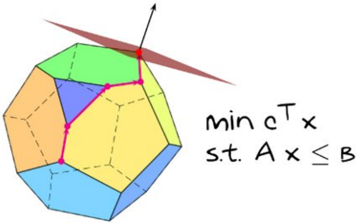
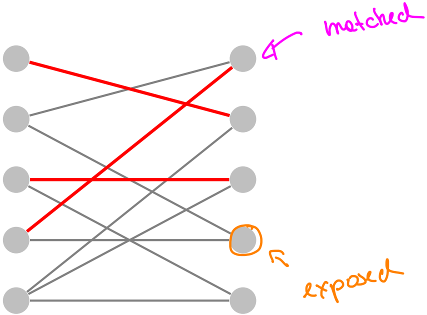
Paths, Cycles and Flows

- Maximum cardinality bipartite matchings
- Augmenting paths
- An $O(m \cdot n)$ algorithm



Exposed and matched nodes



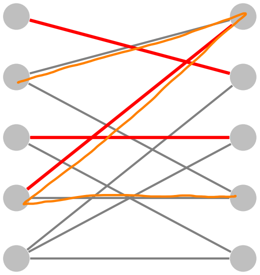
Let $G = (V, E)$ be an undirected bipartite graph.
We are interested in a matching of max. cardinality.

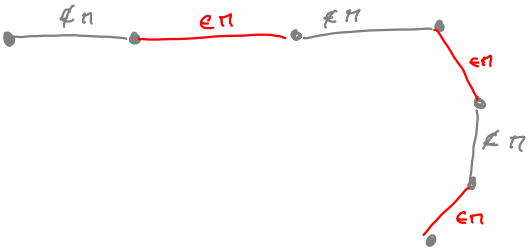
Let $M \subseteq E$ be a matching.

- A vertex that is an endpoint of an edge in M is *matched*.
- A non-matched vertex is *exposed*

Alternating paths

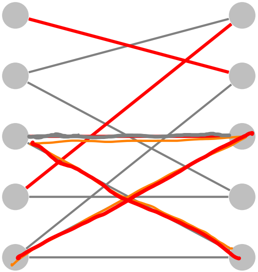
An alternating path with respect to a matching M is a path that alternates between edges in M and edges in $E \setminus M$.

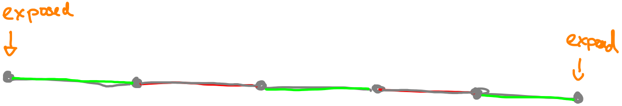




Augmenting paths

An alternating path that starts and ends at exposed nodes is a *augmenting path*.





Quiz: An augmenting path has

○ even

~~○~~ odd

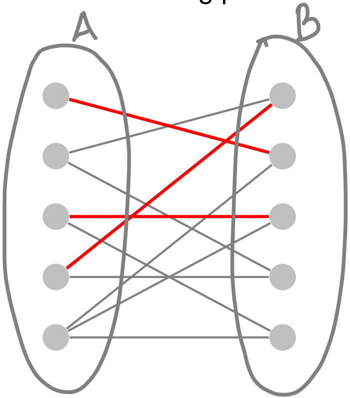
length

Augmenting path: # of non-matching edges

= # of matching edges +1

Augmenting paths

An alternating path that starts and ends at exposed nodes is a *augmenting path*.



Augmenting path is odd.

If one endpoint of an augmenting path is in A, then the other endpoint is in $\boxed{B}$

type A or B or both, in the latter case separated by space

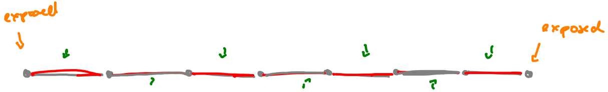
A criterion for maximal cardinality

Theorem

A matching M of a (not necessarily bipartite) graph is of maximum cardinality if and only if there are no augmenting paths with respect to M.

Proof:

"$\Rightarrow$"



$$M' = M \setminus (E(P) \cap M) \cup (E(P) \setminus M) = M \,\Delta\, E(P)$$

$$= (M \cup E(P)) \setminus (M \cap E(P))$$

$$|M'| = |M| + 1$$

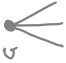
A criterion for maximal cardinality

Theorem

A matching M of a (not necessarily bipartite) graph is of maximum cardinality if and only if there are no augmenting paths with respect to M.

"$\Leftarrow$" Assume there exist Matching M' with $|M'| > |M|$. Consider $G' = (V, M' \Delta M)$.

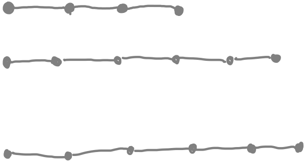
"



$\deg(v) = |\delta(v)|$

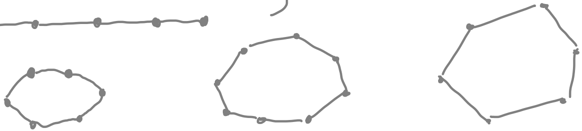
$= 3$ Possible ?

YES  NO 



Node disjoint paths

$\forall v \in V: \deg(v) \leq 2.$ (in G')



Node disjoint cycles.

A criterion for maximal cardinality

Theorem

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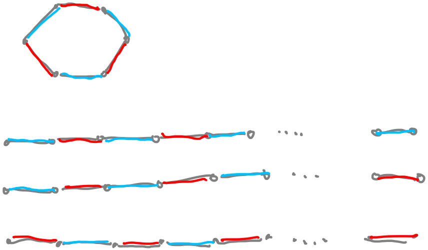
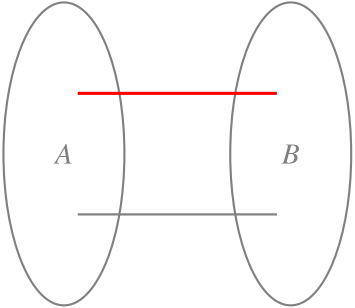


$\exists$ augmenting alternating path

Type 1 + 2

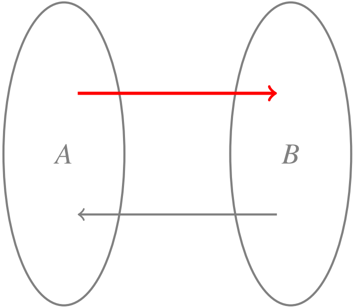
$\lightning$

Computing augmenting paths



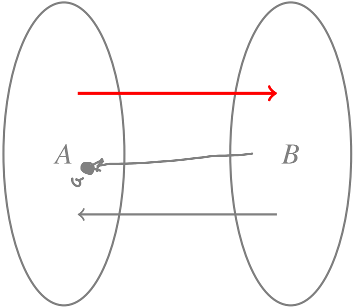
- Turn $G = (A + B, E)$ into a directed graph $D = (V, A)$ as follows.

Computing augmenting paths



- Turn $G = (A + B, E)$ into a directed graph $D = (V, A)$ as follows.
- Direct an edge in the matching from A to B.
- Direct an edge in $E \setminus M$ from B to A.

Computing augmenting paths



- Turn $G = (A + B, E)$ into a directed graph $D = (V, A)$ as follows.
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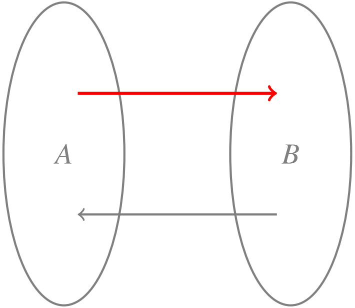
Quiz: Suppose v is exposed and v is in A, what is $|\delta^+(v)|$? $\boxed{0}$

Computing augmenting paths



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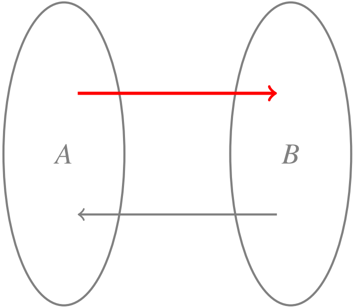
- Find a path in this directed graph between two exposed nodes.

Computing augmenting paths



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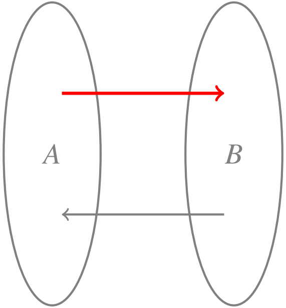
- Direct an edge in the matching from A to B.
- Direct an edge in $E \setminus M$ from B to A.
- Find a path in this directed graph between two exposed nodes.

Quiz: Such a path starts with an exposed node in $\boxed{B}$ and ends in an exposed node in $\boxed{A}$

Type A or B at appropriate place.

Computing augmenting paths (cont.)

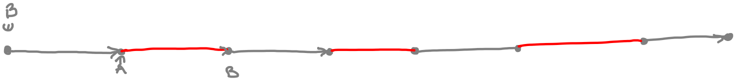


Theorem

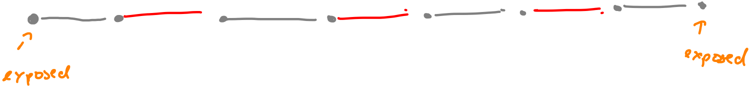
There exists an augmenting path in G for M if and only if there exists a path from an exposed node in B to an exposed node in A in the directed graph D.

Proof:

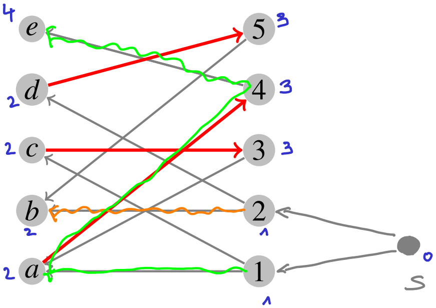
"$\Rightarrow$"



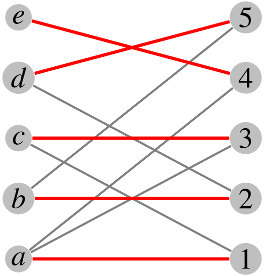
"$\Leftarrow$"



Using BFS to find augmenting paths



Using BFS to find augmenting paths



Algorithm for max. cardinality bipartite matching

$O(|E|)$

$M = \emptyset$
while there exists M-augmenting path
    Update M
return M

$O(|V|)$

Assumption: G has no isolated vertices
($\Rightarrow$ $|E| \geqslant |V|/2$).

Theorem

A maximum cardinality matching in a bipartite graph $G = (V, E)$ can be computed in time $O(|V| \cdot |E|)$