Recursion

Induction

Reading: Why Induction?
10 min

Reading: What is Induction?
10 min

Reading: Arithmetic Series
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Reading: Compound Interest
10 min

Lab: Bernoulli's Inequality
15 min

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10 min

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10 min

Quiz: Puzzle: Connect Points
2 questions

Quiz: Induction
9 questions

Compound Interest

A simple interest deposit is a kind of deposit where you earn $x\%$ of the *initial* deposit each period (day, month or year). A compound interest deposit is a deposit where you earn $x\%$ of *what you already have* each period (day, month or year). Will you get $\$1\,000\,000$ faster starting with $\$1\,000$ and earning 2% every day with compound interest or with simple interest?

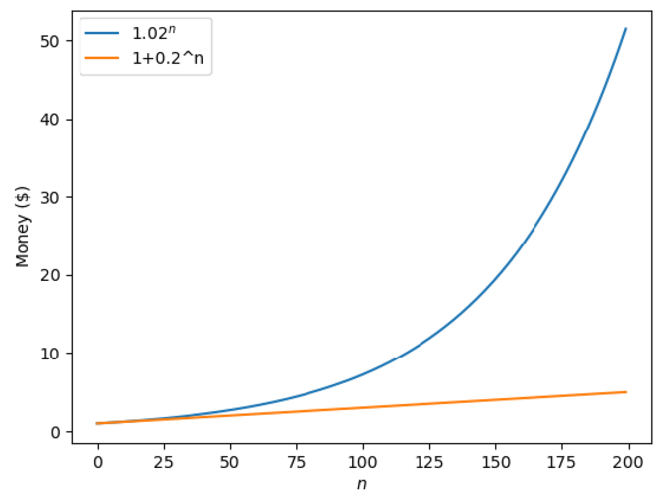
While simple interest will earn you $\$20$ every day, compound interest will give you $\$20$ on the first day, $\$20.4$ on the second day, $\$20.808$ on the third day, and more and more every following day.

In the case of compound interest, if you start with some amount of money, then after n days, this amount is multiplied by 1.02^n. In the case of simple interest, the money is multiplied by $(1+n\cdot 0.02)$. The following code demonstrates the (huge) difference between these two cases for large values of n.

```
import matplotlib.pyplot as plt
import numpy as np

plt.xlabel('$n$')
plt.ylabel('Money ($)')

x = np.arange(200)
plt.plot(x, 1.02 ** x, label='$1.02^n$')
plt.plot(x, 1 + 0.02 * x, label='1+0.2^n')
plt.legend(loc='upper left')
plt.savefig('bernoulli.png')
```



This holds in general: compound interest is always at least as profitable as simple interest. This is known as Bernoulli's inequality, and we will prove it using mathematical induction.

Bernoulli's inequality
For every $x\geq -1$ and every integer $n\geq 0$, it holds that $(1+x)^n\geq 1+xn$.

Proof

The base case $n=0$ holds trivially: $(1+x)^0=1=1+x\cdot 0$. Now we prove the induction step from n to $(n+1)$ for every $n\geq 0$:

$$(1+x)^{n+1}=(1+x)^n\cdot(1+x)\geq(1+xn)\cdot(1+x)$$

where the last inequality is due to induction hypothesis and the fact that an inequality can be multiplied by a non-negative value $(1+x)$, and it will still hold.

Now,

$$(1+x)^{n+1}\geq(1+xn)\cdot(1+x)=1+x(n+1)+x^2n\geq 1+x(n+1)$$

Jacob Bernoulli (1655--1705). (Source: Wikipedia.)

It is instructive to see how exponential functions (such as 1.02^n) grow, and to see how fast they reach large values. For example, how many days of 2% compound interest does it take to get from $\$1\,000$ to $\$1\,000\,000$?

```
def days_to_target(starting_amount, earn_percent,
                   target_amount):
    day = 1
    amount = starting_amount
    daily_factor = (1 + earn_percent / 100.0)
    while amount < target_amount:
        day += 1
        amount = amount * daily_factor
    return day


def print_example(starting_amount, earn_percent,
                  target_amount):
    days = days_to_target(starting_amount, earn_percent,
                          target_amount)
    print(f"If you start with ${starting_amount} "
          f"and earn {earn_percent}% a day,"
          f"\nyou will have more than ${target_amount} "
          f"on day {days}!")


print_example(1000, 2, 1000000)
```

```
If you start with $1000 and earn 2% a day,
you will have more than $1000000 on day 350!
```

Or how much money will I have after a year?

```
def how_much_money(starting_amount, earn_percent, day):
    daily_factor = 1 + (earn_percent / 100.0)
    return starting_amount * (daily_factor ** (day - 1))


def print_example(starting_amount, earn_percent, day):
    money = int(how_much_money(starting_amount,
                               earn_percent, day))

    print(f"If you start with ${starting_amount} "
          f"and earn {earn_percent}% a day,"
          f"\non day {day} you will have "
          f"more than ${money}!")


print_example(1000, 2, 365)
```

```
If you start with $1000 and earn 2% a day,
on day 365 you will have more than $1350400!
```

We encourage you to play with various exponential functions in this python notebook