The ballot problem

W. A. Whitworth, *Messenger of Math*, 1878

In a run-off election between two candidates for class prefect, Jane gets 6 votes and Bob gets 4 votes. The ballots are counted in random order. What is the chance that Jane leads Bob at every step of the count?

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Jane's running total	1	1	2	2	2	3	4	5	5	6
Bob's running total	0	1	1	2	3	3	3	3	4	4

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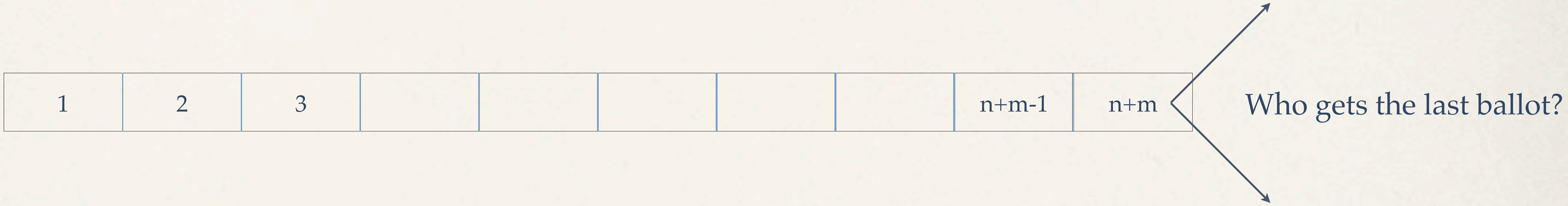

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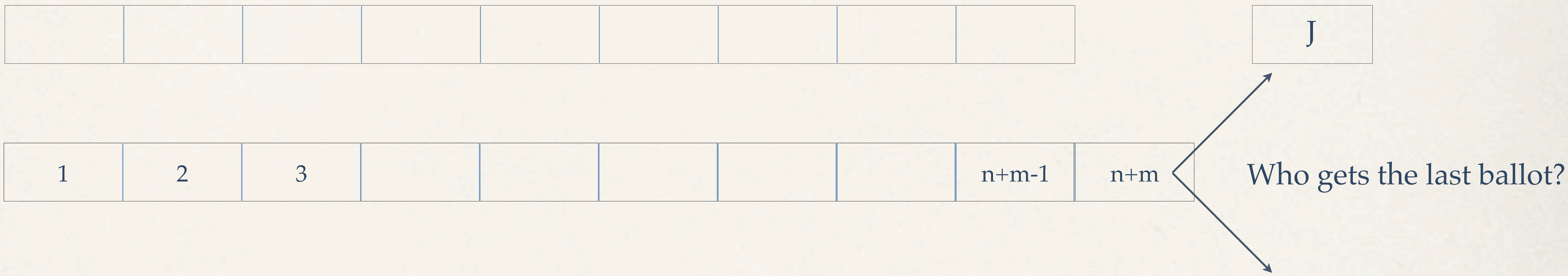

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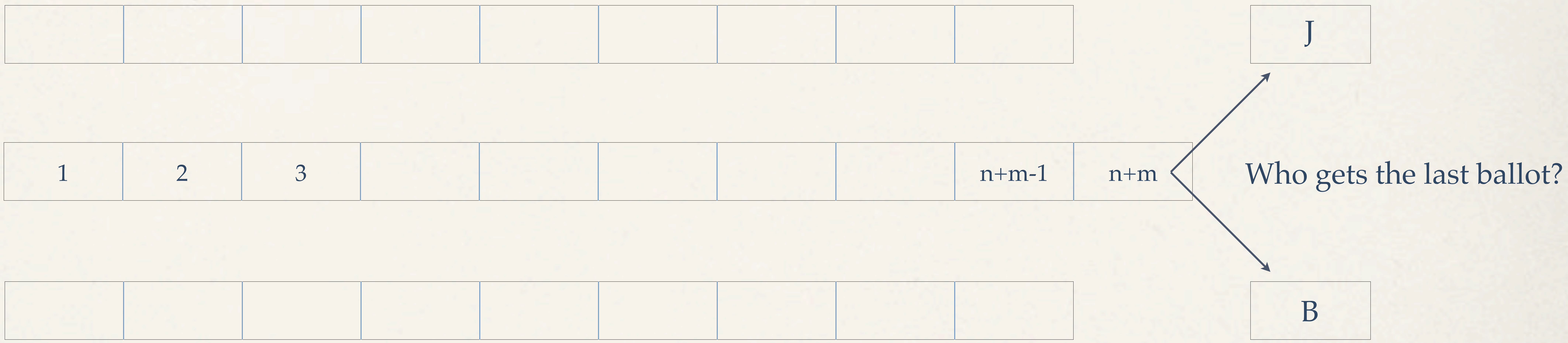

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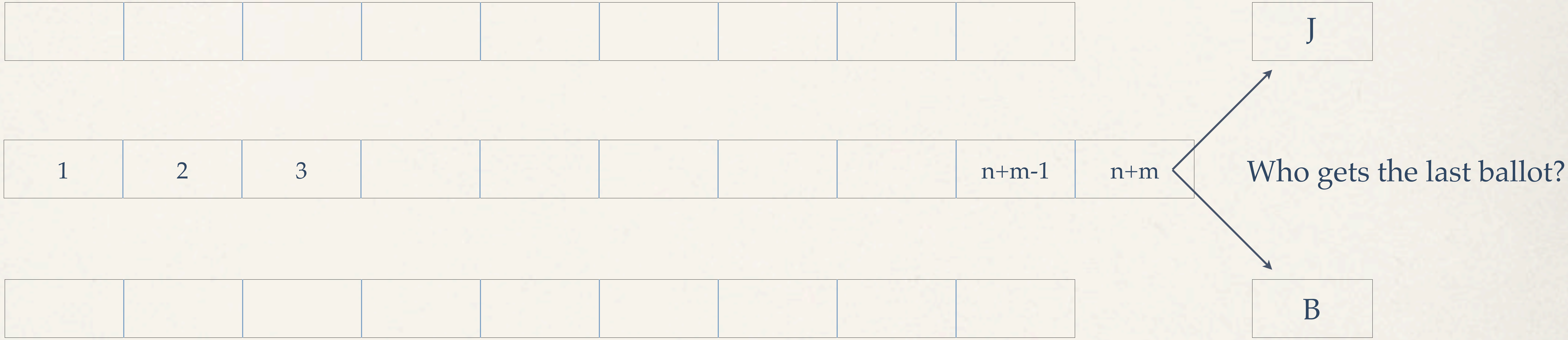

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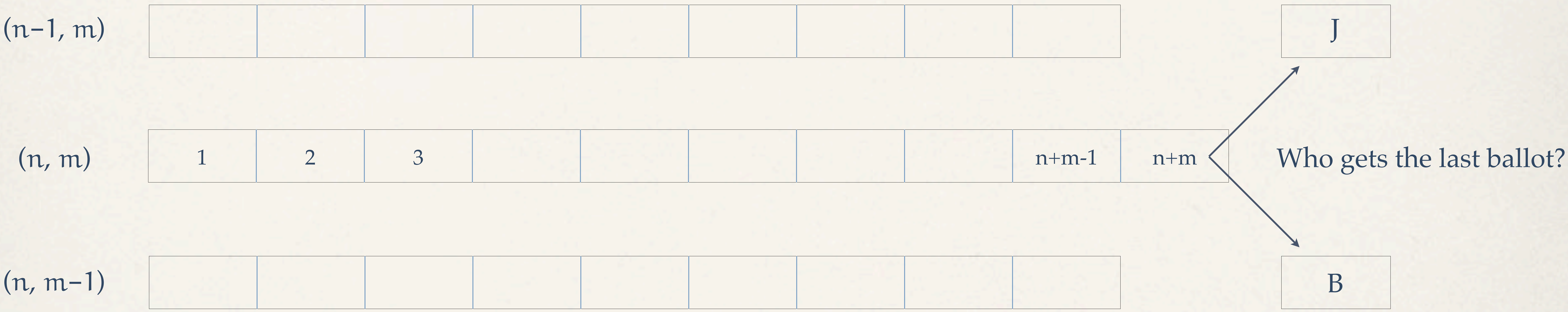
Ballots for: (Jane, Bob)



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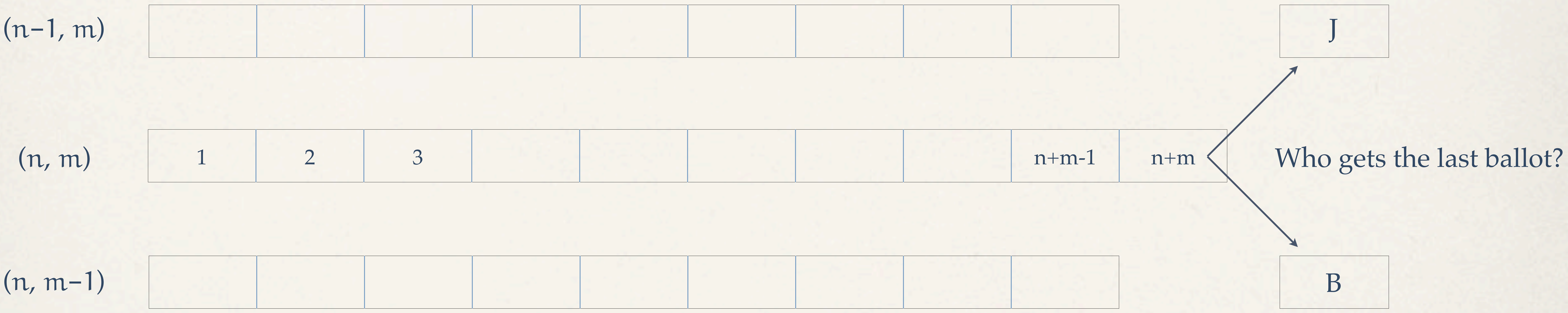

$\mathbf{P}(H) =: P_{n,m}$

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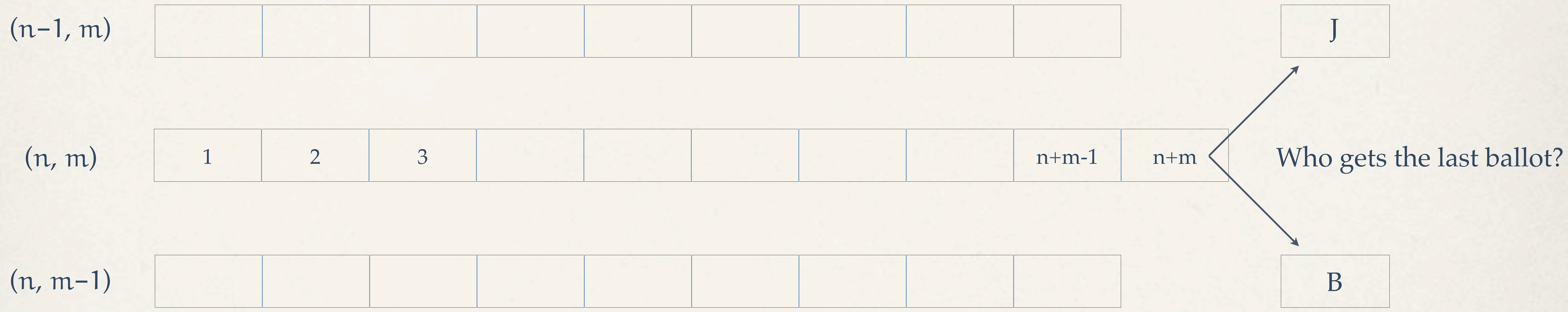

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$$\mathbf{P}(H \mid A) = P_{n-1,m}$$

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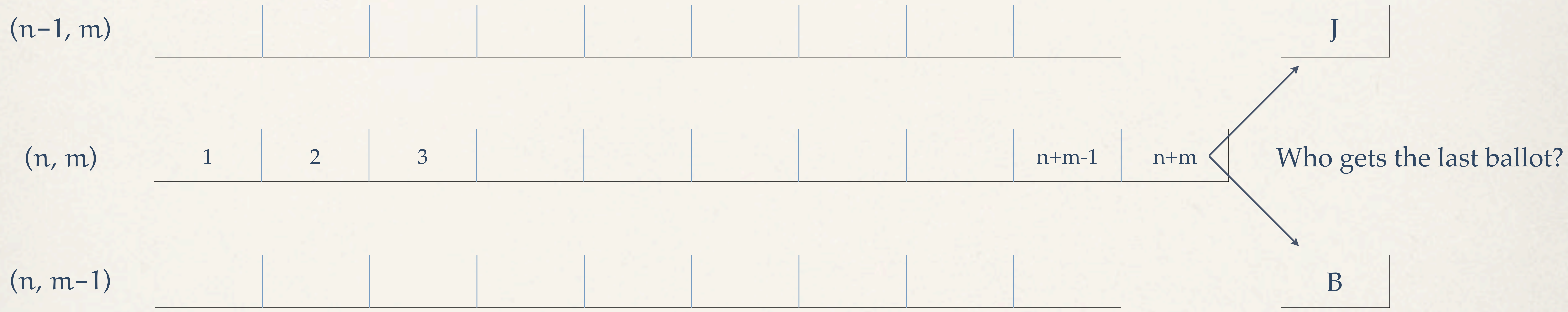

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$\mathbf{P}(H \mid A^{c}) = P_{n,m-1}$

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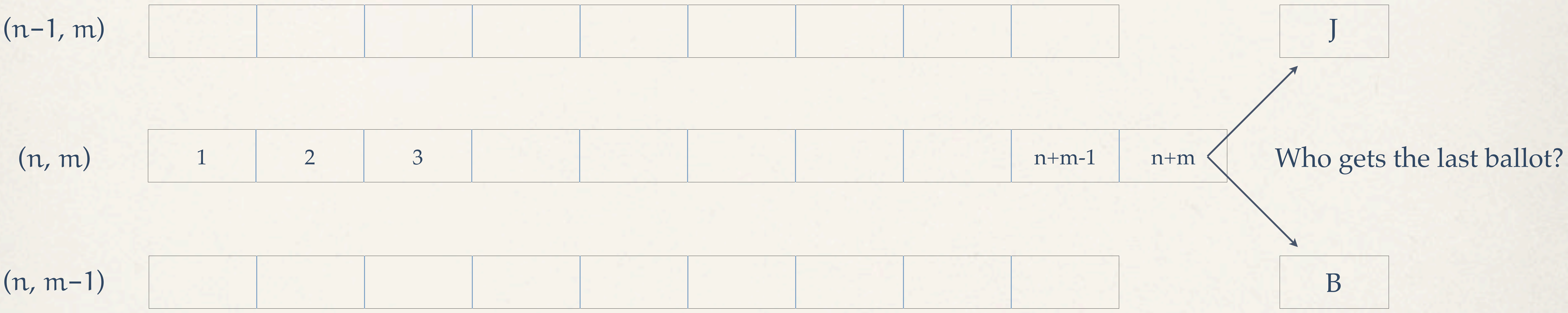

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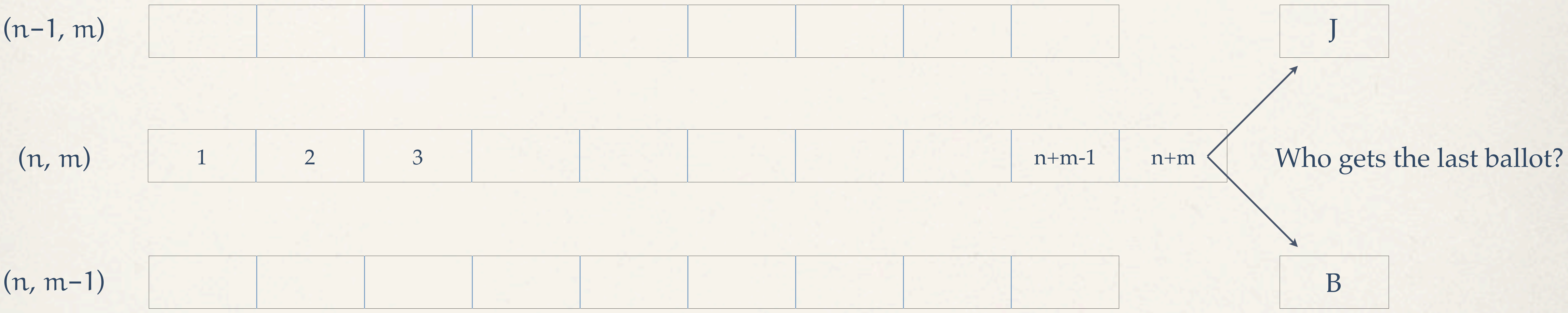

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$$\mathbf{P}(A) = \frac{\binom{n+m-1}{m}}{\binom{n+m}{m}} = \frac{n}{n+m} \qquad \mathbf{P}(A^c) = 1 - \mathbf{P}(A) = \frac{m}{n+m}$$