Lecture 30

Line integrals of vector fields over closed curves

(Relevant section from Stewart, *Calculus, Early Transcendentals*, Sixth Edition: 16.3)

Recall the basic idea of the Generalized Fundamental Theorem of Calculus: If $\mathbf{F}$ is a gradient or conservative vector field – here, we'll simply use the fact that it is a gradient field, i.e., $\mathbf{F} = \vec{\nabla} f$ for some f – and C is a curve that starts at point A and ends at point B, then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \vec{\nabla} f \cdot d\mathbf{r} \\ &= f(B) - f(A). \end{aligned} \tag{1}$$

In other words, the line integral, which could represent the work done by $\mathbf{F}$ if a particle is moved from A to B, depends only on the endpoints A and B and not on the curve from A to B.

Now we consider the following question: What if curve C started at A, travelled around for a while and then ended back at A – in other words, $A = B$? Of course, this means that $f(B) = f(A)$, which would imply that the line integral is zero. You've already encountered such situations in your elementary mechanics course: Suppose that you start with an object lying on a table, pick it up, walk around with it, perhaps going down the stairs, out the building, etc., then enter the building again, come back to the table and place the object on it. If we employ the approximation that the potential energy of the object is $V(y) = mgy$, where, for convenience, $y = 0$ is the table surface, then the net change in potential energy $\Delta V = 0$, so no net work was done by gravity.

In physics and other applications, one is often concerned with line integrals of vector fields over *simple, closed curves* C. "Simple" means nonintersecting. "Closed" means that the curve has no endpoints – you pick any starting point on the curve and you'll eventually arrive back at it. Such line integrals are often denoted as follows

$$\oint_C \mathbf{F} \cdot d\mathbf{r}. \tag{2}$$

One would be tempted to conclude that line integrals of conservative vector fields over closed curves are always zero since we end up where we started from. This is, in fact, often the case. But there are situations in which it is *not* zero. The problem is that we have to revise slightly our definition of a *conservative field*, paying attention to the region D over which we perform the integration. If this region contains **singularities** of the vector field $\mathbf{F}$, i.e., points at which $\mathbf{F}$ or its derivatives do not exist, then line integrals of "supposedly conservative" vector fields $\mathbf{F}$ over closed curves are not necessarily zero. It is possible that these singularities will "interfere" with line integrals over curves that "enclose" them. We'll explain below.

It's convenient to consider two cases. In both cases, we let D denote a region of interest in $\mathbf{R}^n$ over which we wish to consider line integrals of a vector field $\mathbf{F}$. We shall also let C denote a simple, closed curve that lies entirely in D, also assuming that it is piecewise smooth, so that the line integral may be computed (we need the tangent $\mathbf{r}'(t)$ to be defined):

1. **The "nice" case:** Suppose that $\mathbf{F}$ and its derivatives are defined at all points in D. Furthermore, suppose that

$$\text{curl}\,\mathbf{F}(\mathbf{r}) = \vec{\nabla} \times \mathbf{F}(\mathbf{r}) = \mathbf{0} \quad \text{at all points } \mathbf{r} \in D. \tag{3}$$

   Then $\mathbf{F}$ is conservative on D and the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \tag{4}$$

   over **any** simple curve D.

   We have already encountered a number of examples for this case. For example, a couple of lectures ago, we examined the vector field,

$$\mathbf{F}(x, y) = 2x\,\mathbf{i} + 4y\,\mathbf{j} + z\,\mathbf{k}, \tag{5}$$

   It is conservative, since Eq. (3) is satisfied at all points in $\mathbf{R}^3$. We also found that $\mathbf{F} = \vec{\nabla} f$, where

$$f(x, y, z) = x^2 + 2y^2 + \frac{1}{2}z^2. \tag{6}$$

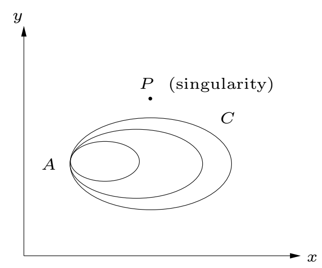
   $\mathbf{F}$ and its derivatives are defined at all points in $\mathbf{R}^3$. Therefore Eq. (4) holds for any closed curve C in any region $D \subset \mathbf{R}^3$.

2. **The "not-so-nice" case:** Suppose that $\mathbf{F}$ and/or its derivatives are undefined at some points in D – we refer to such points as **singularities**. At all other points in D, we assume that

$$\text{curl}\,\mathbf{F}(\mathbf{r}) = \vec{\nabla} \times \mathbf{F}(\mathbf{r}) = \mathbf{0}. \tag{7}$$

   Sub-case No. 2(a): Consider the situation sketched in the figure below. Here, we are looking at a simple closed curve C in the plane. The point P is a singularity of the vector field $\mathbf{F}$. At all other points (x, y), we assume that Eq. (7) holds.

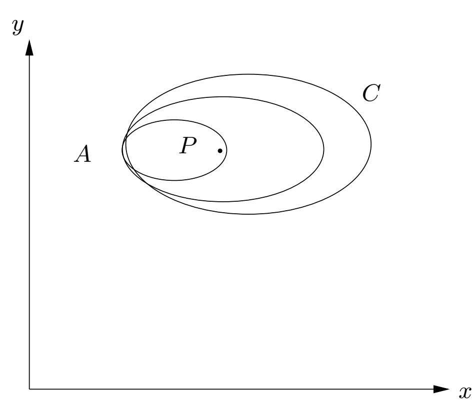
Clearly, the curve C does not enclose the singularity P, so we suspect that it can "avoid" this singularity. Just to be a little more mathematical, take *any* point on C – call it point A. Then start shrinking the curve C continuously to point A, keeping it fixed, as sketched in the figure. If you can perform this shrinking operation until you end up at the single point A **without passing through a singularity**, then the original line integral of $\mathbf{F}$ over the original curve C is zero, i.e., Eq. (4) holds. In the figure below, this can be done – in other words, the singularity P can be avoided.



The singularity P can be avoided during the shrinking of curve C

Therefore, line integral is zero

Sub-case No. 2(b): In the next figure, we consider another curve C which now encloses the singularity P. P. In this situation, we cannot shrink the curve C to point A without crossing the singularity P. In this case, the line integral of $\mathbf{F}$ over the curve C is not necessarily zero even though the curl of the vector field $\mathbf{F}$ is zero everywhere else. Unfortunately, that is all that we can conclude – we would actually have to compute the line integral in some way in order to see if it is zero or not.



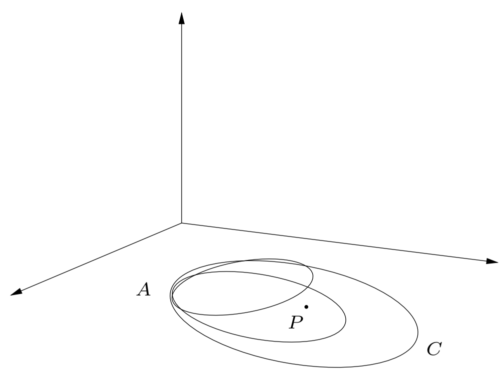
The singularity P cannot be avoided during the shrinking of curve C

Therefore, line integral is not necessarily zero

In the above examples, we were concerned with singularities of *planar* vector fields, i.e., vector fields in $\mathbf{R}^2$. Clearly, if a simple curve C encloses a singular point P, there is no way that we can avoid this singularity as the curve C is shrunk toward any point A on the curve.

Sub-case No. 2(c): On the other hand, if a vector field $\mathbf{F}$ in 3-space, i.e., $\mathbf{R}^3$ has a singular point P, you can imagine that it can be avoided as we shrink a closed curve C. For example, suppose that the singular point P is on the xy-plane, and suppose that our closed curve C is also on the xy-plane, as sketched below. As we shrink the curve C toward a point A, we are allowed to "lift" it up to avoid P during the process. In this case, the line integral of $\mathbf{F}$ over the original curve is zero - assuming, of course, that $\vec{\nabla} \times \mathbf{F} = \mathbf{0}$ at all other points.

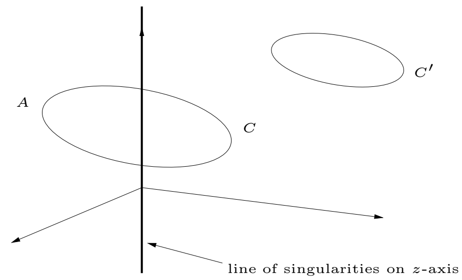
In this sense, the space $\mathbf{R}^3$ is more "forgiving." For this reason, we may state that the gravitational force $\mathbf{F}$ due to a point mass M at the origin is conservative, even though it has a singularity at $(0, 0, 0)$: The line integral of $\mathbf{F}$ over **any** closed curve C, provided that C does not contain the singular point $(0, 0, 0)$ is zero.



In $\mathbf{R}^3$, the singularity at P can be avoided by deforming or "lifting" the curve C as it is being shrunk toward A

Sub-case No. 2(d): But the space $\mathbf{R}^3$ is not totally forgiving! Suppose that we have a vector field – and we'll consider a physically meaningful one in a few moments – that has a line of singularities, for example, on the z-axis. At all other points, $\vec{\nabla} \times \mathbf{F} = \mathbf{0}$. Then, as sketched below, it is impossible for any curve C that encloses the z-axis to be shrunk to a point without crossing the z-axis. Therefore, no conclusions can be made about the line integral of $\mathbf{F}$ over C.

On the other hand, any curve C' that does not enclose the z-axis can be deformed to a single point, implying that the line integral of $\mathbf{F}$ over C' is zero.



In $\mathbf{R}^3$, the singularities on the z-axis cannot be avoided by deforming the curve C as it is being shrunk toward A

On the other hand, curve C' can be shrunk to a point

Connected and simply connected sets

A compact mathematical summary of the above cases can be made in terms of **connected** and **simply connected sets**. In the "nice" case above, where there are no singularities, we are working with simply connected sets – it is possible to shrink closed curves to single points without leaving the set. On the other hand, in the "not-so-nice" case, the singularities of the vector field, where it is not defined, represent "missing" points. In such cases, closed curves cannot be shrunk to single points without leaving the set. The set is then **not** simply connected. Some additional comments on these definitions are presented in an Appendix at the end of this lecture. (The material in the Appendix was **not** presented in class, and is to be considered as supplementary material.)

We now illustrate with an example. Consider the following vector field in $\mathbf{R}^2$:

$$\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2} \ , \ \frac{x}{x^2+y^2} \right). \tag{8}$$

Note that $\mathbf{F}(0,0)$ is undefined. In a recent Problem Set, you showed that $\vec{\nabla} \times \mathbf{F} = \mathbf{0}$ at all points except $(0,0)$, where it is undefined. Let's review this calculation:

$$\begin{aligned}
\vec{\nabla} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/(x^2+y^2) & x/(x^2+y^2) & 0 \end{vmatrix} \\
&= 0\mathbf{i} + 0\mathbf{j} + \left(\frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] - \frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right] \right)\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \right)\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}, \qquad (x,y) \neq (0,0).
\end{aligned}$$

Thus we are led to conclude that $\mathbf{F}$ is a gradient or conservative field over the xy-plane, except at the singular point $(0, 0)$.

From the above discussion, however, we may not conclude that the line integrals of $\mathbf{F}$ over all simple closed curves are zero.

To illustrate, we compute the line integral of $\mathbf{F}$ over the following simple, closed curve: a circle of radius R centered at $(0, 0)$, which we denote as C_R. **The usual convention for line integrals over closed curves in the plane is that the region enclosed by the curve lies to the left – in other words, the path is counterclockwise.**

The circular curve C_R is easily parametrized:

$$\mathbf{r}(t) = (x(t), y(t)) = (R\cos t, R\sin t), \qquad 0 \le t \le 2\pi. \tag{9}$$

Its velocity vector is

$$\mathbf{r}'(t) = (x'(t), y'(t)) = (-R\sin t, R\cos t), \qquad 0 \le t \le 2\pi. \tag{10}$$

Now evaluate $\mathbf{F}$ along the curve:

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \left(-\frac{y(t)}{x(t)^2 + y(t)^2} \, , \, \frac{x(t)}{x(t)^2 + y(t)^2} \right) \\ &= \left(-\frac{R\sin t}{R^2} \, , \, \frac{R\cos t}{R^2} \right) \\ &= \frac{1}{R}(-\sin t, \cos t). \end{aligned} \tag{11}$$

The integrand in the line integral will be

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{r}'(t) = (-\sin t, \cos t) \cdot (\sin t, \cos t) = \sin^2 t + \cos^2 t = 1 \tag{12}$$

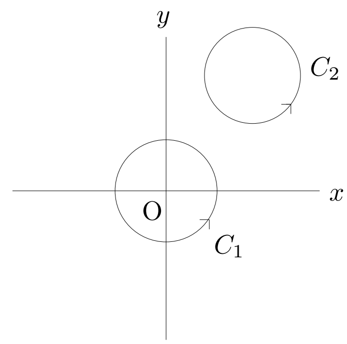
so that

$$\oint_{C_R} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_R} \mathbf{F}(\mathbf{r}(t) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} 1 \, dt = 2\pi. \tag{13}$$

Even though $\mathbf{F}$ is a conservative field, the line integral over a closed curve is nonzero. As discussed above, this is due to the fact that the curve C_R encloses the singular point $(0, 0)$.

We shall simply state the following result for line integrals this vector field: Let C be a simple closed curve in $\mathbf{R}^2$ that does not contain the point $(0, 0)$. Then for the vector field $\mathbf{F}$ defined in Eq. (8),

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \begin{cases} 2\pi, & \text{if } C \text{ encloses (0,0)} \quad (C_1 \text{ below }) \\ 0, & \text{otherwise} \quad (C_2 \text{ below}). \end{cases} \tag{14}$$



For reasons that will become clear in the next lecture, let us now consider the above vector field as a field in $\mathbf{R}^3$, i.e.,

$$\mathbf{F}(x, y, z) = -\frac{y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j} + 0\,\mathbf{k}. \tag{15}$$

In other words, we simply translate this vector field upwards and downwards from the xy-plane into $\mathbf{R}^3$.

We have now produced a vector field in $\mathbf{R}^3$ that has a **line of singularities** along the z-axis. We now have the situation pictured in Sub-case No. 2(d) above. The final result is the following: For a simple, closed curve C in $\mathbf{R}^3$ that does not contain any points from the z-axis,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \begin{cases} 2\pi, & \text{if } C \text{ encloses the } z \text{ axis} \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Appendix: Connected and simply connected sets (not covered in lecture)

The above discussion can be phrased in more mathematical language as follows. For the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \tag{17}$$

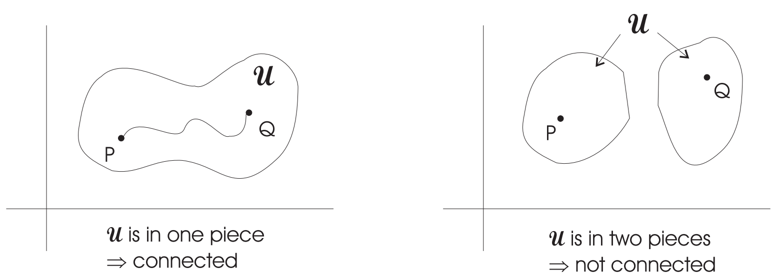
to be independent of path, which includes the consequence that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0, \tag{18}$$

where C is a closed curve, we must have that $\vec{\nabla} \times \mathbf{F} = \mathbf{0}$ over a simply connected set, or at least the region of interest D, over which all integrations are being performed, is a *simply connected set.* Of course, we now have to define "simply connected."

Connected and simply connected sets

Let $U \subseteq \mathbf{R}^n$. Then U is said to be *connected* if any two points $P, Q \in U$ can be joined by a continuous curve C that is contained entirely in U – in other words, all points of C are contained in U.



A set U is connected if it is "in one piece."

Examples: (Verify these results with your own sketches.)

1. The sets $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$ are simply connected.

2. The set $\mathbf{R} - \{0\}$, also written as $\mathbf{R} \backslash \{0\}$, which is obtained by removing the point $x = 0$ from the real line, is not simply connected: It consists of two disconnected pieces, $(-\infty, 0)$ and $(0, \infty)$. You can't connect a point P to the left of 0 to a point Q to the right of 0 with a continuous "curve", in this case a line segment, without going through 0, which is not part of the set.

3. The set $\mathbf{R}^2 \{(0, 0\}$ is connected.

4. On the other hand, the set $\mathbf{R}^2$ – {y-axis} is not connected – the plane is separated into two disconnected sets, $x > 0$ and $x < 0$

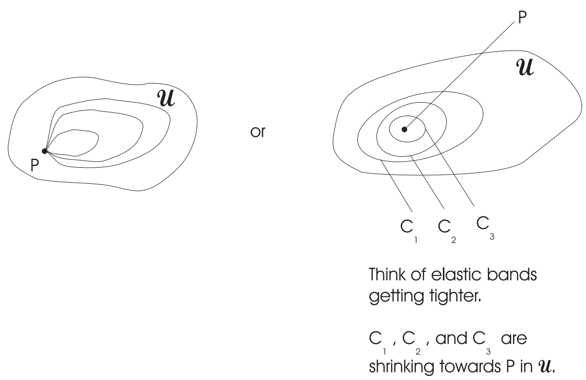
5. The set $\mathbf{R}^2$ – { the circle $x^2 + y^2 = 1$} is not connected. Taking away this circle leaves two disconnected sets – the interior of the circle and the exterior.

6. The set $\mathbf{R}^3 - \{(0, 0, 0)\}$ is connected.

7. The set $\mathbf{R}^3$ – { z-axis } is connected.

8. The set $\mathbf{R}^3$ – { the plane $z = 0$} is not connected.

A set $U \in \mathbf{R}^n$ is *simply connected* if it is connected and if every simple closed curve in U can be continuously shrunk to a point in U without leaving the set. A simply connected set in $\mathbf{R}^2$ cannot have holes.



Examples: (Once again, verify these results with your own sketches.)

1. The sets $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$ are simply connected.

2. The set $\mathbf{R}^2 \backslash \{(0, 0)\}$, obtained by removing the point $(0, 0)$ from $\mathbf{R}^2$ is not simply connected.

3. The set $\mathbf{R}^3 \backslash \{(0, 0, 0)\}$, obtained by removing the point $(0, 0, 0)$ from $\mathbf{R}^3$ is simply connected. (You can avoid the origin when you shrink any curve.)

4. The set $\{(x, y, z) \in \mathbf{R}^3 \mid (x, y) \neq (0, 0)\}$, the set obtained by removing the z-axis from $\mathbf{R}^3$, is not simply connected. Any closed curve that encloses the z-axis cannot be shrunk to a point without leaving the set, i.e., crossing the z-axis. This example will be important in an application to physics that we shall study in the next lecture.

Of course, the z-axis is not special. If you remove any line from $\mathbf{R}^3$, the resulting set is not simply connected.

You can find some additional discussion on connected and simply connected sets in the course textbook by Lovric, pp. 343-344.

Once again, to summarize: If $\mathbf{F} = \vec{\nabla} f$, and both $\mathbf{F}$ and f are defined over a *simply connected set* D, and we restrict all integration to this set, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \tag{19}$$

for any simply closed curve C lying entirely in D.

Lecture 31

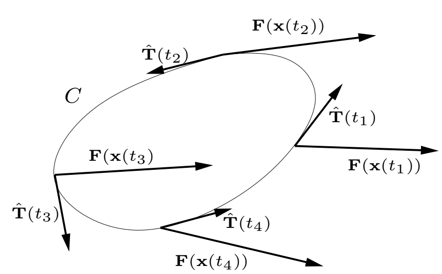
Line integrals of vector-valued functions (cont'd)

Circulation of a vector field around a closed curve C in $\mathbf{R}^2$

In what follows, we consider the line integral of a planar vector field $\mathbf{F}$ around a simple closed curve C in $\mathbf{R}^2$. The convention is that the integration along C is performed in the counterclockwise direction so that the region D enclosed by C lies always to the left of C as we move along the curve. Let's recall that this line integral sums up the projection of the vector field onto the unit tangent vector $\hat{\mathbf{T}}$ to the curve. Assuming that we can parametrize this curve as $\mathbf{r}(t)$, $a \leq t \leq b$,

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{r})(t)) \cdot \mathbf{r}'(t)\, dt \qquad (20) \\
&= \int_a^b \mathbf{F}(\mathbf{r})(t)) \cdot \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} \| \mathbf{r}'(t) \|\, dt \\
&= \int_a^b \mathbf{F}(\mathbf{r})(t)) \cdot \hat{\mathbf{T}}(t)\, ds \\
&= \oint f ds,
\end{aligned}$$

where the scalar-valued function $f(\mathbf{r}(t)) = \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{T}}(t)$ is the projection of $\mathbf{F}$ in the direction of the unit tangent vector to the curve C at $\mathbf{r}(t)$:



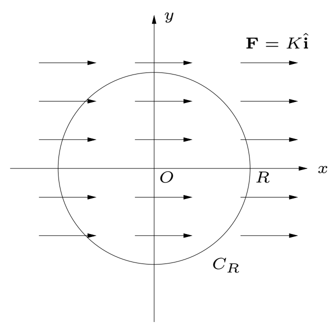
Starting at any point P on the curve C, the orientation of the tangent vector $\hat{\mathbf{T}}$ will change as we travel along C. In one traversal of C, the net rotation of the tangent vector is 2π. This is quite clear when C is a circle. As such, we say that the line integral in (20) is the *circulation of the vector field* $\mathbf{F}$ *around the closed curve* C.

If the vector field $\mathbf{F}$ is roughly parallel over the region D enclosed by curve C, then we expect the line integral to be small in value – in some regions of the curve, $\mathbf{F}$ points in the same direction as $\hat{\mathbf{T}}$ and in others, it points in the opposite direction. In other words, the vector field exhibits very little circulation.

Example 1: Let's consider a very simple case:

$$\mathbf{F}(x, y) = K\hat{\mathbf{i}} \tag{21}$$

and $C = C_R$ is the circle of radius R centered at the origin:



1. Parametrization of curve C_R: $\mathbf{r}(t) = (R\cos t, R\sin t)$, $0 \le t \le 2\pi$.

2. Velocity: $\mathbf{r}'(t) = (-R\sin t, R\cos t)$.

3. Evaluate $\mathbf{F}$ on curve: $\mathbf{F}(\mathbf{r}(t)) = (K, 0)$.

4. Integrand: $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (K, 0) \cdot (-R\sin t, R\cos t) = -KR\sin t$.

Thus

$$\oint_{C_R} \mathbf{F} \cdot d\mathbf{s} = -KR\int_0^{2\pi} \sin t \, dt = 0. \tag{22}$$

As expected, the line integral is zero.

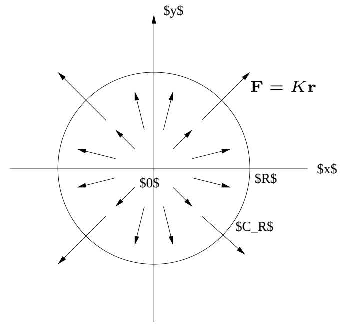
Example 2: Here is another situation that we expect will produce a zero result: $C = C_R$ as before and

$$\mathbf{F}(x, y) = K\mathbf{r} = Kx\hat{\mathbf{i}} + Ky\hat{\mathbf{j}} \tag{23}$$

In this case, the vector $\mathbf{F}$ on the curve is perpendicular to the tangent vector:

1. Parametrization of curve C_R: $\mathbf{r}(t) = (R\cos t, R\sin t)$, $0 \le t \le 2\pi$.

2. Velocity: $\mathbf{r}'(t) = (-R\sin t, R\cos t)$.



3. Evaluate **F** on curve: $\mathbf{F}(\mathbf{r}(t)) = K(x, y) = K(R\cos t, R\sin t)$.

4. Integrand: $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = K(R\cos t, R\sin t) \cdot (-R\sin t, R\cos t) = -KR\cos t\sin t + KR\cos t\sin t = 0$.

Thus

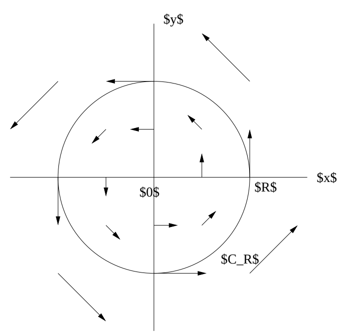
$$\oint_{C_R} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} 0 \; dt = 0. \tag{24}$$

As expected, the line integral is zero.

Example 3: Now consider the vector field

$$\mathbf{F}(x, y) = K(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}) \tag{25}$$

as sketched below.



You will recall that this is the velocity vector field of a disk that is revolving about the origin with angular speed $\omega = K$. In this case, **F** points in the direction of the unit tangent vector $\hat{\mathbf{T}}$ at every point on the curve C_R. As such, we expect a nonzero result:

1. Parametrization of curve C_R: $\mathbf{r}(t) = (R\cos t, R\sin t)$, $0 \le t \le 2\pi$.

2. Velocity: $\mathbf{r}'(t) = (-R\sin t, R\cos t)$.

3. Evaluate $\mathbf{F}$ on curve: $\mathbf{F}(\mathbf{r}(t)) = K(-y, x) = K(-R\sin t, R\cos t)$.

4. Integrand: $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = K(-R\sin t, R\cos t)\cdot(-R\sin t, R\cos t) = KR^2 \sin t \sin t + KR\cos t\cos t = KR$.

Thus

$$\oint_{C_R} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} KR^2\, dt = 2\pi KR^2. \tag{26}$$

As expected, the line integral is nonzero.

Example 4: Finally, we return to the following planar vector field,

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}, \tag{27}$$

which, up to a constant, describes the magnetic field produced by a current in a thin wire – the so-called Biot-Savart effect, to be discussed below. In the previous lecture, we computed the circulation of this vector field around the circle C_R. The result is

$$\oint_{C_R} \mathbf{F}\cdot d\mathbf{s} = 2\pi. \tag{28}$$

A noteworthy aspect of this result is that it is independent of the radius R of the circle.

We then extended this vector field into the space $\mathbf{R}^3$, i.e.,

$$\mathbf{F}(x,y,z) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + 0\mathbf{k}, \tag{29}$$

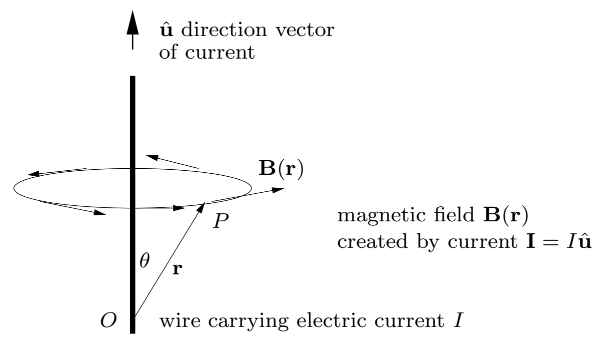
and claimed the following,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \begin{cases} 2\pi, & \text{if } C \text{ encloses the } z \text{ axis} \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Application to Physics: The Biot-Savart effect

The vector field of Example 4 above is important in electromagnetism as we now show. We are concerned with the **Biot-Savart effect**, in which a current of charge flowing in a conducting wire creates a magnetic field surrounding the wire.

In what follows, we assume that the conductor is a straight, thin wire such that the direction of current is given by the unit vector $\hat{\mathbf{u}}$. The current vector is then given by $\mathbf{I} = I\hat{\mathbf{u}}$. The physical situation is sketched below: P is a point of observation with position vector $\mathbf{r}$.



A rough sketch of $\mathbf{B}$ was given in the handout sheets from Lecture 1. Here are three physical facts that are verified by experiment (remember that Physics is an experimental science!):

1. The strength of the magnetic field $\| \mathbf{B} \|$ is proportional to the magnitude I of the current.

2. The strength of the magnetic field $\| \mathbf{B} \|$ diminishes as we move away from the wire. In fact, it diminishes as $1/d$, where d is the distance from the point of measurement P to the wire.

3. The magnetic field vector $\mathbf{B}(\mathbf{r})$ points in a direction that is perpendicular to the current vector $\mathbf{I}$. In fact, flow lines of the magnetic field are circles that lie on planes perpendicular to $\mathbf{I}$.

From electrodynamics, the magnetic field at P is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi} \frac{\hat{\mathbf{u}} \times \mathbf{r}}{\| \hat{\mathbf{u}} \times \mathbf{r} \|^2}. \tag{31}$$

Here, μ_0 denotes the permeability of the vacuum. This equation accounts for the experimental observations given above. Note that

$$\begin{aligned} \| \mathbf{B}(\mathbf{r}) \| &= \frac{\mu_0 I}{2\pi} \frac{\| \hat{\mathbf{u}} \times \mathbf{r} \|}{\| \hat{\mathbf{u}} \times \mathbf{r} \|^2} \\ &= \frac{\mu_0 I}{2\pi} \frac{1}{\| \hat{\mathbf{u}} \times \mathbf{r} \|} \\ &= \frac{\mu_0 I}{2\pi} \frac{1}{\| \mathbf{r} \| \sin\theta} \\ &= \frac{\mu_0 I}{2\pi} \frac{1}{d}, \end{aligned} \tag{32}$$

where $d = r\sin\theta$ is the distance from point P to the wire. We see that the strength of the magnetic field decreases with the distance d and *not* with the square of the distance.

It is convenient to orient the coordinate system so that the wire lies on the z-axis, i.e. $\hat{\mathbf{u}} = \hat{\mathbf{k}}$. Then the current vector $\mathbf{I} = I\hat{\mathbf{k}}$. The magnetic field vector then becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi} \frac{\hat{\mathbf{k}} \times \mathbf{r}}{\| \hat{\mathbf{k}} \times \mathbf{r} \|^2}. \tag{33}$$

We compute the numerator:

$$\begin{aligned} \hat{\mathbf{k}} \times \mathbf{r} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} \\ &= -y\mathbf{i} + x\mathbf{j} \end{aligned} \tag{34}$$

This means that $\| \hat{\mathbf{k}} \times \mathbf{r} \|^2 = x^2 + y^2$ so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi} \left[\frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{i} + 0\mathbf{k} \right], \tag{35}$$

which is the vector field of Example 4 above. Note that $\mathbf{B}$ is undefined for $(x, y) = (0, 0)$, i.e. the z-axis, which is the location of the current-carrying wire. This means that $\mathbf{B}(\mathbf{r})$ is defined on a set that is not simply connected. If C_R denotes the circle $x^2 + y^2 = R^2$, independent of z, then

$$\oint_{C_R} \mathbf{B} \cdot d\mathbf{r} = \frac{\mu_0 I}{2\pi} 2\pi = \mu_0 I. \tag{36}$$

This is the *circulation* of the magnetic field over the circle C_R. Note that it is independent of R. As we move away from the wire, the strength of the magnetic field $\mathbf{B}$ decreases linearly with R. But the length of the closed curve C_R, its circumference, increases linearly with R. These two effects cancel each other in the line integral, which accounts for the independence of R.

Let us recall that the magnetic vector field $\mathbf{B}$ is, apart from a constant, identical in form to the example studied in the previous lecture. We have

$$\vec{\nabla} \times \mathbf{B}(\mathbf{r}) = \mathbf{0}, \qquad (x, y) \neq (0, 0). \tag{37}$$

In other words $\mathbf{B}$ is defined on the set obtained by removing the z-axis from $\mathbf{R}^3$, which is not simply connected. On the other hand, any connected set in $\mathbf{R}^3$ that does not contain the z-axis is simply connected. As a result, we have the following:

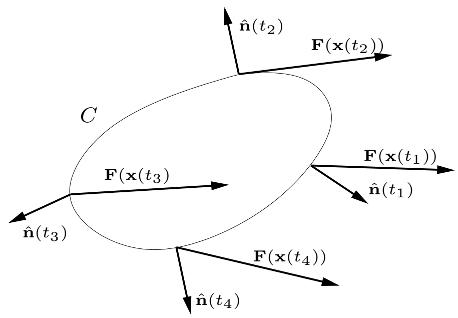
$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \begin{cases} \mu_0 I, & \text{if } C \text{ encircles the } z \text{ axis}, \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

Lecture 32

Line integrals of vector fields over closed curves (cont'd)

Outward flux of a vector field across a closed curve C in $\mathbf{R}^2$

We now return to the line integral in Eq. (20) of the previous lecture, replacing the unit tangent vector $\hat{\mathbf{T}}$ with the *unit outward normal* $\hat{\mathbf{N}}$ to the curve C. This is the unit vector at a point $\mathbf{r}(t)$ on the curve which is perpendicular to the unit tangent vector $\hat{\mathbf{T}}(t)$ and which points outward from the region D enclosed by C, as shown below:



The result is the line integral that we denote as

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) \parallel \mathbf{r}'(t) \parallel \; dt. \tag{39}$$

Clearly, this line integral adds up the projection of the vector field $\mathbf{F}$ onto the outward normal along the curve C. The result is the *total outward flux* of $\mathbf{F}$ across the closed curve C. If $\mathbf{F}$ represents the velocity field of a fluid, then the total outward flux measures the net rate of fluid escape from the region D through the curve C per unit time.

The practical calculation of the outward flux integral is not difficult – the only complication is that you have to determine the outward unit normal vector $\hat{\mathbf{N}}(t)$ to the curve C. This is easily done from a knowledge of the velocity vector $\mathbf{r}'(t)$. You first construct the unit tangent vector as follows:

$$\hat{\mathbf{T}}(t) = \frac{1}{\parallel \mathbf{r}'(t) \parallel}\mathbf{r}'(t) = (T_1(t), T_2(t)). \tag{40}$$

Note that $T_1^2 + T_2^2 = 1$. There are two unit vectors that are perpendicular to $\hat{\mathbf{T}}(t)$ – we'll define them as

$$\hat{\mathbf{N}}_1 = (T_2(t), -T_1(t)), \qquad \hat{\mathbf{N}}_2 = -(T_2(t), -T_1(t)). \tag{41}$$

We choose the vector that points *outward*.

We illustrate with some examples that use the curve C_R employed earlier. The parametrization of the curve and its velocity are given below

$$\mathbf{r}(t) = (R\cos t, R\sin t), \quad \mathbf{r}'(t) = (-R\sin t, R\cos t), \quad 0 \le t \le 2\pi. \tag{42}$$

Thus the speed is given by $\| \mathbf{r}'(t) \| = R$. The unit tangent vector is then

$$\hat{\mathbf{T}}(t) = \frac{1}{\| \mathbf{r}'(t) \|}\mathbf{r}'(t) = \frac{1}{R}(-R\sin t, R\cos t) = (-\sin t, \cos t). \tag{43}$$

Two unit vectors that are perpendicular to $\hat{\mathbf{T}}(t)$ are

$$\hat{\mathbf{N}}_1(t) = (\cos t, \sin t), \qquad \hat{\mathbf{N}}_2(t) = -(\cos t, \sin t). \tag{44}$$

Both of these vectors are normal to the curve C_R. The vector that points outward is $(\cos t, \sin t)$. Thus we set

$$\hat{\mathbf{N}}(t) = (\cos t, \sin t). \tag{45}$$

Of course, this result was expected: In the case of a circle, $\hat{\mathbf{N}}(t)$ is simply the unit vector that points outward from the center, namely, $\hat{\mathbf{r}}(t) = \dfrac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|}$.

Example 1: $\mathbf{F}(x, y) = K\hat{\mathbf{i}}$. We may imagine this vector field to represent a fluid that is travelling on the surface of a table with constant velocity K in the x-direction.

1. Evaluate $\mathbf{F}$ on curve: $\mathbf{F}(\mathbf{r}(t)) = (K, 0)$.
2. Integrand: $\mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) = (K, 0) \cdot (\cos t, \sin t) = K\cos t$.

Thus

$$\oint \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) \| \mathbf{r}'(t) \|\, dt = \int_0^{2\pi} KR\cos t\, dt = 0. \tag{46}$$

The net outward flux of $\mathbf{F} = K\hat{\mathbf{i}}$ through the circle C_R is zero. This result was expected, since the amount of fluid entering the circular region enclosed by curve C_R from the left is balanced by the amount leaving the region on the right (assuming that K is positive).

Example 2: $\mathbf{F}(x, y) = K\mathbf{r} = Kx\hat{\mathbf{i}} + Ky\hat{\mathbf{j}}$.

We expect the outward flux to be positive since this vector field points outward in the direction of the outward normal vector to C_R.

1. Evaluate $\mathbf{F}$ on curve: $\mathbf{F}(\mathbf{r}(t)) = (KR\cos t, KR\sin t)$.

2. Integrand: $\mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) = (KR\cos t, KR\sin t) \cdot (\cos t, \sin t) = KR$.

Thus

$$\oint \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) \parallel \mathbf{r}'(t) \parallel dt = \int_0^{2\pi} KR^2\, dt = 2\pi KR^2. \tag{47}$$

The net outward flux of $\mathbf{F} = K\hat{\mathbf{r}}$ through the circle C_R is $2\pi KR^2$. As expected, this flux is *positive*, representing net outward flow. Also note that the result is dependent upon the radius R of the circle C_R. It seems that the bigger the circle, the more "stuff" is flowing through it.

Example 3: $\mathbf{F}(x, y) = -Ky\hat{\mathbf{i}} + Kx\hat{\mathbf{j}}$.

We expect the outward flux to be zero since this vector field points in the direction of the tangent vector to curve C_R.

1. Evaluate $\mathbf{F}$ on curve: $\mathbf{F}(\mathbf{r}(t)) = (-KR\sin t, KR\cos t)$.

2. Integrand: $\mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) = (-KR\sin t, KR\cos t) \cdot (\cos t, \sin t) = 0$.

Thus

$$\oint \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) \parallel \mathbf{r}'(t) \parallel dt = \int_0^{2\pi} 0\, dt = 0. \tag{48}$$

The net outward flux of $\mathbf{F} = -Ky\hat{\mathbf{i}} + Kx\hat{\mathbf{j}}$ through the circle C_R is 0, as expected.

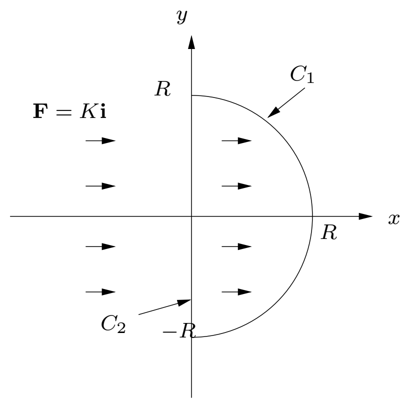
Finally, we mention that the flux of a vector field $\mathbf{F}$ does not necessarily need to be defined over *closed curves* – one may certainly define the flux of a field through a curve C that is not closed: It will be simply

$$\int_C \mathbf{F} \cdot \hat{\mathbf{N}}\, ds, \tag{49}$$

where $\hat{\mathbf{N}}$ denotes the desired unit normal to curve C. In applications, in fact, it may be necessary to break up the computation of the flux of a field through a closed curve C into a sum over its parts C_1.

For example, consider the closed "half-moon" curve that is composed of the circular curve C_R for $x \geq 0$ and the straight line segment $x = 0$, $-R \leq y \leq R$. We now compute the total outward flux of the vector field $\mathbf{F}(x, y) = K\mathbf{i}$ from Example 1.

1. Curve C_1, the "half-moon" part, which may be parametrized as, $\mathbf{r}(t) = (\cos t, \sin t)$, $-\pi/2 \leq t \leq \pi/2$. Much of the necessary work has been done in Example 1 above. The outward flux through



C_1 will be

$$\int_{C_1} \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) \parallel \mathbf{r}'(t) \parallel dt = \int_{-\pi/2}^{\pi/2} KR \cos t\, dt = 2KR. \tag{50}$$

This is the flux of the vector field through the curve C_1.

2. Curve C_2, the segment $-R \leq y \leq R$ on the y-axis. Technically, we should compute this line integral from the point $y = R$ to $y = -R$ so that we continue in a counterclockwise manner on the boundary of the half-moon region. We cannot, however, simply parametrize the curve as $x(t) = 0$, $y(t) = t, R \geq t \geq -R$, since t will then be *decreasing* over the integration. The parameter must *always* be increasing in the integration, in order for the differential dt to be positive. We thus parametrize the curve as $x(t) = 0$, $y(t) = R - t$, $0 \leq t \leq 2R$. In other words $\mathbf{r}(t) = (0, R - t)$, so that $\mathbf{r}'(t) = (0, -1)$ and $\|\mathbf{r}'(t)\| = 1$.

   The outward normal to this curve is the vector $\hat{\mathbf{N}} = -\mathbf{i} = (-1, 0)$. The integrand of the flux integral will then be

$$\mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) = (K, 0) \cdot (-1, 0) = -K. \tag{51}$$

   The flux across curve C_2 will then be

$$\int_{C_2} \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{N}}(t) \|\mathbf{r}'(t)\|\, dt = \int_0^{2R} -K\, dt = -2KR. \tag{52}$$

Therefore the net outward flux through the half-moon region bounded by curves C_1 and C_2 is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = 2KR - 2KR = 0. \tag{53}$$

Classical Integration Theorems in the Plane

In this section, we present two very important results on integration over closed curves in the plane, namely, Green's Theorem and the Divergence Theorem, as a prelude to their important counterparts in $\mathbf{R}^3$ involving surface integrals. Because of the lack of time in the lecture, the theorems were presented very quickly. Examples will be presented in the next lecture.

Green's Theorem in the Plane

(Relevant section from Stewart, *Calculus, Early Transcendentals*, Sixth Edition: 16.4)

Let $\mathbf{F}(x,y) = F_1(x,y)\,\mathbf{i} + F_2(x,y)\,\mathbf{j}$ be a vector field in $\mathbf{R}^2$. Let C be a simple closed (piecewise C^1) curve in $\mathbf{R}^2$ that encloses a simply connected (i.e., "no holes") region $D \subset \mathbf{R}$. Also assume that the partial derivatives $\dfrac{\partial F_2}{\partial x}$ and $\dfrac{\partial F_1}{\partial y}$ exist at all points $(x,y) \in D$. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] dA, \tag{54}$$

where the line integration is performed over C in a counterclockwise direction, with D lying to the left of the path.

Note:

1. The integral on the left is a *line integral* – the circulation of the vector field $\mathbf{F}$ over the closed curve C.
2. The integral on the right is a *double integral* over the region D enclosed by C.

The Divergence Theorem in the Plane

This theorem is discussed in Section 16.4 of the text by Stewart, p. 1067, but only as a special case of Green's Theorem. Unfortunately, its important physical interpretation is not discussed in the text.

Let $\mathbf{F}(x,y) = F_1(x,y)\,\mathbf{i} + F_2(x,y)\,\mathbf{j}$ be a vector field in the plane. Let C be a simple closed (piecewise C^1) curve that encloses a region D. Let $\hat{\mathbf{N}}$ denote the unit outward normal to C assumed to exist at all points on the curve (except perhaps at a finite set of "corners"). Furthermore, assume that the divergence of $\mathbf{F}$ is defined for all points in D, i.e.,

$$\text{div}\,\mathbf{F}(x,y) = \vec{\nabla} \cdot \mathbf{F}(x,y) = \frac{\partial F_1}{\partial x}(x,y) + \frac{\partial F_2}{\partial y}(x,y) \tag{55}$$

is defined for all $(x, y) \in D$.

The Divergence Theorem in the Plane then states that

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = \int\int_D \text{div}\, \mathbf{F}\, dA. \tag{56}$$

The left integral is a *line integral* around the curve C – it measures the *net outward flux* of the vector field $\mathbf{F}$ through the closed curve C. The right integral is a double integration *over the region* D enclosed by C.