# RESTRICTIONS IN REGRESSION MODEL

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Regression analysis is used to establish a relationship via an equation for predicting values of one variable given the value of another variable. It may be used for forecasting to determine future supply, demand, pricing, sales, yield, or some other variable of interest.

Suppose $X_1, X_2, \ldots, X_p$ are the cause of variation in Y, the multiple regression equation of Y on X's to account for this variation is as follows:

$$Y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \ldots\ldots + \beta_k x_{pt} + u_t$$

where $\beta_0$ denotes intercept and $\beta_i$'s $(i = 1,2,\ldots,p)$ are called regression coefficients and u is random error. $\beta_i$ gives average change in Y per unit change in $x_i$ keeping other x's constant. Further,

$$E(u_t) = 0, \quad V(u_t) = E(u_t^2) = \sigma^2, \ Cov(u_t, u_s) = E(u_t u_s) = 0,$$

$$Cov(X_{it}, u_t) = E(X_{it} u_t) = X_{it} E(u_t) = 0.$$

The above regression model in matrix notation can be written as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

Where **Y** is a n x 1 vector of observations, **X** is n x (p+1) matrix corresponding to p independent variables, $\boldsymbol{\beta}$ is (p+1) x 1 vector of parameters or regression coefficients and **u** is the n x 1 vector of errors with $E(u) = 0$ and $E(u\,u') = \sigma^2 I_n$.

A great many problems in econometrics are concerned with the estimation of regression coefficients subject to linear restrictions and the testing of linear restrictions. In demand analysis one may wish to test the homogeneity postulate: for demand functions linear in the logarithms of quantity demanded, absolute prices and money income. This postulate is equivalent to the restriction that the sum of the coefficients of the logarithms of prices and income is zero. Similarly in the estimation of Cobb-Douglas production functions, the hypothesis of constant returns to scale is equivalent to the restriction that the sum of the coefficients of the inputs (in logarithms) is unity.

The simplest type of linear restriction is the specification that one or more regression coefficients are equal to zero or some other constant. Any linear restriction can, by suitable transformation of variables, be reduced to a restriction of that type. In the above examples, this is equivalent to replacing absolute prices and income by prices and income expressed relative to a numaraire, and applying the simple restriction to the coefficient of the logarithm of the numaraire.

The restrictions in the above regression model could be $\beta_1 + \beta_2 = 0$, $\beta_1 = \beta_2$ or $\beta_3 = \beta_5$. The restrictions in the above regression model can be written as

$$\mathbf{R\beta = r}$$

**R** is the q x p ($q \le p$) matrix and **r** is a q x 1 vector of restrictions. For example,

1. $\mathbf{R} = [0 \ldots\ldots \underset{\substack{\uparrow \\ \text{i th position}}}{0\ 1}\ 0 \ldots\ldots 0]$ and $\mathbf{r} = \mathbf{0}$

The null hypothesis here is $H_0 : \beta_i = 0$ .

2. $\mathbf{R} = [0\ 1\ \text{-}1\ 0 \ldots\ldots 0]$ and $\mathbf{r} = \mathbf{0}$

Here $H_0 : \beta_2 - \beta_3 = 0$ or $\beta_2 = \beta_3$ .

3. $\mathbf{R} = [0\ 0\ 1\ 1\ 0 \ldots\ldots 0]$ and $\mathbf{r} = 1$

$H_0 : \beta_3 + \beta_4 = 1$ .

4. $$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 & \ldots\ldots\ldots 0 \\ 0 & 0 & 1 \ldots\ldots\ldots & 0 \\ . & . & . & . \\ 0 & 0 & 0 & \ldots. 1 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \end{bmatrix}$$

results in

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ . \\ \beta_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \end{bmatrix}$$

**Estimation of Parameters under Restrictions**

The method used for restricting the parameter estimates is to introduce a Lagrangian parameter for each restriction. The method consist of minimizing the Lagrangean Function

$$(\mathbf{Y} - \mathbf{X\beta})' (\mathbf{Y} - \mathbf{X\beta}) + \lambda' (\mathbf{r} - \mathbf{R\beta})$$

with respect to β and λ.

The least square estimator under restrictions is

$$\hat{\boldsymbol{\beta}}^* = \hat{\boldsymbol{\beta}} + (\mathbf{X'X})^{-1} \mathbf{R'} \left[\mathbf{R} (\mathbf{X'X})^{-1} \mathbf{R'}\right]^{-1} (\mathbf{r} - \mathbf{R}\hat{\boldsymbol{\beta}})$$

where

$$\hat{\boldsymbol{\beta}} = (\mathbf{X'X})^{-1} \mathbf{X'Y}$$

and

$$V(\hat{\boldsymbol{\beta}}^*) = \sigma^2\,(\mathbf{X}'\mathbf{X})^{-1} - \sigma^2\,(\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{R}'\left[\mathbf{R}\,(\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{R}'\right]^{-1}\mathbf{R}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

The first part of the above variance expression is the variance under unrestricted model. It is thus seen that the variance of parameter estimates is reduced and the parameters are estimated more precisely.

**Test Statistic**
Let the number of restrictions be s, the test statistic is as follows:

$$F = \frac{\left(RSS_r - RSS_u\right)/s}{RSS_u/(n-p)}$$

$RSS_r$ is the residual sum of squares under restricted model and $RSS_u$ is the residual sum of squares under unrestricted model.

**Using SAS**
A RESTRICT statement is used to place restrictions on the parameter estimates in the MODEL preceding it. More than one RESTRICT statement can follow each MODEL statement. Each RESTRICT statement replaces any previous RESTRICT statement. To lift all restrictions on a model, submit a new MODEL statement. If there are several restrictions, separate them with commas. The statement

restrict equation1 = equation2 = equation3;

is equivalent to imposing the two restrictions

restrict equation1 = equation2;

restrict equation2 = equation3;

Each restriction is written as a linear equation and can be written as
equation
or
equation = equation

The form of each **equation** is

$$c_1 \times \text{variable}_1 \pm c_2 \times \text{variable}_2 \pm \ldots \pm c_n \times \text{variable}_n$$

where the $c_j$'s are constants and the $\text{variable}_j$'s are any regressor variables.

When no equal sign appears, the linear combination is set equal to zero. Each variable name mentioned must be a variable in the MODEL statement to which the RESTRICT statement refers. The keyword INTERCEPT can also be used as a variable name, and it refers to the intercept parameter in the regression model.

Note that the parameters associated with the variables are restricted, not the variables themselves. Restrictions should be consistent and not redundant.

Examples of valid RESTRICT statements include the following:

```
restrict x1;
restrict a + b = l;
restrict a = b = c;
restrict a = b, b = c;
restrict 2*f = g + h, intercept + f = 0;
restrict f = g = h = intercept;
```

The third and fourth statements in this list produce identical restrictions. The following restrictions cannot be specified because these restrictions are not consistent:

```
restrict f - g = 0,
f-intercept = 0,
g-intercept = 1;
```

If these restrictions are included in a RESTRICT statement, one of the restrict parameters is set to zero and has zero degrees of freedom, indicating that PROC REG is unable to apply a restriction.

The restrictions usually operate even if the model is not of full rank. Check to ensure that degrees of freedom for each restriction. In addition, the model degrees of freedom should decrease by 1 for each restriction.

The parameter estimates are those that minimize the quadratic criterion (Sum of Squares due to Error, RSS) subject to the restrictions. If a restriction cannot be applied, its parameter value and degrees of freedom are listed as zero.

The method used for restricting the parameter estimates is to introduce a Lagrangian parameter for each restriction. The estimates of these parameters are displayed with test statistics. The t-statistic reported for the Lagrangian parameters does not follow a Student's t-distribution, but its square follows a beta distribution. The p-value for these parameters is computed using the beta distribution.

The Lagrangian parameter $\lambda$ measures the sensitivity of the RSS to the restriction constant. If the restriction constant is changed by a small amount e, the RSS is changed by $2\lambda e$. The t-ratio tests the significance of the restrictions. If $\lambda$ is zero, the restricted estimates are the same as the unrestricted estimates, and a change in the restriction constant in either direction increases the RSS.

RESTRICT statements are ignored if the PCOMIT = or RIDGE = option is specified in the PROC REG statement.

**SAS Syntax**

```
/*Linear Restrictions */
  proc reg data = aids;
  model w_beef = lpb lpp lpc lpt lxp co1 si1 t;
  restrict intercept = 0;
  restrict lpb + lpp = 0;
  run;

/*cross restrictions across equations restrictions*/
proc syslin itsur data = aids;
    b: model w_beef = lpb lpp lpc lpt lxp t ;
    p: model w_pork = lpb lpp lpc lpt lxp t ;
    c: model w_chick = lpb lpp lpc lpt lxp t ;

srestrict
/*symmetry restrictions */
    b.lpp = p.lpb ,
    b.lpc = c.lpb ,
    p.lpc = c.lpp ,
 /* homogeneity restrictions*/
    b.lpb + b.lpp + b.lpc + b.lpt = 0 ,
    p.lpb + p.lpp + p.lpc + p.lpt = 0 ,
    c.lpb + c.lpp + c.lpc + c.lpt = 0;
run;
```

where
w_ denotes share of expenditure of the item in total expenditure
lpb = log of price of beef
lpp = log of price of pork
lpc = log of price of chicken
lpt = log of price of turkey
lxp = log of monthly expenditure adjusted using Stone's price index
t = time trend

For details of the data set and variables, pl. refer to
http://support.sas.com/rnd/app/examples/ets/elasticity/sas.htm

**Illustration (Linear Restriction):** Following is the data on consumption of rice and wheat which depends on the price of rice and wheat along with the income:

| PriceR | PriceW | Income | ConsumptionR | ConsumptionW |
|---|---|---|---|---|
| 12.98 | 0.32 | 10 | 57.7 | 23.67 |
| 14.3 | 2.03 | 6.78 | 59.3 | 32.12 |
| 15.53 | 5.31 | 2.95 | 56.17 | 34.64 |
| 15.13 | 4.74 | 4.2 | 55.77 | 22.5 |
| 15.34 | 7.04 | 2.05 | 51.72 | 40.12 |

| | | | | |
|---|---|---|---|---|
| 17.15 | 5.98 | 0.06 | 60.45 | 48.22 |
| 15.46 | 2.74 | 4.66 | 60.72 | 29 |
| 12.8 | 10.66 | 3.05 | 37.45 | 44.44 |
| 17.04 | 5.13 | 0.26 | 60.97 | 54.32 |
| 13.17 | 2.04 | 8.74 | 55.27 | 36.23 |
| 16.13 | 2.27 | 2.1 | 59.29 | 21.21 |
| 14.34 | 4.08 | 5.55 | 54.03 | 33.33 |
| 12.92 | 2.64 | 9.33 | 53.2 | 61.16 |
| 14.23 | 10.4 | 1.04 | 41.9 | 39 |
| 15.22 | 1.22 | 6.15 | 63.26 | 25.25 |
| 15.74 | 10.61 | 1.69 | 45.8 | 12.7 |
| 14.96 | 4.82 | 4.11 | 58.7 | 31.26 |
| 14.13 | 3.15 | 8.45 | 50.09 | 62.22 |
| 16.39 | 9.7 | 1.71 | 48.89 | 51.23 |
| 16.45 | 3.91 | 2.15 | 62.21 | 26.3 |
| 13.54 | 7.63 | 3.85 | 45.63 | 27.98 |
| 14.2 | 4.47 | 5.11 | 53.92 | 35.5 |
| 15.84 | 5.75 | 2.09 | 55.8 | 31.3 |
| 16.57 | 8.55 | 8.97 | 56.74 | 22.11 |
| 13.32 | 8.59 | 4.01 | 43.15 | 20.2 |
| 15.95 | 8.29 | 0.25 | 50.71 | 34.5 |

```
Data Reg;
Input PriceR PriceW Income ConsumptionR;
Cards;
{The above data to be entered here except for ConsumptionW}

PROC REG
  Model ConsumptionR = PriceR PriceW Income ;
  Restrict PriceR + PriceW = 0;
  run;
quit;
```

**Output**

The SAS System 14:04 Friday, July 1, 2011 1

The REG Procedure
Model: MODEL1
Dependent Variable: ConsumptionR

NOTE: Restrictions have been applied to parameter estimates.

| | |
|---|---|
| Number of Observations Read | 26 |
| Number of Observations Used | 26 |

Analysis of Variance

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 2 | 1001.56870 | 500.78435 | 87.68 | <.0001 |
| Error | 23 | 131.35904 | 5.71126 | | |
| Corrected Total | 25 | 1132.92774 | | | |

| | | | |
|---|---|---|---|
| Root MSE | 2.38982 | R-Square | 0.8841 |
| Dependent Mean | 53.80154 | Adj R-Sq | 0.8740 |
| Coeff Var | 4.44193 | | |

Parameter Estimates

| Variable | DF | Parameter Estimate | Standard Error | t Value | Pr > \|t\| |
|---|---|---|---|---|---|
| Intercept | 1 | 35.39259 | 1.53273 | 23.09 | <.0001 |
| PriceR | 1 | 2.06875 | 0.15841 | 13.06 | <.0001 |
| PriceW | 1 | -2.06875 | 0.15841 | -13.06 | <.0001 |
| Income | 1 | -0.29140 | 0.16628 | -1.75 | 0.0930 |
| RESTRICT | -1 | 32.46817 | 10.46372 | 3.10 | 0.0006* |

* Probability computed using beta distribution.

It is seen that the restriction is significant indicating that the sum total of the coefficients pertaining to price R and price W is not zero.

**Illustration (Restriction on System of Equations):**

```
Data syslin;
input PriceR PriceW Income ConsumptionR ConsumptionW;
cards;
   12.98    0.32     10     57.7   23.67
    14.3    2.03   6.78     59.3   32.12
   15.53    5.31   2.95    56.17   34.64
   15.13    4.74    4.2    55.77    22.5
   15.34    7.04   2.05    51.72   40.12
   17.15    5.98   0.06    60.45   48.22
   15.46    2.74   4.66    60.72      29
    12.8   10.66   3.05    37.45   44.44
   17.04    5.13   0.26    60.97   54.32
   13.17    2.04   8.74    55.27   36.23
   16.13    2.27    2.1    59.29   21.21
   14.34    4.08   5.55    54.03   33.33
```

| | | | | |
|---|---|---|---|---|
| 12.92 | 2.64 | 9.33 | 53.2 | 61.16 |
| 14.23 | 10.4 | 1.04 | 41.9 | 39 |
| 15.22 | 1.22 | 6.15 | 63.26 | 25.25 |
| 15.74 | 10.61 | 1.69 | 45.8 | 12.7 |
| 14.96 | 4.82 | 4.11 | 58.7 | 31.26 |
| 14.13 | 3.15 | 8.45 | 50.09 | 62.22 |
| 16.39 | 9.7 | 1.71 | 48.89 | 51.23 |
| 16.45 | 3.91 | 2.15 | 62.21 | 26.3 |
| 13.54 | 7.63 | 3.85 | 45.63 | 27.98 |
| 14.2 | 4.47 | 5.11 | 53.92 | 35.5 |
| 15.84 | 5.75 | 2.09 | 55.8 | 31.3 |
| 16.57 | 8.55 | 8.97 | 56.74 | 22.11 |
| 13.32 | 8.59 | 4.01 | 43.15 | 20.2 |
| 15.95 | 8.29 | 0.25 | 50.71 | 34.5 |

```
PROC SYSLIN;
a:model ConsumptionR = PriceR PriceW Income;
b:model ConsumptionW = PriceR PriceW Income;
srestrict a.PriceW=b.PriceR;
run;
```

**Output**

The SAS System 14:11 Friday, July 1, 2011 1

The SYSLIN Procedure
Ordinary Least Squares Estimation

Model A
Dependent Variable ConsumptionR

Analysis of Variance

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 3 | 1056.557 | 352.1855 | 101.45 | <.0001 |
| Error | 22 | 76.37112 | 3.471415 | | |
| Corrected Total | 25 | 1132.928 | | | |

| | | | |
|---|---|---|---|
| Root MSE | 1.86317 | R-Square | 0.93259 |
| Dependent Mean | 53.80154 | Adj R-Sq | 0.92340 |
| Coeff Var | 3.46305 | | |

Parameter Estimates

| Variable | DF | Parameter Estimate | Standard Error | t Value | Pr > \|t\| |
|---|---|---|---|---|---|
| Intercept | 1 | 11.02127 | 6.225542 | 1.77 | 0.0905 |
| PriceR | 1 | 3.417049 | 0.360151 | 9.49 | <.0001 |
| PriceW | 1 | -1.71922 | 0.150909 | -11.39 | <.0001 |
| Income | 1 | 0.255105 | 0.188387 | 1.35 | 0.1894 |

| | |
|---|---|
| Model | B |
| Dependent Variable | ConsumptionW |

Analysis of Variance

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 3 | 9.614180 | 3.204727 | 0.02 | 0.9967 |
| Error | 22 | 4009.169 | 182.2350 | | |
| Corrected Total | 25 | 4018.783 | | | |

| | | | |
|---|---|---|---|
| Root MSE | 13.49944 | R-Square | 0.00239 |
| Dependent Mean | 34.63500 | Adj R-Sq | -0.13365 |
| Coeff Var | 38.97630 | | |



The SYSLIN Procedure
Ordinary Least Squares Estimation

Parameter Estimates

| Variable | DF | Parameter Estimate | Standard Error | t Value | Pr > \|t\| |
|---|---|---|---|---|---|
| Intercept | 1 | 64.03599 | 9.811379 | 6.53 | <.0001 |
| PriceR | 1 | -1.71922 | 0.150909 | -11.39 | <.0001 |
| PriceW | 1 | -0.25633 | 1.055147 | -0.24 | 0.8103 |
| Income | 1 | -0.54454 | 1.078554 | -0.50 | 0.6187 |

Parameter Estimates

| Variable | DF | Parameter Estimate | Standard Error | t Value | Pr > \|t\| |
|---|---|---|---|---|---|
| RESTRICT | -1 | -0.11323 | 0.383289 | -0.30 | 0.7753 |

**References**

Draper, N.R. and Smith, H. (1998). Applied Regression Analysis, 3rd Edition, Wiley Pub., New York.

Waterman, M.S (1974). A restricted least squares problem. *Technometrics*, 16, 135-138.