Answers:

1. Dual 2: $max \sum_{j\in C} \alpha_j - \lambda k$

s.t. $\alpha_j - \beta_{ij} \leq c(i,j),\ \forall j \in C\ i \in F$

$\sum_{j\in C} \beta_{ij} \leq \lambda,\ \forall i \in F$

$\alpha_j \geq 0,\ \forall j \in C$

$\beta_{ij} \geq 0,\ \forall j \in C\ i \in F$

2. $3\lambda|S| + \sum_{\text{cluster } C_0} \sum_{j\in C_0} c(i_{C_0}, j) \leq 3. \sum_{j\in C} \alpha_j$

3. $3\lambda k + \sum_{\text{cluster } C_0} \sum_{j\in C_0} c(i_{C_0}, j) \leq 3. \sum_{j\in C} \alpha_j$

=> $\sum_{\text{cluster } C_0} \sum_{j\in C_0} c(i_{C_0}, j) \leq 3(\sum_{j\in C} \alpha_j - \lambda k) \leq 3$ OPT Dual 2

4. $\lambda = \lambda_1 = 0$

In this case, the dual 2 objective function is maximized when the algorithm opens as many facilities as possible (since we have no penalty). Now S contains exactly *k* facilities, so at least *k* of them would be opened by the algorithm.

$\lambda = \lambda_2 = \sum_{j\in C} \sum_{i\in F} c((i,j))$

In this case, the cost of opening a new facility is large, so the algorithm should open as less facility as possible, but it should open at least 1 facility. Hence, only one facility would be opened by the algorithm.

5. From question 2 & 3, we have the following bounds:

cost(S_1) = $\sum_{\text{cluster } C_0 \in S_1} \sum_{j\in C_0} c((i,j)) \leq 3(\sum_{j\in C} \alpha_j^1 - \lambda_1 k)$.

cost(S_2) = $\sum_{\text{cluster } C_0 \in S_2} \sum_{j\in C_0} c((i,j)) \leq 3(\sum_{j\in C} \alpha_j^2 - \lambda_2 k)$.

6. We have the following:

$\lambda_2 - \lambda_1 \leq \epsilon c_{\min}/(3|F|)$, for some fixed $\epsilon > 0$. (Given)

=> $-\lambda_1 \leq -\lambda_2 + \epsilon c_{\min}/(3|F|)$, for some fixed $\epsilon > 0$.

From qustion 5, $cost(S_1) \leq 3\sum_{j\in C} \alpha_j^1 - 3\lambda_1|S_1| \leq 3(\sum_{j\in C} \alpha_j^1 - \lambda_1|S_1|)$

$$\leq 3(\sum_{j\in C} \alpha_j^1 - \lambda_2|S_1|) + \frac{\epsilon c_{\min}|S_1|}{|F|}$$

$$\leq 3(\sum_{j\in C} \alpha_j^1 - \lambda_2|S_1|) + \epsilon c_{\min} \quad (\text{Since } |S_1| \leq |F|)$$

$$\leq 3(\sum_{j\in C} \alpha_j^1 - \lambda_2|S_1|) + \epsilon OPT$$

7. **Proof of lemma 1**

Lemma 1. $\delta_1 \text{cost}(S_1) + \delta_2 \text{cost}(S_2) \le (3+\delta_1\epsilon)$OPT.

Proof: From question 6, we have,

$$\delta_1 \text{cost}(S_1) + \delta_2 \text{cost}(S_2) \le 3\delta_1 \left(\sum_{j\in C} \alpha_j^1 - \lambda_2|S_1|\right) + \delta_1\epsilon\text{OPT} + 3\delta_2\left(\sum_{j\in C} \alpha_j^2 - \lambda_2|S_2|\right).$$

$$= 3(\sum_{j\in C}((\delta_1\alpha_j^1 + \delta_2\alpha_j^2) - \lambda_2(\delta_1|S_1| + \delta_2|S_2|)) + \delta_1\epsilon OPT$$

$$= 3\left(\sum_{j\in C} \tilde{\alpha}_j - \lambda_2 k\right) + \delta_1\epsilon\text{OPT}.$$

$$= 3OPT + \delta_1\epsilon OPT$$

$$= (3+\delta_1\epsilon)\text{OPT}$$

=> $\delta_1 \text{cost}(S_1) + \delta_2 \text{cost}(S_2) \le (3+\delta_1\epsilon)$OPT.

8. Proof of the fact that

S_2 is a solution of cost at most $2(3+\epsilon)$ OPT

$\text{cost}(S_2) \le 2\delta_2 \text{cost}(S_2)$ Since $\delta_2 \ge \frac{1}{2}$

$\le 2(\delta_1 \text{cost}(S_1) + \delta_2 \text{cost}(S_2)) \le 2(3+\delta_1\epsilon)$OPT by question 7

$\le 2(3+\epsilon)$OPT.

9. The probability that the randomized algorithm opens *f1* = $\dfrac{\binom{k-|S_2|}{1}}{\binom{|S_1|-|S_2|}{1}} = \dfrac{k-|S_2|}{|S_1|-|S_2|} = \delta_1$

10. Since i is the facility closest to f_2 (by assumption), we have,

$$c((i,f_2)) \le c((f_1,f_2))$$

11. Proof:

$$c((i,j)) \le c((f_1,f_2)) + c((f_2,j))$$

$\le c((f_1,j)) + c((f_2,j)) + c((f_2,j))$ Since $c((f_1,f_2)) \le c((f_1,j)) + c((f_2,j))$

$$= c_j^1 + 2c_j^2.$$

12. The expected cost for client j $= \Pr(j \text{ assigned to } f_1).c_j^1 + \Pr(j \text{ assigned to } i).c((i,j))$

$= \delta_1 c_j^1 + (1-\delta_1)c((i,j))$ the probability that the randomized algorithm opens f_1 is δ_1

the probability that the randomized algorithm does not open f_1 is $1-\delta_1 = \delta_2$.

$\le \delta_1 c_j^1 + \delta_2(c_j^1 + 2c_j^2)$. (by question 11)

13. The expected cost for client j $\leq \delta_1 c_j^1 + \delta_2(c_j^1 + 2c_j^2) \leq 2(\delta_1 c_j^1 + \delta_2 c_j^2)$, since $\delta_2 \leq \frac{1}{2} \leq \delta_1$.

Hence, we have,

Total cost of S

$$\begin{aligned}
&\leq 2(\delta_1 cost(S_1) + \delta_2 cost(S_2)) \\
&\leq 2(3 + \epsilon\delta_1)OPT \qquad \text{from question 7} \\
&\leq 2(3 + \epsilon)OPT
\end{aligned}$$