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RESEARCH

Nuclear norm and rank minimization

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In many factorization, estimation or approximation methods, the equivalence between minimizing the rank of a matrix X to minimizing its nuclear norm (trace of the singular value matrix) is employed to make the problem convex and thus prone to convex minimization methods.

A minimization problem, over $R^{m \times n}$ involving matrix- rank constrains

$\min_X = \mathrm{rank}(X)$, such that $L(X) = D$ **(low-rank approximation)**

is intractable and of little practical use. The matrix nuclear norm (i.e., the sum of its singular values), represented as $||.||_*$, is the best convex approximation to the rank() function, we can instead solve

$\min_X ||X||_*$, such that $L(X) = D$ **(nuclear norm approximation)**

A proof of the approximation can be found in [1], where , quoting from page 2,:

> *"A recent heuristic introduced in [3] minimizes the nuclear norm, or the sum of the singular values of the matrix, over the affine subset. The nuclear norm is a convex function, can be optimized efficiently, and is the best convex approximation of the rank function over the unit ball of matrices with norm less than one… The nuclear norm heuristic has been observed to produce very low-rank solutions in practice, but a theoretical characterization of when it produces the minimum rank solution has not been previously available. This paper provides the first such mathematical characterization."*

Theorem 2.2 (page 8, The convex envelope of the rank on the set C in the nuclear norm), concludes:

> *nuclear norm of the minimum nuclear norm solution "…providing an upper and lower bound on the optimal rank when the norm of the optimal solution is known. Furthermore,* ***this is the tightest lower bound among all convex lower bounds of the rank function on the set C****".*

On the relationship between the trace and nuclear norm heuristics to matrix rank, quoting from [2], page 2:

> *A popular heuristic for solving rank minimization problems in the controls community is the trace heuristic" where one minimizes the trace of a positive semidenite decision variable instead of the rank. A generalization of this heuristic to non-symmetric matrices introduced by Fazel in [3] minimizes the nuclear norm, or the sum of the singular values of the matrix, over the constraint set. When the matrix variable is symmetric and positive semidenite, this heuristic is equivalent to the trace heuristic, as the trace of a positive semidenite matrix is equal to the sum of its singular values. The nuclear norm is a convex function and can be optimized efficiently via semidenite programming.*

Links and info (nuclear norm, trace and minimization):

[1] B. Recht, M. Fazel, P. A. Parrilo, "Guaranteed Minimum-Rank Solutions of Linear Matrix Equations via Nuclear Norm Minimization (http://arxiv.org/abs/0706.4138)", 2008.
[2] B. Recht, W. Xu, B. Hassibi, "Necessary and Sufficient Conditions for Success of the Nuclear Norm Heuristic for Rank Minimization (http://arxiv.org/abs/0809.1260)",2008.
[3] M. Fazel, "Matrix rank minimization with applications (http://faculty.washington.edu/mfazel/thesis-final.pdf)," Ph.D. dissertation, Stanford University, 2002.

Low-Rank Matrix Recovery and Completion via Convex Optimization (http://perception.csl.illinois.edu/matrix-rank/introduction.html)
Nuclear Norm for Rank Minimization (http://nuit-blanche.blogspot.com/2008/04/compressed-sensing-nuclear-norm-for.html) (nuit-blance)
Trace: What is it good for? (http://nuit-blanche.blogspot.com/2007/10/trace-what-is-it-good-for-how.html)(nuit-blance)

APPROXIMATION **LOW-RANK** **MINIMIZATION** **NUCLEAR NORM** **RANK** **SPARSITY**