Return to independence, an old friend resurfaces

Return to independence, an old friend resurfaces

Independent events: $A_1, A_2, \ldots, A_n$

Return to independence, an old friend resurfaces

Independent events: $A_1, A_2, \ldots, A_n$

$$\mathbf{P}(A_1) = \mathbf{P}(A_2) = \cdots = \mathbf{P}(A_n) = p$$

Return to independence, an old friend resurfaces

Independent events: $A_1, A_2, \ldots, A_n$

$$\mathbf{P}(A_1) = \mathbf{P}(A_2) = \cdots = \mathbf{P}(A_n) = p$$

The inclusion–exclusion sums:

$$S_{k+j} = \sum_{1 \leq i_1 < i_2 < \cdots < i_{k+j} \leq n} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{k+j}})$$

Return to independence, an old friend resurfaces

Independent events: $A_1, A_2, \ldots, A_n$

$$\mathbf{P}(A_1) = \mathbf{P}(A_2) = \cdots = \mathbf{P}(A_n) = p$$

The inclusion–exclusion sums:

$$S_{k+j} = \sum_{1 \leq i_1 < i_2 < \cdots < i_{k+j} \leq n} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{k+j}}) = \binom{n}{k+j} p^{k+j}$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j}$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j}$$

$$X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\mathbf{P}\{X_1 + \cdots + X_n = k\} = \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j}$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\mathbf{P}\{X_1 + \cdots + X_n = k\} = \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j}$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\mathbf{P}\{X_1 + \cdots + X_n = k\} = \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j}$$

$$\binom{k+j}{k}\binom{n}{k+j} = \binom{n}{k}\binom{n-k}{j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j}$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\mathbf{P}\{X_1 + \cdots + X_n = k\} = \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j}$$

$$\binom{k+j}{k}\binom{n}{k+j} = \binom{n}{k}\binom{n-k}{j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{n}{k} \binom{n-k}{j} p^k p^j$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\mathbf{P}\{X_1 + \cdots + X_n = k\} = \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j}$$

$$\binom{k+j}{k}\binom{n}{k+j} = \binom{n}{k}\binom{n-k}{j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{n}{k} \binom{n-k}{j} p^k p^j$$

$$= \binom{n}{k} p^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-p)^j$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\begin{aligned} \mathbf{P}\{X_1 + \cdots + X_n = k\} &= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j} \\ &= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j} \\ &= \sum_{j=0}^{n-k} (-1)^j \binom{n}{k} \binom{n-k}{j} p^k p^j \\ &= \binom{n}{k} p^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-p)^j \end{aligned}$$

$$\binom{k+j}{k}\binom{n}{k+j} = \binom{n}{k}\binom{n-k}{j}$$

The binomial theorem: $(1+x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\begin{aligned} \mathbf{P}\{X_1 + \cdots + X_n = k\} &= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j} \\ &= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j} \\ &= \sum_{j=0}^{n-k} (-1)^j \binom{n}{k} \binom{n-k}{j} p^k p^j \\ &= \binom{n}{k} p^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-p)^j \\ &= \binom{n}{k} p^k (1 + (-p))^{n-k} \end{aligned}$$

$$\binom{k+j}{k}\binom{n}{k+j} = \binom{n}{k}\binom{n-k}{j}$$

The binomial theorem: $(1+x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

indicator for A_j

The number of occurrences of the events A_j:

$$\mathbf{P}\{X_1 + \cdots + X_n = k\} = \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j}$$

$$\binom{k+j}{k}\binom{n}{k+j} = \binom{n}{k}\binom{n-k}{j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j}$$

$$= \sum_{j=0}^{n-k} (-1)^j \binom{n}{k} \binom{n-k}{j} p^k p^j$$

The binomial theorem: $(1+x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j$

$$= \binom{n}{k} p^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-p)^j$$

$$= \binom{n}{k} p^k \bigl(1 + (-p)\bigr)^{n-k} = b_n(k;p)$$

$$S_{k+j} = \binom{n}{k+j} p^{k+j} \qquad X_j = \begin{cases} 1 & \text{if } A_j \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases} \qquad X_1 + \cdots + X_n \sim \text{Binomial}(n, p)$$

indicator for A_j

The number of occurrences of the events A_j:

$$\begin{aligned}
\mathbf{P}\{X_1 + \cdots + X_n = k\} &= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} S_{k+j} \\
&= \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k} \binom{n}{k+j} p^{k+j} \\
&= \sum_{j=0}^{n-k} (-1)^j \binom{n}{k} \binom{n-k}{j} p^k p^j \\
&= \binom{n}{k} p^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-p)^j \\
&= \binom{n}{k} p^k (1 + (-p))^{n-k} = b_n(k;p)
\end{aligned}$$

$$\binom{k+j}{k}\binom{n}{k+j} = \binom{n}{k}\binom{n-k}{j}$$

The binomial theorem: $(1+x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j$