Divisibility

When an integer a is divisible by an integer b? A naive answer will be that we should consider the rational number $\frac{a}{b}$ and if it is an integer, then a is divisible by b. However, this definition refers to a more complex concept of rational numbers. Let us unwrap this explanation and try to understand what we are actually trying to say.

What does it mean that $\frac{a}{b}$ is integer? It means that the denominator cancels out. In other words, we can represent a as a product of b and some integer k: $a = bk$. Then we have $\frac{a}{b} = \frac{bk}{k} = k$. Now, this reformulation only uses a simple notion of multiplication and does not refer to rational numbers. As a result, we arrive at the following definition.

Definition. An integer number a is divisible by an integer number b (or in other words, b divides a), denoted by $b \mid a$, if there is an integer k such that $a = bk$. If a is not divisible by b, we denote this by $b \nmid a$.

The intuition behind this definition is simple (for positive a and b). Suppose we have a objects and we want to split them into equal groups of size b. This is possible if and only if a is divisible by b, and k is the number of the resulting groups.

Problem. Is 15 divisible by 3? Is it divisible by 4? Is it divisible by -5?

For the first question, the answer is positive: we can pick $k = 5$ and have $15 = 3k$. For the second question, the answer is negative: if we pick $k = 3$, we have $4k = 12$, which is too small, and if we pick the next larger integer $k = 4$, we already have $4k = 16$, which is too large. Thus, there is no integer k such that $15 = 4k$.

For the third question, the answer is positive. Indeed, we can pick $k = -3$ and have $15 = (-3) \cdot (-5)$. Note, that k is allowed to be negative (as well as a and b).

Problem. Is -24 divisible by -6? Is it divisible by -5?

The answer to the first question is positive, we can pick $k = 4$ and have $-24 = 4 \cdot (-6)$. The answer to the second question is negative: if we pick $k = 4$, we have $-5k = -20$, which is too large, and if we pick $k = 5$ we have $-5k = -25$, which is already too small.

This is how one checks divisibility in Python.

```
print(15 % 3 == 0)
print(15 % 4 == 0)
print(24 % 6 == 0)
print(-24 % -6 == 0)
```

Run

Reset

```
True
False
True
True
```

Formal definition of divisibility might look strange, indeed, everything seems to be obvious for specific numbers. However, formal definitions allow us to formally prove general properties.

Lemma. If c divides a and b, then c divides $a \pm b$.

Proof. Since c divides a, there is k_1 such that $a = ck_1$. Similarly, there is k_2 such that $b = ck_2$. Then

$$a \pm b = ck_1 \pm ck_2 = c(k_1 \pm k_2).$$

By definition, this means that $a \pm b$ is divisible by c.

Lemma. If $b \mid a$, then for any integer c, we have $b \mid (a \cdot c)$.