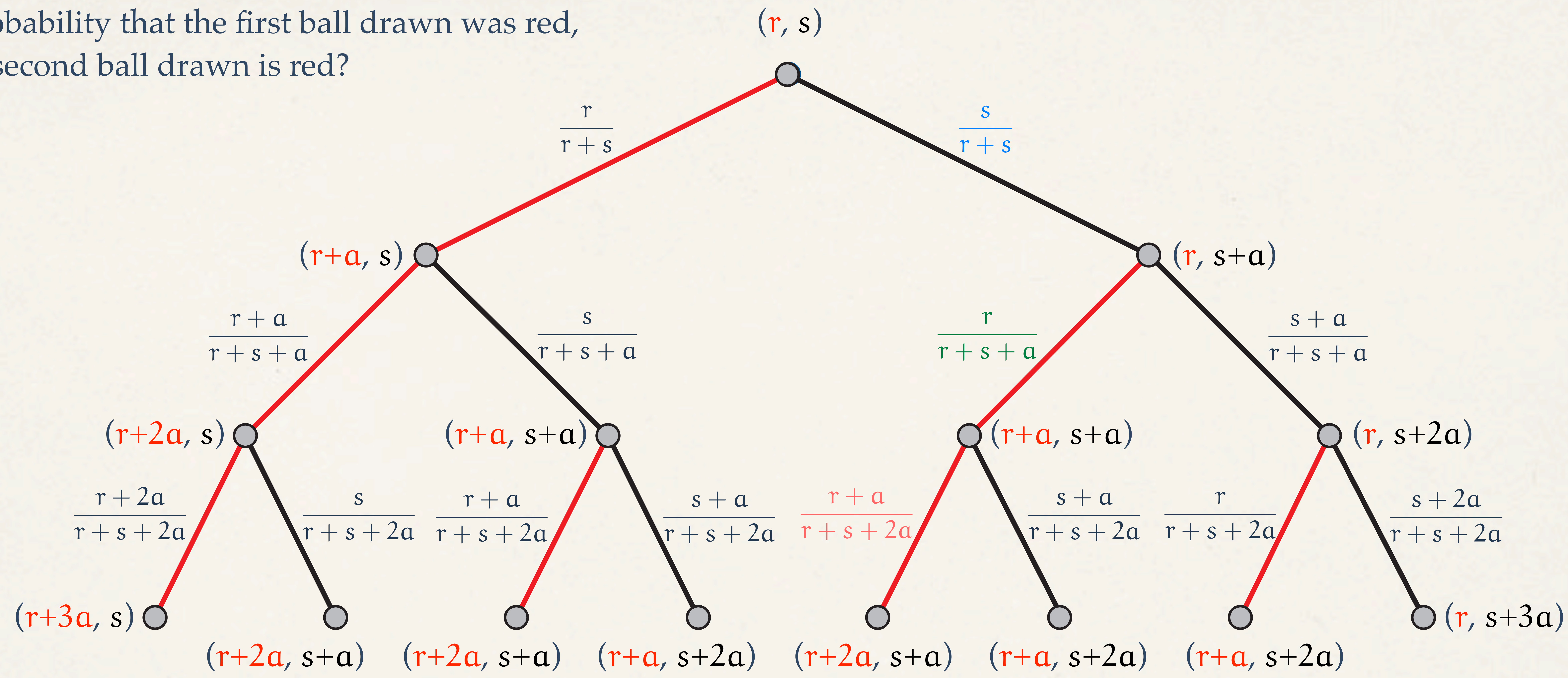
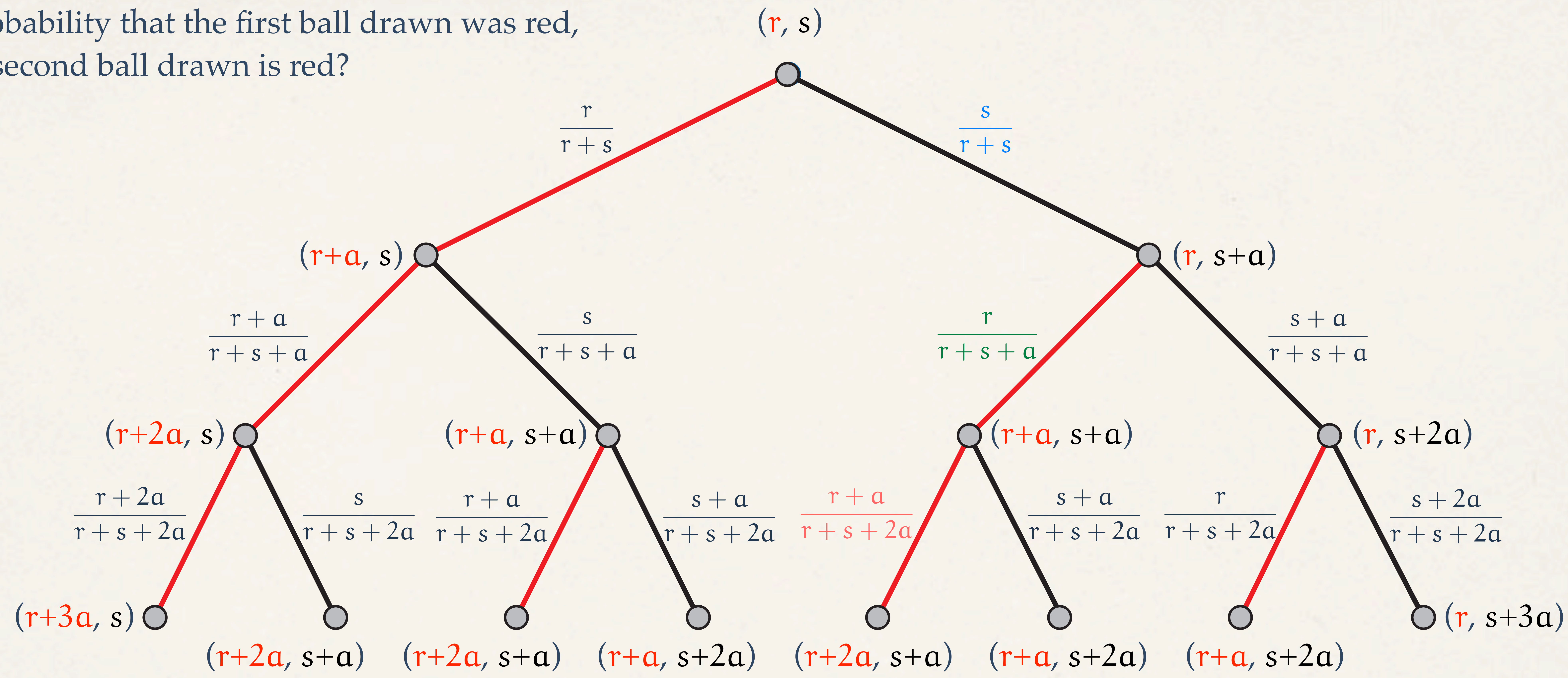
What is the probability that the first ball drawn was red, given that the second ball drawn is red?

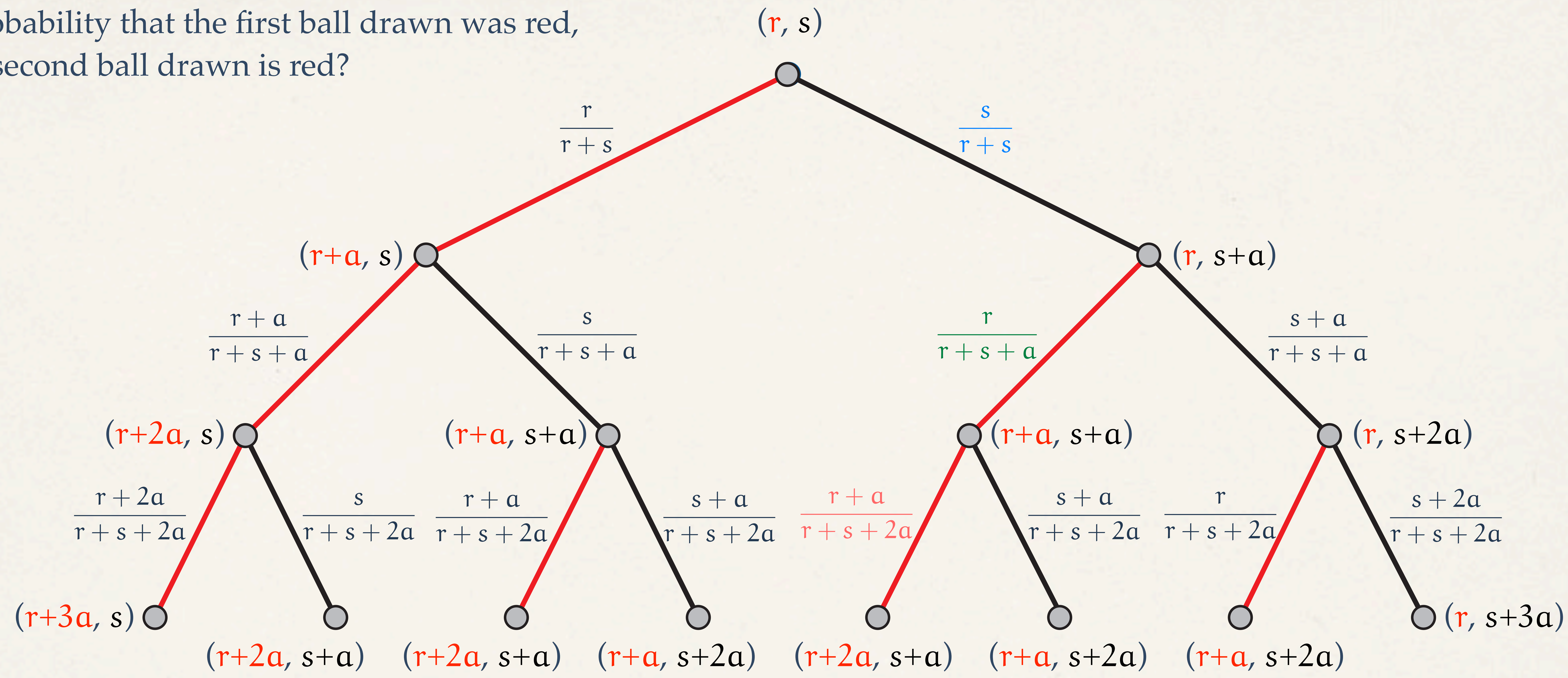


What is the probability that the first ball drawn was red, given that the second ball drawn is red?



First two balls are red

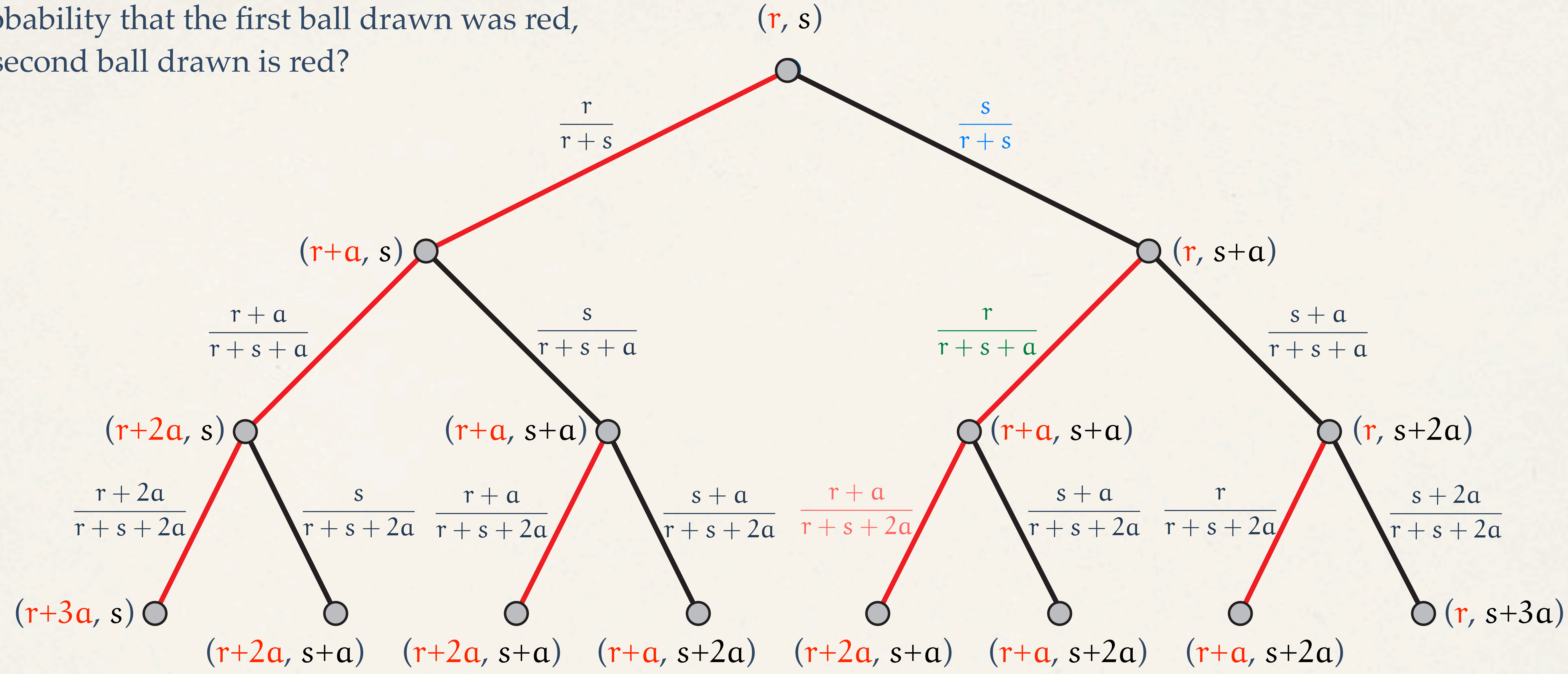
What is the probability that the first ball drawn was red, given that the second ball drawn is red?



First two balls are red

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\, \mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

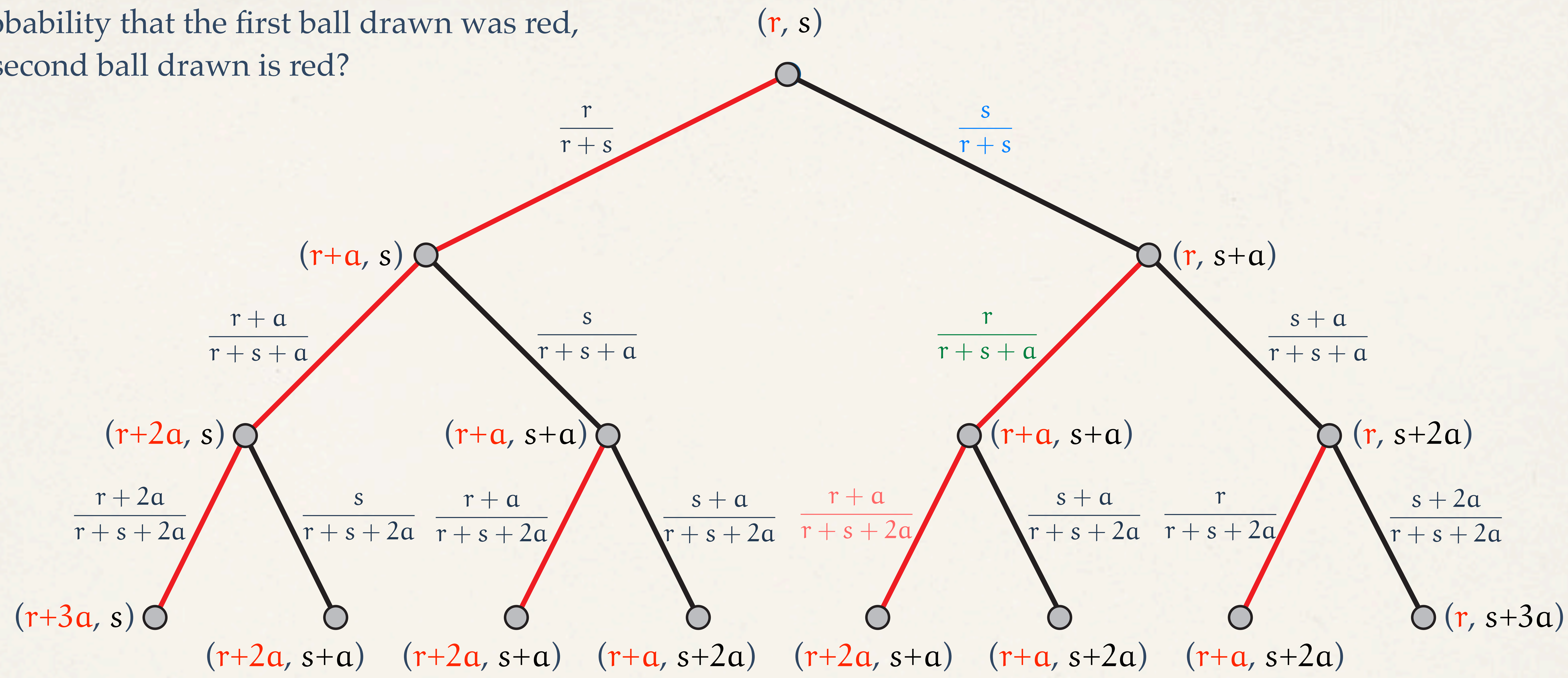
What is the probability that the first ball drawn was red, given that the second ball drawn is red?



First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

What is the probability that the first ball drawn was red, given that the second ball drawn is red?

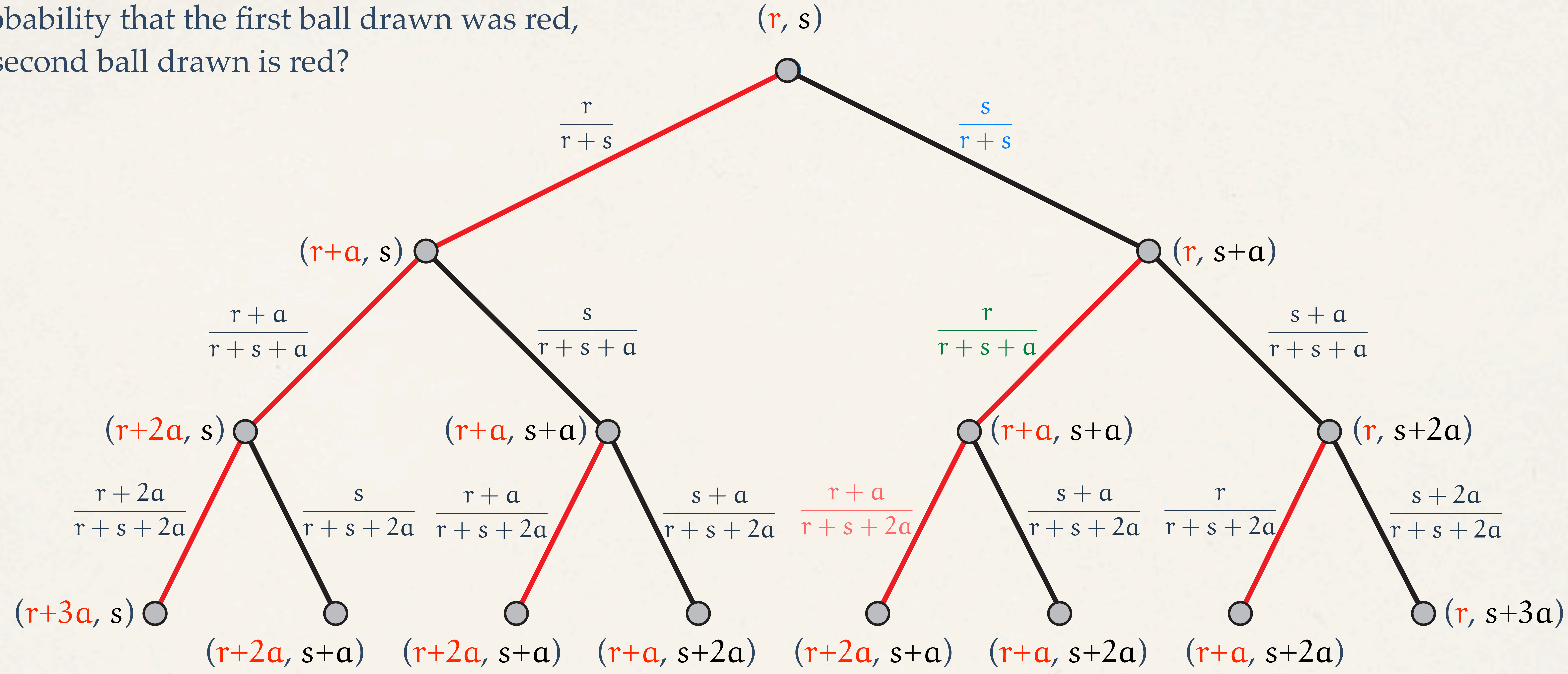


First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



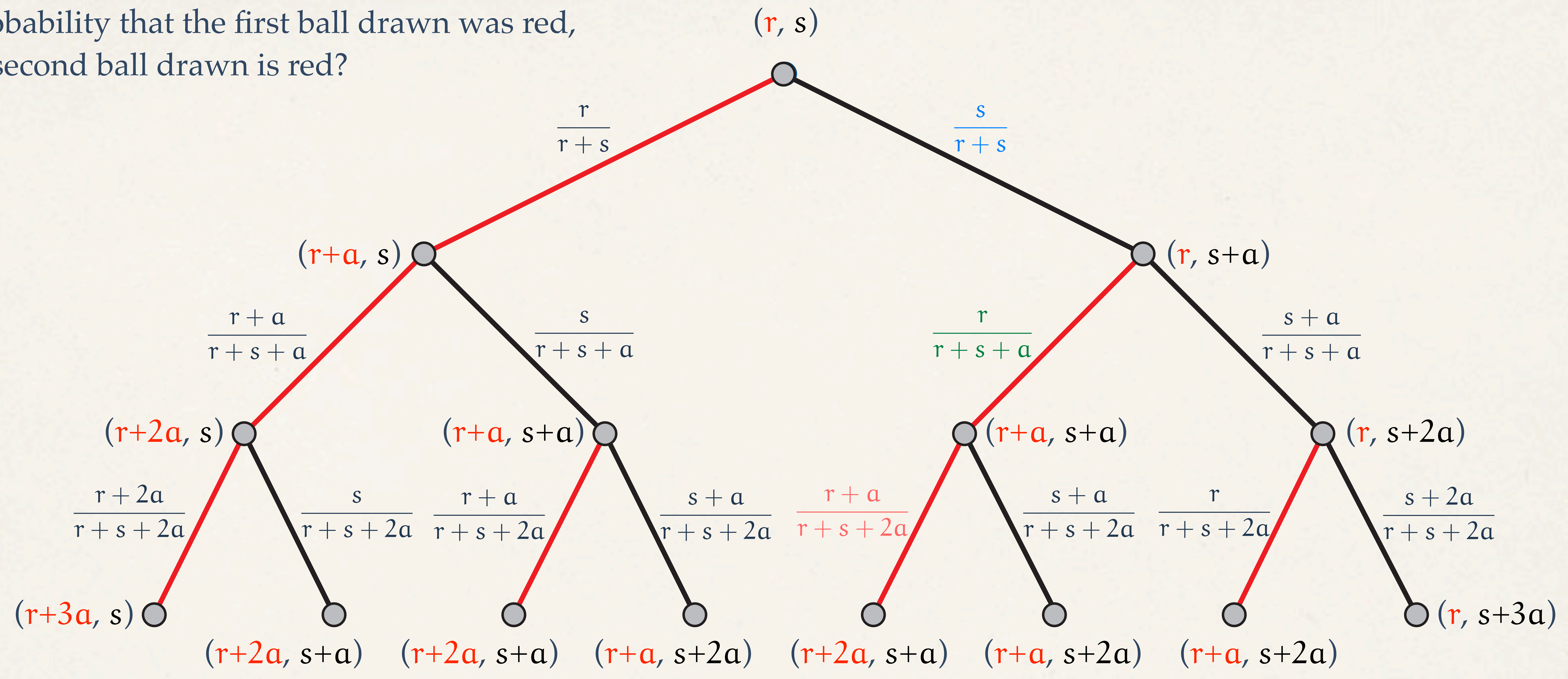
First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1)$$

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



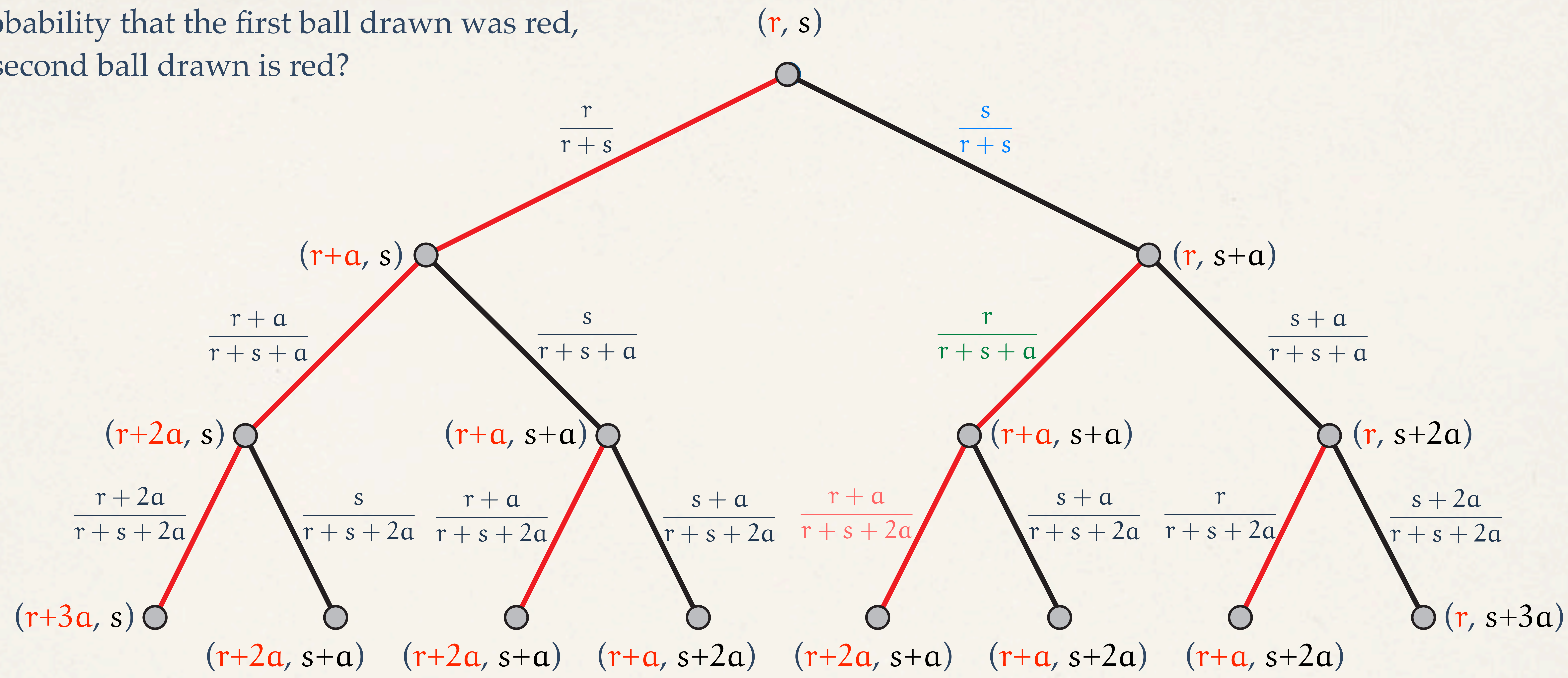
First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red — Total probability

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1)$$

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



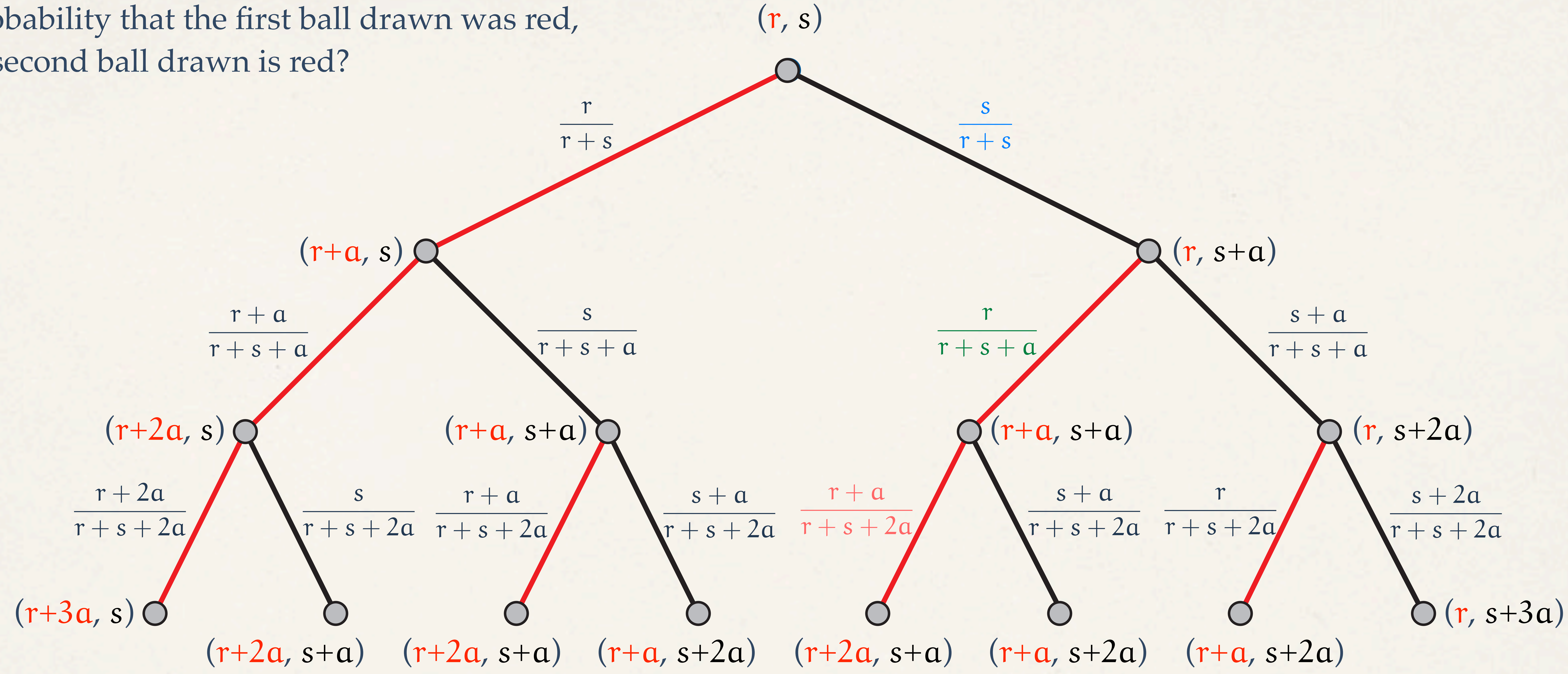
First two balls are red   Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red   Total probability

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r}{r+s+a} \cdot \frac{s}{r+s} = \frac{(r+s+a)r}{(r+s+a)(r+s)} = \frac{r}{r+s}$$

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



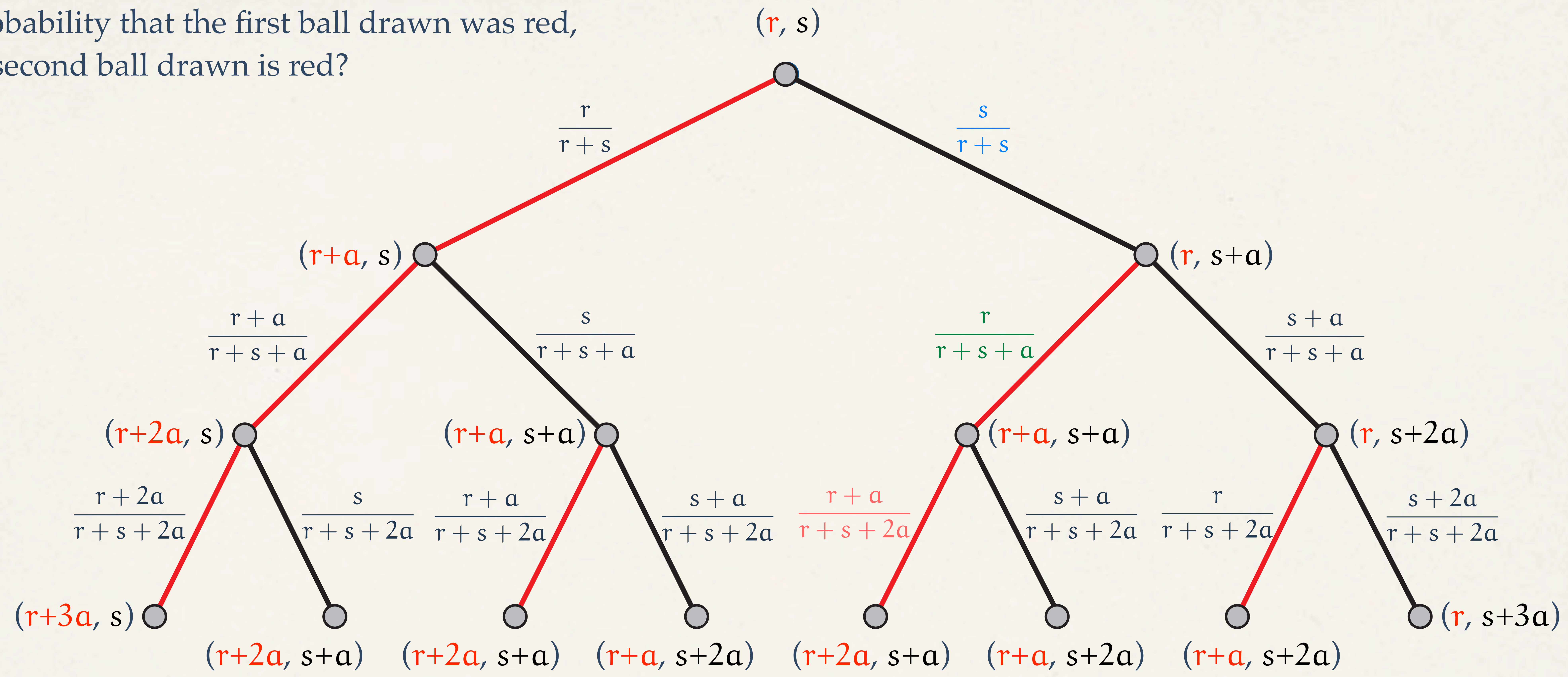
First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red — Total probability

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r}{r+s+a} \cdot \frac{s}{r+s} = \frac{(r+s+a)r}{(r+s+a)(r+s)} = \frac{r}{r+s} = \mathbf{P}(R_1)$$

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



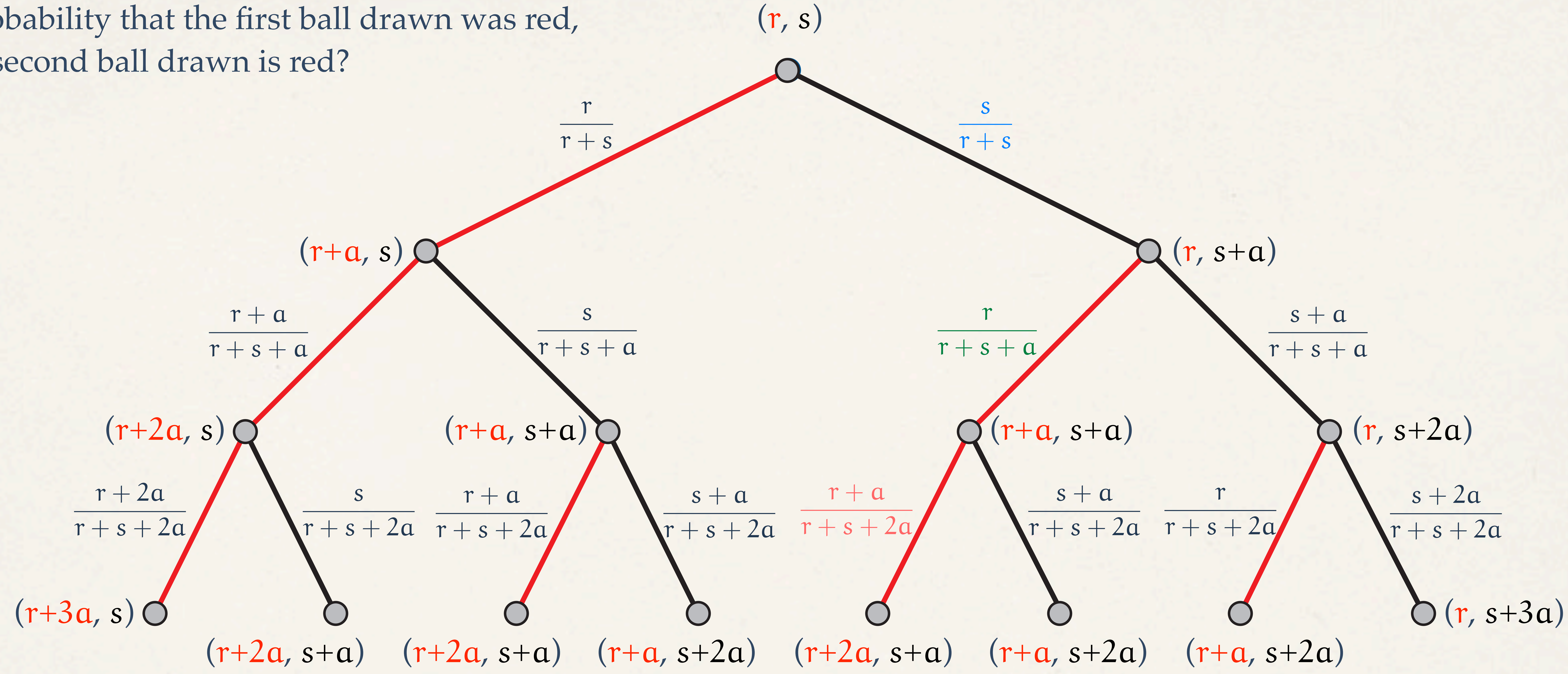
First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red — Total probability — Is this surprising?

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r}{r+s+a} \cdot \frac{s}{r+s} = \frac{(r+s+a)r}{(r+s+a)(r+s)} = \frac{r}{r+s} = \mathbf{P}(R_1)$$

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



First two balls are red — Definition

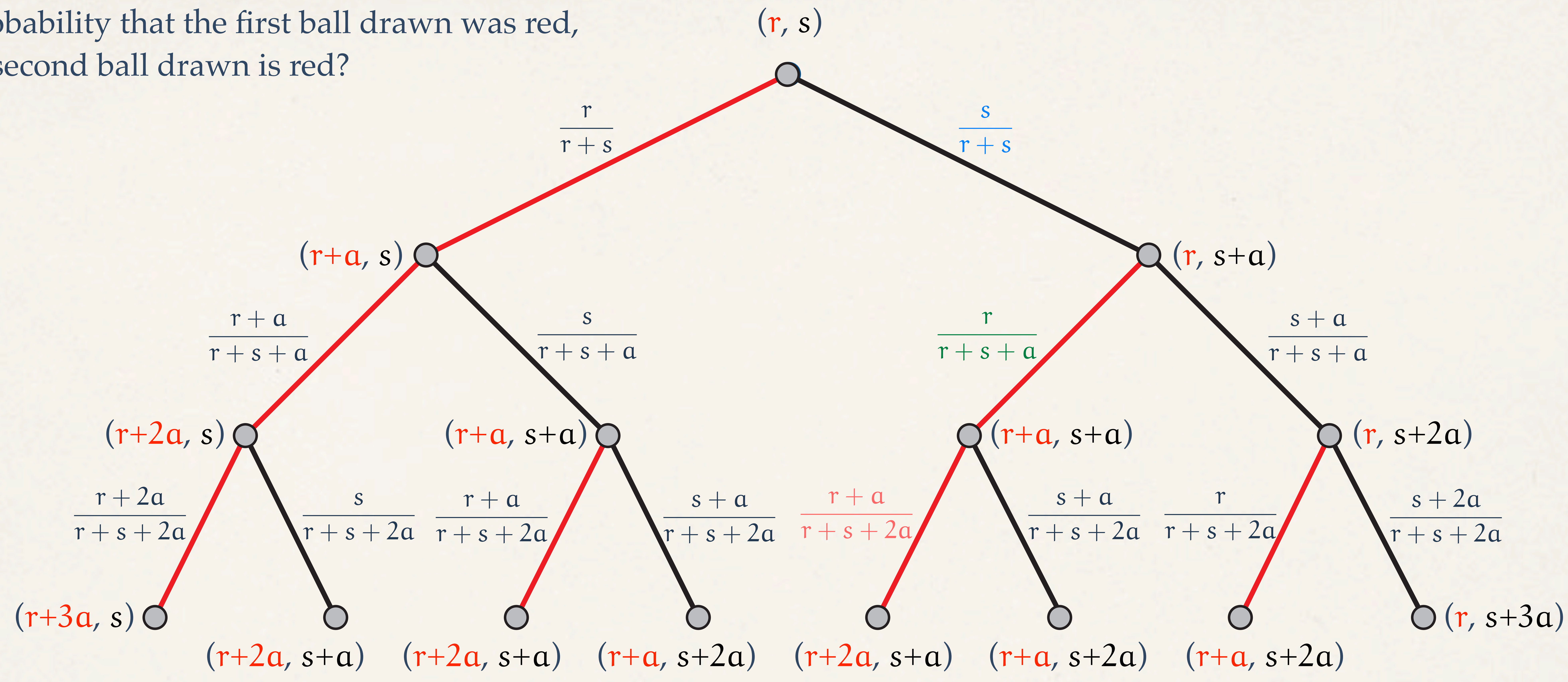
$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red — Total probability — Is this surprising?

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r}{r+s+a} \cdot \frac{s}{r+s} = \frac{(r+s+a)r}{(r+s+a)(r+s)} = \frac{r}{r+s} = \mathbf{P}(R_1)$$

First ball is red given second ball is red

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

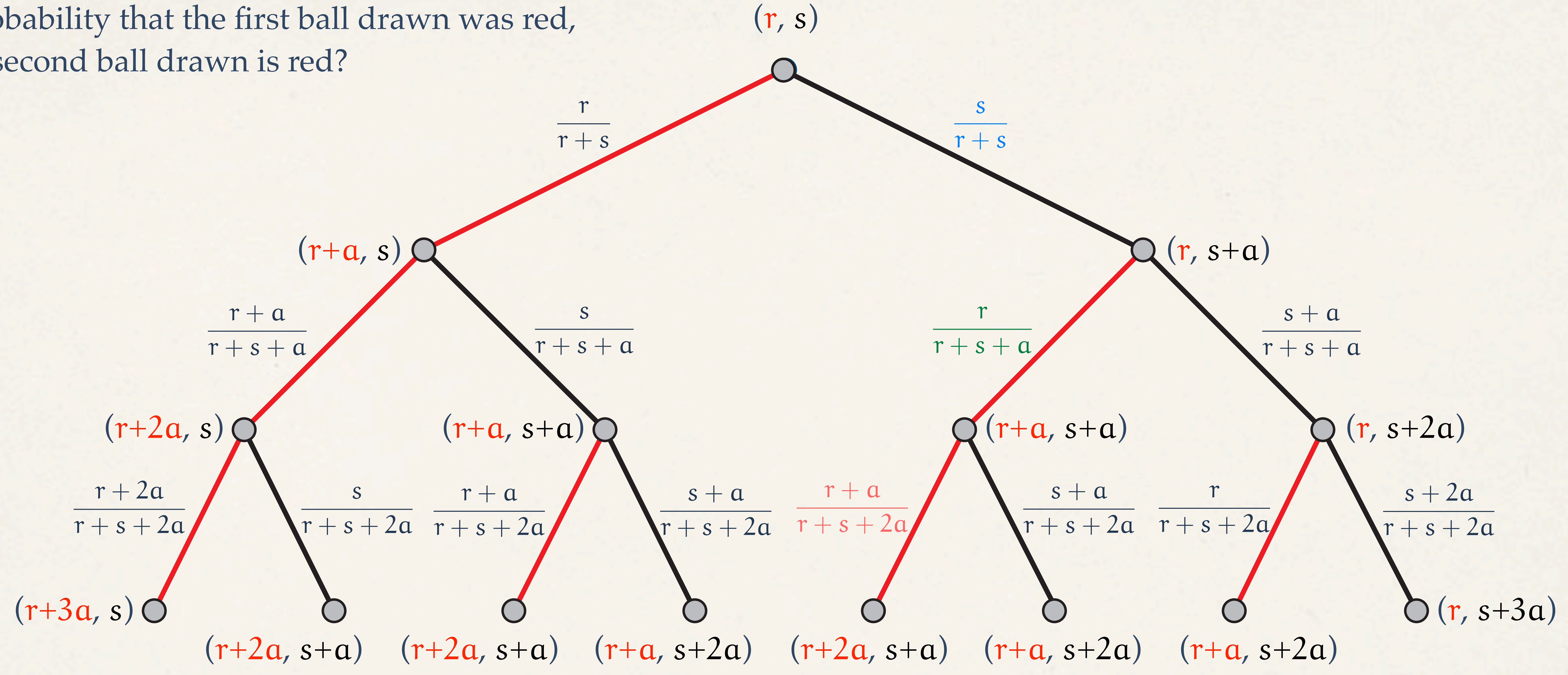
Second ball is red — Total probability — Is this surprising?

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r}{r+s+a} \cdot \frac{s}{r+s} = \frac{(r+s+a)r}{(r+s+a)(r+s)} = \frac{r}{r+s} = \mathbf{P}(R_1)$$

First ball is red given second ball is red

$$\mathbf{P}(R_1 \mid R_2) = \frac{\mathbf{P}(R_1 \cap R_2)}{\mathbf{P}(R_2)} = \frac{(r+a)r}{(r+s+a)(r+s)} \Big/ \frac{r}{r+s} = \frac{r+a}{r+s+a}$$

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red — Total probability — Is this surprising?

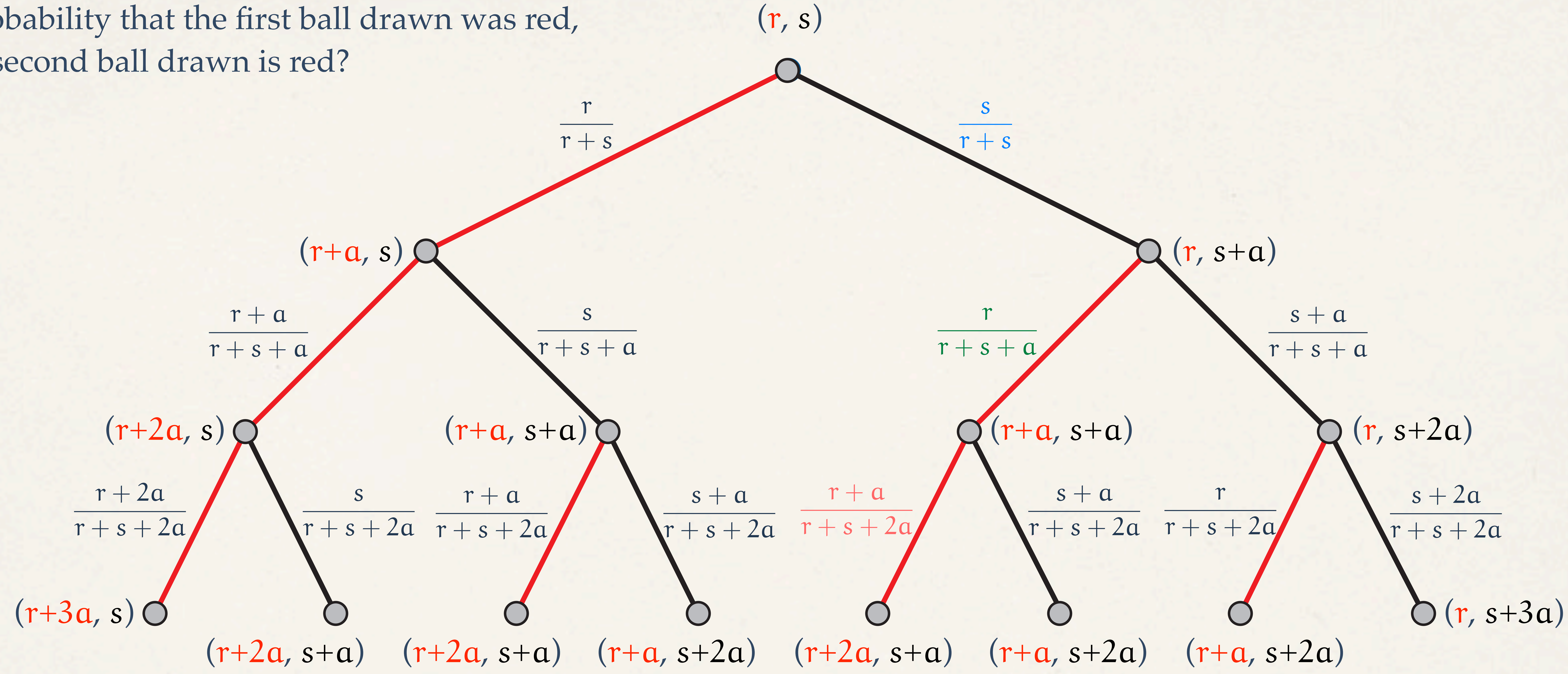
$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r}{r+s+a} \cdot \frac{s}{r+s} = \frac{(r+s+a)r}{(r+s+a)(r+s)} = \frac{r}{r+s} = \mathbf{P}(R_1)$$

First ball is red given second ball is red

$$\mathbf{P}(R_1 \mid R_2) = \frac{\mathbf{P}(R_1 \cap R_2)}{\mathbf{P}(R_2)} = \frac{(r+a)r}{(r+s+a)(r+s)} \Big/ \frac{r}{r+s} = \frac{r+a}{r+s+a}$$

Bayes's rule

What is the probability that the first ball drawn was red, given that the second ball drawn is red?



First two balls are red — Definition

$$\mathbf{P}(R_1 \cap R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Second ball is red — Total probability — Is this surprising?

$$\mathbf{P}(R_2) = \mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) = \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r}{r+s+a} \cdot \frac{s}{r+s} = \frac{(r+s+a)r}{(r+s+a)(r+s)} = \frac{r}{r+s} = \mathbf{P}(R_1)$$

First ball is red given second ball is red

$$\mathbf{P}(R_1 \mid R_2) = \frac{\mathbf{P}(R_1 \cap R_2)}{\mathbf{P}(R_2)} = \frac{(r+a)r}{(r+s+a)(r+s)} \Big/ \frac{r}{r+s} = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\, \mathbf{P}(R_2 \mid R_1)\, \mathbf{P}(R_1)$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Total probability

$$\mathbf{P}(R_2 \cap R_3)) = \mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Total probability

$$\mathbf{P}(R_2 \cap R_3)) = \underbrace{\mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1)}_{\mathbf{P}(R_1 \cap R_2 \cap R_3)} + \underbrace{\mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)}_{\mathbf{P}(B_1 \cap R_2 \cap R_3)}$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Total probability

$$\mathbf{P}(R_2 \cap R_3)) = \underbrace{\mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1)}_{\mathbf{P}(R_1 \cap R_2 \cap R_3)} + \underbrace{\mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)}_{\mathbf{P}(B_1 \cap R_2 \cap R_3)}$$

Chain rule

$$= \mathbf{P}(R_3 \mid R_2 \cap R_1)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_3 \mid R_2 \cap B_1)\,\mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1)$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Total probability

$$\mathbf{P}(R_2 \cap R_3)) = \underbrace{\mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1)}_{\mathbf{P}(R_1 \cap R_2 \cap R_3)} + \underbrace{\mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)}_{\mathbf{P}(B_1 \cap R_2 \cap R_3)}$$

Chain rule

$$= \mathbf{P}(R_3 \mid R_2 \cap R_1)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_3 \mid R_2 \cap B_1)\,\mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1)$$

$$= \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r+a}{r+s+2a} \cdot \frac{r}{r+s+a} \cdot \frac{s}{r+s}$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Total probability

$$\mathbf{P}(R_2 \cap R_3)) = \underbrace{\mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1)}_{\mathbf{P}(R_1 \cap R_2 \cap R_3)} + \underbrace{\mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)}_{\mathbf{P}(B_1 \cap R_2 \cap R_3)}$$

Chain rule

$$= \mathbf{P}(R_3 \mid R_2 \cap R_1)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_3 \mid R_2 \cap B_1)\,\mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1)$$

$$= \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r+a}{r+s+2a} \cdot \frac{r}{r+s+a} \cdot \frac{s}{r+s}$$

$$= \frac{(r+s+2a)(r+a)r}{(r+s+2a)(r+s+a)(r+s)} = \frac{(r+a)r}{(r+s+a)(r+s)}$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

Bayes's rule

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) = \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)} = \frac{(r+2a)(r+a)r}{(r+s+2a)(r+s+a)(r+s)} \Big/ \frac{(r+a)r}{(r+s+a)(r+s)} = \frac{r+2a}{r+s+2a}$$

Chain rule

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) = \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

Total probability

$$\mathbf{P}(R_2 \cap R_3)) = \underbrace{\mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1)}_{\mathbf{P}(R_1 \cap R_2 \cap R_3)} + \underbrace{\mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)}_{\mathbf{P}(B_1 \cap R_2 \cap R_3)}$$

Chain rule

$$= \mathbf{P}(R_3 \mid R_2 \cap R_1)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_3 \mid R_2 \cap B_1)\,\mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1)$$

$$= \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r+a}{r+s+2a} \cdot \frac{r}{r+s+a} \cdot \frac{s}{r+s}$$

$$= \frac{(r+s+2a)(r+a)r}{(r+s+2a)(r+s+a)(r+s)} = \frac{(r+a)r}{(r+s+a)(r+s)}$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) \overset{\text{Bayes's rule}}{=} \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)} = \frac{(r+2a)(r+a)r}{(r+s+2a)(r+s+a)(r+s)} \Bigg/ \frac{(r+a)r}{(r+s+a)(r+s)} = \frac{r+2a}{r+s+2a} > \frac{r+a}{r+s+a} > \frac{r}{r+s}$$

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) \overset{\text{Chain rule}}{=} \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

$$\begin{aligned}
\mathbf{P}(R_2 \cap R_3)) &\overset{\text{Total probability}}{=} \underbrace{\mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1)}_{\mathbf{P}(R_1 \cap R_2 \cap R_3)} + \underbrace{\mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)}_{\mathbf{P}(B_1 \cap R_2 \cap R_3)} \\
&\overset{\text{Chain rule}}{=} \mathbf{P}(R_3 \mid R_2 \cap R_1)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_3 \mid R_2 \cap B_1)\,\mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1) \\
&= \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r+a}{r+s+2a} \cdot \frac{r}{r+s+a} \cdot \frac{s}{r+s} \\
&= \frac{(r+s+2a)(r+a)r}{(r+s+2a)(r+s+a)(r+s)} = \frac{(r+a)r}{(r+s+a)(r+s)}
\end{aligned}$$

$$\mathbf{P}(R_1 \mid R_2) = \frac{r+a}{r+s+a} > \frac{r}{r+s} = \mathbf{P}(R_1)$$

Given that the second and third balls drawn are red, what is the chance that the first ball drawn is red?

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) \overset{\text{Bayes's rule}}{=} \frac{\mathbf{P}(R_1 \cap R_2 \cap R_3)}{\mathbf{P}(R_2 \cap R_3)} = \frac{(r+2a)(r+a)r}{(r+s+2a)(r+s+a)(r+s)} \Big/ \frac{(r+a)r}{(r+s+a)(r+s)} = \frac{r+2a}{r+s+2a} > \frac{r+a}{r+s+a} > \frac{r}{r+s}$$

$$\mathbf{P}(R_1 \mid R_2 \cap R_3) > \mathbf{P}(R_1 \mid R_2) > \mathbf{P}(R_1)$$

$$\mathbf{P}(R_1 \cap R_2 \cap R_3)) \overset{\text{Chain rule}}{=} \mathbf{P}(R_3 \mid R_1 \cap R_2)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) = \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s}$$

$$\mathbf{P}(R_2 \cap R_3)) \overset{\text{Total probability}}{=} \underbrace{\mathbf{P}(R_2 \cap R_3 \mid R_1)\,\mathbf{P}(R_1)}_{\mathbf{P}(R_1 \cap R_2 \cap R_3)} + \underbrace{\mathbf{P}(R_2 \cap R_3 \mid B_1)\,\mathbf{P}(B_1)}_{\mathbf{P}(B_1 \cap R_2 \cap R_3)}$$

Chain rule

$$= \mathbf{P}(R_3 \mid R_2 \cap R_1)\,\mathbf{P}(R_2 \mid R_1)\,\mathbf{P}(R_1) + \mathbf{P}(R_3 \mid R_2 \cap B_1)\,\mathbf{P}(R_2 \mid B_1)\,\mathbf{P}(B_1)$$

$$= \frac{r+2a}{r+s+2a} \cdot \frac{r+a}{r+s+a} \cdot \frac{r}{r+s} + \frac{r+a}{r+s+2a} \cdot \frac{r}{r+s+a} \cdot \frac{s}{r+s}$$

$$= \frac{(r+s+2a)(r+a)r}{(r+s+2a)(r+s+a)(r+s)} = \frac{(r+a)r}{(r+s+a)(r+s)}$$

What if the quantity a is a *negative* integer in Pólya's urn scheme?

- If $a = -1$ the situation is equivalent to selecting a ball and removing it: *sampling without replacement*.
- More generally, if $a = -\gcd(r, s)$, sampling will eventually terminate with all balls removed from the urn.

Which calculations carry through? Do the conditional probability inequalities still hold?