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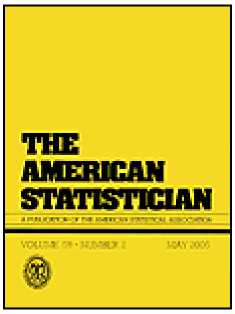

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Generalized Likelihood Ratio Tests and Uniformly Most Powerful Tests

DAVID BIRKES*

For testing a one-sided hypothesis in a one-parameter family of distributions, it is shown that the generalized likelihood ratio (GLR) test coincides with the uniformly most powerful (UMP) test, assuming certain monotonicity properties for the likelihood function. In particular, the equivalence of GLR tests and UMP tests holds for one-parameter exponential families. In addition, the relationship between GLR and UMPU (UMP unbiased) tests is considered when testing two-sided hypotheses.

KEY WORDS: Exponential family; Monotone likelihood ratio property; Unbiased test.

1. INTRODUCTION

A first course in mathematical statistics taught at the level of Mood, Graybill, and Boes (1974) or Bain and Engelhardt (1987) would present the concepts of a uniformly most powerful (UMP) test and of a generalized likelihood ratio (GLR) test. In this article a relationship between these two concepts is demonstrated.

When a GLR test is calculated for a hypothesis for which a UMP test exists, it seems that the GLR test almost always turns out to coincide with the UMP test [but see exercise 24.25 in Kendall and Stuart (1979)]. We show this to be true for tests of one-sided hypotheses in a one-parameter family of distributions under certain conditions on the likelihood function (see Theorem 2 in Sec. 2). In particular, the equivalence of GLR tests and UMP tests is seen to hold for one-parameter exponential families (see Sec. 3). Section 3 considers the relationship between GLR and UMPU (UMP unbiased) tests for two-sided hypotheses.

Suppose that $X_1, \ldots, X_n$ is a random sample from a distribution with density function $f(x; \theta)$ for some real-valued parameter θ in a parameter set Ω. The density func-

We will assume that there is a real-valued sufficient statistic $T(X_1, \ldots, X_n)$. Because of the factorization criterion, likelihood procedures depend on the data only through the sufficient statistic. Rather than the likelihood function of the entire sample, we can use the likelihood function of T, which we will denote by $L(\theta; t)$.

Let us test the one-sided hypothesis $H_0 : \theta \leq b$ versus $H_1 : \theta > b$. Take any $\theta_1 > b$. The Neyman–Pearson lemma says that the most powerful test for testing the simple hypothesis $H_0 : \theta = b$ versus the simple alternative $H_1 : \theta = \theta_1$ is the test that rejects for large values of the likelihood ratio

$$R_s(t) = L(\theta_1; t)/L(b; t).$$

For testing the composite hypothesis $H_0 : \theta \leq b$ versus $H_1 : \theta > b$, the likelihood ratio can be generalized to the ratio

$$\sup_{\theta_1 \in \Omega_1} L(\theta_1; t)/\sup_{\theta_0 \in \Omega_0} L(\theta_0; t),$$

where $\Omega_0 = \{\theta \in \Omega : \theta \leq b\}$ and $\Omega_1 = \{\theta \in \Omega : \theta > b\}$. As a test statistic this ratio is essentially equivalent to the ratio

$$R_g(t) = \sup_{\theta \in \Omega} L(\theta; t)/\sup_{\theta_0 \in \Omega_0} L(\theta_0; t).$$

Letting $\hat{\theta}(t)$ denote the maximum likelihood estimate (MLE) of θ in the full model (parameterized by Ω) and letting $\hat{\theta}_0(t)$ denote the MLE of θ in the hypothesized model (parameterized by Ω_0), we can write

$$R_g(t) = L(\hat{\theta}(t); t)/L(\hat{\theta}_0(t); t).$$

The reciprocal of R_g is called the GLR. A test with rejection region of the form $\{t : R_g(t) > c\}$ or $\{t : R_g(t) \geq c\}$ is called a GLR test. [If $R_g(T)$ is a continuous random variable under all values of θ, then the rejection regions $\{t : R_g(t) > c\}$ and $\{t : R_g(t) \geq c\}$ are equivalent, since they