Facility location

Primal

$$\min \sum_{i} \mathbf{f_i y_i} + \sum_{i,j} \mathbf{c_{ij} x_{ij}} :$$

$$\sum_{i} \mathbf{x_{ij}} \geq 1 \quad \textbf{for all j}$$

$$\mathbf{x_{ij}} \leq \mathbf{y_i} \quad \textbf{for all i, j}$$

$$\mathbf{x_{ij}}, \mathbf{y_i} \geq \mathbf{0}$$

Dual

$$\max \sum_{j} \alpha_{j} :$$

$$\sum_{j} \beta_{ij} \leq \mathbf{f_i} \quad \textbf{for all i}$$

$$\alpha_{j} \leq \beta_{ij} + \mathbf{c_{ij}} \quad \textbf{for all i, j}$$

$$\alpha_{j}, \beta_{ij} \geq \mathbf{0}$$

Algorithm

1. Solve the primal and dual LPs: $y_i^*, x_{ij}^*, \alpha_j^*, \beta_{ij}^*$
2. While some clients are unassigned

   j_C : unassigned client s.t. $\alpha_{j_C}^*$ is min

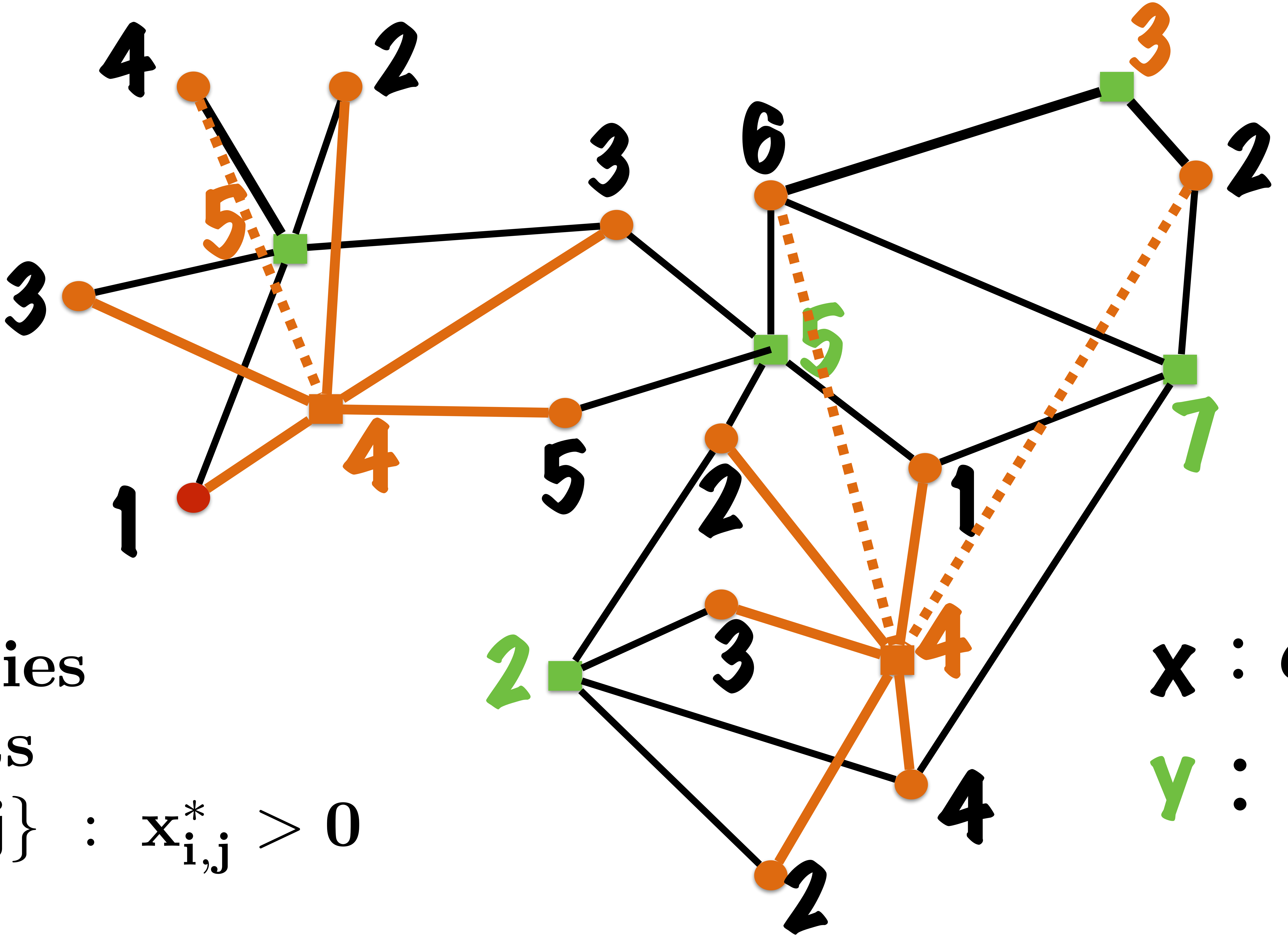
   i_C : cheapest facility s.t. $x_{i_C, j_C}^* > 0$

   open facility i_C

   assign to i_C all unassigned clients s.t.

   there is a facility with $x_{i,j_C} > 0$ and $x_{i,j} > 0$

Service cost: total red length (solid or dotted)



■ : facilities

● : clients

edge $\{i, j\}$: $x^*_{i,j} > 0$

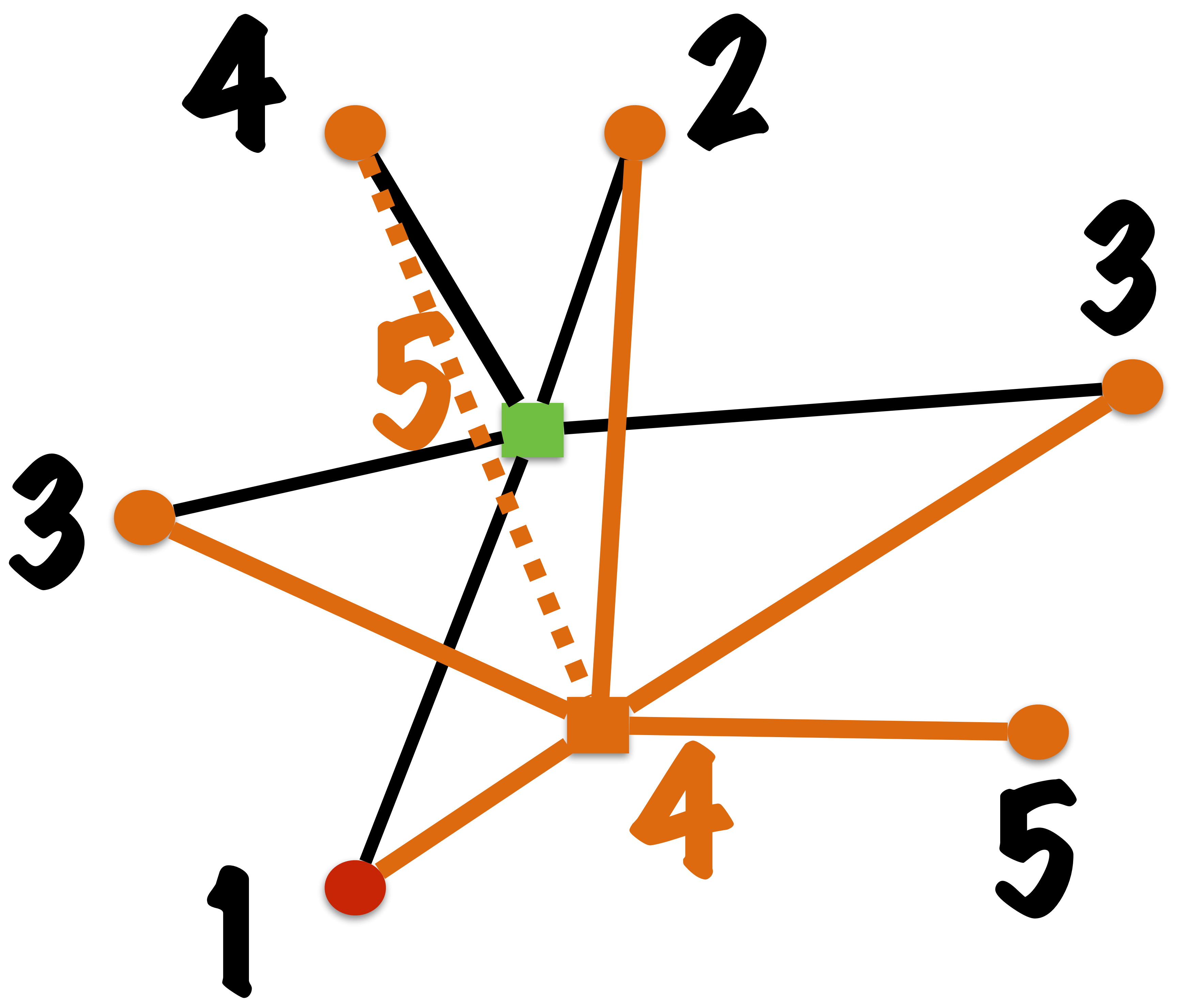
$x : \alpha^*_j$

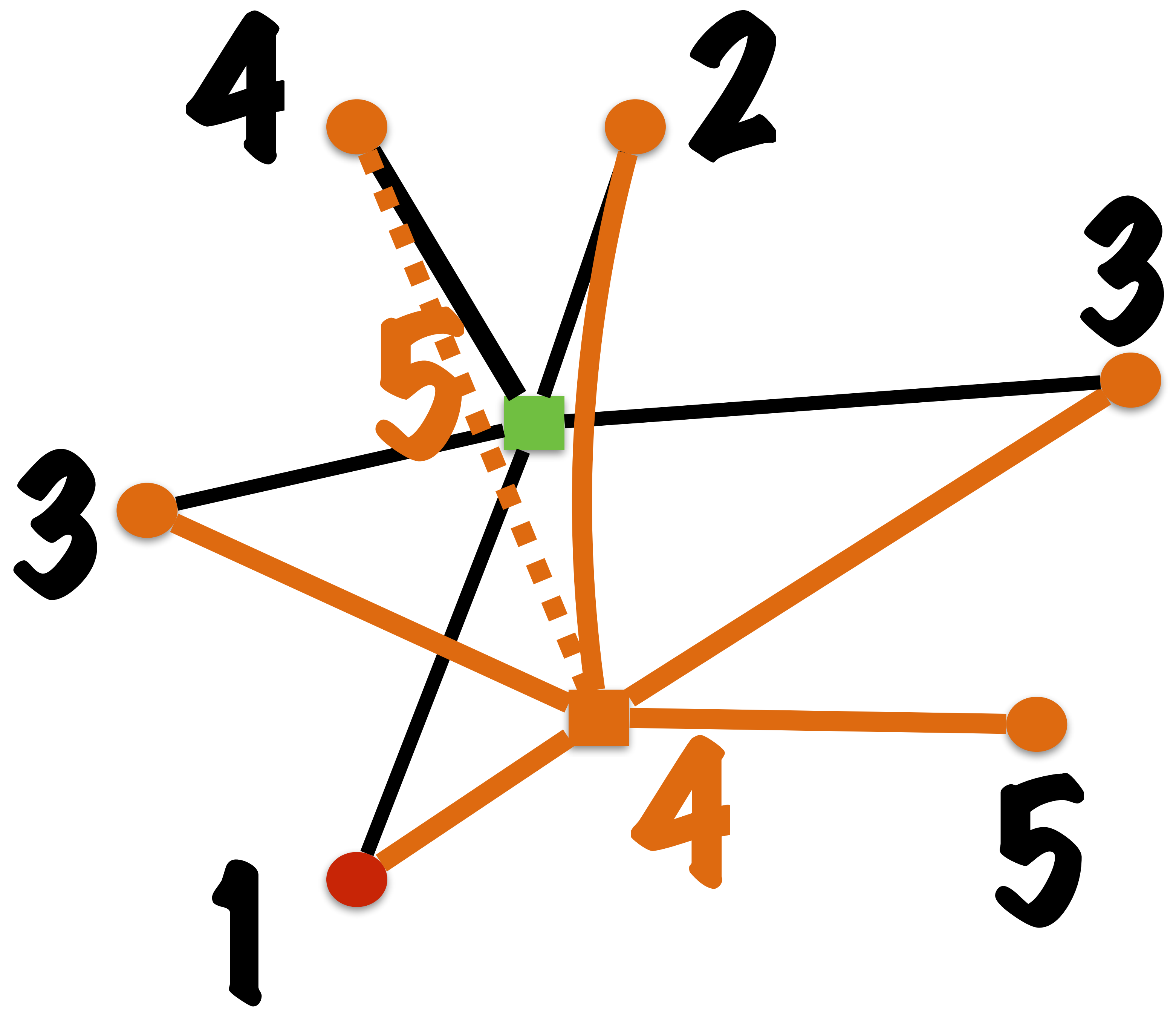
$y : f_i$

Service cost analysis

$$\sum_{\text{Cluster } C} \sum_{j \in C} c_{i_C j}$$

Metric: $c_{i_C,j} \leq c_{i_C,j_C} + c_{i,j_C} + c_{i,j}$





Observation: if $\mathbf{j} \in \mathrm{C}$ **then** $\mathbf{x}^*_{\mathbf{i},\mathbf{j}}, \mathbf{x}^*_{\mathbf{i},\mathbf{j}_\mathrm{C}}, \mathbf{x}^*_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}} > 0$

Complementary slackness:

$$\alpha^*_{\mathbf{j}} = \beta^*_{\mathbf{i},\mathbf{j}} + \mathbf{c}_{\mathbf{i},\mathbf{j}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}}$$
$$\alpha^*_{\mathbf{j}_\mathrm{C}} = \beta^*_{\mathbf{i},\mathbf{j}_\mathrm{C}} + \mathbf{c}_{\mathbf{i},\mathbf{j}_\mathrm{C}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}_\mathrm{C}}$$
$$\alpha^*_{\mathbf{j}_\mathrm{C}} = \beta^*_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}} + \mathbf{c}_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}} \geq \mathbf{c}_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}}$$

Adding: $\mathbf{c_{i_C,j} \leq 2\alpha^*_{j_C} + \alpha^*_j}$

Minimality: $\mathbf{\alpha^*_{j_C} \leq \alpha^*_j}$

So: $\mathbf{c_{i_C,j} \leq 3\alpha^*_j}$

$$\sum_{\text{Cluster } \mathbf{C}} \sum_{\mathbf{j \in C}} \mathbf{c_{i_C j}} \leq \sum_{\text{Cluster } \mathbf{C}} \sum_{\mathbf{j \in C}} \mathbf{3\alpha^*_j}$$

$$= \mathbf{3} \sum_{\mathbf{j}} \alpha^*_{\mathbf{j}}$$

Duality theorem: $\leq \mathbf{3} \cdot \mathrm{OPT}$

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