Feedback — Homework 2

Help Center

Thank you. Your submission for this homework was received.

You submitted this homework on **Mon 16 Mar 2015 2:39 AM PDT**. You got a score of **7.00** out of **7.00**.

Question 1

This problem deals with a TV game show called *Let's Make a Deal* which made its debut on the NBC Television Network on December 30, 1963. This game was very popular for a while and has gained some amount of notoriety because of its apparently paradoxical resolution. You may well be familiar with it. Play along anyway. It is always useful to check analysis against empirical data. The analysis will be provided with the solutions.

As described in the link, there are two strategies you can try in this game: (1) stay with your initial choice, and (2) switch your choice to the only remaining possibility. Try the first strategy ten times by repeatedly clicking the Retry button; then Reset the game and try the second strategy ten times by again repeatedly clicking the Retry button. The game will keep track of your score for each strategy. Click on the Radio button below that best describes your results. There is no right or wrong answer for this segment.

Click here to go to the question.

Your Answer		Score	Explanation
○ The strategy of staying with your initial door choice yielded better results on average over ten trials than the strategy of switching your door choice to the remaining possibility.			
◉ The strategy of staying with your initial door choice yielded worse results on average over ten trials than the strategy of switching your door choice to the remaining possibility.	✔	1.00	
○ The performance of the two strategies was the same over ten trials.			
Total		1.00 / 1.00	

Question Explanation

This explanation is from *The Theory of Probability*, pages 43, 44, Cambridge University Press, 2013.

A popular game show called *Let's Make a Deal* made its debut on the NBC Television Network on December 30, 1963. In the simplest version of the game, a prize is placed behind one of three closed doors and gag items of little or no value (called *{zonks* in the show) are placed behind the other two doors, also closed. Contestants in the show are aware of the fact that one of the three doors conceals a prize of some value but do not know behind which door it is. To begin, a contestant is asked to select a door (but not open it). Once the selection is made, the moderator, the charismatic Monty Hall, opens one of the remaining two doors and displays a zonk behind it. (As at least one of the remaining two doors must conceal a zonk, it is always possible for him to select a door concealing a zonk from the remaining two.) The contestant is now given the option of either staying with her original door selection or switching allegiance to the remaining door in the hope of getting the prize. Once she has made her decision, she opens her final door selection and can take possession of whatever is behind the door. What should her strategy be to maximise her chances of winning the prize?

One is inclined to reflexively believe that the probability of securing a prize is $1/2$ whether the contestant stays with her original choice or picks the remaining door. After all, the prize is behind one of the two unopened doors and it seems eminently reasonable to model the probability of it being behind either door as one-half. A more careful examination shows, however, that this facile argument is flawed and that the contestant stands to gain by always switching to the remaining door.

What is the sample space for the problem? We may begin by supposing that all six permutations of the prize and the two zonks are equally likely. Writing $\star$ for the prize and α and β for the two zonks, the possible arrangements, listed in order by door, are shown in the following table.

$$\star\,\alpha\,\beta \qquad \star\,\beta\,\alpha \qquad \alpha\,\star\,\beta \qquad \beta\,\star\,\alpha \qquad \alpha\,\beta\,\star \qquad \beta\,\alpha\,\star$$

The first two arrangements correspond to the prize being behind door 1, the next two arrangements correspond to the prize being behind door 2, and the last two arrangements correspond to the prize being behind door 3.

Write A for the event that the prize is behind the door initially selected by the contestant and write B for the event that the prize is behind the remaining door after Monty Hall has made his selection. By the symmetry inherent in the problem it makes no matter which door the contestant picks initially and for definiteness let us suppose that she picks the first door. Then we may identify $A = \{\star\,\alpha\,\beta, \star\,\beta\,\alpha\}$ and $\mathbf{P}(A) = \frac{2}{6} = \frac{1}{3}$.

If A occurs then, regardless of the strategy used by Monty Hall for selecting a door to open, the remaining door will conceal a zonk (as the second and third doors both conceal zonks). If, however, A does not occur, then the prize is behind the second or third door and Monty Hall is compelled to select and open the unique door of the two that does not conceal the prize. But then the residual door must conceal the prize. Formally, write C_1, C_2, and C_3 for the events that the prize is behind door 1, door 2, and door 3, respectively. Then C_1, C_2, and C_3 have equal probability and partition the sample space and, moreover, $\mathbf{P}(B \mid C_1) = 0$ and $\mathbf{P}(B \mid C_2) = \mathbf{P}(B \mid C_3) = 1$. It follows that

$$\mathbf{P}(B) = \mathbf{P}(B \mid C_1)\mathbf{P}(C_1) + \mathbf{P}(B \mid C_2)\mathbf{P}(C_2) + \mathbf{P}(B \mid C_3)\mathbf{P}(C_3) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{2}{3}.$$

The student should recognise that we have just made a conditional argument and appealed to total probability! The argument also makes clear that we may identify $B = A^{\complement} = \{\alpha \star \beta, \beta \star \alpha, \alpha\,\beta\star, \beta\,\alpha\star\}$, all these outcomes of the experiment resulting in the prize lying behind the remaining door after Monty Hall has made his selection. It follows again that $\mathbf{P}(B) = 2/3$. The analysis is unaffected by the particular initial choice of the contestant and, in consequence, the strategy of switching, if followed religiously, will unearth the prize two-thirds of the time.

A number of variations on the theme can be proposed based on the moderator's strategy. This example has attained a measure of notoriety due to its apparently paradoxical nature but conditioning on the initial choice makes clear where the fallacy in logic lies.

Question 2

Let $A_1, \dots, A_n$ denote n events, each of positive probability. Identify the chain rule for conditional probabilities from the possibilities given below.

Your Answer		Score	Explanation
$\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cup A_2) \times \cdots \times \mathbf{P}(A_n$			
$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \times \cdots \times \mathbf{P}(A_n$	✔	1.00	
$\mathbf{P}(A_1) = \mathbf{P}(A_1) \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_3 \mid A_1 \cap A_2) \times \cdots \times \mathbf{P}(A_n \mid A_1 \cap A_2 \cap \cdots$			
$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1} \mid A_n) \times \mathbf{P}(A_2 \cap \cdots \cap A_{n-1} \mid$			
$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbf{P}(A_1 \mid A_2) \times \mathbf{P}(A_2 \mid A_3) \times \mathbf{P}(A_3 \mid A_4) \times \cdots \times \mathbf{P}(A_n$			
Total		1.00 / 1.00	

Question Explanation

As intersection is commutative and associative we may reverse the order of intersection without changing the result. By repeated application of the definition of conditional probability, we may now write

$$\begin{aligned}
\mathbf{P}(A_1 \cap A_2 \cap \ldots \cap A_n) &= \mathbf{P}(A_n \cap A_{n-1} \cap \cdots \cap A_2 \cap A_1) \\
&= \mathbf{P}(A_n \mid A_{n-1} \cap \cdots \cap A_2 \cap A_1) \times \mathbf{P}(A_{n-1} \cap \cdots \cap A_2 \cap A_1) \\
&= \mathbf{P}(A_n \mid A_{n-1} \cap \cdots \cap A_2 \cap A_1) \times \mathbf{P}(A_{n-1} \mid A_{n-2} \cap \cdots \cap A_2 \cap A_1) \times \mathbf{P}(A_{n-2} \cap \cdots \cap A_2 \cap A_1) \\
&= \cdots \\
&= \mathbf{P}(A_n \mid A_{n-1} \cap \cdots \cap A_2 \cap A_1) \times \mathbf{P}(A_{n-1} \mid A_{n-2} \cap \cdots \cap A_2 \cap A_1) \times \cdots \times \mathbf{P}(A_2 \mid A_1) \times \mathbf{P}(A_1).
\end{aligned}$$

This just writes the chain rule in reverse. See Lecture 8.1.g.

Question 3

The following prompt should be used for **Questions 3, 4, and 5:**

Your sports team is playing in a knockout tournament along with seven other teams (eight teams in total). The teams are paired up randomly in the first round at the conclusion of which the four winning teams move into the second round (the semifinals), the losing teams being eliminated. The four first round winners are again paired up randomly in the semifinal, the two winners moving on to play the final in the third round, the two losing semifinalists being eliminated. The winner of the final wins the tournament.

Suppose your team has a 60% chance of winning its first round game. If it wins its first round game it has a 50% chance of winning the semifinal. And if it wins the semifinal it has a 30% chance of winning the final and hence the tournament.

What is the probability that your team plays in the final?

Your Answer		Score	Explanation
10%			
50%			
18%			
30%	✔	1.00	
9%			
15%			
Total		1.00 / 1.00	

Question Explanation

The event of interest is so readily related to the given information that we could go directly to a computation of the probability without bothering to even identify the sample space. But it is a good habit to get into to always lend a little thought first to identifying the sample space. One representation is as follows: write L if the team loses a game in a given round and W if it wins the game in that round. Then the sample space is $\Omega = \{\mathrm{L}, \mathrm{WL}, \mathrm{WWL}, \mathrm{WWW}\}$ representing a loss in the first round, a wn in the first round and a loss in the semifinal, wins in the first two rounds and a loss in the final, and wins in all three rounds, respectively.

Identify the events of interest next. Write A, B, and C for the events that your team wins the first round game, wins the second round game, and wins the third round game, respectively. In terms of the sample space, $A = \{\mathrm{WL}, \mathrm{WWL}, \mathrm{WWW}\}$, $B = \{\mathrm{WWL}, \mathrm{WWW}\}$, and $C = \{\mathrm{WWW}\}$. We are given that $\mathbf{P}(A) = 0.6$, $\mathbf{P}(B \mid A) = 0.5$, and $\mathbf{P}(C \mid A \cap B) = 0.3$.

We are asked to evaluate the probability of $A \cap B = \{\mathrm{WWL}, \mathrm{WWW}\}$. By the definition of conditional probability,

$$\mathbf{P}(A \cap B) = \mathbf{P}(B \mid A) \times \mathbf{P}(A) = 0.5 \times 0.6 = 0.3.$$

We conclude that there is a 30% chance of the team reaching the final.

Question 4

What is the probability that your team wins the tournament?

Your Answer		Score	Explanation
15%			
30%			
50%			
9%	✔	1.00	
18%			
20%			
Total		1.00 / 1.00	

Question Explanation

Your team wins the tournament if, and only if, it wins each of its three games. In the notation introduced in the solution of the previous problem, the event that the team wins each of its three games may be identified with the atomic event $A \cap B \cap C = \{\mathrm{WWW}\backslash\}$ and so, by a simple exercise of the chain rule,

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(C \mid A \cap B) \cdot \mathbf{P}(B \mid A) \cdot \mathbf{P}(A) = 0.6 \times 0.5 \times 0.3 = 0.09.$$

Your team hence has a 9% chance of winning the tournament.

Question 5

What is the probability that your team loses in the final?

Your Answer		Score	Explanation
○ 9%			
○ 70%			
○ 18%			
◉ 21%	✔	1.00	
○ 15%			
○ 30%			
Total		1.00 / 1.00	

Question Explanation

Again, in terms of the notation introduced, we may identify the event that your team loses in the final with the singleton set $A \cap B \cap C^{\complement} = \{\mathrm{WWL}\}$. We may compute the desired probability in a couple of ways: leveraging additivity directly or by first computing the atomic probabilities and simply reading out the answer.

Using additivity: We begin with the trite observation that if your team reaches the final then it must either win or lose the tournament. In terms of our notation this leads to the partition $A \cap B = (A \cap B \cap C) \cup (A \cap B \cap C^{\complement})$. Additivity finishes the job:

$$\mathbf{P}(A \cap B \cap C^{\complement}) = \mathbf{P}(A \cap B) - \mathbf{P}(A \cap B \cap C) = 0.3 - 0.09 = 0.21.$$

As the event $A \cap B \cap C^{\complement}$ stands for the event that your team (makes the final and) loses, it follows that your team has a 21% chance of losing in the final

Via the atomic probabilities: The atomic probabilities are implicitly specified in the problem statement. Let us determine them now. This will provide us a sanity check for our earlier answers as well.

We first observe that the event that your team sadly loses in the first round may be identified with $\{\mathrm{L}\} = A^{\complement}$ and so we immediately obtain

$$\mathbf{P}\{\mathrm{L}\} = 1 - \mathbf{P}(A) = 1 - 0.6 = 0.4.$$

We may next identify the event that your team lost in the semifinal with the atom $\{\mathrm{WL}\} = A \setminus B$ and so additivity again gives

$$\mathbf{P}\{\mathrm{WL}\} = \mathbf{P}(A) - \mathbf{P}(A \cap B) = 0.6 - 0.3 = 0.3.$$

The event that the team loses in the final may be identified with $\{\mathrm{WWL}\} = B \setminus C$ and so, by exploiting the facts that $B = A \cap B$ and $C = B \cap C = A \cap B \cap C$ as $C \subseteq B \subseteq A \subseteq \Omega$, additivity once more tells us that

$$\mathbf{P}\{\mathrm{WWL}\} = \mathbf{P}(B) - \mathbf{P}(B \cap C) = \mathbf{P}(A \cap B) - \mathbf{P}(A \cap B \cap C) = 0.3 - 0.09 = 0.21.$$

Finally, as in the previous problem, we may identify the event that your team wins the tournament with the atom $\{\mathrm{WWW}\} = C = A \cap B \cap C$ and so

$$\mathbf{P}\{\mathrm{WWW}\} = \mathbf{P}(A \cap B \cap C) = 0.09.$$

Question 6

The Department of Transportation (DOT) estimates that there is a 1% chance of someone having a car accident on a non-rainy day. This probability increases to 5% if it rains that day. Weather data gathered over a period of ten years estimates that the probability of rain on a typical day is 10%. Rounding to the nearest percent, determine the probability that it rained on a given day *given* that someone had a car accident on that day.

Your Answer		Score	Explanation
5%			
36%	✔	1.00	
48%			
50%			
1%			
Total		1.00 / 1.00	

Question Explanation

The implicit sample space for this problem is determined by the specification, on a given day, of whether there was an accident and whether it was raining. In this space denote by A the event that there was an accident on that day and by B the event that it was raining on that day. We are told that $\mathbf{P}(A \mid B) = 0.05$, $\mathbf{P}(A \mid B^c) = 0.01$, and $\mathbf{P}(B) = 0.1$.

Given $\mathbf{P}(A \mid B)$, we are to find $\mathbf{P}(B \mid A)$. The bridge connecting these two is Bayes's rule:

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \mid B) \cdot \mathbf{P}(B)}{\mathbf{P}(A)}.$$

We may compute $\mathbf{P}(A)$, the unconditional probability of having an accident, by appeal to the law of total probability:

$$\mathbf{P}(A) = \mathbf{P}(A \mid B) \cdot \mathbf{P}(B) + \mathbf{P}(A \mid B^\complement) \cdot \mathbf{P}(B^\complement) = 0.05 \times 0.1 + 0.01 \times 0.9 = 0.005 + 0.009 = 0.014.$$

Leveraging Bayes's rule, we hence obtain

$$\mathbf{P}(B \mid A) = \frac{0.05 \times 0.1}{0.014} = \frac{0.005}{0.014} = \frac{5}{14} \approx 0.36.$$

Question 7

The Snitch in the game of Quidditch is famously elusive. Suppose it is hidden in one of two rooms, equally likely to be in either. The Snitch resists discovery and so if a search is undertaken of the room in which the Snitch is located then the search will uncover the Snitch with probability p and will fail to discover the Snitch with probability $1 - p$. Here p is some specified value between 0 and 1. (Of course, if we search for the Snitch in the wrong room then we certainly won't uncover it.) Suppose that a search for the Snitch in the first room does not discover it. What is the (conditional) probability that the Snitch is hidden in the second room?

Your Answer	Score	Explanation
$\frac{1}{1-p}$		
$1 - p$		
$\frac{1}{2}$		
$\frac{p}{2}$		
$p(1-p)$		

$\frac{1}{2-p}$	✔	1.00
Total		1.00 / 1.00

Question Explanation

The sample points of the experiment are specified by two chance elements:

- The location of the Snitch: room 1 or room 2.
- Whether a search for the Snitch in room 1 discovered it: yes or no.

There are four elements in this sample space and I will leave you to identify them and figure out the atomic probabilities. Let us proceed to the events of interest.

- The target event A:= the Snitch is in room 2.
- The ancillary event (side information) B:= a search for the Snitch in room 1 did not discover it.

We are given the following information:

$$\mathbf{P}(A) = \mathbf{P}(A^{\complement}) = 1/2,$$
$$\mathbf{P}(B \mid A) = 1, \quad \mathbf{P}(B \mid A^{\complement}) = 1 - p.$$

The first line just says that the Snitch is equally likely to be in either room. The first conditional probability on the second line says that, if the Snitch is in room 2, then a search of room 1 will certainly fail to discover it; the second conditional probability on the second line says that, if the Snitch is in room 1, then a search of the room will fail to discover it with probability $1 - p$. We're interested in $\mathbf{P}(A \mid B)$, the probability that the Snitch is in room 2 given that a search of room 1 has failed to uncover it. We're all set for an application of Bayes's rule:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B \mid A)\mathbf{P}(A) + \mathbf{P}(B \mid A^{\complement})\mathbf{P}(A^{\complement})} = \frac{1 \times \frac{1}{2}}{\left(1 \times \frac{1}{2}\right) + \left((1-p) \times \frac{1}{2}\right)} = \frac{1}{2-p}.$$

It is always wise to look back at the question with the answer in hand. Is it reasonable? Well, if $p = 0$ then our says that $\mathbf{P}(A \mid B) = 1/2$: very natural as, if the Snitch can never be discovered then the information from a search of the first room is valueless and the Snitch is equally likely to be in either room. And if $p = 1$ our answer says that $\mathbf{P}(A \mid B) = 1$: if a search of a room containing the Snitch will always uncover it, then the negative evidence from the search of the first room tells us that the Snitch cannot be in the first room, *ergo* it must be in the second room.