Feedback — Final Exam

You submitted this exam on **Sat 27 Apr 2013 3:12 AM PDT -0700**. You got a score of **50.88** out of **69.00**.

Note: Unlike previous assignments you have only **one** attempt at the final exam. Take your time to answer the questions and be completely sure before submitting your answers. Good luck!

Question 1

A company produces and sells two different products. Our goal is to determine the number of units of each product they should produce during one month, assuming that there is an unlimited demand for the products, but there are some constraints on production capacity and budget.

There are 2000 hours of machine time in the month. Producing one unit takes 3 hours of machine time for the first product and 4 hours for the second product. Material and other costs for producing one unit of the first product amount to 2 CHF, while producing one unit of the second product costs 1 CHF. The available budget for production is 600 CHF. The products are sold for 3 CHF and 4 CHF per unit, respectively. Formulate and solve a linear program to maximize the profit subject to the described constraints. What is the profit of the optimum production plan?

Your Answer		Score	Explanation
1400 CHF			
1500 CHF	✓	5.00	
1700 CHF			
1120 CHF			
Total		5.00 / 5.00	

Question Explanation

Let x be the number of units of the first product and let y be the number of units of the second product.

$$\begin{array}{rl} \max & (3-2)x + (4-1)y \\ \text{subject to} & 3x + 4y \leq 2000 \\ & 2x + y \leq 600 \\ & x, y \geq 0 \end{array}$$

This can be simplified to:

$$\begin{aligned} \max\ & x + 3y \\ \text{subject to}\ & 3x + 4y \leq 2000 \\ & 2x + y \leq 600 \\ & x, y \geq 0 \end{aligned}$$

With the help of a small drawing one can see that the optimum solution to this LP is $(0, 500)$ with objective value 1500.

Question 2

A restaurant wants to schedule their waiters. From experience we know that d_i waiters are needed on day i of the week ($1 \leq i \leq 7$). According to the law, the waiters have to work on exactly 5 consecutive days followed by 2 days of rest.

Formulate this as a linear program minimizing the number of waiters employed by the restaurant (ignoring the integrality constraints).

Your Answer		Score	Explanation
◉	✓	3.00	

$$\min \sum_{i=1}^{7} x_i$$

$$\begin{aligned} \text{subject to}\quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 \\ & x_1 + x_2 + x_5 + x_6 + x_7 \geq d_2 \\ & x_1 + x_2 + x_3 + x_6 + x_7 \geq d_3 \\ & x_1 + x_2 + x_3 + x_4 + x_7 \geq d_4 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \geq d_5 \\ & x_2 + x_3 + x_4 + x_5 + x_6 \geq d_6 \\ & x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 \\ & x_1, \ldots, x_7 \geq 0 \end{aligned}$$

○

$$\max \sum_{i=1}^{7} x_i$$

$$\begin{aligned} \text{subject to}\quad & x_1 + x_4 + x_5 + x_6 + x_7 \leq d_1 \\ & x_1 + x_2 + x_5 + x_6 + x_7 \leq d_2 \\ & x_1 + x_2 + x_3 + x_6 + x_7 \leq d_3 \\ & x_1 + x_2 + x_3 + x_4 + x_7 \leq d_4 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \leq d_5 \\ & x_2 + x_3 + x_4 + x_5 + x_6 \leq d_6 \\ & x_3 + x_4 + x_5 + x_6 + x_7 \leq d_7 \\ & x_1, \ldots, x_7 \geq 0 \end{aligned}$$

$$
\begin{aligned}
& \min \sum_{i=1}^{7} x_i \\
\text{subject to} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \le d_1 \\
& x_1 + x_2 + x_5 + x_6 + x_7 \le d_2 \\
& x_1 + x_2 + x_3 + x_6 + x_7 \le d_3 \\
& x_1 + x_2 + x_3 + x_4 + x_7 \le d_4 \\
& x_1 + x_2 + x_3 + x_4 + x_5 \le d_5 \\
& x_2 + x_3 + x_4 + x_5 + x_6 \le d_6 \\
& x_3 + x_4 + x_5 + x_6 + x_7 \le d_7 \\
& x_1, \ldots, x_7 \ge 0
\end{aligned}
$$

Total	3.00 / 3.00

Question Explanation

Let x_i denote the number of waiters that start to work on day i of the week ($1 \le i \le 7$). Since every waiter works for exactly 5 consecutive days, the number of waiters working on day i is $x_i + x_{i-1} + x_{i-2} + x_{i-3} + x_{i-4}$(where the subscripts are chosen modulo 7). Clearly, the objective function is to minimize the sum of the waiters hired over all days of the week.

Question 3

What is the linear program that is equivalent to the following problem?

$$
\begin{aligned}
\min \;\; & 2x + 3\max\{y, 2\} \\
\text{subject to} \;\; & |x+2| + y \le 5
\end{aligned}
$$

Your Answer	Score	Explanation

$$
\begin{aligned}
\min \;\; & 2x - 3h \\
\text{subject to} \;\; & z + y \le 5 \\
& z \ge x + 2 \\
& z \le -x - 2 \\
& h \ge y \\
& h \ge 2
\end{aligned}
$$

✔ 3.00

$$\begin{array}{rl} \min & 2x+3h \\ \text{subject to} & z+y\leq 5 \\ & z\geq x+2 \\ & z\geq -x-2 \\ & h\geq y \\ & h\geq 2 \end{array}$$

○

$$\begin{array}{rl} \min & 2x-3h \\ \text{subject to} & z+y\leq 5 \\ & z\geq x+2 \\ & z\geq -x-2 \\ & h\geq y \\ & h\geq 2 \end{array}$$

○

$$\begin{array}{rl} \min & 2x+3h \\ \text{subject to} & z+y\leq 5 \\ & z\geq x+2 \\ & z\leq -x-2 \\ & h\geq y \\ & h\geq 2 \end{array}$$

Total	3.00 / 3.00

Question Explanation

We replace the term $|x+2|$ with a variable z and two constraints $z\geq x+2$ and $z\geq -x-2$ Furthermore, a new variable h replaces the term $\max\{y,2\}$ and is bound to the constraints $h\geq y$ and $h\geq 2$. Observe that it is crucial at this point that we consider a *minimization* problem. This replacement would not work with a maximization problem.

After formulating this as a linear program, one can easily check that the following holds: Every solution of the original program yields a solution of the linear program with the same objective value. Each solution of the linear program yields a solution for the original problem with at most that objective value.

Question 4

Is the set of optimal solutions of a linear program always a polyhedron?

Your Answer		Score	Explanation
○ No			
◉ Yes	✔	3.00	
Total		3.00 / 3.00	

Question Explanation

Yes. Consider the linear program in standard form $\max\{c^{\mathrm{T}}x : Ax \leq b\}$and let d be the optimal value of the linear program. Then the set of optimal feasible solutions is the set $\{x : Ax \leq b, c^{\mathrm{T}}x \geq d\}$which is a polyhedron.

Question 5

Which of the options is the correct dual of the linear program below?

$$\begin{aligned} &\min c^T x \\ \text{subject to } \; &Ax = b \\ &Cx \geq d \end{aligned}$$

Your Answer		Score	Explanation
○ $\begin{aligned} &\max d^T y_2 - b^T y_1 \\ \text{subject to } &A^T y_1 - C^T y_2 = c \\ &y_2 \geq 0. \end{aligned}$			
◉ $\begin{aligned} &\max d^T y_2 - b^T y_1 \\ \text{subject to } &C^T y_2 - A^T y_1 = c \\ &y_2 \geq 0. \end{aligned}$	✓	5.00	
○ $\begin{aligned} &\max d^T y_2 - b^T y_1 \\ \text{subject to } &C^T y_2 - A^T y_1 = c \\ &y_1, y_2 \geq 0. \end{aligned}$			
○ $\begin{aligned} &\max d^T y_2 - b^T y_1 \\ \text{subject to } &A^T y_1 - C^T y_2 = c \\ &y_1, y_2 \geq 0. \end{aligned}$			
Total		5.00 / 5.00	

Question Explanation

First convert the primal program into standard form

$$-\max -c^T x$$
$$\text{subject to } \begin{pmatrix} A \\ -A \\ -C \end{pmatrix} x \le \begin{pmatrix} b \\ -b \\ -d \end{pmatrix}.$$

Then, form the dual

$$-\min b^T y_1 - b^T y_2 - d^T y_3$$
$$\text{subject to } A^T y_1 - A^T y_2 - C^T y_3 = -c$$
$$y_1, y_2, y_3 \ge 0.$$

Simplify the above dual program by replacing $y_1 - y_2$ by a free variable y_4

$$-\min b^T y_4 - d^T y_3$$
$$\text{subject to } A^T y_4 - C^T y_3 = -c$$
$$y_3 \ge 0.$$

Rename the variables and write the resulting program as a maximization program to get

$$\max d^T y_2 - b^T y_1$$
$$\text{subject to } C^T y_2 - A^T y_1 = c$$
$$y_2 \ge 0.$$

Question 6

Consider the linear program

$$\max \sum_i x_i$$
$$\text{subject to } Ax \le b$$
$$x \ge 0$$

where A is a 0-1 matrix and each column of A has at least $k \ge 1$ entries equal to 1 and all the entries of b are non-negative. Select the least of the following that provides a valid upper bound on the value of the optimal solution to the above linear program.

Your Answer		Score	Explanation
$\frac{2}{k} 1^T b$			
$\frac{1}{k} 1^T b$	✓	3.00	
$k 1^T b$			
$\frac{1}{2k} 1^T b$			
Total		3.00 / 3.00	

Question Explanation

Constructing the dual of the linear program, we see that for the dual program $y_i := \frac{1}{k}$ is a feasible solution which attains a solution value $\frac{1}{k}\ 1^T b$ By using weak duality we can conclude that this is a valid upper bound on the optimal solution to the above linear program.

Question 7

In this question we look at a variant of Farkas' Lemma. If there exists no $x \geq 0$ such that $Ax = b$ then which of the following statements is correct?

Your Answer		Score	Explanation
$\exists y : y \geq 0, A^T y \leq 0, b^T y < 0$			
$\exists y : A^T y \leq 0, b^T y < 0$	✗	0.00	
$\exists y : y \geq 0, A^T y \leq 0, b^T y > 0$			
$\exists y : A^T y \leq 0, b^T y > 0$			
Total		0.00 / 3.00	

Question Explanation

Forming the dual of the feasibility linear program

$$\begin{aligned} \min\ & 0^T x \\ \text{subject to}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

we obtain

$$\begin{aligned} \max\ & b^T y \\ \text{subject to}\ & A^T y \geq 0. \end{aligned}$$

Since the primal program is infeasible and the dual program is feasible ($y = 0$ is a feasible solution) it must be the case that the dual is unbounded and hence we get that $\exists y : A^T y \leq 0, b^T y > 0$

Question 8

Consider the hypercube in $\mathbb{R}^n$ which is the polyhedron

$$P = \{x \in \mathbb{R}^n : 0 \leq x \leq 1\}.$$

We construct another polyhedron as follows. Let $a^{(i)}, b^{(i)} \in \mathbb{R}^n$ ($i = 1, \ldots, n$) be the vectors such that

$$a_j^{(i)} = \begin{cases} 1 & \text{if } i = j \\ \frac{1}{2} & \text{otherwise} \end{cases} \qquad \text{and} \qquad b_j^{(i)} = \begin{cases} 0 & \text{if } i = j \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

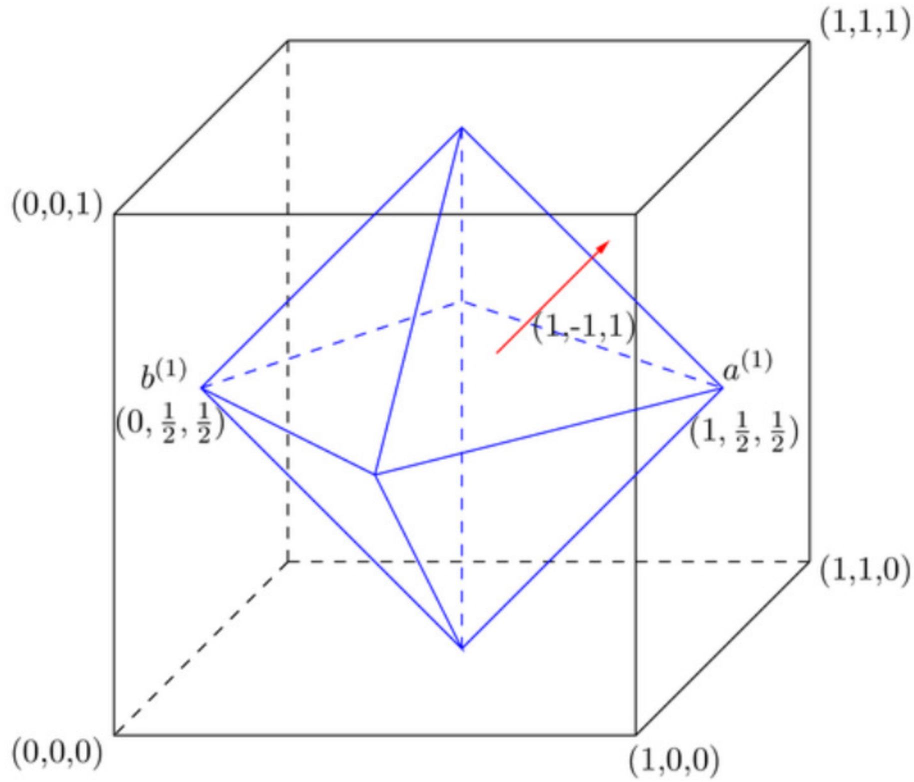
Now, we define the polyhedron $Q \subset \mathbb{R}^n$ as the convex hull of all the vectors $a^{(i)}, b^{(i)}$.

In $\mathbb{R}^3$ the polytopes P and Q would look as indicated in the image, where the red vector indicates the normal vector of one of the hyperplanes corresponding to the inequality description of Q below.

It is straight forward to see that an alternative description of Q is the following

$$Q = \left\{ x \in \mathbb{R}^n : \begin{pmatrix} 1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}^{\mathrm{T}} \cdot b^{(1)} \leq \begin{pmatrix} 1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}^{\mathrm{T}} \cdot x \leq \begin{pmatrix} 1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}^{\mathrm{T}} \cdot a^{(1)} \qquad \text{for all } s_2, \ldots, s_n \in \{-1, +1\} \right\}.$$

Click on all that applies.



Your Answer		Score	Explanation
☐ The hypercube has $2n$ vertices	✔	0.38	
☑ The hypercube has 2^n vertices	✔	0.38	
☑ There are $2n$ non-redundant inequalities in the description of the hypercube.	✔	0.38	
☐ There are 2^n non-redundant inequalities in the description of the hypercube.	✔	0.38	
☑ The hypercube is a non-degenerate polyhedron.	✔	0.38	
☑ The polyhedron Q has $2n$ vertices.	✔	0.38	

☐ The polyhedron Q has 2^n vertices.	✓	0.38
☑ There are 2^n non-redundant inequalities in the description of the polyhedron Q.	✓	0.38
Total		3.00 / 3.00

Question Explanation

The hypercube has 2^n vertices. Namely the set of vectors $(x_1, \dots, x_n)$with $x_i \in \{0, 1\}$ for all $i = 1, \dots, n$ It is clear that each such vector is feasible and the constraint matrix restricted to the set of active inequalities in such a point is a full-rank submatrix of $\binom{I}{-I}$. Hence, the constraint matrix of active constraints has full rank and therefore each such vector is a vertex of the hypercube.

All $2n$ inequalties in the description of the hypercube are non-redundant: Clearly the hypercube is bounded since it is the set $[0, 1]^n$. Discarding one of the inequalities $x_i \leq 1$ results in additional feasible solutions of the form $(0, \dots, 0, x_i, 0, \dots, 0)$with x_i arbitrarily large. The same holds for all the inequalities $x_i \geq 0$.

The hypercube is a non-degenerate polyhedron. At each vertex, there are exactly n inequalities that are tight.

The polyhedron Q has $2n$ vertices, namely $a^{(1)}, \dots, a^{(n)}, b^{(1)}, \dots, b^{(n)}$ All of these vectors are feasible since for any fix $s_2, \dots, s_n \in \{-1, +1\}$we have

$$(1, s_2, \dots, s_n)\, b^{(1)} = 0 + \frac{1}{2}\sum_{j=1}^{n} s_j \leq (1, s_2, \dots, s_n)\, a^{(i)} = \frac{1}{2}\,(1 + s_i) + \frac{1}{2}\sum_{j=1}^{n} s_j$$
$$\leq (1, s_2, \dots, s_n)\, a^{(1)} = 1 + \frac{1}{2}\sum_{j=1}^{n} s_j.$$

and

$$(1, s_2, \dots, s_n) b^{(1)} = 0 + \frac{1}{2}\sum_{j=1}^{n} s_j \leq (1, s_2, \dots, s_n)\, b^{(i)} = \frac{1}{2}\,(1 - s_i) + \frac{1}{2}\sum_{j=1}^{n} s_j$$
$$\leq (1, s_2, \dots, s_n)\, a^{(1)} = 1 + \frac{1}{2}\sum_{j=1}^{n} s_j.$$

The above inequalities further show that at each $a^{(i)}$ and $b^{(i)}$ we have 2^{n-1} active inequalities, respectively, one for each setting of $s_2, \dots, s_n$. (For $a^{(i)}$: if $s_i = -1$ the lower bounds are tight and if $s_i = 1$ the upper bounds are tight. Similarly for $b^{(i)}$.) Since there are n linearly independent vectors among all vectors $(1, s_2, \dots, s_n) (s_2, \dots, s_n \in \{-1, +1\}$ that impose the constraints, for instance

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ -1 \end{pmatrix}, \dots \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

we have that the constraint matrix at $a^{(i)}$ and $b^{(i)}$ of active constraints has full rank, respectively.

All 2^n inequalities in the description of Q are non-redundant. Fix $s_2, \dots, s_n \in \{-1, +1\}$ The point

$$p_i = \begin{cases} \frac{1}{2} + \frac{1}{2(n-1)} & \text{if } i = 1 \\ \frac{1}{2} + \frac{1}{2(n-1)} s_i & \text{otherwise} \end{cases}$$

is not a feasible solution since

$$(1, s_2, \ldots, s_n)p = \frac{1}{2} + \frac{1}{2(n-1)} + \frac{1}{2}\sum_{i=2}^{n} s_i + (n-1)\frac{1}{2(n-1)} > 1 + \frac{1}{2}\sum_{i=2}^{n} s_i = (1, s_2, \ldots, s_n)a^{(i)}.$$

Now, suppose we were deleting the constraint $(1, s_2, \ldots, s_n)x \leq (1, s_2, \ldots, s_n)a^{(1)}$ from the description. Then p becomes feasible. Consider any other constraint for $s'_2, \ldots, s'_n$. We have

$$(1, s'_2, \ldots, s'_n)p = \frac{1}{2} + \frac{1}{2(n-1)} + \frac{1}{2}\sum_{i=2}^{n} s'_i + \frac{1}{2(n-1)}\sum_{i=2}^{n} s'_i s_i.$$

Since $-(n-1) \leq \sum_{i=2}^{n} s'_i s_i \leq n-2$ we have thus

$$(1, s'_2, \ldots, s'_n)p \leq \frac{1}{2} + \frac{1}{2}\sum_{i=2}^{n} s'_i + \frac{1}{2(n-1)} + \frac{n-2}{2(n-1)} = 1 + \frac{1}{2}\sum_{i=2}^{n} s'_i = (1, s_2, \ldots, s_n)a^{(i)} \quad \text{and}$$

$$(1, s'_2, \ldots, s'_n)p \geq \frac{1}{2} + \frac{1}{2}\sum_{i=2}^{n} s'_i + \frac{1}{2(n-1)} - \frac{n-1}{2(n-1)} > 1 + \frac{1}{2}\sum_{i=2}^{n} s'_i = (1, s_2, \ldots, s_n)b^{(i)}.$$

By a similar argument, constraints of the form $(1, s_2, \ldots, s_n)b^{(1)} \leq (1, s_2, \ldots, s_n)x$ can be shown to be non-redundant.

Question 9

Consider the polyhedron defined by the constraints:

$$\begin{aligned} x_1 + 2x_2 - 2x_4 + 2x_5 &\leq 6 \\ x_1 + x_3 + x_4 - x_5 + x_6 &\leq 10 \\ 2x_1 + x_2 &\leq 12 \\ x_1, \ldots, x_6 &\geq 0 \end{aligned}$$

Select all the options that are true.

Your Answer		Score	Explanation
☑ The point $(0, 12, 1, 9, 0, 0)^T$ is a feasible solution	✔	0.50	
☑ The point $(6, 0, 4, 0, 0, 0)^T$ is a basic solution	✔	0.50	
☑ The point $(0, 12, 1, 9, 0, 0)^T$ is a basic solution	✔	0.50	
☐ The point $(0, 12, 1, 9, 0, 0)^T$ is a degenerate basic solution	✔	0.50	
☑ The point $(6, 0, 4, 0, 0, 0)^T$ is a feasible solution	✔	0.50	
☑ The point $(6, 0, 4, 0, 0, 0)^T$ is a degenerate basic solution	✔	0.50	

Total	3.00 / 3.00

Question Explanation

The point $(6,0,4,0,0,0)$is a feasible, degenerate basic solution. Feasibility is easy to check. It is a degenerate basic solution because there are two possible bases for it (i.e. collections of 6 linearly independent active constraints): we can take the first three inequalities along with three of any of the four non-negativity constraints on x_2, x_4, x_5, x_6.

The point $(0,12,1,9,0,0)^T$is a feasible, non-degenerate basic solution. Again feasibility is easy to check. There are exactly 6 constraints that are active at this point (the first 3 equalities along with $x_1, x_5, x_6 \geq 0$), and they are linearly independent, so there is exactly one basis for it.

Question 10

Consider the LP:

$$\begin{array}{lrr} \text{maximize:} & & x+5y \\ \text{subject to:} & (1) & x+3y \leq 30 \\ & (2) & 2x+y \leq 20 \\ & (3) & -4x-5y \leq -8 \\ & (4) & x-y \leq 4 \\ & (5) & -x \leq 5 \\ & (6) & y \leq 10 \end{array}$$

We start the simplex algorithm at the point $x=8, y=4$ This is a vertex, and from it there is only one direction the algorithm can go. What is the computed vector d that represents this direction? (Hint: Find the base B corresponding to vertex $(8,4)^T$, and remember the definition of d.)

Your Answer		Score	Explanation
$d^T = \left(\frac{1}{4}, \frac{1}{2}\right)$			
$d^T = \left(-\frac{1}{3}, \frac{2}{3}\right)$	✓	4.00	
$d^T = \left(\frac{1}{3}, \frac{2}{3}\right)$			
$d^T = \left(-\frac{1}{4}, \frac{1}{2}\right)$			
Total		4.00 / 4.00	

Question Explanation

Only the second and fourth constraints are active at the point $x=(8,4)^T$, so the corresponding basis is $B=\{2,4\}$. The index 2 stays on the basis because $a_2^T c = (2,1)\cdot(1,5)^T = 7 > 0$, and index 4 has to leave the basis because $a_4^T c = (1,-1) \cdot (1,5)^T = -4 < 0$. The vector d is defined such that $a_4^T d = -1$ and $a_2^T d = 0$. Hence, d is the second column of $-A_B^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

Question 11

In the context of the previous exercise, what is the basis of the next visited vertex?

Your Answer		Score	Explanation
○ $\{5,6\}$			
◉ $\{1,2\}$	✓	4.00	
○ $\{1,6\}$			
○ $\{2,6\}$			
Total		4.00 / 4.00	

Question Explanation

We compute the set of indices i corresponding to constraints that may become active when we move in the direction of d: $K = \{i \mid a_i^T d > 0\} = \{1,5,6\}$ One of these indices will replace index 4 and join index 2 in the new basis. We compute the amounts $\varepsilon_i = \frac{b_i - a_i^T x}{a_i^T d}$ that we can move before we hit each each of these hyperplanes, and select the index with the smallest distance: $\varepsilon_1 = \frac{30-(1,3)\cdot(8,4)^T}{(1,3)(-1,3,2/3)^T} = \frac{30-20}{5/3} = 6$ $\varepsilon_5 = \frac{5-(-1,0)\cdot(8,4)^T}{(-1,0)(-1,3,2/3)^T} = \frac{5+8}{1/3} = 39$ $\varepsilon_6 = \frac{10-(0,1)\cdot(8,4)^T}{(0,1)(-1,3,2/3)^T} = \frac{10-4}{2/3} = 9$ The smallest one is ε_1, so the new basis is $\{1,2\}$.

Question 12

What is the best known upper bound for the number of iterations for the simplex method using Bland's pivoting rule? We denote by n the number of variables and by m the number of constraints of the linear programming instance to be solved.

Your Answer		Score	Explanation
○ Polynomial in n and m			
◉ Exponential in n and m	✓	1.00	
Total		1.00 / 1.00	

Question Explanation

The best upper bound for the simplex method using Bland's pivoting rule is exponential in n and m. In fact there are examples, such as the so-called Klee-Minty cube, where an exponential number of iterations is needed to find the optimum solution.

Question 13

What is the maximum height ℓ (i.e. number of layers) of a connected layer family where we choose $n = 2$ elements from ground set of $m = 4$ symbols?

You entered:

16

Preview Help

Your Answer		Score	Explanation
16	✗	0.00	Match failed
Total		0.00 / 3.00	

Question Explanation

A connected layer family of height $\ell = 4$ is possible:

$$\begin{array}{r} (1,2) \\ (1,3),(2,4) \\ (1,4),(2,3) \\ (3,4) \end{array}$$

However, height 6 is clearly not possible for $n = 2$ and $m = 4$ Height 5 is not possible either. Aussming the height was 5, the two topmost layers contain without loss of generality only one element. This implies that one symbol used in the top layer may never be used in any layer below. Thus we have only 3 different symbols left and thus there can not be 4 disjoint layers below the toplayer.

Question 14

A directed graph $G = (V, A)$ is said to be strongly connected if for every pair of distinct vertices u, v in V there exists a path from u to v and v to u. Is it possible to compute in linear time (that is $O(|V| + |A|)$ whether a given directed graph is strongly connected or not?

Your Answer		Score	Explanation
○ No			
◉ Yes	✓	3.00	

Total	3.00 / 3.00

Question Explanation

Yes. Pick an arbitrary vertex $v \in V$. Compute the set R of all vertices reachable from v in G using breadth first search in time $O(|V| + |A|)$ Let $G' = (V, A')$ be the graph obtained from G by reversing the directions of all the arcs. Similarly, compute the set S of all vertices reachable from v in G'. If $R \cap S = V$ then the graph is strongly connected since for any pair of distinct vertices u, w, we can find a path from u to w from the walk obtained by concatenating the path from u to v and v to w (the first path exists since $u \in S$ and the second because $w \in R$). Conversely, if $R \cap S \neq V$ then there exists at least one vertex $u \in V$ that is either not reachable from v because $S \neq V$ or cannot reach v from u because $R \neq V$ which implies that G is not strongly connected.

Question 15

In a directed graph $G = (V, A)$ with rational arc weights and given a subset of vertices $S \subseteq V$ we would like to compute for each vertex $v \in V$, the shortest path distance from the set of vertices S to v. That is, for each vertex $v \in V$, $d(v) := \min_{s \in S} \delta_G(s, v)$ where $\delta_G(s, v)$ is the length of the shortest path from s to v in G. If you used Bellman-Ford for solving this problem (i.e., either finding $d(v)$ for all the vertices or detecting a negative cycle reachable from some vertex in S) then what is the best bound on the time complexity of the algorithm?

Your Answer		Score	Explanation
○ $O(\|V\|\|A\|^2)$			
○ $O(\|V\|\|A\|)$			
◉ $O(\|V\|^2\|A\|)$	✗	0.00	
○ $O(\|V\|^2\|A\|^2)$			
Total		0.00 / 3.00	

Question Explanation

To solve this problem we set the distance labels $d(v)$ for all the vertices $v \in S$ to 0 (and ∞ on all other vertices) before running Bellman-Ford on the graph G. The resulting distance labels that are computed for any vertex $v \in V$ by Bellman-Ford algorithm are exactly $d(v)$ which follows via induction as in the original proof of Bellman-Ford. Since this requires running Bellman-Ford only once the time complexity is $O(|V||A|)$.

Question 16

Consider the LP:

$$\text{maximize: } c^T x$$
$$\text{subject to: } Ax \leq b$$

where A is a matrix of dimensions $m \times n$. Suppose that both this LP and its dual are feasible, and (x^*, y^*) is a pair of optimal solutions, respectively. If we know that y^* has exactly 3 zero components, then we can conclude that the number of constraints of the form $\sum_{i=1}^{n} A_{ji} x_i \leq b_j$ that are active at x^* is

Your Answer		Score	Explanation
◉ At most $m-3$	✘	0.00	
○ At least 3			
○ At most 3			
○ At least $m-3$			
Total		0.00 / 3.00	

Question Explanation

The complementary slackness theorem tells us that for each index $1 \leq j \leq m$, either $y_j = 0$ or $\sum_{i=1}^{n} A_{ji} x_i = b_j$ (i.e. the constraint is active at x^*). This events are not mutually exclusive, so the fact that a component of y^* is zero does not imply anything on the corresponding constraint. But we know that y^* has $m-3$ non-zero components, and for each one of these indices the corresponding constraints must be active at x^*. Therefore there are at least $m-3$ active constraints.

Question 17

Consider the following LP:

$$\begin{aligned}
\text{minimize: } & 4x_1 + 2x_2 + 4x_3 \\
\text{subject to: } & 4x_1 + x_2 + x_3 \geq 10 \\
& x_1 + x_2 + 4x_3 \geq 60 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Formulate the dual LP, draw it on the plane and find the optimal dual point y^* geometrically. Use complementary slackness two times, changing the roles of the primal and the dual LPs, to answer the following questions. (Hint: once you find y^*, you should be able to solve this exercise without writing anything down)

1. How many components of y^* are zero?
2. Apart from the constraints of the form $y_i \geq 0$, how many constraints in the dual are active at y^*?
3. How many components of the optimal primal solution x^* are zero?
4. Apart form the constraints of the form $x_i \geq 0$, how many constraints are active at x^*?

Enter your answers to the above questions as a list of four space-separated integers.

You entered:

3 2 1 1

Your Answer		Score	Explanation
3 2 1 1	✗	0.00	
Total		0.00 / 5.00	

Question Explanation

We will call P and D the primal and dual LPs, respectively. The dual D is:

$$\begin{aligned} \text{maximize: } & 10y_1 + 60y_2 \\ \text{subject to: } & 4y_1 + y_2 \leq 4 \\ & y_1 + y_2 \leq 2 \\ & y_1 + 4y_2 \leq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

It is very easy to draw D and convince ourselves that the unique optimal point is $y^* = (0, 1)^T$. It has 1 zero component (question 1). The first and second constraints of D are not active at y^* (question 2), so complementary slackness tells us that the first and second components of x^* are zero. The vector x^* looks like $x^* = (0, 0, x_3)^T$, but $x_3 \neq 0$ because the all-zero vector is clearly not feasible in P, so x^* has exactly 2 zero components (question 3). Finally, since the second component of y^* is non-zero, complementary slackness tells us that the second constraint of P is active in $x^* = (0, 0, x_3)^T$. We can mentally compute $x_3 = 15$, and check that the first constraint of P is not active in $x^* = (0, 0, 15)^T$ (question 4).

Question 18

In the next questions we will consider the *Knapsack* problem. We are given n items that each have a positive size s_i and non-negative profit p_i per unit ($i = 1, \ldots, n$). Further, we are given integral upper bounds $u_i \geq 1, u_i \in \mathbb{Z}$ on the number of available units of each item. Given a knapsack (or container) of size B we would like to pack items such that the size of the knapsack is respected and the profits are maximized. Note that items need not be packed in units but can also be used in fractional amounts.

The following linear program models this problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_i \, x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} s_i \, x_i \leq B \\ & x_i \leq u_i \qquad (i = 1, \ldots, n) \\ & x_i \geq 0 \qquad (i = 1, \ldots, n) \\ & x \in \mathbb{R}^n \end{aligned}$$

Let P denote the polyhedron defined by the inequalities above. Consider any vertex v of P. How many fractional components does v have at most?

Your Answer		Score	Explanation
Arbitrary			
1	✓	3.00	
0			
n			
Total		3.00 / 3.00	

Question Explanation

A vertex v has n linearly independent constraints that are tight at v. Except for the constraint $\sum_{i=1}^{n} s_i \; x_i \leq B$all the other constraints, when active, force the components of the solution to have some integral value. Hence, there must be $n-1$components of the solution that are integral and therefore there can be at most one fractional component.

Question 19

What is the dual of the linear programming formulation of the knapsack problem (introduced in the previous question)?

Your Answer		Score	Explanation
$$\begin{array}{lll} \min & By + \sum_i u_i z_i & \\ & \text{s.t.} \quad y + s_i z_i \geq p_i & (i = 1, \dots, n) \\ & \quad y \in \mathbb{R}, \quad z \in \mathbb{R}^n & \end{array}$$			
$$\begin{array}{lll} \min & By + \sum_i u_i z_i & \\ & \text{s.t.} \quad s_i y + z_i \geq p_i & (i = 1, \dots, n) \\ & \qquad y \geq 0 & \\ & \qquad z_i \geq 0 & (i = 1, \dots, n) \\ & \quad y \in \mathbb{R}, \quad z \in \mathbb{R}^n & \end{array}$$	✓	3.00	

$$\begin{array}{lll} \min & By & \\ \text{s.t.} & s_i y \geq p_i & (i = 1, \ldots, n) \\ & y \geq 0 & \\ & y \in \mathbb{R} & \end{array}$$

$$\begin{array}{lll} \min & By + \sum_i u_i z_i & \\ \text{s.t.} & y + s_i z_i \geq p_i & (i = 1, \ldots, n) \\ & y \geq 0 & \\ & z_i \geq 0 & (i = 1, \ldots, n) \\ & y \in \mathbb{R}, \quad z \in \mathbb{R}^n & \end{array}$$

Total	3.00 / 3.00

Question Explanation

We form the dual by first transforming the linear program for knapsack into the standard form. The dual program is then immediate.

Question 20

In this exercise we will develop an algorithm that solves the knapsack problem via a primal-dual approach. Consider the following feasible solutions x and (y, z) for the primal and dual program, respectively,

$$x_i = 0 \ (i = 1, \ldots, n) \quad \text{and} \quad y = \max_{i=1,\ldots,n} \frac{p_i}{s_i}, z_i = 0 \ (i = 1, \ldots, n).$$

The algorithm starts with these solutions and proceeds as follows. If $\sum_{i=1}^n s_i x_i < B$ it decreases y until one of the constraints $s_i y + z_i \geq p_i$ becomes tight for some i. It then updates x_i to $\min\{u_i, (B - \sum_{i=1}^n s_i x_i)/s_i\}$ i.e., it uses u_i units of item i if they still fit into the knapsack or fills up the remaining space with this item otherwise. We say that item i was *used*.

Now, as long as $\sum_{i=1}^n s_i x_i < B$ the algorithm keeps

- decreasing y, and
- for every used item i, increases z_i at an s_i times faster rate

until one of the remaining constraints $s_i y + z_i \geq p_i$ becomes tight. It then performs the same update as above and iterates. The algorithm also terminates if $y = 0$.
Let $\bar{x}$ and $(\bar{y}, \bar{z})$ denote the values of x and (y, z) at the end of the algorithm.

Click on all that applies.

Your Answer	Score	Explanation

☐ $\tilde{x}$ is *not* feasible for the knapsack problem.	✓	0.38
☑ $\tilde{x}$ and $(\tilde{y}, \tilde{z})$ fulfill the complementary slackness conditions.	✓	0.38
☑ $(\tilde{y}, \tilde{z})$ is feasible for the dual program.	✓	0.38
☑ $\tilde{x}$ is *not* necessarily a vertex of the polytope defined by the knapsack problem.	✗	0.00
☐ $\tilde{x}$ is a vertex of the polytope defined by the knapsack problem.	✗	0.00
☑ $\tilde{x}$ is feasible for the knapsack problem.	✓	0.38
☐ The algorithm terminates. Assume the items are already sorted according to $\frac{p_i}{s_i}$. Then, the running time is $O(n^2)$.	✗	0.00
☐ $(\tilde{y}, \tilde{z})$ is *not* feasible for the dual program.	✓	0.38
Total		1.88 / 3.00

Question Explanation

In each iteration we decrease y to the next lower fraction of $\frac{p_i}{s_i}$. Hence, there are at most n iterations. In each iteration we increase the z variables for all the used items and set the value of one coordinate of x. This takes at most $O(n)$ time. Hence, a naive implementation of this algorithm takes $O(n^2)$ time. However, the algorithm iterates through the fractions $\frac{p_i}{s_i}$ from the highest to the lowest and whenever an item has this ratio the maximum (still) possible amount of this item is used. Hence, this algorithm is in fact a greedy algorithm that packs the most profitable, per size, items first. Of course, this can easily be implemented in $O(n)$ time once the ratios are sorted.

Clearly, $\tilde{x}$ is feasible for the knapsack problem since items are only packed while $\sum_{i=1}^{n} s_i x_i < B$ and the last item is packed such that $\sum_{i=1}^{n} s_i \tilde{x}_i = B$.

Also $(\tilde{y}, \tilde{z})$ is a feasible solution for the dual program. The algorithm starts with a feasible solution and whenever a constraint goes tight it ensures (by decreasing y and with the adjusted rate increasing z) that these inequalities remain tight over the course of its run.

Furthermore, $\tilde{x}$ and $(\tilde{y}, \tilde{z})$ fulfill the complementary slackness conditions. If $\tilde{y} > 0$ then the algorithm terminated because the knapsack was fully packed, hence $\sum_{i=1}^{n} s_i \tilde{x}_i = B$. If $\tilde{z}_i > 0$ then the algorithm used i. Note that $z_i > 0$ only if the algorithm was able to fully pack all u_i units of item i and proceed to a next iteration (in which z_i was increased from zero). Hence, $x_i = u_i$.

Since $\tilde{x}$ and $(\tilde{y}, \tilde{z})$ are each feasible solutions to the primal and dual, respectively, that satisfy the complementary slackness conditions we know that they are also both optimal.

If possible, the algorithm packs all available units of a used item. Only the last item could be packed fractionally. This means that in $\tilde{x}$ we have $n-1$ tight constraints of the form $0 \le x_i$ or $x_i \le u_i$. The nth tight constraint is $\sum_{i=1}^{n} s_i \tilde{x}_i = B$. All these constraints are linearly independent and hence $\tilde{x}$ is a vertex of the polytope.

Question 21

Consider again the knapsack problem. Click on all that applies.

Your Answer		Score	Explanation
☑ If the ratios $\frac{p_i}{s_i}$ are all pairwise distinct, then the optimal solution is unique.	✔	1.00	
☐ If the profits p_i are all pairwise distinct, then the optimal solution is unique.	✔	1.00	
☐ The optimal solution is unique.	✔	1.00	
Total		3.00 / 3.00	

Question Explanation

In a general knapsack problem, the optimal solution might not be unique. For instance if all ratios $\frac{p_i}{s_i}$ are equal and $\sum_{i=1}^{n} s_i > B$any $0 \leq x \leq u$such that $\sum_{i=1}^{n} s_i x_i = B$is an optimal solution. The same example works even if all profits are pairwise distinct as long as all ratios $\frac{p_i}{s_i}$ are equal.

If the ratios $\frac{p_i}{s_i}$ are pairwise distinct, we can run the above algorithm to obtain an optimal solution. Note, that this algorithm greedily packs the items according to decreasing profit to size ratio. Filling the knapsack with items with smaller profit to size ratio results in a decrease of the objective function.