Erasmus School of Economics

MOOC Econometrics

Lecture P.2 on Building Blocks: Probability Distributions

Erik Kole

Erasmus University Rotterdam

Moments

Let x be a random variable with pdf $f(v)$.

- Mean: first moment, $\mu = E[x]$
- Variance: second centered moment, $\sigma^2 = var[x] = E[(x-\mu)^2]$
- Centered moment of order k

$$\mu_k = E\left[(x-\mu)^k\right]$$

  Often standardized by σ^k.
- Skewness: $\mu_3/\sigma^3 = E\left[(x-\mu)^3\right]/\sigma^3$ (asymmetry)
- Kurtosis: $\mu_4/\sigma^4 = E\left[(x-\mu)^4\right]/\sigma^4$ (tail fatness)

Independence

Independence

Two random variables x, y with joint density $f(v, w)$ and marginal densities $f_x(v)$, $f_y(w)$ are independent if and only if

$$f(v, w) = f_x(v) f_w(v) \quad \text{for all } v, w$$

Consequence: joint probability is product of marginal probabilities

$$P[a \leq x \leq b, c \leq y \leq d] = \int_a^b \int_c^d f(x, w)\, dw\, dv$$
$$= \int_a^b \int_c^d f_x(v) f_w(v)\, dw\, dv \stackrel{*}{=} \int_a^b f_x(v)\, dv \int_c^d f_y(w)\, dw$$
$$= P[a \leq x \leq b] \cdot P[c \leq y \leq d]$$

Independence and expectation

If x and y are independent, the expectation of a product is the product of expectations:

$$E[xy] = E[x]E[y]$$
$$E[g(x)h(y)] = E[g(x)]E[h(y)]$$

Test

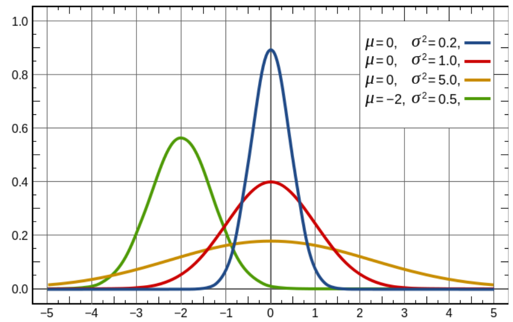
Calculate $\text{cov}[x, y]$ and $E[(x-\mu_x)^2(y-\mu_y)^2]$ when x and y are independent.

Answer

- $\text{cov}(x, y) = E[(x-\mu_x)(y-\mu_y)] = E[x-\mu_x]E[y-\mu_y] = 0.$
- $E[(x-\mu_x)^2(y-\mu_y)^2] = E[(x-\mu_x)^2]E[(y-\mu_y)^2] = \sigma_x^2\sigma_y^2.$

Normal distribution

$$\boxed{x \sim N(\mu, \sigma^2)}: \; f(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\mu)^2}{2\sigma^2}}$$



- $E[x] = \mu$ and $\text{var}[x] = \sigma^2$.
- Bell shaped, symmetric around μ.
- Skewness: 0; Kurtosis: 3.
- $y = (x - \mu)/\sigma$: standard normal distribution, $y \sim N(0, 1)$.

Multivariate normal distribution

$$\underset{(n\times 1)}{x} \text{ multivariate normal: } \quad x \sim N(\underset{(n\times 1)}{\mu}, \underset{(n\times n)}{\Sigma})$$

- $\text{E}[x] = \mu$, $\text{var}[x] = \Sigma$.
- $x_i \sim N(\mu_i, \sigma_i^2)$.
- If $\sigma_{ij} = 0$, x_i and x_j are independent.
- Linear transformations of x remain normal:

$$\underset{(m\times 1)}{y} = \underset{(m\times n)}{A} x + \underset{(m\times 1)}{b}, \quad \text{then } y \sim N(A\mu + b, A\Sigma A')$$

χ^2 (chi-square) distribution

Consider $y_i \sim NID(0, 1)$, $i = 1, 2, \ldots, n$

$$z = y'y = \sum_{i=1}^{n} y_i^2 : \quad y \sim \chi^2(n)$$

- NID: normally and independently distributed
- "z follows a χ^2-distribution with n degrees of freedom."
- $E[z] = E[\sum_{i=1}^n y_i^2] = \sum_{i=1}^n E[y_i^2] = \sum_{i=1}^n 1 = n$

Sequence of random variables

Test

Let $x_i \sim NID(\mu, \sigma^2)$, $i = 1, 2, \ldots, n$. Find the joint distribution of x, and the distributions of

$$y = \sum_{i=1}^{n} x_i \quad \text{and} \quad z = \sum_{i=1}^{n} (x_i - \mu)^2$$

Answer

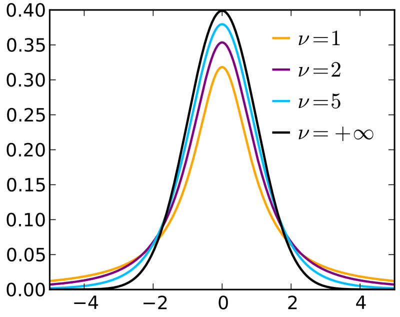
- $x \sim N(\mu\iota, \sigma^2 I)$, as $\sigma_{ij} = 0$ for all $i \neq j$.
- $y = \iota' x$ so $y \sim N(\mu_y, \sigma_y^2)$ with $\mu_y = \iota'\mu\iota = \iota'\iota\mu = n\mu$ and $\sigma_y^2 = \iota'\sigma^2 I\iota = \iota' I \iota\sigma^2 = \iota'\iota\sigma^2 = n\sigma^2$.
- z/σ^2: sum of squared iid normals, so $z/\sigma^2 \sim \chi^2(n)$.

Student's t distribution

Let $y \sim N(0,1)$ and $z \sim \chi^2(\nu)$, y, z independent

$$\frac{y}{\sqrt{z/\nu}} \sim t(\nu)$$

- $t(\nu)$: Student's t distribution with ν degrees of freedom.
- fatter tails than a normal distribution
- converges to a standard normal distribution for $\nu \to \infty$



F-distribution

Let $z_1 \sim \chi^2(d_1)$ and $z_2 \sim \chi^2(d_2)$, z_1 and z_2 independent

$$\frac{z_1/d_1}{z_2/d_2} \sim F(d_1, d_2)$$

- F-distribution with d_1 and d_2 degrees of freedom.

$t-$ and F-distributions

Test

Let $x_1, x_2 \overset{\text{iid}}{\sim} t(n)$. Find distributions for $x_1 + x_2$ and x_1^2.

Answer

$x_1 = \frac{y_1}{\sqrt{z_1/n}}$, with y_1 and z_1 independent; similar for x_2

- $x_1 + x_2 = \frac{y_1}{\sqrt{z_1/n}} + \frac{y_2}{\sqrt{z_2/n}}$, no further simplication
- $x_1^2 = \frac{y_1^2}{z_1/n}$, and $y_1^2 \sim \chi^2(1)$, so $x_1^2 \sim F(1, n)$.

Training Exercise P.2

- Train yourself by making the training exercise (see the website).
- After making this exercise, check your answers by studying the webcast solution (also available on the website).