TEST FLIGHT: THIRD PROBLEM SET SOLUTION

1. Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$$

ANSWER It's false. If $n \geq 2$, then for any m, $3m + 5n \geq 13$, so we need only show that there is no m such that $3m + 5 = 12$, i.e. no m such that $3m = 7$. This is immediate.

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

ANSWER True. Let $n, n+1, n+2, n+3, n+4$ be any five consecutive integers. Then

$$n + (n+1) + (n+2) + (n+3) + (n+4) = 5n + 1 + 2 + 3 + 4 = 5n + 10 = 5(n+2)$$

which proves the result.

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

ANSWER True. Consider the two case n even and n odd separately.

If n is even, say $n = 2k$, then

$$n^2 + n + 1 = 4k^2 + 2k + 1 = 2(2k^2 + k) + 1$$

which is odd.

If n is odd, say $n = 2k + 1$, then

$$n^2+n+1 = (2k+1)^2+(2k+1)+1 = 4k^2+4k+1+2k+1+1 = 4k^2+6k+2+1 = 2(2k^2+3k+1)+1$$

which is odd.

In both cases, $n^2 + n + 1$ is odd.

4. Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

ANSWER Let m be a natural number. By the Division Theorem, there are unique numbers n, r such that $m = 4n + r$, where $0 \leq r < 4$. Thus m is one of $4n, 4n+1, 4n+2$, $4n+3$. Since $4n$ and $4n+2$ are even, if m is odd, the only possibilities are $4n+1$ and $4n+3$.

5. Prove that for any integer n, at least one of the integers $n,\ n+2,\ n+4$ is divisible by 3.

   ANSWER By the Division Theorem, n can be expressed in one of the forms $3q, 3q+1, 3q+2$, for some q. In the first case, n is divisible by 3. In the second case $n+2 = 3q+3 = 3(q+1)$, so $n+2$ is divisible by 3. In the third case $n+4 = 3q+6 = 3(q+2)$, so $n+4$ is divisible by 3.

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

   ANSWER Consider any three numbers of the form $n, n+2, n+4$, where $n > 3$. By the answer to the previous question, one of these numbers is divisible by 3, and hence is not prime.

7. Prove that for any natural number n: $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

   ANSWER Let $S = 2 + 2^2 + 2^3 + \ldots + 2^n$. Then $2S = 2^2 + 2^3 + 2^4 + \ldots + 2^n + 2^{n+1}$. Subtracting the first identity from the second gives $2S - S = 2^{n+1} - 2$. But $2S - S = S$, so this establishes the stated identity.

8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

   ANSWER Let $\epsilon > 0$ be given. By the assumption, we can find an N such that

   $$n \geq N \Rightarrow |a_n - L| < \epsilon/M$$

   Then,

   $$n \geq N \Rightarrow |Ma_n - ML| = M.\,|a_n - L| < M.\epsilon/M = \epsilon$$

   which shows that $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

9. Given a collection $A_n, n = 1, 2, \ldots$ of intervals of the real line, their *intersection* is defined to be

   $$\bigcap_{n=1}^{\infty} A_n = \{x \,|\, (\forall n)(x \in A_n)\}$$

   Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and

   $$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

   Prove that your example has the stated property.

ANSWER Let $A_n = (0, 1/n)$. Clearly, $\bigcap_{n=1}^{\infty} A_n \subseteq A_1 = (0, 1)$. Hence any element of the intersection must be a member of $(0, 1)$. But if $x \in (0, 1)$, we can find a natural number n such that $1/n < x$. Then $x \notin A_n$, so $x \notin \bigcap_{n=1}^{\infty} A_n$. Thus $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

10. Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

    ANSWER Let $A_n = [0, 1/n)$. Clearly, $0 \in \bigcap_{n=1}^{\infty} A_n$. But the same argument as above shows that no other number is in the intersection. Hence $\bigcap_{n=1}^{\infty} A_n = \{0\}$.