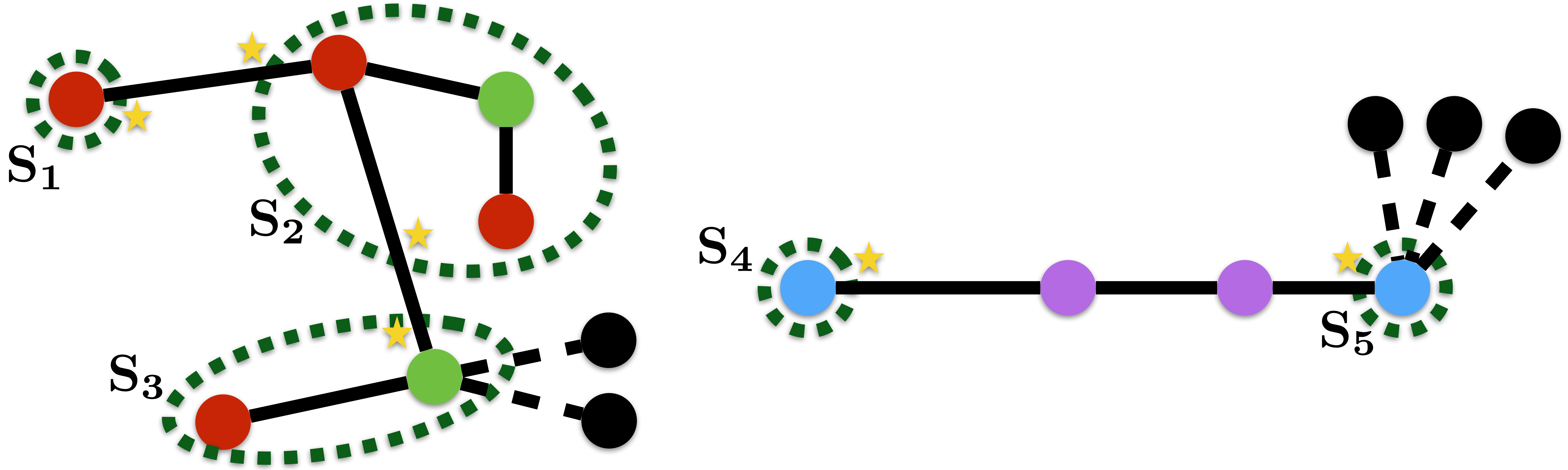
Steiner forest

Lemma

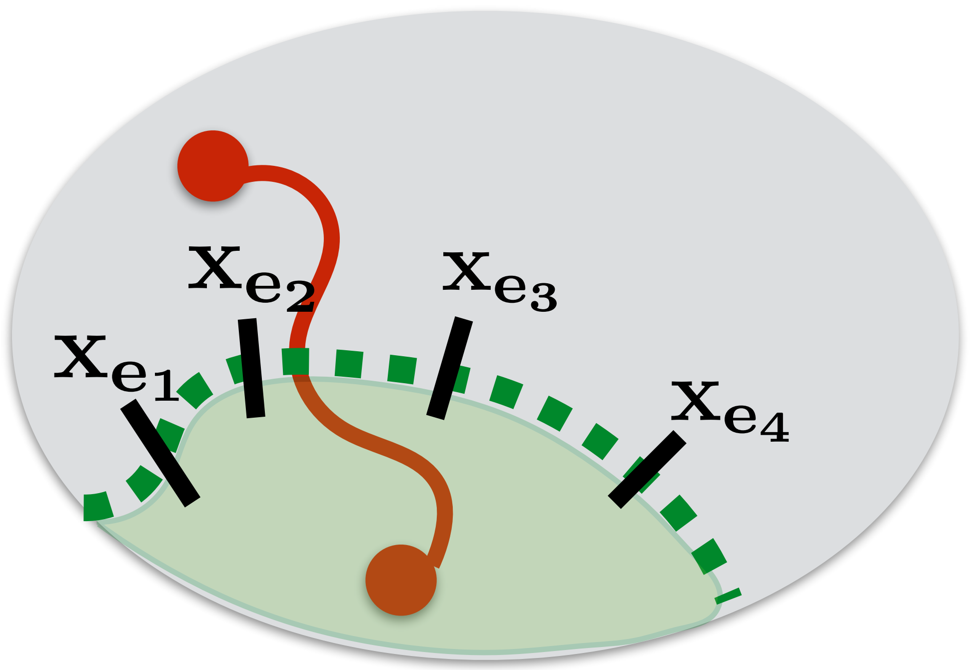
Let F' denote the output set of edges

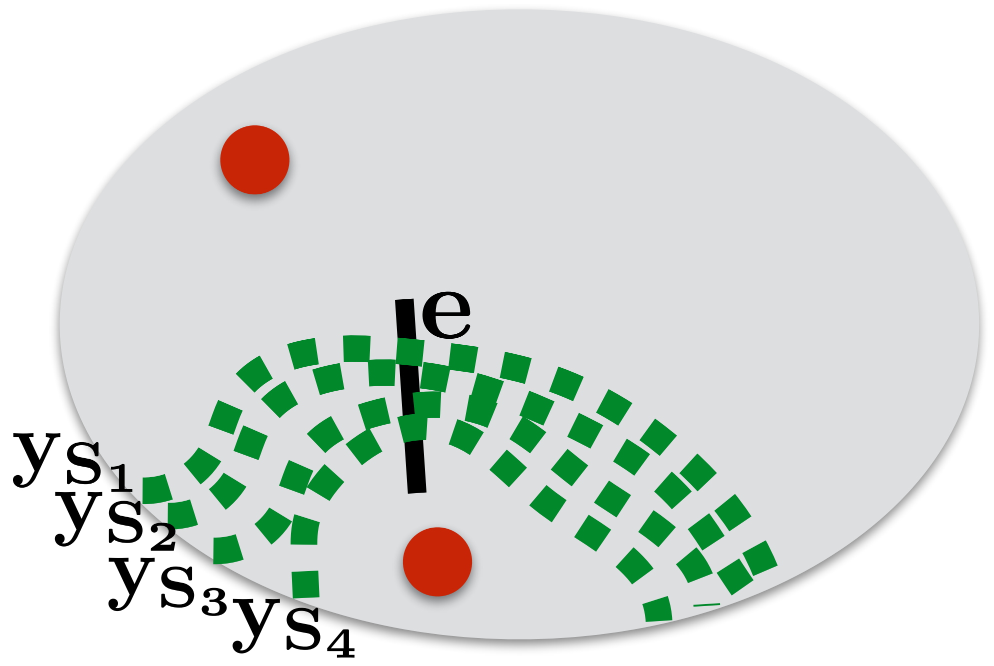
Fix a time of the execution and say a set S is active if its dual variable is being raised.

Then: $$\frac{\sum_{S\ \text{active}} |F' \cap \delta(S)|}{\#(\text{active sets})} \leq 2$$



What properties do F' and active sets have?





Initialization:

$$x \leftarrow 0, y \leftarrow 0$$

Iteration: while x not satisfiable

in parallel, raise every unfrozen y_S with

S minimal

stopped by tight constraint (e)

$$x_e \leftarrow 1$$

freeze y_S in tight constraints

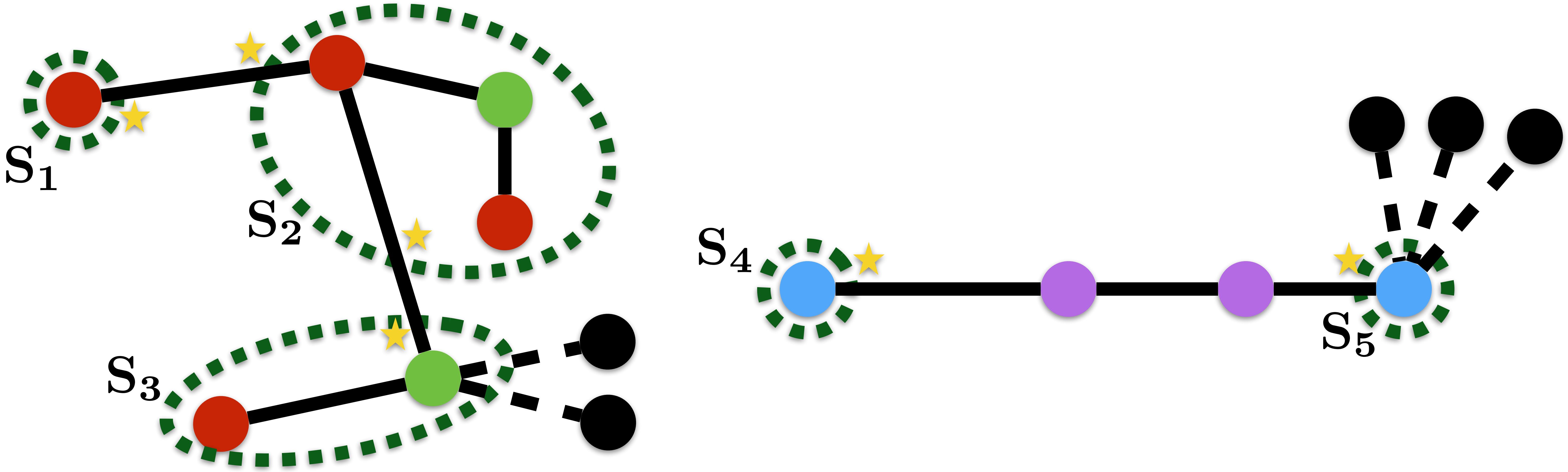
Pruning: let F={edges defined by x}

for each edge e of F in reverse order,

remove e if unnecessary

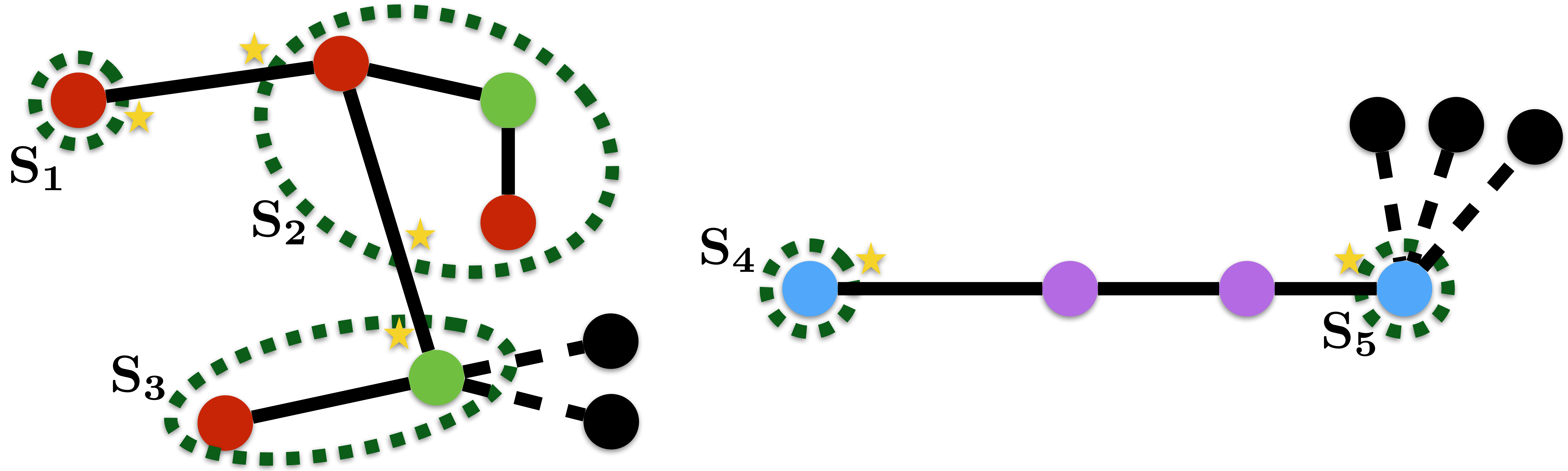
Properties of active sets

active sets are disjoint subsets of nodes containing terminals



Properties of F'

F' is a forest and its leaves are terminals

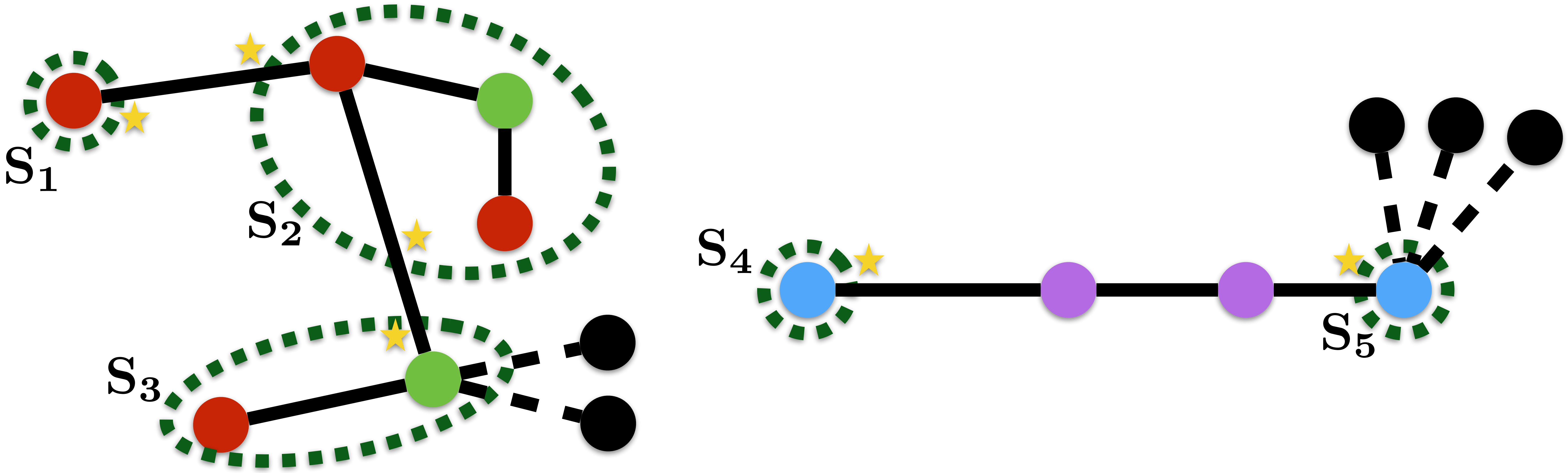


Joint properties of F' and of active sets

Consider a tree of F'

Assume it intersects some active sets

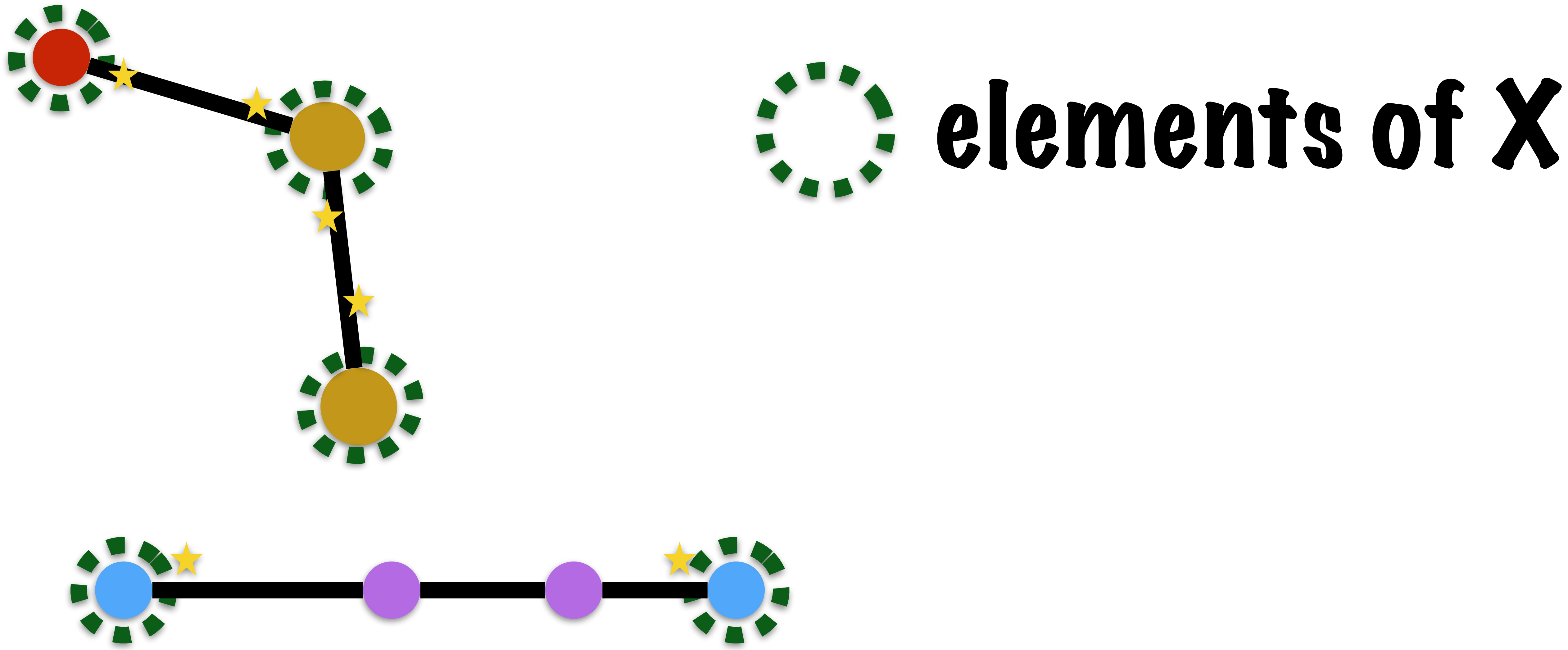
- the active sets are subtrees
- all the leaves are in active sets



Graph theory lemma

Given a tree T and a subset X of its vertices that includes all the leaves:

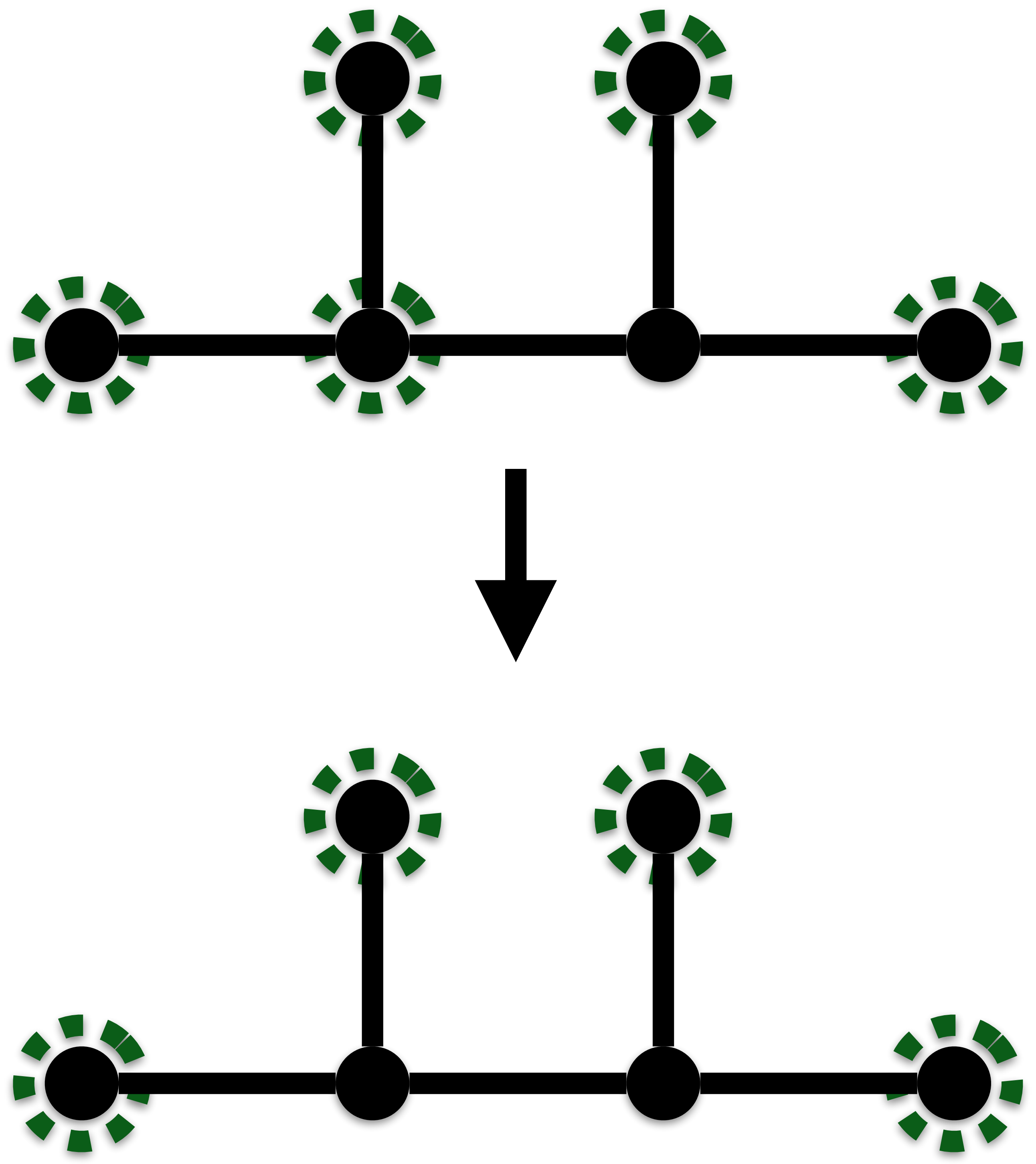
$$\sum_{v \in X} \deg(v) \leq 2 \cdot |X|$$



Graph theory lemma

T tree, X subset of vertices including all leaves:

$$\sum_{v \in X} \deg(v) \leq 2 \cdot |X|$$



Proof: induction

- true for X=V

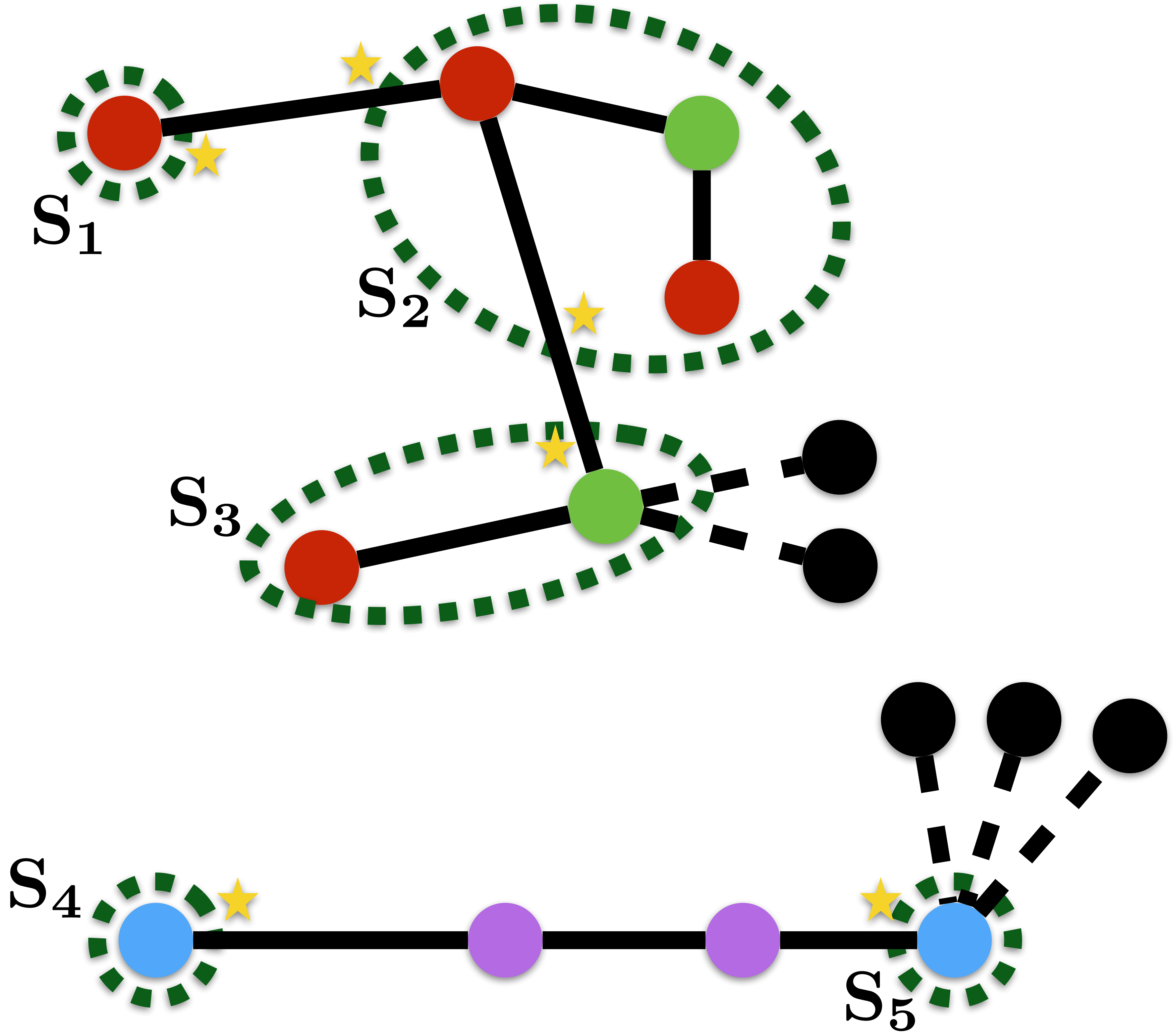
$$\sum_{v \in V} \deg(v) = 2|E| = 2(|V| - 1) < 2 \cdot |V|$$

- if true for X then remains true when removing an internal node from X, **QED**.

Lemma

F' output forest

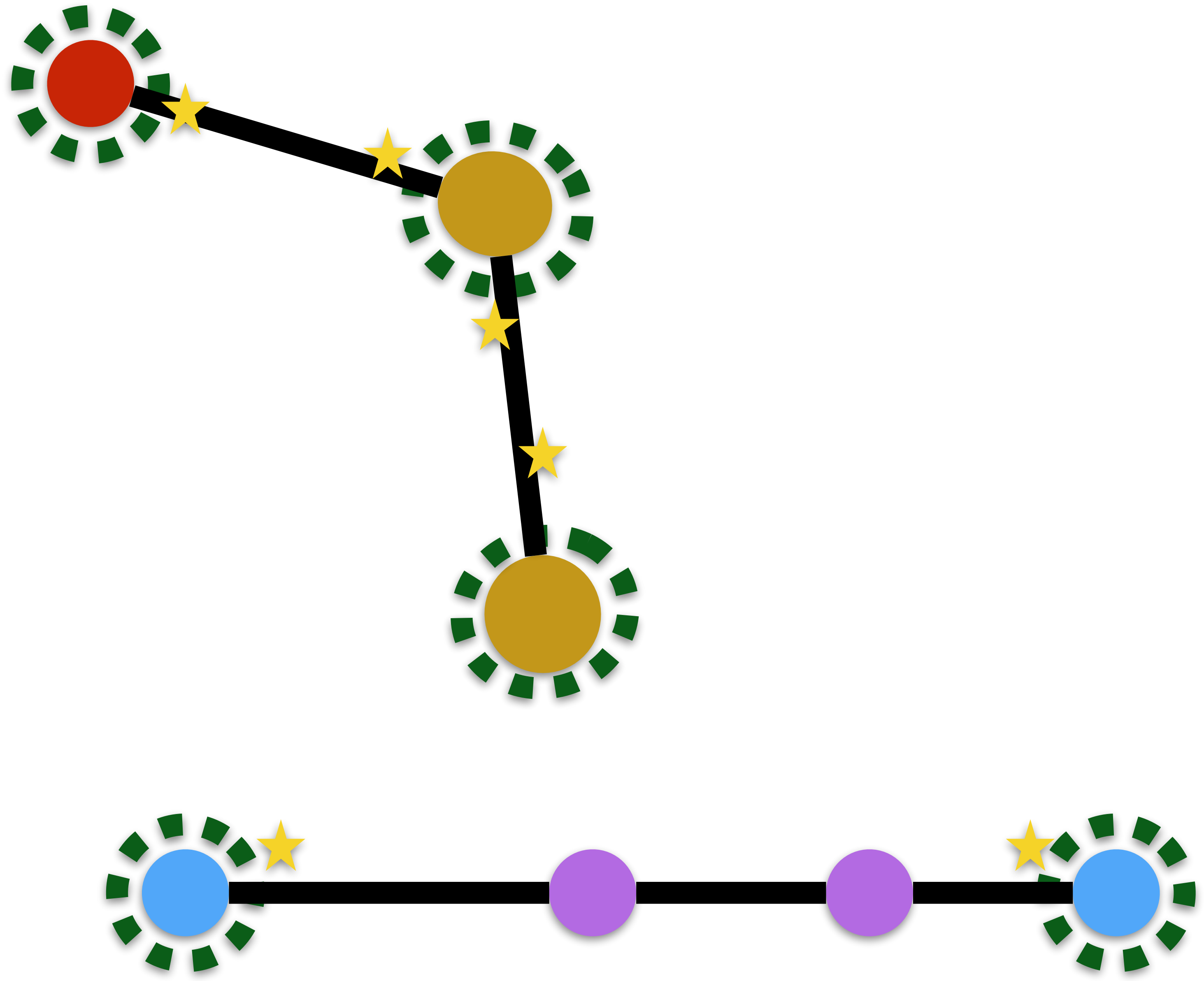
$$\frac{\sum_{S\ \text{active}} |F' \cap \delta(S)|}{\#(\text{active sets})} \leq 2$$



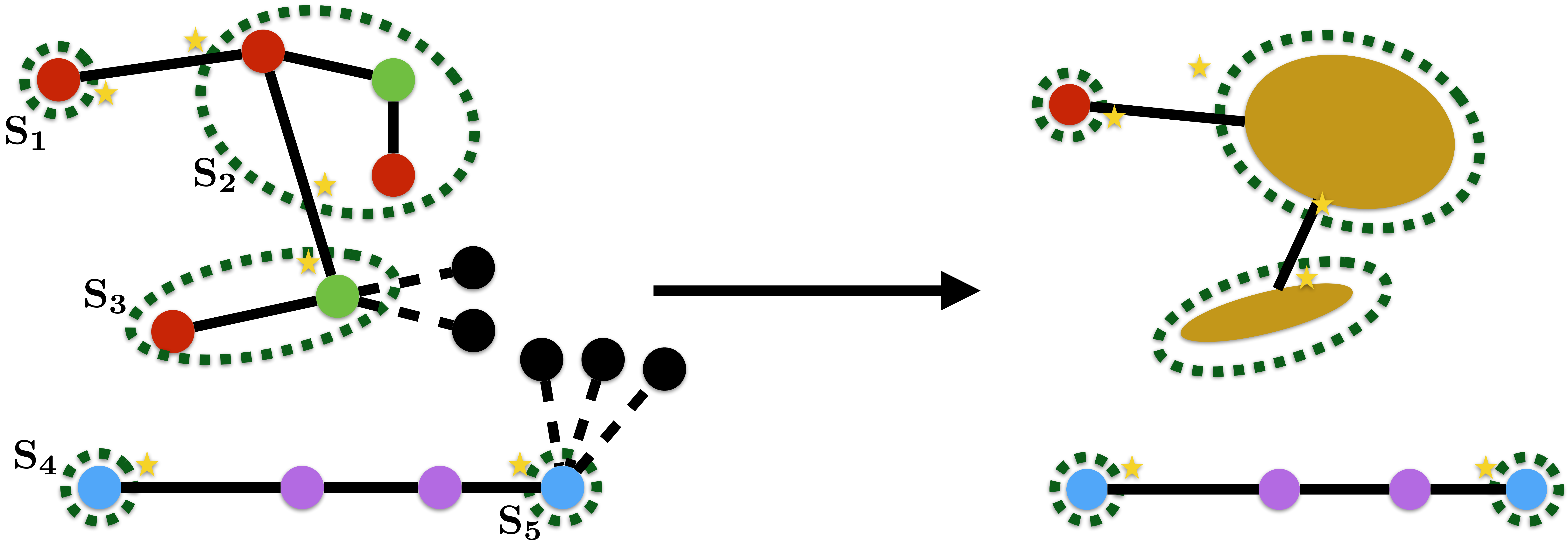
Graph theory lemma

T tree, X subset of vertices including all leaves:

$$\frac{\sum_{v \in X} \deg(v)}{|X|} \leq 2$$



Lemma F' output forest: $\frac{\sum_{S \text{ active}} |F' \cap \delta(S)|}{\#(\text{active sets})} \leq 2$



Contracting active sets does not change

$\sum_{S \text{ active}} |F' \cap \delta(S)|$ **nor** $\#(\text{active sets})$

QED

Result:

a 2-approximation algorithm
for the Steiner forest problem

Comments

- extends: connectivity problems, and beyond
- primal-dual is greedy
- dual gives insight
- post-processing can be necessary
- combinatorial: LPs are guides, not computational tools

Steiner forest