What have we learned?

This module introduced the discrete-time Fourier transform, or DTFT, namely the Fourier analysis tool used to handle infinite-length sequences. We started from an intuitive point of view: as a finite-length signal becomes longer, the fundamental DFT frequencies $\frac{2\pi}{N}k$ become denser and denser in the $[0, 2\pi]$ interval and we can try to replace this countable set by a real-valued frequency value $\omega \in \mathbb{R}$. This leads to the definitions of the DTFT analysis and synthesis formulas

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

The DTFT, as an operator, maps an infinite-length sequence $x[n]$ to a function of real variable ω; by definition, the resulting function is 2π-periodic in ω and we therefore use the notation $X(e^{j\omega})$ to highlight this property, as per standard convention in the DTFT literature.

Another common convention is to use the $[-\pi, \pi]$ interval as the representative interval for the DTFT. Over this interval, positive frequencies between 0 and π are associated to counterclockwise rotations of the underlying complex exponentials; conversely, frequencies between $-\pi$ and 0 are associated to clockwise rotations.

Since the DTFT involves an infinite sum, the question of existence (that is, convergence of the sum) becomes relevant and it can be easily shown that the DTFT exists for all square summable sequences. It also can be shown (but it's not easy) that the DTFT exists for absolutely summable sequences as well.

The DTFT can be formally interpreted as a change of basis in the space $\mathbb{C}^\infty$ where the uncountable set of functions ${e^{j\omega n}}_{\omega\in\mathbb{R}}$ forms an orthogonal basis. Although this space is not a proper vector space, by adding the Dirac delta functional to our set of mathematical tools we can write a formal orthogonality relation between "basis vectors" and we can also extend the DTFT formalism to infinite-energy sequences, as in

$$\mathrm{DTFT}(1) = 2\pi\sum_{k=-\infty}^{\infty}\delta(w - 2\pi k).$$

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