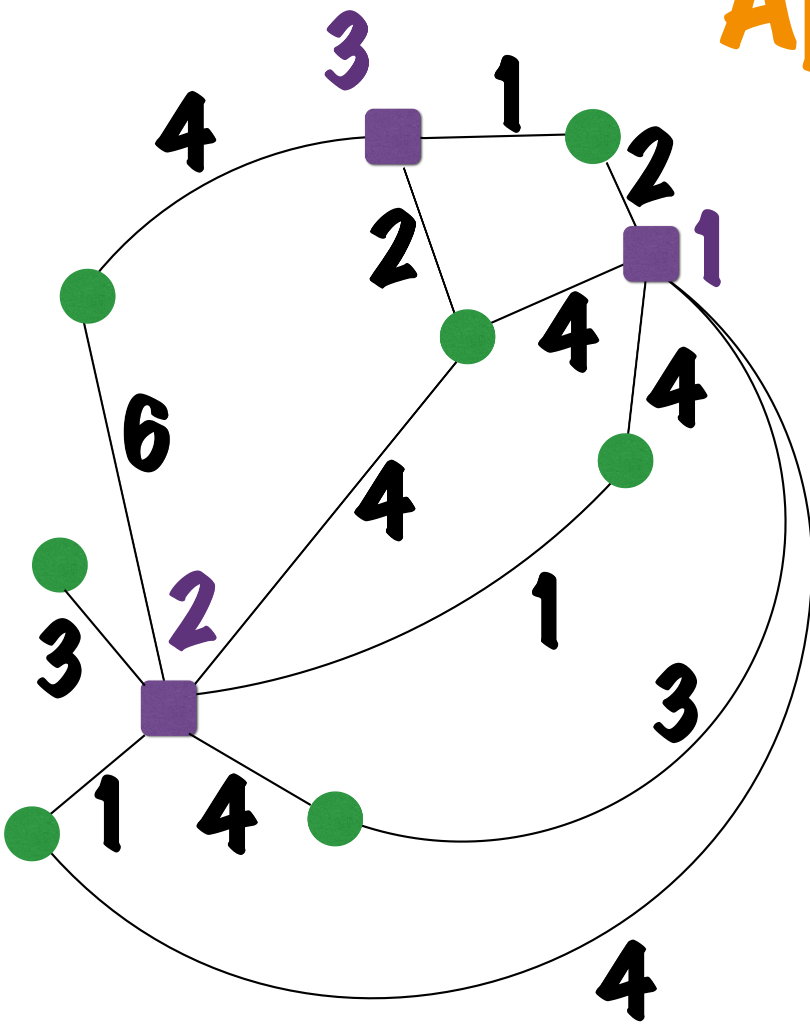
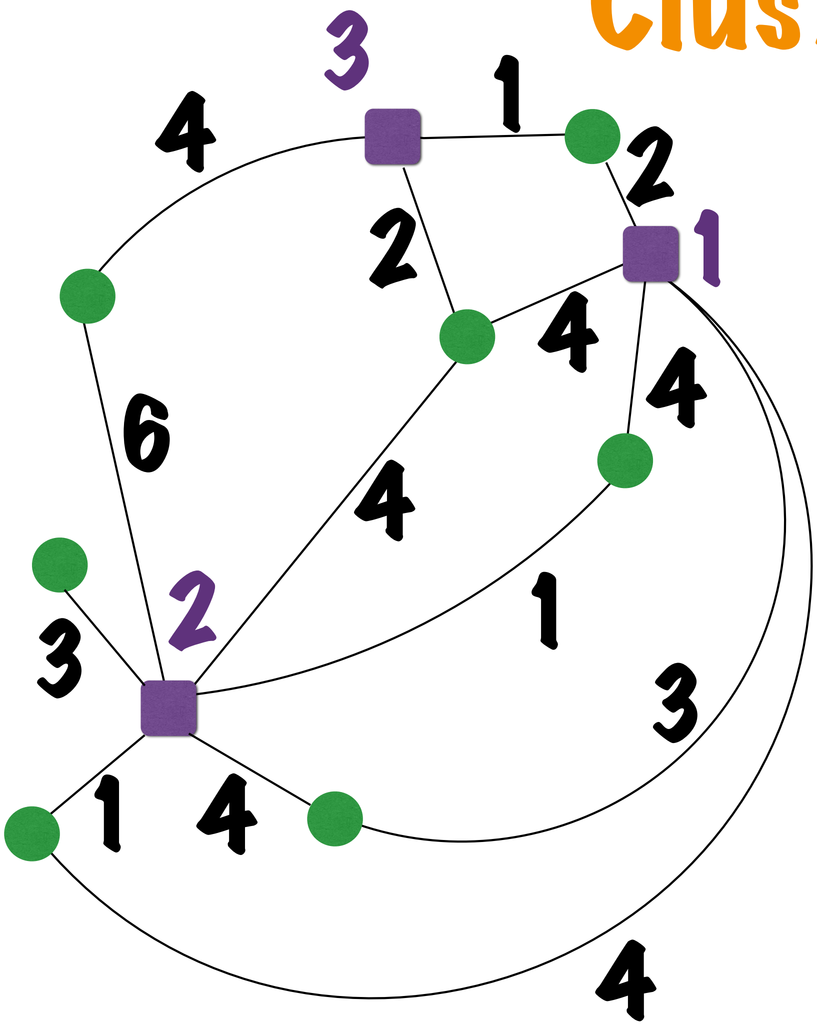
Facility location

Appears in...



Deciding placements of factories, warehouses, schools, hospital.
Deciding how to place servers on the web

Clustering clients

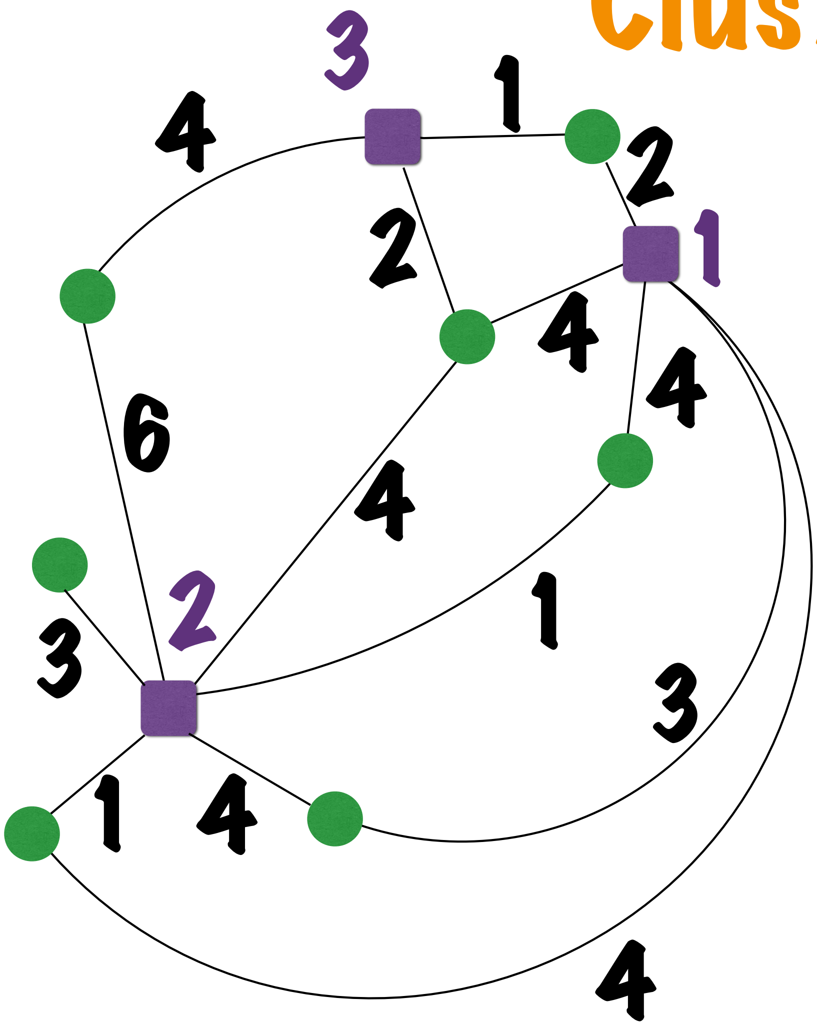


Input

A bipartite graph G with bipartition F (for facilities), C (for clients).

f_i : Cost of opening facility i

c_{ij} : Distance from client j to facility i



Clustering clients

Input

A bipartite graph G with
bipartition F (for facilities),
C (for clients).

Output

Choose a set S of facilities

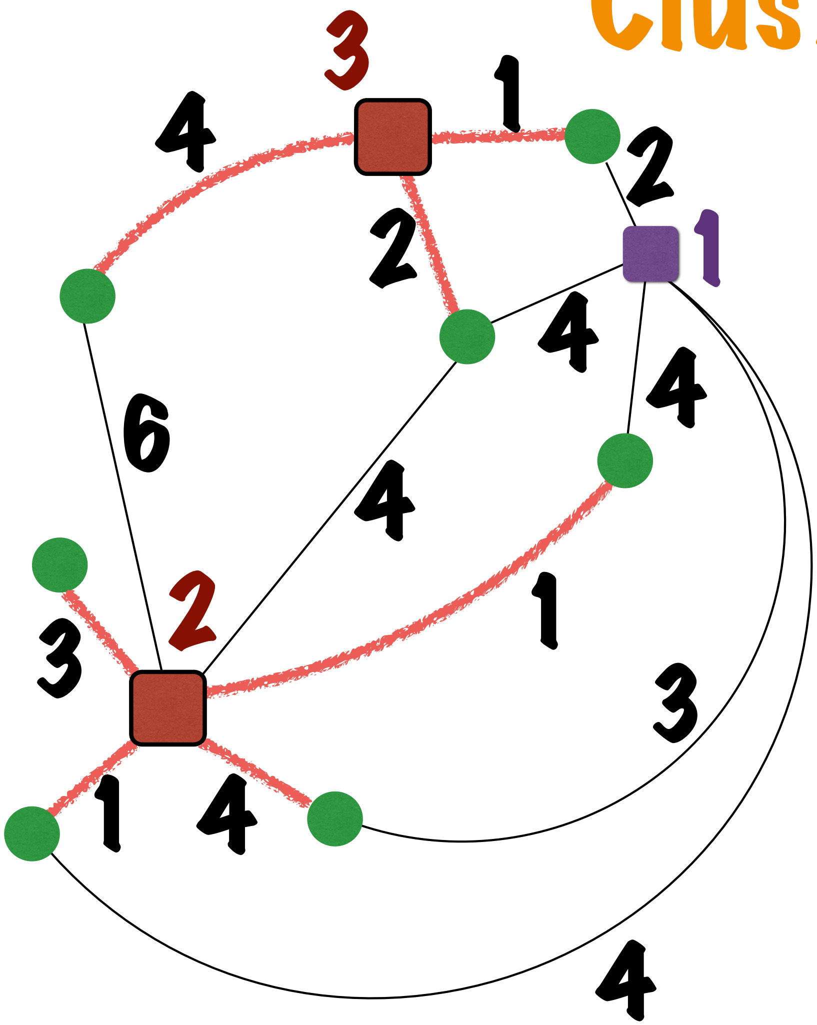
minimize :

cost of the facilities of S

<u>plus</u>

the distances from each client
to the closest facility of S

Clustering clients

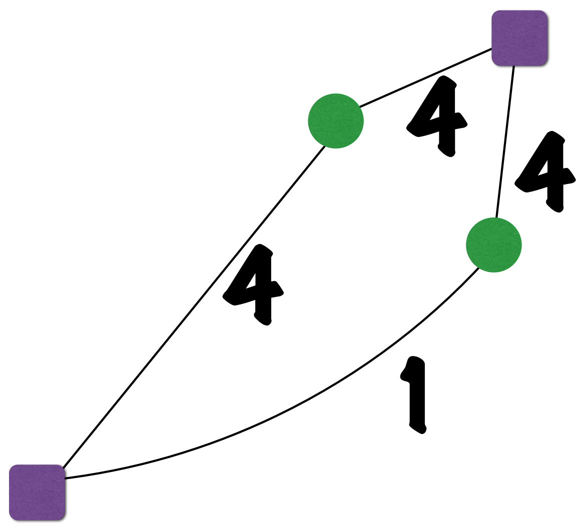


Facility cost = 3 + 2 = 5

Clients cost = 4 + 1 + 2
+1 + 4 + 1 + 3 = 16

Total cost = 21

Metric facility location



Triangle inequality:

$$c_{ij} + c_{j\ell} \geq c_{i\ell}$$

Facility location

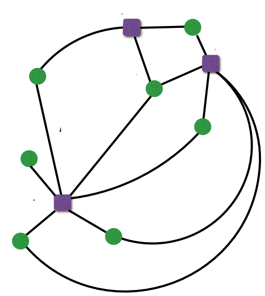
Facility location

IP model

Variables

$x_{ij} = 1$ iff client j is assigned to facility i

$y_i = 1$ iff facility i is open



IP model

Constraints

Each client is assigned to some facility

$$\sum_{i \in F} x_{ij} \geq 1, \qquad j \in C$$

and

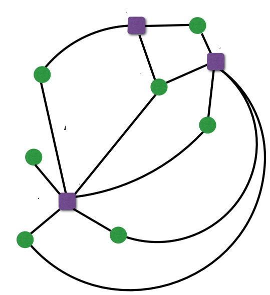
If at least one client is assigned to facility i then facility i has to be open

$$y_i - x_{ij} \geq 0, \qquad j \in C,\ i \in F$$

IP model

Objective

$$\text{minimize} \quad \sum_{i\in F}\sum_{j\in C} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i$$



IP model

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i\in F}\sum_{j\in C} c_{ij}x_{ij} + \sum_{i\in F} f_i y_i & \\
\text{subject to} & \displaystyle\sum_{i\in F} x_{ij} \geq 1, & j \in C \\
& y_i - x_{ij} \geq 0, & j \in C,\ i \in F \\
& y_i \in \{0,1\}, & i \in F \\
& x_{ij} \in \{0,1\}, & j \in C,\ i \in F
\end{array}
$$

Linear programming relaxation

$$\text{minimize} \quad \sum_{i \in F} \sum_{j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i$$

$$\text{subject to} \quad \sum_{i \in F} x_{ij} \geq 1, \qquad j \in C$$

$$y_i - x_{ij} \geq 0, \qquad j \in C,\ i \in F$$

$$y_i \in \{0,1\}, \qquad i \in F \qquad \longrightarrow \qquad 0 \leq y_i \leq 1, \qquad i \in F$$

$$x_{ij} \in \{0,1\}, \qquad j \in C,\ i \in F \qquad \qquad 0 \leq x_{ij} \leq 1, \qquad j \in C,\ i \in F$$

Facility location

Facility location

Linear programming relaxation

Primal : minimize $\sum_{i \in F} \sum_{j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i$

Dual variables :

subject to

$$\sum_{i \in F} x_{ij} \geq 1, \qquad j \in C \qquad \alpha_j$$

$$y_i - x_{ij} \geq 0, \qquad j \in C,\ i \in F \qquad \beta_{ij}$$

$$y_i \geq 0, \qquad i \in F$$

$$x_{ij} \geq 0, \qquad j \in C,\ i \in F$$

Taking the dual

Dual: maximize $\displaystyle\sum_{j \in C} \alpha_j$

Primal variables

subject to

$$\alpha_j - \beta_{ij} \leq c_{ij}, \qquad j \in C,\ i \in F \qquad x_{ij}$$

$$\sum_{j \in C} \beta_{ij} \leq f_i, \qquad i \in F \qquad y_i$$

$$\alpha_j \geq 0, \qquad j \in C$$

$$\beta_{ij} \geq 0, \qquad j \in C,\ i \in F$$

Recall :

Complementary slackness conditions

If x is optimal for (P) and y optimal for (D)

then for every i:

$$c_i = \sum_j a_{ij} y_j \text{ or } x_i = 0$$

and for every j:

$$b_j = \sum_i a_{ij} xi \text{ or } y_j = 0$$

Interpreting the dual

Complementary slackness conditions :

1. $\forall i \in F, j \in C : x_{ij} > 0 \implies \alpha_j - \beta_{ij} = c_{ij}$
2. $\forall i \in F : y_i > 0 \implies \sum_{j \in C} \beta_{ij} = f_i$
3. $\forall j \in C : \alpha_j > 0 \implies \sum_{i \in F} x_{ij} = 1$
4. $\forall i \in F, j \in C : \beta_{ij} > 0 \implies y_i = x_{ij}$

Interpreting the dual

β_{ij} : Contribution of client j for opening facility i

Complementary slackness condition 2 : $\forall i \in F : y_i > 0 \implies \sum_{j \in C} \beta_{ij} = f_i$

If $y_i = 1$ then $\sum_{j \in C} \beta_{ij} = f_i$ we say that facility i is <u>fully paid</u>

Interpreting the dual

Complementary slackness condition 4: $\forall i \in F, j \in C : \beta_{ij} > 0 \implies y_i = x_{ij}$

thus, $\forall i \in F, j \in C : y_i \neq x_{ij} \implies \beta_{ij} = 0$

Recall $y_i - x_{ij} \geq 0,$ **If** $y_i \neq x_{ij}$ **then** $\beta_{ij} = 0$

and so, client j does not contribute to opening any facility except the one it is connected to

Interpreting the dual

Complementary slackness condition 1 : $\forall i \in F, j \in C : x_{ij} > 0 \implies \alpha_j - \beta_{ij} = c_{ij}$

thus, if client j is assigned to facility i we have :

$$\alpha_j - \beta_{ij} = c_{ij}$$

We define α_j as the total price paid by client j.

The total price is made of the use of edge (i,j) c_{ij} and the contribution to the opening cost β_{ij}

Facility location

Facility location

Primal

$$\min \sum_i \mathbf{f_i y_i} + \sum_{i,j} \mathbf{c_{ij} x_{ij}} :$$
$$\sum_i \mathbf{x_{ij}} \geq \mathbf{1} \quad \textbf{for all } \mathbf{j}$$
$$\mathbf{x_{ij}} \leq \mathbf{y_i} \quad \textbf{for all } \mathbf{i, j}$$
$$\mathbf{x_{ij}}, \mathbf{y_i} \geq \mathbf{0}$$

Dual

$$\max \sum_j \alpha_{\mathbf{j}} :$$
$$\sum_j \beta_{\mathbf{ij}} \leq \mathbf{f_i} \quad \textbf{for all } \mathbf{i}$$
$$\alpha_{\mathbf{j}} \leq \beta_{\mathbf{ij}} + \mathbf{c_{ij}} \quad \textbf{for all } \mathbf{i, j}$$
$$\alpha_{\mathbf{j}}, \beta_{\mathbf{ij}} \geq \mathbf{0}$$

Algorithm

1. Solve the primal and dual LPs: $y_i^*, x_{ij}^*, \alpha_j^*, \beta_{ij}^*$
2. While some clients are unassigned

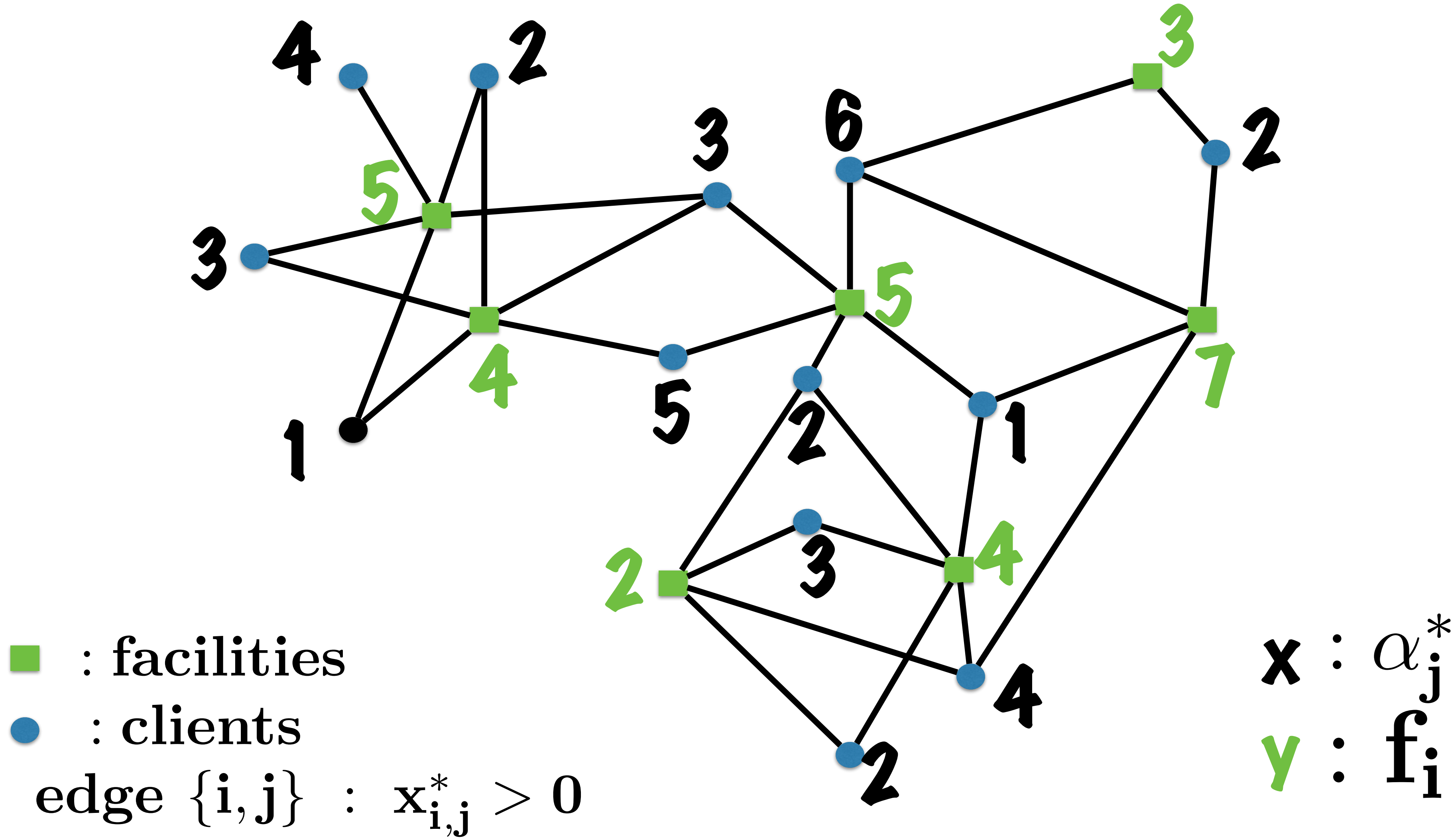
   j_C: unassigned client s.t. $\alpha_{j_C}^*$ is min

   i_C : cheapest facility s.t. $x_{i_C, j_C}^* > 0$

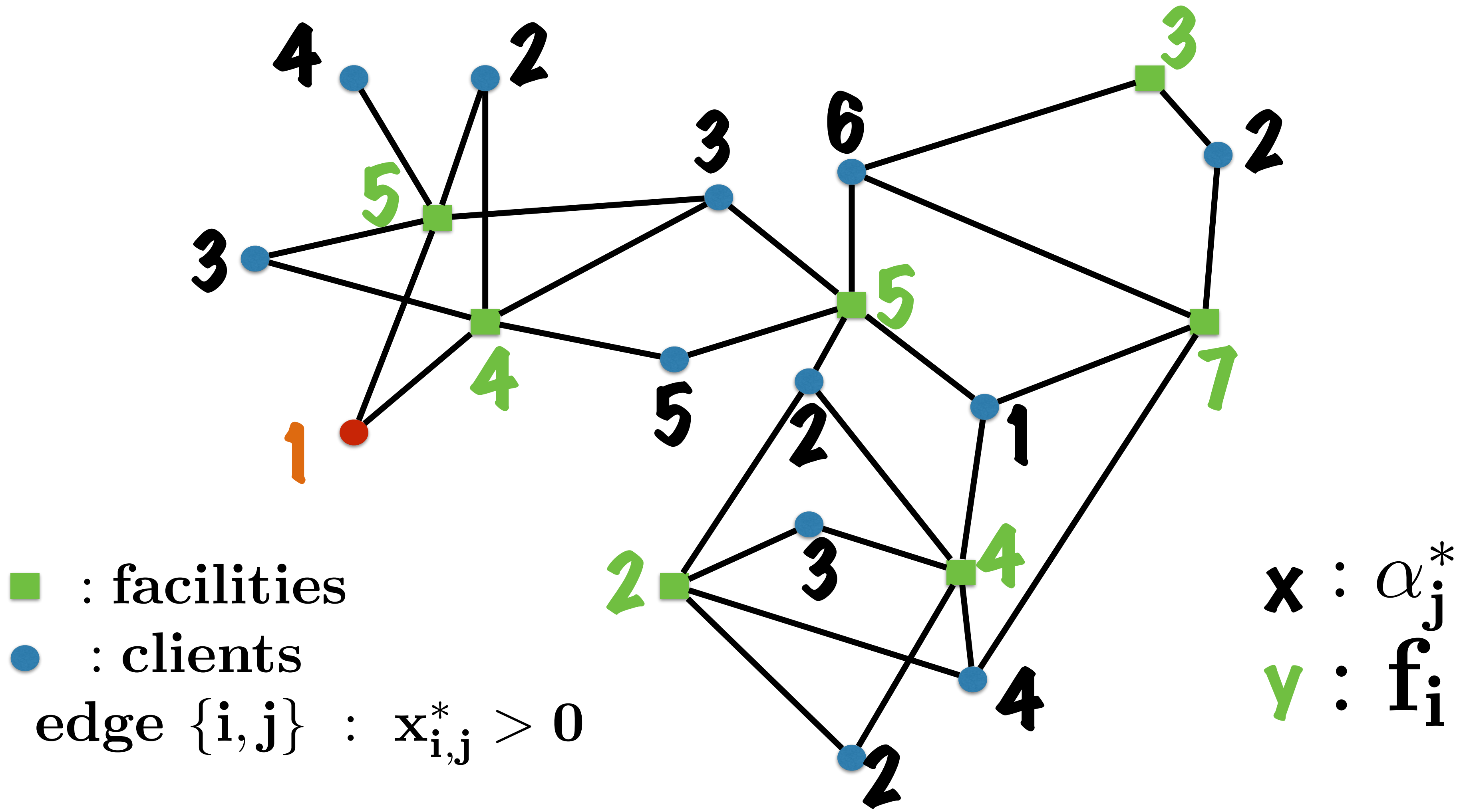
   open facility i_C

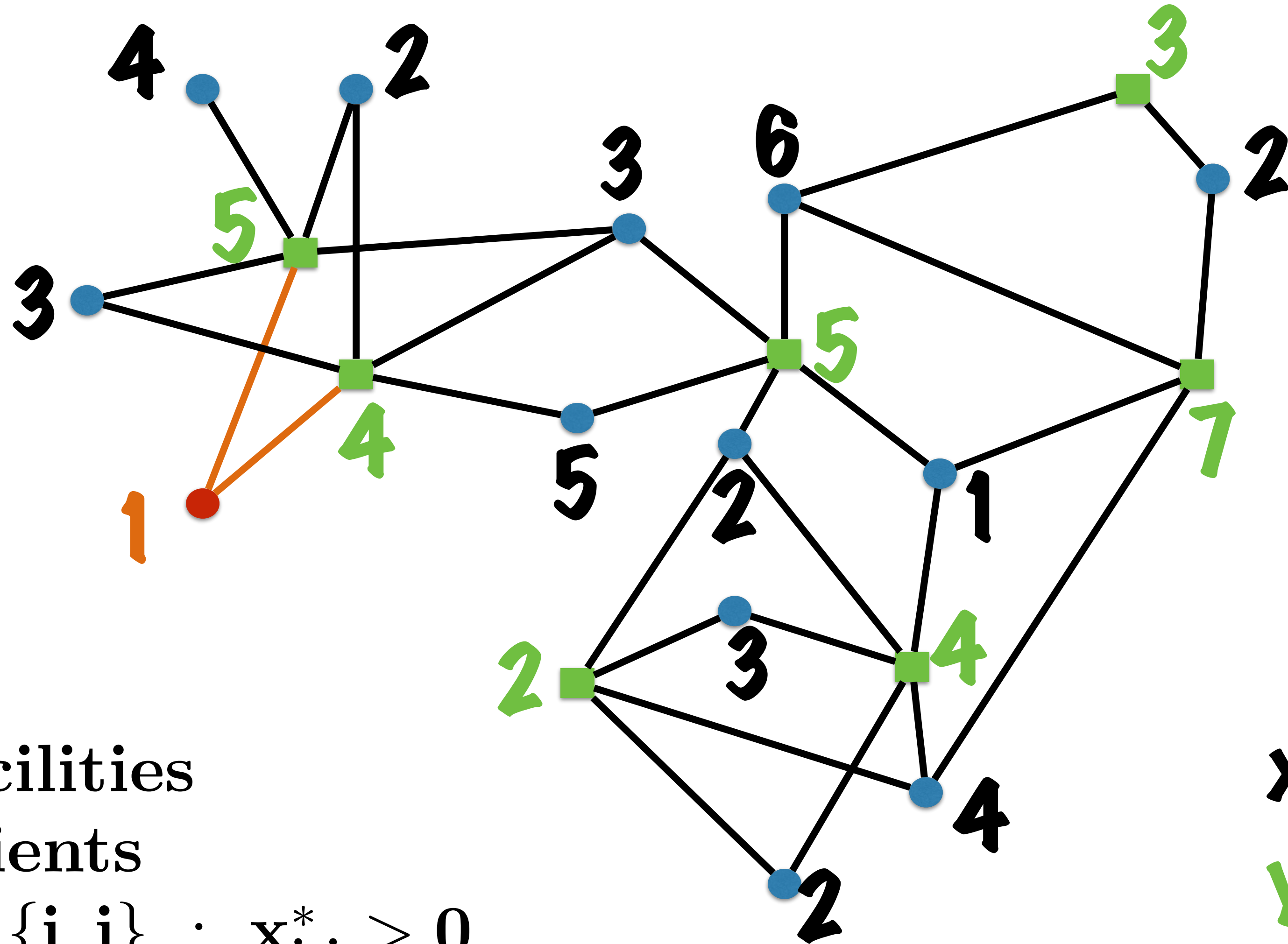
   assign to i_C all unassigned clients s.t.

   there is a facility with $x_{i, j_C} > 0$ and $x_{i,j} > 0$


: facilities
: clients
edge $\{i, j\}$: $x^*_{i,j} > 0$
x : α^*_j
y : f_i

First iteration





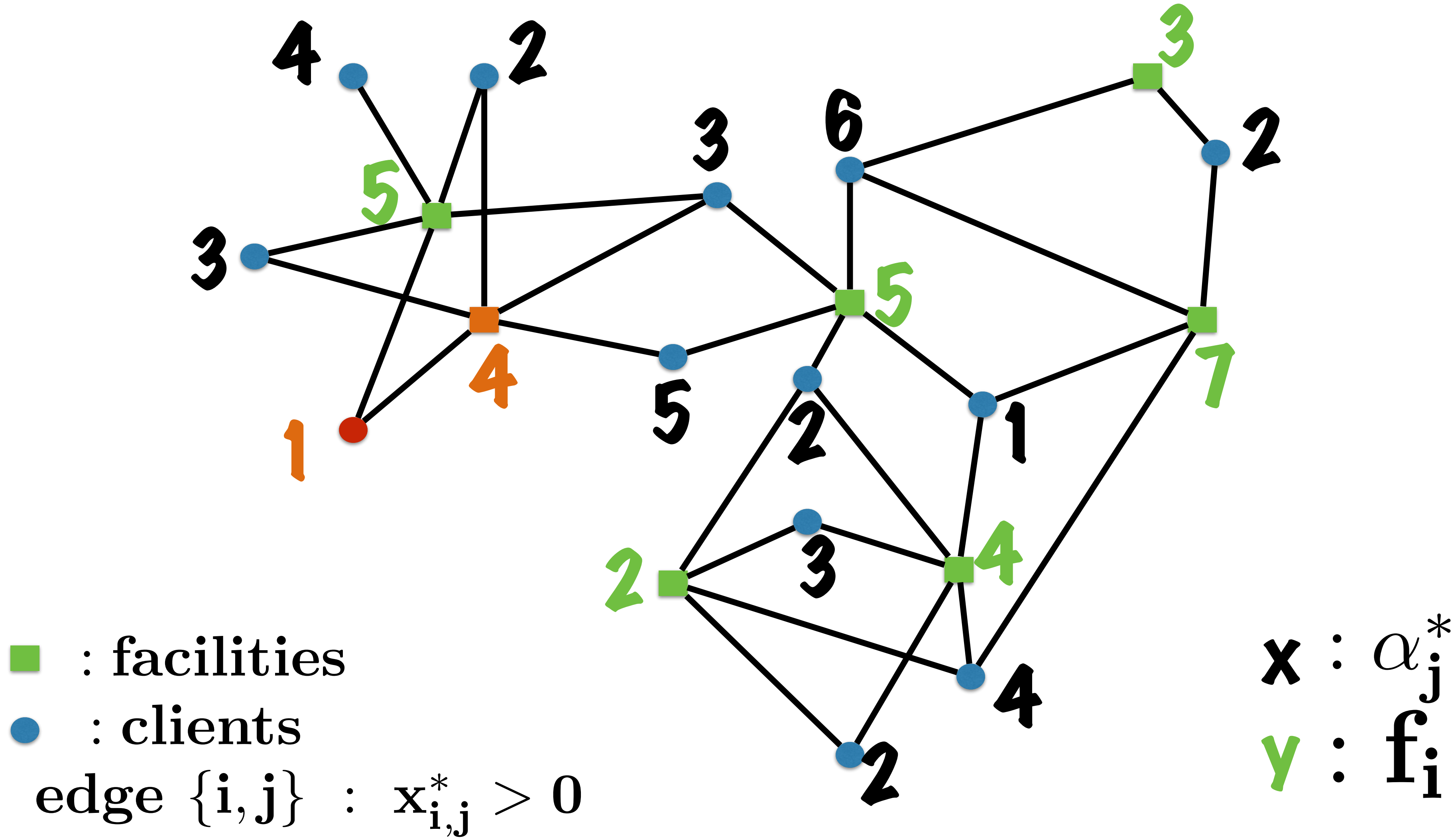
■ : facilities

● : clients

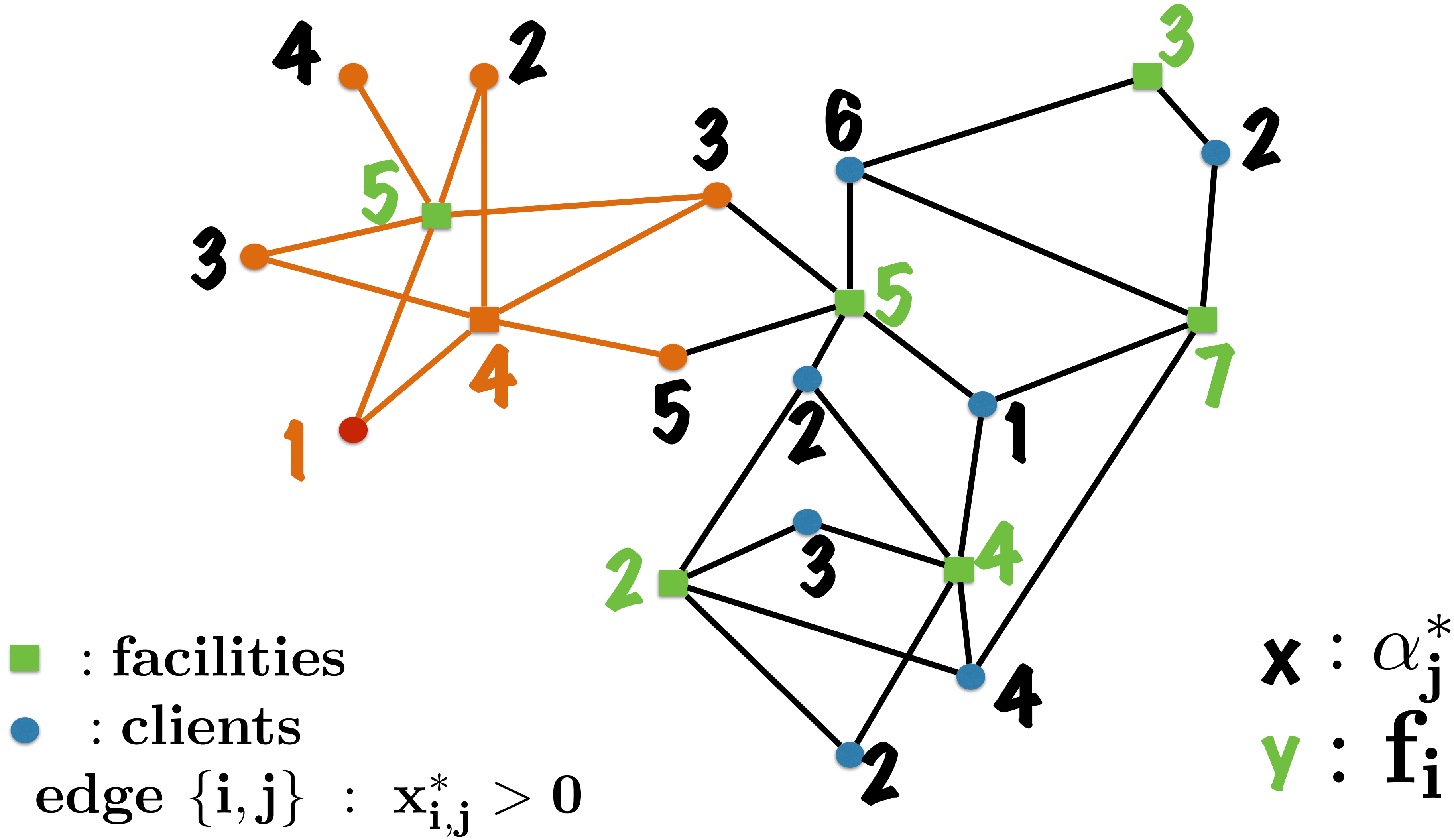
edge $\{i, j\}$: $x^*_{i,j} > 0$

$\mathbf{x}$: α^*_j

$\mathbf{y}$: f_i

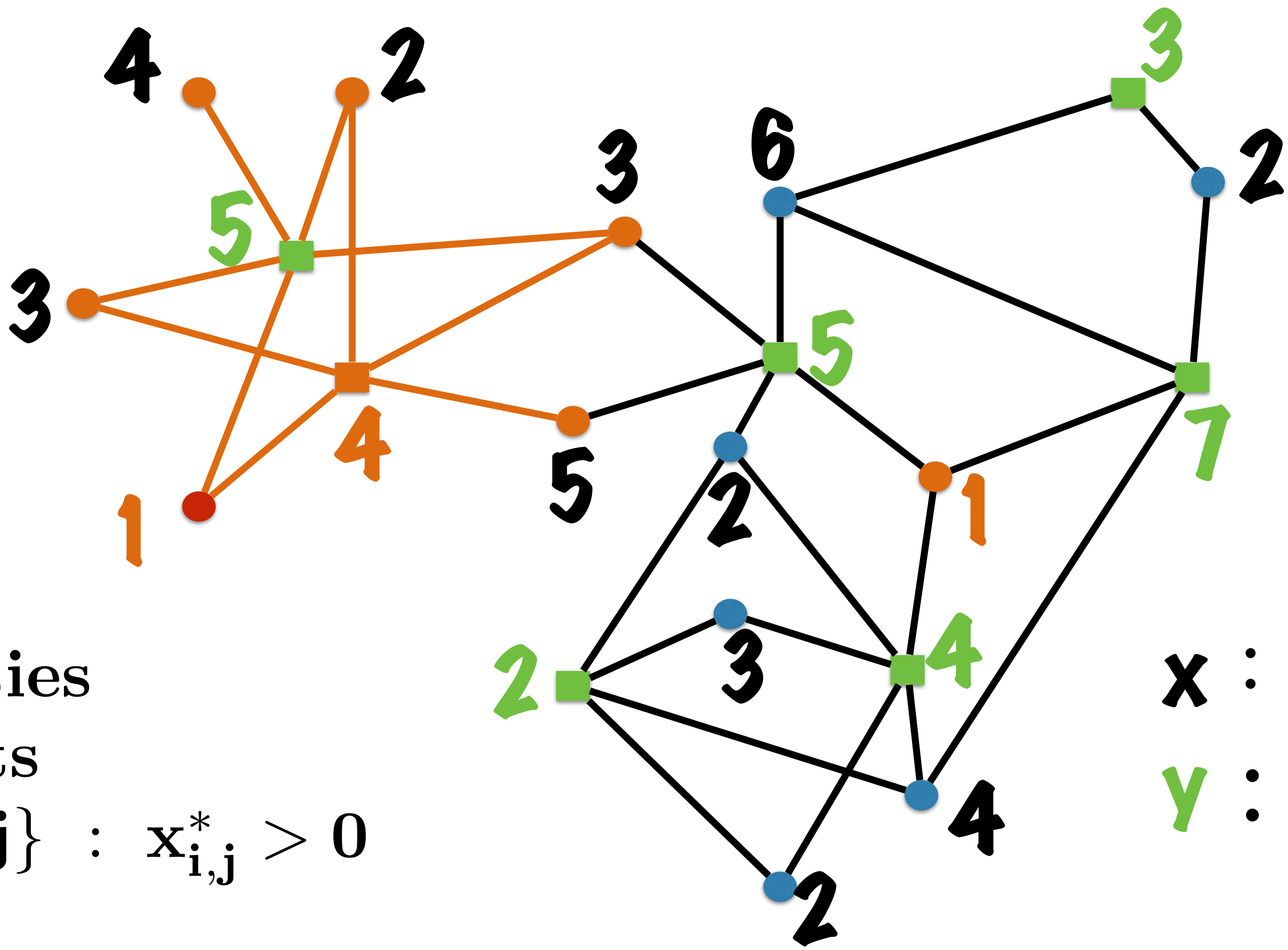


: facilities
: clients
edge $\{i,j\}$: $x^*_{i,j} > 0$
x : α^*_j
y : f_i



: facilities
: clients
edge $\{i, j\}$: $x^*_{i,j} > 0$
x : α^*_j
y : f_i

Second iteration



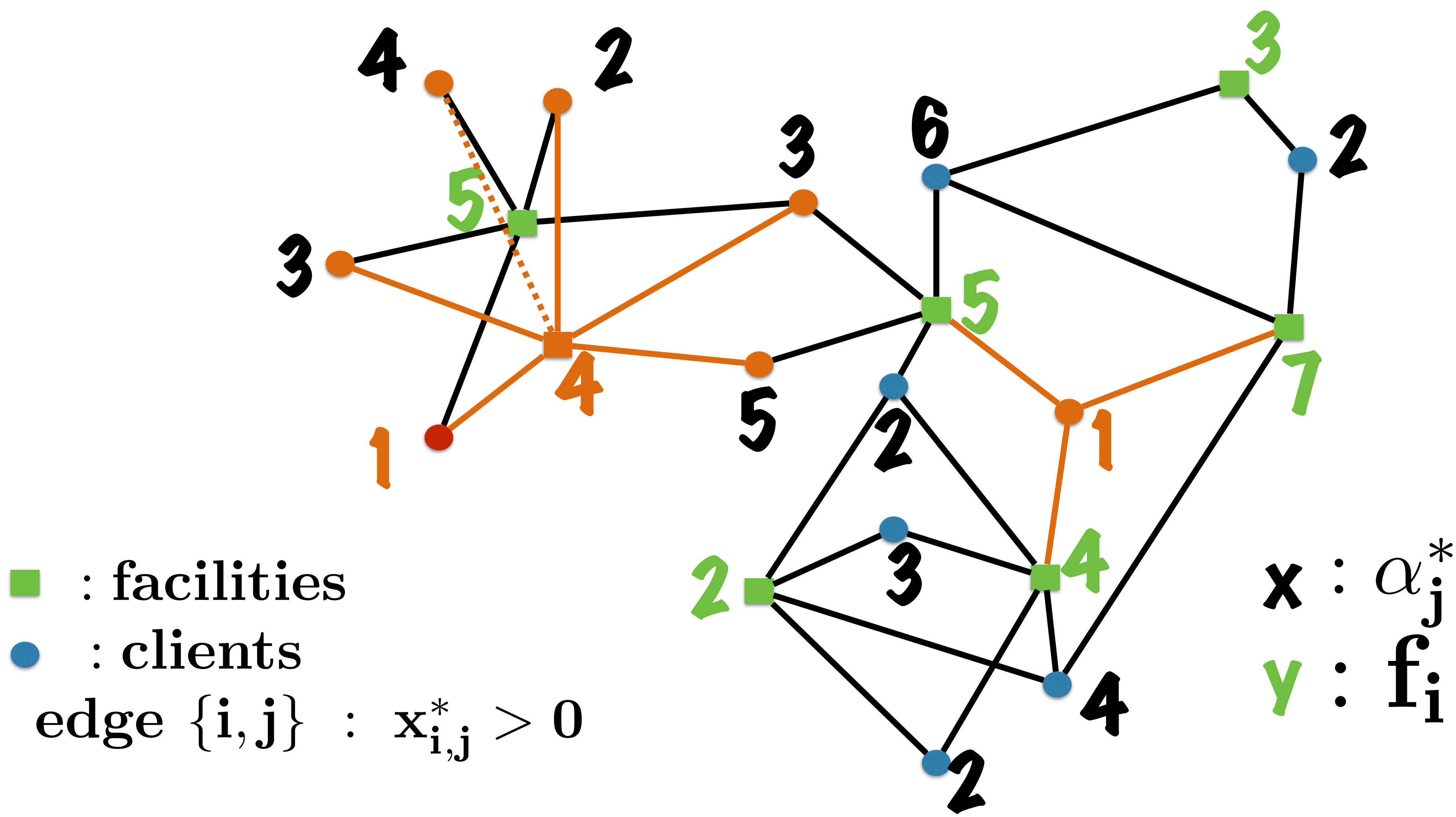
: facilities

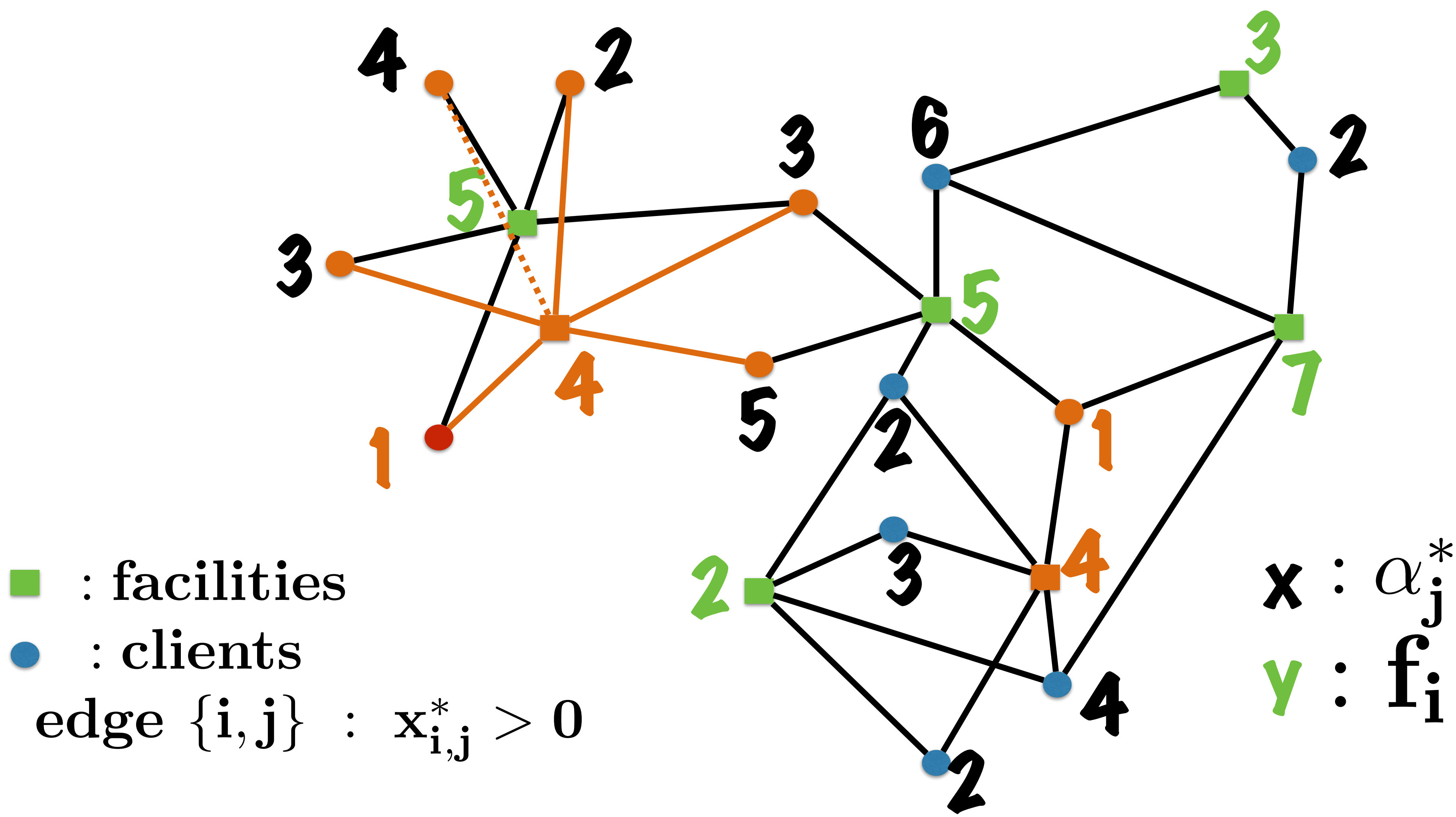
: clients

edge $\{i, j\}$: $x^*_{i,j} > 0$

x : α^*_j

y : f_i


: facilities
: clients
edge $\{i, j\}$: $x^*_{i,j} > 0$
x : α^*_j
y : f_i


: facilities
: clients
edge $\{i,j\}$: $x^*_{i,j} > 0$
x : α^*_j
y : f_i
4
2
3
6
3
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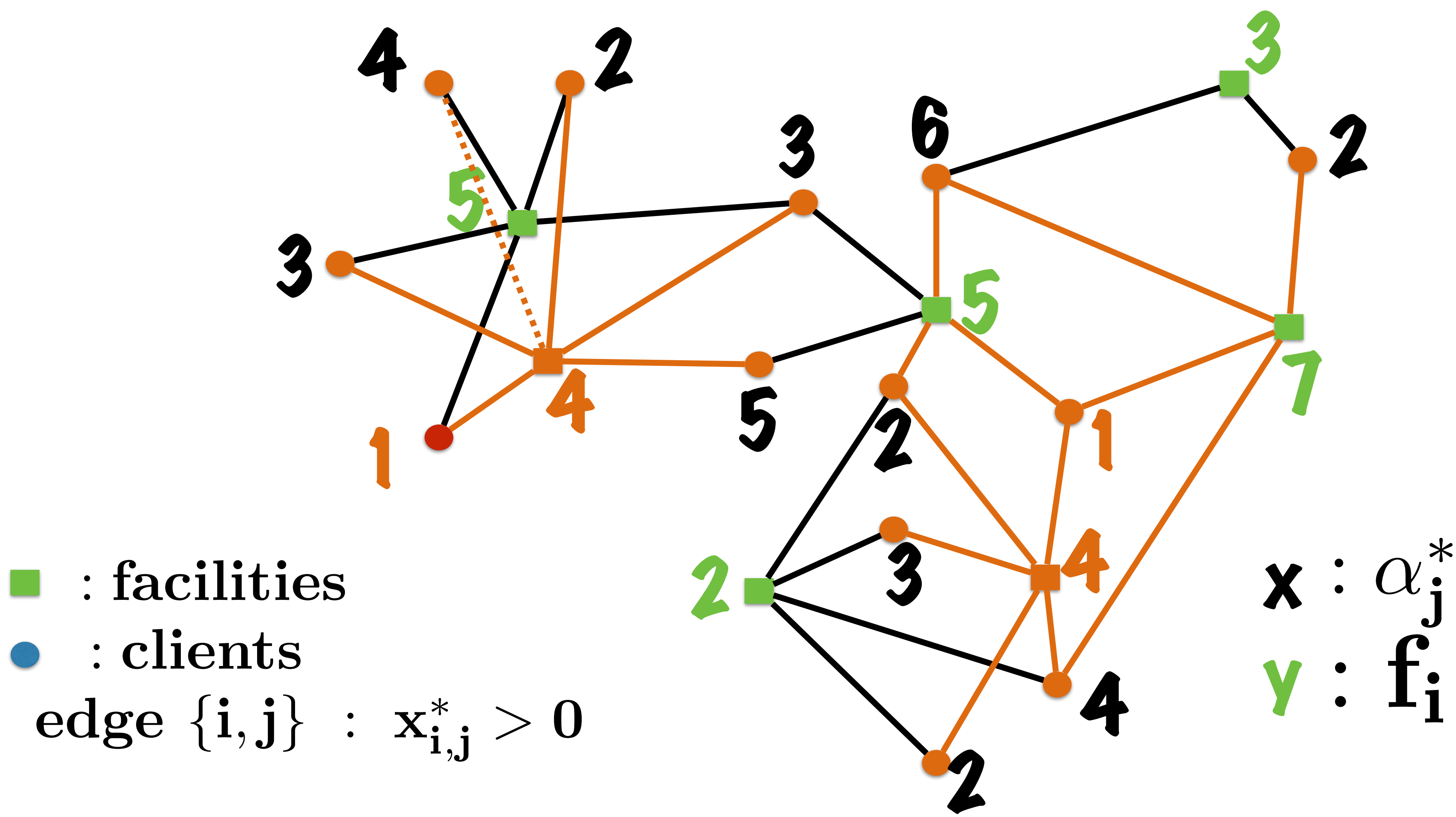
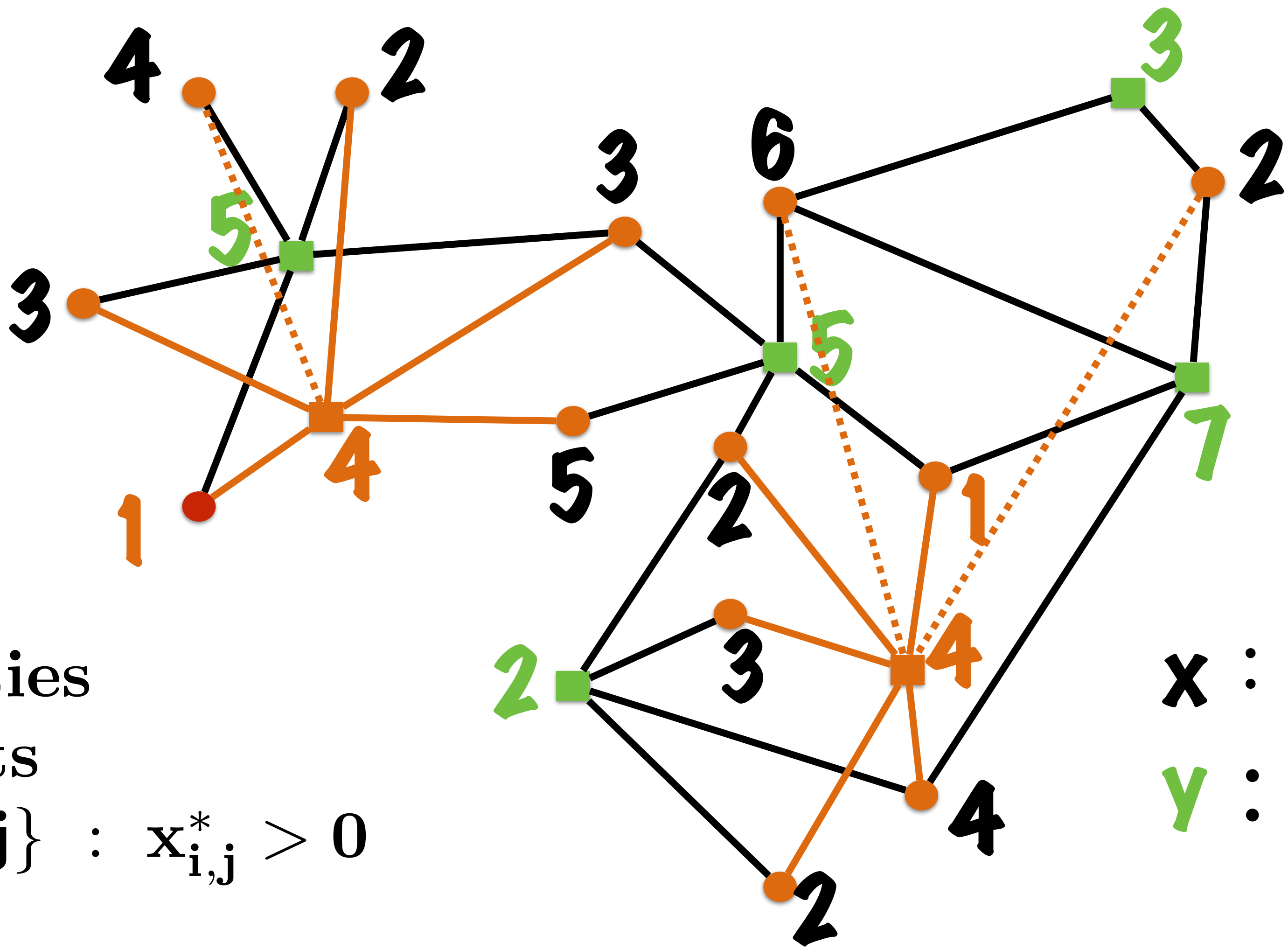
5
3
5
4
5
1
2
1
7
3
4
2
4
2



4
2
3
6
3
2
5
3
5
4
5
7
1
2
1
2
3
4
4
2
: facilities
: clients
edge {i, j} : $x^*_{i,j} > 0$
x : α^*_j
y : f_i

Output



$\blacksquare$: facilities

$\bullet$: clients

edge $\{\mathbf{i},\mathbf{j}\}$: $\mathbf{x}^*_{\mathbf{i},\mathbf{j}} > 0$

x : $\alpha^*_{\mathbf{j}}$

y : $\mathbf{f_i}$

Facility location

Facility location

Primal

$$\min \sum_i f_i y_i + \sum_{i,j} c_{ij} x_{ij} :$$
$$\sum_i x_{ij} \geq 1 \quad \text{for all } j$$
$$x_{ij} \leq y_i \quad \text{for all } i, j$$
$$x_{ij}, y_i \geq 0$$

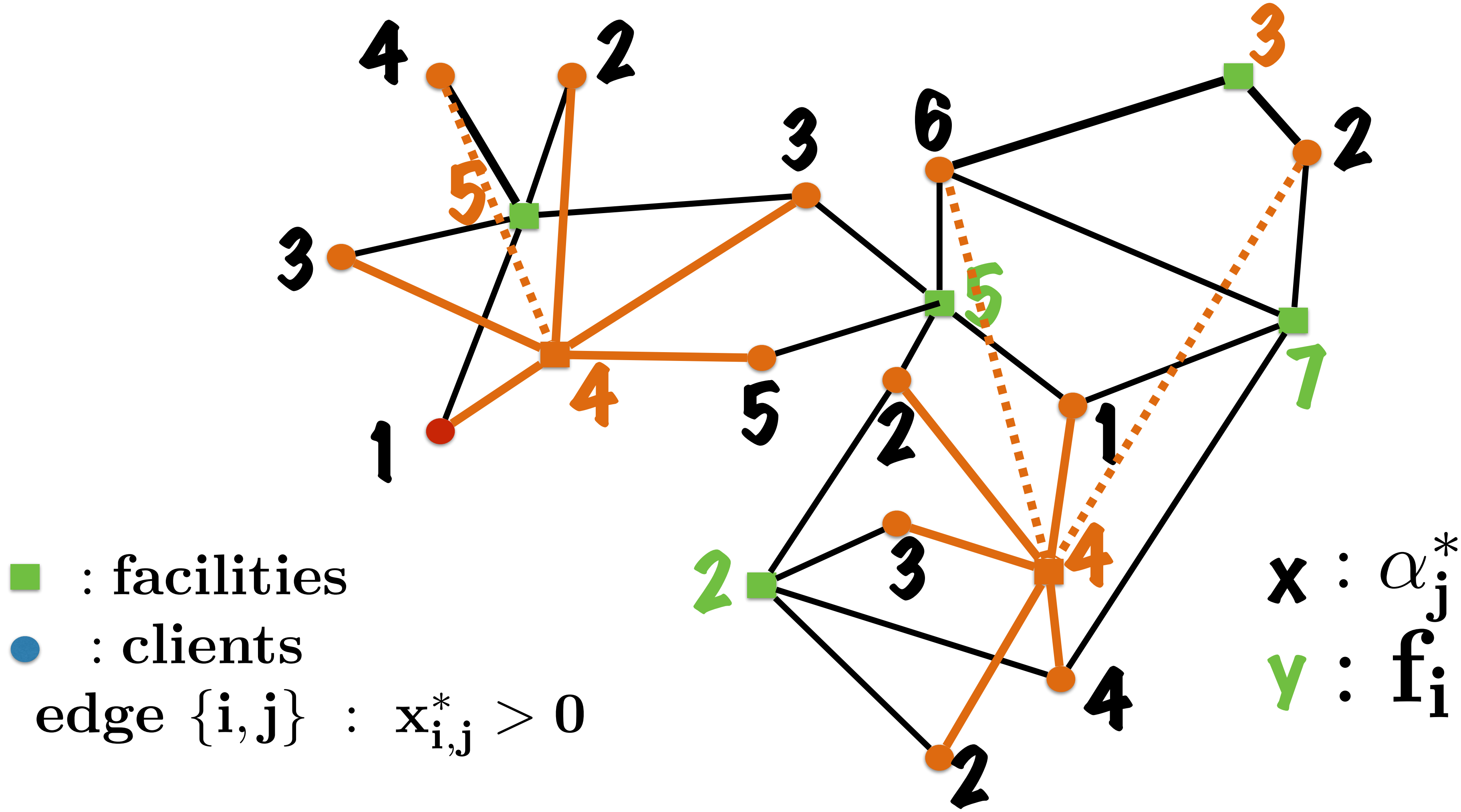
Dual

$$\max \sum_j \alpha_j :$$
$$\sum_j \beta_{ij} \leq f_i \quad \text{for all } i$$
$$\alpha_j \leq \beta_{ij} + c_{ij} \quad \text{for all } i, j$$
$$\alpha_j, \beta_{ij} \geq 0$$

Algorithm

1. Solve the primal and dual LPs: $y_i^*, x_{ij}^*, \alpha_j^*, \beta_{ij}^*$
2. While some clients are unassigned
   j_C: unassigned client s.t. $\alpha_{j_C}^*$ is min
   i_C : cheapest facility s.t. $x_{i_C,j_C}^* > 0$
   open facility i_C
   assign to i_C all unassigned clients s.t.
   there is a facility with $x_{i,j_C} > 0$ and $x_{i,j} > 0$

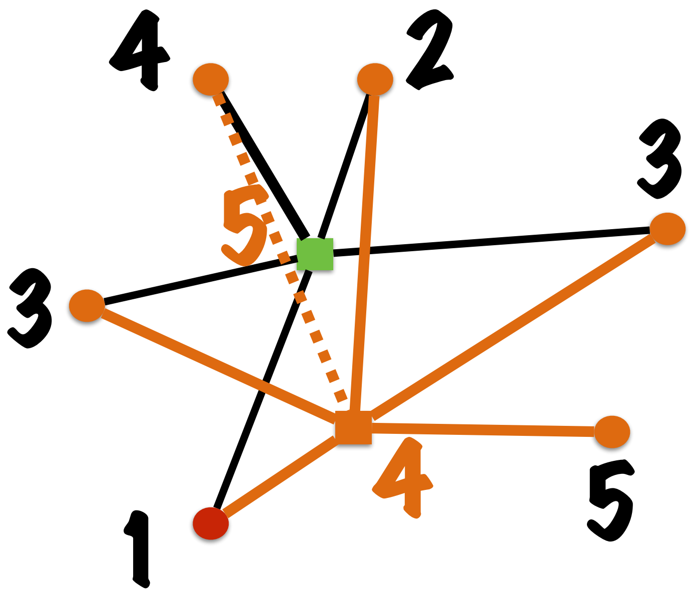
Service cost: total red length (solid or dotted)

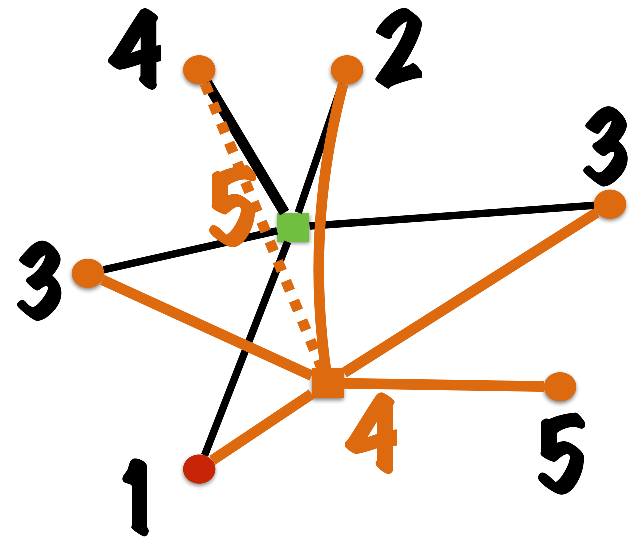


Service cost analysis

$$\sum_{\mathbf{Cluster\ C}} \sum_{\mathbf{j \in C}} \mathbf{c_{i_C j}}$$

Metric: $\mathbf{c_{i_C,j} \leq c_{i_C,j_C} + c_{i,j_C} + c_{i,j}}$





Observation: if $\mathbf{j} \in \mathrm{C}$ **then** $\mathbf{x}^*_{\mathbf{i},\mathbf{j}}, \mathbf{x}^*_{\mathbf{i},\mathbf{j}_\mathrm{C}}, \mathbf{x}^*_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}} > 0$

Complementary slackness:

$$\alpha^*_{\mathbf{j}} = \beta^*_{\mathbf{i},\mathbf{j}} + \mathbf{c}_{\mathbf{i},\mathbf{j}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}}$$
$$\alpha^*_{\mathbf{j}_\mathrm{C}} = \beta^*_{\mathbf{i},\mathbf{j}_\mathrm{C}} + \mathbf{c}_{\mathbf{i},\mathbf{j}_\mathrm{C}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}_\mathrm{C}}$$
$$\alpha^*_{\mathbf{j}_\mathrm{C}} = \beta^*_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}} + \mathbf{c}_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}} \geq \mathbf{c}_{\mathbf{i}_\mathrm{C},\mathbf{j}_\mathrm{C}}$$

Adding: $\mathbf{c_{i_C,j} \leq 2\alpha^*_{jC} + \alpha^*_j}$

Minimality: $\mathbf{\alpha^*_{jC} \leq \alpha^*_j}$

So: $\mathbf{c_{i_C,j} \leq 3\alpha^*_j}$

$$\sum_{\text{Cluster } \mathbf{C}} \sum_{\mathbf{j \in C}} \mathbf{c_{i_C j}} \leq \sum_{\text{Cluster } \mathbf{C}} \sum_{\mathbf{j \in C}} \mathbf{3\alpha^*_j}$$

$$= \mathbf{3} \sum_{\mathbf{j}} \alpha^*_{\mathbf{j}}$$

Duality theorem: $\leq \mathbf{3 \cdot OPT}$

Facility location

Facility location

Primal | **Dual**

$$\min \sum_i \mathbf{f_i y_i} + \sum_{i,j} \mathbf{c_{ij} x_{ij}} :$$
$$\sum_i \mathbf{x_{ij}} \geq \mathbf{1} \quad \textbf{for all j}$$
$$\mathbf{x_{ij}} \leq \mathbf{y_i} \quad \textbf{for all i, j}$$
$$\mathbf{x_{ij}}, \mathbf{y_i} \geq \mathbf{0}$$

$$\max \sum_j \alpha_j :$$
$$\sum_j \beta_{ij} \leq \mathbf{f_i} \quad \textbf{for all i}$$
$$\alpha_j \leq \beta_{ij} + \mathbf{c_{ij}} \quad \textbf{for all i, j}$$
$$\alpha_j, \beta_{ij} \geq \mathbf{0}$$

Algorithm

1. Solve the primal and dual LPs: $y_i^*, x_{ij}^*, \alpha_j^*, \beta_{ij}^*$
2. While some clients are unassigned

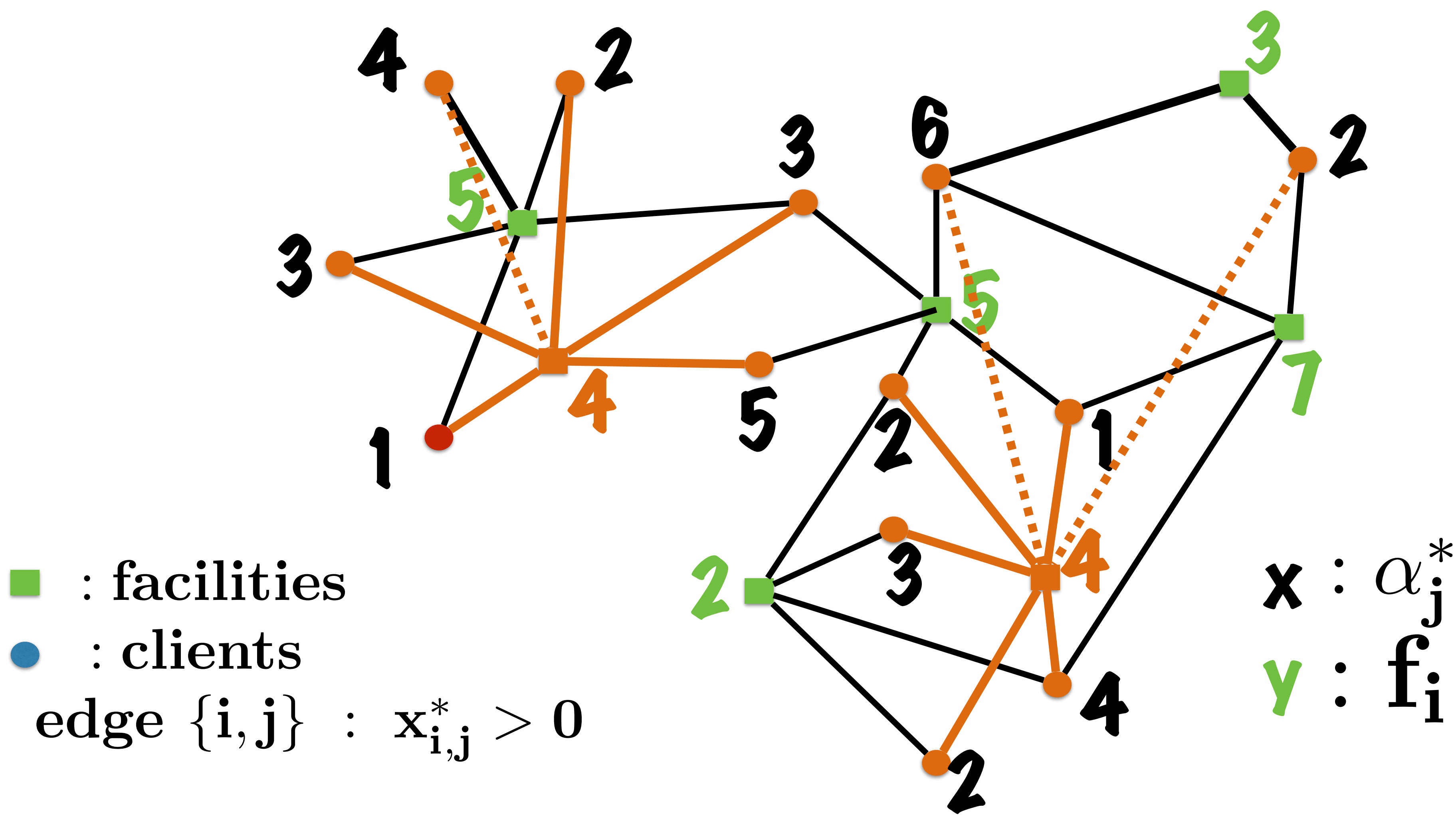
   j_C : unassigned client s.t. $\alpha_{j_C}^*$ is min

   i_C : cheapest facility s.t. $x_{i_C, j_C}^* > 0$

   open facility i_C

   assign to i_C all unassigned clients s.t. there is a facility with $x_{i,j_C}^* > 0$ and $x_{i,j}^* > 0$

Facilities cost: 4+4=8



Facilities cost analysis

$$\sum_{\text{Cluster C}} f_{i_C}$$

Observation: if open i_C then $f_{i_C} = \min\{f_i : x^*_{i,j_C} > 0\}$

$$f_{i_C} \le \sum_{i:x^*_{i,j_C}>0} x^*_{i,j_C} f_i$$

Primal constraint: $x^*_{i,j_C} \le y^*_i$

$$\sum_{\text{Cluster C}} f_{i_C} \le \sum_{\text{Cluster C}} \sum_{i:x^*_{i,j_C}>0} y^*_i f_i$$

$$\sum_{\text{Cluster C}} f_{i_C} \leq \sum_{\text{Cluster C}} \sum_{i:x^*_{i,j_C}>0} y^*_i f_i$$

By algorithm, this is a disjoint sum:

$$\leq \sum_i y^*_i f_i$$

Primal objective: $\leq \mathrm{OPT}$

Together:

$$\sum_{\mathbf{Cluster\ C}} \sum_{\mathbf{j} \in \mathbf{C}} \mathbf{c}_{\mathbf{i}_\mathbf{C},\mathbf{j}} + \sum_{\mathbf{Cluster\ C}} \mathbf{f}_{\mathbf{i}_\mathbf{C}}$$

$$\leq \mathbf{4} \cdot \mathbf{OPT}$$

QED

Facility location

Facility location

Primal

$$\min \sum_i \mathbf{f_i y_i} + \sum_{i,j} \mathbf{c_{ij} x_{ij}} :$$
$$\sum_i \mathbf{x_{ij}} \geq \mathbf{1} \quad \textbf{for all } \mathbf{j}$$
$$\mathbf{x_{ij}} \leq \mathbf{y_i} \quad \textbf{for all } \mathbf{i, j}$$
$$\mathbf{x_{ij}}, \mathbf{y_i} \geq \mathbf{0}$$

Dual

$$\max \sum_j \alpha_j :$$
$$\sum_j \beta_{ij} \leq \mathbf{f_i} \quad \textbf{for all } \mathbf{i}$$
$$\alpha_j \leq \beta_{ij} + \mathbf{c_{ij}} \quad \textbf{for all } \mathbf{i, j}$$
$$\alpha_j, \beta_{ij} \geq \mathbf{0}$$

1. Notation

$\mathbf{i}$ blocked: $\sum_{\mathbf{j}} \beta_{\mathbf{ij}} = \mathbf{f_i}$

$\mathbf{j}$ blocked: $\alpha_{\mathbf{j}} \geq \mathbf{c_{ij}}$ for some blocked $\mathbf{i}$

$$\max \sum_{\mathbf{j}} \alpha_{\mathbf{j}} :$$

$$\sum_{\mathbf{j}} \beta_{\mathbf{ij}} \leq \mathbf{f_i}$$

$$\alpha_{\mathbf{j}} \leq \beta_{\mathbf{ij}} + \mathbf{c_{ij}}$$

$$\alpha_{\mathbf{j}}, \beta_{\mathbf{ij}} \geq \mathbf{0}$$

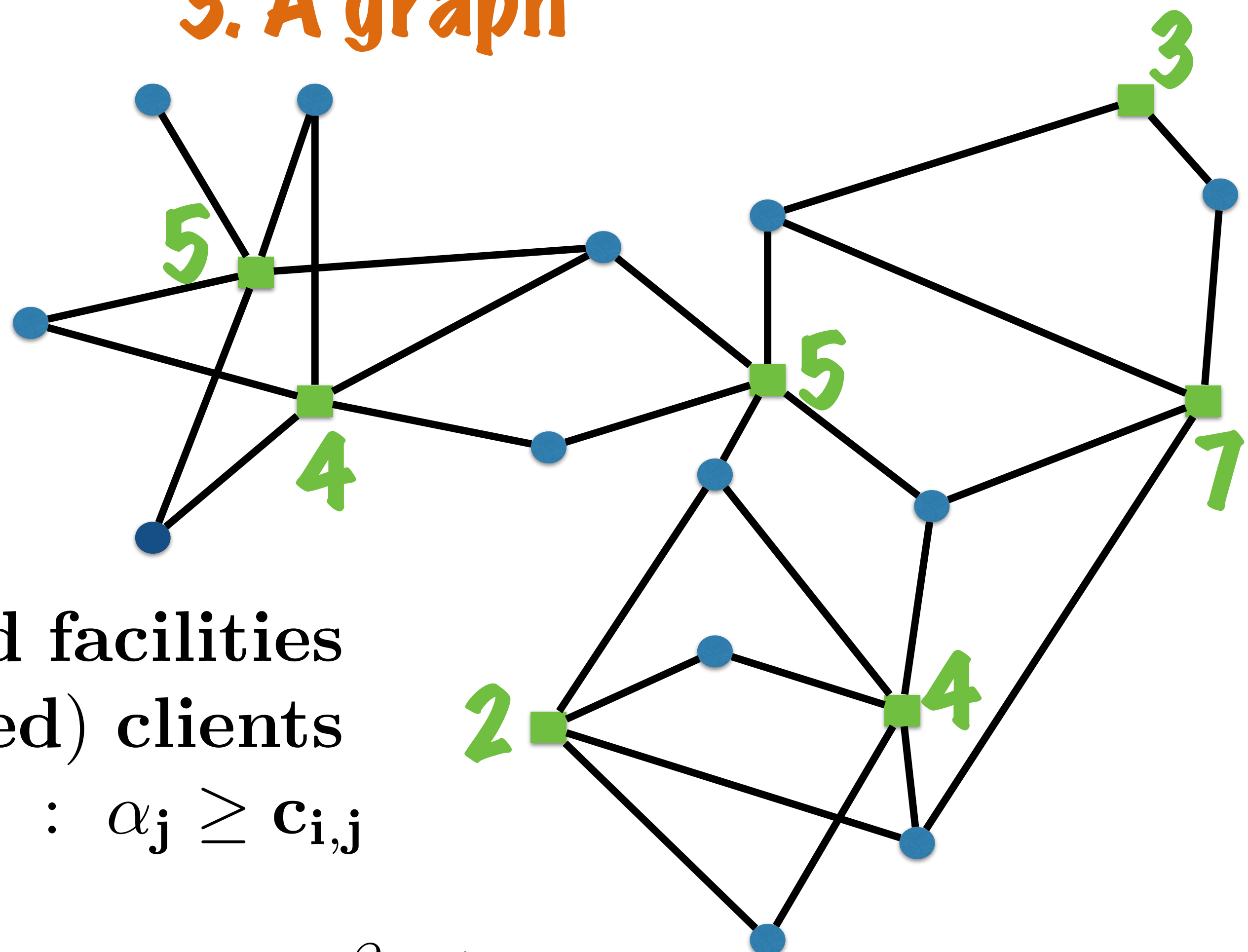
2. A dual solution that grows

$\max \sum_{\mathbf{j}} \alpha_{\mathbf{j}}$:
$\sum_{\mathbf{j}} \beta_{\mathbf{ij}} \leq \mathbf{f_i}$
$\alpha_{\mathbf{j}} \leq \beta_{\mathbf{ij}} + \mathbf{c_{ij}}$
$\alpha_{\mathbf{j}}, \beta_{\mathbf{ij}} \geq \mathbf{0}$

Initialization: $\alpha, \beta \leftarrow \mathbf{0}$
Repeat
in parallel:
raise every unblocked $\alpha_{\mathbf{j}}$
as well as every unblocked $\beta_{\mathbf{ij}}$ s.t.
$\alpha_{\mathbf{j}} \geq \mathbf{c_{ij}}$ for some unblocked $\alpha_{\mathbf{j}}$
Until every $\alpha_{\mathbf{j}}$ is blocked

Fact: $\alpha_{\mathbf{j}} \geq \mathbf{c_{ij}} \Longleftrightarrow \alpha_{\mathbf{j}} = \beta_{\mathbf{ij}} + \mathbf{c_{ij}}$

3. A graph



■ : **blocked facilities**

● : **(blocked) clients**

edge $\{\mathbf{i},\mathbf{j}\}$: $\alpha_{\mathbf{j}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}}$

edge $\{\mathbf{i},\mathbf{j}\} \implies \alpha_{\mathbf{j}} = \beta_{\mathbf{ij}} + \mathbf{c}_{\mathbf{ij}}$

4. Constructing a primal solution

Initialization: facilities are pending,
clients are unassigned

While some clients are unassigned:

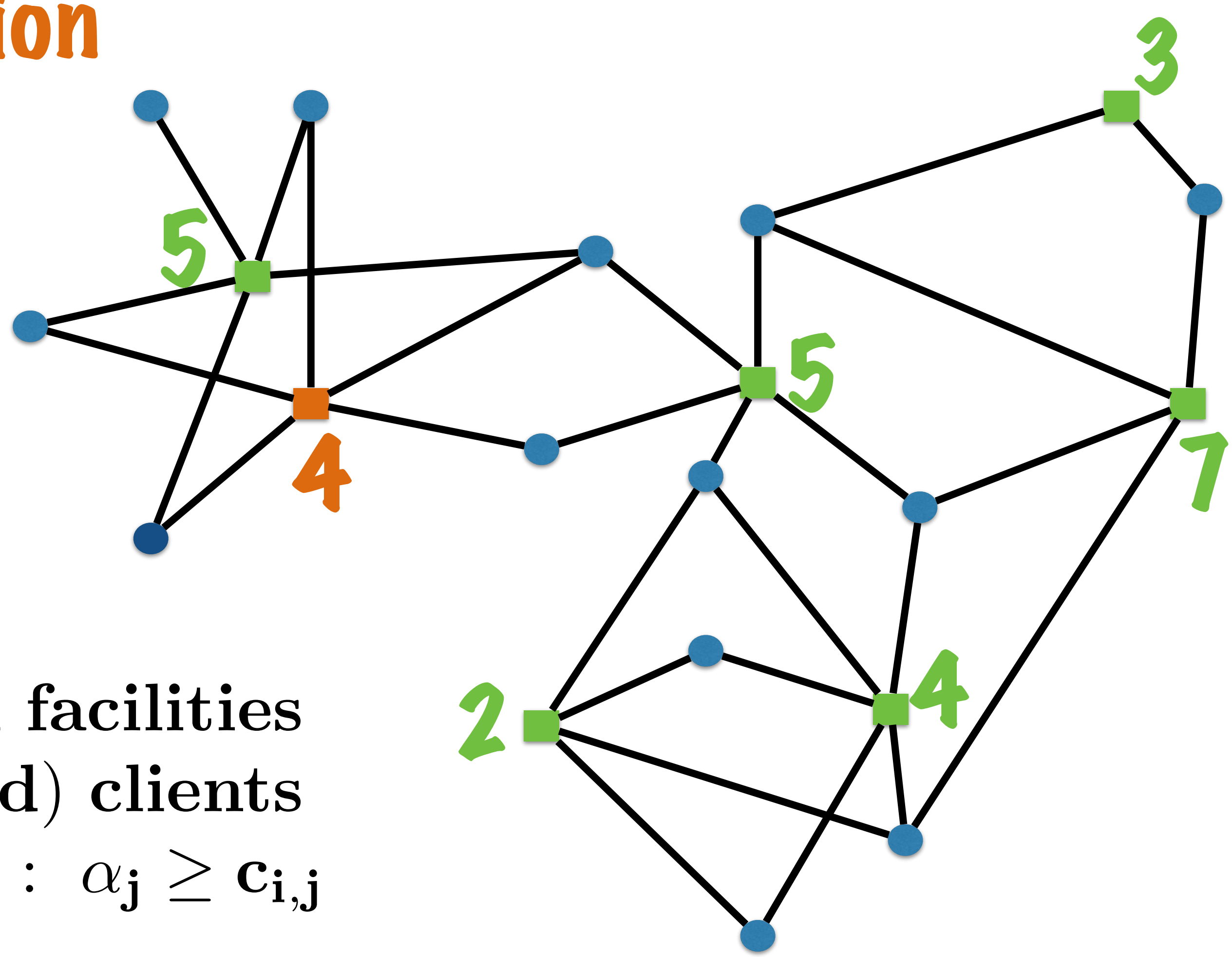
i_C: pending facility that was blocked first

open i_C

close pending facilities within distance 2

assign to i_C unassigned clients within distance 3

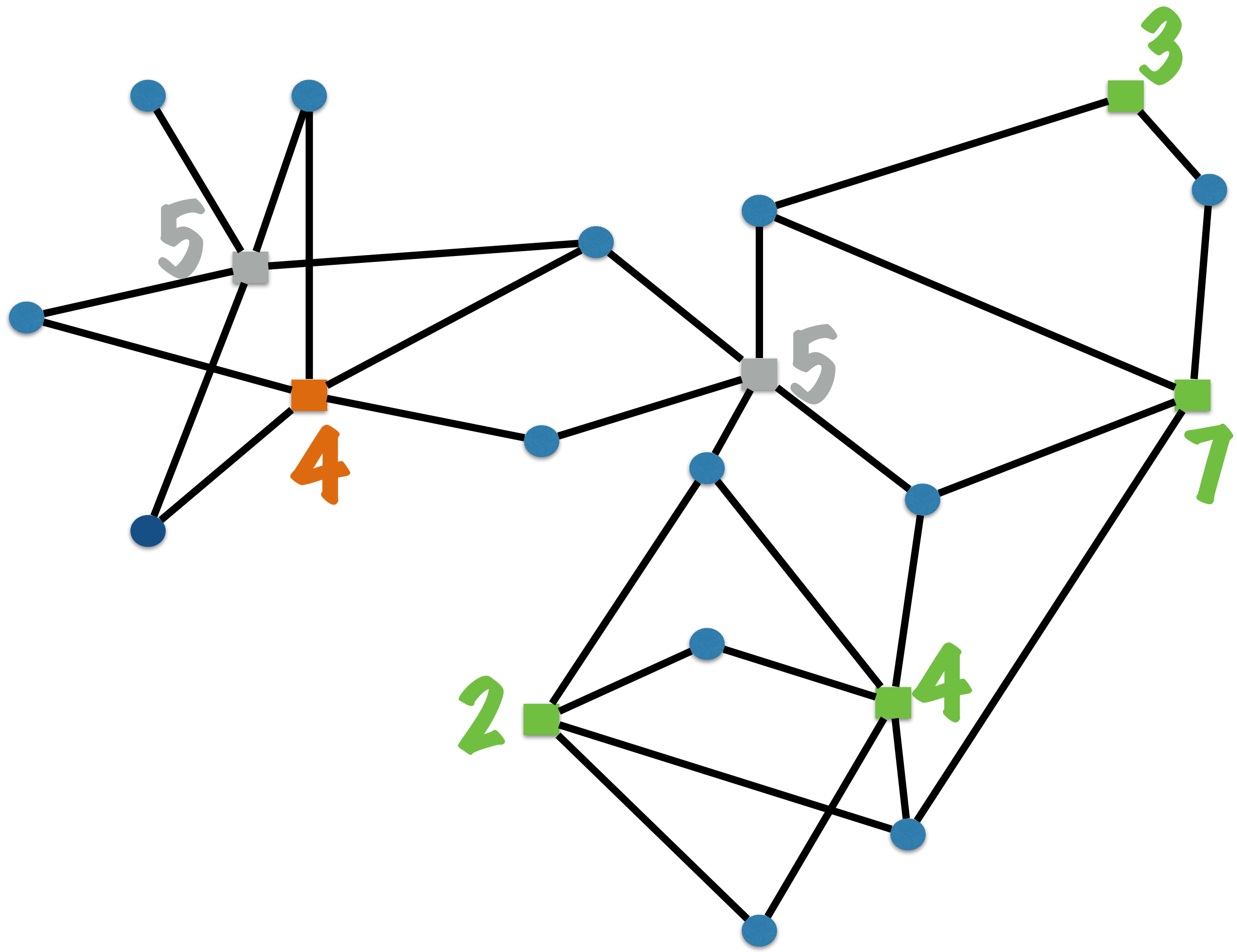
First iteration



■ : **blocked facilities**

● : **(blocked) clients**

edge $\{\mathbf{i},\mathbf{j}\}$: $\alpha_{\mathbf{j}} \geq \mathbf{c}_{\mathbf{i},\mathbf{j}}$


5
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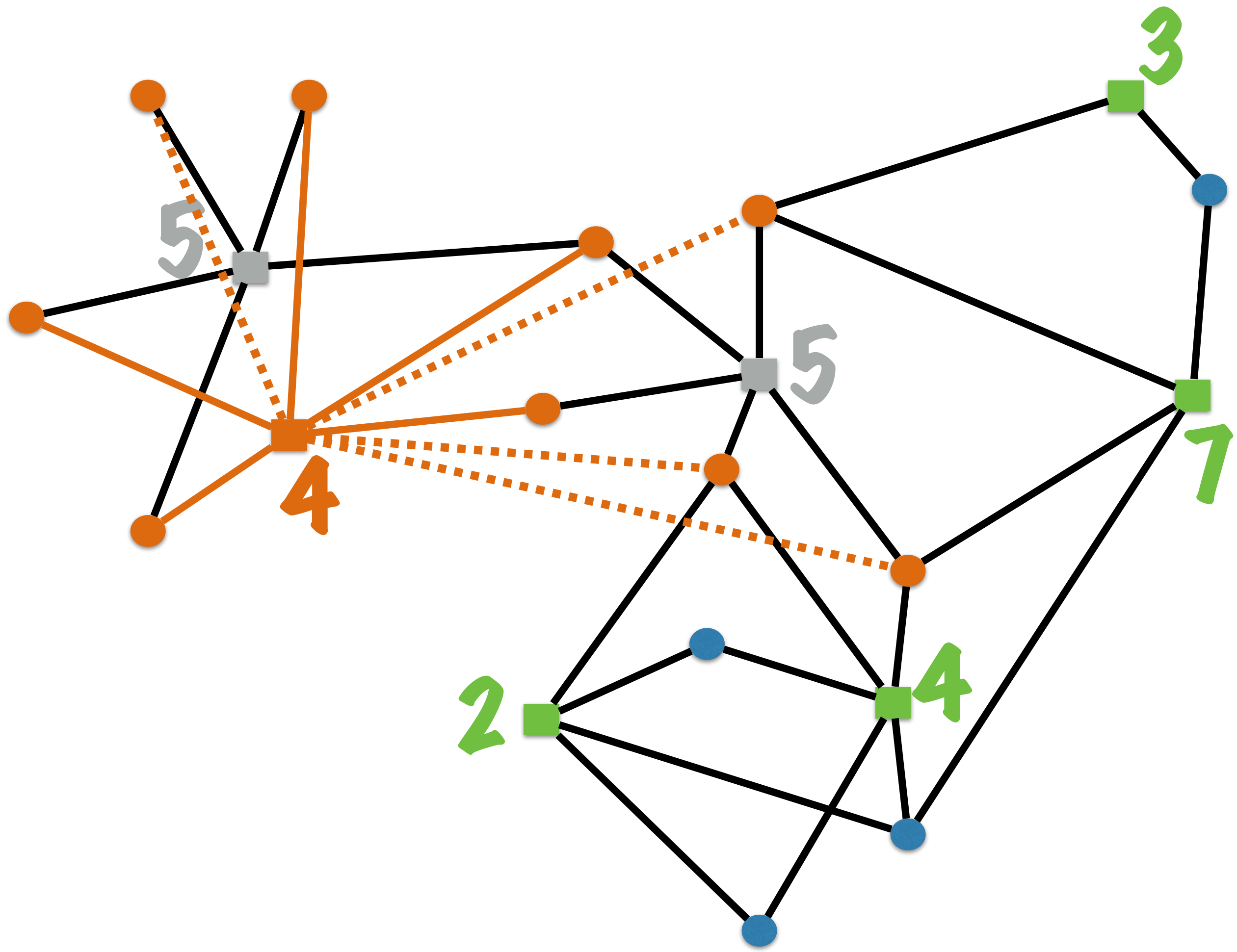
5
5
4
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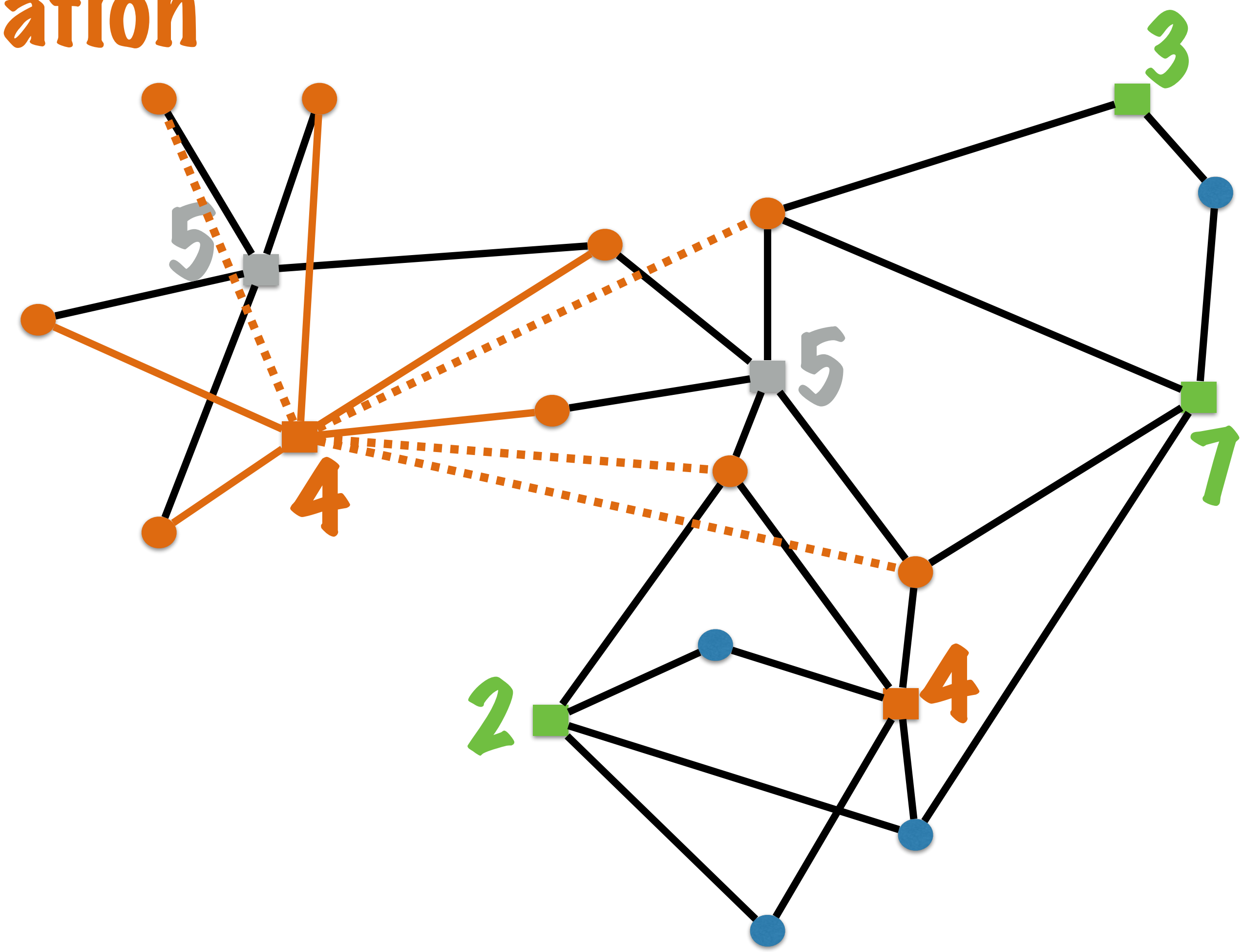
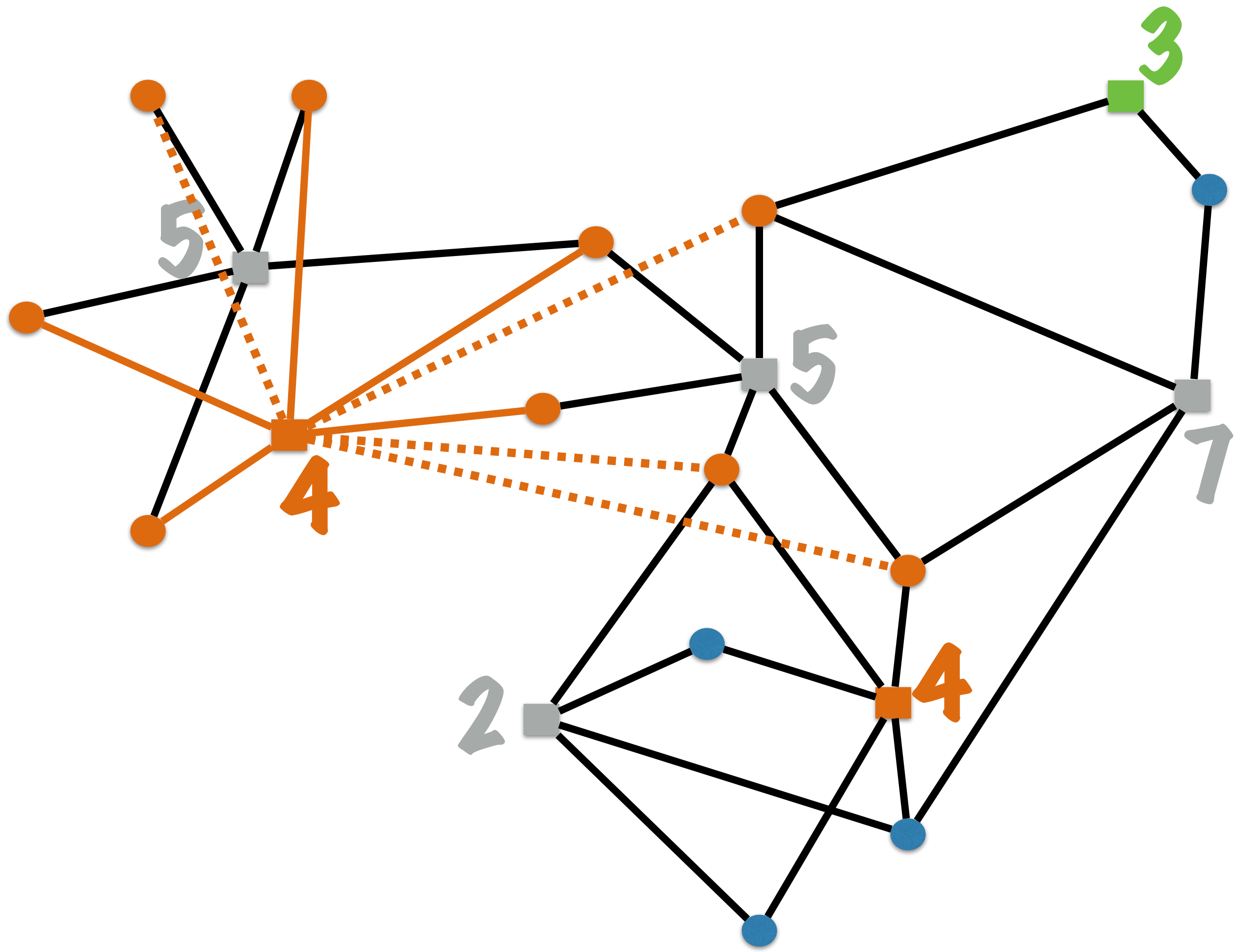
4

Second iteration




3
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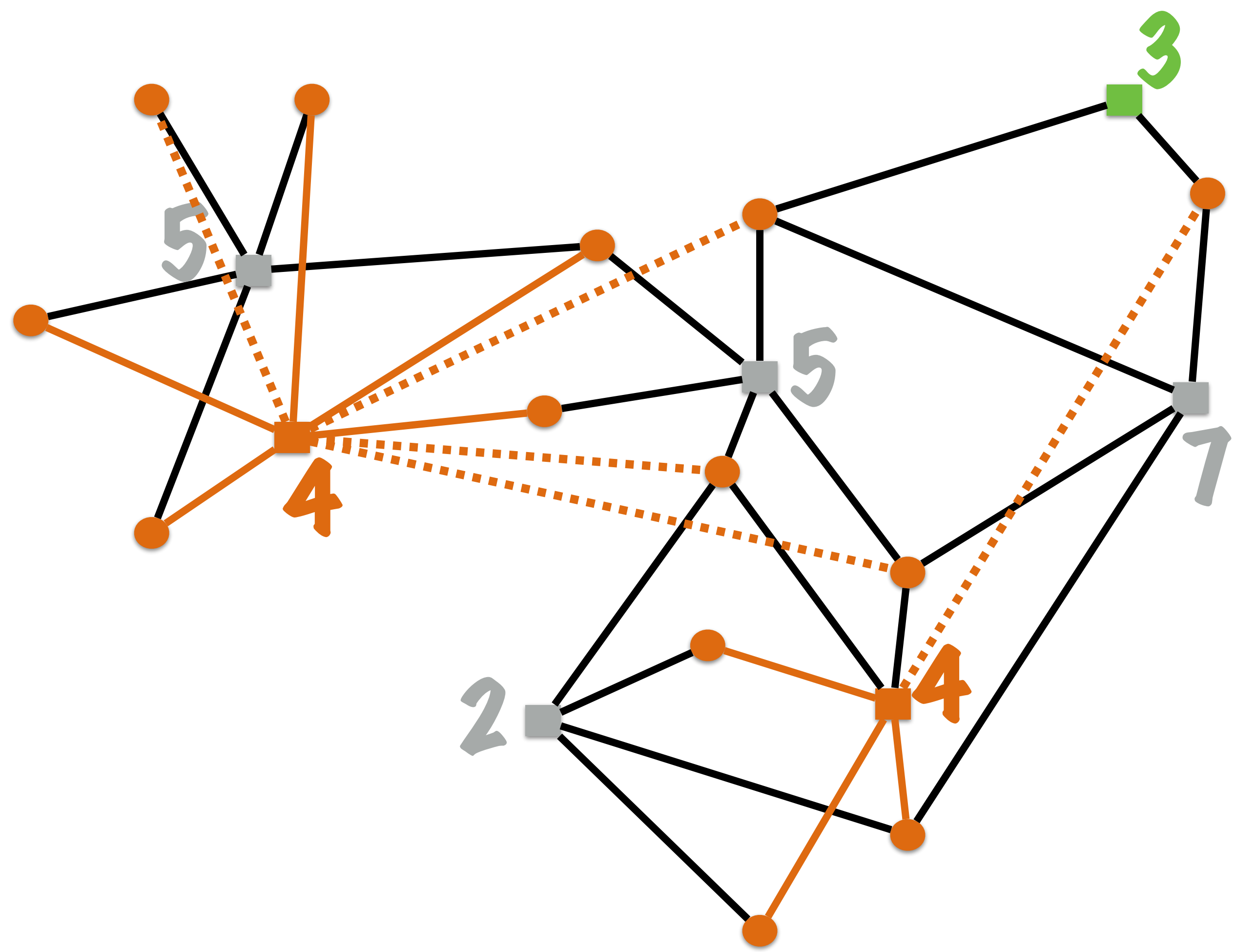
2
4


3
5
5
7
4
4
2

Facility location

Facility location

$\mathbf{i}$ **blocked:** $\sum_{\mathbf{j}} \beta_{\mathbf{ij}} = \mathbf{f_i}$

$\mathbf{j}$ **blocked:** $\alpha_{\mathbf{j}} \geq \mathbf{c_{ij}}$ **for some blocked** $\mathbf{i}$

$$\max \sum_{\mathbf{j}} \alpha_{\mathbf{j}} :$$
$$\sum_{\mathbf{j}} \beta_{\mathbf{ij}} \leq \mathbf{f_i}$$
$$\alpha_{\mathbf{j}} \leq \beta_{\mathbf{ij}} + \mathbf{c_{ij}}$$
$$\alpha_{\mathbf{j}}, \beta_{\mathbf{ij}} \geq \mathbf{0}$$

Initialization: $\alpha, \beta \leftarrow \mathbf{0}$

Repeat

in parallel,

raise every unblocked $\alpha_{\mathbf{j}}$

and every unblocked $\beta_{\mathbf{ij}}$ **s.t.**

$\alpha_{\mathbf{j}} \geq \mathbf{c_{ij}}$ **for some unblocked** $\alpha_{\mathbf{j}}$

Until every $\alpha_{\mathbf{j}}$ **is blocked**

Initialization: facilities are pending,
clients are unassigned

While some clients are unassigned:

i_C: pending facility that was blocked first

open i_C

close pending facilities within distance 2

assign to i_C unassigned clients within distance 3

$$\mathbf{edge}\ \{\mathbf{i},\mathbf{j}\} \implies \alpha_{\mathbf{j}} = \beta_{\mathbf{ij}} + \mathbf{c}_{\mathbf{ij}}$$

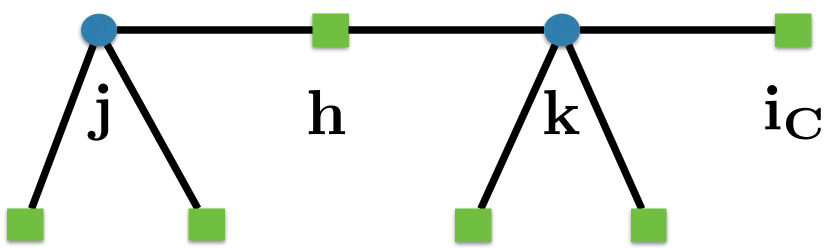
Analysis

Cost= $\sum_{\mathbf{Cluster\ C}} (\mathbf{f}_{\mathbf{i_C}} + \sum_{\mathbf{j} \in \mathbf{C}} \mathbf{c}_{\mathbf{i_C j}})$

$$\mathbf{f}_{\mathbf{i_C}} = \sum_{\mathbf{j\ adjacent\ to\ i_C}} \beta_{\mathbf{i_C j}}$$

$$\mathbf{f}_{\mathbf{i_C}} + \sum_{\mathbf{j\ adjacent\ to\ i_C}} \mathbf{c}_{\mathbf{i_C j}} =$$
$$\sum_{\mathbf{j\ adjacent\ to\ i_C}} \beta_{\mathbf{i_C j}} + \mathbf{c}_{\mathbf{i_C j}} =$$
$$\sum_{\mathbf{j\ adjacent\ to\ i_C}} \alpha_{\mathbf{j}}$$

Clients at distance 3 from $\mathbf{i_C}$



$$\alpha_{\mathbf{j}} \geq \mathbf{c_{hj}}$$
$$\alpha_{\mathbf{k}} \geq \mathbf{c_{hk}}$$
$$\alpha_{\mathbf{k}} \geq \mathbf{c_{i_C k}}$$

$$\mathbf{c_{i_C j}} \leq \mathbf{c_{hj}} + \mathbf{c_{hk}} + \mathbf{c_{i_C k}} \leq \alpha_{\mathbf{j}} + \mathbf{2}\alpha_{\mathbf{k}}$$

Since $\mathbf{i_C}$ was blocked first among contenders:

$$\alpha_{\mathbf{k}} \leq \alpha_{\mathbf{j}}$$

And so: $\mathbf{c_{i_C j}} \leq \mathbf{3} \cdot \alpha_{\mathbf{j}}$

Together:

Cost= $\sum_{\mathbf{Cluster\ C}}(\mathbf{f_{i_C}} + \sum_{\mathbf{j\in C}} \mathbf{c_{i_C j}})$

$$\leq \sum_{\mathbf{Cluster\ C}}(\sum_{\mathbf{j\in C, j\sim i_C}} \alpha_{\mathbf{j}} + \sum_{\mathbf{j\in C, d(j,i_C)=3}} \mathbf{3}\alpha_{\mathbf{j}})$$

$$\leq \sum_{\mathbf{Cluster\ C}} \sum_{\mathbf{j\in C}} \mathbf{3}\alpha_{\mathbf{j}}$$

$$\leq \mathbf{3} \sum_{\mathbf{j}} \alpha_{\mathbf{j}}$$

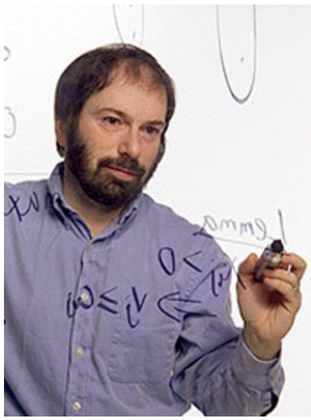
$$\leq \mathbf{3} \cdot \mathbf{OPT}$$

QED

Facility location

Facility location

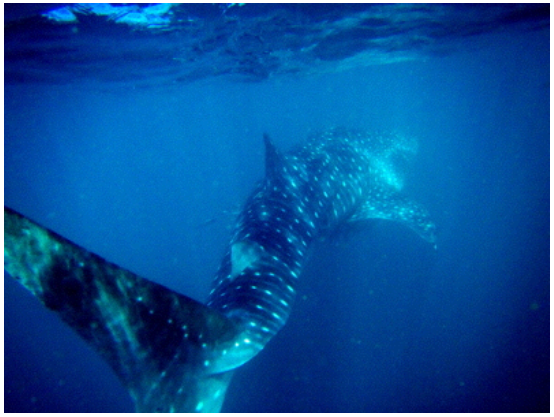
Michel Balinski
LP relaxation (1963)

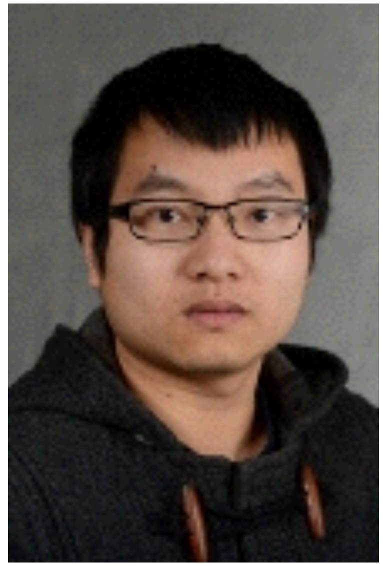
David Shmoys, Eva Tardos, Karen Aardal
4 (really, 3.16) approx (1997)

Kamal Jain
Vijay Vazirani
3 approx (2001)

Hardness



Sudipto Guha, Samir Khuller
Under some Complexity assumption,
lower bound of 1.463...



Shi Li
1.488 approx (2011)

Facility location