An induced space of chance-driven outcomes

Sum of face values	Aggregate pairs	Probability
2	(1,1)	1/36
3	(2,1), (1,2)	2/36
4	(3,1), (2,2), (1,3)	3/36
5	(4,1), (3,2), (2,3), (1,4)	4/36
6	(5,1), (4,2), (3,3), (2,4), (1,5)	5/36
7	(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)	6/36
8	(6,2), (5,3), (4,4), (3,5), (2,6)	5/36
9	(6,3), (5,4), (4,5), (3,6)	4/36
10	(6,4), (5,5), (4,6)	3/36
11	(6,5), (5,6)	2/36
12	(6,6)	1/36

Event	Aggregate outcomes
Win	{7, 11}
Lose	{2, 3, 12}
Continue	{4, 5, 6, 8, 9, 10}

The hugely important principle of additivity:
possibilities add when they are mutually exclusive.

Probabilities:

$$\mathbf{P}\{\text{Win}\} = \mathbf{P}\{7\} + \mathbf{P}\{11\} = \tfrac{6}{36} + \tfrac{2}{36} = \tfrac{2}{9}$$

$$\mathbf{P}\{\text{Lose}\} = \mathbf{P}\{2\} + \mathbf{P}\{3\} + \mathbf{P}\{12\} = \tfrac{1}{36} + \tfrac{2}{36} + \tfrac{1}{36} = \tfrac{1}{9}$$

$$\mathbf{P}\{\text{Continue}\} = 2\left(\tfrac{3}{36} + \tfrac{4}{36} + \tfrac{5}{36}\right) = \tfrac{2}{3}$$

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$$\mathbf{P}\{\geq 4 \text{ tosses}\} = \big(\mathbf{P}\{\text{HTHH}\} + \mathbf{P}\{\text{THTT}\}\big) + \big(\mathbf{P}\{\text{THTHH}\} + \mathbf{P}\{\text{HTHTT}\}\big) + \big(\mathbf{P}\{\text{HTHTHH}\} + \mathbf{P}\{\text{THTHTT}\}\big) + \cdots$$

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$$= 2\left(\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots\right) = 2 \cdot \frac{1}{16}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = 2 \cdot \frac{1}{16} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{4}$$

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Geometric series

Recall: $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$ if $-1 < x < 1$.