Problem 1

A bounded polyhedron $P \subseteq \mathbb{R}^n$ is called a *polytope*. This means there exists a $M \in \mathbb{R}$ such that $P \subseteq [-M, M]^n$.
The convex hull of a set of points $v_1, \dots, v_k \in \mathbb{R}^n$ is defined as

$$\operatorname{conv}(\{v_1, \dots, v_k\}) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \mid 0 \leq \lambda_i \leq 1 \ \forall i = 1, \dots, k, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

In the following let P be a non-empty polytope.

(i) Prove that P has vertices.

(ii) Let $u_1, \dots, u_\ell \in \mathbb{R}^n$ be the vertices of P. Show that $P = \operatorname{conv}(\{u_1, \dots, u_\ell\})$.

Hint: There are multiple ways to solve (ii). To show that $P \subseteq \operatorname{conv}(\{u_1, \dots, u_\ell\})$ one possibility is to suppose that there exists a point $x^* \in P \setminus \operatorname{conv}(\dots)$ and consider the following linear program and its dual:

$$\begin{aligned} \min \quad & 0^{\mathrm{T}} \cdot \lambda \\ \text{s.t.} \quad & \sum_{i=1}^{\ell} \lambda_i u_i = x^* \\ & \sum_{i=1}^{\ell} \lambda_i = 1 \\ & \lambda \geq 0 \\ & \lambda \in \mathbb{R}^\ell \end{aligned}$$

$$\begin{aligned} \max \quad & (x^*)^{\mathrm{T}} \cdot c + \beta \\ \text{s.t.} \quad & u_i^{\mathrm{T}} \cdot c + \beta \leq 0 \quad i = 1, \dots, \ell \\ & c \in \mathbb{R}^n, \beta \in \mathbb{R} \end{aligned}$$

Conclude that the primal problem (the minimization) must be infeasible and that its dual problem (the maximization) is unbounded. From this, yield a contradiction to the fact that there always exists an optimal vertex in P for any linear optimization problem over P.

Problem 2

Let G be a graph and let A be its node-edge incidence matrix. We have seen in class that if G is bipartite then A is totally unimodular. Prove the converse, *i.e.*, if A is totally unimodular then G is bipartite.