What is the expected value of the division of a random variable by a sum of random variables?

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With X_1, X_2 and X_3 being independent random variables, how can I compute $\mathbb{E}\left[\frac{X_1}{X_1+X_2+X_3}\right]$?

Is $\mathbb{E}\left[\frac{X_1}{X_1+X_2+X_3}\right] = \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1]+\mathbb{E}[X_2]+\mathbb{E}[X_3]}$? If not, how is it calculated?

Thank you in advance for any clarification.

2

random-variable expected-value

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asked Mar 24, 2021 at 17:48

Xander

23 2

2 Are you also assuming the variables all have the same distribution? – whuber ♦ Mar 24, 2021 at 18:02

1 Answer

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4

There is a famous "folk theorem" residual to my undergrad classes, namely that

$$\mathbb{E}[X/Y] = \mathbb{E}[X]/\mathbb{E}[Y]$$

and it is often permeates during exams, as it makes computing much easier. Sadly, the equality does not hold in general, even when X and Y are independent (due to Jensen's inequality).

In the case of $\mathbb{E}[X_1/X_1 + X_2 + X_3]$, numerator and denominator are dependent, which usually makes the computation more difficult. However, in the very special case when the three X_i's are iid, $X_i/X_1 + X_2 + X_3$ has the same distribution for all three i's and this leads to an obvious conclusion concerning the expectation of any of them. **Assuming this expectation exists, of course.** A counterexample is provided by a triplet of Normal variables (see Marsaglia's paper in connection).

As a special case where the identity works, take the Dirichlet $\mathcal{D}(\alpha_1, \ldots, \alpha_d)$ distribution, whose expectation is whose expectation is

$$\mathbb{E}[Y_i] = \alpha_i \Big/ \sum_{j=1}^{d} \alpha_j$$

One representation of a Dirichlet random vector $(Y_1, \ldots, Y_d)$ is

$$Y_i = \frac{X_i}{X_1 + \ldots + X_d} \qquad X_i \sim \mathcal{G}(\alpha_i, 1)$$

where the X_i's are independent. In that case,

$$\mathbb{E}[Y_i] = \mathbb{E}[X_i / X_1 + \ldots + X_d] = \mathbb{E}[X_i]/\mathbb{E}[X_1 + \ldots + X_d]$$

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edited Mar 25, 2021 at 8:06

answered Mar 24, 2021 at 18:02

Xi'an
93.5k 9 159 596

1 The identity will work whenever the X_i are *iid* with nonzero expectation, as you basically pointed out in an earlier version of this answer. – whuber ♦ Mar 24, 2021 at 18:29

@whuber: Is this enough to ensure that $X_1/(X_1+X_2)$ has a well-defined expectation? – Xi'an Mar 24, 2021 at 19:51

1 Good question: I don't think so. For instance, let the distribution be uniform on the values $\{-1, 1, 3\}$. $|X_1/(X_1+X_2)|$ equals one divided by zero with a chance of $2/9$. Continuous approximations to this will have comparable problems. Notice that these random variables are (a) bounded and (b) have zero probability to be zero. – whuber ♦ Mar 24, 2021 at 20:55

1 The expression like $X_1/(X_1+X_2)$ also occured in this question stats.stackexchange.com/a/399952 . There is an expression from Hinkley for the case that the X_i are Gaussian distributed (and the expectation will be infinite). In this question stats.stackexchange.com/a/438402 an intuitive view is given for the ratio distribution (and you could do the same for the correlated case). You could also express the distribution of the *angle*. And the expectation of the ratio is the expectation of the tangens of the angle, which becomes infinite when 90 deg has non-zero density. – Sextus Empiricus Mar 25, 2021 at 8:01

1 When the X_i are continuous and non-negative then the division by 0 occurs only in the point $(X_1, X_2) = (0, 0)$ (which has zero probability) and for the other values of X_1, X_2 the value of the ratio $X_1/(X_1+X_2)$ is between 0 and 1, such that the ratio won't have infinite or undefined expectation. – Sextus Empiricus Mar 25, 2021 at 8:18