Feedback — Assignment 4

You submitted this quiz on **Fri 22 Mar 2013 10:21 AM PDT -0700**. You got a score of **30.00** out of **30.00**.

Question 1

Which one of the following is equivalent to the linear program

$$\begin{array}{rl} \text{minimize} & 2x_1 + 3|x_2 - 10| \\ \text{subject to} & |x_1 + 2| + |x_2| \leq 5 \end{array}$$

Hint: Introduce a new variable x_3 to denote $|x_2 - 10|$.

Your Answer	Score	Explanation
○ $\begin{array}{rl} \text{minimize} & 2x_1 + 3x_3 \\ \text{subject to} & x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1 - x_2 \leq 7 \\ & -x_1 - x_2 \leq 3 \\ & x_2 - x_3 \leq 10 \\ & -x_2 - x_3 \leq -10 \end{array}$		
○ $\begin{array}{rl} \text{minimize} & 2x_1 + 3x_3 \\ \text{subject to} & x_1 + x_2 \leq 3 \\ & -x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq 3 \\ & -x_1 - x_2 \leq 7 \\ & x_2 - x_3 \leq -10 \\ & -x_2 - x_3 \leq 10 \end{array}$		
○		

$$\begin{aligned} \text{minimize } & 2x_1 + 3x_3 \\ \text{subject to } & -x_1 + x_2 \leq 3 \\ & x_1 + x_2 \leq 7 \\ & -x_1 - x_2 \leq 3 \\ & x_1 - x_2 \leq 7 \\ & -x_2 - x_3 \leq -10 \\ & x_2 - x_3 \leq 10 \end{aligned}$$

✓ 5.00

$$\begin{aligned} \text{minimize } & 2x_1 + 3x_3 \\ \text{subject to } & x_1 + x_2 \leq 3 \\ & -x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq 3 \\ & -x_1 - x_2 \leq 7 \\ & x_2 - x_3 \leq 10 \\ & -x_2 - x_3 \leq -10 \end{aligned}$$

Total 5.00 / 5.00

Question Explanation

We add a new variable x_3 to our linear program and minimize $2x_1 + 3x_3$ with the constraint that $|x_2 - 10| \leq x_3$, which simplifies to the two inequalities $x_2 - 10 \leq x_3$ and $-x_2 + 10 \leq x_3$. For the constraint, we note that there are four possible ways that the two absolute value functions in the constraint can resolve themselves. Since the constraint specifies that the sum of two non-negative quantities is at most 5, we can add the four constraints corresponding to the four choices of signs $\pm(x_1 + 2) \pm x_2 \leq 5$to our system. Simplifying we get the linear program

$$\begin{aligned} \text{minimize } & 2x_1 + 3x_3 \\ \text{subject to } & x_1 + x_2 \leq 3 \\ & -x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq 3 \\ & -x_1 - x_2 \leq 7 \\ & x_2 - x_3 \leq 10 \\ & -x_2 - x_3 \leq -10 \end{aligned}$$

Question 2

Which of the following statements are correct?

Your Answer		Score	Explanation
☑ $\log(n^2+42) = O(\log n)$	✓	0.75	For $n \geq 7$, $\log(n^2+42) \leq \log(n^2+n^2) = 2\log n$
☐ $100^n = \Omega(n!)$	✓	0.75	Suppose $\exists c > 0$ such that $\forall n \geq n_0 \in \mathbb{N}$, $100^n \geq cn!$. Then this implies that $c \leq 100^n/n!$ for all $n \geq n_0$ leading to the conclusion that $c = 0$. Hence, $100^n = \Omega(n!)$ is not true.
☐ $n^n = O(n!)$	✓	0.75	Suppose $\exists c > 0$ such that $\forall n \geq n_0 \in \mathbb{N}$, $n^n \leq cn!$. Then this implies that $n^n \leq cn! \leq cn^{n-1}$ leading to the conclusion that $n \leq c$ which is false for sufficiently large n. Hence, $n^n = O(n!)$ is not true.
☑ $\sqrt{2}^{\log_2 n} = \Theta(\sqrt{n})$	✓	0.75	$\sqrt{2}^{\log_2 n} = 2^{\log\sqrt{n}} = \sqrt{n}$
Total		3.00 / 3.00	

Question 3

The Euclidean algorithm for computing the greatest common divisor $\gcd(a, b)$ of two integers $a > 0$ and $b \geq 0$ works as follows: if b is 0 then return a otherwise return $\gcd(b, a \text{ rem } b)$ where $a \text{ rem } b$ is the remainder that we obtain on dividing a by b. Knowing that we express the inputs $a, b \in \mathbb{Z}$ in binary encoding, tick all the options from below that are correct. For this problem assume that each of the division steps in the algorithm can be performed in one step.

Your Answer		Score	Explanation
☑ The running time of the Euclidean algorithm is $O(\log \lvert b \rvert)$	✓	0.75	
☐ The Euclidean algorithm runs in time that is logarithmic in the size of the inputs	✓	0.75	
☐ The running time of the Euclidean algorithm is $\Theta(\lvert b \rvert)$	✓	0.75	
☑ The Euclidean algorithm is a linear time algorithm	✓	0.75	
Total		3.00 / 3.00	

Question Explanation

The running time of Euclid's GCD algorithm can be analyzed in the following way: Suppose we perform two iterations of the above algorithm starting from the pair of integers a and b so that we have the following equalities $\mathrm{gcd}(a, b) = \mathrm{gcd}(b, r) = \mathrm{gcd}(r, r')$ where $r = a \text{ rem } b$ and $r' = b \text{ rem } r$. Since r is strictly smaller than b, it follows that $r' < \frac{b}{2}$. So, after two iterations we know that the size of the second argument must have decreased by at least a factor of 2. Since the procedure must terminate after the second argument becomes 0, we can bound the number of steps by $O(\log \lvert b \rvert)$. The size of the inputs here to this algorithm is the number of bits that are required to encode a and b in binary which is $(\lceil \log \lvert a \rvert \rceil + 1) + (\lceil \log \lvert b \rvert \rceil + 1)$. As $O(\log \lvert b \rvert)$ is a linear function of the input size, we have that the algorithm is a linear time algorithm.

Question 4

Suppose we have two integer matrices $A \in \mathbb{Z}^{m \times n}$ and $B \in \mathbb{Z}^{n \times p}$ and we knew that the entries of A and B are bounded by M in absolute value, then which of the following is the best bound that can be given on the bit lengths of the entries in the product of the two matrices AB?

Your Answer	Score	Explanation
○ $O(\log mnM^2)$		

$O(\log mnpM^2)$		
$O(\log mnM^2)$		
$O(\log nM^2)$	✓	3.00
Total		3.00 / 3.00

Question Explanation

Since the i, j-th entry of the product AB is defined as $\sum_{k=1}^{n} A_{ik} B_{kj}$ we have that $(AB)_{ij} \leq nM^2$. Thus, the bit lengths of the entries of AB can be bounded by $O(\log nM^2)$.

Question 5

Suppose we have three matrices with integer entries $A \in \mathbb{Z}^{m \times n}, B \in \mathbb{Z}^{n \times p}, C \in \mathbb{Z}^{p \times q}$, and we wish to evaluate the matrix product ABC. Using the associativity of matrix multiplication, we can multiply the matrices in two ways. If we choose to do it using the method that performs fewer integer multiplications, what would the number of integer multiplications be in that case?

Your Answer		Score	Explanation
$\min\{mq(n+p), nq(m+p)\}$			
$\min\{mq(n+p), mq(n+p)\}$			
$\min\{np(m+q), np(m+q)\}$			
$\min\{mp(n+q), nq(m+p)\}$	✓	3.00	
Total		3.00 / 3.00	

Question Explanation

The two ways of evaluating ABC correspond to $(AB)C$ and $A(BC)$. In the first case, we use mnp multiplications to form the product AB and then use mpq multiplications to form the final result resulting in a total of $mp(n+q)$ integer multiplications. In the second case, arguing similarly, we use $nq(m+p)$ integer multiplications. The answer follows by taking the best of the two.

Question 6

Given a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \leq b\}$ and a point $p \in \mathcal{P}$ such that there is exactly one non–trivial inequality that is active at p can this inequality be redundant? That is, is it possible that the inequality may be removed from the description of $\mathcal{P}$ without altering it?

Your Answer		Score	Explanation
No	✓	5.00	
Yes			
Total		5.00 / 5.00	

Question Explanation

Since we know that every other inequality is not active at the point p, we can then pick an $\epsilon > 0$ small enough such that every point $x \in \mathbb{R}^n$ such that $\|x - p\|_2 \leq \epsilon$ is satisfied by each of the other inequalities. Hence, removing this inequality allows for points that did not originally belong to $\mathcal{P}$ to be included, and hence such an inequality cannot be redundant.

Question 7

The ℓ_1 norm ball in n dimensions (also called the crosspolytope) is defined as $\mathcal{P}_n := \{x \in \mathbb{R}^n \mid |x_1| + \cdots + |x_n| \leq 1\}$ Note that the given description of the polytope is not in the standard form. How many linear inequalities are *necessary* (i.e., not redundant) to describe this polytope (bounded polyhedron)

Your Answer	Score	Explanation
$n!$		
n^2		

○ n^4		
◉ 2^n	✓	5.00
Total		5.00 / 5.00

Question Explanation

Let's call an inequality that is not redundant for a polyhedron a *facet*. In the standard form $\mathcal{P}$ would have 2^n inequalities of the form $\pm x_1 + \cdots \pm x_n \leq 1$ Now, each of these inequalities defines a facet since none of them are redundant. To see this, consider a concrete example with $n = 3$ and take an arbitrary inequality, say, $x_1 - x_2 + x_3 \leq 1$ Think of the point with $(1/3, -1/3, 1/3)$ This point is active at this inequality and is not active at any of the other 7 inequalities. We similarly argue for all n by exhibiting such a point for each inequality. Thus, using the argument from the previous question we have that each of the inequalities defines a facet and is not redundant.

Question 8

The Hirsch conjecture stated (the conjecture has since been proven false) that the diameter $\Delta(n, m)$ of any polyhedron with m facets in n dimensions is at most $m - n$. A Hirsch-sharp polyhedron is an n dimensional polyhedron with m facets whose diameter is equal to $m - n$. Select the ones from the list below that are Hirsch-sharp.

Your Answer		Score	Explanation
☑ The n dimensional unit hypercube (i.e., $\mathcal{P}_n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \; \forall \; i = 1, \ldots, n\}$)	✓	0.60	The n dimensional hypercube has $2n$ facets and its diameter is n which is realized by the shortest path connecting the vertices $(0, \ldots, 0)$ and $(1, \ldots, 1)$
☑ The 3 dimensional pyramid (the three dimensional polyhedron with a square base whose four vertices are connected to a vertex lying	✓	0.60	The diameter of the pyramid is clearly 2, and it

above it by edges)			lies in 3 dimensions and has 5 facets.
☑ Triangle in two dimensions	✔	0.60	The triangle has diameter 1 and it lies in 2 dimensions and has 3 facets.
☐ Hexagon in two dimensions	✔	0.60	The hexagon has diameter 3 and it lies in 2 dimensions and has 6 facets, so $3 \leq 6 - 2 = 4$ does not hold with equality.
☐ Pentagon in two dimensions	✔	0.60	The pentagon has diameter 2 and it lies in 2 dimensions and has 5 facets, so $2 \leq 5 - 2 = 3$ does not hold with equality.
Total		3.00 / 3.00	