Inequality Between Arithmetic and Geometric Mean

The inequality of arithmetic and geometric means states that for any pair of non-negative numbers x and y, their arithmetic mean is no less than their geometric mean:

$$\frac{x+y}{2} \geq \sqrt{xy}\,.$$

This follows from the observation that the square of any number is non-negative.

$$0 \leq (\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{xy}\,,$$

which, after rearranging its terms, implies the original inequality.

In fact, the inequality between the arithmetic and geometric means holds for arbitrarily many positive numbers.

Theorem

For any $x_1, \ldots, x_n > 0$,
$\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$

Proof

For simplicity, first we scale all x_i so that $x_1 x_2 \cdots x_n = 1$. Namely, assume that the geometric mean of these numbers is $c = \sqrt[n]{x_1 x_2 \cdots x_n}$. Then let us divide each x_i by c. Note that both the arithmetic and the geometric mean decrease by a factor of c, therefore, the ratio between the two means does not change. Now, we indeed have $x_1 x_2 \cdots x_n = 1$ without loss of generality.

It remains to show that $x_1 x_2 \cdots x_n = 1$ implies $x_1 + x_2 + \cdots + x_n \geq n$.

We will prove this inequality by induction on n. The base case of $n = 1$ is trivial: $x_1 \geq 1$. For the induction step from $n \geq 1$ to $(n+1)$ we assume that we have $x_1, \ldots, x_{n+1} > 0$ whose product equals 1. In particular, this means that at least one of the numbers is ≤ 1 (indeed, if all numbers are > 1, then their product is > 1). Similarly, at least one of the numbers is ≥ 1. Let us assume that $x_1 \leq 1$ and $x_2 \geq 1$. Then $(x_1 - 1)(x_2 - 1) \leq 0$, which implies that $x_1 + x_2 \geq x_1 x_2 + 1$. In particular, we have that

$$x_1 + x_2 + \cdots + x_n + x_{n+1} \geq 1 + (x_1 x_2) + x_3 + \ldots + x_{n+1}$$

Now, consider n numbers

$$(x_1 x_2), x_3, \ldots, x_{n+1}\,.$$

Since their product is 1, we can apply the induction hypothesis to them: $x_1 x_2 + x_3 + \ldots + x_{n+1} \geq n$\$. We conclude that

$$x_1 + x_2 + \cdots + x_{n+1} \geq 1 + (x_1 x_2) + x_3 + \cdots + x_n \geq 1 + n\,.$$