

A Test of Normality

Textbook Reference: Chapter 14.2
(eighth edition, pages 591–3; seventh edition, pages 624–6).

The calculation of p-values for hypothesis testing typically is based on the assumption that the population distribution is normal. Therefore, a test of the normality assumption may be useful to inspect. A variety of tests of normality have been developed by various statisticians. One of these tests will be described here.

To start, the calculation of descriptive statistics is reviewed.

A data set has the numeric observations: x_1, x_2, \dots, x_n .

Familiar descriptive statistics are the sample mean:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

and the sample variance:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

Now introduce two new statistics.

The sample **skewness** is defined as:

$$S = \frac{1}{n} \cdot \frac{\sum_{i=1}^n (x_i - \bar{x})^3}{(\tilde{\sigma}^2)^{3/2}} \quad \text{where} \quad \tilde{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

Skewness gives a measure of how symmetric the observations are about the mean. For a normal distribution the skewness is **0**. A distribution skewed to the right has positive skewness and a distribution skewed to the left has negative skewness.

The sample **kurtosis** is defined as:

$$K = \frac{1}{n} \cdot \frac{\sum_{i=1}^n (x_i - \bar{x})^4}{(\tilde{\sigma}^2)^2}$$

Kurtosis gives a measure of the thickness in the tails of a probability density function. For a normal distribution the kurtosis is **3**.

Excess kurtosis is defined as:

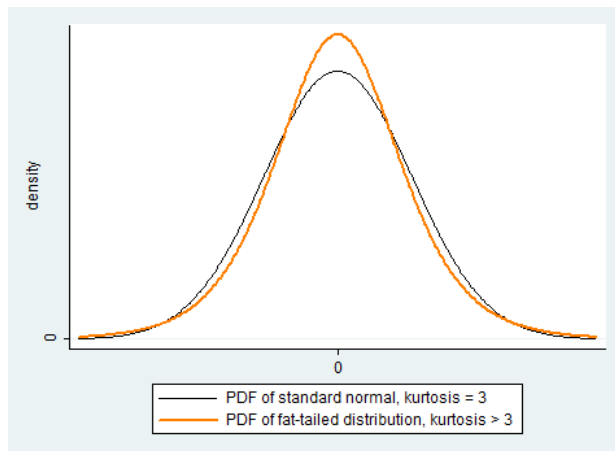
$$EK = K - 3$$

It follows that, for a normal distribution, the excess kurtosis is **0**.

A fat-tailed or thick-tailed distribution has a value for kurtosis that exceeds 3. That is, excess kurtosis is positive.

This is called **leptokurtosis**.

The graph below compares the shape of the probability density function for the standard normal distribution (mean 0 and variance 1) and a 'fat-tailed' distribution, also with mean 0 and variance 1.



Note: the 'fat-tailed' distribution drawn above is the logistic distribution with probability density function:

$$\frac{\exp(-x/b)}{b[1 + \exp(-x/b)]^2} \quad \text{with} \quad b = \sqrt{3}/\pi$$

This distribution has mean 0, variance 1, coefficient of skewness equal to 0, and coefficient of kurtosis equal to 4.2.

The above calculation formula for skewness and kurtosis are considered suitable for 'large samples'.

Formula that incorporate 'small sample' adjustments are available.

The adjusted calculation formula for skewness is:

$$g_1 = \frac{n}{(n-1)(n-2)} \cdot \frac{\sum_{i=1}^n (x_i - \bar{x})^3}{(s^2)^{3/2}}$$

The adjusted calculation formula for excess kurtosis is:

$$g_2 = \frac{n(n+1)}{(n-1)(n-2)(n-3)} \cdot \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{s} \right)^4 - \frac{3(n-1)^2}{(n-2)(n-3)}$$

Microsoft Excel functions are:

SKEW reports skewness using the formula g_1

KURT reports *excess* kurtosis using the formula g_2 .

The Jarque-Bera test for normality is now presented.

Consider testing the null hypothesis:

H₀: normal distribution,
skewness is zero and excess kurtosis is zero;

against the alternative hypothesis:

H₁: non-normal distribution.

The Jarque-Bera test statistic is:

$$JB = n \cdot \left[\frac{S^2}{6} + \frac{(EK)^2}{24} \right]$$

It turns out that this test statistic can be compared with a χ^2 (chi-square) distribution with 2 degrees of freedom.

The null hypothesis of normality is rejected if the calculated test statistic exceeds a critical value from the $\chi^2_{(2)}$ distribution.

The critical values can be found from the Appendix Table for the chi-square distribution as:

Significance Level α	Critical Value
0.10	4.61
0.05	5.99
0.01	9.21

The presentation of this test of normality is valid for 'large samples'. For 'small samples' the decision rule can be viewed as approximate.

Example: A stock market data set has daily percentage returns observed for the year 1997 for two companies – Barrick Gold and Bank of New York. The sample has observations for $n = 253$ trading days.

For each company, an exercise is to test for normality of the daily returns. Various statistics are given in the table below. Both the ‘small sample’ and ‘large sample’ versions of the skewness and excess kurtosis statistics are presented to give emphasis to the methodology.

	Barrick Gold	Bank of NY
‘Small sample’ statistics		
Skewness g_1	0.01	−0.14
Excess Kurtosis g_2	1.38	0.41
‘Large sample’ statistics		
Skewness S	0.01	−0.14
Excess Kurtosis EK	1.33	0.38
Jarque-Bera test for normality – calculated with the ‘large sample’ statistics		
JB test statistic	18.73	2.31
p-value	< 0.0005	0.315

For Barrick Gold, the Jarque-Bera test statistic of 18.73 exceeds the critical values for any reasonable significance level to lead to the conclusion that the daily returns do not follow a normal distribution.

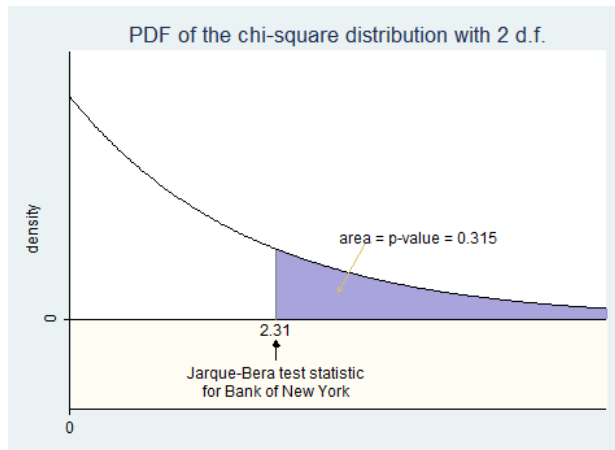
Since the excess kurtosis statistic is greater than zero, the appearance is that the daily returns follow a distribution that features leptokurtosis. Researchers have suggested that the leptokurtosis arises from a pattern of volatility in financial markets where periods of high volatility are followed by periods of relative stability.

A p-value for the test statistic is calculated as a chi-square distribution probability and, with Microsoft Excel, is computed with the function:

CHISQ.DIST.RT(test_statistic, 2)
 ↑
 degrees of freedom

For the Bank of New York, the calculation of a p-value for the Jarque-Bera test statistic is illustrated in the graph below.

A statistical result is that the χ^2 (chi-square) distribution with two degrees of freedom is an exponential distribution.



It is clear that the calculated p-value is greater than any usual significance level (such as $\alpha = 0.10$, 0.05 or 0.01) to suggest that there is no evidence to reject the null hypothesis of a normal distribution for the daily returns of the Bank of New York.