3k+1}}{\sqrt{3k-2}} \\
&= \frac{1}{\sqrt{3k+1}} \cdot \frac{\sqrt{(2k-1)^2(3k+1)}}{\sqrt{(2k)^2(3k-2)}} \\
&= \frac{1}{\sqrt{3k+1}} \cdot \frac{\sqrt{12k^3-8k^2-k+1}}{\sqrt{12k^3-8k^2}} \leq \frac{1}{\sqrt{3k+1}}.
\end{aligned}$$
   Thus the inequality holds for all $n \geq 1$ by induction.

2. Let $a_1/b_1, a_2/b_2, \ldots, a_n/b_n$ be n fractions with $b_i > 0$ for $i = 1, 2, \ldots, n$. Show that the fraction
$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$
   is a number between the largest and smallest of these fractions.

   (Larson 7.1.10)

   Solution: Without loss of generality, assume $a_i/b_i \leq a_{i+1}/b_{i+1}$ for all i when both quantities are defined. For $n = 2$,
$$\begin{aligned}
\frac{a_1}{b_1} = \frac{a_1(b_1+b_2)}{b_1(b_1+b_2)} &= \frac{a_1 + a_1 b_2/b_1}{b_1+b_2} \leq \frac{a_1 + a_2 b_2/b_2}{b_1+b_2} = \frac{a_1+a_2}{b_1+b_2} \\
&= \frac{a_1 b_1/b_1 + a_2}{b_1+b_2} \leq \frac{a_2 b_1/b_2 + a_2}{b_1+b_2} = \frac{a_2(b_1+b_2)}{b_2(b_1+b_2)} = \frac{a_2}{b_2},
\end{aligned}$$
   and so our result holds. Assume our result holds for $n < k$, and consider
$$\frac{a_1 + a_2 + \cdots + a_k}{b_1 + b_2 + \cdots + b_k} = \frac{(a_1 + a_2 + \cdots + a_{k-1}) + a_k}{(b_1 + b_2 + \cdots + b_{k-1}) + b_k}.$$
   By our result for $n = 2$, this fraction will be between $(a_1 + a_2 + \cdots + a_{k-1})/(b_1 + b_2 + \cdots + b_{k-1})$ and a_k/b_k. Applying our result for $n = k-1$, we see that this implies our fraction will be between a_1/b_1 and a_k/b_k, and so our result holds for all n by induction.

3. Prove that $\sqrt[n]{n} < 1 + \sqrt{2/n}$ if n is a positive integer. (Larson 7.1.15)

Solution: Consider

$$(1+\sqrt{2/n})^n = \sum_{j=0}^{n} \binom{n}{j} (\sqrt{2/n})^j \geq 1+n\cdot\sqrt{2/n}+\frac{n(n-1)}{2}\cdot\frac{2}{n} = 1+\sqrt{2n}+(n-1) > n,$$

and so $\sqrt[n]{n} < 1 + \sqrt{2/n}$.

4. Prove that for every integer $n \geq 2$,

$$\prod_{k=1}^{n} \binom{n}{k} \leq \left(\frac{2^n - 2}{n-1}\right)^{n-1} \qquad \text{(Storey)}$$

Solution: Consider

$$\prod_{k=1}^{n} \binom{n}{k} = \prod_{k=1}^{n-1} \binom{n}{k} \leq \left(\frac{1}{n-1}\sum_{j=1}^{n-1}\binom{n}{j}\right)^{n-1} = \left(\frac{2^n - 2}{n-1}\right)^{n-1},$$

where the inequality follows from the $(n-1)$st power of the Arithmetic Mean–Geometric Mean Inequality.

5. Prove that if $a_1, \ldots, a_n$ are real numbers and $a_1 + \cdots + a_n = 1$, then

$$a_1^2 + \cdots + a_n^2 \geq 1/n.$$

Solution: By the Cauchy–Schwartz Inequality,

$$1 = \left(\sum_{j=1}^{n} a_j \cdot 1\right)^2 \leq \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} 1^2 = n \sum_{j=1}^{n} a_j^2,$$

and so our result holds.

6. Prove that the sequence $\{a_n\}$ defined by

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)$$

converges.

Solution: Notice that $1 + 1/2 + \cdots + 1/n$ is a left-hand Riemann sum for $\int_1^{n+1} dx/x = \ln(n+1)$. Graphically, we can see that the sum is an overestimate of the integral, and that the error increases with n and is bounded above by 1. Thus the sequence $\{1 + 1/2 + \cdots + 1/n - \ln(n+1)\} = \{a_n + \ln(n/(n+1))\}$ converges, and so must $\{a_n\}$.

7. Show that for all x,

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!} > 0.$$

Solution: For $x \geq 0$, all terms of the left-hand side are nonnegative, and so the result holds. For $x < 0$, if $x \leq -2n$,

$$\frac{x^{2j-1}}{(2j-1)!} + \frac{x^{2j}}{(2j)!} = \frac{x^{2j-1}}{(2j-1)!}\left(1 + \frac{x}{2j}\right) \geq 0$$

for all $1 \leq j \leq n$, and so the result holds. If $x > -2n$,

$$\frac{x^{2j-1}}{(2j-1)!} + \frac{x^{2j}}{(2j)!} = \frac{x^{2j-1}}{(2j-1)!}\left(1 + \frac{x}{2j}\right) < 0$$

for all $j > n$, and so

$$1 + x + \frac{x^2}{2!} + \cdots \frac{x^{2n}}{(2n)!} > \sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x > 0.$$

Thus our result holds for all x and n.

8. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists. (Putnam, 1998)

Solution: Consider a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. If we reflect this triangle across the x-axis, and then reflect that triangle across the line $y = -x$, we may travel along one edge of each of our three triangles on a path from (a, b) to $(b, -a)$. Notice that this path has total length equal to the perimeter of the triangle. We then see that the perimeter will be minimized if the triangle is chosen so that the path is one straight line segment from (a, b) to $(-b, a)$. Thus, the minimum perimeter is the distance from (a, b) to $(-b, a)$, which is $\sqrt{2a^2 + 2b^2}$.

9. Prove that for every positive n, $n! > (n/e)^n$. (Larson 7.1.12)

Solution: For $n = 1$, the result holds. Assume the result holds for all $k \leq n$, and consider

$$\left(\frac{k+1}{e}\right)^{k+1} = \left(\frac{k}{e}\right)^k \left(\frac{k+1}{k}\right)^k \frac{k+1}{e} < k!\left(1 + \frac{1}{k}\right)^k \frac{k+1}{e} < (k+1)!\frac{e}{e} = (k+1)!,$$

where the second inequality follows from the fact that $(1 + 1/k)^k < e$. Thus, the result holds for all n by induction.

10. Prove that

$$\left(\frac{a+1}{b+1}\right)^{b+1} > \left(\frac{a}{b}\right)^{b}$$

for every $a, b > 0$, $a \neq b$. (Larson 7.4.17)

Solution: Let $f(a) = \dfrac{(a+1)^{b+1}}{a^b}$ and observe that

$$f'(a) = \frac{(b+1)(a+1)^b a^b - ba^{b-1}(a+1)^{b+1}}{a^{2b}} = \frac{a^{b-1}(a+1)^b(a-b)}{a^{2b}} = \frac{(a+1)^b(a-b)}{a^{b+1}}.$$

We then see that $f'(a) < 0$ for $0 < a < b$, $f'(a) = 0$ for $a = b$, and $f'(a) > 0$ for $a > b$. Thus $f(a)$ attains its minimum at $a = b$, and so $a \neq b$ implies $\dfrac{(a+1)^{b+1}}{a^b} > \dfrac{(b+1)^{b+1}}{b^b}$, which implies our result.

11. Find all positive integers $n, k_1, k_2, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$ (Putnam 2005)

Solution by Manjul Bhargava, Kiran Kedlaya, and Lenny Ng, posted at

http://www.unl.edu/amc/a-activities/a7-problems/putnam/-pdf/2005s.pdf

By the arithmetic-harmonic mean inequality or the Cauchy-Schwarz inequality,

$$(k_1 + \cdots + k_n)\left(\frac{1}{k_1} + \cdots + \frac{1}{k_n}\right) \geq n^2.$$

We must thus have $5n - 4 \geq n^2$, so $n \leq 4$. Without loss of generality, we may suppose that $k_1 \leq \cdots \leq k_n$.

If $n = 1$, we must have $k_1 = 1$, which works. Note that hereafter we cannot have $k_1 = 1$.

If $n = 2$, we have $(k_1, k_2) \in \{(2,4), (3,3)\}$, neither of which work.

If $n = 3$, we have $k_1 + k_2 + k_3 = 11$, so $2 \leq k_1 \leq 3$. Hence $(k_1, k_2, k_3) \in$

$\{(2,2,7), (2,3,6), (2,4,5), (3,3,5), (3,4,4)\}$, and only $(2,3,6)$ works.

If $n = 4$, we must have equality in the AM-HM inequality, which only happens when $k_1 = k_2 = k_3 = k_4 = 4$.

Hence the solutions are $n = 1$ and $k_1 = 1$, $n = 3$ and (k_1, k_2, k_3) is a permutation of $(2,3,6)$, and $n = 4$ and $(k_1, k_2, k_3, k_4) = (4,4,4,4)$.