Sums of independent variables

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Partial sum	Expectation	Variance

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$S_1 = X_1$	$\mathbb{E}(S_1) = \mathbb{E}(X_1) = \mu$	$\mathrm{Var}(S_1) = \mathrm{Var}(X_1) = \sigma^2$

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$\vdots$	$\vdots$	$\vdots$
$S_n = S_{n-1} + X_n$	$\mathsf{E}(S_n) = \mathsf{E}(S_{n-1}) + \mathsf{E}(X_n) = n\mu$	$\mathrm{Var}(S_n) = \mathrm{Var}(S_{n-1}) + \mathrm{Var}(X_n) = n\sigma^2$

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Based on our experience with the binomial, we may anticipate that S_n/n concentrates near μ