Maxcut

Quadratic relaxation

$$\max \sum_{\{\mathbf{i},\mathbf{j}\}\in \mathbf{E}} \mathbf{w_{ij}} \frac{-\mathbf{v_i}\cdot\mathbf{v_j}+\mathbf{1}}{\mathbf{2}} \ :$$

$$\mathbf{v_i} \cdot \mathbf{v_i} = \mathbf{1}$$

Linear program

Variables: real numbers
Objective: linear
Constraints: linear equalities

$$\mathbf{x_i}$$

$$\sum \mathbf{c_i x_i}$$

$$\sum_{\mathbf{j}} \mathbf{a_{kj} x_j} = \mathbf{b_k}$$

Vector program

Variables: vectors
Objective: linear in dot products
Constraints: linear equalities in dot products of the vectors

$$\mathbf{v_i}$$

$$\sum \mathbf{c_{ij} v_i \cdot v_j}$$

$$\sum_{\mathbf{ij}}^{(\mathbf{k})} \mathbf{a_{ij}^{(k)} v_i \cdot v_j} = \mathbf{b_k}$$

What's the big deal about vector programs?

Variables: vectors $\mathbf{v_i}$

Objective: linear in dot products $\sum \mathbf{c_{ij} v_i \cdot v_j}$

Constraints: linear equalities in dot products of the vectors $\sum_{\mathbf{ij}}^{(\mathbf{k})} \mathbf{a_{ij}^{(k)} v_i \cdot v_j = b_k}$

Let $\mathbf{y_{ij} = v_i \cdot v_j}$

Objective: min or max $\sum \mathbf{c_{ij} y_{ij}}$

s.t. $\forall \mathbf{k} \; : \sum_{\mathbf{ij}}^{(\mathbf{k})} \mathbf{a_{ij}^{(k)} y_{ij} = b_k}$

and there exist vectors $\mathbf{v_i}$

s.t. $\mathbf{y_{ij} = v_i \cdot v_j}$

Positive semi-definite matrices

Consider a matrix $\mathrm{Y} = (\mathrm{y_{ij}})_{1 \leq \mathrm{i,j} \leq \mathrm{n}}$

Assume it is symmetric: $\mathrm{y_{ij}} = \mathrm{y_{ji}}$

then the following are equivalent:

- there exist vectors $\mathrm{v_i}$ s.t. $\mathrm{y_{ij}} = \mathrm{v_i} \cdot \mathrm{v_j}$
- Y is positive semi-definite
- for all vectors a, $\sum_{\mathrm{ij}} \mathrm{a_i y_{ij} a_j} \geq 0$

Semi-definite programming

Consider a matrix $Y = (y_{ij})_{1 \leq i,j \leq n}$

Objective:

min/max $\sum c_{ij} y_{ij}$

s.t. $\forall k \; : \sum_{ij}^{(k)} a_{ij}^{(k)} y_{ij} = b_k$

$y_{ij} = y_{ji}$

Y positive semi-definite $\qquad \forall a : \; \sum_{ij} a_i y_{ij} a_j \geq 0$

Convex

Theorem

Objective:

min/max $\sum c_{ij} y_{ij}$

s.t. $\forall k \; : \sum_{ij}^{(k)} a_{ij}^{(k)} y_{ij} = b_k$

$y_{ij} = y_{ji}$

Y **positive semi-definite**

Can be "solved" in polynomial time by ellipsoid algorithm

Quadratic relaxation for Maxcut

$$\max \sum_{\{\mathbf{i},\mathbf{j}\}\in \mathbf{E}} \mathbf{w_{ij}} \frac{-\mathbf{v_i}\cdot\mathbf{v_j}+\mathbf{1}}{\mathbf{2}} :$$

$$\mathbf{v_i} \cdot \mathbf{v_i} = \mathbf{1}$$

Can be "solved" in polynomial time
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