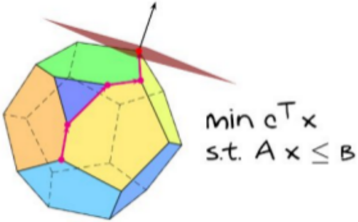
Duality

- Dualizing the dual
- Other forms of duals
- Proving optimality
- Farkas' lemma



The dual of the dual is the primal

$\max c^T x$

$Ax \le b$

$\min\ b^T \cdot y$

$A^T \cdot y = c$

$y \ge 0$

$\approx\ (-)$

$\mathrm{MAX}\ -b^T \cdot y$

$A^T \cdot y \le c$

$-A^T y \le -c$

$-I \cdot y \le 0$

$\approx\ (-)$

$\mathrm{MAX}\ -b^T \cdot y$

$$\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \begin{pmatrix} A^T \\ -A^T \\ -I \end{pmatrix} y \le \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}$$

$(-)$

$$\mathrm{MIN}\ \underbrace{c^T \cdot y_1 - c^T y_2}_{c^T(y_1 - y_2)} + 0^T \cdot y_3$$

$A \cdot y_1 - A \cdot y_2 - y_3 = -b$

$y_1, y_2, y_3 \ge 0$

$$\mathrm{MAX}\ c^T(\overbrace{y_2 - y_1}^{y})$$

$\approx$

$A(y_2 - y_1) + y_3 = b$

$y_1, y_2, y_3 \ge 0$

$\mathrm{MAX}\ c^T \cdot y,\quad A \cdot y \le b$

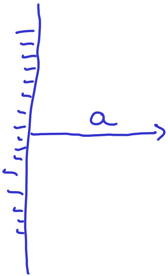
Which combinations are possible?

P \ D	Finite Opt	Unbounded	Infeasible
Finite Opt			
Unbounded			
Infeasible			possible

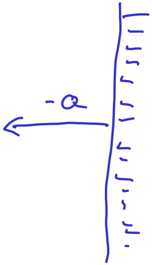
Infeasible primal and dual

max $c^T \cdot x$

$Ax \le b$



$a^T \cdot x \le \beta_1$



$-a^T \cdot x \le \beta_2$

min $b^T \cdot y$

$A^T \cdot y = c$

$y \ge 0$

$c \perp a$

c not even a linear comb. of rows of A.

Example

Dual of $\max\ c^T \cdot x$

$$Ax \le b$$
$$x \ge 0$$

$\max\ c^T \cdot x$

$$\begin{array}{c} y \\ \underbrace{A^T \cdot y - c}_{\ge 0} \end{array} \begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$A^T \cdot y - A^T \cdot y + c = c$$

<u>Dual:</u>

$\min\ b^T \cdot y$

$$A^T \cdot y \ge c$$
$$y \ge 0$$

$\min\ b^T \cdot y$

$$A^T \cdot y = c$$
$$y \ge 0$$

Example

Dual of $\max c^T \cdot x$

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$\min \; b^T \cdot y_1 + 0^T \cdot y_2$$

$$(A^T, -I)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c \qquad \approx \qquad \boxed{\begin{aligned} &\min \; b^T \cdot y_1 \\ &A^T \cdot y_1 \leq c \\ &y_1 \geq 0 \end{aligned}}$$

$$y_1, y_2 \geq 0$$

Proving optimality

$\boxed{\text{LP-solver 1}}$

feasible $x^* \in \mathbb{R}^n$

Says it's optimal

$$\max\ c^T \cdot x$$
$$Ax \leq b$$

$\boxed{\text{LP-solver 2}}$

feasible x^*, y^*

(P) (D)

$$c^T \cdot x^* = b^T \cdot y^*$$

Proof of optimality

Simplex returns x^*, y^*

size of x^* and y^* is polynomial in size of LP.

Farkas' Lemma

A system of inequalities $Ax \leqslant b$ is infeasible if and only if there exists $\lambda \geqslant 0$ such that $\lambda^T A = 0$ and $\lambda^T b = -1$.

Proof:

"$\Leftarrow$" $\quad \underbrace{(\lambda^T \cdot A)}_{=0^T} x \leq \underbrace{\lambda^T \cdot b}_{=-1}$ valid ineq.

$\Rightarrow$ $Ax \leq b$ infeasible

"$\Rightarrow$" $\quad$ MAX $0^T \cdot x$ $\quad$ DUAL: MIN $b^T \cdot y$

$Ax \leq b$ $\quad$ $A^T \cdot y = 0$

$y \geq 0$

$y^* = 0$ feas. sol. $\Rightarrow$ DUAL feas.

DUAL UNBOUNDED !

$\exists y^* \geq 0, \; A^T \cdot y^* = 0, \; b^T \cdot y^* < 0$, $\quad y' = y^* / |b^T \cdot y^*|$

$y' \geq 0, \; A^T \cdot y' = 0, \; b^T \cdot y' = \underbrace{b^T \cdot y^*}_{<0} / |b^T \cdot y^*| = -1$ $\quad \lambda = y'$ $\quad \blacksquare$