4.7: Resonance

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Resonance occurs when the frequency of the inhomogeneous term matches the frequency of the homogeneous solution. To illustrate resonance in its simplest embodiment, we consider the second-order linear inhomogeneous ode

$$\ddot{x} + \omega_0^2 x = f \cos \omega t, \quad x(0) = x_0, \quad \dot{x}(0) = u_0. \tag{4.7.1}$$

Our main goal is to determine what happens to the solution in the limit $\omega \to \omega_0$.

The homogeneous equation has characteristic equation

$$r^2 + \omega_0^2 = 0,$$

so that $r_\pm = \pm i\omega_0$. Therefore,

$$x_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t. \tag{4.7.2}$$

To find a particular solution, we note the absence of a first-derivative term, and simply try

$$x(t) = A \cos \omega t.$$

Upon substitution into the ode, we obtain

$$-\omega^2 A + \omega_0^2 A = f,$$

or

$$A = \frac{f}{\omega_0^2 - \omega^2}.$$

Therefore,

$$x_p(t) = \frac{f}{\omega_0^2 - \omega^2} \cos \omega t.$$

Our general solution is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t,$$

with derivative

$$\dot{x}(t) = \omega_0 (c_2 \cos \omega_0 t - c_1 \sin \omega_0 t) - \frac{f\omega}{\omega_0^2 - \omega^2} \sin \omega t.$$

Initial conditions are satisfied when

$$\begin{aligned} x_0 &= c_1 + \frac{f}{\omega_0^2 - \omega^2}, \\ u_0 &= c_2 \omega_0, \end{aligned}$$

so that

$$c_1 = x_0 - \frac{f}{\omega_0^2 - \omega^2}, \quad c_2 = \frac{u_0}{\omega_0}.$$

Therefore, the solution to the ode that satisfies the initial conditions is

$$\begin{aligned} x(t) &= \left(x_0 - \frac{f}{\omega_0^2 - \omega^2} \right) \cos \omega_0 t + \frac{u_0}{\omega_0} \sin \omega_0 t + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t \\ &= x_0 \cos \omega_0 t + \frac{u_0}{\omega_0} \sin \omega_0 t + \frac{f(\cos \omega t - \cos \omega_0 t)}{\omega_0^2 - \omega^2}, \end{aligned}$$

where we have grouped together terms proportional to the forcing amplitude f.

Resonance occurs in the limit $\omega \to \omega_0$; that is, the frequency of the inhomogeneous term (the external force) matches the frequency of the homogeneous solution (the free oscillation). By L'Hospital's rule, the limit of the term proportional to f is found by differentiating with respect to ω:

$$\begin{aligned}\lim_{\omega\to\omega_0}\frac{f(\cos\omega t-\cos\omega_0 t)}{\omega_0^2-\omega^2} &= \lim_{\omega\to\omega_0}\frac{-ft\sin\omega t}{-2\omega}\\ &= \frac{ft\sin\omega_0 t}{2\omega_0}.\end{aligned} \tag{4.7.3}$$

At resonance, the term proportional to the amplitude f of the inhomogeneous term increases linearly with t, resulting in larger-and-larger amplitudes of oscillation for $x(t)$. In general, if the inhomogeneous term in the differential equation is a solution of the corresponding homogeneous differential equation, then the correct ansatz for the particular solution is a constant times the inhomogeneous term times t.

To illustrate this same example further, we return to the original ode, now assumed to be exactly at resonance,

$$\ddot{x}+\omega_0^2 x = f\cos\omega_0 t,$$

and find a particular solution directly. The particular solution is the real part of the particular solution of

$$\ddot{z}+\omega_0^2 z = fe^{i\omega_0 t}.$$

If we try $z_p = Ce^{i\omega_0 t}$, we obtain $0=f$, showing that the particular solution is not of this form. Because the inhomogeneous term is a solution of the homogeneous equation, we should take as our ansatz

$$z_p = Ate^{i\omega_0 t}.$$

We have

$$\dot{z}_p = Ae^{i\omega_0 t}(1+i\omega_0 t),\quad \ddot{z}_p = Ae^{i\omega_0 t}\left(2i\omega_0-\omega_0^2 t\right);$$

and upon substitution into the ode

$$\begin{aligned}\ddot{z}_p+\omega_0^2 z_p &= Ae^{i\omega_0 t}\left(2i\omega_0-\omega_0^2 t\right)+\omega_0^2 Ate^{i\omega_0 t}\\ &= 2i\omega_0 Ae^{i\omega_0 t}\\ &= fe^{i\omega_0 t}.\end{aligned}$$

Therefore,

$$A=\frac{f}{2i\omega_0},$$

and

$$\begin{aligned}x_p &= \mathrm{Re}\left\{\frac{ft}{2i\omega_0}e^{i\omega_0 t}\right\}\\ &= \frac{ft\sin\omega_0 t}{2\omega_0},\end{aligned}$$

the same result as (4.7.3).

✔ Example 4.7.1

Find a particular solution of $\ddot{x}-3\dot{x}-4x=5e^{-t}$.

Solution

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If we naively try the ansatz

$$x=Ae^{-t},$$

and substitute this into the inhomogeneous differential equation, we obtain

$$A+3A-4A=5,$$

or $0=5$, which is clearly nonsense. Our ansatz therefore fails to find a solution. The cause of this failure is that the corresponding homogeneous equation has solution

$$x_h=c_1e^{4t}+c_2e^{-t},$$

so that the inhomogeneous term $5e^{-t}$ is one of the solutions of the homogeneous equation. To find a particular solution, we should therefore take as our ansatz

$$x = Ate^{-t},$$

with first- and second-derivatives given by

$$\dot{x} = Ae^{-t}(1-t), \quad \ddot{x} = Ae^{-t}(-2+t).$$

Substitution into the differential equation yields

$$Ae^{-t}(-2+t) - 3Ae^{-t}(1-t) - 4Ate^{-t} = 5e^{-t}.$$

The terms containing t cancel out of this equation, resulting in $-5A = 5$, or $A = -1$. Therefore, the particular solution is

$$x_p = -te^{-t}.$$