
Probability

The Theory of Chance, its Historical Context, and its Applications in the Modern World

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probability

Syllabus

I PREAMBLE

- a) The flowering of a mathematical science
- b) The background needed
- c) Table d'hôte: the topics of the course
- d) A word on how to study mathematics

1. Prelude to a theory of chance

1.1. Chance around us

1.2. The amazing aspects of fluctuations and the hot hand phenomenon

- a) The tosses of a coin
- b) Unexpected phenomena in success runs
- c) The hot hand phenomenon and a principled basis for a statistical test
- d) Summary of Tableau 1

2. Combinatorial elements



2.1. *The science of counting*

- a) *The setting: sampling from finite sets*
- b) *Sampling with replacement, powers*
- c) *Sampling without replacement, falling factorials, permutations*
- d) *Subpopulations, binomial coefficients*
- e) *Test your understanding*
- f) *Summary of Tableau 2, Part 1*



2.2. *Basic properties of binomial coefficients*

- a) *Alternative forms for the falling factorial*
- b) *Equivalent expressions for binomial coefficients*
- c) *Pascal's triangle*
- d) *The binomial theorem*
- e) *Test your understanding*
- f) *Summary of Tableau 2, Part 2*

3. Chance in commonplace settings

3.1. Random sampling from finite sets, balls and urns

- a) The birth of a mathematical science
- b) Chevalier de Méré's paradox
- c) Commentary on de Méré's paradox: lessons learnt
- d) The birthday paradox
- e) A generic ball and urn problem
- f) Summary of Tableau 3, Part 1



3.2. Urn models in statistical physics

- a) *Occupancy problems*
- b) *Maxwell–Boltzmann statistics, distinguishable particles, natural particles apparently lack common sense*
- c) *Bose–Einstein statistics, indistinguishable particles, Bosons*
- d) *Fermi–Dirac statistics, the Pauli exclusion principle, Fermions*
- e) *Summary of Tableau 3, Part 2*

3.3. Beyond balls and urns, unequal probabilities and infinite spaces

- a) The first throw in the game of craps
- b) The repeated tosses of a coin
- c) Summary of Tableau 3, Part 3



4. A little set theory

- a) *From finite to infinite sets, number systems*
- b) *Abstract sets, subsets, the universal set, and the empty set*
- c) *Set relations, set inclusions, disjoint sets*
- d) *Set operations, unions, intersections, complements, set and symmetric differences*
- e) *A test of the concepts*
- f) *An algebra of sets, commutative, associative, and distributive properties, de Morgan's laws*
- g) *On how to prove set equality*
- h) *Summary of Tableau 4*

5. The abstract probability space

- a) Towards an axiomatic theory of probability
- b) The abstract sample space
- c) The algebra of events
- d) Lessons from a frequentist's view of chance
- e) The probability measure
- f) Basic properties: the impossible event
- g) Monotonicity, Boole's inequality
- h) Additivity: inclusion and exclusion
- i) *Something to ruminate on*
- j) Additivity once more: partitions
- k) Summary of Tableau 5, the probabilist's trinity



6. Probabilities in simple settings

6.1. Random choice and beyond, discrete spaces

- a) Review: the abstract probability space
- b) Lessons from simple chance settings: De Méré redux, the first throw in craps, the tosses of a coin
- c) Characterising chance in discrete spaces: atoms, mass functions, distributions
- d) The usual culprits: the combinatorial, binomial, Poisson, and geometric distributions
- e) Test your understanding
- f) Summary of Tableau 6, Part 1



6.2. Unexpected ramifications from the repeated toss of a coin, continuous spaces

- a) *An infinite string of coin tosses*
- b) *A non sequitur: dyadic expansions of a real number*
- c) *Coin tosses redux: the continuum emerges, intervals as the carriers of mass*
- d) *The uniform density, sums segue into integrals*
- e) *Probability densities in a continuum sample space, probabilities as areas under a density curve*
- f) *The basic densities: the uniform, exponential, and normal densities*
- g) *Is the normal density properly normalised?*
- h) *Test your understanding; a word from Leibniz*
- i) *Densities in two and more dimensions*

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- j) *A test of the concepts, a two-dimensional normal density, and a slogan*
 - k) *Summary of Tableau 6, Part 2*



7. *Is the outcome of a coin toss really random?*

- a) *A coin toss as a deterministic physical system: whence randomness?*
- b) *What Newton's laws of motion have to say*
- c) *The angle the coin makes when caught*
- d) *The effect of uncertainty in initial conditions*
- e) *Chance in physical systems, Poincaré's principle of arbitrary functions*

III SIDE INFORMATION, INDEPENDENCE

8. **Conditional probability**

8.1. **Characterising side information in a chance experiment, conditional probability**

- a) A familial paradox
- b) The throw of two dice
- c) Conditional probability: the definition
- d) Test your understanding
- e) Is there gender bias in graduate admissions? Simpson's paradox
- f) Intersections, the chain rule for conditional probabilities
- g) Balanced aces in bridge
- h) Summary of Tableau 8, Part 1

8.2. **Additivity and the theorem of total probability**

- a) A recasting of additivity
- b) A random selection of a die
- c) The ballot problem and a useful technique for sequential problems
- d) Additivity, reprised: from partitions to the theorem of total probability
- e) A wager that the sun will rise tomorrow, an urn model and Laplace's law of succession
- f) Back to the future, the Copernican principle
- g) Reversing the direction of conditioning, Bayes's rule for events
- h) Pólya's urn scheme, the spread of contagion
- i) Summary of Tableau 8, Part 2

9. Independence!

9.1. A first look at independent events

- a) Independent possibilities multiply, coin tosses, cards
- b) Independent events and what it implies for conditional probabilities
- c) Test your understanding
- d) Relative ranks, families, again, and an unexpected result
- e) A variation on the theme, conditional independence, a random selection of a coin
- f) Summary of Tableau 9, Part 1

9.2. Repeated independent trials, product spaces

- a) An extended multiplication table
- b) What it means for three events to be independent
- c) Coin tosses constitute independent trials
- d) An example of Bernstein, pairwise independence does not imply independence
- e) Test your understanding, a slogan
- f) Independent families of events: a formidable definition
- g) From independent trials to product spaces and product measure, distinct trials engender independent events
- h) Summary of Tableau 9, Part 2

9.3. Independence: the warp and the woof of the fabric of chance

- a) An application in gambling: the game of craps
- b) An example from genetics: the lasting influence of lethal genes
- c) The hot hand, revisited, and a statistical test
- d) Summary of Tableau 9, Part 3

IV THE BERNOULLI SCHEMA: FROM POLLS TO RARE EVENTS AND LIMIT LAWS

10. From polls to bombs

10.1. A model for polls, the Bernoulli schema, enter the binomial

- a) Dichotomous populations, political parties, invasive species, defective genes, opinions

b) The problem of estimating the sentiment of a large population, the Bernoulli schema

c) Enter the binomial, the distribution of accumulated successes

d) A simple application: psychic tendencies?

e) Return to an urn problem

f) Pepys and Newton, a historical befuddlement

g) Testing sera and vaccines, a statistical test and a slogan

h) A problem in estimation

i) The maximum likelihood principle



j) *A little calculus*

k) Understanding the binomial distribution, the value of a picture

l) Unimodality, the shape of the distribution

m) Expectation, a notion of centre

n) Variance: the concept of spread around a centre

o) Summary of Tableau 10, Part 1

10.2. The Poisson distribution flits in

a) Prelude to a discovery, on the proper size of a jury, from Condorcet to the US Supreme Court

b) Poisson's two-parameter jury model

c) The probability that a jury will convict, outcomes of trials in France 1825–1830

d) A curious discovery of Poisson, startling applications

e) Weird and wonderful observations fitting the Poisson distribution: fatalities in the Prussian army

f) Broadcast authentication

g) Radioactive emissions

h) On the distribution of wars in history

i) Bomb sight: the distribution of bomb hits on London during World War II

j) The Poisson distribution

k) Understanding the nature of Poisson's approximation, unimodality

l) Expectation

m) Variance



n) *Poisson aggregates, the stability of the Poisson, a slogan*

o) *The Poisson process, arrival processes*

- p) Poisson points, waiting times, the exponential distribution reappears*
- q) Inter-arrival times*
- r) Spatial Poisson processes, the distribution of stars*
- s) Summary of Tableau 10, Part 2

11. The fabulous limit laws

11.1. The art of the random sample, Chebyshev's enduring inequality and the magisterial the law of large numbers, why polls work

- a) A model for a poll
- b) Reasons for optimism
- c) The art of the random sample, independence and the subtlety of bias
- d) The dance of error, confidence, and sample size
- e) A binomial estimator, error and confidence
- f) The subtle inequality of Pafnuty Chebyshev, the law of large numbers
- g) Why polls work
- h) How drug testing works
- i) Why drug testing works
- j) Summary of Tableau 11, Part 1



11.2. The law of large numbers in continuous spaces, computation à la Monte Carlo

- a) Questions in the continuum*
- b) Expectation and variance in continuous settings*
- c) Chebyshev, reprised*
- d) Independent trials in the continuum*
- e) The dance of additivity, two slogans*
- f) Sums of independent variables*
- g) The law of large numbers*
- h) Test your understanding, back to arithmetic variables*
- i) The statistical estimation of a mean, the theory of fair games*
- j) A simple computation — and a dimensional tweak to the tale from physics*
- k) A chance-driven computation*
- l) Physics takes a gamble! A visit to Monte Carlo*
- m) Quo vadis? The weak and strong law of large numbers*
- n) Summary of Tableau 11, Part 2*

11.3. The bell curve flits in, why polls *really* work

- a) Revisiting the binomial distribution, a view from the proper centre and scale
- b) Enter the bell curve: the normal density and distribution function
- c) A theorem of de Moivre and Laplace
- d) Why polls *really* work
- e) One curve to rule them all, the central limit theorem
- f) Summary of Tableau 11, Part 3

✓ CENTRAL TENDENCY, PROBABILITY SIEVES, THE POISSON PARADIGM



12. A potpourri of titillating applications

12.1. Central tendency: stock portfolio selection and the curious case of Sir Cyril Burt, psychologist

- a) The scope of the basic limit laws: typical and rare events
- b) Typical events: visions of centrality, concentration and central tendency
- c) The art of making money: portfolios and the stock market
- d) A constantly rebalanced portfolio
- e) The growth of wealth
- f) What the law of large numbers has to say, optimal portfolio selection
- g) Disclaimers and a slogan
- h) Fraud, statistics, and the central limit theorem: the curious case of Sir Cyril Burt, psychologist
- i) Commentary on Burt's figures, a golden rule of Darwin
- j) The distribution of IQ, a bell curve emerges
- k) A chi-squared test, Karl Pearson weighs in, the limitations of analysis
- l) Summary of Tableau 12, Part 1

12.2. Searching for needles in a probability haystack, the method of inclusion and exclusion, Boole's inequality and a probability sieve

- a) Boole's inequality and a probability sieve
- b) A question of existence
- c) Identities via inclusion and exclusion
- d) Le problème des rencontres, matchings
- e) The coupon collector's problem
- f) The Poisson makes an unexpected appearance

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- g) Rare events, revisited*
 - h) Summary of Tableau 12, Part 2*