Facility location

Linear programming relaxation

Primal :

$$\begin{array}{llll}
\text{minimize} & \displaystyle\sum_{i\in F}\sum_{j\in C} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i & & \textbf{Dual variables:} \\
\text{subject to} & \displaystyle\sum_{i\in F} x_{ij} \geq 1, & j \in C & \alpha_j \\
& y_i - x_{ij} \geq 0, & j \in C,\ i \in F & \beta_{ij} \\
& y_i \geq 0, & i \in F & \\
& x_{ij} \geq 0, & j \in C,\ i \in F &
\end{array}$$

Taking the dual

Primal variables

Dual : maximize $\sum_{j \in C} \alpha_j$

$$\begin{array}{llll}
\text{subject to} & \alpha_j - \beta_{ij} \le c_{ij}, & j \in C,\ i \in F & x_{ij} \\
& \sum_{j \in C} \beta_{ij} \le f_i, & i \in F & y_i \\
& \alpha_j \ge 0, & j \in C & \\
& \beta_{ij} \ge 0, & j \in C,\ i \in F &
\end{array}$$

Recall :

Complementary slackness conditions

If x is optimal for (P) and y optimal for (D)

then for every i:

$$c_i = \sum_j a_{ij} y_j \text{ or } x_i = 0$$

and for every j:

$$b_j = \sum_i a_{ij} xi \text{ or } y_j = 0$$

Interpreting the dual

Complementary slackness conditions :

1. $\forall i \in F, j \in C : x_{ij} > 0 \implies \alpha_j - \beta_{ij} = c_{ij}$

2. $\forall i \in F : y_i > 0 \implies \sum_{j \in C} \beta_{ij} = f_i$

3. $\forall j \in C : \alpha_j > 0 \implies \sum_{i \in F} x_{ij} = 1$

4. $\forall i \in F, j \in C : \beta_{ij} > 0 \implies y_i = x_{ij}$

Interpreting the dual

β_{ij} : Contribution of client j for opening facility i

Complementary slackness condition 2 : $\forall i \in F : y_i > 0 \implies \sum_{j \in C} \beta_{ij} = f_i$

If $y_i = 1$ then $\sum_{j \in C} \beta_{ij} = f_i$ we say that facility i is <u>fully paid</u>

Interpreting the dual

Complementary slackness condition 4: $\forall i \in F, j \in C : \beta_{ij} > 0 \implies y_i = x_{ij}$

thus, $\forall i \in F, j \in C : y_i \neq x_{ij} \implies \beta_{ij} = 0$

Recall $y_i - x_{ij} \geq 0,$ **If** $y_i \neq x_{ij}$ **then** $\beta_{ij} = 0$

and so, client j does not contribute to opening any facility except the one it is connected to

Interpreting the dual

Complementary slackness condition 1 : $\forall i \in F, j \in C : x_{ij} > 0 \implies \alpha_j - \beta_{ij} = c_{ij}$

thus, if client j is assigned to facility i we have :

$$\alpha_j - \beta_{ij} = c_{ij}$$

We define α_j **as the total price paid by client j.**

The total price is made of the use of edge (i,j) c_{ij} **and the contribution to the opening cost** β_{ij}

Facility location