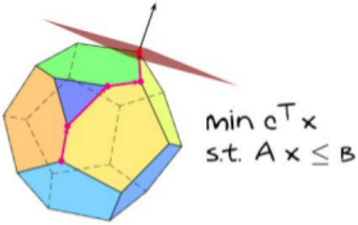
How efficient is the simplex method?

- One iteration of the simplex algorithm



One iteration of the simplex method

$c^T = \lambda_B^T \cdot A_B \quad , \quad \lambda_B^T = c^T \cdot A_B^{-1}$

`while` B is not optimal

Let $i \in B$ be index with $\lambda_i < 0$

Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0,\ j \in B \setminus \{i\}$ and $a_i^T d = -1$

The negative of a column of A_B^{-1}

Determine $K = \{k\colon 1 \leqslant k \leqslant m,\ a_k^T d > 0\}$ $\quad A \cdot d$

`if` $K = \emptyset$

assert LP unbounded

`else`

Let $k \in K$ index where $\min\limits_{k \in K}(b_k - a_k^T x^*)/a_k^T d$ is attained

$\dfrac{b - A \cdot x^*}{A \cdot d}$

update $B := B \setminus \{i\} \cup \{k\}$

One iteration of the simplex method (cont.)

n

$A = m$ d

- Suppose $A \in \mathbb{Q}^{m\times n}$, $c \in \mathbb{Q}^n$, $b \in \mathbb{Q}^m$ (rational data)
- Compute π_B^T $= c^T \cdot A_B^{-1}$ $O(n^2)$
- Compute d (the negative of a column of A_B^{-1}) $O(1)$
- Compute K (by computing $A \cdot d$) $O(m \cdot n)$
- Determine index of element entering the basis (by computing $x^* = A_B^{-1} b_B$, $b - Ax^*$ and Ad)
  $O(n^2)$ $O(m \cdot n)$
- If A_B^{-1} is known, this amounts to a total of: $O(m \cdot n)$

Questions:

- Is the size of A_B^{-1} polynomial in the size of the input (A, b, c)?
- How expensive is it to compute A_B^{-1}?

Matrix inversion

size (A^{-1}) polynomial in size (A)

Quiz:

$$A = \begin{pmatrix} p_{11}/q_{11} & \cdots & p_{1n}/q_{1n} \\ & \cdots & \\ p_{n1}/q_{n1} & \cdots & p_{nn}/q_{nn} \end{pmatrix} \in \mathbb{Q}^{n\times n}.$$

The *size* of the product of denominators $\prod_{i=1,j=1}^{n} q_{ij}$ is

$$\text{size}\left(\prod q_{ij}\right) = \Theta\left(\sum_{ij} \text{size}(q_{ij})\right)$$

- linear in the size of the input
- not polynomial in the size of the input

Matrix inversion

Quiz:

$$A = \begin{pmatrix} p_{11}/q_{11} & \cdots & p_{1n}/q_{1n} \\ & \cdots & \\ p_{n1}/q_{n1} & \cdots & p_{nn}/q_{nn} \end{pmatrix} \in \mathbb{Q}^{n\times n}.$$

The *size* of the product of denominators $\prod_{i=1,j=1}^{n} q_{ij}$ is

Write $A = 1/Q \cdot A'$ where Q is product of denominators and $A' \in \mathbb{Z}^{n\times n}$ $A' = Q \cdot A$

Since $A^{-1} = Q \cdot (A')^{-1}$ invert A' instead of A. W.l.o.g. assume that A is integral.

size (A'^{-1}) = ?

Matrix inversion formula

A^{-1} for $A \in \mathbb{Z}^{n \times n}$

Cramer's Rule

For $A \in \mathbb{R}^{m \times n}$ and $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, A_{ij} denotes the matrix obtained from A by deleting the i-th row and j-th column.

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 9 & 4 & 3 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 9 & 3 \end{pmatrix}$$

Matrix inversion formula:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) & \dots \\ -\det(A_{12}) & \det(A_{22}) & -\det(A_{32}) & \dots \\ \det(A_{13}) & -\det(A_{23}) & \det(A_{33}) & \dots \\ \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

$\in \mathbb{Z}$

How large?

Hadamard bound

$A = (a_1, \dots, a_n)$

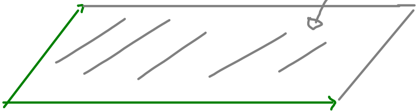
Theorem (Hadamard bound)

$$|\det(A)| \leqslant \prod_{i=1}^{n} \|a_i\|_2 \leqslant n^{n/2} \cdot B^n,$$

$\|a_i\|_2 \leq \left\| \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \right\| = \sqrt{n} \cdot B$

where B is upper bound on absolute values of entries of A.

vol $= |\det(A)|$



vol. largest if vectors orthogonal

$\log(n^{n/2} \cdot B^n) = n/2 \cdot \log(n) + n \cdot \log(B)$

If $A \in \mathbb{Z}^{n\times n}$ is integral then $\mathrm{size}(\det(A)) = O(n \log n + n \cdot \mathrm{size}(B))$. *Polynomial in* $\mathrm{size}(A)$.

The size of the inverse

Corollary

Let $A \in \mathbb{Q}^{n \times n}$ be an invertible matrix. The size of A^{-1} is polynomial in the size of A.

Questions:

- Is the size of A_B^{-1} polynomial in the size of the input (A, b, c)? ✔
- How expensive is it to compute A_B^{-1}?

Updating A_B^{-1}

Suppose basis B is preceded by B' with

$$\begin{aligned} B' &= \{b_1, \ldots, b_{k-1}, & b'_k &, b_{k+1}, \ldots, b_n\} \\ B &= \{b_1, \ldots, b_{k-1}, & b_k &, b_{k+1}, \ldots, b_n\} \end{aligned}$$

Quiz: $A_B \cdot A_{B'}^{-1}$ is of the form

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ v_1 & \cdots & & v_k & \cdots & & v_n \\ & & & & 1 & & 0 \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & & & v_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & & & & \\ & & & v_k & & & \\ & & & & 1 & & \\ & & & \vdots & & \ddots & \\ & & & v_n & & & 1 \end{bmatrix}$$

$v^T = a_k^T \cdot A_{B'}^{-1}$

Updating A_B^{-1} (cont.)

$$
\begin{aligned}
B' &= \{b_1, \ldots, b_{k-1}, \quad b'_k \quad, b_{k+1}, \ldots, b_n\} \\
B &= \{b_1, \ldots, b_{k-1}, \quad b_k \quad, b_{k+1}, \ldots, b_n\}
\end{aligned}
$$

- Compute $a_{b_k}^T \cdot A_{B'}^{-1} = (v_1, \ldots, v_k, \ldots, v_n)$
- A_B^{-1} stems from $A_{B'}^{-1}$ by performing the following operations on $A_{B'}^{-1}$:
  - For each column $i \neq k$: Subtract v_i / v_k times column k from column i $O(n)$ total $O(n^2)$
  - Divide column k by v_k $O(n)$

Amounts to a total number of:

- $O(n)$ arithmetic operations
- $O(n^2)$
- $O(n^3)$

Complexity of one iteration

Theorem

One iteration of the simplex algorithm requires a total number of $O(m \cdot n)$ operations on rational numbers whose size is polynomial in the input size.