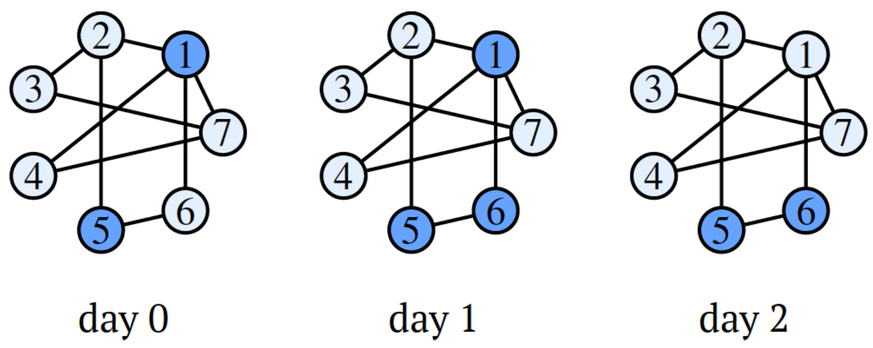
Termination

In this lesson, we use invariants to show that two natural processes terminate quickly. Later we show how to use similar ideas and invariants for estimating the running time of algorithms.

Problem. There are two football teams in a town. Each citizen supports one of the teams. If among someone's friends there are more fans of the other team than of her own, she switches to support the other team. Each day, one such citizen switches. Is it possible that this switching process goes on forever? (For the sake of this problem, we assume that friendship is always mutual and that both the population and the friendship networks do not change.)

The following figure shows a toy example with seven citizens.

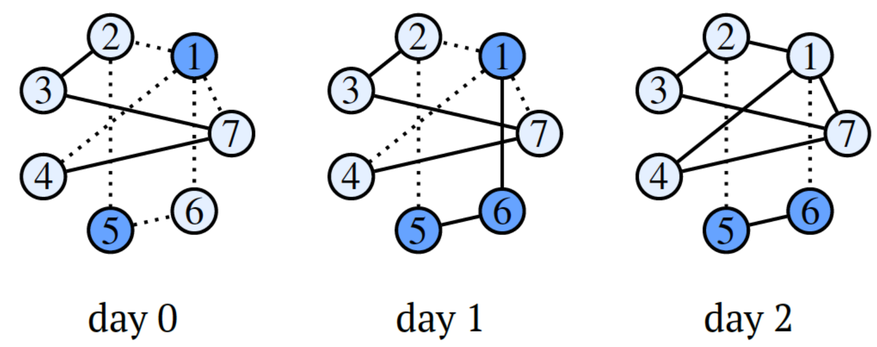


Two individuals are friends if they are connected by a segment in the picture (for example, individuals 3 and 7 are friends whereas individuals 2 and 6 are not). Initially (day 0), individuals 1 and 5 support the first team (call it dark) and all others support the second team (call it light). On the first day, individual 6 realizes that all of her friends (individuals 1 and 5) support the dark team and starts to support it, too. On the second day, individual 1 switches to support the light team as he has three friends supporting the light team (individuals 2, 4, and 7) whereas only one of his friends (individual 6) supports his favorite team. Then, no more further switches are possible and the process stops.

It seems plausible that for some larger size of the population and some other friendship connections the process can go forever. Below, we show that in fact this is impossible. To do this, we find a quantity that decreases each time an individual switches. We want this quantity to be restricted to non-negative integers so that it cannot decrease for infinitely long.

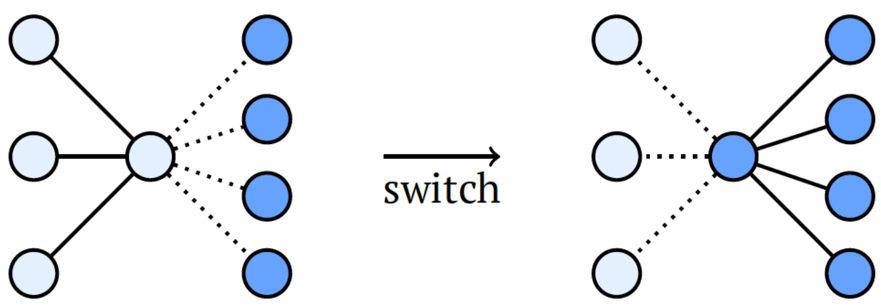
Do you see such a quantity?

Let us call a friendship connection *shaky* if the two friends support different teams. The following figure repeats the previous one with shaky connections shown dotted.



As you see, the number of shaky connections drops after each day: initially (day 0), it is equal to six, on the first day it decreases to four, on the second day it is equal to two. It turns out that this is not a coincidence: for any population and any friendship network, the number of shaky connections decreases whenever an individual switches.

To show this, consider an individual that is about to switch.



We know the reason why she is going to switch: the number of her friends supporting the opposite team is greater than the number of her friends supporting her favorite team. In other words, before the switch, she has more shaky connections than she will have after it.

To complete this argument, it remains to note that the number of shaky connections is a non-negative integer. This means that it cannot decrease for infinitely long and this, in turn, allows us to conclude that the switchingprocess cannot continue forever.