Line Integrals and Green's Theorem

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1 Vector Fields (or vector valued functions)

Vector notation. In 18.04 we will mostly use the notation $(v) = (a, b)$ for vectors. The other common notation $(v) = a\mathbf{i} + b\mathbf{j}$ runs the risk of $\mathbf{i}$ being confused with $i = \sqrt{-1}$ –especially if I forget to make $\mathbf{i}$ boldfaced.

Definition. A vector field (also called called a vector-valued function) is a function $\mathbf{F}(x, y)$ from $\mathbf{R}^2$ to $\mathbf{R}^2$. That is,

$$\mathbf{F}(x, y) = (M(x, y),\ N(x, y)),$$

where M and N are regular functions on the plane. In standard physics notation

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = (M, N)\,.$$

Algebraically, a vector field is nothing more than two ordinary functions of two variables.

Example GT.1. Here are a number of standard examples of vector fields.

(a.1) Force: constant gravitational field $\mathbf{F}(x, y) = (0, -g)$.

(a.2) Velocity:

$$\mathbf{V}(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) = \left(\frac{x}{r^2}, \frac{y}{r^2}\right).$$

(Here r is our usual polar r.) It is a radial vector field, i.e. it points radially away from the origin. It is a shrinking radial field –like water pouring from a source at (0,0).

This vector field exhibits another important feature for us: it is not defined at the origin because the denominator becomes zero there. We will say that $\mathbf{V}$ has a singularity at the origin.

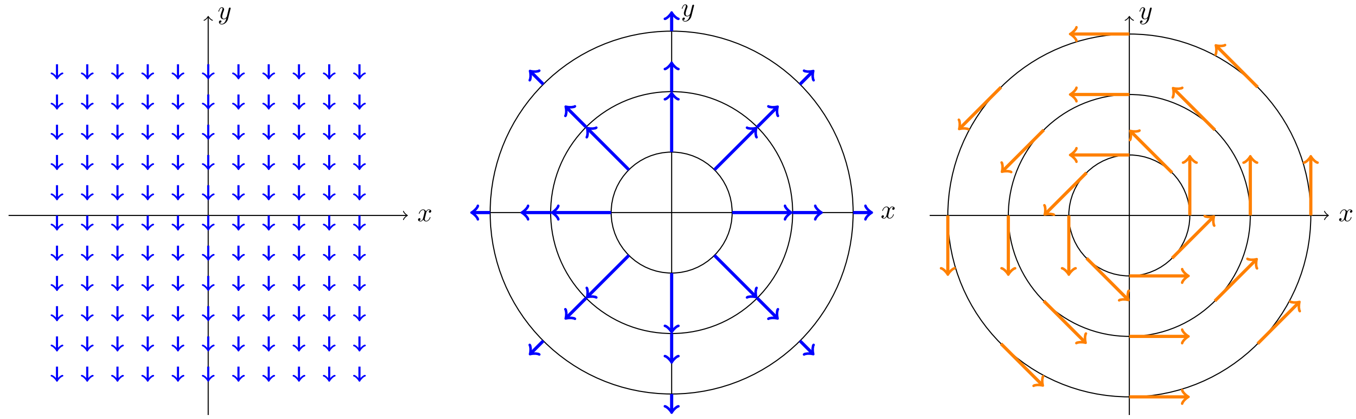
(a.3) Unit tangential field: $\mathbf{F} = (-y, x)/r$. Tangential means tangent to circles centered at the origin. We know it is tangential because it is orthogonal to the radial vector field in (a.2). $\mathbf{F}$ also has a singularity at the origin. We

(a.4) Gradient field: $\mathbf{F} = \nabla f$, e.g., $f(x, y) = xy^2 \Rightarrow \nabla f = \left(y^2, 2xy\right)$.

1.1 Visualization of vector fields

This can be summarized as: draw little arrows in the plane. More specifically, for a field $\mathbf{F}$, at each of a number of points (x, y) draw the vector $\mathbf{F}(x, y)$

Example GT.2. Sketch the vector fields, (a.1), (a.2) and (a.3) from the previous example.



(a.1) Constant vector field (a.2) Shrinking radial field (a.3) Unit tangential field

2 Definition and computation of line integrals along a parametrized curve

Line integrals are also called path or contour integrals.

We need the following ingredients:

A vector field $\mathbf{F}(x, y) = (M, N)$

A parametrized curve C: $\mathbf{r}(t) = (x(t),\, y(t))$, with t running from a to b.

Note: since $\mathbf{r} = (x, y)$, we have $d\mathbf{r} = (dx,\, dy)$.

Definition. The line integral of $\mathbf{F}$ along C is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (M, N) \cdot (dx, dy) = \int_C M\,dx + N\,dy.$$

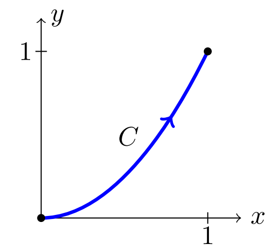
Comment: The notation $\mathbf{F} \cdot d\mathbf{r}$ is common in physics and $M\,dx + N\,dy$ in thermodynamics. (Though everyone uses both notations.)

We'll see what these notations mean in practice with some examples.

Example GT.3. Let $\mathbf{F}(x, y) = \big(x^2 y,\, x - 2y\big)$ and let C be the curve $\mathbf{r}(t) = \big(t,\, t^2\big)$, with t running from 0 to 1. Compute the line integral $I = \displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Do this first using the notation $\displaystyle\int_C M\,dx + N\,dy$. Then repeat the computation using the notation $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

<u>**answer:**</u> First we draw the curve, which is the part of the parabola $y = x^2$ running from $(0, 0)$ to $(1, 1)$.



(i) Using the notation $\int_C M\,dx + N\,dy$.

We have $\mathbf{r} = (x, y)$, so $x = t$, $y = t^2$. In this notation $\mathbf{F} = (M,\, N)$, so $M = x^2 y$ and $N = x - 2y$.

We put everything in terms of t:

$$\begin{aligned} dx &= dt \\ dy &= 2t\,dt \\ M &= (t^2)(t^2) = t^4 \\ N &= t - 2t^2 \end{aligned}$$

Now we can put all of these in the integral. Since t runs from 0 to 1, these are our limits.

$$I = \int_C M\,dx + N\,dy = \int_0^1 t^4\,dt + (t - 2t^2)2t\,dt = \int_0^1 t^4 + 2t^2 - 4t^3\,dt = -\frac{2}{15}.$$

(ii) Using the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Again, we have to put everything in terms of t:

$$\begin{aligned} \mathbf{F} &= (M,\, N) = \left(t^4,\, t - 2t^2\right) \\ \frac{d\mathbf{r}}{dt} &= (1,\, 2t)\,, \quad \text{so } d\,\mathbf{r} = \frac{d\mathbf{r}}{dt}\,dt = (1,\, 2t)\ dt \end{aligned}$$

Thus, $\mathbf{F} \cdot d\mathbf{r} = \left(t^4,\, t - 2t^2\right) \cdot (1,\, 2t)\ dt = t^4 + (t - 2t^2)2t\,dt$. So the integral becomes

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t^4 + (t - 2t^2)2t\,dt.$$

This is exactly the same integral as in method (i).

3 Work done by a force along a curve

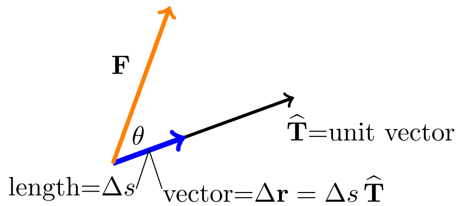
Having seen that line integrals are not unpleasant to compute, we will now try to motivate our interest in doing so. We will see that the work done by a force moving a body along a path is naturally computed as a line integral.

Similar to integrals we've seen before, the work integral will be constructed by dividing the path into little pieces. The work on each piece will come from a basic formula and the total work will be the 'sum' over all the pieces, i.e. an integral.

3.1 Basic formula: work done by a constant force along a small line

We'll start with the simplest situation: a constant force $\mathbf{F}$ pushes a body a distance Δs along a straight line. Our goal is to compute the work done by the force.

The figure shows the force $\mathbf{F}$ which pushes the body a distance Δs along a line in the direction of the unit vector $\widehat{\mathbf{T}}$. The angle between the force $\mathbf{F}$ and the direction $\widehat{\mathbf{T}}$ is θ.



We know from physics that the work done by the force on the body is the component of the force in the direction of motion times the distance moved. That is,

$$\text{work} = |\mathbf{F}|\cos(\theta)\,\Delta s$$

We want to phrase this in terms of vectors. Since $|\widehat{\mathbf{T}}| = 1$ we know $\mathbf{F}\cdot\widehat{\mathbf{T}} = |\mathbf{F}|\cos(\theta)$. Using this in the formula for work we have

$$\text{work} = \mathbf{F}\cdot\widehat{\mathbf{T}}\,\Delta s. \tag{1}$$

Equation 1 is important and we will see it again. For now, we want to make one more substitution. We'll call the vector $\Delta s\,\widehat{\mathbf{T}} = \Delta\mathbf{r}$. This is the displacement of the body. (Note, it is essentially the same as our formula $\frac{ds}{dt}\widehat{\mathbf{T}} = \frac{d\mathbf{r}}{dt}$.) Using this, Equation 1 becomes

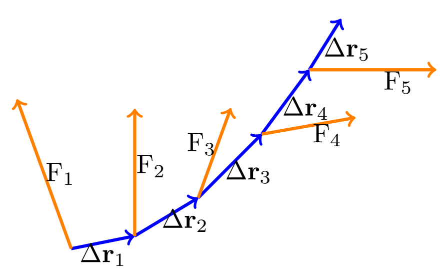
$$\text{work} = \mathbf{F}\cdot\Delta\mathbf{r}. \tag{2}$$

This is the basic work formula that we'll use to compute work along an entire curve

3.2 Work done by a variable force along an entire curve

Now suppose a variable force $\mathbf{F}$ moves a body along a curve C. Our goal is to compute the total work done by the force.

The figure shows the curve broken into 5 small pieces, the jth piece has displacement $\Delta\mathbf{r_j}$. If the pieces are small enough, then the force on the jth piece is approximately constant. This is shown as $\mathbf{F}_j$.



Also, if the pieces are small enough, then each segment is approximately a straight line and the force is approximately constant. So we can apply our basic formula for work and approximate the work done by the force moving the body along the jth piece as

$$\Delta W_j \approx \mathbf{F}_j \cdot \Delta \mathbf{r}_j.$$

The total work is the sum of the work over each piece.

$$\text{total work } = \sum \Delta W_j \approx \sum \mathbf{F}_j \cdot \Delta \mathbf{r}_j.$$

Now, as usual, we let the pieces get infinitesimally small, so the sum becomes an integral and the approximation becomes exact. We get:

$$\text{total work } = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

The subscript C indicates that it is the curve that has been split into pieces. That is, the total work is computed as a line integral of the force over the curve C!

4 Grad, curl and div

Gradient. For a function $f(x, y)$: $\text{grad} f = \boldsymbol{\nabla} f = (f_x, f_y)$.

Curl. For a vector in the plane $\mathbf{F}(x, y) = (M(x, y), N(x, y))$ we define

$$\text{curl}\mathbf{F} = N_x - M_y.$$

NOTE. This is a scalar. In general, the curl of a vector field is another vector field. For vectors fields in the plane the curl is always in the $\widehat{k}$ direction, so we simply drop the $\widehat{k}$ and make curl a scalar. Sometimes it is called the 'baby curl'.

Divergence. The divergence of the vector field $\mathbf{F} = (M, N)$ is

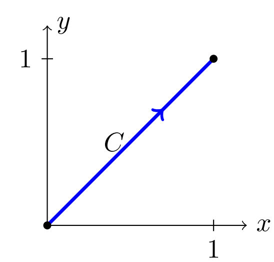
$$\text{div}\mathbf{F} = M_x + N_y.$$

5 Properties of line integrals

In this section we will uncover some properties of line integrals by working some examples.

Example GT.4. First look back at the value found in Example GT.3. Now, use the same vector field as in that example, but, in this case, let C be the straight line from $(0, 0)$ to $(1, 1)$, i.e. same endpoints, but different path. Compute the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

<u>**answer:**</u> As always, start by sketching the curve:



We'll use the notation $\int_C M\,dx + N\,dy$.

Parametrize the curve: $x = t, \quad y = t, \quad$ with t from 0 to 1.

Put everything in terms of t:

$$\begin{aligned} dx &= dt \\ dy &= dt \\ M &= x^2 y = t^3 \\ N &= x - 2y = -t \end{aligned}$$

Now we put this into the integral

$$I = \int_C M\,dx + N\,dy = \int_0^1 t^3\,dt - t\,dt = \int_0^1 t^3 - t\,dt = -\frac{1}{4}.$$

This is a different value from Example GT.3, which leads to the important principle:

Important principle for line integrals. Line integrals over two different paths with the same endpoints may be different.

Example GT.5. Again, look back at the value found in Example GT.3. Now, use the same vector field and curve as Example GT.3 except use the following (different) parametrization of C.

$$x = \sin(t), \quad y = \sin^2(t); \quad 0 \le t \le \pi/2.$$

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

<u>**answer:**</u> We won't sketch the curve it is identical to the one in Example GT.3. Putting everything in terms of t we have

$$\begin{aligned} dx &= \cos(t)\,dt \\ dy &= 2\sin(t)\cos(t)\,dt \\ M &= x^2 y = \sin^2(t)\sin^2(t) = \sin^4(t) \\ N &= x - 2y = \sin(t) - 2\sin^2(t) \end{aligned}$$

We put these in the integral $I = \int_C M\,dx + N\,dy$ and compute

$$\begin{aligned} I &= \int_0^{\pi/2} \sin^4(t)\cos(t)\,dt + (\sin(t) - 2\sin^2(t))2\sin(t)\cos(t)\,dt \\ &= \int_0^{\pi/2} \left(\sin^4(t) + 2\sin^2(t) - 4\sin^3(t)\right)\cos(t)\,dt \\ &\quad \text{(Let } u = \sin(t),\ du = \cos(t)\,dt.) \\ &= \int_0^1 u^4 + 2u^2 - 4u^3\,du \\ &= -\frac{2}{15}. \end{aligned}$$

This is the same value we got in Example GT.3! In fact, the u substitution led to exactly the same integral! This leads us to the important principle:

Important principle for line integrals. The parametrization of the curve doesn't affect the value of line the integral over the curve.

You should note that our work with work make this reasonable, since we developed the line integral abstractly, without any reference to a parametrization.

5.1 List of properties of line integrals

1. Independent of parametrization: The value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the parametrization of C.

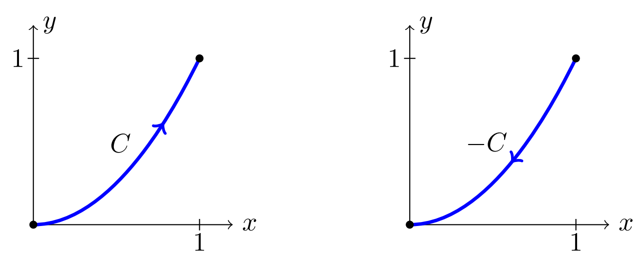
2. Reversing direction on the curve changes the sign: If C is a curve, then we write $-C$ for the same curve traversed in the opposite direction. In this case

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(See the next example.)

Example GT.6. Let C be the curve from Example GT.3. Sketch C and $-C$ and give a parametrization of $-C$.

answer: C follows the parabola $y = x^2$ from (0,0) to (1,1), so the curve $-C$ covers the same section of the parabola, but goes from (1,1) to (0,0), i.e. we reversed the direction of the arrow.



C goes from (0,0) to (1,1) $-C$ goes from (1,1) to (0,0)

The curve C can be parametrized as $\mathbf{r}(t) = \left(t,\, t^2\right)$, with t running from 0 to 1. The easiest way to reverse this is to have t run from 1 to 0

With this parametrization the t limits on the integral are reversed, which, we know from 18.01, changes the sign of the integral.

If you insist on an increasing parameter, we can parametrize $-C$ by

$$\mathbf{r}(u) = \left(1-u,\, (1-u)^2\right), \text{ with } u \text{ runnning from 0 to 1.}$$

3. (Intrinsic formula) We can write the line integral as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

where $\mathbf{T}$ = unit tangent vector to C and ds = differential of arclength.

Reason: We know from our work on parametrized curves that $\dfrac{d\mathbf{r}}{dt} = \mathbf{T}\dfrac{ds}{dt}$. So, $d\mathbf{r} = \mathbf{T}\,ds$. (A comparison of Equations 1 and 2 above, essentially shows the same thing.)

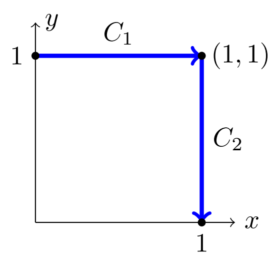
4. If C is a closed curve we use the notation

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C M\,dx + N\,dy.$$

The little circle on the integral sign indicates the curve is closed, i.e. starts and ends at the same point.

6 Rectangular and circular paths

Example GT.7. Evaluate $I = \displaystyle\int_C y\,dx + (x+2y)\,dy$ where C is the rectangular path from (0,1) to (1,0) shown below.



answer: The path C is given in two pieces labeled C_1 and C_2. This means we will have to split the integral into two pieces, i.e.

$$I = \int_C y\,dx + (x+2y)\,dy = \int_{C_1} y\,dx + (x+2y)\,dy + \int_{C_2} y\,dx + (x+2y)\,dy.$$

We'll do the integration one piece at a time. First, $\int_{C_1} y\,dx + (x+2y)\,dy$.

Parametrize C_1: We'll use x as the parameter:

$$x = x,\ y = 1,\ \text{ with } x \text{ running from 0 to 1}$$

Put everything in terms of x:

$$x = x, \quad y = 1, \quad dx = dx, \quad dy = 0, \quad M = y - 1, \quad N \text{ (skip, since } dy = 0).$$

Put this in the integral and compute:

$$\int_{C_1} M\,dx + N\,dy = \int_{C_1} M\,dx = \int_0^1 dx = 1.$$

Next, the integral over C_2.

Parametrize C_2: Use parameter y: $x = 1,\ y = y,\ \ y$ runs from 1 to 0.

Put everything in terms of y:

$$x = 1, \quad y = y, \quad dx = 0, \quad dy = dy, \quad M \text{ (skip, since } dx = 0), \quad N = x + 2y = 1 + 2y$$

Put this in the integral and compute

$$\int_{C_2} M\,dx + N\,dy = \int_{C_2} N\,dy = \int_1^0 1 + 2y\,dy = -2.$$

Adding, the pieces we have $\boxed{I = 1 - 2 = -1.}$

Shorthand. Because $dy = 0$ on C_1 and $dx = 0$ on C_2 we can write

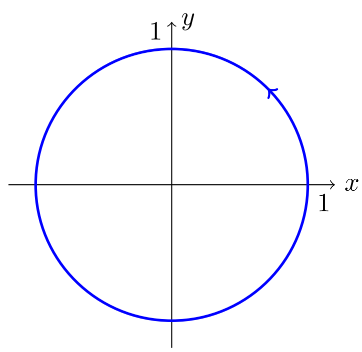
$$\int_{C_1+C_2} M\,dx + N\,dy = \int_{C_1} M\,dx + \int_{C_2} N\,dy.$$

Using the shorthand will save us some writing in the future.

Example GT.8. Evaluate $I = \oint_C -y\,dx + x\,dy$ where C is the unit circle traversed in a counterclockwise (CCW) direction.

<u>**answer:**</u> Parametrization: $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$. So, $dx = \cos(t)\,dt$, $dy = -\sin(t)\,dt$. We get

$$I = \int_0^{2\pi} -\sin t(-\sin t)\,dt + \cos t(\cos t)\,dt = \int_0^{2\pi} dt = 2\pi.$$



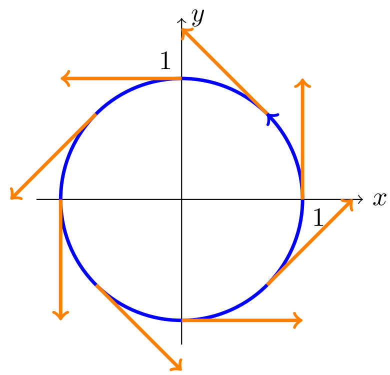
7 Some super-duper, really seriously important examples

In these examples we are going to integrate a tangential field around a closed loop. These will be key computations as we explore Green's theorem and gradient fields.

In the following r is the usual polar distance $r^2 = x^2 + y^2$.

Example GT.9. Let $\mathbf{F} = \left\langle -\frac{y}{r^2}, \frac{x}{r^2} \right\rangle$, and let C be the unit circle traversed in a counterclockwise (CCW) direction. Compute $I = \oint_C \mathbf{F} \cdot d\mathbf{r}$

<u>**answer:**</u> Sketch C and the vector field $\mathbf{F}$.



Parametrize C: $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$.

Put everything in terms of t: (Note, on the unit circle $r = 1$.)

$$dx = \cos(t)\,dt, \quad dy = -\sin(t)\,dt, \quad M = -\frac{y}{r^2} = -\sin(t), \quad N = \frac{x}{r^2} = \cos(t).$$

Put this in the integral and compute:

$$I = \int_0^{2\pi} -\sin(t)(-\sin(t))\,dt + \cos(t)(\cos(t))\,dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = \int_0^{2\pi} dt = 2\pi.$$

Example GT.10. Let $\mathbf{F}$ be the same as the previous example. Let C_2 be the unit circle centered on (2,0) traversed counterclockwise. Compute $I_2 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

answer: Parametrize C_2: $x = 2 + \cos(t)$, $y = \sin(t)$, t from 0 to 2π.

Put everything in terms of t: (Note, r^2 is not constant.)

$$\begin{aligned}
dx &= -\sin(t)\,dt \\
dy &= \cos(t)\,dt \\
r^2 &= x^2 + y^2 = (2 + \cos(t))^2 + \sin^2(t) = 5 + 4\cos(t) \\
M &= -\frac{y}{r^2} = -\frac{\sin(t)}{5 + 4\cos(t)} \\
N &= \frac{x}{r^2} = \frac{2 + \cos(t)}{5 + 4\cos(t)}
\end{aligned}$$

Put this in the integral:

$$I_2 = \int_{C_2} M\,dx + N\,dy = \int_0^{2\pi} \frac{\sin^2(t) + 2\cos(t) + \cos^2(t)}{5 + 4\cos(t)}\,dt = \int_0^{2\pi} \frac{1 + 2\cos(t)}{5 + 4\cos(t)}\,dt$$

Oy! We put this into Wolfram Alpha and found $I_2 = 0$.

Note. We should suspect that the value of 0 is no accident. This is true and we will see it easily once we learn Green's theorem. Avoiding actually computing an integral like this should be motivation enough for us to learn Green's theorem.

18.01 challenge. Compute the integral for I_2. Hints: You can use the substitution $u = \tan(t/2)$ and partial fractions. It's best to use symmetry and compute 2 times the integral from 0 to π.

8 Gradient and conservative fields

We will now focus on the important case where $\mathbf{F}$ is a gradient field. That is, for some function $f(x, y)$,

$$\mathbf{F} = \nabla f = (f_x, f_y).$$

Note. We will learn to call f a potential function for $\mathbf{F}$.

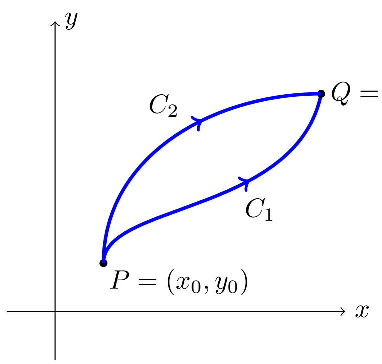
8.1 The fundamental theorem for gradient fields

Theorem GT.11. (fundamental theorem for gradient fields) Suppose that $\mathbf{F} = \nabla f$ is a gradient field and C is any path from point P to point Q. The fundamental theorem for vector fields says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x, y)|_P^Q = f(Q) - f(P) = f(x_1, y_1) - f(x_0, y_0). \tag{3}$$

where $P = (x_0, y_0)$ and $Q = (x_1, y_1)$.

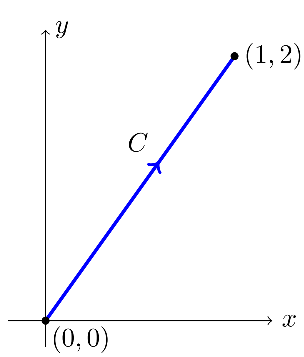
That is, for gradient fields the line integral depends only on the endpoints of the path and is independent of the path taken.



$$\text{If } \mathbf{F} = \nabla f \text{ then } \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P).$$

The proof of the fundamental theorem is given after the next example

Example GT.12. Let $f(x, y) = xy^3 + x^2$ and let C be the curve shown. Compute $\mathbf{F} = \nabla f$ and compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ two ways: (i) directly as a line integral, (ii) using the fundamental theorem.



answer: $\mathbf{F}(x,y) = \nabla f\,(f_x,\, f_y) = \left(y^3 + 2x,\, 3xy^2\right)$

(i) Parametrize C: $\quad x = t, \quad y = 2t, \quad t$ runs from 0 to 1.

Write everything in terms of t:

$$dx = dt, \quad dy = 2\,dt, \quad M = y^3 + 2x = 8t^3 + 2t, \quad M = 3xy^2 = 12t^3.$$

Put all this into the integral and compute:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y^3 + 2x)\,dx + 3xy^2\,dy = \int_0^1 (8t^3 + 2t)\,dt + 12t^3 2\,dt = \int_0^1 32t^3 + 2t\,dt = 9.$$

(ii) By the fundamental theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1,2) - f(0,0) = 9.$$

You decide which method is easier!

8.2 Proof of the fundamental theorem

Proof. The proof uses the definition of line integral together with the chain rule and the usual fundamental theorem of calculus.

We assume the following:
(i) $\mathbf{F} = \nabla f$
(ii) The curve C is parametrized by $\mathbf{r}(t) = (x(t),\, y(t))$, with t running from t_0 to t_1 and $\mathbf{r}(t_0) = P$, $\mathbf{r}(t_1) = Q$.

First recall the for a parametrized curve $\mathbf{r}(t)$ the chain rule says

$$\frac{df(\mathbf{r}(t))}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt}.$$

So,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} \nabla f \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{t_0}^{t_1} \frac{df(\mathbf{r}(t))}{dt}\,dt \\ &= f(\mathbf{r}(t))\big|_{t_0}^{t_1} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)) = f(Q) - f(P) \qquad \text{QED} \end{aligned}$$

8.3 Path independence

Definition. For a vector field $\mathbf{F}$ we say that the line integrals $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ are path independent if for any two points P and Q the integral yields the same value for *every* path connecting P to Q.

From the fundamental theorem we can conclude: if $\mathbf{F} = \nabla f$ is a gradient field, then the integrals $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ are path independent.

The following theorem offers an alternative way to express path independence.

Theorem GT.13. For a given vector field $\mathbf{F}$, the line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ are path independent is equivalent to $\oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C_c.

Proof. To show equivalence we need to show two things:

(i) Path independence implies the line integral around any closed path is 0.

(ii) The line integral around all closed paths is 0 implies path independence.

Proof (i). To start, note that the constant path C_0 where $\mathbf{r}(t) = P_0$, with t running from 0 to 0 has line integral

$$\oint_{C_0} \mathbf{F} \cdot d\mathbf{r} = \int_0^0 \mathbf{F} \cdot \mathbf{0}\, dt = 0.$$

Assume path independence and consider the closed path C_c shown in Figure (i) below. Since both C_c and C_0 have the same start and end points, path independence says the line integrals are the same, i.e.

$$\oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_0} \mathbf{F} \cdot d\mathbf{r} = 0.$$

This proves (i).

(ii) Assume $\oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve. Consider two paths between P and Q as shown in Figure (ii). The curve $C_c = C_1 - C_2$ is a closed curve starting and ending at P. Therefore, by assumption $\oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = 0$. So

$$0 = \int_{C_c} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

This implies $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. That is, the integral is the same for all paths from P to Q, i.e. the line integrals are path independent.

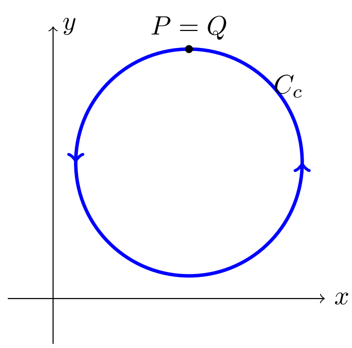


Figure (i)

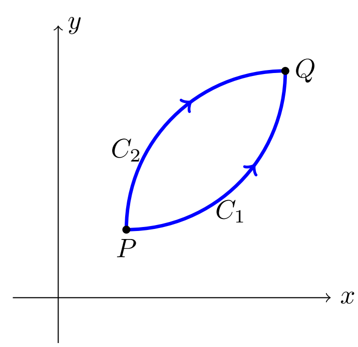


Figure (ii)

8.4 Conservative vector fields

Given a vector field $\mathbf{F}$, Theorem GT.13 in the previous section said that the line integrals of $\mathbf{F}$ were path independent is equivalent to the line integral of $\mathbf{F}$ around any closed path is 0. Following physics terminology, we call such a vector field a conservative vector field.

The fundamental theorem says the if $\mathbf{F}$ is a gradient field: $\mathbf{F} = \boldsymbol{\nabla} f$, then the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent. That is,

$$\boxed{\text{a gradient field is conservative.}}$$

The following theorem says that the converse is also true on connected regions. By a connected region we mean that any two points in the region can be connected by a continuous path that lies entirely in the region.

Theorem GT.14. A conservative field on a connected region is a gradient field.

Proof. Call the region D. We have to show that if we have a conservative field $\mathbf{F} = (M, N)$ on D the there is a potential function f with $\mathbf{F} = \boldsymbol{\nabla} f$.

The easy part will be defining f. The trickier part will be showing that its gradient is $\mathbf{F}$. So, first we define f: Fix a point (x_0, y_0) in D. Then for any point (x, y) in D we define

$$f(x,y) = \int_\gamma \mathbf{F} \cdot d\mathbf{r}, \text{ where } \gamma \text{ is any path from } (x_0, y_0) \text{ to } (x, y).$$

Path independence guarantees that $f(x, y)$ is well defined, i.e. it doesn't depend on the choice of path.

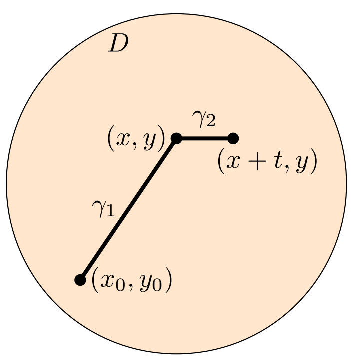
Now we need to show that $\mathbf{F} = \boldsymbol{\nabla} f$, i.e if $\mathbf{F} = (M, N)$, then we need to show that $f_x = M$ and $f_y = N$. We'll show the first case, the case $f_y = N$ is essentially the same. First, note that by definition

$$f_x(x,y) = \left.\frac{df(x+t,y)}{dt}\right|_{t=0}.$$

The function $f(x + t, y)$ is defined in terms of a line integral. So, we need to write down this integral and differentiate it.

In the figure below, $f(x, y)$ is the integral along the path γ_1 from (x_0, y_0) to (x, y) and $f(x + t, y)$ is the integral along $\gamma_1 + \gamma_2$, where γ_2 is the horizontal line segment from (x, y) to $(x + t, y)$. This means that

$$f(x+t,y) = \int_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{r} = f(x,y) + \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{r}.$$



We need to parameterize the horizontal segment γ_2. Notationwise, the letters x, y and t are already taken, so we let $\gamma_2(s) = (u(s), v(s)$, where

$$u(s) = x + s, \; y(s) = y; \quad \text{with } 0 \le s \le t$$

On this segment $du = ds$ and $dy = 0$. So,

$$f(x+t,y) = f(x,y) + \int_0^t M(x+s,y)\,ds$$

The piece $f(x,y)$ is constant as t varies, so the fundamental theorem of calculus says that

$$\frac{df(x+t,y)}{dt} = M(x+t,y).$$

Setting $t = 0$ we have

$$f_x(x,y) = \left.\frac{df(x+t,y)}{dt}\right|_{t=0} = M(x,y).$$

This is exactly what we needed to show! We summarize in a box:

$$\boxed{\text{On a connected region a conservative field is a gradient field.}}$$

Example GT.15. If $\mathbf{F}$ is the electric field of an electric charge it is conservative.

Example GT.16. A gravitational field of a mass is conservative.

9 Potential functions

Definition. If $\mathbf{F} = \nabla f$ is a gradient vector field then we call f a potential function for $\mathbf{F}$.

Note. The usual physics terminology would be to call $-f$ the potential function for $\mathbf{F}$.

This section is devoted to answering two questions.

1. How do we know if a vector field $\mathbf{F}$ is a gradient vector field, i.e. if $\mathbf{F} = \nabla f$ for some potential function f?

2. If it exists, how do we find the potential function f?

9.1 First answers to our questions

Theorem GT.17. Suppose $\mathbf{F} = (M, N)$. We have the following answer to our two questions.

(a) If $\mathbf{F} = \nabla f$, then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, i.e. $M_y = N_x$.

(b) If $M_y = N_x$ in the *whole* plane then $\mathbf{F}$ is a gradient vector field, i.e $\mathbf{F} = \nabla f$ for some potential function f.

(c) If $\mathbf{F}$ is conservative on a connected region, then $\mathbf{F}$ is a gradient field.

Notes. The restriction that $\mathbf{F}$ is defined on the whole plane is too stringent for our needs. Below, we will give what we call the Potential theorem, which only requires that $\mathbf{F}$ be defined and differentiable on what is called a 'simply connected region'.

Proof of (a). If $\mathbf{F} = (M, N) = \nabla f$ then $M = f_x$ and $N = f_y$. This implies,

$$M_y = f_{xy} \text{ and } N_x = f_{yx}, \text{ i.e } M_y = N_x. \text{ QED.}$$

(Provided f has continuous second partials.)

The proof of (b) will be postponed until after we have proved Green's theorem and we can state the Potential theorem. Part (c) is just a restatement of Theorem GT.14. The examples below will show how to find f

Example GT.18. For which values of the constants a and b will $\mathbf{F} = \big(axy, x^2 + by\big)$ be a gradient field?

<u>**answer:**</u> $M_y = ax, \quad N_x = 2x$. To apply the theorem we need $M_y = N_x$ in the entire plane. So, $a = 2$ and b is arbitrary.

Example GT.19. Is $\mathbf{F} = \big((3x^2 + y), \mathrm{e}^x\big)$ conservative?

<u>**answer:**</u> First we check if it is a gradient field: We write $\mathbf{F} = (M, N) = \big(3x^2 + y,\, \mathrm{e}^x\big)$. Then, $M_y = 1, \quad N_x = \mathrm{e}^x$. Since $M_y \neq N_x$, $\mathbf{F}$ is not a gradient field. Now, Theorem GT.17(c) says it can't be conservative.

Example GT.20. Is $\dfrac{(-y, x)}{x^2 + y^2}$ conservative?

<u>**answer:**</u> **NO!** The reasoning is a little trickier than in the previous example. First, it is not hard to compute that

$$M_y = \frac{y^2 - x^2}{(x^2 + y^2)^2} = N_x.$$

BUT, since the field is not defined for all (x, y), Theorem GT.17(b) does not apply. So, all we can say at this point is that we haven't ruled out its being conservative.

To show that it's not conservative we will find a closed path where the line integral is not 0. In fact, we will use our super-duper important example from above.

Let C = unit circle parametrized by $x = \cos(t)$, $y = \sin(t)$.

Writing everything in terms of t:

$$dx = -\sin(t)\,dt, \quad dy = \cos(t)\,dt, \quad M = -\frac{y}{x^2 + y^2} = -\sin(t), \quad N = \frac{x}{x^2 + y^2} = \cos(t).$$

Putting this in the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M\,dx + N\,dy = \oint_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = \int_0^{2\pi} dt = 2\pi$$

Since this is *not* 0, the field is not conservative.

Example GT.21. Is $\mathbf{F} = \dfrac{(x, y)}{x^2 + y^2}$ conservative?

<u>**answer:**</u> Again it is easy to check that $N_x = M_y$, BUT since $\mathbf{F}$ is not defined at $(0, 0)$ Theorem GT.17(b) does not apply. HOWEVER, it turns out that

$$\mathbf{F} = \boldsymbol{\nabla} \ln(\sqrt{x^2 + y^2}) = \boldsymbol{\nabla} \ln r$$

Since $\mathbf{F}$ is a gradient field, it is conservative. (Officially, we should say, $\mathbf{F}$ is conservative on the region consisting of the plane minus the origin.)

9.2 Finding the potential function

We will show two methods for finding the potential functions. In general, for 18.04 the method of integrating along a rectangular path is more relevant.

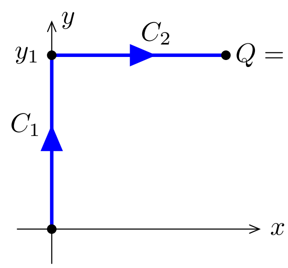
Example GT.22. Show that $\mathbf{F} = \left(3x^2 + 6xy, 3x^2 + 6y\right)$ is conservative and find the potential function f such that $\mathbf{F} = \nabla f$.

answer: We have $\mathbf{F} = (M, N)$, where $M = 3x^2 + 6xy$, $N = 3x^2 + 6y$.

First, $M_y = 6x = N_x$. Since $\mathbf{F}$ is defined for all (x, y) Theorem GT.17 implies $\mathbf{F}$ is a gradient field, hence conservative.

Method 1 for finding f.

Since $\mathbf{F}$ is a gradient field we know $\int_C \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P)$ for any path from P to Q. We make use of this by letting C be a rectangular path from the origin to an arbitrary point $Q = (x_1, y_1)$ (see figure).



Rectangular path from the origin to Q.

We know $\int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1) - f(0, 0)$. So

$$f(x_1, y_1) = \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} + f(0, 0).$$

On the rectangular path shown $dx = 0$ on C_1 and $dy = 0$ on C_2. Therefore,

$$\int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} N\, dy + \int_{C_2} M\, dx.$$

These integrals are straightforward to compute. We do each one separately.

Integral over C_1:
Parametrize C_1: parameter $= y$: $x = 0$, $y = y$, y runs from 0 to y_1.

Put everything we need in terms of the parameter y:

$$dy = dy, \quad N = 3x^2 + 6y = 6y.$$

Put this in the integral and compute

$$\int_{C_1} N\, dy = \int_0^{y_1} 6y\, dy = 3y_1^2.$$

Integral over C_2 :
Parameter $= x$: $x = x$, $y = y_1$, x runs from 0 to x_1.

Put everything we need in terms of the parameter x:

$$dx = dx, \quad M = 3x^2 + 6xy = 3x^2 + 6xy_1$$

Put this in the integral and compute (remember y_1 is a constant):

$$\int_{C_2} M\,dx = \int_0^{x_1} 3x^2 + 6xy_1\,dx = x_1^3 + 3x_1^2 y_1.$$

So,

$$f(x_1, y_1) - f(0,0) = \int_{C_1+C_2} \mathbf{F}\cdot d\mathbf{r} = 3y_1^2 + x_1^3 + 3x_1^2 y_1.$$

Now, we are free to choose any value for $f(0,0)$, i.e. it is an arbitrary constant of integration c. So, dropping the subscripts on x_1 and y_1, we have

$$f(x, y) = 3y^2 + x^3 + 3x^2 y + c.$$

Method 2 for finding f.

We know $M = f_x$ and $N = f_y$. We start by integrating M with respect to x.

$$f_x = 3x^2 + 6xy \;\Rightarrow\; f(x, y) = x^3 + 3x^2 y + g(y).$$

The function $g(y)$ is the 'constant of integration' with respect to x.

Now $f_y = N$, so differentiating our expression for f we get

$$f_y = 3x^2 + g'(y) = N = 3x^2 + 6y.$$

Thus, $g'(y) = 6y$, which implies $g(y) = 3y^2 + c$. Using this in our expression for f, we have

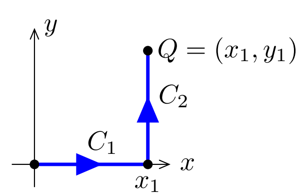
$$f(x, y) = x^3 + 3x^2 y + g(y) = 3x^2 y + 3y^2 + x^3 + c.$$

(Same as method 1.)

Example GT.23. Let $\mathbf{F} = \big((x + y^2), (2xy + 3y^2)\big)$. Show that $\mathbf{F}$ is a gradient field and find the potential function using both methods.

answer: Testing the partials we have: $M_y = 2y = N_x$, $\mathbf{F}$ defined on all (x, y). Thus, by Theorem GT.17, $\mathbf{F}$ is conservative.

Method 1: Use the path shown.



We know $f(x_1, y_1) - f(0,0) = \int_{C_1+C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} M\,dx + \int_{C_2} N\,dy$,

Parametrize C_1: $x = x$, $y = 0$, x runs from 0 to x_1.

In terms of the parameter x along C_1: $dx = dx$, $M = x + y^2 = x$.

Integrating:

$$\int_{C_1} M\,dx = \int_0^{x_1} x\,dx = \frac{x_1^2}{2}.$$

Parametrize C_2: $x = x_1, \quad y = y, \quad y$ runs from 0 to y_1.

In terms of the parameter y along C_2: $dy = dy, \quad N = 2x_1y + 3y^2$.

Integrating:

$$\int_{C_2} N\,dy = \int_0^{y_1} 2x_1y + 3y^2\,dy = x_1y_1^2 + y_1^3.$$

So, $f(x_1, y_1) = \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} + f(0,0) = \frac{x_1^2}{2} + x_1y_1^2 + y_1^3 + f(0,0)$.

Letting $f(0,0) = c$ and dropping the subscripts on x_1, y_1 we have

$$f(x,y) = \frac{x^2}{2} + xy^2 + y^3 + c.$$

Method 2. Since, in 18.04, we are less interested in method 2, we'll leave this to the reader.

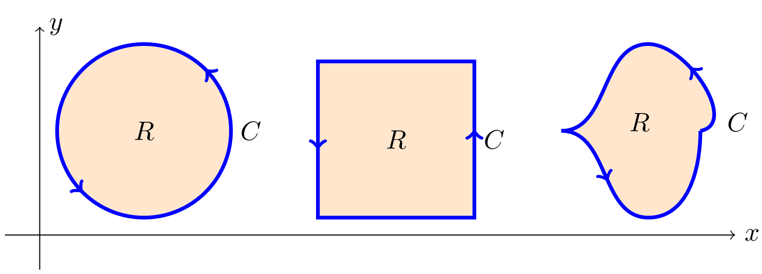
10 Green's theorem

Green's theorem is the one of the big theorems of multivariable calculus. It relates line integrals and area integrals. Using this relation we can often compute a seemingly difficult integral without integration or reduce it to an easy integral. At the this section we will describe how it is analogous to the fundamental theorem of calculus.

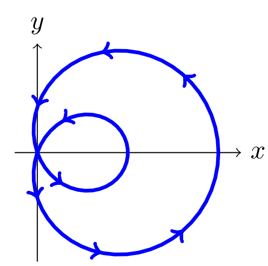
10.1 Simple closed curves

Definition. A simple closed curve is a closed curve with no self-intersection.

A simple closed curve C has a well-defined interior. Call the interior R. We say that C is positively oriented if R is always on the left as you traverse the curve. We call C the boundary of R.



Three positively oriented simple closed curves bounding a region R



Closed but not simple

Note. For smooth curves like the ones shown above the interior is easy to define. For an arbitrary simple closed curves, showing that it has a well-defined interior is more subtle. The theorem that proves this is called the Jordan curve theorem.

10.2 Green's theorem

Theorem GT.24. Green's theorem. Let C be a positively oriented simple closed curve with interior region R. We assume C is piecewise smooth (a few corners are okay). If the vector field $\mathbf{F} = (M, N)$ is defined and differentiable on R then

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA. \tag{4}$$

In two dimensions we define $\text{curl}\mathbf{F} = N_x - M_y$. So, in vector form, Green's theorem is written

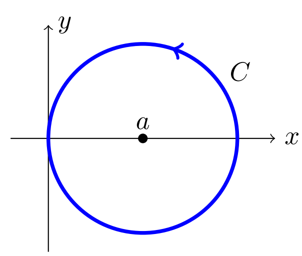
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\,dA.$$

Example GT.25. (Using the right hand side of Equation 4 to find the left hand side.)

Use Green's theorem to compute

$$I = \oint_C 3x^2y^2\,dx + 2x^2(1+xy)\,dy$$

where C is the circle shown.



answer: The line integral is of the form on the left hand side of Green's theorem. So, by Green's theorem we can convert the line integral to an area integral. In the line integral $M = 3x^2y^2$, $N = 2x^2(1+xy)$, so $N_x - M_y = 6x^2y + 4x - 6x^2y = 4x$. Therefore,

$$I = \oint_C 3x^2y^2\,dx + 2x^2(1+xy)\,dy = \iint_R N_x - M_y\,dx\,dy = 4\iint_R x\,dx\,dy.$$

We could compute this directly, but here's a trick. We know the x center of mass is

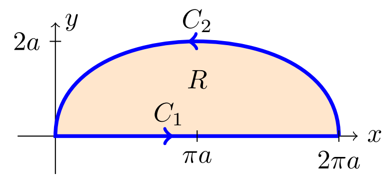
$$x_{cm} = \frac{1}{A}\iint_R x\,dx\,dy = a.$$

Since area $A = \pi a^2$, we have $\displaystyle\iint_R x\,dx\,dy = \pi a^3$, so $\boxed{I = 4\pi a^3.}$

Example GT.26. (Using the left hand side of Equation 4 to find the right hand side.)

Use Green's theorem to find the area under one arch of the cycloid.

answer: Our strategy is to use Green's theorem to replace the area integral with a line integral. The figure shows one arch of the cycloid. The region is under the arch and above the x-axis. The boundary of the region is $C = C_1 + C_2$.



The trick is to use the vector field $\mathbf{F} = (-y, 0)$, so $\text{curl}\mathbf{F} = N_x - M_y = 1$. With this $\mathbf{F}$, Green's theorem says

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C -y\,dx = \iint_R N_x - M_y\,dA = \iint_R dA = \text{ area}$$

That is, the area is equal to the line integral $\displaystyle\oint_C -y\,dx$. We compute the line integral as usual:

Parametrize C_1: $x = x$, $y = 0$, x runs from 0 to $2\pi a$. So,

$$dx = dx, \quad dy = 0, \quad M = -y = 0, \quad N \text{ (skip } N \text{ because } dy = 0).$$

Thus $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$.

Parametrize C_2: $x = a(\theta - \sin(\theta))$, $y = a(1 - \cos(\theta))$, θ runs from 2π to 0. (Note the direction θ runs.) So,

$$dx = a(1-\cos(\theta))\,d\theta, \quad dy \text{ (skip } dy \text{ because } N = 0), \quad M = -y = -a(1-\cos(\theta)), \quad N = 0.$$

Computing the integral over C_2:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} -y\,dx = \int_{2\pi}^{0} -a^2(1 - \cos(\theta))^2\,d\theta = \int_0^{2\pi} a^2(1 - \cos(\theta))^2\,d\theta = 3\pi a^2.$$

(The integral is easily computed by expanding the square and using the half-angle formula.)

Adding the two integrals: $\boxed{\text{the area under one arch of the cycloid is } 3\pi a^2.}$

10.3 Other ways to compute area using line integrals

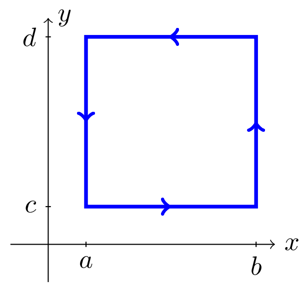
The key to the previous example was that for $\mathbf{F} = (-y, 0)$ we had $\text{curl}\mathbf{F} = N_x - M_y = 1$. There are other vector fields with the property $\text{curl}\mathbf{F} = 1$. For example, $\mathbf{F} = (0, x)$ or $\mathbf{F} = (-y/2, x/2)$. Using Green's theorem we have

$$\text{Area of } R = \oint_C -y\,dx \oint_C x\,dy = \frac{1}{2}\oint_C -y\,dx + x\,dy.$$

Here C is the positively oriented curve that bounds R.

10.4 'Proof' of Green's theorem

(i) First we'll work on the rectangle shown. Later we'll use a lot of rectangles to approximate an arbitrary region.



(ii) We'll simplify a bit and assume $N = 0$. The proof when $N \neq 0$ is essentially the same with a bit more writing.

First we consider the right hand side of Green's theorem (Equation 4). By direct calculation (assuming $N = 0$), the right hand side is

$$\iint_R -M_y \, dA = \int_a^b \int_c^d -\frac{\partial M}{\partial y}(x, y) \, dy \, dx.$$

The inner integral is the integral with respect to y of a derivative with respect to y. That is, we can compute it using the fundamental theorem of calculus.

$$\int_c^d \frac{\partial M}{\partial y}(x, y) \, dy = -M(x, y)|_c^d = -M(x, d) + M(x, c)$$

Putting this into the outer integral we have shown that

$$\iint_R -\frac{\partial M}{\partial y} \, dA = \int_a^b M(x, c) - M(x, d) \, dx. \tag{5}$$

Next we consider the left hand side of Equation 4. We have (remember $N = 0$) to compute $\oint_C M \, dx$. C has four sides we parametrized each one:

$$\begin{aligned} \text{bottom: } & x = x, \ y = c, \ \ x \text{ runs from } a \text{ to } b, \ dx = dx \\ \text{top: } & x = x, \ y = d, \ \ x \text{ runs from } b \text{ to } a, \ dx = dx \\ \text{sides: } & \ \text{skip because } dx = 0. \end{aligned}$$

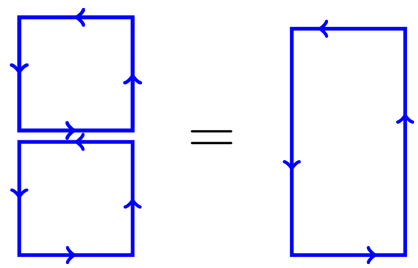
So,

$$\begin{aligned} \oint_C M \, dx &= \int_{bottom} M \, dx + \int_{top} M \, dx \quad \text{(since } dx = 0 \text{ along the sides)} \\ &= \int_a^b M(x, c) \, dx + \int_b^a M(x, d) \, dx = \int_a^b M(x, c) - M(x, d) \, dx. \end{aligned} \tag{6}$$

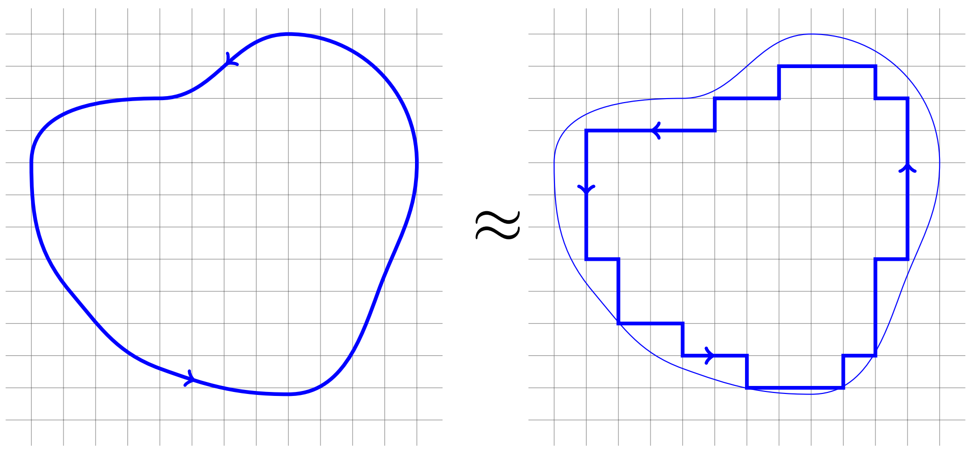
Comparing Equations 5 and 6 we find that we have proved Green's theorem for the rectangle.

Next we'll use rectangles to build up an arbitrary region. We start by stacking two rectangles on top of each other.

For line integrals when adding two rectangles with a common edge the common edges are traversed in opposite directions. So, the sum of the line integrals over the two rectangles equals the line integral over the outside boundary.



Similarly when adding a lot of rectangles: everything cancels except the outside boundary. This extends Green's theorem on a rectangle to Green's theorem on a sum of rectangles. Since any region can be approximated as closely as we want by a sum of rectangles, Green's theorem must hold on arbitrary regions. (See figure below.)



Any region and boundary can be approximated as a sum of rectangles.

11 Analogy to the fundamental theorem of calculus

We saw in the proof of Green's theorem that at one key step we had to integrate $\displaystyle\int \frac{\partial M}{\partial y}\,dy$. To do this we literally used the fundamental theorem. There is another way to view this connection. We will be rather informal in describing it, but it can be made formal and has deep and wide-ranging applications in math and science.

To set up the analogy we recall the fundamental theorem of calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Here's a picture of the domain of integration:

Notice that the left hand side of the fundamental theorem involves the integral of the derivative of F over a region (interval), and the right hand side is a sum of F itself over the boundary (endpoints) of the region.

Likewise, Green's theorem says

$$\iint_R \text{curl}\mathbf{F}\, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

The left hand side involves the integral of a derivative of $\mathbf{F}$ (i.e. $\text{curl}\mathbf{F} = N_x - M_y$) over a region, and the right hand side is an integral (i.e. a 'sum') of $\mathbf{F}$ itself over the boundary of the region. This is exactly the same language we used to describe the fundamental theorem of calculus.

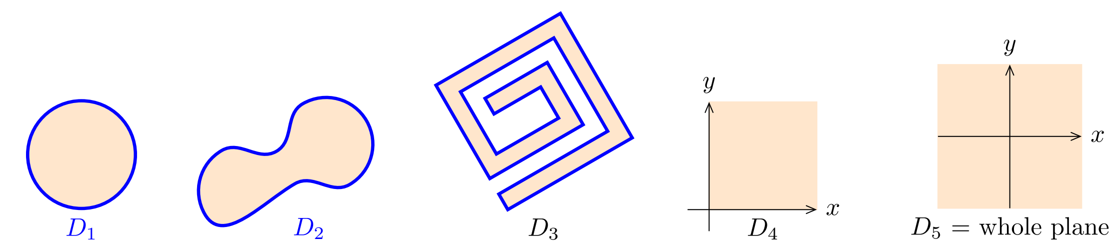
12 Simply connected regions

We will need the topological notion of a simply connected region. We will stick with an informal definition of simply connected that will be sufficient for our purposes.

(For those who are interested: We will assume that a simple closed curve has an inside and an outside. This is intuitive and is easy to show if C is a smooth curve, but turns out to surprisingly hard if we allow C to be strange, e.g. a Koch snowflake.)

Definition: A region D in the plane is called simply connected if, for every simple closed curve that lies entirely in D the interior of C also lies entirely in D.

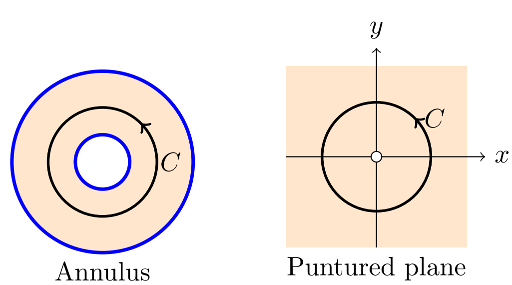
Examples:



D1-D5 are simply connected, since for any simple closed curve inside them its interior is entirely inside the region. This is sometimes phrased as each region has "no holes".

Note. An alternative definition, which works in higher dimensions is that the region is simply connected if any curve in the region can be continuously shrunk to a point without leaving the region.

The regions at below are not simply connected. That is, the interior of the curve C is not entirely in the region.



13 Potential theorem and conservative fields

As an application of Green's theorem we can now give a more complete answer to our question of how to tell if a field is conservative. The theorem does not have a standard name, so we choose to call it the Potential theorem. You should check that it is largely a restatement for simply connected regions of Theorem GT.17 above.

Theorem GT.27. (Potential theorem) Take $\mathbf{F} = (M, N)$ defined and differentiable on a region D.

(a) If $\mathbf{F} = \nabla f$ then $\text{curl}\mathbf{F} = N_x - M_y = 0$.

(b) If D is simply connected and $\text{curl}\mathbf{F} = 0$ on D, then $\mathbf{F} = \nabla f$ for some f.

Notes.

1. We know that on a connected region, being a gradient field is equivalent to being conservative. So we can restate the Potential theorem as: On a simply connected region, $\mathbf{F}$ is conservative is equivalent to $\text{curl}\mathbf{F} = 0$.

2. Recall that once we know work integral is path independent, we can compute the potential function f by picking a base point P_0 in D and letting

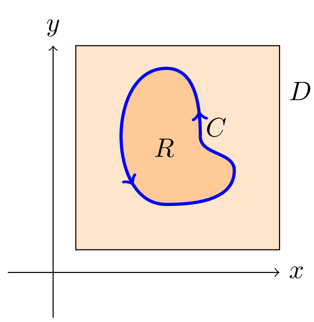
$$f(Q) = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any path in D from P_0 to Q.

Proof of (a): This was proved in Theorem GT.17.

Proof of (b): Suppose C is a simple closed curve in D. Since D is simply connected the interior of C is also in D. Therefore, using Green's theorem we have,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dA = 0.$$



This shows that $\mathbf{F}$ is conservative in D. Therefore by Theorem GT.14, $\mathbf{F}$ is a gradient field.

Summary: Suppose the vector field $\mathbf{F} = (M, N)$ is defined on a simply connected region D. Then, the following statements are equivalent.

(1) $\displaystyle\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is path independent.

(2) $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C.

(3) $\mathbf{F} = \boldsymbol{\nabla} f$ for some f in D

(4) $\mathbf{F}$ is conservative in D.

If $\mathbf{F}$ is continuously differentiable then 1, 2, 3, 4 all imply 5:

(5) $\text{curl}\mathbf{F} = N_x - M_y = 0$ in D

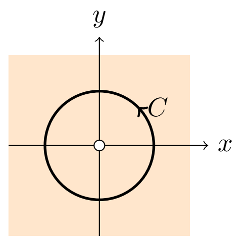
13.1 Why we need simply connected in the Potential theorem

The basic idea is that if there is a hole in D, then $\mathbf{F}$ might not be defined on the interior of C. This is illustrated in the next example.

Example GT.28.(What can go wrong if D is not simply connected.) Here we will repeat the super-duper really important Example GT.9.

Let $\mathbf{F} = \dfrac{(-y, x)}{r^2}$ ("tangential field").

$\mathbf{F}$ is defined on D = plane - (0,0) = punctured plane



Puntured plane

Several times now we have shown that $\text{curl}\mathbf{F} = 0$. (If you've forgotten this, you should recompute it now.) We also know that on any circle C of radius a centered at the origin $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$.

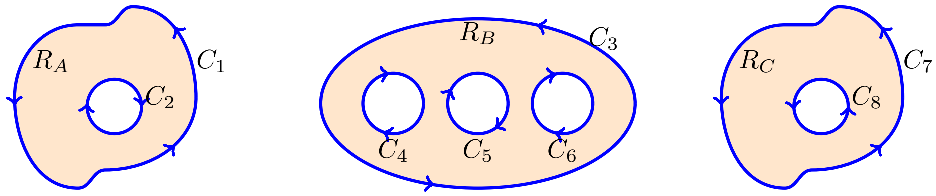
So, the conclusion of the above theorem that $\text{curl}\mathbf{F} = 0$ implies $\mathbf{F}$ is conservative does not hold. The problem is that D is not simply connected and, in fact, $\mathbf{F}$ is not defined on the entire region inside C, so we are not able to apply Green's theorem to conclude that the line integral is 0.

14 Extended Green's theorem

We can extend Green's theorem to a region R which has multiple boundary curves. The figures below show regions bounded by 2 or more curves. You will see that this gives us away to work around singularities in the field $\mathbf{F}$.

14.1 Regions with multiple boundary curves

Consider the following three regions.



The region on the left, R_A is bounded by C_1 and C_2. We say that the boundary is $C_1 + C_2$. Note that the way it is drawn, the region is always to the left as you traverse either boundary curve.

The region on the right, R_C is bounded by C_7 and C_8. We say that the boundary is $C_7 - C_8$. The reason for the minus sign is that the boundary curves should be oriented so that the region is to your left as you traverse the curve. As shown, the region R_C is to the right of C_8, but to the left of $-C_8$.

Likewise, in the middle figure, R_B has boundary $C_3 + C_4 + C_5 + C_6$. You should check that our signs are consistent with the orientation of the curves.

14.2 Extended Green's theorem

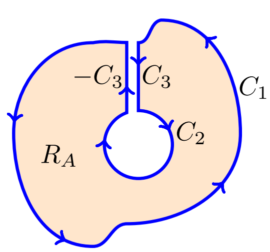
Theorem GT.29. Extended Green's theorem. Suppose R_A is the region in the left-hand figure above then, for any vector field $\mathbf{F}$ differentiable in all of R_A we have

$$\oint_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{R_A} \operatorname{curl}\mathbf{F}\, dx\, dy.$$

Likewise for more than two curves: If R_B has boundary $C_3 + C_4 + C_5 + C_6$ and $\mathbf{F}$ is differentiable on all of R_B then

$$\oint_{C_3+C_4+C_5+C_6} \mathbf{F} \cdot d\mathbf{r} = \iint_{R_B} \operatorname{curl}\mathbf{F}\, dx\, dy.$$

Proof. We will prove the formula for R_A. The case of more than two curves is essentially the same. The key is to make the 'cut' shown in the figure below, so that the resulting curve is simple.



In the figure the curve $C_1 + C_3 + C_2 - C_3$ surrounds the region R_A. (You have to imagine that the cut is infinitesimally wide so C_3 and $-C_3$ are right on top of each other.)

Now the original Green's theorem applies:

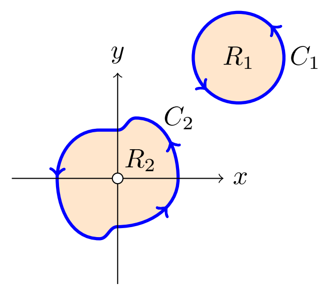
$$\oint_{C_1+C_3+C_2-C_3} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \operatorname{curl}\mathbf{F}\, dx\, dy$$

Since the contributions of C_3 and $-C_3$ will cancel, we have proved the extended form of Green's theorem.

$$\oint_{C_1+C_2} \mathbf{F}\cdot d\mathbf{r} = \iint_R \mathrm{curl}\mathbf{F}\, dx\, dy. \qquad \text{QED}$$

Example GT.30. Again, let $\mathbf{F}$ be the tangential field $\mathbf{F} = \dfrac{(-y,x)}{r^2}$. What values can $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$ take for C a simple closed positively oriented curve that doesn't go through the origin?

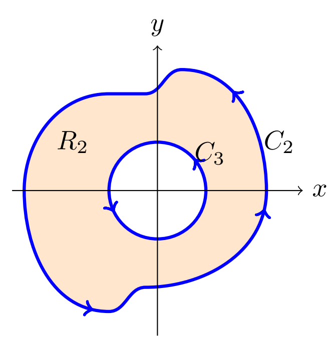
answer: We have two cases (i) C_1 does not go around 0; (ii) C_2 goes around 0



Case (i) We know $\mathbf{F}$ is defined and $\mathrm{curl}\mathbf{F} = 0$ in the entire region inside C_1, so Green's theorem implies

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \iint_R \mathrm{curl}\mathbf{F}\, dx\, dy = 0.$$

Case (ii) We can't apply Green's theorem directly because $\mathbf{F}$ is not defined everywhere inside C_2. Instead, we use the following trick. Let C_3 be a circle centered on the origin and small enough that is entirely inside C_2.



The region R_2 has boundary $C_2 - C_3$ and $\mathbf{F}$ is defined and differentiable in R_2. We know that $\mathrm{curl}\mathbf{F} = 0$ in R_2, so extended Green's theorem implies

$$\int_{C_2-C_3} \mathbf{F}\cdot d\mathbf{r} = \iint_{R_2} \mathrm{curl}\mathbf{F}\, dx\, dy = 0.$$

So $\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_3} \mathbf{F}\cdot d\mathbf{r}$.

Since C_3 is a circle centered on the origin we can compute the line integral directly –we've done this many times already.

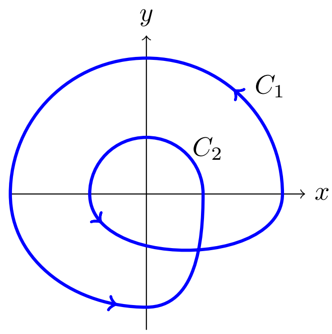
$$\oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = 2\pi.$$

Therefore, for a simple closed curve C and $\mathbf{F}$ as given, the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is either 2π or 0, depending on whether C surrounds the origin or not.

Answer to the question: The only possible values are 0 and 2π.

Example GT.31. Use the same $\mathbf{F}$ as in the previous example. What values can the line integral take if C is not simple.

answer: If C is not simple we can break it into a sum of simple curves.



In the figure, we can think of the entire curve as $C_1 + C_2$. Since each of these curves surrounds the origin we have

$$\int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi + 2\pi = 4\pi.$$

In general, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi n$, where n is the number of times C goes around (0,0) in a counterclockwise direction.

Aside for those who are interested: The integer n is called the *winding number* of C around 0. The number n also equals the number of times C crosses the positive x-axis, counting $+1$ if it crosses from below to above and -1 if it crosses from above to below.

15 One more example

Example GT.32. Let $\mathbf{F} = r^n\,(x, y)$.

For $n \geq 0$, $\mathbf{F}$ is defined on the entire plane. For $n < 0$, $\mathbf{F}$ is defined on the xy-plane minus the origin (the punctured plane).

Use extended Green's theorem to show that $\mathbf{F}$ is conservative on the punctured plane for all integers n. Then, find a potential function.

answer: We start by computing the curl:

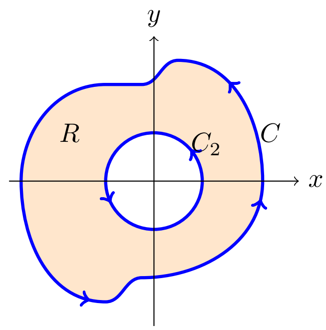
$$M = r^n x \;\Rightarrow\; M_y = nr^{n-2}xy$$
$$N = r^n y \;\Rightarrow\; N_x = nr^{n-2}xy$$

So, curl$\mathbf{F} = N_x - M_y = 0$.

To show that $\mathbf{F}$ is conservative in the punctured plane, we will show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all simple closed curves C that don't go through the origin.

If C is a simple closed curve not around 0 then $\mathbf{F}$ is differentiable on the entire region inside C and Green's theorem implies $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dx\, dy = 0$.

If C is a simple closed curve that surrounds 0, then we can use the extended form of Green's theorem as in Example GT.30



We put a small circle C_2 centered at the origin and inside C.

Since $\mathbf{F}$ is radial, it is orthogonal to C_2. So, on C_2, $\mathbf{F} \cdot d\mathbf{r} = 0$, which implies $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$.

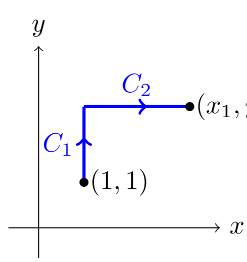
Now, on the region R with boundary $C - C_2$ we can apply the extended Green's theorem

$$\oint_{C-C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\, dx\, dy = 0.$$

Thus, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$, which, as we saw, equals 0.

Thus $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed loops, which implies $\mathbf{F}$ is conservative. QED

To find the potential function we use method 1 over the curve $C = C_1 + C_2$ shown.



The following calculation works for $n \neq -2$. For $n = -2$ everything is the same except we get natural logs instead of powers.

Parametrize C_1 using y: $x = 0, \quad y = y; \quad y$ from 1 to y_1. So,

$dx = 0, \quad dy = dy, \quad$ skip M, since $dx = 0, \quad N = r^n y = y(1+y^2)^{n/2}$. So,

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} M\,dx + N\,dy = \int_1^{y_1} y(1+y^2)^{n/2}\,dy \\
&= \left.\frac{(1+y^2)^{(n+2)/2}}{n+2}\right|_1^{y_1} \qquad (u\text{-substitution: } u = 1+y^2) \\
&= \frac{(1+y_1^2)^{(n+2)/2}}{n+2} - \frac{2^{(n+2)/2}}{n+2}.
\end{aligned}$$

Parametrize C_2 using x: $x = x, \quad y = y_1$; x from 1 to x_1. So,
$dx = dx, \quad dy = 0, \quad M = r^n x = x(x^2+y_1^2)^{n/2} \quad$ skip, N since $dy = 0$. So,

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_2} M\,dx + N\,dy = \int_1^{x_1} x(x^2+y_1^2)^{n/2}\,dx \\
&= \left.\frac{(x^2+y_1^2)^{(n+2)/2}}{n+2}\right|_1^{x_1} \qquad (u\text{-substitution: } u = x^2+y_1^2) \\
&= \frac{(x_1^2+y_1^2)^{(n+2)/2}}{n+2} - \frac{(1+y_1^2)^{(n+2)/2}}{n+2}.
\end{aligned}$$

Adding these we get $f(x_1, y_1) - f(1,1) = \dfrac{(x_1^2+y_1^2)^{(n+2)/2} - 2^{(n+2)/2}}{n+2}$. So,

$$f(x,y) = \frac{r^{n+2}}{n+2} + c. \qquad (\text{If } n = -2 \text{ we get } f(x,y) = \ln r + C.)$$

(Note, we ignored the fact that if (x_1, y_1) is on the negative x-axis we shoud have used a different path that doesn't go through the origin. This isn't really an issue because we know there is a potential function. Because our function f is known to be a potential function everywhere except the negative x-axis, by continuity it also works on the negative x-axis.)