Bayesian Games

PhD Microeconomics II

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Lecture 3 Readings

- Recommended readings
  - Fudenberg and Tirole (1991), pp 209-234
  - MWG (1995), pp 253-257, pp 282-288
- Further reading
  - Gibbons (1992), pp143-163
  - Gintis (2009), pp121-131
  - Hargreaves-Heap & Varoufakis (2004), pp 60-76
  - Myerson (1991), pp 67-83, 127-136, 163-173
  - Osborne (2004), pp 271-357
  - Osborne & Rubinstein (1994), pp 24-30
  - Vega-Redondo (2003), 117-120, 188-204, 217-221
- Seminal contributions
  - Aumann (*Ann. of Stat. 1976*) "Agreeing to disagree"
  - Harsanyi (*Mang. Sci.* 1967-68) "Games with incomplete informat... "
  - Harsanyi (*IJGT* 1973) "Games with randomly perturbed payoffs..."
  - Mertens & Zamir (*IJGT* 1985) "Formulation of Bayesian analysis..."

Motivation

- Recall one of the early assumptions we made in lec 1
  - **Assumption 2:** Players have **complete information**
    - i.e. know all payoff functions
- Is this really true in applications?!
  - Does an employer know how innately productive a worker is?
  - Does the government know people's valuation of a public good?
  - Do you know whether I enjoy being a mean or nice examiner?
  - Does a firm know whether their competitor is low or high cost?
  - Does a car buyer know the true quality of a car?
  - Does an insurance company know how healthy you are?
  - etc...
- Often payoff functions are private information
- **Definition:** A game of **incomplete information** is where at least one payoff function is not known by every player.

Example: Market Entry with Incomplete Information

- Simple market entry game with incomplete information
  - Player 1 (incumbent): $S_1 = \{\text{build new factory, don't build}\}$
  - Player 2 (potential entrant): $S_2 = \{\text{enter, don't enter}\}$
  - Incumbent's cost is private information
  - Sometimes we can use iterative dominance

	E	DE
B	0, -1	2, 0
DB	2, 1	3, 0

cost = high

	E	DE
B	3, -1	5, 0
DB	2, 1	3, 0

cost = low

- 1 always has dominant strategy: B if low cost, DB if high cost
- Let p_H denote 2's prior probability that 1 is high cost
- Considering 2's expected payoff, 2 plays E iff $p_H > 1/2$, solved.

Example: Market Entry with Incomplete Information

- Typically iterative dominance does not get us far

	E	DE
B	0, -1	2, 0
DB	2, 1	3, 0

cost = high

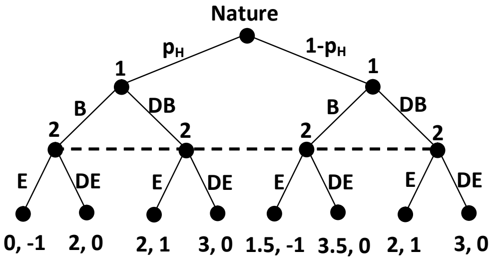
	E	DE
B	1.5, -1	3.5, 0
DB	2, 1	3, 0

cost = low

- *DB* still dom strat if high cost, but no dom strat if low cost
- So 1 has to predict whether or not 2 plays *E*
- Suppose y is the prob 2 plays *E*
- Examining expected payoffs gives, 1 plays *B* iff $y < 1/2$
- So 1 has to predict 2's behaviour to BR and 2 cannot infer 1's actions from knowledge of player 1's possible payoffs alone – stuck!

Harsanyi's Transformation

- Harsanyi (1967/68): transform *incomplete* info into *imperfect* info
  - Incomplete info on costs → Imperfect info about "Nature"'s move



  - Nature chooses Incumbent's "type" (cost)
  - Entrant cannot distinguish between the different types

- **Assumption 8:** Players have **common priors** on moves by Nature
- NE of imperf info game is a *Bayesian Nash Equilibrium*

Bayesian-Nash Equilibrium of Market Entry Game

- Solving for the BNE
  - Know player 1 plays *DB* if high cost, so just consider low cost:
    - Let x be player 1's probability playing B when cost is low
    - Let y be player 2's probability playing E
    - We've already worked out 1's BR

$$x = \begin{cases} 0 & \text{if } y > 1/2 \\ \in [0,1] & \text{if } y = 1/2 \\ 1 & \text{if } y < 1/2 \end{cases}$$

    - Compare 2's expected payoffs to get his BR

$$y = \begin{cases} 0 & \text{if } x > 1/\left[2\left(1-p_H\right)\right] \\ \in [0,1] & \text{if } x = 1/\left[2\left(1-p_H\right)\right] \\ 1 & \text{if } x < 1/\left[2\left(1-p_H\right)\right] \end{cases}$$

  - To identify BNE find (x, y) such that
    - x is optimal for 1 with low cost given 2's strategy
    - y is optimal for 2 given 1's strategy and beliefs p_H
  - Two BNE: $(x = 0, y = 1)\ \forall p_H$; $(x = 1, y = 0)$ iff $p_H \leq \frac{1}{2}$

The Example and Bayesian Games in General

- Reflect on properties of Bayesian games/example
  - Incomplete information game
    - Player 1's cost private info
  - Introduce "Nature" who chooses "type" first$\rightarrow$ imperf info game
    - Nature decided high or low cost for player 1
  - *Ex ante* probabilities same for all players, common prior assumption
    - Both knew p_H and $(1 - p_H)$
  - Nature chooses, endowing players with private info in the *interim*
    - Player 1 found out high or low cost, player 2 didn't
  - i's strategies in transformed game: action for each of i's poss types
    - Player 1 had an action for high cost and an action for low cost
    - Player 2 had 1 action
    - But if player 1 knows his type why a strategy for each of his types?!
    - Similar logic to extensive form's complete contingent plans... For 1 to BR need to think what 2 will do, 2 does not know cost, so thinks probabilistically given p_H about what 1 would do if high and if low cost.

The Example and Bayesian Games in General

- Reflect on properties of Bayesian games/example
  - A Bayesian Nash equilibrium consists of two things:
    1. A strategy profile
    2. Beliefs specified for each player about "types" of others
    - Our two BNE, $(s_1^* \left(cost\right), s_2^*, p_H)$, were

$$\left(\begin{array}{c} DB \text{ if } cost = low \\ DB \text{ if } cost = high \end{array}, E, \forall p_H \right) \& \left(\begin{array}{c} B \text{ if } cost = low \\ DB \text{ if } cost = high \end{array}, DE, p_H \leq \frac{1}{2} \right)$$

  - A BNE requires each player be maximising $E\left[payoff\right]$ given others' strategies and beliefs (accurate on own type and probabilistic on others')
    - 1 knows his type so maximises his $E\left[payoff\right]$ given 2's strategy (which depends on p_H)
    - 2 doesn't know 1's type, so maximises $E\left[payoff\right]$ given p_H and 1's strategy
  - *Ex post* payoff profile depends on full set of choices (including Nature's)

Bayesian Game Description

- Augment game description with "types" – the private info
- Bayesian game description

$$\Gamma^B := \left(N, \{S_i\}, \Theta, p(.), \{u_i(s,\theta)\}\right)$$

  - Players – N
  - Types – Θ, $p(.)$
    - i's *type* $\theta_i \in \Theta_i$, a random variable, realisation known only to i
    - Assume $|\Theta_i|$ is finite
    - $p(\theta_1, ..., \theta_n)$ is the objective joint prob distribution of $\{\theta_i\}_{i\in N}$
    - Assume $p(\theta_i) > 0$ for all $\theta_i \in \Theta_i$
    - Let $p(\theta_{-i} | \theta_i)$ denote i's conditional prob of $-i$'s type given own type
    - Let $\theta = \times_{i\in N}\theta_i$ and $\Theta = \times_{i\in N}\Theta_i$
    - Everything is common knowledge except realised $\{\theta_i\}_{i\in N}$
  - Strategies – S_i
    - Choices over actions (contingent plans for extensive form, later)
    - $s \in S$, usual definitions of strategy profiles
  - Payoffs – $u_i(s,\theta)$

Bayesian Nash Equilibrium

- Harsanyi transformation – Nature chooses θ_i first
  - Strategies can be conditioned on θ_i
    - A pure strategy is a function of type $s_i(\theta_i)$
    - Where $s_i(.) \in S_i^{\Theta_i}$, all maps $\Theta_i \rightarrow S_i$
    - Strategy profiles, $s(.) \in S^{\Theta}$, defined in usual way
  - **Definition:** $s(.)$ is a **Bayes-Nash equilibrium** if for each i

$$s_i(.) \in \arg\max_{s_i' \in S_i^{\Theta_i}} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i})\, u_i\left(s_i'(\theta_i), s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\right).$$

  - Since $p(\theta_i) > 0$ for all i, ex ante and interim conditions equivalent

$$s_i(\theta_i) \in \arg\max_{s' \in S_i} \sum_{\theta_{-i}} p(\theta_{-i} | \theta_i)\, u_i\left(s_i', s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\right).$$

    - i plays BR to conditional distribution of opponents' strategies for each type that he might end up playing (since doesn't know which)
  - BNE existence: appeal to Nash's Theorem, transformed game standard

Another Example: Public Goods Provision

- Public Goods Provision Game
  - $i = \{1, 2\}$, $A_i = \{contribute, don't\}$
  - Payoffs

	C	D
C	$1 - c_1,\ 1 - c_2$	$1 - c_1,\ 1$
D	$1,\ 1 - c_2$	$0, 0$

  - Complete information
    - (Practice for you: find all the NE of this game), for now note:

1. Never efficient to play CC & CC is never an NE (assuming $c_{1,2} \neq 0$)
2. If $c_1 < 1 < c_2$ then CD is efficient & this is the NE (analogous for DC when $c_2 < 1 < c_1$)
3. If $\max\{c_1, c_2\} < 1$ then NE is either CD or DC, provided in both NE so more efficient than DD, not most efficient if higher cost provides

Public Goods Provision with Incomplete Info

- Introduce incomplete information
  - c_i private info
    - Common knowledge that c_i independently drawn from same continuous strictly increase cdf $P(.)$ over $[\underline{c}, \overline{c}]$, where $\underline{c} < 1 < \overline{c}$.
  - Strategies and payoffs
    - i's pure strategy, $s_i(c_i) : [\underline{c}, \overline{c}] \longrightarrow \{0, 1\}$ (contribute or not)
    - i's payoff, $u_i(s_i, s_j, c_i) = \max[s_1, s_2] - c_i s_i$
  - BNE
    - Where $(s_i^*(.), s_j^*(.))$ such that for each i and every possible c_i, $s_i^*(c_i)$ maximises $E_{c_j}[u_i(s_i, s_j^*(c_j), c_i)]$
    - Let $z_j \equiv \Pr(s_j^*(c_j) = 1)$ i.e. eqm prob of opponent contributing
    - i contributes if cost<benefit*$(1 - z_j)$, thus $s_i^*(c_i) = 1$ iff $c_i < 1 - z_j$
    - Thus types $[\underline{c}, c_i^*]$ contribute (empty if $c_i^* < \underline{c}$)
    - Similarly, j contributes iff $c_j \in [\underline{c}, c_j^*]$
    - Since $z_j = \Pr(\underline{c} \le c_j \le c_j^*) = P(c_j^*)$, eqm cutoff satisfies $c_i^* = 1 - P(c_j^*)$
    - Then if unique, c_i^* and c_j^* should satisfy $c^* = 1 - P(1 - P(c^*))$

The Effect of Incomplete Information

- What do the BNE look like? Effect of incomplete cost info?
- Consider two illustrative cases
  - $P(.)$ uniform on $[0,2]$, $P(c) \equiv \frac{c}{2}$
    - c^* is unique, sub $\frac{c}{2}$ into BNE condition, solve c^*, $c^* = \frac{2}{3}$
    - **Result**: Less efficient than complete info
    - Under-contribution: Play D for $c \in (2/3, 1)$ despite ex post benefit $>$ cost, & $1 - P(c^*) = \frac{2}{3}$ chance not supplied by other
    - Over-contribution: $c \in [0, 2/3]$ both play C when only need one to.
  - Some $P(.)$ on $[\underline{c}, \overline{c}]$ where $\underline{c} \geq 1 - P(1)$
    - 2 asymmetric BNE
    - One always plays D, other always plays C for $c \leq 1$
    - e.g. BNE with player 1 always D: $c_1^* = 1 - P(1) \leq \underline{c}$ and $c_2^* = 1$
    - i.e. 1 never contributes because min cost greater than gain; 2 always contributes since otherwise zero probability of provision.
    - **Result**: Efficiency vs uniform unclear, less efficient than complete
    - BNE where 1 always D, if $c_1 < \frac{2}{3}$ & $c_2 > 1$, not provided (is if uniform)
    - BNE where 2 always D, if $c_1 < \frac{3}{4}$ & $c_2 > 1$, provided (not if uniform)
    - Coordination on inefficient BNE possible

The Example and Bayesian Games in General

- Prior beliefs are v important in Bayesian games
  - Note how different the BNE looked with the two different cdfs
- Often assume types drawn independently
  - $p(\theta_{-i} | \theta_i) = p(\theta_{-i})$ here, but types correlated in some games
- Bayesian games often have many BNE
  - Complete info game had 2 NE
  - Incomplete info game had 3 BNE that we identified (may be more)
  - NE are supported by consistent strategies
  - BNE supported by consistent beliefs and strategies
    - Many more possible combinations
- Monotonicity and cut-off rules are common in BNE
  - Despite the differences between the equilibria
  - Equilibrium contribution strategies were monotonic functions of type
  - Contribute up to some type, then don't – cut-off rule
- Information often has efficiency consequences
  - In our game incomplete info reduced efficiency (a common result)

Prior Beliefs

- Where do they come from?
  - Similar to Nash conjectures – e.g. culture, focal, learning etc
  - Could be an objective distribution e.g. male-female ratio in species
- What form "should" they take?
  - Modellers often use uniform or other typical statistical distributions
  - Best to be as general as possible subject to tractability constraints
- Is everyone having the same priors plausible? (Assumption 8)
  - Same info sets and CKR then technically, yes.
  - Aumann (1976): Even if different info sets, still common beliefs:
    - Given CKR, moment players discover holding differing beliefs, incentive to revise beliefs (to incorporate new info)
    - Rational players cannot "agree to disagree"
  - Reasonable?
    - "How many coins in the jar?" vs "Does god exist?"
    - Repeated game vs one-shot game (common *priors* even then?)
    - Costly information transmission
    - Incentives to acquire info – if all free-ride, could agree to disagree!

Interim vs Ex Ante Strict Dominance

- A BNE is an *equilibrium* in the sense that it's predictable
- So player i must
    - Predict player j's strategy choice – to do so...
        - Consider how each $j \neq i$ *thinks* player i will play
        - Consider j's *beliefs* about i's type
- In this prediction process, how should we view types θ_i and θ_i'
    1. A single player making type-contingent decisions at ex ante stage
    2. Two different "individuals" one of which nature will pick to play
- In 1, ex-ante predictions so all types of i predict the same
    - Similar to Harsanyi's original formulation
- In 2, interim predictions so predictions may differ between types of i
    - e.g. genetically determined preferences – ex ante impossible
- No difference for BNE (players have common beliefs)
- But does it matter for strict dominance?

Strict Dominance and the Public Good Game

- Interim strict dominance
  - Given c_i, which of i's strategies are not strictly dominated?
    - D: not strictly dominated (play D if expect j to contribute, $\forall c_i$)
    - If $c_i > 1$ then C strictly dominated for i
    - If $\underline{c} > 1 - P(1)$, no more dominated strategies
    - So, for example, interim dominance permits $c_i \in [\underline{c}, c']$ don't contribute and $c_i \in (c', 1]$ do contribute, former expect j to contribute if $c_j < 1$ and latter expect j never contributes
    - Could not happen in a BNE (cut-off rule/monotonicity in equilibrium)
- Ex ante strict dominance
  - Don't know c_i yet, which of i's strategies is not strictly dominated?
    - Any $s_i(.)$ that has player contribute with prob $z > 0$ and is not a cut-off rule is strictly ex ante dominated by a strategy where player contributes iff $c_i < c'$, where $c' = P(z)$
    - For any s_j, i receives public good with same prob, but his expected cost of provision is strictly lower
    - Intuitively, if i is a single player, then any beliefs of j's strategy that make it attractive to contribute at c' also make it attractive to contribute at $c_i < c'$

Interim vs Ex Ante Strict Dominance

- Generally, more strategies dominated ex ante than interim
  - For a given type-contingent strategy $\hat{\sigma}_1(.)$ of player 1
    - Easier to find $\sigma_1(.)$ satisfying ex ante dominance condition

$$\begin{aligned} & \sum_{\theta_1} p_1(\theta_1) \sum p(\theta_{-i} | \theta_1) u_1(\sigma_1(\theta_1), \sigma_{-1}(\theta_{-1}), \theta) \\ > & \sum_{\theta_1} p_1(\theta_1) \sum_{\theta_{-1}} p(\theta_{-1} | \theta_1) u_1(\hat{\sigma}_1(\theta_1), \sigma_{-1}(\theta_{-1}), \theta) \end{aligned}$$

for all $\sigma_{-1}(.)$, than to find s_1 and θ_1 satisfying interim constraints

$$\begin{aligned} & \sum_{\theta_{-1}} p(\theta_{-1} | \theta_1) u_1(s_1, \sigma_{-1}(\theta_{-1}), \theta) \\ > & \sum_{\theta_{-1}} p(\theta_{-1} | \theta_1) u_1(\hat{\sigma}_1(\theta_1), \sigma_{-1}(\theta_{-1}), \theta) \end{aligned}$$

for all $\sigma_{-1}(.)$.

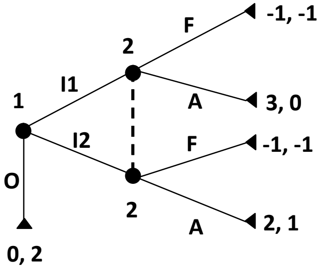
  - Intuition:
    - Interim: Many beliefs (differing by type), need domination for all
    - Ex ante: One belief (same for all types), domination easier

Purification of Mixed Strategy Equilibria

- Critique of using mixed strategies in complete info games
  - People don't flip coins to make decisions!
- Can justify mixed strategies using Bayesian games
  - Complete info game example:
    - $S_i = \{invest, don't\}$; u_i : 1 if only i invests, -1 if both do, 0 if don't
    - Only symmetric NE is playing invest with prob $\frac{1}{2}$
  - Introduce incomplete information
    - If only i invests gets $(1 + \theta_i)$ where θ_i, private, is uniform on $[-\varepsilon, \varepsilon]$
    - BNE: symmetric pure strats $s_i(\theta_i < 0)$=don't and $s_i(\theta_i \geq 0)$=invest
    - Each firm expects other to invest with prob $\frac{1}{2}$, invest iff $\frac{1}{2}(1 + \theta_i) + \frac{1}{2}(-1) \geq 0$, i.e. $\theta_i \geq 0$
    - As $\varepsilon \to 0$, pure strategy BNE $\to$ mixed-strategy NE of complete info
- Known as **purification** of a mixed strategy equilibrium
  - **Result** (Harsanyi, 1973): Any MSNE can be obtained as limit of pure strategy equilibrium in sequence of slightly perturbed games.
    - Intuition: players play pure strats in Bayesian games, don't know opponent's type, so effectively play as if facing a mixed strat

Extensive Form Bayesian Games

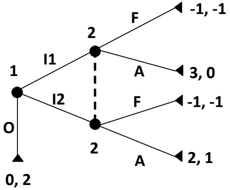
- Only considered incomplete info in strategic form so far
- Extensive form example:



  - 2 pure strat NE: (O, F if enter) & ($I1$, A if enter)
  - First NE doesn't seem reasonable: 2 prefers to accom once enters
  - Subgame Perfection too weak – only 1 subgame, both NE are SPNE!
- Need something stronger for extensive form Bayesian games

Continuation Games vs Subgames

- Subgame perfection has no bite as only 1 well-defined subgame



- **Definition**: A **continuation game,** C of Γ^E, is a subset of Γ^E
  1. Starting at an information set h (may or may not be singleton)
  2. Containing only all sucessor nodes of $x \in h$
  3. If $x' \in C$ and $x'' \in h(x')$, then $x'' \in C$ (i.e. no chopping up info sets)
  4. Adopt other properties (payoff functions, player/action labels) from Γ^E
- How many continuation games in the example?
- To apply the "spirit" of subgame perfection for continuation games
  - Need a probability distribution over nodes in the first info set
  - Then require BNE in every continuation game

- **Definition:** A **system of beliefs** in Γ^E is a probability $\mu(x) \in [0,1]$ for each x in Γ^E such that $\sum_{x \in h} \mu(x) = 1$ for all $h \in H$.
  - i.e. player's assessment of the relative probability of being at each node in that information set conditional upon that info set being reached
- **Definition**: A strategy profile σ in Γ^E is **sequentially rational** at h given a system of beliefs μ if

$$E\left[u_{\iota(h)} \middle| h, \mu, \sigma_{\iota(h)}, \sigma_{-\iota(h)}\right] > E\left[u_{\iota(h)} \middle| h, \mu, \hat{\sigma}_{\iota(h)}, \sigma_{-\iota(h)}\right]$$

for all $\hat{\sigma}_{\iota(h)} \in \Delta\left(S_{\iota(h)}\right)$. If σ satisfies this condition for all $h \in H$, then σ is **sequentially rational given belief system** μ.

  - i.e. no player wants to change strategy once reach an info set given beliefs about what's happened and opponent's strategy

Consistency of Beliefs

- In a weak perfect Bayesian equilibrium
  1. Strategies must be sequentially rational given beliefs
  2. Beliefs must be consistent with strategies
     - Similar to Nash conjectures, i.e. beliefs are correct
     - Like SPNE & NE, require BNE in each continuation game
- Illustrate belief consistency:
  - Assume all i play completely mixed strategies, every h reached with positive prob
  - For each x in a player's h, compute $\Pr(x \mid \sigma)$ then assign conditional prob of being at each x given play has reached that h using Bayes' rule

$$\Pr(x \mid h, \sigma) = \frac{\Pr(x \mid \sigma)}{\sum_{x' \in h} \Pr(x' \mid \sigma)}$$

  - E.g. Previous game if 1 plays $\sigma : \sigma(O) = \frac{1}{4}$, $\sigma(I1) = \frac{1}{2}$ & $\sigma(I2) = \frac{1}{4}$
    - Prob of reaching 2's info set given σ is $\frac{3}{4}$
    - Bayes' rule: prob of being at $I1$ given info set reached is $\frac{2}{3}$ ($I2 = \frac{1}{3}$)
    - 2's should have beliefs $\frac{2}{3}$ and $\frac{1}{3}$ to be consistent with 1's σ

Consistency of Beliefs Off the Path

- What about if players are not playing completely mixed strategies?
- $\exists$ some x not reach with positive probability
- Cannot use Bayes' rule to calculate the probability of reaching these
- Never go off the eqm path, even if played repeatedly, so how do you form beliefs about these x?
- WPBE $=$ agnostic... have whatever beliefs you like off the path!
  - This is why it is "weak" relative to the solution concepts we'll look at next lecture!

Weak Perfect Bayesian vs Nash Equilibrium

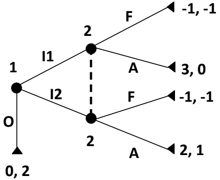
- **Definition**: (σ, μ) is a **weak perfect Bayesian equilibrium** in Γ^E if
  1. σ is sequentially rational given μ
  2. μ is consistent with σ and Bayes' rule on the path, thus any h such that $\Pr(h|\sigma) > 0$ must have

$$\mu(x) = \frac{\Pr(x|\sigma)}{\Pr(h|\sigma)} \text{ for all } x \in h.$$

- Contrast with definition of NE in this context
- **Definition:** σ is an **NE** in Γ^E if $\exists$ some μ such that
  1. σ is sequentially rational given μ for all h such that $\Pr(h|\sigma) > 0$
  2. μ is consistent with σ and Bayes' rule on the path
- Sequential rationality
  - Only required on eqm path by NE
  - Required on and off the path by WPBE
  - Thus $WPBE \subseteq NE$

Applying WPBE to the Market Entry Example

- Applying WPBE to the example:



  - 2 must play "A if enter" in any WPBE
    - This is optimal if get to that info set for any μ
  - Previous NE (O, F if enter) is not WPBE
  - Is other NE, ($I1$, A if enter), WPBE?
    - Need a μ satisfying condition 2 that makes these σ sequentially rational
    - To satisfy condition 2, must set $\mu(I1) = 1$ (info set reached with positive prob given σ)
    - These σ are indeed sequentially rational given that μ
    - Thus ($I1$, A if enter) is a WPBE.

Summary

- Complete information is unrealistic for most applications
- Bayesian games allow us to model incomplete information
- Harsanyi: transform incomplete info game into imperfect info game
  - Then consider the Bayesian-Nash equilibrium
- Beliefs important in Bayesian games, typically assume common priors
- Equilibria characterised by beliefs & strategies so often more equilibria
- Strict dominance: difference between ex ante and interim application
- Mixed strategies can be defended as the limit of perturbed games
- When considering dynamic games, Bayesian-Nash is often too weak
- WPBE requires sequential rationality on and off the path
  - No requirements on beliefs off the path