TEST FLIGHT: FIRST PROBLEM SET SOLUTION

1. Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$$

ANSWER It's true. Let $m = 4, n = 0$. Then $3m + 5n = 12$.

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

ANSWER True. $1 + 2 + 3 + 4 + 5 = 15$, which is divisible by 5.

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

ANSWER We prove it by induction. For $n = 1$, $n^2 + n + 1 = 1 + 1 + 1 = 3$, which is odd.

Suppose $n^2 + n + 1$ is odd. Then

$$(n + 1)^2 + (n + 1) + 1 = n^2 + 2n + 1 + n + 1 + 1 = n^2 + 3n + 2 + 1 = (n + 1)(n + 2) + 1$$

But one of $(n + 1), (n + 2)$ must be even, so $(n + 1)(n + 2)$ is even. Hence $(n + 1)^2 + (n + 1) + 1$ is odd. This proves the result by induction.

4. Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

ANSWER We prove it by induction. For $n = 1$, $4n + 1 = 5$, which is odd.

If it's true for n, then $4(n + 1) + 1 = 4n + 4 + 1 = 4n + 5$ and $4(n + 1) + 3 = 4n + 4 + 3 = 4n + 7$, which are both odd. This proves the result by induction.

5. Prove that for any integer n, at least one of the integers n, $n + 2$, $n + 4$ is divisible by 3.

ANSWER Given m, by the Division Theorem, $m = 4n + q$, where $0 \leq q < 4$. If we divide n by 3, either it divides evenly or it leaves a remainder of 1 or 2. So 3 has to divide one of $n, n + 2, n + 4$.

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

   ANSWER Suppose p, q is a pair of twin primes, where $p > 5$. We show that it is impossible to extend p, q to be a prime triple Let $N = p.q + 1$. Then, either N is prime or else there is a prime r such that $r|N$. It follows that there is no prime that can be added to give a prime triple.

7. Prove that for any natural number n:

$$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

   ANSWER For $n = 1$, the identity reduces to $2 = 2^2 - 2$, which is true.

   Assume it hold for n. Then, adding 2^{n+1} to both sides of the identity,

$$2 + 2^2 + 2^3 + \ldots + 2^n + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1} = 2.2^{n+1} - 2 = 2^{n+2} - 2$$

   This is the identity at $n + 1$. That completes the proof.

8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

   ANSWER By the assumption, we can find an N such that

$$n \geq N \Rightarrow |a_n - L| < \epsilon/M$$

   Then,

$$n \geq N \Rightarrow |Ma_n - ML| = M.\,|a_n - L| < M.\epsilon/M = \epsilon$$

   which shows that $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

9. Given a collection $A_n, n = 1, 2, \ldots$ of intervals of the real line, their *intersection* is defined to be $\bigcap_{n=1}^{\infty} A_n = \{x \,|\, (\forall n)(x \in A_n)\}$. Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

   Prove that your example has the stated property.

   ANSWER Let $A_n = (\frac{1}{n+1}, \frac{1}{n})$. For any $x > 0$, we can find an m such that $1/m < x$, and then $x \notin (\frac{1}{m+1}, \frac{1}{m})$. Hence $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

10. Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

ANSWER Let $A_n = (-1/n, +1/n)$. For any n, $0 \in A_n$, so $0 \in \bigcap_{n=1}^{\infty} A_n$. On the other hand, if $x \neq 0$, then there is an m such that $1/m < |x|$, and for that m, $x \notin A_m$, so $x \notin \bigcap_{n=1}^{\infty} A_n$. Hence $\bigcap_{n=1}^{\infty} A_n = \{0\}$.