The law of large numbers

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Chebyshev's inequality

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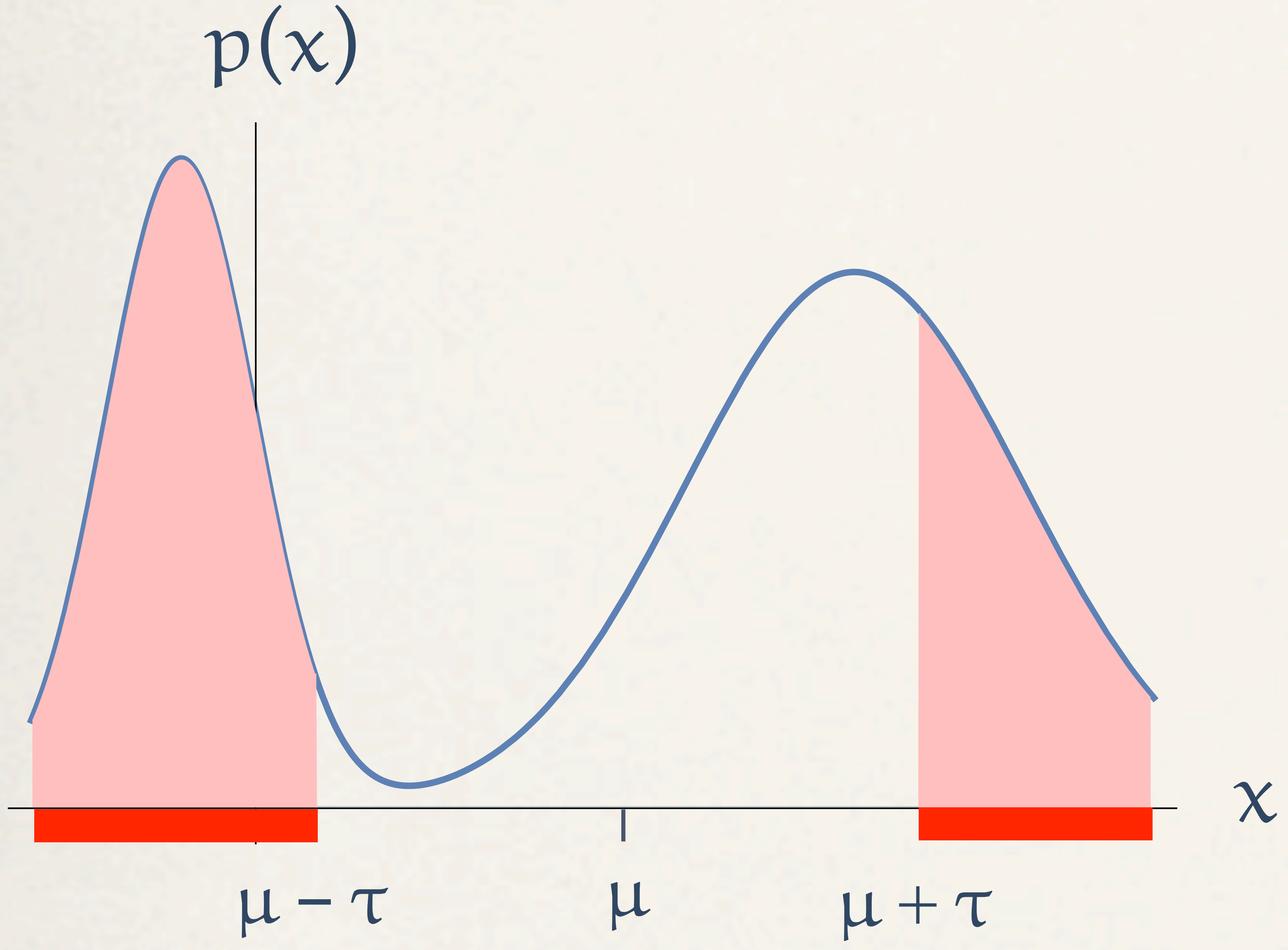

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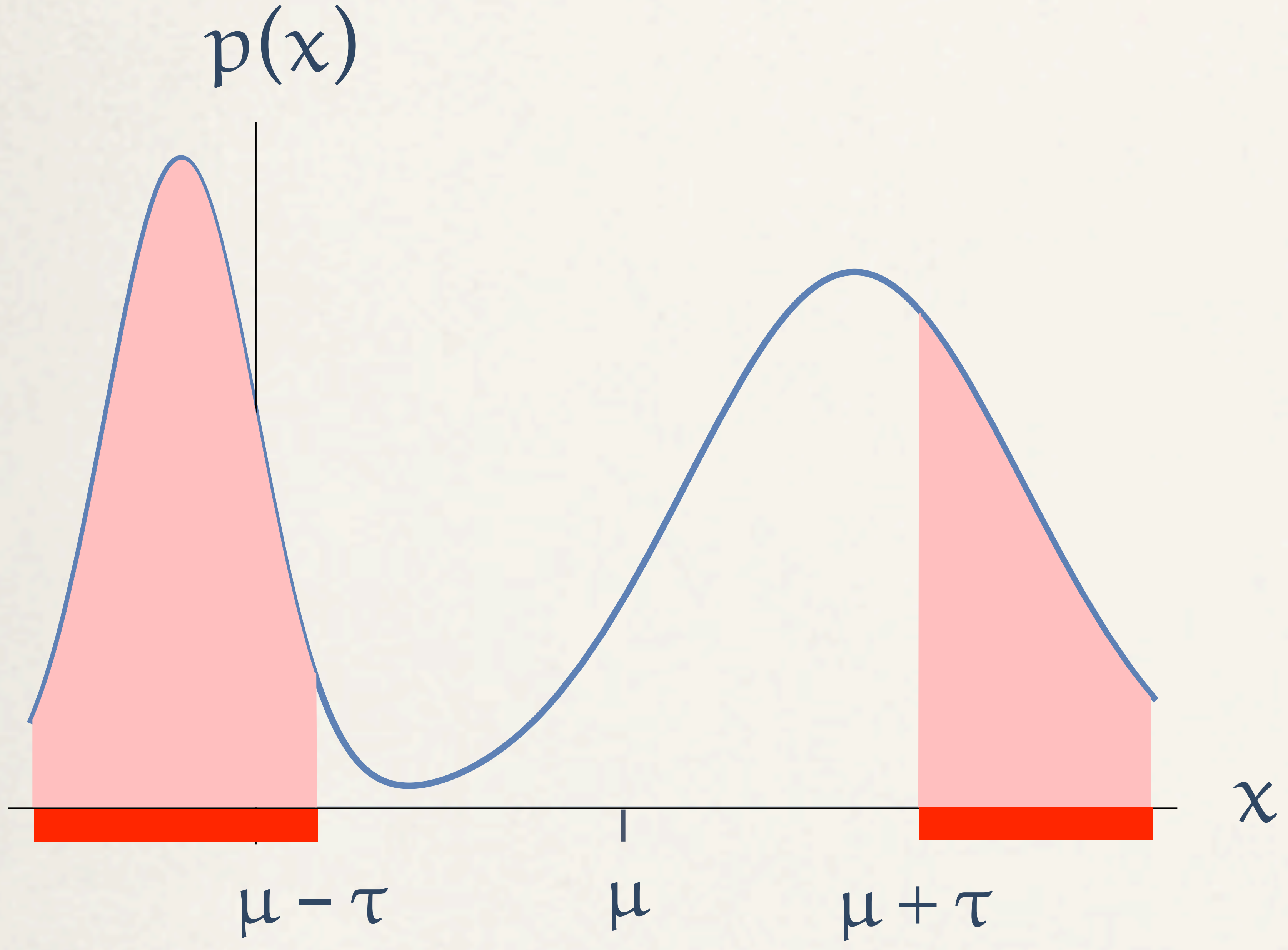
$X \longrightarrow S_n$



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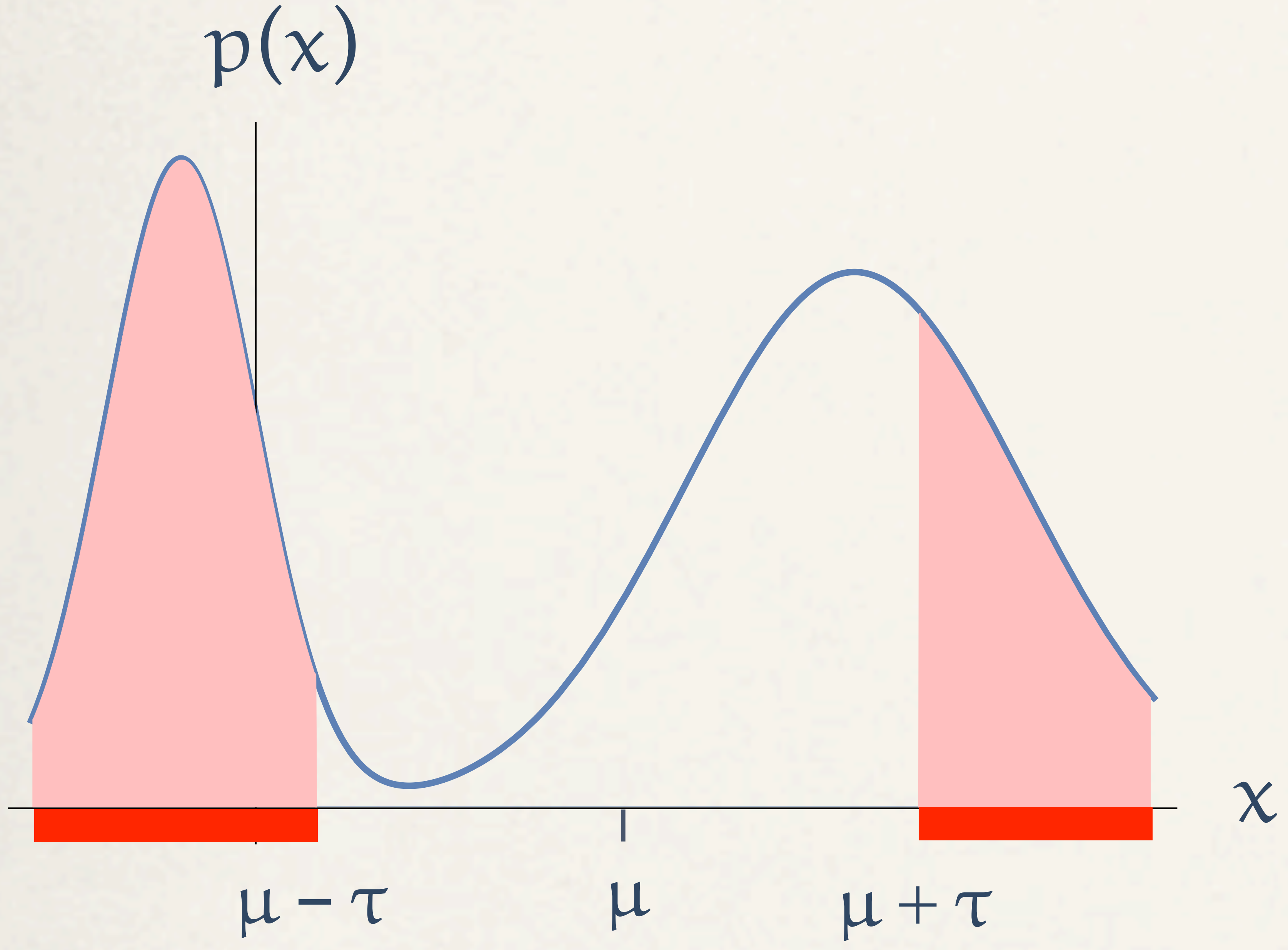
Chebyshev's inequality

$$\mathbf{P}\{|X - \mathbf{E}(X)| > \tau\} \leq \frac{\operatorname{Var}(X)}{\tau^2} \qquad X \longrightarrow S_n \qquad \mathbf{P}\{|S_n - \mathbf{E}(S_n)| > \tau\} \leq \frac{\operatorname{Var}(S_n)}{\tau^2}$$



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Repeated independent trials

$$X, X_1, \dots, X_n, \dots, \quad \mathbf{E}(X) = \mu, \quad \operatorname{Var}(X) = \sigma^2$$

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$$S_n = X_1 + \cdots + X_n$$

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$$\mathbf{E}(S_n) = \mathbf{E}(X_1) + \cdots + \mathbf{E}(X_n) = n\mu$$

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$$\sqrt{\operatorname{Var}(S_n)} = \sqrt{n}\,\sigma$$

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The importance of viewing things in the proper scale:
The expectation increases linearly with n,
but the standard deviation increases only as the square-root of n.

The deviation from the mean

$$\mathbf{P}\{|S_n - \mathbf{E}(S_n)| > \tau\} \le \frac{\operatorname{Var}(S_n)}{\tau^2}$$

$$\mathbf{E}(S_n) = n\mu \qquad \operatorname{Var}(S_n) = n\sigma^2$$

The deviation from the mean

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$$\mathbf{E}(S_n) = n\mu \qquad \mathrm{Var}(S_n) = n\sigma^2$$

$$\mathbf{P}\left\{\left|\frac{S_n}{n} - \mu\right| > \epsilon\right\}$$

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$$\leq \frac{\mathrm{Var}(S_n)}{(n\epsilon)^2}$$

The deviation from the mean

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\mathbf{P}\left\{\left|\frac{S_n}{n} - \mu\right| > \epsilon\right\} &= \mathbf{P}\{|S_n - \overbrace{n\mu}^{\mathbf{E}(S_n)}| > \overbrace{n\epsilon}^{\tau}\} \\
&\le \frac{\mathrm{Var}(S_n)}{(n\epsilon)^2} \\
&= \frac{n\sigma^2}{n^2\epsilon^2}
\end{aligned}$$

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&\le \frac{\mathrm{Var}(S_n)}{(n\epsilon)^2} \\
&= \frac{n\sigma^2}{n^2\epsilon^2} \\
&= \frac{\sigma^2}{n\epsilon^2}
\end{aligned}$$

$$\mathbf{P}\left\{\left|\frac{S_n}{n} - \mu\right| > \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

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Suppose $X, X_1, \dots, X_n, \dots$ represent repeated independent trials with $\mathbf{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, and suppose $\epsilon > 0$ is any fixed small, positive number. Then the partial sums $S_n = X_1 + \cdots + X_n$ are increasingly concentrated at μ and satisfy the asymptotic relation

$$\mathbf{P}\left\{\left|\frac{S_n}{n} - \mu\right| > \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2} \to 0 \qquad (n \to \infty).$$

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Slogan

The probability that S_n/n deviates from its expected value μ *by even a small amount* is small provided the sample size n is sufficiently large.