Summing up Digits

Problem. Is it possible to place signs in the expression

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9$$

to get 100 as a result? What about 2?

Clearly, one gets the largest number by using all $+$ signs. The value in this case is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45\,.$$

Hence, one definitely cannot get 100.

What about getting 2 as a result? Note that our expression contains four even numbers and five odd numbers. This means that no matter how one puts the signs, the result will be odd. Hence, it is impossible to get 2.

Now, it is natural to ask the following question.

Problem. What integers can be obtained by placing signs in the expression

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9\,?$$

We've discussed already that if an integer can be obtained, then it must be between -45 and 45, and it must be odd. Note however that we haven't proved that every such number can be obtained: perhaps, there are other obstacles. It turns out that there are no other obstacles: every odd integer from -45 to 45 can be obtained. We will not discuss why this is true, but you can try to show it yourself.

Problem. Prove that every odd integer from -45 to 45 can be obtained by placing signs in the expression

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9\,.$$

As a hint, consider the following code that tries to represent the given value by placing signs in the expression $1 \pm 2 \pm \cdots \pm n$.

```
def represent(n, value):
    if n == 0 and value == 0:
        return []

    total = sum(range(1, n + 1))

    if abs(value) > total or (total - value) % 2 != 0:
        return False
    if value >= 0:
        return represent(n - 1, value - n) + [n]
    else:
        return represent(n - 1, value + n) + [-n]


for v in (7, 15, 22, 33, 40, 47):
    print(v, end='')
    result = represent(9, v)
    if not result:
        print(' is not representable')
    else:
        print('=', end='')
        for i in result:
            if i < 0:
                print(f'{i}', end='')
            else:
                print(f'+{i}', end='')
        print()
```

Run

Reset

This code proceeds recursively. This suggests that you may want to solve the above problem as follows: first, generalize the statement (so that it states something about every positive integer n rather than just $n = 9$), then prove the generalized statement by induction on n (recall a lesson in Week 3).