Feedback — Homework 3

Help Center

Thank you. Your submission for this homework was received.

You submitted this homework on **Mon 23 Mar 2015 2:16 AM PDT**. You got a score of **7.00** out of **7.00**.

Question 1

Suppose $n \geq 2$ and the events $A_1, A_2, \cdots, A_{n-1}, A_n$ partition the sample space Ω into n sets of positive probability. Which (one or more) of the following statements is true for *every* choice of event B of positive probability?

a. $\mathbf{P}(A_1 \mid B)\,\mathbf{P}(B) = \mathbf{P}(B \mid A_1)\,\mathbf{P}(A_1)$
b. $\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \mid A_i)$
c. $\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \mid A_i)\,\mathbf{P}(A_i)$
d. $\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \cap A_i)\,\mathbf{P}(A_i)$
e. $\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \cup A_i)$
f. $\mathbf{P}(A_1 \mid B) \sum_{i=1}^{n} \mathbf{P}(B \mid A_i)\,\mathbf{P}(A_i) = \mathbf{P}(B \mid A_1)\,\mathbf{P}(A_1)$

Your Answer		Score	Explanation
a, c, and f	✔	1.00	
b and d			
a and c			
d and f			
a, e, and f			
a and e			
Total		1.00 / 1.00	

Question Explanation

We analyse each item separately:

a. True. Exploiting the definition of conditional probability, the left hand side becomes $\mathbf{P}(A_1 \mid B) \cdot \mathbf{P}(B) = \mathbf{P}(A_1 \cap B)$. Similarly, the right hand side reduces to $\mathbf{P}(B \mid A_1) \cdot \mathbf{P}(A_1) = \mathbf{P}(B \cap A_1)$. Therefore, $\mathbf{P}(A_1 \mid B) \cdot \mathbf{P}(B) = \mathbf{P}(B \mid A_1) \cdot \mathbf{P}(A_1) = \mathbf{P}(A_1 \cap B)$.

b. False. Since $\{A_1, \cdots, A_n\}$ is a partition of the sample space, by additivity and normalisation of probability measure, we know that

$$\sum_{i=1}^{n} \mathbf{P}(A_i) = \mathbf{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbf{P}(\Omega) = 1.$$

[See Lecture 5: j]. By boundedness of probability measure [Lecture 5: g] we know that each of the A_i has probability bounded between 0 and 1 and, in particular, none of the A_i can have probability exceeding one. But we can deduce a little more: as we given that $n \geq 2$, we may conclude indeed that $\mathbf{P}(A_i)$ can be equal to 1 for at most one of the A_i (else normalisation would be violated). By the theorem of total probability [Lecture 8.2: g] we conclude that

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \mid A_i) \cdot \mathbf{P}(A_i) < \sum_{i=1}^{n} \mathbf{P}(B \mid A_i).$$

c. True. This is the theorem of total probability. See b.

d. False. By monotonicity,

$$\mathbf{P}(B \cap A_i) = \mathbf{P}(B \mid A_i) \cdot \mathbf{P}(A_i) \leq \mathbf{P}(B \mid A_i)$$

where the inequality is strict for all except at most one A_i (see b.). By the theorem of total probability, it follows that

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \mid A_i) \cdot \mathbf{P}(A_i) > \sum_{i=1}^{n} \mathbf{P}(B \cap A_i) \cdot \mathbf{P}(A_i).$$

e. False. By monotonicity of probability measure [Lecture 5: g], we know that

$$\mathbf{P}(B \cup A_i) \geq \mathbf{P}(B \cap A_i),$$

and, as $n \geq 2$, equality can hold for at most one A_i. By the theorem of total probability once more, it follows that

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \mid A_i) \cdot \mathbf{P}(A_i) = \sum_{i=1}^{n} \mathbf{P}(B \cap A_i) < \sum_{i=1}^{n} \mathbf{P}(B \cup A_i).$$

f. True. By the analysis given for part b., the left-hand side becomes $\mathbf{P}(A_1 \mid B) \cdot \mathbf{P}(B)$ which is equal to $\mathbf{P}(A_1 \cap B)$. The right hand side is $\mathbf{P}(B \mid A_1) \cdot \mathbf{P}(A_1)$ which is equal to $\mathbf{P}(B \cap A_1)$. As intersection is a commutative operation we conclude that

$$\mathbf{P}(A_1 \mid B) \sum_{i=1}^{n} \mathbf{P}(B \mid A_i)\, \mathbf{P}(A_i) = \mathbf{P}(B \mid A_1)\, \mathbf{P}(A_1)$$

as asserted.

Question 2

A given university in a particular country has an equal number of male and female students. You are told that 30% of the male students and 25% of the female students are international students. A student is chosen at random for a campus interview on their experiences in assimilating into the university culture. If the selected student happens to be an international student what is the probability that the student is female?

Your Answer		Score	Explanation
○ $\frac{3}{10}$			
○ $\frac{1}{4}$			
○ $\frac{6}{11}$			
◉ $\frac{5}{11}$	✔	1.00	
○ $\frac{1}{8}$			
○ $\frac{3}{4}$			
Total		1.00 / 1.00	

Question Explanation

The sample space is the space of students, and the sets of male and female students partitions it. We are given a combinatorial setting where all selections of student are equally likely. Denote by F and I the set of female students and the set of international students, respectively. We are given that the fraction of international students on campus is $0.3 \times 0.5 + 0.25 \times 0.5 = 0.275$. For random selection we have $\mathbf{P}(F) = 0.5$ and $\mathbf{P}(I) = 0.275$. Given that the selected student is international, the sample space reduces to the set I and conditional probability comes into play. By Bayes's rule [Lecture 8.2: k],

$$\mathbf{P}(F \mid I) = \frac{\mathbf{P}(I \mid F)\, \mathbf{P}(F)}{\mathbf{P}(I)} = \frac{0.25 \times 0.5}{0.275} = \frac{5}{11}.$$

Question 3

Alice has two eight-sided dice. Her first die, which we call die A, has five red faces and three white faces. Her second die, which we creatively call die B, has three red faces and five white faces. She chooses one of the two dice at random, and then throws the selected die repeatedly. If a red face shows on each of the first two throws, what is the probability of obtaining a red face on the third throw?

Your Answer		Score	Explanation
○ $\frac{17}{32}$			
○ $\frac{5}{8}$			
◉ $\frac{19}{34}$	✔	1.00	
○ $\frac{25}{64}$			
○ $\frac{1}{2}$			
○ $\frac{19}{128}$			
Total		1.00 / 1.00	

Question Explanation

This is slight variation on the themes of Lectures 8.2: b, 8.2: h and 8.2: i. It is not any harder to do the problem in a more general setting and accordingly denote by R_n the event that the nth throw of the selected dies shows red. Observe that the inclusion relation $R_{n+1} \subseteq R_n$ is valid for each n as, if the first $n+1$ throws have shown red face, then surely the first n throws have shown red faces. It follows that $R_{n+1} \cap R_n = R_{n+1}$ and so

$$\mathbf{P}(R_{n+1} \mid R_n) = \frac{\mathbf{P}(R_{n+1} \cap R_n)}{\mathbf{P}(R_n)} = \frac{\mathbf{P}(R_{n+1})}{\mathbf{P}(R_n)}$$

for each $n \geq 1$. In order to evaluate the probability of n red faces in a row, introduce the ancillary event A that the first die is selected. Then, by total probability,

$$\mathbf{P}(R_n) = \mathbf{P}(R_n \mid A)\,\mathbf{P}(A) + \mathbf{P}(R_n \mid A^\complement)\,\mathbf{P}(A^\complement) = \left(\frac{5}{8}\right)^n \cdot \frac{1}{2} + \left(\frac{3}{8}\right)^n \cdot \frac{1}{2} = \frac{1}{2}\left[\left(\frac{5}{8}\right)^n + \left(\frac{3}{8}\right)^n\right]$$

as, *conditioned on the selection of the die*, the successive throws constitute independent trials with success probabilities (chance of red) given by $5/8$ and $3/8$ for the first die and the second die, respectively. It follows that

$$\mathbf{P}(R_{n+1} \mid R_n) = \frac{\frac{1}{2}\left[\left(\frac{5}{8}\right)^{n+1} + \left(\frac{3}{8}\right)^{n+1}\right]}{\frac{1}{2}\left[\left(\frac{5}{8}\right)^{n} + \left(\frac{3}{8}\right)^{n}\right]} = \frac{\left(\frac{5}{8}\right)^{n+1} + \left(\frac{3}{8}\right)^{n+1}}{\left(\frac{5}{8}\right)^{n} + \left(\frac{3}{8}\right)^{n}}.$$

Evaluating the right-hand side for $n = 2$ yields the solution for the given problem:

$$\mathbf{P}(R_3 \mid R_2) = \frac{\left(\frac{5}{8}\right)^{3} + \left(\frac{3}{8}\right)^{3}}{\left(\frac{5}{8}\right)^{2} + \left(\frac{3}{8}\right)^{2}} = \frac{19}{34}.$$

Question 4

The following prompt should be used for **Questions 4, 5, and 6:**

A man possesses five coins: two coins are double-headed, one coin is double-tailed, and two coins are normal (one side head and one side tail). He shuts his eyes, picks a coin at random, and tosses it.

What is the probability that the lower face of the tossed coin is a head? (His eyes are still closed.)

Your Answer		Score	Explanation
$\frac{1}{5}$			
$\frac{2}{5}$			
$\frac{3}{5}$	✔	1.00	
$\frac{3}{10}$			
$\frac{1}{2}$			
$\frac{4}{5}$			
Total		1.00 / 1.00	

Question Explanation

Each sample point requires the specification of a coin and the results of the first two tosses. Let's begin by introducing notation for the events of relevance:

- HH := the coin selected is double-headed.
- TT := the coin selected is double-tailed.
- N := the coin selected is normal.
- L_1 := the lower face of the selected coin shows heads after the first toss.
- U_1 := the upper face of the selected coin shows heads after the first toss.

We will need the event U_1 for the next problem and it is convenient to introduce the notation now.
The problem conditions allow us to infer the following probabilities:

$$\mathbf{P}(HH) = \frac{2}{5}, \qquad \mathbf{P}(TT) = \frac{1}{5}, \qquad \mathbf{P}(N) = \frac{2}{5},$$
$$\mathbf{P}(L_1 \mid HH) = \mathbf{P}(U_1 \mid HH) = 1, \qquad \mathbf{P}(L_1 \mid TT) = \mathbf{P}(U_1 \mid TT) = 0, \qquad \mathbf{P}(L_1 \mid N) = \mathbf{P}(U_1 \mid N) = \frac{1}{2}$$
$$\mathbf{P}(L_1 \cap U_1 \mid HH) = 1, \qquad \mathbf{P}(L_1 \cap U_1 \mid TT) = 0, \qquad \mathbf{P}(L_1 \cap U_1 \mid N) = 0.$$

The events $\{HH, TT, N\}$ partition the sample space and so, by the theorem of total probability [Lecture 8.2: g], we obtain

$$\mathbf{P}(L_1) = \mathbf{P}(L_1 \mid HH)\,\mathbf{P}(HH) + \mathbf{P}(L_1 \mid TT)\,\mathbf{P}(TT) + \mathbf{P}(L_1 \mid N)\,\mathbf{P}(N) = 1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}.$$

Question 5

Continuation of **Question 4**: He opens his eyes and observes that the upper face of the coin that was tossed shows heads. What can we now say about the probability that the lower face of the coin is a head as well?

Your Answer		Score	Explanation
$\frac{2}{5}$			
$\frac{3}{5}$			
$\frac{1}{5}$			
$\frac{2}{3}$	✔	1.00	
$\frac{3}{4}$			
Total		1.00 / 1.00	

Question Explanation

In the notation introduced in the solution to Question 3, we are asked to compute the conditional probability of L_1 given that U_1 has occurred. The setting is ripe for two applications of the theorem of total probability:

$$\mathbf{P}(L_1 \mid U_1) = \frac{\mathbf{P}(L_1 \cap U_1)}{\mathbf{P}(U_1)} = \frac{\mathbf{P}(L_1 \cap U_1 \mid HH)\,\mathbf{P}(HH) + \mathbf{P}(L_1 \cap U_1 \mid TT)\,\mathbf{P}(TT) + \mathbf{P}(L_1 \cap U_1 \mid N)\,\mathbf{P}(N)}{\mathbf{P}(U_1 \mid HH)\,\mathbf{P}(HH) + \mathbf{P}(U_1 \mid TT)\,\mathbf{P}(TT) + \mathbf{P}(U_1 \mid N)\,\mathbf{P}(N)} = \frac{1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + 0 \cdot \frac{2}{5}}{1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5}} = \frac{2}{3}.$$

Question 6

Continuation of **Question 5**: He now closes his eyes again, picks up the coin that was tossed, and tosses it a second time with eyes still closed. What is the probability that the lower face is a head after the second toss?

Your Answer		Score	Explanation
$\frac{3}{5}$			
$\frac{2}{3}$			
$\frac{1}{2}$			
$\frac{4}{5}$			
$\frac{5}{6}$	✔	1.00	
$\frac{7}{12}$			
Total		1.00 / 1.00	

Question Explanation

We will need some more notation to capture the result of the second toss of the coin. In keeping with the notation introduced in **Question 4** identify the event:

- L_2 := the lower face of the selected coin is heads after the second toss.

What additional information is available to us now? When the coin is tossed a second time we now know that the upper face of the selected coin (whichever it is) showed heads after the first toss. In other words, we are given that the event U_1 has occurred and so we are being asked to evaluate the conditional probability $\mathbf{P}(L_2 \mid U_1)$. The key observation is that successive tosses are *conditionally independent once the coin is selected*. Accordingly, we may infer the following conditional probabilities:

$$\mathbf{P}(L_2 \mid HH) = 1, \qquad \mathbf{P}(L_2 \mid TT) = 0, \qquad \mathbf{P}(L_2 \mid N) = \frac{1}{2},$$
$$\mathbf{P}(L_2 \cap U_1 \mid HH) = 1 \times 1 = 1, \qquad \mathbf{P}(L_2 \cap U_1 \mid TT) = 0 \times 0 = 0, \qquad \mathbf{P}(L_2 \cap U_1 \mid N) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

The stage is set for another application of total probability:

$$\mathbf{P}(L_2 \mid U_1) = \frac{\mathbf{P}(L_2 \cap U_1)}{\mathbf{P}(U_1)} = \frac{\mathbf{P}(L_2 \cap U_1 \mid HH)\,\mathbf{P}(HH) + \mathbf{P}(L_2 \cap U_1 \mid TT)\,\mathbf{P}(TT) + \mathbf{P}(L_2 \cap U_1 \mid N)\,\mathbf{P}(N)}{\mathbf{P}(U_1 \mid HH)\,\mathbf{P}(HH) + \mathbf{P}(U_1 \mid TT)\,\mathbf{P}(TT) + \mathbf{P}(U_1 \mid N)\,\mathbf{P}(N)} = \frac{1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{2}{5}}{1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5}} = \frac{5}{6}.$$

The intuition? Once we have seen a head on the first toss, it is conditionally more likely that we are dealing with the double-headed coin and, accordingly, the chance of a head increases.

Question 7

A fair coin is tossed repeatedly. Write T for tails and H for heads and list the outcomes of the tosses sequentially starting with the result of the first toss. As you proceed sequentially through the tosses, what is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H?

Your Answer		Score	Explanation
$\frac{11}{16}$			
$\frac{27}{32}$			
$\frac{1}{2}$			
$\frac{7}{8}$			
$\frac{15}{16}$	✔	1.00	
$\frac{7}{32}$			
Total		1.00 / 1.00	

Question Explanation

Suppose the pattern H, H, H, H occurs for the first time at trial n. If $n = 4$ then the first occurrence of the pattern H, H, H, H certainly precedes that of T, H, H, H. If $n > 4$ then it must be the case that the outcome of the $(n-4)$th trial must be T (else the pattern H, H, H, H would already have been seen on trial $n-1$). But then the pattern T, H, H, H will have been seen at trial $n-1$. It follows that the first occurrence of T, H, H, H precedes that of H, H, H, H whenever the first four trials do not all result in heads. Or, equivalently, the pattern H, H, H, H precedes the pattern T, H, H, H if, and only if, the first four tosses all result in heads. By independent trials, the probability that the first four trials all result in heads is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16},$$

and this hence represents the probability that the pattern H, H, H, H precedes T, H, H, H in the sequence. By additivity, the probability the pattern T, H, H, H occurs before the pattern H, H, H, H is hence

$$1 - \frac{1}{16} = \frac{15}{16}.$$