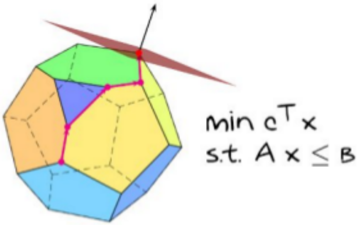
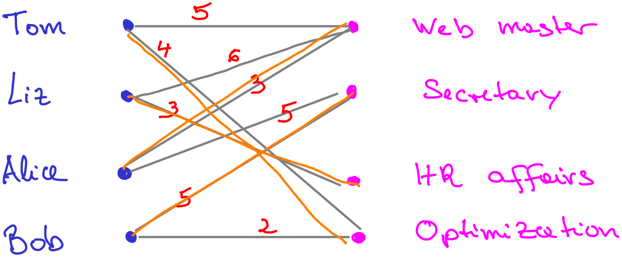
Matchings and vertex covers

- Example
- A min-max relation



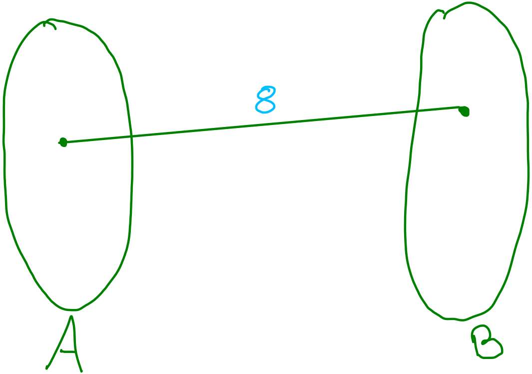
Assigning jobs to employees



Score = 15

Bipartite graphs

A graph $G = (V, E)$ is *bipartite*, if one can partition V into $V = A \dot{\cup} B$ such that each edge $e \in E$ satisfies $|e \cap A| = |e \cap B| = 1$.



Weights:

$w: E \to \mathbb{N}_0$

Notation:

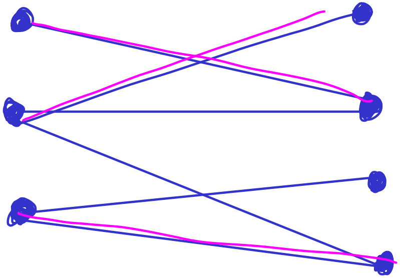
$U \subseteq E$:

$$w(U) = \sum_{e \in U} w_e$$

Matchings

A *matching* is a subset $M \subseteq E$ of the edges such that each $e_1 \neq e_2 \in M$ satisfy $e_1 \cap e_2 = \emptyset$.

The edges in a matching "do not touch".

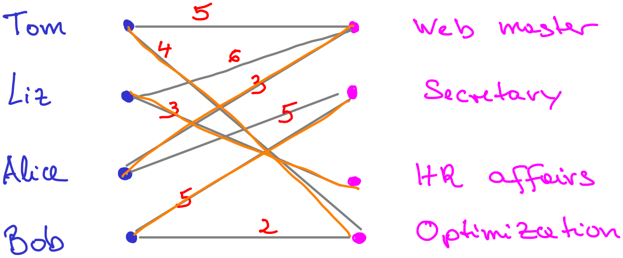


$|M| = 3$

The maximum weight (bipartite) matching problem

Given a (bipartite) graph $G = (V, E)$ and *edge weights* $w : E \to \mathbb{N}_0$, determine a matching $M \subseteq E$ such that

$$w(M) := \sum_{e \in M} w_e \quad \text{is maximal.}$$



$W(M) = 15$

Question:

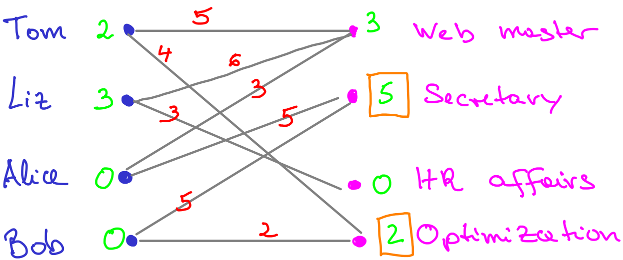
Is M maximal

w-vertex covers

Let $G = (V, E)$ be a graph with edge weights $w : E \to \mathbb{N}_0$. A *w-vertex cover* is a vector $y \in \mathbb{N}_0^{|V|}$ such that

$$\forall uv \in E : \quad y_u + y_v \geqslant w_{uv}.$$

The *value* of a w-vertex cover y is $\sum_{v \in V} y_v$.



$3+4 = 7 \geq 5$

VALUE = 15

w-vertex cover $\geqslant$ weight of matching

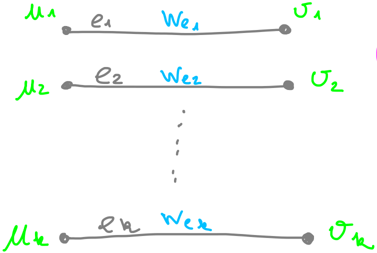
Lemma (Weak duality)

Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{N}_0$ be edge-weights. If M is a matching of G and if y is a w-vertex cover of G, then

$$w(M) \leqslant \sum_{v \in V} y_v.$$

Weight of M

Proof:

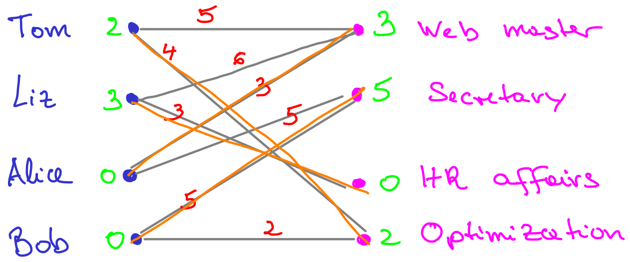


$M = \{e_1, e_2, \dots, e_k\}$

$$\sum_{i=1}^{k} \underbrace{w_{e_i}}_{\leq y_{u_i} + y_{v_i}}$$

$$\leq \sum_{v \in V} y_v \leftarrow \text{VAL. OF W-VERTEX COVER}$$

The job assignment is optimal



$w(M) = 15$

$\leq$

$\sum y_v = 15$

Both! y and M are optimal