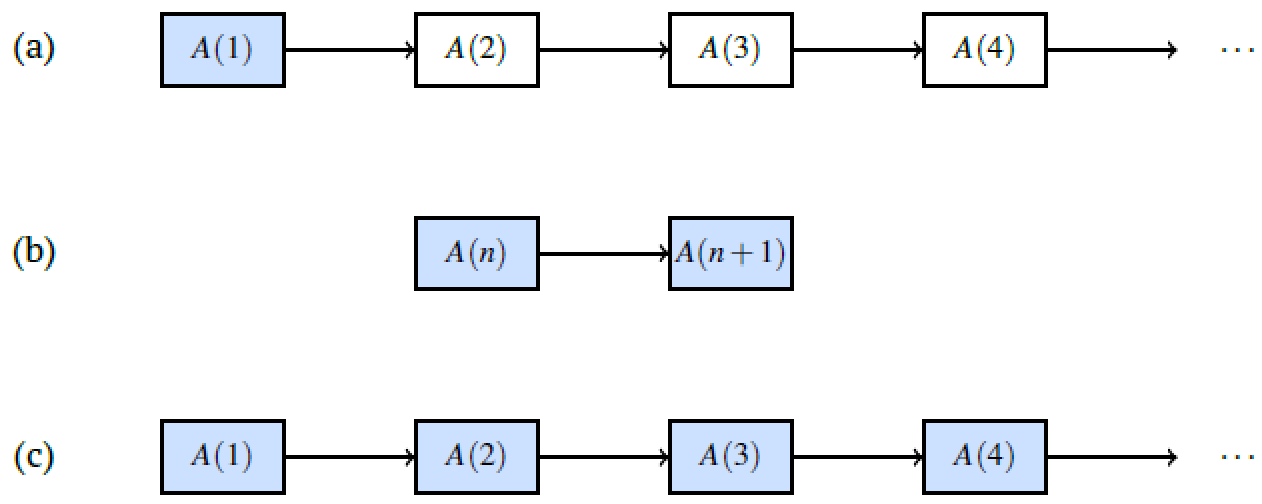
What is Induction?

A mathematical induction proof of the fact that all of the statements $A(1), A(2), A(3)\ldots$ are true, consists of two parts:

1. *The base case*: First, we prove that $A(1)$ is true.
2. *The induction step*: Then, we prove that for every $n \geq 1$, the statement $A(n)$ implies the statement $A(n+1)$ (that is, if $A(n)$ is true, then $A(n+1)$ is true).

Mathematical induction assures that once we (i) proved the base case and (ii) proved the induction step, we have actually proved *all* statements $A(n)$ for $n \geq 1$. See the figure:



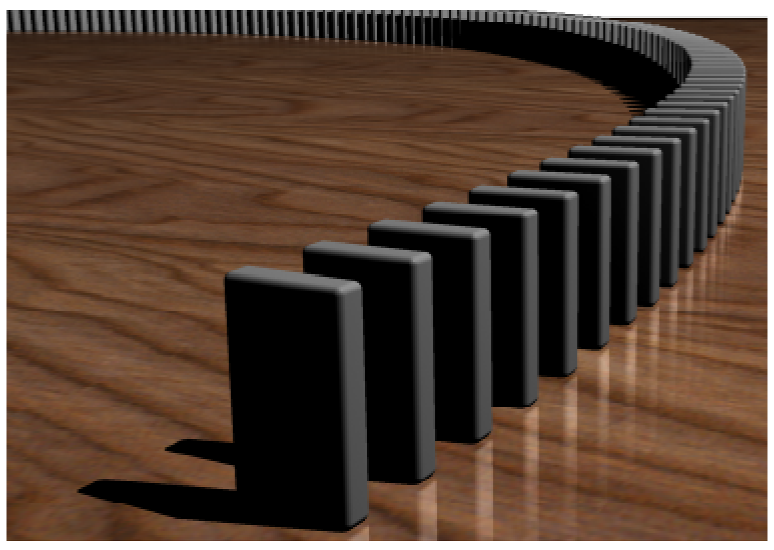
(a) the base case ensures that the first statement is true;
(b) the induction step lets us move from any statement to the following one;
(c) from this, we conclude that all the statements are true.

Thus, a typical induction proof consists of two parts. The first part(usually the easiest one) is to prove the base case. Of course, there is nothing special about $n=1$, and we could start at any other number $n=c$ (say, $c=0$, or $c=15$, or $c=-3$). Either way, we will prove all statements $A(c), A(c+1), A(c+2), \ldots$

The second part of an induction proof is to prove the induction step. Here, we assume that we already know that the statement $A(n)$ is true. This assumption is called the *induction hypothesis*. Assuming the induction hypothesis, we prove the statement $A(n+1)$, and this finishes the proof. Sometimes it comes in handy to use a stronger induction hypothesis : assume that all the previous statements $A(1), \ldots, A(n)$ are true, and use them to prove correctness of $A(n+1)$. This variation of induction proofs is often called *strong induction*.

Let us recall a strong induction proof that we have seen. Consider the following statement $A(n)$: n can be represented as $3k+5l$ where k and l are non-negative integers. Problem Coins states that $A(n)$ is true for all $n \geq 8$. Indeed, for $n=8,9,10$, $A(n)$ is true: $8=3\cdot1+5\cdot1, 9=3\cdot3+5\cdot0, 10=3\cdot0+5\cdot2$This is the base case. For the induction step, assume that $A(8), A(9), \ldots, A(n)$ are all true for some $n \geq 11$. Since $n \geq 11, n-3 \geq 8$ and hence $A(n-3)$ is true. Therefore, $n-3=3\cdot k+5\cdot l$ for some k, l. Thus, $n=3k+5l+3=3(k+1)+5l$.

Mathematical induction can be visualized as a domino effect:given an infinitely long chain of dominoes, each separated by a small distance, it suffices to push the first domino to knock down all dominoes in the chain (see Figure). In fact, this holds due to mathematical induction! Indeed, for every $n \geq 1$, let $A(n)$ be the proposition that the domino number n falls down. Then, we ensure $A(1)$ by pushing the first domino. Since the dominoes are too close to one another, $A(n)$ implies $A(n+1)$: each domino falls after it is knocked over by the previous one.



The domino effect : pushing the first domino knocks down all other dominoes. (Source : Wikipedia).