On how to prove that two sets are equal

On how to prove set equality

$A = B$ means:

(1) every ω in A is also in B (in notation, $A \subseteq B$), and

(2) every ω in B is also in A (in notation, $B \subseteq A$).

To prove set equality it suffices to show that each set is a subset of the other.

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- Suppose ω is in $A \cup (B \cap C)$.
- Then ω is in A or ω is in $B \cap C$.
  - If ω is in A then ω lies in both $A \cup B$ and $A \cup C$.
  - If ω is in $B \cap C$ then ω lies in both B and C and, again, it must lie in both $A \cup B$ and $A \cup C$.
- Then ω is in both $A \cup B$ and $A \cup C$.
- And hence ω is in $(A \cup B) \cap (A \cup C)$.

As every ω in $A \cup (B \cap C)$ is also in $(A \cup B) \cap (A \cup C)$, it follows that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

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Next verify that $(A \cup B) \cap (A \cup C)$ is a subset of $A \cup (B \cap C)$ by working the argument in reverse.

- Suppose ω is in $(A \cup B) \cap (A \cup C)$.
- Then ω is in both $A \cup B$ and $A \cup C$.
- Hence ω is in A or ω is in both B and C, that is to say, in $B \cap C$.
  - If ω is in A then ω certainly lies in $A \cup (B \cap C)$.
  - If ω is in $B \cap C$ then, again, it must lie in $A \cup (B \cap C)$.
- In either case it follows that ω is in $A \cup (B \cap C)$.

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As every ω in $(A \cup B) \cap (A \cup C)$ is also in $A \cup (B \cap C)$, it follows that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

To test your understanding of the process, try to prove De Morgan's laws in this fashion.