Probability measure in discrete spaces

This means finite or countably infinite

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De Méré redux

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If a die is thrown four times,
what are the chances that no ace is seen?

De Méré redux

- *Sample space* Ω: each sample point ω is of the form (k_1, k_2, k_3, k_4) where each of k_1, k_2, k_3, k_4 is an integer from 1 to 6.

$$\Omega = \{ (k_1, k_2, k_3, k_4) : 1 \leq k_1, k_2, k_3, k_4 \leq 6 \} = \{1, 2, 3, 4, 5, 6\}^4$$

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- *Probability measure* $\mathbf{P}$:
  - To each *atom* (singleton set) $\{(k_1, k_2, k_3, k_4)\}$ assign *probability mass*

$$\mathbf{P}\{(k_1, k_2, k_3, k_4)\} = \frac{1}{\operatorname{card} \Omega} = \frac{1}{6^4}$$

  - To the event A assign probability

$$\mathbf{P}(A) = \sum_{(k_1, k_2, k_3, k_4) \in A} \mathbf{P}\{(k_1, k_2, k_3, k_4)\} = \frac{\operatorname{card} A}{\operatorname{card} \Omega} = \frac{5^4}{6^4}$$

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- *Sample space* Ω: each sample point ω is an integer in the range from 2 through 12.

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- *Probability measure* **P**:
  - To each *atom* (singleton set) {k} assign *probability mass*

k	2	3	4	5	6	7	8	9	10	11	12
P{k}	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

  - To the event A assign probability

$$\mathbf{P}(A) = \mathbf{P}\{2, 3, 12\} = \mathbf{P}\{2\} + \mathbf{P}\{3\} + \mathbf{P}\{12\} = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{9}$$

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- *Probability measure* **P**:
  - To each *atom* (singleton set) $\{k\}$ assign *probability mass* $p(k) := \mathbf{P}\{k\} = 2^{-|k|-1}$ (integer $k \neq 0$)
  - To the event A assign probability

$$\mathbf{P}(A) = \sum_{\text{integer } k \text{ with } |k| \geq 3} p(k) = 2\left(\frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots\right) = \frac{2 \cdot 2^{-4}}{1 - 2^{-1}} = \frac{1}{4}$$