Concave function

From Wikipedia, the free encyclopedia

In mathematics, a **concave function** is the negative of a convex function. A concave function is also synonymously called **concave downwards**, **concave down**, **convex upwards**, **convex cap** or **upper convex**.

Contents

- 1 Definition
- 2 Properties
- 3 Examples
- 4 See also
- 5 Notes
- 6 References

Definition

A real-valued function f on an interval (or, more generally, a convex set in vector space) is said to be *concave* if, for any x and y in the interval and for any t in [0,1],

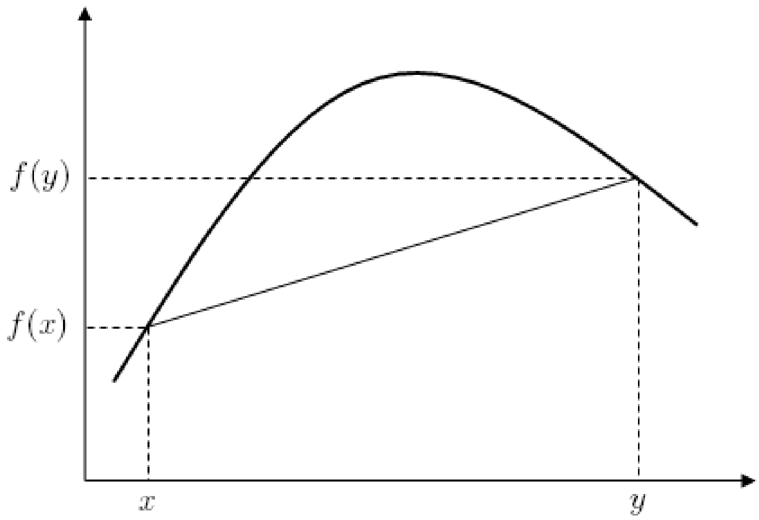
$$f((1-t)x+(t)y) \geq (1-t)f(x)+(t)f(y).$$

A function is called *strictly concave* if

$$f((1-t)x+(t)y) > (1-t)f(x)+(t)f(y)$$

for any t in (0,1) and $x \neq y$.

For a function $f:R \to R$, this definition merely states that for every z between x and y, the point $(z, f(z))$ on the graph of f is above the straight line joining the points $(x, f(x))$ and $(y, f(y))$.



A function *f(x)* is quasiconcave if the upper contour sets of the function $S(a) = \{x : f(x) \geq a\}$ are convex sets.[1]

Properties

A function $f(x)$ is concave over a convex set if and only if the function $-f(x)$ is a convex function over the set.

A differentiable function *f* is concave on an interval if its derivative function f' is monotonically decreasing on that interval: a concave function has a decreasing slope. ("Decreasing" here means non-increasing, rather than strictly decreasing, and thus allows zero slopes.)

For a twice-differentiable function *f*, if the second derivative, $f''(x)$, is positive (or, if the acceleration is positive), then the graph is convex; if $f''(x)$ is negative, then the graph is concave. Points where concavity changes are inflection points.

If a convex (i.e., concave upward) function has a "bottom", any point at the bottom is a minimal extremum. If a concave (i.e., concave downward) function has an "apex", any point at the apex is a maximal extremum.

If $f(x)$ is twice-differentiable, then $f(x)$ is concave if and only if $f''(x)$ is non-positive. If its second derivative is negative then it is strictly concave, but the opposite is not true, as shown by $f(x) = -x^4$.

If *f* is concave and differentiable, then it is bounded above by its first-order Taylor approximation:

$$f(y) \leq f(x) + f'(x)[y - x]$$[2]

A continuous function on *C* is concave if and only if for any *x* and *y* in *C*

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

If a function *f* is concave, and $f(0) \geq 0$, then *f* is subadditive. Proof:

- since *f* is concave, let $y = 0$, $f(tx) = f(tx + (1-t)\cdot 0) \geq tf(x) + (1-t)f(0) \geq tf(x)$
- $f(a)+f(b) = f\left((a+b)\frac{a}{a+b}\right)+f\left((a+b)\frac{b}{a+b}\right) \geq \frac{a}{a+b}f(a+b)+\frac{b}{a+b}f(a+b) = f(a+b)$

Examples

- The functions $f(x) = -x^2$ and $g(x) = \sqrt{x}$ are concave on their domains, as are their second derivatives $f''(x) = -2$ and $g''(x) = -\frac{1}{4x^{1.5}}$ are always negative.
- Any affine function $f(x) = ax + b$ is both (non-strictly) concave and convex.
- The sine function is concave on the interval $[0, \pi]$.
- The function $f(B) = \log|B|$, where $|B|$ is the determinant of a nonnegative-definite matrix *B*, is concave.[3]
- Practical example: rays bending in computation of radiowave attenuation in the atmosphere.

See also

- Concave polygon
- Convex function
- Jensen's inequality
- Logarithmically concave function
- Quasiconcave function

Notes

1. Varian 1992, p. 496.
2. Varian 1992, p. 489.
3. Thomas M. Cover and J. A. Thomas (1988). "Determinant inequalities via information theory". *SIAM Journal on Matrix Analysis and Applications* **9** (3): 384–392. doi:10.1137/0609033 (https://dx.doi.org/10.1137%2F0609033).

References

- Crouzeix, J.-P. (2008). "Quasi-concavity". In Durlauf, Steven N.; Blume, Lawrence E. *The New Palgrave Dictionary of Economics* (http://www.dictionaryofeconomics.com/article?id=pde2008_Q000008) (Second ed.). Palgrave Macmillan. doi:10.1057/9780230226203.1375 (https://dx.doi.org/10.1057%2F9780230226203.1375).
- Rao, Singiresu S. (2009). *Engineering Optimization: Theory and Practice*. John Wiley and Sons. p. 779. ISBN 0-470-18352-7.
- Varian, Hal R. (1992). *Microeconomic Analysis* (Third ed.). W.W. Norton and Company.

Categories: Types of functions | Convex analysis