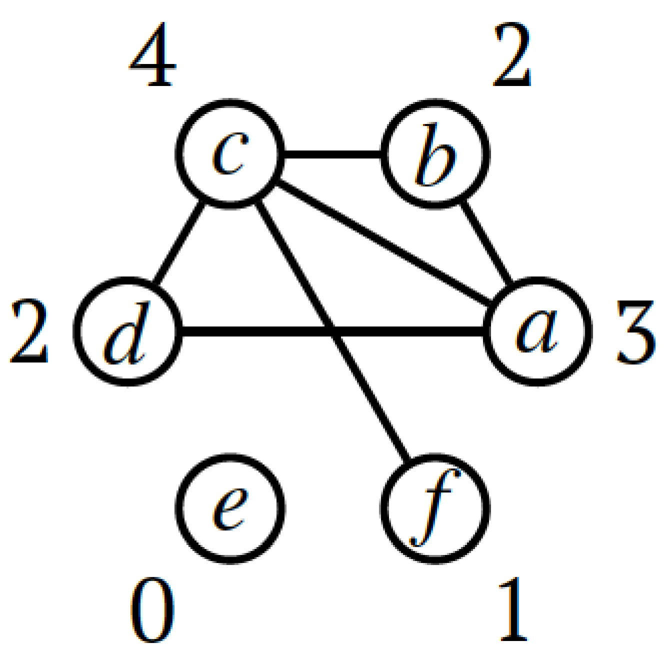
Handshakes

Problem. There are n individuals at a party, and some of them have shaken hands. Prove that there are two individuals who have participated in the same number of handshakes.

The following figure gives an example.



Here there are $n = 6$ people at a party: a, b, c, d, e, f. We join two individuals if they shook hands. Near each individual, her number of handshakes is shown. The individuals b and d have shaken the same number of hands.

In the general case we have n integers (the number of handshakes for each of n individuals) and each of these integers may take any of the values $0, 1, \ldots, n-1$. The pigeonhole principle does not seem to be applicable here: it is perfectly fine if each of the integers $0, 1, \ldots, n-1$ appears once among our n numbers.

But a closer look at the numbers reveals that some two of these numbers cannot appear together.

What are these two integers?

These are the integers 0 and $n-1$: the integer 0 corresponds to an individual who shook no hands, whereas the integer $n-1$ corresponds to an individual who shook all other hands. But these two cases cannot happen simultaneously! Indeed, if an individual shook all other hands, then everybody else shook at least one hand.

This allows us to conclude that there are at most $n-1$ different values among our n integers. The pigeonhole principle implies immediately that some two of these integers are equal.

Summary.

- Proof by contradiction (or reductio ad absurdum) is a basic argument: to prove that a statement is true, one assumes that negation is true and derives a contradiction
- This is one of the most popular proof arguments, and it is usually combined with other proof ideas.