Pipe Flow

Consider the fluid flow through a pipe of circular cross-section radius R with a constant pressure gradient along the pipe length. Define the z-axis to be the symmetry axis down the center of the pipe in the direction of the flowing fluid. The velocity field in a steady flow then takes the form

$$\boldsymbol{u} = u(r)\boldsymbol{k},$$

where $r = \sqrt{x^2 + y^2}$ and u satisfies the Navier-Stokes equation given by

$$\nabla^2 u = -\frac{G}{\nu\rho}.$$

Here, G is the pressure gradient, ν is the kinematic viscosity, and ρ is the fluid density, all assumed to be constant. You may further assume that the interior surface of the pipe has no slip so that the fluid velocity is zero when $r = R$. Solve for the velocity field $u(r)$ in the pipe's cross section using the polar coordinate form for the Laplacian. What is the maximum value of the velocity?