Showing a Normal and a Chi square are independent

Asked 3 years, 7 months ago Modified 1 year, 7 months ago Viewed 1k times

0

Student's t distribution is defined as the ratio of a standard normally distributed random variable and the square root of a Chi-square distributed random variable divided by its degrees of freedom, given that they are independent. In formulas one can write $\frac{Z}{\sqrt{\frac{U}{df}}}$, where Z is $N(0,1)$ and U is χ^2_{df}.

In showing that this statement is true, I arrived at the point in which I have $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ and $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$. Then, following the definition, we would have that

$$\frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}}$$

is distributed as a t_{n-1}. But I am stuck at how to prove than this two random variables are independent between them.We covered a result about independence in the case of two Chi-square random variables and I thought of seeing the standard Normal as the square of a Chi-square random variable but I am afraid of it being mathematically sacrilegious.

Do you have any hint?

distributions self-study normal-distribution t-distribution chi-squared-distribution

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edited Feb 8, 2021 at 16:50
kjetil b halvorsen ♦
67.1k 29 148 494

asked Feb 6, 2019 at 23:36
PhDing
2,520 6 33 60

2 Answers

Sorted by: Highest score (default)

3

This is a brute force solution requiring just multivariable calculus.

It suffices to prove that the sample mean and the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

are independent. Thus, it suffices to prove that the sample mean $\bar{X}$ is independent of the vector

$$(X_1 - \bar{X}, \ldots, X_n - \bar{X}).$$

Moreover, since

$$\begin{aligned}\sum_{i=1}^{n}(X_i - \bar{X}) &= \sum_{i=1}^{n} X_i - \sum_{i=1}^{n}\bar{X} \\ &= n\bar{X} - n\bar{X} \\ &= 0,\end{aligned}$$

and hence

$$X_1 - \bar{X} = -\sum_{i=2}^{n}(X_2 - \bar{X}),$$

it follows that $X_1 - \bar{X}$ can be recovered from just knowing $(X_2 - \bar{X}, \ldots, X_n - \bar{X})$.

> Thus, it suffices to prove that the sample mean $\bar{X}$ is independent from
>
> $$(X_2 - \bar{X}, \ldots, X_n - \bar{X}).$$

Now consider the joint density

$$\begin{aligned}f_{(X_1,\ldots,X_n)}(x_1,\ldots,x_n) &= \left(2\pi\sigma^2\right)^{-n/2}\exp\left(-\sum_{i=1}^{n}\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right) \\ &= \left(2\pi\sigma^2\right)^{-n/2}\exp\left(-\sum_{i=1}^{n}\frac{1}{2}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2 - \frac{n}{2}\left(\frac{\bar{x}-\mu}{\sigma}\right)^2\right) \\ &= \underbrace{\left(2\pi\sigma^2\right)^{-n/2}}_{\text{constant}}\underbrace{\exp\left(-\sum_{i=1}^{n}\frac{1}{2}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2\right)}_{\text{depends only on }(x_2-\bar{x},\ldots,x_n-\bar{x})}\underbrace{\exp\left(-\frac{n}{2}\left(\frac{\bar{x}-\mu}{\sigma}\right)^2\right.}_{\text{depends only on }\bar{x}}\end{aligned}$$

To get from $(X_1, \ldots, X_n)$ to $(\bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})$, consider the diffeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T(x_1, \ldots, x_n) = (\bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}).$$

which is also clearly differentiable). Up to transpose, the Jacobian matrix of T is

$$DT(x_1, \ldots, x_n) = \begin{bmatrix} 1/n & 1/n & 1/n & \cdots & 1/n \\ -1/n & (n-1)/n & -1/n & \cdots & -1/n \\ -1/n & -1/n & (n-1)/n & \cdots & -1/n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & -1/n & \cdots & (n-1)/n. \end{bmatrix},$$

which doesn't depend on $x_1, \ldots, x_n$. Thus, the determinant of DT is some constant C. Now the joint density of $(\bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})$ satisfies

$$f_{(\bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})}(y_1, \ldots, y_n) = |C| f_{(X_1, \ldots, X_n)}(T^{-1}(y_1, \ldots, y_n))$$

which factors as a function of y_1 times a function of $(y_2, \ldots, y_n)$ by what was shown above.

Therefore, $\bar{X}$ and $(X_2 - \bar{X}, \ldots, X_n - \bar{X})$ are independent.

Share Cite Edit Follow Flag edited Feb 7, 2019 at 2:02 answered Feb 7, 2019 at 1:13
Artem Mavrin
3,577 2 16 27

1

I'll provide a hint to your self-study question: A corollary of a classic statistical theorem states that if $\mathbf{x} \sim N_p(\boldsymbol{\mu}, \sigma^2 \boldsymbol{I})$,then $\mathbf{Bx}$ and $\mathbf{x}'\mathbf{Ax}$ are independent if and only if $\mathbf{BA}$ is equal to the zero matrix. So, perhaps you could write the numerator as $\mathbf{Bx}$ and the denominator as $\mathbf{x}'\mathbf{Ax}$ and work from there?

Share Cite Edit Follow Flag answered Feb 7, 2019 at 0:53
StatsStudent
10.5k 4 39 70

I knew a slightly different result but this seems to be the one I need! Thanks! – PhDing Feb 7, 2019 at 4:23

If this hint worked for you, please accept the answer. Thank you! – StatsStudent Feb 7, 2019 at 15:10

Sure, I'll try later! – PhDing Feb 7, 2019 at 15:59

I am fine with the denominator, which can be rewritten as $\frac{X-\mu e}{\sigma}' \frac{ee'}{n-1} \frac{X-\mu e}{\sigma}$, and has the required form. I am stil stuck with the numerator because it is already a Normal, then the only thing I can think of is multiplying by an identity matrix but this is not working. – PhDing Feb 7, 2019 at 20:23

Think about redefining a new variable Z_i. Does that help? – StatsStudent Feb 7, 2019 at 22:20