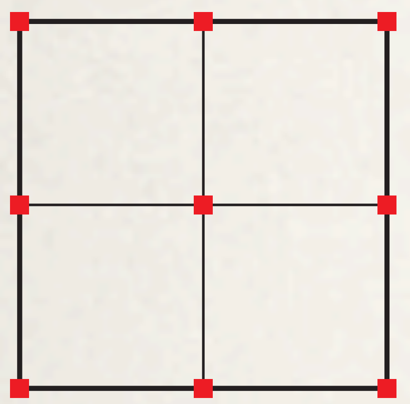
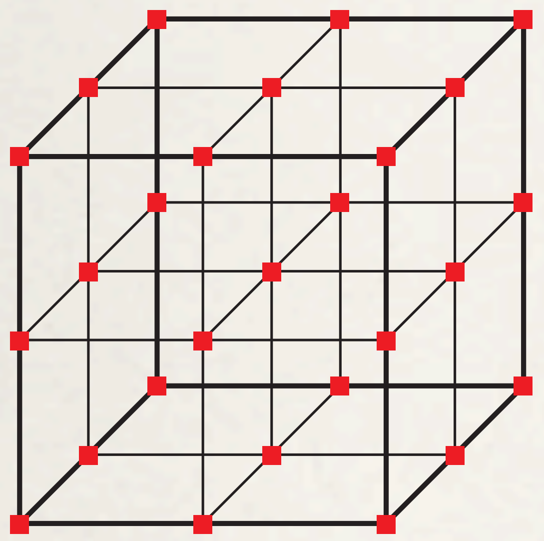
Scaling with dimension

$$J = \int_0^1 f(x)\,dx \approx \frac{1}{6}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + f(1)\right]$$

$$J = \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_D)\, dx_D \cdots dx_1 \approx \sum_{x_1 \in \{0,0.5,1\}} \cdots \sum_{x_D \in \{0,0.5,1\}} a_{x_1,\ldots,x_D} f(x_1, \ldots, x_D)$$





dimension	# computations
1	3
2	9
3	27
$\vdots$	$\vdots$
D	3^D
200	$3^{200} \approx 10^{95}$

This is impossible! Even approximately.

Physics takes a gamble!
Sampling a function at a random point

A problem in numerical integration

A problem in numerical integration

Given:

Aa function $f(x_1, \ldots, x_D)$ of one or more variables on the unit interval (unit square, unit cube, ...), bounded in absolute value by 1.

A problem in numerical integration

Given:

Aa function $f(x_1, \ldots, x_D)$ of one or more variables on the unit interval (unit square, unit cube, …), bounded in absolute value by 1.

Evaluate: $J = \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_D)\, dx_D \cdots dx_1$

A chance-driven computation

A chance-driven computation

Given: a bounded function $f(\boldsymbol{x}) = f(x_1, \ldots, x_D)$ on the unit cube $[0, 1]^D$ with $|f(\boldsymbol{x})| \leq 1$

A chance-driven computation

Given: a bounded function $f(\boldsymbol{x}) = f(x_1, \ldots, x_D)$ on the unit cube $[0,1]^D$ with $|f(\boldsymbol{x})| \leq 1$

Compute: $J = \int \cdots \int_{[0,1]^D} f(\boldsymbol{x})\, d\boldsymbol{x}$

A chance-driven computation

Given: a bounded function $f(\mathbf{x}) = f(x_1, \ldots, x_D)$ on the unit cube $[0,1]^D$ with $|f(\mathbf{x})| \leq 1$

Compute: $J = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x}$

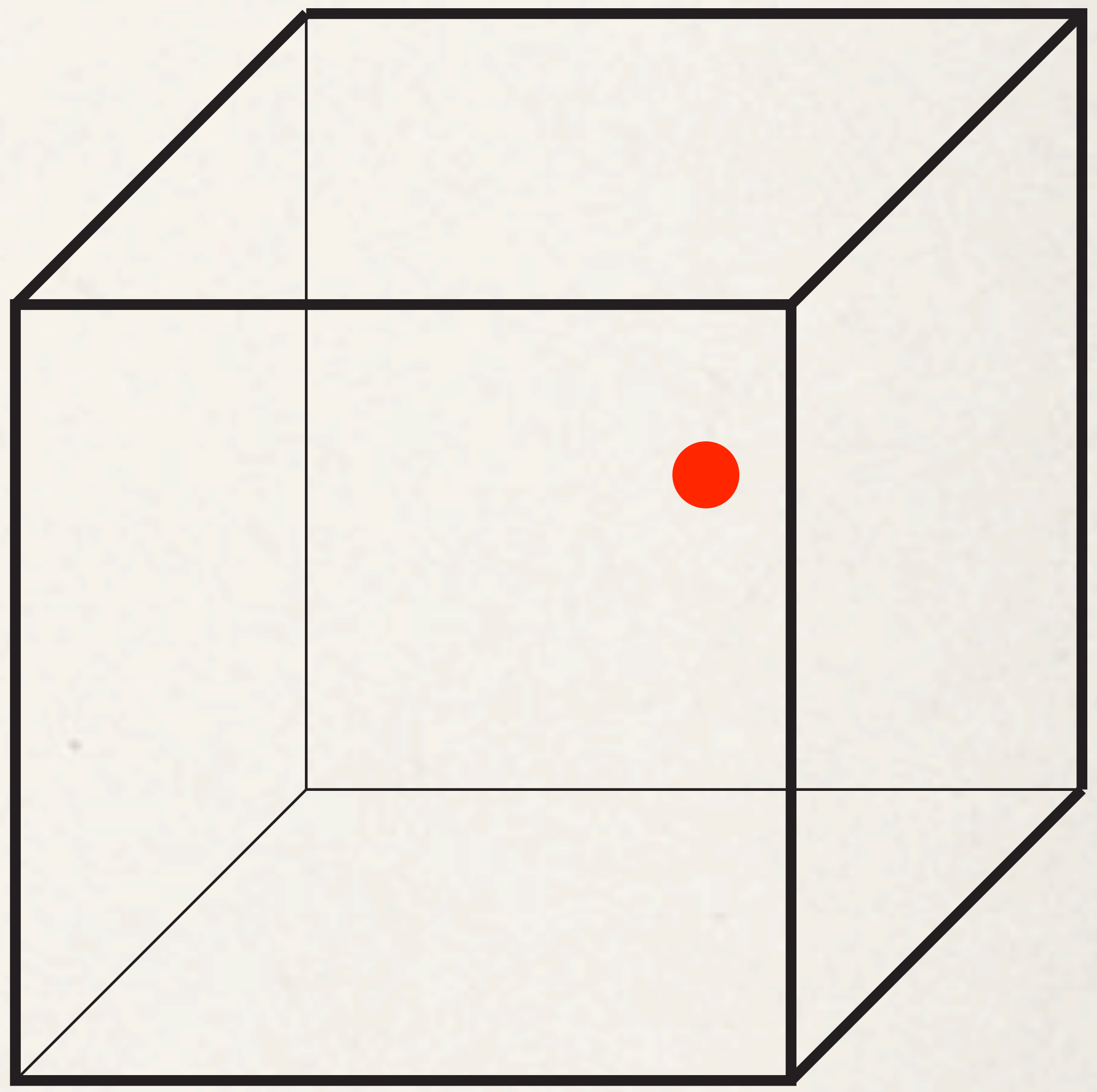
Select $\mathbf{X} = (X_1, \ldots, X_D)$ at random from the D-dimensional cube $[0, 1]^D$

A chance-driven computation

Given: a bounded function $f(\mathbf{x}) = f(x_1, \ldots, x_D)$ on the unit cube $[0,1]^D$ with $|f(\mathbf{x})| \le 1$

Compute: $J = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x}$

Select $\mathbf{X} = (X_1, \ldots, X_D)$ at random from the D-dimensional cube $[0, 1]^D$



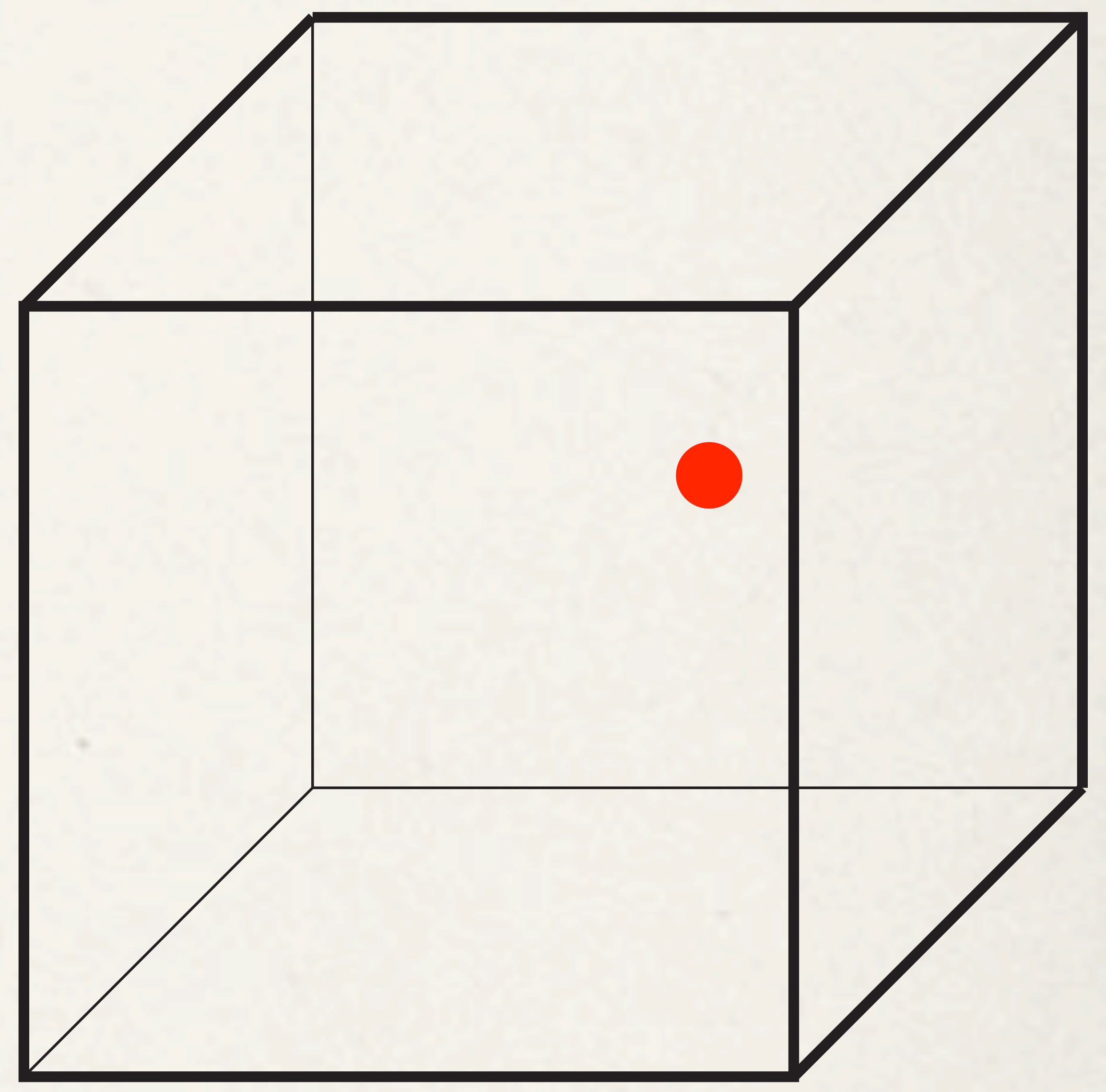
A chance-driven computation

Given: a bounded function $f(\mathbf{x}) = f(x_1, \ldots, x_D)$ on the unit cube $[0,1]^D$ with $|f(\mathbf{x})| \leq 1$

Compute: $J = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x}$

Select $\mathbf{X} = (X_1, \ldots, X_D)$ at random from the D-dimensional cube $[0, 1]^D$

This means: $X_1, \ldots, X_D$ are independent and are each uniformly distributed in the unit interval.



A chance-driven computation

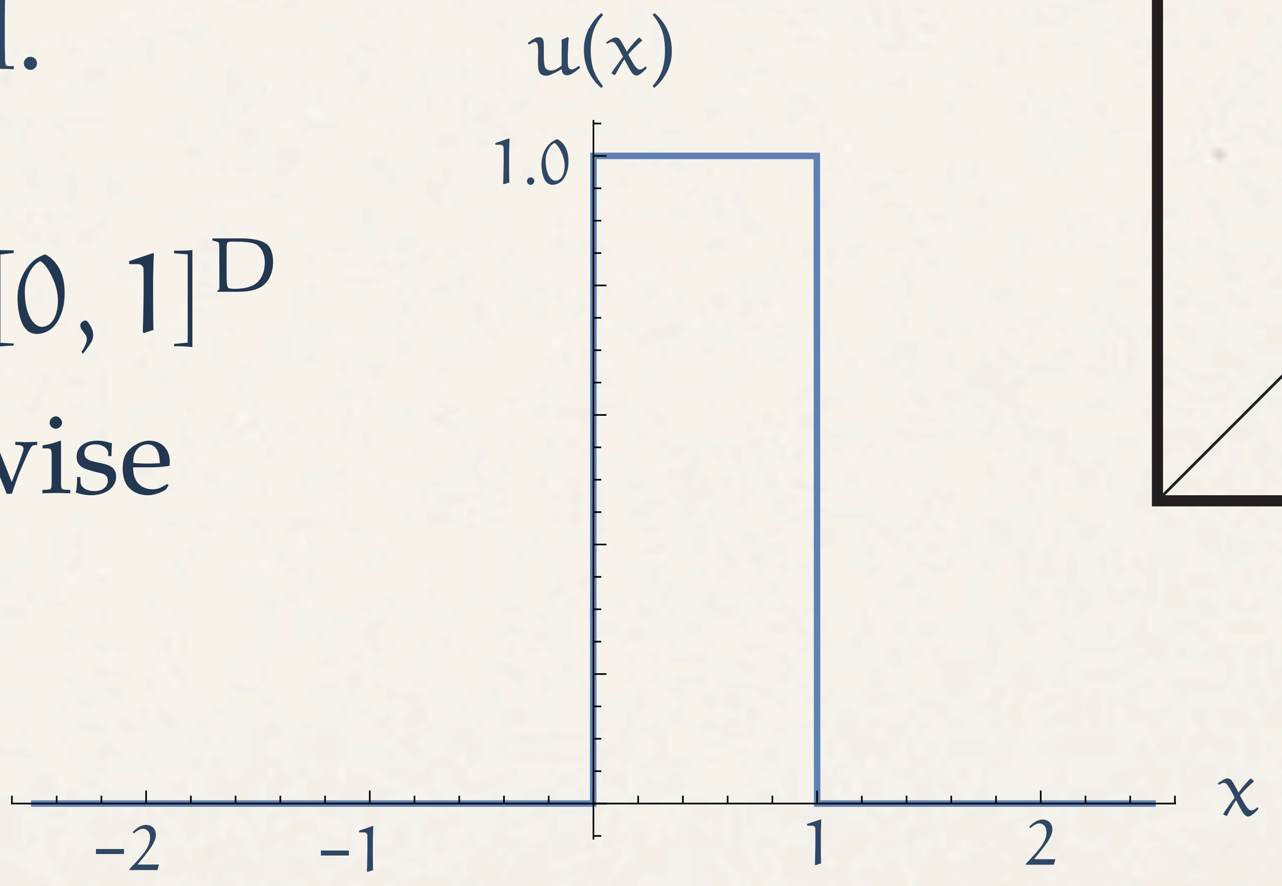
Given: a bounded function $f(\mathbf{x}) = f(x_1, \ldots, x_D)$ on the unit cube $[0,1]^D$ with $|f(\mathbf{x})| \leq 1$

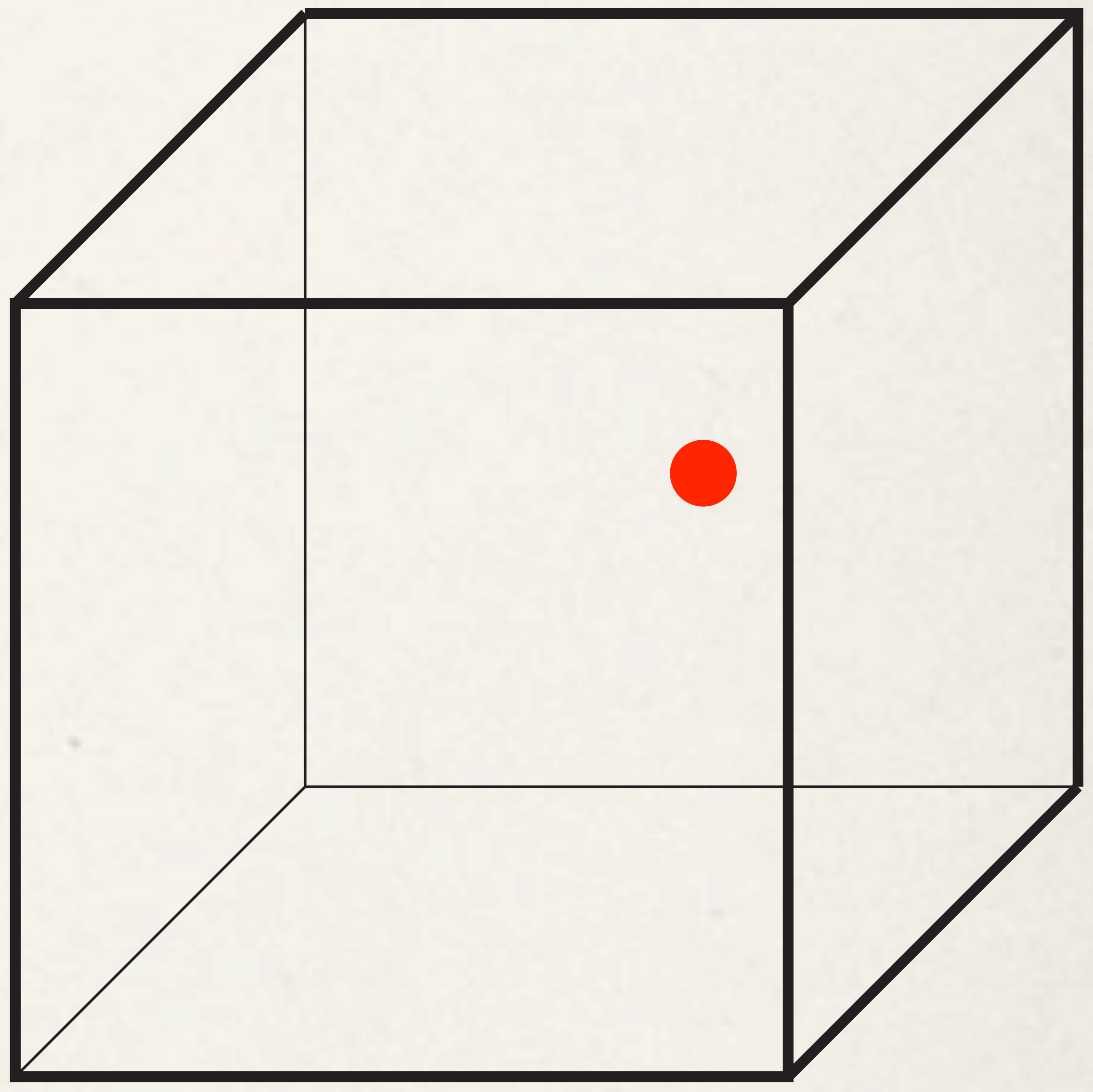
Compute: $$J = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x}$$

Select $\mathbf{X} = (X_1, \ldots, X_D)$ at random from the D-dimensional cube $[0, 1]^D$

This means: $X_1, \ldots, X_D$ are independent and are each uniformly distributed in the unit interval.

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$





A chance-driven computation

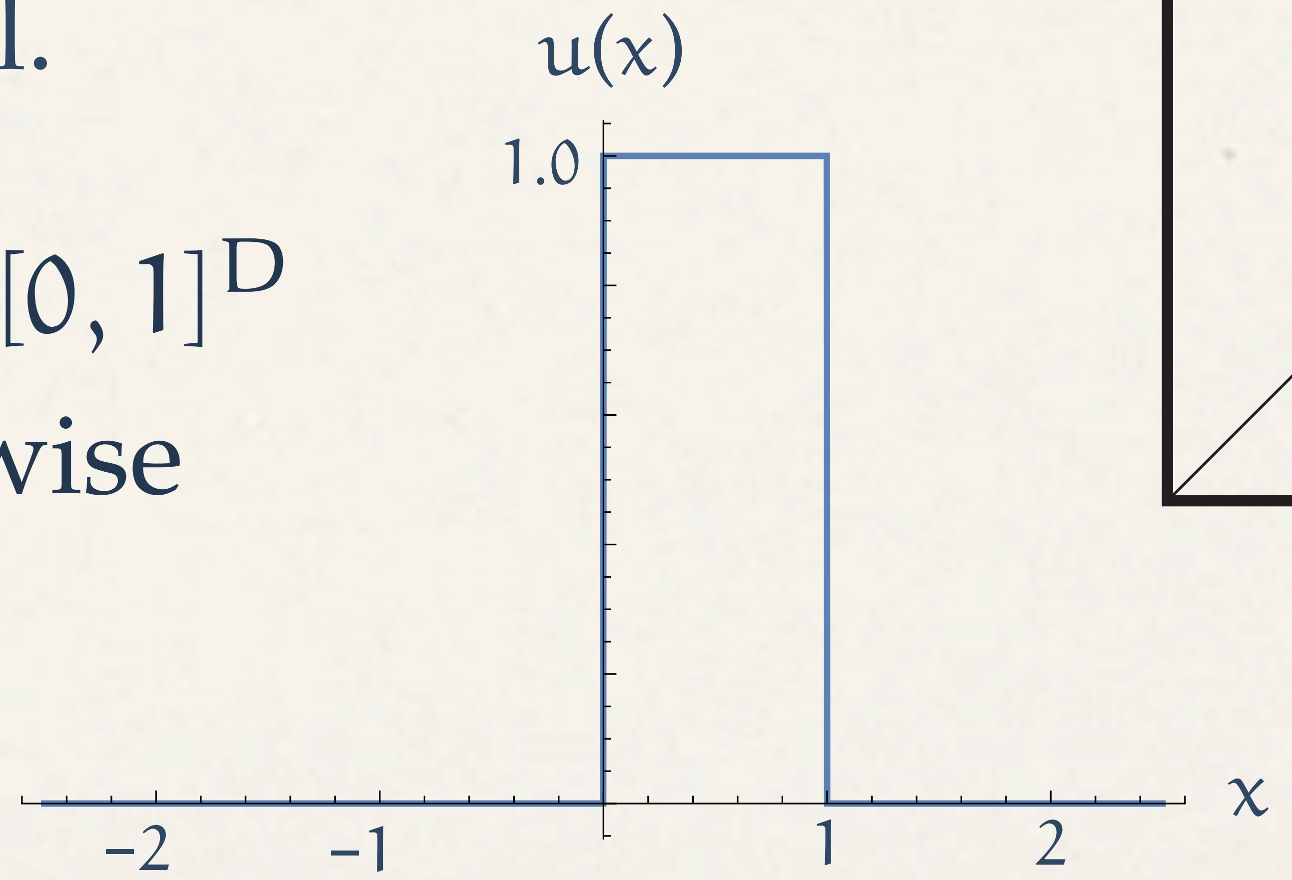
Given: a bounded function $f(\mathbf{x}) = f(x_1, \ldots, x_D)$ on the unit cube $[0,1]^D$ with $|f(\mathbf{x})| \leq 1$

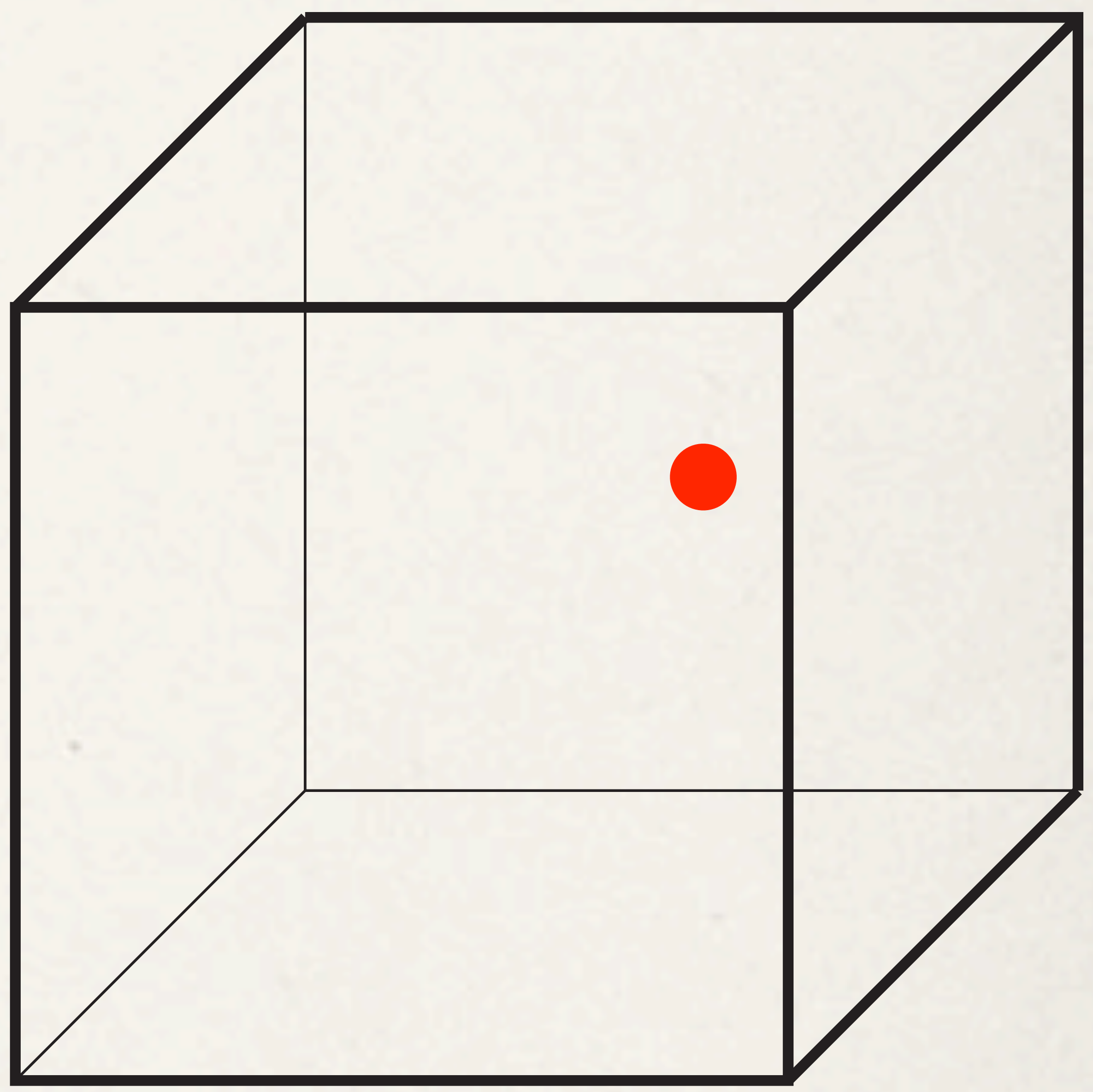
Compute: $J = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x}$

Select $\mathbf{X} = (X_1, \ldots, X_D)$ at random from the D-dimensional cube $[0, 1]^D$

This means: $X_1, \ldots, X_D$ are independent and are each uniformly distributed in the unit interval.

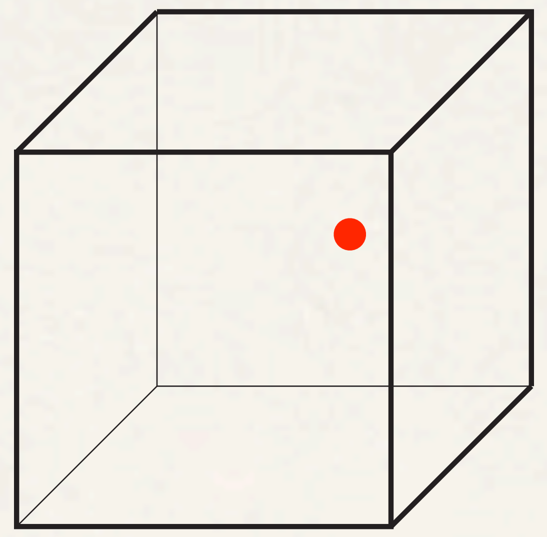
$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



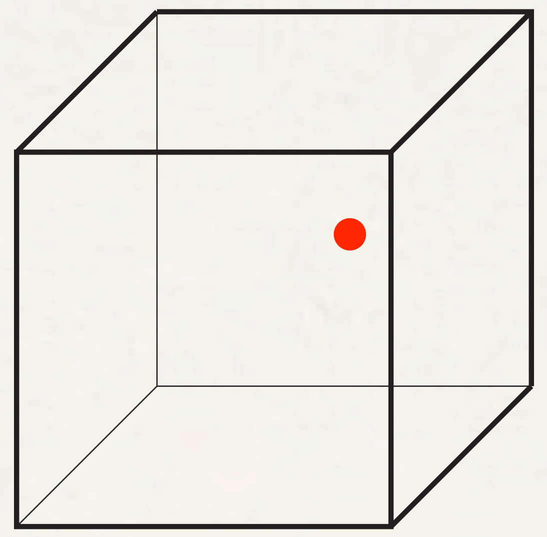


Evaluate $Y = f(\mathbf{X})$

$$\mathbf{X} \sim p(\boldsymbol{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$

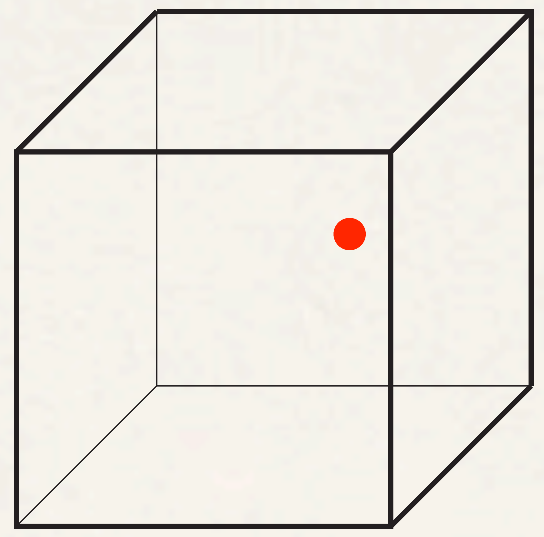


$$\mathbf{X} \sim p(\boldsymbol{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



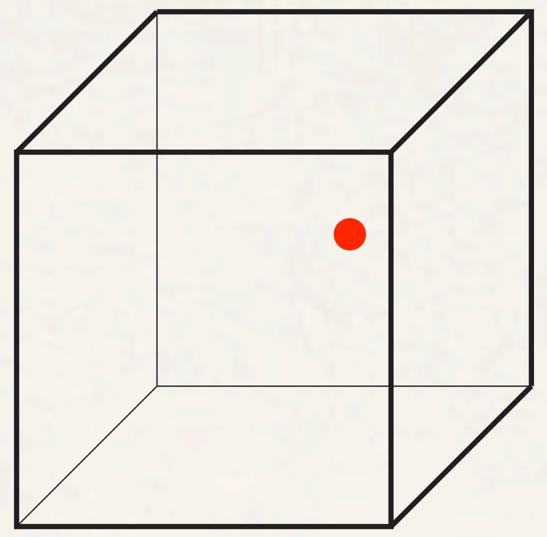
$$E\big(f(\mathbf{X})\big)$$

$$\mathbf{X} \sim p(\boldsymbol{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



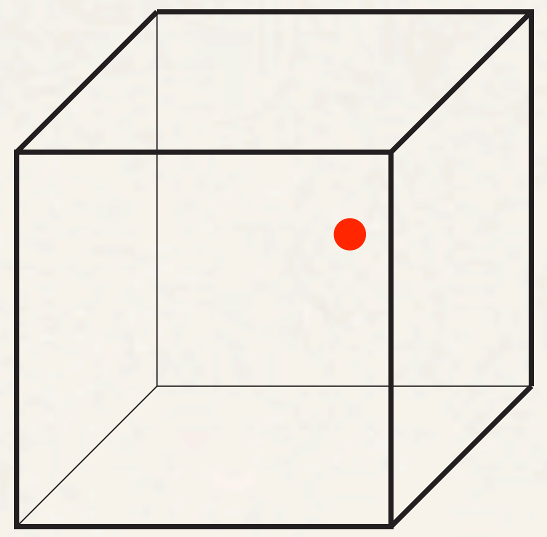
$$\mathrm{E}\big(f(\mathbf{X})\big) = \int \cdots \int f(\boldsymbol{x}) \cdot p(\boldsymbol{x})\, d\boldsymbol{x}$$

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



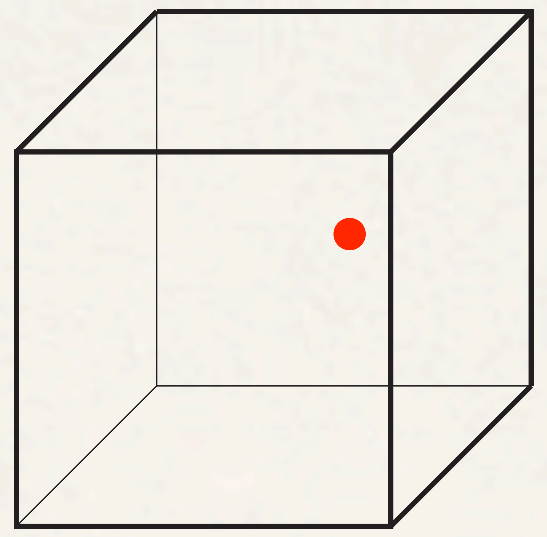
$$\mathbb{E}\big(f(\mathbf{X})\big) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x}) \, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x}) \, d\mathbf{x}$$

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



$$E(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

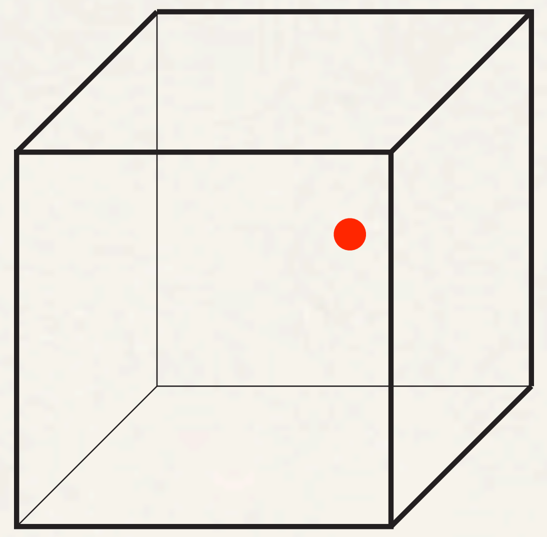
$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



$$\mathrm{E}\big(f(\mathbf{X})\big) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$\mathrm{Var}\big(f(\mathbf{X})\big)$$

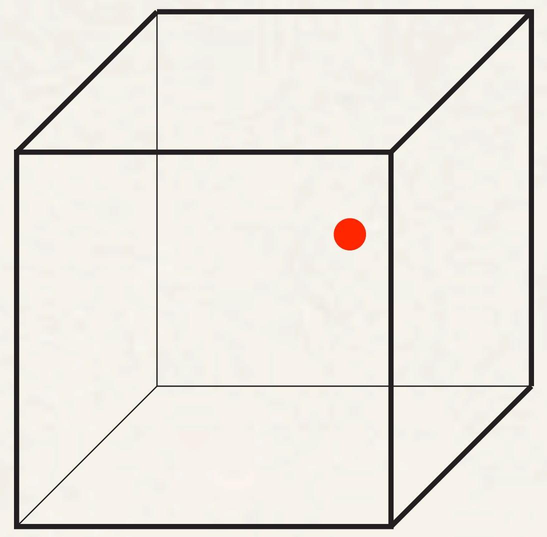
$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



$$E\big(f(\mathbf{X})\big) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$\mathrm{Var}\big(f(\mathbf{X})\big) = \int \cdots \int \left[f(\mathbf{x}) - J\right]^2 \cdot p(\mathbf{x})\, d\mathbf{x}$$

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$

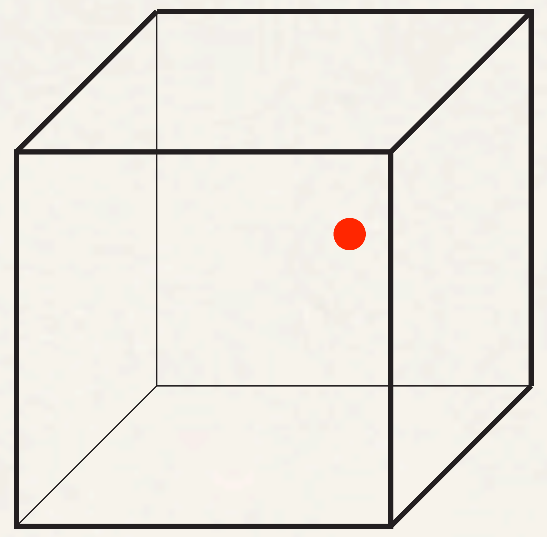


$$E(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$[a-b]^2 = a^2 - 2ab + b^2$$

$$\mathrm{Var}(f(\mathbf{X})) = \int \cdots \int \left[f(\mathbf{x}) - J \right]^2 \cdot p(\mathbf{x})\, d\mathbf{x}$$

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



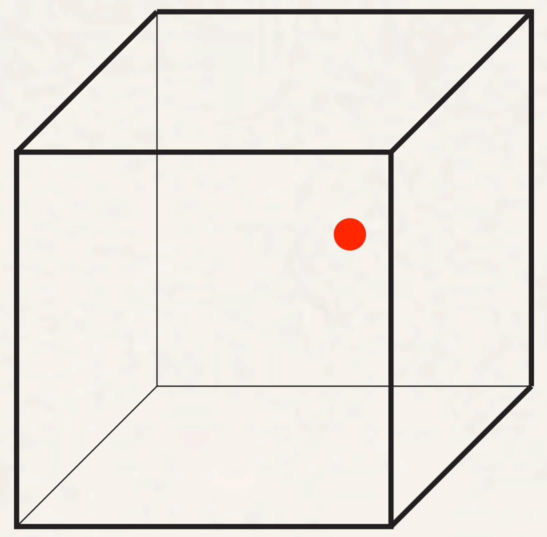
$$E(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$[a-b]^2 = a^2 - 2ab + b^2$$

$$\mathrm{Var}(f(\mathbf{X})) = \int \cdots \int \left[f(\mathbf{x}) - J\right]^2 \cdot p(\mathbf{x})\, d\mathbf{x}$$

$$= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - 2J \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} + J^2 \int \cdots \int p(\mathbf{x})\, d\mathbf{x}$$

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



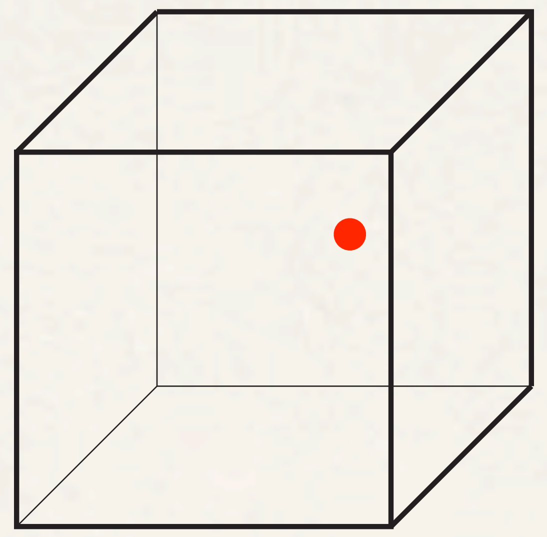
$$\mathrm{E}(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x}) \, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x}) \, d\mathbf{x} = J$$

$$[a-b]^2 = a^2 - 2ab + b^2$$

$$\mathrm{Var}(f(\mathbf{X})) = \int \cdots \int \left[f(\mathbf{x}) - J\right]^2 \cdot p(\mathbf{x}) \, d\mathbf{x}$$

$$= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x}) \, d\mathbf{x} - 2J \overbrace{\int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x}) \, d\mathbf{x}}^{J} + J^2 \int \cdots \int p(\mathbf{x}) \, d\mathbf{x}$$

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$

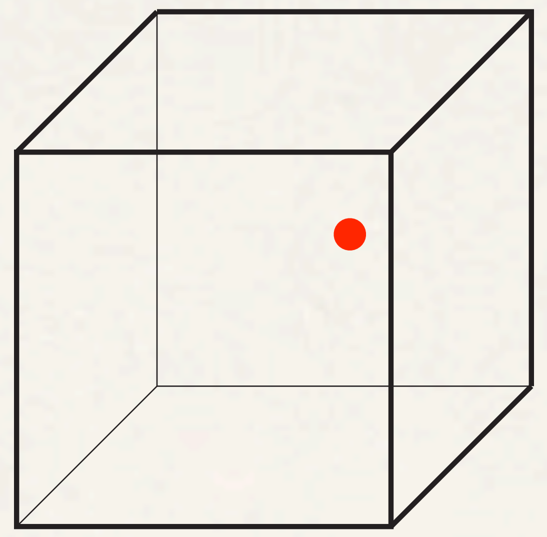


$$E(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$[a-b]^2 = a^2 - 2ab + b^2$$

$$\mathrm{Var}(f(\mathbf{X})) = \int \cdots \int [f(\mathbf{x}) - J]^2 \cdot p(\mathbf{x})\, d\mathbf{x}$$

$$= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - 2J \overbrace{\int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x}}^{J} + J^2 \overbrace{\int \cdots \int p(\mathbf{x})\, d\mathbf{x}}^{1}$$



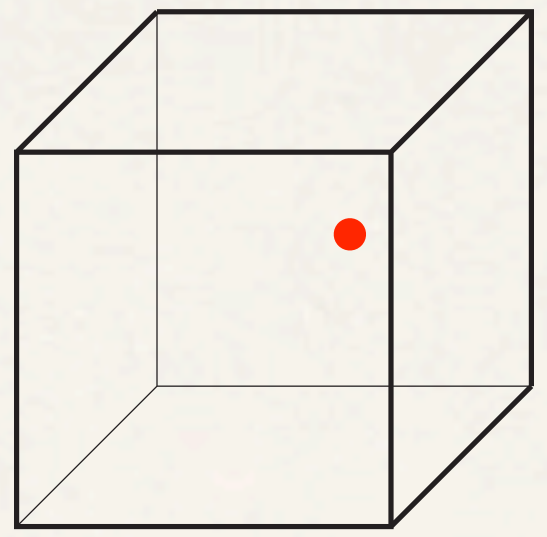
$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$[a-b]^2 = a^2 - 2ab + b^2$$

$$\begin{aligned} \mathrm{Var}(f(\mathbf{X})) &= \int \cdots \int \left[f(\mathbf{x}) - J\right]^2 \cdot p(\mathbf{x})\, d\mathbf{x} \\ &= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - 2J \overbrace{\int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x}}^{J} + J^2 \overbrace{\int \cdots \int p(\mathbf{x})\, d\mathbf{x}}^{1} \\ &= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - J^2 \end{aligned}$$

$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$

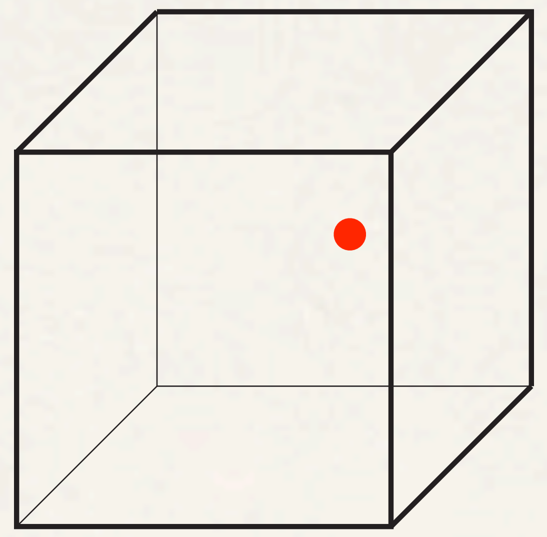


$$\mathbf{E}(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$[a-b]^2 = a^2 - 2ab + b^2$$

$$\begin{aligned} \mathrm{Var}(f(\mathbf{X})) &= \int \cdots \int [f(\mathbf{x}) - J]^2 \cdot p(\mathbf{x})\, d\mathbf{x} \\ &= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - 2J \overbrace{\int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x}}^{J} + J^2 \overbrace{\int \cdots \int p(\mathbf{x})\, d\mathbf{x}}^{1} \\ &= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - J^2 \\ &\leq \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} \end{aligned}$$

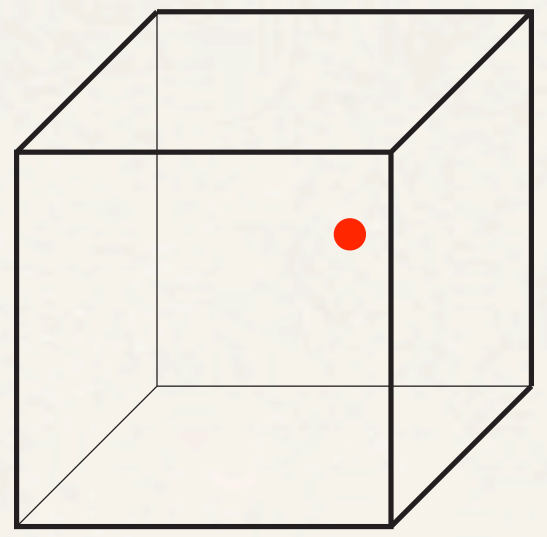
$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$



$$\mathbf{E}(f(\mathbf{X})) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x})\, d\mathbf{x} = J$$

$$[a-b]^2 = a^2 - 2ab + b^2$$

$$\begin{aligned}
\mathrm{Var}(f(\mathbf{X})) &= \int \cdots \int \left[f(\mathbf{x}) - J\right]^2 \cdot p(\mathbf{x})\, d\mathbf{x} \\
&= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - 2J \overbrace{\int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x})\, d\mathbf{x}}^{J} + J^2 \overbrace{\int \cdots \int p(\mathbf{x})\, d\mathbf{x}}^{1} \\
&= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} - J^2 \\
&\leq \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x})\, d\mathbf{x} \\
&\leq \int \cdots \int p(\mathbf{x})\, d\mathbf{x}
\end{aligned}$$



$$\mathbf{X} \sim p(\mathbf{x}) = u(x_1) \times \cdots \times u(x_D) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{E}\big(f(\mathbf{X})\big) = \int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x}) \, d\mathbf{x} = \int \cdots \int_{[0,1]^D} f(\mathbf{x}) \, d\mathbf{x} = J$$

$[a-b]^2 = a^2 - 2ab + b^2$

$$\begin{aligned}
\mathrm{Var}\big(f(\mathbf{X})\big) &= \int \cdots \int \big[f(\mathbf{x}) - J\big]^2 \cdot p(\mathbf{x}) \, d\mathbf{x} \\
&= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x}) \, d\mathbf{x} - 2J \overbrace{\int \cdots \int f(\mathbf{x}) \cdot p(\mathbf{x}) \, d\mathbf{x}}^{J} + J^2 \overbrace{\int \cdots \int p(\mathbf{x}) \, d\mathbf{x}}^{1} \\
&= \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x}) \, d\mathbf{x} - J^2 \\
&\leq \int \cdots \int f(\mathbf{x})^2 \cdot p(\mathbf{x}) \, d\mathbf{x} \\
&\leq \int \cdots \int p(\mathbf{x}) \, d\mathbf{x} \\
&= 1
\end{aligned}$$