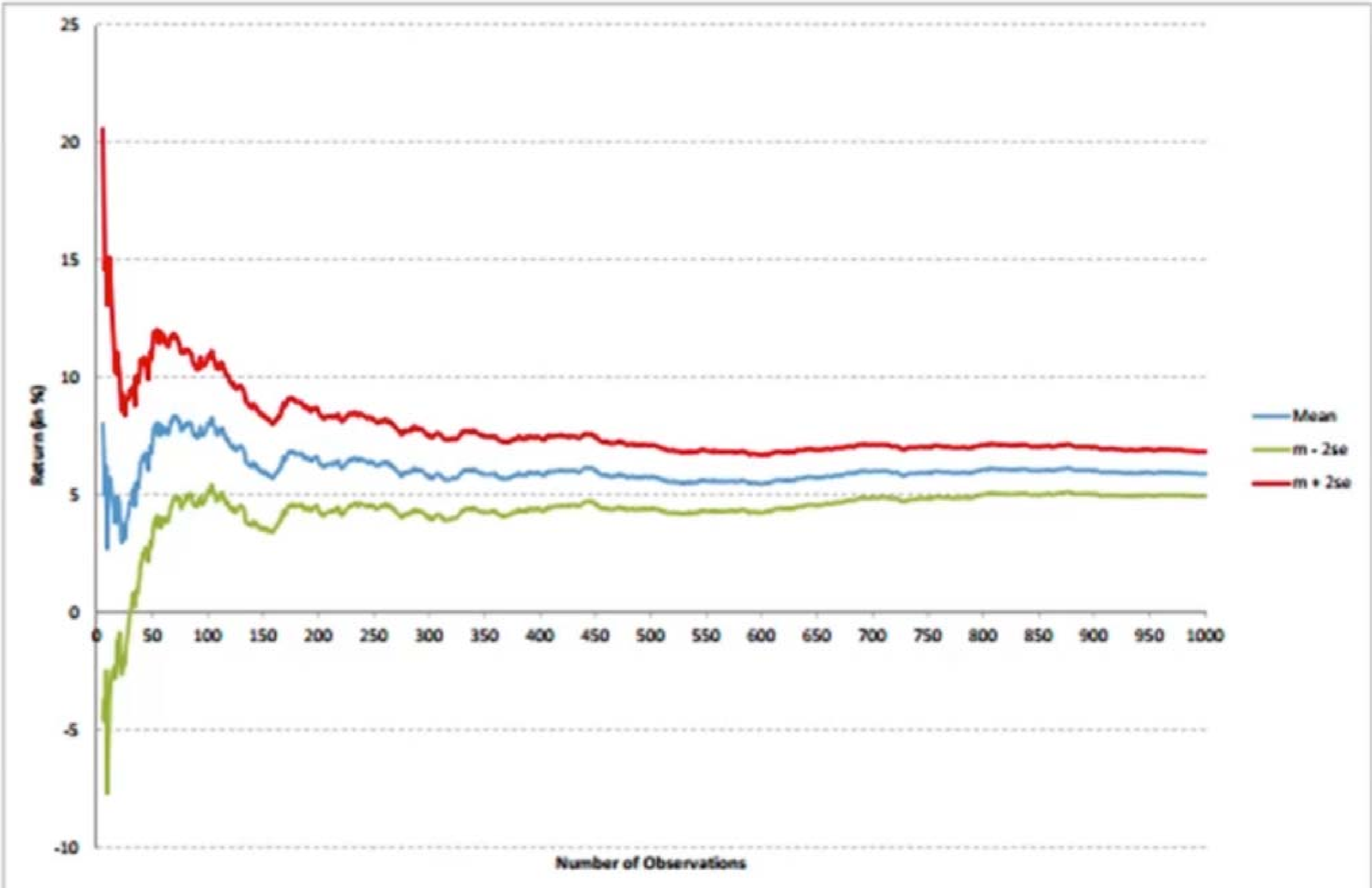
1 a) $se_i = \sqrt{var(m_i)} = \sqrt{\frac{1}{i-1} \sum_{j=1}^{i} (y_j - m_i)^2}$



25
20
15
10
5
0
-5
-10
Return (in %)
0
50
100
150
200
250
300
350
400
450
500
550
600
650
700
750
800
850
900
950
1000
Number of Observations
Mean
m - 2se
m + 2se

b) $se = \sigma/\sqrt{n}$ $\quad 4\sigma/\sqrt{n} = 1 \quad n = 16\sigma^2$

$\sigma = 15\% \quad 16 \cdot 15^2 = 3600$ years

2a) $$E[y] = H\mu = \begin{pmatrix} 1_{n_1} & 0_{n_1} \\ 0_{n_2} & 1_{n_2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1_{n_1}\mu_1 + 0_{n_1}\mu_2 \\ 0_{n_1}\mu_1 + 1_{n_2}\mu_2 \end{pmatrix} = \begin{pmatrix} 1_{n_1}\mu_1 \\ 1_{n_2}\mu_1 \end{pmatrix}$$

$$Var[y] = \sigma^2 I_n$$

b) $$H'H = \begin{pmatrix} 1_{n_1} & 0_{n_1} \\ 0_{n_2} & 1_{n_2} \end{pmatrix}' \begin{pmatrix} 1_{n_1} & 0_{n_1} \\ 0_{n_2} & 1_{n_2} \end{pmatrix} = \begin{pmatrix} 1_{n_1}' & 0_{n_2}' \\ 0_{n_1}' & 1_{n_2}' \end{pmatrix} \begin{pmatrix} 1_{n_1} & 0_{n_1} \\ 0_{n_2} & 1_{n_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1_{n_1}'1_{n_1} + 0_{n_2}'0_{n_2} & 1_{n_1}'0_{n_1} + 0_{n_2}'1_{n_2} \\ 0_{n_1}'1_{n_1} + 1_{n_2}'0_{n_2} & 0_{n_1}'0_{n_1} + 1_{n_2}'1_{n_2} \end{pmatrix} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}$$

$$= T$$

$$TT^{-1} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \begin{pmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{1}{n_2} \end{pmatrix} = \begin{pmatrix} n_1\frac{1}{n_1} + 0 \cdot 0 & n_1 \cdot 0 + 0\frac{1}{n_2} \\ 0 \cdot \frac{1}{n_1} + n_2 \cdot 0 & 0 \cdot 0 + n_2\frac{1}{n_2} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

c) $E(m) = E[T^{-1}H'y] = T^{-1}H'E[y] = T^{-1}H'H\mu = T^{-1}T\mu = \mu$

d) $E[z] = E[y - Hm] = E[y] - HE[m] = H\mu - H\mu = 0_n$

$$My = (I_n - HT^{-1}H')y = I_n y - HT^{-1}H'y = y - H(T^{-1}H'y)$$
$$= y - Hm = z$$

e) $$M = I_n - HT^{-1}H'$$
$$= \begin{pmatrix} I_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & I_{n_2} \end{pmatrix} - \begin{pmatrix} \iota_{n_1} & 0_{n_1} \\ 0_{n_2} & \iota_{n_2} \end{pmatrix} \begin{pmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{1}{n_2} \end{pmatrix} \begin{pmatrix} \iota_{n_1} & 0_{n_1} \\ 0_{n_2} & \iota_{n_2} \end{pmatrix}'$$
$$= \begin{pmatrix} I_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & I_{n_2} \end{pmatrix} - \begin{pmatrix} \iota_{n_1}\frac{1}{n_1} + 0_{n_1}0 & \iota_{n_1}0 + 0_{n_1}\frac{1}{n_2} \\ 0_{n_2}\frac{1}{n_1} + \iota_{n_2}0 & 0_{n_2}0 + \iota_{n_2}\frac{1}{n_2} \end{pmatrix} \begin{pmatrix} \iota'_{n_1} & 0'_{n_2} \\ 0'_{n_1} & \iota'_{n_2} \end{pmatrix}$$
$$= \begin{pmatrix} I_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & I_{n_2} \end{pmatrix} - \begin{pmatrix} \frac{1}{n_1}\iota_{n_1} & 0_{n_1} \\ 0_{n_2} & \frac{1}{n_2}\iota_{n_2} \end{pmatrix} \begin{pmatrix} \iota'_{n_1} & 0'_{n_2} \\ 0'_{n_1} & \iota'_{n_2} \end{pmatrix}$$
$$= \begin{pmatrix} I_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & I_{n_2} \end{pmatrix} - \begin{pmatrix} \frac{1}{n_1}\iota_{n_1}\iota'_{n_1} + 0_{n_1}0'_{n_1} & \frac{1}{n_1}\iota_{n_1}0'_{n_2} + 0_{n_1}\iota'_{n_2} \\ 0_{n_2}\iota'_{n_1} + \frac{1}{n_2}\iota_{n_2}0'_{n_1} & 0_{n_2}0'_{n_2} + \frac{1}{n_2}\iota_{n_2}\iota'_{n_2} \end{pmatrix}$$
$$= \begin{pmatrix} I_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & I_{n_2} \end{pmatrix} - \begin{pmatrix} \frac{1}{n_1}\iota_{n_1}\iota'_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & \frac{1}{n_2}\iota_{n_2}\iota'_{n_2} \end{pmatrix} = \begin{pmatrix} I_{n_1} - \frac{1}{n_1}\iota_{n_1}\iota'_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & I_{n_2} - \frac{1}{n_2}\iota_{n_2}\iota'_{n_2} \end{pmatrix}$$
$$= \begin{pmatrix} M_1 & 0_{n_1,n_2} \\ 0_{n_2,n_1} & M_2 \end{pmatrix}$$

$$M = \begin{pmatrix} M_1 & O_{n_1,n_2} \\ O_{n_2,n_1} & M_2 \end{pmatrix}$$

M_j (1) symmetric
(2) $M_j M_j = M_j$
(3) $tr(M_j) = n_j - 1$

<u>symmetry M</u>

$$M' = \begin{pmatrix} M_1 & O_{n_1,n_2} \\ O_{n_2,n_1} & M_2 \end{pmatrix}' = \begin{pmatrix} M_1' & O_{n_2,n_1}' \\ O_{n_1,n_2}' & M_2' \end{pmatrix} = \begin{pmatrix} M_1 & O_{n_1,n_2} \\ O_{n_2,n_1} & M_2 \end{pmatrix} = M$$

$\Rightarrow M' = M \Rightarrow M$ is symmetric

<u>product MM</u>

$$MM = \begin{pmatrix} M_1 & O_{n_1,n_2} \\ O_{n_2,n_1} & M_2 \end{pmatrix}\begin{pmatrix} M_1 & O_{n_1,n_2} \\ O_{n_2,n_1} & M_2 \end{pmatrix} = \begin{pmatrix} M_1M_1 + O_{n_1,n_2}O_{n_2,n_1} & M_1O_{n_1,n_2} + O_{n_1,n_2}M_2 \\ O_{n_2,n_1}M_1 + M_2O_{n_2,n_1} & O_{n_2,n_1}O_{n_1,n_2} + M_2M_2 \end{pmatrix}$$

$$= \begin{pmatrix} M_1 & O_{n_1,n_2} \\ O_{2,n_1} & M_2 \end{pmatrix} = M$$

<u>trace M</u>

$$tr(M) = tr(M_1) + tr(M_2) = n_1 - 1 + n_2 - 1 = n - 2$$

f) $\text{var}[z] = M \,\text{var}[y]\, M' = M\sigma^2 I M = \sigma^2 M$

$$E[z'z] = E[tr(z'z)] = E[tr(zz')] = tr(E[zz']) = tr(\Sigma_z)$$
$$= \sigma^2 tr(M) = (n-2)\sigma^2$$

g) $E[z'z] = (n-2)\sigma^2$

$$\frac{1}{n-2} z'z = \frac{1}{n-2} y'My = \frac{1}{n-2}(y-Hm)'(y-Hm)$$

3. $\tilde{s}^2 = \frac{n-1}{n} s^2$

$$\text{var}[\tilde{s}^2] = \frac{(n-1)^2}{n} \text{var}[s^2] = \frac{(n-1)^2}{n^2} \frac{2\sigma^4}{n-1} = \frac{2(n-1)}{n^2}\sigma^4$$

$$E[\tilde{s}^2] = \frac{n-1}{n}\sigma^2 \to \sigma^2 \quad \text{for } n \to \infty$$

$$\text{var}[\tilde{s}^2] = \frac{2(n-1)}{n^2}\sigma^4 \to 0 \text{ for } n \to \infty$$