Financial TIme Series Analysis: Part II

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1. Unit root
   - Deterministic trend
   - Stochastic trend
   - Testing for unit root
     - ADF-test (Augmented Dickey-Fuller test)
     - Testing for more than one unit root
   - Segmented trends, structural breaks, and smooth transition
     - Instant occurring: Additive outlier (AO)
     - Smooth transition

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Time series y_t

Deterministic trend:

$$y_t = g(t) + u_t, \tag{1}$$

$t = 1, \ldots, T$ and u_t is a (zero mean) stationary component (ARMA).

$g(t)$ is the trend component, which typically is some sort of polynomial:

$$g(t) = b_0 + b_1 t + b_2 t^2 + \cdots$$

Linear trend: $g(t) = b_0 + bt$.

Economic interpretation: Trend aggregates the unobserved variables affecting y_t (growth due to technological innovations, productivity, population growth, etc.) Trending series is not stationary because $\mathbb{E}[y_t] = g(t)$.

Stationarity is achieved by detrending: I.e.,

$$\tilde{y}_t = y_t - g(t) = u_t$$

is stationary.

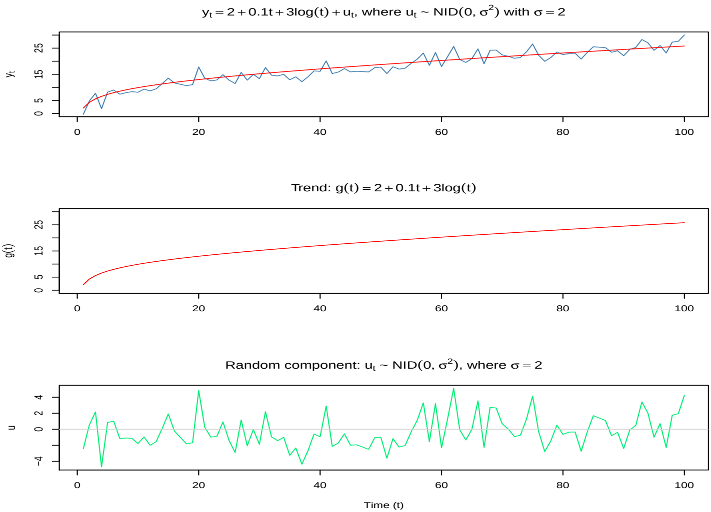
Example 1 (Deterministic trend)

Realization from a process

$$y_t = 2 + 0.1t + 3\log t + u_t$$

$t = 1, \ldots, 100$, where $u_t \sim \mathtt{NID}(0, \sigma^2)$ with $\sigma = 2$.

Deterministic trend



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Stochastic trend:

General AR(p) process

$$y_t = \phi_0 + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + u_t, \tag{2}$$

or using the lag polynomial $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$ ($B^k y_t = y_{t-k}$, back shift operator), we can write shortly

$$\phi(B) y_t = \phi_0 + u_t, \tag{3}$$

where $u_t \sim \mathtt{WN}(0, \sigma_u^2)$.

Consider the the special case of a simple AR(1) model

$$y_t = \delta + \phi y_{t-1} + u_t, \tag{4}$$

where $u_t \sim \mathtt{WN}(0, \sigma_u^2)$.

The stationarity condition is $|\phi| < 1$.

Alternatively, we can write (4)

$$y_t = \delta \sum_{i=0}^{t-1} \phi^i + \sum_{i=0}^{t-1} \phi^i u_{t-i}, \tag{5}$$

where we have assumed that $y_0 = 0$.

Because $|\phi| < 1$, $\phi^i \to 0$ as $i \to \infty$, which implies that the impact of u_{t-i} dies out at exponential rate and the innovation u_t has only a temporary effect on y_t

Consider the special case $\phi = 1$, such that

$$y_t = \delta + y_{t-1} + u_t, \tag{6}$$

which is called a random walk (RW) with drift.

Then the representation in (5) becomes

$$y_t = \delta t + U_t \tag{7}$$

where

$$U_t = \sum_{i=0}^{t-1} u_{t-i} = u_t + u_{t-1} + \cdots + u_1, \tag{8}$$

i.e., the sum of the white noise terms.

Thus, unlike above, the innovations do not die out and u_t has a permanent effect on y_t.

We observe:

$$\mathbb{E}[y_t] = \delta t \tag{9}$$

and

$$\mathrm{var}[y_t] = t\sigma_u^2 \tag{10}$$

which imply that y_t is non-stationary.

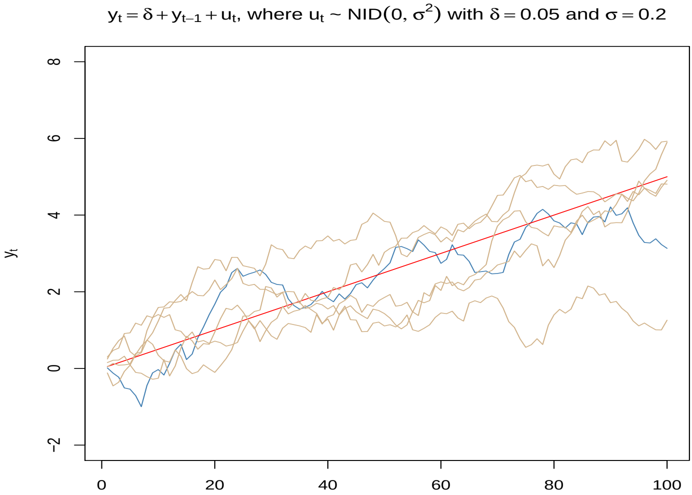
The random walk process in equation (6) (with or without a drift) is a model of *stochastic trend*.

Example 2 (Stochastic trend with drift)

Let $\delta = .1$ in (6), such that

$$y_t = 0.1 + y_t + u_t.$$

Below are few realizations from this process.


$y_t = \delta + y_{t-1} + u_t$, where $u_t \sim NID(0, \sigma^2)$ with $\delta = 0.05$ and $\sigma = 0.2$
y_t
8
6
4
2
0
-2
0
20
40
60
80
100

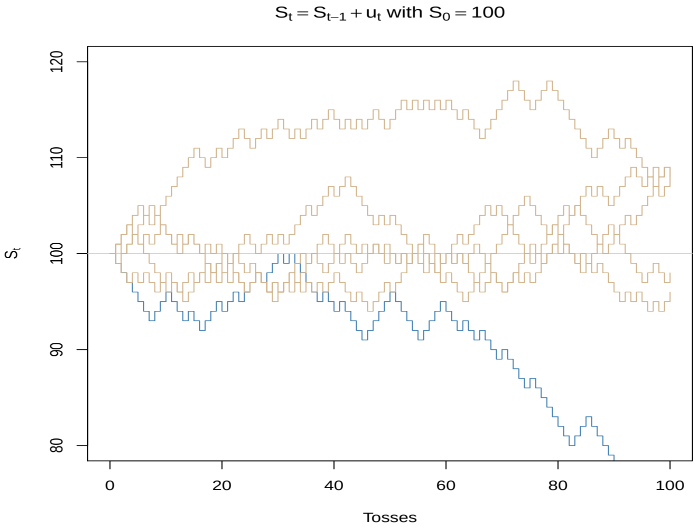
Example 3 (Coin tossing)

Consider a coin tossing game in which head (H) gains 1 euro and tail (T) loses 1 euro. Let the initial capital be 100 and let S_t, $t = 0, 1, \ldots$ denote the amount after t tosses, $S_0 = 100$. The S_t process is

$$S_t = S_{t-1} + u_t, \tag{11}$$

$t = 1, 2, \ldots$, where $S_0 = 100$ and $u_t = 1$ or -1, both with probability $1/2$.

The graph below shows few possible realizations of the process in 100 tosses.


$S_t = S_{t-1} + u_t$ with $S_0 = 100$
S_t
120
110
100
90
80
0
20
40
60
80
100
Tosses

In particular the solution of the polynomial $\phi(B) = 0$, i.e, $1 - \phi B = 0$ the root $1/\phi = 1$ when $\phi = 1$.

It is said the the process $y_t = \delta + y_{t-1} + u_t$ has a *unit root*, which implies that the series is non-stationary.

This kind of process is called *integrated* because it is sum of the (stationary) random components

$$\sum_{i=0}^{t-1} u_{t-i}.$$

Because $\Delta y_t = y_t - y_{t-1} = \delta + u_t$ is stationary, we say that y_t is integrated of order one, denoted $I(1)$.

Generally, if y_t needs to be differences d times before it becomes stationary, it is said that y_t is integrated of order t, denoted as

$$y_t \sim I(d). \tag{12}$$

A stationary series is denoted as $y_t \sim I(0)$.

Because for $y_t \sim I(d)$, $\Delta^d y_t \sim I(0)$, y_t is called *difference stationary*.

In summary, if $\boxed{y_t \sim I(0)}$:

(i) $\mathbb{E}[y_t] = \mu$, for all t.

(ii) An innovation u_t has a temporary effect on y_t.

(iii) $\text{var}[y_t] = \sigma_y^2 < \infty$ for all t.

(iv) $\text{corr}[y_t, y_{t+k}] = \rho_k$ for all t and ρ_k decays (exponentially) as k increases.

If $\boxed{y_t \sim I(1)}$:

(i) $\text{var}[y_t] = \sigma_t^2 \to \infty$ as $t \to \infty$.

(ii) An innovation u_t has a permanent effect on y_t.

(iii) The autocorrelations $\rho_k \to 1$ for all k as $t \to \infty$.

The 'rule' in the Box-Jenkins approach: Difference until the series becomes stationary.

Consequence of over differencing: . . .

Checking for unit roots:

(a) Autocorelation function; see ARIMA

(b) Statistical testing

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Consider

$$y_t = \delta + \theta t + \phi y_{t-1} + u_t, \tag{13}$$

where $u_t \sim I(0)$.

Equation (13) is equivalent to [(Augmented) Dickey-Fuller regression]

$$\Delta y_t = \delta + \theta t + \gamma y_{t-1} + u_t, \tag{14}$$

in which $\gamma = \phi - 1$.

Series y_t is stationary if $|\phi| < 1$.

If $\phi = 1$ (or $\gamma = 0$) then $y_t \sim I(1)$.

In this approach the statistical null hypothesis is: $y_t \sim I(1)$, which in terms of (14) is

$$H_0 : \gamma = 0 \tag{15}$$

with the alternative hypothesis that $y_t \sim I(0)$, or

$$H_1 : \gamma < 0. \tag{16}$$

Given the OLS estimators, the test statistic is the t-ratio

$$t = \frac{\hat{\gamma}}{s_{\hat{\gamma}}}, \tag{17}$$

where $s_{\hat{\gamma}}$ is the standard error of $\hat{\gamma}$.

The null distribution of t in (17) is <u>not</u> the t-distribution!

Depending on the specification of the ADF regression in (14) the test static has different distributions.

No intercept (drift) no trend, i.e., in (14) $\delta = \theta = 0$, such that

$$\Delta y_t = \gamma y_{t-1} + u_t. \tag{18}$$

Drift, no trend, or $\delta \neq 0$ and $\theta = 0$ in (14), such that

$$\Delta y_t = \delta + \gamma y_{t-1} + u_t \tag{19}$$

The model in (17) is the general one allowing both drift and trend.

Note that under the null hypothesis drift (δ)implies linear trend and the trend (θt) in (14) implies quadratic trend.

The finite sample distribution of t is unknown, but its asymptotic distribution is known (under certain assumptions).

Example 4

Unit root in DAX index.

Weekly data 1990-2011 (Jan)

EViews results are given below, R (with `urca` package) example detailed in the class-room.
- log index
autocorrelations
various unit root tests available in EWiews and in R.

```
Null Hypothesis: LOG(CLOSE) has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic based on SIC, MAXLAG=21)
==============================================================
                                         t-Statistic    Prob.*
--------------------------------------------------------------
Augmented Dickey-Fuller test statistic      -1.811545   0.6987
Test critical values:          1% level    -3.966787
                               5% level    -3.414086
                              10% level    -3.129143
==============================================================
*MacKinnon (1996) one-sided p-values.
```

```
Augmented Dickey-Fuller Test Equation
Dependent Variable: D(LOG(CLOSE))
Method: Least Squares
Sample (adjusted): 12/03/1990 1/31/2011
Included observations: 1053 after adjustments

==============================================================
Variable                 Coefficient  Std. Err    t-Stat   Prob.
--------------------------------------------------------------
LOG(CLOSE(-1))             -0.005871  0.003241 -1.811545  0.0703
C                           0.046698  0.024490  1.906820  0.0568
@TREND(11/26/1990)          6.13E-06  5.40E-06  1.133601  0.2572
==============================================================
R-squared             0.003415 Mean dependent var      0.001530
Adjusted R-squared    0.001516 S.D. dependent var      0.031398
S.E. of regression    0.031374 Akaike info criterion  -4.082832
Sum squared resid     1.033542 Schwarz criterion      -4.068703
Log likelihood        2152.611 Hannan-Quinn criter.   -4.077475
F-statistic           1.798834 Durbin-Watson stat      2.043702
Prob(F-statistic)     0.166001
==============================================================
```

The unit root hypothesis is strongly accepted.

Unit root in the first differences?

```
Null Hypothesis: D(LOG(CLOSE)) has a unit root
Exogenous: Constant
Lag Length: 0 (Automatic based on SIC, MAXLAG=21)
==============================================================
                                         t-Statistic    Prob.*
--------------------------------------------------------------
Augmented Dickey-Fuller test statistic      -33.27403   0.0000
Test critical values:           1% level    -3.436348
                                5% level    -2.864077
                               10% level    -2.568172
==============================================================
*MacKinnon (1996) one-sided p-values.
```

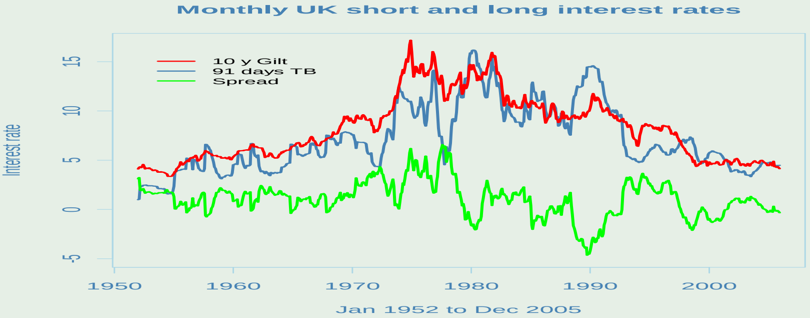
Unit root hypothesis in the returns (log-differences) is clearly rejected.

Conclusion: $\log(\mathtt{DAX}) \sim I(1)$.

Examining the autocorrelation function would lead to the same conclusion.

Example 5

UK interest rate spread.



	Short rate	Long rate	Spread
ADF	-2.7628	-1.5901	-3.835

Critical values for test statistics:

1pct	5pct	10pct
-3.43	-2.86	-2.57

Both unit root hypotheses are accepted at the 5% level.

According to the *expectations hypothesis* (EH) of the term structure of interest rates the long rate is the weighted average of the current and expected rates of the future short rates.

This implies that the spread should be stationary.

Above the unit root hypothesis for spread is rejected, indicating support for the EH.

Other popular unit root tests: Phillips-Perron (PP) (Biometrika, 1988, 335–346), Elliot-Rottenberg-Stock (ERS) (Econometrica, 1996, 813–836)

ERS is supposed be more efficient than the others.

```
          Short rate      Long rate      Spread
ERS        -1.853           -0.8955     -2.3904

Critical values of DF-GLS are:
                 1pct  5pct 10pct
critical values -2.57 -1.94 -1.62
```

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Dickey and Pantula (1987) (*Journal of Business and Economic Statistics*, 455–456)

E.g., presence of at most two unit roots, i.e., $I(2)$.

Use the t-ratio on β_2 from

$$\Delta^2 y_t = \beta_0 + \beta_2 \Delta y_{t-1} + u_t \tag{20}$$

to test two unit roots against one with critical values from the intercept version.

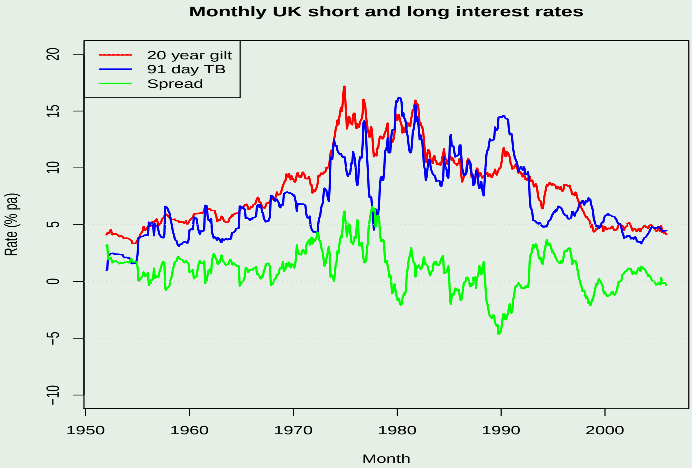
If rejected, proceed to test exactly one unit root with the t-ratio on β_1 from

$$\Delta^2 y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 \Delta y_{t-1} + u_t \tag{21}$$

with critical values from the intercept version (19) of the ADF-test.

Example 6

UK interest rates $I(2)$? (Classroom example).



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Perron (1989, *Econometrica*) generalized the unit root testing (that has possibly a drift and a linear trend) to allow for one time change in the structure at an unknown time T_B (break point), $1 < T_B < T$.

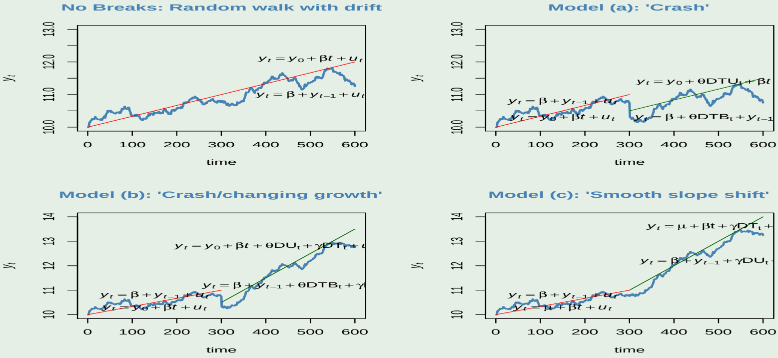
T_B must be determined prior testing.

Occurring instantly/smoothly?

(a) Shift in the intercept of the trend (crash model)
(b) shift in intercept and slope (crash/changing growth)
(c) smooth shift in the slope (joined segments)

Example 7

Certain types of breaks.



Define dummy variables:

$$
\begin{aligned}
\mathtt{DTB}_t = 1,\ \text{if } t = T_B + 1, \quad &= 0 \text{ otherwise} \\
\mathtt{DU}_t = 1,\ \text{if } t > T_B, \quad &= 0 \text{ otherwise} \\
\mathtt{DT}_t = t - T_B,\ \text{if } t > T_B, \quad &= 0 \text{ otherwise.}
\end{aligned}
\tag{22}
$$

Note that $\mathtt{DTB}_t = \Delta\mathtt{DU}_t = \Delta^2\mathtt{DR}_t$.

Null hypothesis models of (a)–(c)

$$y_t = \beta + y_{t-1} + \theta\mathtt{DTB}_t + u_t \tag{23}$$

$$y_t = \beta + y_{t-1} + \theta\mathtt{DTB}_t + \gamma\mathtt{DU}_t + u_t \tag{24}$$

$$y_t = \beta + y_{t-1} + \gamma\mathtt{DT}_t + u_t \tag{25}$$

Trend stationary alternative hypotheses of (a)–(c)

$$y_t = \mu + \beta t + \theta \mathtt{DTB}_t + u_t \tag{26}$$

$$y_t = \mu + \beta t + \theta \mathtt{DU}_t + \gamma \mathtt{DT}_t + u_t \tag{27}$$

$$y_t = \mu + \beta t + \gamma \mathtt{DT}_t + u_t \tag{28}$$

The error term u_t is assumed stationary.

Model (a): One time change by magnitude θ

Model (b): Null: Changing growth, drift changes from β to $\beta + \theta\gamma$ at time $T_B + 1$ and then to $\beta + \gamma$ afterwards. Alternative: Intercept changes by θ and slope by γ at $T_B + 1$.

Model (c): Null: Drift changes to $\beta + \gamma$. Alternative: Both segments of the trends are equal at T_B.

Testing for unit root in these circumstances is bit trickier than in the ordinary case.

Basically it consists of four steps (Perron 1989, *Econometrica*).

Step 1: Calculate detrended series $\tilde{y}_t$. For example in the case (a) $\tilde{y}_t = y_t - \hat{\mu} - \hat{\beta} t - \hat{\theta}\mathtt{DU}_t$, where the parameters are estimates by OLS.

Step 2: Test unit root using the t-statistic for $\phi = 1$ in the regression [cases (a) and (b)]

$$\tilde{y}_t = \sum_{i=0}^{k} \omega_i \mathtt{DBT}_{t-i} + \phi \tilde{y}_{t-1} + \sum_{i=1}^{k} c_i \Delta \tilde{y}_{t-i} + e_t. \tag{29}$$

and in case (c)

$$\tilde{y}_t = \phi \tilde{y}_{t-1} + \sum_{i=1}^{k} c_i \Delta \tilde{y}_{t-i} + e_t. \tag{30}$$

Step 3: Compute the set of t-statistics for all possible breaks and select the date for the break T_B for which the t-statistic is minimized.

Step 4: Compare the selected t-value of Step 3 to appropriate critical value (Tables given e.g. in Vogelsang and Perron 1998 *International Economic Review*).

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Rather than instantaneous break (alternative hypothesis), the trend can change smoothly. In such a case one possibility to model it is to utilize logistic smooth transition regression (LSTR):

Alternative hypotheses of (a)–(c)

$$y_t = \mu_1 + \mu_2 S_t(\gamma, m) + u_t \tag{31}$$
$$y_t = \mu_1 + \beta_1 t + \mu_2 S_t(\gamma, m) + u_t \tag{32}$$
$$y_t = \mu_1 + \beta_1 t + \mu_2 S_t(\gamma, m) + \beta_2 t S_t(\gamma, m) + u_t, \tag{33}$$

where

$$S_t(\gamma, m) = \frac{1}{1 + \exp(-\gamma(t - mT))}. \tag{34}$$

$0 \le S_t(\gamma, m) \le 1$. Parameter m determines the timing of the transition midpoint.

For $\gamma > 0$, $S_{-\infty}(\gamma, m) = 0$, $S_{\infty}(\gamma, m) = 1$, and $S_{mT}(\gamma, m) = 0.5$.

γ determines the speed of transmission, for $\gamma = 0$, $S_t(0, m) = 0.5$.

LSTAR models are estimated by nonlinear least squares (NLS) and unit root is tested by ADF applied to the residuals of the NLS regression (more details in Laybourne, Newbold, and Vougas 1998 *Journal of Time Series Analysis*).