Aces in bridge

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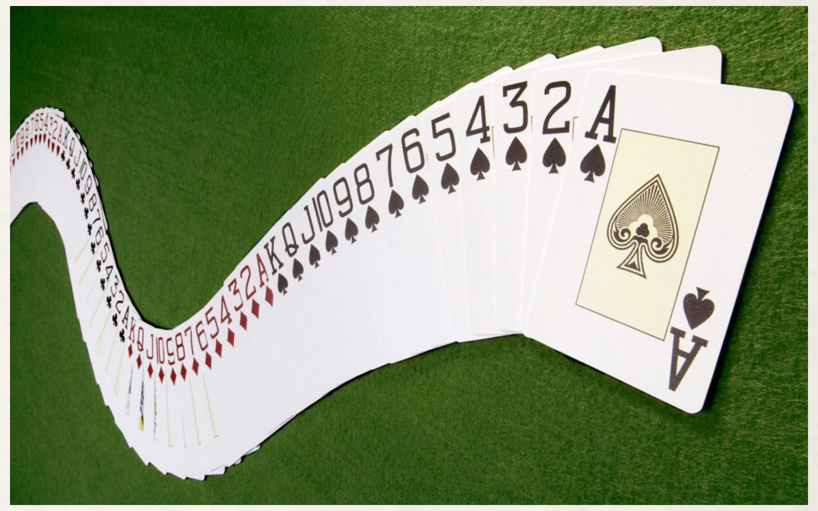

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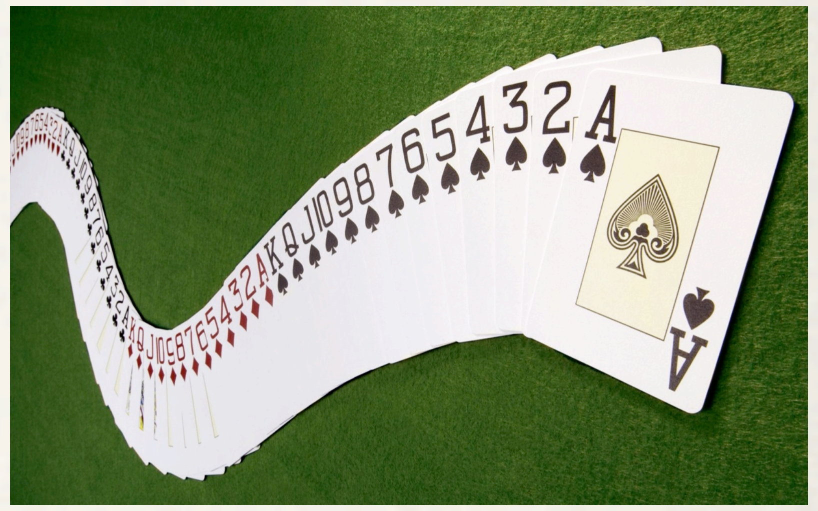

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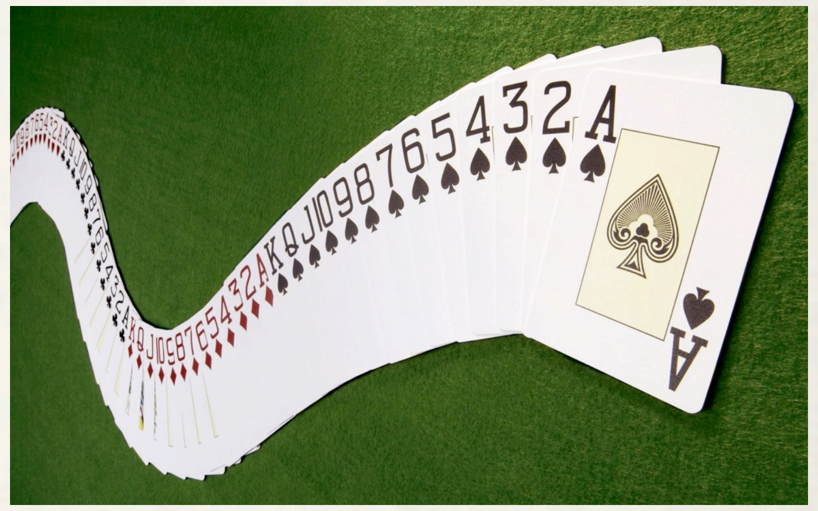

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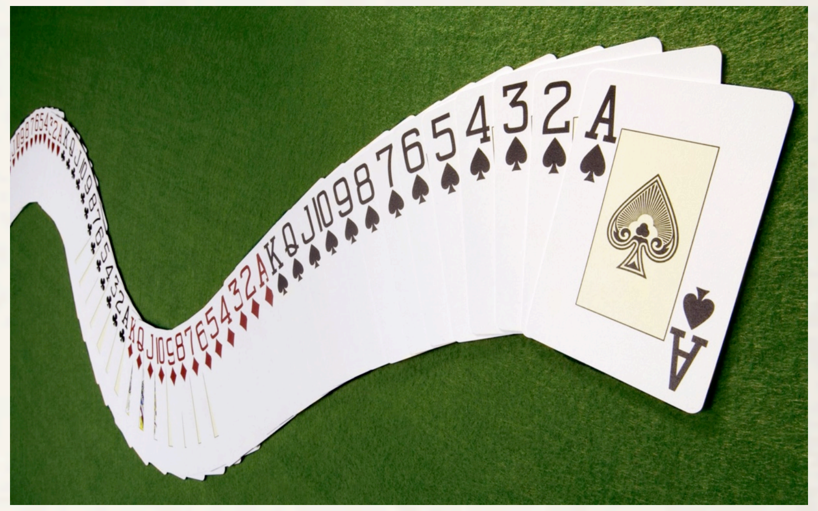

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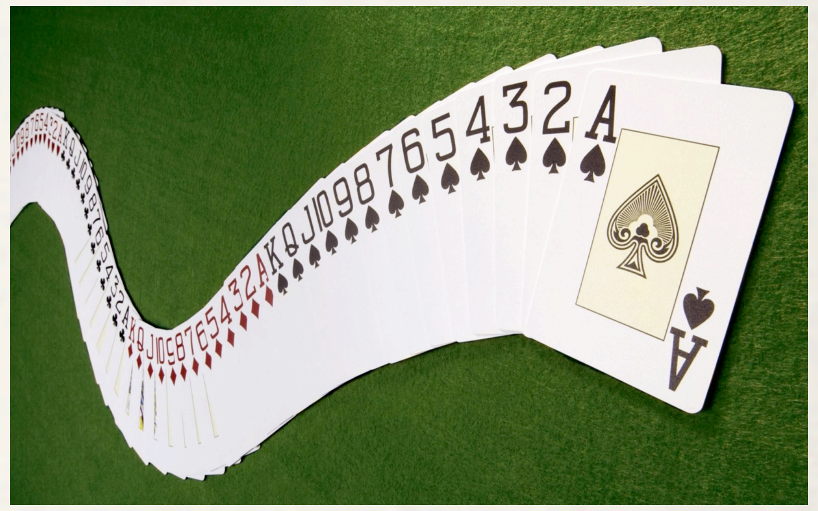

$$A_{\spadesuit\heartsuit\diamondsuit\clubsuit} \subseteq A_{\spadesuit\heartsuit\diamondsuit} \subseteq A_{\spadesuit\heartsuit} \subseteq A_\spadesuit = \Omega$$

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\end{aligned}$$

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&= \frac{13}{49} \times \frac{26}{50} \times \frac{39}{51} \times \frac{52}{52} \approx 0.1055
\end{aligned}$$