A curious discovery of Poisson

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Testing the quality of the approximation

k	0	1	2	3	4
$b_{20}(k;\ 0.15)$	0.039	0.137	0.229	0.243	0.182
$b_{60}(k;\ 0.05)$	0.046	0.145	0.226	0.230	0.172
$b_{100}(k;\ 0.03)$	0.048	0.147	0.225	0.227	0.171
Po(k; 3)	0.050	0.149	0.224	0.224	0.168

$$b_n(k;p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$Po(k;\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\lambda = np = 3$$