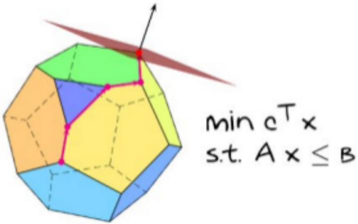
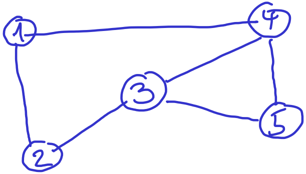
How efficient is the simplex method?

- Graphs of polyhedra
- and their diameter



Graphs

An *undirected graph* $G = (V, E)$ consists of a finite set V of *nodes* or *vertices* and a set E of *edges*, where each edge $e \in E$ is a two-element subset of vertices, $e = \{u, v\}$, $u \neq v \in V$. We also write $e = uv$.

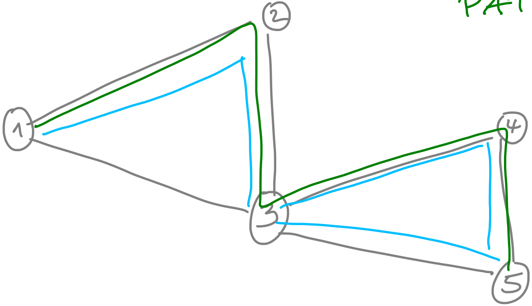


$V = \{1, 2, 3, 4, 5\}$

$E = \{\{1,2\}, \{1,4\}, \{3,2\}, \{3,4\}, \{3,5\}, \{4,5\}\}$

Walks and paths

A *walk* from node $i_1 \in V$ to $i_t \in V$ is a sequence $i_1, i_2, \dots, i_t$ of nodes such that $i_k i_{k+1} \in E$, $k = 1, \dots, t-1$. A walk is called a *path* if it has no repeated nodes.



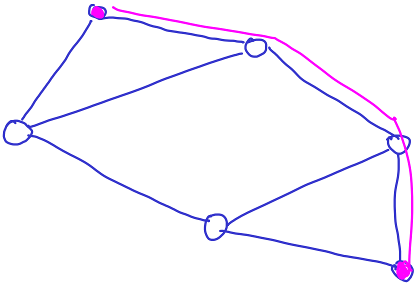
PATH: 1, 2, 3, 4, 5

WALK: 1, 2, 3, 5, 4, 3

Distance and diameter

The *distance* of $u, v \in V$ is the smallest t such that there exists a path $i_0, \ldots, i_t$ in G with $i_0 = u$ and $i_t = v$.

The *diameter* of G is the largest distance of two nodes of G.

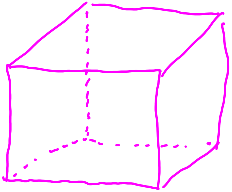


What is the diameter of this graph?

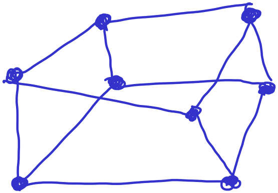
$\boxed{3}$

Graphs of polyhedra

A polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ with vertices defines a graph $G_P = (V, E)$ as follows. The set of nodes V is the set of vertices of P and $v_1 v_2 \in E$ iff v_1 and v_2 are adjacent.



The diameter of G_P is the diameter of P



Diameter and simplex algorithm

If a version of the simplex algorithm requires only a polynomial number of iterations (in n and m), then the diameter of each polyhedral graph is polynomial.

Linear programming, simplex algorithm and diameter

Linear Program

$$\max\{c^T x \colon x \in \mathbb{R}^n,\, Ax \leqslant b\}$$

- Simplex algorithm walks along edges of graph G_P of $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$.
- *Big mystery*: Is there a version of Simplex requiring a polynomial number of iterations?
- Necessary condition: The *diameter* of the graph G_P must be polynomial.
- We define $\Delta(n, m)$: *Largest diameter* of a graph G_P of a polyhedron $P \subseteq \mathbb{R}^n$ described by m inequalities.

Theorem (Kalai and Kleitman 1992)

$\Delta(n, m) \leqslant m^{1+\log n}$

An important property of G_P

$\underbrace{\mathbb{1}^T \cdot A_B \cdot y^*}_{\leq b_B} + \pi \cdot \underbrace{\mathbb{1}^T \cdot A_C \cdot y^*}_{\leq b_C} < \mathbb{1}^T \cdot b_B + \pi \cdot \mathbb{1}^T \cdot b_C$

some are strict !!!

$A_B x \leq b_B$

Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ be a polyhedron with vertices. Then G_P is connected.

Furthermore, for each pair of vertices u, v there exists a path connecting u and v such that each inequality of $Ax \leqslant b$ active at both u and v is also active at each vertex of that path.

$u \rightsquigarrow v$

Proof: $B \subseteq \{1, \dots, m\}$ basis associated with v

$C \subseteq \{1, \dots, m\}$ ineq. active at both u and v.

$c^T = \mathbb{1}^T \cdot A_B + \pi \cdot \mathbb{1}^T \cdot A_C$, $c^T \cdot x \leq \mathbb{1}^T \cdot b_B + \pi \cdot \mathbb{1}^T \cdot b_C$

large number! > 0

valid for P and active at v

and inactive at any other $y^* \neq v \in P$. unique opt. sol.

Simplex: $u \rightsquigarrow v$

An important property of G_P

$\exists\, \varepsilon > 0$ s.th. for each vertex w with $A_C \cdot w \le b_C$

some strict.

one has $1\!\!1^T \cdot A_C \cdot w \le 1\!\!1^T \cdot b_C - \varepsilon$ $\qquad 1\!\!1^T \cdot A_C \cdot w < 1\!\!1^T \cdot b_C$

M large enough s.th. for each such vertex w:

$$c^T \cdot w = 1\!\!1^T \cdot A_B \cdot w + M \cdot 1\!\!1^T \cdot A_C \cdot w \le 1\!\!1^T \cdot A_B \cdot w + \cancel{M \cdot 1\!\!1^T \cdot b_C} - M \cdot \varepsilon$$

$$< c^T \cdot u = 1\!\!1^T \cdot A_B \cdot u + \cancel{M \cdot \underbrace{1\!\!1^T \cdot A_C \cdot u}_{= 1\!\!1^T \cdot b_C}} \qquad \underset{u}{\rightsquigarrow}$$

Simplex does not visit such a w ∎

Degeneracy

Theorem

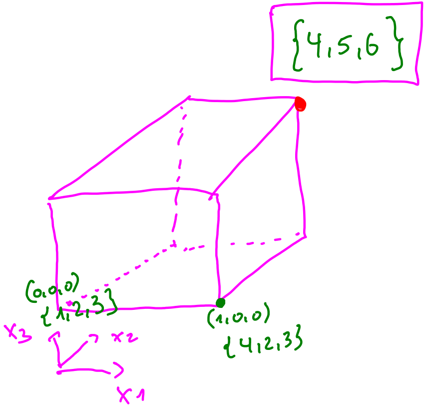
$\Delta(n, m)$ is attained at a non-degenerate polyhedron.

Vertices $\longleftrightarrow$ feasible bases

Identify vertices with their feasible basis

$[m] = \{1, \dots, m\}$

$G_P = (V, E)$ is graph with $V \subseteq \binom{[m]}{n}$.



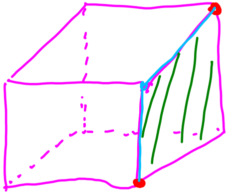
1	$x_1 \geq 0$
2	$x_2 \geq 0$
3	$x_3 \geq 0$
4	$x_1 \leq 1$
5	$x_2 \leq 1$
6	$x_3 \leq 1$

The connectivity condition

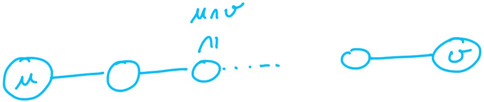
$\in \binom{[m]}{n}$

$G_P = (V, E)$ with $V \subseteq \binom{[m]}{n}$ is a graph such that for each $u, v \in V$, there exists a path $i_0, i_1, \dots, i_t$ in G such that $u = i_0$ and $v = i_t$ and

$$i_j \supseteq u \cap v, \;\; j = 1, \dots, t-1$$



$x_1 \leq 1$



P non-degenerate.