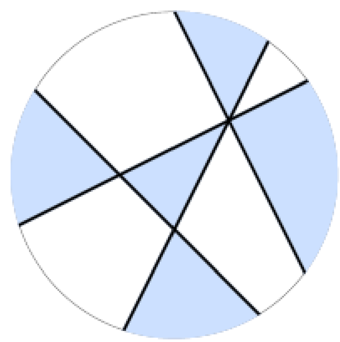
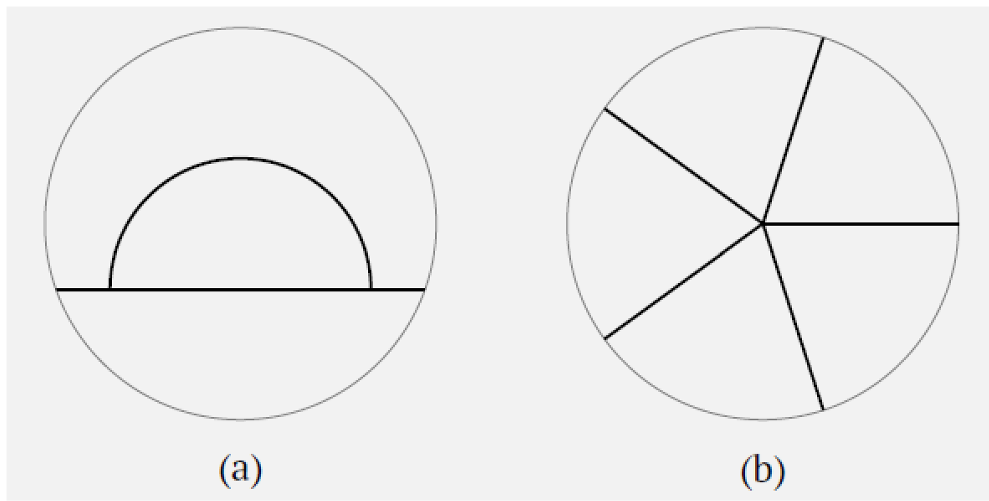
Plane Coloring

Problem

The plane is partitioned into several regions by straight (infinitely long) lines. Prove that the plane can be colored in just two colors so that neighboring regions have different colors.

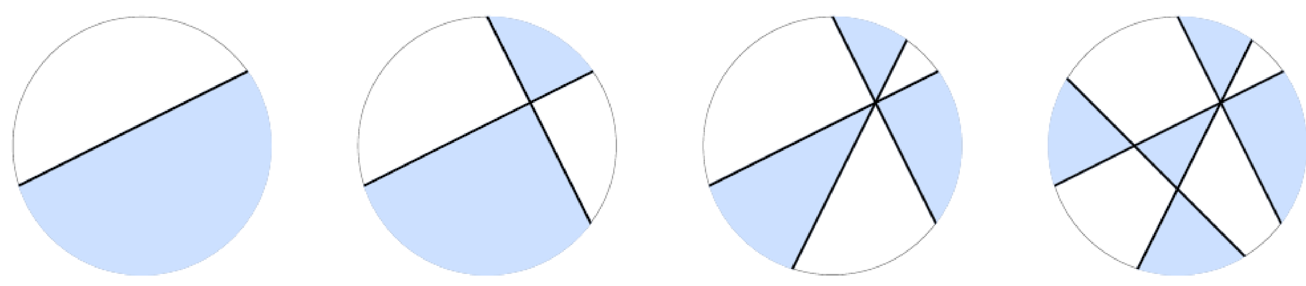


Note that it is crucial that all lines in this problem are *straight* and *infinitely long*. Indeed, the figure below shows that if the lines are not straight (a), or if the lines are not infinitely long in both directions (b), then more than two colors may be needed.



In fact, for regions bounded by any (for example, curvy and finite) lines, one can always use just four colors. This is the famous four color theorem, which to this day only has proofs involving a large computer search!

Let us apply mathematical induction to the plane coloring problem. We want to show that regions formed by n lines can be colored in two colors so that neighboring regions have different colors. We prove this by induction on n. For the base case $n = 1$, we can take a coloring where the two regions have different colors. For the induction step from n to $n + 1$, let us assume that we have a proper coloring of the plane with n lines, and that we are adding one more line L, see the picture:



An example of the inductive coloring from the proof. After adding a new line L, we keep colors of all regions on one side of L, and switch colors of all regions on the other side of L.

Now, all regions on one side of L keep their colors, while all regions on the other side of L switch their colors. Consider two neighboring regions R_1 and R_2. If their common border is not a part of L, then they had different colors in the old coloring, and they still have different colors in the new coloring. On the other hand, if their common border belongs to L, then they used to be one region and had the same color, but one of the regions switched color.

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