Putnam Training

Session 1 : Thursday 30^{th} September

Sequences and Series

INEQUALITIES

The triangle inequality:

$$1. \quad |x+y| \le |x| + |y|$$

$$2. \quad \big||x| - |y|\big| \le |x-y|$$

The Cauchy-Schwarz inequality:

$$1. \quad \sum x_i^2 \cdot \sum y_i^2 \ge \left(\sum x_i y_i\right)^2$$

$$2. \quad \sum x_i^2 \ge \frac{1}{n}\left(\sum x_i\right)^2$$

The Jensen inequality:

$$\textit{If } \sum \lambda_i = 1,\ \lambda_i \ge 0 \ \textit{ and } f \textit{ is convex then}$$

$$f\left(\sum \lambda_i x_i\right) \ge \sum \lambda_i f(x_i)$$

The inequality of arithmetic and geometric means:

$$\frac{1}{n}\sum x_i \ge \sqrt[n]{\prod x_i}$$

LIMITS OF SEQUENCES AND RECURRENCES

Limits of sequences

1. *Explicit sequence* $a_n = f(n)$
2. *Recursively defined sequence* $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_1)$

Monotone convergence theorem

If a sequence is increasing and bounded above, then it converges to its least upper bound.

Solving linear recursions

If $b_k a_n + b_{k-1} a_{n-1} + \cdots + b_0 a_{n-k} = 0$ *then solve the characteristic equation*

$$b_k x^k + b_{k-1} x^{k-1} + \cdots + b_0 = 0$$

with roots $x_0, x_1, \ldots, x_{k-1}$. *If they are different then*

$$a_n = c_0 x_0^n + c_1 x_1^n + \ldots + c_{k-1} x_{k-1}^n$$

where the c_i *are constants determined by the initial conditions* $a_0 = \alpha_0, a_1 = \alpha_1, \ldots, a_{k-1} = \alpha_{k-1}$.

Fibonacci Numbers

The Fibonacci numbers satisfy

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = F_2 = 1$$

Write down $x^2 = x + 1$ and use the quadratic formula to get

$$x = \frac{1 \pm \sqrt{5}}{2}$$

So we know

$$F_n = c_0 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_1 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Since $F_0 = F_1 = 2$ we get

$$1 = c_0 \left(\frac{1+\sqrt{5}}{2} \right)^2 + c_1 \left(\frac{1-\sqrt{5}}{2} \right)^2$$

$$1 = c_0 \left(\frac{1+\sqrt{5}}{2} \right)^1 + c_1 \left(\frac{1-\sqrt{5}}{2} \right)^1$$

so $c_0 = \frac{1}{\sqrt{5}}, c_1 = \frac{-1}{\sqrt{5}}$ and

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Limits and Recurrence Equations

Defining $a_n = f(a_{n-1})$, if a_n is increasing and bounded then (MCT) it has a limit L satisfying

$$L = f(L)$$

Continued fractions and iterated functions

1. $$a_n = b_0 + \cfrac{b_1}{1 + \cfrac{b_2}{1 + \cfrac{b_3}{1 + \cdots + \cfrac{b_{n-1}}{1 + b_n}}}}$$

2. $$a_n = b_0 + c_1\sqrt{b_1 + c_2\sqrt{b_2 + \cdots + c_n\sqrt{b_n}}}$$

CALCULUS AND SERIES

Stirling's Formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\theta} \qquad \frac{1}{12n+1} < \theta < \frac{1}{12n}$$

Taylor series:

1. $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}, -1 < x \leq 1$
2. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
3. $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
4. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) x^k}{k!}$

Exponential inequalities:

1. $$\prod (1 + x_i) \le e^{\sum x_i}$$
2. $$\prod (1 + x_i) \ge e^{\sum x_i - \frac{1}{2}\sum x_i^2}$$

Identities

1. $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, |x| < 1$$
2. $$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$
3. $$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$
4. $$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Riemann sums

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\tfrac{k}{n}) = \int_0^1 f(x)dx$$

Telescoping sums

$$\sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0$$

Convergence tests

1. Comparison test. Comparing a sum to a known convergent or divergent series
2. Integral test. Compare with an integral.
3. Ratio test. Ratio of consecutive terms larger than 1 implies divergence.

Harmonic sums

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges
2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

Rationality

$$\sqrt{2}, e, \pi, \ln 2 \ \textit{are irrational}$$

The sum of reciprocals of a fast enough growing sequence is usually irrational such as

$$\sum_{n=1}^{\infty} 2^{-n^2}$$

Diophantine approximation

For every real number α and given a positive integer n, there exists integers $q \leq n$ and $p \leq \lceil \alpha n \rceil$ such that

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{qn}$$

Calculus on sums

Under appropriate convergence criteria,

1. $\sum \lim_{x \to a} f(x) = \lim_{x \to a} \sum f'(x)$
2. $\sum f'(x) = \frac{d}{dx} \sum f'(x)$
3. $\sum \int_0^x f(t) dt = \int_0^x \left(\sum f(t) \right) dt$

WARMUP PROBLEMS

Problem 1. Solve the recurrence

$$a_n = 2a_{n-1} + a_{n-3} \qquad a_0 = a_1 = 1$$

Solution. Write down

$$x^2 - 2x - 3 = 0$$
$$x = -1, x = 3$$

and therefore

$$a_n = c_0(-1)^n + c_1 3^n$$

Using the initial conditions

$$1 = c_0 + c_1$$
$$1 = -c_0 + 3c_1$$
$$c_1 = c_0 = \frac{1}{2}$$

So we get

$$a_n = \frac{1}{2}(-1)^n + \frac{1}{2}3^n \qquad \checkmark$$

Problem 2. Find the 100^{th} digit after the decimal space in

$$(1+\sqrt{2})^{2010}$$

Solution. This is part of a solution to a recurrence. Ignore the denominator and expand:

$$(x-1-\sqrt{2})(x-1+\sqrt{2}) = x^2 - 2x - 1$$

So the recurrence was

$$a_n - 2a_{n-1} = a_{n-2}$$

If the initial conditions are $a_0 = a_1 = 1$ we get

$$a_n = \tfrac{1}{2}(1+\sqrt{2})^n + \tfrac{1}{2}(1-\sqrt{2})^n$$

The second term for $n = 2010$ has only zeroes in the first 100 places after the decimal point since it is very small, less than $2^{-2010} < 10^{-500}$. The left side is an integer. So that means

$$\tfrac{1}{2}(1+\sqrt{2})^n$$

has 9s in the first at least 500 locations after the decimal point and therefore there is a 9 at position 100 after the decimal point in $(1+\sqrt{2})^n$. ✓

Problem 3. Determine

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$$

Solution. Experiment shows that the answer should be 3.

Consider

$$f(x)=\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+\cdots}}}$$

It is not hard to see that $\frac{x+1}{2}\le f(x)\le 2(x+1)$ [prove it] and

$$f(x)^2 = xf(x+1)+1$$

Putting the inequalities into the equation we get

$$\begin{aligned}
\frac{x(x+1)}{2}+1 &\le f(x)^2 \le 2x(x+1)+1\\
\frac{(x+1)^2}{2} &\le f(x)^2 \le 2(x+1)^2\\
\frac{x+1}{\sqrt{2}} &\le f(x) \le \sqrt{2}(x+1)
\end{aligned}$$

Repeating gives for any $n\in\mathbb{N}$

$$2^{-1/2^n}(x+1)\le f(x)\le 2^{1/2^n}(x+1)$$

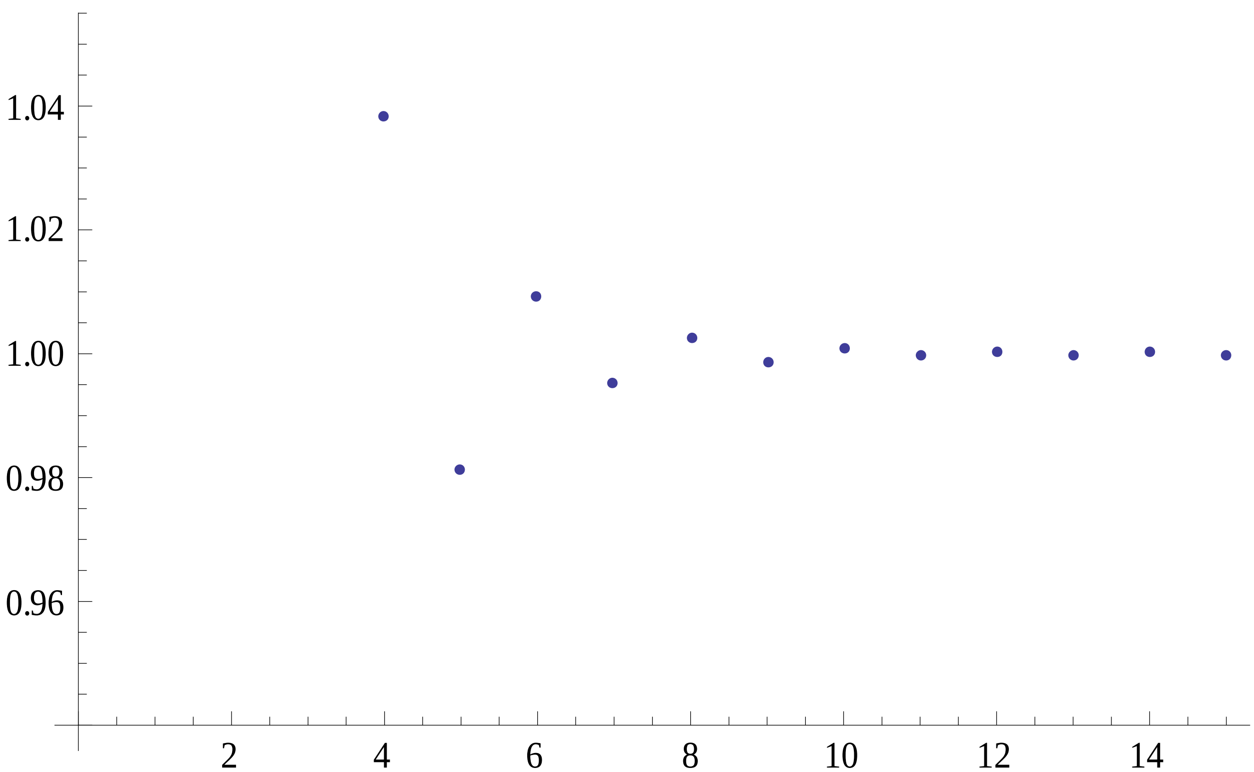
and this means $f(x)=x+1$. So the answer is $f(2)=3$. ✓

Problem 4. Find the limit of a_n when

$$2a_n a_{n-1} = a_{n-1} + 1 \quad \text{and} \quad a_0 = c$$

Solution. If it has a limit L then

$$2L^2 = L + 1 \Rightarrow L = 1 \text{ or } L = -\frac{1}{2}$$



$c = 1$

But we have to prove there is a limit. We can't use monotone convergence since evidently the sequence is not monotone.

By the way we can write

$$a_n = \frac{1}{2} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{2} + \cdots + \cfrac{1}{\frac{1}{2} + \frac{1}{2c}}}}$$

Check then that $a_n = -\frac{1}{2}$ always when $c = -\frac{1}{2}$. So the limit is

$$L = -\frac{1}{2} \quad \text{when } c = -\frac{1}{2}$$

For other values of c, try to show

1. a_{2n} is monotone, positive and bounded
2. a_{2n-1} is monotone, positive and bounded
3. $a_n / a_{n-1} \to 1$

It follows that $\lim a_{2n} = \lim a_{2n-1}$ exist by the monotone convergence theorem. The limits are positive since all the terms are positive. So we find that the limit is

$$L = 1 \quad \text{when } c \neq -\frac{1}{2}$$

Remark. Actually the solution to the recurrence is

$$a_n = \frac{-(-1)^n + 2^n + c(-1)^n + c2^{n+1}}{-2(-1)^n - 2^n + 2c(-1)^n + 2^{n+1}c}$$

but we do not have a method for finding this solution.