Test your understanding

$$(p+q)^n = \binom{n}{n}p^n + \binom{n}{n-1}p^{n-1}q + \cdots + \binom{n}{1}pq^{n-1} + \binom{n}{0}q^n = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$$

Discover three combinatorial identities by examining what the binomial theorem says about:

$$(1+1)^n$$

$$(1-1)^n$$

$$\left(\frac{1}{3}+\frac{2}{3}\right)^n$$

$$(1+1)^n$$

$$2^n = (1+1)^n = 1 + n + \binom{n}{2} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n}$$

$$(1 - 1)^n$$

$$0 = (1-1)^n = 1 - n + \binom{n}{2} - \cdots + (-1)^k \binom{n}{k} + \cdots + (-1)^n \binom{n}{n}$$

$$\left(\frac{1}{3}+\frac{2}{3}\right)^n$$

$$1 = \left(\frac{1}{3} + \frac{2}{3}\right)^n = \frac{2^n}{3^n} + n\frac{2^{n-1}}{3^n} + \binom{n}{2}\frac{2^{n-2}}{3^n} + \cdots + \binom{n}{k}\frac{2^{n-k}}{3^n} + \cdots + \binom{n}{n}\frac{1}{3^n}$$

— or —

$$3^n = 2^n + n2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{k}2^{n-k} + \cdots + \binom{n}{n}$$

$$(p+q)^n = \binom{n}{n}p^n + \binom{n}{n-1}p^{n-1}q + \cdots + \binom{n}{1}pq^{n-1} + \binom{n}{0}q^n = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$$

$$2^n = 1 + n + \binom{n}{2} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n}$$

$$0 = 1 - n + \binom{n}{2} - \cdots + (-1)^k\binom{n}{k} + \cdots + (-1)^n\binom{n}{n}$$

$$3^n = 2^n + n2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{k}2^{n-k} + \cdots + \binom{n}{n}$$