Chebyshev, reprised

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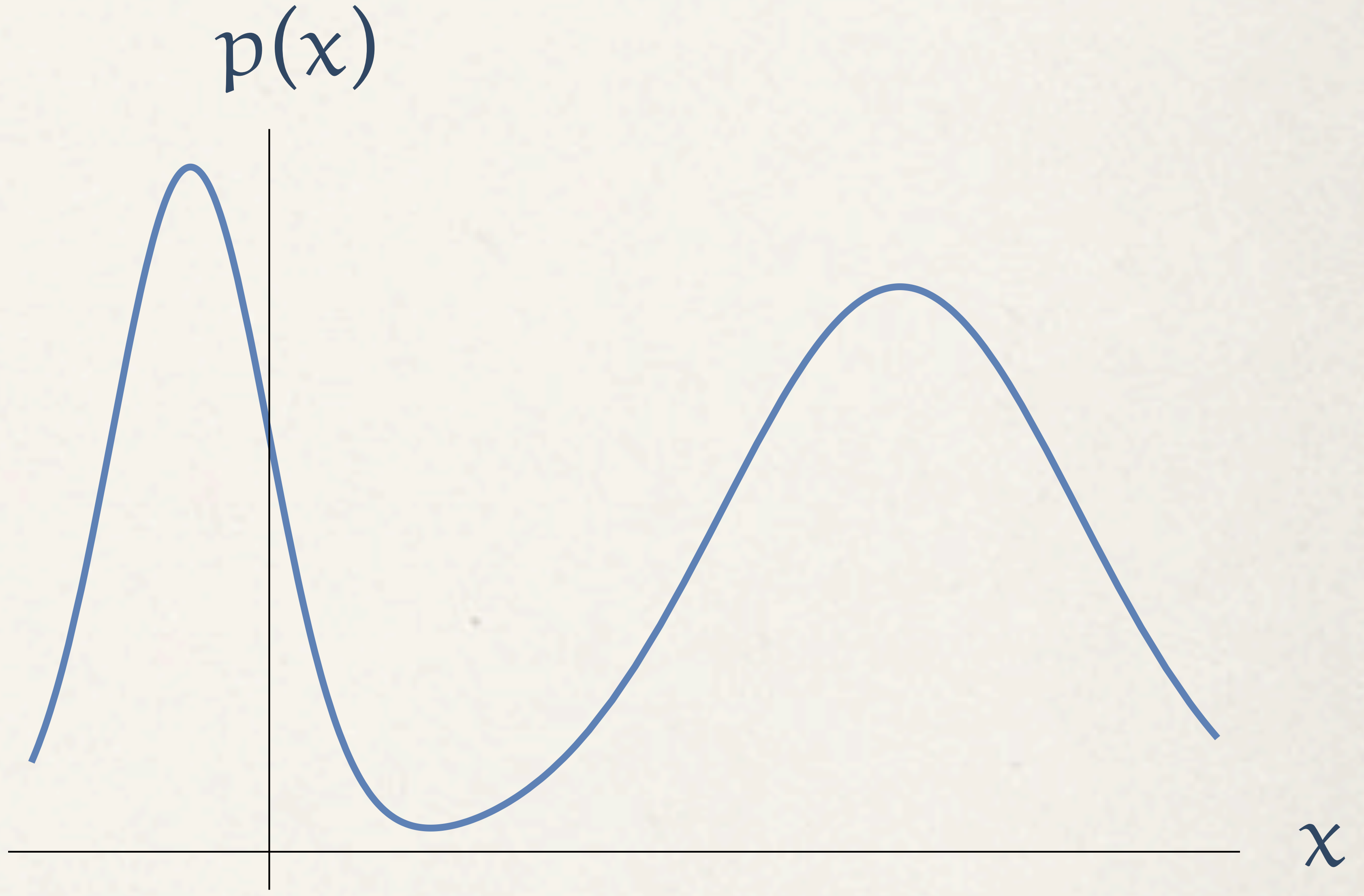

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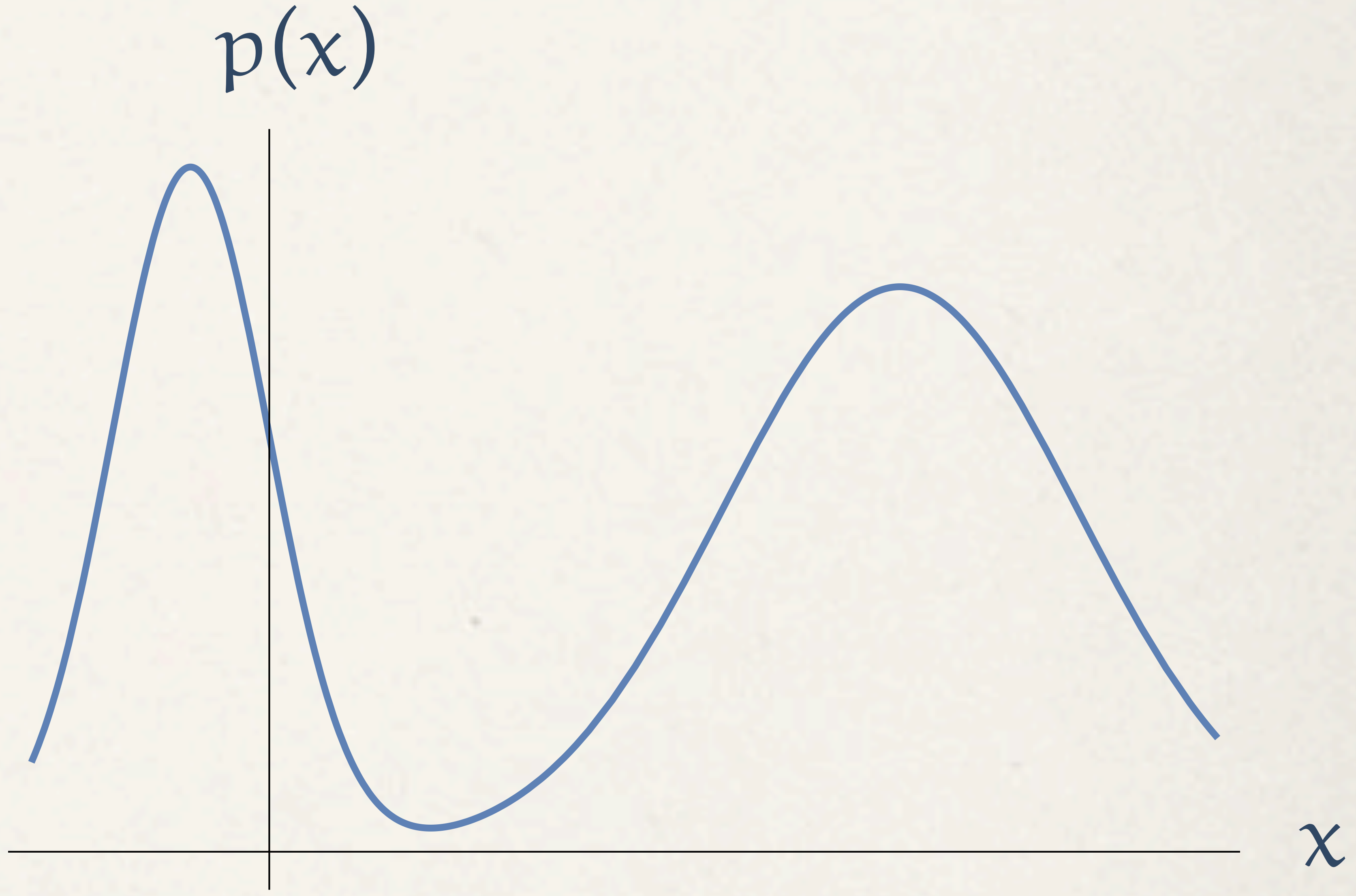

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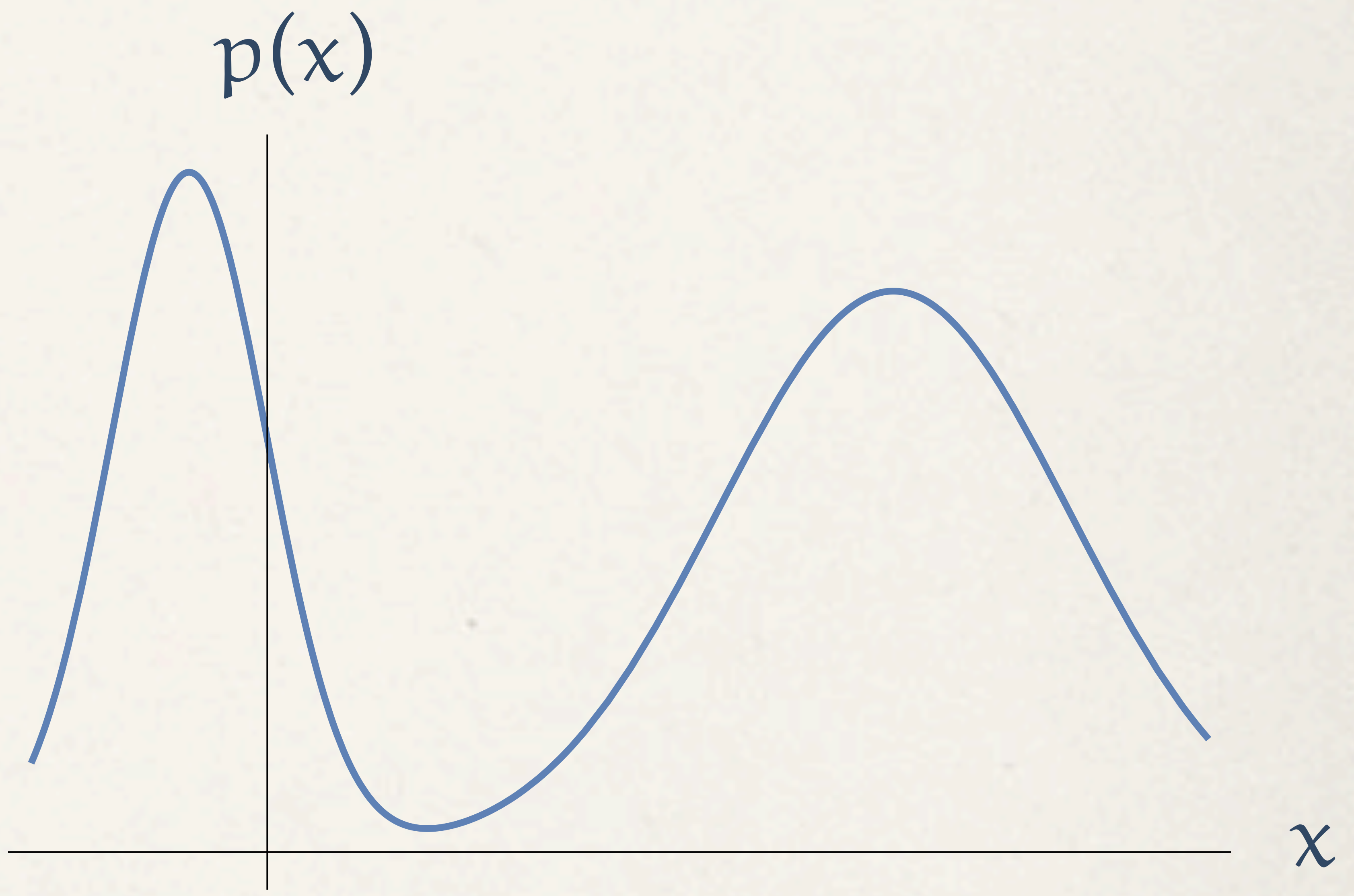

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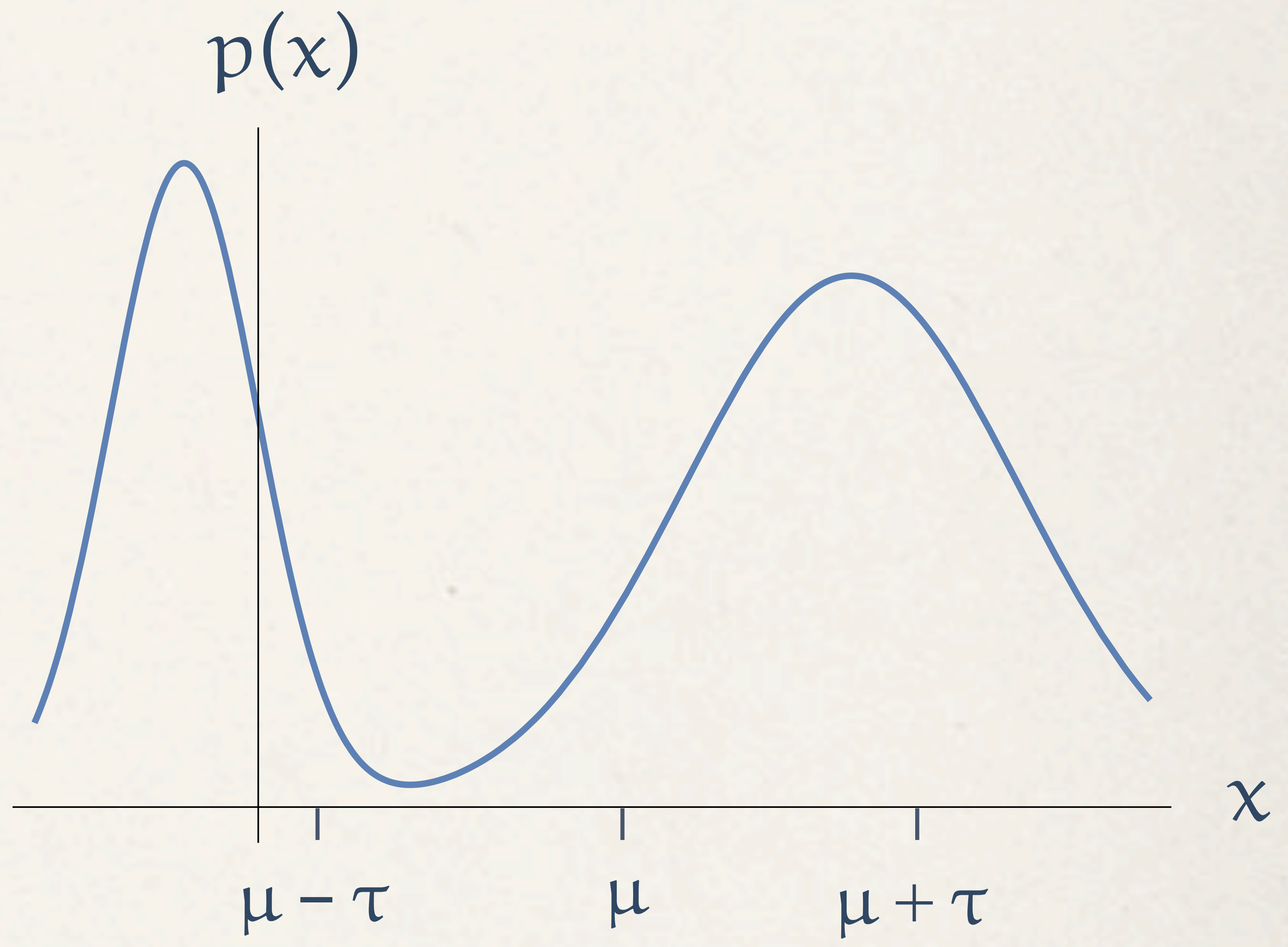

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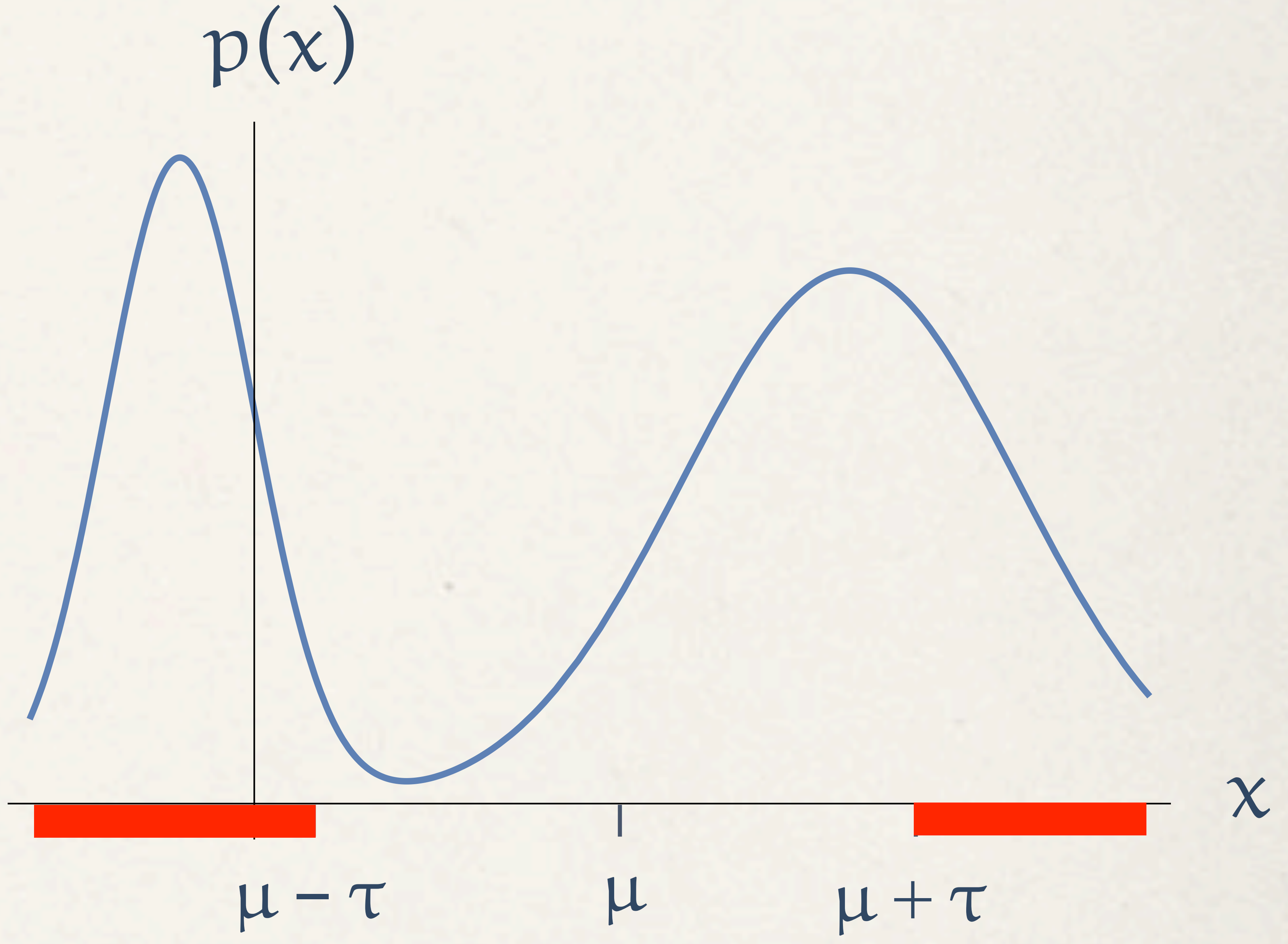

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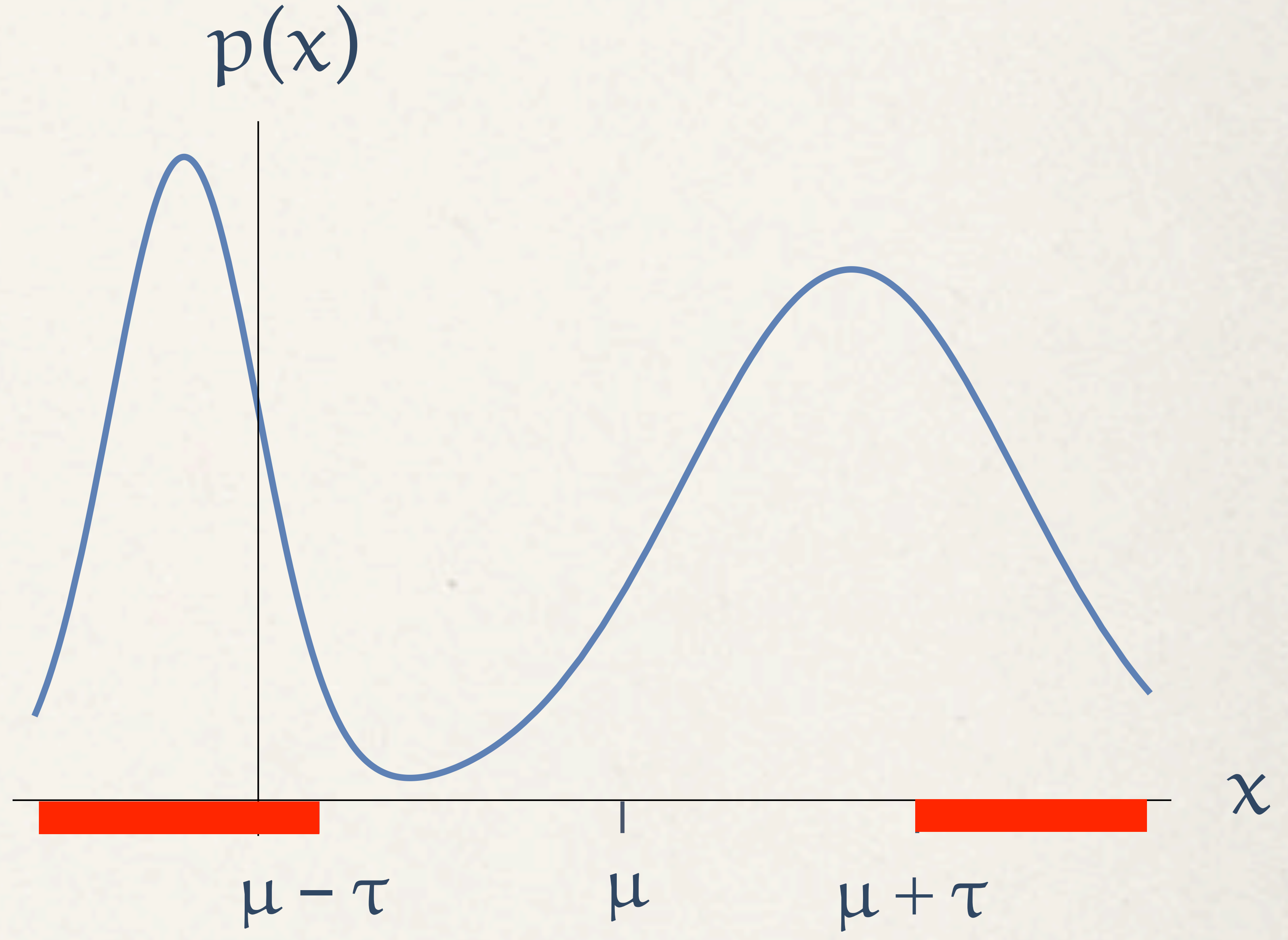

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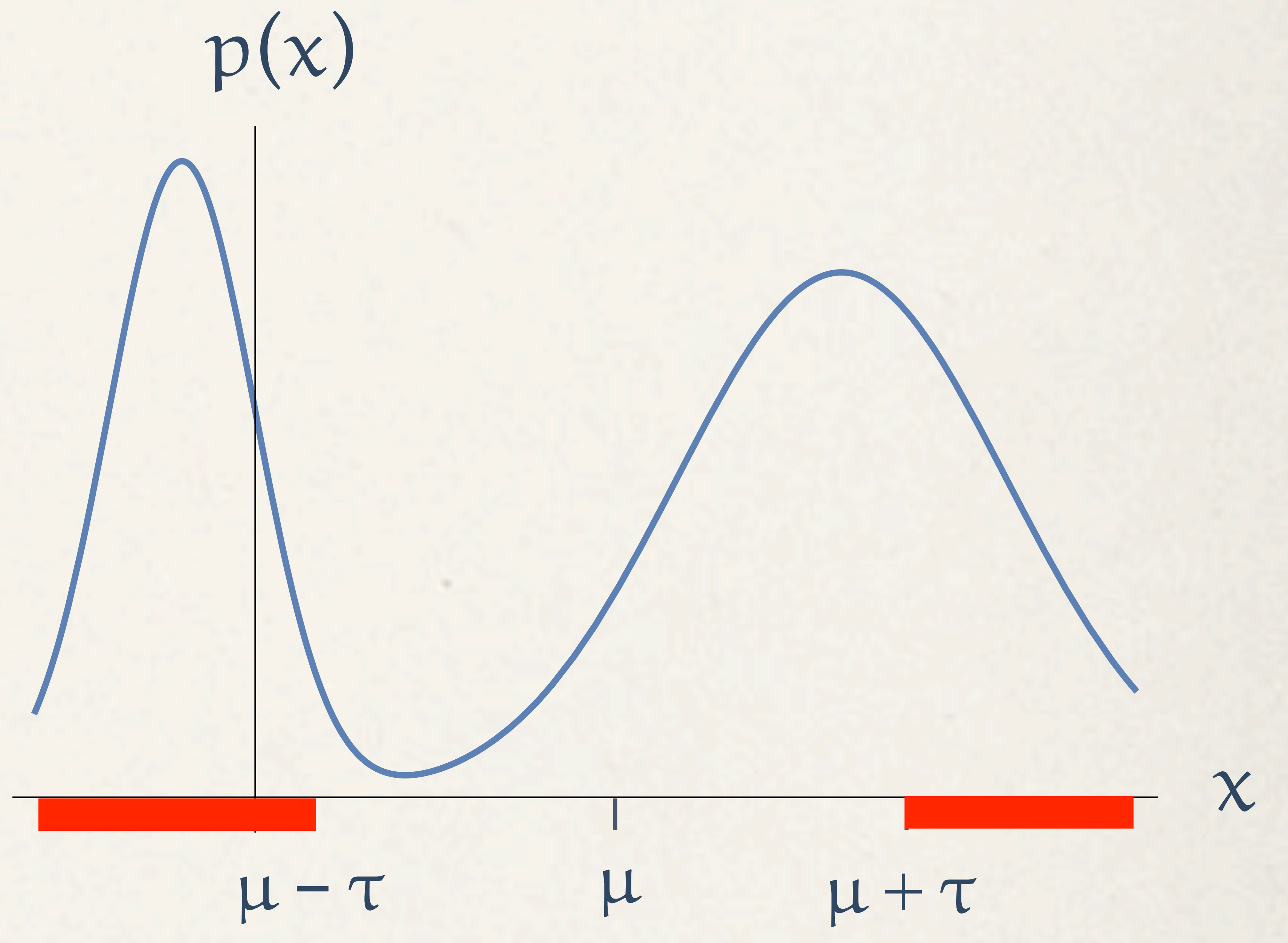

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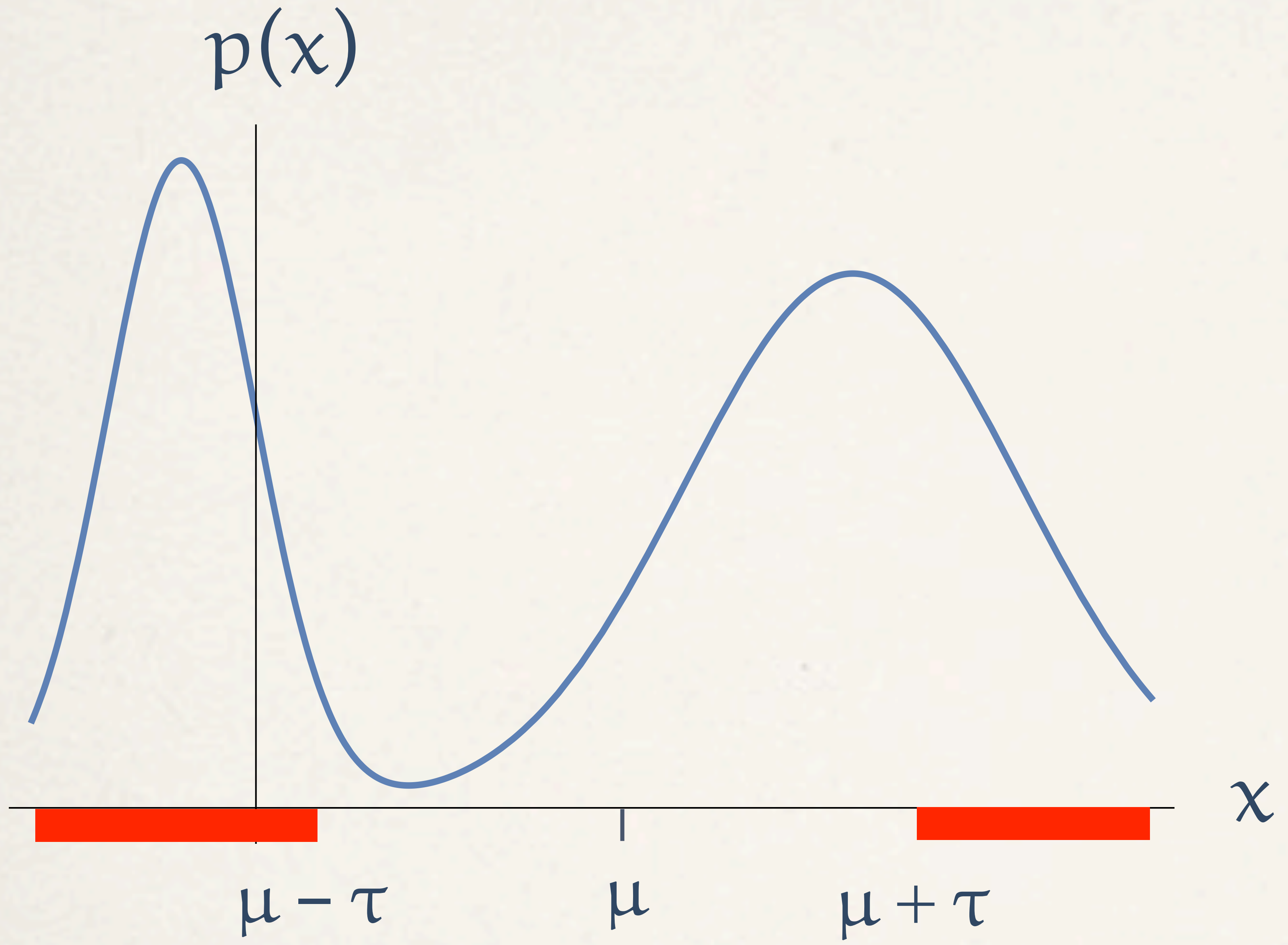

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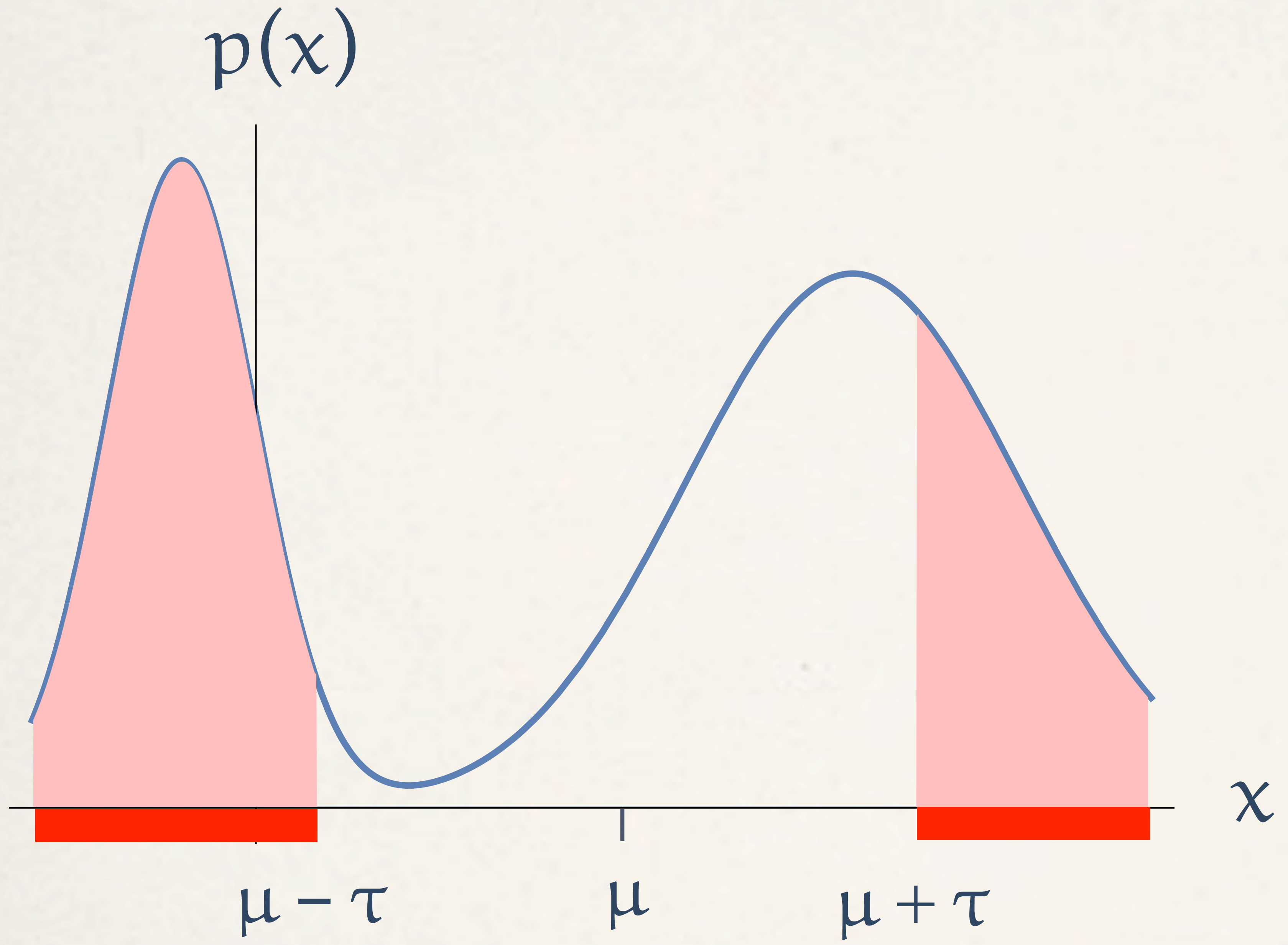

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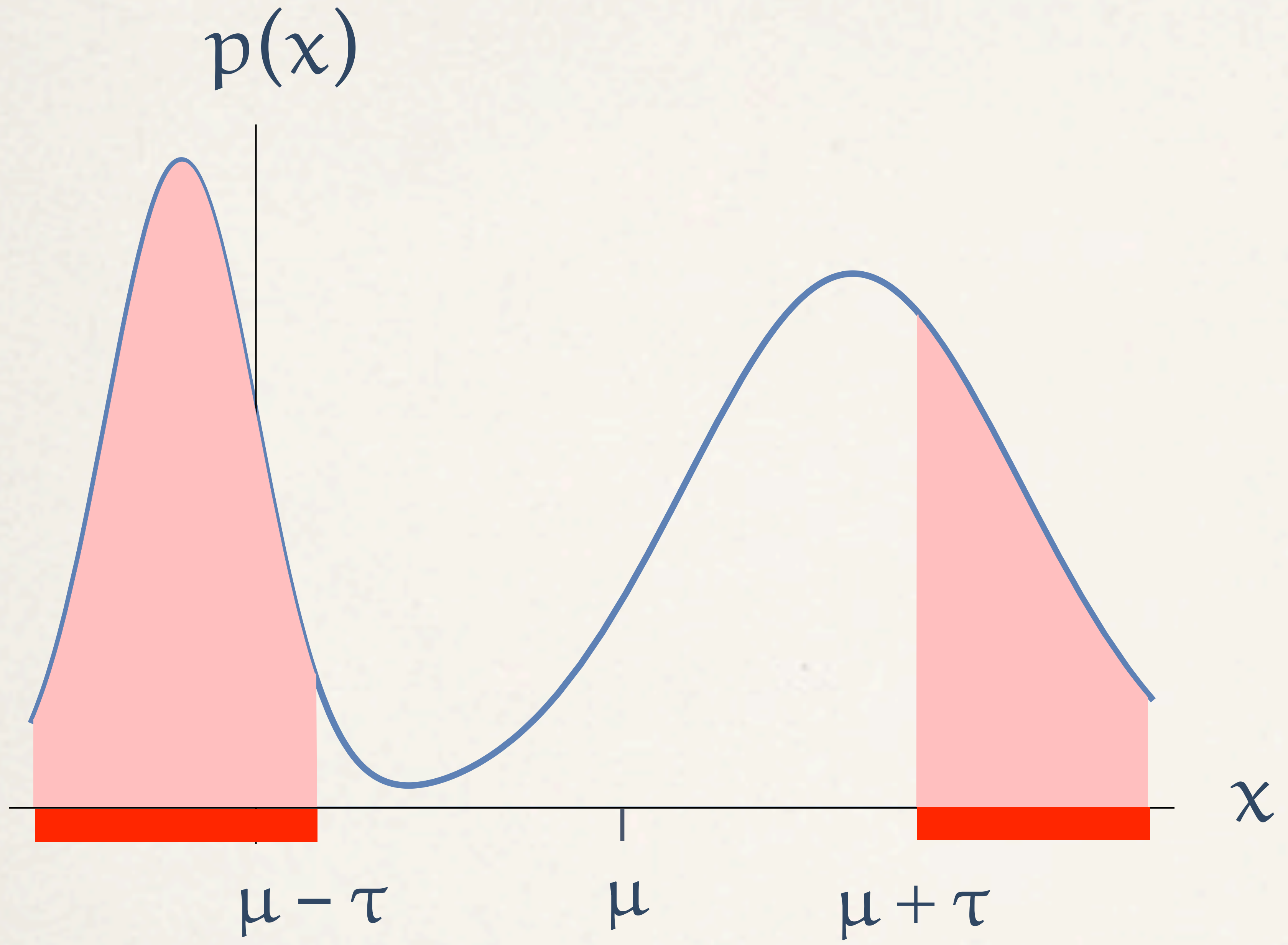

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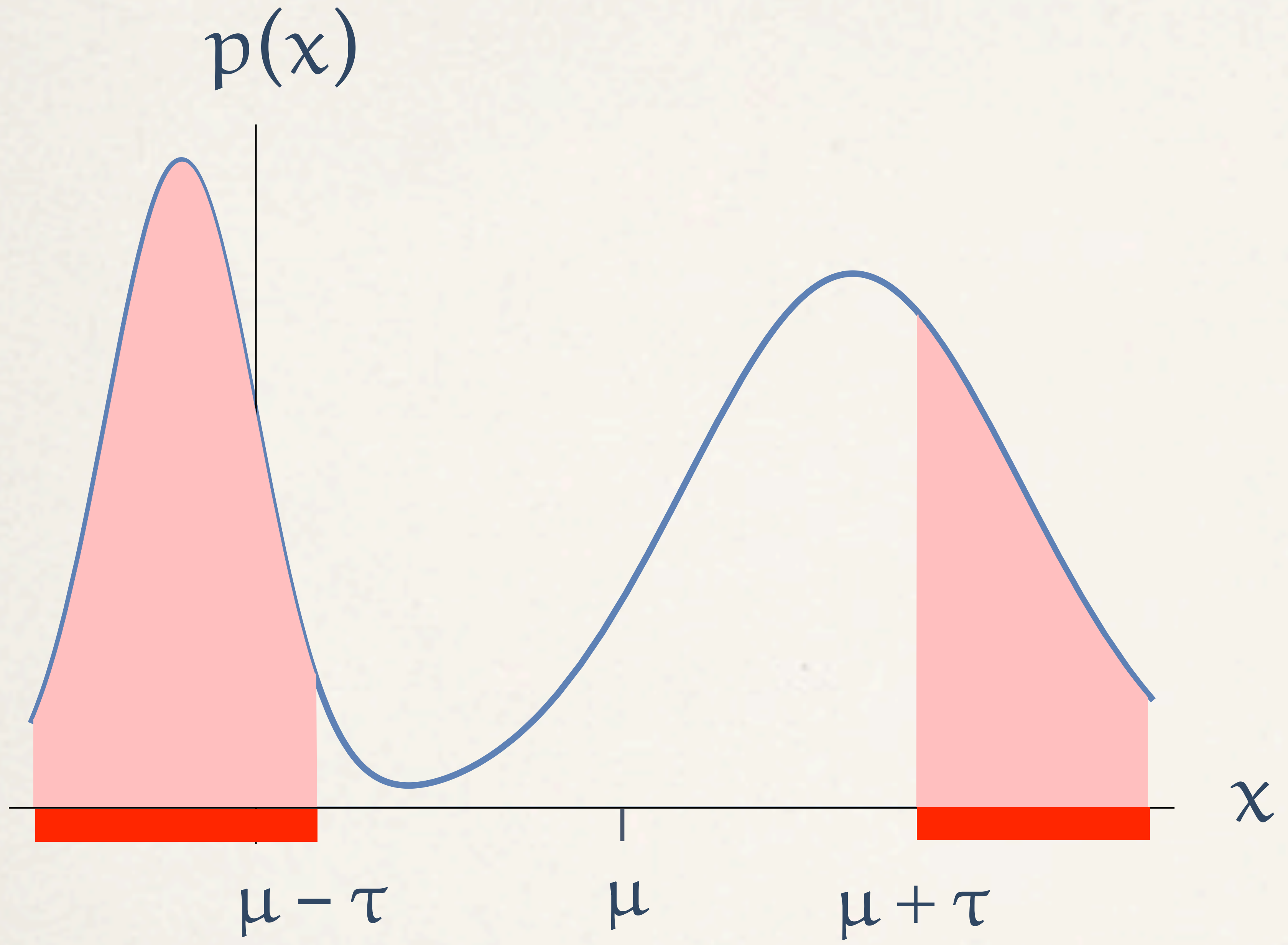

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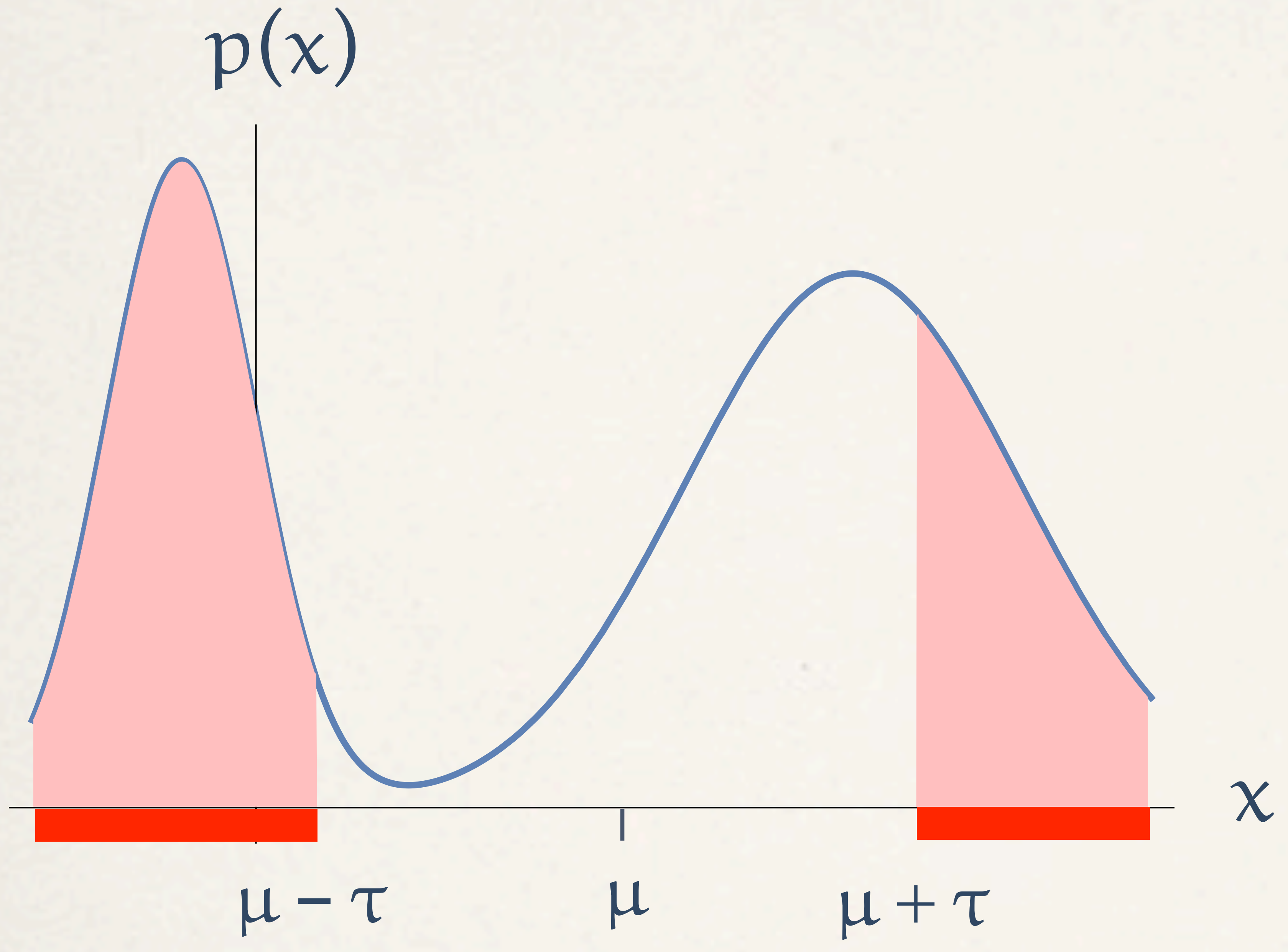

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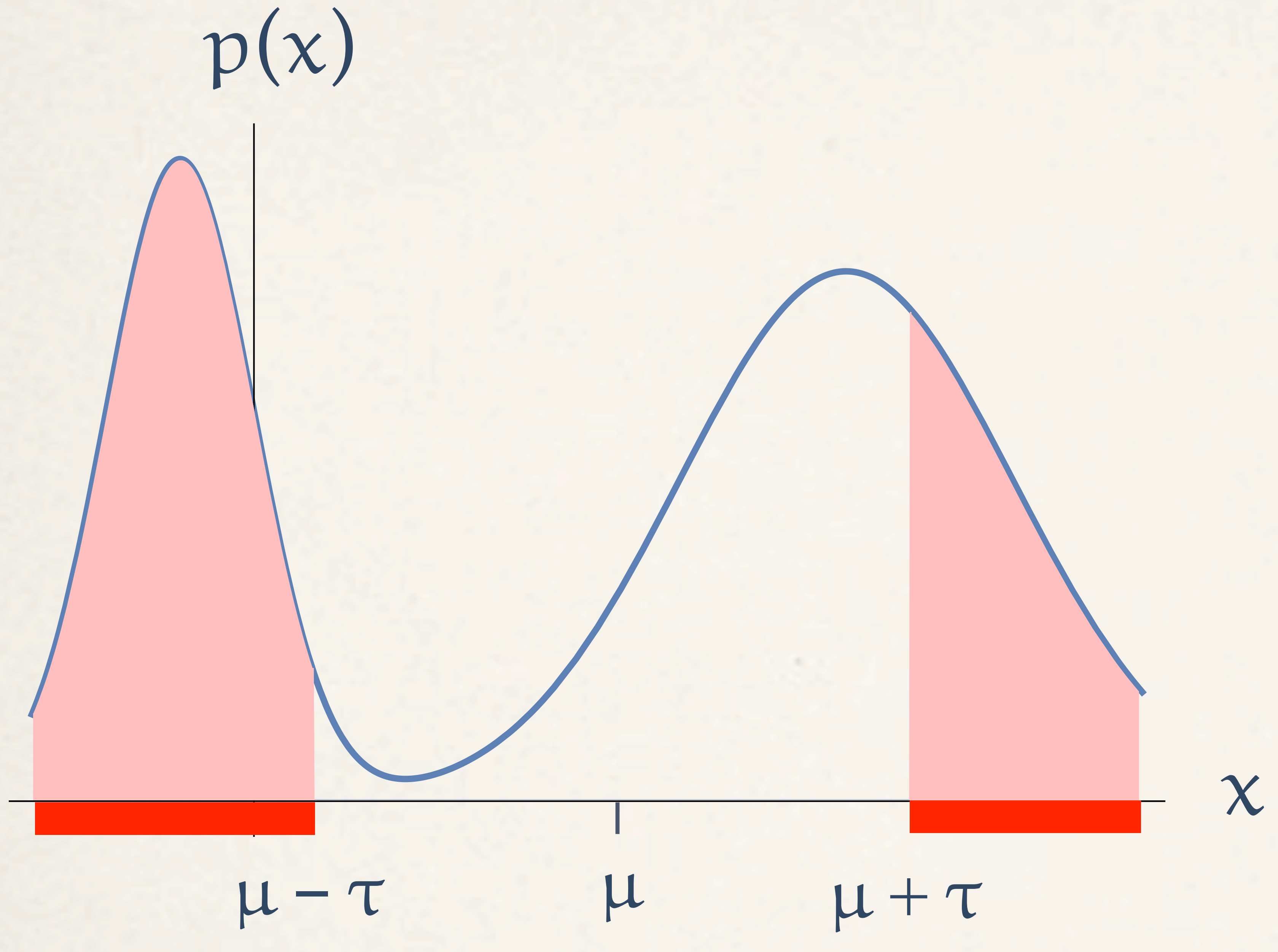

$$\mathbf{P}\{|X - \mu| > \tau\} = \int_{\{x:|x-\mu|>\tau\}} p(x)\,dx$$

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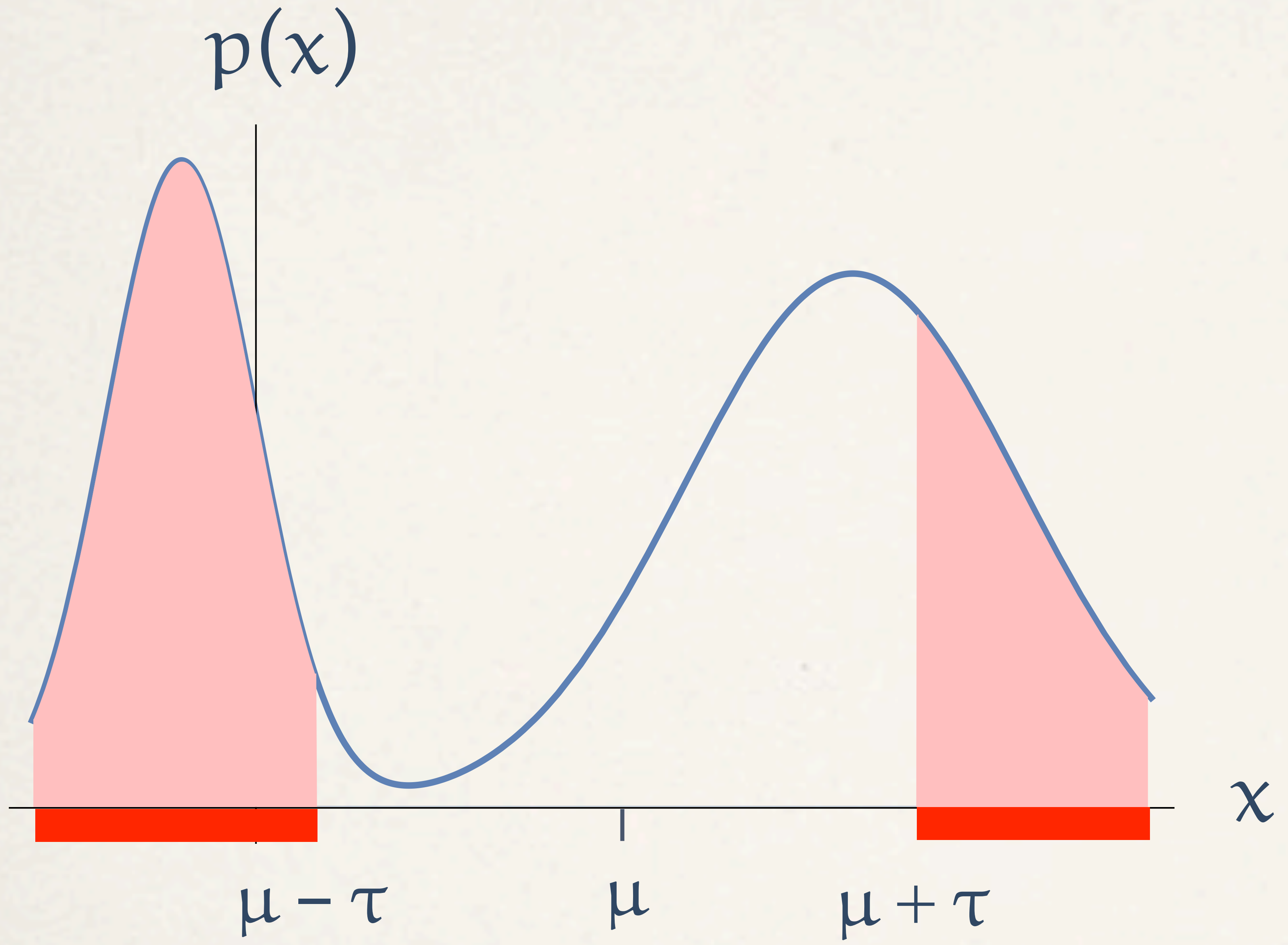

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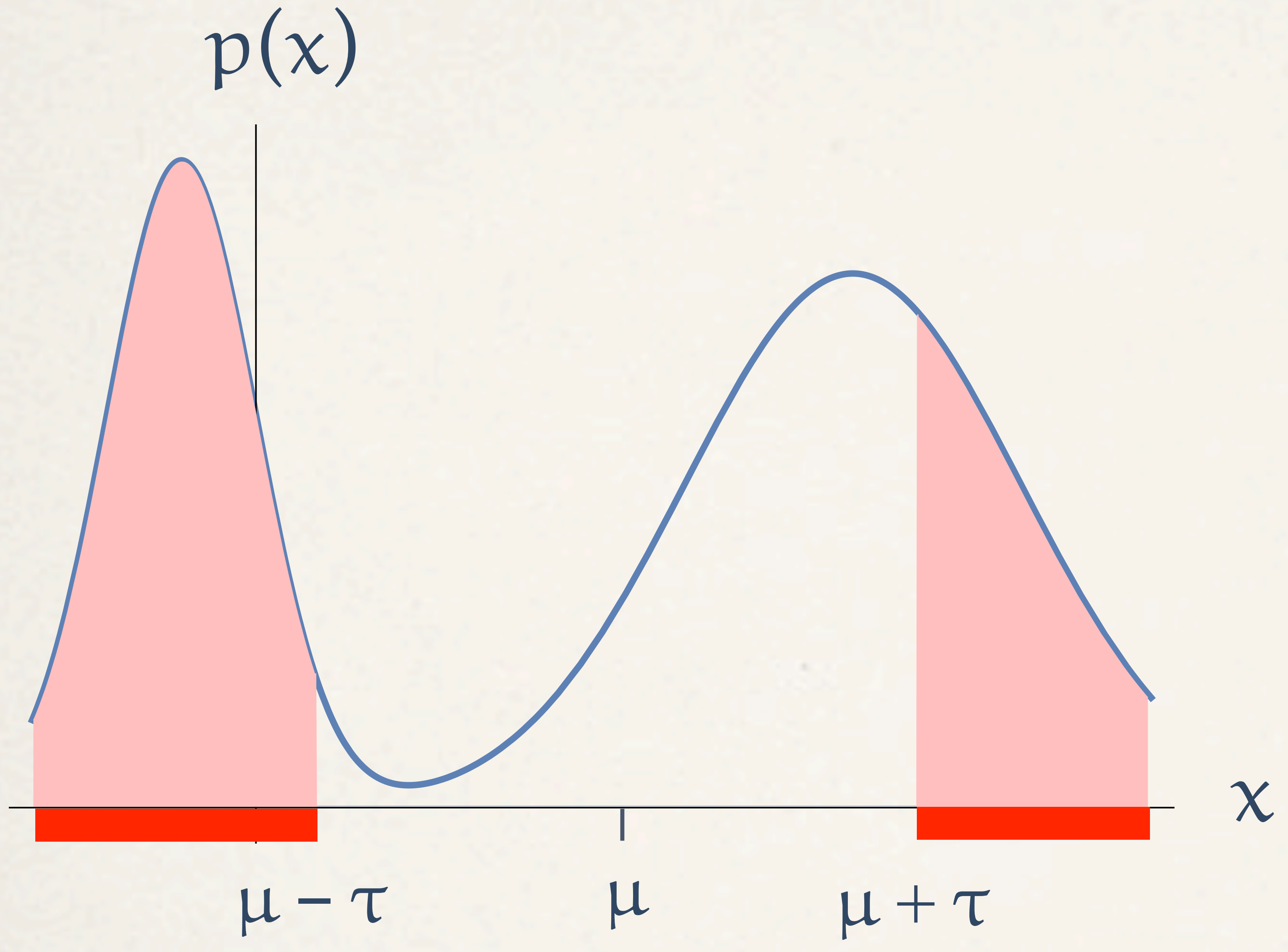

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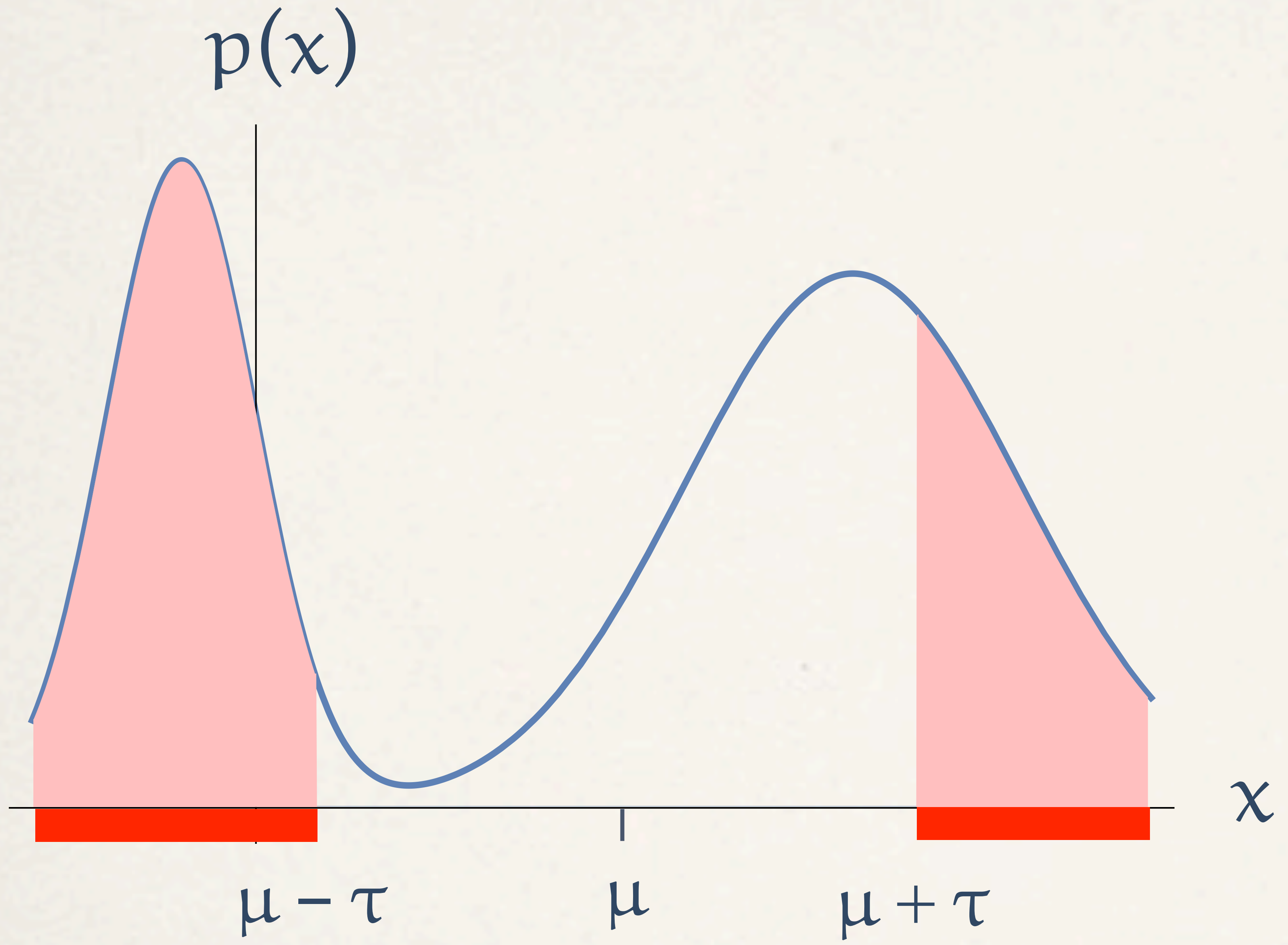

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Slogan

The probability that X deviates from its expected value by more than three standard deviations is small.