The Nullspace of a Matrix

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The Nullspace of a Matrix

The solution sets of homogeneous linear systems provide an important source of vector spaces. Let A be an m by n matrix, and consider the homogeneous system

$$A\mathbf{x} = \mathbf{0}$$

Since A is m by n, the set of all vectors $\mathbf{x}$ which satisfy this equation forms a subset of $\mathbf{R}^n$. (This subset is nonempty, since it clearly contains the zero vector: $\mathbf{x} = \mathbf{0}$ always satisfies $A\mathbf{x} = \mathbf{0}$.) This subset actually forms a subspace of $\mathbf{R}^n$, called the **nullspace** of the matrix A and denoted $N(A)$. To prove that $N(A)$ is a subspace of $\mathbf{R}^n$, closure under both addition and scalar multiplication must be established. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $N(A)$, then, by definition, $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. Adding these equations yields

$$A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} \quad \Rightarrow \quad A(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{0} \quad \Rightarrow \quad \mathbf{x}_1 + \mathbf{x}_2 \in N(A)$$

which verifies closure under addition. Next, if $\mathbf{x}$ is in $N(A)$, then $A\mathbf{x} = \mathbf{0}$, so if k is any scalar,

$$k(A\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad A(k\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad k\mathbf{x} \in N(A)$$

verifying closure under scalar multiplication. Thus, the solution set of a homogeneous linear system forms a vector space. Note carefully that if the system is *not* homogeneous, then the set of solutions is *not* a vector space since the set will not contain the zero vector.

Example 1: The plane P in Example 7, given by $2x + y - 3z = 0$, was shown to be a subspace of $\mathbf{R}^3$. Another proof that this defines a subspace of $\mathbf{R}^3$ follows from the observation that $2x + y - 3z = 0$ is equivalent to the homogeneous system

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

where A is the 1 x 3 matrix [2 1 −3]. P is the nullspace of A.

Example 2: The set of solutions of the homogeneous system

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$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

forms a subspace of $\mathbf{R}^n$ for some n. State the value of n and explicitly determine this subspace.

Since the coefficient matrix is 2 by 4, **x** must be a 4-vector. Thus, n = 4: The nullspace of this matrix is a subspace of $\mathbf{R}^4$. To determine this subspace, the equation is solved by first row-reducing the given matrix:

$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \xrightarrow[(-1)r_1]{2r_1 \text{ added to } r_2} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$

Therefore, the system is equivalent to

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that is,

$$\begin{aligned} x_1 - x_2 - 2x_3 - 4x_4 &= 0 \\ 2x_2 + 5x_3 + x_4 &= 0 \end{aligned}$$

If you let x_3 and x_4 be free variables, the second equation directly above implies

$$x_2 = -\tfrac{1}{2}(5x_3 + x_4)$$

Substituting this result into the other equation determines x_1:

$$\begin{aligned} x_1 - [-\tfrac{1}{2}(5x_3 + x_4)] - 2x_3 - 4x_4 &= 0 \\ x_1 &= -\tfrac{1}{2}(x_3 - 7x_4) \end{aligned}$$

Therefore, the set of solutions of the given homogeneous system can be written as

$$\left\{ \begin{bmatrix} -\tfrac{1}{2}(x_3 - 7x_4) \\ -\tfrac{1}{2}(5x_3 + x_4) \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbf{R} \right\}$$

which is a subspace of $\mathbf{R}^4$. This is the nullspace of the matrix

$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix}$$

Example 3: Find the nullspace of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

By definition, the nullspace of *A* consists of all vectors **x** such that *A* **x** = **0**. Perform the following elementary row operations on *A*,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{\mathbf{r}_1 \leftrightarrow \mathbf{r}_2} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{-2\mathbf{r}_1 \text{ added to } \mathbf{r}_2} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

to conclude that *A* **x** = **0** is equivalent to the simpler system

$$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second row implies that $x_2 = 0$, and back-substituting this into the first row implies that $x_1 = 0$ also. Since the only solution of *A* **x** = **0** is **x** = **0**, the nullspace of *A* consists of the zero vector alone. This subspace, { **0**}, is called the **trivial subspace** (of $\mathbf{R}^2$).

Example 4: Find the nullspace of the matrix

$$B = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

To solve *B* **x** = **0**, begin by row-reducing *B*:

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \xrightarrow{2\mathbf{r}_1 \text{ added to } \mathbf{r}_2} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

The system $B\mathbf{x} = \mathbf{0}$ is therefore equivalent to the simpler system

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the bottom row of this coefficient matrix contains only zeros, x_2 can be taken as a free variable. The first row then gives $2x_1 + x_2 = 0 \;\Rightarrow\; x_1 = -\frac{1}{2}x_2$ so any vector of the form

$$\begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix}$$

satisfies $B\mathbf{x} = \mathbf{0}$. The collection of all such vectors is the nullspace of B, a subspace of $\mathbf{R}^2$:

$$N(B) = \left\{ \begin{bmatrix} -\frac{1}{2}x \\ x \end{bmatrix} : x \in \mathbf{R} \right\}$$

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