The Navier-Stokes Equation

The incompressible Navier-Stokes equation governing fluid flow is given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u},$$

with $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0.$ Here, ρ and ν are fluid density and viscosity.

(a) By taking the divergence of the Navier-Stokes equation, derive the following equation for the pressure in terms of the velocity field:

$$\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}.$$

(b) By taking the curl of the Navier-Stokes equation, and defining the vorticity as $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$, derive the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

You can use all the vector identities presented in these lecture notes, but you will need to prove that

$$\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}.$$