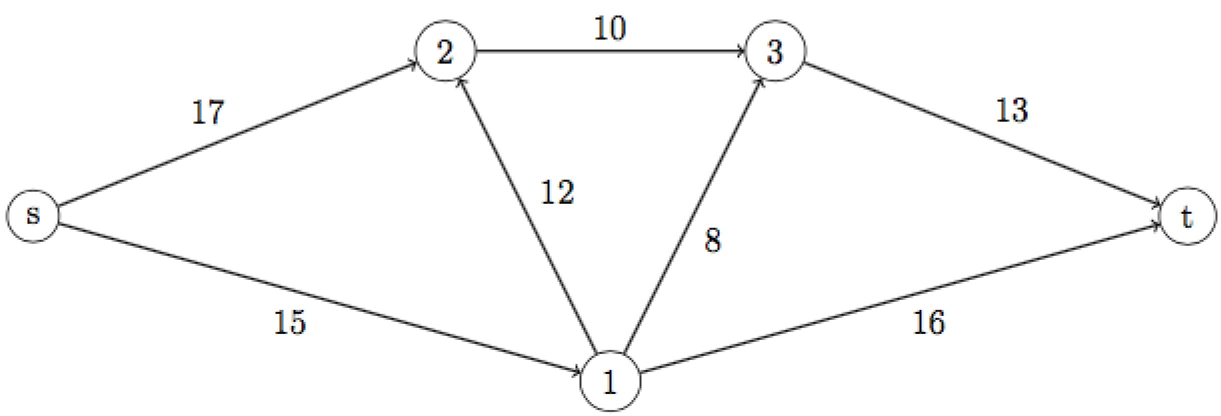
Maximum Flow Problems

John Mitchell

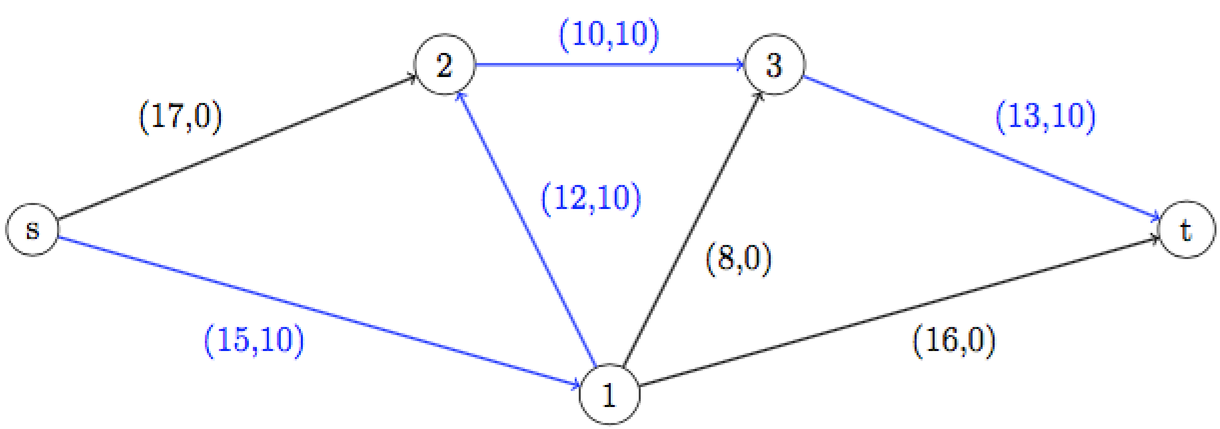
The Ford-Fulkerson augmenting flow algorithm can be used to find the maximum flow from a source to a sink in a directed graph $G = (V,E)$. Each arc $(i,j) \in E$ has a capacity of u_{ij}. We find paths from the source to the sink along which the flow can be increased. The paths might include arcs facing in the reverse direction from the path; flow is decreased on these arcs.

Example

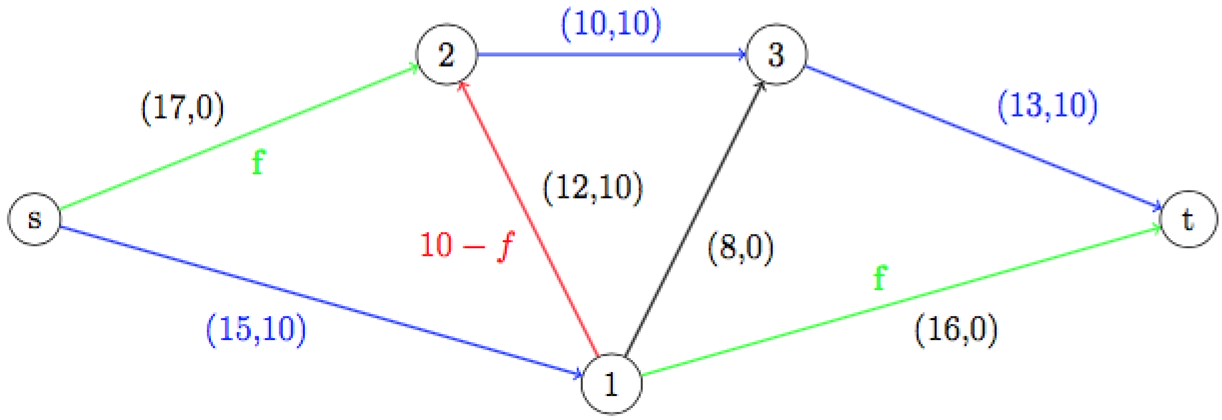
This graph has a source s, a destination or sink t, and three transshipment nodes. The capacities of the edges are listed in the figure.



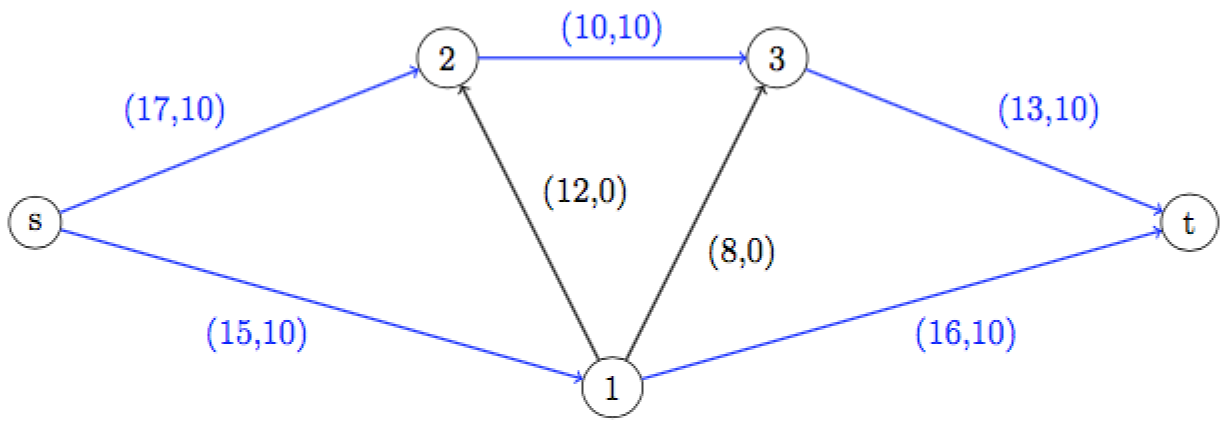
There is no initial flow. One augmenting path is $s \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow t$. 10 units can be pushed along this arc, at which point arc $(2, 3)$ is saturated. This results in the following situation. The labels on each arc represent (capacity,flow).



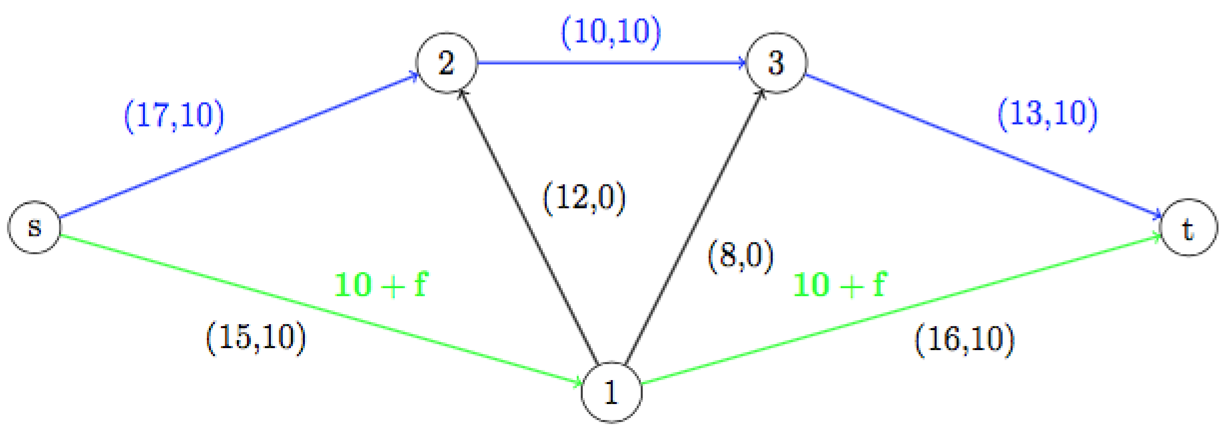
There are now augmenting paths in the graph that result in a reduction of flow on arc (1,2). One possible augmenting path is $s \rightarrow 2 \rightarrow 1 \rightarrow t$. The path is indicated by green forward arcs and red reverse arcs and the flow on the path is denoted f.



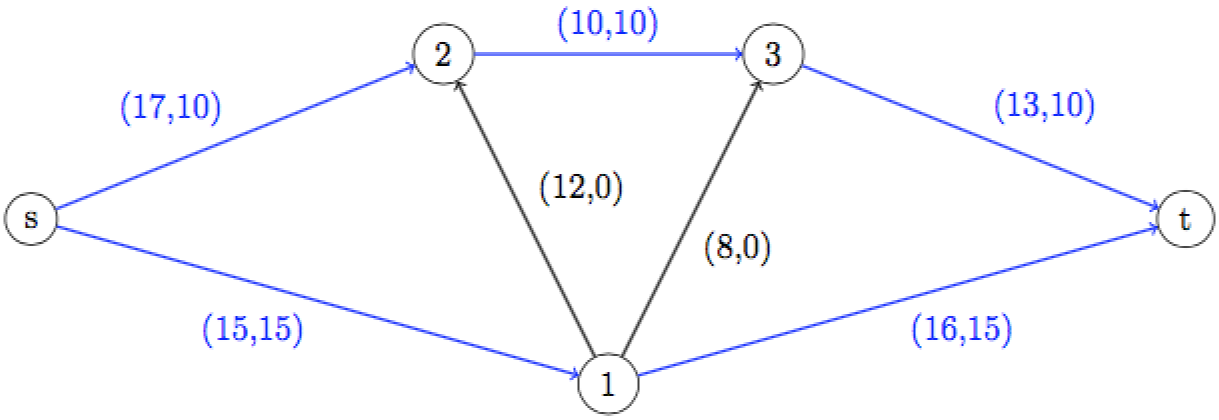
The maximum possible value for the flow is $f = 10$, giving the overall flow below.



There is another augmenting path in the graph, $s \longrightarrow 1 \longrightarrow t$, with both arcs used in the forward direction. The path is indicated by green forward arcs and the flow on the path is denoted f.



The maximum possible value for the flow is $f = 5$, giving the overall flow below.



If we try to augment flow further, we cannot push flow along the arc $(s, 1)$. We can push flow along $(s, 2)$, but no further: arc $(2, 3)$ is saturated, and the arc $(1, 2)$ entering node 2 is empty.

So this flow is **optimal**.

We can divide the graph into two sets based on the flow:

- Nodes to which we can push flow from s: s itself, and also node 2. Let $S_s := \{s, 2\}$.
- Nodes to which we cannot push flow from s: nodes $1, 3, t$. Let $S_t := \{1, 3, t\}$.

The arcs between S_s and S_t constitute an (s,t)-**cut** in the graph: the source is on one side of the cut and the sink is on the other. All flow from s to t has to flow across this cut, and so there is no way to increase the flow further, because all the arcs in the cut pointing from S_s to S_t are saturated and all the arcs pointing in the reverse direction are empty. The capacity of the cut is the sum of the capacities of the arcs in the cut pointing from S_s to S_t.

It is a fundamental result that **Max Flow** = **Min Cut**. This result can be proved using LP duality. In our example problem, the max flow problem can be written as the following linear program, using a variable x_{ts} to represent the total flow from s to t:

$$
\begin{array}{lrcl}
\max_x & x_{ts} & & \\
\text{s.t.} & x_{s1} + x_{s2} - x_{ts} & = & 0 \\
& -x_{s1} + x_{12} + x_{13} + x_{14} & = & 0 \\
& -x_{s2} - x_{12} + x_{23} & = & 0 \\
& -x_{13} - x_{23} + x_{3t} & = & 0 \\
& -x_{1t} - x_{3t} + x_{ts} & = & 0 \\
& 0 \leq x_{s1} & \leq & 15 \\
& 0 \leq x_{s2} & \leq & 17 \\
& 0 \leq x_{12} & \leq & 12 \\
& 0 \leq x_{13} & \leq & 8 \\
& 0 \leq x_{1t} & \leq & 16 \\
& 0 \leq x_{23} & \leq & 10 \\
& 0 \leq x_{3t} & \leq & 13
\end{array}
$$

In the dual LP, we have variables y_i for each vertex i, and variables w_{ij} corresponding to the upper bounds on each flow x_{ij}:

$$
\begin{array}{llcccccccl}
\min_{y,w} & \multicolumn{9}{l}{15w_{s1} + 17w_{s2} + 12w_{12} + 8w_{13} + 16w_{1t} + 10w_{23} + 13w_{3t}} \\
\text{s.t.} & y_s & - & y_1 & + & w_{s1} & \geq & 0 & & \\
 & y_s & - & y_2 & + & w_{s2} & \geq & 0 & & \\
 & y_1 & - & y_2 & + & w_{12} & \geq & 0 & & \\
 & y_1 & - & y_3 & + & w_{13} & \geq & 0 & & \\
 & y_1 & - & y_t & + & w_{1t} & \geq & 0 & & \\
 & y_2 & - & y_3 & + & w_{23} & \geq & 0 & & \\
 & y_3 & - & y_t & + & w_{32} & \geq & 0 & & \\
 & y_t & - & y_s & & & = & 1 & & \\
 & & & & & w_{ij} & \geq & 0 & \text{for all arcs (i,j)} &
\end{array}
$$

(*s,t*)-cuts correspond to feasible solutions to this linear program:

$$y_i = 0 \text{ if } i \in S_s, \quad y_i = 1 \text{ if } i \in S_t,$$

and then set

$$w_{ij} = \max\{0, y_j - y_i\}.$$

The value of such a solution in the dual LP is exactly the capacity of the cut. Using the cut we found earlier, $S_s = \{s, 2\}$, $S_t = \{1, 3, t\}$, we get

$$y_s = y_2 = 0, \quad y_1 = y_3 = y_t = 1$$

and

$$w_{s1} = w_{23} = 1, \quad w_{s2} = w_{12} = w_{13} = w_{1t} = w_{3t} = 0,$$

with value 25, the capacity of the cut and also the maximum flow.