TEST FLIGHT: SECOND PROBLEM SET SOLUTION

1. Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$$

ANSWER It's false. We need only look at values of m from 1 to 3 (since $3 \times 4 = 12$, which already gives the right-hand side) and values of n from 1 to 2 (since $5 \times 3 = 15 \geq 12$). If you calculate $3m + 5n$ for the six possible pairs in this range, you find that the answer is never 12. This proves the result.

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

ANSWER False. Let $n, n+1, n+2, n+3, n+4$ be any five consecutive integers. Then

$$n + (n+1) + (n+2) + (n+3) + (n+4) = 5n + 1 + 2 + 3 + 4 = 5n + 8 = 5(n+1) + 3$$

which is not a multiple of 5 since in the Division Theorem it leaves a remainder of 3.

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

ANSWER For any n, $n^2 + n + 1 = n(n+1) + 1$. But $n(n+1)$ is always even (since one of $n, n+1$ is even and the other odd). Hence $n(n+1)$ is always odd, as claimed.

4. Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

ANSWER This is not true. For example, if $n = -1$, which is an integer, then $4n + 1 = -3$ and $4n + 3 = -1$. But -3 and -1 are not natural numbers.

5. Prove that for any integer n, at least one of the integers n, $n+2$, $n+4$ is divisible by 3.

ANSWER n can be expressed in one of the forms $3q, 3q+1, 3q+2$, for some q.

In the first case, n is divisible by 3.

In the second case $n + 2 = 3q + 3 = 3(q+1)$, so $n + 2$ is divisible by 3.

In the third case $n + 4 = 3q + 6 = 3(q+2)$, so $n + 4$ is divisible by 3.

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

   ANSWER Let $n, n+2, n+4$ be any three successive natural numbers, where $n > 3$. I show that 3 divides one of these numbers. If 3 does not divide n, then by the Division Theorem, $n = 3q+1$ or $n = 3q+2$, for some q. In the first case, $n+2 = 3q+3$, so $3|n$, and in the second case $n+4 = 3q+6$, so again $3|n$. Thus 3 must divide one of the three numbers, which means they cannot all be prime.

7. Prove that for any natural number n:

$$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

   ANSWER We prove the result by induction. For $n = 1$, the identity reduces to $2 = 2^2 - 2$, which is true. Assume it hold for n. Then,

$$2 + 2^2 + 2^3 + \ldots + 2^n + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1} = 2.2^{n+1} - 2 = 2^{n+2} - 2$$

   This is the identity at $n + 1$. The result follows by induction.

8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

   ANSWER Pick $\epsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, there is an N such that a_n is within a distance of ϵ/M of L whenever $n > N$. For any such n, Ma_n is within a distance $M(\epsilon/M) = \epsilon$ of ML. Hence $\{Ma_n\}_{n=1}^{\infty}$ tends to ML as n tends to ∞.

9. Given a collection $A_n, n = 1, 2, \ldots$ of intervals of the real line, their *intersection* is defined to be $\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$. Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

   Prove that your example has the stated property.

   ANSWER Take the sequence $(0, 1)$, $(0, 1/2)$. $(0, 1/4)$, $(0, 1/8), \ldots$ That is, $A_n = (0, 1/2^{n-1})$. Since $\{1/2^n\}_{n=1}^{\infty}$ tends to 0 as $n \to \infty$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

10. Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

ANSWER Take $A_n = [0, 1/2^n]$. Then $0 \in A_n$ for all n, so $0 \in \bigcap_{n=1}^{\infty} A_n$. By the same argument as in question 9 above, it follows that $\bigcap_{n=1}^{\infty} A_n = \{0\}$.