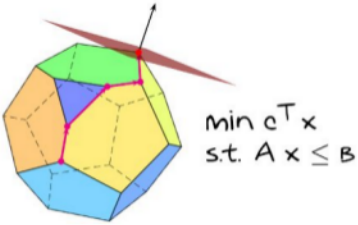
Duality

- Upper bounds
- The dual linear program
- Weak and strong duality



Upper bounds

$$\text{MAX } c^T \cdot x \qquad A \in \mathbb{R}^{m \times n}$$
$$Ax \le b \qquad b \in \mathbb{R}^m, c \in \mathbb{R}^n$$

$\lambda \in \mathbb{R}^m$ and $\lambda \ge 0$, then $(\lambda^T \cdot A)\, x \le \lambda^T \cdot b$ valid for

$P = \{ x \in \mathbb{R}^n : Ax \le b \}$

If in addition $\lambda^T \cdot A = c^T$, then $c^T \cdot x \le \lambda^T \cdot b$ VALID FOR ALL FEAS. SOL!

$\lambda^T \cdot b$ IS upper bound on obj. values of feasible solutions

Quiz

Given $\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}$. Which linear program is the one that yields the best (minimum) valid upper bound on the objective values of feasible solutions?

MIN $b^T \cdot \lambda$

$A^T \cdot \lambda = c$

Min $b^T \cdot \lambda$

$A^T \cdot \lambda = c$

$\lambda \geq 0$

$\lambda \in \mathbb{R}^m$ feasible

$\lambda \geq 0$

$\lambda^T \cdot A = c$

Thus $c^T \cdot x \leq \underline{\lambda^T \cdot b}$

VALID FOR ALL feas. sol. of

The dual linear program

Given $\max \{c^T \cdot x \;:\; x \in \mathbb{R}^n, Ax \leq b\}$, the dual linear program is

$$\min \; b^T \cdot y$$
$$A^T y = c$$
$$y \geq 0$$

$A \in \mathbb{R}^{m \times n}$

$y \in \mathbb{R}^{\boxed{m}}$

$c^T \cdot x \leq \lambda^T \cdot b$

Quiz

$\forall \alpha \geq 0,\ (-\alpha, \alpha, \alpha)$ feasible

OB. VAL $= 2 \cdot \alpha$

unbounded

Identify the dual of

$$\max \quad x_1 + 2x_2 + x_3$$

$$x_1 + x_2 \leq 1$$

$$x_1 + x_3 \leq 4$$

$c^T = (1, 2, 1)$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$\min \ b^T \cdot y$

$A^T \cdot y = c$

$y \geq 0$

infeasible

$$\min \quad y_1 + 4y_2$$

$$y_1 + y_2 = 1$$

$$y_1 = 2$$

$$y_2 = 1$$

$$y_1, y_2 \geq 0$$

$$\min \quad y_1 + y_2$$

$$y_1 + y_2 = 1$$

$$y_1 \leq 2$$

$$y_2 \leq 1$$

$$y_1, y_2 \geq 0$$

Weak duality

Theorem (Weak duality)

Consider a linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}$ and its dual $\min\{b^T y : y \in \mathbb{R}^m, A^T y = c, y \geqslant 0\}$. If $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$ are primal and dual feasible respectively, then $c^T x^* \leqslant b^T y^*$.

Proof: $c^T \cdot x \leq b^T \cdot y^*$ Is valid for all feas. sol. of primal.

$\blacksquare$

Strong duality

Theorem (Strong duality)

Consider a linear program $\max\{c^T x : x \in \mathbb{R}^n,\ Ax \leqslant b\}$ and its dual $\min\{b^T y : y \in \mathbb{R}^m,\ A^T y = c,\ y \geqslant 0\}$. If the primal is feasible and bounded, then there exist a primal feasible x^* and a dual feasible y^* with $c^T x^* = b^T y^*$.

$c^T x^* = b^T \cdot y^*$ $\qquad$ $c^T x \leq b^T \cdot y^*$ ← dual feas.

$\Rightarrow$ Mean to prove optimality of primal solutions

Proof: CASE 1: A has full col. rank.

Simplex $\rightsquigarrow$ Optimal Basis $B \subseteq \{1, \dots, m\}$

Strong duality (cont.)

Optimality of B: $\lambda \in \mathbb{R}^m$ s.t. $\lambda_B^T \cdot A_B = c^T$, $\lambda_i = 0$ $\forall i \notin B$

is ≥ 0. $\Rightarrow$ λ dual feas.

$\lambda^T \cdot A = c^T$

$\lambda \geq 0$

$x^* = A_B^{-1} \cdot b_B$ current primal solution.

$$c^T \cdot x^* = \lambda_B^T \cdot A_B x^* = \lambda_B^T \cdot \underbrace{A_B \cdot A_B^{-1}}_{I} \cdot b_B = \lambda_B^T \cdot b_B$$

$$= \lambda^T \cdot b$$

Strong duality (cont.)

CASE 2: $\operatorname{rank}(A) < n$

$$\max c^T x \qquad \max c^T(x_1 - x_2)$$
$$Ax \le b \qquad A(x_1 - x_2) \le b$$
$$x_1, x_2 \ge 0$$

$$\begin{array}{l} \lambda_1 \{ \\ \lambda_2 \{ \\ \lambda_3 \{ \end{array} \begin{pmatrix} A & -A \\ -I & 0 \\ 0 & -I \end{pmatrix}, \quad (c^T, -c^T), \quad \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}$$

Simplex: $(x_1^* - x_2^*)$

$$\lambda^T = (\lambda_1^T, \lambda_2^T, \lambda_3^T) \ge 0$$

$$\left.\begin{array}{l} \lambda_1^T \cdot A - \lambda_2^T = c^T \Rightarrow \lambda_1^T A \ge c^T \\ \lambda_1^T \cdot (-A) - \lambda_3^T = -c^T \Rightarrow -\lambda_1^T \cdot A \ge -c^T \end{array}\right\} \Rightarrow \lambda_1^T \cdot A = c^T$$

$\Rightarrow \lambda_1$ is dual feasible

$$c^T(x_1^* - x_2^*) = \lambda^T \cdot \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} = \lambda_1^T \cdot b$$

$\blacksquare$