Probability and Statistics: To p, or not to p?

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2.4 Bayesian updating

> 'When the facts change, I change my mind. What do you do, sir?'
> *John Maynard Keynes.*

Bayesian updating is the act of updating your (probabilistic) beliefs in light of new information. Formally named after Thomas Bayes (1701–61), for two events A and B, the simplest form of **Bayes' theorem** is:

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}.$$

Example

Consider the probability distribution of the score on a fair die:

$X = x$	1	2	3	4	5	6
$P(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6

Suppose we define the event A to be 'roll a 6'. Unconditionally, i.e. *a priori* (before we receive any additional information), we have:

$$P(A) = P(X = 6) = \frac{1}{6}.$$

Now let us suppose we are told that the event:

$$B = \text{even score} = \{2, 4, 6\}$$

has occurred (where $P(B) = 1/2$), which means we can effectively revise our sample space, S^*, by eliminating 1, 3 and 5 (the odd scores), such that:

$$S^* = \{\cancel{1},\ 2,\ \cancel{3},\ 4,\ \cancel{5},\ 6\} = \{2, 4, 6\}.$$

So now the revised sample space contains three equally likely outcomes (instead of the original six), so the **Bayesian updated probability** (known as a **conditional probability** or *a posteriori* probability) is:

$$P(A \mid B) = \frac{1}{3}$$

where '|' can be read as 'given', hence $A \mid B$ means 'A given B'.

Deriving this result formally using Bayes' theorem, we already have $P(A) = 1/6$ and also $P(B) = 1/2$, so we just need $P(B \mid A)$, which is the probability of an even score given a score of 6. Since 6 is an even score, $P(B \mid A) = 1$. Hence:

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)} = \frac{1 \times 1/6}{1/2} = \frac{2}{6} = \frac{1}{3}.$$

Suppose instead we consider the case where we are told that an odd score was obtained. Since even scores and odd scores are **mutually exclusive** (they cannot occur simultaneously) and **collectively exhaustive** (a die score must be even or odd), then we can view this as the *complementary* event, denoted B^c, such that:

$$B^c = \text{odd score} = \{1, 3, 5\} \quad \text{and} \quad P(B^c) = 1 - P(B) = 1/2.$$

So, given an odd score, what is the conditional probability of obtaining a 6? Intuitively, this is zero (an impossible event), and we can verify this with Bayes' theorem:

$$P(A \mid B^c) = \frac{P(B^c \mid A)\, P(A)}{P(B^c)} = \frac{0 \times 1/6}{1/2} = 0$$

where, clearly, we have $P(B^c \mid A) = 0$ (since 6 is an even, not odd, score, so it is impossible to obtain an odd score given the score is 6).

Example

Suppose that 1 in 10,000 people (0.01%) has a particular disease. A diagnostic test for the disease has 99% **sensitivity** (if a person has the disease, the test will give a positive result with a probability of 0.99). The test has 99% **specificity** (if a person does not have the disease, the test will give a negative result with a probability of 0.99).

Let B denote the presence of the disease, and B^c denote no disease. Let A denote a positive test result. We want to calculate $P(A)$.

The probabilities we need are $P(B) = 0.0001$, $P(B^c) = 0.9999$, $P(A \mid B) = 0.99$ and also $P(A \mid B^c) = 0.01$, and hence:

$$\begin{aligned} P(A) &= P(A \mid B)\, P(B) + P(A \mid B^c)\, P(B^c) \\ &= 0.99 \times 0.0001 + 0.01 \times 0.9999 \\ &= 0.010098. \end{aligned}$$