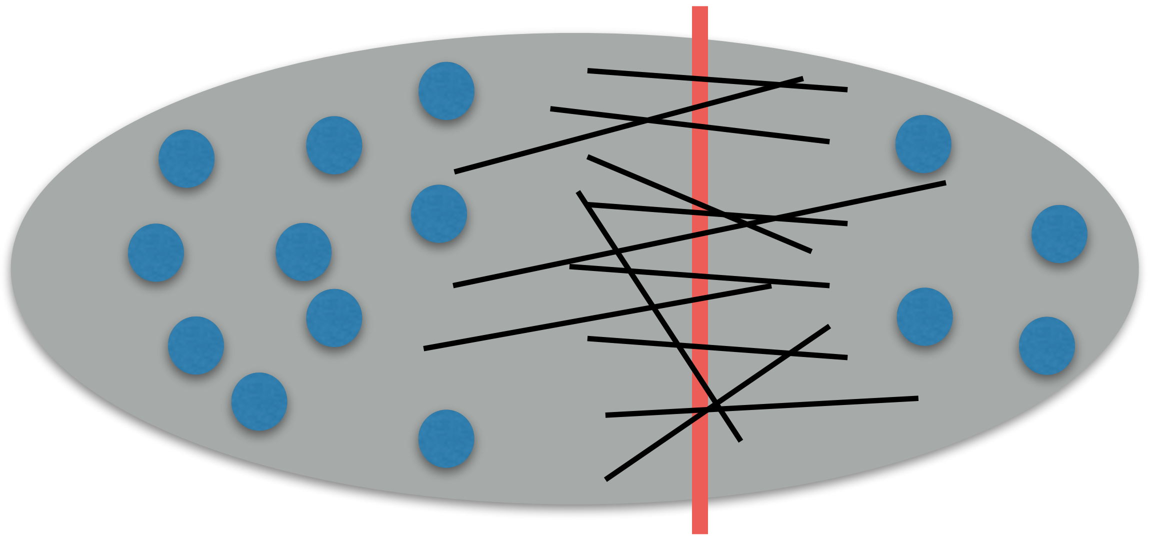
Maxcut

Partition graph
to maximize
edge weight across cut

Can we do better than 2?



Integer programming formulation

One variable per edge:

$$\mathrm{x_{ij}} = \begin{cases} 1 & \text{if crosses cut} \\ 0 & \text{otherwise} \end{cases}$$

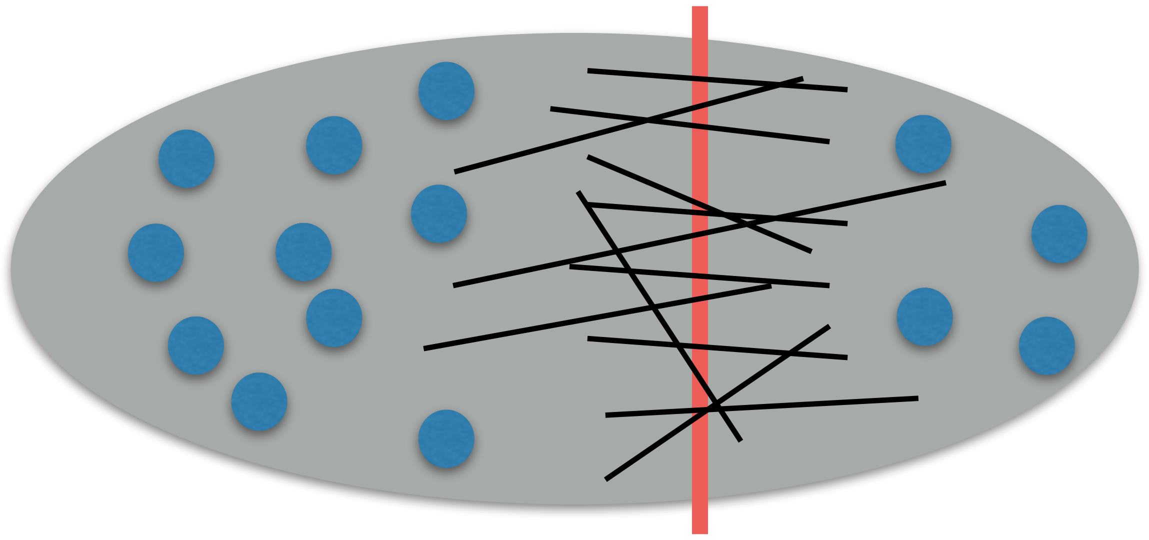
Objective:

$$\max \sum_{\{\mathrm{i,j}\}\in \mathrm{E}} \mathrm{w_{ij}x_{ij}}$$

How do we represent cuts?

Partition graph
to maximize
edge weight across cut

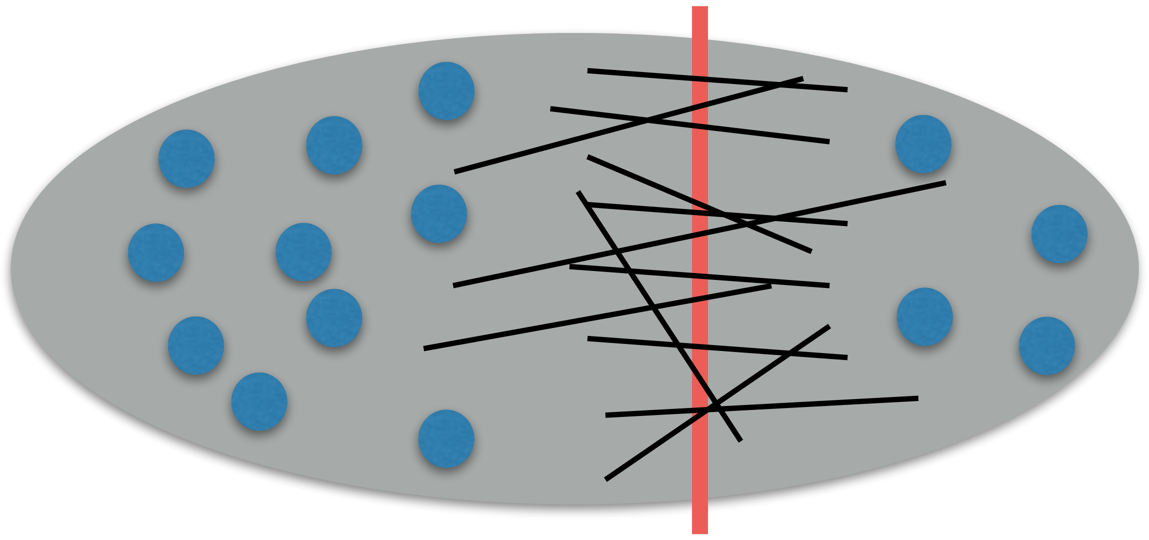
Cuts



Given $(x_e)_{e \in E}$, existence of partition {S,V-S}?

Idea: One variable x_{ij} per vertex pair
instead of per edge

Cut properties ?

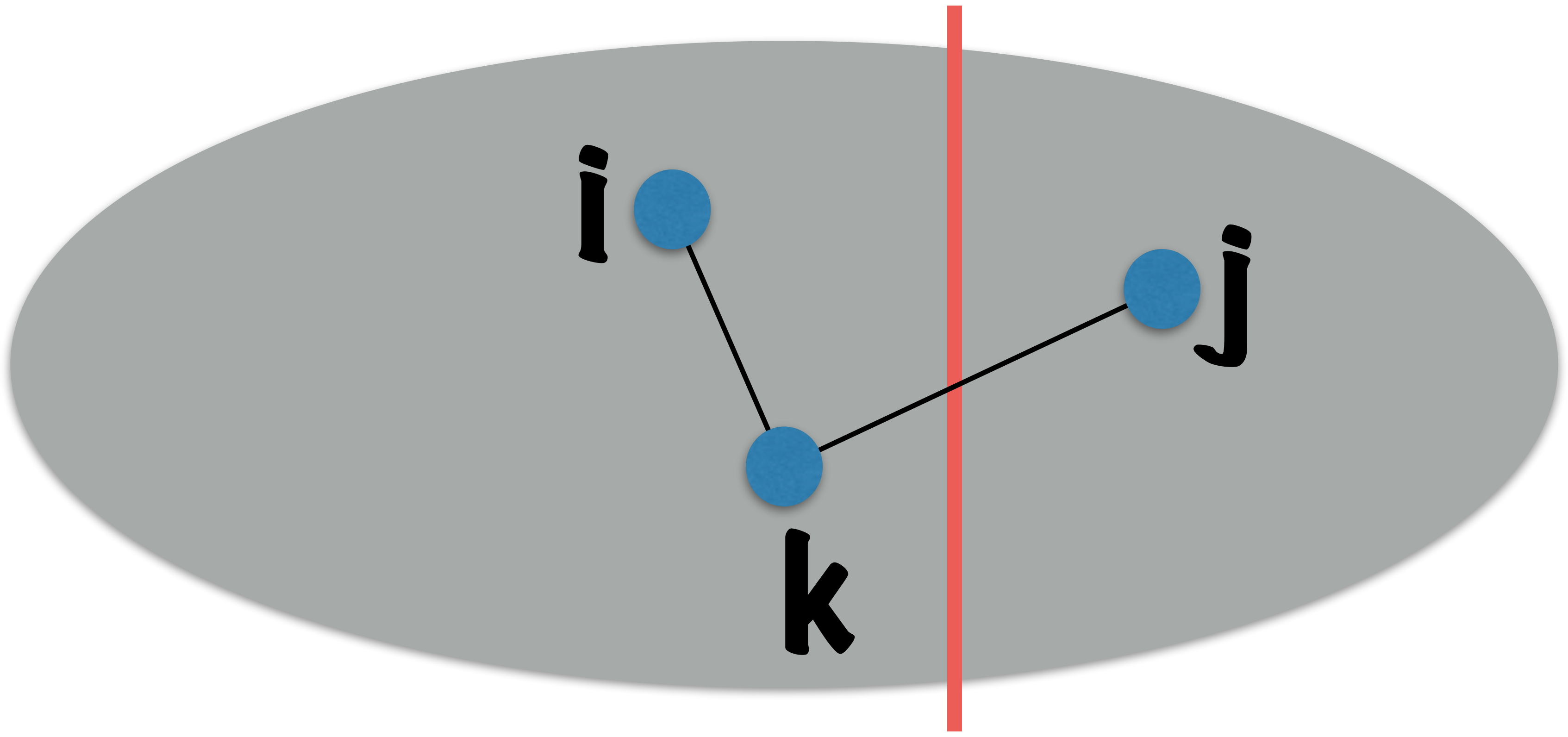


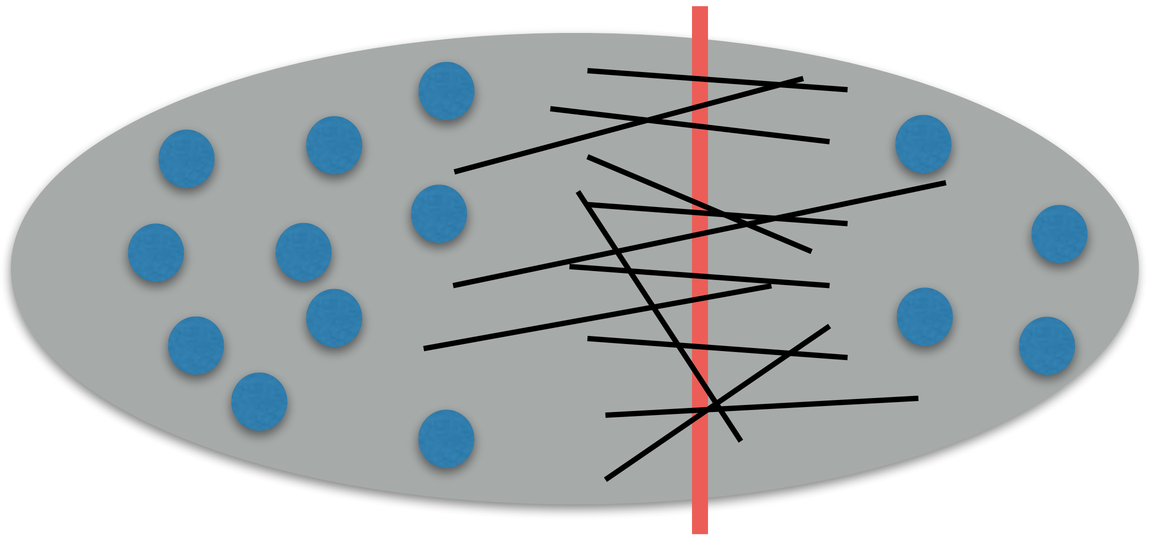
implicit symmetry : $\mathbf{x_{ij} = x_{ji}}$

Cut properties

$$\mathbf{x_{ij}} = \begin{cases} 1 & \textbf{if crosses cut} \\ 0 & \textbf{otherwise} \end{cases}$$

$$\mathbf{x_{ij} \leq x_{ik} + x_{kj}}$$

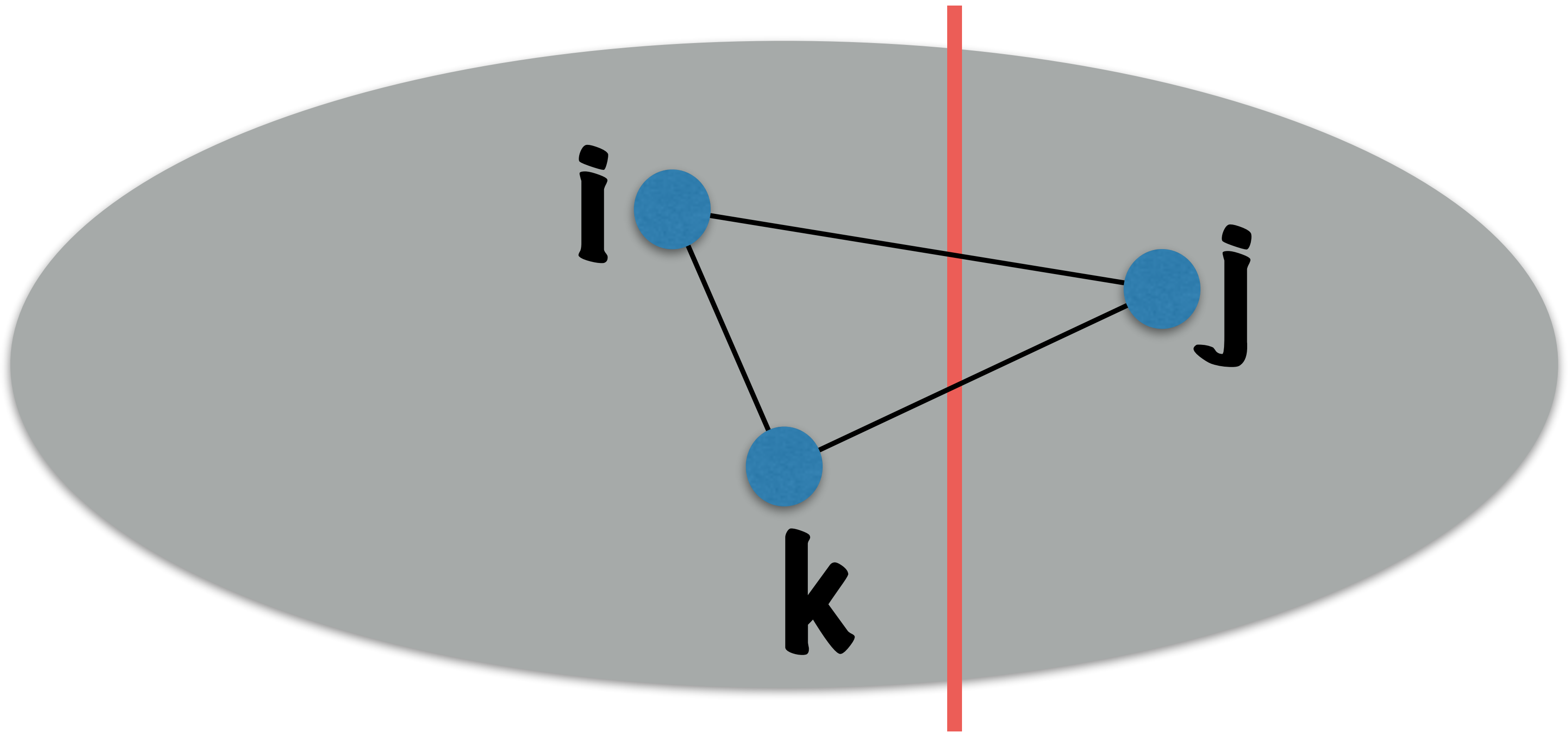


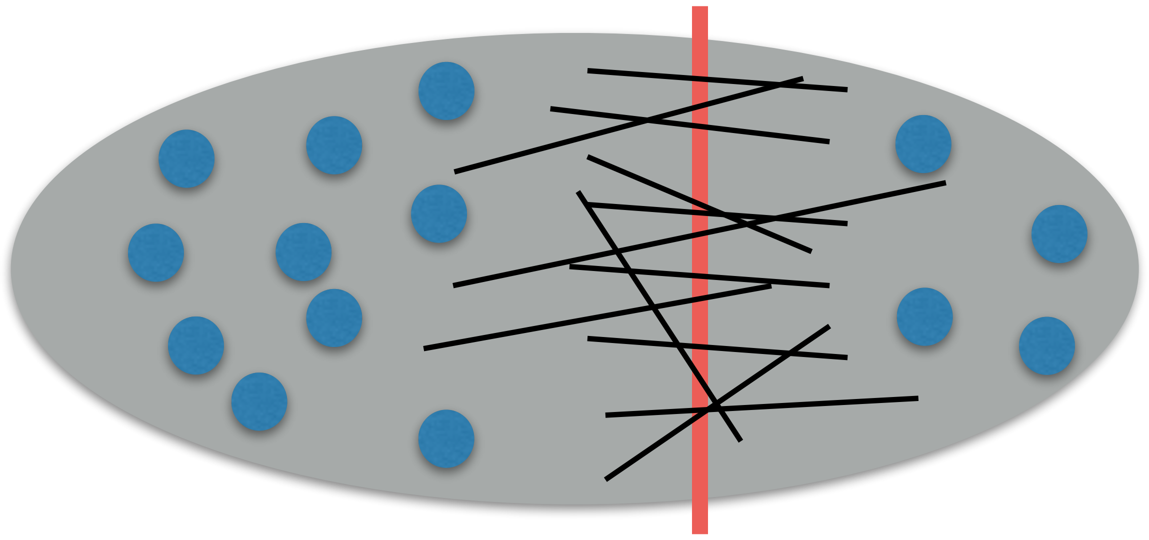


Cut properties

$$\mathbf{x_{ij}} = \begin{cases} 1 & \textbf{if crosses cut} \\ 0 & \textbf{otherwise} \end{cases}$$

$$\mathbf{x_{ij}} + \mathbf{x_{jk}} + \mathbf{x_{ki}} \leq \mathbf{2}$$





Lemma

$$\mathbf{x_{ij}} \leq \mathbf{x_{ik}} + \mathbf{x_{kj}}$$
$$\mathbf{x_{ij}} + \mathbf{x_{jk}} + \mathbf{x_{ki}} \leq 2 \iff$$
$$\mathbf{x_{ij}} \in \{0, 1\}$$

There exists a cut S
s.t.
$\mathbf{x_{ij}} = 1$
iff pair $\mathrm{i, j}$
crosses cut (S,V-S)

$\Longleftarrow$ is clear

$\Longrightarrow$ needs a proof

$$x_{ij} \le x_{ik} + x_{kj}$$
$$x_{ij} + x_{jk} + x_{ki} \le 2$$
$$x_{ij} \in \{0, 1\}$$

Proof: let $S \leftarrow \{1\} \cup \{i : x_{1i} = 0\}$

$x_{ij}, i \in S, j \notin S :$ $\quad \underbrace{x_{1j}}_{=1} \le \underbrace{x_{1i}}_{=0} + x_{ij}$

$x_{ij}, i \in S, j \in S :$ $\quad x_{ij} \le \underbrace{x_{i1}}_{=0} + \underbrace{x_{1j}}_{=0}$

$x_{ij}, i \notin S, j \notin S :$ $\quad x_{ij} + \underbrace{x_{1i}}_{=1} + \underbrace{x_{1j}}_{=1} \le 2$

QED

LP relaxation for Maxcut

Symmetric variables $\mathbf{x_{ij}}$ **for** $\mathbf{i, j} \in \mathbf{V}$

$$\max \sum_{\{\mathbf{i,j}\}\in\mathbf{E}} \mathbf{w_{ij}} \; \mathbf{x_{ij}} \; :$$

$$\mathbf{x_{ij}} \leq \mathbf{x_{ik}} + \mathbf{x_{kj}}$$

$$\mathbf{x_{ij}} + \mathbf{x_{jk}} + \mathbf{x_{ki}} \leq \mathbf{2}$$

$$\mathbf{0} \leq \mathbf{x_{ij}} \leq \mathbf{1}$$

Maxcut