Feedback — Homework 5

Help Center

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You submitted this homework on **Mon 6 Apr 2015 11:13 AM PDT**. You got a score of **7.00** out of **7.00**.

Question 1

We have seen some of the strange aspects of fluctuations in our explorations of the hot hand phenomenon and again in the ballot problem of W. A. Whitworth that we had discussed in Lectures 8.2:d, e, f. This problem takes you on an exploration of another fascinating facet of fluctuations. The link below will take you to the simulation.

Click here to go to the question.

Run the simulation ten times, ponder the results you get, and report them by selecting one of the three radio buttons below.

Your Answer		Score	Explanation
For your 10 realisations of random walks over 2400 steps, the average time of the last return to the origin was no larger than 800.			
For your 10 realisations of random walks over 2400 steps, the average time of the last return to the origin was between 801 and 1600.			
For your 10 realisations of random walks over 2400 steps, the average time of the last return to the origin was larger than 1600.	✔	1.00	
Total		1.00 / 1.00	

Question Explanation

The strange features of fluctuations were remarked by Paul Lévy in 1939 with the discovery of what is called the *arc sine law*. Suppose S_n denotes the position of a random walk after n steps. Lévy discovered the following limit law when the number of steps, n, is large: *for every fixed* $0 < t < 1$*, the probability that the last return to the origin of the walk* S_n *occurs on or before step* tn *is asymptotically given by* $\frac{2}{\pi}\arcsin\sqrt{t}$ *as* $n \to \infty$. I will not provide the proof here; it's a little too involved for a first course in the subject. (Those who are interested will find an elementary analysis in pages 244 – 252 and remarkable generalisations and sophistications in pages 550 – 557 of *The Theory of Probability*.) What is key here is that the arcsine function given on the right increases most rapidly near 0 and 1 and increases much more gradually over the long middle segment. A curious reader may wish to plot the function $f(t) = \frac{2}{\pi}\arcsin(\sqrt{t}\,)$ as t varies from 0 to 1 to get a feel for it. For instance, $f(0.25) = 0.3\dot{3}$ and $f(0.75) = 1 - 0.3\dot{3}$ so that, in the middle half interval for $0.25 < t < 0.75$, the function only increases by one-thirds. in particular, there is 33% chance that the last return to the origin occurs in the first quarter of the walk; a 33% chance that it occurs in the last quarter of the walk; and only a 33% chance that it occurs in the broad, middle half of the walk. You may have been struck in your simulations by the paucity of zero crossings in the walk: we may have been led by untrained intuition to expect a lot of zero crossings as, over the long run, $+1$s and -1s should be roughly in balance. Lévy's theorem explains the seeming paradox.

The following clarifying example and surprising theorem are excerpted from *The Theory of Probability*.

An example: the tortoise and the hare. Two players, say A and B, engage in a series of competitive events and a running total is kept of who is ahead. If over a sequence of 100 games player A gets into the lead early and never relinquishes it is that clear evidence of her superiority? Similar questions can be posed about, say, IQ tests: should Suzie be considered to be a stronger student than Johnny if she jumps out into the lead and stays there over a hundred tests? (The relevance of the ballot problem of Lectures 8.2: d, e, and f should be clear in this context.)

One can judge the statistical validity of the conclusion that A is the stronger player by considering a parallel experiment in which a fair coin is tossed repeatedly and a running total of the excess of successes over failures kept. Common intuition leads us to expect frequent changes in lead if the coin is fair. An observation hence that A is ahead of B over, say, the last half or the last two-thirds of the duration of the experiment may be taken in support of the premise that A is the superior player (that is, she is playing with a bent coin favouring success). But intuition is not a reliable guide in itself and does not substitute for a careful mathematical analysis.

The probability that in a sequence of n fair coin tosses the last equalisation of heads and tails occurred at or before step tn with heads leading thereafter is given asymptotically by Lévy's theorem to be approximately equal to $B(t) = \frac{2}{\pi}\arcsin\sqrt{t}$. As $B(t)$ is symmetric about $t = 1/2$, the probability that one player or the other is in the lead over the entire last half of the game is $1/2$. *There is thus a 50% chance that one player or the other leads throughout the second half of the game.*

It is intuitively even more disturbing that the probability that one player stays in the lead through the last $(1-t)n$ trials remains large even when t is quite small: the probability is approximately one-third that one player stays in the lead through the last three-quarters of the game; and about 20% of the time one player leads through the last 90% of the game. Thus, it is not at all unlikely that one player will lead the second through practically the entire duration of a game even when wins and losses are completely chance-dependent.

Here is another, somewhat more sophisticated additional conclusion for the student who has enough mathematical background. Let q denote the probability that the walk *never* returns to the origin. Let K_n denote the position of the last return to the origin in a walk over n steps. Then, for any n, we must have $q \leq \mathbf{P}\{K_n > tn\}$ (this is monotonicity of probability measure, Lecture 5: g), and, by allowing n to become large, we see that, for any fixed $0 < t < 1$, we must have $q \leq 1 - \frac{2}{\pi}\arcsin\sqrt{t}$. By selecting t close to 1, the right-hand side may be made as small as we wish and so we conclude that $q = 0$ identically, and the random walk will return to the origin with probability one. With each return to the origin, however, the situation resets and a new random walk progresses which ultimately returns to the origin and restarts the process anew. By induction there must be an infinity of returns to the origin and we obtain the following *theorem*:

A random walk on the line returns to the origin infinitely often with probability one.

A result which may explain the difficulty in weaning a drinker from the demon drink.

Question 2

Two biased coins whose success probabilities are 0.6 and 0.7, respectively, are tossed once each, the trials assumed to be independent. Determine the probability that exactly one of the tosses results in a success.

Your Answer		Score	Explanation
0.12			
0.42			
0.48			
0.54			
0.28			
0.46	✔	1.00	
Total		1.00 / 1.00	

Question Explanation

A first principles approach: Write X_1 and X_2 for the results of the tosses of each of the coins. Then X_1 represents a Bernoulli trial with success probability 0.6 while X_2 represents a Bernoulli trial with success probability 0.7. The sample space may be identified with the collection of pairs $\Omega = \{(0,0),(1,0),(0,1),(1,1)\}$ where we write 1 for success and 0 for failure. We are equipped with product measure because the trials are independent. The event that exactly one of the tosses results in a success may now be identified with the subset $A = \{(1,0),(0,1)\}$ and so

$$\mathbf{P}(A) = \mathbf{P}\{X_1 = 1, X_2 = 0\} + \mathbf{P}\{X_1 = 0, X_2 = 1\} = \mathbf{P}\{X_1 = 1\}\mathbf{P}\{X_2 = 0\} + \mathbf{P}\{X_1 = 0\}\mathbf{P}\{X_2 = 1\} = 0.6(1-0.7) + (1-0.6)0.7 = 0.46.$$

And for folks who revel in formalism: Denote by A_1 and A_2 the events that the first coin shows success and the second coin shows success, respectively. These two events are independent per the slogan of Lecture 9.2:i. This, of course, means that $\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1)\mathbf{P}(A_2)$. As A is the symmetric difference of the two sets, the problem asks us to compute

$$\mathbf{P}(A) = \mathbf{P}(A_1 \,\Delta\, A_2) = \mathbf{P}(A_1 \setminus A_2) + \mathbf{P}(A_2 \setminus A_1).$$

In words, this is equal to the probability that exactly one of A_1 or A_2 occur. But by leveraging additivity and independence, we see that

$$\begin{aligned}\mathbf{P}(A_1 \setminus A_2) &= \mathbf{P}(A_1) - \mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) - \mathbf{P}(A_1)\mathbf{P}(A_2) = 0.6 - 0.6 \times 0.7 = 0.18,\\ \mathbf{P}(A_2 \setminus A_1) &= \mathbf{P}(A_2) - \mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_2) - \mathbf{P}(A_2)\mathbf{P}(A_1) = 0.7 - 0.7 \times 0.6 = 0.28.\end{aligned}$$

It follows that the desired probability is given by $\mathbf{P}(A) = 0.18 + 0.28 = 0.46$.

Question 3

A fair coin with faces labelled $+1$ and -1 is tossed thrice. Let X_1, X_2, and X_3 denote the results of the three tosses. Suppose $S_3 = X_1 + X_2 + X_3$. Determine $\mathbf{P}\{S_3 = 1\}$.

Your Answer		Score	Explanation
$\frac{1}{4}$			
$\frac{3}{4}$			
$\frac{3}{8}$	✔	1.00	
$\frac{1}{2}$			
$\frac{1}{8}$			
$\frac{5}{8}$			
Total		1.00 / 1.00	

Question Explanation

A first principles approach: The sample space is the space of binary triples (k_1, k_2, k_3) where each of the elements of the triple is -1 or $+1$. The probability measure is combinatorial with each triple having probability $1/8$. The event $\{S_3 = 1\}$ occurs if, and only if, there are two $+1$s and one -1 in the triple. It follows that

$$\mathbf{P}\{S_3 = 1\} = \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{8}.$$

Leveraging the binomial distribution: We can identify the experiment with a sequence of three Bernoulli trials via the replacements $Z_1 = (X_1 + 1)/2$, $Z_2 = (X_2 + 1)/2$, and $Z_3 = (X_3 + 1)/2$. We see that when X_j takes value $+1$, then Z_j take values 1, and when X_j takes value -1 then Z_j takes value 0. In other words, Z_1, Z_2, and Z_3 are ordinary Bernoulli trials. And we know that $Z_1 + Z_2 + Z_3$ represents the accumulated number of successes in three tosses of a fair coin, hence has the associated binomial distribution. We simply relate S_3 to $Z_1 + Z_2 + Z_3$ and read out the answer:

$$\mathbf{P}\{S_3 = 1\} = \mathbf{P}\{X_1 + X_2 + X_3 = 1\} = \mathbf{P}\{(2Z_1 - 1) + (2Z_2 - 1) + (2Z_3 - 1) = 1\}$$
$$= \mathbf{P}\{2(Z_1 + Z_2 + Z_3) = 4\} = \mathbf{P}\{Z_1 + Z_2 + Z_3 = 2\} = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}.$$

Question 4

A large and tedious book called *The Theory of Probability* contains 830 pages, each page containing a large number of words (the author apparently has a tiresome fondness for language). In spite of the best efforts of the publishing staff and the author a small number of word misprints have crept into the volume and after publication the book is found to have 83 word misprints. Now the author would be mortified indeed if the introductory page itself contained two or more misprints. Estimate the probability of this unfortunate event, that the first page of the introduction contains two or more misprints, and select the answer closest to it. Bear in mind that, in view of the lectures this week, there is a natural probability model for this setting which you are expected to use.

Your Answer		Score	Explanation
○ 0.004524			
◉ 0.004679	✔	1.00	
○ $\frac{1}{90}$			
○ 0.000154653			
○ 0.01			
○ 0.001			
Total		1.00 / 1.00	

Question Explanation

In view of Lectures 10.2:e – l, we think of misprints as rare events, that is to say, Bernoulli trials with a small success probability p. The number of words in the introductory page is large, say, n. The number of misprints in the introductory page is hence governed by a binomial distribution corresponding to n Bernoulli trials with "success" probability p. We model this as approximated by a Poisson distribution with parameter $\lambda = np$.

The given data tells us that the average number of misprints per page is $83/830 = 0.1$ and so we have the natural estimate $\lambda = 0.1$. The probability that there are exactly k misprints on the page is now estimated by the Poisson probability

$$p(k; 0.1) = e^{-0.1}\frac{0.1^k}{k!}$$

and so, the probability that there are two or more misprints may be estimated by

$$1 - p(0; 0.1) - p(1; 0.1) = 1 - e^{-0.1} - 0.1e^{-0.1} = 0.004679$$

truncated to six places. The author need not be excessively worried about this happening.

Question 5

Three highways pass through a county. On any given day, the number of accidents occurring on these highways in the county are independent Poisson random variables with parameters 0.2, 0.3, and 0.5, respectively. What is the probability that exactly two accidents occur in total on these highways in the county today?

Your Answer		Score	Explanation
○ 0.125528			
○ 0.069897			
◉ 0.18394	✔	1.00	

- 0.114043
- 0.264241
- 0.17802

Total	1.00 / 1.00

Question Explanation

Write X_1, X_2, and X_3 for the number of accidents today in each of three highways passing through the county. We are dealing with a product space of triples (k_1, k_2, k_3) whose elements are non-negative integers. The probability measure is a product measure of Poisson probabilities with parameters 0.2, 0.3, and 0.5, respectively, so that

$$\mathbf{P}\{(k_1,k_2,k_3)\} = \mathbf{P}\{X_1 = k_1, X_2 = k_2, X_3 = k_3\} = \mathbf{P}\{X_1 = k_1\}\mathbf{P}\{X_2 = k_2\}\mathbf{P}\{X_3 = k_3\}$$
$$= e^{-0.2}\,\frac{(0.2)^{k_1}}{k_1!}\cdot e^{-0.3}\,\frac{(0.3)^{k_2}}{k_2!}\cdot e^{-0.5}\,\frac{(0.5)^{k_3}}{k_3!} = e^{-1}\,\frac{0.2^{k_1}0.3^{k_2}0.5^{k_3}}{k_1!k_2!k_3!}.$$

1. The direct approach: The event that there are exactly two accidents corresponds to the set $A = \{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}$. By additivity,

$$\mathbf{P}(A) = e^{-1}\left(\frac{0.2^2}{2} + \frac{0.3^2}{2} + \frac{0.5^2}{2} + 0.2\times 0.3 + 0.2\times 0.5 + 0.3\times 0.5\right) = 0.18394.$$

2. An approach using the stability of the Poisson: A very slightly shorter argument is available to you if you have braved the dangerous bend Lecture 10.2: p. Write $S = X_1 + X_2 + X_3$. Then S represents the total number of accidents on the highways on the given day. As S is a sum of independent Poisson variables, by the stability result for the Poisson distribution, S itself has a Poisson distribution with parameter $0.2 + 0.3 + 0.5 = 1$. Accordingly,

$$\mathbf{P}(A) = \mathbf{P}\{S = 2\} = e^{-1}\,\frac{1^2}{2!} = \frac{1}{2e} = 0.18394.$$

Question 6

What is the probability that a game of craps does not terminate in the first three throws? Select the most accurate answer among those given below.

[See Lectures 9.3:a through 9.3:d for the rules of the game and an analysis.]

Your Answer		Score	Explanation
$\frac{2}{3}$			
0.343621	✔	1.00	
0.247054			
0.270707			
$\frac{1}{3}$			
0.478395			
Total		1.00 / 1.00	

Question Explanation

The probability space for this problem was described in Lecture 9.3:a. For brevity, write A for the event that the game does not terminate in the first three throws. The game will terminate on the first throw if the outcome is 2, 3, 7, 11, or 12. Accordingly, A occurs if, and only if, the first throw results in a value $k \in \{4,5,6,8,9,10\}$ and the next two throws result in neither k nor 7. The probability, conditioned on the event that the first throw resulted in a value k, that the next two throws are neither k nor 7 is given in the notation of Lecture 9.3:a by

$$(1 - p_k - p_7)^2$$

as the trials are independent. Accordingly, the probability that the first three throws start with k and don't lead to termination is givenby

$$p_k \cdot (1 - p_k - p_7)^2.$$

We resort to additivity and sum over all $k \in \{4,5,6,8,9,10\}$ to obtain the desired probability:

$$\mathbf{P}(A) = p_4\cdot(1-p_4-p_7)^2 + p_5\cdot(1-p_5-p_7)^2 + p_6\cdot(1-p_6-p_7)^2 + p_8\cdot(1-p_8-p_7)^2 + p_9\cdot(1-p_9-p_7)^2 + p_{10}\cdot(1-p_{10}-p_7)^2$$
$$= 2\left[\frac{3}{36}\left(1-\frac{3}{36}-\frac{6}{36}\right)^2 + \frac{4}{36}\left(1-\frac{4}{36}-\frac{6}{36}\right)^2 + \frac{5}{36}\left(1-\frac{5}{36}-\frac{6}{36}\right)^2\right] = \frac{167}{486} = 0.343621.$$

Roughly speaking, slightly more than one out of three craps games will take four or more throws to terminate.

Another way of approaching the problem is to first calculate the probability of termination in one, two, or three steps. Write T_1, T_2, and T_3 for these three events. Arguing as in Lectures 9.3a–d, we obtain

$$\mathbf{P}(T_1) = p_2 + p_3 + p_7 + p_{11} + p_{12} = \frac{1}{36} + \frac{2}{36} + \frac{6}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{3} = 0.333333,$$

as we terminate in one step if, and only if, we throw 2, 3, 7, 11, or 12. We terminate in two steps if we first throw some $k \in \{4,5,6,8,9,10\}$ and then on the second throw obtain k or 7. Accordingly,

$$\begin{aligned}\mathbf{P}(T_2) &= p_4(p_4+p_7) + p_5(p_5+p_7) + p_6(p_6+p_7) + p_8(p_8+p_7) + p_9(p_9+p_7) + p_{10}(p_{10}+p_7)\\ &= 2\left[\frac{3}{36}\left(\frac{3}{36}+\frac{6}{36}\right) + \frac{4}{36}\left(\frac{4}{36}+\frac{6}{36}\right) + \frac{5}{36}\left(\frac{5}{36}+\frac{6}{36}\right)\right] = \frac{61}{324} = 0.188272.\end{aligned}$$

And we terminate on the third throw if we first throw some $k \in \{4,5,6,8,9,10\}$, then obtain neither k nor 7 on the second throw, and finally obtain either k or 7 on the third throw. Accordingly,

$$\begin{aligned}\mathbf{P}(T_3) &= p_4(1-p_4-p_7)(p_4+p_7) + p_5(1-p_5-p_7)(p_5+p_7) + p_6(1-p_6-p_7)(p_6+p_7)\\ &\quad + p_8(1-p_8-p_7)(p_8+p_7) + p_9(1-p_9-p_7)(p_9+p_7) + p_{10}(1-p_{10}-p_7)(p_{10}+p_7)\\ &= 2\left[\frac{3}{36}\left(1-\frac{3}{36}-\frac{6}{36}\right)\left(\frac{3}{36}+\frac{6}{36}\right) + \frac{4}{36}\left(1-\frac{4}{36}-\frac{6}{36}\right)\left(\frac{4}{36}+\frac{6}{36}\right) + \frac{5}{36}\left(1-\frac{5}{36}-\frac{6}{36}\right)\left(\frac{5}{36}+\frac{6}{36}\right)\right] = \frac{131}{972} = 0.134774.\end{aligned}$$

As $A^\complement = T_1 \cup T_2 \cup T_3$, by additivity,

$$\mathbf{P}(A^\complement) = \mathbf{P}(T_1) + \mathbf{P}(T_2) + \mathbf{P}(T_3) = \frac{1}{3} + \frac{61}{324} + \frac{131}{972} = \frac{319}{486} = 0.656379.$$

One more appeal to additivity, finishes off and we obtain

$$\mathbf{P}(A) = 1 - \mathbf{P}(A^\complement) = 1 - \frac{319}{486} = \frac{167}{486} = 0.343621.$$

Question 7

A businessman purchases perishable commodities in the wholesale market at $10 each and sells them on the retail market at $15 each. He hence makes a profit of $5 for every unit he sells. But the commodities are perishable and so any units unsold at the end of the day are discarded and he incurs a loss of $10 on each unit he discards. (I've chosen toy numbers just to keep the arithmetic simple; the important thing is the principle behind the problem.) Long experience has taught the businessman that his daily demand D has a binomial distribution with parameters $n = 5$ and $p = 2/3$. In other words, his demand on a given day satisfies

$$\mathbf{P}\{D = k\} = b_5(k; 2/3) = \binom{5}{k}\left(\frac{2}{3}\right)^k\left(1-\frac{2}{3}\right)^{5-k} \qquad (k = 0, 1, \ldots, 5).$$

[The notation for the binomial distribution was introduced in Lecture 10.1:c.] The businessman has to purchase the commodities ahead of the daily demand though he does not know exactly what the demand will be. How many items should he purchase so as to maximise his expected profit? *Hint*: This is not a difficult problem but you will need to think clearly. Suppose the businessman purchases c commodities. If the demand D happens to exceed c then he will sell all his purchased commodities (and there will be excess demand that he cannot satisfy). If the demand D is no larger than c then he can sell D units but will have to discard $c - D$ units at the end of the day. His profit (or loss) is a chance-driven arithmetic variable, say, X which depends hence on the number c of commodities purchased. If his profit (or loss) X for a given value of c has a mass function

$$\mathbf{P}\{X = m\} = p(m)$$

then the expected profit for that value of c is given by

$$\mathbf{E}(X) = \sum_m mp(m).$$

Evaluate this for $c = 0, 1, 2, \ldots$ and select that value of c for which the expectation is largest. (Should he ever consider buying more than five units?)

Your Answer		Score	Explanation
○ 1			
○ 2			
◉ 3	✔	1.00	
○ 4			
○ 0			
○ 5			
Total		1.00 / 1.00	

Question Explanation

The profit $X = X(D; c)$ is a function of both the demand D and the number c of units purchased on that day. In view of the hint, we may write out the profit explicitly as a piecewise function given by

$$X = \begin{cases} 15D - 10c & \text{if } D \le c, \\ 15c - 10c & \text{if } D > c. \end{cases}$$

For each given value of c, the profit X inherits a mass function from D: if $0 \le k \le c$ then the profit takes value $X = 15k - 10c$ with probability $b_5(k; 2/3)$; if $c + 1 \le k \le 5$ then the profit takes value $X = 15c - 10c = 5c$ with probability $b_5(k; 2/3)$. The expected profit is hence a function of c given by

$$f(c) = \sum_{k=0}^{c}(15k - 10c)b_5(k; 2/3) + \sum_{k=c+1}^{5}(15c - 10c)b_5(k; 2/3).$$

We can certainly evaluate this numerically for every value of c. The expected profits rounded to two decimal places are shown in the table below.

c	0	1	2	3	4	5
$f(c)$	0	4.94	9.26	11.11	8.02	0

By inspection we deduce that the largest expected profit is obtained for $c = 3$.

Looking back at the problem, this is certainly intuitive. If he purchases too few commodities then he loses a profit opportunity in the excess demand that he does not satisfy; if he purchases too many commodities then he incurs a loss because he has to discard a lot of them. So c should be somewhere near the expected demand (if the profit is not excessively large — in which case we would gamble more — or very small — in which case we would be very conservative). But we know for the binomial that the expected demand is $5 \times 2/3 = 3.3\dot{3}$. It is satisfying that this is exactly where our calculations land us.