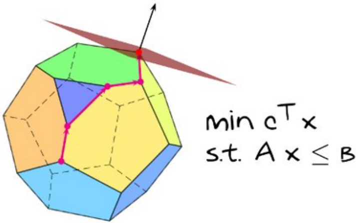
Linear programming

- Definition of linear programming
- Some useful notation



What is a linear program?

$f: \mathbb{R}^n \to \mathbb{R}$

$f(x) = d^T \cdot x = d_1 x_1 + \dots + d_n x_n$

linear function $\quad d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$

A linear program consists of a linear objective function

$$c_1 x_1 + \cdots + c_n x_n$$

$= c^T \cdot x$

and linear inequalities

$$\begin{array}{ccc} a_{11}x_1 + & \cdots & + a_{1n}x_n \leqslant b_1 \\ & \vdots & \\ a_{m1}x_1 + & \cdots & + a_{mn}x_n \leqslant b_m. \end{array}$$

(linear) constraints

VARIABLES

Find $\underline{x_1, \ldots, x_n} \in \mathbb{R}$ of maximum objective function value among all those $x_1, \ldots, x_n \in \mathbb{R}$ satisfying the linear inequalities.

Back to introductory example

$$
\begin{array}{lrcl}
\text{maximize} & 100 \cdot x_1 + 125 \cdot x_2 & & \\
\text{such that:} & 3 \cdot x_1 + 6 \cdot x_2 & \leqslant & 30 \\
& 8 \cdot x_1 + 4 \cdot x_2 & \leqslant & 44 \\
& x_1 & \leqslant & 5 \\
& x_2 & \leqslant & 4 \\
& x_1 & \geqslant & 0 \quad \Leftrightarrow \quad -x_1 \leq 0 \\
& x_2 & \geqslant & 0 \quad \Leftrightarrow \quad -x_2 \leq 0
\end{array}
$$

$x \in \mathbb{R}^n$ satisfies $a^T \cdot x \geq \beta$, $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ if and only if

x satisfies $-a^T \cdot x \leq -\beta$

Matrix notation

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{array}{llllll} \max & c_1x_1 & + \cdots + & c_nx_n & & \\ \text{s.t.:} & a_{11}x_1 & + \cdots + & a_{1n}x_n & \leqslant & b_1 \\ & & \vdots & & & \\ & a_{m1}x_1 & + \cdots + & a_{mn}x_n & \leqslant & b_m. \end{array}$$

$$A \in \mathbb{R}^{m \times n}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\max \{ c^T \cdot x : x \in \mathbb{R}^n, A \cdot x \leq b \}$$

$$A x \leq b$$

Back to introductory example

$$
\begin{array}{lrcl}
\max & 100 \cdot x_1 + 125 \cdot x_2 & & \\
\text{s. t.:} & 3 \cdot x_1 + 6 \cdot x_2 & \leqslant & 30 \\
 & 8 \cdot x_1 + 4 \cdot x_2 & \leqslant & 44 \\
 & x_1 & \leqslant & 5 \\
 & x_2 & \leqslant & 4 \\
 & x_1 & \geqslant & 0 \\
 & x_2 & \geqslant & 0
\end{array}
$$

$c^T = (100, 125)$

$b^T = (30, 44, 5, 4, 0, 0)$

$$
A = \begin{pmatrix} 3 & 6 \\ 8 & 4 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}
$$

Max vs. Min

$\min\{5,4,-3\} = -3$

$= -\max\{-5,-4,3\} = -3$

$\max\ c^T \cdot x$

If $\min\{c^T \cdot x : x \in \mathbb{R}^n, Ax \le b\}$ exists, then

$= -\max\{-c^T x : x \in \mathbb{R}^n, Ax \le b\}$

Quiz

What are A, b and c in the matrix-notation $\max\{c^T x : x \in \mathbb{R}^n, \quad Ax \leqslant b\}$ of the following linear program:

$$
\begin{array}{lrcl}
\min & 2 \cdot x_1 + 3 \cdot x_2 & & \\
\text{s.t.:} & 2 \cdot x_1 + x_2 & \geqslant & 2 \\
 & x_1 + x_2 & \leqslant & 10 \\
 & x_1 & \geqslant & 0 \\
 & x_2 & \geqslant & 0
\end{array}
$$

$(\cdot -1)$

$\rightarrow \; -2x_1 - x_2 \leq -2$

$$A = \begin{pmatrix} \boxed{-2} & \boxed{-1} \\ \boxed{1} & \boxed{1} \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$b = \begin{pmatrix} \boxed{-2} \\ \boxed{10} \\ \boxed{0} \\ \boxed{0} \end{pmatrix} \qquad c = \begin{pmatrix} \boxed{-2} \\ \boxed{-3} \end{pmatrix}$$

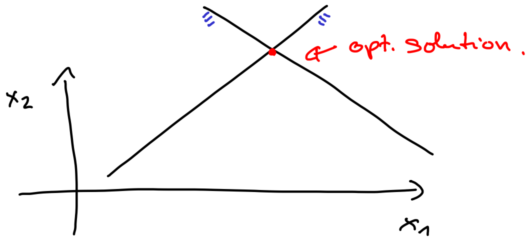
Feasible solutions

A point $x \in \mathbb{R}^n$ is called *feasible*, if x satisfies all linear inequalities. If there are feasible solutions of a linear program, then the linear program is called *feasible*.

Optimal solutions

A feasible $x \in \mathbb{R}^n$ is an *optimal solution* of ~~the~~ a linear program if $c^T x \geqslant c^T y$ for all feasible $y \in \mathbb{R}^n$.

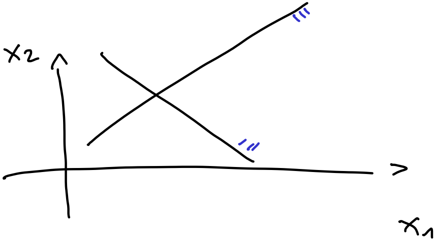
$\max \{ c^T \cdot x : x \in \mathbb{R}^n, Ax \leq b \}$



max x_2

Bounded linear program

A linear program is *bounded* if there exists a constant $M \in \mathbb{R}$ such that $c^T x \leqslant M$ holds for all feasible $x \in \mathbb{R}^n$.



LP unbounded

max x_1

Quiz

The linear program

$$\begin{array}{lrl} \max & x_1 & \\ \text{s.t.:} & x_1 + x_2 & \leqslant 1 \\ & x_1 & \geqslant 1 \end{array}$$

- is infeasible
- is feasible
- is bounded unbounded

$\forall k \geq 1$: $(k, -k+1)$ is feas.

$M \in \mathbb{R}$ $k = \max\{M+1, 1\}$

$k \geq 1$ and

$(k, -k+1)$ is feas.

$> M$