Time Series Solution Sketches Sheet 2 HT 2010

1. Consider the process $X_t = a\cos(\lambda t + \Theta)$, where Θ is uniformly distributed on $(0, 2\pi)$, and where a and λ are constants. Is this process stationary? Find the autocorrelations and the spectrum of X_t.

   [To find the autocorrelations you may want to use the identity $\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$.]

   Solution: For $X_t = a\cos(\lambda t + \Theta)$ we need to consider the joint distributions of $(X(t_1), \ldots, X(t_k))$ and of $(X(t_1+\tau), \ldots, X(t_k+\tau))$. Since shifting time by t is equivalent to shifting Θ by λt, and since Θ is uniform on $(0, 2\pi)$, these two joint distributions are the same, and so X_t is stationary. For the mean,

$$E(X_t) = aE(\cos(\lambda t + \Theta)) = \frac{a}{2\pi}\int_0^{2\pi} \cos(\lambda t + \theta)\, d\theta = \frac{a}{2\pi}[\sin(\lambda t + \theta)]_0^{2\pi} = 0.$$

   For the autocovariance function,

$$\begin{aligned}\gamma_t &= E(X_t X_0) = a^2 E(\cos(\Theta)\cos(\lambda t + \Theta)) = a^2 E\left[\tfrac{1}{2}\{\cos(\lambda t + 2\Theta) + \cos(\lambda t)\}\right] \\ &= \tfrac{a^2}{2}\left[\tfrac{1}{2\pi}\int_0^{2\pi} \cos(\lambda t + 2\theta) + \cos(\lambda t)\, d\theta\right] = \tfrac{a^2}{2}\cos(\lambda t) = \tfrac{a^2}{4}\left(e^{it\lambda} + e^{-it\lambda}\right).\end{aligned}$$

   In particular $\gamma_0 = \frac{a^2}{2}$ and $\rho_t = \cos(\lambda t)$. The spectrum is F where $\gamma_t = \int_{-\pi}^{\pi} e^{it\omega}\, dF(\omega)$.

   Recall from lectures: If

$$\gamma_h = \frac{\sigma^2}{2}\left(e^{-i\lambda_0 h} + e^{i\lambda_0 h}\right)$$

   then the spectrum is

$$F(\lambda) = \begin{cases} 0 & \text{if } \lambda < -\lambda_0 \\ \frac{\sigma^2}{2} & \text{if } -\lambda_0 \le \lambda < \lambda_0 \\ \sigma^2 & \text{if } \lambda \ge \lambda_0. \end{cases}$$

   Hence here

$$F(\omega) = \begin{cases} 0 & \text{if } \omega < -\lambda \\ \frac{a^2}{4} & \text{if } -\lambda \le \omega < \lambda \\ \frac{a^2}{2} & \text{if } \omega \ge \lambda. \end{cases}$$

2. Find the Yule-Walker equations for the AR(2) process $X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + \epsilon_t$ where $\epsilon_t \sim \mathrm{WN}(0, \sigma^2)$. Hence show that this process has autocorrelation function $\rho_k = \frac{16}{21}\left(\frac{2}{3}\right)^{|k|} + \frac{5}{21}\left(-\frac{1}{3}\right)^{|k|}$.

   [To solve an equation of the form $a\rho_k + b\rho_{k-1} + c\rho_{k-2} = 0$, try $\rho_k = A\lambda^k$ for some constants A and λ: solve the resulting quadratic equation for λ and deduce that ρ_k is of the form $\rho_k = A\lambda_1^k + B\lambda_2^k$ where A and B are constants.]

   Solution: The Yule-Walker equations are

$$\rho_k = \tfrac{1}{3}\rho_{k-1} + \tfrac{2}{9}\rho_{k-2}.$$

   So as in the hint, to solve $\rho_k - \frac{1}{3}\rho_{k-1} - \frac{2}{9}\rho_{k-2} = 0$ try $\rho_k = A\lambda^k$. Substituting this into the above equation, and cancelling a factor of λ^{k-2}, we get

$$\lambda^2 - \tfrac{1}{3}\lambda - \tfrac{2}{9} = 0$$

   which has roots $\lambda = \frac{2}{3}$ and $\lambda = -\frac{1}{3}$, so $\rho_k = A(\frac{2}{3})^k + B(-\frac{1}{3})^k$.

   We also require $\rho_0 = 1$ and $\rho_1 = \frac{1}{3} + \frac{2}{9}\rho_1$. Hence we can solve for A and B: $A = \frac{16}{21}$ and $B = \frac{5}{21}$. So

$$\rho_k = \tfrac{16}{21}(\tfrac{2}{3})^k + \tfrac{5}{21}(-\tfrac{1}{3})^k.$$

3. Let $\{Y_t\}$ be a stationary process with mean zero and let a and b be constants.

   (a) If $X_t = a + bt + s_t + Y_t$ where s_t is a seasonal component with period 12, show that $\nabla\nabla_{12}X_t = (1-B)(1-B^{12})X_t$ is stationary.

   (b) If $X_t = (a + bt)s_t + Y_t$ where s_t is again a seasonal component with period 12, show that $\nabla_{12}^2 X_t = (1-B^{12})(1-B^{12})X_t$ is stationary.

Solution:

(a) We have

$$\nabla X_t = a + bt + s_t + Y_t - [a + b(t-1) + s_{t-1} + Y_{t-1}] = b + s_t - s_{t-1} + Y_t - Y_{t-1}$$

and

$$\begin{aligned}\nabla\nabla_{12}X_t &= b + s_t - s_{t-1} + Y_t - Y_{t-1} - [b + s_{t-12} - s_{t-13} + Y_{t-12} - Y_{t-13}] \\ &= Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}\end{aligned}$$

and this is a stationary process since Y_t is stationary. (We have used the fact that $s_t = s_{t-12}$ for all t.)

(b) We have

$$\nabla_{12}X_t = (a + bt)s_t + Y_t - [(a + b(t-12))s_{t-12} + Y_{t-12}] = Y_t + 12bs_{t-12} - Y_{t-12}$$

and

$$\nabla_{12}^2 X_t = Y_t + 12bs_{t-12} - Y_{t-12} - [Y_{t-12} + 12bs_{t-24} - Y_{t-24}] = Y_t - 2Y_{t-12} + Y_{t-24}$$

and this is stationary since Y_t is stationary (again using $s_t = s_{t-12}$ for all t.)

4. Consider the univariate state-space model given by state conditions $X_0 = W_0$, $X_t = X_{t-1} + W_t$, and observations $Y_t = X_t + V_t$, $t = 1, 2, \ldots$, where V_t and W_t are independent, Gaussian, white noise processes with $\text{var}(V_t) = \sigma_V^2$ and $\text{var}(W_t) = \sigma_W^2$. Show that the data follow an ARIMA(0,1,1) model, that is, ∇Y_t follows an MA(1) model. Include in your answer an expression for the autocorrelation function of ∇Y_t in terms of σ_V^2 and σ_W^2.

Solution:

$$\nabla Y_t = Y_t - Y_{t-1} = (X_t + V_t) - (X_{t-1} + V_{t-1}) = X_t - X_{t-1} + V_t - V_{t-1} = W_t + V_t - V_{t-1}$$

and so ∇Y_t is an MA(1). To make the connection with MA(1) more transparent, note that $\epsilon_t = V_t + W_t$ gives a mean zero white noise series with variance $\sigma_\epsilon^2 = \sigma_V^2 + \sigma_W^2$. Thus ϵ_t has the same distribution as $\sqrt{\frac{\sigma_V^2+\sigma_W^2}{\sigma_V^2}}V_t$.

Putting $\beta = -\sqrt{\frac{\sigma_V^2}{\sigma_V^2+\sigma_W^2}}$ thus gives that, in distribution, $V_t + W_t - V_{t-1} = \epsilon_t + \beta\epsilon_{t-1}$.

Note that for identifiability we usually require $|\beta| < 1$; which is satisfied here.

As V_t, V_{t-1} and W_t are independent,

$$\gamma_0 = Var(\nabla Y_t) = \sigma_W^2 + 2\sigma_V^2.$$

Furthermore,

$$\gamma_1 = Cov(\nabla Y_t, \nabla Y_{t+1}) = Cov(W_t + V_t - V_{t-1}, W_{t+1} + V_{t+1} - V_t) = -\sigma_V^2,$$

and, from the independence, $\gamma_k = 0$ for $|k| \geq 2$. Hence the acf is $\rho_0 = 1$, $\rho_1 = -\frac{\sigma_V^2}{\sigma_W^2+2\sigma_V^2}$, and $\rho_k = 0$ for $|k| \geq 2$.