DSP-2 (DFS & DFT) (S)

II. Discrete Fourier series

2007 Syllabus*: Properties of discrete Fourier series, DFS representation of periodic sequences, Discrete Fourier transforms, Properties of DFT, Linear convolution of sequences using DFT, Computation of DFT, Relation between z-transform and DFS.*

Contents:

(1) The *Fourier series* (FS) of a continuous-time periodic signal, *x(t)*, with fundamental period T_0, is given by the *synthesis equation*

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

The Fourier coefficients, X_k, are given by the *analysis equation*

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j2\pi k F_0 t} dt$$

The fundamental frequency, F_0 (Hz), and the period, T_0 (seconds), are related by $F_0 = 1/T_0$.

(2) The *Fourier transform* (FT) of a continuous-time aperiodic signal, *x(t)*, is given by the *analysis equation*

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt \qquad \text{or} \qquad X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Here Ω and F are analog frequencies, with $\Omega = 2\pi F$. The *inverse Fourier transform* is given by the *synthesis equation*

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF \qquad \text{or} \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

(3) The *Fourier series* (DTFS/DFS) for a discrete-time periodic signal (periodic sequence), *x(n)*, with fundamental period N is given by the *synthesis equation*

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi k n/N}, \quad 0 \le n \le N-1$$

The Fourier coefficient X_k are given by the *analysis equation*

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi k n/N}, \quad 0 \le k \le N-1$$

This is called the **discrete-time Fourier series (DTFS)** or just **discrete Fourier series (DFS)** for short. The sequence of coefficients, X_k, also is periodic with period N.

These two equations are derived below.

(Note that if the factor $(1/N)$ is associated with *x(n)* rather than with X_k the two DFS equations are identical to the two DFT equations which are derived below in their *standard form*.)

(4) The *Fourier transform* (DTFT) of a finite energy discrete-time aperiodic signal (aperiodic sequence), *x(n)*, is given by the *analysis equation* (some write $X(e^{j\omega})$ instead of $X(\omega)$)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Fourier transform of a discrete-time signal is this: For continuous time signals the Fourier transform, and hence the spectrum of the signal, have a frequency range $(-\infty, \infty)$; in contrast, for a discrete-time signal the frequency range of the DTFT is unique over the interval of $(-\pi, \pi)$ or, equivalently, $(0, 2\pi)$.

Since $X(\omega)$ is a periodic function of the frequency variable ω, it has a Fourier series expansion; in fact, the Fourier coefficients are the $x(n)$ values.

2.2 Discrete Fourier series

Let $x(n)$ be a real periodic discrete-time sequence of period N. If $x(n)$ can be expressed as a weighted sum of complex exponentials, the response of a linear system to $x(n)$ is easily determined by superposition. By analogy with the Fourier series representation of a periodic *continuous-time* signal, we can expect that we can obtain a similar representation for the periodic *discrete-time* sequence $x(n)$. That is, we seek a representation for $x(n)$ of the form

$$x(n) = \sum_k X_k \, e^{jk\omega_0 n} \qquad \text{for all } n$$

Here X_k are the Fourier coefficients and $\omega_0 = 2\pi/N$ is the fundamental (digital) frequency (as $\Omega_0 = 2\pi/T_0 = 2\pi F_0$ is in the case of continuous-time Fourier series). With $k\omega_0 = \omega_k = k\left(\frac{2\pi}{N}\right)$, the above is also written

$$x(n) = \sum_k X_k \, e^{j\omega_k n} \quad \text{or} \quad \sum_k X_k \, e^{jk2\pi n/N}$$

The function $e^{jk2\pi n/N}$ is periodic in k with a periodicity of N and there are only N distinct functions in the set $\left\{e^{jk2\pi n/N}\right\}$ corresponding to $k = 0, 1, 2, ..., N-1$. Thus the representation for $x(n)$ contains only N terms (as opposed to infinitely many terms in the continuous-time case)

$$x(n) = \sum_{k=<N>} X_k \, e^{jk2\pi n/N}$$

The summation can be done over any N consecutive values of k, indicated by the summation index $k = <N>$. For the most part, however, we shall consider the range $0 \le k \le N-1$, and the representation for $x(n)$ is then written as

$$x(n) = \sum_{k=0}^{N-1} X_k \, e^{jk2\pi n/N} \quad \text{for all } n$$

This equation is the **discrete-time Fourier series (DTFS)** or just **discrete Fourier series (DFS)** of the periodic sequence $x(n)$ with coefficients X_k.

The coefficients X_k or $X(k)$ are given by (we skip the algebra – S&S)

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi k n/N}, \qquad 0 \le k \le N-1 \qquad \rightarrow \text{(B)}$$

equally spaced in angle, of the z-transform of *one period* of $x(n)$. Let $x_1(n)$ represent one period of $x(n)$. That is,

$$x_1(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Then $X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = \sum_{n=0}^{N-1} x_1(n) z^{-n}$, and $X(k) = X_1(z)\big|_{z=e^{j2\pi k/N}}$. This then corresponds to sampling the z-transform $X_1(z)$ at N points equally spaced in angle around the unit circle, with the first such sample occurring at $z = 1$. (Note that the periodic sequence $x(n)$ cannot be represented by its z-transform since there is no value of z for which the z-transform will converge. However, $x_1(n)$ does have a z-transform.)

2.3 Properties of discrete Fourier series

Properties of discrete Fourier series (DFS) for periodic sequences The following notation is used:

p = periodic; e = even; o = odd
$W_N = e^{-j2\pi/N}$
Re [.] = Real part of
Im [.] = Imaginary part of
|.| = Magnitude of
Arg (.) = Argument of

The following properties should be noted.

	Sequence	DFS
1	$x_p(n+m)$	$W_N^{-km} X_p(k)$
2	$x_p^*(n)$	$X_p^*(-k)$
3	$x_p^*(-n)$	$X_p^*(k)$
4	Re $[x_p(n)]$	$X_{pe}(k)$
5	j Im $[x_p(n)]$	$X_{po}(k)$

Example 2.3.1 Show that DFS $\{x_p(n+m)\} = W_N^{-km} X_p(k)$.

Solution We have

$$\text{DFS } \{x_p(n+m)\} = \sum_{n=0}^{N-1} x_p(n+m) W_N^{kn}$$

Set $n+m = \lambda$ so that $n = \lambda - m$ and the limits $n = 0$ to $N-1$ become $\lambda = m$ to $N-1+m$. Then the RHS becomes

Example 2.3.2 Show that DFS $\{x_p^*(n)\} = X_p^*(-k)$.

Solution We have

$$\text{DFS}\ \{x_p^*(n)\} = \sum_{n=0}^{N-1} x_p^*(n)\, W_N^{kn} = \left\{\left(\sum_{n=0}^{N-1} x_p^*(n) W_N^{kn}\right)^*\right\}^*$$

$$= \left\{\sum_{n=0}^{N-1} x_p(n) W_N^{-kn}\right\}^* = \{X_p(-k)\}^* = X_p^*(-k) \qquad \text{QED}$$

Based on the properties above we can show that for a real periodic sequence $x_p(n)$, the following symmetry properties of the discrete Fourier series hold:

1. Re $[X_p(k)]$ = Re $[X_p(-k)]$
2. Im $[X_p(k)]$ = –Im $[X_p(-k)]$
3. $|X_p(k)| = |X_p(-k)|$
4. arg $X_p(k)$ = – arg $X_p(-k)$

2.4 The discrete Fourier transform (DFT)

(Omit) The discrete Fourier transform (DFT) derived from the Fourier series The exponential Fourier series of a continuous time periodic signal $x(t)$ with fundamental period T_0 is given by the synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{j2\pi k F_0 t} \qquad \rightarrow (1)$$

where the Fourier coefficients X_k are given by the analysis equation

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j2\pi k F_0 t}\, dt \qquad \rightarrow (2)$$

with the fundamental frequency F_0 and the period T_0 related by F_0 (Hz) = $1/T_0$ (sec).

To obtain finite-sum approximations for the above two equations, consider the analog periodic signal $x(t)$ shown in Figure and its sampled version $x_s(nT)$. Using $x_s(nT)$, we can approximate the integral for X_k by the sum

$$X_k = \frac{1}{T_0}\sum_{n=0}^{N-1} x_s(nT)\, e^{-j2\pi k F_0 nT}\, T, \qquad k = 0, 1, \ldots, N-1$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi k\, n/N}, \qquad k = 0, 1, \ldots, N-1$$

where we used the relation $F_0 T = 1/N$, and approximated dt (or Δt) by T, and have used the shorthand notation $x(n) = x_s(nT)$. (This procedure is similar to that used in a typical introduction to integral calculus).

Fourier Series
Fourier Transform

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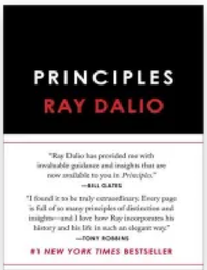

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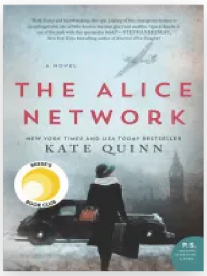

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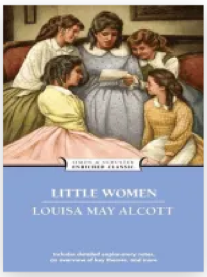

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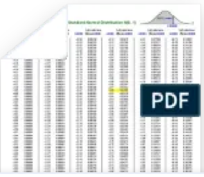

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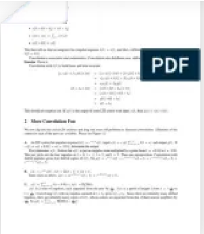

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