Facility location

i blocked: $\sum_j \beta_{ij} = f_i$

j blocked: $\alpha_j \geq c_{ij}$ for some blocked i

$\max \sum_j \alpha_j :$
$\sum_j \beta_{ij} \leq f_i$
$\alpha_j \leq \beta_{ij} + c_{ij}$
$\alpha_j, \beta_{ij} \geq 0$

Initialization: $\alpha, \beta \leftarrow 0$

Repeat

in parallel,

raise every unblocked α_j

and every unblocked β_{ij} s.t.

$\alpha_j \geq c_{ij}$ for some unblocked α_j

Until every α_j is blocked

Initialization: facilities are pending,
clients are unassigned

While some clients are unassigned:
i_C: pending facility that was blocked first
open i_C
close pending facilities within distance 2
assign to i_C unassigned clients within distance 3

$$\text{edge } \{i,j\} \implies \alpha_j = \beta_{ij} + c_{ij}$$

Analysis

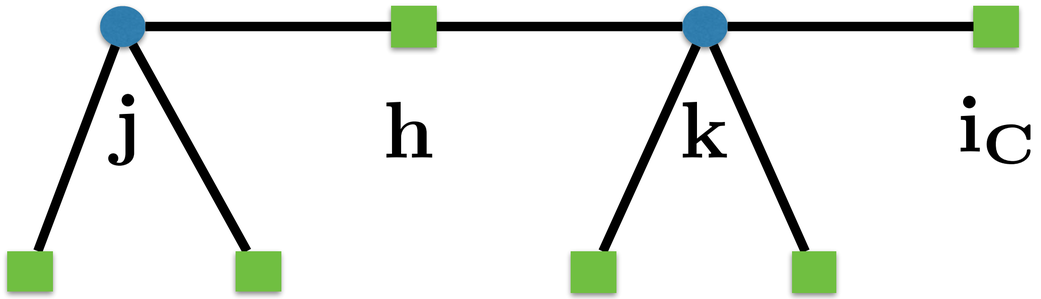
$$\text{Cost} = \sum_{\text{Cluster } C} (f_{i_C} + \sum_{j \in C} c_{i_C j})$$

$$f_{i_C} = \sum_{j \text{ adjacent to } i_C} \beta_{i_C j}$$

j $\quad$ i_C

$$f_{i_C} + \sum_{j \text{ adjacent to } i_C} c_{i_C j} =$$
$$\sum_{j \text{ adjacent to } i_C} \beta_{i_C j} + c_{i_C j} =$$
$$\sum_{j \text{ adjacent to } i_C} \alpha_j$$

Clients at distance 3 from i_C



$$\alpha_j \geq c_{hj}$$
$$\alpha_k \geq c_{hk}$$
$$\alpha_k \geq c_{i_C k}$$

$$c_{i_C j} \leq c_{hj} + c_{hk} + c_{i_C k} \leq \alpha_j + 2\alpha_k$$

Since i_C was blocked first among contenders:

$$\alpha_k \leq \alpha_j$$

And so: $c_{i_C j} \leq 3 \cdot \alpha_j$

Together:

$$\textbf{Cost} = \sum_{\text{Cluster } C} (f_{i_C} + \sum_{j \in C} c_{i_C j})$$

$$\leq \sum_{\text{Cluster } C} (\sum_{j \in C, j \sim i_C} \alpha_j + \sum_{j \in C, d(j, i_C) = 3} 3\alpha_j)$$

$$\leq \sum_{\text{Cluster } C} \sum_{j \in C} 3\alpha_j$$

$$\leq 3 \sum_j \alpha_j$$

$$\leq 3 \cdot \mathrm{OPT}$$

QED

Facility location