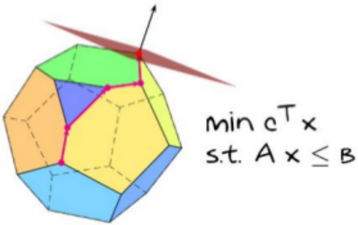
Strong duality for bipartite graphs

- Totally unimodular matrices
- Proving strong duality in the bipartite case



Strong duality: Proof idea

Max. weight matching **Min. w-vertex cover**

$$OPT_{IP} \underset{=}{\leqslant} OPT_{LP} = OPT_{LP} \underset{=}{\leqslant} OPT_{IP}$$

holds if G is bipartite

Totally unimodular matrices

A matrix $A \in \{0, \pm 1\}$ is *totally unimodular*, if the determinant of each square sub-matrix of A is equal to $0, \pm 1$.

Example:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Which of the two matrices are TU?

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$$

Node-edge incidence matrices of bipartite graphs

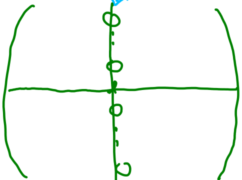
Theorem

Let $G = (V, E)$ be a bipartite graph. The node-edge incidence matrix A^G of G is totally unimodular.

Proof: (by induction on k, B is $k \times k$ sub-matrix of A^G)

$k = 1$ $\quad B = 0, \pm 1 \quad \Rightarrow \det(B) = 0, \pm 1$

$k > 1$: CASE 1: B has column with exactly one "1":

$B =$ 

develop det. along this column!

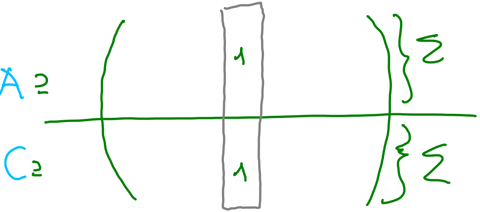
$\Rightarrow \det(B) = (\pm 1) \cdot \det(B')$

$(k-1) \times (k-1)$ sub-matrix

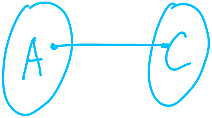
Node-edge incidence matrices of bipartite graphs

CASE 2: Each column of B contains exactly 2 "1"s:

ORDER ROWS of B such that vertices $V = A \dot{\cup} C$ from bi-partition A are on top. (possibly multiplying det by -1)



$$\det(B) = \boxed{0}$$



Totally unimodular matrices and integer programs

$\max\{c^T.x : Ax \leq b, x \geq 0, x \in \mathbb{R}^n\} \overline{\geq} \max\{c^T.x : Ax \leq b, x \geq 0, x \in \mathbb{Z}^n\}$

Theorem

If $A \in \mathbb{Z}^{m \times n}$ is totally unimodular and $b \in \mathbb{Z}^m$, then every vertex of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leqslant b,\ x \geqslant 0\}$ is integral.

Proof:

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$B \subseteq \{1, \dots, m+n\}$ basis, then

$B = \underset{\substack{\cap\vert \\ \{1,\dots,m\}}}{B_1} \cup \underset{\substack{\cap\vert \\ \{m+1,\dots,n\}}}{B_2}$

$B_2 = \{i+m : I\}$

$I \subseteq \{1, \dots, n\}$

$x^*_{\bar{I}}$ unique solution of

Quizz: $x^*_{\bar{I}} = \boxed{0}$

$\{1,\dots,n\} \setminus I = \bar{I}$

$\tilde{A} \cdot x = \tilde{b}$, where $\tilde{A}$ is

$k \times k$ Sub-matrix of A and $\tilde{b}$ is vector having k of the components of b

Quiz: $k = |\bar{I}|$ $\cancel{\bigcirc}$ $k = n - |\bar{I}|$ $\cancel{\bigcirc}$ $k = |\bar{I}|$ $\bigcirc$

Totally unimodular matrices and integer programs

Using the matrix inversion formula $\tilde{A}^{-1} = \frac{1}{\det(\tilde{A})} \cdot \operatorname{adj}(\tilde{A})$

$\in \{\pm 1\}$

$$\operatorname{adj}(\tilde{A}) = \begin{pmatrix} \det(\tilde{A}_{11}) & -\det(\tilde{A}_{21}) & \cdots \\ -\det(\tilde{A}_{12}) & \det(\tilde{A}_{22}) & \cdots \\ \vdots & & \end{pmatrix}$$

$\in \{0, \pm 1\}$

integer matrix

$$x^*_I = \tilde{A}^{-1} \cdot \tilde{b} \in \mathbb{Z}^{|I|}$$

$\in \mathbb{Z}^{|I| \times |I|}$

$\blacksquare$

Totally unimodular matrices and integer programs (cont.)

Corollary

If $A \in \mathbb{Z}^{m\times n}$ is totally unimodular, $b \in \mathbb{Z}^m$, and if $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b,\ x \geqslant 0\}$ is bounded, then

$$\max\{c^T x \colon x \in \mathbb{R}^{\mathbf{n}},\ Ax \leqslant b,\ x \geqslant 0\} = \max\{c^T x \colon x \in \mathbb{Z}^{\mathbf{n}},\ Ax \leqslant b,\ x \geqslant 0\}.$$

Proof:

" $\geq$ " is clear but opt vertex is integral $\Rightarrow$ " $\leq$ "

$\blacksquare$

Strong duality in the bipartite case

Theorem (Egerváry 1931)

Let $G = (V, E)$ be a bipartite graph and let $w : E \to \mathbb{N}_0$ be edge-weights. The maximum weight of a matching is equal to the minimum value of a w-vertex cover.

Proof:

Max. weight match — Min value w-vertex cover

$$\mathrm{OPT}_{IP} \overset{\text{A^G is TU}}{=} \mathrm{OPT}_{LP} \longleftarrow \underset{\text{Strong duality}}{=} \longrightarrow \mathrm{OPT}_{LP} \overset{\text{$(A^G)^T$ is TU}}{=} \mathrm{OPT}_{IP}$$

A^G is TU

Strong duality

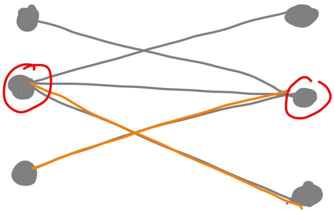
$(A^G)^T$ is TU

$\mathrm{OPT}_{IP} = \mathrm{OPT}_{IP}$

$\square$

König's theorem

A *vertex cover* of a graph $G = (V, E)$ is a subset $U \subseteq V$ such that $e \cap U \neq \emptyset$ for each $e \in E$.



W- vertex cover for

$\omega = 1$

Quiz: Prove that max. cardinality of matching $= 2$

Theorem (König 1931)

Let $G = (V, E)$ be a bipartite graph. The maximum cardinality of a matching of G is equal to the minimum cardinality of a vertex cover of G.