Probability and Statistics: To p, or not to p?

Module Leader: Dr James Abdey

4.6 Confidence intervals

A **point estimate** (such as a sample mean, $\bar{x}$) is our 'best guess' of an unknown population parameter (such as a population mean, μ) based on sample data. Although:

$$\mathrm{E}(\bar{X}) = \mu$$

meaning that *on average* the sample mean is equal to the population mean, as it is based on a sample there is some **uncertainty (imprecision)** in the accuracy of the estimate. Different random samples would tend to lead to different observed sample means. **Confidence intervals** communicate the level of imprecision by converting a point estimate into an interval estimate.

Formally, an $x\%$ confidence interval **covers** the unknown parameter with $x\%$ probability **over repeated samples**. The shorter the confidence interval, the more reliable the estimate.

As we shall see, this is achievable by:

- reducing the level of confidence (undesirable)
- increasing the sample size (costly).

If we assume we have either i. **known σ**, or ii. **unknown σ but a large sample size**, say $n \geq 50$, then the formulae for the **endpoints** of a confidence interval for a single mean are:

$$\textbf{i.} \quad \bar{x} \pm z \times \frac{\sigma}{\sqrt{n}} \quad \textbf{and} \quad \textbf{ii.} \quad \bar{x} \pm z \times \frac{s}{\sqrt{n}}.$$

Here $\bar{x}$ is the sample mean, σ is the population standard deviation, s is the sample standard deviation, n is the sample size and z is the **confidence coefficient**, reflecting the confidence level.

Influences on the margin of error

More simply, we can view the confidence interval for a mean as:

$$\textbf{best guess} \pm \textbf{margin of error}$$

where $\bar{x}$ is the best guess, and the margin of error is:

$$\text{i.} \quad z \times \frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \text{ii.} \quad z \times \frac{s}{\sqrt{n}}.$$

Therefore, we see that there are **three influences on the size of the margin of error** (and hence on the width of the confidence interval). Specifically:

- other things equal, larger sample sizes improve the precision of the point estimate, hence the confidence interval becomes shorter, so:

$$\text{as } n \uparrow \quad \Rightarrow \quad \text{margin of error } \downarrow \quad \Rightarrow \quad \text{width } \downarrow$$

- other things equal, σ (or s) reflects the amount of variation in the population so a larger standard deviation means more uncertainty in the representativeness of a random sample, hence the confidence interval becomes longer, so:

$$\text{as } \sigma \uparrow \quad \Rightarrow \quad \text{margin of error } \uparrow \quad \Rightarrow \quad \text{width } \uparrow$$

- other things equal, a greater level of confidence equates to a larger confidence coefficient, hence the confidence interval becomes longer, so:

$$\text{as confidence level } \uparrow \quad \Rightarrow \quad \text{margin of error } \uparrow \quad \Rightarrow \quad \text{width } \uparrow .$$

Confidence coefficients

For a 95% confidence interval, $z = 1.96$, leading to:

$$\text{i.} \quad \bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \text{ii.} \quad \bar{x} \pm 1.96 \times \frac{s}{\sqrt{n}}.$$

Other levels of confidence pose no problem, but require a different confidence coefficient. For **large n**, we obtain this coefficient from the **standard normal distribution**.

- For 90% confidence, use the confidence coefficient $z = 1.645$.
- For 95% confidence, use the confidence coefficient $z = 1.960$.
- For 99% confidence, use the confidence coefficient $z = 2.576$.

Example

A company producing designer label jeans carries out a sampling exercise in order to estimate the average price which all retailers are charging for the jeans.

A random sample of retailers, with an assumed $\sigma = 3.25$ gave the following summary statistics:

$$\bar{x} = £25.75 \quad \text{and} \quad n = 60.$$

A 95% confidence interval for the mean retailer's price of the jeans is:

$$25.75 \pm 1.96 \times \frac{3.25}{\sqrt{60}} \quad \Rightarrow \quad (£24.93,\ £26.57).$$

Note how if the same $\bar{x}$ was obtained from a random sample of $n = 100$, then the 95% confidence interval becomes shorter:

$$25.75 \pm 1.96 \times \frac{3.25}{\sqrt{100}} \quad \Rightarrow \quad (£25.11,\ £26.39).$$

For the original sample size of $n = 60$, if instead we had assumed $\sigma = 3.75$, then the 95% confidence interval becomes longer:

$$25.75 \pm 1.96 \times \frac{3.75}{\sqrt{60}} \quad \Rightarrow \quad (£24.80,\ £26.70).$$

For the original sample size of $n = 60$ and assumed $\sigma = 3.25$, then a 99% confidence interval becomes longer:

$$25.75 \pm 2.576 \times \frac{3.25}{\sqrt{60}} \quad \Rightarrow \quad (£24.67,\ £26.83).$$

See how z, σ and n each affect the width of the confidence interval as expected.