The Scientist and Engineer's Guide to Digital Signal Processing

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Chapter 10: Fourier Transform Properties

Linearity of the Fourier Transform

The Fourier Transform is *linear,* that is, it possesses the properties of *homogeneity* and *additivity*. This is true for all four members of the Fourier transform family (Fourier transform, Fourier Series, DFT, and DTFT).

Figure 10-1 provides an example of how homogeneity is a property of the Fourier transform. Figure (a) shows an arbitrary time domain signal, with the corresponding frequency spectrum shown in (b). We will call these two signals: *x*[ ] and *X*[ ], respectively. *Homogeneity* means that a change in amplitude in one domain produces an identical change in amplitude in the other domain. This should make intuitive sense: when the amplitude of a time domain waveform is changed, the amplitude of the sine and cosine waves making up that waveform must also change by an equal amount.

In mathematical form, if *x*[ ] and *X*[ ] are a Fourier Transform pair, then *kx*[ ] and *kX*[ ] are also a Fourier Transform pair, for any constant *k*. If the frequency domain is represented in *rectangular* notation, *kX*[ ] means that both the real part and the imaginary part are multiplied by *k*. If the frequency domain is represented in *polar* notation, *kX*[ ] means that the magnitude is multiplied by *k*, while the phase remains unchanged.

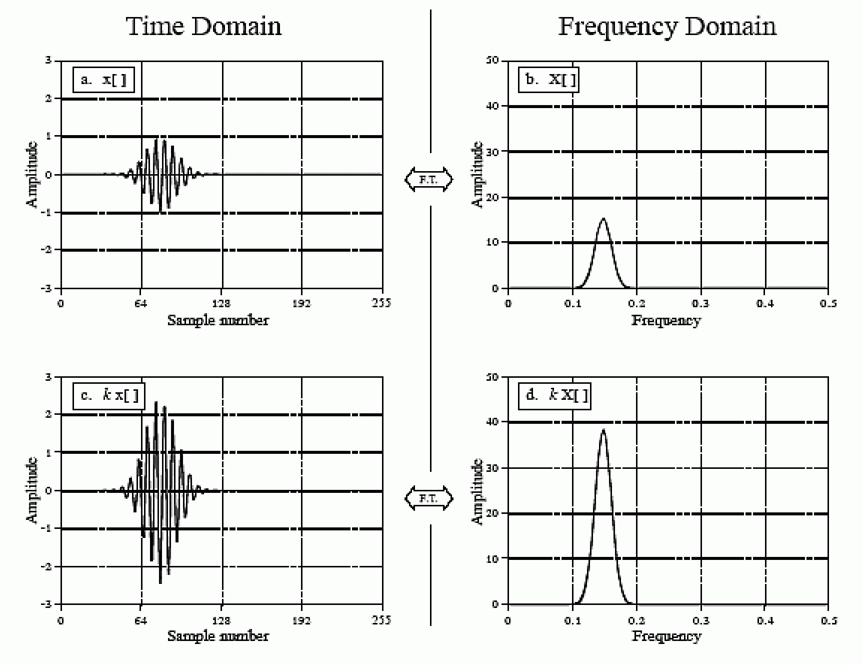

FIGURE 10-1
Homogeneity of the Fourier transform. If the amplitude is changed in one domain, it is changed by the same amount in the other domain. In other words, *scaling* in one domain corresponds to *scaling* in the other domain.

Additivity of the Fourier transform means that *addition* in one domain corresponds to *addition* in the other domain. An example of this is shown in Fig. 10-2. In this illustration, (a) and (b) are signals in the time domain called $x_1[n]$ and $x_2[n]$, respectively. Adding these signals produces a third time domain signal called $x_3[n]$, shown in (c). Each of these three signals has a frequency spectrum consisting of a real and an imaginary part, shown in (d) through (i). Since the two time domain signals *add* to produce the third time domain signal, the two corresponding spectra *add* to produce the third spectrum. Frequency spectra are added in rectangular notation by adding the real parts to the real parts and the imaginary parts to the imaginary parts. If: $x_1[n] + x_2[n] = x_3[n]$, then: $ReX_1[f] + ReX_2[f] = ReX_3[f]$ and $ImX_1[f] + ImX_2[f] = ImX_3[f]$. Think of this in terms of cosine and sine waves. All the cosine waves add (the real parts) and all the sine waves add (the imaginary parts) with no interaction between the two.

Frequency spectra in polar form cannot be directly added; they must be converted into rectangular notation, added, and then reconverted back to

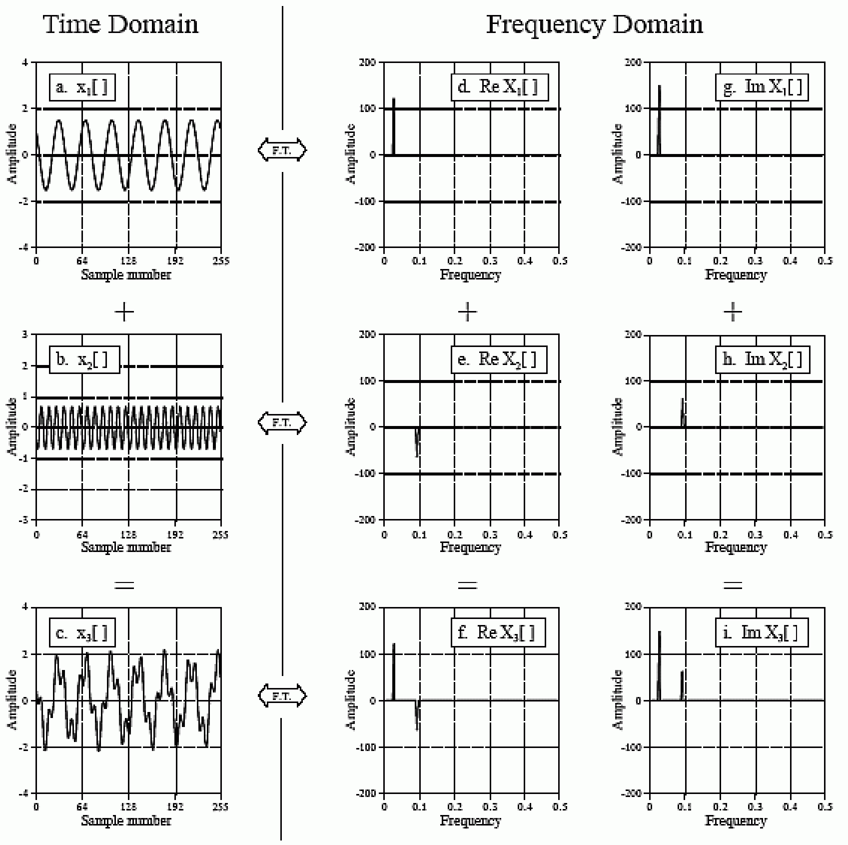

FIGURE 10-2
Additivity of the Fourier transform. Adding two or more signals in one domain results in the corresponding signals being added in the other domain. In this illustration, the time domain signals in (a) and (b) are added to produce the signal in (c). This results in the corresponding real and imaginary parts of the frequency spectra being added.

polar form. This can also be understood in terms of how sinusoids behave. Imagine adding two sinusoids having the same frequency, but with different amplitudes (A_1 and A_2) and phases (φ_1 and φ_2). If the two phases happen to be same ($\varphi_1 = \varphi_2$), the amplitudes will add ($A_1 + A_2$) when the sinusoids are added. However, if the two phases happen to be exactly opposite ($\varphi_1 = -\varphi_2$), the amplitudes will *subtract* ($A_1 - A_2$) when the sinusoids are added. The point is, when sinusoids (or spectra) are in polar form, they *cannot* be added by simply adding the magnitudes and phases.

In spite of being linear, the Fourier transform is *not* shift invariant. In other words, a shift in the time domain *does not* correspond to a shift in the frequency domain. This is the topic of the next section.

Next Section: Characteristics of the Phase