QUESTION 4

Theorem For any natural number n,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

Proof: By induction.

For $n = 1$, the left-hand side is $\frac{1}{1.2} = \frac{1}{2}$ and the right-hand side is $\frac{1}{2}$, so the identity is valid for $n = 1$.

Assume the identity holds for n. Then:

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}
\end{aligned}$$

This is the identity for $n + 1$. Hence, by induction, the theorem is proved.