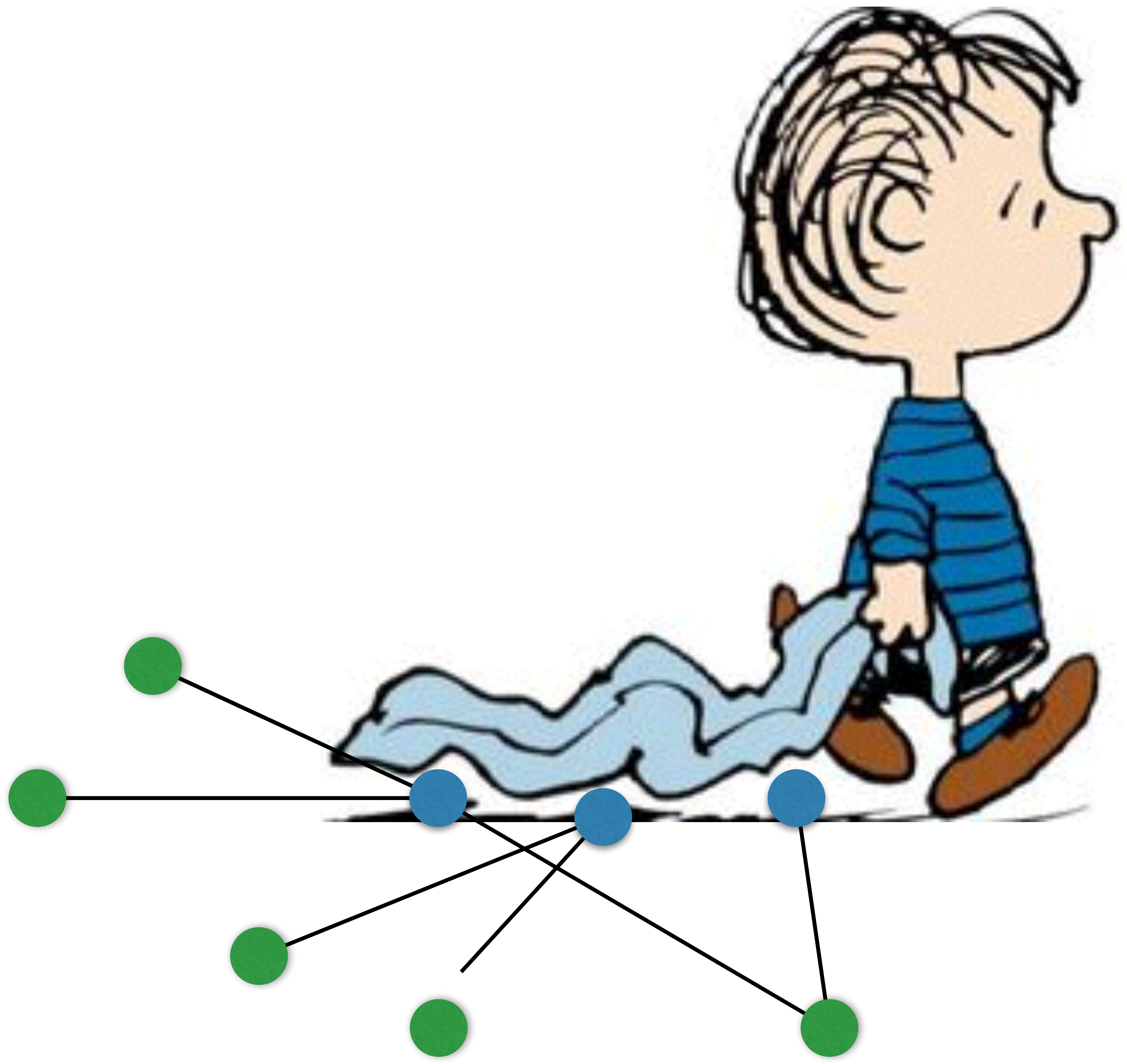
Primal-dual algorithm for Vertex Cover



$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u^*}$$
$$\mathbf{x_u^*} + \mathbf{x_v^*} \geq 1$$
$$\mathbf{0} \leq \mathbf{x_u^*} \leq 1$$

Primal $\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$

$$\mathbf{x_u} + \mathbf{x_v} \geq \mathbf{1} \quad \forall \mathbf{uv} \in \mathbf{E}$$

$$\mathbf{x_u} \geq \mathbf{0} \quad \forall \mathbf{u} \in \mathbf{V}$$

$$\begin{array}{cc} & \begin{array}{ccccc} (\mathbf{y}) & (\mathbf{u}) & (\mathbf{v}) & (\mathbf{x}) & \end{array} \\ \begin{array}{r} (\mathbf{e} = \mathbf{uv}) \\ \\ (\mathbf{e}' = \mathbf{ux}) \\ \\ (\mathbf{e}'' = \mathbf{uy}) \end{array} & \left(\begin{array}{ccccc} \cdots & 0\ 1\ 0 & \cdots 0\ 1\ 0 & \cdots & \cdots 0 \\ \cdots & 0\ 0\ 0 & \cdots & \cdots & \cdots 0 \\ \cdots & 0\ 1\ 0 & \cdots & 0\ 1\ 0 & \cdots 0 \\ \cdots & 0\ 0\ 0 & \cdots & \cdots & \cdots 0 \\ \cdots 0\ 1\ 0 \cdots & 0\ 1\ 0 & \cdots & \cdots & \cdots 0 \end{array} \right) \end{array}$$

Dual $\max \sum_{\mathbf{e}} \mathbf{y_e}$

$$\sum_{\mathbf{e}:\mathbf{u} \in \mathbf{e}} \mathbf{y_e} \leq \mathbf{w_u} \quad \forall \mathbf{u} \in \mathbf{V}$$

$$\mathbf{y_e} \geq \mathbf{0} \quad \forall \mathbf{e} \in \mathbf{E}$$

“Construct” integer solutions x for (P), y for (D)

$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$$
$$\mathbf{x_u + x_v \geq 1 \quad \forall uv \in E}$$
$$\mathbf{x_u \geq 0 \quad \forall u \in V}$$

$$\max \sum_{\mathbf{e}} \mathbf{y_e}$$
$$\sum_{\mathbf{e:u \in e}} \mathbf{y_e \leq w_u \quad \forall u \in V}$$
$$\mathbf{y_e \geq 0 \quad \forall e \in E}$$

Start with: $\mathbf{x} = (0, \cdots, 0), \mathbf{y} = (0, \cdots, 0)$

$\mathbf{x}$ **has low value but is not feasible**
$\mathbf{y}$ **is feasible but has low value**

“Construct” integer solutions x for (P), y for (D)

$$\min \sum_{\mathbf{u}} \mathbf{w_u x_u}$$
$$\mathbf{x_u} + \mathbf{x_v} \geq 1 \ \ \forall \mathbf{uv} \in \mathbf{E}$$
$$\mathbf{x_u} \geq 0 \ \ \forall \mathbf{u} \in \mathbf{V}$$

$$\max \sum_{\mathbf{e}} \mathbf{y_e}$$
$$\sum_{\mathbf{e:u \in e}} \mathbf{y_e} \leq \mathbf{w_u} \ \ \forall \mathbf{u} \in \mathbf{V}$$
$$\mathbf{y_e} \geq 0 \ \ \forall \mathbf{e} \in \mathbf{E}$$

Start with: $\mathbf{x} = (0, \cdots, 0), \mathbf{y} = (0, \cdots, 0)$

Repeat:

pick $\mathrm{e} = \mathrm{uv}$ such that $\mathbf{x_u} + \mathbf{x_v} < 1$

increase $\mathrm{y_e}$ until

$$\sum_{\mathbf{f:u \in f}} \mathbf{y_f} = \mathbf{w_u} \ \text{ or } \ \sum_{\mathbf{f:v \in f}} \mathbf{y_f} = \mathbf{w_v}$$

first case: $\mathbf{x_u} \leftarrow 1$ second case: $\mathbf{x_v} \leftarrow 1$

$$\min \sum_u w_u x_u$$
$$x_u + x_v \geq 1 \quad \forall uv \in E$$
$$x_u \geq 0 \quad \forall u \in V$$

$$\max \sum_e y_e$$
$$\sum_{e:u\in e} y_e \leq w_u \quad \forall u \in V$$
$$y_e \geq 0 \quad \forall e \in E$$

Repeat:

pick $e = uv$ such that $x_u + x_v < 1$

increase y_e until

$$\sum_{f:u\in f} y_f = w_u \quad \text{or} \quad \sum_{f:v\in f} y_f = w_v$$

first case: $x_u \leftarrow 1$ second case: $x_v \leftarrow 1$

Invariants: y remains feasible throughout

x has fewer and fewer violated constraints

In the end: both are feasible

Repeat:

pick $e = uv$ such that $x_u + x_v < 1$

increase y_e until

$$\sum_{f:u\in f} y_f = w_u \text{ or } \sum_{f:v\in f} y_f = w_v$$

first case: $x_u \leftarrow 1$ second case: $x_v \leftarrow 1$

Invariant: for every $u \in V$

$$\sum_{f:u\in f} y_f = w_u \text{ or } x_u = 0$$

and for every $e = uv \in E$

$$y_e = 0 \text{ or } x_u + x_v = 1 \text{ or } x_u + x_v = 2$$

$$\min \textstyle\sum_{\mathbf{u}} \mathbf{w_u x_u}$$
$$\mathbf{x_u + x_v \geq 1} \ \ \forall \mathbf{uv \in E}$$
$$\mathbf{x_u \geq 0} \ \ \forall \mathbf{u \in V}$$

$$\max \textstyle\sum_{\mathbf{e}} \mathbf{y_e}$$
$$\textstyle\sum_{\mathbf{e:u\in e}} \mathbf{y_e \leq w_u} \ \ \forall \mathbf{u \in V}$$
$$\mathbf{y_e \geq 0} \ \ \forall \mathbf{e \in E}$$

$$\begin{aligned} \textstyle\sum_{\mathbf{u}} \mathbf{w_u x_u} &= \textstyle\sum_{\mathbf{u:x_u \neq 0}} \mathbf{w_u x_u} \\ &= \textstyle\sum_{\mathbf{u:x_u\neq 0}} \sum_{\mathbf{v:uv\in E}} \mathbf{y_{uv} x_u} \\ &= \textstyle\sum_{\mathbf{u}} \sum_{\mathbf{e=uv\in E}} \mathbf{y_e x_u} \\ &= \textstyle\sum_{\mathbf{e}} (\sum_{\mathbf{u\in e}} \mathbf{x_u}) \mathbf{y_e} \\ &\leq \textstyle\sum_{\mathbf{e}} \mathbf{2y_e} \\ &= \mathbf{2} \textstyle\sum_{\mathbf{e}} \mathbf{y_e} \leq \mathbf{2 \cdot OPT} \end{aligned}$$

Primal-dual algorithm for vertex cover

Repeat:

pick $e = uv$ such that $x_u + x_v < 1$

increase y_e until

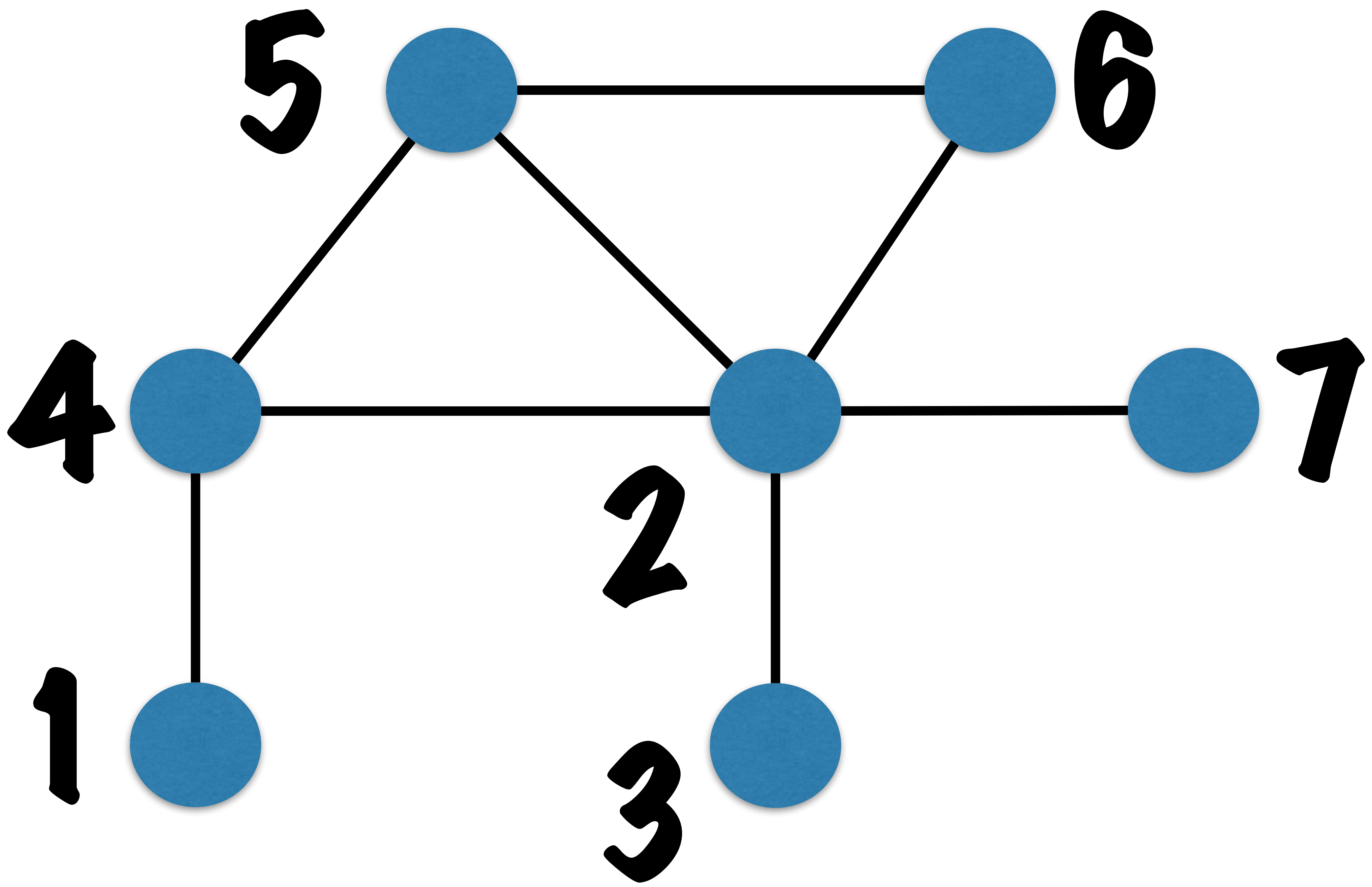
$$\sum_{f:u \in f} y_f = w_u \text{ or } \sum_{f:v \in f} y_f = w_v$$

first case: $x_u \leftarrow 1$ second case: $x_v \leftarrow 1$

Theorem: the primal-dual algorithm for vertex cover is a 2-approximation

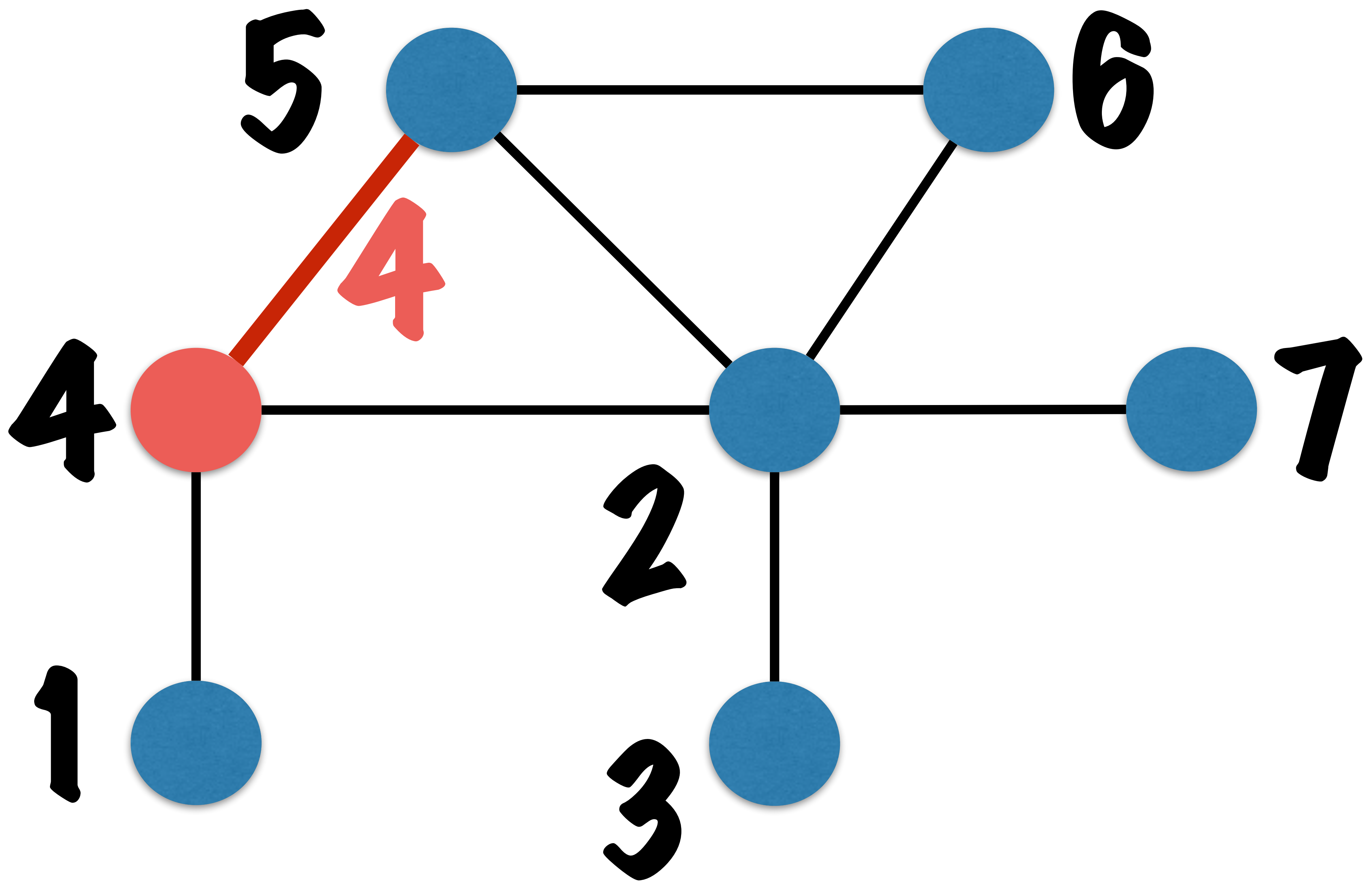
Example

vertex weights



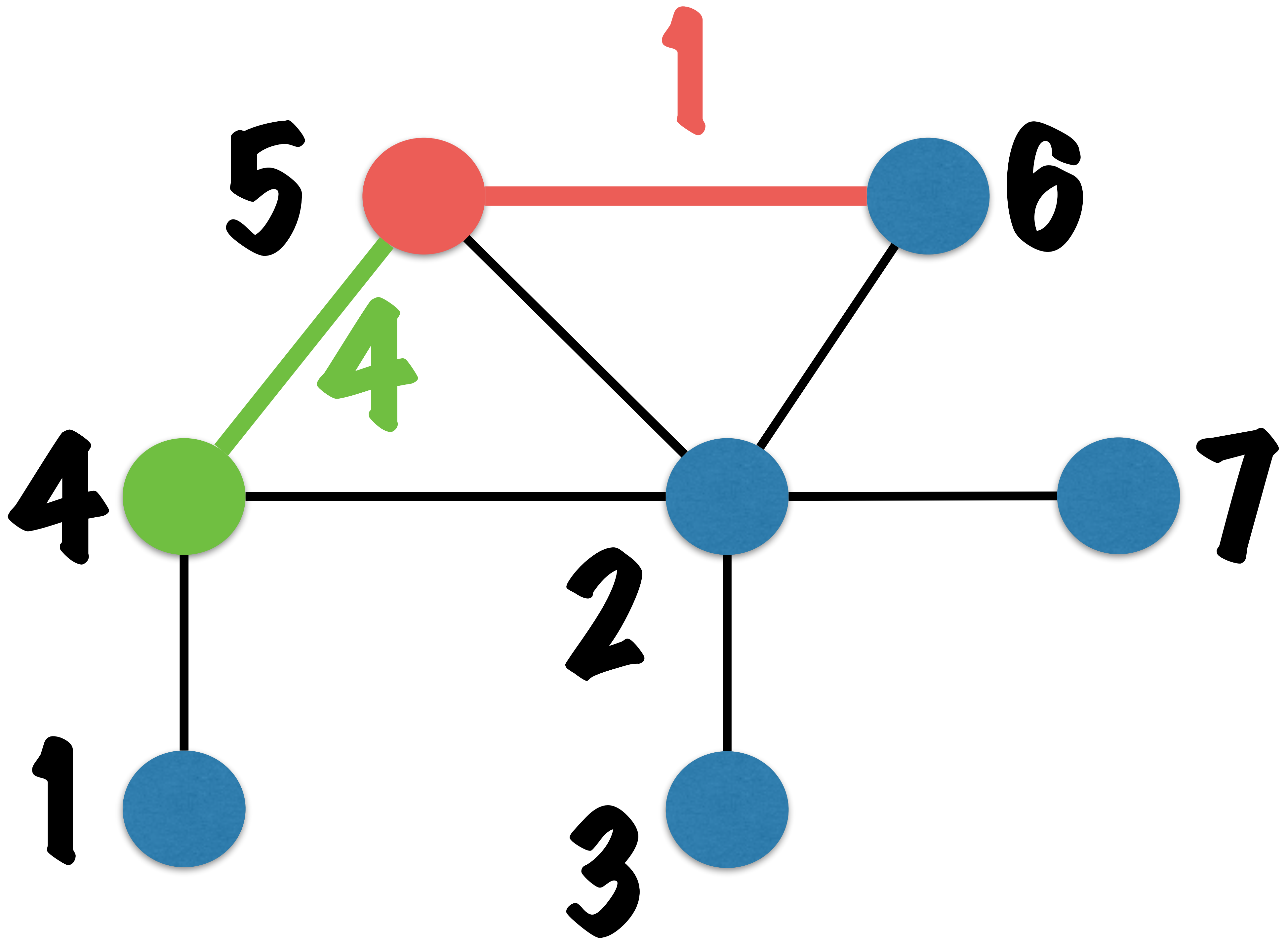
Example

vertex weights



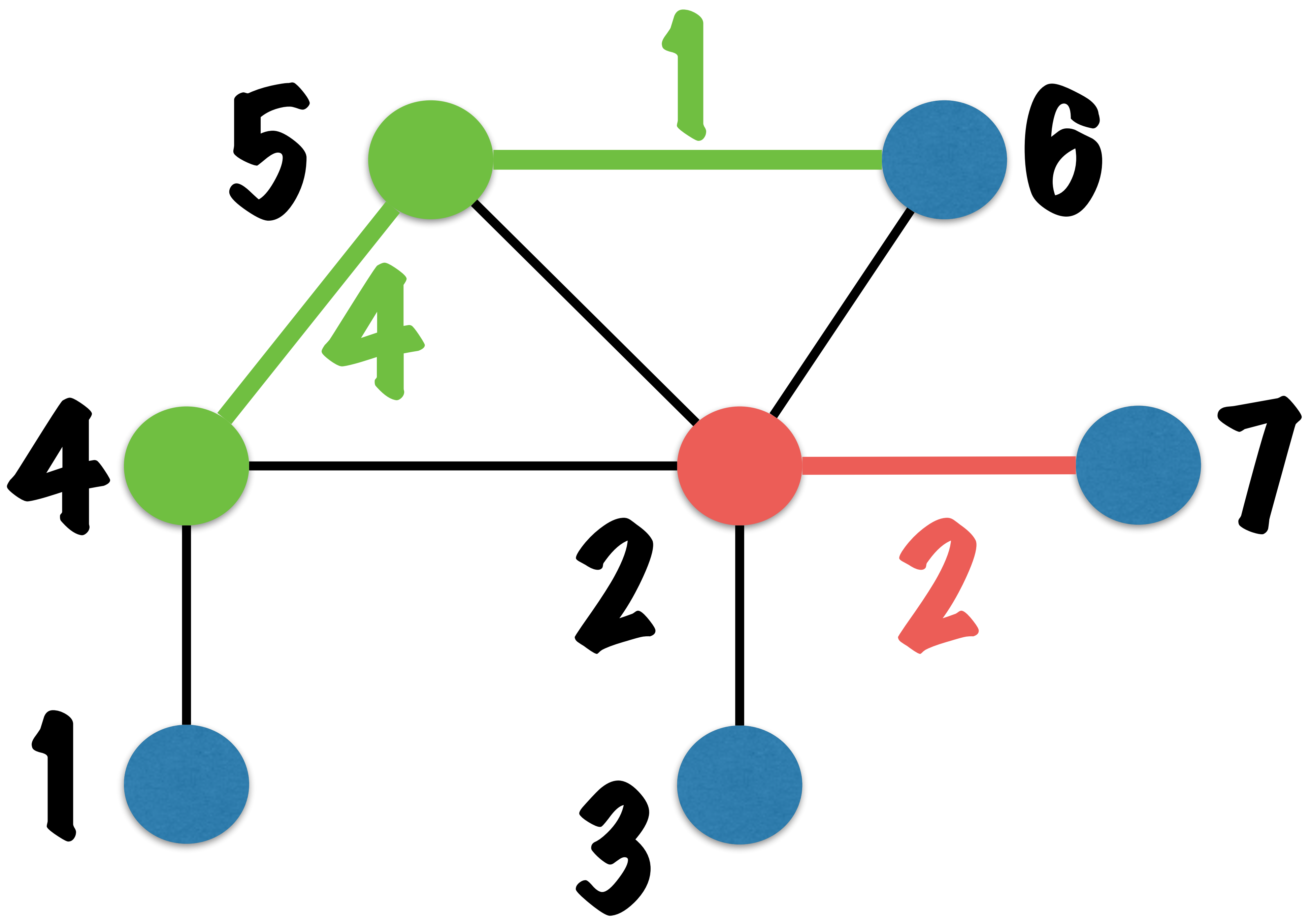
Example

vertex weights



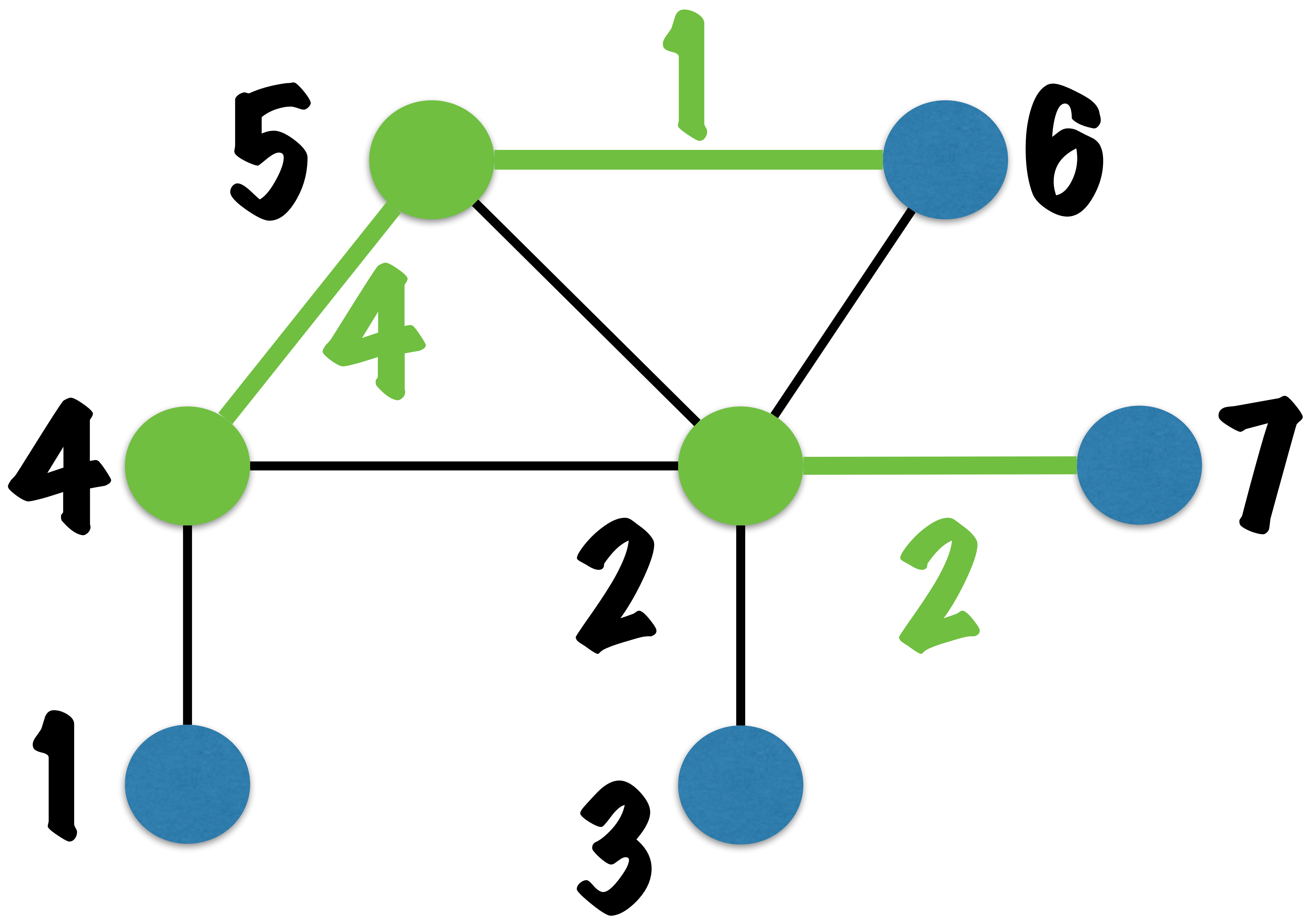
Example

vertex weights



Example

vertex weights



Output {2,4,5}
OPT={1,2,5}