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> Placing the most ancient epoch of history at five thousand years ago, or at 1,826,213 days, and the sun having risen constantly in the interval at each revolution of twenty-four hours, it is a bet of 1,826,214 to 1 that it will rise again tomorrow.
> — P. S. Laplace, *Théorie Analytique des Probabilités* (1812).

Non-sequitur: an urn model

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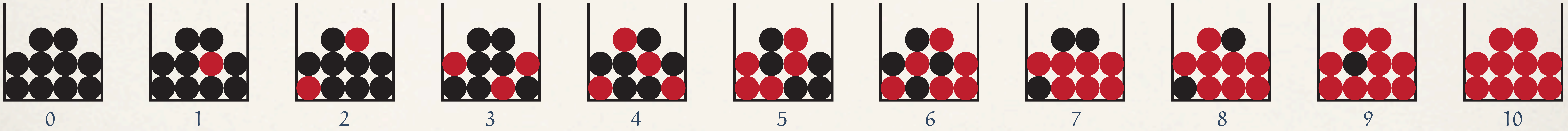

Non-sequitur: an urn model

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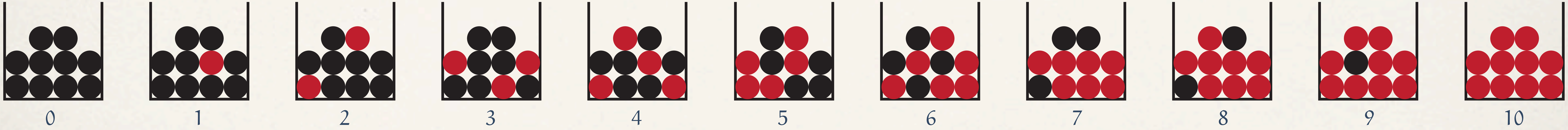

- Given $N + 1$ urns labelled $0, 1, \ldots, N$, each containing N balls: urn k contains k red balls and $N - k$ black balls.
  - An urn is first selected at random from the $N + 1$ urns.
  - Balls are repeatedly drawn with replacement from the chosen urn.
- Given that r red balls were drawn in a row, what is the chance that the $(r+1)$th draw results in a red ball?

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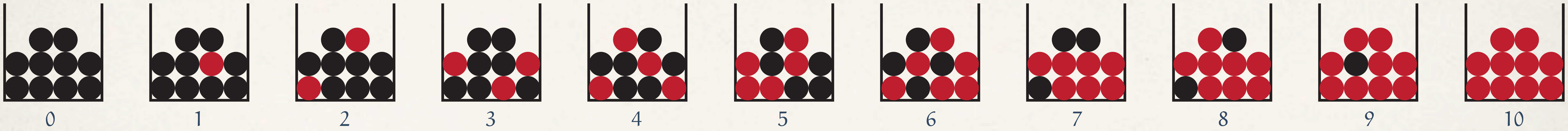
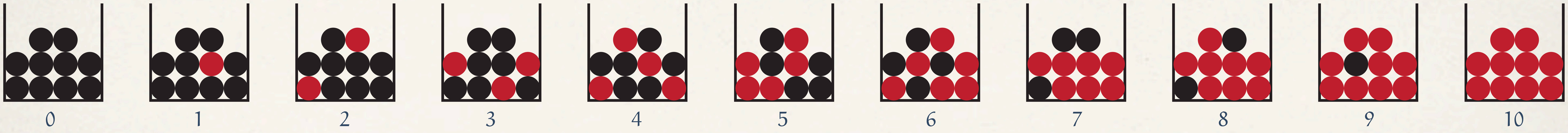
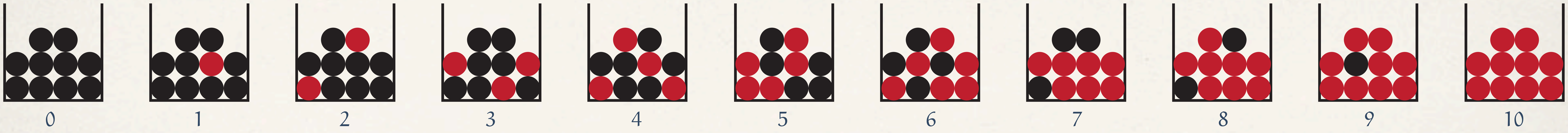
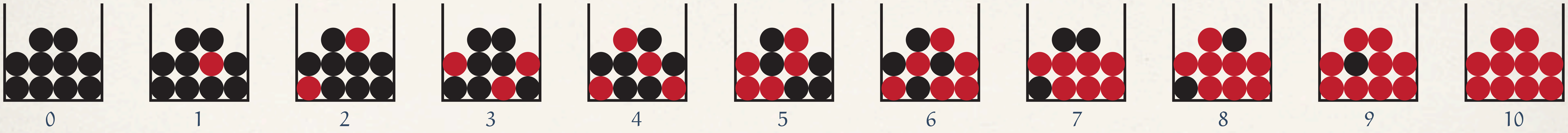



0
1
2
3
4
5
6
7
8
9
10



- *Sample space* Ω: set of sequences (urn; 1st ball, …, rth ball, (r+1)th ball); the sample points are of the form $\omega = (x; y_1, \ldots, y_r, y_{r+1})$ where $x \in \{0, 1, \ldots, N\}$ and $y_1, \ldots, y_r, y_{r+1} \in$ {red, black}.



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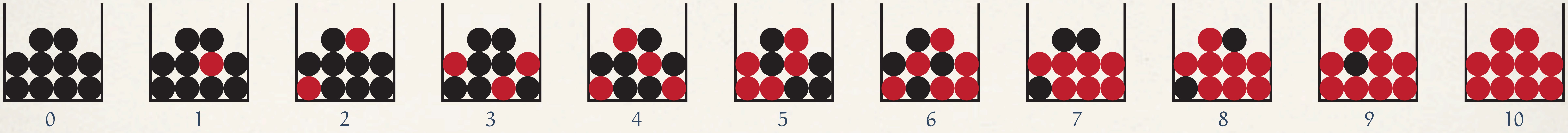

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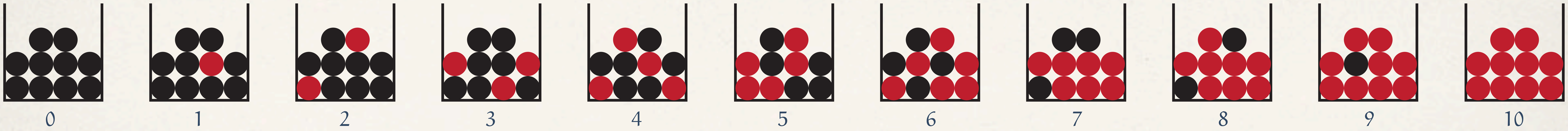

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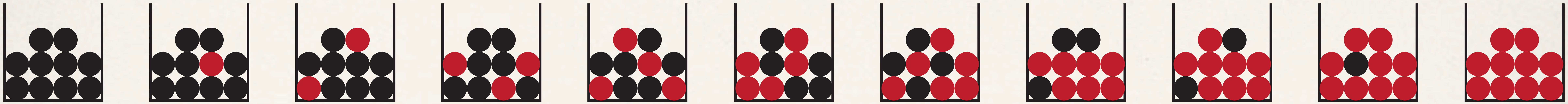

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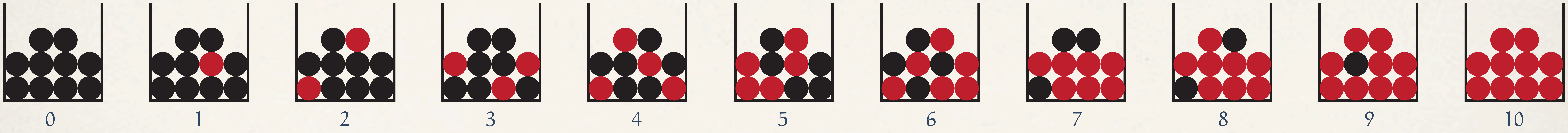

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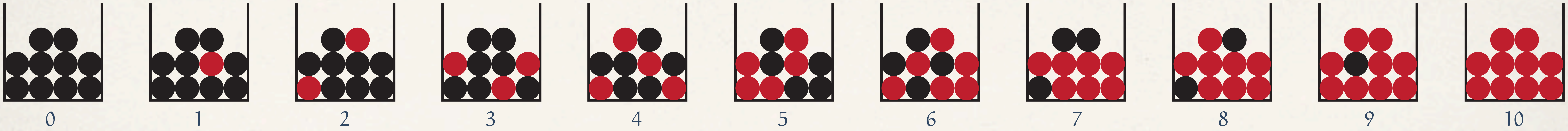

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  - Conditional probabilities for given urn: $\mathbf{P}(H_k \mid A_j) = j^k / N^k$.