Notes: The Definite Integral

Riemann Sums

The **net area** under the graph of f is the difference of areas $A_1 - A_2$, where A_1 is the area of the region above the x-axis and below the graph of f and A_2 is the area of the region below the x-axis and above the graph of f.

Divide the interval $[a, b]$ into n subintervals with endpoints $x_0 = a, x_1, \cdots, x_{n-1}, x_n = b$. $\Delta x_i = x_i - x_{i-1}$ is the **length** of the ith subinterval. A **sample point** is any point $x_i^\star$ chosen from the ith subinterval $[x_{i-1}, x_i]$.

The sum $\sum_{i=1}^{n} f(x_i^\star)\Delta x_i$ is called a **Riemann sum**. It is an approximation of the net area under the graph of f on $[a, b]$ by n rectangles.

There are a few particular ways to choose sample points:

- A **left-endpoint sum** chooses the left endpoint x_{i-1} of each interval $[x_{i-1}, x_i]$ as the sample point.
- A **right-endpoint sum** chooses the right endpoint x_i of each interval $[x_{i-1}, x_i]$ as the sample point.
- A **midpoint sum** chooses the midpoint $\dfrac{x_{i-1} + x_i}{2}$ of each interval $[x_{i-1}, x_i]$ as the sample point.
- A **lower sum** is taken by choosing each sample point $x_i^\star$ so that $f(x_i^\star)$ is the minimum value of f in $[x_{i-1}, x_1]$.
- An **upper sum** is taken by choosing each sample point $x_i^\star$ so that $f(x_i^\star)$ is the maximum value of f in $[x_{i-1}, x_i]$.

Visit this interactive Desmos graph to see an example of Riemann sums.

If a function is monotonic, then certain sums are overestimates or underestimates:

- For a decreasing function, a left endpoint sum is an overestimate and a right endpoint sum is an underestimate.