Feedback — Homework 4

Help Center

Thank you. Your submission for this homework was received.

You submitted this homework on **Fri 3 Apr 2015 12:50 PM PDT**. You got a score of **7.00** out of **7.00**.

Question 1

A fair coin is tossed repeatedly. What is the probability that at least two heads have occurred through the fifth toss?

Your Answer		Score	Explanation
$\frac{5}{16}$			
$\binom{4}{2}2^{-4}$			
$\frac{25}{32}$			
2^{-5}			
$\frac{13}{16}$	✔	1.00	
$\frac{7}{16}$			
Total		1.00 / 1.00	

Question Explanation

The complement of the event in question is easy to handle as this corresponds to either no heads or exactly one head occurring through five tosses. If there are no heads then we have a sequence of five tails in a row and, as the tosses are independent, this has probability

$$\left(\tfrac{1}{2}\right)^5 = 2^{-5}.$$

If there is exactly one head then it can be in any one of five positions leading to the outcomes HTTTT, THTTT, TTHTT, TTTHT, or TTTTH. Each of these sequences has probability 2^{-5} (again by appeal to independence, heads and tails having equal probability $1/2$ as the coin is fair) and so the probability of exactly one head in five tosses is $5 \cdot 2^{-5}$. By additivity, the probability of obtaining fewer than two heads in the first five tosses is

$$2^{-5} + 5 \cdot 2^{-5} = 6 \cdot 2^{-5} = \frac{3}{16}.$$

By additivity, once more, the probability of the complementary event that two or more heads is obtained in the first five tosses is given by

$$1 - \frac{3}{16} = \frac{13}{16}.$$

An alternative approach is informative. What is the probability of obtaining *exactly* k heads in the first five tosses? Well, the locations of these k heads in the sequence can be selected in $\binom{5}{k}$ ways, the remaining $5-k$ locations all being occupied by tails. We have a setting of repeated independent trials with each toss resulting in heads and tails with equal probability $1/2$. Accordingly, any given sequence of outcomes for the five tosses has probability 2^{-5} and so the probability of exactly k heads in five tosses is given by

$$\binom{5}{k}2^{-5}.$$

The probability of at least two heads is obtained, via additivity, by summing over values of k ranging from 2 to 5 and so, the probability that there are at least two heads in five tosses is given by

$$\binom{5}{2}2^{-5} + \binom{5}{3}2^{-5} + \binom{5}{4}2^{-5} + \binom{5}{5}2^{-5} = \left[\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right]2^{-5} = [10 + 10 + 5 + 1]2^{-5} = \frac{26}{32} = \frac{13}{16}.$$

We have just discovered the august *binomial distribution* which is the subject of Tableau 10.1.

Question 2

A fair coin is tossed repeatedly. What is the probability that the second head occurs on the fifth toss?

Your Answer		Score	Explanation
$\binom{4}{3}2^{-4}$			
2^{-3}	✔	1.00	
2^{-4}			
$\binom{5}{2}2^{-5}$			
$1-(5)2^{-5}$			
$\binom{4}{2}2^{-4}$			
Total		1.00 / 1.00	

Question Explanation

It is easy to write down the sample points that trigger the event of interest explicitly: HTTTH, THTTH, TTHTH, TTTHH. As the trials are independent, the probability that the second head occurs on the fifth toss is $4\times 2^{-5}=2^{-3}$.

More generally, suppose $1\le r\le n$. We toss a coin with success (heads) probability p repeatedly stopping at the rth success. What is the probability that we terminate on the nth toss? The key idea to keep in mind is that if the sequence terminates on the nth toss then it must be the case that the nth toss indeed resulted in the rth success as we stop immediately the moment we achieve r successes. It follows then that the first $n-1$ tosses must contain $r-1$ successes in any order whatsoever with the remaining $(n-1)-(r-1)=n-r$ trials resulting in failures. As the trials are independent, the probability that the first $n-1$ tosses result in exactly $r-1$ heads (in any order) is hence given by

$$\binom{n-1}{r-1}p^{r-1}(1-p)^{n-r}.$$

The nth trial is independent of the first $n-1$ trials and must result in a success if the sequence is to be terminated. It follows that the probability that the sequence terminates at trial n with the rth success is given by

$$\binom{n-1}{r-1}p^{r-1}(1-p)^{n-r}\cdot p=\binom{n-1}{r-1}p^{r}(1-p)^{n-r}\qquad(n\ge r).$$

This is called the *negative binomial distribution*. It comes into play when there is a waiting time for an event to occur.

Question 3

The following prompt should be used for **Questions 3 and 4**.

Suppose A and B are independent events with positive probabilities, $\mathbf{P}(A)>0$ and $\mathbf{P}(B)>0$. Consider the following statements:

a. $\mathbf{P}(A\mid B)=\mathbf{P}(B\mid A)$.
b. The two events A and B are disjoint.
c. $\mathbf{P}(A\cup B)=\mathbf{P}(B\mid A)+\mathbf{P}(A)\bigl(1-\mathbf{P}(B)\bigr)$.
d. $\mathbf{P}(A\cup B)=\mathbf{P}(A\mid B)+\mathbf{P}(B\mid A)$.
e. $\mathbf{P}\bigl(B^{\complement}\mid A\bigr)=1-\mathbf{P}(B\mid A)$.

Which of the succeeding options identifies *all* of the preceding statements that are *never* true under the given conditions?

Your Answer	Score	Explanation
a, b, and c		

a, b, and d		
d and e		
a		
b, c, and d		
b and d	✔	1.00
Total		1.00 / 1.00

Question Explanation

The statements b and d are never true; the others can be true. Let us analyse each of the statements in turn:

a. By the definition of independence, we get $\mathbf{P}(B \mid A) = \mathbf{P}(A)$ and $\mathbf{P}(A \mid B) = \mathbf{P}(A)$. In other words, conditioning on an independent event won't change the probability of the event. So the statement reduces to $\mathbf{P}(A) = \mathbf{P}(B)$ which **can be true** for the given conditions.

b. Suppose A and B are disjoint; in other words, $\mathbf{P}(A \cap B) = 0$. On the other hand, the independence of the events tells us that $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B) > 0$ as each of A and B is given to have positive probability. We have a contradiction. It follows that statement b is **never true** for the given conditions.

c. By inclusion and exclusion (Tableau 5:h), the left-hand side may be expressed in the form

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A)\,\mathbf{P}(B),$$

the final step following because A and B are given to be independent. Again, the independence of A and B tells us that $\mathbf{P}(B \mid A) = \mathbf{P}(B)$ (see Tableau 9.1:b) and so the right-hand side is given by

$$\mathbf{P}(B \mid A) + \mathbf{P}(A)\big(1 - \mathbf{P}(B)\big) = \mathbf{P}(B) + \mathbf{P}(A) - \mathbf{P}(A)\,\mathbf{P}(B).$$

It follows that statement c is **always true** under the given conditions.

d. By independence, the right-hand side becomes

$$\mathbf{P}(B \mid A) + \mathbf{P}(A \mid B) = \mathbf{P}(A) + \mathbf{P}(B).$$

But, as we have seen from part c, the left-hand side is given by

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A)\ \mathbf{P}(B) > \mathbf{P}(A) + \mathbf{P}(B)$$

as we are given that $\mathbf{P}(A) > 0$ and $\mathbf{P}(B) > 0$. So statement d is **never true** under the given conditions. You should note that under the condition that A and B are independent, statement d is just statement b in another guise.

e. The independence of A and B implies the independence of $A^{\complement}$ and B (Tableau 9.1:b). It follows that both the identities $\mathbf{P}(B \mid A) = \mathbf{P}(B)$ and $\mathbf{P}(B^{\complement} \mid A) = \mathbf{P}(B^{\complement})$ hold and so

$$\mathbf{P}(B^{\complement} \mid A) = \mathbf{P}(B^{\complement}) = 1 - \mathbf{P}(B) = 1 - \mathbf{P}(B \mid A).$$

We conclude that statement e is **always true** under the given conditions.

Question 4

Which of the succeeding options identifies *all* the statements in the prompt for Question 3 that are *always* true under the given conditions?

Your Answer		Score	Explanation
a			
d and e			
b and c			
c and e	✔	1.00	
a, c, and e			
a and e			

Total	1.00 / 1.00

Question Explanation

Under the conditions given in the prompt, statements c and e are always true; the remaining statements are either false or only sometimes true. The question-level solution to the previous problem has the details of the analysis.

Question 5

The following prompt should be used for **Questions 5 and 6**.

In a sanitised genetic model, suppose eye colour is governed by a gene existing in two variants (alleles) *B* and *b*. The brown colour gene variant *B* is dominant; the blue colour gene variant *b* is recessive. The genotype of an individual is determined by one of the three gene pairings *BB, Bb*, or *bb* (gene ordering does not matter). Individuals with genotypes *BB* or *Bb* will have brown eyes though the latter genotype carries the blue gene and can propagate it; individuals with genotype *bb* have blue eyes. Suppose that for both sexes in the parental generation the genotypes *BB* and *bb* each occur with frequency $1/4$, while genotype *Bb* occurs with frequency $1/2$. We consider a setting of random selection and mating where parental genotypes are randomly selected and each parent contributes one randomly selected gene from their genotype to the child's genotype. [Tableaux 9.3:e through 9.3:k provide definitions, background, and ways to think about such problems.]

Darius and his wife Atossa are expecting a son who will be named Xerxes. Without knowing Darius's or Atosa's eye color, what is the probability that Xerxes will have blue eyes? [*Hint:* What is the sample space? A graphical representation as in Tableau 9.3:h makes matters transparent.]

Your Answer		Score	Explanation
◉ $\frac{1}{4}$	✔	1.00	
○ $\frac{3}{16}$			
○ $\frac{1}{3}$			
○ $\frac{1}{2}$			
○ $\frac{4}{9}$			
○ 1			
Total		1.00 / 1.00	

Question Explanation

The chance-dependent elements in the problem are: Darius's genotype, Atossa's genotype, the gene selection leading to Xerxes's genotype, and the gene selection leading to Acahimenes's genotype. Write D, At, X, and Ac, respectively, for the four genotypes.

By random selection, the parental genotype combinations have *a priori* probabilities given by:

D, At	*a priori* probability
BB, BB	$1/16$
BB, Bb	$1/8$
BB,bb	$1/16$
Bb, BB	$1/8$
Bb, Bb	$1/4$
Bb, bb	$1/8$
bb, BB	$1/16$
bb, Bb	$1/8$
bb, bb	$1/16$

By virtue of random mating, the conditional probabilities in the filial generation, given the parental genotype combinations, are given in the following table:

D, At	BB	Bb	bb
BB, BB	1	0	0
BB, Bb	1/2	1/2	0
BB,bb	0	1	0
Bb, BB	1/2	1/2	0
Bb, Bb	1/4	1/2	1/4
Bb, bb	0	1/2	1/2
bb, BB	0	1	0
bb, Bb	0	1/2	1/2
bb, bb	0	0	1

Total probability now allows us to quickly write down the probability that Xerxes will have blue eyes. Skipping the parental genotype combinations for which the conditional probability of a blue-eyed child is zero to avoid unnecessarily complicating our equation, we see that

$$\begin{aligned}\mathbf{P}\{\mathrm{X}=bb\} &= \mathbf{P}\{\mathrm{X}=bb \mid \mathrm{D}=Bb, \mathrm{At}=Bb\}\,\mathbf{P}\{\mathrm{D}=Bb, \mathrm{At}=Bb\}+\mathbf{P}\{\mathrm{X}=bb \mid \mathrm{D}=Bb, \mathrm{At}=bb\}\,\mathbf{P}\{\mathrm{D}=Bb, \mathrm{At}=bb\}\\ &\quad+\mathbf{P}\{\mathrm{X}=bb \mid \mathrm{D}=bb, \mathrm{At}=Bb\}\,\mathbf{P}\{\mathrm{D}=bb, \mathrm{At}=Bb\}+\mathbf{P}\{\mathrm{X}=bb \mid \mathrm{D}=bb, \mathrm{At}=bb\}\,\mathbf{P}\{\mathrm{D}=bb, \mathrm{At}=bb\}\\ &=\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{8}+\frac{1}{2}\cdot\frac{1}{8}+1\cdot\frac{1}{16}=\frac{4}{16}=\frac{1}{4}.\end{aligned}$$

Take a good hard look at our answer. Do you notice anything interesting? Here it is: the genotype frequency is *unchanged* from parental to filial genotype. We've discovered a law named after the British mathematician G. H. Hardy. Let me borrow some text from *The Theory of Probability* to describe what we have discovered.

> *Hardy's law*: Random mating within an arbitrary parent population will produce within one generation an approximately stationary genotype distribution with gene frequencies unchanged from that of the parent population.

Hardy's law should not be read to imply an immutability in gene frequencies. The key word is "approximate". While there is no systematic drift in any direction after the first generation, neither is there any restoring force which seeks to hold parental gene frequencies fixed. In spite of the stabilising influence of Hardy's law from generation to generation, chance fluctuations will ultimately doom the population to one of boring homogeneity if there are no changes imposed from without on the system. Fortunately, mutations, the introduction of new genetic material from outside, and many other effects mitigate against the adoption of a uniform party line (i.e., absorption into a homozygous population) in practice. *Vive la difference!*

Question 6

We are told that Xerxes has brilliant blue eyes but do not have information on the eye colours of Darius and Atossa who are now expecting another child who will be named Achaimenes. What is the probability that Achaimenes will also have blue eyes? [History buffs will recognise that the story line for this problem is borrowed from an ancient Persian lineage.]

Your Answer		Score	Explanation
$\frac{4}{9}$			
1			
$\frac{1}{2}$			
$\frac{2}{3}$			
$\frac{1}{4}$			
$\frac{9}{16}$	✔	1.00	
Total		1.00 / 1.00	

Question Explanation

Continuing with the notation and tables introduced in the solution to Question 5, conditioned on the event {X = *bb*}, neither parent can have the *BB* genotype as is clear from the last column of the table of conditional probabilities for the filial genotype given the parental genotypes. Thus, conditioned on the event {X = *bb*}, the only parental combinations that are feasible are (D, At) = (*Bb, Bb*) or (*Bb, bb*) or (*bb, Bb*) or (*bb, bb*). We may compute the *a posteriori* probabilities of these possibilities via Bayes's rule for events [Lecture 8.2:k]:

$$\mathbf{P}\{\mathrm{D}=Bb,\mathrm{At}=Bb\mid \mathrm{X}=bb\}=\frac{\mathbf{P}\{\mathrm{X}=bb\mid \mathrm{D}=Bb,\mathrm{At}=Bb\}\,\mathbf{P}\{\mathrm{D}=Bb,\mathrm{At}=Bb\}}{\mathbf{P}\{\mathrm{X}=bb\}}=\frac{\frac{1}{4}\times\frac{1}{4}}{\frac{1}{4}}=\frac{1}{4},$$

$$\mathbf{P}\{\mathrm{D}=Bb,\mathrm{At}=bb\mid \mathrm{X}=bb\}=\frac{\mathbf{P}\{\mathrm{X}=bb\mid \mathrm{D}=Bb,\mathrm{At}=bb\}\,\mathbf{P}\{\mathrm{D}=Bb,\mathrm{At}=bb\}}{\mathbf{P}\{\mathrm{X}=bb\}}=\frac{\frac{1}{2}\times\frac{1}{8}}{\frac{1}{4}}=\frac{1}{4},$$

$$\mathbf{P}\{\mathrm{D}=bb,\mathrm{At}=Bb\mid \mathrm{X}=bb\}=\frac{\mathbf{P}\{\mathrm{X}=bb\mid \mathrm{D}=bb,\mathrm{At}=Bb\}\,\mathbf{P}\{\mathrm{D}=bb,\mathrm{At}=Bb\}}{\mathbf{P}\{\mathrm{X}=bb\}}=\frac{\frac{1}{2}\times\frac{1}{8}}{\frac{1}{4}}=\frac{1}{4},$$

$$\mathbf{P}\{\mathrm{D}=bb,\mathrm{At}=bb\mid \mathrm{X}=bb\}=\frac{\mathbf{P}\{\mathrm{X}=bb\mid \mathrm{D}=bb,\mathrm{At}=bb\}\,\mathbf{P}\{\mathrm{D}=bb,\mathrm{At}=bb\}}{\mathbf{P}\{\mathrm{X}=bb\}}=\frac{1\times\frac{1}{16}}{\frac{1}{4}}=\frac{1}{4},$$

These calculations may be summarised in the following table of *a posteriori* probabilities for the genotype pairing of the parents given that Xerxes has blue eyes:

D, At	*a posteriori* probability of parental genotypes given that $\mathrm{X}=bb$
Bb, Bb	$1/4$
Bb, bb	$1/4$
bb, Bb	$1/4$
bb, bb	$1/4$

Given that Xerxes has blue eyes, the parental genotype combinations contributing to the sample space have reduced to just the four possibilities with *a posteriori* probabilities given above. Again under conditions of random mating, the conditional probabilities for Achaimenes's genotype, given these parental genotype pairs, are now given by

D, At	BB	Bb	bb
Bb, Bb	$1/4$	$1/2$	$1/4$
Bb, bb	0	$1/2$	$1/2$
bb, Bb	0	$1/2$	$1/2$
bb, bb	0	0	1

This, of course, is just an extraction of these four rows from the original table of conditional probabilities given in the solution to Question 5.

A version of total probability when events are conditioned upon a common event now allows us to quickly write down the probability that Xerxes will have blue eyes. Skipping the parental genotype combinations for which the conditional probability of a blue-eyed child is zero to avoid unnecessarily complicating our equation, we see that

$$\begin{aligned}\mathbf{P}\{\mathrm{Ac}=bb\mid \mathrm{X}=bb\} &= \mathbf{P}\{\mathrm{Ac}=bb\mid \mathrm{D}=Bb,\mathrm{At}=Bb,\mathrm{X}=bb\}\,\mathbf{P}\{\mathrm{D}=Bb,\mathrm{At}=Bb\mid \mathrm{X}=bb\}\\ &\quad+\mathbf{P}\{\mathrm{Ac}=bb\mid \mathrm{D}=Bb,\mathrm{At}=bb,\mathrm{X}=bb\}\,\mathbf{P}\{\mathrm{D}=Bb,\mathrm{At}=bb\mid \mathrm{X}=bb\}\\ &\quad+\mathbf{P}\{\mathrm{Ac}=bb\mid \mathrm{D}=bb,\mathrm{At}=Bb,\mathrm{X}=bb\}\,\mathbf{P}\{\mathrm{D}=bb,\mathrm{At}=Bb\mid \mathrm{X}=bb\}\\ &\quad+\mathbf{P}\{\mathrm{Ac}=bb\mid \mathrm{D}=bb,\mathrm{At}=bb,\mathrm{X}=bb\}\,\mathbf{P}\{\mathrm{D}=bb,\mathrm{At}=bb\mid \mathrm{X}=bb\}\\ &=\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}+1\cdot\frac{1}{4}=\frac{9}{16}.\end{aligned}$$

Is this in the direction of what we might expect? The information that his brother had blue eyes eliminates *BB* parental genotype combinations. Roughly speaking, the odds of blue genes in the parental mix are now improved. The student may be reminded of Laplace's law of succession.

Question 7

Between May 15 and July 17 of 1951, Joe DiMaggio set a Major League Baseball record with a streak of 56 consecutive games in which he had at least one hit. For many baseball purists, this is the greatest record in sports. Let's do a simplified preliminary analysis of a streak of this nature in this problem. If you don't know the rules of baseball, don't worry. We won't need the details.

Suppose that our hero plays in 56 consecutive games in each of which he gets four "at bats". His chances of getting a "hit" on any given "at bat" is 0.325. Analyse the problem by modelling his sequence of at bats as a sequence of coin tosses with a hit on an at bat counting as a success in a coin toss. What is the probability that he gets at least one hit in each of the 56 games he plays?

[This portion is not required and is intended for those who would like to test their understanding of streaks based on the lectures. Write a computer programme based upon the recurrences developed in the lectures to determine the probability of obtaining at least one streak of 56 consecutive games with at least one hit per game over a 154 game season. Based on what you discover, does DiMaggio's streak appear really unusual?]

Your Answer		Score	Explanation
0.0111566			
2.76065×10^{-10}			
3.14159×10^{-7}			
4.6288×10^{-28}			
5.05663×10^{-23}			
2.19316×10^{-6}	✔	1.00	
Total		1.00 / 1.00	

Question Explanation

The probability that our intrepid hero doesn't get a hit on an at bat is $1 - 0.325$. As the at bats constitute repeated independent trials, the probability that he doesn't get any hits in a given game (in which he has four at bats) is given by

$$(1 - 0.325)^4.$$

By additivity, the probability that he gets at least one hit in a given game is given by

$$p = 1 - (1 - 0.325)^4.$$

Each game represents an independent trial with success probability p. (The at bats are partitioned into 56 disjoint sub-collections, each containing four at bats. The event that he gets at least one hit in a given game is determined solely by the four at bats corresponding to that game.) It follows that the probability that he gets at least one hit in each of 56 successive games is

$$p^{56} = \left(1 - (1 - 0.325)^4\right)^{56} = 2.19316 \times 10^{-6}.$$