Item Navigation

More Induction Examples

Problem

Prove that for every $n \geq 1$,

$$\sum_{i=1}^{n-1} i \cdot (n-i) = \frac{(n-1)n(n+1)}{6} .$$

The induction base case is $n = 1$, where both sides of the formula equal 0. For the induction step, assume that this formula holds for $n \geq 1$, and let us prove this statement for $n+1$, simply by plugging in $(n+1)$ everywhere that we previously saw n.

$$1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1 = \frac{n(n+1)(n+2)}{6} .$$

Indeed,

$1 \cdot n + 2 \cdot (n-1) + \cdots + n \cdot 1 = 1 \cdot (n-1) + 2 \cdot (n-2) + \cdots + (n-1) \cdot 1 + 1 + 2 + \ldots + (n-1) + n = \frac{(n-1)n(n+1)}{6} + 1 + 2 + \ldots + (n-1) + n = \frac{(n-1)n(n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$

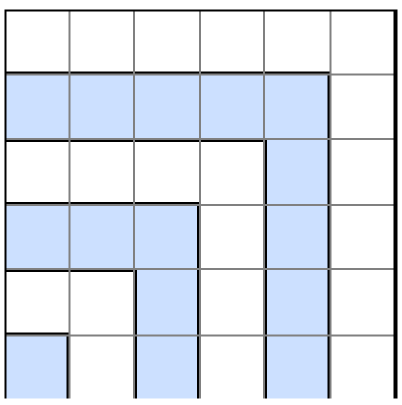
Here we used the induction hypothesis and the formula for arithmetic series from a previous section.

Practice proofs by induction on the following problems.

Problem

Prove that the following equalities hold for every $n > 0$.

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$ (see the picture for geometric proof)
2. $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2$
3. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq \frac{n}{2} + 1$



As it often happens with induction proofs, the induction proof convinces us that the formula is correct, but gives no clues as to where the formula is actually coming from. For this particular case, there is again a neat geometric visualization of the formula.

Completed Go to next item