6.3: Common Fourier Series

Introduction

Once one has obtained a solid understanding of the fundamentals of **Fourier series analysis** and the **General Derivation of the Fourier Coefficients**, it is useful to have an understanding of the common signals used in Fourier Series Signal Approximation.

Deriving the Fourier Coefficients

Consider a square wave $f(x)$ of length 1. Over the range [0,1), this can be written as

$$x(t)=\begin{cases}1 & t \leq \frac{1}{2} \\ -1 & t > \frac{1}{2}\end{cases}. \tag{6.3.1}$$

Fourier series approximation of a square wave

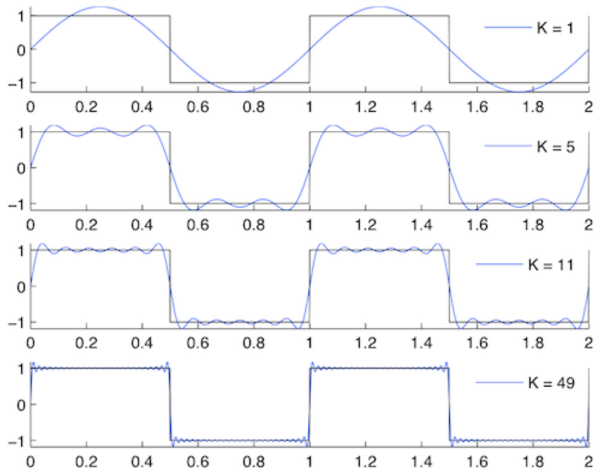

Figure 6.3.1: Fourier series approximation to $sq(t)$. The number of terms in the Fourier sum is indicated in each plot, and the square wave is shown as a dashed line over two periods.

Real Even Signals

Given that the square wave is a real and even signal,

- $f(t)=f(-t)$ EVEN
- $f(t)=f^*(t)$ REAL

therefore,

- $c_n=c_{-n}$ EVEN
- $c_n=c_n^*$ REAL

Consider this mathematical question intuitively: Can a discontinuous function, like the square wave, be expressed as a sum, even an infinite one, of continuous signals? One should at least be suspicious, and in fact, it can't be thus expressed.

The extraneous peaks in the square wave's Fourier series **never** disappear; they are termed **Gibb's phenomenon** after the American physicist Josiah Willard Gibbs. They occur whenever the signal is discontinuous, and will always be present whenever the signal has jumps.

Deriving the Fourier Coefficients for Other Signals

The Square wave is the standard example, but other important signals are also useful to analyze, and these are included here.

Constant Waveform

This signal is relatively self-explanatory: the time-varying portion of the Fourier Coefficient is taken out, and we are left simply with a constant function over all time.

$$x(t)=1 \tag{6.3.2}$$

Sinusoid Waveform

With this signal, only a specific frequency of time-varying Coefficient is chosen (given that the Fourier Series equation includes a sine wave, this is intuitive), and all others are filtered out, and this single time-varying coefficient will exactly match the desired signal.

$$x(t)=\sin(\pi t) \tag{6.3.3}$$

Triangle Waveform

$$x(t)=\begin{cases} t & t\leq 1/4 \\ 2-4t & 1/4\leq t\leq 3/4 \\ -7/4+4t & 3/4\leq t\leq 1 \end{cases} \tag{6.3.4}$$

This is a more complex form of signal approximation to the square wave. Because of the **Symmetry Properties** of the Fourier Series, the triangle wave is a real and odd signal, as opposed to the real and even square wave signal. This means that

- $f(t)=-f(-t)$ ODD
- $f(t)=f^{*}(t)$ REAL

therefore,

- $c_n=-c_{-n}$
- $c_n=-c_n^{*}$ IMAGINARY

Fourier series approximation of a triangle wave

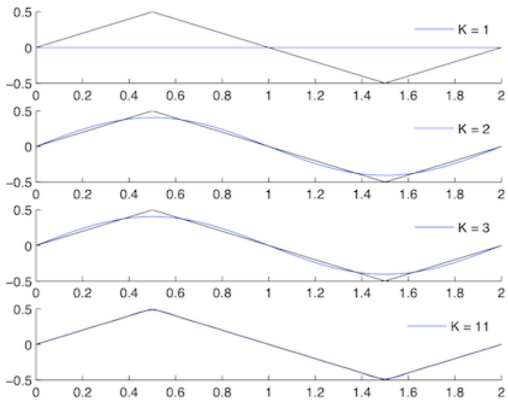

Figure 6.3.2

Sawtooth Waveform

$$x(t)=t-\mathrm{Floor}(t) \tag{6.3.5}$$

Because of the **Symmetry Properties** of the Fourier Series, the sawtooth wave can be defined as a real and odd signal, as opposed to the real and even square wave signal. This has important implications for the Fourier Coefficients.

Fourier series approximation of a sawtooth wave

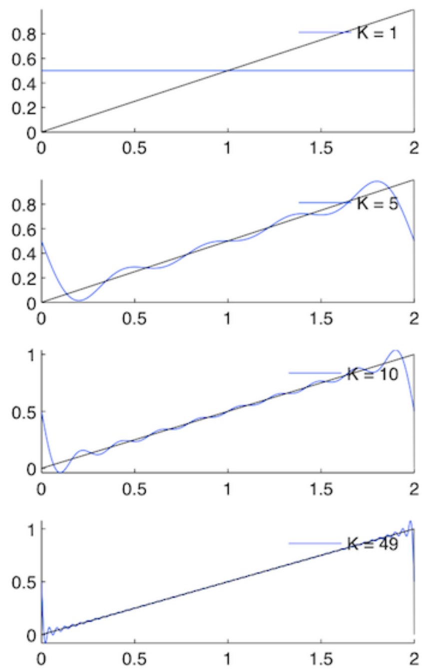

Figure 6.3.3

Fourier Series Approximation VI

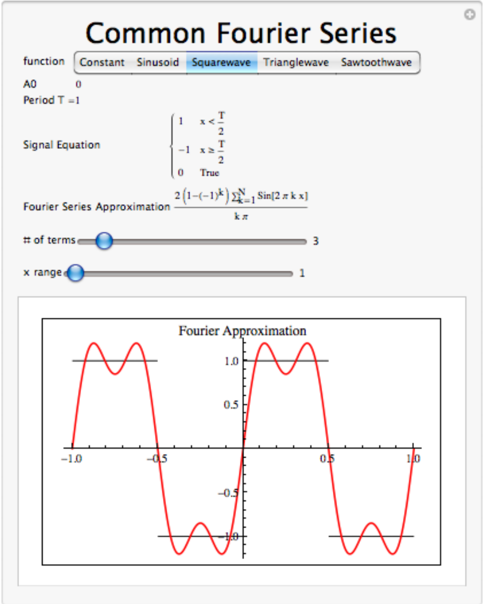

Figure 6.3.4: Interact (when online) with a Mathematica CDF demonstrating the common Fourier Series. To download, right click and save file as .cdf.

Summary

To summarize, a great deal of variety exists among the common Fourier Transforms. A summary table is provided here with the essential information.

Table 6.3.1: Common Continuous-Time Fourier Series

Description	Time Domain Signal for $t \in [0,1)$	Frequency Domain Signal
Constant Waveform	$x(t)=1$	$c_k=\begin{cases}1 & k=0\\0 & k\neq 0\end{cases}$
Sinusoid Waveform	$x(t)=\sin(\pi t)$	$c_k=\begin{cases}1/2 & k=\pm 1\\0 & k\neq \pm 1\end{cases}$
Square Waveform	$x(t)=\begin{cases}1 & t\leq 1/2\\-1 & t>1/2\end{cases}$	$c_k=\begin{cases}4/\pi k & \text{k odd}\\0 & \text{k even}\end{cases}$
Triangle Waveform	$x(t)=\begin{cases}t & t\leq 1/2\\1-t & t>1/2\end{cases}$	$c_k=\begin{cases}-8\sin(k\pi)/2)/(\pi k)^2 & \text{k odd}\\0 & \text{k even}\end{cases}$

Description	Time Domain Signal for $t \in [0,1)$	Frequency Domain Signal
Sawtooth Waveform	$x(t) = t/2$	$c_k = \begin{cases} 0.5 & k = 0 \\ -1/\pi k & k \neq 0 \end{cases}$

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