Complex Eigenvalues

Today we consider how to deal with complex eigenvalues in a linear homogeneous system of first order equations. We will also look back briefly at how what we have done with systems recapitulates what we did with second order equations.

1. Complex Eigenvalues

2. Second Order Equations as Systems

1 Complex Eigenvalues

We know that to solve a system of n equations (written in matrix form as $\mathbf{x}' = A\mathbf{x}$), we must find n linearly independent solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$. In the case where A has n real and distinct eigenvalues, we have already solved the system by using the solutions $e^{\lambda_i t}\mathbf{v}_i$, where λ_i and $\mathbf{v}_i$ are the eigenvalues and eigenvectors of A.
We now consider the case where A has complex eigenvalues.
We will assume that A has only real entries. Then the characteristic polynomial $|A - rI|$ has real coefficients, and therefore any eigenvalues occur in conjugate pairs:

$$r = a + bi \quad \text{and} \quad \overline{r} = a - bi$$

Only slightly more surprising is the fact that the *eigenvectors* also occur in conjugate pairs. For example, suppose we have eigenvalue r with eigenvector $\mathbf{v}$. Then they satisfy the equation

$$(A - rI)\mathbf{v} = \mathbf{0}$$

Now if we take the complex conjugate of both sides, and note that both A and I have only real entries, we get

$$(A - \overline{r}I)\overline{\mathbf{v}} = \mathbf{0}$$

Therefore, an eigenvector associated with $\overline{r}$ is $\overline{\mathbf{v}}$! If we have a solution $e^{rt}\mathbf{v}$, we also have its conjugate $e^{\overline{r}t}\overline{\mathbf{v}}$, and this means that we also have its real and imaginary parts, since

$$\mathrm{Re}(\mathbf{x}) = \frac{1}{2}(\mathbf{x} + \overline{\mathbf{x}}) \quad \text{and} \quad \mathrm{Im}(\mathbf{x}) = -\frac{i}{2}(\mathbf{x} - \overline{\mathbf{x}})$$

Now let us write the eigenvector split into real and imaginary parts, as

$$\mathbf{v} = \mathbf{a} + \mathbf{b}i$$

(Note that **a** and **b** are real vectors.) If we also write our eigenvalues with real and imaginary parts as $r = \lambda + \mu i$, then one solution can be rewritten as follows:

$$\begin{aligned}(\mathbf{a} + \mathbf{b}i)e^{(\lambda+\mu i)t} &= (\mathbf{a} + \mathbf{b}i)e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \\ &= e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)) + ie^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t))\end{aligned}$$

Of course, we also have the complex conjugate of this solution. Therefore, we can get both the real and imaginary parts as solutions. So we have found two *real* solutions:

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)) \quad \text{and} \quad \mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t))$$

Example:

Solve the system

$$\mathbf{x}' = \begin{pmatrix} 6 & -13 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

First we find the eigenvalues of the matrix A in $\mathbf{x}' = A\mathbf{x}$:

$$\begin{vmatrix} 6-\lambda & -13 \\ 1 & -\lambda \end{vmatrix} = (6-\lambda)(-\lambda) + 13 = \lambda^2 - 6\lambda + 13 = 0$$

Solving for λ yields

$$\lambda = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

We only need to find the eigenvector associated with *one* of these eigenvalues. Let's find the eigenvector for $\lambda = 3 + 2i$ by solving $(A - \lambda I)\mathbf{v} = 0$. We row-reduce the augmented matrix

$$\left(\begin{array}{cc|c} 6-(3+2i) & -13 & 0 \\ 1 & -(3+2i) & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|c} 3-2i & -13 & 0 \\ 1 & -3-2i & 0 \end{array}\right).$$

A useful trick to convert a complex value into a real value is to multiply by the complex conjugate, so to get rid of the complex number in the first column of row one, let us multiply by the conjugate $3+2i$. Then $(3-2i)(3+2i) = 9-4i^2 = 9+4 = 13$, and we get

$$\left(\begin{array}{cc|c} 3-2i & -13 & 0 \\ 1 & -3-2i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 13 & -39-26i & 0 \\ 1 & -3-2i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3-2i & 0 \\ 1 & -3-2i & 0 \end{array}\right)$$

after also dividing through by 13 on row one. Then we can subtract row one from row two, and we end the row reduction with:

$$\left(\begin{array}{cc|c} 1 & -3-2i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

We note that we now have $v_1 + (-3-2i)v_2 = 0$ in the first row, and nothing in the second row. (Note that as expected, we have eliminated at least one row in solving for our eigenvectors.)
So we have $v_1 = (3+2i)v_2$, and v_2 is a free variable. Let's assign $v_2 = 1$, and then we have the eigenvalue/eigenvector pair

$$\lambda = 3+2i, \quad \text{and} \quad \begin{pmatrix} 3+2i \\ 1 \end{pmatrix}.$$

So we get a solution of the form

$$e^{(3+2i)t}\begin{pmatrix} 3+2i \\ 1 \end{pmatrix} = e^{3t}e^{2it}\begin{pmatrix} 3+2i \\ 1 \end{pmatrix} = e^{3t}(\cos(2t) + i\sin(2t))\begin{pmatrix} 3+2i \\ 1 \end{pmatrix}$$

(Remember: $e^{it} =$)
Multiplying through and separating into real and imaginary parts yields

$$\begin{pmatrix} 3e^{3t}\cos(2t) - 2e^{3t}\sin(2t) + i\left[3e^{3t}\sin(2t) + 2e^{3t}\cos(2t)\right] \\ e^{3t}\cos(2t) + ie^{3t}\sin(2t) \end{pmatrix} =$$

$$\begin{pmatrix} 3e^{3t}\cos(2t) - 2e^{3t}\sin(2t) \\ e^{3t}\cos(2t) \end{pmatrix} + i\begin{pmatrix} 3e^{3t}\sin(2t) + 2e^{3t}\cos(2t) \\ e^{3t}\sin(2t) \end{pmatrix}$$

$$\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} + i\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

We know that the real and imaginary parts are both solutions, so our general solution is

$$c_1\begin{pmatrix} 3e^{3t}\cos(2t) - 2e^{3t}\sin(2t) \\ e^{3t}\cos(2t) \end{pmatrix} + c_2\begin{pmatrix} 3e^{3t}\sin(2t) + 2e^{3t}\cos(2t) \\ e^{3t}\sin(2t) \end{pmatrix}$$

If we wish to set an initial condition, such as $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, we can solve for c_1 and c_2:

$$\begin{aligned} c_1 \begin{pmatrix} 3e^0 \cos(0) - 2e^0 \sin(0) \\ e^0 \cos(0) \end{pmatrix} + c_2 \begin{pmatrix} 3e^0 \sin(0) + 2e^0 \cos(0) \\ e^0 \sin(0) \end{pmatrix} &= \\ c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \begin{pmatrix} 5 \\ 3 \end{pmatrix} \end{aligned}$$

which gives us the following augmented matrix:

$$\left(\begin{array}{cc|c} 3 & 2 & 5 \\ 1 & 0 & 3 \end{array} \right).$$

Row reduction leads to

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array} \right)$$

so $c_1 = 3$ and $c_2 = -2$ are the required constants.

Example:

Solve the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Eigenvalues:

Eigenvectors:

General Solution:

$$\mathbf{x} =$$

Solving the initial condition: $(\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix})$

Solution:

$$\mathbf{x} =$$

or

$$x_1(t) =$$

$$x_2(t) =$$

2 Second Order Equations as Systems

We know that any order n equation can be converted to a system of n first order equations. Let's see what happens when we use this approach to solve a second order equation.

Example:

Solve $y'' + 2y' - 3y = 0$.
We know the characteristic equation is $r^2 + 2r - 3 = 0$, which has roots $r = -3$ and $r = 1$. Thus we know the general solution is

$$y(t) = c_1 e^{-3t} + c_2 e^t$$

If we first convert to a system, we set $x_1 = y$, $x_2 = y'$, and get the following:

$$\begin{aligned} x_1'(t) &= x_2 \\ x_2'(t) &= 3x_1 - 2x_2 \end{aligned}$$

or

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$$

We find our eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ 3 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) + 2 = \lambda^2 + 2\lambda - 3 = 0$$

Thus $\lambda = -3$ and $\lambda = 1$ are the eigenvalues.

We find our eigenvectors:

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \mathbf{x} = 0$$

We have the single relation $-3x_1 = x_2$, so we can use $(1, -3)^T$. For $\lambda = 1$, we solve

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \mathbf{x} = 0$$

Here we get $x_1 = x_2$, so we use $(1, 1)^T$. Thus, our general solution is

$$c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

or

$$\begin{aligned} x_1(t) &= c_1 e^{-3t} + c_2 e^t \\ x_2(t) &= -3c_1 e^{-3t} + c_2 e^t \end{aligned}$$

Since $x_1 = y$, we see that we have obtained the same solution as we did before.

Example:

Solve $y'' + 4y = 0$.
We know that the characteristic equation is $r^2 + 4 = 0$, so $r = \pm 2i$. Thus our general solution is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

If we convert this to a system, we let $x_1 = y$ and $x_2 = y'$ to get

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 \end{aligned}$$

or

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}$$

We get our eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

So our eigenvalues are $\lambda = \pm 2i$. We can then find eigenvectors: If $\lambda = 2i$, we get

$$\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \mathbf{x} = 0$$

Solving this, we get eigenvector $(-i, 2)^T$. The eigenvector for $\lambda = -2i$ is then the conjugate, $(i, 2)^T$. So expanding the solution corresponding to $\lambda = 2i$ and $(-i, 2)^T$ into real and imaginary parts yields

$$(\cos(2t) + i\sin(2t))\begin{pmatrix} -i \\ 2 \end{pmatrix} = \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} + i\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}.$$

So our solution is

$$c_1\begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} + c_2\begin{pmatrix} -\cos(2t) \\ 2\sin(2t) \end{pmatrix}$$

The first row gives $c_1\sin(2t) - c_2\cos(2t)$. Since c_2 could be any value, this is equivalent to the answer we would get from solving the second order equation previously.