Last Time

- Discrete time stopping times
- Optional stopping theorem
- Gambler's ruin problem

Today's lecture: Section 4.3.2

Continuous Time Stopping Time

- A random variable τ taking values in $[0, \infty) \cup \{+\infty\}$ is a **stopping time** with respect to filtration $\{\mathcal{F}_t, t \geq 0\}$ if

$$\{\tau \leq t\} = \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0$$

- If $\{\mathcal{F}_t\}$ is right-continuous then τ is a continuous time stopping time for filtration $\{\mathcal{F}_t\}$ if and only if

$$\{\tau < t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0$$

- If θ, τ, τ_1, τ_2,... are all stopping times for the same filtration then the following are also stopping times:

$$\min(\theta, \tau),\ \max(\theta, \tau),\ \theta + \tau,\ \sup_{n \geq 1} \tau_n,\ \inf_{n \geq 1} \tau_n$$

Hitting Times

- Let $\{X_t\}$ be an RCLL process adapted to a right-continuous filtration $\{\mathcal{F}_t\}$
- For a Borel set B let

$$\tau_B = \inf\{t \geq 0 : X_t \in B\}$$

  with $\tau_B = \infty$ if $X_t \notin B$ for all $t \geq 0$
- If B is open then τ_B is an $\{\mathcal{F}_t\}$-stopping time
- If B is closed and $\{X_t\}$ has continuous paths then τ_B is an $\{\mathcal{F}_t\}$-stopping time

Stopped Process

- Suppose $\{X_t\}$ is a SP and τ is a stopping time for filtration $\{\mathcal{F}_t\}$
- The **stopped (at time τ) process** $\{X_{t\wedge\tau}, t \geq 0\}$ is defined by

$$X_{t\wedge\tau} = \begin{cases} X_t & t \leq \tau \\ X_\tau & t > \tau \end{cases}$$

- If $\{(X_t, \mathcal{F}_t)\}$ is a RCLL sub-martingale and τ is an $\{\mathcal{F}_t\}$-stopping time then $\{(X_{t\wedge\tau}, \mathcal{F}_t)\}$ is a sub-martingale. In particular, $\mathbb{E}(X_{t\wedge\tau}) \geq \mathbb{E}(X_0)$.
- If $\{(X_t, \mathcal{F}_t)\}$ is a RCLL martingale and τ is an $\{\mathcal{F}_t\}$-stopping time then $\{(X_{t\wedge\tau}, \mathcal{F}_t)\}$ is a martingale. In particular, $\mathbb{E}(X_{t\wedge\tau}) = \mathbb{E}(X_0)$.

Doob's Optional Stopping Theorem

- Suppose $\{(X_t, \mathcal{F}_t)\}$ is an RCLL sub-martingale and τ is an $\{\mathcal{F}_t\}$-stopping time. If
  - $\tau < \infty$ a.s., and
  - $\{X_{t\wedge\tau}, t \geq 0\}$ is uniformly integrable
- Then $\mathbb{E}(X_\tau) \geq \mathbb{E}(X_0)$.
- If instead $\{(X_t, \mathcal{F}_t)\}$ is a martingale then $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.

Example: Exercise 4.3.18

- Let $\{W_t\}$ be a Brownian motion with $\{\mathcal{F}_t\}$ its canonical filtration.
- Fix $a > 0$ and $b > 0$ and define

$$\tau_{a,b} = \inf\{t \geq 0 : W_t \notin (-a, b)\}$$

- Can use optional sampling applied to W_t to show

$$\mathbb{P}(W_\tau = -a) = b/(b+a)$$

- Can use optimal sampling applied to the martingale $Y_t = W_t^2 - t$ to show $\mathbb{E}(\tau_{a,b}) = ab$
- Can use optimal sampling applied to the *exponential martingale* $\exp(\sigma W_t - \sigma^2 t/2)$ (for $\sigma > 0$) to find $\mathbb{E}(e^{-\theta\tau_{b,b}})$ for $\theta > 0$

Stopped σ-field

- Let τ be a $\{\mathcal{F}_t\}$-stopping time. The **stopped σ-field $\mathcal{F}_\tau$** is the collection of all events $A \in \mathcal{F}$ such that

$$A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0$$

- $\mathcal{F}_\tau$ is a σ-field
- The RV τ is $\mathcal{F}_\tau$-measurable
- If τ and S are $\{\mathcal{F}_t\}$-stopping times with $S \leq \tau$ a.s. then $\mathcal{F}_S \subseteq \mathcal{F}_\tau$
- If $\{X_t\}$ is adapted to $\{\mathcal{F}_t\}$ then X_τ is $\mathcal{F}_\tau$-measurable
- **Example 4.3.23:** On $\Omega = \{HH, HT, TH, TT\}$ let $\tau = 1$ if the first coin is H and $\tau = 2$ otherwise. Then $\mathcal{F}_\tau = \sigma(\{TH\}, \{TT\}, \{HT, HH\})$.

Optional Sampling Theorem

- Suppose $\{(X_t, \mathcal{F}_t)\}$ is a RCLL sub-martingale and τ, S are $\{\mathcal{F}_t\}$-stopping times with $S \leq \tau$ a.s.,
- If either $\tau \leq C$ for some constant $C < \infty$, or $\{X_t\}$ is uniformly integrable
- Then
$$\mathbb{E}(X_\tau | \mathcal{F}_S) \geq X_S$$
- In particular, $\mathbb{E}(X_\tau) \geq \mathbb{E}(X_S) \geq \mathbb{E}(X_0)$
- In the case where $\{(X_t, \mathcal{F}_t)\}$ is a martingale then
$$\mathbb{E}(X_\tau | \mathcal{F}_S) = X_S,$$
and $\mathbb{E}(X_\tau) = \mathbb{E}(X_S) = \mathbb{E}(X_0)$