Section 16.4 : Line Integrals Of Vector Fields

In the previous two sections we looked at line integrals of functions. In this section we are going to evaluate line integrals of vector fields. We'll start with the vector field,

$$\vec F\left(x,y,z\right) = P\left(x,y,z\right)\vec i + Q\left(x,y,z\right)\vec j + R\left(x,y,z\right)\vec k$$

and the three-dimensional, smooth curve given by

$$\vec r\left(t\right) = x\left(t\right)\vec i + y\left(t\right)\vec j + z\left(t\right)\vec k \qquad a \le t \le b$$

The **line integral of $\vec F$ along C** is

$$\int_C \vec F \centerdot d\vec r = \int_a^b \vec F\left(\vec r\left(t\right)\right) \centerdot \vec r\,'\left(t\right)\,dt$$

Note the notation in the integral on the left side. That really is a **dot product** of the vector field and the differential really is a vector. Also, $\vec F\left(\vec r\left(t\right)\right)$ is a shorthand for,

$$\vec F\left(\vec r\left(t\right)\right) = \vec F\left(x\left(t\right),y\left(t\right),z\left(t\right)\right)$$

We can also write line integrals of vector fields as a line integral with respect to arc length as follows,

$$\int_C \vec F \centerdot d\vec r = \int_C \vec F \centerdot \vec T\,ds$$

where $\vec T\left(t\right)$ is the unit tangent vector and is given by,

$$\vec T\left(t\right) = \frac{\vec r\,'\left(t\right)}{\left\|\vec r\,'\left(t\right)\right\|}$$

If we use our knowledge on how to compute line integrals with respect to arc length we can see that this second form is equivalent to the first form given above.

$$\begin{aligned}\int_C \vec F \centerdot d\vec r &= \int_C \vec F \centerdot \vec T\,ds\\ &= \int_a^b \vec F\left(\vec r\left(t\right)\right) \centerdot \frac{\vec r\,'\left(t\right)}{\left\|\vec r\,'\left(t\right)\right\|}\left\|\vec r\,'\left(t\right)\right\|\,dt\\ &= \int_a^b \vec F\left(\vec r\left(t\right)\right) \centerdot \vec r\,'\left(t\right)\,dt\end{aligned}$$

In general, we use the first form to compute these line integral as it is usually much easier to use. Let's take a look at a couple of examples.

Example 1 Evaluate $\displaystyle\int_C \vec F \centerdot d\vec r$ where $\vec F\left(x,y,z\right) = 8x^2yz\,\vec i + 5z\,\vec j - 4x\,y\,\vec k$ and C is the curve given by $\vec r\left(t\right) = t\,\vec i + t^2\,\vec j + t^3\,\vec k$, $0 \le t \le 1$.

Show Solution

Example 2 Evaluate $\displaystyle\int_C \vec F \centerdot d\vec r$ where $\vec F\left(x,y,z\right) = x\,z\,\vec i - y\,z\,\vec k$ and C is the line segment from $\left(-1,2,0\right)$ to $\left(3,0,1\right)$.

Hide Solution

We'll first need the parameterization of the line segment. We saw how to get the parameterization of line segments in the first **section** on line integrals. We've been using the two dimensional version of this over the last couple of sections. Here is the parameterization for the line.

$$\begin{aligned}\vec r\left(t\right) &= \left(1-t\right)\left\langle -1,2,0\right\rangle + t\left\langle 3,0,1\right\rangle\\ &= \left\langle 4t-1,2-2t,t\right\rangle, \qquad\qquad 0 \le t \le 1\end{aligned}$$

So, let's get the vector field evaluated along the curve.

$$\begin{aligned}\vec F\left(\vec r\left(t\right)\right) &= \left(4t-1\right)\left(t\right)\,\vec i - \left(2-2t\right)\left(t\right)\,\vec k\\ &= \left(4t^2-t\right)\vec i - \left(2t-2t^2\right)\vec k\end{aligned}$$

Now we need the derivative of the parameterization.

$$\vec r\,'\left(t\right) = \left\langle 4,-2,1\right\rangle$$

The dot product is then,

$$\vec F\left(\vec r\left(t\right)\right) \centerdot \vec r\,'\left(t\right) = 4\left(4t^2-t\right) - \left(2t-2t^2\right) = 18t^2 - 6t$$

The line integral becomes,

$$\begin{aligned}\int_C \vec F \centerdot d\vec r &= \int_0^1 18t^2 - 6t\,dt\\ &= \left.\left(6t^3 - 3t^2\right)\right|_0^1\\ &= 3\end{aligned}$$

Let's close this section out by doing one of these in general to get a nice relationship between line integrals of vector fields and line integrals with respect to x, y, and z.

Given the vector field $\vec F\left(x,y,z\right) = P\,\vec i + Q\,\vec j + R\,\vec k$ and the curve C parameterized by $\vec r\left(t\right) = x\left(t\right)\vec i + y\left(t\right)\vec j + z\left(t\right)\vec k$, $a \le t \le b$ the line integral is,

$$\begin{aligned}\int_C \vec F \centerdot d\vec r &= \int_a^b \left(P\,\vec i + Q\,\vec j + R\,\vec k\right) \centerdot \left(x'\,\vec i + y'\,\vec j + z'\,\vec k\right)\,dt\\ &= \int_a^b Px' + Qy' + Rz'\,dt\\ &= \int_a^b Px'\,dt + \int_a^b Qy'\,dt + \int_a^b Rz'\,dt\\ &= \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz\\ &= \int_C P\,dx + Q\,dy + R\,dz\end{aligned}$$

So, we see that,

$$\int_C \vec F \centerdot d\vec r = \int_C P\,dx + Q\,dy + R\,dz$$

Note that this gives us another method for evaluating line integrals of vector fields.

This also allows us to say the following about reversing the direction of the path with line integrals of vector fields.

Fact

$$\int_{-C} \vec F \centerdot d\vec r = -\int_C \vec F \centerdot d\vec r$$

This should make some sense given that we know that this is true for line integrals with respect to x, y, and/or z and that line integrals of vector fields can be defined in terms of line integrals with respect to x, y, and z.