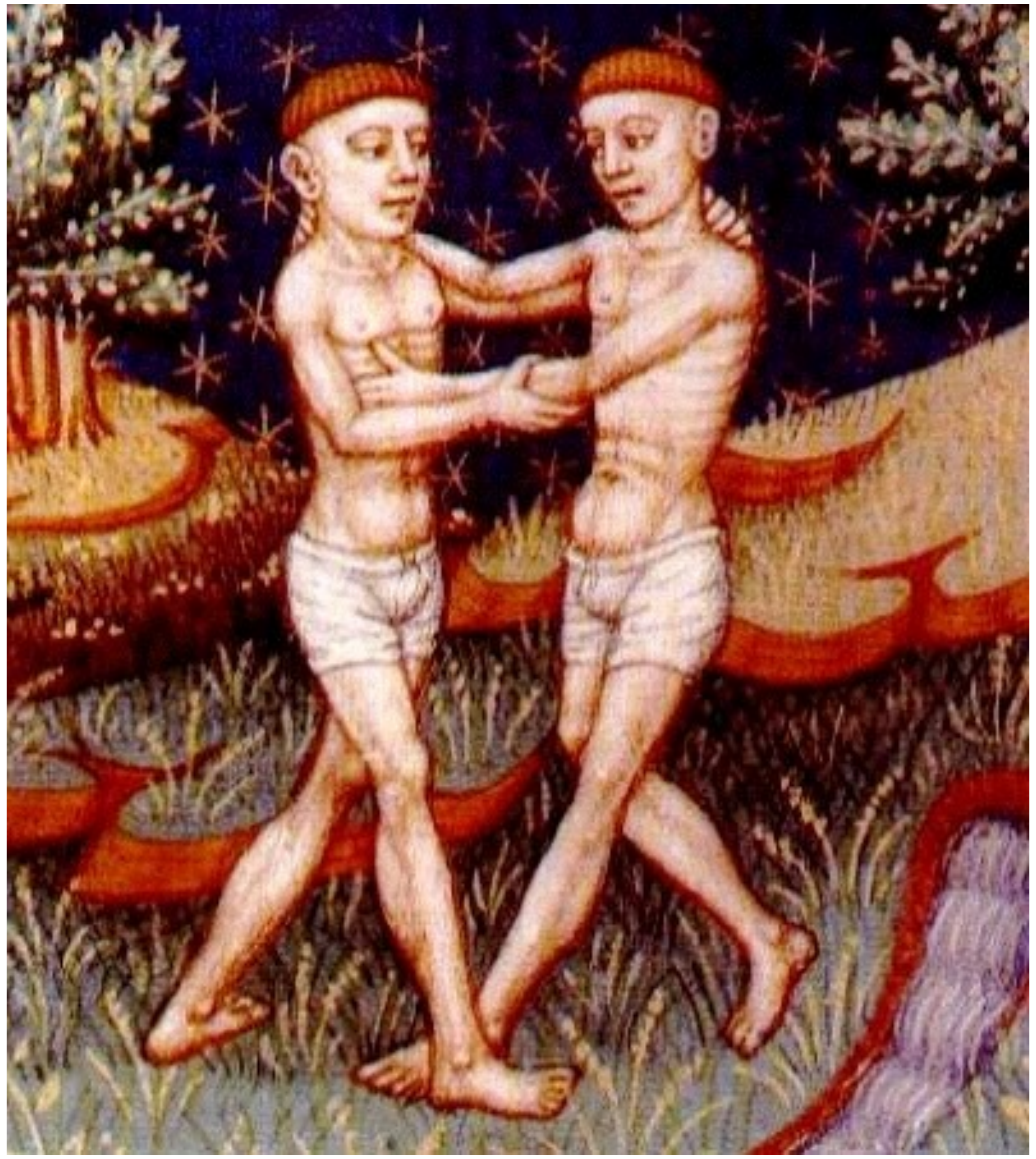

Strong duality Theorem in general

(P)

$$\min \mathbf{c} \cdot \mathbf{x} :$$
$$\mathbf{A}\mathbf{x} \geq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

(D)

$$\max \mathbf{b} \cdot \mathbf{y} :$$
$$\mathbf{A}^{\mathbf{T}}\mathbf{y} \leq \mathbf{c}$$
$$\mathbf{y} \geq 0$$

Four possible cases:

- (P) is empty, (D) has value $+\infty$
- (D) is empty, (P) has value $-\infty$
- value(P)=value(D)
- [(P) and (D) empty]

Proof of the weak duality theorem

$$\max \mathbf{c} \cdot \mathbf{x} : \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \quad \mathbf{x} \ge \mathbf{0}$$

$$\le$$

$$\min \mathbf{b} \cdot \mathbf{y} : \quad \mathbf{A}^{\mathbf{T}}\mathbf{y} \ge \mathbf{c} \quad \mathbf{y} \ge \mathbf{0}$$

(P)

$$\begin{aligned}
&\max \mathbf{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n} : \\
&\mathbf{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1} \\
&\mathbf{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2} \\
&\cdots \\
&\mathbf{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m} \\
&\mathbf{x_1, x_2, \ldots, x_n \ge 0}
\end{aligned}$$

(D)

$$\begin{aligned}
&\min \mathbf{b_1 y_1 + b_2 y_2 + \cdots + b_m y_m} : \\
&\mathbf{a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1} \\
&\mathbf{a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_m \ge c_2} \\
&\cdots \\
&\mathbf{a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n} \\
&\mathbf{y_1, y_2, \ldots, y_m \ge 0}
\end{aligned}$$

(P)

$$\max \mathbf{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n} :$$
$$\mathbf{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1}$$
$$\mathbf{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2}$$
$$\cdots$$
$$\mathbf{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m}$$
$$\mathbf{x_1, x_2, \ldots, x_n \geq 0}$$

(D)

$$\min \mathbf{b_1 y_1 + b_2 y_2 + \cdots + b_m y_m} :$$
$$\mathbf{a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1}$$
$$\mathbf{a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_m \geq c_2}$$
$$\cdots$$
$$\mathbf{a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n}$$
$$\mathbf{y_1, y_2, \ldots, y_m \geq 0}$$

Take x feasible for (P), y feasible for (D)
Must prove:

$$\mathbf{c_1 x_2 + \cdots + c_n x_n \leq b_1 y_1 + \cdots + b_m y_m}$$

Must prove: $\mathbf{c_1x_2 + \cdots + c_nx_n \leq b_1y_1 + \cdots + b_my_m}$

1. Use constraints of (D)

(D)

$$
\begin{aligned}
&\min \mathbf{b_1y_1 + b_2y_2 + \cdots + b_my_m} : \\
&\mathbf{a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1} \\
&\mathbf{a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_m \geq c_2} \\
&\cdots \\
&\mathbf{a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n} \\
&\mathbf{y_1, y_2, \ldots, y_m \geq 0}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{c_1x_1 + \cdots + c_nx_n \leq} \\
&\mathbf{(a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m)x_1 + \cdots +} \\
&\mathbf{(a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m)x_n}
\end{aligned}
$$

2. Invert summations

$$(\mathbf{a_{11}y_1} + \mathbf{a_{21}y_2} + \cdots + \mathbf{a_{m1}y_m})\mathbf{x_1} + \cdots +$$
$$(\mathbf{a_{1n}y_1} + \mathbf{a_{2n}y_2} + \cdots + \mathbf{a_{mn}y_m})\mathbf{x_n} =$$
$$(\mathbf{a_{11}x_1} + \cdots + \mathbf{a_{1n}x_n})\mathbf{y_1} + \cdots +$$
$$(\mathbf{a_{m1}x_1} + \cdots + \mathbf{a_{mn}x_n})\mathbf{y_m}$$

3. Use constraints of (P)

(P)

$$
\begin{aligned}
&\max \mathbf{c_1x_1} + \mathbf{c_2x_2} + \cdots + \mathbf{c_nx_n} : \\
&\mathbf{a_{11}x_1} + \mathbf{a_{12}x_2} + \cdots + \mathbf{a_{1n}x_n} \le \mathbf{b_1} \\
&\mathbf{a_{21}x_1} + \mathbf{a_{22}x_2} + \cdots + \mathbf{a_{2n}x_n} \le \mathbf{b_2} \\
&\cdots \\
&\mathbf{a_{m1}x_1} + \mathbf{a_{m2}x_2} + \cdots + \mathbf{a_{mn}x_n} \le \mathbf{b_m} \\
&\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n} \ge \mathbf{0}
\end{aligned}
$$

$$
(\mathbf{a_{11}x_1} + \cdots + \mathbf{a_{1n}x_n})\mathbf{y_1} + \cdots + (\mathbf{a_{m1}x_1} + \cdots + \mathbf{a_{mn}x_n})\mathbf{y_m} \le \mathbf{b_1y_1} + \cdots + \mathbf{b_my_m}
$$

In summary

$$\begin{aligned}
&\max \mathbf{c_1x_1 + c_2x_2 + \cdots + c_nx_n} :\\
&\mathbf{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1}\\
&\mathbf{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2}\\
&\cdots\\
&\mathbf{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m}\\
&\mathbf{x_1, x_2, \ldots, x_n \ge 0}
\end{aligned}$$

$$\begin{aligned}
&\min \mathbf{b_1y_1 + b_2y_2 + \cdots + b_my_m} :\\
&\mathbf{a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1}\\
&\mathbf{a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_m \ge c_2}\\
&\cdots\\
&\mathbf{a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n}\\
&\mathbf{y_1, y_2, \ldots, y_m \ge 0}
\end{aligned}$$

Given x feasible for (P), y feasible for (D):

$$\mathbf{c_1x_1 + \cdots + c_nx_n \le b_1y_1 + \cdots + b_my_m}$$

So:

$$\max\{\mathbf{c_1x_1 + \cdots + c_nx_n : x \in (P)}\} \le \min\{\mathbf{b_1y_1 + \cdots + b_my_m : y \in (D)}\}$$

QED