University of Illinois **Fall 2012**

ECE 313: Problem Set 5: Problems and Solutions
Law of total probability, Bayes' formula and binary hypothesis testing

Due: Wednesday, October 3 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.10 and 2.11

1. **[Grabbing chocolates]**
Suppose that you have two bags with white and dark chocolates. Bag 1 has two white chocolates and six dark chocolates. Bag 2 has four white chocolates and two dark chocolates. You choose one bag at random, both being equally likely, and you grab five chocolates with replacement (you grab one, look at it, put it back, repeat) from the chosen bag. Let A be the event that you grab three white chocolates and two dark chocolates.

   (a) Find $P(A)$.
   Solution: Let B_i be the event that bag $i \in \{1, 2\}$ is chosen, then $P(B_1) = P(B_2) = 1/2$. If bag 1 is chosen, the probability of grabbing a white chocolate is $\frac{2}{2+6} = \frac{1}{4}$. If bag 2 is chosen, the probability of grabbing a white chocolate is $\frac{4}{4+2} = \frac{2}{3}$. Then, using the law of total probability,

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = \frac{1}{2}(P(A|B_1) + P(A|B_2)) \\ &= \frac{1}{2}\left(\binom{5}{3}\left(\frac{1}{4}\right)^3\left(1-\frac{1}{4}\right)^{5-3} + \binom{5}{3}\left(\frac{2}{3}\right)^3\left(1-\frac{2}{3}\right)^{5-3}\right) = \frac{103,790}{497,664} \approx 0.2086 \end{aligned}$$

   (b) Find the probability that you chose bag 1 given that event A occurred.
   Solution: Using Bayes' formula,

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{\binom{5}{3}\left(\frac{1}{4}\right)^3\left(1-\frac{1}{4}\right)^{5-3}\frac{1}{2}}{\frac{103,790}{497,664}} = \frac{22,394,880}{106,280,960} \approx 0.2107$$

2. **[Matching genes]**
The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes; and if one is a brown-eyed gene and the other is a blue-eyed gene, then the person will have brown eyes (the brown-eyed gene is dominant over the blue-eyed gene). A newborn child independently receives one eye gene from each of its parents, and the gene it receives from a parent is equally likely to be either one of the genes of that parent. Suppose Dilbert and both of his parents have brown eyes, but Dilbert's sister has blue eyes.

   (a) Find the probability that Dilbert has a blue-eyed gene.
   Solution: Let B denote the blue-eyed gene and let N denote the brown-eyed gene. Dilbert's sister must have the gene pair BB because she has blue eyes, while Dilbert's parents must both have one gene of each color in order for them to have brown eyes

and also pass along the blue-eyed gene to their daughter. Then, there are three possible gene pairs Dilbert can have because he has brown eyes: BN, NB, NN, where the first gene comes from his father and the second from his mother. Let D denote the event that Dilbert has one blue-eyed gene, then $P(D) = \frac{2}{3}$.

(b) Suppose that Dilbert's wife has blue eyes. Find the probability that their first child will have blue eyes.

Solution: Let C_1 be the event that their first child has blue eyes. Then, using the law of total probability,

$$P(C_1) = P(C_1|D)P(D) + P(C_1|D^c)P(D^c) = \frac{1}{2}\frac{2}{3} + 0\left(1 - \frac{2}{3}\right) = \frac{1}{3}$$

because Dilbert's wife will surely pass the B gene (she has the gene pair BB because she has blue eyes), so the eye color of the child depends only on the gene that Dilbert passes.

(c) If their first child has brown eyes, what is the probability that their second child will also have brown eyes?

Solution: Let C_2 be the event that their second child has blue eyes. Then, using the definition of conditional probability and also the law of total probability,

$$\begin{aligned} P(C_2^c|C_1^c) &= \frac{P(C_2^c C_1^c)}{P(C_1^c)} = \frac{P(C_2^c C_1^c|D)P(D) + P(C_2^c C_1^c|D^c)P(D^c)}{1 - P(C_1)} \\ &= \frac{P(C_2^c|D)P(C_1^c|D)P(D) + P(C_2^c|D^c)P(C_1^c|D^c)P(D^c)}{1 - P(C_1)} \\ &= \frac{\frac{1}{2}\frac{1}{2}\frac{2}{3} + (1)(1)\left(1 - \frac{2}{3}\right)}{1 - \frac{1}{3}} = \frac{\frac{3}{6}}{\frac{2}{3}} = \frac{3}{4} \end{aligned}$$

because given the gene pair of Dilbert (D or D^c), the eye color of each child depends only on the random choice of the gene from Dilbert, which is independent from one child to the next.

3. **[Bayes' rule and total probability]**

Suppose you have two fair coins and you flip them both. Let X be the number of heads that show. Then, you grab a standard deck of 52 cards and draw X cards with replacement. Let Y be the number of aces you draw.

(a) Find $P\{Y = 1|X = 2\}$.

Solution: Each card drawn has a probability $1/13$ of being an ace because there are 13 possible ranks $A, 2, 3, \ldots, 10, J, Q, K$, and each draw is independent of the others because of the replacement. Therefore, $Y \sim Binomial(X, 1/13)$, and $P\{Y = 1|X = 2\} = \binom{2}{1}\left(\frac{1}{13}\right)^1\left(1 - \frac{1}{13}\right)^{2-1} = \frac{24}{169} \approx 0.1420$.

(b) Find the pmf of Y.

Solution: Each coin has a probability $1/2$ of being heads, independently of each other, so $X \sim Binomial(2, 1/2)$. Using this fact, along with the law of total probability and the fact that $Y \sim Binomial(X, 1/13)$, we obtain for $i = 0, 1, 2$:

$$\begin{aligned} p_Y(i) &= P\{Y = i|X = 0\}P\{X = 0\} + P\{Y = i|X = 1\}P\{X = 1\} \\ &\quad + P\{Y = i|X = 2\}P\{X = 2\} \end{aligned}$$

$$
\begin{aligned}
p_Y(0) &= [1]\left[\binom{2}{0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{2-0}\right] \\
&+ \left[\binom{1}{0}\left(\frac{1}{13}\right)^0\left(1-\frac{1}{13}\right)^{1-0}\right]\left[\binom{2}{1}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{2-1}\right] \\
&+ \left[\binom{2}{0}\left(\frac{1}{13}\right)^0\left(1-\frac{1}{13}\right)^{2-0}\right]\left[\binom{2}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{2-2}\right] = \frac{625}{676} \approx 0.9246
\end{aligned}
$$

$$
\begin{aligned}
p_Y(1) &= [0]\left[\binom{2}{0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{2-0}\right] \\
&+ \left[\binom{1}{1}\left(\frac{1}{13}\right)^1\left(1-\frac{1}{13}\right)^{1-1}\right]\left[\binom{2}{1}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{2-1}\right] \\
&+ \left[\binom{2}{1}\left(\frac{1}{13}\right)^1\left(1-\frac{1}{13}\right)^{2-1}\right]\left[\binom{2}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{2-2}\right] = \frac{50}{676} \approx 0.074
\end{aligned}
$$

$$
\begin{aligned}
p_Y(2) &= [0]\left[\binom{2}{0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{2-0}\right] + [0]\left[\binom{2}{1}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{2-1}\right] \\
&+ \left[\binom{2}{2}\left(\frac{1}{13}\right)^2\left(1-\frac{1}{13}\right)^{2-2}\right]\left[\binom{2}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{2-2}\right] = \frac{1}{676} \approx 0.0015
\end{aligned}
$$

(c) Find $P\{X = 2|Y = 1\}$.
Solution: Using Bayes' formula,

$$
\begin{aligned}
P\{X = 2|Y = 1\} &= \frac{P\{Y = 1|X = 2\}P\{X = 2\}}{P\{Y = 1\}} \\
&= \frac{\left[\binom{2}{1}\left(\frac{1}{13}\right)^1\left(1-\frac{1}{13}\right)^{2-1}\right]\left[\binom{2}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{2-2}\right]}{\frac{50}{576}} = \frac{16224}{33800} = \frac{12}{25} \approx 0.48
\end{aligned}
$$

4. **[Calvin tries to call Hobbes]**
Calvin is trying to call Hobbes but he doesn't know if the number he's calling is his home phone or his work phone. The probability that Hobbes answers at his home phone is p_0, whereas the probability that Hobbes answers at his work phone is p_1, with $0 < p_0 < p_1 < 1$. Calvin calls the number he has once a day until Hobbes finally picks up. Let X denote the number of times Calvin has to call up until Hobbes finally answers (the answered call included in the count). Assume that the successive call attempts are independent trials of an experiment, so that $X \sim Geometric(p_i)$, under hypothesis H_i. Assume that $2\pi_0 = 3\pi_1$.

(a) Find the ML decision rule in this binary hypothesis testing. Assume ties are broken in favor of H_1.
Solution: The likelihood ratio in this case is given by

$$
\Lambda(k) = \frac{P\{X = k|H_1\}}{P\{X = k|H_0\}} = \frac{p_1(1-p_1)^{k-1}}{p_0(1-p_0)^{k-1}}
$$

The ML rules in favor of H_1 if $\Lambda(k) \geq 1$. Therefore, to rule in favor of H_1,

$$\begin{aligned}
\Lambda(k) &\geq 1 \\
\frac{p_1(1-p_1)^{k-1}}{p_0(1-p_0)^{k-1}} &\geq 1 \\
\ln\left(\frac{p_1(1-p_1)^{k-1}}{p_0(1-p_0)^{k-1}}\right) &\geq \ln 1 \\
\ln\left(\frac{p_1}{p_0}\right) + (k-1)\ln\left(\frac{1-p_1}{1-p_0}\right) &\geq 0 \\
k &\overset{(*)}{\leq} \frac{-\ln\left(\frac{p_1}{p_0}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)} + 1 \\
k &\leq 1 + \frac{\ln\left(\frac{p_0}{p_1}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)}
\end{aligned}$$

where $(*)$ follows because $p_0 < p_1$ so that $\ln\left(\frac{1-p_1}{1-p_0}\right) < 0$.

(b) Find p_{MD}, p_{FA}, and p_e under the ML decision rule assuming $p_0 = e^{-1}$, $p_1 = 1 - e^{-1}$.
Solution: Notice that with the given values of p_0 and p_1, the ML rule becomes ruling in favor of H_1 if $X \leq 2$. Therefore,

$$\begin{aligned}
p_{MD} &= P\{\text{declare } H_0 \text{ true}|H_1\} = P\{X > 2|H_1\} = 1 - P\{X \leq 2|H_1\} \\
&= 1 - \left(p_1(1-p_1)^{1-1} + p_1(1-p_1)^{2-1}\right) = 1 - \left((1-e^{-1}) + (1-e^{-1})e^{-1}\right) \\
&= e^{-2} \approx 0.1353
\end{aligned}$$

$$\begin{aligned}
p_{FA} &= P\{\text{declare } H_1 \text{ true}|H_0\} = P\{X \leq 2|H_0\} = p_0(1-p_0)^{1-1} + p_0(1-p_0)^{2-1} \\
&= e^{-1} + e^{-1}(1-e^{-1}) = 2e^{-1} - e^{-2} \approx 0.6004
\end{aligned}$$

To calculate the average error probability we use the fact that in this case $2\pi_0 = 3\pi_1$ so that $\pi_0 = 3/5$ and $\pi_1 = 2/5$.

$$p_e = \pi_0 p_{FA} + \pi_1 p_{MD} = \frac{3}{5}\left(2e^{-1} - e^{-2}\right) + \frac{2}{5}e^{-2} = \frac{6}{5}e^{-1} - \frac{1}{5}e^{-2} \approx 0.4144$$

(c) Find the MAP decision rule in this binary hypothesis testing. Assume ties are broken in favor of H_1, and leave p_0 and p_1 as variables (do not use the values given in part (b)).
Solution: In this case, the MAP rules in favor of H_1 if $\Lambda(k) \geq \frac{\pi_0}{\pi_1}$. Therefore, to rule in favor of H_1,

$$
\begin{aligned}
\Lambda(k) &\geq \frac{\pi_0}{\pi_1} \\
\frac{p_1(1-p_1)^{k-1}}{p_0(1-p_0)^{k-1}} &\geq \frac{\pi_0}{\pi_1} \\
\ln\left(\frac{p_1(1-p_1)^{k-1}}{p_0(1-p_0)^{k-1}}\right) &\geq \ln\left(\frac{\pi_0}{\pi_1}\right) \\
\ln\left(\frac{p_1}{p_0}\right) + (k-1)\ln\left(\frac{1-p_1}{1-p_0}\right) &\geq \ln\left(\frac{\pi_0}{\pi_1}\right) \\
k &\stackrel{(*)}{\leq} \frac{\ln\left(\frac{\pi_0}{\pi_1}\right) - \ln\left(\frac{p_1}{p_0}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)} + 1 \\
k &\leq 1 + \frac{\ln\left(\frac{\pi_0 p_0}{\pi_1 p_1}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)}
\end{aligned}
$$

where again $(*)$ follows because $p_0 < p_1$ so that $\ln\left(\frac{1-p_1}{1-p_0}\right) < 0$.

(d) Find p_{MD}, p_{FA}, and p_e under the MAP decision rule assuming $p_0 = e^{-1}$, $p_1 = 1 - e^{-1}$.

Solution: Notice that with the given values of p_0 and p_1, the MAP rule becomes ruling in favor of H_1 if $X \leq 2 + \frac{\ln 3 - \ln 2}{-1 - \ln(1-e^{-1})} \approx 1.251$. Therefore, MAP rules in favor of H_1 if $X \leq 1$. Hence,

$$
\begin{aligned}
p_{MD} &= P\{\text{declare } H_0 \text{ true}|H_1\} = P\{X > 1|H_1\} = 1 - P\{X \leq 1|H_1\} \\
&= 1 - p_1(1-p_1)^{1-1} = 1 - (1 - e^{-1}) = e^{-1} \approx 0.3679
\end{aligned}
$$

$$
p_{FA} = P\{\text{declare } H_1 \text{ true}|H_0\} = P\{X \leq 1|H_0\} = p_0(1-p_0)^{1-1} = e^{-1} \approx 0.3679
$$

And,

$$
p_e = \pi_0 p_{FA} + \pi_1 p_{MD} = e^{-1}(\pi_0 + \pi_1) = e^{-1} \approx 0.3679
$$

As expected, p_e for the MAP rule is less than or equal to p_e for the ML rule.

5. [**Telephone game**]

Suppose that in a game of telephone, the message to be passed on (sent) can be one of three equally likely messages (m_1, m_2, m_3). As the message passes from one end to the other, the message can get distorted and the message received at the other end can be different than the one sent. Suppose that if m_1 is sent, the conditional probabilities of the received message being m_1, m_2, m_3 are $0.7, 0.2, 0.1$, respectively. If m_2 is sent, the conditional probabilities of the received message being m_1, m_2, m_3 are $0.05, 0.9, 0.05$, respectively. If m_3 is sent, the conditional probabilities of the received message being m_1, m_2, m_3 are $0.15, 0.05, 0.8$, respectively.

(a) For $i = 1, 2, 3$, find the probability that message m_i is received.

Solution: For $i = 0, 1, 2$, let S_i denote the event that message m_i is sent, and let R_i denote the event that message m_i is received. Using the law of total probability we obtain

$$
\begin{aligned}
P(R_i) &= P(R_i|S_1)P(S_1) \ + \ P(R_i|S_2)P(S_2) \ + \ P(R_i|S_3)P(S_3) \\
&= \frac{1}{3}\left(P(R_i|S_1) \ + \ P(R_i|S_2) \ + \ P(R_i|S_3)\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P(R_1) &= \frac{1}{3}(0.7 + 0.05 + 0.15) = \frac{18}{60} \approx 0.3 \\
P(R_2) &= \frac{1}{3}(0.2 + 0.9 + 0.05) = \frac{23}{60} \approx 0.3833 \\
P(R_3) &= \frac{1}{3}(0.1 + 0.05 + 0.8) = \frac{19}{60} \approx 0.3167
\end{aligned}
$$

(b) For $i = 1, 2, 3$, find the probability that message m_i was sent given that message m_i is received.

Solution: Bayes' rule applied here yields

$$
P(S_i|R_i) = \frac{P(R_i|S_i)P(S_i)}{P(R_i)} = \frac{1}{3}\frac{P(R_i|S_i)}{P(R_i)}
$$

Hence,

$$
\begin{aligned}
P(S_1|R_1) &= \frac{1}{3}\frac{0.7}{\frac{18}{60}} = \frac{7}{9} \approx 0.7777 \\
P(S_2|R_2) &= \frac{1}{3}\frac{0.9}{\frac{23}{60}} = \frac{18}{23} \approx 0.7826 \\
P(S_3|R_3) &= \frac{1}{3}\frac{0.8}{\frac{19}{60}} = \frac{16}{19} \approx 0.8421
\end{aligned}
$$

(c) We want to set up a hypothesis testing to decide which message was sent based on the received message, with H_i being the hypothesis that message m_i was sent, for $i = 1, 2, 3$. Obtain the likelihood matrix for this ternary hypothesis testing.

Solution: Let H_i be the hypothesis that message m_i is sent. Then, the likelihood matrix becomes

	receive m_j		
	1	2	3
H_1	0.7	0.2	0.1
H_2	0.05	0.9	0.05
H_3	0.15	0.05	0.8

(d) How many different possible decision rules are there in this ternary hypothesis testing?

Solution: For each received message one can decide among three different sent messages, therefore, there are $3^3 = 27$ possible decision rules.

(e) Find the ML rule for this ternary hypothesis testing.

Solution: In the ML rule, given a received message, one decides in favor of the most likely sent message, which means that for each column in the likelihood matrix, one chooses the row with the largest value. Hence, in this case, for $i = 1, 2, 3$, if m_i is received, m_i is chosen by the ML rule.

(f) Find the probability of error for the ML rule in this ternary hypothesis testing.

Solution: In this case, one makes an error if the rule chooses a different message than the one sent. For $i = 0, 1, 2$, let S_i denote the event that message m_i is sent, and let R_i denote the event that message m_i is received. Using the law of total probability we obtain

$$\begin{aligned}
P\{error\} &= P\{error|S_1\}P(S_1) \;+\; P\{error|S_2\}P(S_2) \;+\; P\{error|S_3\}P(S_3) \\
&= \frac{1}{3}\left(P\{error|S_1\} + P\{error|S_2\} + P\{error|S_3\}\right) \\
&= \frac{1}{3}\left(P\{R_2 \cup R_3|S_1\} + P\{R_1 \cup R_3|S_2\} + P\{R_1 \cup R_2|S_3\}\right) \\
&= \frac{1}{3}\left(0.2 + 0.1 + 0.05 + 0.05 + 0.15 + 0.08\right) = \frac{1}{5},
\end{aligned}$$

which is the same as the sum of all the unselected entries in the likelihood matrix.

(g) Find the MAP rule for this ternary hypothesis testing.

Solution: The prior probabilities are equal for all hypothesis because the messages are equally likely to be sent, therefore the MAP rule is the same as the ML rule. That is, for $i = 1, 2, 3$, if m_i is received, m_i is chosen by the MAP rule.

(h) Find the probability of error for the MAP rule in this ternary hypothesis testing.

Solution: The prior probabilities are equal for all hypothesis, therefore the error probability for the the MAP rule is the same as the ML rule. Hence $P\{error\} = \frac{1}{5}$.