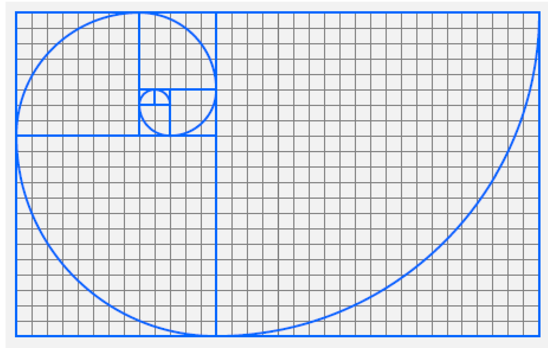
What Can Go Wrong with Induction?

The Fibonacci sequence
$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \ldots$

is defined as: $F_0 = 1, F_1 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for all $n \geq 2$.

This sequence is one of the most popular integer sequences with applications not only in mathematics and computer science, but also in music and even nature! Read more on Wikipedia or the Online Encyclopedia of Integer Sequences.

The picture below shows a tiling of a rectangle by squares with side lengths $1, 1, 2, 3, 5, 8, 13, 21$ and a spiral through their corners.



The growth factor of the spiral is the *golden ratio*:
$\phi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$
that also reflects the growth rate of F_n.

The exact value of F_n is given by Binet's formula:
$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$
where $\psi = 1 - \phi = -\phi^{-1}$.

Interestingly, the formula involves irrational numbers though F_n is an integer. A similar closed-form expression can be derived for any sequence defined recursively (with constant coefficients).

Find a mistake in the following proof.

Problem:

Prove that $F_n \geq 2^{n/2}$ for all $n \geq 6$.

Solution:

The proof is by induction. The base case $n = 6$ holds trivially: $F_6 = 8 = 2^{6/2}$. Let us prove the induction step

$$F_n = F_{n-2} + F_{n-1} \geq 2^{\frac{n-2}{2}} + 2^{\frac{n-1}{2}}$$

where the last inequality is due to induction hypothesis. Now,

$$F_n \geq 2^{\frac{n-2}{2}} + 2^{\frac{n-1}{2}} = 2^{\frac{n-2}{2}} \cdot (1 + \sqrt{2}) \geq 2^{\frac{n-2}{2}} \cdot 2 = 2^{\frac{n}{2}}$$

The issue with this solution is quite subtle: we prove the base case only for one value of n, but we use it as if we proved it for at least two values of n. It is easier to see this for specific values of n. In the base case, we only proved that $F_n \geq 2^{n/2}$ for $n = 6$. When we prove this inequality for $n = 7$, the induction step assumes that the inequality holds for $n = 6$ *and* $n = 5$. Which is not true! We have only verified the inequality for $n = 6$. (If it held for $n = 5$ too, then the whole induction proof would be correct.)

This teaches us an important lesson:

If the induction step assumes the correctness of the statement for several smaller values of n, then we have to examine several values of n in the base case. For example, if the induction step assumes that the statement holds for n and $n - 1$, and proves the correctness of the statement for $n + 1$, then the induction base must cover at least two consecutive values of n.

While the proof above is flawed, the statement is actually correct: $F_n \geq 2^{n/2}$ for all $n \geq 6$. One way to prove this is to check the base cases $n = 6, n = 7$, and then repeat the previous proof of the induction step. This flaw in the proof may seem insignificant, but in fact it is very dangerous. For example, the same (flawed) argument could be used to prove the following incorrect statement:

F_n is even for all $n \geq 6$:

This holds for $n = 6$, and if F_{n-2} and F_{n-1} are even, then $F_n = F_{n-2} + F_{n-1}$ would be even too.

This statement is not even correct! We will see more examples of such flaws below.

Find a mistake in the following proof:

Problem:

Prove that any $n \geq 1$ points on the plane lie on a line.

Solution:

Of course, for any point there is a line containing it, so the base case **n=1** holds. For the induction step from n to $n + 1$, consider points $p_1, p_2, \ldots, p_{n+1}$. By the induction hypothesis, there is a line L_1 containing the points $p_1, p_2, \ldots, p_n$. Similarly, there is a line L_2 containing the points $p_2, \ldots, p_{n+1}$. Since L_1 and L_2 both contain all the points $p_2, p_3, \ldots, p_n$, L_1 and L_2 are the same line. Therefore, all $n + 1$ points lie on one line.

It is not true that all points lie on a line. For example, consider three points of a (non-degenerate) triangle, there is no line passing through all three of them.

Again, let us carefully examine what assumptions the proof of the induction step requires. We find a line L_1 passing through the first n points, and we can indeed find such a line by the induction hypothesis. Similarly, we can indeed find L_2. If both these two lines passed through at least two (distinct) points, then they would indeed coincide. They have $n - 1$ points in common: $p_2, \ldots, p_n$. But for $n = 2$, this gives just a single common point.

Hence, for $n = 2$ we cannot conclude that L_1 and L_2 are the same line, and cannot finish the proof of the induction step. This induction proof failed only because we could not finish the proof of the case $n = 2$. While the induction step would indeed work for $n > 2$, there is no base case to base this proof on.

Find a mistake in the following proof.

Problem:

Prove that all horses are of the same color.

Solution:
We prove this statement by induction on n - the number of horses. The base case of $n = 1$ is indeed trivial. Assuming that the induction hypothesis holds for any set of n horses, we prove the statement for $n + 1$ horses. Let us consider the first n horses. By the induction hypothesis they all are of the same color. Now let us take the last n horses, similarly, they are of the same color. But then the $n - 1$ horses in the middle are of the same color as the first n horses and the last n horses. This implies that all $n + 1$ horses are of the same color.

See other examples of horse paradox on Wikipedia.

The next problem is similar to the previous one. Try to carefully examine what assumptions the induction step of the proof makes, and find a mistake in this proof.

Find a mistake in the following proof.

Problem:

Prove that for any integer $n \geq 0, 5n = 0$.

Solution:

The base case holds trivially: $n = 0$ implies that $5n = 0$. For the induction step, we assume that the statement holds for all numbers 0 through n, and we want to prove the statement for $n + 1$. First, we write $n + 1 = i + j$ where $i, j \leq n$. Then
$5(n + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0$

Again, this proof "almost works", the induction step only fails for the case of $n = 1$. The induction step assumes that $n + 1$ can be written as a sum $n + 1 = i + j$ where $i, j \leq n$. But this is not true for $n = 0$: no sum of numbers $i, j \leq 0$ would give $n + 1 = 1$.

Final notes

Key points about mathematical induction:

- Mathematical induction is used to prove that some statements $A(i)$ hold for all values of i.
- An induction proof consists of two parts: the base case and the induction step.
- The base case assures that $A(i)$ holds for some (not necessarily small) values of i.
- The base case and the induction step must be consistent: if the induction step uses $A(n)$ and $A(n-1)$, then the base case should cover at least 2 consecutive values of n.
- The induction step for proving $A(n+1)$ can use $A(n)$ or even all $A(i)$ for $i \leq n$. The latter is called strong induction.
- Sometimes proving a stronger statement by induction may be easier.