Facility location

Primal

$$\min \sum_i \mathbf{f_i y_i} + \sum_{i,j} \mathbf{c_{ij} x_{ij}} :$$
$$\sum_i \mathbf{x_{ij}} \geq 1 \quad \textbf{for all j}$$
$$\mathbf{x_{ij}} \leq \mathbf{y_i} \quad \textbf{for all i, j}$$
$$\mathbf{x_{ij}}, \mathbf{y_i} \geq \mathbf{0}$$

Dual

$$\max \sum_j \alpha_j :$$
$$\sum_j \beta_{ij} \leq \mathbf{f_i} \quad \textbf{for all i}$$
$$\alpha_j \leq \beta_{ij} + \mathbf{c_{ij}} \quad \textbf{for all i, j}$$
$$\alpha_j, \beta_{ij} \geq \mathbf{0}$$

Algorithm

1. Solve the primal and dual LPs: $y_i^*, x_{ij}^*, \alpha_j^*, \beta_{ij}^*$
2. While some clients are unassigned

   j_C : unassigned client s.t. $\alpha_{j_C}^*$ is min

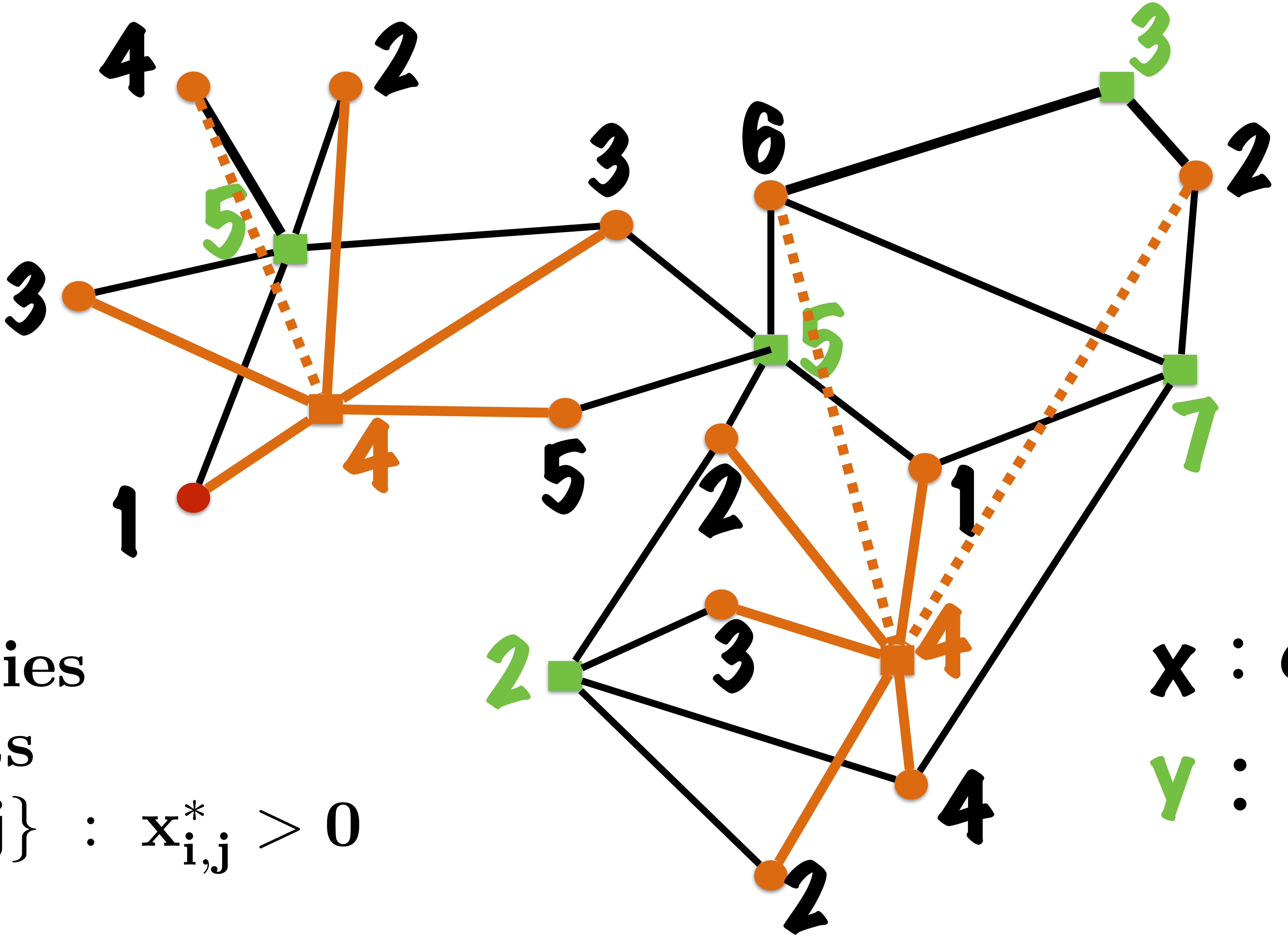
   i_C : cheapest facility s.t. $x_{i_C, j_C}^* > 0$

   open facility i_C

   assign to i_C all unassigned clients s.t.

   there is a facility with $x_{i, j_C}^* > 0$ and $x_{i,j}^* > 0$

Facilities cost: 4+4=8



: facilities

: clients

edge $\{i, j\}$: $x^*_{i,j} > 0$

x : α^*_j

y : f_i

Facilities cost analysis

$$\sum_{\text{Cluster } C} f_{i_C}$$

Observation: if open i_C then $f_{i_C} = \min\{f_i : x^*_{i,j_C} > 0\}$

$$f_{i_C} \leq \sum_{i : x^*_{i,j_C} > 0} x^*_{i,j_C} f_i$$

Primal constraint: $x^*_{i,j_C} \leq y^*_i$

$$\sum_{\text{Cluster } C} f_{i_C} \leq \sum_{\text{Cluster } C} \sum_{i : x^*_{i,j_C} > 0} y^*_i f_i$$

$$\sum_{\text{Cluster C}} f_{i_C} \leq \sum_{\text{Cluster C}} \sum_{i:x^*_{i,j_C}>0} y^*_i f_i$$

By algorithm, this is a disjoint sum:

$$\leq \sum_i y^*_i f_i$$

Primal objective: $\leq \mathrm{OPT}$

Together:

$$\sum_{\mathbf{Cluster\ C}} \sum_{\mathbf{j \in C}} \mathbf{c}_{\mathbf{i_C, j}} + \sum_{\mathbf{Cluster\ C}} \mathbf{f}_{\mathbf{i_C}}$$

$$\leq 4 \cdot \mathbf{OPT}$$

QED

Facility location