QUESTION 6

The *Fibonacci sequence* is defined iteratively by setting $F_1 = F_2 = 1$ and thereafter letting $F_{n+2} = F_n + F_{n+1}$.

Theorem For any natural number n,

$$F_n \geq \left(\frac{3}{2}\right)^{n-2}$$

Proof: We have $F_1 = 1 \geq \frac{2}{3} = \left(\frac{3}{2}\right)^{-1}$ and $F_2 = 1 = \left(\frac{3}{2}\right)^{0}$, so the inequality is valid for $n = 1, 2$.

Now assume the inequality holds for n, where $n \geq 2$. Then:

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&\geq \left(\frac{3}{2}\right)^{n-2} + \left(\frac{3}{2}\right)^{n-3} \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{3}{2}+1\right), \text{ by algebra} \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{5}{2}\right) \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{10}{4}\right) \\
&\geq \left(\frac{3}{2}\right)^{n-3}\left(\frac{9}{4}\right) \\
&= \left(\frac{3}{2}\right)^{n-3}\left(\frac{3}{2}\right)^{2} \\
&= \left(\frac{3}{2}\right)^{n-1}
\end{aligned}$$

which establishes the inequality for $n+1$.