Homework 10

Problem 23
Give formal proofs corresponding to the valid steps discussedn Problem 16, page 103

1. Modus Tollens: From $A \to B$ and $\neg B$ infer $\neg A$

1.	$A \to B$	
2.	$\neg B$	$/ \therefore \neg A$
3.	A	
4.	B	$\to$ Elim 1, 3
5.	$B \wedge \neg B$	$\wedge$ Intro 4, 2
6.	$\neg A$	$\neg$ Intro 3-5

2. Strengthening the Antecedent: From $B \to C$ infer $(A \wedge B) \to C$

1.	$B \to C$	$/ \therefore (A \wedge B) \to C$
2.	$A \wedge B$	
3.	B	$\wedge$ Elim 2
4.	C	$\to$ Elim 1, 3
5.	$(A \wedge B) \to C$	$\wedge$ Intro 2-4

3. Weakening the Consequent: From $A \to B$ infer $A \to (B \vee C)$

1.	$A \to B$	
2.	A	
3.	B	$\to$ Elim 1, 2
4.	$B \vee C$	$\vee$ Intro 3
5.	$A \to (B \vee C)$	$\to$ Intro 2-4

4. Constructive Dilemma: From $A \vee B$, $A \to C$, and $B \to D$ infer $C \vee D$

1.	$A \vee B$	
2.	$A \to C$	
3.	$B \to D$	
4.	A	
5.	C	$\to$ Elim 2, 4
6.	$C \vee D$	$\vee$ Intro 5
7.	B	
8.	D	$\to$ Elim 3, 7
9.	$C \vee D$	$\vee$ Intro 8
10.	$C \vee D$	$\vee$ Elim 1, 4-6, 7-9

5. Transitivity of the biconditional: From $A \leftrightarrow B$ and $B \leftrightarrow C$ infer $A \leftrightarrow C$

1. $A \leftrightarrow B$	
2. $B \leftrightarrow C$	
3. A	
4. B	$\leftrightarrow$ Elim 1, 3
5. C	$\leftrightarrow$ Elim 2, 4
6. $A \to C$	$\to$ Intro 3-5
7. C	
8. B	$\leftrightarrow$ Elim 2, 7
9. A	$\leftrightarrow$ Elim 1, 8
10. $C \to A$	$\to$ Intro 7-9
11. $A \leftrightarrow C$	$\leftrightarrow$ Intro 3-6, 7-10

Problem 24

Give formal proofs of the following

1. $A \to (B \to A)$ from no premises

1. A	
2. B	
3. A	Reit 1
4. $B \to A$	$\to$ Intro 2-3
5. $A \to (B \to A)$	$\to$ Intro 1-4

To prove something with no premises means that we must start off with an assumption. We work this problem by noticing that the main connective is a $\to$. Given that, we know that we can use a $\to$ Intro to prove the main connective and therefore introduce the antecedent as our assumption. The second step of the proof is done by noticing that the other connective is a $\to$ and can be solved in the same way. Having assumed B (the antecedent in the inside $\to$), all we have to do is restate A to show that A follows from B. having shown that A follows from B, we have shown $B \to A$. Because we were able to get this only by assuming A, we have shown that $A \to (B \to A)$.

2. $(A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$ from no premises

1. $\underline{(A \to (B \to C))}$	
2. $\underline{A \wedge B}$	
3. A	$\wedge$ Elim 2
4. $B \to C$	$\to$ Elim 1, 3
5. B	$\wedge$ Elim 2
6. C	$\to$ Elim 4, 5
7. $(A \wedge B) \to C$	$\to$ Intro 2-6
8. $(A \to (B \to C)) \to ((A \wedge B) \to C)$	$\to$ Intro 1-7
9. $\underline{(A \wedge B) \to C}$	
10. $\underline{A}$	
11. $\underline{B}$	
12. $A \wedge B$	$\wedge$ Intro 10, 11
13 C	$\to$ Elim 9, 12
14. $B \to C$	$\to$ Intro 11-13
15. $A \to (B \to C)$	$\to$ Intro 10-14
16. $((A \wedge B) \to C) \to (A \to (B \to C))$	$\to$ Intro 9-15
17. $(A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$	$\leftrightarrow$ Intro 1-8, 9-16

The major connective here is a biconditional, so we know that we need to use a $\leftrightarrow$ Intro. The way a $\leftrightarrow$ Intro works is by proving first that the left side implies the right side (by using a $\to$ Intro, assuming the left side, and proving the right side) and then by proving that the right side implies the left side (by using a $\to$ Intro, assuming the right side, and proving the left side). From here, it's a pretty straightforward proof that it similar to the last one.

3. $C \wedge D$ from premises $A \vee (B \wedge C)$, $\neg E$, $(A \vee B) \to (D \vee E)$, and $\neg A$

1. $A \vee (B \wedge C)$	
2. $\neg E$	
3. $(A \vee B) \to (D \vee E)$	
4. $\neg A$	
5. A	
6. $\neg(B \wedge C)$	
7. $A \wedge \neg A$	$\wedge$ Intro 5, 4
8. $B \wedge C$	$\neg$ Intro 6-7
9. $B \wedge C$	
10. $B \wedge C$	Reit 9
11. $B \wedge C$	$\vee$ Elim 1, 5-8, 9-10
12. C	$\wedge$ Elim 11
13. B	$\wedge$ Elim 11
14. $A \vee B$	$\vee$ Intro 13
15. $D \vee E$	$\to$ Elim 3, 14
16. $\neg D$	
17. $\neg D \wedge \neg E$	$\wedge$ Intro 16, 2
18. $\neg(D \vee E)$	DeM 17
19. $(D \vee E) \wedge \neg(D \vee E)$	$\wedge$ Intro 15, 18
20. D	$\neg$ Intro 16-19
21. $C \wedge D$	$\wedge$ Intro 12, 20

There are a couple of ways of doing this proof. What I have done here is to use a $\vee$ Elim to prove that $B \wedge C$ follows from $A \vee (B \wedge C)$. [This is because $\neg A$ is a premise, so we know that $B \wedge C$ must be true.] So, line 5 is the left side of the $\vee$ from line 1 and I use a $\neg$ Elim to prove that $B \wedge C$ follows from it (lines 6-7), and line 9 is the right side of the $\vee$ from line 1. Now that we've shown that $B \wedge C$ (line 11), we know that C follows ($\wedge$ Elim). This is half of what we want in our conclusion. Now all we need to do is to get D, which I have done by using a $\neg$ Intro (lines 16-19). Finally, now that we have gotten C by itself (line 12) and D by itself (line 20), we know that $C \wedge D$ (by $\wedge$ Intro).

Problem 25

The book wants you to prove two equivalences, which you will be able to cite in later homework using our method of citing previous theorms. Once you know them, you can then use them in $\leftrightarrow$ Elim (in system *F* and *F'*) and in Gen Sub (in system *F'*).

1. Prove $(P \to Q) \leftrightarrow (\neg P \vee Q)$

1. $P \to Q$	
2. $\neg(\neg P \vee Q)$	
3. $P \wedge \neg Q$	DeM 2
4. P	$\wedge$ Elim 3
5. $\neg Q$	$\wedge$ Elim 3
6. Q	$\to$ Elim 1, 4
7. $Q \wedge \neg Q$	$\wedge$ Intro 5, 6
8. $\neg P \vee Q$	$\neg$ Intro 2-7
9. $(P \to Q) \to (\neg P \vee Q)$	$\to$ Intro 1-8
10. $\neg P \vee Q$	
11. P	
12. $\neg Q$	
13. $P \wedge \neg Q$	$\wedge$ Intro 11, 12
14. $\neg(\neg P \vee Q)$	DeM 13
15. $(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	$\wedge$ Intro 10, 14
16. Q	$\neg$ Intro 12-15
17. $P \to Q$	$\to$ Intro 11-16
18. $(\neg P \vee Q) \to (P \to Q)$	$\to$ Intro 10-17
19. $(P \to Q) \leftrightarrow (\neg P \vee Q)$	$\leftrightarrow$ Intro 1-9, 10-18

This is a biconditional introduction so we will need two conditional introductions, one in which we prove that the right follows from the assumption of the left and another in which we prove that the left follows from the assumption of the right. The first half is taken care of by means of a $\neg$ Intro. I assume the opposite of what I want to prove $\neg(\neg P \vee Q)$, prove a contradiction $(Q \wedge \neg Q)$, and therefore conclude $\neg P \vee Q$.

The second half involves proving a conditional $(P \to Q)$, so I use a conditional introduction, which involves assuming the antecedent (P) and proving the consequent (Q). I actually prove Q by using a $\neg$ Intro. Thus, I assume the opposite of what I want to prove $(\neg Q)$, prove a contradiction, and therefore conclude Q. Having shown that $(P \to Q) \to (\neg P \vee Q)$ and $(\neg P \vee Q) \to (P \to Q)$, we know that $(P \to Q) \leftrightarrow (\neg P \vee Q)$, by $\leftrightarrow$ Intro.

Problem 25, #2

1. $\neg(P \rightarrow Q)$	
2. $\neg(P \wedge \neg Q)$	
3. $\neg P \vee Q$	
4. $\neg P$	
5. P	
6. $\neg Q$	
7. $P \wedge \neg P$	$\wedge$ Intro 5, 4
8. Q	$\neg$ Intro 6-7
9. $P \rightarrow Q$	$\rightarrow$ Intro 5-8
10. Q	
11. P	
12. Q	Reit 10
13. $P \rightarrow Q$	$\rightarrow$ Intro 11-12
14. $P \rightarrow Q$	$\vee$ Elim 3, 4-9, 10-13
15. $(P \rightarrow Q) \wedge \neg(P \rightarrow Q)$	$\wedge$ Intro 14, 1
16. $P \wedge \neg Q$	$\neg$ Intro 2-15
17. $\neg(P \rightarrow Q) \rightarrow (P \wedge \neg Q)$	$\rightarrow$ Intro 1-11
18. $P \wedge \neg Q$	
19. $P \rightarrow Q$	
20. P	$\wedge$ Elim 18
21. Q	$\rightarrow$ Elim 19, 20
22. $\neg Q$	$\wedge$ Elim 18
23. $Q \wedge \neg Q$	$\wedge$ Intro 21, 22
24. $\neg(P \rightarrow Q)$	$\neg$ Intro 19-23
25. $(P \wedge \neg Q) \rightarrow \neg(P \rightarrow Q)$	$\rightarrow$ Intro 18-24
26. $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$	$\leftrightarrow$ Intro 1-17, 18-26

Problem 26

Give a formal proof of Cube(a) $\leftrightarrow$ Small(a) from the following set of premises:

To make this proof simpler, I'm going to cite a previous theorem. Rather than referring you to the book, I'll just prove it.

1. $P \vee Q \vee R$	
2. $\neg P$	
3. $\neg(Q \vee R)$	
4. $\neg P \wedge \neg(Q \vee R)$	$\wedge$ Intro 2, 3
5. $\neg(P \vee (Q \vee R))$	DeM 4
6. $\neg(P \vee Q \vee R)$	(Assoc $\vee$) *It is unnecessary to state this rule*
7. $(P \vee Q \vee R) \wedge \neg(P \vee Q \vee R)$	$\wedge$ Intro 1, 6
8. $Q \vee R$	$\neg$ Intro 3-7

1. Cube(a) $\vee$ Dodec(a) $\vee$ Tet(a)	
2. Small(a) $\vee$ Medium(a) $\vee$ Large(a)	
3. Medium(a) $\leftrightarrow$ Dodec(a)	
4. Tet(a) $\leftrightarrow$ Large(a)	
5. Cube(a)	
6. $\neg$Small(a)	
7. Medium(a) $\vee$ Large(a)	Prev Thm (See above)
8. Medium(a)	
9. Dodec(a)	$\leftrightarrow$ Elim 3, 8
10. Dodec(a) $\vee$ Tet(a)	$\vee$ Intro 9
11. Large(a)	
12. Tet(a)	$\leftrightarrow$ Elim 4, 11
13. Dodec(a) $\vee$ Tet(a)	$\vee$ Intro 12
14. Dodec(a) $\vee$ Tet(a)	$\vee$ Elim 7, 8-10, 11-13
15. Cube(a) $\wedge$ (Dodec(a) $\vee$ Tet(a))	$\wedge$ Intro 5, 14 [Note that this is a contradiction in the general sense]
16. Small(a)	$\neg$ Intro 6-15
17. Cube(a) $\rightarrow$ Small(a)	$\rightarrow$ Intro 5-16
18. Small(a)	
19. $\neg$Cube(a)	
20. Dodec(a) $\vee$ Tet(a)	Prev Thm (See above)
21. Dodec(a)	
22. Medium(a)	$\leftrightarrow$ Elim 3, 21
23. Medium(a) $\vee$ Large(a)	$\vee$ Intro 22
24. Tet(a)	
25. Large(a)	$\leftrightarrow$ Elim 4, 24
26. Medium(a) $\vee$ Large(a)	$\vee$ Intro 25
27. Medium(a) $\vee$ Large(a)	$\vee$ Elim 20, 21-23, 24-26
28. Small(a) $\wedge$ Medium(a) $\vee$ Large(a)	$\wedge$ Intro 18, 27 [Note that this is a contradiction]
29. Cube(a)	$\neg$ Intro 19-28
30. Small(a) $\rightarrow$ Cube(a)	$\rightarrow$ Intro 18-29
31. Cube(a) $\leftrightarrow$ Small(a)	$\leftrightarrow$ Intro 5-17, 18-30

Note that lines 15 and 28 use the more general form of a contradiction discussed on page 72 of the book and used in problem 45 from the previous chapter. The form is not quite the same, although we can make it explicit using distribution.

Problem 27

F' modified proof by contradiction to allow multiple assumptions. We can also do this with conditional introduction. Give the statement of conditional introduction and use it to prove $(A \wedge B) \to C$ from $A \to C$.

First, the rule:

Conditional Introduction ($\to$ Intro)

P
$\underline{Q}$
.
.
.
S
> $(P \wedge Q) \to S$

What we have done here is we've taken the statement of Conditional Introduction (see page 306) and modified it to state a rule that if we can prove R by assuming P and Q, we will have shown that $(P \wedge Q) \to S$

Now we'll use it in a proof

1. $\underline{A \to C}$	
2. A	
3. $\underline{B}$	
4. C	$\to$ Elim 1, 2
5. $(A \wedge B) \to C$	$\to$ Intro 2-4

What's going on here is that because we modified our rule, A and B can both be assumptions, indicated by the fact that they are above the Fitch bar. (The first horizontal Fitch bar indicates that $A \to C$ is a premise.) We prove C in the same way we did in a previous problem. Thus, by having proven C by assuming A and B, we have shown $(A \wedge B) \to C$ by citing our modified rule.

Problem 28

State a valid rule for Disjunction Elimination that allows for subproofs with multiple premises.

Problem 29

Give formal proofs of the following in system *F'*.

1. Prove $A \leftrightarrow \neg B$ from premises $A \vee B \vee C$, $B \rightarrow (A \rightarrow \neg C)$, and $A \leftrightarrow C$.

1. $A \vee B$	
2. $B \rightarrow (A \rightarrow \neg C)$	
3. $A \leftrightarrow C$	
4. A	
5. $B \rightarrow (A \rightarrow \neg A)$	Gen Sub 2, 3
6. B	
7. $A \rightarrow \neg A$	$\rightarrow$ Elim 5, 6
8. $\neg A$	$\rightarrow$ Elim 7, 4
9. $\perp$	$\perp$ Intro 4, 8
10. $\neg B$	$\neg$ Intro 6-9
11. $\neg B$	
12. $A \vee B$	
13. $\neg A$	
14. $\neg A \wedge \neg B$	$\wedge$ Intro 13, 11
15. $\neg(A \vee B)$	DeM 14
16. $\perp$	$\perp$ Intro 13-16
17. A	$\neg$ Intro 13-16
18. $A \leftrightarrow \neg B$	

#2

1. $\neg A \rightarrow B$	
2. $C \rightarrow (D \vee E)$	
3. $D \rightarrow \neg C$	
4. $\underline{A \rightarrow \neg E}$	/ $\therefore$ $C \rightarrow B$
5. $\underline{C}$	
6. $D \vee E$	$\rightarrow$ Elim 2, 5
7. $\underline{\neg D}$	
8. $\neg C$	$\rightarrow$ Elim 3, 7
9. $\perp$	$\perp$ Intro 5, 8
10. $\neg D$	$\neg$ Intro 7-9
11. $\underline{\neg E}$	
12. $\neg D \wedge \neg E$	$\wedge$ Intro 10, 11
13. $\neg(D \vee E)$	DeM 12
14. $\perp$	$\perp$ Intro 6, 13
15. E	
16. $\underline{A}$	
17. $\neg E$	$\rightarrow$ Intro 4, 16
18. $\perp$	$\perp$ Intro 15, 17
19. $\neg A$	$\neg$ Intro 16-18
20. B	$\rightarrow$ Elim 1, 19
21. $C \rightarrow B$	$\rightarrow$ Intro 5-20

AX:likes(X,X)
 ~likes(a,b)
        a=b
         likes(a,a)
         likes(a,b)
 a=b => likes(a,b)
        a=b
        ~likes(a,b)
 a=b => ~likes(a,b)
 ~a=b

1.f(X,Y) & f(Y,Z) => f(X,Z)   Premise
2.EY:(AX:(h(X) => f(X,Y)) & AZ:(r(Z) => f(Y,Z))) Premise
3.AX:(h(X) => f(X,Y)) & AZ:(r(Z) => f(Y,Z))   Assumption
4.AX:(h(X) => f(X,Y))   And Elimination: 3
5.AZ:(r(Z) => f(Y,Z)) And Elimination: 3
6.h(X) => f(X,Y) Universal Elimination: 4
7.r(Z) => f(Y,Z) Universal Elimination: 5
8.h(X) & r(Z)   Assumption
9.h(X) And Elimination: 8

10.r(Z) And Elimination: 8
11.f(X,Y) Implication Elimination: 6, 9
12.f(Y,Z) Implication Elimination: 7, 10
13.f(X,Y) & f(Y,Z) And Introduction: 11, 12
14.f(X,Z) Implication Elimination: 1, 13
15.h(X) & r(Z) => f(X,Z) Implication Introduction: 14
16. AZ:(h(X) & r(Z) => f(X,Z)) Universal Introduction: 15
17. AX:AZ:(h(X) & r(Z) => f(X,Z))   Universal Introduction: 16
18. AX:(h(X) => f(X,Y)) & AZ:(r(Z) => f(Y,Z)) => AX:AZ:(h(X) & r(Z) => f(X,Z)) Implication Introduction: 17
19.AY:(AX:(h(X) => f(X,Y)) & AZ:(r(Z) => f(Y,Z)) => AX:AZ:(h(X) & r(Z) => f(X,Z))) Universal Introduction: 18
20.AX:AZ:(h(X) & r(Z) => f(X,Z))

1. falls(a)   Premise
2. AX:(falls(X) => falls(s(s(X)))) Premise
3.AX:(falls(s(X)) => falls(X)) Premise
4.falls(X) Assumption
5.falls(X) => falls(s(s(X))) Universal Elimination: 2
6.falls(s(s(X))) Implication Elimination: 5, 4
7.falls(s(s(X))) => falls(s(X))Universal Elimination: 3
8.falls(s(X))Implication Elimination: 7, 6
9.falls(X) => falls(s(X)) Implication Introduction: 8
10.AX:(falls(X) => falls(s(X)))Universal Introduction: 9
11.AX:falls(X)

1.p(0) Premise
2.AX:(p(X) => p(f(X))) Premise
3.AX:(p(f(X)) => p(g(X))) Premise
4. p(X) Assumption
5.p(X) => p(f(X)) Universal Elimination: 2
6.p(f(X)) Implication Elimination: 5, 4
7.p(f(X)) => p(g(X)) Universal Elimination: 3
8.p(g(X)) Implication Elimination: 7, 6
9.p(X) => p(g(X)) Implication Introduction: 8
10.AX:(p(X) => p(g(X))) Universal Introduction: 9
11.AX:p(X) Induction: 1, 2, 10

1.AX:(p(X) => q(X)) Premise
2.EX:p(X) Premise
3.p(X) Assumption
4.p(X) => q(X) Universal Elimination: 1
5.q(X) Implication Elimination: 4, 3
6.EX:p(X) Reiteration: 2
7.EX:q(X) Existential Introduction: 5
8.p(X) => EX:q(X) Implication Introduction: 7

9.AX:(p(X) => EX:q(X)) Universal Introduction: 8
10.EX:q(X) Existential Elimination: 2, 9