Algebra
Applied Mathematics
Calculus and Analysis
Discrete Mathematics
Foundations of Mathematics
Geometry
History and Terminology
Number Theory
Probability and Statistics
Recreational Mathematics
Topology
Alphabetical Index
New in MathWorld

Hermite-Gauss Quadrature

Hermite-Gauss quadrature, also called Hermite quadrature, is a Gaussian quadrature over the interval $(-\infty, \infty)$ with weighting function $W(x) = e^{-x^2}$ (Abramowitz and Stegun 1972, p. 890). The abscissas for quadrature order n are given by the roots x_i of the Hermite polynomials $H_n(x)$, which occur symmetrically about 0. The weights are

$$w_i = -\frac{A_{n+1}\gamma_n}{A_n H_n'(x_i) H_{n+1}(x_i)} \tag{1}$$

$$= \frac{A_n}{A_{n-1}} \frac{\gamma_{n-1}}{H_{n-1}(x_i) H_n'(x_i)}, \tag{2}$$

where A_n is the coefficient of x^n in $H_n(x)$. For Hermite polynomials,

$$A_n = 2^n, \tag{3}$$

so

$$\frac{A_{n+1}}{A_n} = 2. \tag{4}$$

Additionally,

$$\gamma_n = \sqrt{\pi}\, 2^n\, n!, \tag{5}$$

so

$$w_i = -\frac{2^{n+1}\, n!\, \sqrt{\pi}}{H_{n+1}(x_i) H_n'(x_i)} \tag{6}$$

$$= \frac{2^n (n-1)!\, \sqrt{\pi}}{H_{n-1}(x_i) H_n'(x_i)} \tag{7}$$

$$= \frac{2^{n+1}\, n!\, \sqrt{\pi}}{[H_n'(x_i)]^2} \tag{8}$$

$$= \frac{2^{n+1}\, n!\, \sqrt{\pi}}{[H_{n+1}(x_i)]^2} \tag{9}$$

$$= \frac{2^{n-1}\, n!\, \sqrt{\pi}}{n^2 [H_{n-1}(x_i)]^2}, \tag{10}$$

where (8) and (9) follow using the recurrence relation

$$H_n'(x) = 2n H_{n-1}(x) = 2x H_n(x) - H_{n+1}(x) \tag{11}$$

to obtain

$$H_n'(x_i) = 2n H_{n-1}(x_i) = -H_{n+1}(x_i), \tag{12}$$

and (10) is from Abramowitz and Stegun (1972 p. 890).

The error term is

$$E = \frac{n!\, \sqrt{\pi}}{2^n (2n)!} f^{(2n)}(\xi). \tag{13}$$

Beyer (1987) gives a table of abscissas and weights up to $n = 12$.

n	x_i	w_i
2	± 0.707107	0.886227
3	0	1.18164
	± 1.22474	0.295409
4	± 0.524648	0.804914
	± 1.65068	0.0813128
5	0	0.945309
	± 0.958572	0.393619
	± 2.02018	0.0199532

The abscissas and weights can be computed analytically for small n.

n	x_i	w_i
2	$\pm \frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{\pi}$
3	0	$\frac{2}{3}\sqrt{\pi}$
	$\pm \frac{1}{2}\sqrt{6}$	$\frac{1}{6}\sqrt{\pi}$
4	$\pm \sqrt{\frac{3-\sqrt{6}}{2}}$	$\frac{\sqrt{\pi}}{4(3-\sqrt{6})}$
	$\pm \sqrt{\frac{3+\sqrt{6}}{2}}$	$\frac{\sqrt{\pi}}{4(3+\sqrt{6})}$

More things to try: approximate zero | area between y=sinc(x) and the x-axis from x=-4pi to 4pi | factoradic form of the permutation (3 1 2 5 4)

REFERENCES

Abramowitz, M. and Stegun, I. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 890, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 464, 1987.

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp. 327-330, 1956.

REFERENCED ON WOLFRAM|ALPHA

Hermite-Gauss Quadrature

CITE THIS AS:

Weisstein, Eric W. "Hermite-Gauss Quadrature." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/Hermite-GaussQuadrature.html

SUBJECT CLASSIFICATIONS

Applied Mathematics › Numerical Methods › Root-Finding ›