1. $x = \mu + \sigma y \qquad y = (x-\mu)/\sigma$

$P(-1 \le y \le 1) = P(-1 \le \frac{x-\mu}{\sigma} \le 1) = P(\mu - \sigma \le x \le \mu + \sigma) \approx 0.68$

$P(-2 \le y \le 2) = P(-2 \le \frac{x-\mu}{\sigma} \le 2) = P(\mu - 2\sigma \le x \le \mu + 2\sigma) \approx 0.95$

2. $y_i \sim N(0,1) \qquad z = \sum_{i=1}^{n} y_i^2$

$$\begin{aligned}
\text{var}(z) &= E\left[\left(\sum_{i=1}^{n} y_i^2 - n\right)^2\right] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} y_i^2 y_j^2\right] - 2n\,E\left[\sum_{i=1}^{n} y_i^2\right] + n^2 \\
&= E\left[\sum_{i=1}^{n} y_i^4\right] + 2E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_i^2 y_j^2\right] - n^2 \\
&= \sum_{i=1}^{n} E[y_i^4] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E[y_i^2]\,E[y_j^2] - n^2 \\
&= 3n + 2\cdot\tfrac{1}{2}n(n-1) - n^2 \\
&= 3n + n^2 - n - n^2 \\
&= 2n
\end{aligned}$$

$E[y_i^4] = 3 \qquad y_i$ and y_j independent

3. $z_i = \sum_{i=1}^{k_i} y_{ij}^2 \qquad z_1 + z_2 = \sum_{i=1}^{2}\sum_{j=1}^{k_i} y_{ij}^2$

$z_1 + z_2 \sim \chi^2(k_1 + k_2)$

4. a) $E[y] = E[L^{-1}(x-\mu)] = L^{-1}(E[x]-\mu) = L^{-1}(\mu-\mu) = 0$

$$\mathrm{var}[y] = \mathrm{var}[L^{-1}(x-\mu)] = L^{-1}\mathrm{var}[x](L^{-1})'$$
$$= L^{-1}\Sigma(L^{-1})' = L^{-1}LL'(L^{-1})'$$
$$= I\,L'(L')^{-1} = I\,I$$
$$= I$$

$y \sim N(0, I)$

b) $$z = (x-\mu)'\Sigma^{-1}(x-\mu) = (x-\mu)'(LL')^{-1}(x-\mu)$$
$$= (x-\mu)'(L')^{-1}L^{-1}(x-\mu) = (x-\mu)'(L^{-1})'L^{-1}(x-\mu)$$
$$= (L^{-1}(x-\mu))^{-1}L^{-1}(x-\mu)$$

$z \sim \chi^2(n)$

5. $$\frac{x'x}{n} = \frac{1}{n}\left(\frac{1}{\sqrt{2/k}}y\right)'\frac{1}{\sqrt{2/k}}y = \frac{1}{n}\frac{1}{\sqrt{2/k}}y'y\frac{1}{\sqrt{2/k}}$$
$$= \frac{y'y/n}{z/k} \quad \begin{matrix}\sim \chi^2(n) \\ \sim \chi^2(k)\end{matrix}$$

$$\Rightarrow \frac{x'x}{n} \sim F(n,k)$$