What is the general equation of the ellipse that is not in the origin and rotated by an angle?

I have the equation not in the center, i.e.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

But what will be the equation once it is rotated?

(geometry) (analytic-geometry)

edited Jun 21 '13 at 13:19
enzotib
5,514 2 12 30

asked Jun 21 '13 at 13:08
andikat dennis
164 1 4 12

3 en.wikipedia.org/wiki/Rotation_of_axes – lab bhattacharjee Jun 21 '13 at 15:08

5 Answers

After a lot of mistakes I finally got the correct equation for my problem:-

$$\frac{((x-h)\cos(A)+(y-k)\sin(A))^2}{(a^2)}+\frac{((x-h)\sin(A)-(y-k)\cos(A))^2}{(b^2)}=1$$

edited Nov 8 '13 at 16:10

community wiki
2 revs, 2 users 80%
andikat dennis

This form is really elegant.Should be in text-books. (Axis rotation / A) relation varying them between $0, \pi$ can be appended. – Narasimham Jun 1 '16 at 7:02

The equation you gave can be converted to the parametric form:

$$x=h+a\cos\theta \quad ; \quad y=k+b\sin\theta$$

If we let $\mathbf{x}_0=(h,k)$ denote the center, then this can also be written as

$$\mathbf{x}=\mathbf{x}_0+(a\cos\theta)\mathbf{e}_1+(b\sin\theta)\mathbf{e}_2$$

where $\mathbf{e}_1=(1,0)$ and $\mathbf{e}_2=(0,1)$.

To rotate this curve, choose a pair of mutually orthogonal unit vectors $\mathbf{u}$ and $\mathbf{v}$, and then

$$\mathbf{x}=\mathbf{x}_0+(a\cos\theta)\mathbf{u}+(b\sin\theta)\mathbf{v}$$

One way to define the $\mathbf{u}$ and $\mathbf{v}$ is:

$$\mathbf{u}=(\cos\alpha,\sin\alpha) \quad ; \quad \mathbf{v}=(-\sin\alpha,\cos\alpha)$$

This will give you an ellipse that's rotated by an angle α, with center still at the point $\mathbf{x}_0=(h,k)$.

If you prefer an implicit equation, rather than parametric ones, then any rotated ellipse (or, indeed, any rotated conic section curve) can be represented by a general second-degree equation of the form

$$ax^2+by^2+cxy+dx+ey+f=0$$

The problem with this, though, is that the geometric meaning of the coefficients a, b, c, d, e, f is not very clear.

There are further details on this page.

Addition. Borrowing from rschwieb's solution ...

Since you seem to want a single implicit equation, proceed as follows. Let $c = \sqrt{a^2 - b^2}$. Then the foci of the rotated ellipse are at $\mathbf{x}_0 + c\mathbf{u}$ and $\mathbf{x}_0 - c\mathbf{u}$. Using the "pins and string" definition of an ellipse, which is described here, its equation is

$$\|\mathbf{x} - (\mathbf{x}_0 + c\mathbf{u})\| + \|\mathbf{x} - (\mathbf{x}_0 - c\mathbf{u})\| = \text{constant}$$

This is equivalent to the one given by rschwieb. If you plug $\mathbf{u} = (\cos\alpha, \sin\alpha)$ into this, and expand everything, you'll get a single implicit equation.

The details are messy (which is probably why no-one wants to actually write everything out for you).

edited Jun 22 '13 at 2:51 answered Jun 21 '13 at 13:23
bubba 23.8k 3 23 61

Another option is to use the geometric definition of an ellipse as the set of points whose sum distance to the foci is constant.

If the foci are at (a, b) and (a', b'), and the sum distance is C, you get:

$$\sqrt{(x-a)^2 + (y-b)^2} + \sqrt{(x-a')^2 + (y-b')^2} = C$$

answered Jun 21 '13 at 13:34
rschwieb 83.6k 10 73 183

So will the final equation be $((h + a * cos(T))^2)/(a^2) + ((k + b * sin(T))^2)/(b^2) = 1$– andikat dennis Jun 21 '13 at 13:45

If you take the time to translate this answer into terms of the center and the minor axes, then yes, it will line up with the other answers. I just wanted to provide an alternate approach to the "get there from normal ellipse equations" approach given above. – rschwieb Jun 21 '13 at 14:01

As stated, using the definition for center of an ellipse as the intersection of its axes of symmetry, your equation for an ellipse is centered at (h, k), but it is not rotated, i.e. the axes of symmetry are parallel to the x and y axes.

If this were not true, you would have a cross-product term involving $x \times y$. If you had such a term, you could calculate the counterclockwise rotation angle α required in order to eliminate the cross-product term (and thereby make the axes of symmetry parallel to the x and y axes).

One way is to use the formula

$$\cot 2\alpha = \frac{A - C}{B},$$

where α is the counterclockwise rotation angle, A is the coefficient of x^2, B the coefficient of the cross-product term $x \times y$, and C is the coefficient of y^2.

In order to apply the rotation once you know α, you can find new coordinates x', y' in terms of x, y via $x' = x\cos\alpha - y\sin\alpha$ and $y' = x\sin\alpha + y\cos\alpha$

Source: Calculus and Analytic Geometry, by George Thomas (paraphrased).

edited Jun 22 '13 at 3:05 answered Jun 21 '13 at 14:26
bubba 23.8k 3 23 61 Greg B. Hill 856 5 19

If you want the center to be (h, k)

- **first** apply a general rotation of coordinates transformation to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

  to rotate the axes to whatever angle you desire.
- **then** translate the center to (h, k) by replacing the new x and y by $(x - h)$ and $(y - k)$.

edited Jun 21 '13 at 13:25 answered Jun 21 '13 at 13:21
enzotib 5,514 2 12 30 Dilip Sarwate 17.4k 1 21 57

@enzotib Thanks for the edit! I was not paying attention to the formatting while I was typing. – Dilip Sarwate Jun 21 '13 at 13:27

only a missing $ – enzotib Jun 21 '13 at 13:28

Is it possible to combine these two as one equation? – andikat dennis Jun 21 '13 at 13:52

"Is it possible to combine these two as one equation?" Yes. Work out the details for yourself or use the methods suggested by rschweib or bubba. – Dilip Sarwate Jun 21 '13 at 15:01