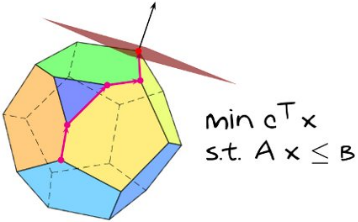
Linear programming

- Linear algebra vs. linear optimization
- Fitting a line
- Classification



Linear algebra vs. linear programming

Solving a linear system

Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, determine $x \in \mathbb{R}^n$ such that $Ax = b$ or assert that such an x does not exist.

- Gaussian algorithm

$$\max c^T \cdot x \quad \text{s.t. } A \cdot x = b \qquad \begin{cases} Ax \le b \\ -Ax \le -b \end{cases}$$

LP

$\vee$ |

Solving systems of linear equations

Kernel and image

$A \in \mathbb{R}^{m \times n}$

$\ker(A) = \{ x \in \mathbb{R}^n : A \cdot x = 0 \} \subseteq \mathbb{R}^n$

$\mathrm{im}(A) = \{ A \cdot x : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

Quiz

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The linear program

$b \in \mathrm{im}(A) \Rightarrow \exists x^* \in \mathbb{R}^n$

$$\max\{c^T x : x \in \mathbb{R}^n, Ax = b\}$$ s.t. $A \cdot x^* = b$

$A \cdot (x^* + \underset{\in \mathbb{R}}{\lambda} \cdot c) = \underbrace{A \cdot x^*}_{=b} + \underbrace{\lambda \cdot A \cdot C}_{=0} = b$

is feasible and unbounded if

$b \in \mathbb{R}^m$, $\ker(A) \subseteq \mathbb{R}^n$

- $b \in \ker(A)$
- $b \in \mathrm{im}(A)$
- $b \in \mathrm{im}(A)$ and $c \in \ker(A) \setminus \{0\}$

$\Pi \in \mathbb{R}$:

$c^T (x^* + \lambda \cdot c)$

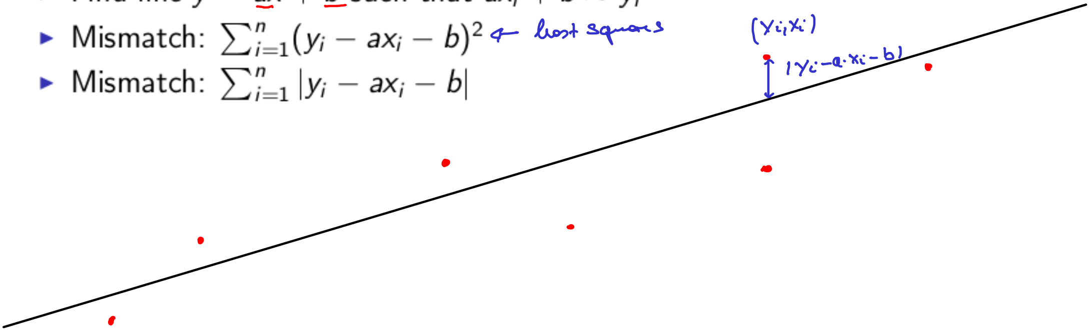
$= c^T \cdot x^* + \lambda \cdot \underbrace{c^T \cdot c}_{>0}$

$\lambda \cdot c^T \cdot c + c^T \cdot x^* > \Pi$

$\Leftrightarrow \quad \lambda > \dfrac{\Pi - c^T \cdot x^*}{c^T \cdot c}$

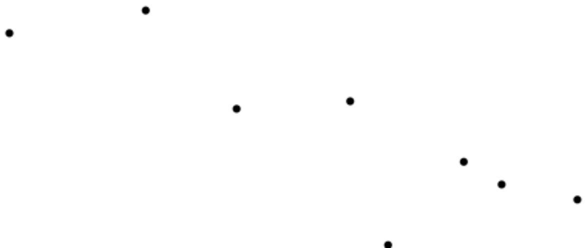
Fitting a line

- Given points $(y_i, x_i) \in \mathbb{R}^2$ $i = 1, \ldots, n$
- Find line $y = ax + b$ such that $ax_i + b \approx y_i$
- Mismatch: $\sum_{i=1}^{n} (y_i - ax_i - b)^2$ ← least squares
- Mismatch: $\sum_{i=1}^{n} |y_i - ax_i - b|$



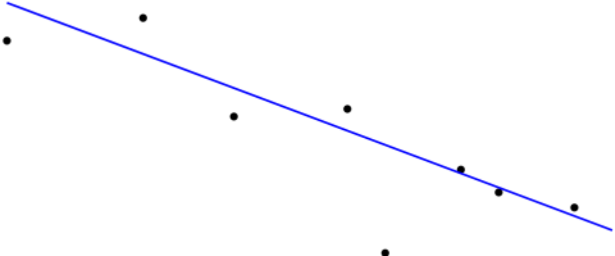
Fitting a line

$$\min \sum_{i=1}^{n} |y_i - \underline{a}x_i - \underline{b}|$$
$$a, b \in \mathbb{R}$$



Fitting a line

$$\min_{a, b \in \mathbb{R}} \sum_{i=1}^{n} |y_i - ax_i - b|$$

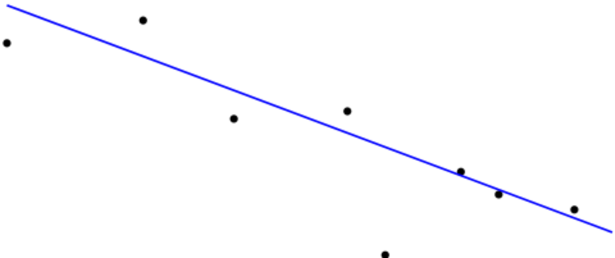


Fitting a line

$$\min \sum_{i=1}^{n} |y_i - ax_i - b| \quad a, b \in \mathbb{R}$$

Idea: Model absolute value $|y_i - ax_i - b|$ as smallest h_i satisfying

$$\begin{aligned} h_i &\geqslant y_i - ax_i - b \\ h_i &\geqslant -(y_i - ax_i - b) \end{aligned}$$



Fitting a line

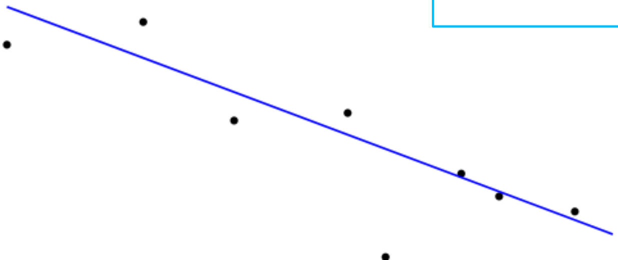
$$\min \sum_{i=1}^{n} |y_i - ax_i - b| \quad a, b \in \mathbb{R}$$

Idea: Model absolute value $|y_i - ax_i - b|$ as smallest h_i satisfying

$$\begin{aligned} h_i &\geqslant y_i - ax_i - b \\ h_i &\geqslant -(y_i - ax_i - b) \end{aligned}$$

$$\begin{array}{lll} \min & \sum_{i=1}^{n} \underline{h_i} & \\ \text{s.t.:} & h_i \geqslant y_i - \underline{a}x_i - \underline{b}, & i = 1, \ldots, n \\ & h_i \geqslant -y_i + ax_i + b, & i = 1, \ldots, n \end{array}$$

<u>VARS:</u> $h_1, \ldots, h_n, a, b$



Fitting a line

Quiz: In example below, we have 8 points. How many variables?

- 8
- 9
- 10

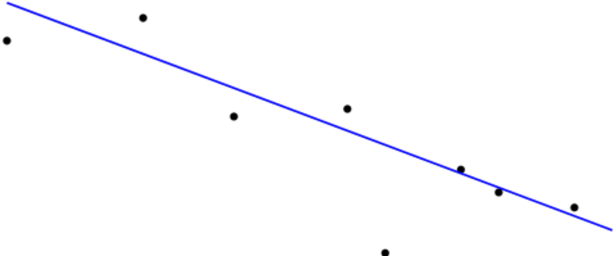
Idea: Model absolute value $|y_i - ax_i - b|$ as smallest h_i satisfying

$$\begin{aligned} h_i &\geqslant y_i - ax_i - b \\ h_i &\geqslant -(y_i - ax_i - b) \end{aligned}$$

$h_1, \ldots, h_8, a, b$

$$\begin{array}{lll} \min & \sum_{i=1}^{n = 8} h_i & \\ \text{s.t.:} \quad h_i \geqslant y_i - ax_i - b, & i = 1, \ldots, n \\ \phantom{\text{s.t.:}} \quad h_i \geqslant -y_i + ax_i + b, & i = 1, \ldots, n \end{array}$$

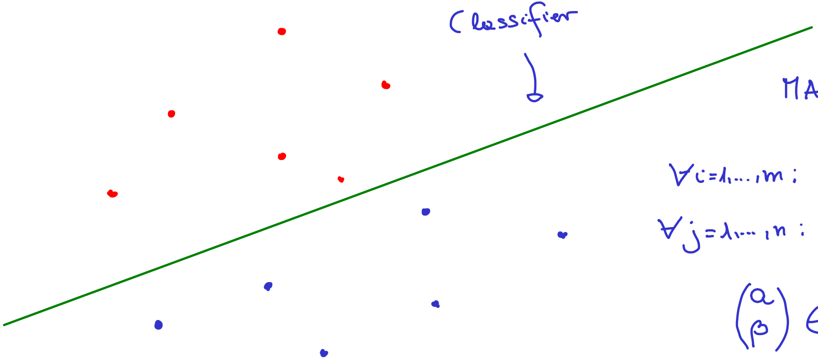
alltogether: 10 VARS.



Classification

- Given m red points $x_1, \ldots, x_m \in \mathbb{R}^k$ and n blue points $y_1, \ldots, y_n \in \mathbb{R}^k$
- Determine $\underline{a \in \mathbb{R}^k \text{ and } \beta} \in \mathbb{R}$ such that

$$a^T x_i \underline{> \beta,} \quad i = 1, \ldots, m \quad \text{and} \quad \underline{a^T y_j < \beta,} \quad j = 1, \ldots, n$$



$$\text{MAX } 0^T \cdot \begin{pmatrix} a \\ \beta \end{pmatrix}$$

$$\forall i = 1, \ldots, m: \quad a^T \cdot x_i \geq \beta + 1$$

$$\forall j = 1, \ldots, n: \quad a^T \cdot y_j \leq \beta - 1$$

$$\begin{pmatrix} a \\ \beta \end{pmatrix} \in \mathbb{R}^{k+1}$$

The LP is feasible if and only if there exists a classifier.

"$\Rightarrow$" ✓

"$\Leftarrow$" Let $a \in \mathbb{R}^k$, $\beta \in \mathbb{R}$ be a classifier

$(\ast \frac{1}{\varepsilon})$

$i = 1, \dots, m: \quad a^T \cdot x_i \geq \beta + \varepsilon$ $\quad \exists \varepsilon > 0$

$j = 1, \dots, n: \quad a^T \cdot y_j \leq \beta - \varepsilon \leftarrow (\ast \frac{1}{\varepsilon})$

$i = 1, \dots, m: \quad \frac{1}{\varepsilon} a^T \cdot x_i \geq \frac{1}{\varepsilon} \cdot \beta + 1$

$j = 1, \dots, n: \quad \frac{1}{\varepsilon} a^T \cdot y_j \leq \frac{1}{\varepsilon} \beta - 1$

$\Rightarrow \begin{pmatrix} \frac{1}{\varepsilon} a \\ \frac{1}{\varepsilon} \beta \end{pmatrix}$ feasible sol. of LP

Quiz

$\max \; 0$ $\quad -x_i^T \cdot a + \beta \le -1$

$a^T \cdot x_i \ge \beta + 1 \quad i = 1, \dots, m$

$a^T \cdot y_j \le \beta - 1 \quad j = 1, \dots, n$

$y_j^T \cdot a - \beta \le -1$

MAX $c^T \cdot x$

$Ax \le b$

$x \in \mathbb{R}^n$

What is A in matrix formulation

$\max \{ c^T \cdot z : z \in \mathbb{R}^{k+1}, A z \le b \}$

$$\begin{pmatrix} x_1^T & -1 \\ \vdots & \\ x_m^T & -1 \\ y_1^T & -1 \\ \vdots & \\ y_n^T & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -x_1^T & 1 \\ \vdots & \\ -x_m^T & 1 \\ y_1^T & -1 \\ \vdots & \\ y_n^T & -1 \end{pmatrix}$$