Notes: The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

Part 1. If f is continuous on $[a, b]$ then the function g defined by $g(x) = \int_a^x f(t)dt, a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on $[a, b]$ and $g'(x) = f(x)$.

Part 2. If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any *antiderivative* of f, that is, a function such that $F' = f$.

Using FTC, Part 1

Part 1 of the Fundamental Theorem of Calculus simplifies differentiation of complex functions.

For example, let's differentiate $g(x) = \int_0^x \sqrt{1 + t^2 dt}$ without the FTC. First, we must evaluate the integral. For this we refer to a table of integrals (see section 1.6).

$$\begin{aligned} g(x) = \int_0^x \sqrt{1 + t^2 dt} &= \frac{t}{2}\sqrt{1 + t^2 + \frac{1}{2}\ln(t + \sqrt{1 + t^2})]_0^x} \\ &= \frac{x}{2}\sqrt{1 + x^2 + \frac{1}{2}\ln(x + \sqrt{1 + x^2})} \end{aligned}$$

Then, we must differentiate this expression with respect to x.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{x}{2}\sqrt{1 + x^2 + \frac{1}{2}\ln(x + \sqrt{1 + x^2})}\right) \\ &= \frac{1}{2}\sqrt{1 + x^2 + \frac{x}{2} \cdot \frac{1}{2}(1 + x^2)^{-1/2} \cdot (2x) + \frac{1}{2} \cdot \frac{1}{x + \sqrt{1 + x^2}} \cdot (1 + \frac{1}{2}2x(1 + x^2)^{-}} \\ &= \frac{1}{2}\sqrt{1 + x^2 + \frac{x^2}{2\sqrt{1 + x^2}} + \frac{1}{2(x + \sqrt{1 + x^2})} \cdot \frac{\sqrt{1 + x^2 + x}}{\sqrt{1 + x^2}}} \\ &= \frac{\sqrt{1 + x^2(\sqrt{1 + x^2})(x + \sqrt{1 + x^2}) + x^2(x + \sqrt{1 + x^2}) + \sqrt{1 + x^2 + x}}}{2\sqrt{1 + x^2(x + \sqrt{1 + x^2})}} \\ &= \frac{2(x + \sqrt{1 + x^2})(1 + x^2)}{2\sqrt{1 + x^2(x + \sqrt{1 + x^2})}} = \boxed{\sqrt{1 + x^2}} \end{aligned}$$

But, since $f(t) = \sqrt{1 + t^2}$ is continuous when $t \geq 0$, we can use the FTC part 1 to bypass all these calculations. The derivative is the integrand evaluated at $t = x$.