Maxcut

LP relaxation for Max-Cut

Symmetric variables x_{ij} **for** $i, j \in V$

$$\max \sum_{\{i,j\} \in E} w_{ij} x_{ij}$$

$$x_{ij} \leq x_{ik} + x_{kj}, \quad \text{for all } i, j, k \in V,$$

$$x_{ij} + x_{jk} + x_{ki} \leq 2, \quad \text{for all } i, j, k \in V.$$

Q: Can we do better than 2?
A: Not using this LP ...

Theorem: There exists a graph G such that LP(G)/maxcut(G) > 2- (a little bit).

Idea: Find a graph G such that:

Property 1
LP(G) at least
#{edges}·0.99

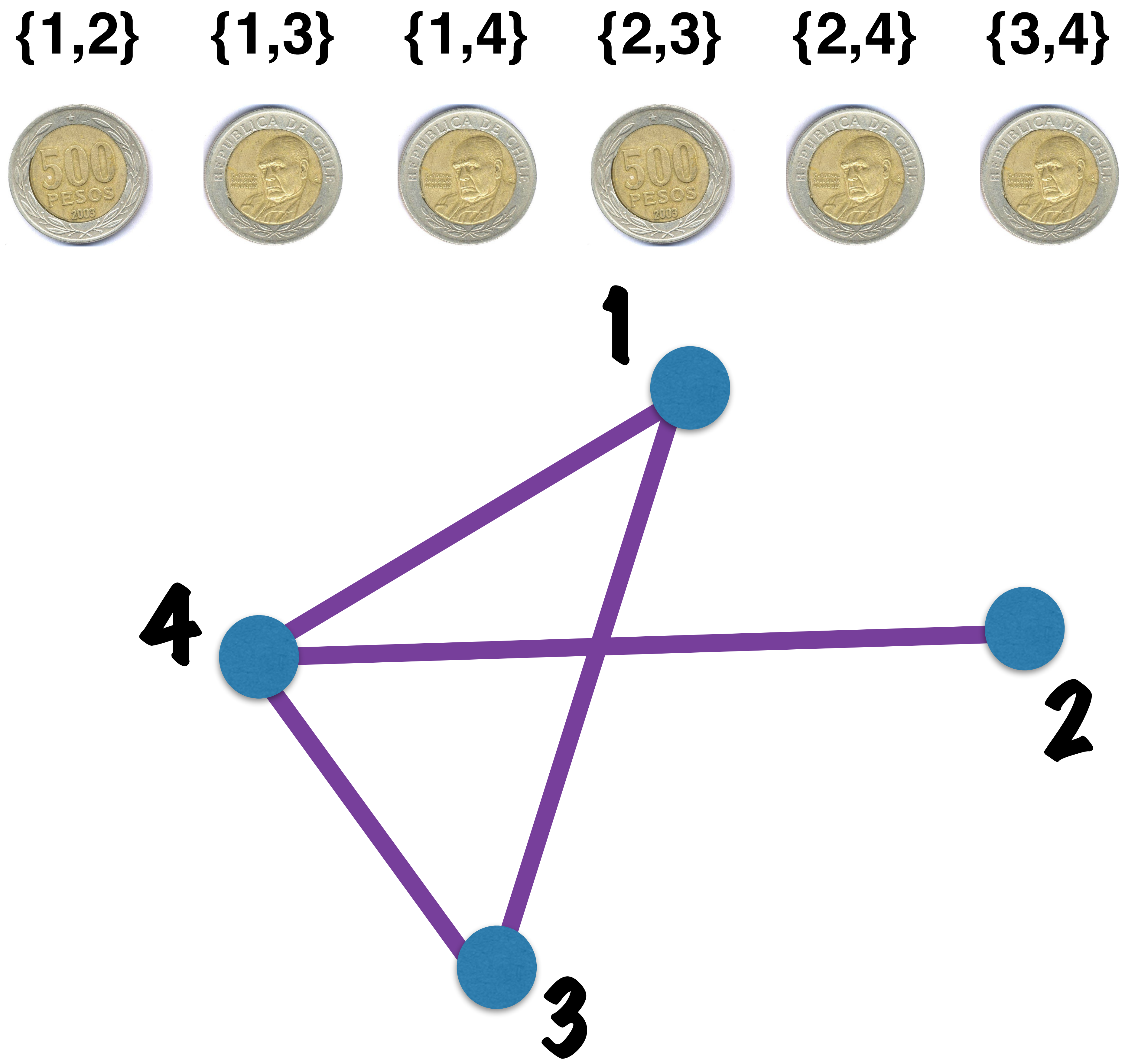
and

Property 2
maxcut(G) at most
#{edges}· 0.51

then LP(G)/maxcut(G) > 0.99/0.51 = 1.9...

How do we find G? Random graphs

G(n,p): graphs with n nodes, each edge drawn independently with probability p.



For each u,v in V:

head w.p. p ⟶ add {u,v} to G,

tail w.p. 1-p ⟶ we do not.

Lemma: if all cycles in G have length at least $g = 100 \log(100)$, then

$$LP(G) > 0.99 \cdot \#\{\text{edges of } G\}$$

Property 1

LP(G) at least $\#\{edges\} \cdot 0.99$

$G \sim G(n, C/n)$

Markov's Inequality

$\Pr(\#\{\text{cycles of length at most } g-1\} > n^{1/2}) < 0.5$

G': graph after breaking these cycles in G by removing at most $n^{1/2}$ edges.

Hence, $LP(G') > 0.99 \cdot \#\{\text{edges of } G'\}$

G' satisfies Property 1 !

Property 2

maxcut(G') at most #{edges}$\cdot$ 0.51

$G \sim G(n, C/n)$

maxcut(G')<0.51$\cdot$ #{edges of G'}
with probability at least 0.8

Putting things together ...

Pr(G satisfies Property 1 and Property 2)

$> 1 - \Pr(G \text{ does not satisfy Property 1}) - \Pr(G \text{ does not satisfy Property 2})$

$> 1 - 0.5 - 0.2$

$= 0.3$

Hence, there exists a graph G' satisfying both properties.

$LP(G')/maxcut(G') > 0.99/0.51 = 1.9...$

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