Feedback — Homework 1

Help Center

Thank you. Your submission for this homework was received.

You submitted this homework on **Sun 15 Mar 2015 1:27 AM PDT**. You got a score of **7.00** out of **7.00**.

Question 1

We have seen that, starting from a few intuitive axioms, we can infer a variety of additional properties of probability measure. The results we have seen include:

1. If Ω denotes the underlying sample space then $\mathbf{P}(\Omega) = 1$.
2. $\mathbf{P}(\emptyset) = 0$.
3. For any event E, we have $\mathbf{P}(E) \geq 0$.
4. For any event E, we have $\mathbf{P}(E) \leq 1$.
5. For any two events E and F where $E \subseteq F$, we have $\mathbf{P}(E) \leq \mathbf{P}(F)$.
6. For any event E, we have $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$.
7. For any two events E and F, we have $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$.
8. For any family of events $E_1, E_2, \ldots, E_n$, we have $\mathbf{P}(E_1 \cup E_2 \cup \cdots \cup E_n) \leq \mathbf{P}(E_1) + P(E_2) + \cdots + \mathbf{P}(E_n)$.
9. If $E_1, E_2, \ldots, E_n, \ldots$ is a finite or countably infinite collection of pairwise disjoint events, then $\mathbf{P}(E_1 \cup E_2 \cup \cdots \cup E_n \cup \cdots) = \mathbf{P}(E_1) + \mathbf{P}(E_2) + \cdots + \mathbf{P}(E_n) + \cdots$.

Which of these properties are the axioms from which all the other properties are derived?

Your Answer		Score	Explanation
1, 3, 8			
2, 4, 6, 8			
1, 3, 4, 7			
1, 3, 9	✔	1.00	
2, 4, 9			
Total		1.00 / 1.00	

Question Explanation

Properties 1, 3, and 9 are axioms.

The axioms of probability measure were outlined in Tableau 5. Refer to the video lecture for Tableau 5.k or the associated slides for a summary.

Question 2

You are given two (distinguishable) dice; identify them as A and B. Die A has three red faces and five white faces while die B has five red faces and three white faces. You flip a fair coin once. If it shows heads then you pick die A and throw it once. If, however, the coin shows up tails then you pick die B and throw it once. Which of the following options completely characterises a sample point (that is to say, an outcome) of the sample space in this chance experiment?

Your Answer		Score	Explanation
Die B shows up red.			
The coin lands on heads.			
The coin lands on tails and die B shows up white.	✔	1.00	
The coin lands on tails, die B shows up white, and die A shows up red.			
The coin lands on heads and die B shows up red.			
Total		1.00 / 1.00	

Question Explanation

The specification of a sample point must eliminate all chance-related uncertainty in the experiment while conforming to the conditions under which the experiment is to be performed. In this case there are two chance-driven elements: the selection of the die and the result of throwing it. The first is completely specified by specifying the outcome of the toss of a coin; the second is completely specified by specifying the colour of the face of the die that shows on throwing it. Implicit here is the understanding that we do not distinguish between faces of the same colour on a given die. (The experiment as given here is equivalent to one where we replace the dice by two coins, each with one red face and one white face, the first coin A biased three-in-eight towards red, the second coin B biased five-in-eight towards tails. The selection of which of these coins is selected and tossed is determined by the result of the toss of a fair coin.) A different sample space arises if the coloured faces of the dice are distinguishable. But, as the list of possible answers indicates, this is not the setting that is implicitly considered here. When modelling a chance experiment we will have to make principled choices based upon context taking into account all the information that is available.

Returning to our problem, an answer that the coin lands heads cannot be considered a satisfactory sample point because it leaves open the question of what happened to the throw of the chosen die (A in this case). Nor can an answer that the coin lands on heads and die B shows up red be considered satisfactory because the conditions of the experiment say that if the coin lands on heads then it is die A that is thrown. Similarly, the assertion that both dice show white is not satisfactory as only one die is thrown under the conditions of the experiment. The assertion that the coin lands on tails and die B shows up white does however satisfy our requirements that all chance-related uncertainty is eliminated by this specification and it conforms to the underlying conditions of the experiment.

As one of the posters has pointed out, the option "Die *B* shows up red" may also be considered a satisfactory sample point as it implicitly also specifies the outcome of the coin toss. For the reasons provided earlier, an explicit identification of all the elements comprising the sample space is preferable, at least at a first entry into the subject, but the answer is certainly viable. We've given retroactive credit for folks who selected this as the right option.

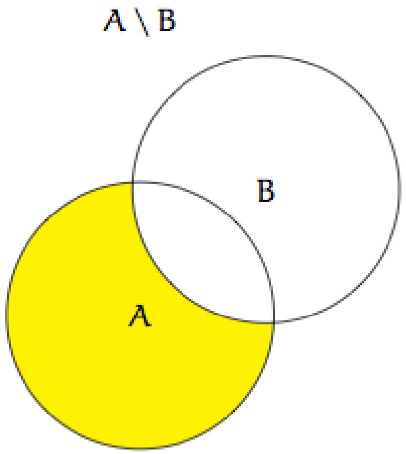
Question 3

The following prompt should be used for **Questions 3 and 4:**

At a university, 70% of the students are international, 20% are majoring in mathematics, and 10% are both international *and* majoring in mathematics. Let A be the event that a student chosen at random is international, and let B be the event that a student chosen at random is majoring in mathematics. In mathematical notation this means: $\mathbf{P}(A) = 0.7$, $\mathbf{P}(B) = 0.2$, and $\mathbf{P}(A \cap B) = 0.1$.

Determine $\mathbf{P}(A \setminus B)$, i.e., the probability that a student chosen at random is international but *not* majoring in mathematics.

[Recall that the set difference $A \setminus B$ is the collection of sample points that are in A and *not* in B, visualised schematically as the shaded region in the Venn diagram given below.]



Your Answer		Score	Explanation
0.1			
0.5			
0.6	✔	1.00	

0.8

0.9

Total 1.00 / 1.00

Question Explanation

This is an exercise in additivity.

We may partition A into the union of two disjoint sets via the decomposition

$$A = (A \cap B) \cup (A \setminus B).$$

[The Venn diagram suggests the validity of this decomposition. To prove this formally, first argue that $(A \cap B)$ and $(A \setminus B)$ are disjoint, and then show that the sets on the left and the right are the same by showing that any element of the set on the left is contained in the right, and vice versa. This is not a difficult exercise and you may wish to quickly undertake it to make sure you understand the process.]

It follows by additivity that

$$\mathbf{P}(A) = \mathbf{P}(A \cap B) + \mathbf{P}(A \setminus B).$$

By rearranging the equation, we find

$$\mathbf{P}(A \setminus B) = \mathbf{P}(A) - \mathbf{P}(A \cap B) = 0.7 - 0.1 = 0.6.$$

Question 4

Determine $\mathbf{P}(A \cup B)$.

Your Answer		Score	Explanation
0.3			
0.5			
0.9			
0.2			
0.8	✔	1.00	
Total		1.00 / 1.00	

Question Explanation

The inclusion and exclusion identity in the case of two events tells us that

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 0.7 + 0.2 - 0.1 = 0.8.$$

If you'd like a quick refresher, see lecture 5.h and the summary lecture 5.k.

Question 5

The following prompt should be used for **Questions 5 and 6:**

You are going to a concert with three friends. You have four tickets which assign you to the same row of twelve seats but the tickets do not assign an exact seat number. You arrive together only to see that eight of the seats in your row are already occupied, but, luckily, the remaining four seats in your assigned row form a block of adjacent seats so you and your friends can sit together. Model the situation as a chance experiment in which the eight early arrivals take their seats at random, and the event of interest is that, with the eight early arrivals having taken their seats, the remaining four seats form a block of adjacent seats.

Identify the sample space and the number of elements in it in this chance experiment.

Your Answer		Score	Explanation
The $\binom{12}{8}8!$ possible ways in which eight of the 12 seats may be occupied.			
The $12!$ possible ways in which the total 12 seats may be arranged in a row.			
The $\binom{12}{8}$ possible ways in which eight of the 12 seats may be occupied.	✔	1.00	
The $8!$ possible ways in which the eight occupied seats may be arranged in a row.			
The $4!$ possible ways in which the four empty seats may be arranged in a row.			
Total		1.00 / 1.00	

Question Explanation

There are 12 seats in the row, the eight early arrivals taking eight of them. Label the seats in some order from left to right and identify which eight of the twelve seats is found to be occupied when you arrive. If we write 1 for "occupied" and 0 for "free" then, for example, 100011111011 represents a generic sample point with the only free seats being 2, 3, 4, and 10. As the specification of who sits where is not relevant to the problem, only which seats are free, this corresponds to specifying all such sequences of 0s and 1s of length 12 containing exactly eight 1s. We may enumerate them by systematically identifying all ways of selecting eight locations from 12 and it follows that there are $\binom{12}{8} = \binom{12}{4}$ ways of specifying which seats are occupied and which are free at the moment you arrive with your friends.

Question 6

Let A be the event that the four empty seats are all next to each other. Determine the probability of A for the given model of randomness. Is the arrangement observed surprising for this model? If so this might be an indication of non-randomness in seat choice.

Your Answer		Score	Explanation
○ The probability of the four empty seats being next to each other is $\frac{2}{3}$ and the arrangement seen is not at all surprising as it would occur on average twice in three times.			
○ The probability of the four empty seats being next to each other is $\frac{1}{3}$ and the observed arrangement is not surprising as it could occur on average once in three times.			
○ The probability of the four empty seats being next to each other is $\frac{9}{126}$ and the observed arrangement is surprising because the calculated probability for event A is small.			
◉ The probability of the four empty seats being next to each other is $\frac{1}{55}$ and the observed arrangement is surprising because the calculated probability for event A is small.	✔	1.00	
Total		1.00 / 1.00	

Question Explanation

When a question is asked, the first step in solving that problem is to enumerate the information that is provided in that problem. What information do we have from the above question?

We have 4 **tickets**.

There are 12 **seats** to choose from.

There are 8 **occupied seats**.

Next we need to identify what is asked. *Rewording the problem tells us that we want to find the probability that we can find 4 **seats next to each other** out of the 12 **possible seats***.

So let us write out a possible outcome favourable to this event:

Occupied, Occupied, Occupied, Free, Free, Free, Free, Occupied, Occupied, Occupied, Occupied, Occupied.

Or, in the succinct notation introduced in the solution of the previous problem of 1 for "occupied" and 0 for "free", 111000011111. The set of outcomes that are favourable to A are those that contain a sequence of four contiguous 0s in a sequence of twelve 1s and 0s containing exactly eight 1s (or, equivalently, four 0s). Since the four 0s have to form an unbroken string, their location is completely characterised by where the *first* of the four 0s is located. A little thought shows that the first 0 (free seat) can be located at positions 1, 2, 3, ..., 9. Once it is positioned, say at location i, then we know that the next four positions $i+1$, $i+2$,

$i+3$, and $i+4$ all have to have 0s; the remaining positions all have 1s. (In the example given above, the first 0 occurs at location 4 so that the next three 0s are at locations 5, 6, and 7; all other locations are occupied by 1s.) It follows that there are exactly 9 ways of specifying where the sequence of four contiguous 0s lies and so A has exactly nine elements: $\mathrm{card}(A) = 9$.

As we saw in the previous problem, the sample space has $\binom{12}{8}$ elements in it. As all arrangements of occupied seats are presumed to be equally likely (this is a combinatorial setting), we obtain

$$\mathbf{P}(A) = \frac{9}{\binom{12}{8}} = \frac{1}{55} = 0.01818\cdots.$$

Is this arrangement surprising and therefore indicative of non-randomness?

The probability $\mathbf{P}(A) = 0.01818\cdots$ is quite small. In our model the chance of finding an arrangement with four adjacent free seats only occurs about two percent of the time. We conclude that the observed phenomenon is quite unlikely to happen by sheer chance. This suggests that the seats that were found to be occupied were not selected randomly. (Perhaps the eight early arrivals formed one or more parties of individuals who know each other.)

Question 7

The social disease of keeping up with the Joneses may be parodied in this invented game of catch-up on a social ladder, the height on which indicates social position. Two protagonists are each provided with a six-sided fair die the faces of which show the numbers 1, 2, 3, 4, 5, 6. Both social climbers start at the foot of the social ladder. Begin the game by having the first player roll her die and move as many steps up the ladder as show on her die face. The second player, envious of the progress of her rival, takes her turn next, rolls her die, and moves up the ladder as many steps as show on her die face. The game now progresses in turns. At each turn, *whichever player is lower on the ladder* rolls her die and moves up as many steps as indicated on her die face. The game terminates at the *first* instant when both players end up on the same ladder step. (At which point, the two rivals realise the futility of it all and as the social honours are now even they resolve to remain friends and give up social competition.) Call each throw of a die, from either player, a turn. To illustrate, consider the following sequence with players named A and B: player A throws 3 (she is now on rung 3); player B takes her turn and throws 1 (she is now on rung 1); player B takes another turn as she is below player A and throws 4 (she passes player A and is now on rung 5); player A is now below player B and so takes her turn and throws 2 (she is now also on rung 5). For this sequence there have been 4 turns and the game terminates with both players on rung 5. Suppose n represents any positive integer larger than 1. What is the probability that the game terminates in n turns?

Your Answer	Score	Explanation

- $\frac{5^n}{6^n}$
- 6^{-n}
- $\frac{5^{n-2}}{6^{n-1}}$ ✔ 1.00
- $\left(\frac{1}{6}\right)^{n-1} \cdot \frac{5}{6}$
- 2^{-n}

Total 1.00 / 1.00

Question Explanation

Suppose the game terminates in n turns. How many sequences are there that are favourable for this event? Well, the first turn can take any of 6 values and the last turn can take only 1 value as the person below has to throw exactly what is needed to reach the rung of the person above. That leaves the $n-2$ turns between the first and the last turns. As the game does not terminate on any of these intermediate turns, it must be the case that, on each throw of a die on the intermediate turns, the person who is lower on the ladder can throw any number *except* the one number that will get her on the same rung as the person above her. This means that there are 5 possibilities for each of the intermediate throws and so there are 5^{n-2} ways in which these throws can be made without terminating the game. There are thus $6 \cdot 5^{n-2} \cdot 1$ sequences of n turns that are favourable to the event that the game terminates on the nth throw. But there are 6^n possibilities in total for n throws of a die and so the probability that the game terminates on the nth throw is $\frac{6 \cdot 5^{n-2}}{6^n} = \frac{5^{n-2}}{6^{n-1}}$. As you can see, only counting arguments were necessary here but if you have viewed Lecture 6.1d you may recognise that you can rewrite this expression in the form $\left(\frac{5}{6}\right)^{n-2} \cdot \frac{1}{6}$. This is the *geometric distribution with success probability* $\frac{1}{6}$.