A variation on the theme of independence: conditional independence

Conditionally independent events

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Definition

We say that events A and B in a probability space are conditionally independent given an event C if

$$\mathbf{P}(A \cap B \mid C) = \mathbf{P}(A \mid C) \times \mathbf{P}(B \mid C).$$

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A and B are not (unconditionally) independent.

Slogan

Conditional independence does not imply (unconditional) independence, or vice versa.