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On Cayley Digraphs That Do Not Have Hamiltonian Paths

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Abstract

We construct an infinite family $\{\overrightarrow{\mathrm{Cay}}(G_i; a_i; b_i)\}$ of connected, 2-generated Cayley digraphs that do not have hamiltonian paths, such that the orders of the generators a_i and b_i are unbounded. We also prove that if G is any finite group with $|[G, G]| \leq 3$, then every connected Cayley digraph on G has a hamiltonian path (but the conclusion does not always hold when $|[G, G]| = 4$ or 5).

the commutator subgroup of G has order p. More precisely, $G = \mathbb{Z}_m \ltimes \mathbb{Z}_p$ is a semidirect product of two cyclic groups, so G is metacyclic.

Remark 4. Here are some related open questions and other comments.

(1) The above results show that connected Cayley digraphs on solvable groups do not always have hamiltonian paths. On the other hand, it is an open question whether connected Cayley digraphs on *nilpotent* groups always have hamiltonian paths. (See [3] for recent results on the nilpotent case.)
(2) The above results always produce a digraph with an even number of vertices. Do there exist infinitely many connected Cayley digraphs of odd order that do not have hamiltonian paths?
(3) We conjecture that the assumption "$p \equiv 3 \pmod 4$" can be eliminated from the statement of Theorem 3. On the other hand, it is necessary to require that $p > 3$ (see Corollary 16).
(4) If G is abelian, then it is easy to show that every connected Cayley digraph on G has a hamiltonian path. However, some abelian Cayley digraphs do not have a hamiltonian cycle. See Section 5 for more discussion of this.
(5) The proof of Theorem 3 appears in Section 3, after some preliminaries in Section 2.

2. Preliminaries

We recall some standard notation, terminology, and basic facts.

Notation. Let G be a group, and let H be a subgroup of G. (All groups in this paper are assumed to be finite.)

(i) e is the identity element of G;
(ii) $x^g = g^{-1}xg$, for $x, g \in G$;
(iii) we write $H \trianglelefteq G$ to say that H is a normal subgroup of G;
(iv) $H^G = \langle h^g \mid h \in H, g \in G \rangle$ is the normal closure of H in G, so $H^G \trianglelefteq G$.

Definition 5. Let S be a subset of the group G.

(i) $H = \langle SS^{-1} \rangle$ is the arc-forcing subgroup, where $SS^{-1} = \{st^{-1} \mid s, t \in S\}$.
(ii) For any $a \in S$, $a^{-1}H$ is called the terminal coset. (This is independent of the choice of a.)

(i) A vertex $g \in G$ travels by s if L contains the directed edge $g \to gs$.
(ii) A subset X of G travels by s if every element of X travels by s.

Lemma 8 (see Housman [4, p. 82]). *Suppose L is a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G;a,b)$, with initial vertex e, and let $H = \langle ab^{-1}\rangle$ be the arc-forcing subgroup. Then,*

(1) *the terminal vertex of L belongs to the terminal coset $a^{-1}H$,*
(2) *each regular coset either travels by a or travels by b.*

3. Proof of Theorem 3

Let

(i) α be an even number that is relatively prime to $(p-1)/2$, with $\alpha > n$;
(ii) β a multiple of $(p-1)/2$ that is relatively prime to α, with $\beta > n$;
(iii) $\bar{a}$ a generator of $\mathbb{Z}_\alpha$;
(iv) $\bar{b}$ a generator of $\mathbb{Z}_\beta$;
(v) z a generator of $\mathbb{Z}_p$;
(vi) r a primitive root modulo p;
(vii) $G = (\mathbb{Z}_\alpha \times \mathbb{Z}_\beta) \ltimes \mathbb{Z}_p$, where $z^{\bar{a}} = z^{-1}$ and $z^{\bar{b}} = z^{r^2}$;
(viii) $a = \bar{a}z$, so $|a| = \alpha$, and a inverts $\mathbb{Z}_p$;
(ix) $b = \bar{b}z$, so $|b| = \beta$, and b acts on $\mathbb{Z}_p$ via an automorphism of order $(p-1)/2$;
(x) $H = \langle ab^{-1}\rangle = \langle \bar{a}\,\bar{b}^{-1}\rangle = \mathbb{Z}_\alpha \times \mathbb{Z}_\beta$.

Suppose L is a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G;a,b)$. This will lead to a contradiction.

It is well known (and easy to see) that Cayley digraphs are vertex-transitive, so there is no harm in assuming that the initial vertex of L is e. Note that

(i) the terminal coset is $a^{-1}H = z^{-1}H$;

Therefore $\#\mathscr{B}_e \le (p-1)/2 \le p-2$, so we can choose two cosets z^iH and z^jH that do not belong to $\mathscr{B}_e$.

Recall that, by definition, $z'H$ is not the terminal coset $z^{-1}H$, so $z'z$ is a nontrivial element of $\mathbb{Z}_p$. Then, since $\mathbb{Z}_p^\times = \langle -1, r^2\rangle$, we can choose some $h \in \langle \overline{a}, \overline{b}\rangle = H$, such that $(z^{j-i})^h = z'z$. Now, since

$$z^iH, z^jH \notin \mathscr{B}_e,$$
$$z^{-1}h^{-1}z^{j-i} \in z^{-1}\left(z^{j-i}\right)^h H = z^{-1}\left(z'z\right)H = z'H, \quad (7)$$

we may multiply on the left by $g = z^{-1}h^{-1}z^{-i}$ to see that

$$z^{-1}H, z'H \notin \mathscr{B}_g. \quad (8)$$

Therefore, no element of $\mathscr{B}_g$ is either the terminal coset or the regular coset that travels by a. This means that every coset in $\mathscr{B}_g$ travels by b, so L contains the cycle $[g](b^\beta)$, which contradicts the fact that L is a (hamiltonian) path.

Case 2. Assume at least two regular cosets travel by a in L. Let z^iH and z^jH be two regular cosets that both travel by a. Since $\mathbb{Z}_p^\times = \langle -1, r^2\rangle$, we can choose some $h \in \langle \overline{a}, \overline{b}\rangle = H$, such that $(z^{-1})^h = z^{j-i}$.

Note that $z^ih^{-1}a^k$ travels by a, for every $k \in \mathbb{Z}$.

(i) If $k = 2\ell$ is even, then

so

$$a^k = (\overline{a}z)^{2\ell} = (\overline{a}z\overline{a}z)^\ell$$
$$= \left(\overline{a}^2 z^{\overline{a}} z\right)^\ell = \left(\overline{a}^2 z^{-1} z\right)^\ell = \overline{a}^{2\ell} \in H, \quad (9)$$

$z^ih^{-1}a^k \in z^iH$ travels by a.

(ii) If $k = 2\ell + 1$ is odd, then

so

$$a^k = (\overline{a}z)^{2\ell+1} = (\overline{a}z)^{2\ell}(\overline{a}z) = \overline{a}^{2\ell}(\overline{a}z) = \overline{a}^k z, \quad (10)$$

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ve
ls

$$z^ih^{-1}a^k = z^ih^{-1}\left(\overline{a}^k z\right) = z^ih^{-1}z^{-1}\overline{a}^k$$
$$= z^i(z^{-1})^h h^{-1}\overline{a}^k \in z^i\left(z^{j-i}\right)H = z^jH \quad (11)$$

Proof. Since $[G,G] \subseteq N$, we know that G/N is an abelian group, so there is a hamiltonian path $(s_i)_{i=1}^{m}$ in $\overrightarrow{\mathrm{Cay}}(G/N;S)$ (see Proposition 19 below). Also, by assumption, there is a hamiltonian path $(t_j)_{j=1}^{n}$ in $\overrightarrow{\mathrm{Cay}}(N;a,b)$. Then

$$\left(\left(\left(t_j\right)_{j=1}^{n}, s_i\right)_{i=1}^{m}, \left(t_j\right)_{j=1}^{n}\right) \tag{12}$$

is a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G;S)$.

Definition 10. If K is a subgroup of G, then $K \setminus \overrightarrow{\mathrm{Cay}}(G;S)$ denotes the digraph whose vertices are the right cosets of K in G and with a directed edge $Kg \rightarrow Kgs$ for each $g \in G$ and $s \in S$. Note that $K \setminus \overrightarrow{\mathrm{Cay}}(G;S) = \overrightarrow{\mathrm{Cay}}(G/K;S)$ if $K \trianglelefteq G$.

Lemma 11 ("Skewed-Generators Argument," cf. [3, Lem. 2.6], [5, Lem. 5.1]). *Assume*

(i) *S is a generating set for the group G;*
(ii) *K is a subgroup of G, such that every connected Cayley digraph on K has a hamiltonian path;*
(iii) *$(s_i)_{i=1}^{n}$ is a hamiltonian cycle in $K \setminus \overrightarrow{Cay}(G;S)$;*
(iv) *$\langle S s_2 s_3 \cdots s_n \rangle = K$.*

Then $\overrightarrow{Cay}(G;S)$ has a hamiltonian path.

Proof. Since $\langle S s_2 s_3 \cdots s_n \rangle = K$, we know that $\overrightarrow{\mathrm{Cay}}(K; S s_2 s_3 \cdots s_n)$ is connected, so, by assumption, it has a hamiltonian path $(t_j s_2 s_3 \cdots s_n)_{j=1}^{m}$. Then

$$\left(\left(t_j, (s_i)_{i=2}^{n}\right)_{j=1}^{m-1}, t_m, (s_i)_{i=2}^{n-1}\right) \tag{13}$$

is a hamiltonian path in $\overrightarrow{\mathrm{Cay}}(G;S)$.

Lemma 12. *Assume*

(i) *S is a generating set of G, with arc-forcing subgroup $H = \langle S S^{-1} \rangle$;*
(ii) *there is a hamiltonian path in every connected Cayley digraph on H^G;*
(iii) *either $H = H^G$, or H is contained in a unique maximal subgroup of H^G.*

Then $\overrightarrow{Cay}(G;S)$ has a hamiltonian path.

The final hypothesis of the preceding lemma is automatically satisfied when $[G,G]$ is cyclic of prime-power order.

Lemma 13. *If $[G,G]$ is cyclic of order p^k, where p is prime, and H is any subgroup of G, then either $H = H^G$ or H is contained in a unique maximal subgroup of H^G.*

Proof. Note that the normal closure H^G is the (unique) smallest normal subgroup of G that contains H. Therefore, $H^G \subseteq H\ [G,G]$ (since $H\ [G,G]$ is normal in G). This implies that if M is any proper subgroup of H^G that contains H, then

$$M = H \cdot (M \cap [G,G]) \subseteq H \cdot \left(H^G \cap [G,G]\right)^p. \tag{16}$$

Therefore, $H \cdot (H^G \cap [G,G])^p$ is the unique maximal subgroup of H^G that contains M.

The following known result handles the case where G is nilpotent.

Theorem 14 (see Morris [3]). *Assume G is nilpotent, and S generates G. If either*

(1) *$\#S \leq 2$ or*
(2) *$|[G,G]| = p^k$, where p is prime and $k \in \mathbb{N}$,*

then $\overrightarrow{Cay}(G;S)$ has a hamiltonian path.

We now state the main result of this section.

Theorem 15. *Suppose*

(i) *$[G,G]$ is cyclic of prime-power order,*
(ii) *every element of G either centralizes $[G,G]$ or inverts it.*

Then every connected Cayley digraph on G has a hamiltonian path.

Proof. Let S be a generating set for G. Write $[G,G] = \mathbb{Z}_{p^k}$ for some p and k. Since every minimal generating set of $\mathbb{Z}_{p^k}$ has only one element, there exist $a, b \in S$, such that $\langle [a,b] \rangle = [G,G]$. Then, by Lemma 9, we may assume $S = \{a, b\}$.

Let $H = \langle ba^{-1} \rangle$ be the arc-forcing subgroup. We may assume $H^G = G$, for otherwise we could assume, by induction on

Example 18. Let $G = \mathbb{Z}_{12} \ltimes \mathbb{Z}_5 = \langle h \rangle \ltimes \langle z \rangle$, where $z^h = z^3$. Then $|[G,G]| = 5$, and the Cayley digraph $\overrightarrow{\mathrm{Cay}}(G; h^2 z, h^3 z)$ is connected but does not have a hamiltonian path.

Proof. A computer search can confirm the nonexistence of a hamiltonian path very quickly, but, for completeness, we provide a human-readable proof.

Let $a = h^2 z = z^4 h^2$ and $b = h^3 z = z^3 h^3$. The argument in Case 2 of the proof of Theorem 3 shows that no more than one regular coset travels by a in any hamiltonian path. On the other hand, since a hamiltonian path cannot contain any cycle of the form $[g](b^4)$, we know that at least $\lfloor (|G| - 1)/4 \rfloor = 14$ vertices must travel by a. Since $|ab^{-1}| = 12 < 14$, this implies that some regular coset travels by a. So exactly one regular coset travels by a in any hamiltonian path.

For $0 \le i \le 3$ and $0 \le m \le 11$, let $L_{i,m}$ be the spanning subdigraph of $\overrightarrow{\mathrm{Cay}}(G; a, b)$ in which

(i) all vertices have outvalence 1, except $b^{-1}(ab^{-1})^m = z^4 h^{9-m}$, which has outvalence 0;
(ii) the vertices in the regular coset $z^i H$ travel by a;
(iii) a vertex $b^{-1} h^{-j} = z^4 h^{9-j}$ in the terminal coset travels by a if $0 \le j < m$;
(iv) all other vertices travel by b.

An observation of D. Housman [7, Lem. 6.4(b)] tells us that if L is a hamiltonian path from e to $b^{-1}(ab^{-1})^m$, in which $z^i H$ is the regular coset that travels by a, then $L = L_{i,m}$. Thus, from the conclusion of the preceding paragraph, we see that every hamiltonian path (with initial vertex e) must be equal to $L_{i,m}$, for some i and m.

However, $L_{i,m}$ is not a (hamiltonian) path. More precisely, for each possible value of i and m, the following list displays a cycle that is contained in $L_{i,m}$:

(i) if $i = 0$ and $0 \le m \le 8$,

$$z^2 h^3 \xrightarrow{b} zh^6 \xrightarrow{b} z^3 h^9 \xrightarrow{b} z^4 \xrightarrow{b} z^2 h^3; \tag{19}$$

(ii) if $i = 0$ and $9 \le m \le 11$,

$$h^2 \xrightarrow{a} zh^4 \xrightarrow{b} z^4 h^7 \xrightarrow{a} zh^9 \xrightarrow{b} z^2 \tag{20}$$

$$h \longrightarrow z h \longrightarrow zh \longrightarrow z h \longrightarrow h; \tag{25}$$

(viii) if $i = 3$ and $m = 11$,

$$\begin{aligned} z^3h^2 &\xrightarrow{a} z^4h^4 \xrightarrow{a} z^3h^6 \xrightarrow{a} z^4h^8 \\ &\xrightarrow{a} z^3h^{10} \xrightarrow{a} z^4 \xrightarrow{a} z^3h^2. \end{aligned} \tag{26}$$

Since $L_{i,m}$ is never a hamiltonian path, we conclude that $\overrightarrow{\text{Cay}}(G; a, b)$ does not have a hamiltonian path.

5. Nonhamiltonian Cayley Digraphs on Abelian Groups

When G is abelian, it is easy to find a hamiltonian path in $\overrightarrow{\text{Cay}}(G; S)$.

Proposition 19 (see [6, Thm. 3.1]). *Every connected Cayley digraph on any abelian group has a hamiltonian path.*

On the other hand, it follows from Lemma 8(2) that sometimes there is no hamiltonian cycle.

Proposition 20 (see Rankin [8, Thm. 4]). *Assume $G = \langle a, b\rangle$ is abelian. Then there is a hamiltonian cycle in $\overrightarrow{Cay}(G; a, b)$ if and only if there exist $k, \ell \geq 0$, such that $\langle a^k b^\ell\rangle = \langle ab^{-1}\rangle$, and $k + \ell = |G : \langle ab^{-1}\rangle|$.*

Example 21. If $\gcd(a, n) > 1$ and $\gcd(a + 1, n) > 1$, then $\overrightarrow{\text{Cay}}(\mathbb{Z}_n; a, a + 1)$ does not have a hamiltonian cycle.

All of the non-hamiltonian Cayley digraphs provided by Proposition 20 are 2-generated. However, a few 3-generated examples are also known. Specifically, the following result lists (up to isomorphism) the only known examples of connected, non-hamiltonian Cayley digraphs $\overrightarrow{\text{Cay}}(G; S)$, such that $\#S > 2$ (and $e \notin S$).

Theorem 22 (see Locke and Witte [9]). *The following Cayley digraphs do not have hamiltonian cycles:*

(1) $\overrightarrow{Cay}(\mathbb{Z}_{12k}; 6k, 6k + 2, 6k + 3)$, *for any* $k \in \mathbb{Z}^+$;
(2) $\overrightarrow{Cay}(\mathbb{Z}_{2k}; a, b, b + k)$, *for* $a, b, k \in \mathbb{Z}^+$, *such that certain technical conditions (Remark 23) are satisfied.*

Remark 23. The precise conditions in (2) are (i) either a or k is odd, (ii) either a is even or b and k are both even, (iii) $\gcd(a - b, k) = 1$, (iv) $\gcd(a, 2k) \neq 1$, and (v) $\gcd(b, k) \neq 1$.

$a' - b' \in \langle a-b \rangle$, it is easy to see that $k' = k''$, but we do not need this fact.)

Claim. H_0 has an odd number of connected components. Arguing as in the proof of [9, Lem. 4.1] (except that, as before, Case 1 is when $k \notin \langle a \rangle$ and Case 2 is when $k \in \langle a \rangle$), we see that the number of connected components in H_0 is

$$\begin{gathered} |G : \langle a,k \rangle| + |G : \langle b,k \rangle| \quad \text{if } k \notin \langle a \rangle, \\ |G : \langle b,k \rangle| \quad \text{if } k \in \langle a \rangle. \end{gathered} \tag{27}$$

Since $\langle a' - b' \rangle = \langle a-b \rangle$, we know that one of a' and b' is an even multiple of $a-b$, and the other is an odd multiple. (Otherwise, the difference would be an even multiple of $a-b$, so it would not generate $\langle a-b \rangle$.) Thus, one of $|G : \langle a,k \rangle|$ and $|G : \langle b,k \rangle|$ is even, and the other is odd. So $|G : \langle a,k \rangle| + |G : \langle b,k \rangle|$ is odd. This establishes the claim if $k \notin \langle a \rangle$.

We may now assume $k \in \langle a \rangle$. This implies that the element a' has odd order (and k' must be nontrivial, but we do not need this fact). This means that a' is an even multiple of $a-b$, so b' must be an odd multiple of $a-b$ (since $\langle a' - b' \rangle = \langle a-b \rangle$). Therefore, $|\langle a-b \rangle : \langle b' \rangle|$ is odd, which means $|G : \langle b,k \rangle|$ is odd. This completes the proof of the claim.

Now, if $|G : \langle b,k \rangle|$ is odd, we can apply a very slight modification of the argument in case 4 of the proof of [9, Thm. 4.1($\Leftarrow$)]. (Subcase 4.1 is when $k \notin \langle a \rangle$ and subcase 4.2 is when $k \in \langle a \rangle$.) We conclude that $\overrightarrow{\mathrm{Cay}}(G; a, b, b+k)$ has a hamiltonian cycle, as desired.

Finally, if $|G : \langle b,k \rangle|$ is even, then more substantial modifications to the argument in [9] are required. For convenience, let $m = |G : \langle a,k \rangle|$. Note that, since $|G : \langle b,k \rangle|$ is even, the proof of the claim shows that m is odd and $k \notin \langle a \rangle$.

Define H_0' as in subcase 4.1 of [9, Thm. 4.1($\Leftarrow$)] (with G in the place of $\mathbb{Z}_{2k}$ and replacing $\gcd(b,k)$ with $|G : \langle b,k \rangle|$). Let $H_1 = H_0'$, and inductively construct, for $1 \le i \le (m+1)/2$, an element H_i of $\mathscr{E}$, such that

$$\begin{gathered} \{ v \mid z_v = 0,\ 0 \le y_v \le 2i-2 \} \\ \cup \ \{ v \mid z_v = 1,\ x_v = 0 \text{ or } 1 \ (\mathrm{mod}\ |G : \langle b,k \rangle|) \} \end{gathered} \tag{28}$$

is a component of H_i, and all other components are components of H_0. The construction of H_i from H_{i-1} is the same as in subcase 4.1, but with $2i$ replaced by $2i-1$.

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