Learning Objectives

2.7.1 Convert from cylindrical to rectangular coordinates.
2.7.2 Convert from rectangular to cylindrical coordinates.
2.7.3 Convert from spherical to rectangular coordinates.
2.7.4 Convert from rectangular to spherical coordinates.

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

DEFINITION

In the **cylindrical coordinate system**, a point in space (Figure 2.89) is represented by the ordered triple (r, θ, z) , where

- (r, θ) are the polar coordinates of the point's projection in the *xy*-plane
- z is the usual $z\text{-coordinate}$ in the Cartesian coordinate system

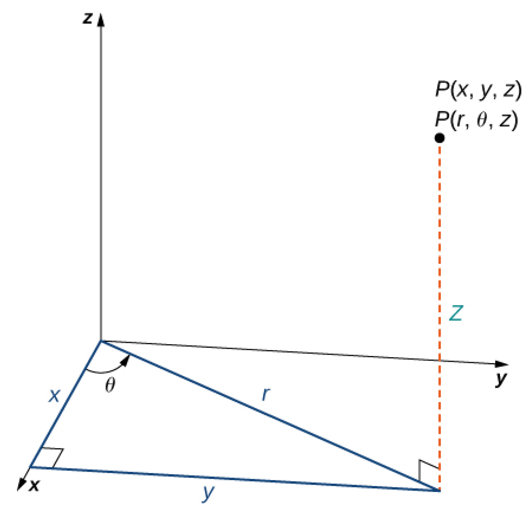


Figure 2.89 The right triangle lies in the *xy*-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive *x*-axis and the hypotenuse. The
z-coordinate describes the location of the point above or below the *xy*-plane.

In the *xy*-plane, the right triangle shown in Figure 2.89 provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

THEOREM 2.15

Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:

$$\begin{aligned} x &= r\cos\theta && \text{These equations are used to convert from} \\ y &= r\sin\theta && \text{cylindrical coordinates to rectangular} \\ z &= z && \text{coordinates.} \\ &\text{and} && \\ r^2 &= x^2+y^2 && \text{These equations are used to convert from} \\ \tan\theta &= \tfrac{y}{x} && \text{rectangular coordinates to cylindrical} \\ z &= z && \text{coordinates.} \end{aligned}$$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan\theta = \frac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the *xy*-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\frac{\pi}{2}$, $\frac{3\pi}{2}$, or 0, depending on the value of y.

Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point P in the *xy*-plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in the following figure.

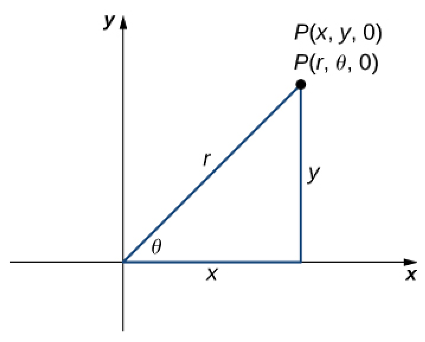


Figure 2.90 The Pythagorean theorem provides equation $r^2 = x^2 + y^2$. Right-triangle relationships tell us that $x = r\cos\theta$, $y = r\sin\theta$, and $\tan\theta = y/x$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c$, $y = c$, or $z = c$ are all planes. Planes of these forms are parallel to the *yz*-plane, the *xz*-plane, and the *xy*-plane, respectively. When we convert to cylindrical coordinates, the *z*-coordinate does not change. Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the *xy*-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the *z*-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the *x*-axis, which gives us a half-plane that starts at the *z*-axis (Figure 2.91 and Figure 2.92).

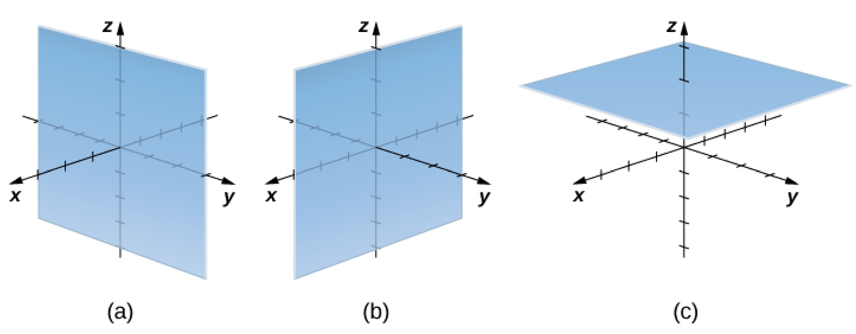


Figure 2.91 In rectangular coordinates, (a) surfaces of the form $x = c$ are planes parallel to the *yz*-plane, (b) surfaces of the form $y = c$ are planes parallel to the *xz*-plane, and (c) surfaces of the form $z = c$ are planes parallel to the *xy*-plane.

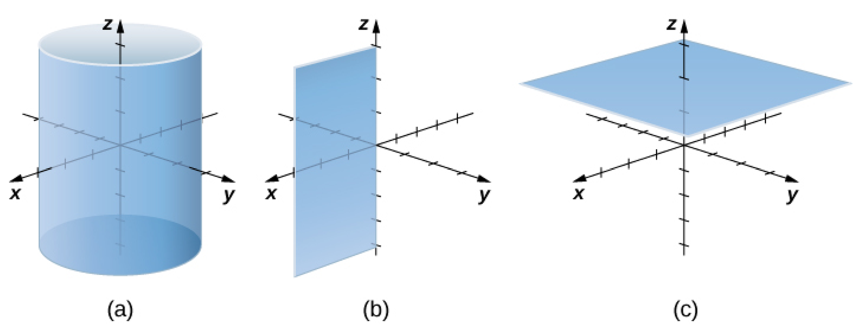


Figure 2.92 In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius c, (b) surfaces of the form $\theta = c$ are half-planes at angle c from the *x*-axis, and (c) surfaces of the form $z = c$ are planes parallel to the *xy*-plane.

EXAMPLE 2.60

Converting from Cylindrical to Rectangular Coordinates

Plot the point with cylindrical coordinates $\left(4, \frac{2\pi}{3}, -2\right)$ and express its location in rectangular coordinates.

[Show/Hide Solution]

CHECKPOINT 2.55

Point R has cylindrical coordinates $\left(5, \frac{\pi}{6}, 4\right)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

If this process seems familiar, it is with good reason. This is exactly the same process that we followed in Introduction to Parametric Equations and Polar Coordinates to convert from polar coordinates to two-dimensional rectangular coordinates.

EXAMPLE 2.61

Converting from Rectangular to Cylindrical Coordinates

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

[Show/Hide Solution]

CHECKPOINT 2.56

Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2 + y^2 = 25$ in the Cartesian system can be represented by cylindrical equation $r = 5$.

EXAMPLE 2.62

Identifying Surfaces in the Cylindrical Coordinate System

Describe the surfaces with the given cylindrical equations.

a. $\theta = \frac{\pi}{4}$
b. $r^2 + z^2 = 9$
c. $z = r$

[Show/Hide Solution]

CHECKPOINT 2.57

Describe the surface with cylindrical equation $r = 6$.

Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, location of a point in space is described using two distances $(r \text{ and } z)$ and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

DEFINITION

In the **spherical coordinate system**, a point P in space (Figure 2.97) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive *z*-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

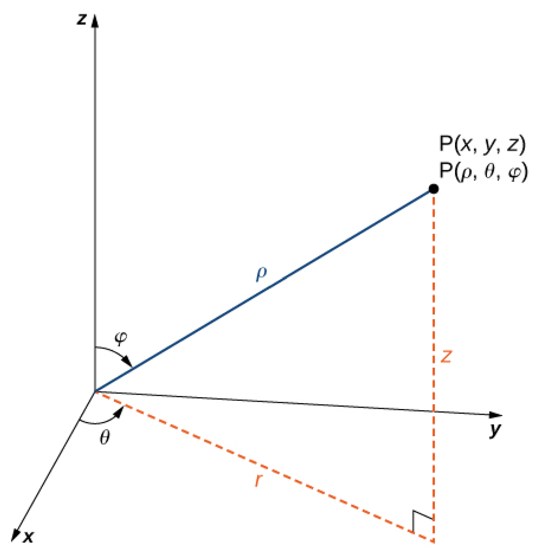


Figure 2.97 The relationship among spherical, rectangular, and cylindrical coordinates.

By convention, the origin is represented as $(0, 0, 0)$ in spherical coordinates.

THEOREM 2.16

Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z) and spherical coordinates (ρ, θ, φ) of a point are related as follows:

$$\begin{array}{rcll} x & = & \rho \sin \varphi \cos \theta & \text{These equations are used to convert from} \\ y & = & \rho \sin \varphi \sin \theta & \text{spherical coordinates to rectangular} \\ z & = & \rho \cos \varphi & \text{coordinates.} \\ & \text{and} & & \\ \rho^2 & = & x^2 + y^2 + z^2 & \text{These equations are used to convert from} \\ \tan \theta & = & \frac{y}{x} & \text{rectangular coordinates to spherical} \\ \varphi & = & \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right). & \text{coordinates.} \end{array}$$

If a point has cylindrical coordinates (r, θ, z), then these equations define the relationship between cylindrical and spherical coordinates.

$$\begin{array}{rcll} r & = & \rho \sin \varphi & \text{These equations are used to convert from} \\ \theta & = & \theta & \text{spherical coordinates to cylindrical} \\ z & = & \rho \cos \varphi & \text{coordinates.} \\ & \text{and} & & \\ \rho & = & \sqrt{r^2 + z^2} & \text{These equations are used to convert from} \\ \theta & = & \theta & \text{cylindrical coordinates to spherical} \\ \varphi & = & \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right) & \text{coordinates.} \end{array}$$

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at Figure 2.98, it is easy to see that $r = \rho \sin \varphi$. Then, looking at the triangle in the *xy*-plane with r as its hypotenuse, we have $x = r \cos \theta = \rho \sin \varphi \cos \theta$. The derivation of the formula for y is similar. Figure 2.96 also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2 + z^2}$ (from the first equation) yields $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan \theta = \frac{y}{x}$ to choose the correct value of θ.

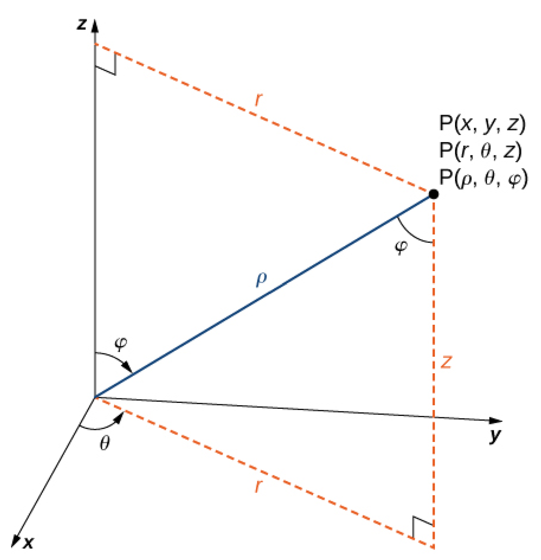


Figure 2.98 The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\varphi = c$. The points on these surfaces are at a fixed angle from the *z*-axis and form a half-cone (Figure 2.99).

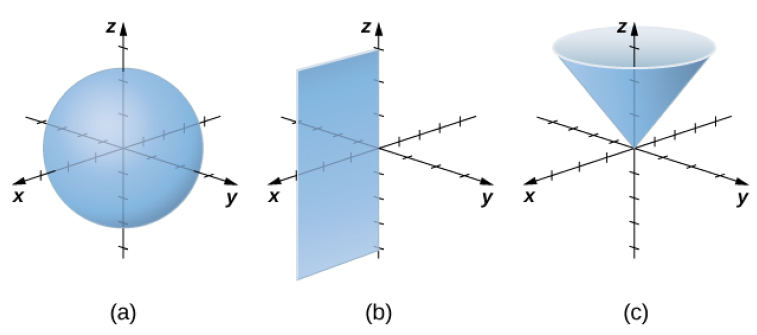


Figure 2.99 In spherical coordinates, surfaces of the form $\rho = c$ are spheres of radius ρ (a), surfaces of the form $\theta = c$ are half-planes at an angle θ from the *x*-axis (b), and surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the *z*-axis (c).

EXAMPLE 2.63

Converting from Spherical Coordinates

Plot the point with spherical coordinates $\left(8, \frac{\pi}{3}, \frac{\pi}{6}\right)$ and express its location in both rectangular and cylindrical coordinates.

[Show/Hide Solution]

CHECKPOINT 2.58

Plot the point with spherical coordinates $\left(2, -\frac{5\pi}{6}, \frac{\pi}{6}\right)$ and describe its location in both rectangular and cylindrical coordinates.

EXAMPLE 2.64

Converting from Rectangular Coordinates

Convert the rectangular coordinates $\left(-1, 1, \sqrt{6}\right)$ to both spherical and cylindrical coordinates.

[Show/Hide Solution]

EXAMPLE 2.65

Identifying Surfaces in the Spherical Coordinate System

Describe the surfaces with the given spherical equations.

a. $\theta = \frac{\pi}{3}$
b. $\varphi = \frac{5\pi}{6}$
c. $\rho = 6$
d. $\rho = \sin \theta \sin \varphi$

[Show/Hide Solution]

CHECKPOINT 2.59

Describe the surfaces defined by the following equations.

a. $\rho = 13$
b. $\theta = \frac{2\pi}{3}$
c. $\varphi = \frac{\pi}{4}$

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. A sphere that has Cartesian equation $x^2 + y^2 + z^2 = c^2$ has the simple equation $\rho = c$ in spherical coordinates.

In geography, latitude and longitude are used to describe locations on Earth's surface, as shown in Figure 2.104. Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth. Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees.

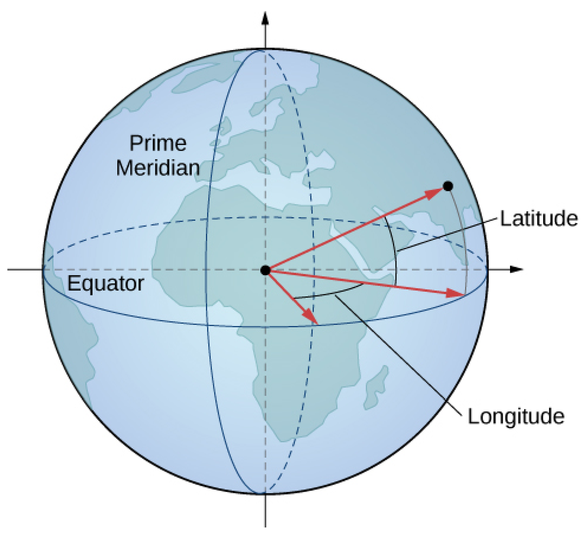


Figure 2.104 In the latitude–longitude system, angles describe the location of a point on Earth relative to the equator and the prime meridian.

Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive *z*-axis. The prime meridian represents the trace of the surface as it intersects the *xz*-plane. The equator is the trace of the sphere intersecting the *xy*-plane.

EXAMPLE 2.66

Converting Latitude and Longitude to Spherical Coordinates

The latitude of Columbus, Ohio, is 40° N and the longitude is 83° W, which means that Columbus is 40° north of the equator. Imagine a ray from the center of Earth through Columbus and a ray from the center of Earth through the equator directly south of Columbus. The measure of the angle formed by the rays is 40°. In the same way, measuring from the prime meridian, Columbus lies 83° to the west. Express the location of Columbus in spherical coordinates.

[Show/Hide Solution]

CHECKPOINT 2.60

Sydney, Australia is at 34°S and 151°E. Express Sydney's location in spherical coordinates.

Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

EXAMPLE 2.67

Choosing the Best Coordinate System

In each of the following situations, we determine which coordinate system is most appropriate and describe how we would orient the coordinate axes. There could be more than one right answer for how the axes should be oriented, but we select an orientation that makes sense in the context of the problem. *Note*: There is not enough information to set up or solve these problems; we simply select the coordinate system (Figure 2.105).

a. Find the center of gravity of a bowling ball.
b. Determine the velocity of a submarine subjected to an ocean current.
c. Calculate the pressure in a conical water tank.
d. Find the volume of oil flowing through a pipeline.
e. Determine the amount of leather required to make a football.

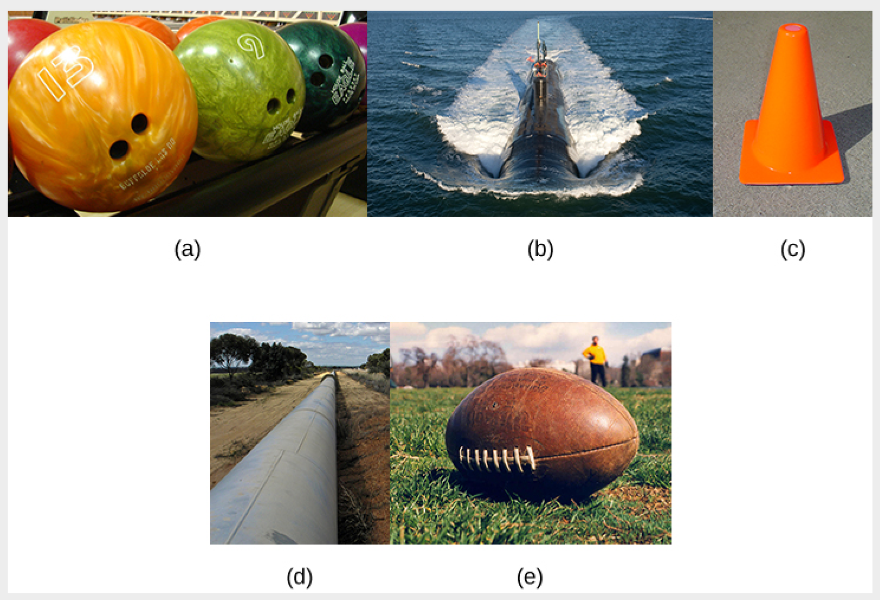
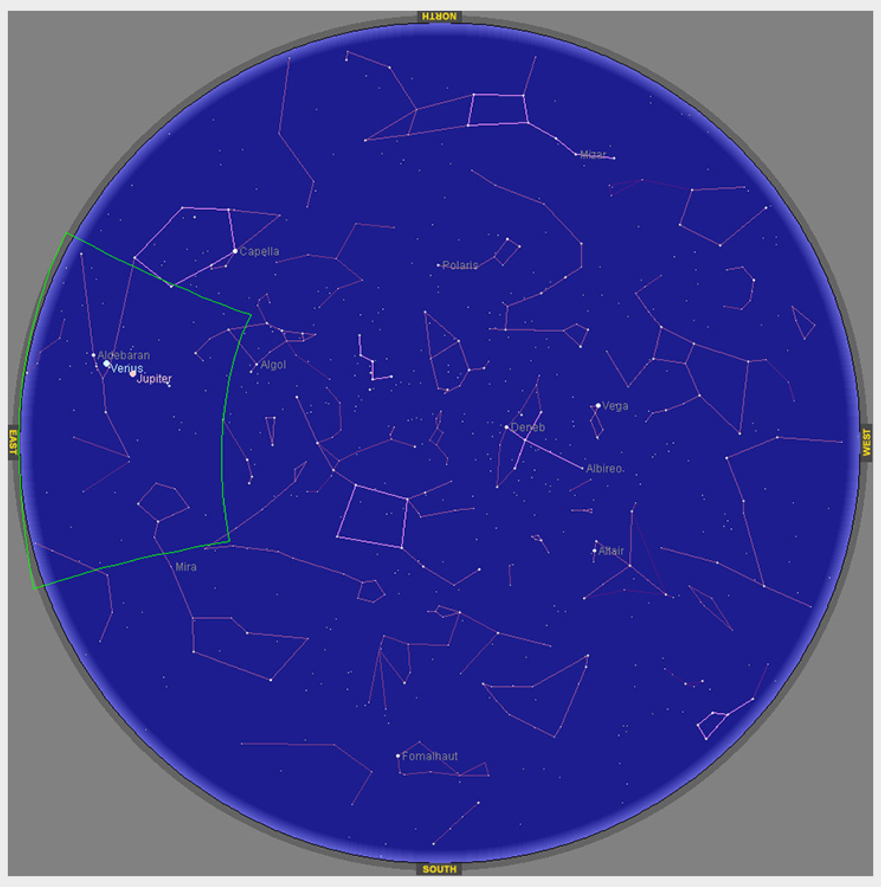


Figure 2.105 (credit: (a) modification of work by scl hua, Wikimedia, (b) modification of work by DVIDSHUB, Flickr, (c) modification of work by Michael Malak, Wikimedia, (d) modification of work by Sean Mack, Wikimedia, (e) modification of work by Elvert Barnes, Flickr)

[Show/Hide Solution]

CHECKPOINT 2.61

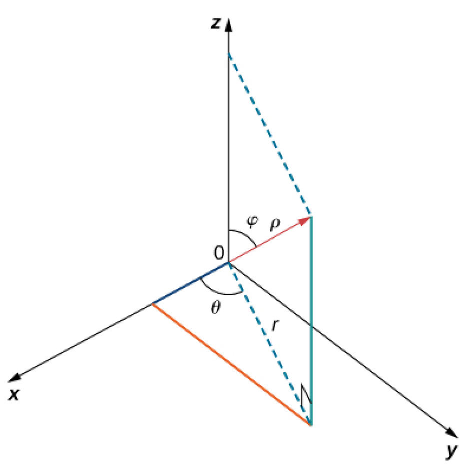
Which coordinate system is most appropriate for creating a star map, as viewed from Earth (see the following figure)?



How should we orient the coordinate axes?

Section 2.7 Exercises

Use the following figure as an aid in identifying the relationship between the rectangular, cylindrical, and spherical coordinate systems.



For the following exercises, the cylindrical coordinates (r, θ, z) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

363. $\left(4, \frac{\pi}{6}, 3\right)$

364. $\left(3, \frac{\pi}{3}, 5\right)$

365. $\left(4, \frac{7\pi}{6}, 3\right)$

366. $(2, \pi, -4)$

For the following exercises, the rectangular coordinates (x, y, z) of a point are given. Find the cylindrical coordinates (r, θ, z) of the point.

367. $\left(1, \sqrt{3}, 2\right)$

368. $(1, 1, 5)$

369. $(3, -3, 7)$

370. $\left(-2\sqrt{2}, 2\sqrt{2}, 4\right)$

For the following exercises, the equation of a surface in cylindrical coordinates is given.

Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

371. **[T]** $r = 4$

372. **[T]** $z = r^2 \cos^2 \theta$

373. **[T]** $r^2 \cos(2\theta) + z^2 + 1 = 0$

374. **[T]** $r = 3 \sin \theta$

375. **[T]** $r = 2 \cos \theta$

376. **[T]** $r^2 + z^2 = 5$

377. **[T]** $r = 2 \sec \theta$

378. **[T]** $r = 3 \csc \theta$

For the following exercises, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in cylindrical coordinates.

379. $z = 3$

380. $x = 6$

381. $x^2 + y^2 + z^2 = 9$

382. $y = 2x^2$

383. $x^2 + y^2 - 16x = 0$

384. $x^2 + y^2 - 3\sqrt{x^2 + y^2} + 2 = 0$

For the following exercises, the spherical coordinates (ρ, θ, φ) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

385. $(3, 0, \pi)$

386. $\left(1, \frac{\pi}{6}, \frac{\pi}{6}\right)$

387. $\left(12, -\frac{\pi}{4}, \frac{\pi}{4}\right)$

388. $\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right)$

For the following exercises, the rectangular coordinates (x, y, z) of a point are given. Find the spherical coordinates (ρ, θ, φ) of the point. Express the measure of the angles in degrees rounded to the nearest integer.

389. $(4, 0, 0)$

390. $(-1, 2, 1)$

391. $(0, 3, 0)$

392. $\left(-2, 2\sqrt{3}, 4\right)$

For the following exercises, the equation of a surface in spherical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

393. **[T]** $\rho = 3$

394. **[T]** $\varphi = \frac{\pi}{3}$

395. **[T]** $\rho = 2\cos\varphi$

396. **[T]** $\rho = 4\csc\varphi$

397. **[T]** $\varphi = \frac{\pi}{2}$

398. **[T]** $\rho = 6\csc\varphi\sec\theta$

For the following exercises, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in spherical coordinates. Identify the surface.

399. $x^2 + y^2 - 3z^2 = 0,\ z \neq 0$

400. $x^2 + y^2 + z^2 - 4z = 0$

401. $z = 6$

402. $x^2 + y^2 = 9$

For the following exercises, the cylindrical coordinates of a point are given. Find its associated spherical coordinates, with the measure of the angle φ in radians rounded to four decimal places.

403. **[T]** $\left(1, \frac{\pi}{4}, 3\right)$

404. **[T]** $(5, \pi, 12)$

405. $\left(3, \frac{\pi}{2}, 3\right)$

406. $\left(3, -\frac{\pi}{6}, 3\right)$

For the following exercises, the spherical coordinates of a point are given. Find its associated cylindrical coordinates.

407. $\left(2, -\frac{\pi}{4}, \frac{\pi}{2}\right)$

408. $\left(4, \frac{\pi}{4}, \frac{\pi}{6}\right)$

409. $\left(8, \frac{\pi}{3}, \frac{\pi}{2}\right)$

410. $\left(9, -\frac{\pi}{6}, \frac{\pi}{3}\right)$

For the following exercises, find the most suitable system of coordinates to describe the solids.

411. The solid situated in the first octant with a vertex at the origin and enclosed by a cube of edge length a, where $a > 0$

412. A spherical shell determined by the region between two concentric spheres centered at the origin, of radii of a and b, respectively, where $b > a > 0$

413. A solid inside sphere $x^2 + y^2 + z^2 = 9$ and outside cylinder $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

414. A cylindrical shell of height 10 determined by the region between two cylinders with the same center, parallel rulings, and radii of 2 and 5, respectively

415. **[T]** Use a CAS to graph the region between elliptic paraboloid $z = x^2 + y^2$ and cone $x^2 + y^2 - z^2 = 0$. Then describe the region in cylindrical coordinates.

416. **[T]** Use a CAS to graph in spherical coordinates the "ice cream-cone region" situated above the *xy*-plane between sphere $x^2 + y^2 + z^2 = 4$ and elliptical cone $x^2 + y^2 - z^2 = 0$.

417. Washington, DC, is located at 39° N and 77° W (see the following figure). Assume the radius of Earth is 4000 mi. Express the location of Washington, DC, in spherical coordinates.

418. San Francisco is located at 37.78°N and 122.42°W. Assume the radius of Earth is 4000 mi. Express the location of San Francisco in spherical coordinates.

419. Find the latitude and longitude of Rio de Janeiro if its spherical coordinates are $(4000, -43.17^\circ, 102.91^\circ)$.

420. Find the latitude and longitude of Berlin if its spherical coordinates are $(4000, 13.38^\circ, 37.48^\circ)$.

421. **[T]** Consider the torus of equation $\left(x^2 + y^2 + z^2 + R^2 - r^2\right)^2 = 4R^2\left(x^2 + y^2\right)$, where $R \geq r > 0$.
  a. Write the equation of the torus in spherical coordinates.
  b. If $R = r$, the surface is called a *horn torus*. Show that the equation of a horn torus in spherical coordinates is $\rho = 2R\sin\varphi$.
  c. Use a CAS to graph the horn torus with $R = r = 2$ in spherical coordinates.

422. **[T]** The "bumpy sphere" with an equation in spherical coordinates is $\rho = a + b\cos(m\theta)\sin(n\varphi)$, with $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$, where a and b are positive numbers and m and n are positive integers, may be used in applied mathematics to model tumor growth.
  a. Show that the "bumpy sphere" is contained inside a sphere of equation $\rho = a + b$. Find the values of θ and φ at which the two surfaces intersect.
  b. Use a CAS to graph the surface for $a = 14$, $b = 2$, $m = 4$, and $n = 6$ along with sphere $\rho = a + b$.
  c. Find the equation of the intersection curve of the surface at b. with the cone $\varphi = \frac{\pi}{12}$. Graph the intersection curve in the plane of intersection.