1. Proof by contradiction

Po-Shen Loh

CMU Putnam Seminar, Fall 2012

1 Classical results

1. The real number $\sqrt{2}$ is irrational.
2. The set of real numbers is uncountable.

2 Problems

Putnam 1952/A1. The polynomial $p(x)$ has all integral coefficients. The leading coefficient, the constant term, and $p(1)$ are all odd. Show that $p(x)$ has no rational roots.

Putnam 1962/A6. X is a subset of the rationals which is closed under addition and multiplication, and it does not contain 0. For any rational $x \neq 0$, exactly one of x or $-x$ is in X. Show that X is the set of all positive rationals.

Putnam 1965/B5. Show that a graph with $2n$ points and $n^2 + 1$ edges necessarily contains a 3-cycle, but that we can find a graph with $2n$ points and n^2 edges without a 3-cycle. A 3-cycle is a collection of 3 vertices x, y, z, such that xy, yz, and zx are all edges in the graph.

Putnam 1952/B6. A, B, C are points of a fixed ellipse E. Show that the area of ABC is maximized if and only if the centroid of ABC is at the center of E. The centroid of a triangle is its center of mass, which also happens to lie at the intersection of its three medians.

Putnam 1964/B6. D is a disk. Show that we cannot find congruent sets A, B with $A \cap B = \emptyset$, and $A \cup B = D$. More formally, D is the closed unit disk, including boundary, i.e., all points (x, y) satisfying $x^2 + y^2 \leq 1$. We must show that it is impossible to choose a subset A of D such that there is a geometric transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is a bijection from A to $D \setminus A$. A geometric transformation is a composition of rotations, translations, and reflections.

Putnam 1958/B5. S is an infinite set of points in the plane. The distance between any two points of S is integral. Prove that S is a subset of a straight line.

Putnam 1964/A6. S is a finite set of collinear points. Let k be the maximum distance between any two points of S. Given a pair of points of S a distance $d < k$ apart, we can find another pair of points of S also a distance d apart. Prove that if two pairs of points of S are distances a and b apart, then a/b is rational.

Putnam 1964/B3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that for each $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. (That limit corresponds to sending evaluating $f(\alpha), f(2\alpha), f(3\alpha), \ldots$ and finding the limit of the sequence.) Prove that $\lim_{x\to\infty} f(x) = 0$, where now this limit corresponds to sending x to ∞ along the real axis. That is, for every $\epsilon > 0$, there is a T such that for all real numbers $x > T$, we have $|f(x)| < \epsilon$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.

3. Polynomials

Po-Shen Loh

CMU Putnam Seminar, Fall 2012

1 Classical results

1. Find a nice expression for the derivative of the polynomial $(x-1)(x-2)(x-3)^2$.
2. Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial which satisfies $p(-x) = p(x)$ for every real x. Prove that $a_i = 0$ for every odd i.

2 Problems

Putnam 1958/A1. Show that the real polynomial $\sum_0^n a_i x^i$ has at least one real root if $\sum \frac{a_i}{i+1} = 0$.

Putnam 1959/A1. Prove that we can find a real polynomial $p(y)$ such that $p(x-1/x) = x^n - 1/x^n$ (where n is a positive integer) iff n is odd.

Putnam 1938/A3. The roots of $x^3 + ax^2 + bx + c = 0$ are α, β, and γ. Find the cubic whose roots are α^3, β^3, and γ^3.

Putnam 1940/A6. Let $p(x)$ be a polynomial with real coefficients, and let $r(x)$ be the polynomial defined by the derivative $r(x) = p'(x)$. Suppose that there are positive integers a and b for which $r^a(x)$ divides $p^b(x)$ as polynomials. Prove that for some real numbers A and α, and for some integer n, we have $p(x) = A(x-\alpha)^n$.

Putnam 1947/B4. $p(z) = z^2 + az + b$ has complex coefficients. $|p(z)| = 1$ on the unit circle $|z| = 1$. Show that $a = b = 0$.

Putnam 1956/B7. Let $p(z)$ and $q(z)$ be complex polynomials with the same set of roots (but possibly different multiplicities). Suppose that $p(z)+1$ and $q(z)+1$ also have the same set of roots. Show that $p(z) = q(z)$.

Putnam 1957/A4. Let $p(z)$ be a polynomial of degree n with complex coefficients. Its roots (in the complex plane) can be covered by a disk of radius r. Show that for any complex k, the roots of $np(z) - kp'(z)$ can be covered by a disk of radius $r + |k|$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.

PUTNAM PROBLEM SOLVING SEMINAR
WEEK 7: GRAPH THEORY

DIANE MACLAGAN

The Rules. There are way too many problems here to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Try to turn the first problems into graph problems (if they don't start that way). Think about degrees of vertices. Generalize. Try small and extreme cases. Look for patterns. Use induction. Eat pizza. Work in groups. Use lots of paper. Talk it over. Choose effective notation. Don't give up after five minutes.

The Problems. *The problems are VERY APPROXIMATELY ordered from "easiest" to "hardest."*

1. At a dinner party people shake hands as they are introduced. Not everyone shakes hands with everyone else (some of them already know each other!). Show that there have to be two people who shake hands the same number of times. Show that the number of people who have shaken hands an odd number of times is even.

2. The *adjacency matrix* of a graph G with n vertices is the $n \times n$ matrix A with $A_{ij} = 1$ if there is an edge joining v_i and v_j, and $A_{ij} = 0$ otherwise. What does A^2 mean? A^k?

3. A graph G is *simple* if it has no loops or multiple edges. It is *connected* if you can walk from any vertex to any other vertex along edges. If $\{v_1, \ldots, v_k\}$ is a subset of the vertices of G, the induced subgraph is the graph whose vertices are $\{v_1, \ldots, v_k\}$ and whose edges are the edges of G with both endpoints in $\{v_1, \ldots, v_k\}$.

(1) Let G be a simple graph with no isolated vertices (vertices of degree 0) and no induced subgraphs with exactly two edges. Show that G is the complete graph.
(2) The graph C_4 is the cycle with four vertices (a square). The graph P_4 is the path with four vertices (a line). Let G be a connected simple graph that does not have P_4 or C_4 as induced subgraphs. Prove that G has a vertex adjacent to all other vertices.

4. In the country of Jetlaggia it is possible to travel by air between any two of the main cities; if there is not a direct flight there is at least an indirect flight passing through other cities on the way. A *path* is an air route between two different cities that passes through no intermediate (or start or end) city more than once. The *length* of a path is the total number of cities on it, counting its endpoint but not its starting point. Let M be the maximum of

Date: Monday, December 1, 2003.

all path lengths in Jetlaggia. Prove that any two paths of length M must have at least one city in common.

5. How many people do we need to have at a party to ensure that there are always at least three people all of whom know each other, or three people none of whom know each other? What if three is changed to four? (Hint below).

6. Prove that for any five points in the plane with no three on a line, there are always four which form a convex quadrilateral without the fifth point in the interior.

This is known as the "Happy Ending Problem". It was first observed by Esther Klein, and was generalized later by George Szekeres and Paul Erdos showing that for every k there is some number N for which if there are more than N points in the plane with no three on a line then there is some convex k-gon. The "Happy Ending" is that Klein and Szekeres later married.

7. A set of 1990 people is divided into non-intersecting subsets in such a way that

(1) no one in a subset knows all the others in the subset;
(2) among any three people in a subset there are always at least two who do not know each other;
(3) for any two people in a subset who do not know each other, there is exactly one person in the same subset knowing both of them.

Prove that within each subset every person has the same number of acquaintances. What is the maximum possible number of subsets? (APMO 1990)

8. *Instant Insanity* is a puzzle consisting of four cubes with faces coloured red, blue, green and yellow. Each cube has at least one face of each colour, and the puzzle is to stack them in a tower so that all four colours appear on all four sides of the tower. The specific colours are in Figure 1 below.

9. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is 1.

10. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares. (Putnam, 2001). (Hint below).

Hints:

- For question 5 with four the answer is $\frac{1}{2}(2003 - 1967)$ (obscured so you can't look at it by accident).

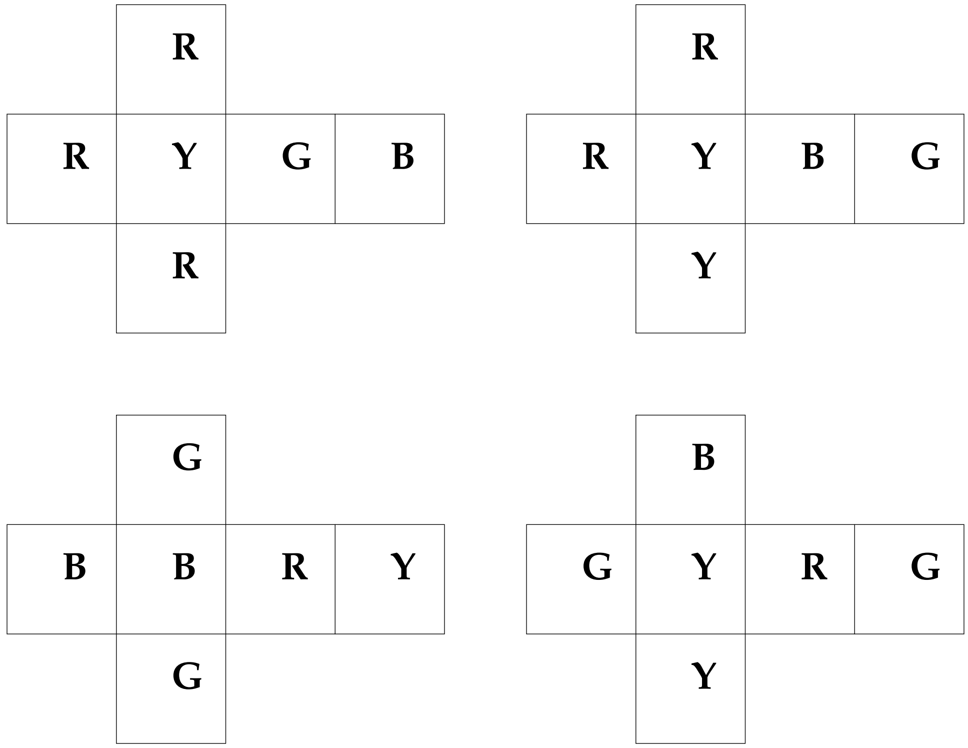


FIGURE 1. Nets for the cubes in the Instant Insanity puzzle

- For question 10, once you've turned it into graph theory, find out what we know about matchings.

A1/A2/B1/B2 Problems from the 1998–2000 Putnams

As a final practice for the Putnam, here are some actual Putnam problems.

1998A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1998A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

1998B1. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

1998B2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1999A1. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

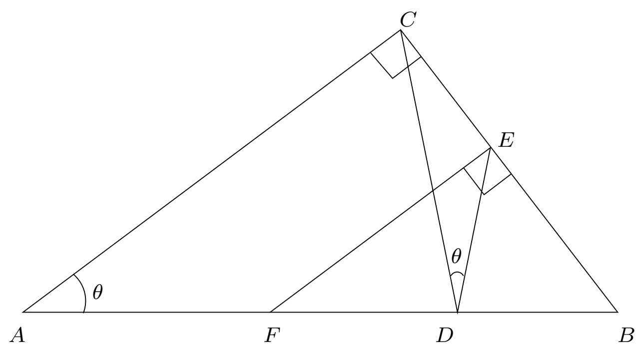
1999A2. Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

1999B1. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$. [Here $|PQ|$ denotes the length of the line segment PQ.]

1999B2. Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots. [The roots may be either real or complex.]

2000A1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?



2000A2. Prove that there exist infinitely many integers n such that n, $n+1$, $n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

2000B1. Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

2000B2. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

This handout can be found at

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4. Calculus

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CMU Putnam Seminar, Fall 2012

1 Classical results

1. Let $f : [0, 1] \to \mathbb{R}$ be a monotone increasing function, and let $g : [0, 1] \to \mathbb{R}$ be a monotone decreasing function. Show that $\int_0^1 f(x)g(x)dx \le \int_0^1 f(x)dx \int_0^1 g(x)dx$, i.e., that the expected value of the product of two negatively correlated random variables is at most the product of their expected values.

2 Problems

Putnam 1946/A2. Given functions $f, g : \mathbb{R} \to \mathbb{R}$, and $x \in \mathbb{R}$, let $I(fg)$ denote the function which maps x to $\int_1^x f(t)g(t)dt$. Prove that whenever $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are real polynomials, the polynomial

$$I(ac)I(bd) - I(ad)I(bc)$$

is divisible by $(x - 1)^4$.

Putnam 1947/B1. Let $f : [1, \infty) \to \mathbb{R}$ be a differentiable function which satisfies $f'(x) = \frac{1}{x^2+f(x)^2}$ and $f(1) = 1$. Show that as $x \to \infty$, $f(x)$ tends to a limit which is less than $1 + \frac{\pi}{4}$.

Putnam 1958/A5. Show that there is at most one continuous function $f : [0, 1]^2 \to \mathbb{R}$ satisfying $f(x, y) = 1 + \int_0^x \int_0^y f(s, t)dtds$.

Putnam 1958/B4. Let S be a spherical shell of radius 1, i.e., the set of points satisfying $x^2 + y^2 + z^2 = 1$. Find the average straight line distance between two points of S.

Putnam 1946/A1. Let $p(x)$ be a real polynomial of degree at most 2, which satisfies $|p(x)| \le 1$ for all $-1 \le x \le 1$. Show that $|p'(x)| \le 4$ for all $-1 \le x \le 1$.

Putnam 1947/B2. Let K be a positive real number, and let $f : [0, 1] \to \mathbb{R}$ be a differentiable function whose derivative satisfies $|f'(x)| \le K$ for all $0 \le x \le 1$. Prove that

$$\left| \int_0^1 f(x)dx - \sum_{i=1}^n \frac{f(i/n)}{n} \right| \le \frac{K}{n} .$$

Putnam 1957/B3. Let $f : [0, 1] \to \mathbb{R}^+$ be a monotone decreasing continuous function. Show that

$$\int_0^1 f(x)dx \int_0^1 xf(x)^2dx \le \int_0^1 xf(x)dx \int_0^1 f(x)^2dx .$$

Putnam 1958/B7. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which satisfies $\int_0^1 x^n f(x)dx = 0$ for all non-negative integers n. Prove that f is the zero function.

6. Inequalities

Po-Shen Loh

CMU Putnam Seminar, Fall 2012

1 Classical results

Smoothing principle. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then if $x + y = x' + y'$ but x' and y' are closer together, we have

$$f(x') + f(y') \le f(x) + f(y) .$$

Furthermore, if f is strictly convex, then the inequality is strict.

Jensen. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$,

$$f\left(\frac{a_1 + \cdots + a_n}{n}\right) \le \frac{f(a_1) + \cdots + f(a_n)}{n} .$$

Compactness. If D is a compact set and $f : D \to \mathbb{R}$ is continuous, then f achieves a maximum on D, i.e., there is at point $x \in D$ such that for all $y \in D$, $f(x) \ge f(y)$.

AM-GM. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Then

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + \cdots + a_n}{n} ,$$

with equality if and only if all a_i are equal.

Cauchy-Schwarz. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then

$$\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) ,$$

with equality only if the sequences $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional.

Dirichlet approximation. For any real number r and any positive integer N, there are integers a and b with $1 \le b \le N$ which satisfy

$$\left|r - \frac{a}{b}\right| < \frac{1}{b^2} .$$

2 Problems

Putnam 1950/B1. Let P_1, P_2, $\ldots$, P_n be points on a line, not necessarily distinct. Which points P on the line minimize the sum of distances $\sum_i |PP_i|$?

Irish Olympiad 1998/7a. Prove that for all positive real numbers a, b, c, the following holds:

$$\frac{9}{a+b+c} \le 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) .$$

Putnam 1940/B7. Given $n > 8$, let $a = \sqrt{n}$ and $b = \sqrt{n+1}$. Which is greater, a^b or b^a?

Putnam 1951/B3. Show that $\log\left(1 + \frac{1}{x}\right) > \frac{1}{1+x}$ for $x > 0$.

Putnam 1946/A1. Let $p(x)$ be a real polynomial of degree at most 2, which satisfies $|p(x)| \leq 1$ for all $-1 \leq x \leq 1$. Show that $|p'(x)| \leq 4$ for all $-1 \leq x \leq 1$.

Putnam 1946/A4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in \mathbb{R}$. Show that f is constant.

Putnam 1949/B3. Let C be a closed plane curve with the property that every pair of points in C are at distance at most 1 apart. Show that we can find a disk of radius $\frac{1}{\sqrt{3}}$ which contains C.

Putnam 1947/B3. Let O be the origin $(0,0)$, and let C be the line segment $\{(x,y) : x \in [1,3], y = 1\}$. Let K be the curve $\{P : \text{for some } Q \in C,\ P \text{ lies on } OQ \text{ and } PQ = 0.01\}$. Let k be the length of the curve K. Is k greater or less than 2?

Putnam 1949/B1. Show that for any rational $0 < \frac{a}{b} < 1$, we have $\left|\frac{a}{b} - \frac{1}{\sqrt{2}}\right| > \frac{1}{4b^2}$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.

7. Convergence

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CMU Putnam Seminar, Fall 2012

1 Classical results

Monotonicity. Every bounded monotone real sequence $a_1, a_2, \ldots$ converges to a limit.

Cauchy sequence (definition). A sequence $a_1, a_2, \ldots$ is called a *Cauchy sequence* if for every $\epsilon > 0$, there is a positive integer N such that for all $i, j > N$, we have $|a_i - a_j| < \epsilon$. The real and complex number systems have the property that every Cauchy sequence converges to a limit, which is a number in the system.

Absolute convergence. Let $z_1, z_2, \ldots$ be a sequence of complex numbers, for which $\sum_i |z_i|$ converges. Then $\sum_i z_i$ converges as well.

Abel summation. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences, and let B_k denote $\sum_{i=1}^k b_i$ for every k. Then

$$\sum_{i=1}^{n} a_i b_i = a_n B_n - \sum_{i=1}^{n-1} B_i (a_{i+1} - a_i) .$$

Classical. Prove that the sequence $\sqrt{7}$, $\sqrt{7+\sqrt{7}}$, $\sqrt{7+\sqrt{7+\sqrt{7}}}$, $\ldots$ converges, and determine its limit. This is often denoted as $\sqrt{7+\sqrt{7+\sqrt{7+\cdots}}}$.

2 Problems

Putnam 1940/A7. Show that if $\sum_{i=1}^\infty a_i^2$ and $\sum_{i=1}^\infty b_i^2$ both converge, then so does $\sum_{i=1}^\infty (a_i - b_i)^p$, for every $p \geq 2$.

Putnam 1964/B1. Let $a_1, a_2, \ldots$ be positive integers such that $\sum \frac{1}{a_i}$ converges. For each n, let b_n denote the number of positive integers i for which $a_i \leq n$. Prove that $\lim_{n\to\infty} \frac{b_n}{n} = 0$.

Putnam 1951/A7. Let $a_1, a_2, \ldots$ be a sequence of real numbers for which the sum $\sum_{i=1}^\infty a_i$ converges. Show that the sum $\sum_{i=1}^\infty \frac{a_i}{i}$ also converges.

Putnam 1952/B5. Let a_i be a monotonically decreasing sequence of positive real numbers, for which $\sum_{i=1}^\infty a_i$ converges. Show that $\sum_{i=1}^\infty i(a_i - a_{i+1})$ also converges.

Putnam 1952/B7. Let α be an arbitrary real number. Define $a_1 = \alpha$, and for all $n \geq 1$, let $a_{n+1} = \cos a_n$. Prove that a_n converges to a limit, and that this limit does not depend on α.

Putnam 1953/A6. Prove that the sequence $\sqrt{7}$, $\sqrt{7-\sqrt{7}}$, $\sqrt{7-\sqrt{7+\sqrt{7}}}$, $\sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}$, $\ldots$, converges, and determine its limit.

Putnam 1949/A3. Let $z_1, z_2, \ldots$ be nonzero complex numbers with the property that $|z_i - z_j| > 1$ for all i, j. Prove that $\sum \frac{1}{z_i^3}$ converges.

Putnam 1949/B5. Let a_i be a sequence of positive real numbers. Show that $\limsup \left(\frac{a_1+a_{n+1}}{a_n}\right)^n \geq e$.

VTRMC 1998/5. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers, for which $\sum_{i=1}^{\infty} \frac{1}{a_i}$ converges. For every n, let $b_n = \frac{a_1+\cdots+a_n}{n}$. Show that $\sum_{i=1}^{\infty} \frac{1}{b_n}$ also converges.

3 Homework

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10. Combinatorics

Po-Shen Loh

CMU Putnam Seminar, Fall 2012

1 Classical results

Erdős-Ko-Rado. Let $\mathcal{F}$ be a family of k-element subsets of $\{1, 2, \ldots, n\}$, with the property that every pair of members of $\mathcal{F}$ has nonempty intersection. Then the size of $\mathcal{F}$ is at most $\binom{n-1}{k-1}$.

Lucas. Let n and k be non-negative integers, with base-p expansions $n = (n_t n_{t-1} \ldots n_0)_{(p)}$ and $k = (k_t k_{t-1} \ldots k_0)_{(p)}$, respectively. Then

$$\binom{n}{k} \equiv \binom{n_t}{k_t} \times \binom{n_{t-1}}{k_{t-1}} \times \cdots \times \binom{n_0}{k_0} \pmod{p}.$$

2 Problems

Putnam 1958/B2. Let X be a subset of $\{1, 2, 3, \ldots, 2n\}$ with $n + 1$ elements. Show that we can find $a, b \in X$ with a dividing b.

Putnam 1954/A2. Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = \frac{1}{\sqrt{2}}$. Is this result true for a smaller k?

Putnam 1964/B2. Let S be a finite set, and suppose that a collection $\mathcal{F}$ of subsets of S has the property that any two members of $\mathcal{F}$ have at least one element in common, but $\mathcal{F}$ cannot be extended (while keeping this property). Prove that $\mathcal{F}$ contains just half of the subsets of S.

Putnam 1957/B4. Show that the number of ways of representing n as an ordered sum of 1's and 2's equals the number of ways of representing $n + 2$ as an ordered sum of integers greater than 1. For example: $4 = 1+1+1+1 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2$ (5 ways) and $6 = 4+2 = 2+4 = 3+3 = 2+2+2$ (5 ways).

Putnam 1956/A7. Show that for any given positive integer n, the number of odd $\binom{n}{m}$ with $0 \le m \le n$ is a power of 2.

Putnam 1958/B6. A graph has n vertices $\{1, 2, \ldots, n\}$ and a complete set of edges. Each edge is oriented, as either $i \to j$ or $j \to i$. Show that we can find a permutation of the vertices a_i so that $a_1 \to a_2 \to a_3 \to \cdots \to a_n$.

Putnam 1958/B7. Let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, \ldots, n$. Call a_i a "big" integer if $a_i > a_j$ for all $j > i$. Find the mean number of "big" integers over all permutations on the first n integers.

Putnam 1958/B3. In a tournament of n players, every pair of players plays once. There are no draws. Player i wins w_i games. Prove that we can find three players i, j, k such that i beats j, j beats k and k beats i iff $\sum_{t=1}^{n} w_t^2 < \frac{(n-1)n(2n-1)}{6}$.

Putnam 1955/B5. Let n be a positive integer. Suppose we have an infinite sequence of 0's and 1's is such that it only contains n different blocks of n consecutive terms. Show that it is eventually periodic.

12. Integer polynomials

Po-Shen Loh

CMU Putnam Seminar, Fall 2012

1 Classical results

Vandermonde determinant. Let $a_0, a_1, \ldots, a_n$ be distinct numbers, and let $b_0, b_1, \ldots, b_n$ be arbitrary (possibly equal to each other or to any of the a_i). Then there is a unique polynomial $p(x) = c_n x^n + \cdots + c_0$ such that $p(a_i) = b_i$ for all $0 \le i \le n$.

Lagrange interpolation. An expression for the above polynomial is

$$p(x) = \sum_{i=0}^{n} \frac{b_i}{\prod_{j \neq i}(a_i - a_j)} \prod_{j \neq i}(x - a_j) .$$

Fermat's Last Theorem. The equation $x^n + y^n = z^n$ has no positive integer solutions (x, y, z, n) with $n \ge 3$.

2 Problems

Putnam 1940/A1. Let $p(x)$ be a polynomial with integer coefficients. Suppose that for some positive integer c, none of $p(1), p(2), \ldots, p(c)$ are divisible by c. Prove that $p(b)$ is not zero for any integer b.

Putnam 1947/B5. Let $p(x)$ be the polynomial $(x-a)(x-b)(x-c)(x-d)$. Assume $p(x) = 0$ has four distinct integral roots and that $p(x) = 4$ has an integral root k. Show that k is the mean of a, b, c, d.

Putnam 1953/B2. Let $p(x)$ be a real polynomial of degree n such that $p(m)$ is integral for all integers m. Show that if k is a coefficient of $p(x)$, then $n!k$ is an integer.

Putnam 1940/B5. Find all rational triples (a, b, c) for which a, b, c are the roots of $x^3 + ax^2 + bx + c = 0$.

Putnam 1955/A6. For what positive integers n does the polynomial $p(x) = x^n + (2+x)^n + (2-x)^n$ have a rational root?

Putnam 1950/A6. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and suppose that each a_n is 0 or 1.

(1) Show that if $f(1/2)$ is rational, then $f(x)$ has the form $p(x)/q(x)$ for some integer polynomials $p(x)$ and $q(x)$.

(2) Show that if $f(1/2)$ is not rational, then $f(x)$ does not have the form $p(x)/q(x)$ for any integer polynomials $p(x)$ and $q(x)$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.

Homework 1
Solutions

Problem 1. Let n, k be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

(IMC B1, 2008)

Proof. Let $f(x) = x^{2k} - x^k + 1$, $g(x) = x^{2k} + x^k + 1$ and $h(x) = x^{2n} + x^n + 1$.

Using De Moivre's formula ($(\cos x + i \sin x)^n = \cos nx + i \sin nx$) notice that $x_1 := \cos \frac{\pi}{3k} + i \sin \frac{\pi}{3k}$ is a root of $f(x)$, so $h(x_1) = 0$ (since $f(x)|h(x)$). Letting $\alpha = \frac{\pi n}{3k}$ we obtain:

$$0 = x_1^{2n} + x_1^n + 1 = (\cos 2\alpha + i \sin 2\alpha) + (\cos \alpha + i \sin \alpha) + 1,$$

which is equivalent to $(2\cos\alpha + 1)(\cos\alpha + i \sin\alpha) = 0$, i.e., $\alpha = \pm\frac{2\pi}{3} + 2\pi a$, where $a \in \mathbb{Z}$.

Moreover, $g(x) = \frac{x^{3k}-1}{x^k-1}$ so every root of $g(x)$ is of the form $x_2 = \cos\frac{2\pi t}{3k} + i \sin\frac{2\pi t}{3k}$, where $t = 3b \pm 1$, $b \in \mathbb{Z}$. We want to prove that any root x_2 of g is also a root of h. This follows from:

$$\begin{aligned} h(x_2) = x_2^{2n} + x_2^n + 1 &= (\cos 4\alpha t + i \sin 4\alpha t) + (\cos 2\alpha t + i \sin 2\alpha t) + 1 \\ &= (2\cos 2\alpha t + 1)(\cos 2\alpha t + i \sin 2\alpha t), \end{aligned}$$

and $2\cos 2\alpha t + 1 = 2\cos 2(\pm\frac{2\pi}{3} + 2\pi a)t + 1 = 2\cos\frac{4\pi t}{3} + 1 = 2\cos\frac{4\pi}{3}(3b \pm 1) + 1 = 0$.

□

Problem 2. Find all polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ ($a_n \neq 0$) satisfying the following two conditions:

- $\{a_0, a_1, \ldots, a_n\}$ is a permutation of the set $\{0, 1, \ldots, n\}$;
- all the roots of $P(x)$ are rational numbers.

(IMC A4, 2005)

Solution. The coefficients of P are non-negative, thus $P(x) > 0$ for every $x > 0$, implying that P does not have any positive roots. Since P has only rational roots, we can write it as $P(x) = \prod_{k=1}^{n}(b_k x + c_k)$, where $b_k > 0$, $c_k \geq 0$ are integers. If $c_k = 0$ for two different values of k then we'd get that $a_0 = a_1 = 0$. Thus $c_k > 0$ in at least $n-1$ cases; so if we plug in $x = 1$:

$$P(1) = a_n + \ldots + a_0 = \frac{n(n+1)}{2} = \prod_{k=1}^{n}(b_k + c_k) \geq 2^{n-1}.$$

The inequality $n(n+1) \geq 2^n$ forces $n \leq 4$. In addition, if $a_0 \neq 0$ then $a_k = 0$ for some $1 \leq k \leq n-1$, and by Viete's formula we get that

$$\sum_{1 \leq i_1 < i_{n-k} \leq n} \frac{c_{i_1}}{b_{i_1}} \cdots \frac{c_{i_{n-k}}}{b_{i_{n-k}}} = \frac{a_k}{a_n} = 0,$$

which is a contradiction, since the left side is positive. Consequently, $a_0 = 0$.

An easy check shows that there are only 5 possible polynomials: $P(x) = x$, $P(x) = x^2 + 2x$, $P(x) = 2x^2 + x$, $P(x) = x^3 + 3x^2 + 2x$, and $P(x) = 2x^3 + 3x^2 + x$. □

Problem 3. Suppose that a, A, b, B, c, C are real numbers, $a, A \neq 0$, such that for all $x \in \mathbb{R}$

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|. \tag{1}$$

Prove that

$$|b^2 - 4ac| \leq |B^2 - 4AC|.$$

(Putnam A4, 2003)

Proof. WLOG, assume that $a > 0$ (otherwise we can replace (a, b, c) with $(-a, -b, -c)$, in which cases the quantities $|ax^2 + bx + c|$ and $|b^2 - 4ac|$ do not change), and similarly $A > 0$. Since $\lim_{x\to\infty} |ax^2 + bx + c|/x^2 = a$ and $\lim_{x\to\infty} |Ax^2 + Bx + C|/x^2 = A$, we get that $A \geq a > 0$.

Let $d = b^2 - 4ac$ and $D = B^2 - 4AC$. We distinguish 3 cases:

(i) If $D > 0$ then Ax^2+Bx+C has two distinct real roots r_1 and r_2, which by (1) must also be the two roots of $ax^2 + bx + c$ (in particular $d > 0$). Then $D^2 = A^2(r_1 - r_2)^2 \geq a^2(r_1 - r_2)^2 = d^2$, and hence $D \geq d$.

(ii) If $D \leq 0$ and $d \leq 0$, then (1) reads as $Ax^2 + Bx + C \geq ax^2 + bx + c$, for all x. Taking $x = \frac{-B}{2A}$ gives $-\frac{D}{4A} = Ax^2 + Bx + C \geq ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{d}{4a} \geq \frac{-d}{4a}$. Thus $-d \leq -\frac{a}{A}D \leq -D$.

(iii) If $D \leq 0$ and $d > 0$ then by (1): $Ax^2 + Bx + C \geq \pm(ax^2 + bx + c)$, for all x. Then $(A\pm a)x^2+(B\pm b)x+(C\pm c) \geq 0$, for all x. Therefore $(B+b)^2 \leq 4(A+a)(C+c)$ and $(B-b)^2 \leq 4(A-a)(C-c)$. Summing up the previous two relations yields $2(B^2-4AC)+2(b^2-4ac) \leq 0$, and hence $-D \geq d$.

□

Problem 4. Say that a polynomial with real coefficients in two variables x, y is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

(Putnam B4, 2009)

Solution. The average value of a polynomial $P(x, y)$ around a circle of radius r can be written is polar coordinates as

$$I(P, r) = \frac{1}{2\pi} \int_0^{2\pi} P(r\cos\theta, r\sin\theta)d\theta,$$

which is a polynomial in r of degree n, where $n = \deg P$. Accordingly, it makes sense to split the polynomial into its homogeneous parts.

Let H_i be the space of homogeneous polynomials in (x, y) of degree i, for $0 \leq i \leq 2009$. A polynomial $P = \sum_{i=0}^{2009} P_i$ with $P_i \in H_i$ is balanced iff $I(P, r) = 0$ for each $r > 0$, or equivalently

$$\int_0^{2\pi} \sum_{i=0}^{2009} P_i(r\cos\theta, r\sin\theta)d\theta = 0 \Longleftrightarrow \sum_{i=0}^{2009} \left(\int_0^{2009} P_i(\cos\theta, \sin\theta)d\theta \right) r^i = 0.$$

A polynomial in r is identically 0 only when all its coefficients are 0, thus P is balanced iff

$$\int_0^{2009} P_i(\cos\theta, \sin\theta)d\theta = 0, \forall\ 0 \leq i \leq 2009.$$

Consider the linear transformation $L_i : H_i \to \mathbb{R}$ defined by

$$L_i(f) = \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta.$$

If $V_i = \ker L_i$ then $V = \bigoplus_{i=0}^{2009} V_i$. Note that $\dim H_i = i + 1$ as $\{x^i, x^{i-1}y, \ldots, y^i\}$ is a spanning set. Now, when i is odd then $L_i(x^j y^{i-j}) = 0$, so $\dim V_i = \dim H_i = i + 1$. However, when i is even we can find a polynomial with non-zero image (for example, $f(x, y) = (x^2 + y^2)^{i/2}$), so by the nullity-rank theorem $\dim V_i = \dim H_i - 1 = i$. In conclusion,

$$\dim V = \sum_{i=0}^{2009} \dim V_i = \sum_{i=0}^{2009} (i + 1) - \sum_{\substack{i=0 \\ i\text{-even}}}^{2009} 1 = \frac{2010 \cdot 2011}{2} - \frac{2010}{2} = 2020050.$$

$\square$

Homework 2
Solutions

Problem 1. (A1/2008) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(x,y) + f(y,z) + f(z,x) = 0, \ \forall \ x,y,z \in \mathbb{R}.$$

Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x,y) = g(x) - g(y), \forall \ x,y \in \mathbb{R}.$$

Proof. Plugging in $x = y = z = 0$ shows that $f(0,0) = 0$. Now, if $y = z = 0$ we get $f(x,0) = -f(0,x)$. Thus, for $z = 0$ we obtain

$$f(x,y) = -f(y,0) - f(0,x) = f(x,0) - f(y,0),$$

so we can define $g(x) = f(x,0)$. □

Problem 2. (IMC A1/2006) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove or disprove each of the following statements:

(a) If f is continuous and range$(f) = \mathbb{R}$ then f is monotonic.

(b) If f is monotonic and range$(f) = \mathbb{R}$ then f is continuous.

(c) If f is monotonic and f is continuous then range$(f) = \mathbb{R}$.

Solution. (a) False. For example, take $f(x) = x^3 - x$. Clearly f is continuous and range$(f) = \mathbb{R}$, but $f(0) > f(\frac{1}{2}) < f(1)$, which shows that f is not monotonic.

(b) True. WLOG, we can assume that f is increasing (otherwise, one can consider $-f$). Fix $a \in \mathbb{R}$, then $\lim_{a^-} f \leq \lim_{a^+} f$. If the two limits are equal then the function is continuous at a. Otherwise, if $\lim_{a^-} f = b < \lim_{a^+} f = c$ then $f(x) \geq c$, for all $x > a$ and $f(x) \leq b$, for all $x < a$. In that case range$(f) \subset (-\infty, b) \cup \{f(a)\} \cup (c, \infty)$, which is not all of $\mathbb{R}$.

(c) False. The function $f(x) = \arctan(x)$ is monotonic and continuous, but range$(f) = (-\frac{\pi}{2}, \frac{\pi}{2}) \neq \mathbb{R}$. □

Problem 3. (IMC B1/2007) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b ?

Solution. It does not. For example, if $f(x) = e^x$ then the translation $(x,y) \to (x - \ln c, y)$ maps the graph of f to the graph of $cf(x) = e^{x + \ln c}$. □

Problem 4. (IMC A1/2008) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational whenever $x - y$ is rational.

Solution. Consider $g(x) = f(x+1) - f(x)$. This is a continuous function which takes only rational values. We claim that g is constant. Indeed, if g takes two distinct values a and b ($a < b$) then we can find and irrational number $s \in (a, b)$ (recall that the irrationals are dense in $\mathbb{R}$). Then by the Intermediate Value Theorem $\exists\, x_0$ such that $g(x_0) = s$, which is a contradiction. Consequently, $g(x)$ is constant.

Set $f(x+1) - f(x) = f(1) - f(0) = r \in \mathbb{Q}$ and $f(0) = s$. Then $f(1) = r + s$ and by induction $f(n) = nr + s$, for every positive integer n. Similarly, $f(-1) = -r + s$ and $f(-n) = -nr + s$.

By the same reasoning, for every $q = \frac{m}{n} \in \mathbb{Q}$ the function $f(x+q) - f(x)$ is constant (depending on q), say c. As above, one can show that $f(kq) = kc + s$, $\forall\, k \in \mathbb{Z}$. In particular, $f(m) = f(nq) = nc + s$. However $f(m) = mr + s$, implying that $c = \frac{mr}{n}$. Therefore $f(\frac{m}{n}) = \frac{m}{n} r + s$, or equivalently, $f(x) = xr + s$, $\forall\, x \in \mathbb{Q}$. The continuity of f extends this formula for all $x \in \mathbb{R}$.

Finally, it is easy to check that all functions f of the form $f(x) = rx + s$, where $r \in \mathbb{Q}$ and $s \in \mathbb{R}$, satisfy the statement.

$\square$

Homework 4
Solutions

Problem 1. (Putnam B1, 2010) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \ldots = m$$

for every positive integer m?

Solution. If there were such a sequence, then the equation would also hold for $m = 2, 3, 4$. In that case

$$\begin{aligned} 0 &= 4 - 2 \cdot 3 + 2 \\ &= \sum_{i=1}^{\infty} a_i^4 - 2 \cdot \sum_{i=1}^{\infty} a_i^3 + \sum_{i=1}^{\infty} a_i^2 \\ &= \sum_{i=1}^{\infty} a_i^2 (a_i - 1)^2. \end{aligned}$$

Thus for each i, $a_i = 0$ or $a_i = 1$, so $a_i = a_i^2$. However, $1 = \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_i^2 = 2$ is a contradiction. □

Problem 2. (Putnam A3, 1999) Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

Proof. Clearly

$$\frac{1}{1 - 2x - x^2} = \frac{1}{2\sqrt{2}} \left(\frac{1 + \sqrt{2}}{1 - (1 + \sqrt{2})x} - \frac{1 - \sqrt{2}}{1 - (1 - \sqrt{2})x} \right) \text{ and } \frac{1}{1 - (1 \pm \sqrt{2})x} = \sum_{n=0}^{\infty} (1 \pm \sqrt{2})^n x^n.$$

Therefore

$$a_n = \frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1} \right).$$

Then it is straightforward to check that $a_n^2 + a_{n+1}^2 = a_{2n+2}$. □

Problem 3. (Putnam A3, 2004) Define a sequence $\{u_n\}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!.$$

Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Proof. We show by induction on $n \geq 3$ that

$$u_n = (n-1)!! := \begin{cases} (n-1)(n-3)\dots 3\cdot 1 & \text{if } n \text{ is even;} \\ (n-1)(n-3)\dots 4\cdot 2 & \text{if } n \text{ is odd.} \end{cases}$$

This is clearly true for the first several cases $u_3 = 2$, $u_4 = 3$, $u_5 = 4\cdot 2$, $u_6 = 5\cdot 3$. Assume that $u_n = (n-1)!!$, $u_{n+1} = n!!$ and $u_{n+2} = (n+1)!!$, then

$$u_{n+3} = \frac{n! + u_{n+1}u_{n+2}}{u_n} = \frac{n! + n!!(n+1)!!}{(n-1)!!} = \frac{(n-1)!!n!! + n!!(n-1)!!(n+1)}{(n-1)!!} = n!!(n+2) = (n+2)!!.$$

□

Problem 4. (Putnam B3, 2001) For every positive integer n let $\langle n\rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n}.$$

Solution. Note that $\langle n\rangle = k$ iff $k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2} \iff (k-1)k < n \leq k(k+1)$, implying that

$$n \in \{k^2 - k + 1, \dots, k^2 + k\}.$$

Thus the set

$$A := \{n - \langle n\rangle : n \geq 1\} = \{(k-1)^2, \dots k^2 : k \geq 1\}$$

contains all non-negative integers with the positive squares repeated twice. Similarly, the set

$$B := \{n + \langle n\rangle : n \geq 1\} = \{k^2 + 1, \dots (k+1)^2 - 1 : k \geq 1\}$$

contains all the positive integers with the positive squares omitted. Therefore every positive integer appears exactly twice in $A \cup B$, so

$$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^{n-\langle n\rangle}} + \sum_{n=1}^{\infty} \frac{1}{2^{n+\langle n\rangle}} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{2^n} = 3.$$

□

Homework 6
Solutions

Problem 1. Consider $S = \{1, 2, \ldots, 2n\}$ and let $M \subset S$ be a subset of cardinality $|M| = n + 1$. Prove that

(a) there exist two elements $a, b \in M$ such that a and b are relatively prime.

(b) (B2/1958) there exist two distinct elements $a, b \in M$, such that a divides b.

Is the above still true if $|M| = n$?

Solution. (a) Consider the sets $M_k = \{2k-1, 2k\}$, for $k = 1, 2, \ldots, n$. Since these n sets partition S, the Pigeonhole Principle implies that M contains at least one of them, say T_k. But then M contains two consecutive integers (the elements of T_k), which are relatively prime.

If $|M| = n$ a possible counterexample is $M = \{2, 4, \ldots, 2n\}$, the subset of even numbers.

(b) Consider the sets $T_k = S \cap \{2^i(2k-1) \mid i = 0, 1, 2, \ldots\}$, for $k = 1, 2, \ldots, n$. Every element $s \in S$ can be uniquely written as $s = 2^i \cdot (2k-1)$, for some nonnegative integer i and some positive integer $1 \leq k \leq n$. Hence $T_1, \ldots, T_n$ partition S. By the Pigeonhole Principle $M \cap T_k$ contains at least two distinct elements (say a and b, $a < b$) for some k. Then $a = 2^i(2k-1)$ and $b = 2^j(2k-1)$, so a divides b.

If $|M| = n$ a possible counterexample is $M = \{n+1, n+2, \ldots, 2n\}$.

□

Problem 2. (A1/2003) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$?

For example, if $n = 4$, there are four ways: $1+1+1+1$, $1+1+1$, $2+2$, 4.

Solution. Answer: there are n ways.

We use induction on n. For $n = 1, 2$ the statement is obvious, so assume $n \geq 3$. Note that every solution $(a_1, a_2, \ldots, a_k)$ for n corresponds to a solution $(a_2, \ldots, a_k, a_1 + 1)$ for $n + 1$, where $a_1 + 1 > 1$. Conversely, every solution $(b_1, b_2, \ldots, b_k)$ with $b_k > 1$ for $(n+1)$ corresponds to a solution $(b_k - 1, b_1, \ldots, b_{k-1})$ for n. Thus there is a one-to-one correspondence between the solutions for n and the solutions for $n + 1$ in which not all of the a_i's are 1 (and this gives one additional solution). Consequently, by induction, the number of solutions for $n + 1$ is $n + 1$.

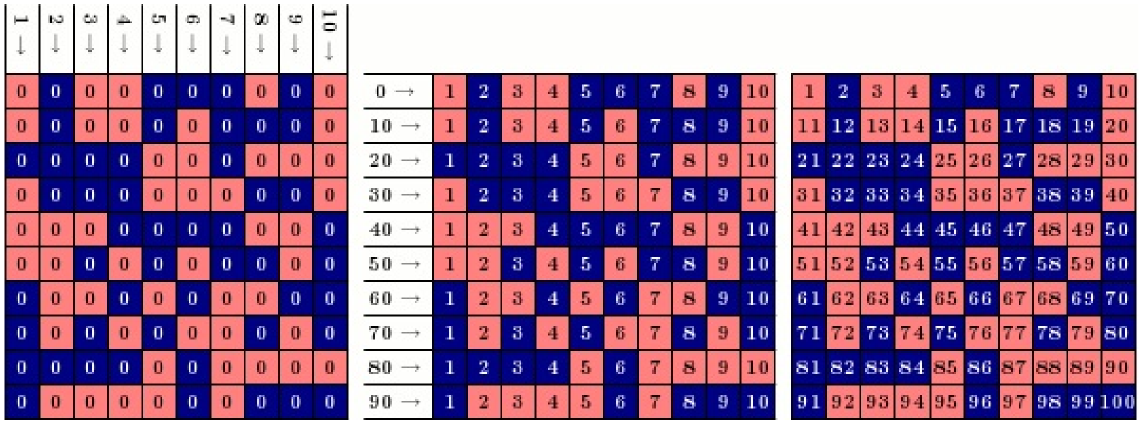
□

Problem 3. (B1/2001) Let n be an *even* positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of a $n \times n$ grid, from left to right, so that the k-th row is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (e.g., a checkerboard coloring is one possibility). Prove that for any coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Solution (by Zachary Abel). Start with a $n \times n$ grid that has zero written in every square. Consider an arbitrary coloring of the squares of the grid so that half of the squares in each row and in each column are red and the other half are black. Obviously the sum of the numbers on the red squares equals the sum of the numbers on the black squares (they are both zero). Now notice that by adding j, $1 \leq j \leq n$ to every square of column j the two sums remain equal. After we have done that, we can similarly modify the rows: add $(k-1)n$ to every square of row k. Here is an example for $n = 10$:



In the end we obtain the "original" grid from the statement, and at each step (starting with the "zero" grid) the two sums are equal. □

Problem 4. (B1/1996) Define a *selfish* set to be a set which has its own cardinality as an element. Find the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Solution. Let $f(n)$ be the number of minimal selfish subsets of $\{1, 2, \ldots, n\}$. We'll show that $f(n)$ is the n-th Fibonacci number.

Note that a selfish set is minimal iff its smallest element is equal to its cardinality (if the set contains an element smaller than its cardinality, one could obtain a smaller selfish subset by deleting some elements). Hence every minimal selfish subset of $\{1, 2, \ldots, n\}$ of cardinality $k \geq 1$, is obtained by choosing a $k-1$-element subset of $\{k+1, k+2, \ldots, n\}$ and adding $\{k\}$ to it, which can be done in $\binom{n-k}{k-1}$ ways (in particular, this shows that there are no selfish sets of size $k > \lfloor \frac{n+1}{2} \rfloor$).

Now if $n = 2t$ then

$$f(n) + f(n+1) = \sum_{k=1}^{t} \binom{n-k}{k-1} + \sum_{k=1}^{t+1} \binom{n+1-k}{k-1}$$
$$= \left(\sum_{k=2}^{t+1} \binom{2t+1-k}{k-2} + \sum_{k=2}^{t+1} \binom{2t+1-k}{k-1} \right) + 1$$
$$= \sum_{k=2}^{t+1} \binom{2t+2-k}{k-1} + 1$$
$$= \sum_{k=1}^{t+1} \binom{n+2-k}{k-1} = f(n+2).$$

Similarly, if $n = 2t+1$ then

$$f(n) + f(n+1) = \sum_{k=1}^{t+1} \binom{n-k}{k-1} + \sum_{k=1}^{t+1} \binom{n+1-k}{k-1}$$
$$= \left(\sum_{k=2}^{t+2} \binom{2t+2-k}{k-2} + \sum_{k=2}^{t+1} \binom{2t+2-k}{k-1} \right) + 1$$
$$= 1 + \sum_{k=2}^{t+1} \binom{2t+3-k}{k-1} + 1$$
$$= \sum_{k=1}^{t+2} \binom{n+2-k}{k-1} = f(n+2),$$

and since $f(1) = f(2) = 1$ we conclude that $f(n)$ is the n-th Fibonacci number. □

Homework 7
Solutions

Problem 1. (Putnam B2/2000) Prove that

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all integers $1 \le m \le n$.

Proof. Let $d = \gcd(m,n)$ we claim that $\exists\ x, y \in \mathbb{Z}$ such that $xm + yn = d$. Indeed, write $m = dm'$ and $n = dn'$, with $\gcd(m', n') = 1$. Consider the numbers $i \cdot n'$ for $i = 1, 2, \ldots, (m'-1)$. Clearly, all these numbers have different remainders modulo m' (if $i_1 n' \equiv i_2 n' \mod m'$ then $m' | n'(i_1 - i_2)$ and so $i_1 = i_2$), and none of these remainders is 0. Thus, except for 0, we get a complete list of remainders modulo m'. In particular, 1 occurs in this list for some i, so that $in' \equiv 1 \mod m'$. In that case $in' - 1 = m'j$, for some $j \in Z$. Taking $y = i$, $x = -j$ and multiplying by d yields $xm + yn = d$.

Then
$$\frac{\gcd(m,n)}{n}\binom{n}{m} = \frac{d}{n}\binom{n}{m} = \frac{xm+yn}{n}\binom{n}{m} = x\binom{n-1}{m-1} + y\binom{n}{m} \in \mathbb{Z}.$$

□

Problem 2. (Putnam B1/1969) Let n be a positive integer such that $n+1$ is divisible by 24. Prove that the sum of all the divisors of n is divisible by 24.

Proof. Notice that we can group all the divisors of n into disjoint pairs of the form $(d, \frac{n}{d})$, with $d|n$. We claim that the sum of the numbers of every such pair (and thus the sum of all the divisors) is divisible by 24, i.e., for any divisor d of n we have: $24 | d + \frac{n}{d}$. We are given that the claim holds for $d = 1$, so we can assume that $d \ge 2$. Every perfect square is congruent to 0 or 1 modulo 3, and to $0, 1$ or 4 modulo 8. However, since $24 | n+1$ we get that $\gcd(24, n) = 1$ so $\gcd(24, d) = 1$ and, in particular, $d \not\equiv 0 \mod 3$ and $d \not\equiv 0 \mod 2$. Consequently, $d^2 \equiv 1 \mod 24$ and thus $d + \frac{n}{d} = \frac{d^2+n}{d} \equiv 1 - 1 = 0$ mod 24, as desired. □

Problem 3. (IMO 2005) Prove that for any prime p there exists a positive integer n such that $2^n + 3^n + 6^n - 1$ is divisible by p.

Proof. We'll assume that $p > 3$ (for $p = 2$ one can choose $n = 1$ and for $p = 3$ one can choose $n = 2$). Then by Fermat's Little Theorem we know that $a^{p-1} \equiv 1 \mod p$, for any $a \in \{2, 3, 6\}$. Hence $3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} \equiv 3 + 2 + 1 = 6 \mod p$. Equivalently, $6(2^{p-2} + 3^{p-2} + 6^{p-2}) \equiv 6 \mod p$ or $p | 2^{p-2} + 3^{p-2} + 6^{p-2} - 1$, so for $p > 3$ one can choose $n = p - 2$. □

Problem 4. (Putnam B3/1973) Consider an integer $n > 1$ with the property that the polynomial $k^2 - k + n$ is a prime for all $0 \leq k \leq n - 1$. (For example, $n = 5$ and $n = 41$ have this property.) Show that there is exactly one triple of integers (a, b, c) satisfying the conditions:

$$\begin{aligned} b^2 - 4ac &= 1 - 4n, \\ 0 < a &\leq c, \\ -a \leq b &< a. \end{aligned}$$

Proof. If (a, b, c) is a triple satisfying the statement then b must be odd, so $|b| = 2t - 1$ for some $t \geq 1$. Note that $1 - 4n = b^2 - 4ac \leq a^2 - 4a^2 = -3a^2 \Longrightarrow \frac{3a^2+1}{4} \leq n$. So if $t > n$ then we get that $\frac{3a^2+1}{4} \leq n < t = \frac{|b|+1}{2} \leq \frac{a+1}{2}$, or equivalently, $3a^2 + 1 < 2(a + 1) \Longleftrightarrow (a - 1)(3a + 1) < 0$, which is a contradiction. Thus $1 \leq t \leq n - 1$.

In addition, $1 - 4n = b^2 - 4ac = (2t - 1)^2 - 4ac$ is equivalent to

$$t^2 - t + n = ac.$$

Since t is in the allowable range we conclude that ac must be a prime. This forces $a = 1$ and hence $b = -1$ (as $-a \leq b < a$ and b is odd). Finally, we obtain $c = n$ and consequently the triple $(1, -1, n)$ (which verifies the statement) is the only possible one. $\square$

Homework 8
Solutions

Problem 1. (Putnam A1/2001) Consider a set Σ with a binary operation $\star$ on Σ (i.e., for each $x, y \in \Sigma$, $x \star y \in \Sigma$). If $(x \star y) \star x = y$, $\forall\ x, y \in \Sigma$, prove that $x \star (y \star x) = y$, $\forall\ x, y \in \Sigma$.

Proof. Note that $x \star (y \star x) = ((y \star x) \star y) \star (y \star x) = y$.

□

Problem 2. (Putnam A1/1995) Let S be a set of real numbers which is closed under multiplication (i.e., if $a, b \in S$ then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Proof. Assume that there are elements $t_1, t_2 \in T$ such that $t_1 t_2 \in U$, and $u_1, u_2 \in U$ such that $u_1 u_2 \in T$. Then $t_1 t_2 (u_1 u_2) \in T$, while $(t_1 t_2) u_1 u_2 \in U$, which is a contradiction.

□

Problem 3. (Putnam B2/1989) Let S be a nonempty set with an associative binary operation such that $xy = xz$ implies $y = z$ and $yx = zx$ implies $y = z$. Assume that for every $a \in S$ the set $\{a^n : n = 1, 2, 3, \dots\}$ is finite. Must S be a group?

Solution. Choose an element $a \in S$. Since the set $\{a^n : n = 1, 2, 3, \dots\}$ is finite, we conclude that $\exists\ m, n,\ m \geq n + 2$, such that $a^m = a^n$. Set $e = a^{m-n}$. Then $a^m x = a^n e x$, $\forall\ x \in S$, so by the left cancellation property we get $x = ex$, and similarly $x = xe$. Thus $x = ex = xe$, which shows that e is an identity. In addition, $e = a^{m-n} = a a^{m-n-1} = a^{m-n-1} a$, so a^{m-n-1} is an inverse of a. Consequently, S is a group.

□

Problem 4. (Putnam A4/1997) Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element $a \in G$ such that the map $\psi(x) := a\phi(x)$ is a homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

Proof. Since we want $\psi(e) = e$, it is natural to take $a = \phi^{-1}(e)$. We'll show that this choice of a makes the map ψ a homomorphism.
First notice that $\phi(e)$ commutes with $\phi(g)$,$\forall g \in G$. Indeed, one can see that by cancelling $\phi(g^{-1})$ in the identity

$$\phi(e)\phi(g)\phi(g^{-1}) = \phi(g)\phi(e)\phi(g^{-1}).$$

This implies that $\phi^{-1}(e)$ also commutes with $\phi(g)$, $\forall g \in G$, and hence

$$\phi(x)\phi(y)\phi(y^{-1}x^{-1}) = \phi(e)\phi(xy)\phi(y^{-1}x^{-1}) \Longrightarrow \phi^{-1}(e)\phi(x)\phi^{-1}(e)\phi(y) = \phi^{-1}(e)\phi(xy),$$

which shows that $\psi(x)\psi(y) = \psi(xy)$. □

Homework 9
Solutions

Problem 1. (Putnam A1/1951) Let A be a real 4×4 skew-symmetric matrix (i.e., $A^t = -A$). Prove that $\det A \geq 0$.

Proof. We claim that all eigenvalues of A are pure imaginary. Indeed, let λ be an eigenvalue with an eigenvector v, i.e., $Av = \lambda v$. By conjugating (recall that A is real) we get $A\overline{v} = \overline{\lambda}\overline{v}$. This means that

$$(A\overline{v})^t v = (\overline{\lambda}\overline{v})^t v = \overline{\lambda}||v||^2.$$

However, the left side of the above identity is equal to $\overline{v}^t A^t v = \overline{v}^t(-Av) = -\lambda||v||^2$. Consequently $-\lambda = \overline{\lambda}$, showing that λ is pure imaginary.

Finally, the eigenvalues are the roots of the characteristic polynomial of A. Since they are complex numbers, they must occur in conjugate pairs. Moreover, the determinant is the product of the eigenvalues so it is of the form $\prod \lambda\overline{\lambda} = \prod |\lambda|^2 \geq 0$.

□

Remark. *Of course, one can directly compute by hand the determinant of a 4×4 matrix. The above solution holds for any real $n \times n$ skew-symmetric matrix A. In particular, it implies that if n is odd then* $\det(A) = 0$ *(can you see why?).*

Problem 2. (Putnam A2/2008) Alice and Bob play a game in which they take turn filling entries of an initially empty 2008×2008 array. Alice plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alice wins if the determinant of the resulting matrix is nonzero; Bob wins if it is zero. Which player has a winning strategy?

Solution. Divide the array into 1×2 dominoes. Once Alice places a number, Bob places the same number in the other half of the same domino (note that Alice always plays in an empty domino because Bob always completes half-filled dominoes). In the end, the first two columns are identical, so Bob wins.

□

Problem 3. (Putnam A3/2009) Let d_n be the determinant of the $n \times n$ matrix whose entries, from the left to the right and from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \det \begin{pmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{pmatrix}$$

The argument of cos is in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

Solution. Recall that $\cos t + \cos(t+2) = 2\cos 1 \cdot \cos(t+1)$, which means that for $n \geq 3$, if we add the third column to the first column (this doesn't change d_n), we obtain a nonzero multiple of the second column. Hence $d_n = 0$ for $n \geq 3$, and therefore the desired limit is also 0.

□

Problem 4. (IMC A3/1997) Let A and B be real $n \times n$ matrices such that $A^2 + B^2 = AB$. Prove that if $BA - AB$ is an invertible matrix then n is divisible by 3.

Proof. Let $\zeta = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ be a primitive 3-rd root of unity (so $\overline{\zeta} = \zeta^2 = -(1+\zeta)$). Consider $C = A + \zeta B$, then

$$C \cdot \overline{C} = (A + \zeta B)(A + \overline{\zeta} B) = A^2 + B^2 + \overline{\zeta} AB + \zeta BA = AB - (1 + \zeta)AB + \zeta BA = \zeta(BA - AB).$$

Since $C \cdot \overline{C}$ has real entries, its determinant is also real. Thus $\det(\zeta(BA - AB)) = \zeta^n \det(BA - AB) \in \mathbb{R}$, and this determinant can not be 0 (since $BA - AB$ is invertible). Therefore $\zeta^n \in \mathbb{R}$, so $3|n$.

□

14. Calculus and Linear Algebra

Po-Shen Loh

CMU Putnam Seminar, Fall 2012

1 Warm-up

Putnam 2012/A0. When and where is the Putnam?

1913 entrance exam to Carnegie Institute of Technology (Math). A spherical triangle has angles of $70°$, $90°$, and $100°$, and the underlying sphere has radius 10. What is the area of the spherical triangle?

1913 entrance exam to CIT (English). What is the feminine form of the noun "duck"?

2 Problems

Putnam 1941/A2. Define $f(x) = \int_0^x \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} dt$. Calculate the n-th derivative $f^{(n)}(x)$.

Putnam 1942/A3. Does $\sum_{n \geq 0} \frac{n! k^n}{(n+1)^n}$ converge or diverge for $k = \frac{19}{7}$?

Putnam 1941/B3. Let y_1 and y_2 be any two linearly independent solutions to the differential equation $y'' + p(x)y' + q(x)y = 0$. Let $z = y_1 y_2$. Find the differential equation satisfied by z.

Putnam 1955/B2. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, with f'' continuous and $f(0) = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x)/x$ for $x \neq 0$, and $g(0) = f'(0)$. Show that g is differentiable and that g' is continuous.

Putnam 1949/A6. Show that $\prod_{n=1}^{\infty} \frac{1+2\cos(2z/3^n)}{3} = \frac{\sin z}{z}$ for all complex z.

Putnam 1948/B6. Take the origin O of the complex plane to be the vertex of a cube, so that OA, OB, OC are edges of the cube (with A, B, C possibly lying in the third dimension, outside the complex plane). Let the feet of the perpendiculars from A, B, C to the complex plane be the complex numbers u, v, w. Show that $u^2 + v^2 + w^2 = 0$.

Putnam 1948/A5. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the n-th roots of unity. Find

$$\prod_{i<j} (\omega_i - \omega_j)^2 .$$

Putnam 1940/B6. The $n \times n$ matrix (m_{ij}) is defined as $m_{ij} = a_i a_j$ for $i \neq j$, and $a_i^2 + k$ for $i = j$. Show that $\det(m_{ij})$ is divisible by k^{n-1} and find its other factor.

3 No homework

Please do not submit write-ups for any problems. There is no homework for next week. There is no next week. Do not pass Go, do not collect \$200.

Putnam Training Exercise
Induction and Recurrences
(Answers)

October 15th, 2013

1. Prove that $n! > 2^n$ for all $n \geq 4$.

- *Answer:* We prove it by induction. The basis step corresponds to $n = 4$, and in this case certainly we have $4! > 2^4$ $(24 > 16)$. Next, for the induction step, assume the inequality holds for some value of $n \geq 4$, i.e., we assume $n! > 2^n$, and look at what happens for $n + 1$:

$$(n+1)! = n!\,(n+1) \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{>} 2^n(n+1) > 2^n \cdot 2 = 2^{n+1}\,.$$

Hence the inequality also holds for $n + 1$. Consequently it holds for every $n \geq 4$.

2. Consider the sequence a_n defined by recursion $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ $(n \geq 1)$. Prove that $a_n < 2$ for every $n \geq 1$.

- *Answer:* It is indeed true for $a_1 = \sqrt{2}$. Next, if we assume that $0 < a_n < 2$, then $0 < a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = \sqrt{4} = 2$.

3. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.

- *Answer:* We prove it by strong induction. First we notice that the result is true for $n = 0$ $(F_0 = 0 < 1 = 2^0)$, and $n = 1$ $(F_1 = 1 < 2 = 2^1)$. Next, for the inductive step, assume that $n \geq 1$ and assume that the claim is true, i.e. $F_k < 2^k$, for every k such that $0 \leq k \leq n$. Then we must prove that the result is also true for $n + 1$. In fact:

$$F_{n+1} = F_n + F_{n-1} \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{<} 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}\,,$$

and we are done.

4. Let a_n be the following expression with n nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}.$$

Prove that $a_n = 2\cos\frac{\pi}{2^{n+1}}$.

- *Answer:* Note that a_n can be defined recursively like this: $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. We proceed by induction.
For $n = 1$ we have in fact $a_1 = \sqrt{2}$, and $2\cos\frac{\pi}{4} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$.
Next, assuming the result is true for some $n \geq 1$, we have

$$\begin{aligned} a_{n+1} = \sqrt{2 + a_n} &= \sqrt{2 + 2\cos\frac{\pi}{2^{n+1}}} \\ &= \sqrt{2 + 2\cos 2\frac{\pi}{2^{n+2}}} \\ &= \sqrt{2 + 2(2\cos^2\frac{\pi}{2^{n+2}} - 1)} \\ &= \sqrt{\cos^2\frac{\pi}{2^{n+2}}} \\ &= \cos\frac{\pi}{2^{n+2}}\,, \end{aligned}$$

where we have used the duplication formula $\cos\alpha = 2\cos^2\alpha - 1$.

5. Find the maximum number $R(n)$ of regions in which the plane can be divided by n straight lines.

- *Answer:* By experimentation we easily find:

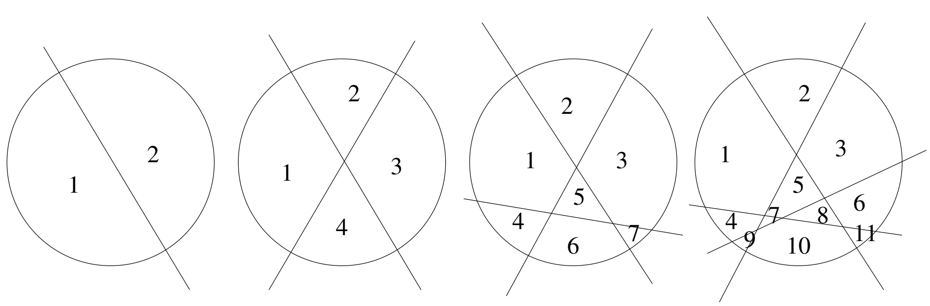


FIGURE 1. Plane regions.

n	1	2	3	4	...
$R(n)$	2	4	7	11	...

A formula that fits the first few cases is $R(n) = (n^2 + n + 2)/2$. We will prove by induction that it works for all $n \geq 1$. For $n = 1$ we have $R(1) = 2 = (1^2 + 1 + 2)/2$, which is correct. Next assume that the property is true for some positive integer n, i.e.:

$$R(n) = \frac{n^2 + n + 2}{2}\,.$$

We must prove that it is also true for $n + 1$, i.e.,

$$R(n+1) = \frac{(n+1)^2 + (n+1) + 2}{2} = \frac{n^2 + 3n + 4}{2}\,.$$

So lets look at what happens when we introduce the $(n+1)$th straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n+1$ segments by those intersection points. Each of those $n+1$ segments divides a previous region into two regions, so the number of regions increases by $n+1$. Hence:

$$R(n+1) = S(n) + n + 1\,.$$

But by induction hypothesis, $R(n) = (n^2 + n + 2)/2$, hence:

$$R(n+1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2}\,.$$

QED.

6. We divide the plane into regions using straight lines. Prove that those regions can be colored with two colors so that no two regions that share a boundary have the same color.

\- *Answer:* We prove it by induction in the number n of lines. For $n = 1$ we will have two regions, and we can color them with just two colors, say one in red and the other one in blue. Next assume that the regions obtained after dividing the plane with n lines can always be colored with two colors, red and blue, so that no two regions that share a boundary have the same color. We need to prove that such kind of coloring is also possible after dividing the plane with $n + 1$ lines. So assume that the plane divided by n lines has been colored in the desired way. After we introduce the $(n+1)$th line we need to recolor the plane to make sure that the new coloring still verifies that no two regions that share a boundary have the same color. We do it in the following way. The $(n + 1)$th line divides the plane into two half-planes. We keep intact the colors in all the regions that lie in one half-plane, and reverse the colors (change red to blue and blue to red) in all the regions of the other half-plane. So if two regions share a boundary and both lie in the same half-plane, they will still have different colors. Otherwise, if they share a boundary but are in opposite half-planes, then they are separated by the $(n + 1)$th line; which means they were part of the same region, so had the same color, and must have acquired different colors after recoloring.

7. Define a *domino* to be a 1×2 rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?

- *Answer:* Let x_n be the number of tilings of an $n \times 2$ rectangle by dominoes. We easily find $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the rightmost domino vertically and tile the rest of the rectangle in x_{n-1} ways, or we can place two horizontal dominoes to the right and tile the rest in x_{n-2} ways, so $x_n = x_{n-1} + x_{n-2}$. So the answer is the shifted Fibonacci sequence, $x_n = F_{n+1}$.

8. Let α, β be two (real or complex) numbers, and define the sequence $a_n = \alpha^n + \beta^n$ $(n = 1, 2, 3, \dots)$. Assume that a_1 and a_2 are integers. Prove that $2^{\lfloor \frac{n-1}{2} \rfloor} a_n$ is an integer for every $n \geq 1$.

- *Answer:* We have that α and β are the roots of the polynomial

$$(x - \alpha)(x - \beta) = x^2 - sx + p\,,$$

where $s = \alpha + \beta$, $p = \alpha\beta$.
We have that $s = a_1$ is an integer. Also, $2p = a_1^2 - a_2$ is an integer. The given sequence verifies the recurrence

$$a_{n+2} = s\, a_{n+1} - p\, a_n\,,$$

hence

$$2^{\lfloor \frac{n+1}{2} \rfloor} a_{n+2} = s\, 2^{\lfloor \frac{n+1}{2} \rfloor} a_{n+1} - 2p\, 2^{\lfloor \frac{n-1}{2} \rfloor} a_n\,.$$

From here we get the desired result by induction.

9. Suppose that $x_0 = 18$, $x_{n+1} = \dfrac{10x_n}{3} - x_{n-1}$ $(n \geq 1)$, and assume that the sequence $\{x_n\}$ converges to some real number. Find x_1.

- *Answer:* The general solution for the recurrence can be expressed using the roots of its characteristic polynomial

$$x^2 - \frac{10x}{3} + 1 = 0\,.$$

The roots are 3 and $1/3$, hence a general solution is $x_n = A \cdot 3^n + B \cdot 3^{-n}$. If the sequence converges then $A = 0$, and the condition $x_0 = 18$ yields $B = 18$, hence the sequence is $x_n = 18 \cdot 3^{-n}$, the limit is 0, and $x_1 = 18/3 = 6$.

10. Ackermann's function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined for every $m, n \geq 0$ by the following double recurrence:

(a) $A(0, n) = n + 1$ for every $n \geq 0$.
(b) $A(m, 0) = A(m - 1, 1)$ if $m \geq 1$.
(c) $A(m, n) = A(m - 1, A(m, n - 1))$ if $m, n \geq 1$.

Find $A(5,0)$.

- *Answer:* The answer is $A(5,0) = 2^{16} - 3 = 65533$.

To prove it we start be getting close form expressions for $A(m,n)$, $0 \le n \le 3$.

(a) For $m = 0$, we have $A(0,n) = n + 1$ by definition.
(b) For $m = 1$, we prove by induction that $A(1,n) = n + 2$. In fact, $A(1,0) = A(0,1) = 2$, and if $n \ge 1$ then $A(1,n) = A(0, A(1, n-1)) = A(0, n+1) = n+2$ for $n \ge 1$.
(c) For $m = 2$, we see also by induction that $A(2,n) = 2n + 3$. In fact, base case: $A(2,0) = A(1,1) = 3$, and induction step: $A(2,n) = A(1, A(2, n-1)) = A(1, 2n+1) = 2n+3$ for $n \ge 1$.
(d) For $m = 3$, we use induction again to show $A(3,n) = 2^{n+3} - 3$. In fact, $A(3,0) = A(2,1) = 5$, $A(3,n) = A(2, A(3, n-1)) = A(2, 2^{n+2} - 3) = 2 \cdot (2^{n+2} - 3) + 3 = 2^{n+3} - 3$.

Finally, we have $A(5,0) = A(4,1) = A(3, A(4,0))) = A(3, A(3,1))) = A(3, 2^4 - 3) = A(3,13) = 2^{16} - 3 = 65533$, Q.E.D.

Putnam Training Exercise
Number Theory and Congruences
(Answers)

October 22nd, 2013

1. Can the sum of the digits of a square be (a) 3, (b) 1977?

- *Answer:*
(a) No, a square divisible by 3 is also divisible by 9.
(b) Same argument.

2. Show that if $a^2 + b^2 = c^2$, then $3|ab$.

- *Answer:* For any integer n we have that n^2 only can be 0 or 1 mod 3. So if 3 does not divide a or b they must be 1 mod 3, and their sum will be 2 modulo 3, which cannot be a square.

3. Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every possible integer n.

- *Answer:* That is equivalent to proving that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime for every n. These are two possible ways to show it:

- Assume a prime p divides $n^3 + 2n = n(n^2 + 2)$. Then it must divide n or $n^2 + 2$. Writing $n^4 + 3n^2 + 1 = n^2(n^2 + 3) + 1 = (n^2 + 1)(n^2 + 2) - 1$ we see that p cannot divide $n^4 + 3n^2 + 1$ in either case.

- The following identity

$$(n^2 + 1)(n^4 + 3n^2 + 1) - (n^3 + 2n)^2 = 1$$

(which can be checked algebraically) shows that any common factor of $n^4 + 3n^2 + 1$ and $n^3 + 2n$ should divide 1, so their gcd is always 1. (Note: if you are wondering how I arrived to that identity, I just used the Euclidean algorithm on the two given polynomials.)

4. Prove that there are infinitely many prime numbers of the form $4n + 3$.

- *Answer:* Assume that the set of primes of the form $4n+3$ is finite. Let P be their product. Consider the number $N = P^2 - 2$. Note that the square of an odd number is of the form $4n+1$, hence P^2 is of the form $4n+1$ and N will be of the form $4n+3$.

Now, if all prime factors of N where of the form $4n+1$, N would be of the form $4n+1$, so N must have some prime factor p of the form $4n+3$. So it must be one of the primes in the product P, hence p divides $N - P^2 = 2$, which is impossible.

5. Show that there exist 2013 consecutive numbers, each of which is divisible by the cube of some integer greater than 1.

- *Answer:* Pick 2013 different prime numbers $p_1, p_2, \dots, p_{2013}$ (we can do that because the set of prime numbers is infinite) and solve the following system of 2013 congruences:

$$\begin{cases} x \equiv 0 \pmod{p_1^3} \\ x \equiv -1 \pmod{p_2^3} \\ x \equiv -2 \pmod{p_3^3} \\ \quad\cdots \\ x \equiv -2012 \pmod{p_{2013}^3} \end{cases}$$

According to the Chinese Remainder Theorem, that system of congruences has a solution x (modulo $M = p_1^3 \dots p_{2013}^3$). For $k = 1, \dots, 2013$ we have that $x + k \equiv 0 \pmod{p_k^3}$, hence $x + k$ is in fact a multiple of p_k^3.

6. (USAMO, 1979) Find all non-negative integral solutions $(n_1, n_2, \dots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599\,.$$

- *Answer:* We look at the equation modulo 16. First we notice that $n^4 \equiv 0$ or $1 \pmod{16}$ depending on whether n is even or odd. On the other hand $1599 \equiv 15 \pmod{16}$. So the equation can be satisfied only if the number of odd terms in the LHS is 15 modulo 16, but that is impossible because there are only 14 terms in the LHS. Hence the equation has no solution.

7. Let $f(n)$ denote the sum of the digits of n. Let $N = 4444^{4444}$. Find $f(f(f(N)))$.

- *Answer:* Since each digit cannot be greater than 9, we have that $f(n) \le 9 \cdot (1 + \log_{10} n)$, so in particular $f(N) \le 9 \cdot (1 + 4444 \cdot \log_{10} 4444) < 9 \cdot (1 + 4444 \cdot 4) = 159993$. Next we have $f(f(N)) \le 9 \cdot 6 = 54$. Finally among numbers not greater than 54, the one with the greatest sum of the digits is 49, hence $f(f(f(N))) \le 4 + 9 = 13$.

Next we use that $n \equiv f(n) \pmod 9$. Since $4444 \equiv 7 \pmod 9$, then

$$4444^{4444} \equiv 7^{4444} \pmod 9\,.$$

We notice that the sequence $7^n \bmod 9$ for $n = 0, 1, 2, \dots$ is $1, 7, 4, 1, 7, 4, \dots$, with period 3. Since $4444 \equiv 1 \pmod 3$, we have $7^{4444} \equiv 7^1 \pmod 9$, hence $f(f(f(N))) \equiv 7 \pmod 9$. The only positive integer not greater than 13 that is congruent with 7 modulo 9 is 7, hence $f(f(f(N))) = 7$.

8. Do there exist 2 irrational numbers a and b greater than 1 such that $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for every positive integers m, n?

- *Answer:* The answer is affirmative. Let $a = \sqrt{6}$ and $b = \sqrt{3}$. Assume $\lfloor a^m \rfloor = \lfloor b^n \rfloor = k$ for some positive integers m, n. Then, $k^2 \leq 6^m < (k+1)^2 = k^2 + 2k + 1$, and $k^2 \leq 3^n < (k+1)^2 = k^2 + 2k + 1$. Hence, subtracting the inequalities and taking into account that $n > m$:

$$2k \geq |6^m - 3^n| = 3^m |2^m - 3^{n-m}| \geq 3^m .$$

Hence $\dfrac{9^m}{4} \leq k^2 \leq 6^m$, which implies $\frac{1}{4} \leq \left(\frac{2}{3}\right)^m$. This holds only for $m = 1, 2, 3$. This values of m can be ruled out by checking the values of

$$\lfloor a \rfloor = 2, \quad \lfloor a^2 \rfloor = 6, \quad \lfloor a^3 \rfloor = 14,$$

$$\lfloor b \rfloor = 1, \quad \lfloor b^2 \rfloor = 3, \quad \lfloor b^3 \rfloor = 5, \quad \lfloor b^4 \rfloor = 9, \quad \lfloor b^5 \rfloor = 15 .$$

Hence, $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for every positive integers m, n.

9. Prove that there are no primes in the following infinite sequence of numbers:

$$1001,\ 1001001,\ 1001001001,\ 1001001001001, \ldots$$

- *Answer:* Each of the given numbers can be written

$$1 + 1000 + 1000^2 + \cdots + 1000^n = p_n(10^3)$$

where $p_n(x) = 1 + x + x^2 + \cdots + x^n$, $n = 1, 2, 3, \ldots$. We have $(x-1)p_n(x) = x^{n+1} - 1$. If we set $x = 10^3$, we get:

$$999 \cdot p_n(10^3) = 10^{3(n+1)} - 1 = (10^{n+1} - 1)(10^{2(n+1)} + 10^{n+1} + 1) .$$

If $p_n(10^3)$ were prime it should divide one of the factors on the RHS. It cannot divide $10^{n+1} - 1$, because this factor is less than $p_n(10^3)$, so $p_n(10^3)$ must divide the other factor. Hence $10^{n+1} - 1$ must divide 999, but this is impossible for $n > 2$. In only remains to check the cases $n = 1$ and $n = 2$. But $1001 = 7 \cdot 11 \cdot 13$, and $1001001 = 3 \cdot 333667$, so they are not prime either.

10. The digital root of a number is the (single digit) value obtained by repeatedly adding the (base 10) digits of the number, then the digits of the sum, and so on until obtaining a single digit—e.g. the digital root of 65,536 is 7, because $6 + 5 + 5 + 3 + 6 = 25$ and $2 + 5 = 7$. Consider the sequence $a_n =$ integer part of $10^n \pi$, i.e.,

$$a_1 = 31\,, \quad a_2 = 314\,, \quad a_3 = 3141\,, \quad a_4 = 31415\,, \quad a_5 = 314159\,, \quad \ldots$$

and let b_n be the sequence

$$b_1 = a_1\,, \quad b_2 = a_1^{a_2}\,, \quad b_3 = a_1^{a_2^{a_3}}\,, \quad b_4 = a_1^{a_2^{a_3^{a_4}}}\,, \quad \ldots$$

Find the digital root of b_{10^6}.

- *Answer:* In spite of its apparent complexity this problem is very easy, because the digital root of b_n becomes a constant very quickly. First note that the digital root of a number a is just the reminder r of a modulo 9, and the digital root of a^n will be the remainder of r^n modulo 9.

For $a_1 = 31$ we have

digital root of a_1 = digital root of $31 = 4$;

digital root of a_1^2 = digital root of $4^2 = 7$;

digital root of a_1^3 = digital root of $4^3 = 1$;

digital root of a_1^4 = digital root of $4^4 = 4$;

and from here on it repeats with period 3, so the digital root of a_1^n is 1, 4, and 7 for remainder modulo 3 of n equal to 0, 1, and 2 respectively.

Next, we have $a_2 = 314 \equiv 2 \pmod 3$, $a_2^2 \equiv 2^2 \equiv 1 \pmod 3$, $a_2^3 \equiv 2^3 \equiv 2 \pmod 3$, and repeating with period 2, so the reminder of a_2^n depends only on the parity of n, with $a_2^n \equiv 1 \pmod 3$ if n is even, and $a_2^n \equiv 2 \pmod 3$ if n is odd.

And we are done because a_3 is odd, and the exponent of a_2 in the power tower defining b_n for every $n \geq 3$ is odd, so the reminder modulo 3 of the exponent of a_1 will be 2, and the reminder modulo 9 of b_n will be 7 for every $n \geq 3$.

Hence, the answer is 7.

Putnam Training Exercise
Polynomials
(Answers)

October 29th, 2013

1. Find a polynomial with integer coefficients whose zeros include $\sqrt{2}+\sqrt{5}$.

- *Answer:* If $x=\sqrt{2}+\sqrt{5}$ then

$$\begin{aligned} x^2 &= 7+2\sqrt{10}\,, \\ x^2-7 &= 2\sqrt{10}\,, \\ (x^2-7)^2 &= 40\,, \\ x^4-14x^2+9 &= 0\,. \end{aligned}$$

Hence the desired polynomial is x^4-14x^2+9.

2. Prove that $(2+\sqrt{5})^{1/3}-(-2+\sqrt{5})^{1/3}$ is rational.

- *Answer:* Let $\alpha=(2+\sqrt{5})^{1/3}-(-2+\sqrt{5})^{1/3}$. By raising to the third power, expanding and simplifying we get that α verifies the following polynomial equation:

$$\alpha^3+3\alpha-4=0\,.$$

We have $x^3+3x-4=(x-1)(x^2+x+4)$. The second factor has no real roots, hence x^3+3x-4 has only one real root equal to 1, i.e., $\alpha=1$.

3. If $a,b,c>0$, is it possible that each of the polynomials $P(x)=ax^2+bx+c$, $Q(x)=cx^2+ax+b$, $R(x)=bx^2+cx+a$ has two real roots?

- *Answer:* The answer is No. If $P(x)$ has two real roots we would have $b^2>4ac$. Analogously for $R(x)$ and $Q(x)$ we should have $a^2>4cb$, and $c^2>4ab$ respectively. Multiplying the inequalities we get $a^2b^2c^2>64a^2b^2c^2$, which is impossible.

4. Suppose that α, β, and γ are real numbers such that

$$\begin{aligned} \alpha+\beta+\gamma &= 2\,, \\ \alpha^2+\beta^2+\gamma^2 &= 14\,, \\ \alpha^3+\beta^3+\gamma^3 &= 17\,. \end{aligned}$$

Find $\alpha\beta\gamma$.

- *Answer:* Writing the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ, we have

$$\begin{aligned} \alpha+\beta+\gamma &= s\,, \\ \alpha^2+\beta^2+\gamma^2 &= s^2-2q\,, \\ \alpha^3+\beta^3+\gamma^3 &= s^3-3qs+3p\,, \end{aligned}$$

where $s=\alpha+\beta+\gamma$, $q=\alpha\beta+\beta\gamma+\alpha\gamma$, $p=\alpha\beta\gamma$.

So we have $s=2$, and from the second given equation get $q=-5$. Finally from the third equation we get $p=-7$. So, this is the answer, $\alpha\beta\gamma=-7$.

5. Show that $(1+x+\cdots+x^n)^2-x^n$ is the product of two polynomials.

- *Answer:* Letting $A_{n-1}=1+x+\cdots+x^{n-1}$, we have

$$\begin{aligned} (1+x+\cdots+x^n)^2-x^n &= (A_{n-1}+x^n)^2-x^n \\ &= A_{n-1}^2+2A_{n-1}x^n+x^{2n}-x^n \\ &= A_{n-1}^2+2A_{n-1}x^n+(x^n-1)x^n \\ &= A_{n-1}^2+2A_{n-1}x^n+A_{n-1}(x-1)x^n \\ &= A_{n-1}(A_{n-1}+2x^n+(x-1)x^n) \\ &= A_{n-1}(A_{n-1}+x^n+x^{n+1}) \\ &= (1+x+\cdots+x^{n-1})(1+x+\cdots+x^{n+1})\,. \end{aligned}$$

6. Is it possible to write the polynomial $f(x)=x^{105}-9$ as the product of two polynomials of degree less than 105 with integer coefficients?

- *Answer:* By contradiction. Assume $f(x)=g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Let $\alpha_1,\dots,\alpha_k$ the (complex) roots of $g(x)$. For each $j=1,\dots,k$ we have $\alpha_j^{105}=9$, hence $|\alpha_j|=\sqrt[105]{9}$, and $|\alpha_1\alpha_2\cdots\alpha_k|=(\sqrt[105]{9})^k=$ the absolute value of the constant term of $g(x)$ (an integer.) But $(\sqrt[105]{9})^k=\sqrt[105]{3^{2k}}$ cannot be an integer.

7. Find all prime numbers p that can be written $p=x^4+4y^4$, where x,y are positive integers.

- *Answer:* The answer is $p=5$. By Sophie Germain's Identity we have

$$x^4+4y^4=(x^2+2y^2+2xy)(x^2+2y^2-2xy)=[(x+y)^2+y^2][(x-y)^2+y^2]\,,$$

which can be prime only if $x=y=1$.

8. Determine all polynomials such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

- *Answer:* The answer is $P(x) = x$.

In order to prove this we show that $P(x)$ equals x for infinitely many values of x. In fact, let a_n the sequence $0, 1, 2, 5, 26, 677, \ldots,$ defined recursively $a_0 = 0$, and $a_{n+1} = a_n^2 + 1$ for $n \geq 0$. We prove by induction that $P(a_n) = a_n$ for every $n = 0, 1, 2, \ldots$. In the basis case, $n = 0$, we have $P(0) = 0$. For the induction step assume $n \geq 1$, $P(a_n) = a_n$. Then $P(a_{n+1}) = P(a_n^2 + 1) = P(a_n)^2 + 1 = a_n^2 + 1 = a_{n+1}$.

Since in fact $P(x)$ coincides with x for infinitely many values of x, we must have $P(x) = x$ identically.

9. Prove that there is no polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$,..., are all prime numbers.

- *Answer:* By contradiction. We have that $a_0 = P(0)$ must be a prime number. Also, $P(ka_0)$ is a multiple of a_0 for every $k = 0, 1, 2, \ldots$, but if $P(ka_0)$ is prime then $P(ka_0) = a_0$ for every $k \geq 0$. This implies that the polynomial $Q(x) = P(a_0 x) - a_0$ has infinitely many roots, so it is identically zero, and $P(a_0 x) = a_0$, contradicting the hypothesis that P is of degree at least 1.

10. Two players A and B play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B win the game?

- *Answer:* The answer is affirmative, B can in fact guess the polynomial—call it $f(x) = a_0 + a_1 x^2 + a_2 x^2 + \cdots a_n x^n$. By asking A to evaluate it at 1, B gets an upper bound $f(1) = a_0 + a_1 + a_2 + \cdots a_n = M$ for the coefficients of the polynomial. Then, for any integer $N > M$, the coefficients of the polynomial are just the digits of $f(N) = a_0 + a_1 N^2 + a_2 N^2 + \cdots a_n N^n$ in base N.

Putnam Training Exercise
Inequalities
(Answers)

November 5th, 2013

1. The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$ What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?

- *Answer:* We have that $n!$ is increasing for $n \geq 1$, i.e., $1 \leq n < m \implies n! < m!$ So $1999! > 2000 \implies (1999!)! > 2000! \implies ((1999!)!)! > (2000!)! \implies \dots \implies 1999!^{(2000)} > 2000!^{(1999)}$.

2. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

- *Answer:* Using the Arithmetic Mean-Geometric Mean Inequality on each factor of the LHS we get

$$\left(\frac{a^2b + b^2c + c^2a}{3}\right)\left(\frac{ab^2 + bc^2 + ca^2}{3}\right) \geq \left(\sqrt[3]{a^3b^3c^3}\right)\left(\sqrt[3]{a^3b^3c^3}\right) = a^2b^2c^2\,.$$

Multiplying by 9 we get the desired inequality.

Another solution consists of using the Cauchy-Schwarz inequality:

$$\begin{aligned}(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) &= \\ \left((a\sqrt{b})^2 + (b\sqrt{c})^2 + (c\sqrt{a})^2\right)&\left((\sqrt{b}\,c)^2 + (\sqrt{c}\,a)^2 + (\sqrt{a}\,b)^2\right) \\ &\geq (abc + abc + abc)^2 \\ &= 9a^2b^2c^2\,.\end{aligned}$$

3. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

- *Answer:* y the power means inequality:

$$\underbrace{\frac{a+b+c}{3}}_{M^1(a,b,c)} \geq \underbrace{\left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^2}_{M^{1/2}(a,b,c)}$$

From here the desired result follows.

4. Find the maximum value of $f(x) = \sin^4(x) + \cos^4 x$ for $x \in \mathbb{R}$.

- *Answer:* The answer is 1. Since $|\sin x| \leq 1$ we have $\sin^4(x) \leq \sin^2(x)$, and analogously $\cos^4(x) \leq \cos^2(x)$. Hence $f(x) = \sin^4(x) + \cos^4 x \leq \sin^2(x) + \cos^2 x = 1$. On the other hand the value 1 s attained e.g. at $x = 0$.

5. Find the minimum value of the function $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$, where $x_1, x_2, \dots, x_n$ are positive real numbers such that $x_1 x_2 \cdots x_n = 1$.

- *Answer:* By the Arithmetic Mean-Geometric Mean Inequality

$$1 = \sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

Hence $f(x_1, x_2, \dots, x_n) \geq n$. On the other hand $f(1, 1, \dots, 1) = n$, so the minimum value is n.

6. If $x, y, z > 0$, and $x + y + z = 1$, find the minimum value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

- *Answer:* By the Arithmetic Mean-Harmonic Mean inequality:

$$\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x + y + z}{3} = \frac{1}{3},$$

hence

$$9 \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

On the other hand for $x = y = z = 1/3$ the sum is 9, so the minimum value is 9.

7. Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:

$$\frac{a\,A + b\,B + c\,C}{a + b + c} \geq \frac{\pi}{3}.$$

- *Answer:* Assume $a \leq b \leq c$, $A \leq B \leq C$. Then

$$\begin{aligned} 0 &\leq (a - b)(A - B) + (a - c)(A - C) + (b - c)(B - C) \\ &= 3(aA + bB + cC) - (a + b + c)(A + B + C). \end{aligned}$$

Using $A + B + C = \pi$ and dividing by $3(a + b + c)$ we get the desired result.

Remark: We could have used also Chebyshev's Inequality:

$$\frac{a\,A + b\,B + c\,C}{3} \geq \left(\frac{a+b+c}{3}\right)\left(\frac{A+B+C}{3}\right).$$

8. Find the positive solutions of the system of equations

$$x_1 + \frac{1}{x_2} = 4\,, \quad x_2 + \frac{1}{x_3} = 1\,, \ldots\,, \quad x_{99} + \frac{1}{x_{100}} = 4\,, \quad x_{100} + \frac{1}{x_1} = 1\,.$$

- *Answer:* By the Geometric Mean-Arithmetic Mean inequality

$$x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}\,, \ldots, x_{100} + \frac{1}{x_1} \geq 2\sqrt{\frac{x_{100}}{x_1}}\,.$$

Multiplying we get

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{100} + \frac{1}{x_1}\right) \geq 2^{100}\,.$$

From the system of equations we get

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{100} + \frac{1}{x_1}\right) = 2^{100}\,,$$

so all those inequalities are equalities, i.e.,

$$x_1 + \frac{1}{x_2} = 2\sqrt{\frac{x_1}{x_2}} \quad \Longrightarrow \quad \left(\sqrt{x_1} - \frac{1}{\sqrt{x_2}}\right)^2 = 0 \quad \Longrightarrow \quad x_1 = \frac{1}{x_2}\,,$$

and analogously: $x_2 = 1/x_3$, $\ldots$, $x_{100} = 1/x_1$. Hence $x_1 = 1/x_2$, $x_2 = 1/x_3$, $\ldots$, $x_{100} = 1/x_1$, and from here we get $x_1 = 2$, $x_2 = 1/2, \ldots$, $x_{99} = 2$, $x_{100} = 1/2$.

9. (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

- *Answer:* The given inequality is equivalent to

$$\frac{(m+n)!}{m!\,n!}\,m^m n^n = \binom{m+n}{n} m^m n^n < (m+n)^{m+n}\,,$$

which is obviously true because the binomial expansion of $(m+n)^{m+n}$ includes the term on the left plus other terms.

10. Let $a_i > 0$, $i = 1, \ldots, n$, and $s = a_1 + \cdots + a_n$. Prove

$$\frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \cdots + \frac{a_n}{s - a_n} \geq \frac{n}{n-1}\,.$$

- *Answer:* By the rearrangement inequality we have for $k = 2, 3, \cdots, n$:

$$\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \geq \frac{a_k}{s-a_1} + \frac{a_{k+1}}{s-a_2} + \cdots + \frac{a_{k-1}}{s-a_n},$$

were the numerators on the right hand side are a cyclic permutation of $a_1, \cdots, a_n$ (assume $a_{n+i} = a_i$). Adding those $n-1$ inequalities we get

$$(n-1)\left(\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n}\right) \geq \frac{s-a_1}{s-a_1} + \frac{s-a_2}{s-a_2} + \cdots + \frac{s-a_n}{s-a_n} = n,$$

and the result follows.

PUTNAM TRAINING SESSIONS
IMPORTANT CONCEPTS, PART V
POLYNOMIALS

1. **Division algorithm for polynomials**

   If $F(x)$ and $G(x)$ are polynomials over a field K (for example, K might be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$), there exist unique polynomials $Q(x)$ and $R(x)$ over the field K such that
   $$F(x) = Q(x)G(x) + R(x),$$
   where $R(x) = 0$ or $\deg R(x) < \deg G(x)$.

2. **Factor Theorem**

   If $F(x)$ is a polynomial over a field K, an element a of K is a root of $F(x) = 0$ if and only if $x - a$ is a factor of $F(x)$.

3. **Identity Theorem**

   Let $F(x)$ and $G(x)$ be polynomials of degree $\leq n$ over a field K. Suppose that $x_1, \ldots, x_{n+1}$ are distinct elements of K such that $F(x_i) = G(x_i)$ for all $1 \leq i \leq n+1$. Then $F(x) = G(x)$ for all x.

4. **Rational Root Theorem**

   If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients, and if the rational r/s (r and s are coprime) is a root of $P(x) = 0$, then $r|a_0$ and $s|a_n$.

5. **Gauss' Lemma**

   Let $P(x)$ be a polynomial with integer coefficients. If $P(x)$ can be factored into a product of two polynomials with rational coefficients, then $P(x)$ can be factored into a product of two polynomials with integer coefficients.

6. **Eisenstein Irreducibility Criterion**

   Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients. If there exists a prime p such that: $p \nmid a_n$, $p|a_i$ for all $0 \leq i \leq n-1$, and $p^2 \nmid a_0$, then $P(x)$ is irreducible over $\mathbb{Q}$ (which means that $P(x)$ can not be factored into polynomials with rational coefficients of degree ≥ 1.)

1999 UIUC Undergraduate Math Contest

Solutions

Problem 1.

Let a_n denote the integer closest to $\sqrt{n}$. (For example, $a_1 = a_2 = 1$ and $a_3 = a_4 = 2$ since $\sqrt{1} = 1$, $\sqrt{2} = 1.41\ldots$, $\sqrt{3} = 1.73\ldots$, and $\sqrt{4} = 2$.) Evaluate the sum

$$S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{1980}}.$$

Solution. For any positive integer k, a_n is equal to k, if and only if $\sqrt{n}$ lies between $k - 1/2$ and $k + 1/2$, i.e., if and and only if n lies between $k^2 - k + 1/4$ and $k^2 + k + 1/4$. Since there are exactly $2k$ integer values in this range, and since $1980 = 44 \cdot 45 = 44^2 + 44$, it follows that $S = \sum_{k=1}^{44} (1/k) \cdot 2k = 88$.

Problem 2.

Let ABC be a triangle, and let BD and CE denote the angle-bisectors at B and C. Show that if BD and CE have the same length, then the triangle is isosceles (that is, the sides AB and AC have the same length).

Solution. Let $a = BC$, $b = AC$, $c = AB$ denote the three sides of the triangle, β and γ the angles of the triangle at B and C, and $d = BD = CE$ the (common) length of the angle-bisectors at these points. The area $\mathbf{A}$ of the triangle ABC is, on the one hand, $\mathbf{A} = (1/2)ac\sin\beta$. On the other hand, splitting ABC into the triangles BCD and BDA, which have areas $(1/2)ad\sin(\beta/2)$ and $(1/2)dc\sin(\beta/2)$, respectively, we obtain $\mathbf{A} = (1/2)d(a + c)\sin(\beta/2)$. Setting the two expressions for A equal and using the double angle formula $\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$, it follows that (1) $2\cos(\beta/2) = d(1/a + 1/c)$. Similarly, interchanging the roles of B and C, we obtain (2) $2\cos(\gamma/2) = d(1/a + 1/b)$. If we now assume that b and c are not equal, say (without loss of generality) $b < c$, then $\beta < \gamma$ and so $\cos(\beta/2) > \cos(\gamma/2)$. However, by (1) and (2) this would imply $1/c > 1/b$, contradicting the assumption $b < c$. Hence b and c must be equal as claimed.

Problem 3.

Let a sequence $\{x_n\}$ be given by $x_1 = 1$ and $x_{n+1} = x_n^2 + x_n$ for $n = 1, 2, 3, \ldots$. Let $y_n = 1/(1 + x_n)$ and let $S_n = \sum_{k=1}^{n} y_k$ and $P_n = \prod_{k=1}^{n} y_k$ denote, respectively, the sum and the product of the first n terms of the sequence $\{y_k\}$. Evaluate $P_n + S_n$ for $n = 1, 2, 3, \ldots$.

Solution. From the given recurrence we obtain $x_{n+1} = x_n/y_n$, so that $y_n = x_n/x_{n+1}$ for all n. Hence $P_n = \prod_{k=1}^{n}(x_k/x_{k+1}) = x_1/x_{n+1} = 1/x_{n+1}$ for all n. Moreover, from the identity

$$y_n = \frac{1}{1 + x_n} = \frac{1}{x_n} - \frac{1}{(1 + x_n)x_n} = \frac{1}{x_n} - \frac{1}{x_{n+1}},$$

we see that $S_n = \sum_{k=1}^n (1/x_k - 1/x_{k+1}) = 1/x_1 - 1/x_{n+1} = 1 - 1/x_{n+1}$. Hence $P_n + S_n = 1$ for all n.

Problem 4.

Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2}^{\,x_n}$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges and find its limit.

Solution. Since $x_1 = \sqrt{2} < 2$ and if $x_n < 2$ then $x_{n+1} = \sqrt{2}^{\,x_n} < \sqrt{2}^{\,2} = 2$, it follows by induction that (1) $x_n < 2$ for all n. Thus, the sequence $\{x_n\}$ is bounded from above. Next let $f(x) = \sqrt{2}^{\,x} - x$. Then $f'(x) = \sqrt{2}^x \log\sqrt{2} - 1 < 2\log\sqrt{2} - 1 < 0$ for $x < 2$, so $f(x)$ is decreasing for $x < 2$, and since $f(2) = 0$, this implies $f(x) > 0$, or equivalently $\sqrt{2}^x > x$, for $x < 2$. In view of (1), it follows that $x_{n+1} = \sqrt{2}^{\,x_n} > x_n$ for all n. Hence the sequence $\{x_n\}$ is monotone increasing and bounded from above and therefore must be convergent. Let L denote the limit of this sequence. By (1) we have (2) $L = \lim_{n\to\infty} x_n \leq 2$, and letting $n \to \infty$ on both sides of the recurrence $x_{n+1} = \sqrt{2}^{\,x_n}$, we obtain $L = \sqrt{2}^L$ or (3) $f(L) = 0$. Since $f(2) = 0$, $L = 2$ is a solution to (3). Moreover, $L = 2$ is the only solution satisfying (2), since $f(x)$ is decreasing for $x < 2$. Hence the limit of the sequence $\{x_n\}$ is 2.

Problem 5.

Prove that the series

$$\frac{1}{1} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \dots$$

converges and evaluate its sum.

Solution. Let S_n denote the sum of the first n terms of this sequence. Then

$$S_n = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{[n/3]} \frac{3}{3k} = \sum_{k=[n/3]+1}^{n} \frac{1}{k}.$$

Let T_n denote the latter sum. Comparing this sum with an integral we see that

$$\log 3 - \log(1 + \frac{3}{n}) = \int_{n/3+1}^{n} \frac{1}{x} dx \leq T_n \leq \int_{n/3-1}^{n} \frac{1}{x} dx = \log 3 - \log(1 - \frac{3}{n})$$

Since $\log(1 \pm 3/n) \to 0$ as $n \to \infty$, we conclude that T_n, and therefore S_n, converges, and has limit $\log 3$. Hence the given infinite series converges with sum $\log 3$.

Problem 6.

Given positive integers n and m with $n \geq 2m$, let $f(n, m)$ be the number of binary sequences of length n (i.e., strings $a_1 a_2 \dots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. Find a simple formula for $f(n, m)$.

Solution. Every sequence of the required form can be written as $B_1C_101B_2C_201\ldots01B_{m+1}C_{m+1}$, where each B_i is a block of 1's and each C_i a block of 0's, with empty blocks being allowed, and the sum of the lengths of the blocks B_i and C_i is $n-2m$. Moreover, the sequence is uniquely determined by the $(2m+2)$-tuple (1) $(b_1, c_1, b_2, c_2, \ldots, b_{m+1}, c_{m+1})$ where b_i and c_i denote the number of elements in the blocks B_i and C_i, respectively. Conversely, any tuple of the form (1) with nonnegative integers b_i and c_i satisfying $\sum_{i=1}^{m+1}(b_i+c_i) = (n-2m)$ determines a sequence of the required type. Hence the number of such sequences is equal to the number of ways one can write $2n-m$ as a sum of $2m+2$) nonnegative integers, with order taken into account. The latter problem is equivalent to counting the number of ways of choosing $2n-m$ donuts from $2m+2$ varieties, a well-known combinatorial problem whose answer is given by the binomial coefficient $\binom{a}{b}$ with $a = (n-2m)+(2m+2)-1 = n+1$ and $b = (2m+2)-1 = 2m+1$. Hence $f(n,m) = \binom{n+1}{2m+1}$.

Convexity

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1 Warm-up

1. Prove that there is an integer N such that no matter how N points are placed in the plane, with no 3 collinear, some 10 of them form the vertices of a convex polygon.

2. Let 9 points P_1, P_2, ..., P_9 be given on a line. Determine all points X which minimize the sum of distances

$$P_1X + P_2X + \cdots + P_9X \,.$$

2 Tools

Recall that a set S (say in the plane) is *convex* if for any $x, y \in S$, the line segment with endpoints x and y is completely contained in S. It turns out that this concept is very useful in the theory of functions.

Definition. *We say that a function $f(x)$ is* ***convex*** *on the interval I when the set $\{(x, y) : x \in I, y \geq f(x)\}$ is convex. On the other hand, if the set $\{(x, y) : x \in I, y \leq f(x)\}$ is convex, then we say that f is* ***concave****. Note that it is possible for f to be neither convex nor concave. We say that the convexity/concavity is* ***strict*** *if the graph of $f(x)$ over the interval I contains no straight line segments.*

Remark. Plugging in the definition of set-theoretic convexity, we find the following equivalent definition. The function f is convex on the interval I iff for every $a, b \in I$, the line segment between the points $(a, f(a))$ and $(b, f(b))$ is always above or on the curve f. Analogously, f is concave iff the line segment always lies below or on the curve. This definition is illustrated in Figure 1.

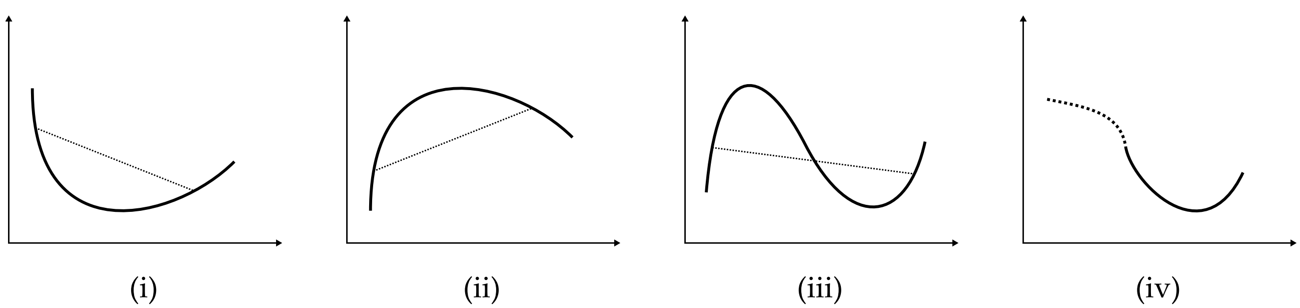


Figure 1: The function in (i) is convex, (ii) is concave, and (iii) is neither. In each diagram, the dotted line segments represent a sample line segment as in the definition of convexity. However, note that a function that fails to be globally convex/concave can be convex/concave on parts of their domains. For example, the function in (iv) is convex on the part where it is solid and concave on the part where it is dotted.

After the above remark, the following famous and useful inequality should be quite believable and easy to remember.

Theorem. *(Basic version of Jensen's Inequality) Let $f(x)$ be **convex** on the interval I. Then for any $a, b \in I$,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}.$$

*On the other hand, if $f(x)$ is **concave** on I, then we have the reverse inequality for all $a, b \in I$:*

$$f\left(\frac{a+b}{2}\right) \geq \frac{f(a)+f(b)}{2}.$$

Remark. This can be interpreted as "for convex f, the average of f exceeds f of the average," which motivates the general form of Jensen's inequality:

Theorem. *(Jensen's Inequality) Let $f(x)$ be **convex** on the interval I. Then for any $x_1, \ldots, x_t \in I$,*

$$f(\textit{average of } \{x_i\}) \leq \textit{average of } \{f(x_i)\}.$$

*If f were **concave** instead, then the inequality would be reversed.*

Remark. We wrote the above inequality without explicitly stating what "average" meant. This is to allow weighted averages, subject to the condition that the same weight pattern is used on both the LHS and RHS.

2.1 That's great, but how do I prove that a function is convex?

1. If you know calculus, take the second derivative. It is a well-known fact that if the second derivative $f''(x)$ is ≥ 0 for all x in an interval I, then f is **convex** on I. On the other hand, if $f(x) \leq 0$ for all $x \in I$, then f is **concave** on I.

2. By the above test or by inspection, here are some basic functions that you should safely be able to claim are convex/concave.

   - Constant functions $f(x) = c$ are both **convex** and **concave**.
   - Powers of x: $f(x) = x^r$ with $r \geq 1$ are **convex** *on the interval* $0 < x < \infty$, and with $0 < r \leq 1$ are **concave** on that same interval. (Note that $f(x) = x$ is both convex and concave!)
   - Reciprocal powers: $f(x) = \frac{1}{x^r}$ are **convex** *on the interval* $0 < x < \infty$ for all powers $r > 0$. For negative odd integers r, $f(x)$ is **concave** *on the interval* $-\infty < x < 0$, and for negative even integers r, $f(x)$ is **convex** *on the interval* $-\infty < x < 0$.
   - The logarithm $f(x) = \log x$ is **concave** *on the interval* $0 < x < \infty$, and the exponential $f(x) = e^x$ is **convex** everywhere.

3. $f(x)$ is convex iff $-f(x)$ is concave.

4. You can combine basic convex functions to build more complicated convex functions.

   - If $f(x)$ is convex, then $g(x) = c \cdot f(x)$ is also convex for any *positive* constant multiplier c.
   - If $f(x)$ is convex, then $g(x) = f(ax+b)$ is also convex for *any* constants $a, b \in \mathbb{R}$. But the interval of convexity will change: for example, if $f(x)$ were convex on $0 < x < 1$ and we had $a = 5, b = 2$, then $g(x)$ would be convex on $2 < x < 7$.
   - If $f(x)$ and $g(x)$ are convex, then their sum $h(x) = f(x) + g(x)$ is convex.

5. If you are brave, you can invoke the oft-used claim "by observation," e.g,

$$f(x) = |x| \textit{ is convex by observation.}$$

The above statement might actually pass, depending on the grader, but the following desparate statement definitely will not:

$$f(x) = \frac{(x+1)^2}{2x^2+(3-x)^2} \textit{ is concave by observation.}$$

(It is false anyway, so that would ruin your credibility...)

Now prove the following results.

1. $f(x) = \frac{1}{1-x}$ is convex on $-\infty < x < 1$.

2. For any constant $c \in \mathbb{R}$, $f(x) = \frac{x^2}{c-x}$ is convex on $-\infty < x < c$.

   Solution:
   $$\frac{x^2}{c-x} = -\frac{x^2}{x-c} = -\left[x + c + \frac{c^2}{x-c}\right] = -x - c - \frac{c^2}{x-c}$$

   Each summand is convex.

3. $f(x) = \frac{x(x-1)}{2}$ is convex.

   Solution: Expand: $x^2/2 - x/2$. This is sum of two convex functions.

4. $f(x) = \frac{x(x-1)\cdots(x-r+1)}{r!}$ is convex on $r - 1 < x < \infty$.

   Solution: Forget about the $r!$ factor. Think of $f(x)$ as a product of r linear terms. Then by the product rule, f' is the sum of r products, each consisting of $r-1$ linear terms. For example, one of the terms would be $x(x-1)(x-3)(x-4)\cdots(x-r+1)$; this corresponds to the $x-2$ factor missing. But then the second derivative f'' is a sum of even more products, but each product consists of $r-2$ linear terms (with two factors missing). But for $x > r$, every factor is > 0, and we are taking a sum of products of them, so $f'' > 0$.

2.2 Endpoints of convex functions

- If $f(x)$ is **convex** on the interval $a \le x \le b$, then $f(x)$ attains a maximum, and that value is either $f(a)$ or $f(b)$.

- If $f(x)$ is **concave** on the interval $a \le x \le b$, then $f(x)$ attains a minimum, and that value is either $f(a)$ or $f(b)$.

2.3 Smoothing and unsmoothing

Theorem. *Let $f(x)$ be **convex** on the interval I. Suppose $a < b$ are both in I, and suppose $\epsilon > 0$ is a real number for which $a + \epsilon \le b - \epsilon$. Then $f(a) + f(b) \ge f(a+\epsilon) + f(b-\epsilon)$. If f is **strictly convex**, then the inequality is strict.*

This means that:

- For **convex** functions f, we can decrease the sum $f(a) + f(b)$ by "smoothing" a and b together, and increase the sum by "unsmoothing" a and b apart.

- For **concave** functions f, we can increase the sum $f(a) + f(b)$ by "smoothing" a and b together, and decrease the sum by "unsmoothing" a and b apart.

- In all of the above statements, if the convexity/concavity is **strict**, then the increasing/decreasing is strict as well.

This "smoothing principle" gives another way to draw conclusions about the assignments to the variables which bring the LHS and RHS closest together (i.e., sharpening the inequality). Hopefully, this process will give us a simpler inequality to prove.

Warning. Be careful to ensure that your smoothing process terminates. For example, if we are trying to prove that $(a+b+c)/3 \geq \sqrt[3]{abc}$ for $a, b, c \geq 0$, we can observe that if we smooth any pair of variables together, then the LHS remains constant while the RHS increases. Therefore, a naïve smoothing procedure would be:

> *As long as there are two unequal variables, smooth them both together into their arithmetic mean. At the end, we will have $a = b = c$, and the RHS will be exactly equal to the LHS, so we are done.*

Unfortunately, "the end" may take infinitely long to occur, if the initial values of a, b, c are unfavorable! Instead, one could use a smoothing argument for which each iteration increases the number of a, b, c equal to their arithmetic mean $(a+b+c)/3$.

2.4 Compactness

Definition. *Let $f(x_1, \ldots, x_t)$ be a function with domain D. We say that the function* ***attains a maximum on*** *D at some assignment $(x_i) = (c_i)$ if $f(c_1, \ldots, c_t)$ is greater than or equal to every other value $f(x_1, \ldots, x_t)$ with $(x_1, \ldots, x_t) \in D$.*

Remark. Not every function has a maximum! Consider, for example, the function $1/x$ on the domain $0 < x < \infty$, or even the function x on the domain $0 < x < 1$.

Definition. *Let D be a subset of $\mathbb{R}^n$.*

- *If D "includes its boundary", then we say that D is* ***closed****.*
- *If there is some finite radius r for which D is contained within the ball of radius r around the origin, then we say that D is* ***bounded****.*
- *If D is both closed and bounded, then[1] we say D is* ***compact****.*

Theorem. *Let f be a* ***continuous*** *function defined over a domain D which is* ***compact****. Then f attains a maximum on D, and also attains a minimum on D.*

3 Problems

1. (India 1995, from Kiran) Let $x_1, \ldots, x_n$ be positive numbers summing to 1. Prove that

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}.$$

   Solution: Done immediately by Jensen, just need to prove that $x/\sqrt{1-x}$ is convex on the interval $0 < x < 1$. Use the substitution $t = 1 - x$, and prove convexity (on the same interval) of $(1-t)/\sqrt{t} = 1/\sqrt{t} + (-\sqrt{t})$. But this is the sum of two convex functions, hence convex!

2. Let a, b, c be positive real numbers with $a + b + c \geq 1$. Prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{1}{2}.$$

   Solution: Let $S = a+b+c$. Then the fractions are of the form $\frac{x^2}{S-x}$, which is convex by the exercise in the previous section. Hence LHS is $\geq 3 \cdot \frac{(S/3)^2}{(2/3)S} = S/2$, and we assumed that $S \geq 1$.

[1] Strictly speaking, this is not the proper definition of "compactness," but rather is a consequence of the Heine-Borel Theorem.

3. Let $m \geq n$ be positive integers. Prove that every graph with m edges and n vertices has at least $\frac{m^2}{n}$ "V-shapes," which are defined to be $(x, \{y, z\})$, where $\{y, z\}$ is an unordered pair of distinct vertices, both of which are adjacent to x.

   Solution: The number is exactly $\sum \binom{d_i}{2}$, where the d_i are the degrees. Average degree is $2m/n$, and $\binom{x}{2}$ is convex, so Jensen implies that this is $\geq n\binom{2m/n}{2} \geq n(2m/n)(2m/n-1)/2 \geq m^2/n$.

4. (Application of Kövári-Sós-Turán.) Given n points in the plane, prove that the number of pairs of points which are distance 1 apart is at most $n^{3/2}$.

   Extra credit: Improve the bound to $100n^{4/3}$.

   Mega bonus: Improve the bound to $100^{100}n^{1.3333333333333333333333333333333333333333333333333333333333333333333333333}$.

   Solution: Zarankiewicz counting for $K_{2,3}$. Assume we have average degree at least $2n^{1/2}$ and no $K_{2,3}$. For each pair of vertices u, v, let the codegree $d(u, v)$ be their number of common neighbors. By assumption, all of these are at most 2, so their sum is at most $\binom{n}{2} \cdot 2 \leq n^2$.

   Now consider an arbitrary vertex x. It contributes to exactly $\binom{d(x)}{2}$ codegrees. Since $\binom{t}{2}$ is quadratic and convex, the total contribution to codegrees is

   $$\sum_x \binom{d(x)}{2} \geq n \cdot \binom{\overline{d}}{2}.$$

   Since we assumed $\overline{d} \geq 2n^{1/2}$, we get a total contribution of at least roughly $2n^2$, contradiction.

5. (Bulgaria, 1995) Let $n \geq 2$ and $0 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, n$. Show that

   $$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots x_nx_1) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

   Bonus: determine when there is equality.

   Solution: This is actually linear in each variable separately. Therefore, we can iteratively unsmooth each variable to an endpoint $\{0, 1\}$. Rewrite LHS as

   $$x_1(1 - x_2) + x_2(1 - x_3) + \cdots x_n(1 - x_1).$$

   With $\{0, 1\}$-assignments, each term is zero unless $x_i = 1$ and $x_{i+1} = 0$, in which case the term is 1. This is a "1, 0" pattern in consecutive variables when reading the x_i cyclically. Clearly, the maximum possible number of "1, 0" patterns is the claimed RHS.

6. (USAMO 1980/5) Show that for all real numbers $0 \leq a, b, c \leq 1$,

   $$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

   Solution: Consider b, c as constant, and see what happens if we perturb a. Then, the LHS is a convex function of a, because first term is linear, 2nd/3rd terms are of the form $\frac{\cdot}{x+\cdot}$, and the third term is a purely linear function of a. Hence LHS is maximized at one of the extreme values of a (either 0 or 1), so we may assume it is one of these. Similarly, $b, c \in \{0, 1\}$. This gives 8 assignmets of (a, b, c) to check, and they all work.

7. (Zvezda 1998) Prove for all reals $a, b, c \geq 0$:

   $$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

   Solution: If we smooth, say, a and b together, then LHS is invariant, and RHS is of the form $c\sqrt{ab} + \sqrt{c}\sqrt{ab}(\sqrt{a} + \sqrt{b})$. But $\sqrt{ab}$ will grow, as will $\sqrt{a} + \sqrt{b}$. Smooth until each element hits the arithmetic mean. Therefore, RHS will grow, so we may assume that all are equal. This corresponds to $(3t)^2/3 \geq 3t^2$, which is indeed true.

8. (T. Mildorf's *Inequalities*, problem 3) Let $a_1, \ldots, a_n \geq 0$ be real numbers summing to 1. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n \leq \frac{1}{4}.$$

   Solution: Observe that if any of the a_i are zero, then we could increase the LHS by deleting the zero entries (thus reducing the value of n). For example, if $n = 4$ and the sequence of a_i's was $(0.5, 0, 0.4, 0.1)$, then the corresponding LHS with $n = 3$ and a sequence of $(0.5, 0.4, 0.1)$ would be higher. Therefore, we may assume that the sequence of a_i has no zeros.

   Next, consider unsmoothing the pair corresponding to a_1 and a_3. Observe that a_1 is only multiplied by a_2, but a_3 is multiplied by $a_2 + a_4 > a_2$ if $n \geq 4$, because we just showed all $a_i > 0$. Therefore, we may increase the LHS by pushing a_1 all the way to zero, and giving its mass to a_3. Repeat this process (at most $n < \infty$ times) until $n \leq 3$.

   Finally, since we only have a_1, a_2, and a_3 left, the LHS is simply equal to $a_2(a_1 + a_3) = a_2(1 - a_2)$ because they sum to 1. Clearly, this has maximum value $1/4$. (There are also two other cases which correspond to ending up at $n = 2$ or $n = 1$, but those are trivial.)

9. (Hong Kong 2000) Let $a_1 \leq \cdots \leq a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$. Show that

$$a_1^2 + \cdots + a_n^2 + na_1a_n \leq 0.$$

   Solution: If there are at least two intermediate a_i, neither of which are equal to a_1 or a_n, then we can unsmooth them and increase LHS. So we may assume that there are x of the a_i equal to some $a = a_1$, at most one of the a_i equal to some b, and $n - x$ or $n - 1 - x$ of the a_i equal to some c.

   Case 1 (when there is no b): using $xa + (n - x)c = 0$, we solve and get $x = nc/(c - a)$. Then $xa^2 + (n - x)c^2 = -nac$, exactly what we needed.

   Case 2 (when there is one b): using $xa + b + (n-1-x)c = 0$, we solve and get $x = [b + (n-1)c]/(c-a)$. Then $xa^2 + b^2 + (n-1-x)c^2 = -(n-1)ac + b(-a-c) + b^2$. It suffices to show that $b(-a-c) + b^2 \leq -ac$. If $b \geq 0$, then use $b \leq c$ to get $b^2 \leq bc$, hence $b(-a-c) + b^2 \leq -ab \leq -ac$, final inequality using that $a \leq 0$ and $b \leq c$. On the other hand, if $b \leq 0$, then use $b \geq a$ to get $b^2 \leq ab$, hence $b(-a-c) + b^2 \leq -bc \leq -ac$, final inequality using that $c \geq 0$ and $b \geq a$.

10. (IMO 1984) For $x, y, z > 0$ and $x + y + z = 1$, prove that $xy + yz + xz - 2xyz \leq 7/27$.

    Solution: Smooth with the following expression: $x(y + z) + yz(1 - 2x)$. Now, if $x \leq 1/2$, then we can push y and z together. The mushing algorithm is as follows: first, if there is one of them that is greater than $1/2$, pick any other one and mush the other two until all are within $1/2$. Next we will be allowed to mush with any variable taking the place of x. Pick the middle term to be x; then by contradiction, the other two terms must be on opposite sides of $1/3$. Hence we can mush to get one of them to be $1/3$. Finally, use the $1/3$ for x and mush the other two into $1/3$. Plugging in, we get $7/27$.

11. (MOP 2008/6) Prove for real numbers $x \geq y \geq 1$:

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} \geq \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}.$$

12. (MOP Team Contest 2008) Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\prod x_i = 1$. Prove:

$$\sum \frac{1}{n - 1 + x_i} \leq 1.$$

13. (MOP 1998/5/5) Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be a sequence of real numbers. Prove that:

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}).$$

Solution: Since the inequality is homogeneous, we can normalize the a_k so that $a_1 = 1$. (If they are all zero, it is trivial anyway.) Now define the random variable X such that $P(X \geq k) = \sqrt{a_k}$. Then STS

$$\sqrt{\mathbb{E}[\min\{X_1, X_2\}]} \leq \mathbb{E}[\sqrt{X}],$$

where X, X_1, X_2 are i.i.d. Prove by induction on n. Base case is if $n = 1$, trivial. Now if you go to $n+1$ by shifting q amount of probability from $P(X = n)$ to $P(X = n+1)$, RHS will increase by exactly $q(\sqrt{n+1} - \sqrt{n})$. Yet LHS increases by exactly q^2 under the square root. Now since the probability shifted from $P(X = n)$, the square root was originally at least $q^2 n$. In the worst case, the LHS increases by $\sqrt{q^2 n + q^2} - \sqrt{q^2 n}$, which equals the RHS increase.

14. (Putnam 1957/B3.) Let $f : [0,1] \to \mathbb{R}^+$ be a monotone decreasing continuous function. Show that

$$\int_0^1 f(x)dx \int_0^1 xf(x)^2 dx \leq \int_0^1 xf(x)dx \int_0^1 f(x)^2 dx\,.$$

Solution: Apply the next problem, changing the measure to $dy = f(x)dx$, normalized to be a probability measure. Then x and $f(x)$ are anticorrelated, and Chebyshev applies with respect to that measure.

15. (Chebyshev.) Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be two sequences of nonnegative real numbers. Show that

$$\frac{a_1 b_1 + \cdots + a_n b_n}{n} \leq \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n}\,,$$

i.e., that the expected value of the product of two negatively correlated random variables is at most the product of their expected values.

Solution: Smoothing: push a_i closer together, and obviously the LHS increases while maintaining RHS. Also, this can be shown by considering $\int_0^1 \int_0^1 (f(x) - g(x))(f(y) - g(y))dxdy$, as this is always ≤ 0 due to the anticorrelation, and we use Fubini on the terms which mix x and y.

4 Real problems

1. (L., Lubetzky.) Let $g_0(s), g_1(s), g_2(s), \ldots$ be an infinite family of functions defined recursively by:

$$\begin{cases} g_0(s) = & e^{-s}\,, \\ g_{t+1}(s) = & \dfrac{1}{2\alpha}\left[g_t(\alpha s)^2 - g_t(\alpha s + \tfrac{1}{2})g_t(\alpha s) + g_t(\alpha s + \tfrac{1}{2}) + g_t(\tfrac{1}{2})g_t(\alpha s)\right] \text{ where } \alpha = \tfrac{1}{2}[1 + g_t(\tfrac{1}{2})]\,. \end{cases}$$

Prove that the sequence $g_t(1/2)$ converges to 1 as $t \to \infty$.

2. (L., Pikhurko, Sudakov.) Fix an integer $q \geq 2$ and a real parameter γ. Consider the following objective and constraint functions:

$$\textsc{obj}(\boldsymbol{\alpha}) := \sum_{A \neq \emptyset} \alpha_A \log |A|\,; \qquad \textsc{v}(\boldsymbol{\alpha}) := \sum_{A \neq \emptyset} \alpha_A, \quad \textsc{e}(\boldsymbol{\alpha}) := \sum_{A \cap B = \emptyset} \alpha_A \alpha_B.$$

The vector $\boldsymbol{\alpha}$ has $2^q - 1$ coordinates $\alpha_A \in \mathbb{R}$ indexed by the nonempty subsets $A \subset [q]$, and the sum in $\textsc{e}(\boldsymbol{\alpha})$ runs over *unordered* pairs of disjoint nonempty sets $\{A, B\}$. Let $\textsc{Feas}(\gamma)$ be the *feasible set* of vectors defined by the constraints $\boldsymbol{\alpha} \geq 0$, $\textsc{v}(\boldsymbol{\alpha}) = 1$, and $\textsc{e}(\boldsymbol{\alpha}) \geq \gamma$. In terms of q and γ, determine the maximum possible value $\textsc{obj}(\boldsymbol{\alpha})$ can take for any $\boldsymbol{\alpha} \in \textsc{Feas}(\gamma)$.

Graph theory

Po-Shen Loh

June 2013

1 Problems and famous results

1. (Putnam 1957/A5.) Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.

   Solution: Show that if there is any vertex of degree ≥ 3 in the unit distance graph, then it has a neighbor of degree 1 in the unit distance graph. Pulling off that neighbor by induction solves the problem, or else all degrees are ≤ 2, at which point the edge bound follows.

2. Every tournament (complete graph with each edge oriented in some direction) contains a Hamiltonian directed path (hitting every vertex exactly once).

3. (Romania 2006.) Each edge of a polyhedron is oriented with an arrow such that every vertex has at least one edge directed toward it, and at least one edge directed away from it. Show that some face of the polyhedron has its boundary edges coherently oriented in a circular direction.

   Solution: A directed cycle in the graph exists by simply following out-edges until we repeat vertices. If it has stuff inside it, then one can cut the cycle with a directed path, and then there is a shorter directed cycle. Compare the sizes of cycles by the number of faces they contain.

4. (Monotone paths.) Show that for every even n, it is possible to label the edges of K_n with the distinct integers $1, 2, \ldots, \binom{n}{2}$, in such a way that no increasing walk contains more than $n-1$ edges. An increasing walk is a sequence of vertices $v_0, v_1, \ldots, v_t$ such that the labels of the edges $v_i v_{i+1}$ increase with i. The vertices $v_0, \ldots, v_t$ are not required to be distinct—that is the difference between the definitions of walks and paths.

5. (Sweden 2010.) A town has $3n$ citizens. Any two persons in the town have at least one common friend in this same town. Show that one can choose a group consisting of n citizens such that every person of the remaining $2n$ citizens has at least one friend in this group of n.

   Solution: The codegree condition implies that the diameter of the graph is at most 2. We prove that every n-vertex graph with diameter ≤ 2 has a dominating set (a subset S of vertices such that every other vertex is either in, or has a neighbor in S) of size only $\leq \sqrt{n \log n} + 1$. To see this, let $p = \sqrt{\frac{\log n}{n}}$.

   Observe that since the diameter is at most 2, if any vertex has degree $\leq np$, then its neighborhood already is a dominating set of suitable size. Therefore, we may assume that all vertices have degree strictly greater than np. It feels "easy" to find a small dominating set in this graph because all degrees are high. Consider a random sample of np vertices (selected uniformly at random, with replacement), and let S be their union. Note that $|S| \leq np$. Now the probability that a particular fixed vertex v fails to have a neighbor in S is strictly less than $(1-p)^{np}$, because we need each of np independent samples to miss the neighborhood of v. This is at most $e^{-np^2} \leq e^{-\log n} = n^{-1}$. Therefore, a union bound over the n choices of v produces the result.

6. (Bondy 1.5.9.) There are n points in the plane such that every pair of points has distance ≥ 1. Show that there are at most $3n$ (unordered) pairs of points that span distance exactly 1 each.

   Solution: The unit distance graph is planar.

7. (Prüfer.) A graph with vertex set $\{1, \ldots, n\}$ is a *spanning tree* if it is a tree which includes all of those n vertices. It turns out that there is a surprisingly beautiful formula for the number of spanning trees on $\{1, \ldots, n\}$: it is just n^{n-2}.

8. Let n be an even integer. It is possible to partition the edges of K_n into exactly $n-1$ perfect matchings. (In this context of non-bipartite graphs, a perfect matching is a collection of $n/2$ edges that touch every vertex exactly once.) We can interpret that as a way to run a round-robin sports tournament among n teams: on each of $n-1$ days, the n teams pair up according to the day's perfect matching, and each of the $n/2$ edges tells who plays who that day. There are $n/2$ simultaneous games on each of the $n-1$ days.

   On each of the $n-1$ days, there are $n/2$ winning teams from the $n/2$ games. So, there are $n/2$ winners of Day 1, $n/2$ winners of Day 2, ..., and $n/2$ winners of Day $(n-1)$. Prove that no matter how the $\binom{n}{2}$ individual games turned out, it is always possible (after all of the games) to select one team who was a winner of Day 1, one team who was a winner of Day 2, ..., and one team who was a winner of Day $(n-1)$, such that we don't pick the same team twice. Note that since there are n teams in total, this selection will always leave exactly one team out.

Graph theory

Po-Shen Loh

June 2013

1 Basic results

We begin by collecting some basic facts which can be proved via "bare-hands" techniques.

1. The sum of all of the degrees is equal to twice the number of edges. Deduce that the number of odd-degree vertices is always an even number.

   Solution: By counting in two ways, we see that the sum of all degrees equals twice the number of edges.

2. A graph is called *bipartite* if it is possible to separate the vertices into two groups, such that all of the graph's edges only cross between the groups (no edge has both endpoints in the same group). Prove that this property holds if and only if the graph has no cycles of odd length.

   Solution: Separate into connected components. For each, choose a special vertex, and color based on parity of length of shortest path from that special vertex.

3. Every connected graph with all degrees even has an *Eulerian circuit*, i.e., a walk that traverses each edge exactly once.

   Solution: Start walking from a vertex v_1 without repeating any edges, and observe that by the parity condition, the walk can only get stuck at v_1, so we get one cycle. If we still have more edges left to hit, connectivity implies that some vertex v_2 on our current walk is adjacent to an unused edge, so start the process again from v_2. Splice the two walks together at v_2, and repeat until done.

4. Suppose that a graph has at least as many edges as vertices. Show that it contains a cycle.

   Solution: As long as there are vertices with degree exactly 1, delete both the vertex and its incident edge. Also delete all isolated vertices. These operations preserve $E \geq V$, but we can never delete everything because once $V = 1$, E must be 0, so we can never get down to only 1 vertex or less.

   Therefore we end up with a nonempty graph with all degrees ≥ 2, and by taking a walk around and eventually hitting itself, we get a cycle.

5. Suppose that the graph G has all degrees at most Δ. Prove that it is possible to color the vertices of G using $\leq \Delta + 1$ colors, such that no pair of adjacent vertices receives the same color.

   Solution: Consider the greedy algorithm for coloring vertices.

6. Let G_1, G_2, G_3 be three (possibly overlapping) graphs on the same vertex set, and suppose that G_1 can be properly colored with 2 colors, G_2 can be properly colored with 3 colors, and G_3 can be properly colored with 4 colors. Let G be the graph on the same vertex set, formed by taking the union of the edges appearing in G_1, G_2, G_3. Prove that G can be properly colored with 24 colors.

   Solution: Product coloring.

2 Matching

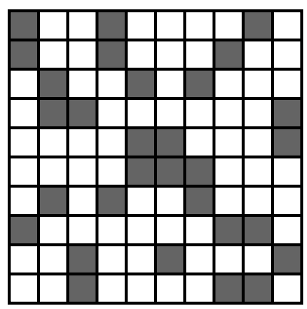
Consider a bipartite graph $G = (V, E)$ with partition $V = A \cup B$. A *matching* is a collection of edges which have no endpoints in common. We say that *A has a perfect matching to B* if there is a matching which hits every vertex in A.

Theorem. *(Hall's Marriage Theorem) For any set $S \subset A$, let $N(S)$ denote the set of vertices (necessarily in B) which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for* ***every*** *$S \subset A$.*

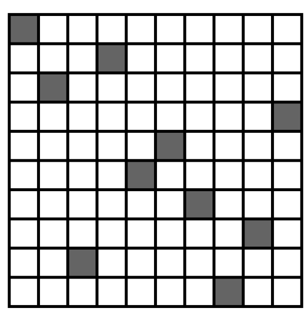
This has traditionally been called the "marriage" theorem because of the possible interpretation of edges as "acceptable" pairings, with the objective of maximizing the number of pairings. In real life, however, perhaps there may be varying degrees of "acceptability." This may be formalized by giving each vertex (in both parts) an ordering of its incident edges. Then, a matching M is called *unstable* if there is an edge $e = ab \notin M$ for which both a and b both prefer the edge e to their current partner (according to M).

Theorem. *(Stable Marriage Theorem) A stable matching always exists, for every bipartite graph and every collection of preference orderings.*

1. You are given a 10×10 grid, with the property that in every row, exactly 3 squares are shaded, and in every column, exactly 3 squares are shaded. An example is below.



Prove that there must always be a shaded *transversal*, i.e., a choice of 10 shaded squares such that no two selected squares are in the same row or column. An example is below.



Solution: Hall's theorem.

2. (Hall with deficiency.) In a bipartite graph, every subset S of the left side has $|N(S)| \geq |S| - 1$. Prove that there is an almost-perfect matching, in the sense that there is a matching which involves all but at most one vertex of the left side.

3. (Multi-Hall.) In a bipartite graph, every subset S of the left side has $|N(S)| \geq 2|S|$. Prove that there is a perfect 1-to-2 matching, in the sense that each vertex of the left is matched to a pair of vertices on the right, and all of the pairs on the right are disjoint.

4. Every k-regular bipartite graph contains a perfect matching.

5. (Diestel 2.11.) Suppose that a bipartite graph has bipartition $A \cup B$, and for every edge ab with $a \in A$ and $b \in B$, we have $\deg(a) \geq \deg(b)$. Prove that there is a perfect matching from A to B.

3 Planarity

When we represent graphs by drawing them in the plane, we draw edges as curves, permitting intersections. If a graph has the property that it can be drawn in the plane without any intersecting edges, then it is called *planar*. Here is the tip of the iceberg. One of the most famous results on planar graphs is the Four-Color Theorem, which says that every planar graph can be properly colored using only four colors. But perhaps the most useful planarity theorem in Olympiad problems is the Euler Formula:

Theorem. *Every connected planar graph satisfies $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces.*

Solution: Actually prove that $V - E + F = 1 + C$, where C is the number of connected components. Each connecting curve is piecewise-linear, and if we add vertices at the corners, this will keep $V - E$ invariant. Now we have a planar graph where all connecting curves are straight line segments.

Then induction on $E + V$. True when $E = 0$, because $F = 1$ and $V = C$. If there is a leaf (vertex of degree 1), delete both the vertex and its single incident edge, and $V - E$ remains invariant. If there are no leaves, then every edge is part of a cycle. Delete an arbitrary edge, and that will drop E by 1, but also drop F by 1 because the edge was part of a cycle boundary, and now that has merged two previously distinct faces.

Now, use the theorem to solve the following problems:

1. Prove that K_5 is not planar.

2. Prove that $K_{3,3}$ is not planar.

3. Prove that $K_{4,4}$ is not planar.

4. Prove that every planar graph can be properly colored using at most 6 colors.

The Euler criterion immediately implies that every connected graph has at least $E - (3V - 6)$ crossings. As it turns out, one can do much better:

Theorem. *(Ajtai, Chvátal, Newborn, Szemeredi; Leighton.) Every connected graph with $E \geq 4V$ has at least $\frac{E^3}{64V^2}$ crossings.*

4 Ramsey theory

Complete disorder is impossible.

— T. S. Motzkin, on the theme of Ramsey Theory.

The *Ramsey Number* $R(s, t)$ is the minimum integer n for which **every** red-blue coloring of the edges of K_n contains a completely red K_s or a completely blue K_t. Ramsey's Theorem states that $R(s, t)$ is always finite, and we will prove this in the first exercise below. The interesting question in this field is to find upper and lower bounds for these numbers, as well as for quantities defined in a similar spirit.

1. Prove by induction that $R(s, t) \leq \binom{s+t-2}{s-1}$. Note that in particular, $R(3, 3) \leq 6$.

Solution: Observe that $R(s,t) \le R(s-1,t)+R(s,t-1)$, because if we have that many vertices, then if we select one vertex, then it cannot simultaneously have $< R(s-1,t)$ red neighbors and $< R(s,t-1)$ blue neighbors, so we can inductively build either a red K_s or a blue K_t. But

$$\binom{(s-1)+t-2}{(s-2)} + \binom{s+(t-1)-2}{s-1} = \binom{s+t-2}{s-1},$$

because in Pascal's Triangle the sum of two adjacent guys in a row equals the guy directly below them in the next row.

2. Show that $R(t,t) \le 2^{2t}$. Then show that $R(t,t) > 2^{t/2}$ for $t \ge 3$, i.e., there is a red-blue coloring of the edges of the complete graph on $2^{t/2}$, such that there are no monochromatic K_t.

   Solution: The first bound follows immediately from the Erdős-Szekeres bound. The second is an application of the probabilistic method. Let $n = 2^{t/2}$, and consider a random coloring of the edges of K_n, where each edge independently receives its color with equal probabilities. For each set S of t vertices, define the event E_S to be when all $\binom{t}{2}$ edges in S are the same color. It suffices to show that $\mathbb{P}[\text{some } E_S \text{ occurs}] < 1$. But by the union bound, the LHS is

$$\begin{aligned}\binom{n}{t}\cdot\left(2\cdot 2^{-\binom{t}{2}}\right) &\le \frac{n^t}{t!}\cdot 2\cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= \frac{\left(2^{t/2}\right)^t}{t!}\cdot 2\cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= 2\cdot\frac{2^{t/2}}{t!}.\end{aligned}$$

   This final quantity is less than 1 for all $t \ge 3$.

3. (IMO 1964/4.) Seventeen people correspond by mail with one another—each one with all the rest. In their letters only 3 different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least 3 people who write to each other about the same topic.

   Solution: This is asking us to prove that the 3-color Ramsey Number $R(3,3,3)$ is ≤ 17. By the same observation as in the previous problem, $R(a,b,c) \le R(a-1,b,c)+R(a,b-1,c)+R(a,b,c-1)-1$. Then using symmetry, $R(3,3,3) \le 3R(3,3,2)-1$. It suffices to show that $R(3,3,2) \le 6$. But this is immediate, because if we have 6 vertices, if we even use the 3rd color on a single edge, we already get a K_2. So we cannot use the 3rd color. But then from above, we know $R(3,3) \le 6$, so we are done.

5 Blue

1. (Putnam 1957/A5.) Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are at distance 1 apart.

   Solution: Show that if there is any vertex of degree ≥ 3 in the unit distance graph, then it has a neighbor of degree 1 in the unit distance graph. Pulling off that neighbor by induction solves the problem, or else all degrees are ≤ 2, at which point the edge bound follows.

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   Observe that since the diameter is at most 2, if any vertex has degree $\leq np$, then its neighborhood already is a dominating set of suitable size. Therefore, we may assume that all vertices have degree strictly greater than np. It feels "easy" to find a small dominating set in this graph because all degrees are high. Consider a random sample of np vertices (selected uniformly at random, with replacement), and let S be their union. Note that $|S| \leq np$. Now the probability that a particular fixed vertex v fails to have a neighbor in S is strictly less than $(1-p)^{np}$, because we need each of np independent samples to miss the neighborhood of v. This is at most $e^{-np^2} \leq e^{-\log n} = n^{-1}$. Therefore, a union bound over the n choices of v produces the result.

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   On each of the $n-1$ days, there are $n/2$ winning teams from the $n/2$ games. So, there are $n/2$ winners of Day 1, $n/2$ winners of Day 2, $\ldots$, and $n/2$ winners of Day $(n-1)$. Prove that no matter how the $\binom{n}{2}$ individual games turned out, it is always possible (after all of the games) to select one team who was a winner of Day 1, one team who was a winner of Day 2, $\ldots$, and one team who was a winner of Day $(n-1)$, such that we don't pick the same team twice. Note that since there are n teams in total, this selection will always leave exactly one team out.

Putnam notes//The harmonic series

Almost the first divergent series (other than something like $\sum_{n=1}^{\infty} n$) that everybody sees is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Occasionally, we see problems that are based on the proof of this fact, so we will show something slightly more general; a simple variation on this has appeared on the Putnam.

PROBLEM 1. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, and that $a_n \leq a_{2n-1} + a_{2n}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

Of course the harmonic series satisfies this condition. Solution: We show that, for any $m \geq 1$ $\sum_{n=1}^{2^m} a_n \geq a_1 + ma_2$. This will do it, of course (by the Archimedean property of the reals). Actually, we see that $\sum_{n=2^m+1}^{2^{m+1}} a_n = \sum_{k=2^{m-1}+1}^{2^m} (a_{2k-1} + a_{2k}) \geq \sum_{k=2^{m-1}+1}^{2^m} a_k$ for any $m \geq 1$. If we assume as an induction hypothesis that this latter is $\geq a_2$, we get that each of the partial sums from $n = 2^m + 1$ to 2^{m+1} is $\geq a_2$. Since $\sum_{n=1}^{2^m} a_n = a_1 + a_2 + \sum_{n=2^1+1}^{2^2} a_n + \sum_{n=2^2+1}^{2^3} a_n + \cdots + \sum_{n=2^{m-1}+1}^{2^m} a_n$, this is at least $a_1 + ma_2$, as promised.

A slight variation, left to you:

PROBLEM 2. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, and that $a_n \leq a_{3n-2} + a_{3n-1} + a_{3n}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

More frequently, we see problems which use the result. These can come unexpectedly. For instance,

PROBLEM 3. (B3, Putnam 1985) Let $f : \mathcal{N} \times \mathcal{N} \longrightarrow \mathcal{N}$ be a function that takes each pair of positive integers to a positive integer. Suppose that f is onto and 8-to-one; that is, for each positive integer m, there are exactly 8 pairs (j, k) such that $f(j, k) = m$. show that there is a pair (j, k) such that $f(j, k) > jk$.

This is a slight paraphrase, but exactly the same question. Hard to see how the harmonic series comes in, isn't it? But just watch. Incidentally, as we will see there are two red herring in this problem; the number 8 and the specification that f is strictly 8-to-1.

Solution: Choose N such that $\sum_{n=1}^{N} \frac{1}{n} > 8$. For each $1 \leq j \leq N$, and $k \leq \frac{N!}{j}$, if we want to have $f(j, k) \leq jk$, we must have $f(j, k) \leq N!$ obviously. If this were possible, we would have $\sum_{n=1}^{N} \frac{N!}{n} > 8(N!)$ pairs (j, k) with $f(j, k) \leq N!$ But this is impossible if each value of f occurs not more than 8 times. (Each $m \leq N!$ can only occur 8 times, giving a total of $8(N!)$ values to dole out to all those pairs.)

Or, how about

PROBLEM 4. (Putnam ?) For any integer $n \geq 3$, we let D_n be the determinant of the $(n-2) \times (n-2)$ matrix which has the entries $3, 4, \ldots, n$ on the diagonal (in that order) and all other entries 1. Does the sequence $\frac{D_n}{(n-1)!}$ converge?

Solution: As is probably no surprise in this context, D_n is in fact $(n-1)![1 + \frac{1}{2} + \cdots + \frac{1}{n-1}]$, so the answer is no.

To calculate D_n, we start row-reducing the matrix. If we label the kth row R_k and let $R_k^* = R_k - R_{k+1}$ for each $1 \leq k \leq n-3$, the resulting matrix has the same determinant as the original. R_k^* has all but two entries zero; the diagonal entry drops to $k+1$, and just to the right of that, we have $-(k+2)$. The bottom row R_{n-2} has not, as yet, changed.

Ex gratia, for $n = 6$, the original matrix is $\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 6 \end{pmatrix}$ and after these row operations, it becomes $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 1 & 1 & 1 & 6 \end{pmatrix}$.

This is not yet upper triangular, but we row-reduce it further to systematically eliminate those 1's on the bottom row. To get rid of the first one, we replace R_{n-2} by $R_{n-2,1} = R_{n-2} - \frac{1}{2}R_1^*$. The 1 in the bottom left corner becomes a zero, the 1 next to it becomes $1 + \frac{3}{2} = 3(\frac{1}{2} + \frac{1}{3})$. The rest of the 1's and the n on the diagonal are left untouched so far.

Next replace $R_{n-2,1}$ by $R_{n-2,2} = R_{n-2,1} - (\frac{1}{2} + \frac{1}{3})R_2^*$. The first two entries in this row are zeroes, the third is now $4(\frac{1}{2} + \frac{1}{3} + \frac{1}{4})$. In the case above, we get first $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 0 & 3(\frac{1}{2} + \frac{1}{3}) & 1 & 6 \end{pmatrix}$ and then $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 4(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) & 6 \end{pmatrix}$.

We proceed like this (inductively, natch) for each $k \leq n-4$; if we have $R_{n-2,k-1}$ we replace it by $R_{n-2,k} = R_{n-2,k-1} - (\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1})R_k^*$. What occurs when we do this is that we replace the first nonzero entry of $R_{n-2,k-1}$ by zero, and the next 1 gets replaced by $1 + (k+2)(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1}) = (k+2)(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+2})$.

$R_{n-2,n-4}$ will be all zeroes, except its two right-most elements will be $(n-2)(\frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n-2})$ and n. Our last matrix above is this tage for $n = 6$. One more row operation, setting $R_{n-2,n-3} = R_{n-2,n-4} - (\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-2})R_{n-3}^*$, gives us an upper-triangular matrix, with entries 2,3,. . . ,$n-2$ and — in the corner — $n + (n-1)(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-2}) = (n-1)(1 + \frac{1}{2} + \cdots + \frac{1}{n-1})$. The 4×4 case is $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 5(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}) \end{pmatrix}$. Clearly, this matrix has determinant $(n-1)!(1+\frac{1}{2}+\cdots+\frac{1}{n-1})$; but our row operation shave not changed the value of the determinant.

As noted above, once we have D_n the problem becomes trivial — given that you know the harmonic series diverges.

Actually, I don't remember exactly whether the problem on the Putnam asked about the sequence $\frac{D_n}{(n-1)!}$ or about $\frac{D_n}{n!}$ / The latter sequence *does* converge, to 0 in fact, because it is $\frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) < \frac{1+\ln n}{n}$. An easy application of l'Hôpital's rule now finishes this problem.

One of the dangers of a problem like this is keeping track of the exact numbers (e.g., it's easy to think that $D_n = n!(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ or some such thing). This is partly why I carried the example along.

MY PUTNAM PROBLEMS

These are the problems I proposed when I was on the Putnam Problem Committee for the 1984–86 Putnam Exams. Problems intended to be A1 or B1 (and therefore relatively easy) are marked accordingly. The problems marked with asterisks actually appeared on the Putnam Exam (possibly reworded). — R. Stanley

1. (A1 or B1 problem) Given that
$$\int_0^1 \frac{\log(1+x)}{x}dx = \frac{\pi^2}{12},$$
evaluate
$$\int_0^1 \int_0^y \frac{\log(1+x)}{x}dx\,dy.$$

2.* (A1 or B1 problem) Let B be an $a\times b\times c$ brick. Let C_1 be the set of all points p in $\mathbb{R}^3$ such that the distance from p to C (i.e., the minimum distance between p and a point of C) is at most one. Find the volume of C_1.

3.* (A1 or B1 problem) If n is a positive integer, then define
$$f(n) = 1! + 2! + \cdots + n!.$$
Find polynomials $P(n)$ and $Q(n)$ such that
$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$
for all $n \geq 1$.

4. (A1 or B1 problem) Let C be a circle of radius 1, and let D be a diameter of C. Let P be the set of all points inside or on C such that p is closer to D than it is to the circumference of C. Find a rational number r such that the area of P is r.

5. Let n be a positive integer, let $0 \leq j < n$, and let $f_n(j)$ be the number of subsets S of the set $\{0, 1, \ldots, n-1\}$ such that the sum of the elements

of S gives a remainder of j upon division by n. (By convention, the sum of the elements of the empty set is 0.) Prove or disprove:

$$f_n(j) \leq f_n(0),$$

for all $n \geq 1$ and all $0 \leq j < n$.

6. Let P be the set of all real polynomials all of whose coefficients are either 0 or 1. Find

$$\inf\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(\alpha) = 0\}$$

and

$$\sup\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(0) = 1 \text{ and } f(\alpha) = 0\}.$$

Here inf denotes infinum (greatest lower bound) and sup denotes supremum (least upper bound).

Somewhat more difficult:

$$\sup\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(i\alpha) = 0\},$$

where $i^2 = -1$.

7. Let n be a positive integer, and let X_n be the set of all $n \times n$ matrices whose entries are $+1$ or -1. Call a nonempty subset S of X_n *full* if whenever $A \in S$, then any matrix obtained from A by multiplying a row or column by -1 also belongs to S. Let $w(A)$ denote the number of entries of A equal to 1. Find, as a function of n,

$$\max_S \frac{1}{|S|} \sum_{A \in S} w(A)^3,$$

where S ranges over all full subsets of X_n. (Express your answer as a polynomial in n.)

8.* Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leq 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral

$$\iiint_R x^1\, y^9\, z^8\, w^4\, dx\, dy\, dz$$

in the form $a!\, b!\, c!\, d!/n!$, where a, b, c, d, and n are positive integers.

9.* Let n be a positive integer, and let $f(n)$ denote the last nonzero digit in the decimal expansion of $n!$. For instance, $f(5) = 2$.

(a) Show that if $a_1, a_2, \ldots, a_k$ are distinct positive integers, then $f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k})$ depends only on the sum $a_1 + a_2 + \cdots + a_k$.

(b) Assuming (a), we can define $g(s) = f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k})$, where $s = a_1 + a_2 + \cdots + a_k$. Find the least positive integer p for which

$$g(s) = g(s+p), \text{ for all } s \geq 1,$$

or else show that no such p exists.

10.* A *transversal* of an $n \times n$ matrix is a set of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry of A is either -1, 1, or 0.

(b) All transversals of A have the same sum of their elements.

Find a formula for $f(n)$ of the form

$$a_1 \cdot b_1^n + a_2 \cdot b_2^n + a_3 \cdot b_3^n + a_4,$$

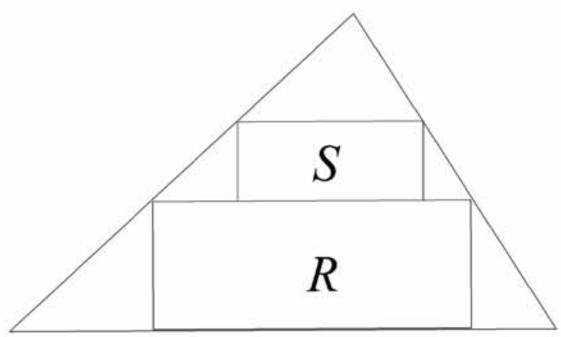
where a_i, b_i are rational numbers.

Easier version (not on Putnam Exam):

(a) Each entry of A is either 0 or 1.

(b) All transversals of A have the same number of 1's.

11.* Let T be a triangle and R, S rectangles inscribed in T as shown:

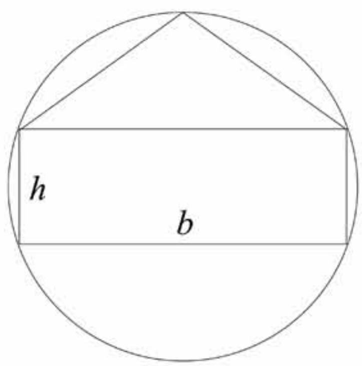


Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where T ranges over all triangles and R, S over all rectangles as above, and where A denotes area.

12.* (A1 or B1 problem) Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.



For what value of h do the rectangle and triangle have the same area?

13.* If $p(x) = \sum_{i=0}^{m} a_i x^i$ is a polynomial with real coefficients a_i, then set

$$, (p(x)) = \sum_{i=0}^{m} a_i^2.$$

Let $f(x) = 3x^2+7x+2$. Find (with proof) a polynomial $g(x)$ satisfying

$$g(0) = 1, \text{ and}$$

$$, (f(x)^n) = , (g(x)^n) \text{ for every integer } n \geq 1.$$

14.* Define polynomials $f_n(x)$ for $n \geq 0$ by

$$\begin{aligned} f_0(x) &= 1 \\ f'_{n+1}(x) &= (n+1) f_n(x+1), \ n \geq 0 \\ f_n(0) &= 0, \ n \geq 1. \end{aligned}$$

Find (with proof) the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Variation (not on Putnam Exam): $f_0(x) = 1$, $f_{n+1}(x) = xf_n(x) + f'_n(x)$. Find $f_{2n}(0)$.

15. Define
$$c(k, n) = \cos \frac{\pi k}{n} + \sqrt{1 + \cos^2 \frac{\pi k}{n}}.$$
Find (with proof) all positive integers n satisfying
$$c(1, n) = c(2, n)c(3, n).$$

16. Let R be a ring (not necessarily with identity). Suppose that there exists a nonzero element x of R satisfying
$$x^4 + x = 2x^3.$$
Prove or disprove: There exists a nonzero element y of R satisfying $y^2 = y$.

17. Find the largest real number λ for which there exists a 10×10 matrix $A = (a_{ij})$, with each entry a_{ij} equal to 0 or 1, and with exactly 84 0's, and there exists a nonzero column vector x of length 10 with real entries, such that $Ax = \lambda x$.

18. Choose two points p and q independently and uniformly from the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in the (x, y)-plane. What is the probability that there exists a circle C contained entirely within the first quadrant $x \geq 0$, $y \geq 0$ such that C contains x and y in its interior? Express your answer in the form $1 - (a + b\pi)(c + d\sqrt{e})$ for rational numbers a, b, c, d, e.

19.* (A1 or B1 problem) Let k be the smallest positive integer with the following property:

> There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$ has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

NOTE. The original version of this problem was considerably more difficult (and was not intended for A1 or B1). It was as follows:

Let $P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0$ be a monic polynomial of degree eleven with real coefficients a_i, with $a_0 \neq 0$. Suppose that all the zeros of $P(x)$ are real, i.e., if α is a complex number such that $P(\alpha) = 0$, then α is real. Find (with proof) the least possible number of nonzero coefficients of $P(x)$ (including the coefficient 1 of x^{11}).

20. Find (with proof) the largest integer k for which there exist three 9-element subsets X_1, X_2, X_3 of real numbers and k triples (a_1, a_2, a_3) satisfying $a_i \in X_i$ and $a_1 + a_2 + a_3 = 0$.

21. Let
$$S = \sum \frac{1}{m^2n^2},$$
where the sum ranges over all pairs (m, n) of positive integers such that the largest power of 2 dividing m is different from the largest power of 2 dividing n. Express S in the form $\alpha\pi^k$, where k is an integer and α is rational. You may assume the formula
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

22. Let a and b be nonnegative integers with binary expansions $a = a_0 + 2a_1 + \cdots$ and $b = b_0 + 2b_1 + \cdots$ (so $a_i, b_i = 0$ or 1), and define
$$a \wedge b = a_0b_0 + 2a_1b_1 + 4a_2b_2 + \cdots = \sum 2^i a_i b_i.$$
Given an integer $n \geq 0$, define $f(n)$ to be the number of pairs (a, b) of nonnegative integers satisfying $n = a + b + (a \wedge b)$. Find a polynomial $P(x)$ for which
$$\sum_{n=0}^{\infty} f(n)x^n = \prod_{k=0}^{\infty} P\left(x^{2^k}\right), \quad |x| < 1,$$
or show that no such $P(x)$ exists.

23. Given $v = (v_1, \ldots, v_n)$ where each $v_i = 0$ or 1, let $f(v)$ be the number of even numbers among the n numbers

$$v_1+v_2+v_3, v_2+v_3+v_4, \ldots, v_{n-2}+v_{n-1}+v_n, v_{n-1}+v_n+v_1, v_n+v_1+v_2.$$

For which positive integers n is the following true: for all $0 \le k \le n$, exactly $\binom{n}{k}$ vectors of the 2^n vectors $v \in \{0,1\}^n$ satisfy $f(v) = k$?

24. Let p be a prime number. Let c_k denote the coefficient of x^{2k} in the polynomial $(1 + x + x^3 + x^4)^k$. Find the remainder when the number $\sum_{k=0}^{p-1}(-1)^k c_k$ is divided by p. Your answer should depend only on the remainder obtained when p is divided by some fixed number n (independent of p).

25. Let $x(t)$ and $y(t)$ be real-valued functions of the real variable t satisfying the differential equations

$$\begin{aligned} \frac{dx}{dt} &= -xt + 3xy - 2t^2 + 1 \\ \frac{dy}{dt} &= xt + yt + 2t^2 - 1, \end{aligned}$$

with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

26.* Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers with $1 \le b_1 < b_2 < \cdots < b_n$. Suppose that there is a polynomial $f(x)$ satisfying

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Express $f(1)$ in terms of $b_1, \ldots, b_n$ and n (but independent from $a_1, \ldots, a_n$).

27. Given positive integers n and i, let x be the unique real number $\ge i$ satisfying $x^{x-i} = n$. Define $f(n,i) = (x+1)^{x-i}$, and set $f(0,i) = 0$ for all i. Suppose that $a_1, a_2, \ldots$ is a nonnegative integer sequence satisfying $a_{i+1} \le f(a_i, i)$ for all $i \ge 1$. Prove or disprove: a_i is a polynomial function of i for i sufficiently large.

28. Let $0 \leq x \leq 1$. Let the binary expansion of x be

$$x = a_1 2^{-1} + a_2 2^{-2} + \cdots$$

(where, say, we never choose the expansion ending in infinitely many 1's). Define

$$f(x) = a_1 3^{-1} + a_2 3^{-2} + \cdots.$$

In other words, write x in binary and read x in ternary. Evaluate $\int_0^1 f(x)dx$.

29.* Let $f(x,y,z) = x^2+y^2+z^2+xyz$. Let $p(x,y,z)$, $q(x,y,z)$, and $r(x,y,z)$ be polynomials satisfying

$$f(p(x,y,z), q(x,y,z), r(x,y,z)) = f(x,y,z).$$

Prove or disprove: (p,q,r) consists of some permutation of $(\pm x, \pm y, \pm z)$, where the number of minus signs is even.

30. Let

$$\frac{1}{1-x-y-z-6(xy+xz+yz)} = \sum_{r,s,t=0}^{\infty} f(r,s,t)x^r y^s z^t$$

(convergent for $|x|, |y|, |z|$ sufficiently small). Find the largest real number R for which the power series

$$F(u) = \sum_{n=0}^{\infty} f(n,n,n)u^n$$

converges for all $|u| < R$.

Problems on Polynomials

1. **Parity/congruence problems.** For these problems, try to use parity/congruences.

   (a) Let a, b, c be *odd* integers. Without using the quadratic formula, show that the polynomial $P(x) = ax^2 + bx + c$ has no integer roots. (Hint: Parity!)

   (b) Show that there exists no polynomial $P(x)$ with integer coefficients such that $P(1) = 4$ and $P(6) = 5$. (Hint: Congruences!)

   (c) If $P(x)$ is a polynomial with integer coefficients such that $P(3) = 10$, find an integer $x \neq 3$ such that $P(x)$ is divisible by 10.

2. **Roots and factors of polynomials.** For these problems try to apply the results about factors and roots of polynomials.

   (a) Let $P(x) = x^5 + 2x^4 - 3x^3 - 4x^2 + 5x - 1$. *Without calculation*, determine if $P(x)$ is divisible by $x - 1$.

   (b) Let $Q(x) = x^5 + 2x^4 - 3x^3 - 4x^2 + 5x + 1$ (i.e., $Q(x) = P(x) + 2$, where $P(x)$ is as above.) *Without calculations* determine if $Q(x)$ is divisible by $x^2 + x - 2$.

   (c) Find a degree n polynomial $P(x)$ that satisfies $P(k) = 0$ for $k = 1, 2, 3, \ldots, n$ and has leading coefficient 1.

3. **Roots and factors of a polynomial, II. More challenging problems.**

   (a) Let $P(x)$ be a polynomial of degree n satisfying $P(0) = 1$ and $P(k) = k$ for $k = 1, \dots, n$. Find $P(n+1)$.

   (b) Let $P(x)$ be a polynomial with integral coefficients. Suppose that there exist four distinct integers a_i ($i = 1, \dots, 4$) such that $P(a_i) = 5$ for $i = 1, \dots, 4$. Show that there exists no integer k such that $P(k) = 8$.

   (c) Let a, b, c be distinct real numbers, and suppose $P(x)$ is a polynomial that leaves remainders a, b, and c, when divided by $x - a$, $x - b$, and $x - c$, respectively. What is the remainder when $P(x)$ is divided by $(x - a)(x - b)(x - c)$?

4. **Congruences, II: More challenging problems.**

   (a) Let $P(x)$ be a polynomial with integer coefficients, and suppose that $P(x)$ has an integer root. Show that, for every positive integer m, at least one of the numbers $P(1), P(2), \ldots, P(m)$ is divisible by m.

   (b) Let $P(x)$ be a non-constant polynomial with integer coefficients and such that $|P(0)| \geq 2$. Show that there are infinitely many integers x such that $P(x)$ is composite.

   (c) Show that the result remains true without the assumption $|P(0)| \geq 2$, i.e., in the cases (a) $P(0) = 0$ (easy!) and (b) $P(0) = \pm 1$ (harder!). Thus, any non-constant polynomial with integer coefficients takes on infinitely many composite values.

Fun/Challenge Problems of the Week: A Polynomial Oracle

It is well-known that a polynomial of degree n is completely determined by its values at $n+1$ distinct points. Moreover, the number $n+1$ here is best-possible—knowing only n values is not sufficient to determine the polynomial. It may come as a surprise therefore that, if one restricts to polynomials with nonnegative integer coefficients, then the knowledge of only two cleverly chosen values, both at integers, is enough to uniquely determine the polynomial, regardless of its degree. This is the subject of this week's Fun/Challenge Problem:

> Suppose $P(x)$ is an unknown polynomial, of unknown degree, with nonnegative integer coefficients. Your goal is to determine this polynomial. You have access to an oracle that, given an integer n, spits out $P(n)$, the value of the polynomial at n. However, the oracle charges a fee for each such computation, so you want to minimize the number of computations you ask the oracle to do. Show that it is possible to uniquely determine the polynomial after only two consultations of the oracle.

Happy Problemsolving!

PREPARATION NOTES FOR THE PUTNAM COMPETITION

Note that the questions and solutions for the Putnam examinations are published in the *American Mathematical Monthly.* Generally, you may find them close to the end of each volume, in the October or November issue. Back issues of the *Monthly* can be found in the mathematics library (SS 622).

NUMBER THEORY

Putnam problems

1998-A-4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

1998-B-5. Let N be the positive integer with 1998 decimal digits, all of them 1; that is, $N = 1111 \cdots 11$ (1998 digits). Find the thousandth digit after the decomal point of $\sqrt{N}$.

1998-B-6. Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

1997-A-5. Let N_n denote the number of ordered $n-$tuples of positive integers $(a_1, a_2, \cdots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

1997-B-3. For each positive integer n write the sum $\sum_{m=1}^{n} \frac{1}{m}$ in the form $\frac{p_n}{q_n}$ where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

1997-B-5. Prove that for $n \geq 2$,

$$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n \equiv \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n-1 \pmod{n} .$$

1996-A-5. If p is a prime number greater than 3, and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.
(For example, $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 7 + 21 + 35 + 35 = 2 \cdot 7^2$.)

1995-A-3. The number $d_1d_2 \cdots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2 \cdots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1d_2 \cdots d_9$ by the corresponding digit e_i $(1 \leq i \leq 9)$ is divisible by 7. The number $f_1f_2 \cdots f_9$ is related to $e_1e_2 \cdots e_9$ in the same way; that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1d_2 \cdots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

1995-A-4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels x_1, x_2, $\cdots$, x_n satisfy

$$\sum_{i=1}^{k} x_i \leq k - 1 \qquad \text{for} \quad k = 1, 2, \cdots n .$$

1994-B-1. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \cdots, \}$. a is **within** n of b if $b - n \leq a \leq b + n$.)

1994-B-6. For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \leq a, b, c, d \leq 99$,

$$n_2 + n_b \equiv n_c + n_d \pmod{10100}$$

implies $\{a, b\} = \{c, d\}$.

1993-A-4. Let $x_1, x_2, \cdots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \cdots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

1993-B-1. Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994} \quad .$$

1993-B-5. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

1993-B-6. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$, and replace them with $2x$ and $y - x$.

1992-A-3. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$.

1992-A-5. For each positive integer n, let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}, \quad \text{for} \quad 0 \leq j \leq m-1 \; .$$

1989-A-1. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

1988-B-1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \cdots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

1988-B-6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \cdots\}$.

1987-A-2. The sequence of digits

$$1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,0\,1\,1\,1\,2\,1\,3\,1\,4\,1\,5\,1\,6\,1\,7\,1\,8\,1\,9\,2\,0\,2\,1 \cdots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in hich the $m-$digit numbers are placed, define $f(n)$ to be m. For example $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two digit integer 55. Find, with proof, $f(1987)$.

Other problems

1. (a) Let m be a positive integer. Prove that the number of digits used in writing down the numbers from 1 up to 10^m using ordinary decimal digits is equal to the number of zeros required in writing down the numbers from 1 up to 10^{m+1}.

   (b) Suppose that the numbers from 1 to n are written down using ordinary base-10 digits. Let $h(n)$ be the number of zeros used. Thus, $h(5) = 0$, $h(11) = 1$, $h(87) = 8$ and $h(306) = 57$. Does there exists an integer k such that $h(n) > n$ for every integer n exceeding k? If not, provide a proof; if so, give a specific value.

REAL NUMBERS

Putnam problems

1998-B-5. Let N be a positive integer with 1998 decimal digits, all of them 1; that is, $N = 1111 \cdots 11$ (1998 digits). Find the thousandth digits after the decomal point of $\sqrt{N}$.

1997-B-1. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$S_n = \sum_{m=1}^{6n-1} \min \left(\left\{ \frac{m}{6n} \right\}, \left\{ \frac{m}{3n} \right\} \right) .$$

(Here, min (a, b) denotes the minimum of a and b.)

1995-A-1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (notnecessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

1995-B-6. For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \cdots\}.$$

Prove that $\{1, 2, 3, \cdots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$. [As usual $\lfloor x \rfloor$ is the greatest integer $\leq x$.]

1994-A-5. Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{r_{1994}} , \qquad \text{with} \;\; i_1 < i_2 < \cdots < i_{1994} \;\; .$$

Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

1990-A-2. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \cdots)$? Justify your answer.

1990-A-4. Consider a paper punch that can be centered at any point of the plane, and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

Other problems

1. Let τ be the "golden ratio", *i.e.*, τ is the positive real for which $\tau^2 = \tau + 1$. Prove that, for each positive integer n,

$$\lfloor \tau \lfloor \tau n \rfloor \rfloor + 1 = \lfloor \tau^2 n \rfloor \quad .$$

INEQUALITIES

Putnam problems

1998-B-1. Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

1998-B-2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the $x-$axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1996-B-2. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}} \quad .$$

1996-B-3. Given that $\{x_1, x_2, \cdots, x_n\} = \{1, 2, \cdots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1 \ .$$

1988-B-2. Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Other problems

1. If $x > 1$, prove that

$$x > (1 + \frac{1}{x})^{x-1} \quad .$$

2. How many permutations $\{x_1, x_2, \cdots, x_n\}$ of $\{1, 2, \cdots, n\}$ are there such that the cyclic sum

$$\sum_{i=1}^{n} |x_i - x_{i+1}|$$

is (a) maximum? (b) minimum? (Note that $x_{n+1} = x_1$.) [CM 2018]

3. Let a, b, c, d be distinct real numbers for which

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \qquad \text{and} \qquad ac = bd \quad .$$

Find the maximum value of

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \ .$$

[CM 2020]

4. Let $n \geq m \geq 1$ and $x \geq y \geq 0$. Suppose that $x^{n+1} + y^{n+1} \leq x^m - y^m$. Prove that $x^n + y^n \leq 1$. [CM 2044]

5. Show that for positive reals a. b. c,

$$3 \max\{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}, \frac{b}{a} + \frac{c}{b} + \frac{a}{c}\} \geq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) .$$

[CM 2064]

6. Find all values of λ for which

$$2(x^3 + y^3 + z^3) + 3(1 + 3\lambda)xyz \geq (1 + \lambda)(x + y + z)(yz + zx + xy)$$

holds for all positive real x, y, z. [CM 2105]

7. Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left[\sum_{i=1}^{n} (a_i + b_i)\right]\left[\sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i}\right]$$

for any positive real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$. [CM 2113]

8. Let $a, b > 1$. Prove that

$$\prod_{k=1}^{n} (ak + b^{k-1}) \leq \prod_{k=1}^{n} (ak + b^{n-k}) \quad .$$

[CM 2145]

9. Suppose that a, b, c are real and that $|ax^2 + bx + c| \leq 1$ for $-1 \leq x \leq 1$. Prove that $|cx^2 + bx + a| \leq 2$ for $-1 \leq x \leq 1$. [CM 2153]

10. Let n be a positive integer and let t be a positive real. Suppose that $x_n = (1/n)(1 + t + t^2 + \cdots + t^{n-1})$. Show that, for each pair r, s of positive integers, there is a positive integer m for which $x_r x_s \leq x_m$. [CM 2159]

11. Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots b_n > 0$ and n be a positive integers. Prove that

$$\sqrt[n]{\prod_{k=1}^{n} (a_k + b_k)} \geq \sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt[n]{\prod_{k=1}^{n} b_k} \quad .$$

[CM 2176]

12. Prove that the inequality

$$\left(\frac{ab + ac + ad + bc + bd + cd}{6}\right)^{\frac{1}{2}} \geq \left(\frac{abc + abd + acd + bcd}{4}\right)^{\frac{1}{3}}$$

holds for any positive reals a, b, c, d.

13. Let $a > 0$, $0 \leq x_1, x_2, \cdots, x_n \leq a$ and n be an integer exceeding 1. Suppose that

$$x_1 x_2 \cdots x_n = (a - x_1)^2 (a - x_2)^2 \cdots (a - x_n)^2 \ .$$

Determine the maximum possible value of the product. [CM 1781]

14. Prove that, if n and m are positive integers for which $n \geq m^2 \geq 16$, then $2^n \geq n^m$. [CM 2163]

15. Let $x, y, z \geq 0$ with $x + y + z = 1$. For real numbers a, b, determine the maximum value of $c = c(a, b)$ for which $a + bxy \geq c(yz + zx + xy)$. [CM 2172]

16. Let $0 < x, y < 1$. Prove that the minimum of $x^2 + xy + y^2$, $x^2 + x(y-1) + (y-1)^2$, $(x-1)^2 + (x-1)y + y^2$ and $(x-1)^2 + (x-1)(y-1) + (y-1)^2$ does not exceed $1/3$.

17. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

SEQUENCES, SERIES AND RECURRENCES

Notes

1. $x_{n+1} = ax_n$ has the general solution $x_n = x_1 a^{n-1}$.
2. $x_{n+1} = x_n + b$ has the general solution $x_n = x_1 + (n-1)b$.
3. $x_{n+1} = ax_n + b$ (with $a \neq 1$) can be rewritten $x_{n+1} + k = a(x_n + k)$ where $(a-1)k = b$ and so reduces to the recurrence 1.
4. $x_{n+1} = ax_n + bx_{n-1}$ has different general solution depending on the discriminant of the characteristic polynomial $t^2 - at - b$.
   (a) If $a^2 - 4b \neq 0$ and the distinct roots of the characteristic polynomial are r_1 and r_2, then the general solution of the recurrence is
   $$x_n = c_1 r_1^n + c_2 r_2^n$$
   where the constants c_1 and c_2 are chosen so that
   $$x_1 = c_1 r_1 + c_2 r_2 \qquad \text{and} \qquad x_2 = c_1 r_1^2 + c_2 r_2^2 \quad .$$
   (b) If $a^2 - 4b = 0$ and r is the double root of the characteristic polynomial, then
   $$x_n = (c_1 n + c_0) r^n$$
   where c_1 and c_0 are chosen so that
   $$x_1 = (c_1 + c_0) r \qquad \text{and} \qquad x_2 = (2c_1 + c_0) r^2 \quad .$$
5. $x_{n+1} = (1-s)x_n + sx_{n-1} + r$ can be rewritten $x_{n+1} - x_n = -s(x_n - x_{n-1}) + r$ and solved by a previous method for $x_{n+1} - x_n$.
6. $x_{n+1} = ax_n + bx_{n-1} + c$ where $a + b \neq 1$ can be rewritten $(x_{n+1} + k) = a(x_n + k) + b(x_{n-1} + k)$ where $(a + b - 1)k = c$ and solved for $x_n + k$.
7. The general homogeneous linear recursion has the form
   $$x_{n+k} = a_{k-1} x_{n+k-1} + \cdots + a_1 x_{n+1} + a_0 \quad .$$
   Its characteristic polynomials is
   $$t^k - a_{k-1} t^{k-1} - \cdots - a_1 t - a_0 \quad .$$
   Let r be a root of this polynomial of multiplicity m; then the nth term of the recurrence is a linear combination of terms of the type
   $$(c_{m-1} r^{m-1} + \cdots + c_1 r + c_0) r^n \quad .$$

1998-A-4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

1998-B-4. Find necessaary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0 \ .$$

1997-A-6. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1} \ .$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

1994-A-1. Suppose that a sequence $a_1, a_2, a_3, \cdots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

1994-A-5. Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

1993-A-2. Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1$$

for $n = 1, 2, 3, \cdots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

1993-A-6. The infinite sequence of $2's$ and $3's$

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \cdots$$

has the property that, if one forms a second sequence that records the number of $3's$ between successive $2's$, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

1992-A-1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions
(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

1992-A-5. For each positive integer n, let

$$a_n = \begin{cases} 0, & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1, & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}$$

for $0 \leq j \leq m-1$.

1991-B-1. For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

1990-A-1. Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3} \quad .$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

1988-B-4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is

$$\sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \ .$$

1985-A-4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

1979-A-3. Let x_1, x_2, x_3, $\cdots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-2}} \qquad \text{for} \quad n = 3, 4, 5, \cdots \quad .$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

1975-B-6. Show that, if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then
(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and
(b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

Other problems

1. Let n be an even positive integer and let $x_1, x_2, \cdots, x_n$ be n positive reals. Define

$$f(x_1, \cdots, x_n) = \left(\frac{x_2 + x_3}{2}, \frac{x_2 + x_3}{2}, \frac{x_4 + x_5}{2}, \frac{x_4 + x_5}{2}, \cdots, \frac{x_n + x_1}{2}, \frac{x_n + x_1}{2} \right) \quad .$$

   Determine

$$\lim_{k \to \infty} f^k(x_1, \cdots, x_n)$$

   where f^k denotes the kth iterate of f, *i.e.*, $f^k = f \circ f^{k-1}$ for $k \geq 2$.

2. Let $0 < x_0 < 1$ and, for $n \geq 0$, $x_{n+1} = x_n(1 - x_n)$. Prove that $\sum x_n$ diverges while $\sum x_n^2$ converges. Discuss the behaviour of nx_n as $n \to \infty$.

3. (a) Consider a finite family of arithmetic progressions of integers, each extending infinitely in both directions. Each two of the progressions have a term in common. Prove that all progressions have a term in common.

(b) Can the assumption that all terms be integers be dropped?

4. A class of sequences is defined by

$$S_1 = \{1,1\}$$
$$S_2 = \{1,2,1\}$$
$$S_3 = \{1,3,2,3,1\}$$
$$S_4 = \{1,4,3,5,2,5,3,4,1\}$$

and for integers $n \geq 3$, if

$$S_n = \{a_1, a_2, \cdots, a_{m-1}, a_m\} \quad ,$$

then

$$S_{n+1} = \{a_1, a_1+a_2, a_2, a_2+a_3, \cdots, a_{m-1}, a_{m-1}+a_m, a_m\} \quad .$$

How many terms in S_n are equal to n?

5. Prove of disprove, where $i^2 = -1$, that

$$\frac{1}{4i}\sum\{i^k \tan(\frac{k\pi}{4n}) : 1 \leq k \leq 4n, gcd(n,k) = 1\}$$

is an integer. [CM 2129]

CALCULUS, ANALYSIS

Putnam problems

1998-A-3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0 \ .$$

1997-A-3. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdots 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right) dx \ .$$

1996-A-6. Let $c \geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbf{R}$. [Note: $\mathbf{R}$ is the set of real numbers.]

1995-A-2. For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}}\right) dx$$

converge?

1994-A-2. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the $x-$axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the $y-$axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

1994-B-3. Find the set of all real numbers k with the following property:

For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

1994-B-5. For any real number α, define the function f_α by $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \le k \le n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2) \ .$$

($\lfloor x \rfloor$ denotes the greatest integer $\le x$, and $f_\alpha^k = f_\alpha \circ \cdots \circ f_\alpha$ is the $k-$fold composition of f_α.)

1993-A-1. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.

1993-A-5. Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{1}{100}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

is a rational number.

1993-B-4. The function $K(x, y)$ is positive and continuous for $0 \le x \le 1$, $0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all x, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x, y)dy = g(x) \qquad \text{and} \qquad \int_0^1 g(y)K(x, y)dy = f(x) \ .$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

1992-A-2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1 + x)^\alpha$. Evaluate

$$\int_0^1 C(-y-1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992} \right) dy \ .$$

1992-A-4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f(\frac{1}{n}) = \frac{n^2}{n^2 + 1}, \qquad n = 1, 2, 3, \cdots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \cdots$.

1992-B-3. For any pair (x, y) of real numbers, a sequence $(a_n(x, y))_{n \ge 0}$ is defined as follows:

$$a_0(x, y) = x$$

$$a_{n+1}(x, y) = \frac{(a_n(x, y))^2 + y^2}{2} \ , \qquad \text{for all} \ \ n \ge 0 \ .$$

Find the area of the region

$$\{(x,y)|(a_n(x,y))_{n\geq 0} \text{ converges}\}$$

.

1992-B-4. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with x^3-x. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x}{x^3-x}\right)=\frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

Other problems

1. Let $f(x)$ be a continuously differentiable real function defined on the closed interval $[0,1]$ for which

$$\int_0^1 f(x)dx = 0 \quad .$$

   Prove that

$$2\int_0^1 f(x)^2 dx \leq \int_0^1 |f'(x)|dx \cdot \int_0^1 |f(x)|dx \quad .$$

2. Let $0<x,y<1$. Prove that the minimum of x^2+xy+y^2, $x^2+x(y-1)+(y-1)^2$, $(x-1)^2+(x-1)y+y^2$ and $(x-1)^2+(x-1)(y-1)+(y-1)^2$ does not exceed $1/3$.

3. Integrate

$$\int \tan^2(x-a)\tan^2(x-b)dx \quad .$$

4. Integrate

$$\int_0^{\frac{\pi}{2}} \frac{x\cos x\sin x}{\cos^4 x+\sin^4 x}dx \ .$$

5. Let $O=(0,0)$ and $Q=(1,0)$. Find the point P on the line with equation $y=x+1$ for which the angle OPQ is a maximum.

DIFFERENTIAL EQUATIONS

First Order Equations

1. Linear $y'+p(x)y=q(x)$
   Muliply through by the integrating factor $\exp(\int p(x)dx)$ to obtain

$$(y\exp(\int p(x)dx))' = q(x)\exp(\int p(x)dx) \quad .$$

2. Separation of variables $y'=f(x)g(y)$
   Put in the form $dy/g(y)=f(x)dx$ and integrate both sides.

3. Homogeneous $y'=f(x,y)$ where $f(tx,ty)=f(x,y)$.
   Let $y=ux$, $y'=u'x+u$ to get $u'x+u=f(1,u)$.

4. Fractional linear
$$y' = \frac{ax+by}{cx+dy} \quad .$$
Do as in case 3, or introduce an auxiliary variable t and convert to a system
$$\frac{dy}{dt} = ax+by \qquad\qquad \frac{dx}{dt} = cx+dy$$
and try $x = c_1 e^{\lambda t}$, $y = c_2 e^{\lambda t}$. If this yields two distinct values for λ, the ratio $c_1 : c_2$ can be found.If there is only one value of λ, try $x = (c_1 + c_2 t)e^{\lambda t}$, $y = (c_3 + c_4 t)e^{\lambda t}$.

5. Modified fractional linear
$$y' = \frac{ax+by+c}{hx+ky+r} \quad .$$
(i) If $ak - bh \neq 0$, choose p, q so that $ap + bq + c = 0$, $hp + kq + r = 0$ and make a change of variables: $X = x - p$, $Y = y - q$.
(ii) If $ak - bh = 0$, set $u = ax + by + c$ to obtain a separation of variables equation in y and u.

6. General linear exact equation
$$P(x,y)dx + Q(x,y)dy = 0$$
where
$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 \quad .$$
The equation has a solution of the form $f(x,y) = c$, where $\partial f/\partial x = P$ and $\partial f/\partial y = Q$. We have
$$f(x,y) = \int_{x_0}^{x} P(u,x)du + \int_{y_0}^{y} Q(x_0,v)dv$$
or
$$f(x,y) = \int_{y_0}^{y} Q(x,v)dv + \int_{x_0}^{x} P(u,y_0)du$$
where (x_0, y_0) is any point.

7. General linear inexact equation
$$P(x,y)dx + Q(x,y)dy = 0$$
where
$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \neq 0 \quad .$$
We need to find an "integrating factor" $h(x,y)$ to satisfy
$$h(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) + P\frac{\partial h}{\partial y} - Q\frac{\partial h}{\partial x} = 0 \quad .$$
There is no general method for equations of this type, but one can try assuming that h is a function of x alone, y alone, xy, x/y or y/x. If h can be found, multiply the equation through by h and proceed as in Case 6.

8. Riccati equation $y' = f_0(x) + f_1(x)y + f_2(x)y^2$ where $f_2(x) \neq 0$.
If a solution y_0 is known, set $y = y_0 + (1/u)$ to get a first order linear equation in u and x.

9. Bernoulli equation $y' + p(x)y = q(x)y^n$.
If $n = 0$, use Case 1. If $n = 1$, use Case 2. If $n = 2$, consider Case 8. If $n \neq 0, 1$, set $u = y^{1-n}$ to get a first order equation in u and x.

Linear equations of nth order with constant coefficients

A linear equation of the nth degree with constant coefficients has the form

$$c_n y^{(n)} + c_{n-1} y^{(n-1)} + \cdots + c_2 y'' + c_1 y' + c_0 y = q(x)$$

where the c_i are constants and y is an unknown function of x. The general solution of such an equation is the sum of two parts:

the complementary function (general solution of the homogeneous equation formed by taking $q(x) = 0$);

a particular integral (any solution of the given equation).

The complementary function is the sum of terms of the form

$$(a_{r-1} x^{r-1} + a_{r-2} x^{r-2} + \cdots + a_2 t^2 + a_1 t + a_0) e^{\lambda x}$$

where λ is a root of multiplicity r of the auxiliary polynomial

$$c_n t^n + c_{n-1} t^{n-1} + \cdots + c_2 t^2 + c_1 t + c_0$$

and a_i are arbitrary constants.

A particular integral can be found when $q(x)$ can be written as the sum of terms of the type $h(x)e^{\rho x}$ where $h(x)$ is a polynomial and ρ is complex. This includes the cases $q(x) = h(x)e^{\alpha x} \sin \beta x$ and $q(x) = h(x)e^{\alpha x} \cos \beta x$, with α and β real. When the c_i and the coefficients of $h(x)$ are real, solve with $q(x) = h(x)e^{(\alpha + i\beta)}x$ and take the real or imaginary parts, respectively, of the solution obtained.

Operational calculus: Let $Du = u'$. The left side of the equation can be written $p(D)y = q(x)$ where $p(t) = c_n t^n + \cdots + c_1 t + c_0$. For any polynomial $p(t)$, we have the operational rules

$$p(D)e^{rx} = p(r)e^{rx} \qquad \text{and} \qquad p(D)(ue^{rx}) = e^{rx} p(D + r)u \quad .$$

The following examples illustrate how a particular integral can be obtained without having to deal with special cases or undetermined coefficients:

(i) $y'' + 2y' + 2y = 2e^{-x} \sin x$.

The solution of this equation is the imaginary part of the solution of the following equation

$$(D^2 + 2D + 2)y = 2e^{(-1+i)x} \quad .$$

We try for a particular integral of the form $y = ue^{(-1+i)x}$, where the exponent agrees with the exponent on the right side of the equation. Then, factoring the polynomial in D and substituting for y, we obtain:

$$(D + 1 - i)(D + 1 + i)(ue^{(-1+i)x}) = 2e^{(-1+i)x} \quad .$$

Bringing the exponent through and cancelling it, we get

$$D(D + 2i)u = (D + 2i)Du = 2 \qquad (*)$$

Differentiate:

$$(D + 2i)D^2 u = 0 \qquad (**) \; .$$

Any u for which (*) and (**) hold will yields a particular integral. Choose u such that $u'' = 0$. Then (**) holds. To satisfy (*), we need $2iDu = 2$ or $Du = -i$. Hence take $u(x) = -ix$. A particular integral of the complex equation is $-ix \exp((-1 + i)x)$, and a particular integral of the original equation is

$$\text{Im } (-ixe^{(-1+i)x}) = -xe^{-x} \cos x \quad .$$

(ii) $y'' - 4y' + 4y = 8x^2 e^{2x} \sin 2x$

A particular integral of this equation is the imaginary part of the particular integral of the equation

$$(D-2)^2 y = 8x^2 e^{(2+2i)x}.$$

Let $y = ue^{(2+2i)x}$. Then, bring the exponential through the operator as before and cancelling, we get

$$(D+2i)^2 u = (D^2 + 4iD - 4)u = 8x^2 \quad (1)$$

$$(D^2 + 4iD - 4)Du = 16x \quad (2)$$

$$(D^2 + 4iD - 4)D^2 u = 16 \quad (3)$$

$$(D^2 + 4iD - 4)D^3 u = 0 \quad (4) .$$

Let $D^3 u = 0$ to satisfy (4). Then (3) requires $-4D^2 u = 16$ or $D^2 u = -4$. Now, (2) requires $-16i - 4Du = 16x$ or $Du = -16x - 16i$. Finally, to make (1) hold, we need $u = -2x^2 - 4ix + 3$. The solution to (ii) is thus

$$y = \mathrm{Im}(-2x^2 - 4ix + 3)e^{(2+2i)x} \quad .$$

Putnam questions

1997-B-2. Let f be a twice differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x) \ ,$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

1995-A-5. Let $x_1, x_2, \cdots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\cdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} \geq 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \cdots, x_n$ necessarily linearly dependent?

1989-B-3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a nonnegative integer, define

$$\mu_n = \int_0^\infty x^n f(x) dx$$

(sometimes called the nth moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that this limit is 0 only if $\mu_0 = 0$.

1988-A-2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

GEOMETRY

Putnam problems

1998-A-1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1998-A-2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the $x-$axis and B be the area of the region lying to the right of the $y-$axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position of s.

1998-A-5. Let $\mathfrak{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, D_2, \cdots, D_n$ in $\mathfrak{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E \ .$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

1998-A-6. Let A, B, C denote distinct points with integer points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1 \ ,$$

then A, B, C are three vertices of a square. Here $[XY]$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

1998-B-2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the $x-$axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

1998-B-3. Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P a regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α and β are real numbers.

1997-A-1. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

1997-B-6. The dissection of the $3 - 4 - 5$ triangle shown below has diameter $5/2$.

Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between [airs of points belonging to the same part.)

1996-A-1. Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed into that rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangles.

1996-A-2. Let C_1 and C_2 be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exist points x on C_1 and Y on C_2 such that M is the midpoint of the lines segment XY.

1996-B-6. Let $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0) \quad .$$

1995-B-2. An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin(x/a)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

Other problems

1. A large sheet of paper is ruled by horizontal and vertical lines spaced a distance of 1 cm. A clear plastic sheet is free to slide on top of it. Some ink has been spilled on the sheet, making one or more blots. The total area of these blots is less than 1 sq. cm. Prove that the plastic sheet can be positioned so that none of the intersections of the horizontal and vertical lines is covered by any of the blots.

2. A rectangular sheet of paper is laid upon a second rectangular sheet of identical size as indicated in the diagram. Prove that the second sheet covers at least half the area of the first sheet.

3. Three plane mirrors that meet at a point are mutually perpendicular. A ray of light reflects off each mirror exactly once in succession. Prove that the initial and final directions of the ray are parallel and opposite.

4. Let $ABCD$ be a square with E the midpoint of CD. The vertex B is folded up to E and the page is flattened to produce a straight crease. If the fold along this crease also takes A to F, prove that EF intersects AD in a point that trisects the side AD.

5. A closed curve is drawn in the plane which may intersect itself any finite number of times. The curve passes through each point of self-intersection exactly twice. Suppose that the points of self-intersection

are labelled A, B, C, $\cdots$. Beginning at any (nonintersection) point on the curve, trace along the curve recording each intersection point in turn as you pass through it, until you return to the starting point. Each intersection point will be recorded exactly twice. Prove that between the two occurrences of any intersection point are *evenly* many points of intersection.

For the example below, starting at P and proceeding to the right, we encounter the points $ABCADFGEFGEDBC$ in order before returning to P.

6. Let A, B, C be three points in the plane, any pair of which are unit distance apart. For each point P, we can determine a triple of nonnegative real numbers (u, v, w), where u, v, w are the respective lengths of PA, PB, PC. Of course, not every triple of nonnegative reals arise in this way, and when two of the numbers are given, there are at most finitely many possibilities for the third. This suggests that there must be a relationship among them.

   (a) Find a polynomial equation that must be satisfied by u, v and w as described above.

   (b) If we take fixed values of u and v and regard the equation in (a) as a polynomial in w, analyze the character of its roots and relate this to the geometry of the situation.

7. A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

   and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

GROUP THEORY AND AXIOMATICS

The following concepts should be reviewed: group, order of groups and elements, cyclic group, conjugate elements, commute, homomorphism, isomorphism, subgroup, factor group, right and left cosets.

Lagrange's Theorem: The order of a finite group is exactly divisible by the order of any subgroup and by the order of any element of the group.

A group of prime order is necessarily commutative and has no proper subgroups.

A subset S of a group G is a set of *generators* for G iff every element of G can be written as a product of elements in S and their inverses. A *relation* is an equation satisfied by one or more elements of the group. Many Putnam problems are based on the possibility that some relations along with the axioms will imply other relations.

Putnam problems

1997-A-4. Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element a in G such that $\psi(x) = a\phi(x)$ is a homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all x and y in G).

1996-A-4. Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that:

1. $(a, b, c) \in S$ if and only if $(b, c, a) \in S$,

2. $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$,

3. (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

Prove that there exists a one-to-one function $g : A \to \mathbf{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. [Note: $\mathbf{R}$ is the set of real numbers.]

1989-B-2. Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \cdots\}$ is finite. Must S be a group?

1978-A-4. A "bypass" operation on a set S is a mapping from $S \times S$ to S with the property

$$B(B(w, x), B(y, z)) = B(w, z)$$

for all w, x, y, z in S.

(a) Prove that $B(a, b) = c$ implies $B(c, c) = c$ when B is a bypass.

(b) Prove that $B(a, b) = c$ implies $B(a, x) = B(c, x)$ for all x in S when B is a bypass.

(c) Construct a table for a bypass operation B on a finite set S with the following three properties: (i) $B(x, x) = x$ for all x in S. (ii) There exists d and e in S with $B(d, e) = d \neq e$. (iii) There exists f and g in S with $B(f, g) \neq f$.

1977-B-6. Let H be a subgroup with h elements in a group G.Suppose that G has an element a such that, for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products $x_1ax_2a \cdots x_na$, with n a positive integer and the x_i in H.

(a) Show that P is a finite set.

(b) Show that, in fact, P has no more than $3h^2$ elements.

1976-B-2. Suppose that G is a group generated by elements A and B, that is, every element of G can be written as a finite "word" $A^{n_1}B^{n_2}A^{n_3} \cdots B^{n_k}$, where $n_1, n_2, \cdots n_k$ are any integers, and $A^0 = B^0 = 1$, as usual. Also, suppose that

$$A^4 = B^7 = ABA^{-1}B = 1 \ , \quad A^2 \neq 1 \ , \quad \text{and} \quad B \neq 1 \ .$$

(a) How many elements of G are of the form C^2 with C in G?

(b) Write each such square as a word in A and B.

1975-B-1. In the additive group of ordered pairs of integers (m, n) (with addition defined componentwise), consider the subgroup H generated by the three elements

$$(3, 8) \qquad (4, -1) \qquad (5, 4) \quad .$$

Then H has another set of generators of the form

$$(1, b) \qquad (0, a)$$

for some integers a, b with $a > 0$. Find a.

1972-B-3. Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove $B = 1$.

1969-B-2. Show that a finite group can not be the union of two of its proper subgroups. Does the statement remain true if "two" is replaced by "three"?

1968-B-2. A is a subset of a finite group G, and A contains more than one half of the elements of G. Prove that each element of G is the product of two elements of A.

Other problems

1. A set S is nonnegative real numbers is said to be *closed under* $\pm$ iff, for each x, y in S, either $x + y$ or $|x - y|$ belongs to S. For instance, if $\alpha > 0$ and n is a positive integer, then the set

$$S(n, \alpha) \equiv \{0, \alpha, 2\alpha, \cdots, n\alpha\}$$

has the property. Show that every finite set closed under $\pm$ is either $\{0\}$, is of the form $S(n, \alpha)$, or has exactly four elements.

2. S is a set with a distinguished element u upon which an operation $+$ is defined that, for all $a, b, c \in S$, satisfies these axioms:
   (a) $a + u = a$;
   (b) $a + a = u$;
   (c) $(a + c) + (b + c) = a + b$.

   Define $a * b = a + (u + b)$. Prove that, for all $a, b, c \in S$,

$$(a * b) * c = a * (b * c) \quad .$$

3. Suppose that a and b are two elements of a group satisfying $ba = ab^2$, $b \neq 1$ and $a^{31} = 1$. Determine the order of b.

FIELDS

Putnam problems

1987-B-6. Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2 - 1)/2$ distinct nonzero elements of F with the property that for each $\alpha \neq 0$ in F, exactly one of α and $-\alpha$ is in S. Let N be the number of elements in the intersection $S \cap \{2\alpha : \alpha \in S\}$. Prove that N is even.

1979-B-3. Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible (*i.e.*, nonfactorable) polynomial over F of the form

$$x^2 + bx + c \qquad\qquad b, c \in F \quad .$$

For how many elements k in F is $p(x) + k$ irreducible over F?

ALGEBRA

Putnam problems

1997-B-4. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1 \quad >$$

1995-B-4. Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}} \quad .$$

Express your answer in the form $(a + b\sqrt{c})/d$, where a, b, c, d are integers.

1993-B-2. For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j} \binom{n}{k-2j} \ ,$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$ and $\binom{a}{b} = 0$ otherwise.)

Other problems

1. Solve the equation

$$\sqrt{x+5} = 5 - x^2 \quad .$$

2. The numbers $1, 2, 3, \cdots$ are placed in a triangular array and certain observations concerning row sums are made as indicated below:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 3 & & & & \\
 & & & 4 & & 5 & & 6 & & & \\
 & & 7 & & 8 & & 9 & & 10 & & \\
 & 11 & & 12 & & 13 & & 14 & & 15 & \\
16 & & 17 & & 18 & & 19 & & 20 & & 21
\end{array}$$

$$\begin{aligned}
1 &= (0+1)(0^2+1^2) \\
5 &= 1^2 + 2^2 \\
15 &= (1+2)(1^2+2^2) \\
34 &= 2 \times (1^2 + 4^2) \\
65 &= (2+3)(2^2+3^2) \\
111 &= 3 \times (1^2 + 6^2)
\end{aligned}$$

$$1 = 1^4 \qquad\qquad 1 + 15 = 2^4 \qquad\qquad 1 + 15 + 65 = 3^4$$

$$5 = 1 + 2^2 = 1(1^2 + 2^2)$$

$$5 + 34 = 3 + 6^2 = (1+2)(2^2 + 3^2)$$

$$5 + 34 + 111 = 6 + 12^2 = (1+2+3)(3^2+4^2)$$

Formulate and prove generalizations of these observations.

3. Let n be a positive integer and x a real number not equal to a positive integer. Prove that

$$\frac{n}{x} + \frac{n(n-1)}{x(x-1)} + \frac{n(n-1)(n-2)}{x(x-1)(x-2)} + \cdots + \frac{n(n-1)(n-2)\cdot\cdots\cdot 1}{x(x-1)(x-2)\cdots(x-n+1)} = \frac{n}{x-n+1} \quad .$$

4. Determine a value of the parameter θ so that

$$f(x) \equiv \cos^2 x + \cos^2(x+\theta) - \cos x \cos(x+\theta)$$

is a constant function of x.

MATRICES, DETERMINANTS AND LINEAR ALGEBRA

1996-B-4. For any square matrix A, we can define $\sin A$ by the usual power series

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1} \quad .$$

Prove or disprove: There exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix} \quad .$$

1995-B-3. To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

1994-A-4. Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

1994-B-4. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \qquad \text{and} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

Show that $\lim_{n\to\infty} d_n = \infty$.

1992-B-5. Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{matrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ . & . & . & . & \cdots & . \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{matrix}$$

Is the set

$$\{\frac{D_n}{n!} : n \geq 2$$

bounded?

1992-B-6. Let M be a set of real $n \times n$ matrices such that

(i) $I \in M$, where I is the $n \times n$ identity matrix;
(ii) if $A \in M$ and $B \in M$, then either $AB \in M$ and $-AB \in M$, but not both;
(iii) if $A \in M$ and $B \in M$, then either $AB = BA$ or $AB = -BA$;
(iv) if $A \in M$ and $A \neq I$, then there is at least one $B \in M$ such that $AB = -BA$.

Prove that M contains at most n^2 matrices.

1991-A-2. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

1990-A-5. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

1986-A-4. A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:
(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.
(b) The sum of the n entries of a transversal is the same for all transversals of A.
An example of such a matric A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad .$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4 \quad .$$

where the a_i's and b_i's are rational numbers.

1986-B-6. Suppose that A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, M^t is the transpose of M. Prove that $A^tD - C^tB = I$.

1985-B-6. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \le i \le r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \text{tr } (M_i) = 0$, where tr (A) denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.

COMBINATORICS

Putnam problems

1997-A-2. Players $1, 2, 3, \cdots, n$ are seated around a table and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who then passes two players to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

1996-A-3. Suppose that each of twenty students has made a choice of anywhere from zero to six courses from a total of six courses offered. Prove or disprove: There are five students and two courses such that all five have chosen both courses or all five have chosen neither.

1996-B-1. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with a proof, the number of subsets of $\{1, 2, \cdots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets if selfish.

1996-B-5. Given a finite string S of symbol X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \le \Delta(T) \le 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

1995-A-4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels

$x_1, x_2, \cdots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \qquad \text{for} \quad k = 1, 2, \cdots, n \ .$$

1995-B-1. For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

1995-B-5. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

a. one bean from a heap, provided at least two beans are left behing in that heap, **or**

b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

1994-A-3. Show that if the points of an isosceles right triangle of side length 1 are earch colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

1994-A-6. Let f_1, f_2, $\cdots$, f_{10} be bijections of the set of integers such that for each integer n, there is some composition $f_{1_1} \circ f_{1_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathfrak{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}} : e_i = 0 \ \text{ or } \ 1 \ \text{ for } \ 1 \leq i \leq 10\}$$

(f_i^0 is the identity function and $f_i^1 = f_i$). Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathfrak{F}$ map A to itself.

1993-A-3. Let $\mathfrak{P}_\mathfrak{n}$ be the set of subsets of $\{1,2,\cdots,n\}$. Let $c(n,m)$ be the number of functions $f : \mathfrak{P}_n \to \{1,2,\cdots.m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n \ .$$

1992-B-1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

Other problems

1. Let n and k be positive integers. Determine the number of ways of choosing k numbers from $\{1,2,\cdots,n\}$ so that no three consecutive numbers appear in any choice.

2. There are n safes and n keys. Each key opens exactly one safe and each safe is opened by exactly one key. The keys are locked in the safes at random, with one key in each safe. k of the safes are broken open and the keys inside retrieved. What is the probability that the remaining safes can be opened with keys?

3. A class with at least 35 students goes on a cruise. Seven small boats are hired, each capable of carrying 300 kilograms. The combined weight of the students is 1800 kilograms. It is determined that any group of 35 students can fit into the boats without exceeding the individual capacity of any of them. Prove that it is unnecessary to prevent any student from taking the cruise.

4. Each can of K_9-Food costs 3 dollars and contains 10 units of protein and 10 units of carbohydrate. Each can of Pooch-Mooch costs 5 dollars and has 10 units of protein, 20 units of carbohydrate and 5 units of vitamins. Each puppy needs a daily feed with 45 units of protein, 60 units of carbohydrate and 5 units of vitamins. How would you feed adequately 10 puppies for 10 days in the cheapest way?

5. A rectangle is partitioned into smaller rectangles (not necessarily congruent to one another) with sides parallel to to those of the large rectangle. Each small rectangle has at least one side an integer number of units long. Prove that the large rectangle does also.

6. During a long speech, each member of the audience fell asleep exactly twice. For any pair of auditors, there was a moment when both of them were asleep. Prove that there must have been a moment during the speech when at least a third of the audience were asleep.

7. A group of students with ages ranging from 17 to 23, inclusive, with at least one student of each age, represents 11 universities. Prove that there are at least 5 students such that each has more members of the group of the same age than members from the same university.

8. On a $2n \times 2n$ chessboard, $3n$ squares are chosen at random. Prove that n rooks (castles) can be placed on the board so that each chosen square is either occupied by a rook or under attack from at least one rook. (Note that each rook can attack any square in the same row or column which is visible from the rook.)

9. At a party, every two people greet each other in exactly one of four ways (nodding, shaking hands, kissing, hugging). Candy kisses Randy, but not Sandy. For every three people, their three pairwise greetings are either all the same or all different. What is the maximum number of people at the party?

PROBABILITY

1995-A-6. Suppose that each of n people writes down the numbers 1, 2, 3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that, for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

1993-B-2. Consider the following game played with a deck fof $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing the number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. If we assume optimal strategy by both A and B, what is the probability that A wins?

1993-B-3. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

1992-A-6. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

100 MATHEMATICAL PROBLEMS

COMPILED BY YANG WANG

The following problems are compiled from various sources, particularly from

- D. J. Newman, *A Problem Seminar*, Springer-Verlag, 1982.
- Kenneth S. Williams, with Kenneth Hardy, *The Red Book of Mathematical Problems*, Dover Publications, 1997.
- Kenneth Hardy,Kenneth S. Williams, *The Green Book of Mathematical Problems*, Dover Publications, 1997.
- Problems from the past William Lowell Putnam Mathematical Competitions.

Most of the problems listed here require no advanced mathematical background to solve, and they range from fairly easy to moderately difficult. I have deliberately avoided including very easy and very difficult problems. Nevertheless if you have not had experience solving mathematical problems you may find many of them challenging. The problems have *NOT* being sorted according to the degree of difficulty.

Problem 1 Prove that the equation

$$y^2 = x^3 + 23$$

has no integer solutions.

Problem 2 Evaluate the limit

$$L = \lim_{n\to\infty} \frac{n}{2^n} \sum_{k=1}^{n} \frac{2^k}{k}.$$

Problem 3 Evaluate the integral

$$I = \int_0^1 \ln x \ln(1-x)\; dx.$$

Problem 4 Solve the recurrence relation

$$\sum_{k=1}^{n} \binom{n}{k} a(k) = \frac{n}{n+1}, \;\; n = 1, 2, \dots \; .$$

Problem 5 Let

$$a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, \;\; n = 0, 1, 2, \dots \; .$$

Does the series $\sum_{n=0}^{\infty} a_n$ converge, and if so, what is its sum?

Problem 6 Let $a_1, \ldots, a_m$ be m (≥ 2) real numbers. Set

$$A_n = a_1 + a_2 + \cdots + a_n, \ n = 1, 2, \ldots, m.$$

Prove that

$$\sum_{n=2}^{m} \left(\frac{A_n}{n}\right)^2 \leq 12 \sum_{n=1}^{m} a_n^2.$$

Problem 7 Let a and b be coprime positive integers. For k a positive integer, let $N(k)$ denote the number of integral solutions to the equation

$$ax + by = k, \ x \geq 0, \ y \geq 0.$$

Evaluate the limit

$$L = \lim_{k\to\infty} \frac{N(k)}{k}.$$

Problem 8 Let $a_1, \ldots, a_m$ be m (≥ 1) real numbers which are such that $\sum_{n=1}^{m} a_n \neq 0$. Prove the inequality

$$\Big(\sum_{n=1}^{m} n a_n^2\Big)^2 \Big/ \Big(\sum_{n=1}^{m} a_n^2\Big)^2 > \frac{1}{2\sqrt{m}}.$$

Problem 9 Evaluate the infinite series

$$\sum_{n=1}^{\infty} \tan^{-1}\Big(\frac{2}{n^2}\Big).$$

Problem 10 Let $p_1, \ldots, p_n$ denote n distinct integers and let $f_n(x)$ be a polynomial of degree n given by

$$f_n(x) = (x - p_1)(x - p_2) \cdots (x - p_n).$$

Prove that the polynomial $g_n(x) = f_n^2(x) + 1$ cannot be expressed as the product of two non-constant polynomials with integral coefficients.

Problem 11 Let $f(x)$ be a monic polynomial of degree $n \geq 1$ with complex coefficients. Let $x_1, \ldots, x_n$ be the n complex roots of $f(x)$. The discriminant $D(f)$ of $f(x)$ is the complex number

$$D(f) = \Pi_{1\leq i<j\leq n}(x_i - x_j)^2.$$

Express the discriminant of $f(x^2)$ in terms of $D(f)$.

Problem 12 Prove that for each positive integer n there exists a circle in the xy-plane which contains exactly n lattice points inside.

Problem 13 Let n be a given non-negative integer. Determine the number of solutions of the equation

$$x + 2y + 2z = n$$

in non-negative integers x, y, z.

Problem 14 Let n be a fixed integer ≥ 2. Determine all functions $f(x)$, which are boounded for $0 < x < a$, and which satisfy the functional equation

$$f(x) = \frac{1}{n^2}\Big(f(\frac{x}{n}) + f(\frac{x+a}{n}) + \cdots + f(\frac{x+(n-1)a}{n})\Big).$$

Problem 15 Evaluate the limit

$$L = \lim_{y \to 0} \frac{1}{y} \int_0^{\pi} \tan(y \sin x)\ dx.$$

Problem 16 Let $\epsilon \in (0, 1)$. Prove that there are infinitely many integers n for which $\cos n \geq 1 - \epsilon$.

Problem 17 Determine all differentiable functions $f(x)$ such that

$$f(x) + f(y) = f\Big(\frac{x+y}{1-xy}\Big)$$

for all real x and y with $xy \neq 1$.

Problem 18 If x and y are rational numbers such that $\tan \pi x = y$, prove that $x = k/4$ for some integer k not congruent to 2 (mod4).

Problem 19 The sequence $x_0, x_1, x_2 \dots$ is defined by the recurrence

$$x_0 = 0,\ x_1 = 1,\ x_{n+1} = \frac{x_n + nx_{n-1}}{n+1},\ n > 1.$$

Determine $L = \lim_{n\to\infty} x_n$.

Problem 20 Find the sum of the series

$$S = 1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + \cdots\ .$$

Problem 21 Prove that

$$\frac{1}{n+1}\binom{2n}{n}$$

is an integer for all $n > 0$.

Problem 22 Let $x > 1$. Find the sum of the series

$$S = \sum_{n=0}^{\infty} \frac{2^n}{x^{2^n} + 1}.$$

Problem 23 Let k be an integer. Prove that the formal power series

$$\sqrt{1+kx} = 1 + a_1 x + a_2 x^2 + \cdots$$

has integral coefficients if and only if k is divisible by 4.

Problem 24 Find the sum of the series

$$\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \cdots .$$

Problem 25 A cross-country racer runs a 10-mile race in 50 minutes. Prove that somewhere along the course the racer ran 2 miles in exactly 10 minutes.

Problem 26 Determine the inverse of the $n \times n$ matrix

$$S = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix},$$

where $n \geq 2$.

Problem 27 Determine 2 matrices B and C with integral entries such that

$$\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = B^3 + C^3.$$

Problem 28 Find two non-congruent similar triangles with sides of integral length having the lengths of two sides of one triangle equal to the lengths of two sides of the other.

Problem 29 Set $J_n = \{1, 2, \dots, n\}$. For each non-empty subset S of J_n define

$$w(S) = \max_{s \in S} s - \min_{s \in S} s.$$

Determine the average of $w(S)$ over all non-empty subsets S of J_n.

Problem 30 Prove that the number of odd binomial coefficients in each row of Pascal's triangle is a power of 2.

Problem 31 Prove that the polynomial

$$f(x) = x^n + x^3 + x^2 + x + 5$$

is irreducible over $\mathbb{Z}$ for $n \geq 4$.

Problem 32 Prove that $\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \sqrt{11}$.

Problem 33 For $x > 1$ determine the sum of the series

$$\frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \cdots .$$

Problem 34 Let a_k be a sequence of positive numbers. Define $S_n = \sum_{k=1}^{n} a_k$. Prove that $\sum_{n=1}^{\infty} a_n/S_n^2$ converges. (In fact, $\sum_{n=1}^{\infty} a_n/S_n^{\alpha}$ converges for all $\alpha > 1$.)

Problem 35 Let $f(x)$ be a continuous function on $[0, a]$, where $a > 0$, such that $f(x) + f(a - x)$ does not vanish on $[0, a]$. Evaluate the integral

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)}\, dx.$$

Problem 36 For $\varepsilon > 0$ evaluate the limit

$$\lim_{x\to\infty} x^{1-\varepsilon} \int_x^{x+1} \sin(t^2)\, dt.$$

Problem 37 Prove that the equation

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24$$

has no integer solution.

Problem 38 Let $x_0 \geq 0$ be fixed, and a, b satisfy

$$\sqrt{b} < a < 2\sqrt{b}.$$

Define x_n recursively by

$$x_n = \frac{ax_{n-1} + b}{x_{n-1} + a}, \ n = 1, 2, 3, \ldots .$$

Prove that $\lim_{n\to\infty} x_n$ exists and evaluate it.

Problem 39 Prove that for any integer $m \geq 0$ the sum

$$S_m(n) = \sum_{k=1}^{n} k^{2m+1}$$

is a polynomial in $n(n+1)$.

Problem 40 Determine a function $f(n)$ such that the n^{th} term of the sequence

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5, \ldots$$

is given by $\lfloor f(n) \rfloor$.

Problem 41 Let $a_1, a_2, \ldots, a_n$ be given. Find the least value of $x_1^2 + x_2^2 + \cdots + x_n^2$ given that

$$a_1x_1 + a_2 + x_2 + \cdots + a_nx_n = 1.$$

Problem 42 Evaluate the infinite series

$$S = 1 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + \cdots .$$

Problem 43 Let $F(x)$ be a differential function such that $F'(a-x) = F'(x)$ for all x in $[0, a]$. Evaluate $\int_0^a F(x)\,dx$ and give an example of such a function $F(x)$.

Problem 44 Determine the real function whose power series is

$$\frac{x^3}{3!} + \frac{x^9}{9!} + \frac{x^{15}}{15!} + \cdots .$$

Problem 45 Determine the value of the integral

$$I_n = \int_0^\pi \left(\frac{\sin nx}{\sin x}\right)^2 dx$$

for all positive integer n.

Problem 46 Let $f(x)$ be a continuous function on $[0, a]$, where $a > 0$, such that $f(x)f(a-x) = 1$. Prove that there are infinitely many such functions, and evaluate the integral

$$\int_0^a \frac{dx}{1 + f(x)}.$$

Problem 47 Let $p > 0$ be a real number and let $n \geq 0$ be an integer. Evaluate

$$u_n(p) = \int_0^\infty e^{-px} \sin^n x \, dx.$$

Problem 48 Evaluate $\displaystyle\sum_{k=0}^{n-2} 2^k \tan \frac{\pi}{2^{n-k}}$ for all $n \geq 2$.

Problem 49 Let $k \geq 2$ be a fixed integer. For $n \geq 1$ define

$$a_n = \begin{cases} 1, & \text{if } n \text{ is not a multiple of } k \\ -(k-1), & \text{if } n \text{ is a multiple of } k. \end{cases}$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n}$.

Problem 50 The length of two altitudes of a triangle are h and k, where $h \neq k$. Find an upper and a lower bound for the length of the third altitude in terms of h and k.

Problem 51 Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}$.

Problem 52 Let $x > 2$ and $A_n = (a_{ij})$ be the $n \times n$ matrix where

$$a_{ij} = \begin{cases} x, & \text{if } i = j \\ 1, & \text{if } |i - j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate $D_n = \det(A_n)$.

Problem 53 Detremine a necessary and sufficient condition for the equation

$$\begin{cases} x + y + z &= A, \\ x^2 + y^2 + z^2 &= B, \\ x^3 + y^3 + z^3 &= C, \end{cases}$$

to have a solution with at least one of x, y, z equal to 0.

Problem 54 Let S be a set with n elements. Determine an explicit formula for the number $A(n)$ of subsets of S whose cardinality is a multiple of 3.

Problem 55 Let S be the set of all composite positive odd integers less than 79.

(a) Show that S may be written as the union of three (not necessarily disjoint) arithmetic progressions.
(b) Show that S cannot be written as the union of two arithmetic progressions.

Problem 56 Let a, b be fixed positive integers. Find the general solution to the recurrence relation

$$x_0 = 0,\ x_{n+1} = x_n + a + \sqrt{b^2 + 4ax_n}.$$

Problem 57 Let $\varepsilon > 0$ Around every point of integer coordinates draw a circle of radius ε. Prove that every straight line through the origin must intersect an infinitely many circles.

Problem 58 Let p and q be distinct primes. Let S be the sequence

$$\{p^m q^n :\ m, n = 0, 1, 2, 3, \dots\}$$

arranged in increasing order. For any pair of nonnegative integers (a, b) give an explicit formula for the position of $p^a q^b$ in the sequence using a, b, p, q.

Problem 59 Let $f(x)$ be the unique differentiable real function satisfying

$$f^{2n+1}(x) + f(x) - x = 0.$$

Evaluate he integral

$$\int_0^x f(t)\ dt.$$

Problem 60 Evaluate the double integral

$$\int_0^1 \int_0^1 \frac{dxdy}{1-xy}.$$

Problem 61 Prove that the sum of two consecutive odd primes is the product of at least three (possibly repeated) prime factors.

Problem 62 If four distinct points lie in the plane such that any three of them can be covered by a disk of radius one, prove that the four points can be covered by a disk of radius one.

Problem 63 Let G be the group generated by a and b subject to the relation $aba = b^3$ and $b^5 = 1$. Prove that G is Abelian.

Problem 64 Let $u(x)$ be a non-trivial solution of the differential equation

$$u'' + pu = 0,$$

defined on the interval $[1, \infty)$, where $p = p(x)$ is continuous on I. Prove that u has only finitely many zeros in any interval $[a, b]$, $1 \le a < b$.

Problem 65 Let M be a 3×3 matrix with entries chosen randomly from $\{0, 1\}$. What is the probability that $\det(M)$ is odd?

Problem 66 Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \ldots)$?

Problem 67 Prove that any convex pentagon whose vertices (no three of which are colinear) have integer coordinates must have area at least $5/2$.

Problem 68 Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are need to remove every point?

Problem 69 If A, B are square matrices such that $ABAB = 0$, does it follow that $BABA = 0$?

Problem 70 Find all real-valued continuously differentiable fnctions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(x))^2 + (f'(x))^2]\, dt + 1990.$$

Problem 71 Let S be a set of 2×2 integer matrices whose entries a_{ij} (1) are all squares of integers and, (2) satisfy $a_{ij} \le 200$. Show that if S has more than 50387 $(15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

Problem 72 Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

Problem 73 Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2 B = B^2 A$, can $A^2 + B^2$ be invertible?

Problem 74 Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 < y < 1$.

Problem 75 For each integer $n > 0$ let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is the sequence eventually constant?

Problem 76 Suppose that f and g are non-constant and differentiable real functions defined on $(-\infty, \infty)$. For all x, y we have

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $f^2(x) + g^2(x) = 1$ for all x.

Problem 77 (Putnam 91) Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

Problem 78 (Putnam 92) Prove that $f(n) = 1 - n$ is the only integer valued function defined on the integers that satisfies the following conditions.

(1) $f(f(n)) = n$ for all integer n;
(2) $f(f(n+2)+2) = n$ for all integer n;
(3) $f(0) = 1$.

Problem 79 (Putnam 92) Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series of $(1+x)^\alpha$ about $x = 0$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

Problem 80 (Putnam 92) For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, such that

$$(x^2 + y^2)^m = (xy)^n.$$

Problem 81 (Putnam 92) Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

Problem 82 (Putnam 92) Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

Problem 83 (Putnam 65) At a party, assume that no boy dances with every girl but each girl dances with at least one boy. prove that there are two couples gb and $g'b'$ which dance whereas b doesn't dance with g' and b' doesn't dance with g.

Problem 84 (Putnam 65) Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\Big\{\frac{\pi}{2n}(x_1 + x_2 + \cdots + x_n)\Big\} dx_1\, dx_2 \cdots dx_n.$$

Problem 85 (Putnam 65) Evaluate Prove that there are exactly three right-angled triangles whose sides are integers while the area is numerically equal to twice the perimeter.

Problem 86 (Putnam 66) Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

Problem 87 (Putnam 66) Prove that among any ten consecutive integers at least one is relative prime to each of the others.

Problem 88 (Putnam 67) Let $f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$, where a_j are real numbers. Given that $|f(x)| \leq |\sin(x)|$ for all real x, prove that

$$|a_1 + 2a_2 + \cdots + na_n| \leq 1.$$

Problem 89 (Putnam 68) Assume that a 60^o angle cannot be trisected with ruler and compass alone. Prove that for any positive integer n on angle of $60^o/n$ can be trisected using ruler and compass alone.

Problem 90 Shuffle an ordinary deck of 52 playing cards. On the average, how far from the top will the first ace be?

Problem 91 Akira, Betul and Cleopatra fight a 3-way pistol duel. All three know that Akira's chance of hitting any target is 0.5, while Betul *never* misses, and Cleopatra has a 0.8 chance of hitting any target. The way the duel works is that each person is to fire at their choice of target. The order of firing is determined by a random drawing, and the firing proceeds cyclically in that order (unless someone is hit, in that case this person doesn't shoot and the turn goes to the next person). The duel ends when only one person is left unhit. What is the optimal strategy for each of them? Who is most likely to survive given that everyone adopts the optimal strategy? What is the probability that Akira will be the survivor?

Problem 92 Let a, b, c, d be non-negative real numbers. Prove that

$$\sqrt{a+b+c}+\sqrt{b+c+d}+\sqrt{c+a+b}+\sqrt{d+a+b} \geq 3\sqrt{a+b+c+d}.$$

Problem 93 Mr. and Mrs. Smith went to a party attended by 15 other couples. Various handshakes took place during the party. In the end, Mrs. Smith asked each person at the party how many handshakes did they have. To her surprise, each person gave a different answer. How many hand shakes did Mr. Smith have? (Here we assume that no person shakes hand with his/her spouse and of course, himself/herself.)

Problem 94 Suppose that $a + 1/a \in \mathbb{Q}$. Prove that $a^n + 1/a^n \in \mathbb{Q}$ for all integer $n > 0$.

Problem 95 Suppose that the sequence of integers $\{a_n\}$ satisfies

$$a_0 = 0,\ a_1 = a_2 = 1,\ \frac{a_{n+1} - 3a_n + a_{n-1}}{2} = (-1)^n.$$

Prove that a_n is a perfect square.

Problem 96 24 chairs are evenly spaced around a circular table on which are name cards for 24 guests. The guests failed to notice these cards until they have sat down, and it turns out that no one is sitting in fornt of his/her own card. Prove that the table can be rotated so that at least two of these guests are simultaneously correctly seated. (A much harder questions is: Can the table be rotated so that at least 3 guests are simultaneously seated correctly?)

Problem 97 Let A be any set of 51 distinct integers chosen from $1, 2, 3, \ldots, 100$. Prove that there must be two distinct integers in A such that one divides the other.

Problem 98 Given a positive integer n, show that there exists a positive integer containing only the digits 0 and 1 (in decimal notation), and which is divisible by n.

Problem 99 Let x_1, x_2, $\ldots, x_{20}$ be integers. Prove that some of them have sum divisible by 20.

Problem 100 Prove that from a set of 10 distinct two-digit numbers (in base 10), it is possible to select two disjoint subsets whose members have the same sum.

Math Contest Sampler

Below are some problems from recent Putnam exams and local math contests, arranged roughly in increasing order of difficulty. For solutions to these problems, and a large collection of additional contest problems, visit the UI Math Contest Website:

`http://www.math.illinois.edu/contests.html`

Problem 1, UI Freshman Math Contest 2012. Determine, with proof, whether there exists a power of 2 whose decimal representation ends in the digits 2012.

Problem 1, UI Freshman Math Contest 2011. Let x be the number whose decimal expansion consists of the sequence of natural numbers written next to each other, i.e., $x = 0.12345678910111213\ldots$.

(a) Determine the 2011th digit after the decimal point of x.

(b) Prove that x is irrational.

Problem A1, Putnam 2012. Let $d_1, d_2, \ldots, d_{12}$ be 12 real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Problem 5, UI Mock Putnam Exam 2008. Let $a_1, a_2, \ldots, a_{65}$ be 65 positive integers, none of which has a prime factor greater than 13. Prove that, for some i, j with $i \neq j$, the product $a_i a_j$ is a perfect square.

Problem 6, UI Undergraduate Math Contest 2012. Call a positive integer defective if its decimal representation does not contain all ten digits $0, 1, 2, \ldots, 9$. Thus, for example, the number 3141592653589 is defective (since it does not contain the digits 7 and 0), but the number 31415926535897932384626433832795028 is not defective (since it contains each of the digits $0, 1, \ldots, 9$). Let D denote the set of defective numbers. Determine, with proof, whether the sum of reciprocals of the numbers in D converges or diverges.

PUTNAM PROBLEMS

SEQUENCES, SERIES AND RECURRENCES

Notes

1. $x_{n+1} = ax_n$ has the general solution $x_n = x_1 a^{n-1}$.
2. $x_{n+1} = x_n + b$ has the general solution $x_n = x_1 + (n-1)b$.
3. $x_{n+1} = ax_n + b$ (with $a \neq 1$) can be rewritten $x_{n+1} + k = a(x_n + k)$ where $(a-1)k = b$ and so reduces to the recurrence 1.
4. $x_{n+1} = ax_n + bx_{n-1}$ has different general solution depending on the discriminant of the characteristic polynomial $t^2 - at - b$.
   (a) If $a^2 - 4b \neq 0$ and the distinct roots of the characteristic polynomial are r_1 and r_2, then the general solution of the recurrence is
   $$x_n = c_1 r_1^n + c_2 r_2^n$$
   where the constants c_1 and c_2 are chosen so that
   $$x_1 = c_1 r_1 + c_2 r_2 \qquad \text{and} \qquad x_2 = c_1 r_1^2 + c_2 r_2^2 \ .$$
   (b) If $a^2 - 4b = 0$ and r is the double root of the characteristic polynomial, then
   $$x_n = (c_1 n + c_0) r^n$$
   where c_1 and c_0 are chosen so that
   $$x_1 = (c_1 + c_0) r \qquad \text{and} \qquad x_2 = (2c_1 + c_0) r^2 \ .$$
5. $x_{n+1} = (1-s)x_n + sx_{n-1} + r$ can be rewritten $x_{n+1} - x_n = -s(x_n - x_{n-1}) + r$ and solved by a previous method for $x_{n+1} - x_n$.
6. $x_{n+1} = ax_n + bx_{n-1} + c$ where $a + b \neq 1$ can be rewritten $(x_{n+1} + k) = a(x_n + k) + b(x_{n-1} + k)$ where $(a + b - 1)k = c$ and solved for $x_n + k$.
7. The general homogeneous linear recursion has the form
   $$x_{n+k} = a_{k-1} x_{n+k-1} + \cdots + a_1 x_{n+1} + a_0 \ .$$
   Its characteristic polynomials is
   $$t^k - a_{k-1} t^{k-1} - \cdots - a_1 t - a_0 \ .$$
   Let r be a root of this polynomial of multiplicity m; then the nth term of the recurrence is a linear combination of terms of the type
   $$(c_{m-1} r^{m-1} + \cdots + c_1 r + c_0) r^n \ .$$

Putnam questions

2009-B-6. Prove that for every positive integer n, there is a sequence $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonegative integer k or of the form b mod c for some earlier positive terms b and c. (Here b mod c denotes the remainder when b is divided by c, so $0 \leq (b \text{mod} c) < c$.)

2007-B-3. Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\leq a$,)

2006-A-3. Let $1, 2, 3, \cdots, 2005, 2006, 2007, 2009, 2012, 2016, \cdots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \cdots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

2006-B-6. Let k be an integer greater than 1. Suppose $a_k > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n \geq 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k} .$$

2004-A-3. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

2002-A-5. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \cdots \right\} .$$

2001-B-3. For any positive integer n let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} .$$

2001-B-6. Assume that $\{a_n\}_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-1} + a_{n+i} < 2a_n$$

for $i = 1, 2, \cdots, n-1$?

2000-A-1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \cdots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

2000-A-6. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \cdots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

1999-A-3. Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n .$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m .$$

1999-A-4. Sum the series
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n3^m+m3^n)} \ .$$

1999-A-6. The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_1=1$, $a_2=2$, $a_3=24$, and for $n\geq 4$,
$$a_n=\frac{6a_{n-1}^2a_{n-3}-8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} \ .$$
Show that, for all n, a_n is an integer multiple of n.

1999-B-3. Let $A=\{(x,y):0\leq x,y\leq 1\}$. For $(x,y)\in A$, let
$$S(x,y)=\sum_{\frac{1}{2}\leq\frac{m}{n}\leq 2}x^my^n \ ,$$
where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate
$$\lim\{(1-xy^2)(1-x^2y)S(x,y):(x,y)\longrightarrow(1,1),(x,y)\in A\} \ .$$

1998-A-4. Let $A_1=0$ and $A_2=1$. For $n>2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3=A_2A_1=10$, $A_4=A_3A_2=101$, $A_5=A_4A_3=10110$, and so forth. Determine all n such that 11 divides A_n.

1998-B-4. Find necessaary and sufficient conditions on positive integers m and n so that
$$\sum_{i=0}^{mn-1}(-1)^{\lfloor i/m\rfloor+\lfloor i/n\rfloor}=0 \ .$$

1997-A-6. For a positive integer n and any real number c, define x_k recursively by $x_0=0$, $x_1=1$, and for $k\geq 0$,
$$x_{k+2}=\frac{cx_{k+1}-(n-k)x_k}{k+1} \ .$$
Fix n and then take c to be the largest value for which $x_{n+1}=0$. Find x_k in terms of n and k, $1\leq k\leq n$.

1994-A-1. Suppose that a sequence $a_1,a_2,a_3,\cdots$ satisfies $0<a_n\leq a_{2n}+a_{2n+1}$ for all $n\geq 1$. Prove that the series $\sum_{n=1}^{\infty}a_n$ diverges.

1994-A-5. Let $(r_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty}r_n=0$. Let S be the set of numbers representable as a sum
$$r_{i_1}+r_{i_2}+\cdots+r_{i_{1994}}$$
with $i_1<i_2<\cdots<i_{1994}$. Show that every nonempty interval (a,b) contains a nonempty subinterval (c,d) that does not intersect S.

1993-A-2. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that
$$x_n^2-x_{n-1}x_{n+1}=1$$
for $n=1,2,3,\cdots$. Prove that there exists a real number a such that $x_{n+1}=ax_n-x_{n-1}$ for all $n\geq 1$.

1993-A-6. The infinite sequence of $2's$ and $3's$
$$2,3,3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,3,2,\cdots$$

has the property that, if one forms a second sequence that records the number of $3's$ between successive $2's$, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

1992-A-1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions
(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

1992-A-5. For each positive integer n, let

$$a_n = \begin{cases} 0, & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1, & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}$$

for $0 \le j \le m-1$.

1991-B-1. For each integer $n \ge 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \le n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \ge 0$. For what positive integers A is this sequence eventually constant?

1990-A-1. Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \ge 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3} \quad .$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

1988-B-4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is

$$\sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \ .$$

1985-A-4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \ge 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

1979-A-3. Let x_1, x_2, x_3, $\cdots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-2}} \qquad \text{for} \quad n = 3, 4, 5, \cdots \quad .$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

1975-B-6. Show that, if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then
(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and
(b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

Putnam Training Exercise
Induction and Recurrences

October 15th, 2013

1. Prove that $n! > 2^n$ for all $n \geq 4$.

2. Consider the sequence a_n defined by recursion $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ $(n \geq 1)$. Prove that $a_n < 2$ for every $n \geq 1$.

3. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.

4. Let a_n be the following expression with n nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}\,.$$

Prove that $a_n = 2 \cos \frac{\pi}{2^{n+1}}$.

5. Find the maximum number $R(n)$ of regions in which the plane can be divided by n straight lines.

6. We divide the plane into regions using straight lines. Prove that those regions can be colored with two colors so that no two regions that share a boundary have the same color.

7. Define a *domino* to be a 1×2 rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?

8. Let α, β be two (real or complex) numbers, and define the sequence $a_n = \alpha^n + \beta^n$ $(n = 1, 2, 3, \ldots)$. Assume that a_1 and a_2 are integers. Prove that $2^{\lfloor \frac{n-1}{2} \rfloor} a_n$ is an integer for every $n \geq 1$.

9. Suppose that $x_0 = 18$, $x_{n+1} = \dfrac{10x_n}{3} - x_{n-1}$ $(n \geq 1)$, and assume that the sequence $\{x_n\}$ converges to some real number. Find x_1.

10. Ackermann's function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined for every $m, n \geq 0$ by the following double recurrence:

(a) $A(0, n) = n + 1$ for every $n \geq 0$.
(b) $A(m, 0) = A(m - 1, 1)$ if $m \geq 1$.
(c) $A(m, n) = A(m - 1, A(m, n - 1))$ if $m, n \geq 1$.

Find $A(5, 0)$.

Putnam Training Exercise
Number Theory and Congruences

October 22nd, 2013

1. Can the sum of the digits of a square be (a) 3, (b) 1977?

2. Show that if $a^2 + b^2 = c^2$, then $3|ab$.

3. Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every possible integer n.

4. Prove that there are infinitely many prime numbers of the form $4n + 3$.

5. Show that there exist 2013 consecutive numbers, each of which is divisible by the cube of some integer greater than 1.

6. (USAMO, 1979) Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599\,.$$

7. Let $f(n)$ denote the sum of the digits of n. Let $N = 4444^{4444}$. Find $f(f(f(N)))$.

8. Do there exist 2 irrational numbers a and b greater than 1 such that $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for every positive integers m, n?

9. Prove that there are no primes in the following infinite sequence of numbers:

$$1001,\ 1001001,\ 1001001001,\ 1001001001001, \ldots$$

10. The digital root of a number is the (single digit) value obtained by repeatedly adding the (base 10) digits of the number, then the digits of the sum, and so on until obtaining a single digit—e.g. the digital root of 65,536 is 7, because $6 + 5 + 5 + 3 + 6 = 25$ and $2 + 5 = 7$. Consider the sequence $a_n =$ integer part of $10^n\pi$, i.e.,

$$a_1 = 31\,, \quad a_2 = 314\,, \quad a_3 = 3141\,, \quad a_4 = 31415\,, \quad a_5 = 314159\,, \quad \ldots$$

and let b_n be the sequence

$$b_1 = a_1\,, \quad b_2 = a_1^{a_2}\,, \quad b_3 = a_1^{a_2^{a_3}}\,, \quad b_4 = a_1^{a_2^{a_3^{a_4}}}\,, \quad \ldots$$

Find the digital root of b_{10^6}.

Putnam Training Exercise
Polynomials

October 29th, 2013

1. Find a polynomial with integer coefficients whose zeros include $\sqrt{2}+\sqrt{5}$.

2. Prove that $(2+\sqrt{5})^{1/3}-(-2+\sqrt{5})^{1/3}$ is rational.

3. If $a,b,c>0$, is it possible that each of the polynomials $P(x)=ax^2+bx+c$, $Q(x)=cx^2+ax+b$, $R(x)=bx^2+cx+a$ has two real roots?

4. Suppose that α, β, and γ are real numbers such that

$$\begin{aligned}\alpha+\beta+\gamma&=2\,,\\ \alpha^2+\beta^2+\gamma^2&=14\,,\\ \alpha^3+\beta^3+\gamma^3&=17\,.\end{aligned}$$

Find $\alpha\beta\gamma$.

5. Show that $(1+x+\cdots+x^n)^2-x^n$ is the product of two polynomials.

6. Is it possible to write the polynomial $f(x)=x^{105}-9$ as the product of two polynomials of degree less than 105 with integer coefficients?

7. Find all prime numbers p that can be written $p=x^4+4y^4$, where x,y are positive integers.

8. Determine all polynomials such that $P(0)=0$ and $P(x^2+1)=P(x)^2+1$.

9. Prove that there is no polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$,..., are all prime numbers.

10. Two players A and B play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B win the game?

Putnam Training Exercise
Inequalities

November 5th, 2013

1. The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$ What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?

2. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

3. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

4. Find the maximum value of $f(x) = \sin^4(x) + \cos^4 x$ for $x \in \mathbb{R}$.

5. Find the minimum value of the function $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, where $x_1, x_2, \ldots, x_n$ are positive real numbers such that $x_1 x_2 \cdots x_n = 1$.

6. If $x, y, z > 0$, and $x + y + z = 1$, find the minimum value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\,.$$

7. Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:

$$\frac{a\,A + b\,B + c\,C}{a + b + c} \geq \frac{\pi}{3}\,.$$

8. Find the positive solutions of the system of equations

$$x_1 + \frac{1}{x_2} = 4\,, \quad x_2 + \frac{1}{x_3} = 1\,, \ldots\,, \quad x_{99} + \frac{1}{x_{100}} = 4\,, \quad x_{100} + \frac{1}{x_1} = 1\,.$$

9. (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

10. Let $a_i > 0$, $i = 1, \ldots, n$, and $s = a_1 + \cdots + a_n$. Prove

$$\frac{a_1}{s - a_1} + \frac{a_2}{s - a_2} + \cdots + \frac{a_n}{s - a_n} \geq \frac{n}{n-1}\,.$$

R-VI. Polynomials

Po-Shen Loh

1 July 2004

1 Warm-Ups

1. Consider the cubic equation $ax^3 + bx^2 + cx + d = 0$. The roots are

$$\begin{aligned} x &= \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &- \frac{b}{3a}. \end{aligned}$$

   Prove that no such general formula exists for a quintic equation.

2 Theory

Thanks to Elgin Johnston (1997) for these theorems.

Rational Root Theorem Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients. Then any rational solution r/s (expressed in lowest terms) must have $r|a_0$ and $s|a_n$.

Descartes's Rule of Signs Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients. Then the number of positive roots is equal to $N - 2k$, where N is the number of sign changes in the coefficient list (ignoring zeros), and k is some nonnegative integer.

Eisenstein's Irreducibility Criterion Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients and let q be a prime. If q is a factor of each of $a_{n-1}, a_{n-2}, \ldots, a_0$, but q is not a factor of a_n, and q^2 is not a factor of a_0, then $p(x)$ is irreducible over the rationals.

Einstein's Theory of Relativity Unfortunately, this topic is beyond the scope of this program.

Gauss's Theorem If $p(x)$ has integer coefficients and $p(x)$ can be factored over the rationals, then $p(x)$ can be factored over the integers.

Lagrange Interpolation Suppose we want a degree-n polynomial that passes through a set of $n+1$ points: $\{(x_i, y_i)\}_{i=0}^n$. Then the polynomial is:

$$p(x) = \sum_{i=0}^{n} \frac{y_i}{\text{normalization}} (x - x_0)(x - x_1) \cdots (x - x_n),$$

where the i-th "normalization" factor is the product of all the terms $(x_i - x_j)$ that have $j \neq i$.

3 Problems

Thanks to Elgin Johnston (1997) for most of these problems.

1. (Crux Math., June/July 1978) Show that $n^4 - 20n^2 + 4$ is composite when n is any integer.

   Solution: Factor as difference of two squares. Prove that neither factor can be ± 1.

2. (St. Petersburg City Math Olympiad 1998/14) Find all polynomials $P(x, y)$ in two variables such that for any x and y, $P(x + y, y - x) = P(x, y)$.

   Solution: Clearly constant polynomials work. Also, $P(x, y) = P(x + y, y - x) = P(2y, -2x) = P(16x, 16y)$. Suppose we have a nontrivial polynomial. Then on the unit circle, it is bounded because we can just look at the fixed coefficients. Yet along each ray $y = tx$, we get a polynomial whose translate has infinitely many zeros, so it must be constant. Hence P is constant along all rays, implying that P is bounded by its max on the unit circle, hence bounded everywhere. Now suppose maximum degree of y is N. Study the polynomial $P(z^{N+1}, z)$. The leading coeff of this is equal to the leading coeff of $P(x, y)$ when sorted with respect to x as more important. Since the z-poly is also bounded everywhere, it too must be constant, implying that the leading term is a constant.

3. (Putnam, May 1977) Determine all solutions of the system

$$\begin{aligned} x + y + z &= w \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w}. \end{aligned}$$

   Solution: Given solutions x, y, z, construct 3-degree polynomial $P(t) = (t - x)(t - y)(t - z)$. Then $P(t) = t^3 - wt^2 + At - Aw = (t^2 + A)(t - w)$. In particular, roots are w and a pair of opposites.

4. (Crux Math., April 1979) Determine the triples of integers (x, y, z) satisfying the equation

$$x^3 + y^3 + z^3 = (x + y + z)^3.$$

   Solution: Move z^3 to RHS and factor as $x^3 \pm y^3$. We get $(x + y) = 0$ or $(y + z)(z + x) = 0$. So two are opposites.

5. (USSR Olympiad) Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every positive integer n.

   Solution: Use Euclidean algorithm for GCD. $(n^3 + 2n)n = n^4 + 2n^2$, so difference to denominator is $n^2 + 1$. Yet that's relatively prime to $n(n^2 + 2)$.

6. (Po, 2004) Prove that $x^4 - x^3 - 3x^2 + 5x + 1$ is irreducible.

   Solution: Eisenstein with substitution $x \mapsto x + 1$.

7. (Canadian Olympiad, 1970) Let $P(x)$ be a polynomial with integral coefficients. Suppose there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k with $P(k) = 8$.

   Solution: Drop it down to 4 zeros, and check whether one value can be 3. Factor as $P(x) = (x - a)(x - b)(x - c)(x - d)R(x)$; then substitute k. 3 is prime, but we'll get at most two ± 1 terms from the $(x - \alpha)$ product.

8. (Monthly, October 1962) Prove that every polynomial over the complex numbers has a nonzero polynomial multiple whose exponents are all divisible by 10^9.

   Solution: Factor polynomial as $a(x - r_1)(x - r_2) \cdots (x - r_n)$. Then the desired polynomial is $a(x^P - r_1^P) \cdots (x^P - r_n^P)$, where $P = 10^9$. Each factor divides the corresponding factor.

9. (Elgin, MOP 1997) For which n is the polynomial $1+x^2+x^4+\cdots+x^{2n-2}$ divisible by the polynomial $1+x+x^2+\cdots+x^{n-1}$?

   Solution: Observe:

$$
\begin{aligned}
(x^2-1)(1+x^2+x^4+\cdots+x^{2n-2}) &= x^{2n}-1 \\
(x-1)(1+x+x^2+\cdots+x^{n-1}) &= x^n-1 \\
(x+1)(1+x^2+x^4+\cdots+x^{2n-2}) &= (x^n+1)(1+x+x^2+\cdots+x^{n-1}).
\end{aligned}
$$

   So if the quotient is $Q(x)$, then $Q(x)(x+1)=x^n+1$. This happens iff -1 is a root of x^n+1, which is iff n is odd.

10. (Czech-Slovak Match, 1998/1) A polynomial $P(x)$ of degree $n \geq 5$ with integer coefficients and n distinct integer roots is given. Find all integer roots of $P(P(x))$ given that 0 is a root of $P(x)$.

    Solution: Answer: just the roots of $P(x)$. Proof: write $P(x)=x(x-r_1)(x-r_2)\cdots(x-r_N)$. Suppose we have another integer root r; then $r(r-r_1)\cdots(r-r_N)=r_k$ for some k. Since degree is at least 5, this means that we have $2r(r-r_k)$ dividing r_k. Simple analysis shows that r is between 0 and r_k; more analysis shows that we just need to defuse the case of $2ab \mid a+b$. Assume $a \leq b$. Now if $a=1$, only solution is $b=1$, but then we already used ± 1 in the factors, so we actually have to have $12r(r-r_k)$ dividing r_k, no good. If $a>1$, then $2ab > 2b \geq a+b$, contradiction.

11. (Hungarian Olympiad, 1899) Let r and s be the roots of

$$x^2-(a+d)x+(ad-bc)=0.$$

    Prove that r^3 and s^3 are the roots of

$$y^2-(a^3+d^3+3abc+3bcd)y+(ad-bc)^3=0.$$

    Hint: use Linear Algebra.

    Solution: r and s are the eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

    The y equation is the characteristic polynomial of the cube of that matrix.

12. (Hungarian Olympiad, 1981) Show that there is only one natural number n such that $2^8+2^{11}+2^n$ is a perfect square.

    Solution: $2^8+2^{11}=48^2$. So, need to have 2^n as difference of squares N^2-48^2. Hence $(N+48)$, $(N-48)$ are both powers of 2. Their difference is 96. Difference between two powers of 2 is of the form $2^M(2^N-1)$. Uniquely set to 2^7-2^5.

13. (MOP 97/9/3) Let $S=\{s_1,s_2,\ldots,s_n\}$ be a set of n distinct complex numbers, for some $n \geq 9$, exactly $n-3$ of which are real. Prove that there are at most two quadratic polynomials $f(z)$ with complex coefficients such that $f(S)=S$ (that is, f permutes the elements of S).

14. (MOP 97/9/1) Let $P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ be a nonzero polynomial with integer coefficients such that $P(r)=P(s)=0$ for some integers r and s, with $0<r<s$. Prove that $a_k \leq -s$ for some k.

Putnam Σ.2

Po-Shen Loh

2 September 2012

1 Problems

Putnam 1996/A4. Let S be the set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

1. $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
2. $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;
3. (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

Prove that there exists a one-to-one function g from A to R such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. Note: R is the set of real numbers.

Putnam 1996/A5. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

Putnam 1996/A6. Let $c > 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : R \to R$ such that $f(x) = f(x^2 + c)$ for all $x \in R$. Note that R denotes the set of real numbers.

Putnam $\Sigma.3$

Po-Shen Loh

9 September 2012

1 Problems

Putnam 1996/B4. For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

Putnam 1996/B5. Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \le \Delta(T) \le 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

Putnam 1996/B6. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers x and y such that

$$(a_1, b_1) x^{a_1} y^{b_1} + (a_2, b_2) x^{a_2} y^{b_2} + \cdots$$
$$+ (a_n, b_n) x^{a_n} y^{b_n} = (0, 0).$$

Putnam Training

Session 1 : Thursday 30$^{\text{th}}$ September

Sequences and Series

INEQUALITIES

The triangle inequality:

$$1.\quad |x+y| \le |x| + |y|$$

$$2.\quad \big||x| - |y|\big| \le |x-y|$$

The Cauchy-Schwarz inequality:

$$1.\quad \sum x_i^2 \cdot \sum y_i^2 \ge \left(\sum x_i y_i\right)^2$$

$$2.\quad \sum x_i^2 \ge \frac{1}{n}\left(\sum x_i\right)^2$$

The Jensen inequality:

$$\textit{If } \sum \lambda_i = 1,\ \lambda_i \ge 0 \ \textit{ and } f \textit{ is convex then}$$

$$f\left(\sum \lambda_i x_i\right) \ge \sum \lambda_i f(x_i)$$

The inequality of arithmetic and geometric means:

$$\frac{1}{n}\sum x_i \ge \sqrt[n]{\prod x_i}$$

LIMITS OF SEQUENCES AND RECURRENCES

Limits of sequences

1. *Explicit sequence* $a_n = f(n)$
2. *Recursively defined sequence* $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_1)$

Monotone convergence theorem

If a sequence is increasing and bounded above, then it converges to its least upper bound.

Solving linear recursions

If $b_k a_n + b_{k-1} a_{n-1} + \cdots + b_0 a_{n-k} = 0$ *then solve the characteristic equation*

$$b_k x^k + b_{k-1} x^{k-1} + \cdots + b_0 = 0$$

with roots $x_0, x_1, \ldots, x_{k-1}$. *If they are different then*

$$a_n = c_0 x_0^n + c_1 x_1^n + \ldots + c_{k-1} x_{k-1}^n$$

where the c_i *are constants determined by the initial conditions* $a_0 = \alpha_0, a_1 = \alpha_1, \ldots, a_{k-1} = \alpha_{k-1}$.

Fibonacci Numbers

The Fibonacci numbers satisfy

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = F_2 = 1$$

Write down $x^2 = x + 1$ and use the quadratic formula to get

$$x = \frac{1 \pm \sqrt{5}}{2}$$

So we know

$$F_n = c_0 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_1 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Since $F_0 = F_1 = 2$ we get

$$1 = c_0 \left(\frac{1+\sqrt{5}}{2} \right)^2 + c_1 \left(\frac{1-\sqrt{5}}{2} \right)^2$$

$$1 = c_0 \left(\frac{1+\sqrt{5}}{2} \right)^1 + c_1 \left(\frac{1-\sqrt{5}}{2} \right)^1$$

so $c_0 = \frac{1}{\sqrt{5}}, c_1 = \frac{-1}{\sqrt{5}}$ and

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Limits and Recurrence Equations

Defining $a_n = f(a_{n-1})$, if a_n is increasing and bounded then (MCT) it has a limit L satisfying

$$L = f(L)$$

Continued fractions and iterated functions

1. $$a_n = b_0 + \cfrac{b_1}{1 + \cfrac{b_2}{1 + \cfrac{b_3}{1 + \cdots + \cfrac{b_{n-1}}{1 + b_n}}}}$$

2. $$a_n = b_0 + c_1\sqrt{b_1 + c_2\sqrt{b_2 + \cdots + c_n\sqrt{b_n}}}$$

CALCULUS AND SERIES

Stirling's Formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\theta} \qquad \frac{1}{12n+1} < \theta < \frac{1}{12n}$$

Taylor series:

1. $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}, -1 < x \leq 1$
2. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
3. $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
4. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) x^k}{k!}$

Exponential inequalities:

1. $$\prod (1 + x_i) \le e^{\sum x_i}$$
2. $$\prod (1 + x_i) \ge e^{\sum x_i - \frac{1}{2}\sum x_i^2}$$

Identities

1. $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, |x| < 1$$
2. $$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$
3. $$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$
4. $$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Riemann sums

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\tfrac{k}{n}) = \int_0^1 f(x)dx$$

Telescoping sums

$$\sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0$$

Convergence tests

1. Comparison test. Comparing a sum to a known convergent or divergent series
2. Integral test. Compare with an integral.
3. Ratio test. Ratio of consecutive terms larger than 1 implies divergence.

Harmonic sums

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges
2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

Rationality

$$\sqrt{2}, e, \pi, \ln 2 \ \textit{are irrational}$$

The sum of reciprocals of a fast enough growing sequence is usually irrational such as

$$\sum_{n=1}^{\infty} 2^{-n^2}$$

Diophantine approximation

For every real number α and given a positive integer n, there exists integers $q \le n$ and $p \le \lceil \alpha n \rceil$ such that

$$\left| \alpha - \frac{p}{q} \right| \le \frac{1}{qn}$$

Calculus on sums

Under appropriate convergence criteria,

1. $\sum \lim_{x \to a} f(x) = \lim_{x \to a} \sum f'(x)$
2. $\sum f'(x) = \frac{d}{dx} \sum f'(x)$
3. $\sum \int_0^x f(t)dt = \int_0^x \left(\sum f(t) \right) dt$

WARMUP PROBLEMS

Problem 1. Solve the recurrence

$$a_n = 2a_{n-1} + a_{n-3} \qquad a_0 = a_1 = 1$$

Solution. Write down

$$x^2 - 2x - 3 = 0$$
$$x = -1, x = 3$$

and therefore

$$a_n = c_0(-1)^n + c_1 3^n$$

Using the initial conditions

$$1 = c_0 + c_1$$
$$1 = -c_0 + 3c_1$$
$$c_1 = c_0 = \frac{1}{2}$$

So we get

$$a_n = \frac{1}{2}(-1)^n + \frac{1}{2}3^n \qquad \checkmark$$

Problem 2. Find the 100[th] digit after the decimal space in

$$(1+\sqrt{2})^{2010}$$

Solution. This is part of a solution to a recurrence. Ignore the denominator and expand:

$$(x-1-\sqrt{2})(x-1+\sqrt{2}) = x^2-2x-1$$

So the recurrence was

$$a_n - 2a_{n-1} = a_{n-2}$$

If the initial conditions are $a_0 = a_1 = 1$ we get

$$a_n = \tfrac{1}{2}(1+\sqrt{2})^n + \tfrac{1}{2}(1-\sqrt{2})^n$$

The second term for $n = 2010$ has only zeroes in the first 100 places after the decimal point since it is very small, less than $2^{-2010} < 10^{-500}$. The left side is an integer. So that means

$$\tfrac{1}{2}(1+\sqrt{2})^n$$

has 9s in the first at least 500 locations after the decimal point and therefore there is a 9 at position 100 after the decimal point in $(1+\sqrt{2})^n$. ✓

Problem 3. Determine

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$$

Solution. Experiment shows that the answer should be 3.

Consider

$$f(x) = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+\cdots}}}$$

It is not hard to see that $\frac{x+1}{2} \le f(x) \le 2(x+1)$ [prove it] and

$$f(x)^2 = xf(x+1)+1$$

Putting the inequalities into the equation we get

$$\begin{aligned}\frac{x(x+1)}{2}+1 \le f(x)^2 \le 2x(x+1)+1\\ \frac{(x+1)^2}{2} \le f(x)^2 \le 2(x+1)^2\\ \frac{x+1}{\sqrt{2}} \le f(x) \le \sqrt{2}(x+1)\end{aligned}$$

Repeating gives for any $n \in \mathbb{N}$

$$2^{-1/2^n}(x+1) \le f(x) \le 2^{1/2^n}(x+1)$$

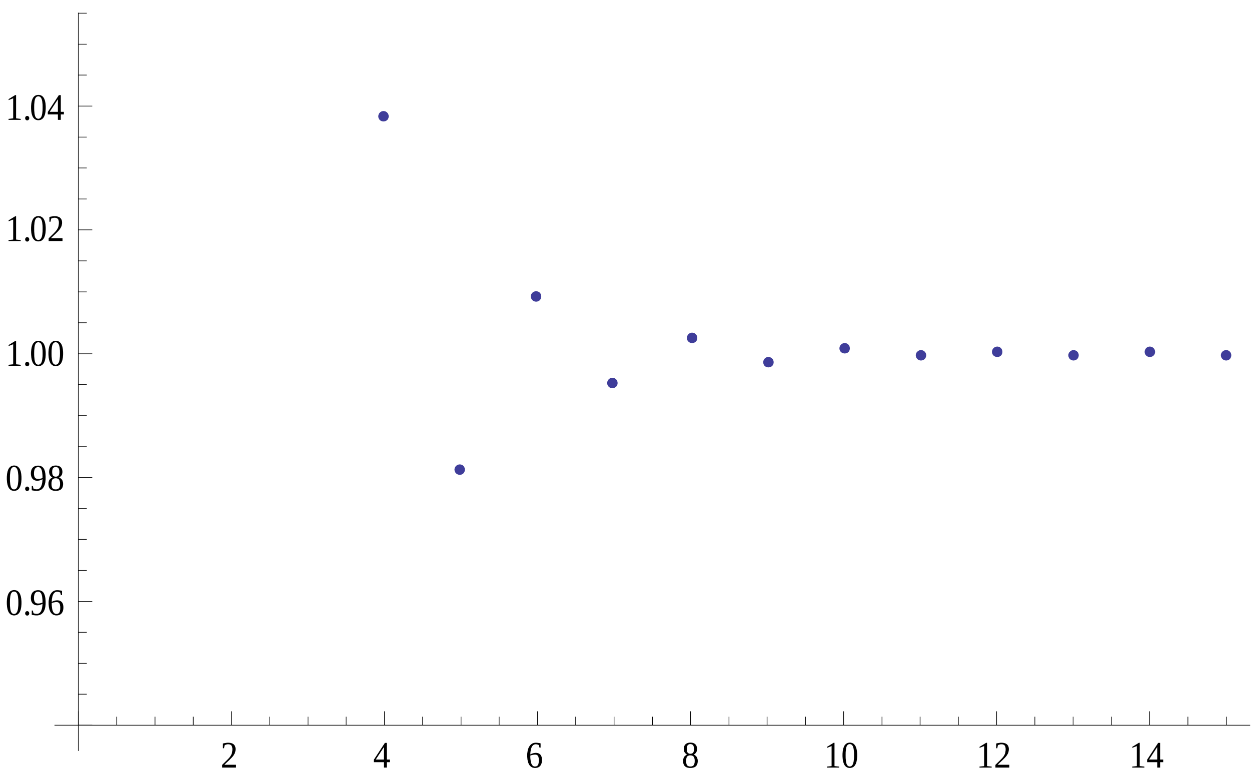
and this means $f(x) = x+1$. So the answer is $f(2) = 3$. ✓

Problem 4. Find the limit of a_n when

$$2a_n a_{n-1} = a_{n-1} + 1 \quad \text{and} \quad a_0 = c$$

Solution. If it has a limit L then

$$2L^2 = L + 1 \Rightarrow L = 1 \text{ or } L = -\frac{1}{2}$$



$c = 1$

But we have to prove there is a limit. We can't use monotone convergence since evidently the sequence is not monotone.

By the way we can write

$$a_n = \frac{1}{2} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{2} + \cdots + \cfrac{1}{\frac{1}{2} + \frac{1}{2c}}}}$$

Check then that $a_n = -\frac{1}{2}$ always when $c = -\frac{1}{2}$. So the limit is

$$L = -\tfrac{1}{2} \quad \text{when } c = -\tfrac{1}{2}$$

For other values of c, try to show

1. a_{2n} is monotone, positive and bounded
2. a_{2n-1} is monotone, positive and bounded
3. $a_n / a_{n-1} \to 1$

It follows that $\lim a_{2n} = \lim a_{2n-1}$ exist by the monotone convergence theorem. The limits are positive since all the terms are positive. So we find that the limit is

$$L = 1 \quad \text{when } c \neq -\tfrac{1}{2}$$

Remark. Actually the solution to the recurrence is

$$a_n = \frac{-(-1)^n + 2^n + c(-1)^n + c2^{n+1}}{-2(-1)^n - 2^n + 2c(-1)^n + 2^{n+1}c}$$

but we do not have a method for finding this solution.

Chris Street
Solved Putnam Exam practice problems

Problem, 1932 A-2. Determine all polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2+1$ and $P(0) = 0$.

Solution. The only such polynomial is x, the identity polynomial.

Proof. Let $P(x)$ be such a polynomial.

Define the inductive sequence $i_0 = 0, i_n = i_{n-1}^2 + 1$. We make two observations about this sequence - first, that it is strictly increasing and therefore the i_k's are distinct. Secondly, for all j, $P(i_j) = i_j$. This is easily shown by a basic induction: $P(i_0) = P(0) = 0$, by hypothesis, and if $P(i_m) = i_m$, then $P(i_{m+1}) = P(i_m^2+1) = P(i_m)^2 + 1 = i_m^2 + 1 = i_{m+1}$.

Now, consider the polynomial $Q(x) = P(x) - x$. Let $n = \deg Q(x)$. Suppose $n \geq 1$. Then $i_0, i_1 \cdots i_n$ are $(n+1)$ distinct zeros of $Q(x)$. This is a contradiction of the fundamental theorem of algebra.

Thus $n = 0$, and $Q(x)$ is a constant polynomial. Since we know $Q(0) = P(0) - 0 = 0$, it follows that $Q(x)$ is the polynomial identically equal to zero, and the claim is established. ■

Problem, 1977 A-4. For $0 < x < 1$, express

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}}$$

as a rational function of x.

Solution. It is $f(x) = \frac{x}{1-x}$.

Proof. We desire to find

$$\lim_{k\to\infty} S_k$$

where

$$S_k = \sum_{n=0}^{k} \frac{x^{2^n}}{1-x^{2^{n+1}}}$$

First, I desire to show that this sequence S converges. This is easily seen by the fact that, for $0 < x < 1$,

$$S_k = \sum_{n=0}^{k} \frac{x^{2^n}}{1-x^{2^{n+1}}} \leq \sum_{n=1}^{2^k} \frac{x^n}{1-x^{2n}} \leq \sum_{n=1}^{2^k} \frac{x^n}{1-x} = \frac{1}{1-x}\sum_{n=1}^{2^k} x^n = \frac{1-x^{2^k+1}}{(1-x)^2} - \frac{1}{1-x}.$$

and the fact that S_k is strictly increasing for $0 < x < 1$. Now that I know that S converges, then I know that the subsequence $T_m = S_{2m+1}$ also converges and to the same limit. I write

$$\begin{aligned}
T_m &= \sum_{n=0}^{m} \frac{x^{2^{2n}}}{1-x^{2^{2n+1}}} + \frac{x^{2^{2n+1}}}{1-x^{2^{2n+2}}} \\
&= \sum_{n=0}^{m} \frac{x^K}{1-x^{2K}} + \frac{x^{2K}}{1-x^{4K}}, \text{where } K = 2^{2n} \\
&= \sum_{n=0}^{m} \frac{x^K(1+x^{2K}) + x^{2K}}{1-x^{4K}} \\
&= \sum_{n=0}^{m} \frac{x^K + x^{2K} + x^{3K}}{1-x^{4K}} \\
&= \sum_{n=0}^{m} \frac{1 + x^K + x^{2K} + x^{3K} - 1}{(1 + x^K + x^{2K} + x^{3K})(1-x^K)} \\
&= \sum_{n=0}^{m} \frac{1}{1-x^K} - \frac{1}{1-x^{4K}}.
\end{aligned}$$

Substituting back $K = 2^{2n}$, we have

$$T_m = \sum_{n=0}^{m} \frac{1}{1-x^{2^{2n}}} - \frac{1}{1-x^{2^{2n+2}}}.$$

From this, we see that we have a telescoping sum, and thus

$$T_m = \frac{1}{1-x} - \frac{1}{1-x^{2^{2m+2}}}.$$

Taking the limit as $m \to \infty$, we see that the second term goes to 1 (since $0 < x < 1$), and thus we have

$$\lim_{m\to\infty} T_m = \lim_{k\to\infty} S_k = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

■

Problem, B-1 1977. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}.$$

Solution. Write

$$P_m = \prod_{n=2}^{m} \frac{n^3-1}{n^3+1} = \prod_{n=2}^{m} \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}.$$

We desire $\lim_{m\to\infty} P_m$.

Now we are going to write the k'th term in the product as

$$\frac{a_k b_k}{c_k d_k}, \text{ where}$$

$$a_k = k-1, b_k = k^2+k+1, c_k = k+1, d_k = k^2-k+1.$$

Notice that $a_k = c_{k-2}$ and $d_k = b_{k-1}$. Thus

$$\begin{aligned} P_m = \prod_{n=2}^{m} \frac{n^3-1}{n^3+1} &= \frac{a_2 b_2}{c_2 d_2}\frac{a_3 b_3}{c_3 d_3}\frac{a_4 b_4}{c_4 d_4}\cdots\frac{a_{m-2} b_{m-2}}{c_{m-2} d_{m-2}}\frac{a_{m-1} b_{m-1}}{c_{m-1} d_{m-1}}\frac{a_m b_m}{c_m d_m} \\ &= \frac{a_2 b_2}{c_2 d_2}\frac{a_3 b_3}{c_3 b_2}\frac{c_2 b_4}{c_4 b_3}\cdots\frac{c_{m-4} b_{m-2}}{c_{m-2} b_{m-3}}\frac{c_{m-3} b_{m-1}}{c_{m-1} b_{m-2}}\frac{c_{m-2} b_m}{c_m b_{m-1}} \\ &= \frac{a_2 a_3 b_m}{d_2 c_{m-1} c_m} \\ &= \frac{2}{3}\frac{m^2+m+1}{m^2+m} \end{aligned}$$

Therefore $\lim_{m\to\infty} P_m = \lim_{m\to\infty} \frac{2}{3}\frac{m^2+m+1}{m^2+m} = \frac{2}{3}$. ■

Problem, B-5 1968. Let p be a prime number. Let J_p be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose entries are chosen from the set $\{0, 1, 2, \cdots, p-1\}$ and which satisfy the conditions $a + d \equiv 1 \mod p$ and $ad - bc \equiv 0 \mod p$. Determine how many members J_p has.

Solution. I do my work in the field Z_p, of integers mod p. We desire to count all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in Z_p, a + d = 1, ad = bc$.

We fix a. This forces $d = 1 - a$. Our second equation becomes $a(a-1) = bc$. Now, how many solutions does this equation have?

If $a(a-1) = 0$, then either $a = 0$ or $a = 1$. In either case, the solutions are $b = 0$ or $c = 0$, which yields $2p - 1$ solutions.

If $a(a-1) \neq 0$, then $bc = m$ with $m \neq 0$. Since Z_p is cyclic under $+$ of order p, $\forall c \neq 0 \in Z_p$ $\exists$ exactly one $b \in Z_p$ s.t. $bc = m$. This gives a total of $p - 1$ solutions.

Thus, all in all, there are $2(2p-1) + (p-2)(p-1) = p^2 + p$ solutions, and so $|J_p| = p^2 + p$. ■

Problem, A-1 1965. At a party, assume that no boy dances with every girl but each girl dances with at least one boy. Prove that there are two couples gb and $g'b'$ which dance, whereas b does not dance with g' nor does g dance with b'.

Solution. Let b be the boy that dances with the maximal number of girls. (We are here assuming a finite dance floor.) Let g' be a girl that does not dance with b. Let b' be a boy that g' does dance with.

Now, there exists a girl g that does dance with b, but does not dance with b'. For if not, b' dances with at least one more girl than b does, a contradiction to our assumption.

Now we have found the b, g, b', g' that solve the problem. ■

Problem, A-1 1983. How many positive integers n are there such that n is an exact divisor of at least one of the numbers $10^{40}, 20^{30}$?

Solution. Call A the set of positive divisors of 10^{40}, and B the set of positive divisors of 20^{30}.

$10^{40} = 2^{40}5^{40}$, so all divisors have the form 2^i5^j, with i and j running independently from 0 to 40. Thus $|A| = 41 \times 41 = 1,681$.

$20^{30} = 2^{60}5^{30}$, and again all divisors have the form 2^i5^j, with i from 0 to 61 and j from 0 to 31. Thus $|B| = 61 \times 31 = 1,891$.

Now, to determine $|A \bigcap B|$, we look at the divisors of $\gcd(10^{40}, 20^{30}) = 2^{40}5^{30}$. This has $41 \times 31 = 1,271$ divisors, thus $|A \bigcap B| = 1,271$.

So the desired quantity, $|A \bigcup B|$, is $|A| + |B| - |A \bigcap B| = 1,681 + 1,891 - 1,271 = 2,301$. ■

Problem, A-3 1967. Consider polynomial forms $ax^2 + bx + c$ with integer coefficients which have two distinct zeros in the open interval $0 < x < 1$. Exhibit with a proof the least positive integer value of a for which such a polynomial exists.

Solution. What we want is

$$0 < -b + \sqrt{b^2 - 4ac} < 2a, 0 < -b - \sqrt{b^2 - 4ac} < 2a.$$

Manipulating the inequalities gives

$$\begin{aligned} -2a <& 2\sqrt{b^2 - 4ac} < 2a \\ -a <& \sqrt{b^2 - 4ac} < a \\ 0 <& \sqrt{b^2 - 4ac} < a \\ 0 <& b^2 - 4ac < a^2. \end{aligned}$$

Also we want

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \neq 1,$$

thus

$$b^2 - 4ac \neq (2a + b)^2.$$

We make note that we desire $c \neq 0$ to guarantee no root at 0.

We write $ax^2 + bx + c = ax^2 - a(r_1 + r_2)x + c$, where r_1, r_2 are our roots. We have $0 < r_1 + r_2 < 2$. Also $r_1 + r_2$ is rational (it is $\frac{-b}{2a}$.) Thus say $r_1 + r_2 = \frac{p}{q}$, a fraction in lowest terms. b is an integer, so $q|ap$, thus $q|a$ since $(p, q) = 1$.

Reiterating, we have formulated the following constraints on the problem:

$$\begin{aligned}
0 <& a^2\left(\frac{p}{q}\right)^2 - 4ac < a^2,\\
a^2\left(\frac{p}{q}\right)^2 - 4ac &\neq (2a - a\frac{p}{q})^2,\\
(p,q) = 1, 0 <& \frac{p}{q} < 2, q|a, c \neq 0.
\end{aligned}$$

where I have back-substituted $b = -a\frac{p}{q}$ into our previous inequalities.

This greatly limits the number of possibilites for (a, q, p) solution triples. For a from 1 to 4, the possibilities are $(1,1,1)$, $(2,1,1)$, $(2,2,1)$, $(2,2,3)$, $(3,1,1)$, $(3,3,1)$, $(3,3,2)$, $(3,3,4)$, $(3,3,5)$, $(4,1,1)$, $(4,2,1)$, $(4,2,3)$, $(4,4,1)$, $(4,4,3)$, $(4,4,5)$, $(4,4,7)$. We can limit these further since if (a, q, p) does not satisfy the conditions, then $(b, q, p), b > a$ will not either. But none of the cases satisfy all of the constraints.

However, when $a = 5$, the polynomial $5x^2 - 5x + 1$ has roots $(\frac{1}{2} \pm \frac{\sqrt{5}}{10})$, which are both in $(0, 1)$. Hence 5 is the smallest such a that works. ■

Problem, B-2 1965. Suppose n players play a round-robin tournament (ie, every player plays every other player exactly once.) Each game results in a win or loss for a player: there are no ties. Let w_k be the number of wins by player k, and let l_k be the number of losses by player k. Show that

$$\sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} l_i^2$$

Proof. Let G be the total number of games played (this number is $\binom{n}{2}$, of course, but that is not important here.) The number of games played by each player is $(n-1)$, so we can write that, for all i, $w_i = (n-1) - l_i$. Then we have

$$\begin{aligned}
\sum_{i=1}^{n} w_i^2 &= \sum_{i=1}^{n} w_i((n-1) - l_i)\\
&= (n-1)\sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i l_i\\
&= (n-1)G - \sum_{i=1}^{n} w_i l_i\\
&= (n-1)\sum_{i=1}^{n} l_i - \sum_{i=1}^{n} l_i w_i\\
&= \sum_{i=1}^{n} l_i((n-1) - w_i) = \sum_{i=1}^{n} l_i^2.
\end{aligned}$$

■

Problem, B-4 1967. We have a hallway with n lockers, labeled 1 through n. The lockers have two possible states, open and closed. Initially they are all closed. The first kid walking down the hallway flips every locker to the opposite state (that is, he opens them all). The 2nd kid flips the locker door 2 and every other locker door after that. The kth kid flips the state of every kth locker door. After infinitely many kids have done this, which locker doors are closed and which are open?

Solution. Take locker number n. It began closed, and will be flipped $\sigma(n)$ times, where $\sigma(n)$ is the number of positive integer divisors of n. It will finish up closed if and only if $\sigma(n)$ is an even number, and will finish up open if and only if $\sigma(n)$ is odd.

Suppose n has the prime factorization $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$, the p_m's prime. Then

$$\sigma(n) = \prod_{m=1}^{k} (i_m + 1).$$

Notice that $\sigma(n)$ is odd if and only if each of the i_m's is even. Thus, $\sigma(n)$ is odd if and only if n is a perfect square. Therefore, the open lockers will be exactly those whose number is a perfect square: that is, the 1st, 4th, 9th, etc. All other doors will be closed. ■

Alternative proof of fact used above. (Without using unique factorization) The divisors of n come in pairs: that is, if p is an integer divisor of n, then n/p is an integer divisor of n. Hence, the number of divisors for a given n will be even *unless* for some divisor p, $p = n/p$ - which is to say, $n = p^2$. ■

Problem, A-1 1977. Consider all lines that meet the graph of

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points, say $(x_i, y_i), i = 1, 2, 3, 4$. Show that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line, and find its value.

Solution. Remember that a polynomial with real coefficients may be written as

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = c_n (x - r_1)(x - r_2) \cdots (x - r_{n-1})(x - r_n),$$

where $r_1, r_2, \cdots r_n$ are the roots of the polynomial in $\mathbb{C}$, with multiplicities. Multiplying out and equating coefficients gives us the useful identity

$$c_{n-1} = c_n(-r_1 - r_2 - \cdots - r_{n-1} - r_n).$$

To apply this fact to our problem, let's call the line which intersects the polynomial in question $L(x) = mx + b$ (we know it has a finite slope m, since no vertical line will meet the function in more than one place.) Now notice that x_1, x_2, x_3, x_4 are roots of the polynomial $y - L(x) = 2x^4 + 7x^3 + (3-m)x - 5 - b$. Since this polynomial is of degree 4, and each of $x_1 \cdots x_4$ are distinct, these are all of the roots.

Thus, we have

$$7 = 2(-x_1 - x_2 - x_3 - x_4)$$

which gives that

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = -\frac{7}{8}.$$

Notice that this value is entirely independent of the values of m or b. ■

Problem, A-1 1978. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \cdots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

Solution. Consider the disjoint sets $A_1 = \{1\}, A_2 = \{52\}, A_3 = \{4, 100\}, A_4 = \{7, 97\}, A_5 = \{10, 94\}, \cdots, A_{17} = \{46, 58\}, A_{18} = \{49, 55\}$. The union of these sets gives you the complete geometric progression. Also notice that the sum of the elements in each of $A_3, A_4 \cdots A_{18}$ is 104.

At most 2 elements of A can be chosen from the sets A_1 and A_2. Hence at least 18 are chosen from the 16 different sets $A_3, A_4 \cdots A_{18}$. Then there is a set $A_j, j \geq 3$, such that both elements of A_j are chosen. (Indeed, there are at least two such sets, but we only need one.) But these two elements add to 104. ■

Problem, A-2 1988. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a non-zero function g defined on (a, b) such that the wrong product rule is true for x in (a, b).

Solution. Let $g(x) = e^x(2x-1)^{1/2}$, differentiable on the open interval $(\frac{1}{2}, \infty)$. Now

$$\begin{aligned}
f(x) &= e^{x^2} \\
g(x) &= e^x(2x-1)^{1/2} \\
f'(x) &= 2xe^{x^2} \\
g'(x) &= e^x(2x-1)^{-1/2} + e^x(2x-1)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
(f(x)g(x))' &= e^{x^2}(e^x(2x-1)^{-1/2} + e^x(2x-1)^{1/2}) + 2xe^{x^2}e^x(2x-1)^{1/2} \\
&= e^{x^2}e^x(2x-1)^{-1/2} - (1-2x)e^{x^2}e^x(2x-1)^{-1/2} + 2xe^{x^2}e^x(2x-1)^{1/2} \\
&= 2xe^{x^2}e^x(2x-1)^{-1/2} + 2xe^{x^2}e^x(2x-1)^{1/2}.
\end{aligned}$$

This is equal to

$$\begin{aligned} f'(x)g'(x) &= 2xe^{x^2}(e^x(2x-1)^{-1/2} + e^x(2x-1)^{1/2}) \\ &= 2xe^{x^2}e^x(2x-1)^{-1/2} + 2xe^{x^2}e^x(2x-1)^{1/2} \end{aligned}$$

Hence we have found a $g(x)$ that satisfies the requirements, and we are done. ■

The sketch work. The answer was of course not pulled out of ether, but this work was done on scratch paper to determine the correct answer.

What we want is

$$[e^{x^2}g(x)]' = [e^{x^2}]'g'(x),$$

or

$$\begin{aligned} 2xe^{x^2}g(x) + e^{x^2}g'(x) &= 2xe^{x^2}g'(x) \\ 2xe^{x^2}g(x) + e^{x^2}g'(x) - 2xe^{x^2}g'(x) &= 0 \\ e^{x^2}(2xg(x) + (1-2x)g'(x)) &= 0 \end{aligned}$$

Since e^{x^2} is always positive for real x, this equation is only true if

$$2xg(x) + (1-2x)g'(x) = 0.$$

Solving for $g'(x)$ yields

$$g'(x) = \left(1 + \frac{1}{2x-1}\right)g(x).$$

This is a simple differential equation; notice that if $G(x)$ is an antiderivative of $\left(1 + \frac{1}{2x-1}\right)$, then a solution is

$$g(x) = e^{G(X)}.$$

An antiderivative of $\left(1 + \frac{1}{2x-1}\right)$ is $x + \frac{1}{2}\log(2x-1)$. Thus a solution is

$$g(x) = e^{x+\frac{1}{2}\log(2x+1)} = e^x(2x-1)^{1/2}.$$

Now we are ready to pull this function out of our hat to solve the problem! ■

Problem, A-2, 1987. The sequence

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9, 2, 0, \cdots\}$$

is obtained by writing the positive integers in order. If the 10^n'th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

Solution. I will define the function $g(n)$ as follows: $g(n)$ gives the number of elements in the sequence after the number n has been placed in the sequence. For example, $g(1) = 1, g(9) = 9, g(10) = 11, g(101) = 195$.

To find a formula for $g(n)$ and to solve the problem, we will write, instead of $g(n)$, $g(10^k + l)$, where 10^k is the largest power of 10 less than or equal to n, and l is $n - 10^k$. Then, if k satisfies $g(10^k + l) = 10^{1987}$, then $f(1987) = k + 1$.

To write a closed form expression for $g(10^k + l)$, we notice that $10^k + l$ of the numbers from 1 to $10^k + l$ have at least 1 digit. $10^k + l - 9$ of the numbers from 1 to $10^k + 1$ have at least 2 digits. $10^k + l - 99$ of the numbers from 1 to $10^k + 1$ have at least 3 digits, and so on. Thus an expression for $g(10^k + 1)$ is

$$g(10^k + l) = (k + 1)l + \sum_{i=0}^{k} (10^k - 10^i + 1)$$

which, evaluating the geometric sum and simplifying, gives

$$g(10^k + l) = \frac{9(k+1)10^k - 10^{k+1} + 1}{9} + (k + 1)(l + 1).$$

Now we solve

$$\begin{aligned} 10^{1987} &= \frac{9(k+1)10^k - 10^{k+1} + 1}{9} + (k + 1)(l + 1) \\ 10^{1987} &= (k + 1)10^k + (k + 1)(l + 1) + \frac{1 - 10^k}{9} \end{aligned}$$

Now set $k = 1983$. This gives on the right side

$$1.984 \times 10^{1986} + 1984(l + 1) + \frac{1}{9} - \frac{10}{9} \times 10^{1982}$$

Setting this equal to 10^{1987}, we find that

$$8.016 \times 10^{1986} + \frac{10}{9} 10^{1982} - \frac{1}{9} = 1984(l + 1).$$

Now we see that $1984(l + 1)$ has to equal approximately 8.016×10^{1986} to make the equation true. Since l may range from 0 to $9 \times 10^{1986} - 1$, we see that l can be chosen appropriately. Thus $k = 1983$, and $f(1987) = 1984$. ■

Problem, A-5 1988. Prove that there exists a unique function from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x)$$

and $f(x) > 0$ for all $x > 0$.

Solution. First, notice that $f(x) = 2x$ is a solution: certainly $2x > 0$ for $x > 0$, and $f(f(x)) = 4x = 6x - 2x = 6x - f(x)$.

Now we must prove that no other function works.

So suppose that another function, $g(x)$, satisfies these conditions. Suppose that, at $x = k$, $g(k) = 2k + c$. Now, what is $g^{(n)}(k)$, where $g^{(n)}$ is the function iterated n times (ie, $g^{(1)}(k) = g(k), g^{(2)}(k) = g(g(k))$, etc.)? I claim that $g^{(n)}(k) = a_n k + b_n c$, where a_n and b_n are sequences defined by

$$\begin{aligned} a_0 &= 1, a_1 = 2 \\ b_0 &= 0, b_1 = 1 \\ a_n &= 6a_{n-2} - a_{n-1}, b_2 = 6b_{n-2} - b_{n-1}, \text{ for } n \geq 2 \end{aligned}$$

This is easily proved by an induction using the formula $g^{(n)} = 6g^{(n-2)} - g^{(n-1)}$. Similarly, it is easy to show that $a_i = 2^i$ for all i. Now, for all odd $n \geq 1, b_n > 0$, and for all even $n \geq 2, b_n < 0$. To show this, note that $b_1 = 1, b_2 = -1$. Now suppose that this holds for b_m, b_{m+1}, m some odd number greater than 1. Then, $b_{m+2} = 6b_m - b_{m+1}$ - a positive minus a negative, which is positive. Similarly, $b_{m+3} = 6b_{m+1} - b_{m+2}$ - a negative minus a positive. Thus, the result is established.

Along with this, note that $|b_n| > 6|b_{n-2}|$. Since $b_1 = 1$ and $b_2 = -1$, $b_{2m+1} > 6^m$ and $b_{2m+2} < -(6^m)$. Thus

$$\lim_{n\to\infty} \frac{-a_n}{b_n} = \lim_{n\to\infty} -\left(\frac{2}{\sqrt{6}}\right)^n = 0.$$

Now, since $a_n k + b_n c > 0$, we have $c > (-a_n)/b_n k$ for b_n positive, and $c < (-a_n)/b_n k$ for b_n negative. We have seen that $(-a_n)/b_n \to 0$ as $n \to \infty$, thus c is bounded above by a sequence whose limit is 0, and bounded below by a sequence whose limit is zero, and so must equal 0 itself. Thus $g(x) = f(x)$, and this proves the uniqueness of the solution. ■

Problem, B-2 1966. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

Solution. Call the numbers $n, n+1, n+2, \cdots, n+9$. Suppose that the statement is not true: that is, for every $n+i, 0 \leq i \leq 9$, there exists a $n+j, 0 \leq j \leq 9$, such that $(n+i, n+j) \neq 1$ and $i \neq j$.

Then $n+i$ and $n+j$ share a common prime divisor, p. Let $n+i = pm_1, n+j = pm_2$. Then $|(n+i) - (n+j)| = |i-j| = |p(m_1 - m_2)|$. Since $|i-j| \leq 9$, p is one of 2, 3, 5, or 7.

Therefore, it follows that every element of $\{n, n+1, \cdots, n+9\}$ is divisible by at least one of 2, 3, 5, or 7.

Let M_a be the subset of $\{n, n+1, \cdots, n+9\}$ containing all the members divisible by a.

Then $|M_2| \leq 5, |M_3| \leq 4, |M_5| \leq 2, |M_7| \leq 2$.

Notice that $|M_6| \geq 1$, and that if $|M_3| = 4$, then $|M_6| = 2$. (2 of the 4 divisors of 3 must be even.) Thus $|M_3| - |M_6| \leq 2$.

Also, if $|M_5| = 2$, then $|M_{10}| = 1$ (one of the divisors of 5 is even.) Thus $|M_5| - |M_{10}| \leq 1$.

Similarly, if $|M_7| = 2$, then $|M_{14}| = 1$. Thus $|M_7| - |M_{14}| \leq 1$.
Now this gives

$$|M_2 \cup M_3 \cup M_5 \cup M_7| \leq |M_2| + (|M_3| - |M_6|) + (|M_5| - |M_{10}|) + (|M_7| - |M_{14}|).$$

But the quantity on the left is 10, and the quantity on the right is $\leq 5+2+1+1 = 9$, thus we have a contradiction $10 \leq 9$. Therefore, it must be that for some $n+i$ no such j exists, and thus this $n+i$ is relatively prime to every other element of the set. ■

Problem, B-4 1960. Consider the arithmetic progression $a, a+d, a+2d \cdots$, where a and d are positive integers. For any positive integer k, prove that the progression has either no exact kth powers, or infinitely many.

Solution. It suffices to prove that, if one kth power is in the progression, there exists a larger kth power in the progression.

Thus, suppose that $j^k = a + id$ for some positive integers j, i. We will show that there exists an integer $b > 0$ and an integer $m > 0$ such that $(j+b)^k = a + id + md$.

Write

$$\begin{aligned}
(j+b)^k &= \sum_{q=0}^{k} \binom{k}{q} j^q b^{k-q} \\
&= j^k + \sum_{q=0}^{k-1} \binom{k}{q} j^q b^{k-q} \\
&= a + id + \sum_{q=0}^{k-1} \binom{k}{q} j^q b^{k-q} \\
&= a + id + b\sum_{q=0}^{k-1} \binom{k}{q} j^q b^{k-q-1}
\end{aligned}$$

Now notice that if we set $b = d$, then we have

$$(j+d)^k = a + id + md$$

with $m = \sum_{q=0}^{k-1} \binom{k}{q} j^q d^{k-q-1}$, the sum of positive integers and therefore a positive integer itself, and we are done. ■

Problem, A-1 1961. The graph of the equation $x^y = y^x$ in the first quadrant (i.e., the region where $x > 0$ and $y > 0$) consists of a straight line and a curve. Find the coordinates of the intersection point of the line and the curve.

Solution. Let $y = cx, c > 0$. Then we desire to find pairs (x, c) that satisfy

$$x^{cx} = (cx)^x.$$

These quantities are positive, so we can take logarithms. We write:

$$\begin{aligned} x^{cx} &= (cx)^x \\ cx \log x &= x \log(cx) \\ cx \log x &= x(\log x + \log c) \\ x(\log x + \log c - c \log x) &= 0. \end{aligned}$$

Since $x > 0$, this is true iff

$$\begin{aligned} \log x + \log c - c \log x &= 0 \\ (1 - c) \log x &= -\log c. \end{aligned}$$

Notice that if $c = 1$, the above equation becomes vacuous. This only says that $c = 1$ is a solution for any x, or equivalently an (x, y) pair satisfying $x = y$ is a (trivial) solution to the problem. Here is our straight line set of solutions.

To continue, we suppose that $c \neq 1$. Then it is proper to divide by $(c - 1)$:

$$\log x = \frac{\log c}{c - 1}, c \neq 1.$$

This is the equation of the solution curve, giving x in terms of c. By dividing by $(c - 1)$, we eliminated the solution line and created a singular point at the intersection point. So now we want to find what the limiting value of x is as this curve approaches the $c = 1$ solution line. Hence we find

$$\lim_{c \to 1} \frac{\log c}{c - 1} = 1$$

by L'Hospital's rule. Thus, at the intersection point, $\log x = 1$, or $x = e$. Since this point is on the line $y = x$, $y = e$ also, and so the intersection point is (e, e). ■

Problem, A-2 2001. You have coins $C_1, C_2, \cdots, C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Solution. Define O_n to be the probability that the number of heads for n coins is odd, and E_n the probability that the number of heads is even. Then we have the relations

$$\begin{aligned} O_n + E_n &= 1, && \text{for all } n \\ O_n &= \frac{1}{2n+1} E_{n-1} + \frac{2n}{2n+1} O_{n-1}, && \text{for } n \geq 2. \end{aligned}$$

Rewriting the second equation using the first, we obtain

$$\begin{aligned} O_n &= \frac{1}{2n+1}(1 - O_{n-1}) + \frac{2n}{2n+1} O_{n-1} \\ O_n &= \frac{1}{2n+1} + \frac{2n-1}{2n+1} O_{n+1}. \end{aligned}$$

Now I make the claim that $O_n = \frac{n}{2n+1}$ for all n. We verify this by induction on n, the number of coins:

For $n = 1$ we can easily compute $O_1 = \frac{1}{3}$, hence the claim is valid for this value of n. Assume that the claim is true for some $m \geq 1$. Then, by our recurrence relation,

$$O_{m+1} = \frac{1}{2m+3} + \frac{2m+1}{2m+3} O_m$$
$$O_{m+1} = \frac{1}{2m+3} + \frac{2m+1}{2m+3}\frac{m}{2m+1}, \text{by inductive hypothesis,}$$
$$O_{m+1} = \frac{1}{2m+3} + \frac{m}{2m+3} = \frac{m+1}{2m+3},$$

which has the desired form. This completes the induction and proves the claim.

Thus the desired probability, written as a rational function of n, is $O_n = \frac{n}{2n+1}$. ■

Problem, A-5 2001. Prove that there exist unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001.$$

Solution. Any such a satisfies the polynomial

$$x^{n+1} - (x+1)^n - 2001$$

which has constant coefficient -2002 and leading coefficient 1. Thus, by the Rational Root Theorem and the fact that a is a positive integer, a is a positive integer divisor of $2002 = 2 \times 7 \times 11 \times 13$.

Also, $a \neq 1001$, since $1001^n \equiv 1 \mod 10$ for all n, and 1002^n is not divisible by 10 for any n - thus, $1001^n - 1002^{n-1} \not\equiv 1 \mod 10$ for all n.

Exactly the same observation shows that $a \neq 11, a \neq 91, a \neq 1$.

Now $a \neq 14$, since $14^n \equiv 2 \mod 6$ for odd n, and $14^n \equiv 4 \mod 6$ for even $n > 0$. $15^n \equiv 3 \mod 6$ for all $n > 0$. However, $2001 \equiv 3 \mod 6$. $2 - 3 \not\equiv 3 \mod 6$ and $4 - 3 \not\equiv 3 \mod 6$.

The situation $\mod 6$ is the same for 26 and for 182, thus $a \neq 26$, $a \neq 182$.

$a \neq 2$, for if $2^{n+1} - 3^n = 2001$ for some $n > 0$, then $2^{n+1} = 3(3^{n-1} + 667)$, which cannot be since 3 does not divide any power of 2.

$a \neq 143$, since $143^n \equiv 8 \mod 9$ for odd n and $143^n \equiv 1 \mod 9$ for even n. $144 \equiv 0 \mod 9$. However, $2001 \equiv 3 \mod 9$, and this is neither 8 nor 1.

$a \neq 154$, since $n = 1$ is not a solution, $154^n \equiv 0 \mod 8$ for all $n > 2$, and $155^n \equiv 5, 1 \mod 8$ depending on the parity of n. $2001 \equiv 1 \mod 8$ and neither $(0-1)$ nor $(0-5)$ is congruent to $1 \mod 8$.

Similarly, considering 2002 and 2003 $\mod 8$ yields $a \neq 2002$.

$a \neq 77$, since $77^n \equiv 1, 2 \mod 3$ depending on the parity of n, and $78^n \equiv 0 \mod 3$. However, $2001 \equiv 0 \mod 3$, and neither 1 nor 2 is congruent to $0 \mod 3$.

$a \neq 22$, since $22^n \equiv 4 \mod 12$ for $n > 1$, and $23^n \equiv 1, 11 \mod 12$ depending on the parity of n. $2001 \equiv 9 \mod 12$ and neither $(4-1)$ nor $(4-11) = (4+1)$ are congruent to $9 \mod 12$.

$a \neq 7$, since if $n > 0$ is even, $7^n \equiv 1 \mod 12$ and $8^{n-1} \equiv 8 \mod 12$. If n is odd, $7^n \equiv 7 \mod 12$ and $8^{n-1} \equiv 4 \mod 12$. $2001 \equiv 9 \mod 12$. In neither case is $7^{n+1} - 8^n \equiv 9 \mod 12$.

$a \neq 286$, since $286^n \equiv 1 \mod 15$ for all n, and $287^n \equiv 1, 2, 4, 8 \mod 15$ depending on the value of $n \mod 4$. $2001 \equiv 6 \mod 15$, and none of $(1-1), (1-2), (1-4), (1-8)$ is congruent to $6 \mod 15$.

Thus we have accounted for all of the divisors of 2002 except 13. $13^{n+1} - 14^n \equiv 1 \mod 10$ if and only if $n \equiv 2 \mod 4$. Also, $13^{n+1} - 14^n \equiv 10 \mod 11$ if and only if $n \equiv 2 \mod 11$. Thus possible solutions are $n = 2, 46, 90 \cdots$. Sure enough, $13^3 - 14^2 = 2001$, and we know that a is unique. To show that no other n is a solution, we need only note that $14^{46} > 13^{47}$. Of course, this is difficult to verify directly by hand, but may be proven by many methods of estimation. One (very easy) way to do it, using the fact that $(1 + \frac{1}{n})^{n+1} > e$ for all n, is:

$$\left(\frac{14}{13}\right)^{46} = \left(\left(1 + \frac{1}{13}\right)^{14}\right)^3 \left(\frac{14}{13}\right)^4 > e^3 > \left(\frac{5}{2}\right)^3 > 13.$$

Thus the value of n is unique as well, and the single solution in positive integers is

$13^3 - 14^2 = 2001$.

■

Problem, B-1, 2001. Let n be an even positive integer. Write the numbers $1, 2 \cdots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, ..., (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility.) Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Solution. Let a valid coloring be given.

Define $\alpha_{ij} = -1$ if the square in row i, column j is colored black, and $\alpha_{ij} = +1$ if the square in row i, column j is colored red. Then the problem stated is equivalent to the following:

Show

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}((i-1)n + j) = 0,$$

where

$$\sum_{i=1}^{n} \alpha_{ij} = 0 \quad \text{for any } j,$$
$$\sum_{j=1}^{n} \alpha_{ij} = 0 \quad \text{for any } i.$$

To show this, we now simply evaluate the sum:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}((i-1)n+j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} in - \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} n + \sum_{i=1}^{n}\sum_{j=1}^{n} j\alpha_{ij} \\
&= n\sum_{i=1}^{n} i \sum_{j=1}^{n} \alpha_{ij} - n\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} + \sum_{j=1}^{n} j \sum_{i=1}^{n} \alpha_{ij} \\
&= n\sum_{i=1}^{n}(i \times 0) - n\sum_{i=1}^{n} 0 + \sum_{j=1}^{n}(j \times 0) \\
&= 0,
\end{aligned}$$

and we are done. ■

Problem, B-3 2001. For any positive integer n let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Solution. Suppose i is an integer such that $k^2 \le i \le (k+1)^2$, k an integer. Since $(k+\frac{1}{2})^2 = k^2 + k + \frac{1}{4}$, if $i > k^2 + k$, then $\langle i \rangle = k + 1$. Otherwise, $\langle i \rangle = k$.

From this we see that the set of all integers i such that $\langle i \rangle = k$ is the set of i satisfying $(k-1)^2 + (k-1) + 1 \le i \le k^2 + k$, or $k^2 - k + 1 \le i \le k^2 + k$.

If we take the sum over just those i such that $\langle i \rangle = k$, we get

$$\sum_{i=k^2-k+1}^{k^2+k} \frac{2^k + 2^{-k}}{2^i} = (2^k + 2^{-k}) \sum_{i=k^2-k+1}^{k^2+k} \left(\frac{1}{2}\right)^i.$$

This is a geometric series; evaluating we obtain:

$$(2^k + 2^{-k}) \left(\frac{1 - \left(\frac{1}{2}\right)^{k^2+k+1}}{1 - \frac{1}{2}} - \frac{1 - \left(\frac{1}{2}\right)^{k^2-k+1}}{1 - \frac{1}{2}} \right)$$

Simplifying we obtain:

$$2^{-k^2+2k} - 2^{-k^2-2k}.$$

Now the desired sum is:

$$\sum_{k=1}^{\infty}(2^{-k^2+2k} - 2^{-k^2-2k})$$

How do we evaluate this? All we need do is notice that $-k^2 - 2k = -(k+2)^2 + 2(k+2)$, and thus the sum telescopes:

$$(2^1 - 2^{-3}) + (2^0 - 2^{-8}) + (2^{-3} - 2^{-15}) + (2^{-8} - 2^{-24}) \cdots$$

...and the only terms remaining in the sum as $n \to \infty$ are $2^0 + 2^1 = 3$. Thus the desired sum is 3. ■

Problem, A-3 1968. Prove that a list can be made of all the subsets of a finite set in such a way that

(i.) The empty set is first in the list,

(ii.) each subset occurs exactly once, and

(iii.) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

Solution. For a set of n elements, the problem is equivalent to this one: can we write a list of all strings of length n on the alphabet $\{0, 1\}$, where each string occurs once and only once, the first string is the string of n zeroes, and each string in the list thereafter may be obtained from the previous one by toggling exactly one of the characters (that is, changing one 0 to a 1, or one 1 to a 0.)

To show how such a list is constructed, we will induct on n. If $n = 1$, then clearly the only viable list is $0, 1$. Now suppose a list can be constructed for $n = m, m \geq 1$. Then to construct a list for $m + 1$ objects do the following:

For the first half of the list, copy down the list for m elements, adding a 0 to the front of each element. For the second half of the list, copy down the list for m elements *backwards,* adding a 1 to the front of each element.

Does this list satisfy the requirements? The string of $m+1$ zeroes is first on the list. Each possible string appears exactly once. From inductive hypothesis, each half of the list has the property that each successive element is obtained by changing exactly one of the characters. The only thing to prove is that this property holds between the last element of the first half and the first element of the second half. But clearly it does, since the first element of the second half is the last element of the first half with the leading 0 changed to a 1. Thus we have constructed a list which works, and by the induction we can do so for any n. ■

Problem, B-2 1961. Let a, b be given positive real numbers with $a < b$. If two points are selected at random from a straight line of length b, what is the probability that the distance between them is at least a?

Solution. Without loss of generality, we will assume that the straight line is the interval $[0, b]$. Suppose the two points p, q are at least distance a apart.

There are three cases: the first, $p < a$, in which $q \in [p+a, b]$. The total area in which q can fall is $b - p - a$.

The second is when $a \leq p \leq b-a$, in which either $q \in [0, p-a]$ or $q \in [p+a, b]$. The total area is $b - 2a$.

The third case is when $b - a < p$, in which case $q \in [0, p - a]$. The total area here is $p - a$.

Call the event that p is as in the first case C_1, in the second case C_2, and in the third case C_3. Then the total probability is

$$P(C_1)E(C_1) + P(C_2)E(C_2) + P(C_3)E(C_3)$$

where $E(C_i)$ is the expected probability of picking an acceptable q given p in case i. In terms of interval areas, this expression can be written as

$$\frac{a}{b}\frac{1}{a}\int_0^a \frac{(b-p-a)}{b}dp + \frac{b-2a}{b}\frac{1}{b-2a}\int_a^{b-a}\frac{(b-2a)}{b}dp + \frac{a}{b}\frac{1}{a}\int_{b-a}^{b}\frac{p-a}{b}dp.$$

which we simplify to

$$\frac{1}{b^2}\int_0^a (b-p-a)dp + \frac{b-2a}{b^2}\int_a^{b-a} dp + \frac{1}{b^2}\int_{b-a}^{b}(p-a)dp.$$

which evaluates to

$$\frac{1}{b^2}\left(ab - \frac{a^2}{2} - a^2\right) + \frac{(b-2a)^2}{b^2} + \frac{1}{b^2}\left(\frac{b^2}{2} - ab - \frac{(b-a)^2}{2} + a(b-a)\right)$$

or, simplified,

$$\frac{(b-a)^2}{b^2} = P(|p-q| \geq a).$$

■

Problem, A-5 1956. Given n objects arranged in a row, a subset of these objects is called *unfriendly* if no two of its elements is consecutive. Show that the number of unfriendly subsets each having k elements is $\binom{n-k+1}{k}$.

Solution. I will show a one-to-one correspondence between the set of unfriendly k-subsets and the set of k-subsets of $n-k+1$ objects. The result then follows.

We have the objects labelled $1, 2, \cdots, n$.

Take a given unfriendly k-subset S. Write it as $S = \{e_0, e_1, \cdots, e_{k-2}, e_{k-1}\}$, where the e_i are the numbers of the elements in the subset. Order the e_i so that $e_0 < e_1 < e_2 < \cdots < e_{k-1}$. Then, for each $e_i, i \geq 1$, $e_i > e_{i-1} + 1$ (since no objects are consecutive). Each set of e_i then uniquely defines a unfriendly k-subset.

Now define the set $S' = \{e_0, e_1 - 1, e_2 - 2, \cdots, e_{k-1} - (k-1)\}$. Each successive element is larger than the previous, and the very largest an element may be is $(n-(k-1)) = (n-k+1)$. Thus S' is a k-subset of $\{1, \cdots, n-k+1\}$. Clearly for two different unfriendly k-subsets we will obtain two different S' in this fashion.

If we are given a k-subset K of $\{1,\cdots,n-k+1\}$, written as $\{f_0,f_1,\cdots,f_{k-1}\}$, with $f_0<f_1<f_2<\cdots<f_{k-1}$, then we can construct the new set $K'=\{f_0,f_1+1,f_2+2\cdots f_{k-1}+(k-1)\}$. Clearly this is an unfriendly subset of $\{1,2,3,\cdots,n\}$ with k elements. Taking two different such subsets results in two different unfriendly subsets.

Thus the one-to-one relationship is demonstrated, and the claim is proven. ■

Problem, A-1 1975. Supposing that an integer n is the sum of two triangular numbers,

$$n=\frac{a^2+a}{2}+\frac{b^2+b}{2},$$

write $4n+1$ as the sum of two squares, $4n+1=x^2+y^2$, and show how x and y can be expressed in terms of a and b.

Show that, conversely, if $4n+1=x^2+y^2$, then n is the sum of two triangular numbers.

Solution. If n is the sum of two triangular numbers, $n=\frac{a^2+a}{2}+\frac{b^2+b}{2}$, then $4n+1=x^2+y^2$, where $x=a+b+1$ and $y=b-a$. Verifying this claim is simply a matter of algebra.

So now I concentrate on the second part. Since $4n+1$ is odd, it follows that one of x,y is even and one of x,y is odd. So just suppose, without loss of generality, that x is even and y is odd, and write $x=2l, y=2m+1$, with l,m integers. Then, $4n+1=4l^2+4m^2+4m+1$ and $n=l^2+m^2+m$.

Thus it's all good if I prove that, for any integers l,m, l^2+m^2+m is the sum of two triangular numbers. Let $c=l+m$ and $d=l-m$. Then, $l^2+m^2+m=\frac{c^2+c}{2}+\frac{d^2+d}{2}$ and we are done. ■

Sketch work. The proof above is succinct and one hundred percent correct, but perhaps it is a bit odd how I came up with the relations for a,b,c,d to pull out of my hat at the appropriate times.

For the first part, for example, I suggested to myself a relationship of the form $x=c_1a+c_2b+c_3$, with c_1,c_2,c_3 some constants. Producing three examples and solving the system of equations was enough to theorize the exact formula for x and a formula for y quickly followed. My answer to the second part was produced the same way. Here is the (slightly messy) algebra verifying them:

$$\begin{aligned}4n+1=2a^2+2a+2b^2+2b+1=\\(a^2+2a+2ab+b^2+2b+1)+(b^2-2ab+a^2)&=(a+b+1)^2+(b-a)^2\\&=x^2+y^2,\text{ and}\end{aligned}$$

$$\begin{aligned}\frac{(l+m)(l+m+1)}{2}+\frac{(l-m)(l-m+1)}{2}&=\frac{1}{2}(2l^2+2m^2+2m)\\&=l^2+m^2+m.\end{aligned}$$

■

Problem, A-5 1969. Let $u(t)$ be a continuous function in the system of differential equations

$$\frac{dx}{dt} = -2y + u(t), \frac{dy}{dt} = -2x + u(t).$$

Show that, regardless of the choice of $u(t)$, the solution of the system which satisfies $x = x_0, y = y_0$ at $t = 0$ will never pass through $(0,0)$ unless $x_0 = y_0$. When $x_0 = y_0$, show that for any positive value t_0 of t, it is possible to choose $u(t)$ so the solution is at $(0,0)$ when $t = t_0$.

Solution. Combining the two equations in the system, we have

$$x' - y' = 2(x - y).$$

So let $Q(t) = x(t) - y(t)$. Then the above becomes $Q'(t) = 2Q(t)$, with the general solution $Q(t) = ce^{2t}$, with c a constant. Using the initial conditions, $c = x_0 - y_0$. Notice that unless $c = x_0 - y_0 = 0$, $Q(t)$ is never 0 for any t. Thus, unless $x_0 = y_0$, $x(t) \neq y(t)$ for all t, and the solution will never pass through the point $(0,0)$.

That finishes the first part of the problem. Now, to tackle the second.

If $x_0 = y_0$, then $Q(t) = 0$ for all t and thus $x(t) = y(t)$. Then, the first equation in the system becomes

$$\frac{dx}{dt} = -2x + u(t).$$

Now, we are given $t_0 \neq 0$. Pick

$$u(t) = 2x_0 - \frac{x_0}{t_0}(2t + 1).$$

This is a continuous u for $t_0 \neq 0$. In this case, the solution for $x(t)$ is

$$x(t) = (1 - \frac{t}{t_0})x_0.$$

At $t = 0$, $x(t) = x_0 = y_0 = y(t)$, and at $t = t_0$, $x(t) = y(t) = 0$. Thus we have shown the existence of an appropriate $u(t)$ and are done. ■

Problem, A-6 1973. Prove that it is impossible for seven distinct straight lines to be situated in the Euclidean plane so as to have at least six points where exactly three of these lines intersect and at least four points where exactly two of these lines intersect.

Solution 1. This is the pretty solution. Any two nonparallel lines in the Euclidean plane intersect in exactly one point. Thus there are at most $\binom{7}{2} =$ 21 points of intersection.

In a place where exactly two lines intersect, there are $\binom{2}{2} = 1$ of these points used up.

In a place where exactly three lines intersect, there are $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$ of these points used up.

So then, to have six three-intersection points and four two-intersection points, we require $3 \times 6 + 4 \times 1 = 22$ intersection points. However, we have only 21. ■

Solution 2. This is a longer and perhaps less elegant solution, but does show how one can hammer at a problem with the Pigeonhole Principle until it roughly resembles something trivial...

1. We refer to the lines as $L_1, L_2, \cdots L_7$.

2. Two nonparallel lines in the plane intersect in exactly one point.

3. A corollary to (2): if L_i and L_j meet in a 2-intersection, they do not meet in a 3-intersection, and vice versa. Neither can a pair of lines meet in two different 3-intersections.

4. Also from (3), if a line is in 3 different 3-intersections, then it intersects all 6 of the other lines.

5. No line can appear in 4 different 3-intersections, since then one other line appears with it in 2 3-intersections, violating (3).

6. There are 6 3-intersections, involving 18 lines. There are 7 different lines, thus 4 lines appear in at least 3 3-intersections each. By (5), these 4 lines appear in exactly 3 3-intersections.

7. Call these four lines L_1, L_2, L_3, L_4. Each of these lines intersects every other line, by (4).

8. From (7), that means that all four of our 2-intersections must involve lines chosen from L_5, L_6, L_7. But only three possible choices exist: L_5L_6, L_5L_7, L_6L_7.

Therefore the seven solution lines do not exist. ■

Problem, A-4 1998. Define the sequence a_n as follows: $a_0 = 0, a_1 = 1$, and a_{n+2} is obtained by writing the digits of a_{n+1} immediately followed by the digits of a_n. When is a_n divisible by 11?

Solution. First, it is clear that the number of digits in a_n is F_n, the nth Fibonnaci number. The number F_n is even iff n is divisible by 3.

Remember the divisibility test for 11: a base 10 integer number is divisible by 11 iff, when one begins with the first digit of the number, subtracts from it the second digit, adds to that the third digit, subtracts the fourth, and so on, once finished, obtains a multiple of 11 as the sum. For example 1331 is divisible by 11 since $1 - 3 + 3 - 1 = 0$.

Let b_n be the count obtained in this fashion for a_n (it is actually 11 minus the number's remainder modulus 11.) Thus $b_0 = 0, b_1 = 1, b_2 = 1, b_3 = 2$ and so on.

Clearly, either $b_{n+2} = b_{n+1} + b_n$ or $b_{n+2} = b_{n+1} - b_n$, depending on whether there are an even or odd number of digits in a_{n+1}. If the number is even, it is the former expression, and if the number is odd, it is the latter expression. This only depends on the remainder of n modulus 3.

From this we can see that if we ever have $b_n = b_{n-3k}$ and $b_{n-1} = b_{n-1-3k}$, we have $b_j = b_{j-3k}$ for all following j.

Simple computation shows that $b_7 = 0 = b_1$ and $b_8 = 1 = b_2$. Hence we begin to cycle forever after that. No n_i with i from 1 to 6 is 0. Thus, a_n is divisible by 11 if and only if $n = 6k + 1$ for some integer k. ■

Problem, B-6 1998. Show that for any integers a, b, c, we can find a positive integer n such that $n^3 + an^2 + bn + c$ is not a perfect square.

Solution. Recall that a perfect square q is always congruent to 0 or 1 mod 4. Thus it is sufficient if we produce an n such that the above expression is not congruent to either 0 or 1 $\mod 4$.

If $a + b + c \equiv 1, 2 \mod 4$, set $n = 1$. Then $n^3 + an^2 + bn + c \equiv 1 + a + b + c \mod 4$ and we are done.

Otherwise, if $c \equiv 2, 3 \mod 4$, set $n = 4$. Then $n^3 + an^2 + bn + c \equiv c \mod 4$ and we are done.

Otherwise, if $b \equiv 1, 3 \mod 4$, set $n = 2$. Then $n^3 + an^2 + bn + c \equiv 2b + c \mod 4$. Since $c \equiv 0, 1 \mod 4$, we are done.

Now, if none of the above work, then $b \equiv 0, 2 \mod 4$, $c \equiv 0, 1 \mod 4$, and $a + b + c \equiv 0, 3 \mod 4$. Set $n = 3$. Then $n^3 + an^2 + bn + c \equiv 3 + 2b + a + b + c \mod 4$. This is congruent to either 2 or 3 $\mod 4$. We are done! ■

PUTNAM TRAINING PROBLEMS, 2011

(Last updated: October 21, 2011)

REMARK. This is a list of problems discussed during the training sessions of the NU Putnam team and arranged by subjects. The document has three parts, the first one contains the problems, the second one hints, and the solutions are in the third part. —Miguel A. Lerma

EXERCISES

1. **Induction.**

1.1. Prove that $n! > 2^n$ for all $n \geq 4$.

1.2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

1.3. Let a and b two distinct integers, and n any positive integer. Prove that $a^n - b^n$ is divisible by $a - b$.

1.4. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.

1.5. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer.

1.6. Find the maximum number $R(n)$ of regions in which the plane can be divided by n straight lines.

1.7. We divide the plane into regions using straight lines. Prove that those regions can be colored with two colors so that no two regions that share a boundary have the same color.

1.8. A great circle is a circle drawn on a sphere that is an "equator", i.e., its center is also the center of the sphere. There are n great circles on a sphere, no three of which meet at any point. They divide the sphere into how many regions?

1.9. We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5¢ and 12¢ stamps. Prove that we can perform the task if $n \geq 44$.

1.10. A chessboard is a 8×8 grid (64 squares arranged in 8 rows and 8 columns), but here we will call "chessboard" any $m \times m$ square grid. We call *defective* a chessboard if one of its squares is missing. Prove that any $2^n \times 2^n$ $(n \geq 1)$ defective chessboard can be tiled (completely covered without overlapping) with L-shaped trominos occupying exactly 3 squares, like this  .

1.11. This is a modified version of the game of Nim (in the following we assume that there is an unlimited supply of tokens.) Two players arrange several piles of tokens in a row. By turns each of them takes one token from one of the piles and adds at will as many tokens as he or she wishes to piles placed to the left of the pile from which the token was taken. Assuming that the game ever finishes, the player that takes the last token wins. Prove that, no matter how they play, the game will eventually end after finitely many steps.

1.12. Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

1.13. Prove that for every $n \geq 2$, the expansion of $(1+x+x^2)^n$ contains at least one even coefficient.

1.14. We define recursively the *Ulam numbers* by setting $u_1 = 1$, $u_2 = 2$, and for each subsequent integer n, we set n equal to the next Ulam number if it can be written uniquely as the sum of two different Ulam numbers; e.g.: $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, etc. Prove that there are infinitely many Ulam numbers.

1.15. Prove Bernoulli's inequality, which states that if $x > -1$, $x \neq 0$ and n is a positive integer greater than 1, then $(1+x)^n > 1 + nx$.

2. **Inequalities.**

2.1. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

2.2. Prove that $n! < \left(\dfrac{n+1}{2}\right)^n$, for $n = 2, 3, 4, \ldots,$

2.3. If $0 < p$, $0 < q$, and $p + q < 1$, prove that $(px + qy)^2 \leq px^2 + qy^2$.

2.4. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

2.5. Let $x, y, z > 0$ with $xyz = 1$. Prove that $x + y + z \leq x^2 + y^2 + z^2$.

2.6. Show that

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2} \quad \geq \sqrt{(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2}$$

2.7. Find the minimum value of the function $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, where $x_1, x_2, \ldots, x_n$ are positive real numbers such that $x_1 x_2 \cdots x_n = 1$.

2.8. Let $x, y, z \geq 0$ with $xyz = 1$. Find the minimum of

$$S = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\,.$$

2.9. If $x, y, z > 0$, and $x + y + z = 1$, find the minimum value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\,.$$

2.10. Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:

$$\frac{a\,A + b\,B + c\,C}{a+b+c} \geq \frac{\pi}{3}\,.$$

2.11. (Putnam, 2003) Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}$$

2.12. The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$ What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?

2.13. Which is larger, 1999^{1999} or 2000^{1998}?

2.14. Prove that there are no positive integers a, b such that $b^2 + b + 1 = a^2$.

2.15. (Inspired in Putnam 1968, B6) Prove that a polynomial with only real roots and all coefficients equal to ± 1 has degree at most 3.

2.16. (Putnam 1984) Find the minimum value of

$$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

2.17. Show that $\displaystyle\frac{1}{\sqrt{4n}} \leq \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}$.

2.18. (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

2.19. Let $a_1, a_2, \dots, a_n$ be a sequence of positive numbers, and let $b_1, b_2, \dots, b_n$ be any permutation of the first sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n\,.$$

2.20. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ increasing sequences of real numbers, and let $x_1, x_2, \dots, x_n$ be any permutation of $b_1, b_2, \dots, b_n$. Show that

$$\sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} a_i x_i \,.$$

2.21. Prove that the p-mean tends to the geometric mean as p approaches zero. In other other words, if $a_1, \dots, a_n$ are positive real numbers, then

$$\lim_{p\to 0} \left(\frac{1}{n} \sum_{k=1}^{n} a_k^p \right)^{1/p} = \left(\prod_{k=1}^{n} a_k \right)^{1/n}$$

2.22. If a, b, and c are the sides of a triangle, prove that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \ge 3 \,.$$

2.23. Here we use Knuth's up-arrow notation: $a \uparrow b = a^b$, $a \uparrow\uparrow b = \underbrace{a \uparrow (a \uparrow (\dots \uparrow a))}_{b \text{ copies of } a}$, so e.g. $2 \uparrow\uparrow 3 = 2 \uparrow (2 \uparrow 2)) = 2^{2^2}$. What is larger, $2 \uparrow\uparrow 2011$ or $3 \uparrow\uparrow 2010$?

2.24. Prove that $e^{1/e} + e^{1/\pi} \ge 2e^{1/3}$.

2.25. Prove that the function $f(x) = \sum_{i=1}^{n} (x - a_i)^2$ attains its minimum value at $x = \overline{a} = \frac{a_1 + \dots + a_n}{n}$.

2.26. Find the positive solutions of the system of equations

$$x_1 + \frac{1}{x_2} = 4 \,, \quad x_2 + \frac{1}{x_3} = 1 \,, \dots , \quad x_{99} + \frac{1}{x_{100}} = 4 \,, \quad x_{100} + \frac{1}{x_1} = 1 \,.$$

2.27. Prove that if the numbers a, b, and c satisfy the inequalities $|a-b| \ge |c|$, $|b-c| \ge |a|$, $|c-a| \ge |b|$, then one of those numbers is the sum of the other two.

2.28. Find the minimum of $\sin^3 x/\cos x + \cos^3 x/\sin x$, $0 < x < \pi/2$.

2.29. Let $a_i > 0$, $i = 1, \dots, n$, and $s = a_1 + \dots + a_n$. Prove

$$\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \dots + \frac{a_n}{s-a_n} \ge \frac{n}{n-1} \,.$$

3. Number Theory.

3.1. Show that the sum of two consecutive primes is never twice a prime.

3.2. Can the sum of the digits of a square be (a) 3, (b) 1977?

3.3. Prove that there are infinitely many prime numbers of the form $4n + 3$.

3.4. Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every possible integer n.

3.5. Let $p(x)$ be a non-constant polynomial such that $p(n)$ is an integer for every positive integer n. Prove that $p(n)$ is composite for infinitely many positive integers n. (This proves that there is no polynomial yielding only prime numbers.)

3.6. Prove that two consecutive Fibonacci numbers are always relatively prime.

3.7. Show that if $a^2 + b^2 = c^2$, then $3|ab$.

3.8. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ can never be an integer for $n \geq 2$.

3.9. Let $f(n)$ denote the sum of the digits of n. Let $N = 4444^{4444}$. Find $f(f(f(N)))$.

3.10. Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of an integer.

3.11. Find all triples of positive integers (a, b, c) such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 2\,.$$

3.12. Find all positive integer solutions to $abc - 2 = a + b + c$.

3.13. (USAMO, 1979) Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599\,.$$

3.14. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that for some $k > 0$, F_k is a multiple of $10^{10^{10^{10}}}$.

3.15. Do there exist 2 irrational numbers a and b greater than 1 such that $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for every positive integers m, n?

3.16. The numbers 2^{2005} and 5^{2005} are written one after the other (in decimal notation). How many digits are written altogether?

3.17. If p and $p^2 + 2$, are primes show that $p^3 + 2$ is prime.

3.18. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not prime.

3.19. Let m and n be positive integers such that $m < \lfloor \sqrt{n} + \frac{1}{2} \rfloor$. Prove that $m + \frac{1}{2} < \sqrt{n}$.

3.20. Prove that the function $f(n) = \lfloor n + \sqrt{n} + 1/2 \rfloor$ $(n = 1, 2, 3, \ldots)$ misses exactly the squares.

3.21. Prove that there are no primes in the following infinite sequence of numbers:

$$1001,\ 1001001,\ 1001001001,\ 1001001001001, \ldots$$

3.22. (Putnam 1975, A1.) For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$. For what b_0 do we have b_i constant for sufficiently large i?

3.23. Let $a_n = 10 + n^2$ for $n \geq 1$. For each n, let d_n denote the gcd of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

3.24. Suppose that the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y$, $2x + 2y + 1$, $3x + 3y + 1$ are all perfect squares.

3.25. If $2n + 1$ and $3n + 1$ are both perfect squares, prove that n is divisible by 40.

3.26. How many zeros does 1000! ends with?

3.27. For how many k is the binomial coefficient $\binom{100}{k}$ odd?

3.28. Let n be a positive integer. Suppose that 2^n and 5^n *begin* with the same digit. What is the digit?

3.29. Prove that there are no four consecutive non-zero binomial coefficients $\binom{n}{r}$, $\binom{n}{r+1}$, $\binom{n}{r+2}$, $\binom{n}{r+3}$ in arithmetic progression.

3.30. (Putnam 1995, A1) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.

3.31. (Putnam 2003, A1) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

3.32. (Putnam 2001, B-1) Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from right to left is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n\,.$$

Color the squares of the grid so that half the squares in each row and in each column are read and the other half are black (a chalkboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers in the black squares.

3.33. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?

3.34. Prove that if n is an integer greater than 1, then n does not divide $2^n - 1$.

3.35. The digital root of a number is the (single digit) value obtained by repeatedly adding the (base 10) digits of the number, then the digits of the sum, and so on until obtaining a single digit—e.g. the digital root of 65,536 is 7, because $6+5+5+3+6 = 25$ and $2+5 = 7$. Consider the sequence $a_n =$ integer part of $10^n\pi$, i.e.,

$$a_1 = 31\,, \quad a_2 = 314\,, \quad a_3 = 3141\,, \quad a_4 = 31415\,, \quad a_5 = 314159\,, \quad \dots$$

and let b_n be the sequence

$$b_1 = a_1\,, \quad b_2 = a_1^{a_2}\,, \quad b_3 = a_1^{a_2^{a_3}}\,, \quad b_4 = a_1^{a_2^{a_3^{a_4}}}\,, \quad \dots$$

Find the digital root of b_{10^6}.

4. **Polynomials.**

4.1. Find a polynomial with integral coefficients whose zeros include $\sqrt{2}+\sqrt{5}$.

4.2. Let $p(x)$ be a polynomial with integer coefficients. Assume that $p(a) = p(b) = p(c) = -1$, where a, b, c are three different integers. Prove that $p(x)$ has no integral zeros.

4.3. Prove that the sum

$$\sqrt{1001^2+1} + \sqrt{1002^2+1} + \cdots + \sqrt{2000^2+1}$$

is irrational.

4.4. (USAMO 1975) If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, \dots, n$, determine $P(n+1)$.

4.5. (USAMO 1984) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

4.6. Let n be an even positive integer, and let $p(x)$ be an n-degree polynomial such that $p(-k) = p(k)$ for $k = 1, 2, \dots, n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

4.7. Let $p(x)$ be a polynomial with integer coefficients satisfying that $p(0)$ and $p(1)$ are odd. Show that p has no integer zeros.

4.8. (USAMO 1976) If $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4+x^3+x^2+x+1)S(x)$$

prove that $x-1$ is a factor of $P(x)$.

4.9. Let a, b, c distinct integers. Can the polynomial $(x-a)(x-b)(x-c)-1$ be factored into the product of two polynomials with integer coefficients?

4.10. Let $p_1, p_2, \ldots, p_n$ distinct integers and let $f(x)$ be the polynomial of degree n given by

$$f(x) = (x - p_1)(x - p_2) \cdots (x - p_n) .$$

Prove that the polynomial

$$g(x) = (f(x))^2 + 1$$

cannot be expressed as the product of two non-constant polynomials with integral coefficients.

4.11. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

4.12. Does there exist a polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$?.

4.13. It possible to write the polynomial $f(x) = x^{105} - 9$ as the product of two polynomials of degree less than 105 with integer coefficients?

4.14. Find all prime numbers p that can be written $p = x^4 + 4y^4$, where x, y are positive integers.

4.15. (Canada, 1970) Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integral coefficients. Suppose that there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k with $P(k) = 8$.

4.16. Show that $(1 + x + \cdots + x^n)^2 - x^n$ is the product of two polynomials.

4.17. Let $f(x)$ be a polynomial with real coefficients, and suppose that $f(x) + f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

4.18. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix}$

4.19. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^4 & x^4 & y^4 & z^4 \end{vmatrix}$

4.20. Do there exist polynomials a, b, c, d such that $1 + xy + x^2y^2 = a(x)b(y) + c(x)d(y)$?

4.21. Determine all polynomials such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

4.22. Consider the lines that meet the graph

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points $P_i = [x_i, y_i]$, $i = 1, 2, 3, 4$. Prove that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line, and compute its value.

4.23. Let k be the smallest positive integer for which there exist distinct integer a, b, c, d, e such that

$$(x - a)(x - b)(x - c)(x - d)(x - e)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers for which this minimum k is achieved.

4.24. Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \le 13x^2$.

4.25. (Putnam 1999, A1) Find polynomials $f(x)$, $g(x)$, and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1\,, & \text{if } x < -1\,, \\ 3x + 2\,, & \text{if } -1 \le x \le 0\,, \\ -2x + 2\,, & \text{if } x > 0\,. \end{cases}$$

4.26. Suppose that α, β, and γ are real numbers such that

$$\begin{aligned} \alpha + \beta + \gamma &= 2\,, \\ \alpha^2 + \beta^2 + \gamma^2 &= 14\,, \\ \alpha^3 + \beta^3 + \gamma^3 &= 17\,. \end{aligned}$$

Find $\alpha\beta\gamma$.

4.27. Prove that $(2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3}$ is rational.

4.28. Two players A and B play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B win the game?

4.29. Let $f(x)$ a polynomial with real coefficients, and suppose that $f(x) + f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

4.30. If $a, b, c > 0$, is it possible that each of the polynomials $P(x) = ax^2 + bx + c$, $Q(x) = cx^2 + ax + b$, $R(x) = bx^2 + cx + a$ has two real roots?

4.31. Let $f(x)$ and g(x) be nonzero polynomials with real coefficients such that $f(x^2 + x + 1) = f(x)g(x)$. Show that $f(x)$ has even degree.

4.32. Prove that there is no polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$,…, are all prime numbers.

5. **Complex Numbers.**

5.1. Let m and n two integers such that each can be expressed as the sum of two perfect squares. Prove that mn has this property as well. For instance $17 = 4^2 + 1^2$, $13 = 2^2 + 3^2$, and $17 \cdot 13 = 221 = 14^2 + 5^2$.

5.2. Prove that $\displaystyle\sum_{k=0}^{n} \sin k = \frac{\sin \frac{n}{2} \sin \frac{n+1}{2}}{\sin \frac{1}{2}}$.

5.3. Show that if z is a complex number such that $z + 1/z = 2\cos a$, then for any integer n, $z^n + 1/z^n = 2\cos na$.

5.4. Factor $p(z) = z^5 + z + 1$.

5.5. Find a close-form expression for $\displaystyle\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$.

5.6. Consider a regular n-gon which is inscribed in a circle with radius 1. What is the product of the lengths of all $n(n-1)/2$ diagonals of the polygon (this includes the sides of the n-gon).

5.7. (Putnam 1991, B2) Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y

$$f(x+y) = f(x)\,f(y) - g(x)\,g(y)$$
$$g(x+y) = f(x)\,g(y) + g(x)\,f(y)$$

If $f'(0) = 0$ prove that $f(x)^2 + g(x)^2 = 1$ for all x.

5.8. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided that one also changes the state of every dth bulb after it (where d is a divisor of n strictly less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

5.9. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that $a^2 + b^2 + u^2 + v^2 + au + bv \geq \sqrt{3}$.

6. **Generating Functions.**

6.1. Prove that for any positive integer n

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1},$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ (binomial coefficient).

6.2. Prove that for any positive integer n

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

6.3. Prove that for any positive integers $k \leq m, n$,

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{m+n}{k}.$$

6.4. Let F_n be the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$, defined recursively $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2.$$

6.5. Find a recurrence for the sequence $u_n =$ number of nonnegative solutions of

$$2a + 5b = n.$$

6.6. How many different sequences are there that satisfy all the following conditions:
(a) The items of the sequences are the digits 0–9.
(b) The length of the sequences is 6 (e.g. 061030)
(c) Repetitions are allowed.
(d) The sum of the items is exactly 10 (e.g. 111322).

6.7. (Leningrad Mathematical Olympiad 1991) A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for any $k = 1, 2, 3, \ldots, p$. For instance the sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $a_5 = 3$, $a_6 = 2$ is 3-balanced because $a_1 + a_4 = 1 + 4 = 5$, $a_2 + a_5 = 2 + 3 = 5$, $a_3 + a_6 = 3 + 2 = 5$. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.

7. **Recurrences.**

7.1. Find the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive elements of $\{1, 2, \ldots, n\}$.

7.2. Determine the maximum number of regions in the plane that are determined by n "vee"s. A "vee" is two rays which meet at a point. The angle between them is any positive number.

7.3. Define a *domino* to be a 1×2 rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?

7.4. (Putnam 1996) Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$

which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets are selfish.

7.5. Let $a_1, a_2, \ldots, a_n$ be an ordered sequence of n distinct objects. A *derangement* of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is $1, 2, 3, 4$, then $2, 4, 3, 1$ is not a derangement, but $2, 1, 4, 3$ is. Let D_n denote the number of derangements of an n-element sequence. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2})\,.$$

7.6. Let α, β be two (real or complex) numbers, and define the sequence $a_n = \alpha^n + \beta^n$ ($n = 1, 2, 3, \ldots$). Assume that a_1 and a_2 are integers. Prove that $2^{\lfloor \frac{n-1}{2} \rfloor} a_n$ is an integer for every $n \geq 1$.

7.7. Suppose that $x_0 = 18$, $x_{n+1} = \dfrac{10x_n}{3} - x_{n-1}$, and that the sequence $\{x_n\}$ converges to some real number. Find x_1.

8. **Calculus.**

8.1. Believe it or not the following function is constant in an interval $[a, b]$. Find that interval and the constant value of the function.

$$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}\,.$$

8.2. Find the value of the following infinitely nested radical

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}\,.$$

8.3. (Putnam 1995) Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}}$$

Express your answer in the form $(a + b\sqrt{c})/d$, where a, b, c, d, are integers.

8.4. (Putnam 1992) Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f(\tfrac{1}{n}) = \frac{n^2}{n^2+1}\,, \qquad n = 1, 2, 3, \ldots$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$.

8.5. Compute $\displaystyle\lim_{n\to\infty} \left\{\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}\right\}$.

8.6. Compute $\lim_{n\to\infty} \left\{ \prod_{k=1}^{n} \left(1+\frac{k}{n}\right) \right\}^{1/n}$.

8.7. (Putnam 1997) Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6}\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2}\right) dx\,.$$

8.8. (Putnam 1990) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \dots)$? (In other words, is it possible to find integers n and m such that $\sqrt[3]{n} - \sqrt[3]{m}$ is as close as we wish to $\sqrt{2}$?)

8.9. (Leningrad Mathematical Olympiad, 1988) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x)\cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

8.10. Let $f : [0,1] \to \mathbb{R}$ continuous, and suppose that $f(0) = f(1)$. Show that there is a value $x \in [0, 1998/1999]$ satisfying $f(x) = f(x + 1/1999)$.

8.11. For which real numbers c is $(e^x + e^{-x})/2 \le e^{cx^2}$ for all real x?

8.12. Does there exist a positive sequence a_n such that $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty 1/(n^2 a_n)$ are convergent?

9. **Pigeonhole Principle.**

9.1. Prove that any $(n+1)$-element subset of $\{1, 2, \dots, 2n\}$ contains two integers that are relatively prime.

9.2. Prove that if we select $n+1$ numbers from the set $S = \{1, 2, 3, \dots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

9.3. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, \dots, 100\}$. Prove that there must be two distinct integers in A whose sum if 104.

9.4. Let A be the set of all 8-digit numbers in base 3 (so they are written with the digits 0,1,2 only), including those with leading zeroes such as 00120010. Prove that given 4 elements from A, two of them must coincide in at least 2 places.

9.5. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

9.6. (Putnam, 2006-B2.) Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1}.$$

9.7. (IMO 1972.) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

9.8. Prove that among any seven real numbers $y_1, \dots, y_7$, there are two such that

$$0 \le \frac{y_i - y_j}{1 + y_i y_j} \le \frac{1}{\sqrt{3}}.$$

9.9. Prove that among five different integers there are always three with sum divisible by 3.

9.10. Prove that there exist an integer n such that the first four digits of 2^n are $2, 0, 0, 9$.

9.11. Prove that every convex polyhedron has at least two faces with the same number of edges.

10. **Telescoping.**

10.1. Prove that $\displaystyle \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = 9$.

10.2. (Putnam 1984) Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

10.3. (Putnam 1977) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

10.4. Evaluate the infinite series: $\displaystyle \sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}$.

11. **Symmetries.**

11.1. A spherical, 3-dimensional planet has center at $(0, 0, 0)$ and radius 20. At any point of the surface of this planet, the temperature is $T(x, y, z) = (x + y)^2 + (y - z)^2$ degrees. What is the average temperature of the surface of this planet?

11.2. (Putnam 1980) Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$.

11.3. Consider the following two-player game. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny which is already on the table. The table starts out with no pennies. The last player who makes a legal move wins. Does the first player have a winning strategy?

12. **Inclusion-Exclusion.**

12.1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

12.2. Imagine that you are going to give n kids ice-cream cones, one cone per kid, and there are k different flavors available. Assuming that no flavor gets mixed, find the number of ways we can give out the cones using all k flavors.

12.3. Let $a_1, a_2, \dots, a_n$ an ordered sequence of n distinct objects. A *derangement* of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is $\{1,2,3,4\}$, then $\{2,4,3,1\}$ is not a derangement, but $\{2,1,4,3\}$ is. Let D_n denote the number of derangements of an n-element sequence. Show that

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

13. **Combinatorics and Probability.**

13.1. Prove that the number of subsets of $\{1,2,\dots,n\}$ with odd cardinality is equal to the number of subsets of even cardinality.

13.2. Find the number of subsets of $\{1,2,\dots,n\}$ that contain no two consecutive elements of $\{1,2,\dots,n\}$.

13.3. Peter tosses 25 fair coins and John tosses 20 fair coins. What is the probability that they get the same number of heads?

13.4. From where he stands, one step toward the cliff would send a drunken man over the edge. He takes random steps, either toward or away from the cliff. At any step his probability of taking a step away is p, of a step toward the cliff $1-p$. Find his chance of escaping the cliff as a function of p.

13.5. Two real numbers X and Y are chosen at random in the interval $(0,1)$. Compute the probability that the closest integer to X/Y is odd. Express the answer in the form $r + s\pi$, where r and s are rational numbers.

13.6. On the unit circle centered at the origin ($x^2 + y^2 = 1$) we pick three points at random. We cut the circle into three arcs at those points. What is the expected length of the arc containing the point $(1, 0)$?

13.7. In a laboratory a handful of thin 9-inch glass rods had one tip marked with a blue dot and the other with a red. When the laboratory assistant tripped and dropped them onto the concrete floor, many broke into three pieces. For these, what was the average length of the fragment with the blue dot?

13.8. We pick n points at random on a circle. What is the probability that the center of the circle will be in the convex polygon with vertices at those points?

14. **Miscellany.**

14.1. (Putnam 1986) What is the units (i.e., rightmost) digit of $\left\lfloor \dfrac{10^{20000}}{10^{100}+3} \right\rfloor$?

14.2. (IMO 1975) Prove that there are infinitely many points on the unit circle $x^2+y^2 = 1$ such that the distance between any two of them is a rational number.

14.3. (Putnam 1988) Prove that if we paint every point of the plane in one of three colors, there will be two points one inch apart with the same color. Is this result necessarily true if we replace "three" by "nine"?

14.4. Imagine an infinite chessboard that contains a positive integer in each square. If the value of each square is equal to the average of its four neighbors to the north, south, west and east, prove that the values in all the squares are equal.

14.5. (Putnam 1990) Consider a paper punch that can be centered at any point of the plane and that, when operated, removes precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

14.6. (Putnam 1984) Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!\,.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

14.7. (Putnam 1974) Call a set of positive integers "conspiratorial" if no three of them are pairwise relatively prime. What is the largest number of elements in any conspiratorial subset of integers 1 through 16?

14.8. (Putnam 1984) Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F,

(i) $x * z = y * z$ implies $x = y$ (right cancellation holds), and
(ii) $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

14.9. (Putnam 1995) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

14.10. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as average of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

14.11. Suppose that a sequence $a_1, a_2, a_3, \dots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

14.12. On a table there is a row of fifty coins, of various denominations (the denominations could be of any values). Alice picks a coin from one of the ends and puts it in her pocket, then Bob chooses a coin from one of the ends and puts it in his pocket, and the alternation continues until Bob pockets the last coin. Prove that Alice can play so that she guarantees at least as much money as Bob.

14.13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \circ f$ has a fixed point, i.e., there is some real number x_0 such that $f(f(x_0)) = x_0$. Prove that f also has a fixed point.

Hints

1.1. —

1.2. For the induction step, rewrite $2^{2(n+1)} - 1$ as a sum of two terms that are divisible by 3.

1.3. For the inductive step assume that step $a^n - b^n$ is divisible by $a - b$ and rewrite $a^{n+1} - b^{n+1}$ as a sum of two terms, one of them involving $a^n - b^n$ and the other one being a multiple of $a - b$.

1.4. Strong induction.

1.5. Rewrite $r^{n+1} + 1/r^{n+1}$ in terms of $r^k + 1/r^k$ with $k \leq n$.

1.6. How many regions can be intersected by the $(n + 1)$th line?

1.7. Color a plane divided with n of lines in the desired way, and think how to recolor it after introducing the $(n + 1)$th line.

1.8. How many regions can be intersected the by $(n + 1)$th circle?

1.9. We have $1 = 5 \cdot (-7) + 12 \cdot 6 = 5 \cdot 5 + 12 \cdot (-2)$. Also, prove that if $n = 5x + 12y \geq 44$, then either $x \geq 7$ or $y \geq 2$.

1.10. For the inductive step, consider a $2^{n+1} \times 2^{n+1}$ defective chessboard and divide it into four $2^n \times 2^n$ chessboards. One of them is defective. Can the other three be made defective by placing strategically an L?

1.11. Use induction on the number of piles.

1.12. The numbers 8 and 9 form one such pair. Given a pair $(n, n + 1)$ of consecutive square-fulls, find some way to build another pair of consecutive square-fulls.

1.13. Look at oddness/evenness of the four lowest degree terms of the expansion.

1.14. Assume that the first m Ulam numbers have already been found, and determine how the next Ulam number (if it exists) can be determined.

1.15. We have $(1 + x)^{n+1} = (1 + x)^n(1 + x)$.

2.1. One way to solve this problem is by using the Arithmetic Mean-Geometric Mean inequality on each factor of the left hand side.

2.2. Apply the Arithmetic Mean-Geometric Mean inequality to the set of numbers $1, 2, \ldots, n$.

2.3. Power means inequality with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$.

2.4. Power means inequality.

2.5. —

2.6. This problem can be solved by using Minkowski's inequality, but another way to look at it is by an appropriate geometrical interpretation of the terms (as distances between points of the plane.)

2.7. Many minimization or maximization problems are inequalities in disguise. The solution usually consists of "guessing" the maximum or minimum value of the function, and then proving that it is in fact maximum or minimum. In this case, given the symmetry of the function a good guess is $f(1, 1, \dots, 1) = n$, so try to prove $f(x_1, x_2, \dots, x_n) \geq n$. Use the Arithmetic Mean-Geometric Mean inequality on $x_1, \dots, x_n$.

2.8. Apply the Cauchy-Schwarz inequality to the vectors $(\frac{x}{\sqrt{y+z}}, \frac{y}{\sqrt{z+x}}, \frac{z}{\sqrt{x+y}})$ and (u, v, w), and choose appropriate values for u, v, w.

2.9. Arithmetic-Harmonic Mean inequality.

2.10. Assume $a \leq b \leq c$, $A \leq B \leq C$, and use Chebyshev's Inequality.

2.11. Divide by the right hand side and use the Arithmetic Mean-Geometric Mean inequality on both terms of the left.

2.12. Note that $n!$ is increasing ($n < m \implies n! < m!$)

2.13. Look at the function $f(x) = (1999 - x) \ln (1999 + x)$.

2.14. The numbers b^2 and $(b + 1)^2$ are consecutive squares.

2.15. Use the Arithmetic Mean-Geometric Mean inequality on the squares of the roots of the polynomial.

2.16. Think geometrically. Interpret the given expression as the square of the distance between two points in the plane. The problem becomes that of finding the minimum distance between two curves.

2.17. Consider the expressions $P = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2n-1}{2n}\right)$ and $Q = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n-2}{2n-1}\right)$. Note that $\frac{k-1}{k} < \frac{k}{k+1}$, for $k = 1, 2, \dots$.

2.18. Look at the binomial expansion of $(m + n)^{m+n}$.

2.19. Arithmetic Mean-Geometric Mean inequality.

2.20. Try first the cases $n = 1$ and $n = 2$. Then use induction.

2.21. Take logarithms and use L'Hôpital.

2.22. Set $x = b + c - a$, $y = c + a - b$, $z = a + b - c$.

2.23. We have $2^{2^2} = 16 < 27 = 3^3$.

2.24. Show that $f(x) = e^{1/x}$ for $x > 0$ is decreasing and convex.

2.25. Prove that $f(x) - f(\overline{a}) \geq 0$.

2.26. By the AM-GM inequality we have $x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}, \ldots$ Try to prove that those inequalities is actually equalities.

2.27. Square both sides of those inequalities.

2.28. Rearrangement inequality.

2.29. Rearrangement inequality.

3.1. Contradiction.

3.2. If s is the sum of the digits of a number n, then $n - s$ is divisible by 9.

3.3. Assume that there are finitely many primes of the form $4n+3$, call P their product, and try to obtain a contradiction similar to the one in Euclid's proof of the infinitude of primes.

3.4. Prove that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime.

3.5. Prove that $p(k)$ divides $p(p(k) + k)$.

3.6. Induction.

3.7. Study the equation modulo 3.

3.8. Call the sum S and find the maximum power of 2 dividing each side of the equality

$$n!S = \sum_{k=1}^{n} \frac{n!}{k}\,.$$

3.9. $f(n) \equiv n \pmod 9$.

3.10. Chinese Remainder Theorem.

3.11. The minimum of a, b, c cannot be very large.

3.12. Try changing variables $x = a + 1$, $y = b + 1$, $z = c + 1$.

3.13. Study the equation modulo 16.

3.14. Use the Pigeonhole Principle to prove that the sequence of pairs (F_n, F_{n+1}) is eventually periodic modulo $N = 10^{10^{10^{10}}}$.

3.15. Try $a = \sqrt{6}$, $b = \sqrt{3}$.

3.16. —

3.17. If p is an odd number not divisible by 3, then $p^2 \equiv \pm 1 \pmod{6}$.

3.18. Sophie Germain's identity: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

3.19. The number $\sqrt{n}$ is irrational or an integer.

3.20. If $m \neq \lfloor n + \sqrt{n} + 1/2 \rfloor$, what can we say about m?

3.21. Each of the given numbers can be written $p_n(10^3)$, where $p_n(x) = 1+x+x^2+\cdots+x^n$, $n = 1, 2, 3, \dots$.

3.22. Study the cases $b_k =$ perfect square, and $b_k =$ not a perfect square. What can we deduce about b_{k+1} being or not being a perfect square in each case?

3.23. $\gcd(a, b) = \gcd(a, b - a)$.

3.24. What is $(x - y)(2x + 2y + 1)$ and $(x - y)(3x + 3y + 1)$?

3.25. Think modulo 5 and modulo 8.

3.26. Think of 1000! as a product of prime factors and count the number of 2's and the number of 5's in it.

3.27. Find the exponent of 2 in the prime factorization of $\binom{n}{k}$.

3.28. If N begins with digit a then $a \cdot 10^k \leq N < (a + 1) \cdot 10^k$.

3.29. The desired sequence of binomial numbers must have a constant difference.

3.30. Induction. The base case is $1 = 2^0 3^0$. The induction step depends on the parity of n. If n is even, divide by 2. If it is odd, subtract a suitable power of 3.

3.31. If $0 < k \leq n$, is there any such sum with exactly k terms? How many?

3.32. Interpret the grid as a 'sum' of two grids, one with the terms of the form $(k - 1)n$, and the other one with the terms of the form $1, \dots, n$.

3.33. Each of the given numbers can be written $p_n(10^2)$, where $p_n(x) = 1+x+x^2+\cdots+x^n$, $n = 1, 2, 3, \dots$.

3.34. If n is prime Fermat's Little Theorem yields the result. Otherwise let p be the smallest prime divisor of n. . .

3.35. The digital root of a number is its reminder modulo 9. Then show that a_1^n ($n = 1, 2, 3, \cdots$) modulo 9 is periodic.

4.1. Call $x = \sqrt{2} + \sqrt{5}$ and eliminate the radicals.

4.2. Factor $p(x) + 1$.

4.3. Prove that the sum is the root of a monic polynomial but not an integer.

4.4. Look at the polynomial $Q(x) = (x + 1)P(x) - x$.

4.5. Use the relationship between zeros and coefficients of a polynomial.

4.6. The $(n - 1)$-degree polynomial $p(x) - p(-x)$ vanishes at n different points.

4.7. For each integer k study the parity of $p(k)$ depending on the parity of k.

4.8. We must prove that $P(1) = 0$. See what happens by replacing x with fifth roots of unity.

4.9. Assume $(x - a)(x - b)(x - c) - 1 = p(x)q(x)$, and look at the possible values of $p(x)$ and $q(x)$ for $x = a, b, c$.

4.10. Assume $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Prove that the can be assumed to be positive for every x and $h(p_i) = k(p_i) = 1$, $i = 1, \dots, n$. Deduce that both are of degree n and determine their form. Get a contradiction by equating coefficients in $g(x)$ and $h(x)k(x)$.

4.11. The remainder will be a second degree polynomial. Plug the roots of $x^3 - x$.

4.12. Find the value of $f(n)$ for n integer.

4.13. Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Look at the product of the roots of $g(x)$

4.14. Sophie Germain's Identity.

4.15. We have that a, b, c, d are distinct roots of $P(x) - 5$.

4.16. One way to solve this problem is by letting $A_{n-1} = 1 + x + \cdots + x^{n-1}$ and doing some algebra.

4.17. Study the behavior of $f(x)$ as $x \to \pm\infty$. Also determine the number of roots of $f(x)$.

4.18. Expand the determinant along the last column and find its zeros as a polynomial in z.

4.19. Expand the determinant along the last column and find its zeros as a polynomial in z.

4.20. Write the given condition in matrix form and give each of x and y three different values.

4.21. Find some polynomial that coincides with $P(x)$ for infinitely many values of x.

4.22. Find intersection points solving a system of equations.

4.23. The numbers a, b, c, d, e are the roots of the given polynomial. How are the roots of a fifth-degree polynomial with exactly 1,2,... non-zero coefficients?

4.24. Find first the set of x verifying the constrain.

4.25. Try with first degree polynomials. Some of those polynomials must change sign precisely at $x = -1$ and $x = 0$. Recall that $|u| = \pm u$ depending on whether $u \geq 0$ or $u < 0$.

4.26. Write the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ.

4.27. Find a polynomial with integer coefficients with that number as one of its roots.

4.28. What happens if B has an upper bound for the coefficients of the polynomials?

4.29. How could $f(x)$ become zero, and how many times? From the behavior of $f(x) + f'(x)$, what can we conclude about the leading coefficient and degree of $f(x)$?

4.30. What conditions must the coefficients satisfy for a second degree polynomial to have two real roots?

4.31. Prove that f(x) cannot have real roots.

4.32. We have that $a_0 = P(0)$ must be a prime number.

5.1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

5.2. The left hand side of the equality is the imaginary part of $\displaystyle\sum_{k=0}^{n} e^{ik}$.

5.3. What are the possible values of z?

5.4. If $\omega = e^{2\pi i/3}$ then ω and ω^2 are two roots of $p(z)$.

5.5. Write $\sin t = (e^{ti} - e^{-ti})/2i$.

5.6. Assume the vertices of the n-gon placed on the complex plane at the nth roots of unity.

5.7. Look at the function $h(x) = f(x) + ig(x)$.

5.8. Assume the lights placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$.

5.9. Hint Let $z_1 = a - bi$, $z_2 = u + vi$. We have $|z_1|^2 = a^2 + b^2$, $|z_2| = u^2 + v^2$, $\Re(z_1 z_2) = au + bv$, $\Im(z_1 z_2) = 1$, and must prove $|z_1|^2 + |z_2|^2 + \Re(z_1 z_2) \geq \sqrt{3}$.

6.1. Expand and differentiate $(1 + x)^n$.

6.2. Expand both sides of $(1+x)^n(1+x)^n = (1+x)^{2n}$ and look at the coefficient of x^n.

6.3. Expand both sides of $(1 + x)^m(1 + x)^n = (1 + x)^{m+n}$ and look at the coefficient of x^j.

6.4. Look at the generating function of the Fibonacci sequence.

6.5. Find the generating function of the sequence u_n = number of nonnegative solutions of $2a + 5b = n$.

6.6. The answer equals the coefficient of x^{10} in the expansion of $(1 + x + x^2 + \cdots + x^9)^6$, but that coefficient is very hard to find directly. Try some simplification.

6.7. Look at the polynomial $P(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{50} x^{49}$, and at its values at 3rd, 5th,...roots of unity.

7.1. The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n, and the ones containing n.

7.2. The $(n + 1)$th "vee" divides the existing regions into how many further regions?

7.3. The tilings of a $n\times 2$ rectangle by dominoes can be divided into two classes depending on whether we place the rightmost domino vertically or horizontally.

7.4. The minimal selfish subsets of $\{1, 2, \dots, n\}$ can be divided into two classes depending on whether they contain n or not.

7.5. Assume that $b_1, b_2, \dots, b_n$ is a derangement of the sequence $a_1, a_2, \dots, a_n$. How many possible values can b_n have? Once we have fixed the value of b_n, divide the possible derangements into two appropriate classes.

7.6. Find a recurrence for a_n.

7.7. Find a general solution to the recurrence and determine for which value(s) of x_1 the sequence converges.

8.1. $\sqrt{u^2} = |u|$.

8.2. Find the limit of the sequence $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ $(n \geq 1)$.

8.3. Call the limit L. Find some equation verified by L.

8.4. Justify that the desired derivatives must coincide with those of the function $g(x) = 1/(1 + x^2)$.

8.5. Compare the sum to some integral of the form $\int_a^b \frac{1}{x}\, dx$.

8.6. Take logarithms. Interpret the resulting expression as a Riemann sum.

8.7. Interpret the first series is as a Maclaurin series. Interchange integration and summation with the second series (don't forget to justify why the interchange is "legitimate".)

8.8. In fact *any* real number r is the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$. We want $r \approx \sqrt[3]{n} - \sqrt[3]{m}$, i.e., $r + \sqrt[3]{m} \approx \sqrt[3]{n}$. Note that $\sqrt[3]{n+1} - \sqrt[3]{n} \to 0$ as $n \to \infty$.

8.9. If $y \in f(\mathbb{R})$ what is $f(y)$?

8.10. Consider the function $g(x) = f(x) - f(x + 1/999)$. Use the intermediate value theorem.

8.11. Compare Taylor expansions.

8.12. If they were convergent their sum would be convergent too.

9.1. Divide the set into n subsets each of which has only pairwise relatively prime numbers.

9.2. Divide the set into n subsets each of which contains only numbers which are multiple or divisor of the other ones.

9.3. Look at pairs of numbers in that sequence whose sum is precisely 104. Those pairs may not cover the whole progression, but that can be fixed...

9.4. Prove that for each $k = 1, 2, \dots, 8$, at least 2 of the elements given coincide at place k. Consider a pair of elements which coincide at place 1, another pair of elements which coincide at place 2, and so on. How many pairs of elements do we have?

9.5. Consider the sequences a_i = number of games played from the 1st through the jth day of the month, and $b_j = a_j + 14$. Put them together and use the pigeonhole principle to prove that two elements must be the equal.

9.6. Consider the fractional part of sums of the form $s_i = x_1 + \cdots + x_i$.

9.7. Consider the number of different subsets of a ten-element set, and the possible number of sums of at most ten two-digit numbers.

9.8. Write $y_i = \tan x_i$, with $-\frac{\pi}{2} \leq x_i \leq \frac{\pi}{2}$ $(i = 1, \dots, 7)$. Find appropriate "boxes" for the x_is in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$

9.9. Classify the numbers by their reminder when divided by 3.

9.10. We must prove that there are positive integers n, k such that

$$2009 \cdot 10^k \leq 2^n < 2010 \cdot 10^k \,.$$

9.11. Look at the face with the maximum number of edges and its neighbors.

10.1. Rationalize and telescope.

10.2. Try to re-write the nth term of the sum as $\frac{A_k}{3^k - 2^k} - \frac{B_k}{3^{k+1} - 2^{k+1}}$.

10.3. If you write a few terms of the product you will notice a lot of cancellations. Factor the numerator and denominator of the nth term of the product and cancel all possible factors from $k = 2$ to $k = N$. You get an expression in N. Find its limit as $N \to \infty$.

10.4. Write the nth term as a sum of two partial fractions.

11.1. Start by symmetrizing the given function:

$$f(x, y, z) = T(x, y, z) + T(y, z, x) + T(z, x, y) \,.$$

11.2. Look at the expression $f(x) + f(\frac{\pi}{2} - x)$.

11.3. What kind of symmetry can the first player take advantage of?

12.1. —

12.2. Find the number of distributions of ice-cream cones without the restriction "using all k flavors". Then remove the distributions in which at least one of the flavors is unused.

12.3. If P_i is the set of permutations fixing element a_i, then the set of non-derangements are the elements of the $P_1 \cup P_2 \cup \cdots \cup P_n$.

13.1. Find the numbers and subtract. Or find a bijection between the subsets with odd cardinality and those with even cardinality.

13.2. Find a bijection between the k-element subsets of $\{1, 2, \dots, n\}$ with no consecutive elements and all k-element subsets of $\{1, 2, \dots, n - k + 1\}$.

13.3. The probability of John getting n heads is the same as that of he getting n tails.

13.4. Consider what happens after the first step, and in which ways the man can reach the edge from there.

13.5. Look at the area of the set of points verifying the condition.

13.6. The lengths of the three arcs have identical distributions.

13.7. The lengths of the three pieces have identical distributions.

13.8. Find the probability of the polygon *not* containing the center of the circle.

14.1. Compare to $\dfrac{10^{20000} - 3^{200}}{10^{100} + 3}$.

14.2. If $\cos u$, $\sin u$, $\cos v$, and $\sin v$ are rational, so are $\cos(u + v)$ and $\sin(u + v)$.

14.3. Contradiction.

14.4. Since the values are positive integers, one of them must be the smallest one. What are the values of the neighbors of a square with minimum value?

14.5. Try punches at $(0, 0)$, $(\pm\alpha, 0)$, $\dots$ for some appropriate α.

14.6. Note that $\frac{f(n+2)-f(n+1)}{f(n+1)-f(n)}$ is a very simple polynomial in n.

14.7. Start by finding some subset T of S as large as possible and such that any three elements of it are pairwise relatively prime.

14.8. Try a binary operation that depends only on the first element: $x * y = \phi(x)$.

14.9. How many different values of $\pi(x)$ are possible?

14.10. Find a set S attaining the minimum cardinality for A_S.

14.11. Group the terms of the sequence appropriately.

14.12. Is it possible for Alice to force Bob into taking coins only from odd-numbered or even-numbered positions?

14.13. If f has not fixed point then $f(x) - x$ is never zero, and f being continous, $f(x) - x$ will have the same sign for every x.

Solutions

1.1. We prove it by induction. The basis step corresponds to $n = 4$, and in this case certainly we have $4! > 2^4$ $(24 > 16)$. Next, for the induction step, assume the inequality holds for some value of $n \geq 4$, i.e., we assume $n! > 2^n$, and look at what happens for $n + 1$:

$$(n+1)! = n!\,(n+1) \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{>} 2^n(n+1) > 2^n \cdot 2 = 2^{n+1}\,.$$

Hence the inequality also holds for $n + 1$. Consequently it holds for every $n \geq 4$.

1.2. For the basis step, we have that for $n = 1$ indeed $2^{2\cdot 1} - 1 = 4 - 1 = 3$ is divisible by 3. Next, for the inductive step, assume that $n \geq 1$ and $2^{2n} - 1$ is divisible by 3. We must prove that $2^{2(n+1)} - 1$ is also divisible by 3. We have

$$2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \cdot 2^{2n} - 1 = 3 \cdot 2^{2n} + (2^{2n} - 1)\,.$$

In the last expression the last term is divisible by 3 by induction hypothesis, and the first term is also a multiple of 3, so the whole expression is divisible by 3 and we are done.

1.3. By induction. For $n = 1$ we have that $a^1 - b^1 = a - b$ is indeed divisible by $a - b$. Next, for the inductive step, assume that $a^n - b^n$ is divisible by $a - b$. We must prove that $a^{n+1} - b^{n+1}$ is also divisible by $a - b$. In fact:

$$a^{n+1} - b^{n+1} = (a - b)\,a^n + b\,(a^n - b^n)\,.$$

On the right hand side the first term is a multiple of $a - b$, and the second term is divisible by $a - b$ by induction hypothesis, so the whole expression is divisible by $a - b$.

1.4. We prove it by strong induction. First we notice that the result is true for $n = 0$ $(F_0 = 0 < 1 = 2^0)$, and $n = 1$ $(F_1 = 1 < 2 = 2^1)$. Next, for the inductive step, assume that $n \geq 1$ and assume that the claim is true, i.e. $F_k < 2^k$, for every k such that $0 \leq k \leq n$. Then we must prove that the result is also true for $n + 1$. In fact:

$$F_{n+1} = F_n + F_{n-1} \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{<} 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}\,,$$

and we are done.

1.5. We prove it by induction. For $n = 1$ the expression is indeed an integer. For $n = 2$ we have that $r^2 + 1/r^2 = (r + 1/r)^2 - 2$ is also an integer. Next assume that $n > 2$ and that the expression is an integer for $n - 1$ and n. Then we have

$$\left(r^{n+1} + \frac{1}{r^{n+1}}\right) = \left(r^n + \frac{1}{r^n}\right)\left(r + \frac{1}{r}\right) - \left(r^{n-1} + \frac{1}{r^{n-1}}\right)\,,$$

hence the expression is also an integer for $n + 1$.

1.6. By experimentation we easily find:

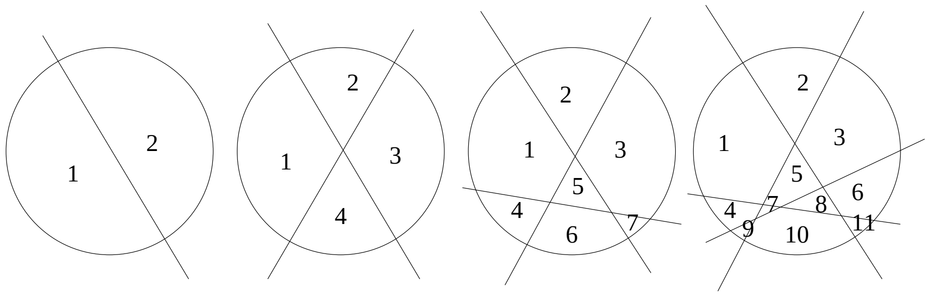


FIGURE 1. Plane regions.

n	1	2	3	4	...
$R(n)$	2	4	7	11	...

A formula that fits the first few cases is $R(n) = (n^2 + n + 2)/2$. We will prove by induction that it works for all $n \geq 1$. For $n = 1$ we have $R(1) = 2 = (1^2 + 1 + 2)/2$, which is correct. Next assume that the property is true for some positive integer n, i.e.:

$$R(n) = \frac{n^2 + n + 2}{2} .$$

We must prove that it is also true for $n + 1$, i.e.,

$$R(n+1) = \frac{(n+1)^2 + (n+1) + 2}{2} = \frac{n^2 + 3n + 4}{2} .$$

So lets look at what happens when we introduce the $(n + 1)$th straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n + 1$ segments by those intersection points. Each of those $n + 1$ segments divides a previous region into two regions, so the number of regions increases by $n + 1$. Hence:

$$R(n + 1) = S(n) + n + 1 .$$

But by induction hypothesis, $R(n) = (n^2 + n + 2)/2$, hence:

$$R(n+1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2} .$$

QED.

1.7. We prove it by induction in the number n of lines. For $n = 1$ we will have two regions, and we can color them with just two colors, say one in red and the other one in blue. Next assume that the regions obtained after dividing the plane with n lines can always be colored with two colors, red and blue, so that no two regions that share a boundary have the same color. We need to prove that such kind of coloring is also possible after dividing the plane with $n+1$ lines. So assume that the plane divided by n lines has been colored in the desired way. After we introduce the

$(n+1)$th line we need to recolor the plane to make sure that the new coloring still verifies that no two regions that share a boundary have the same color. We do it in the following way. The $(n+1)$th line divides the plane into two half-planes. We keep intact the colors in all the regions that lie in one half-plane, and reverse the colors (change red to blue and blue to red) in all the regions of the other half-plane. So if two regions share a boundary and both lie in the same half-plane, they will still have different colors. Otherwise, if they share a boundary but are in opposite half-planes, then they are separated by the $(n+1)$th line; which means they were part of the same region, so had the same color, and must have acquired different colors after recoloring.

1.8. The answer is $f(n) = n^2 - n + 2$. The proof is by induction. For $n = 1$ we get $f(1) = 2$, which is indeed correct. Then we must prove that if $f(n) = n^2 - n + 2$ then $f(n+1) = (n+1)^2 - (n+1) + 2$. In fact, the $(n+1)$th great circle meets each of the other great circles in two points each, so $2n$ points in total, which divide the circle into $2n$ arcs. Each of these arcs divides a region into two, so the number of regions grow by $2n$ after introducing the $(n+1)$th circle. Consequently $f(n+1) = n^2 - n + 2 + 2n = n^2 + n + 2 = (n+1)^2 - (n+1) + 2$, QED.

1.9. We proceed by induction. For the basis step, i.e. $n = 44$, we can use four 5¢ stamps and two 12¢ stamps, so that $5 \cdot 4 + 12 \cdot 2 = 44$. Next, for the induction step, assume that for a given $n \geq 44$ the task is possible by using x 5¢ stamps and y 12¢ stamps, i.e, $n = 5x + 12y$. We must now prove that we can find some combination of x' 5¢ stamps and y' 12¢ stamps so that $n+1 = 5x' + 12y'$. First note that either $x \geq 7$ or $y \geq 2$ — otherwise we would have $x \leq 6$ and $y \leq 1$, hence $n \leq 5 \cdot 6 + 12 \cdot 1 = 42 < 44$, contradicting the hypothesis that $n \geq 44$. So we consider the two cases:

1. If $x \geq 7$, then we can accomplish the goal by setting $x' = x - 7$ and $y' = y + 6$:
$$5x' + 12y' = 5(x-7) + 12(y+6) = 5x + 12y + 1 = n + 1\,.$$
2. On the other hand, if $y \geq 2$ then, we can do it by setting $x' = x + 5$ and $y' = y - 2$:
$$5x' + 12y' = 5(x+5) + 12(y-2) = 5x + 12y + 1 = n + 1\,.$$

1.10. We prove it by induction on n. For $n = 1$ the defective chessboard consists of just a single L and the tiling is trivial. Next, for the inductive step, assume that a $2^n \times 2^n$ defective chessboard can be tiled with L's. Now, given a $2^{n+1} \times 2^{n+1}$ defective chessboard, we can divide it into four $2^n \times 2^n$ chessboards as shown in the figure. One of them will have a square missing and will be defective, so it can be tiled with L's. Then we place an L covering exactly one corner of each of the other $2^n \times 2^n$ chessboards (see figure). The remaining part of each of those chessboards is like a defective chessboard and can be tiled in the desired way too. So the whole $2^{n+1} \times 2^{n+1}$ defective chessboard can be tiled with L's.

1.11. We use induction on the number n of piles. For $n = 1$ we have only one pile, and since each player must take at least one token from that pile, the number of tokes in

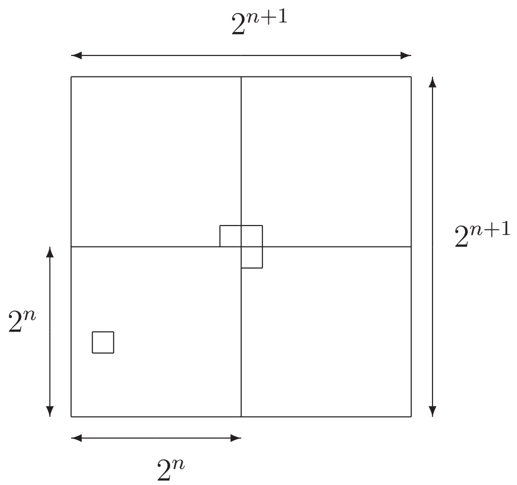


FIGURE 2. A $2^{n+1} \times 2^{n+1}$ defective chessboard.

it will decrease at each move until it is empty. Next, for the induction step, assume that the game with n piles must end eventually. We will prove that the same is true for $n+1$ piles. First note that the players cannot keep taking tokens only from the first n piles, since by induction hypothesis the game with n piles eventually ends. So sooner or later one player must take a token from the $(n+1)$th pile. It does not matter how many tokes he or she adds to the other n piles after that, it is still true that the players cannot keep taking tokens only from the first n piles forever, so eventually someone will take another token from the $(n+1)$th pile. Consequently, the number of tokens in that pile will continue decreasing until it is empty. After that we will have only n piles left, and by induction hypothesis the game will end in finitely many steps after that.[1]

1.12. The numbers 8 and 9 are a pair of consecutive square-fulls. Next, if n and $n+1$ are square-full, so are $4n(n+1)$ and $4n(n+1)+1=(2n+1)^2$.

1.13. For $n=2,3,4,5,6$ we have:

$$\begin{aligned}
(1+x+x^2)^2 &= 1+2x+3x^2+2x^3+x^4\\
(1+x+x^2)^3 &= 1+3x+6x^2+7x^3+\cdots\\
(1+x+x^2)^4 &= 1+4x+10x^2+16x^3+\cdots\\
(1+x+x^2)^5 &= 1+5x+15x^2+30x^3+\cdots\\
(1+x+x^2)^6 &= 1+6x+21x^2+50x^3+\cdots
\end{aligned}$$

[1] An alternate proof based on properties of ordinal numbers is as follows (requires some advanced set-theoretical knowledge.) Here ω = first infinite ordinal number, i.e., the first ordinal after the sequence of natural numbers $0,1,2,3,\dots$. Let the ordinal number $\alpha = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{n-1}\omega^{n-1}$ represent a configuration of n piles with $a_0, a_1, \dots, a_{n-1}$ tokens respectively (read from left to right.) After a move the ordinal number representing the configuration of tokens always decreases. Every decreasing sequence of ordinals numbers is finite. Hence the result.

In general, if $(1+x+x^2)^n = a + bx + cx^2 + dx^3 + \cdots$, then

$$(1+x+x^2)^{n+1} = a + (a+b)x + (a+b+c)x^2 + (b+c+d)x^3 + \cdots ,$$

hence the first four coefficients of $(1+x+x^2)^{n+1}$ depend only on the first four coefficients of $(1+x+x^2)^n$. The same is true is we write the coefficients modulo 2, i.e., as "0" if they are even, or "1" if they are odd. So, if we call $q_n(x) = (1+x+x^2)^n$ with the coefficients written modulo 2, we have

$$\begin{aligned}
q_1(x) &= 1 + 1x + 1x^2 \\
q_2(x) &= 1 + 0x + 1x^2 + 0x^3 + 1x^4 \\
q_3(x) &= 1 + 1x + 0x^2 + 1x^3 + \cdots \\
q_4(x) &= 1 + 0x + 0x^2 + 0x^3 + \cdots \\
q_5(x) &= 1 + 1x + 1x^2 + 0x^3 + \cdots \\
q_6(x) &= 1 + 0x + 1x^2 + 0x^3 + \cdots
\end{aligned}$$

We notice that the first four coefficients of $q_6(x)$ coincide with those of $q_2(x)$, and since these first four coefficients determine the first four coefficients of each subsequent polynomial of the sequence, they will repeat periodically so that those of $q_n(x)$ will always coincide with those of q_{n+4}. Since for $n = 2, 3, 4, 5$ at least one of the first four coefficients of $q_n(x)$ is 0 (equivalently, at least one of the first four coefficients of $(1+x+x^2)^n$ is even), the same will hold for all subsequent values of n.

1.14. Let $U_m = \{u_1, u_2, \dots, u_m\}$ $(m \geq 2)$ be the first m Ulam numbers (written in increasing order). Let S_m be the set of integers greater than u_m that can be written uniquely as the sum of two different Ulam numbers from U_m. The next Ulam number u_{m+1} is precisely the minimum element of S_m, unless S_m is empty, but it is not because $u_{m-1} + u_m \in S_m$.

1.15. By induction. For the base case $n = 2$ the inequality is $(1+x)^2 > 1 + 2x$, obviously true because $(1+x)^2 - (1+2x) = x^2 > 0$. For the induction step, assume that the inequality is true for n, i.e., $(1+x)^n > 1 + nx$. Then, for $n+1$ we have

$$(1+x)^{n+1} = (1+x)^n(1+x) > (1+nx)(1+x) = \\ 1 + (n+1)x + x^2 > 1 + (n+1)x\,,$$

and the inequality is also true for $n+1$.

2.1. Using the Arithmetic Mean-Geometric Mean Inequality on each factor of the LHS we get

$$\left(\frac{a^2b + b^2c + c^2a}{3}\right)\left(\frac{ab^2 + bc^2 + ca^2}{3}\right) \geq \left(\sqrt[3]{a^3b^3c^3}\right)\left(\sqrt[3]{a^3b^3c^3}\right) = a^2b^2c^2\,.$$

Multiplying by 9 we get the desired inequality.

Another solution consists of using the Cauchy-Schwarz inequality:

$$\begin{aligned}(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) = \\ \left((a\sqrt{b})^2 + (b\sqrt{c})^2 + (c\sqrt{a})^2\right)\left((\sqrt{b}\,c)^2 + (\sqrt{c}\,a)^2 + (\sqrt{a}\,b)^2\right) \\ \geq (abc + abc + abc)^2 \\ = 9a^2b^2c^2\,.\end{aligned}$$

2.2. This result is the Arithmetic Mean-Geometric Mean applied to the set of numbers $1, 2, \dots, n$:

$$\sqrt[n]{1 \cdot 2 \cdot \cdots \cdot n} < \frac{1 + 2 + \cdots + n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}\,.$$

Raising both sides to the nth power we get the desired result.

2.3. The simplest solution consists of using the weighted power means inequality to the (weighted) arithmetic and quadratic means of x and y with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$:

$$\frac{p}{p+q}\,x + \frac{q}{p+q}\,y \leq \sqrt{\frac{p}{p+q}\,x^2 + \frac{q}{p+q}\,y^2}\,,$$

hence

$$(px + qy)^2 \leq (p + q)(px^2 + qy^2)\,.$$

Or we can use the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}(px + qy)^2 &= (\sqrt{p}\sqrt{p}\,x + \sqrt{q}\sqrt{q}\,y)^2 \\ &\leq \left(\{\sqrt{p}\}^2 + \{\sqrt{q}\}^2\right)\left(\{\sqrt{p}\,x\}^2 + \{\sqrt{q}\,y\}^2\right) \quad \text{(Cauchy-Schwarz)} \\ &= (p + q)(px^2 + qy^2)\,.\end{aligned}$$

Finally we use $p + q \leq 1$ to obtain the desired result.

2.4. By the power means inequality:

$$\underbrace{\frac{a + b + c}{3}}_{M^1(a,b,c)} \geq \underbrace{\left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^2}_{M^{1/2}(a,b,c)}$$

From here the desired result follows.

2.5. We have:

$$\begin{aligned}x + y + z &= (x + y + z)\sqrt[3]{xyz} && (xyz = 1) \\ &\leq \frac{(x + y + z)^2}{3} && \text{(AM-GM inequality)} \\ &\leq x^2 + y^2 + z^2\,. && \text{(power means inequality)}\end{aligned}$$

2.6. The result can be obtained by using Minkowski's inequality repeatedly:

$$\begin{aligned}\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+\cdots+\sqrt{a_n^2+b_n^2} &\geq \sqrt{(a_1+a_2)^2+(b_1+b_2)^2}+\cdots+\sqrt{a_n^2+b_n^2}\\ &\geq \sqrt{(a_1+a_2+a_3)^2+(b_1+b_2+b_3)^2}+\cdots\\ &\qquad +\sqrt{a_n^2+b_n^2}\\ &\cdots\\ &\geq \sqrt{(a_1+a_2+\cdots+a_n)^2+(b_1+b_2+\cdots+b_n)^2}\end{aligned}$$

Another way to think about it is geometrically. Consider a sequence of points in the plane $P_k=(x_k,y_k)$, $k=0,\dots,n$, such that

$$(x_k,y_k)=(x_{k-1}+a_k,y_{k-1}+b_k) \qquad \text{for } k=1,\dots,n\,.$$

Then the left hand side of the inequality is the sum of the distances between two consecutive points, while the right hand side is the distance between the first one and the last one:

$$d(P_0,P_1)+d(P_1,P_2)+\cdots+d(P_{n-1},P_n)\leq d(P_0,P_n)\,.$$

2.7. By the Arithmetic Mean-Geometric Mean Inequality

$$1=\sqrt[n]{x_1x_2\dots x_n}\leq\frac{x_1+x_2+\cdots+x_n}{n}\,,$$

Hence $f(x_1,x_2,\dots,x_n)\geq n$. On the other hand $f(1,1,\dots,1)=n$, so the minimum value is n.

2.8. For $x=y=z=1$ we see that $S=3/2$. We will prove that in fact $3/2$ is the minimum value of S by showing that $S\geq 3/2$.
Note that

$$S=\left(\frac{x}{\sqrt{y+z}}\right)^2+\left(\frac{y}{\sqrt{z+x}}\right)^2+\left(\frac{z}{\sqrt{x+y}}\right)^2.$$

Hence by the Cauchy-Schwarz inequality:

$$S\cdot(u^2+v^2+w^2)\geq\left(\frac{xu}{\sqrt{y+z}}+\frac{yv}{\sqrt{z+x}}+\frac{zw}{\sqrt{x+y}}\right)^2.$$

Writing $u=\sqrt{y+z}$, $v=\sqrt{z+x}$, $w=\sqrt{x+y}$ we get

$$S\cdot 2(x+y+z)\geq(x+y+z)^2\,,$$

hence, dividing by $2(x+y+z)$ and using the Arithmetic Mean-Geometric Mean inequality:

$$S\geq\frac{1}{2}(x+y+z)\geq\frac{1}{2}\cdot 3\sqrt[3]{xyz}=\frac{3}{2}\,.$$

2.9. By the Arithmetic Mean-Harmonic Mean inequality:

$$\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\leq\frac{x+y+z}{3}=\frac{1}{3}\,,$$

hence

$$9 \le \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\,.$$

On the other hand for $x = y = z = 1/3$ the sum is 9, so the minimum value is 9.

2.10. Assume $a \le b \le c$, $A \le B \le C$. Then

$$\begin{aligned} 0 &\le (a-b)(A-B) + (a-c)(A-C) + (b-c)(B-C) \\ &= 3(aA + bB + cC) - (a+b+c)(A+B+C)\,. \end{aligned}$$

Using $A + B + C = \pi$ and dividing by $3(a + b + c)$ we get the desired result.
- *Remark:* We could have used also Chebyshev's Inequality:

$$\frac{a\,A + b\,B + c\,C}{3} \ge \left(\frac{a+b+c}{3}\right)\left(\frac{A+B+C}{3}\right).$$

2.11. Assume $a_i+b_i > 0$ for each i (otherwise both sides are zero). Then by the Arithmetic Mean-Geometric Mean inequality

$$\left(\frac{a_1 \cdots a_n}{(a_1+b_1)\cdots(a_n+b_n)}\right)^{1/n} \le \frac{1}{n}\left(\frac{a_1}{a_1+b_1} + \cdots + \frac{a_n}{a_n+b_n}\,,\right)$$

and similarly with the roles of a and b reversed. Adding both inequalities and clearing denominators we get the desired result.
(*Remark:* The result is known as *superadditivity* of the *geometric mean.*)

2.12. We have that $n!$ is increasing for $n \ge 1$, i.e., $1 \le n < m \implies n! < m!$ So $1999! > 2000 \implies (1999!)! > 2000! \implies ((1999!)!)! > (2000!)! \implies \dots \implies 1999!^{(2000)} > 2000!^{(1999)}$.

2.13. Consider the function $f(x) = (1999 - x)\ln(1999 + x)$. Its derivative is $f'(x) = -\ln(1999+x) + \dfrac{1999-x}{1999+x}$, which is negative for $0 \le x \le 1$, because in that interval

$$\frac{1999-x}{1999+x} \le 1 = \ln e < \ln(1999+x)\,.$$

Hence f is decreasing in $[0, 1]$ and $f(0) > f(1)$, i.e., $1999 \ln 1999 > 1998 \ln 2000$. Consequently $1999^{1999} > 2000^{1998}$.

2.14. We have $b^2 < \underbrace{b^2 + b + 1}_{a^2} < b^2 + 2b + 1 = (b+1)^2$. But b^2 and $(b+1)^2$ are consecutive squares, so there cannot be a square strictly between them.

2.15. We may assume that the leading coefficient is $+1$. The sum of the squares of the roots of $x^n + a_1x^{n-1} + \cdots + a_n$ is $a_1^2 - 2a_2$. The product of the squares of the roots is a_n^2. Using the Arithmetic Mean-Geometric Mean inequality we have

$$\frac{a_1^2 - 2a_2}{n} \ge \sqrt[n]{a_n^2}\,.$$

Since the coefficients are ± 1 that inequality is $(1 \pm 2)/n \ge 1$, hence $n \le 3$.

Remark: $x^3 - x^2 - x + 1 = (x+1)(x-1)^2$ is an example of 3th degree polynomial with all coefficients equal to ± 1 and only real roots.

2.16. The given function is the square of the distance between a point of the quarter of circle $x^2+y^2 = 2$ in the open first quadrant and a point of the half hyperbola $xy = 9$ in that quadrant. The tangents to the curves at $(1,1)$ and $(3,3)$ separate the curves, and both are perpendicular to $x = y$, so those points are at the minimum distance, and the answer is $(3-1)^2 + (3-1)^2 = 8$.

2.17. Let

$$P = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2n-1}{2n}\right), \quad Q = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n-2}{2n-1}\right).$$

We have $PQ = \dfrac{1}{2n}$. Also $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \cdots < \frac{2n-1}{2n}$, hence $2P \geq Q$, so $2P^2 \geq PQ = \dfrac{1}{2n}$, and from here we get $P \geq \dfrac{1}{\sqrt{4n}}$.

On the other hand we have $P < Q\dfrac{2n}{2n+1} < Q$, hence $P^2 < PQ = \dfrac{1}{2n}$, and from here $P < \dfrac{1}{\sqrt{2n}}$.

2.18. The given inequality is equivalent to

$$\frac{(m+n)!}{m!\,n!}\,m^m n^n = \binom{m+n}{n} m^m n^n < (m+n)^{m+n},$$

which is obviously true because the binomial expansion of $(m+n)^{m+n}$ includes the term on the left plus other terms.

2.19. Using the Arithmetic Mean-Geometric Mean inequality we get:

$$\frac{1}{n}\left\{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right\} \geq \sqrt[n]{\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}\cdots\frac{a_n}{b_n}} = 1.$$

From here the desired result follows.

2.20. We prove it by induction. For $n = 1$ the result is trivial, and for $n = 2$ it is a simple consequence of the following:

$$0 \leq (a_2 - a_1)(b_2 - b_1) = (a_1b_1 + a_2b_2) - (a_1b_2 + a_2b_1).$$

Next assume that the result is true for some $n \geq 2$. We will prove that is is true for $n+1$. There are two possibilities:

1. If $x_{n+1} = b_{n+1}$, then we can apply the induction hypothesis to the n first terms of the sum and we are done.

2. If $x_{n+1} \neq b_{n+1}$, then $x_j = b_{n+1}$ for some $j \neq n+1$, and $x_{n+1} = b_k$ for some $k \neq n+1$. Hence:

$$\sum_{i=1}^{n+1} a_i x_i = \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i x_i + a_j x_j + a_{n+1} x_{n+1}$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i x_i + a_j b_{n+1} + a_{n+1} b_k$$

(using the inequality for the two-term increasing sequences a_j, a_{n+1} and b_k, b_{n+1})

$$\leq \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i x_i + a_j b_k + a_{n+1} b_{n+1}\,.$$

This reduces the problem to case 1.

2.21. We have

$$\ln \left(\frac{1}{n} \sum_{k=1}^{n} a_k^p \right)^{1/p} = \frac{\ln \left(\frac{1}{n} \sum_{k=1}^{n} a_k^p \right)}{p}\,.$$

Also, $a_k \to 1$ as $p \to 0$, hence numerator and denominator tend to zero as p approaches zero. Using L'Hôpital we get

$$\lim_{p \to 0} \frac{\ln \left(\frac{1}{n} \sum_{k=1}^{n} a_k^p \right)}{p} = \lim_{p \to 0} \frac{\sum_{k=1}^{n} a_k^p \ln a_k}{\sum_{k=1}^{n} a_k^p} = \frac{\sum_{k=1}^{n} \ln a_k}{n} = \ln \left(\prod_{k=1}^{n} a_k \right)^{1/n}\,.$$

From here the desired result follows.

2.22. Set $x = b+c-a$, $y = c+a-b$, $z = a+b-c$. The triangle inequality implies that x, y, and z are positive. Furthermore, $a = (y+z)/2$, $b = (z+x)/2$, and $c = (x+y)/2$. The LHS of the inequality becomes:

$$\frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z} = \frac{1}{2} \left(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{x}{z} + \frac{z}{x} \right) \geq 3\,.$$

2.23. We have that $2 \uparrow\uparrow 3 = 2^{2^2} = 16 < 27 = 3^3 = 3 \uparrow\uparrow 2$. Then using $a \uparrow\uparrow (n+1) = a^{a \uparrow\uparrow n}$ we get $2 \uparrow\uparrow (n+1) < 3 \uparrow\uparrow n$ for $n \geq 2$, and from here it follows that $2 \uparrow\uparrow 2011 < 3 \uparrow\uparrow 2010$.

2.24. Consider the function $f(x) = e^{1/x}$ for $x > 0$. We have $f'(x) = -\frac{1/x}{e^{1/x}} < 0$, $f''(x) = e^{1/x}(\frac{2}{x^3} + \frac{1}{x^4}) > 0$, hence f is decreasing and convex.
By convexity, we have

$$\frac{1}{2}(f(e) + f(\pi)) \geq f\left(\tfrac{e+\pi}{2}\right)\,.$$

On the other hand we have $(e+\pi)/2 < 3$, and since f is decreasing, $f(\frac{e+\pi}{2}) > f(3)$, and from here the result follows.

2.25. We have:

$$\begin{aligned} f(x) - f(\overline{a}) &= \sum_{i=1}^{n}(x-a_i)^2 - \sum_{i=1}^{n}(\overline{a}-a_i)^2 \\ &= \sum_{i=1}^{n}\left\{(x-a_i)^2 - (\overline{a}-a_i)^2\right\} \\ &= \sum_{i=1}^{n}(x^2 - 2a_i x - \overline{a}^2 + 2a_i\overline{a}) \\ &= nx^2 - 2n\overline{a}x + n\overline{a}^2 \\ &= n(x-\overline{a})^2 \geq 0\,, \end{aligned}$$

hence $f(x) \geq f(\overline{a})$ for every x.

2.26. By the Geometric Mean-Arithmetic Mean inequality

$$x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}, \dots, x_{100} + \frac{1}{x_1} \geq 2\sqrt{\frac{x_{100}}{x_1}}\,.$$

Multiplying we get

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{100} + \frac{1}{x_1}\right) \geq 2^{100}\,.$$

From the system of equations we get

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{100} + \frac{1}{x_1}\right) = 2^{100}\,,$$

so all those inequalities are equalities, i.e.,

$$x_1 + \frac{1}{x_2} = 2\sqrt{\frac{x_1}{x_2}} \quad\Longrightarrow\quad \left(\sqrt{x_1} - \frac{1}{\sqrt{x_2}}\right)^2 = 0 \quad\Longrightarrow\quad x_1 = \frac{1}{x_2}\,,$$

and analogously: $x_2 = 1/x_3, \dots, x_{100} = 1/x_1$. Hence $x_1 = 1/x_2$, $x_2 = 1/x_3$, $\dots$, $x_{100} = 1/x_1$, and from here we get $x_1 = 2$, $x_2 = 1/2, \dots$, $x_{99} = 2$, $x_{100} = 1/2$.

2.27. Squaring the inequalities and moving their left hand sides to the right we get

$$\begin{aligned} 0 &\geq c^2 - (a-b)^2 = (c+a-b)(c-a+b) \\ 0 &\geq a^2 - (b-c)^2 = (a+b-c)(a-b+c) \\ 0 &\geq b^2 - (c-a)^2 = (b+c-a)(b-c+a)\,. \end{aligned}$$

Multiplying them together we get:

$$0 \geq (a+b-c)^2(a-b+c)^2(-a+b+c)^2\,,$$

hence, one of the factors must be zero.

2.28. The answer is 1. In fact, the sequences $(\sin^3 x, \cos^3 x)$ and $(1/\sin x, 1/\cos x)$ are oppositely sorted, hence by the rearrangement inequality:

$$\begin{aligned}\sin^3 x/\cos x + \cos^3 x/\sin x &\geq \sin^3 x/\sin x + \cos^3 x/\cos x \\ &= \sin^2 x + \cos^2 x = 1\,.\end{aligned}$$

Equality is attained at $x = \pi/4$.

2.29. By the rearrangement inequality we have for $k = 2, 3, \cdots, n$:

$$\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \geq \frac{a_1}{s-a_k} + \frac{a_2}{s-a_{k+1}} + \cdots + \frac{a_n}{s-a_{k-1}}\,,$$

were the denominators on the right hand side are a cyclic permutation of $s - a_1, \cdots, s - a_n$. Adding those $n - 1$ inequalities we get the desired result.

3.1. If p and q are consecutive primes and $p+q = 2r$, then $r = (p+q)/2$ and $p < r < q$, but there are no primes between p and q.

3.2. (a) No, a square divisible by 3 is also divisible by 9.
(b) Same argument.

3.3. Assume that the set of primes of the form $4n + 3$ is finite. Let P be their product. Consider the number $N = P^2 - 2$. Note that the square of an odd number is of the form $4n+1$, hence P^2 is of the form $4n+1$ and N will be of the form $4n+3$. Now, if all prime factors of N where of the form $4n + 1$, N would be of the form $4n + 1$, so N must have some prime factor p of the form $4n + 3$. So it must be one of the primes in the product P, hence p divides $N - P^2 = 2$, which is impossible.

3.4. That is equivalent to proving that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime for every n. These are two possible ways to show it:
- Assume a prime p divides $n^3 + 2n = n(n^2 + 2)$. Then it must divide n or $n^2 + 2$. Writing $n^4 + 3n^2 + 1 = n^2(n^2 + 3) + 1 = (n^2 + 1)(n^2 + 2) - 1$ we see that p cannot divide $n^4 + 3n^2 + 1$ in either case.
- The following identity

$$(n^2 + 1)(n^4 + 3n^2 + 1) - (n^3 + 2n)^2 = 1$$

(which can be checked algebraically) shows that any common factor of $n^4 + 3n^2 + 1$ and $n^3 + 2n$ should divide 1, so their gcd is always 1. (Note: if you are wondering how I arrived to that identity, I just used the Euclidean algorithm on the two given polynomials.)

3.5. Assume $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, with $a_n \neq 0$. We will assume WLOG that $a_n > 0$, so that $p(k) > 0$ for every k large enough—otherwise we can use the argument below with $-p(x)$ instead of $p(x)$.
We have

$$p(p(k) + k) = \sum_{i=0}^{n} a_i\,[p(k) + k]^i\ .$$

For each term of that sum we have that

$$a_i\,[p(k)+k]^i = [\text{multiple of } p(k)] + a_i k^i\,,$$

and the sum of the $a_i k^i$ is precisely $p(k)$, so $p(p(k)+k)$ is a multiple of $p(k)$. It remains only to note that $p(p(k)+k) \neq p(k)$ for infinitely many positive integers k, otherwise $p(p(x)+x)$ and $p(x)$ would be the same polynomial, which is easily ruled out for non constant $p(x)$.

3.6. This can be proved easily by induction. Base case: $F_1 = 1$ and $F_2 = 1$ are in fact relatively prime. Induction Step: we must prove that if F_n and F_{n+1} are relatively prime then so are F_{n+1} and F_{n+2}. But this follows from the recursive definition of the Fibonacci sequence: $F_n + F_{n+1} = F_{n+2}$; any common factor of F_{n+1} and F_{n+2} would be also a factor of F_n, and consequently it would be a common factor of F_n and F_{n+1} (which by induction hypothesis are relatively prime.)

3.7. For any integer n we have that n^2 only can be 0 or 1 mod 3. So if 3 does not divide a or b they must be 1 mod 3, and their sum will be 2 modulo 3, which cannot be a square.

3.8. Assume the sum S is an integer. Let 2^i be the maximum power of 2 dividing n, and let 2^j be the maximum power of 2 dividing $n!$ Then

$$\frac{n!}{2^j}2^i S = \sum_{k=1}^{n} \frac{n!}{k2^{j-i}}\,.$$

For $n \geq 2$ the left hand side is an even number. In the right hand side all the terms of the sum are even integers except the one for $k = 2^i$ which is an odd integer, so the sum must be odd. Hence we have an even number equal to an odd number, which is impossible.

3.9. Since each digit cannot be greater than 9, we have that $f(n) \leq 9\cdot(1+\log_{10} n)$, so in particular $f(N) \leq 9\cdot(1+4444\cdot\log_{10}4444) < 9\cdot(1+4444\cdot 4) = 159993$. Next we have $f(f(N)) \leq 9\cdot 6 = 54$. Finally among numbers not greater than 54, the one with the greatest sum of the digits is 49, hence $f(f(f(N))) \leq 4+9 = 13$.
Next we use that $n \equiv f(n) \pmod 9$. Since $4444 \equiv 7 \pmod 9$, then

$$4444^{4444} \equiv 7^{4444} \pmod 9\,.$$

We notice that the sequence $7^n \bmod 9$ for $n = 0,1,2,\dots$ is $1,7,4,1,7,4,\dots$, with period 3. Since $4444 \equiv 1 \pmod 3$, we have $7^{4444} \equiv 7^1 \pmod 9$, hence $f(f(f(N))) \equiv 7 \pmod 9$. The only positive integer not greater than 13 that is congruent with 7 modulo 9 is 7, hence $f(f(f(N))) = 7$.

3.10. Pick 1999 different prime numbers $p_1, p_2, \dots, p_{1999}$ (we can do that because the set of prime numbers is infinite) and solve the following system of 1999 congruences:

$$\begin{cases} x \equiv 0 \pmod{p_1^3} \\ x \equiv -1 \pmod{p_2^3} \\ x \equiv -2 \pmod{p_3^3} \\ \quad \dots \\ x \equiv -1998 \pmod{p_{1999}^3} \end{cases}$$

According to the Chinese Remainder Theorem, that system of congruences has a solution x (modulo $M = p_1^3 \dots p_{1999}^3$). For $k = 1, \dots, 1999$ we have that $x + k \equiv 0 \pmod{p_k^3}$, hence $x + k$ is in fact a multiple of p_k^3.

3.11. Assume $a \ge b \ge c$. Then

$$2 = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) \le \left(1 + \frac{1}{c}\right)^3 .$$

From here we get that $c < 4$, so its only possible values are $c = 1, 2, 3$.
For $c = 1$ we get $(1 + 1/c) = 2$, hence

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = 1 ,$$

which is impossible.
For $c = 2$ we have $(1 + 1/c) = 3/2$, hence

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = \frac{4}{3} ,$$

and from here we get

$$a = \frac{3(b+1)}{b-3} ,$$

with solutions $(a, b) = (15, 4)$, $(9, 5)$ and $(7, 6)$.
Finally for $c = 3$ we have $1 + 1/c = 1 + 1/3 = 4/3$, hence

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = \frac{3}{2} .$$

So

$$a = \frac{2(b+1)}{b-2} .$$

The solutions are $(a, b) = (8, 3)$ and $(5, 4)$.
So the complete set of solutions verifying $a \ge b \ge c$ are

$$(a, b, c) = (15, 4, 2),\ (9, 5, 2),\ (7, 6, 2),\ (8, 3, 3),\ (5, 4, 3) .$$

The rest of the triples verifying the given equation can be obtained by permutations of a, b, c.

3.12. The change of variables $x = a + 1$, $y = b + 1$, $z = c + 1$, transforms the equation into the following one:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 .$$

Assuming $x \le y \le z$ we have that $x \le 3 \le z$.
For $x = 1$ the equation becomes

$$\frac{1}{y} + \frac{1}{z} = 0 .$$

which is impossible.
For $x = 2$ we have

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{2} ,$$

or

$$z = \frac{2y}{y-2} ,$$

with solutions $(y, z) = (3, 6)$ and $(4, 4)$.
For $x = 3$ the only possibility is $(y, z) = (3, 3)$.
So the list of solutions is

$$(x, y, z) = (2, 3, 6),\ (2, 4, 4),\ (3, 3, 3) ,$$

and the ones obtained by permuting x, y, z.
With the original variables the solutions are (except for permutations of variables);

$$(a, b, c) = (1, 2, 5),\ (1, 3, 3),\ (2, 2, 2) .$$

3.13. We look at the equation modulo 16. First we notice that $n^4 \equiv 0$ or $1 \pmod{16}$ depending on whether n is even or odd. On the other hand $1599 \equiv 15 \pmod{16}$. So the equation can be satisfied only if the number of odd terms in the LHS is 15 modulo 16, but that is impossible because there are only 14 terms in the LHS. Hence the equation has no solution.

3.14. Call $N = 10^{10^{10^{10}}}$, and consider the sequence a_n = remainder of dividing F_n by N. Since there are only N^2 pairs of non-negative integers less than N, there must be two identical pairs $(a_i, a_{i+1}) = (a_j, a_{j+1})$ for some $0 \le i < j$. Let $k = j - i$. Since $a_{n+2} = a_{n+1} + a_n$ and $a_{n-1} = a_{n+1} - a_n$, by induction we get that $a_n = a_{n+k}$ for every $n \ge 0$, so in particular $a_k = a_0 = 0$, and this implies that F_k is a multiple of N. (In fact since there are $N^2 + 1$ pairs (a_i, a_{i+1}), for $i = 0, 1, \dots, N^2$, we can add the restriction $0 \le i < j \le N$ above and get that the result is true for some k such that $0 < k \le N^2$.)

3.15. The answer is affirmative. Let $a = \sqrt{6}$ and $b = \sqrt{3}$. Assume $\lfloor a^m \rfloor = \lfloor b^n \rfloor = k$ for some positive integers m, n. Then, $k^2 \le 6^m < (k+1)^2 = k^2 + 2k + 1$, and $k^2 \le 3^n < (k+1)^2 = k^2 + 2k + 1$. Hence, subtracting the inequalities and taking into account that $n > m$:

$$2k \ge |6^m - 3^n| = 3^m |2^m - 3^{n-m}| \ge 3^m .$$

Hence $\dfrac{9^m}{4} \le k^2 \le 6^m$, which implies $\frac{1}{4} \le \left(\frac{2}{3}\right)^m$. This holds only for $m = 1, 2, 3$. This values of m can be ruled out by checking the values of

$$\lfloor a \rfloor = 2, \quad \lfloor a^2 \rfloor = 6, \quad \lfloor a^3 \rfloor = 14,$$

$$\lfloor b \rfloor = 1, \quad \lfloor b^2 \rfloor = 3, \quad \lfloor b^3 \rfloor = 5, \quad \lfloor b^4 \rfloor = 9, \quad \lfloor b^5 \rfloor = 15\,.$$

Hence, $\lfloor a^m \rfloor \ne \lfloor b^n \rfloor$ for every positive integers m, n.

3.16. There are integers k, r such that $10^k < 2^{2005} < 10^{k+1}$ and $10^r < 5^{2005} < 10^{r+1}$. Hence $10^{k+r} < 10^{2005} < 10^{k+r+2}$, $k + r + 1 = 2005$. Now the number of digits in 2^{2005} is $k + 1$, and the number of digits in 5^{2005} is $r + 1$. Hence the total number of digits is 2^{2005} and 5^{2005} is $k + r + 2 = 2006$.

3.17. For $p = 2$, $p^2 + 2 = 6$ is not prime.
For $p = 3$, $p^2 + 2 = 11$, and $p^3 + 2 = 29$ are all prime and the statement is true.
For prime $p > 3$ we have that p is an odd number not divisible by 3, so it is congruent to ± 1 modulo 6. Hence $p^2 + 2 \equiv 3 \pmod 6$ is multiple of 3 and cannot be prime.

3.18. If n is even then $n^4 + 4^n$ is even and greater than 2, so it cannot be prime.
If n is odd, then $n = 2k+1$ for some integer k, hence $n^4 + 4^n = n^4 + 4 \cdot (2^k)^4$. Next, use Sophie Germain's identity: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

3.19. From the hypothesis we have that $m + 1 \le \lfloor \sqrt{n} + \frac{1}{2} \rfloor \le \sqrt{n} + \frac{1}{2}$. But the second inequality must be strict because $\sqrt{n}$ is irrational or an integer, and consequently $\sqrt{n} + \frac{1}{2}$ cannot be an integer. From here the desired result follows.

3.20. Assume $m \ne \lfloor n + \sqrt{n} + 1/2 \rfloor$ for every $n = 1, 2, 3, \dots$ Then for some n, $f(n) < m < f(n+1)$. The first inequality implies

$$n + \sqrt{n} + \frac{1}{2} < m\,.$$

The second inequality implies $m + 1 \le f(n+1)$, and

$$m + 1 < n + 1 + \sqrt{n+1} + \frac{1}{2}$$

(Note that equality is impossible because the right hand side cannot be an integer.) Hence

$$\sqrt{n} < m - n - \frac{1}{2} < \sqrt{n+1}\,,$$

$$n < (m-n)^2 - (m-n) + \frac{1}{4} < n + 1$$

$$n - \frac{1}{4} < (m-n)^2 - (m-n) < n + \frac{3}{4}$$

$$(m-n)^2 - (m-n) = n\,.$$

$$m = (m-n)^2\,.$$

So, m is a square.

We are not done yet, since we still must prove that $f(n)$ misses *all* the squares. To do so we use a counting argument. Among all positive integers $\leq k^2+k$ there are exactly k squares, and exactly k^2 integers of the form $f(n) = \lfloor n + \sqrt{n} + 1/2 \rfloor$. Hence $f(n)$ is the nth non square.

Another way to express it: in the set $A(k) = \{1, 2, 3, \dots, k^2+k\}$ consider the two subsets $S(k) =$ squares in $A(k)$, and $N(k) =$ integers of the form $f(n) = \lfloor n + \sqrt{n} + 1/2 \rfloor$ in $A(k)$. The set $S(k)$ has k elements, $N(k)$ has k^2 elements, and $A(k) = S(k) \cup N(k)$. Since

$$\underbrace{|S(k) \cup N(k)|}_{k^2+k} = \underbrace{|S(k)|}_{k} + \underbrace{|N(k)|}_{k^2} - |S(k) \cap N(k)|$$

we get that $|S(k) \cap N(k)| = 0$, i.e, $S(k) \cap N(k)$ must be empty.

3.21. Each of the given numbers can be written

$$1 + 1000 + 1000^2 + \cdots + 1000^n = p_n(10^3)$$

where $p_n(x) = 1 + x + x^2 + \cdots + x^n$, $n = 1, 2, 3, \dots$. We have $(x-1)p_n(x) = x^{n+1} - 1$. If we set $x = 10^3$, we get:

$$999 \cdot p_n(10^3) = 10^{3(n+1)} - 1 = (10^{n+1} - 1)(10^{2(n+1)} + 10^{n+1} + 1)\,.$$

If $p_n(10^3)$ were prime it should divide one of the factors on the RHS. It cannot divide $10^{n+1} - 1$, because this factor is less than $p_n(10^3)$, so $p_n(10^3)$ must divide the other factor. Hence $10^{n+1} - 1$ must divide 999, but this is impossible for $n > 2$. In only remains to check the cases $n = 1$ and $n = 2$. But $1001 = 7 \cdot 11 \cdot 13$, and $1001001 = 3 \cdot 333667$, so they are not prime either.

3.22. We will prove that the sequence is eventually constant if and only if b_0 is a perfect square.

The "if" part is trivial, because if b_k is a perfect square then $d(b_k) = 0$, and $b_{k+1} = b_k$. For the "only if" part assume that b_k is *not* a perfect square. Then, suppose that $r^2 < b_k < (r+1)^2$. Then, $d(b_k) = b_k - r^2$ is in the interval $[1, 2r]$, so $b_{k+1} = r^2 + 2d(b_k)$ is greater than r^2 but less than $(r+2)^2$, and not equal to $(r+1)^2$ by parity. Thus b_{k+1} is also not a perfect square, and is greater than b_k. So, if b_0 is not a perfect square, no b_k is a perfect square and the sequence diverges to infinity.

3.23. The answer is 41. In fact, we have:

$$\gcd(a_n, a_{n+1}) = \gcd(a_n, a_{n+1} - a_n) = \gcd(n^2 + 10, 2n + 1) = \cdots$$

(since $2n + 1$ is odd we can multiply the other argument by 4 without altering the gcd)

$$\begin{aligned} \cdots = \gcd(4n^2 + 40, 2n + 1) &= \gcd((2n + 1)(2n - 1) + 41, 2n + 1) \\ &= \gcd(41, 2n + 1) \leq 41\,. \end{aligned}$$

The maximum value is attained e.g. at $n = 20$.

3.24. The given condition implies:

$$(x-y)(2x+2y+1)=y^2\,.$$

Since the right hand side is a square, to prove that the two factors on the left hand side are also squares it suffices to prove that they are relatively prime. In fact, if p if a prime number dividing $x-y$ then it divides y^2 and consequently it divides y. So p also divides x, and $x+y$. But then it cannot divide $2x+2y+1$.
An analogous reasoning works using the following relation, also implied the given condition:

$$(x-y)(3x+3y+1)=x^2\,.$$

3.25. It suffices to prove that n is a multiple of 5 and 8, in other words, that $n \equiv 0 \pmod 5$, and $n \equiv 0 \pmod 8$.
We first think modulo 5. Perfect squares can be congruent to 0, 1, or 4 modulo 5 only. We have:

$$\begin{aligned}
2n+1\equiv 0 \pmod 5 \quad &\Longrightarrow \quad n\equiv 2 \pmod 5\\
2n+1\equiv 1 \pmod 5 \quad &\Longrightarrow \quad n\equiv 0 \pmod 5\\
2n+1\equiv 4 \pmod 5 \quad &\Longrightarrow \quad n\equiv 4 \pmod 5\\
3n+1\equiv 0 \pmod 5 \quad &\Longrightarrow \quad n\equiv 3 \pmod 5\\
3n+1\equiv 1 \pmod 5 \quad &\Longrightarrow \quad n\equiv 0 \pmod 5\\
3n+1\equiv 4 \pmod 5 \quad &\Longrightarrow \quad n\equiv 1 \pmod 5\,.
\end{aligned}$$

So the only possibility that can make both $2n+1$ and $3n+1$ perfect squares is $n \equiv 0 \pmod 5$, i.e., n is a multiple of 5.
Next, we think modulo 8. Perfect squares can only be congruent to 0, 1, or 4 modulo 8, and we have:

$$\begin{aligned}
3n+1\equiv 0 \pmod 8 \quad &\Longrightarrow \quad n\equiv 5 \pmod 8\\
3n+1\equiv 1 \pmod 8 \quad &\Longrightarrow \quad n\equiv 0 \pmod 8\\
3n+1\equiv 4 \pmod 8 \quad &\Longrightarrow \quad n\equiv 1 \pmod 8\,.
\end{aligned}$$

The possibilities $n \equiv 5 \pmod 8$ and $n \equiv 1 \pmod 8$ can be ruled out because n must be even. In fact, if $2n+1=a^2$, then a is odd, and $2n=a^2-1=(a+1)(a-1)$. Since a is odd we have that $a-1$ and $a+1$ are even, so $2n$ must be a multiple of 4, consequently n is even. So, we have that the only possibility is $n \equiv 0 \pmod 8$, i.e., n is a multiple of 8.
Since n is a multiple of 5 and 8, it must be indeed a multiple of 40, QED.

3.26. The prime factorization of 1000! contains more 2's than 5's, so the number of zeros at the end of 1000! will equal the exponent of 5. That will be equal to the number of multiples of 5 in the sequence $1,2,3,\dots,1000$, plus the number of multiples of $5^2=25$, plus the number of multiples of $5^3=125$, and the multiples of $5^4=625$, in total:

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 200 + 40 + 8 + 1 = 249\,.$$

So 1000! ends with 249 zeros.

3.27. The answer is 8.
More generally, for any given positive integer n, the number of binomial coefficients $\binom{n}{k}$ that are odd equals 2 raised to the number of 1's in the binary representation of n—so, for $n = 100$, with binary representation 1100100 (three 1's), the answer is $2^3 = 8$. We prove it by induction in the number s of 1's in the binary representation of n.
- Basis step: If $s = 1$, then n is a power of 2, say $n = 2^r$. Next, we use that the exponent of a prime number p in the prime factorization of $m!$ is

$$\sum_{i\geq 1} \left\lfloor \frac{m}{p^i} \right\rfloor ,$$

where $\lfloor x \rfloor$ = greatest integer $\leq x$. Since $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the exponent of 2 in the prime factorization of $\binom{2^r}{k}$ is

$$\sum_{i\geq 1} \left\lfloor \frac{2^r}{2^i} \right\rfloor - \sum_{i\geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i\geq 1} \left\lfloor \frac{2^r - k}{2^i} \right\rfloor = \sum_{i=1}^{r} \left\lceil \frac{k}{2^i} \right\rceil - \sum_{i=1}^{r} \left\lfloor \frac{k}{2^i} \right\rfloor ,$$

where $\lceil x \rceil$ = least integer $\geq x$. The right hand side is the number of values of i in the interval from 1 to r for which $\frac{k}{2^i}$ is not an integer. If $k = 0$ or $k = 2^r$ then the expression is 0, i.e., $\binom{2^r}{k}$ is odd. Otherwise, for $0 < k < 2^r$, the right hand side is strictly positive (at least $k/2^r$ is not an integer), and in that case $\binom{2^r}{k}$ is even. So the number of values of k for which $\binom{2^r}{k}$ is odd is $2 = 2^1$. This sets the basis step of the induction process.
- Induction step: Assume the statement is true for a given $s \geq 1$, and assume that the number of 1's in the binary representation of n is $s + 1$, so n can be written $n = 2^r + n'$, where $0 < n' < 2^r$ and n' has s 1's in its binary representation. By induction hypothesis the number of values of k for which $\binom{n'}{k}$ is odd is 2^s. We must prove that the number of values of k for which $\binom{n}{k} = \binom{2^r+n'}{k}$ is odd is 2^{s+1}. To do so we will study the parity of $\binom{n}{k}$ in three intervals, namely $0 \leq k \leq n'$, $n' < k < 2^r$, and $2^r \leq k \leq n$.
(1) For every k such that $0 \leq k \leq n'$, $\binom{n'}{k}$ and $\binom{n}{k}$ have the same parity. In fact, using again the above formula to determine the exponent of 2 in the prime

factorization of binomial coefficients, we get

$$\begin{aligned}\sum_{i\geq 1}\left\lfloor\frac{n}{2^i}\right\rfloor-\sum_{i\geq 1}\left\lfloor\frac{k}{2^i}\right\rfloor-\sum_{i\geq 1}\left\lfloor\frac{n-k}{2^i}\right\rfloor &= \sum_{i\geq 1}\left\lfloor\frac{2^r+n'}{2^i}\right\rfloor-\sum_{i\geq 1}\left\lfloor\frac{k}{2^i}\right\rfloor \\ &\quad -\sum_{i\geq 1}\left\lfloor\frac{2^r+n'-k}{2^i}\right\rfloor \\ &= \sum_{i=1}^{r}\left(2^{r-i}+\left\lfloor\frac{n'}{2^i}\right\rfloor\right)-\sum_{i=1}^{r}\left\lfloor\frac{k}{2^i}\right\rfloor \\ &\quad -\sum_{i=1}^{r}\left(2^{r-i}+\left\lfloor\frac{n'-k}{2^i}\right\rfloor\right) \\ &= \sum_{i\geq 1}\left\lfloor\frac{n'}{2^i}\right\rfloor-\sum_{i\geq 1}\left\lfloor\frac{k}{2^i}\right\rfloor-\sum_{i\geq 1}\left\lfloor\frac{n'-k}{2^i}\right\rfloor .\end{aligned}$$

Hence, the number of values of k in the interval from 0 to n' for which $\binom{n}{k}$ is odd is 2^s.
(2) If $n' < k < 2^r$, then $\binom{n}{k}$ is even. In fact, we have that the power of 2 in the prime factorization of $\binom{n}{k}$ is:

$$\sum_{i\geq 1}\left\lfloor\frac{n}{2^i}\right\rfloor-\sum_{i\geq 1}\left\lfloor\frac{k}{2^i}\right\rfloor-\sum_{i\geq 1}\left\lfloor\frac{n-k}{2^i}\right\rfloor=\sum_{i=1}^{r}\left(\left\lfloor\frac{n}{2^i}\right\rfloor-\left\lfloor\frac{k}{2^i}\right\rfloor-\left\lfloor\frac{n-k}{2^i}\right\rfloor\right).$$

using $\lfloor x\rfloor+\lfloor y\rfloor\leq\lfloor x+y\rfloor$, and given that $n/2^i=k/2^i+(n-k)/2^i$, we see that all terms of the sum on the right hand side are nonnegative, and all we have to show is that at least one of them is strictly positive. That can be accomplished by taking $i=r$. In fact, we have $2^r<n<2^{r+1}$, hence $1<n/2^r<2$, $\lfloor n/2^r\rfloor=1$. Also, $0<k<2^r$, hence $0<k/2^r<1$, $\lfloor k/2^r\rfloor=0$. And $2^r=n-n'>n-k>0$, so $0<(n-k)/2^r<1$, $\lfloor(n-k)/2^r\rfloor=0$. Hence,

$$\left\lfloor\frac{n}{2^r}\right\rfloor-\left\lfloor\frac{k}{2^r}\right\rfloor-\left\lfloor\frac{n-k}{2^r}\right\rfloor=1-0-0=1>0\,.$$

(3) If $2^r\leq k\leq n$, then letting $k'=n-k$ we have that $0\leq k'\leq n'$, and $\binom{n}{k}=\binom{n}{k'}$, and by (1), the number of values of k in the interval from 2^r to n for which $\binom{n}{k}$ is odd is 2^s.
The three results (1), (2) and (3) combined show that the number of values of k for which $\binom{n}{k}$ is odd is $2\cdot 2^s=2^{s+1}$. This completes the induction step, and the result is proved.

3.28. The answer is 3.
Note that $2^5=32$, $5^5=3125$, so 3 is in fact a solution. We will prove that it is the only solution.

Let d be the common digit at the beginning of 2^n and 5^n. Then

$$d \cdot 10^r \le 2^n < (d+1) \cdot 10^r \,,$$
$$d \cdot 10^s \le 5^n < (d+1) \cdot 10^s$$

for some integers r, s. Multiplying the inequalities we get

$$d^2 10^{r+s} \le 10^n < (d+1)^2 10^{r+s} \,,$$
$$d^2 \le 10^{n-r-s} < (d+1)^2 \,,$$

so d is such that between d^2 and $(d+1)^2$ there must be a power of 10. The only possible solutions are $d = 1$ and $d = 3$. The case $d = 1$ can be ruled out because that would imply $n = r + s$, and from the inequalities above would get

$$5^r \le 2^s < 2 \cdot 5^r \,,$$
$$2^s \le 5^r < 2 \cdot 2^s \,,$$

hence $2^s = 5^r$, which is impossible unless $r = s = 0$ (implying $n = 0$).
Hence, the only possibility is $d = 3$.

3.29. Assume that the given binomial coefficients are in arithmetic progression. Multiplying each binomial number by $(k+3)!(n-k)!$ and simplifying we get that the following numbers are also in arithmetic progression:

$$(k+1)(k+2)(k+3),$$
$$(n-k)(k+2)(k+3),$$
$$(n-k)(n-k-1)(k+3),$$
$$(n-k)(n-k-1)(n-k-2)\,.$$

Their differences are

$$(n-2k-1)(k+2)(k+3),$$
$$(n-k)(n-2k-3)(k+3),$$
$$(n-k)(n-k-1)(n-2k-5)\,.$$

Writing that they must be equal we get a system of two equations:

$$\begin{cases} n^2 - 4kn - 5n + 4k^2 + 8k + 2 = 0 \\ n^2 - 4kn - 9n + 4k^2 + 16k + 14 = 0 \end{cases}$$

Subtracting both equations we get

$$4n - 8k - 12 = 0\,,$$

i.e., $n = 2k + 3$, so the four binomial numbers should be of the form

$$\binom{2k+3}{k}, \quad \binom{2k+3}{k+1}, \quad \binom{2k+3}{k+2}, \quad \binom{2k+3}{k+3}.$$

However

$$\binom{2k+3}{k} < \binom{2k+3}{k+1} = \binom{2k+3}{k+2} > \binom{2k+3}{k+3},$$

so they cannot be in arithmetic progression.

- *Remark:* There are sets of three consecutive binomial numbers in arithmetic progression, e.g.: $\binom{7}{1} = 7$, $\binom{7}{2} = 21$, $\binom{7}{3} = 35$.

3.30. We proceed by induction, with base case $1 = 2^0 3^0$. Suppose all integers less than $n-1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n. If n is odd, put $m = \lfloor \log_3 n \rfloor$, so that $3^m \le n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n - 3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then

$$n = 3^m + 2s_1 + \cdots + 2s_k,$$

and clearly none of the $2s_i$ divide each other or 3^m. Moreover, since $2s_i \le n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

3.31. There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k = 1, \dots, n$ (and clearly no others). To see this, note that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$, then

$$\begin{aligned} ka_1 &= a_1 + a_1 + \cdots + a_1 \\ &\le n \le a_1 + (a_1 + 1) + \cdots + (a_1 + 1) \\ &= ka_1 + k - 1. \end{aligned}$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1 + a_2 + \cdots + a_k$: if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i - 1)$. The possible values of i are $1, \dots, k$, and exactly one of these sums comes out equal to n, proving our claim.

3.32. Let R (resp. B) denote the set of red (resp. black) squares in such a coloring, and for $s \in R \cup B$, let $f(s)n + g(s) + 1$ denote the number written in square s, where $0 \le f(s), g(s) \le n - 1$. Then it is clear that the value of $f(s)$ depends only on the row of s, while the value of $g(s)$ depends only on the column of s. Since every row contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} f(s) = \sum_{s \in B} f(s).$$

Similarly, because every column contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} g(s) = \sum_{s \in B} g(s).$$

It follows that

$$\sum_{s \in R} f(s)n + g(s) + 1 = \sum_{s \in B} f(s)n + g(s) + 1,$$

as desired.

3.33. The answer is only 101.
Each of the given numbers can be written

$$1 + 100 + 100^2 + \cdots + 100^n = p_n(10^2)\,,$$

where $p_n(x) = 1+x+x^2+\cdots+x^n$, $n = 1, 2, 3, \ldots$. We have $(x-1)P_n(x) = x^{n+1}-1$. If we set $x = 10^2$, we get

$$99 \cdot p_n(10^2) = 10^{2(n+1)} - 1 = (10^{n+1} - 1)(10^{n+1} + 1)\,.$$

If $p_n(10^2)$ is prime it must divide one of the factors of the RHS. It cannot divide $10^{n+1} - 1$ because this factor is less than $p_n(10^2)$, so $p_n(10^2)$ must divide the other factor. Hence $10^{n+1} - 1$ must divide 99. This is impossible for $n \geq 2$. In only remains to check the case $n = 1$. In this case we have $p_1(10^2) = 101$, which is prime.

3.34. By contradiction. Assume n divides $2^n - 1$ (note that this implies that n is odd). Let p be the smallest prime divisor of n, and let $n = p^k m$, where p does not divide m. Since n is odd we have that $p \neq 2$. By Fermat's Little Theorem we have $2^{p-1} \equiv 1 \pmod{p}$. Also by Fermat's Little Theorem, $(2^{mp^{k-1}})^{p-1} \equiv 1 \pmod{p}$, hence $2^n = 2^{p^k m} = (2^{p^{k-1}m})^{p-1} \cdot 2^{p^{k-1}m} \equiv 2^{p^{k-1}m} \pmod{p}$. Repeating the argument we get $2^n = 2^{p^k m} \equiv 2^{p^{k-1}m} \equiv 2^{p^{k-2}m} \equiv \cdots \equiv 2^m \pmod{p}$. Since by hypothesis $2^n \equiv 1 \pmod{p}$, we have that $2^m \equiv 1 \pmod{p}$.
Next we use that if $2^a \equiv 1 \pmod{p}$, and $2^b \equiv 1 \pmod{p}$, then $2^{\gcd(a,b)} \equiv 1 \pmod{p}$. If $g = \gcd(n, p-1)$, then we must have $2^g \equiv 1 \pmod{p}$. But since p is the smallest prime divisor of n, and all prime divisors of $p - 1$ are less than p, we have that n and p do not have common prime divisors, so $g = 1$, and consequently $2^g = 2$, contradicting $2^g \equiv 1 \pmod{p}$.

3.35. In spite of its apparent complexity this problem is very easy, because the digital root of b_n becomes a constant very quickly. First note that the digital root of a number a is just the reminder r of a modulo 9, and the digital root of a^n will be the remainder of r^n modulo 9.
For $a_1 = 31$ we have
digital root of a_1 = digital root of $31 = 4$;
digital root of a_1^2 = digital root of $4^2 = 7$;
digital root of a_1^3 = digital root of $4^3 = 1$;
digital root of a_1^4 = digital root of $4^4 = 4$;
and from here on it repeats with period 3, so the digital root of a_1^n is 1, 4, and 7 for remainder modulo 3 of n equal to 0, 1, and 2 respectively.
Next, we have $a_2 = 314 \equiv 2 \pmod{3}$, $a_2^2 \equiv 2^2 \equiv 1 \pmod{3}$, $a_2^3 \equiv 2^3 \equiv 2 \pmod{3}$, and repeating with period 2, so the reminder of a_2^n depends only on the parity of n, with $a_2^n \equiv 1 \pmod{3}$ if n is even, and $a_2^n \equiv 2 \pmod{3}$ if n is odd.
And we are done because a_3 is odd, and the exponent of a_2 in the power tower defining b_n for every $n \geq 3$ is odd, so the reminder modulo 3 of the exponent of a_1 will be 2, and the reminder modulo 9 of b_n will be 7 for every $n \geq 3$.
Hence, the answer is 7.

4.1. If $x = \sqrt{2} + \sqrt{5}$ then

$$\begin{aligned} x^2 &= 7 + 2\sqrt{10}\,, \\ x^2 - 7 &= 2\sqrt{10}\,, \\ (x^2 - 7)^2 &= 40\,, \\ x^4 - 14x^2 + 9 &= 0\,. \end{aligned}$$

Hence the desired polynomial is $x^4 - 14x^2 + 9$.

4.2. We have that $p(x)+1$ has zeros at a, b, and c, hence $p(x)+1 = (x-a)(x-b)(x-c)q(x)$. If p had an integral zero d we would have

$$(d-a)(d-b)(d-c)q(d) = 1\,,$$

where $d-a$, $d-b$, and $d-c$ are distinct integers. But that is impossible, because 1 has only two possible factors, 1 and -1.

4.3. We prove it by showing that the sum is the root of a monic polynomial but not an integer—so by the rational roots theorem it must be irrational.
First we notice that $n < \sqrt{n^2+1} < n + 1/n$, hence the given sum is of the form

$$S = 1001 + \theta_1 + 1002 + \theta_2 + \cdots + 2000 + \theta_{1000}$$

were $0 < \theta_i < 1/1001$, consequently

$$0 < \theta_1 + \theta_2 + \cdots + \theta_{1000} < 1\,,$$

so S is not an integer.
Now we must prove that S is the root of a monic polynomial. More generally we will prove that a sum of the form

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

where the a_i's are positive integers, is the root of a monic polynomial.[2] This can be proved by induction on n. For $n = 1$, $\sqrt{a_1}$ is the root of the monic polynomial $x^2 - a_1$. Next assume that $y = \sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$ is a zero of a monic polynomial $P(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_0$. We will find a polynomial that has $z = y + \sqrt{a_{n+1}}$ as a zero. We have

$$0 = P(y) = P(z - \sqrt{a_{n+1}}) = (z - \sqrt{a_{n+1}})^r + c_{r-1}(z - \sqrt{a_{n+1}})^{r-1} + \cdots + c_0\,.$$

Expanding the parentheses and grouping the terms that contain $\sqrt{a_{n+1}}$:

$$0 = P(z - \sqrt{a_{n+1}}) = z^r + Q(z) + \sqrt{a_{n+1}}\,R(z)\,.$$

[2]The result can be obtained also by resorting to a known theorem on *algebraic integers* ("algebraic integer" is the mathematical term used to designate a root of a monic polynomial.) It is known that algebraic integers form a mathematical structure called *ring*, basically meaning that the sum, difference and product of two algebraic integers is an algebraic integer. Now, if a_i and k_i are positive integers, then $\sqrt[k_i]{a_i}$ is an algebraic integer, because it is a root of the monic polynomial $x^{k_i} - a_i$. Next, since the sum or difference of algebraic integers is an algebraic integer then $\pm \sqrt[k_1]{a_1} \pm \sqrt[k_2]{a_2} \pm \cdots \pm \sqrt[k_n]{a_n}$ is in fact an algebraic integer (note that the roots do not need to be square roots, and the signs can be combined in any way.)

Putting radicals on one side and squaring

$$(z^r + Q(z))^2 = a_{n+1}(R(z))^2 \,,$$

so

$$T(x) = (x^r + Q(x))^2 - a_{n+1}(R(x))^2$$

is a monic polynomial with z as a root.

4.4. Consider the following polynomial:

$$Q(x) = (x+1)P(x) - x \,.$$

We have that $Q(k) = 0$ for $k = 0, 1, 2, \dots, n$, hence, by the *Factor theorem*,

$$Q(x) = Cx(x-1)(x-2)\dots(x-n)\,,$$

where C is a constant to be determined. Plugging $x = -1$ we get

$$Q(-1) = C(-1)(-2)\cdots(-(n+1))\,.$$

On the other hand $Q(-1) = 0 \cdot P(-1) - (-1) = 1$, hence $C = \dfrac{(-1)^{n+1}}{(n+1)!}$.
Next, plugging in $x = n+1$ we get

$$(n+2)P(n+1) - (n+1) = C(n+1)! = \frac{(-1)^{n+1}}{(n+1)!}(n+1)! = (-1)^{n+1}\,,$$

hence

$$\boxed{P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}}\,.$$

4.5. Let the zeros be a, b, c, d. The relationship between zeros and coefficients yields

$$\begin{aligned} a+b+c+d &= 18 \\ ab+ac+ad+bc+bd+cd &= k \\ abc+abd+acd+bcd &= -200 \\ abcd &= -1984\,. \end{aligned}$$

Assume $ab = -32$ and let $u = a+b$, $v = c+d$, $w = cd$. Then

$$\begin{aligned} u+v &= 18 \\ -32+uv+w &= k \\ -32v+uw &= -200 \\ -32w &= -1984\,. \end{aligned}$$

From the last equation we get $w = 62$, and replacing in the other equations we easily get $u = 4$, $v = 14$. Hence

$$k = -32 + 4 \cdot 14 + 62 = \boxed{86}\,.$$

4.6. Let $p(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n$. The $(n-1)$-degree polynomial $p(x)-p(-x) = 2(a_1x + a_3x^3 + \cdots + a_{n-1}x^{n-1})$ vanishes at n different points, hence, it must be identically null, i.e., $a_1 = a_3 = \cdots = a_{n-1} = 0$. Hence $p(x) = a_0 + a_2x^2 + a_4x^4 \cdots + a_nx^n$, and $q(x) = a_0 + a_2x + a_4x^2 \cdots + a_nx^{n/2}$.

4.7. If k is an even integer we have $p(k) \equiv p(0) \pmod 2$, and if it is odd then $p(k) \equiv p(1) \pmod 2$. Since $p(0)$ and $p(1)$ are odd we have $p(k) \equiv 1 \pmod 2$ for every integer k, so $p(k)$ cannot be zero.

4.8. We must prove that $P(1) = 0$. Consider the four complex numbers $\rho_k = e^{2\pi ik/5}$, $k = 1, 2, 3, 4$. All of them verify $\rho_k^5 = 1$, so together with 1 they are the roots of $x^5 - 1$. Since $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$ then the ρ_k's are the roots of $x^4 + x^3 + x^2 + x + 1$. So

$$P(1) + \rho_k Q(1) + \rho_k^2 R(1) = 0\,.$$

Adding for $k = 1, 2, 3, 4$ and taking into account that the numbers ρ_k^2 are the ρ_k's in a different order we get

$$P(1) = 0\,.$$

4.9. The answer is no. We prove it by contradiction. Assume $(x-a)(x-b)(x-c)-1 = p(x)q(x)$, where p is linear and q is quadratic. Then $p(a)q(a) = p(b)q(b) = p(c)q(c) = -1$. If the coefficients of p and q must integers they can take only integer values, so in each product one of the factor must be 1 and the other one is -1. Hence either $p(x)$ takes the value 1 twice or it takes the value -1 twice. But a 1st degree polynomial cannot take the same value twice.

4.10. We prove it by contradiction. Suppose $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Since $g(x) > 0$ for every x, $h(x)$ and $k(x)$ cannot have real roots, so they cannot change signs and we may suppose $h(x) > 0$ and $k(x) > 0$ for every x. Since $g(p_i) = 1$ for $i = 1, \dots, n$, we have $h(p_i) = k(p_i) = 1$, $i = 1, \dots, n$. If either $h(x)$ or $k(x)$ had degree less than n, it would constant, against the hypothesis, so they must be of degree n. Also we know that $h(x) - 1$ and $k(x) - 1$ are zero for $x = p_i$, $i = 1, \dots, n$, so their roots are precisely $p_1, \dots, p_n$, and we can write

$$h(x) = 1 + a(x - p_1)\cdots(x - p_n)$$
$$k(x) = 1 + b(x - p_1)\cdots(x - p_n)\,,$$

where a and b are integers. So we have

$$(x - p_1)^2 \cdots (x - p_n)^2 + 1 = \\ 1 + (a + b)(x - p_1)\cdots(x - p_n) + ab(x - p_1)^2 \cdots (x - p_n)^2\,.$$

Hence

$$\begin{cases} a+b & = 0 \\ ab & = 1\,, \end{cases}$$

which is impossible, because there are no integers a, b verifying those equations.

4.11. Assume the quotient is $q(x)$ and the remainder is $r(x) = ax^2 + bx + c$. Then

$$x^{81} + x^{49} + x^{25} + x^9 + x = q(x)(x^3 - x) + r(x)\,,$$

Plugging in the values $x = -1, 0, 1$ we get $r(-1) = -5$, $r(0) = 0$, $r(1) = 5$. From here we get $a = c = 0$, $b = 5$, hence the remainder is $r(x) = 5x$.

4.12. For positive integer n we have $f(n) = \frac{n}{n+1} f(n-1) = \frac{n-1}{n+1} f(n-2) = \cdots = 0 \cdot f(-1) = 0$. Hence $f(x)$ has infinitely many zeros, and must be identically zero $f(x) \equiv 0$.

4.13. By contradiction. Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Let $\alpha_1, \dots, \alpha_k$ the (complex) roots of $g(x)$. For each $j = 1, \dots, k$ we have $\alpha_j^{105} = 9$, hence $|\alpha_j| = \sqrt[105]{9}$, and $|\alpha_1 \alpha_2 \cdots \alpha_k| = (\sqrt[105]{9})^k =$ the absolute value of the constant term of $g(x)$ (an integer.) But $(\sqrt[105]{9})^k = \sqrt[105]{3^{2k}}$ cannot be an integer.

4.14. The answer is $p = 5$. By Sophie Germain's Identity we have

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy) = [(x+y)^2 + y^2][(x-y)^2 + y^2]\,,$$

which can be prime only if $x = y = 1$.

4.15. We have that a, b, c, d are distinct roots of $P(x) - 5$, hence

$$P(x) - 5 = g(x)(x-a)(x-b)(x-c)(x-d)\,,$$

where $g(x)$ is a polynomial with integral coefficients. If $P(k) = 8$ then

$$g(x)(x-a)(x-b)(x-c)(x-d) = 3\,,$$

but 3 is a prime number, so all the factors on the left but one must be ± 1. So among the numbers $(x-a)$, $(x-b)$, $(x-c)$, $(x-b)$, there are either two 1's or two -1's, which implies that a, b, c, d cannot be all distinct, a contradiction.

4.16. Calling $A_{n-1} = 1 + x + \cdots + x^{n-1}$, we have

$$\begin{aligned}
(1 + x + \cdots + x^n)^2 - x^n &= (A_{n-1} + x^n)^2 - x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + x^{2n} - x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + (x^n - 1)x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + A_{n-1}(x-1)x^n \\
&= A_{n-1}(A_{n-1} + 2x^n + (x-1)x^n) \\
&= A_{n-1}(A_{n-1} + x^n + x^{n+1}) \\
&= (1 + x + \cdots + x^{n-1})(1 + x + \cdots + x^{n+1})\,.
\end{aligned}$$

4.17. Since $f(x)$ and $f(x) + f'(x)$ have the same leading coefficient, the limit of $f(x)$ as $x \to \pm\infty$ must be equal to that of $f(x) + f'(x)$, i.e., $+\infty$.
Note that f cannot have multiple real roots, because at any of those roots both f and f' would vanish, contradicting the hypothesis. So all real roots of f, if any, must be simple roots.

Since $f(x) \to +\infty$ for both $x \to \infty$ and $x \to -\infty$, it must have an even number of real roots (if any): $x_1 < x_2 < \cdots < x_{2n}$. Note that between x_1 and x_2, $f(x)$ must be negative, and by Rolle's theorem its derivative must be zero at some intermediate point $a \in (x_1, x_2)$, hence $f(a)+f'(a) = f(a) < 0$, again contradicting the hypothesis. Consequently, $f(x)$ has no real roots, and does not change sign at any point, which implies $f(x) > 0$ for all x.

4.18. This is a particular case of the well known Vandermonde determinant, but here we will find its value using arguments from polynomial theory. Expanding the determinant along the last column using Laplace formula we get

$$a_0(w,x,y) + a_1(w,x,y)z + a_2(w,x,y)z^2 + a_3(w,x,y)z^3\,,$$

where $a_i(w,x,y)$ is the cofactor of z^i.
Since the determinant vanishes when two columns are equal, that polynomial in z has zeros at $z = w$, $z = x$, $z = y$, hence it must be of the form

$$a_3(w,x,y,z)(z-y)(z-x)(z-w)\,.$$

Note that $a_3(w,x,y) = \begin{vmatrix} 1 & 1 & 1 \\ w & x & y \\ w^2 & x^2 & y^2 \end{vmatrix}$, which can be computed in an analogous way:

$$\begin{aligned} a_3(w,x,y) &= b_2(w,x)(y-x)(y-w)\,, \\ b_2(w,x) &= c_1(w)(x-w)\,, \\ c_1(w) &= 1\,. \end{aligned}$$

Hence the value of the given determinant is

$$(z-y)(z-x)(z-w)(y-x)(y-w)(x-w)\,.$$

4.19. Expanding the determinant along the last column using Laplace formula we get

$$a_0(w,x,y) + a_1(w,x,y)z + a_2(w,x,y)z^2 + a_4(w,x,y)z^4\,,$$

where $a_i(w,x,y)$ is the cofactor of z^i. In particular $a_4(w,x,y) = (y-x)(y-w)(x-w)$ by Vandermonde formula.
Since the determinant vanishes when two columns are equal, that polynomial in z has zeros at $z = w$, $z = x$, $z = y$, hence it must be of the form

$$\begin{aligned} &a_4(w,x,y,z)(z-y)(z-x)(z-w)\,b(w,x,y,z) = \\ &\qquad (z-y)(z-x)(z-w)(y-x)(y-w)(x-w)\,b(w,x,y,z)\,, \end{aligned}$$

where $b(w,x,y,z)$ is some first degree homogeneous polynomial in w,x,y,z. Note that the value of $b(w,x,y,z)$ won't change by permutations of its arguments, so $b(w,x,y,z)$ is symmetric, and all its coefficients must be equal, hence $b(w,x,y,z) = k \cdot (w+x+y+z)$ for some constant k. The value of k can be found by computing the determinant for particular values of w,x,y,z, say $w = 0$, $x = 1$, $y = 2$, $z = 3$, and we obtain $k = 1$. Hence the value of the determinant is

$$(z-y)(z-x)(z-w)(y-x)(y-w)(x-w)(w+x+y+z)\,.$$

4.20. The answer is no.
We can write the condition in matrix form in the following way:

$$\begin{pmatrix}1 & x & x^2\end{pmatrix}\begin{pmatrix}1\\ y\\ y^2\end{pmatrix} = \begin{pmatrix}a(x) & c(x)\end{pmatrix}\begin{pmatrix}b(y)\\ d(y)\end{pmatrix}.$$

By assigning values $x = 0, 1, 2$, $y = 0, 1, 2$, we obtain the following identity:

$$\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 2 & 4\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 1 & 4\end{pmatrix} = \begin{pmatrix}a(0) & c(0)\\ a(1) & c(1)\\ a(2) & c(2)\end{pmatrix}\begin{pmatrix}b(0) & b(1) & b(2)\\ d(0) & d(1) & d(2)\end{pmatrix}.$$

The product on the left hand side yields a matrix of rank 3, while the right hand side has rank at most 2, contradiction.

4.21. The answer is $P(x) = x$.
In order to prove this we show that $P(x)$ equals x for infinitely many values of x. In fact, let a_n the sequence $0, 1, 2, 5, 26, 677, \dots$, defined recursively $a_0 = 0$, and $a_{n+1} = a_n^2 + 1$ for $n \geq 0$. We prove by induction that $P(a_n) = a_n$ for every $n = 0, 1, 2, \dots$. In the basis case, $n = 0$, we have $P(0) = 0$. For the induction step assume $n \geq 1$, $P(a_n) = a_n$. Then $P(a_{n+1}) = P(a_n^2+1) = P(a_n)^2+1 = a_n^2+1 = a_{n+1}$. Since in fact $P(x)$ coincides with x for infinitely many values of x, we must have $P(x) = x$ identically.

4.22. Given a line $y = mx + b$, the intersection points with the given curve can be computed by solving the following system of equations

$$\begin{cases} y = 2x^4 + 7x^3 + 3x - 5 \\ y = mx + b\,. \end{cases}$$

Subtracting we get $2x^4 + 7x^3 + 3(x - m) - 5 - b = 0$. If the line intersects the curve in four different points, that polynomial will have four distinct roots x_1, x_2, x_3, x_4, and their sum will be minus the coefficient of x^3 divided by the coefficient of x^4, i.e., $-7/2$, hence

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = -\frac{7}{8}\,.$$

4.23. The answer is $k = 3$, and an example is

$$(x + 2)(x + 1)x(x - 1)(x - 2) = x^5 - 5x^3 + 4x\,,$$

where $\{-2, -1, 0, 1, 2\}$ is the desired set of integers.
To complete the argument we must prove that k cannot be less than 3. It cannot be 1 because in that case the polynomial would be x^5, with all its five roots equal zero (note that a, b, c, d, e are the roots of the polynomial, and by hypothesis they must be distinct integers). Assume now that $k = 2$. Then the polynomial would be of the form $x^5 + nx^i = x^i(x^{5-i} + n)$, n a nonzero integer, $0 \leq i \leq 4$. If $i \geq 2$ then the polynomial would have two or more roots equal zero, contradicting the hypothesis. If $i = 1$ then the roots of the polynomial would be 0, and the roots of $x^4 + n$, at least

two of which are non-real complex roots. If $i = 0$ then the polynomial is $x^5 + n$, which has four non-real complex roots.

4.24. The zeros of $x^4 - 13x^2 + 36$ are $x = \pm 2$ and ± 3, hence the condition is equivalent to $x \in [-3, -2] \cup [2, 3]$. On the other hand $f'(x) = 3x^2 - 3$, with zeros at $x = \pm 1$. This implies that $f(x)$ is monotonic on $[-3, -2]$ and $[2, 3]$, hence (with the given constrain) its maximum value can be attained only at the boundaries of those intervals. Computing $f(-3) = -18$, $f(-2) = -2$, $f(2) = 2$, $f(3) = 18$, we get that the desired maximum is 18.

4.25. Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x - 3|)/2 \\ &\qquad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x - 3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x - 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

4.26. Writing the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ, we have

$$\begin{aligned} \alpha + \beta + \gamma &= s\,, \\ \alpha^2 + \beta^2 + \gamma^2 &= s^2 - 2q\,, \\ \alpha^3 + \beta^3 + \gamma^3 &= s^3 - 3qs + 3p\,, \end{aligned}$$

where $s = \alpha + \beta + \gamma$, $q = \alpha\beta + \beta\gamma + \alpha\gamma$, $p = \alpha\beta\gamma$.
So we have $s = 2$, and from the second given equation get $q = -5$. Finally from the third equation we get $p = -7$. So, this is the answer, $\alpha\beta\gamma = -7$.
(Note: this is not needed to solve the problem, but by solving the equation $x^3 - sx^2 + qs - p = x^3 - 2x^2 + 5x + 7 = 0$ we find that the three numbers α, β, and γ are approx. 2.891954442, −2.064434534, and 1.172480094.)

4.27. Let $\alpha = (2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3}$. By raising to the third power, expanding and simplifying we get that α verifies the following polynomial equation:

$$\alpha^3 + 3\alpha - 4 = 0\,.$$

We have $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$. The second factor has no real roots, hence $x^3 + 3x - 4$ has only one real root equal to 1, i.e., $\alpha = 1$.

4.28. The answer is affirmative, B can in fact guess the polynomial—call it $f(x) = a_0 + a_1x^2 + a_2x^2 + \cdots a_nx^n$. By asking A to evaluate it at 1, B gets an upper bound $f(1) = a_0 + a_1 + a_2 + \cdots a_n = M$ for the coefficients of the polynomial. Then, for any integer $N > M$, the coefficients of the polynomial are just the digits of $f(N) = a_0 + a_1N^2 + a_2N^2 + \cdots a_nN^n$ in base N.

4.29. If $f(x_0) = 0$ at some point x_0, then by hypothesis we would have $f'(x_0) > 0$, and f would be (strictly) increasing at x_0. This implies:
(1) If the polynomial f becomes zero at some point x_0, then $f(x) > 0$ for every $x > x_0$, and $f(x) < 0$ for every $x < x_0$.
Writing $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have $f(x) + f'(x) = a_nx^n + (a_{n-1} + na_n)x^{n-1} + \cdots + (a_0 + a_1)$. Given that that $f(x) + f'(x) > 0$ for all x, we deduce that $a_n = \lim_{n\to\infty} \frac{f(x)+f'(x)}{x^n} > 0$. On the other hand n must be even, otherwise $f(x) + f'(x)$ would become negative as $x \to -\infty$. Hence $f(x) > 0$ for $|x|$ large enough. By (1) this rules out the possibility of $f(x)$ becoming zero at some point x_0, and so it must be always positive.
Remark: The statement is not true for functions in general, e.g., $f(x) = -e^{-2x}$ verifies $f(x) + f'(x) = e^{-2x} > 0$, but $f(x) < 0$ for every x.

4.30. The answer is No. If $P(x)$ has two real roots we would have $b^2 > 4ac$. Analogously for $R(x)$ and $Q(x)$ we should have $a^2 > 4cb$, and $c^2 > 4ab$ respectively. Multiplying the inequalities we get $a^2b^2c^2 > 64a^2b^2c^2$, which is impossible.

4.31. First we prove (by contradiction) that $f(x)$ has no real roots. In fact, if x_1 is a real root of $f(x)$, then we have that $x_2 = x_1^2 + x_1 + 1$ is also a real root of $f(x)$, because $f(x_1^2 + x_1 + 1) = f(x_1)g(x_1) = 0$. But $x_1^2 + 1 > 0$, hence $x_2 = x_1^2 + x_1 + 1 > x_1$. Repeating the reasoning we get that $x_3 = x_2^2 + x_2 + 1$ is another root of $f(x)$ greater than x_2, and so on, so we get an infinite increasing sequence of roots of $f(x)$, which is impossible. Consequently $f(x)$ must have even degree, because all odd degree polynomials with real coefficients have at least one real root. Q.E.D.
Note: An example of a polynomial with the desired property is: $f(x) = x^2 + 1$, $f(x^2 + x + 1) = (x^2 + 1)(x^2 + 2x + 2)$.
Remark: The result is not generally true for polynomials with complex coefficients—counterexample: $f(x) = x + i$, $f(x^2 + x + 1) = x^2 + x + 1 + i = (x + i)(x + 1 - i)$.

4.32. By contradiction. We have that $a_0 = P(0)$ must be a prime number. Also, $P(ka_0)$ is a multiple of a_0 for every $k = 0, 1, 2, \dots$, but if $P(ka_0)$ is prime then $P(ka_0) = a_0$ for every $k \geq 0$. This implies that the polynomial $Q(x) = P(a_0x) - a_0$ has infinitely many roots, so it is identically zero, and $P(a_0x) = a_0$, contradicting the hypothesis that P is of degree at least 1.

5.1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$. We have

$$|z|^2 = |a + bi|^2|c + di|^2 = (a^2 + b^2)(c^2 + d^2) = mn\,,$$

and

$$|z|^2 = (ac - bd)^2 + (ad + bc)^2\,,$$

so mn is also in fact a sum of two perfect squares.

5.2. The left hand side of the equality is the imaginary part of

$$\sum_{k=0}^{n} e^{ik} = \frac{e^{i(n+1)}-1}{e^i-1} = \frac{e^{i(n+1/2)}-e^{-i/2}}{e^{i/2}-e^{-i/2}} = \frac{\cos\left(n+\frac{1}{2}\right)-\cos\frac{1}{2}+i\{\sin\left(n+\frac{1}{2}\right)+\sin\frac{1}{2}\}}{2i\sin\frac{1}{2}}\,.$$

The imaginary part of that expression is

$$\frac{\cos\frac{1}{2}-\cos\left(n+\frac{1}{2}\right)}{2\sin\frac{1}{2}} = \frac{\sin\frac{n}{2}\sin\frac{n+1}{2}}{\sin\frac{1}{2}}$$

5.3. We have that $z = e^{\pm ia}$, so $z + 1/z = e^{ia} + e^{-ia} = 2\cos a$, hence:

$$z^n + 1/z^n = e^{ina} + e^{-ina} = 2\cos na\,.$$

5.4. Factoring a polynomial is easier to accomplish if we can find its roots. In this case we will look for roots that are roots of unity $e^{2k\pi i/n}$:

$$p(e^{2k\pi i/n}) = e^{10k\pi i/n} + e^{2k\pi i/n} + 1\,.$$

The three terms of that expression are complex numbers placed on the unit circle at the vertices of an equilateral triangle for $n = 3$ and $k = 1, 2$, so if $\omega = e^{2k\pi i/3}$, then ω and ω^2 are roots of $p(z)$, hence $p(z)$ is divisible by $(z-\omega)(z-\omega^2) = z^2+z+1$. By long division we find that the other factor is $z^3 - z^2 + 1$, hence:

$$p(z) = (z^2+z+1)(z^3-z^2+1)\,.$$

5.5. Write $\sin t = (e^{ti} - e^{-ti})/2i$ and consider the polynomial

$$p(x) = \prod_{k=1}^{n-1}(x - e^{2\pi ik/n})\,.$$

We have:

$$P = \prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \prod_{k=1}^{n-1}\frac{e^{\pi ik/n}-e^{-\pi ik/n}}{2i} = \frac{e^{-\pi i(n-1)/2}}{(2i)^{n-1}}\prod_{k=1}^{n-1}(e^{2\pi ik/n}-1) = \frac{p(1)}{2^{n-1}}\,.$$

On the other hand the roots of $p(x)$ are all nth roots of 1 except 1, so $(x-1)p(x) = x^n - 1$, and

$$p(x) = \frac{x^n-1}{x-1} = 1 + x + x^2 + \cdots + x^{n-1}\,.$$

Consequently $p(1) = n$, and $P = \dfrac{n}{2^{n-1}}$.

5.6. Assume the vertices of the n-gon placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Then the length of the diagonal connecting vertices j and k is $|\zeta^i - \zeta^k|$, and the desired product can be written

$$P = \prod_{0\le j<k<n}|\zeta^j - \zeta^k|\,.$$

By symmetry we obtain the same product if we replace the condition $j < k$ with $k < j$, and multiplying both expressions together we get:

$$P^2 = \prod_{\substack{0\le j,k<n \\ j\neq k}} |\zeta^j - \zeta^k| = |\zeta^j| \prod_{\substack{0\le j,k<n \\ j\neq k}} |1 - \zeta^{k-j}| \,.$$

Note that $|\zeta^j| = 1$, and for each k, $r = k-j$ takes all non-zero values from $k-n+1$ to k. Since $\zeta^r = \zeta^{r+n}$ we may assume that r ranges from 1 to $n-1$, so we can rewrite the product like this:

$$P^2 = \left(\prod_{r=1}^{n-1} |1-\zeta^r|\right)^n \,.$$

Next consider the polynomial

$$p(x) = \prod_{r=1}^{n-1} (x - \zeta^r) \,.$$

Its roots are the same roots of $x^n - 1$ except 1, hence $x^n - 1 = (x-1)p(x)$ and

$$p(x) = \frac{x^n-1}{x-1} = 1 + x + x^2 + \cdots + x^{n-1} \,,$$

hence

$$\prod_{r=1}^{n-1} (1-\zeta^r) = p(1) = n \,.$$

consequently $P^2 = n^n$, and $P = n^{n/2}$.

5.7. Define $h(x) = f(x) + ig(x)$. Then h is differentiable and $h'(0) = bi$ for some $b \in \mathbb{R}$. The given equations can be reinterpret as $h(x+y) = h(x)h(y)$. Differentiating respect to y and substituting $y = 0$ we get $h'(x) = h(x)h'(0) = bi \cdot h(x)$, so $h(x) = Ce^{bix}$ for some $C \in \mathbb{C}$. From $h(0+0) = h(0)h(0)$ we get $C = C^2$. If $C = 0$ then $h = 0$ and f and g would be constant, contradicting the hypothesis. Thus $C = 1$. Finally, for any $x \in \mathbb{R}$,

$$f(x)^2 + g(x)^2 = |h(x)|^2 = |e^{bix}|^2 = 1 \,.$$

5.8. Assume the lights placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Without loss of generality we may assume that the light at 1 is initially on. Now, if $d < n$ is a divisor of n and the lights $\zeta^a, \zeta^{a+d}, \zeta^{a+2d}, \dots, \zeta^{a+(\frac{n}{n}-1)d}$ have the same state, then we can change the state of this n/d lights. The sum of these is

$$\zeta^a + \zeta^{a+d} + \zeta^{a+2d} + \cdots + \zeta^{a+(\frac{n}{n}-1)d} = \zeta^a \left(\frac{1-\zeta^n}{1-\zeta^d}\right) = \zeta^a \left(\frac{1-1}{1-\zeta^d}\right) = 0 \,.$$

So if we add up all the roots that are "on", the sum will never change. The original sum was 1, and the goal is to get all the lights turned on. That sum will be

$$1+\zeta+\zeta^2+\cdots+\zeta^{n-1}=\frac{1-\zeta^n}{1-\zeta}=0\neq 1\,.$$

Hence we can never turn on all the lights.

5.9. Let $z_1=a-bi$, $z_2=u+vi$. Then $|z_1|^2=a^2+b^2$, $|z_2|=u^2+v^2$, $\Re(z_1z_2)=au+bv$, $\Im(z_1z_2)=1$. On the other hand:

$$|z_1z_2|^2=\Re(z_1z_2)^2+\Im(z_1z_2)^2=\Re(z_1z_2)^2+1\,.$$

Now for any real t,

$$(t\sqrt{3}+1)^2\geq 0 \implies 3t^2+1\geq -2t\sqrt{3} \implies 4t^2+4\geq(\sqrt{3}-t)^2\,.$$

Hence

$$(|z_1|^2+|z_2|^2)^2\geq 4|z_1z_2|^2=4\big(\Re(z_1z_2)^2+1\big)\geq\big(\sqrt{3}-\Re(z_1z_2)\big)^2\,.$$

So, $|z_1|^2+|z_2|^2\geq\sqrt{3}-\Re(z_1z_2)$. Or $|z_1|^2+|z_2|^2+\Re(z_1z_2)\geq\sqrt{3}$, as required.

6.1. We have

$$\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n=(1+x)^n\,.$$

Differentiating respect to x:

$$\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\cdots+n\binom{n}{n}=n(1+x)^{n-1}\,.$$

Plugging in $x=1$ we get the desired identity

6.2. The desired expression states the equality between the coefficient of x^n in each of the following expansions:

$$(1+x)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^k\,,$$

and

$$\{(1+x)^n\}^2=\left\{\sum_{k=0}^{n}\binom{n}{k}x^k\right\}^2=\sum_{k=0}^{n}\sum_{i+j=k}\binom{n}{i}\binom{n}{j}x^k\,.$$

Taking into account that $\binom{n}{j}=\binom{n}{n-j}$, for $k=n$ we get

$$\sum_{i+j=n}\binom{n}{i}\binom{n}{j}=\sum_{i+j=n}\binom{n}{i}\binom{n}{n-j}=\sum_{i=1}^{n}\binom{n}{i}^2\,,$$

and that must be equal to the coefficient of x^n in $(1+x)^{2n}$, which is $\binom{2n}{n}$.

6.3. This is just a generalization of the previous problem. We have

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k \,,$$

and

$$\begin{aligned}(1+x)^m(1+x)^n &= \left\{\sum_{i=0}^{m} \binom{m}{i} x^i\right\} \left\{\sum_{j=0}^{n} \binom{n}{j} x^j\right\} \\ &= \sum_{k=0}^{m+n} \sum_{\substack{i+j=k \\ 0\le i,j\le k}} \binom{n}{j}\binom{m}{i} x^k \,.\end{aligned}$$

The coefficient of x^k must be the same on both sides, so:

$$\binom{m+n}{k} = \sum_{\substack{i+j=k \\ 0\le i,j\le k}} \binom{n}{j}\binom{m}{i} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} \,,$$

where we replace $i = k - j$ in the last step.

6.4. The generating function for the Fibonacci sequence is

$$0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots = \frac{x}{1-x-x^2} \,.$$

The desired sum is the left hand side with $x = 1/2$, hence its value is

$$0 + \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}-\frac{1}{2^2}} = \boxed{2}\,.$$

6.5. The generating function of u_n is the following:

$$\begin{aligned}f(x) &= u_0 + u_1 x + u_2 x^2 + \cdots \\ &= (1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots) \\ &= \frac{1}{1-x^2}\frac{1}{1-x^5} \\ &= \frac{1}{1-x^2-x^5+x^7} \,.\end{aligned}$$

Hence

$$1 = (1 - x^2 - x^5 + x^7)(u_0 + u_1 x + u_2 x^2 + \cdots)\,.$$

From here we get that $1 \cdot u_0 = 1$, hence $u_0 = 1$. Similarly $1 \cdot u_1 = 0$, hence $u_1 = 0$, etc., so we get $u_0 = u_2 = u_4 = u_5 = u_7 = 1$, and $u_1 = u_3 = 0$. Then for $k > 7$ the coefficient of x^k of the product must be

$$u_k - u_{k-2} - u_{k-5} + u_{k-7} = 0\,.$$

So we get the following recursive relation for the terms of the sequence:

$$u_k = u_{k-2} + u_{k-5} - u_{k-7}\,,$$

together with the initial conditions $u_0 = u_2 = u_4 = u_5 = u_7 = 1$, and $u_1 = u_3 = 0$.

6.6. The answer equals the coefficient of x^{10} in the expansion of

$$(1 + x + x^2 + \cdots + x^9)^6 .$$

Since $1+x+x^2+\cdots = 1/(1-x)$ the answer can be obtained also from the coefficient of x^{10} in the Maclaurin series of $1/(1-x)^6 = (1-x)^{-6}$. Since that includes six sequences of the form $0, 0, \cdots, 10, \cdots, 0$ we need to subtract 6, so the final answer is

$$\begin{aligned}\binom{-6}{10} - 6 &= \frac{(-6)(-7)(-8)(-9)(-10)(-11)(-12)(-13)(-14)(-15)}{10!} - 6 \\ &= 3003 - 6 = \boxed{2997} .\end{aligned}$$

6.7. Consider the polynomial $P(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{50} x^{49}$. If r is a 3rd root of unity different from 1 then $P(r) = c(1 + r + r^2)$, where $c = a_k + a_{k+3} + a_{k+6} + \cdots$ But $1+r+r^2 = (r^3-1)/(r-1) = 0$, so $P(r) = 0$. Analogous reasoning shows that $P(r) = 0$ for each 5th, 7th, 11th, 13th, 17th root of unity r different from 1. Since there are respectively $2 + 4 + 6 + 10 + 12 + 16 = 50$ such roots of unity we have that $P(r)$ is zero for 50 different values of r. But a 49-degree polynomial has only 49 roots, so $P(x)$ must be identically zero.

7.1. Let $f(n)$ be that number. Then we easily find $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 5, \ldots$ suggesting that $f(n) = F_{n+2}$ (shifted Fibonacci sequence). We prove this by showing that $f(n)$ verifies the same recurrence as the Fibonacci sequence. The subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n, and the ones containing n. The number of the ones not containing n is just $f(n-1)$. On the other hand the ones containing n cannot contain $n-1$, so their number equals $f(n-2)$. Hence $f(n) = f(n-1) + f(n-2)$, QED.

7.2. Let x_n be the number of regions in the plane determined by n "vee"s. Then $x_1 = 2$, and $x_{n+1} = x_n + 4n + 1$. We justify the recursion by noticing that the $(n+1)$th "vee" intersects each of the other "vee"s at 4 points, so it is divided into $4n+1$ pieces, and each piece divides one of the existing regions of the plane into two, increasing the total number of regions by $4n+1$. So the answer is

$$x_n = 2 + (4+1) + (4 \cdot 2 + 1) + \cdots + (4 \cdot (n-1) + 1) = 2n^2 - n + 1 .$$

7.3. Let x_n be the number of tilings of an $n \times 2$ rectangle by dominoes. We easily find $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the rightmost domino vertically and tile the rest of the rectangle in x_{n-1} ways, or we can place two horizontal dominoes to the right and tile the rest in x_{n-2} ways, so $x_n = x_{n-1} + x_{n-2}$. So the answer is the shifted Fibonacci sequence, $x_n = F_{n+1}$.

7.4. Let f_n denote the number of minimal selfish subsets of $\{1, 2, \ldots, n\}$. For $n = 1$ we have that the only selfish set of $\{1\}$ is $\{1\}$, and it is minimal. For $n = 2$ we have two selfish sets, namely $\{1\}$ and $\{1, 2\}$, but only $\{1\}$ is minimal. So $f_1 = 1$ and $f_2 = 1$. For $n > 2$ the number of minimal selfish subsets of $\{1, 2, \ldots, n\}$ not containing n is equal to f_{n-1}. On the other hand, for any minimal selfish set containing n, by removing n from the set and subtracting 1 from each remaining element we obtain a minimal selfish subset of $\{1, 2, \ldots, n\}$. Conversely, any minimal selfish subset of $\{1, 2, \ldots, n-2\}$ gives raise to a minimal selfish subset of $\{1, 2, \ldots, n\}$ containing n by the inverse procedure. Hence the number of minimal selfish subsets of $\{1, 2, \ldots, n\}$ containing n is f_{n-2}. Thus $f_n = f_{n-1} + f_{n-2}$, which together with $f_1 = f_2 = 1$ implies that $f_n = F_n$ (nth Fibonacci number.)

7.5. Assume that $b_1, b_2, \ldots, b_n$ is a derangement of the sequence $a_1, a_2, \ldots, a_n$. The element b_n can be any of $a_1, \ldots, a_{n-1}$, so there are $n - 1$ possibilities for its value. Once we have fixed the value of $b_n = a_k$ for some $k = 1, \ldots, n - 1$, the derangement can be of one of two classes: either $b_k = a_n$, or $b_k \neq a_n$. The first class coincides with the derangements of $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n-1}$, and there are D_{n-2} of them. The second class coincides with the derangements of $a_1, \ldots, a_{n-1}$ with a_k replaced with a_n, and there are D_{n-1} of them.

7.6. We have that α and β are the roots of the polynomial

$$(x - \alpha)(x - \beta) = x^2 - sx + p\,,$$

where $s = \alpha + \beta$, $p = \alpha\beta$.

We have that $s = a_1$ is an integer. Also, $2p = a_1^2 - a_2$ is an integer. The given sequence verifies the recurrence

$$a_{n+2} = s\,a_{n+1} - p\,a_n\,,$$

hence

$$2^{\lfloor\frac{n+1}{2}\rfloor} a_{n+2} = s\,2^{\lfloor\frac{n+1}{2}\rfloor} a_{n+1} - 2p\,2^{\lfloor\frac{n-1}{2}\rfloor} a_n\,.$$

From here we get the desired result by induction.

7.7. The general solution for the recurrence can be expressed using the roots of its characteristic polynomial

$$x^2 - \frac{10x}{3} + 1 = 0\,.$$

The roots are 3 and $1/3$, hence a general solution is $x_n = A \cdot 3^n + B \cdot 3^{-n}$. If the sequence converges then $A = 0$, and the condition $x_0 = 18$ yields $B = 18$, hence the sequence is $x_n = 18 \cdot 3^{-n}$, the limit is 0, and $x_1 = 18/3 = 6$.

8.1. The solution is based on the fact that $\sqrt{u^2} = |u|$. Letting $u = 1 \pm \sqrt{x-1}$ we have that $u^2 = x \pm 2\sqrt{x-1}$, hence the given function turns out to be:

$$f(x) = |1 + \sqrt{x-1}| + |1 - \sqrt{x-1}|\,,$$

Defined for $x \geq 1$.

The expression $1+\sqrt{x-1}$ is always positive, hence $|1+\sqrt{x-1}| = 1+\sqrt{x-1}$. On the other hand $|1-\sqrt{x-1}| = 1-\sqrt{x-1}$ if $1-\sqrt{x-1} \geq 0$ and $|1-\sqrt{x-1}| = -1+\sqrt{x-1}$ if $1-\sqrt{(x-1)} \leq 0$, hence

$$f(x) = \begin{cases} 1+\sqrt{x-1}+1-\sqrt{x-1} = 2 & \text{if } 1-\sqrt{x-1} \geq 0 \\ 1+\sqrt{x-1}-1+\sqrt{x-1} = 2\sqrt{x-1} & \text{if } 1-\sqrt{x-1} < 0 \end{cases}$$

So the function is equal to 2 if $1-\sqrt{x-1} \geq 0$, which happens for $1 \leq x \leq 2$. So $f(x) = 2$ (constant) in $[1, 2]$.

8.2. The desired value is the limit of the following sequence:

$$\begin{aligned} a_1 &= \sqrt{2} \\ a_2 &= \sqrt{2+\sqrt{2}} \\ a_3 &= \sqrt{2+\sqrt{2+\sqrt{2}}} \\ &\dots \end{aligned}$$

defined by the recursion $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$ $(n \geq 1)$.
First we must prove that the given sequence has a limit. To that end we prove

1. The sequence is bounded. More specifically, $0 < a_n < 2$ for every $n = 1, 2, \dots$. This can be proved by induction. It is indeed true for $a_1 = \sqrt{2}$. Next, if we assume that $0 < a_n < 2$, then $0 < a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = \sqrt{4} = 2$.
2. The sequence is increasing. In fact: $a_{n+1}^2 = 2+a_n > a_n+a_n = 2a_n > a_n^2$, hence $a_{n+1} > a_n$.

According to the *Monotonic Sequence Theorem*, every bounded monotonic (increasing or decreasing) sequence has a limit, hence a_n must have in fact a limit $L = \lim_{n\to\infty} a_n$.
Now that we know that the sequence has a limit L, by taking limits in the recursive relation $a_{n+1} = \sqrt{2+a_n}$, we get $L = \sqrt{2+L}$, hence $L^2 - L - 2 = 0$, so $L = 2$ or -1. Since $a_n > 0$ then $L \geq 0$, hence $L = 2$. Consequently:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}} = 2\,.$$

8.3. We will prove that the answer is $(3+\sqrt{5})/2$.
The value of the infinite continued fraction is the limit L of the sequence defined recursively $x_0 = 2207$, $x_{n+1} = 2207 - 1/x_n$, which exists because the sequence is decreasing (induction). Taking limits in both sides we get that $L = 2007 - 1/L$. Since $x_n > 1$ for all n (also proved by induction), we have that $L \geq 1$. If we call the answer r we have $r^8 = L$, so $r^8 + 1/r^8 = 2207$. Then $(r^4+1/r^4)^2 = r^8 + 2 + 1/r^8 = 2 + 2207 = 2209$, hence $r^4 + 1/r^4 = \sqrt{2209} = 47$. Analogously, $(r^2+1/r^2)^2 = r^4 + 2 + 1/r^4 = 2 + 47 = 49$, so $r^2 + 1/r^2 = \sqrt{49} = 7$. And

$(r+1/r)^2 = r^2+2+1/r^2 = 2+7 = 9$, so $r+1/r = \sqrt{9} = 3$. From here we get $r^2-3r+1=0$, hence $r = (3\pm\sqrt{5})/2$, but $r = L^{1/8} \geq 1$, so $r = (3+\sqrt{5})/2$.

8.4. That function coincides with $g(x) = 1/(1+x^2)$ at the points $x = 1/n$, and the derivatives of g at zero can be obtained from its Maclaurin series $g(x) = 1 - x^2 + x^4 - x^6 + \cdots$, namely $g^{(2k)}(0) = (-1)^k k!$ and $g^{(2k+1)}(0) = 0$. In order to prove that the result applies to f too we have to study their difference $h(x) = f(x) - g(x)$. We have that $h(x)$ is infinitely differentiable. Also $h(1/n) = 0$ for $n = 1, 2, 3, \dots$, hence $h(0) = \lim_{n\to\infty} h(1/n) = 0$. By Rolle's theorem, $h'(x)$ has zeros between the zeros of $h(x)$, hence $h'(0)$ is the limit of a sequence of zeros, so $h'(0) = 0$. The same is true about all derivatives of h at zero. This implies that $f^{(k)}(0) = g^{(k)}(0)$ for every $k = 1, 2, 3, \dots$, hence $f^{(2k)}(0) = (-1)^k k!$ and $f^{(2k+1)}(0) = 0$.

8.5. By looking at the graph of the function $y = 1/x$ we can see that

$$\int_n^{2n} \frac{1}{x}\,dx < \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} < \int_{n-1}^{2n-1} \frac{1}{x}\,dx\,.$$

We have

$$\int_n^{2n} \frac{1}{x}\,dx = \ln(2n) - \ln n = \ln 2\,,$$

$$\int_{n-1}^{2n-1} \frac{1}{x}\,dx = \ln(2n-1) - \ln(n-1) = \ln\left\{\frac{2n-1}{n-1}\right\} \underset{n\to\infty}{\longrightarrow} \ln 2\,.$$

Hence by the Squeeze Theorem, the desired limit is $\ln 2$.

8.6. Let P be the limit. Then

$$\ln(P) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \ln\left(1 + \frac{k}{n}\right)$$

That sum is a Riemann sum for the following integral:

$$\int_0^1 \ln(1+x)\,dx = [(1+x)(\ln(1+x) - 1)]_0^1 = 2\ln 2 - 1\,.$$

Hence $P = e^{2\ln 2 - 1} = 4/e$.

8.7. The series on the left is $xe^{-x^2/2}$. Since the terms of the second sum are non-negative, we can interchange the sum and integral:

$$\int_0^\infty xe^{-x^2/2} \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2}\,dx = \sum_{n=0}^{\infty} \int_0^\infty xe^{-x^2/2} \frac{x^{2n}}{2^{2n}(n!)^2}\,dx$$

The term for $n = 0$ is

$$\int_0^\infty xe^{-x^2/2}\,dx = \left[-e^{-x^2/2}\right]_0^\infty = 0 - (-1) = 1\,.$$

Next, for $n \geq 1$, integrating by parts:

$$\int_0^\infty x^{2n} \left(xe^{-x^2/2}\right) dx = \underbrace{\left[-x^{2n}e^{-x^2/2}\right]_0^\infty}_{0} + 2n \int_0^\infty x^{2(n-1)} \left(xe^{-x^2/2}\right) dx \,.$$

Thus, by induction

$$\int_0^\infty x^{2n} \left(xe^{-x^2/2}\right) dx = 2 \cdot 4 \cdot 6 \cdots 2n \,.$$

Hence the integral is

$$\sum_{n=0}^\infty \frac{1}{2^n n!} = e^{1/2} = \sqrt{e} \,.$$

8.8. The answer is affirmative, in fact *any* real number r is the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$. First assume

$$\sqrt[3]{n} \leq r + \sqrt[3]{m} < \sqrt[3]{n+1} \,,$$

which can be accomplished by taking $n = \lfloor (r + \sqrt[3]{m})^3 \rfloor$. We have

$$\begin{aligned} 0 \leq r - (\sqrt[3]{n} - \sqrt[3]{m}) &< \sqrt[3]{n+1} - \sqrt[3]{n} \\ &= \frac{1}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}} \,. \end{aligned}$$

Since the last expression tends to 0 as $n \to \infty$, we have that

$$r = \lim_{m \to \infty} \left\{ \sqrt[3]{\lfloor (r + \sqrt[3]{m})^3 \rfloor} - \sqrt[3]{m} \right\} \,.$$

8.9. From $f(f(x)) = 1/f(x)$ we get that $f(y) = 1/y$ for all $y \in f(\mathbb{R})$. Hence $f(999) = 1/999$. Since f is continuous it takes all possible values between $1/999$ and 999, in particular $500 \in f(\mathbb{R})$. Hence $f(500) = 1/500$.

8.10. Consider the function $g : [0, 1998/1999] \to \mathbb{R}$, $g(x) = f(x) - f(x + 1/999)$. Then g is continuous on $[0, 1998/1999]$, and verifies

$$\sum_{k=0}^{1998} g(k/1999) = f(1) - f(0) = 0 \,.$$

Since the sum is zero it is impossible that all its terms are positive or all are negative, so either one is zero, or there are two consecutive terms with opposite signs. In the former case, $g(k/1999) = 0$ for some k, so $f(k/1999) = f((k+1)/1999)$ and we are done. Otherwise, if there are two consecutive terms $g(k/1999)$ and $g((k+1)/1999)$ with different signs, then for some $x \in [k/1999, (k+1)/1999]$ we have $g(x) = 0$, hence $f(x) = f(x + 1/1999)$, and we are also done.

8.11. The answer is $c \geq 1/2$.

In fact, the given inequality can be written like this:

$$e^{cx^2} - \frac{e^x + e^{-x}}{2} \geq 0\,.$$

The Taylor expansion of the left hand side is

$$e^{cx^2} - \frac{e^x + e^{-x}}{2} = \left(c - \frac{1}{2}\right)x^2 + \left(\frac{c^2}{2!} - \frac{1}{4!}\right)x^4 + \left(\frac{c^3}{3!} - \frac{1}{6!}\right)x^6 + \cdots$$

We see that for $c \geq 1/2$ all the coefficients are non-negative, and the inequality holds.

On the other hand, if $c < 1/2$ we have

$$\lim_{x\to 0} \frac{e^{cx^2} - \frac{e^x+e^{-x}}{2}}{x^2} = c - \frac{1}{2} < 0\,,$$

so in a neighborhood of 0 the numerator must become negative, and the inequality does not hold.

8.12. There is no such sequence. If they were convergent their sum would be convergent too, but by the AM-GM inequality we have:

$$\sum_{n=1}^{\infty}\left(a_n + \frac{1}{n^2 a_n}\right) \geq \sum_{n=1}^{\infty}\frac{2}{n} = \infty\,.$$

9.1. We divide the set into n-classes $\{1,2\},\{3,4\},\dots,\{2n-1,2n\}$. By the pigeonhole principle, given $n+1$ elements, at least two of them will be in the same class, $\{2k-1,2k\}$ $(1 \leq k \leq n)$. But $2k-1$ and $2k$ are relatively prime because their difference is 1.

9.2. For each odd number $\alpha = 2k-1$, $k = 1,\dots,n$, let C_α be the set of elements x in S such that $x = 2^i\alpha$ for some i. The sets $C_1, C_3, \dots, C_{2n-1}$ are a classification of S into n classes. By the pigeonhole principle, given $n+1$ elements of S, at least two of them will be in the same class. But any two elements of the same class C_α verify that one is a multiple of the other one.

9.3. The given set can be divided into 18 subsets $\{1\}$, $\{4,100\}$, $\{7,97\}$, $\{10,94\}$,..., $\{49,55\}$, $\{52\}$. By the pigeonhole principle two of the numbers will be in the same set, and all 2-element subsets shown verify that the sum of their elements is 104.

9.4. For $k = 1,2,\dots,8$, look at the digit used in place k for each of the 4 given elements. Since there are only 3 available digits, two of the elements will use the same digit in place k, so they coincide at that place. Hence at each place, there are at least two elements that coincide at that place. Pick any pair of such elements for each of the 8 places. Since there are 8 places we will have 8 pairs of elements, but there are only $\binom{4}{2} = 6$ two-element subsets in a 4-element set, so two of the pairs will be the same pair, and the elements of that pair will coincide in two different places.

9.5. Let a_j the number of games played from the 1st through the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Likewise, $b_j = a_j + 14$, $j = 1, \ldots, 30$ is also an increasing sequence of distinct positive integers with $15 \leq b_j \leq 59$. The 60 positive integers $a_1, \ldots, a_{30}, b_1, \ldots, b_{30}$ are all less than or equal to 59, so by the pigeonhole principle two of them must be equal. Since the a_j's are all distinct integers, and so are the b_j's, there must be indices i and j such that $a_i = b_j = a_j + 14$. Hence $a_i - a_j = 14$, i.e., exactly 14 games were played from day $j + 1$ through day i.

9.6. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $i = 0, \ldots, n$, put $s_i = x_1 + \cdots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \ldots, \{s_n\}$ into ascending order, and call the result $t_0, \ldots, t_n$. Since $0 = t_0 \leq \cdots \leq t_n < 1$, the differences

$$t_1 - t_0, \ldots, t_n - t_{n-1}, 1 - t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n + 1)$; if it is anything other than $1 - t_n$, it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \leq i < j \leq n$. Put $S = \{x_{i+1}, \ldots, x_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$; then

$$\begin{aligned} \left| m + \sum_{s \in S} s \right| &= |m + s_j - s_i| \\ &= |\{s_j\} - \{s_i\}| \\ &\leq \frac{1}{n+1}, \end{aligned}$$

as desired. In case $1 - t_n \leq 1/(n + 1)$, we take $S = \{x_1, \ldots, x_n\}$ and $m = -\lceil s_n \rceil$, and again obtain the desired conclusion.

9.7. A set of 10 elements has $2^{10} - 1 = 1023$ non-empty subsets. The possible sums of at most ten two-digit numbers cannot be larger than $10 \cdot 99 = 990$. There are more subsets than possible sums, so two different subsets S_1 and S_2 must have the same sum. If $S_1 \cap S_2 = \emptyset$ then we are done. Otherwise remove the common elements and we get two non-intersecting subsets with the same sum.

9.8. Writing $y_i = \tan x_i$, with $-\frac{\pi}{2} \leq x_i \leq \frac{\pi}{2}$ $(i = 1, \ldots, 7)$, we have that

$$\frac{y_i - y_j}{1 + y_i y_j} = \tan(x_i - x_j),$$

so all we need is to do is prove that there are x_i, x_j such that $0 \leq x_i - x_j \leq \frac{\pi}{6}$. To do so we divide the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ into 6 subintervals each of length $\frac{\pi}{6}$. By the box principle, two of the x_is will be in the same subinterval, and their difference will be not larger than $\frac{\pi}{6}$, as required.

9.9. Classify the numbers by their reminder when divided by 3. Either three of them will yield the same reminder, and their sum will be a multiple of 3, or there will be at least a number x_r for each possible reminder $r = 0, 1, 2$, and their sum $x_0 + x_1 + x_2$ will be a multiple of 3 too.

9.10. We must prove that there are positive integers n, k such that

$$2009 \cdot 10^k \leq 2^n < 2010 \cdot 10^k \,.$$

That double inequality is equivalent to

$$\log_{10}(2009) + k \leq n \log_{10}(2) < \log_{10}(2010) + k \,.$$

where $\log_{10}$ represents the decimal logarithm. Writing $\alpha = \log_{10}(2009) - 3$, $\beta = \log_{10}(2010) - 3$, we have $0 < \alpha < \beta < 1$, and the problem amounts to showing that for some integer n, the fractional part of $n \log_{10}(2)$ is in the interval $[\alpha, \beta)$. This is true because $\log_{10}(2)$ is irrational, and the integer multiples of an irrational number are dense modulo 1 (their fractional parts are dense in the interval $[0, 1)$).

9.11. Let F be the face with the largest number m of edges. Then for the $m + 1$ faces consisting of F and its m neighbors the possible number of edges are $3, 4, \ldots, m$. These are only $m - 2$ possibilities, hence the number of edges must occur more than once.

10.1. After rationalizing we get a telescopic sum:

$$\begin{aligned} \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} &= (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{100}-\sqrt{99}) \\ &= 10 - 1 = 9 \,. \end{aligned}$$

10.2. We have

$$\frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} = \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}} \,.$$

So this is a telescopic sum:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}} \right\} \\ &= \lim_{n\to\infty} \left\{ 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}} \right\} \\ &= 3 - 1 = 2 \,. \end{aligned}$$

10.3. This is a telescopic product:

$$\frac{n^3 - 1}{n^3 + 1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} = \frac{(n-1)\{n\,(n+1)+1\}}{(n+1)\{(n-1)\,n+1\}} \,,$$

hence

$$\begin{aligned}\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1} &= \lim_{N\to\infty}\prod_{n=2}^{N}\frac{(n-1)\{n\,(n+1)+1\}}{(n+1)\{(n-1)\,n+1\}} \\ &= \lim_{N\to\infty}\frac{2\{N(N+1)+1\}}{3N(N+1)} = \frac{2}{3}\,.\end{aligned}$$

10.4. We have

$$\begin{aligned}\frac{n}{n^4+n^2+1} &= \frac{n}{(n^2+1)^2-n^2} \\ &= \frac{1/2}{n^2-n+1} - \frac{1/2}{n^2+n+1} \\ &= \frac{1/2}{(n-1)\,n+1} - \frac{1/2}{n\,(n+1)+1}\,.\end{aligned}$$

So

$$\begin{aligned}\sum_{n=0}^{N}\frac{n}{n^4+n^2+1} &= \frac{1/2}{(-1)\cdot 0+1} - \frac{1/2}{0\cdot 1+1} + \frac{1/2}{0\cdot 1+1} - \frac{1/2}{1\cdot 2+1} + \cdots \\ &\qquad \cdots + \frac{1/2}{(N-1)\,N+1} - \frac{1/2}{N\,(N+1)+1} \\ &= \frac{1}{2} - \frac{1/2}{N\,(N+1)+1} \quad \underset{N\to\infty}{\longrightarrow} \quad \frac{1}{2}\,.\end{aligned}$$

Hence, the sum is 1/2.

11.1. Consider the function

$$f(x,y,z) = T(x,y,z) + T(y,z,x) + T(z,x,y) \quad = 4x^2 + 4y^2 + 4z^2\,.$$

On the surface of the planet that function is constant and equal to $4\cdot 20^2 = 1600$, and its average on the surface of the planet is 1600. Since the equation of a sphere with center in $(0,0,0)$ is invariant by rotation of coordinates, the three terms $T(x,y,z)$, $T(y,z,x)$, $T(z,x,y)$ have the same average value $\overline{T}$ on the surface of the planet, hence $1600 = 3\overline{T}$, and $\overline{T} = 1600/3$.

11.2. Writing $\alpha = \sqrt{2}$, the integrand $f(x) = 1/(1+\tan^\alpha x)$ verifies the following symmetry:

$$\begin{aligned}f(x) + f(\tfrac{\pi}{2}-x) &= \frac{1}{1+\tan^\alpha x} + \frac{1}{1+\cot^\alpha x} \\ &= \frac{1}{1+\tan^\alpha x} + \frac{\tan^\alpha x}{1+\tan^\alpha x} \\ &= 1\,.\end{aligned}$$

On the other hand, making the substitution $u = \frac{\pi}{2} - x$:

$$\int_0^{\frac{\pi}{2}} f(\tfrac{\pi}{2} - x)\,dx = -\int_{\frac{\pi}{2}}^0 f(u)\,du = \int_0^{\frac{\pi}{2}} f(x)\,dx = I\,,$$

where I is the desired integral. So:

$$2I = \int_0^{\frac{\pi}{2}} \left\{f(x) + f(\tfrac{\pi}{2} - x)\right\}\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}\,.$$

Hence $I = \dfrac{\pi}{4}$.

11.3. The first player does have a winning strategy: place the first penny exactly on the center of the table, and then after the second player places a penny, place the next penny in a symmetric position respect to the center of the table. After each of the first player's move the configuration of pennies on the table will have radial symmetry, so if the second player can still place a penny somewhere on the table, the radially symmetric position respect to the center of the table will still not be occupied and the first player will also be able to place a penny there.

12.1. Let A be the set of positive integers no exceeding 1000 that are divisible by 7, and let B the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of positive integers not exceeding 1000 that are divisible by 7 or 11. The number of elements in A is $|A| = \left\lfloor \dfrac{1000}{7} \right\rfloor = 142$. The number of elements in B is $|B| = \left\lfloor \dfrac{1000}{11} \right\rfloor = 90$. The set of positive integers not exceeding 1000 that are divisible by 7 *and* 11 is $A \cap B$, and the number of elements in there is $|A \cap B| = \left\lfloor \dfrac{1000}{7 \cdot 11} \right\rfloor = 12$. Finally, by the inclusion-exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220\,.$$

12.2. (This is equivalent to finding the number of onto functions from a n-element set to an k-element set.) If we remove the restriction "using all k flavors" then the first child can receive an ice-cream of any of the k available flavors, the same is true for the second child, and the third, etc. Hence the number of ways will be the product $k \cdot k \cdots k = k^n$.

Now we need to eliminate the distributions of ice-cream cones in which at least one of the flavors is unused. So let's call A_i = set of distributions of ice-creams in which at least the ith flavor is never used. We want to find the number of elements in the union of the A_i's (and later subtract it from k^n). According to the Principle of Inclusion-Exclusion that number is the sum of the elements in each of the A_i's, minus the sum of the elements of all possible intersections of two of the A_i's, plus the sum of the elements in all possible intersections of three of those sets, and so on. We have:

$|A_i| = (k-1)^n$ ($k-1$ flavors distributed among n children)

$|A_i \cap A_j| = (k-2)^n$ ($k-2$ flavors among n children)

$|\text{any triple intersection}| = (k-3)^n$ ($k-3$ flavors among n children)

and so on. On the other hand there are k sets A_i, $\binom{k}{2}$ intersections of two sets, $\binom{k}{3}$ intersections of three sets, etc. Hence the number of distributions of flavors that miss some flavor is

$$\binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n - \cdots \pm \binom{k}{k}0^n\,,$$

and the number of distributions of flavors that do not miss any flavor is k^n minus the above sum, i.e.:

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots \mp \binom{k}{k}0^n = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n\,.$$

12.3. Let's denote P_i the set of permutations fixing element a_i. The set of non-derangements are the elements of the union $P_1 \cup P_2 \cup \cdots \cup P_n$, and its number can be found using the inclusion-exclusion principle:

$$|P_1 \cup P_2 \cup \cdots \cup P_n| = \sum_i |P_i| - \sum_{i\neq j}|P_i \cap P_j| + \sum_{i\neq j\neq k\neq i}|P_i \cap P_j \cap P_k| - \cdots$$

Each term of that expression is the number of permutations fixing a certain number of elements. The number of permutations that fix m given elements is $(n-m)!$, and since there are $\binom{n}{m}$ ways of picking those m elements, the corresponding sum is $\binom{n}{m}(n-m)! = \frac{n!}{m!}$. Adding and subtracting from the total number of permutations $n!$, we get

$$D_n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n\frac{n!}{n!} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right)\,.$$

13.1. - *First Solution:* The number of subsets of $\{1, 2, \dots, n\}$ with odd cardinality is

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

The number of subsets of even cardinality is cardinality is

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

The difference is

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = (1-1)^n = 0\,.$$

- *Second Solution:* We define a bijection between the subsets with odd cardinality and those with even cardinality in the following way: if S is a subset with an odd number of elements we map it to $S' = S \cup \{1\}$ if $1 \notin S$, or $S' = S \setminus \{1\}$ if $1 \in S$.

13.2. (*Note:* see the section about recurrences for an alternate solution—here we use a combinatorial argument.)
We will prove that the number of k-element subsets of $\{1, 2, \dots, n\}$ with no consecutive elements equals the number of all k-element subsets of $\{1, 2, \dots, n-k+1\}$. To do so we define a 1-to-1 correspondence between both kinds of subsets in the following way: to each subset $\{a_1, a_2, \dots, a_k\}$ ($a_1 < a_2 < \cdots < a_k$) of $\{1, 2, \dots, n\}$ without consecutive elements we assign the subset $\{a_1, a_2-1, \dots, a_i-i+1, \dots, a_k-k+1\}$ of $\{1, 2, \dots, n-k+1\}$. We see that the mapping is in fact a bijection, with the inverse defined $\{b_1, b_2, \dots, b_i, \dots, b_k\} \mapsto \{b_1, b_2+1, \dots, b_i+i-1, \dots, b_k+k-1\}$. Hence, the number of k-element subsets of $\{1, 2, \dots, n\}$ with no consecutive elements is $\binom{n-k+1}{k}$. Note that the formula is valid also for $k = 0$ and $k = 1$.
Hence, the total number of subsets of $\{1, 2, \dots, n\}$ with no consecutive elements is the sum

$$\sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k+1}{k}.$$

This sum is known to be equal to the shifted Fibonacci number F_{n+1}.

13.3. The probability of John getting n heads is the same as getting n tails. So the problem is equivalent to asking the probability of John getting as many tails as the number of heads gotten by Peter, and that is the same as both getting jointly a total of 20 heads. So the probability asked is the same as that of getting 20 heads after tossing $25 + 20 = 45$ coins, i.e.:

$$\binom{45}{20} 2^{-45}.$$

(That is $0.09009314767\dots$)

13.4. Let x be the distance from the man to the edge measured in steps. For $n > 0$, let P_n the probability that the drunken man ends up over the edge when he starts at $x = n$ steps from the cliff. Then $P_1 = (1-p) + pP_2$. We now rewrite P_2 in the following way. Paths from $x = 2$ to $x = 0$ can be broken into two parts: a path that goes from $x = 2$ to $x = 1$ for the first time, and a path that goes from $x = 1$ to $x = 0$. The probability of the latter is P_1, because the situation is exactly the same as at the beginning. The probability of the former is also P_1, because the structure of problem is identical to the original one with x increased by 1. Since both probabilities are independent, we have $P_2 = P_1^2$. Hence

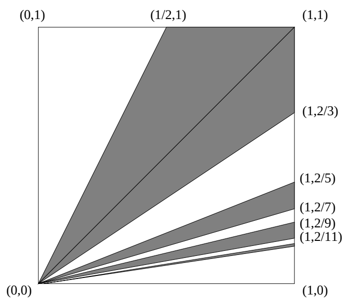
$$P_1 = (1-p) + pP_1^2.$$

Solving this equation we get two solutions, namely $P_1 = 1$ and $P_1 = \dfrac{1-p}{p}$.

We now need to determine which solution goes with each value of p. For $p = 1/2$ both solutions agree, and then $P_1 = 1$. For $p = 0$ we have $P_1 = 1$, and when $p = 1$, $P_1 = 0$, because the man always walks away from the cliff. For $0 < p < 1/2$ the second solution is impossible, so we must have $P_1 = 1$. For $1/2 < p \leq 1$ we have that the second solution is strictly less than 1. By continuity P_1 cannot take both values 1 and $\dfrac{1-p}{p}$ on the interval $(1/2, 1]$, so since $P_1 = 0$ for $p = 1$, we must have $P_1 = \dfrac{1-p}{p}$ on that interval. Hence, the probability of escaping the cliff is

$$1 - P_1 = \begin{cases} 0 & \text{if } 0 \leq p \leq \frac{1}{2}\,, \\ 2 - \dfrac{1}{p} & \text{if } \frac{1}{2} < p \leq 1\,. \end{cases}$$

13.5. The set $\{(X, Y) \mid X, Y \in (0, 1)\}$ is the unit square square with corners in $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, whose area is 1. The desired probability will be the area of the subset of points (X, Y) in that square such that the closest integer to X/Y is odd. The condition "the closest integer to X/Y is odd" is equivalent to $|X/Y - (2n + 1)| < 1/2$ for some non-negative integer n, or equivalently, $2n + 1/2 < X/Y < 2n + 3/2$. That set of points is the space in the unit square between the lines $Y = \frac{2}{4n+1}X$ and $Y = \frac{2}{4n+3}X$. That area can be decomposed into triangles and computed geometrically (see figure.)



For $n = 0$ the area is $1/4 + 1/3$. For $n \geq 1$ it is $\frac{1}{4n+1} - \frac{1}{4n+3}$. Hence the total area is

$$P = \frac{1}{4} + \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

We can find the sum of that series using the Gregory-Leibniz series:

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}\,,$$

and we get

$$P = -\frac{1}{4} + \frac{\pi}{4} = 0.5353981635\ldots\,.$$

13.6. Let L_1, L_2, and L_3 the lengths of those three arcs. We have $L_1 + L_2 + L_3 = 2\pi$. On the other hand the expected value of several random variables is additive:

$$E[L_1 + L_2 + L_3] = E[L_1] + E[L_2] + E[L_3]\,.$$

By symmetry $E[L_1] = E[L_2] = E[L_3]$, and the sum must be 2π, hence each expected value is $\frac{2\pi}{3}$. So, this is the answer, the expected value of the arc containing the point $(1,0)$ is $\frac{2\pi}{3}$.

13.7. The problem is equivalent to dropping two random points on an interval of length 9 inches. By identifying the two endpoints of the interval the problem becomes identical to dividing a circle of length 9 at three points chosen at random. The expected values of their lengths must add to 9 inches, and by symmetry they should be the same, so each expected value must be 3 inches. Hence, this is the answer, the average length of the fragment with the blue dot will be 3 inches.

13.8. Label the points $x_0, x_1, x_2, \dots, x_n$ ($x_n \equiv x_0$.) Then the center of the circle will *not* be in the polygon if and only if one of the arcs defined by two consecutive points (measured counterclockwise) is greater than π. Let E_k ($k = 0, \dots, n-1$) be the event "the arc from x_k to the point next to x_k (counterclockwise) is larger than π." The probability of each E_k is obviously $\frac{1}{2^{n-1}}$, because for it to happen all points other than x_k must lie in the same half-circle ending at x_k. On the other hand, the events $E_0, E_1, \dots, E_{n-1}$ are incompatible, i.e., no two of them can happen at the same time. Then, the probability of one of them happening is the sum of the probabilities:

$$P(E_0 \text{ or } E_1 \text{ or } \cdots \text{ or } E_{n-1}) = P(E_0) + P(E_1) + \cdots + P(E_{n-1}) = \frac{n}{2^{n-1}}\,.$$

Hence, the desired probability is $1 - \dfrac{n}{2^{n-1}}$.

14.1. Let $$I = \frac{10^{20000} - 3^{200}}{10^{100} + 3} = \frac{(10^{100})^{200} - 3^{200}}{10^{100} + 3} = (10^{100})^{199} - (10^{100})^{198} \cdot 3 + \cdots + 10^{100} \cdot 3^{198} - 3^{199}$$
so I is an integer. On the other hand since $\frac{3^{200}}{10^{100}+3} < 1$ we have that $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor = I$. Finally the rightmost digit of I can be found as the 1-digit number congruent to $-3^{199} \pmod{10}$. The sequence $3^n \bmod 10$ has period 4 and $199 = 3 + 4 \cdot 49$, hence $-3^{199} \bmod 10 = -3^3 \bmod 10 = -27 \bmod 10 = 3$. Hence the units digit of I is 3.

14.2. Let α be any (say the smallest) acute angle of a right triangle with sides 3, 4 and 5 (or any other Pythagorean triple). Next, place an infinite sequence of points on the unit circle at coordinates $(\cos(2n\alpha), \sin(2n\alpha))$, $n = 0, 1, 2, \dots$ (The sequence contains in fact infinitely many points because α cannot be a rational multiple of π.) The distance from $(\cos(2n\alpha), \sin(2n\alpha))$ to $(\cos(2m\alpha), \sin(2m\alpha))$ is $2\sin(|n-m|\alpha)$, so all we need to prove is that $\sin(k\alpha)$ is rational for any k. This can be done by induction using that $\sin\alpha$ and $\cos\alpha$ are rational, and if $\sin u$, $\cos u$, $\sin v$ and $\cos(v)$ are all rational so are $\sin(u+v) = \sin u \cos v + \cos u \sin v$ and $\cos(u+v) = \cos u \cos v - \sin u \sin v$.

14.3. We can prove the first part by way of contradiction. Assume that we have colored the points of the plane with three colors such that any two points at distance 1 have different colors. Consider any two points A and B at distance $\sqrt{3}$ (see figure 3). The circles of radius 1 and center A and B meet at two points P and Q, forming equilateral triangles APQ and BPQ. Since the vertices of each triangle must have different colors that forces A and B to have the same color. So any two points at distance $\sqrt{3}$ have the same color. Next consider a triangle DCE with $CD = CE = \sqrt{3}$ and $DE = 1$. The points D and E must have the same color as C, but since they are at distance 1 they should have different colors, so we get a contradiction.

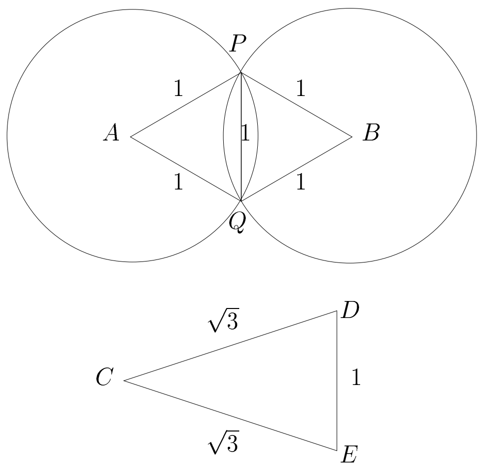


FIGURE 3

For the second part, if we replace "three" by "nine" then we can color the plane with nine different colors so that any two points at distance 1 have different colors: we can arrange them periodically in a grid of squares of size $2/3 \times 2/3$ as shown in figure 4. If two points P and Q have the same color then either they belong to the same square and $PQ < (2/3)\sqrt{2} < 1$, or they belong to different squares and $PQ \geq 4/3 > 1$.

14.4. Since the values are positive integers, one of them, say n, will be the smallest one. Look at any square with that value n. Since the values of its four neighbors must be at least n and their average is n, all four will have value n. By the same reasoning the neighbors of these must be n too, and so on, so all the squares must have the same value n.

14.5. One or two points are obviously insufficient, but three can do it. Choose $\alpha \in \mathbb{R}$ so that α^2 is irrational, for example $\alpha = \sqrt[3]{2}$. Use punches at $A = (-\alpha, 0)$, $B = (0, 0)$,

A	B	C	A	B	C
D	E	F	D	E	F
G	H	I	G	H	I
A	B	C	A	B	C
D	E	F	D	E	F

FIGURE 4

and $C = (\alpha, 0)$. If $P = (x, y)$ then

$$AP^2 - 2BP^2 + CP^2 = (x+\alpha)^2 + y^2 - 2(x^2 + y^2) + (x-\alpha)^2 + y^2 = 2\alpha^2$$

is irrational, so AP, BP, CP cannot all be rational.

14.6. We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)(f(n+1) - f(n)),$$

hence

$$\begin{aligned} f(n+2) &= (n+2)(f(n+1) - f(n)) + f(n+1) \\ &= (n+3)f(n+1) - (n+2)f(n), \end{aligned}$$

and we can take $P(x) = x + 3$, $Q(x) = -x - 2$.

14.7. A conspiratorial subset of $S = \{1, 2, \dots, 16\}$ has at most two elements from $T = \{1, 2, 3, 5, 7, 11, 13\}$, so it has at most $2 + 16 - 7 = 11$ numbers. On the other hand all elements of $S \setminus T = \{4, 6, 8, 9, 10, 12, 14, 15, 16\}$ are multiple of either 2 or 3, so adding 2 and 3 we obtain the following 11-element conspiratorial subset:

$$\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}.$$

Hence the answer is 11.

14.8. The statement is true. Let ϕ any bijection on F with no fixed points ($\phi(x) \neq x$ for every x), and set $x * y = \phi(x)$. Then

(i) $x * z = y * z$ means $\phi(x) = \phi(y)$, and this implies $x = y$ because ϕ is a bijection.

(ii) We have $x * (y * z) = \phi(x)$ and $(x * y) * z = \phi(\phi(x))$, which cannot be equal because that would imply than $\phi(x)$ is a fixed point of ϕ.

14.9. For a given partition π of $\{1,2,3,4,5,6,7,8,9\}$, no more than three different values of $\pi(x)$ are possible (four would require one part each of size at least 1,2,3,4, and that's already more than 9 elements). If no such x, y exist, each pair $(\pi(x), \pi'(x))$ occurs for at most 1 element of x, and since there are only 3×3 possible pairs, each must occur exactly once. In particular, each value of $\pi(x)$ must occur 3 times. However, clearly any given value of $\pi(x)$ occurs $k\pi(x)$ times, where k is the number of distinct parts of that size. Thus $\pi(x)$ can occur 3 times only if it equals 1 or 3, but we have three distinct values for which it occurs, contradiction.

14.10. The answer is $2n-3$.
Note that this number is attained with $S = \{1,2,3,\dots,n\}$ because

$$2A_S = \{3,4,\dots,n,n+1,n+2,\dots,2n-1\}$$

has cardinality $2n-3$. It remains to prove that $2n-3$ is in fact minimum. That is so because for any $A = \{a_1, a_2, \dots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ we have that

$$\underbrace{a_1 + a_2 < a_1 + a_3 < a_1 + a_4 < \cdots < a_1 + a_n}_{n-1} < \underbrace{a_2 + a_n < \cdots < a_{n-1} + a_n}_{n-2}$$

are $2n-3$ distinct numbers.

14.11. We can group the terms of the sequence in the following way:

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1}_{b_0} + \underbrace{(a_2 + a_3)}_{b_1} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{b_2} + \cdots + \underbrace{(a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})}_{b_k} + \cdots$$

The condition implies that $b_k \le b_{k+1}$ for every $k \ge 0$, hence the sequence diverges.

14.12. Alice adds the values of the coins in odd positions 1st, 3rd, 5th, etc., getting a sum S_{odd}. Then she does the same with the coins placed in even positions 2nd, 4th, 6th, etc., and gets a sum S_{even}. Assume that $S_{odd} \ge S_{even}$. Then she will pick all the coins in odd positions, forcing Bob to pick only coins in the even positions. To do so she stars by picking the coin in position 1, so Bob can pick only the coins in position 2 or 50. If he picks the coin in position 2, Alice will the pick coin in position 3, if he picks the coin in position 50 she picks the coin in position 49, and so on, with Alice always picking the coin at the same side as the coin picked by Bob.
If $S_{odd} \le S_{even}$, then Alice will use a similar strategy ensuring that she will end up picking all the coins in the even positions, and forcing Bob to pick the coins in the odd positions—this time she will pick first the 50th coin, and then at each step she will pick a coin at the same side as the coin picked by Bob.

14.13. By contradiction. If the equality $f(x) = x$ never holds then $f(x) > x$ for every x, or $f(x) < x$ for every x. Then $f(f(x)) > f(x) > x$ for every x, or $f(f(x)) < f(x) < x$ for every x, contradicting the hypothesis that $f \circ f$ has a fixed point.

PUTNAM TRAINING
EASY PUTNAM PROBLEMS

(Last updated: February 26, 2014)

REMARK. This is a list of exercises on Easy Putnam Problems —Miguel A. Lerma

EXERCISES

1. **2013-A1.** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

2. **2013-B1.** For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n + 1) = (-1)^n c(n)$. Find the value of
$$\sum_{n=1}^{2013} c(n)c(n+2)\,.$$

3. **2012-A1.** Let $d_1, d_2, \dots, d_{12}$ be real numbers in the interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i,d_j, d_k are the side lengths of an acute triangle.

4. **2012-B1.** Let S be the class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:
   (i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S.
   (ii) If $f(x)$ and $g(s)$ are in S, then functions $f(x) + g(x)$ and $f(g(x))$ are in S;
   (iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.
   Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

5. **2011-B1.** Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that
$$\varepsilon < |h\sqrt{m} - k\sqrt{n}| < 2\varepsilon.$$

6. **2010-A1.** Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

7. **2010-B1.** Is there an infinite sequence of real numbers $a_1, a_2, a_3, \dots$ such that
$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

8. 2010-B2. Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

9. 2009-A1. Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

10. 2009-B1. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

11. 2008-A1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

12. 2008-A2. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

13. 2008-B1. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

14. 2007-A1. Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

15. 2007-B1. Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$. [Note: one must assume f is nonconstant.]

16. 2006-A1. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

17. 2006-B2. Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

18. 2005-A1. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

19. 2005-B1. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.)

20. 2004-A1. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

21. 2004-B2. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

22. 2003-A1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

23. 2002-A1. Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

24. 2002-A2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

25. 2001-A1. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

26. 2000-A2. Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

27. 1999-A1. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

28. 1998-A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

29. 1997-A5. Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \dots, a_n)$ such that $1/a_1 + 1/a_2 + \dots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

Hints

1. If the the numbers of the faces having a common vertex have different numbers, what can we say about their sum?

2. Telescoping.

3. Three numbers $0 < a \leq b \leq c$ are the side lengths of an acute triangle precisely if $a^2 + b^2 > c^2$.

4. $e^{u+v} = e^u e^v$.

5. There is some rational number between $\frac{3\varepsilon}{hk}$ and $\frac{4\varepsilon}{hk}$.

6. For k to be as large as possible the "boxes" must be "small".

7. One approach is to use the Cauchy-Schwartz inequality for some appropriately chosen values of k. Another approach is to look at how the LHS grows with m depending on the values of the a_k.

8. By looking at Pythagorean triples we get a reasonable conjecture about what the smallest possible value of AB could be. Then use $|AC - BC| \leq AB$, with equality if and only if A, B, C are collinear.

9. Find relations among the values of the function at nine points forming a 2 by 2 square grid.

10. Induction.

11. Try successively $(x, y, z) = (0, 0, 0)$, $(x, y, z) = (x, 0, 0)$, $(x, y, z) = (x, y, 0)$.

12. Try either getting two equal rows, or all rows summing zero.

13. How can we find the center of a circle if we are given some points on that circle?

14. Considered different cases depending on how each curve intersects the line $y = x$.

15. Consider $f(f(n) + 1) \bmod f(n)$.

16. Change to cylindrical coordinates.

17. Pigeonhole Principle.

18. Induction. The base case is $1 = 2^0 3^0$. The induction step depends on the parity of n. If n is even, divide by 2. If it is odd, subtract a suitable power of 3.

19. Note that $\lfloor 2a \rfloor = 2\lfloor a \rfloor$ or $\lfloor 2a \rfloor = 2\lfloor a \rfloor + 1$ depending on whether the fractional part of a is in $[0, 0.5)$ or $[0.5, 1)$.

20. Assume that $S(N)$ jumps abruptly from less than $4/5$ to more that $4/5$ at some point and find a contradiction.

21. Rewrite the inequality $\dfrac{(m+n)!}{m!\,n!}\, m^m\, n^n < (m+n)^{m+n}$.

22. If $0 < k \leq n$, is there any such sum with exactly k terms? How many?

23. Differentiate $P_n(x)/(x^k - 1)^{n+1}$ and get a relation between $P_n(1)$ and $P_{n+1}(1)$.

24. Draw a great circle through two of the points.

25. Replace a by $b * a$.

26. Show that the equation $x^2 - y^2 = z^2 + 1$ has infinitely many integer solutions. Set $n = y^2 + z^2$.

27. Try with first degree polynomials. Some of those polynomials must change sign precisely at $x = -1$ and $x = 0$. Recall that $|u| = \pm u$ depending on whether $u \geq 0$ or $u < 0$.

28. Consider the plane containing both the axis of the cone and two opposite vertices of the cube's bottom face.

29. Discard solutions coming in pairs, such as the ones for which $a_1 \neq a_2$; so we may assume $a_1 = a_2$.

Solutions

1. If the numbers on the faces having a common vertex v have different numbers $a_0 < a_1 < a_2 < a_3 < a_4$, then $a_k \geq k$ for $k = 0, 1, 2, 3, 4$, and $S_v = a_0 + a_1 + a_2 + a_3 + a_4 \geq 0 + 1 + 2 + 3 + 4 = 10$. Adding over the 12 vertices we get $\sum_v S_v \geq 12 \cdot 10 = 120$. In that sum each number occurs three times, one per each vertex of the face, so the sum of the numbers written on the faces of the icosahedron will be greater than or equal to $12/3 = 40$, contradicting the hypothesis that it is 39.

2. We have:

$$\begin{aligned} c(2n)c(2n+2) + c(2n+1)c(2n+3) &= c(n)c(n+1) + (-1)^n c(n)(-1)^{n+1} c(n+1) \\ &= c(n)c(n+1) - c(n)c(n+1) = 0\,, \end{aligned}$$

so each term in an even position cancels with the next term, and the sum telescopes:

$$\begin{aligned} \sum_{n=1}^{2013} c(n)c(n+2) &= \underbrace{c(1)}_{1}\,\underbrace{c(3)}_{-1} + \underbrace{\sum_{n=2}^{2013} c(n)c(n+2)}_{0} \\ &= -1\,. \end{aligned}$$

3. Assume without loss of generality that $1 < d_1 \leq d_2 \leq \cdots \leq d_{12} < 12$.
Note that three numbers $0 < a \leq b \leq c$ are the side lengths of an acute triangle precisely if $a^2 + b^2 > c^2$, so if not such three indices exist we would have $d_i^2 + d_{i+1}^2 \leq d_{i+2}^2$ for $i = 1, \dots, 10$. Consequently $1 < d_1, d_2$, $d_3^3 \geq d_1^2 + d_2^2 > 2$, $d_4^3 \geq d_2^2 + d_3^2 > 3$, and analogously $d_5^2 > 5$, $d_6^2 > 8$, $d_7^2 > 13$, $d_8^2 > 21$, $d_9^2 > 34$, $d_{10}^2 > 55$, $d_{11}^2 > 89$, $d_{12}^2 > 144$, but this last inequality implies $d_{12} > 12$, which is a contradiction.

4. Assume $f(x)$ and $g(x)$ are in S. Then by (ii) the following function is in S:

$$\begin{aligned} f_2(f(x)) + f_2(g(x)) &= \ln\left(f(x) + 1\right) + \ln\left(g(x) + 1\right) \\ &= \ln\left((f(x)+1)(g(x)+1)\right) \\ &= \ln\left(f(x)g(x) + f(x) + g(x) + 1\right). \end{aligned}$$

Next, by (i) the following function is in S:

$$\begin{aligned} f_1(\ln\left(f(x)g(x) + f(x) + g(x) + 1\right)) &= e^{\ln\left(f(x)g(x)+f(x)+g(x)+1\right)} - 1 \\ &= f(x)g(x) + f(x) + g(x)\,. \end{aligned}$$

Finally we can apply (iii), subtract $f(x)$, then $g(x)$, and we get that the following function is in S:

$$(f(x)g(x) + f(x) + g(x)) - f(x) - g(x) = f(x)g(x)\,,$$

where the use of (iii) is justified because the final difference $f(x)g(x)$ is non-negative.

5. Since the rational numbers are dense in the reals, we can find positive integers a, b such that

$$\frac{3\varepsilon}{hk} < \frac{b}{a} < \frac{4\varepsilon}{hk}.$$

By choosing a and b large enough we can also ensure that $3a^2 > b$. We then have

$$\frac{\varepsilon}{hk} < \frac{b}{3a} < \frac{b}{\sqrt{a^2+b}+a} = \sqrt{a^2+b} - a$$

and

$$\sqrt{a^2+b} - a = \frac{b}{\sqrt{a^2+b}+a} \leq \frac{b}{2a} < 2\frac{\varepsilon}{hk}.$$

We may then take $m = k^2(a^2+b), n = h^2a^2$.

6. The largest such k is $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$. For n even, this value is achieved by the partition

$$\{1, n\}, \{2, n-1\}, \ldots;$$

for n odd, it is achieved by the partition

$$\{n\}, \{1, n-1\}, \{2, n-2\}, \ldots.$$

One way to see that this is optimal is to note that the common sum can never be less than n, since n itself belongs to one of the boxes. This implies that $k \leq (1+\cdots+n)/n = (n+1)/2$. Another argument is that if $k > (n+1)/2$, then there would have to be two boxes with one number each (by the pigeonhole principle), but such boxes could not have the same sum.

7. - *First solution:* No such sequence exists. If it did, then the Cauchy-Schwartz inequality would imply

$$\begin{aligned} 8 &= (a_1^2 + a_2^2 + \cdots)(a_1^4 + a_2^4 + \cdots) \\ &\geq (a_1^3 + a_2^3 + \cdots)^2 = 9, \end{aligned}$$

contradiction.

- *Second solution:* Suppose that such a sequence exists. If $a_k^2 \in [0,1]$ for all k, then $a_k^4 \leq a_k^2$ for all k, and so

$$4 = a_1^4 + a_2^4 + \cdots \leq a_1^2 + a_2^2 + \cdots = 2,$$

contradiction. There thus exists a positive integer k for which $a_k^2 > 1$. However, in this case, for m large, $a_k^{2m} > 2m$ and so $a_1^{2m} + a_2^{2m} + \cdots \neq 2m$.

8. The smallest distance is 3, achieved by $A = (0,0)$, $B = (3,0)$, $C = (0,4)$. To check this, it suffices to check that AB cannot equal 1 or 2. (It cannot equal 0 because if two of the points were to coincide, the three points would be collinear.)

The triangle inequality implies that $|AC - BC| \leq AB$, with equality if and only if A, B, C are collinear. If $AB = 1$, we may assume without loss of generality that $A = (0,0)$, $B = (1,0)$. To avoid collinearity, we must have $AC = BC$, but this forces $C = (1/2, y)$ for some $y \in \mathbb{R}$, a contradiction.

If $AB = 2$, then we may assume without loss of generality that $A = (0,0), B = (2,0)$. The triangle inequality implies $|AC - BC| \in \{0, 1\}$. Also, for $C = (x,y)$, $AC^2 = x^2 + y^2$ and $BC^2 = (2-x)^2 + y^2$ have the same parity; it follows that $AC = BC$. Hence $c = (1, y)$ for some $y \in \mathbb{R}$, so y^2 and $y^2 + 1 = BC^2$ are consecutive perfect squares. This can only happen for $y = 0$, but then A, B, C are collinear, a contradiction again.

9. Yes, it does follow. Let P be any point in the plane. Let $ABCD$ be any square with center P. Let E, F, G, H be the midpoints of the segments AB, BC, CD, DA, respectively. The function f must satisfy the equations

$$\begin{aligned}
0 &= f(A) + f(B) + f(C) + f(D) \\
0 &= f(E) + f(F) + f(G) + f(H) \\
0 &= f(A) + f(E) + f(P) + f(H) \\
0 &= f(B) + f(F) + f(P) + f(E) \\
0 &= f(C) + f(G) + f(P) + f(F) \\
0 &= f(D) + f(H) + f(P) + f(G).
\end{aligned}$$

If we add the last four equations, then subtract the first equation and twice the second equation, we obtain $0 = 4f(P)$, whence $f(P) = 0$.

10. Every positive rational number can be uniquely written in lowest terms as a/b for a, b positive integers. We prove the statement in the problem by induction on the largest prime dividing either a or b (where this is considered to be 1 if $a = b = 1$). For the base case, we can write $1/1 = 2!/2!$. For a general a/b, let p be the largest prime dividing either a or b; then $a/b = p^k a'/b'$ for some $k \neq 0$ and positive integers a', b' whose largest prime factors are strictly less than p. We now have $a/b = (p!)^k \frac{a'}{(p-1)!b'}$, and all prime factors of a' and $(p-1)!b'$ are strictly less than p. By the induction assumption, $\frac{a'}{(p-1)!b'}$ can be written as a quotient of products of prime factorials, and so $a/b = (p!)^k \frac{a'}{(p-1)!b'}$ can as well. This completes the induction.

11. The function $g(x) = f(x, 0)$ works. Substituting $(x, y, z) = (0, 0, 0)$ into the given functional equation yields $f(0, 0) = 0$, whence substituting $(x, y, z) = (x, 0, 0)$ yields $f(x, 0) + f(0, x) = 0$. Finally, substituting $(x, y, z) = (x, y, 0)$ yields $f(x, y) = -f(y, 0) - f(0, x) = g(x) - g(y)$.

12. **First solution:** Pair each entry of the first row with the entry directly below it in the second row. If Alan ever writes a number in one of the first two rows, Barbara writes the same number in the other entry in the pair. If Alan writes a number anywhere other than the first two rows, Barbara does likewise. At the end, the resulting matrix will have two identical rows, so its determinant will be zero.
Second solution: Whenever Alan writes a number x in an entry in some row, Barbara writes $-x$ in some other entry in the same row. At the end, the resulting matrix will have all rows summing to zero, so it cannot have full rank.

13. There are at most two such points. For example, the points $(0, 0)$ and $(1, 0)$ lie on a circle with center $(1/2, x)$ for any real number x, not necessarily rational.
On the other hand, with three point A, B, C, we could find the center of the circle as the intersection of the perpendicular bisectors of the segments AB and BC. If A, B, and C are rational, the middle points of AB and BC will be rational, the bisectors

will be rational lines (representable by equations with rational coefficients), and their intersection will be rational.

14. The only such α are $2/3, 3/2, (13 \pm \sqrt{601})/12$.
In fact, let C_1 and C_2 be the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$, respectively, and let L be the line $y = x$. We consider three cases.
If C_1 is tangent to L, then the point of tangency (x, x) satisfies

$$2\alpha x + \alpha = 1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

by symmetry, C_2 is tangent to L there, so C_1 and C_2 are tangent. Writing $\alpha = 1/(2x+1)$ in the first equation and substituting into the second, we must have

$$x = \frac{x^2 + x}{2x+1} + \frac{1}{24},$$

which simplifies to $0 = 24x^2 - 2x - 1 = (6x+1)(4x-1)$, or $x \in \{1/4, -1/6\}$. This yields $\alpha = 1/(2x+1) \in \{2/3, 3/2\}$. If C_1 does not intersect L, then C_1 and C_2 are separated by L and so cannot be tangent.
If C_1 intersects L in two distinct points P_1, P_2, then it is not tangent to L at either point. Suppose at one of these points, say P_1, the tangent to C_1 is perpendicular to L; then by symmetry, the same will be true of C_2, so C_1 and C_2 will be tangent at P_1. In this case, the point $P_1 = (x, x)$ satisfies

$$2\alpha x + \alpha = -1, \qquad x = \alpha x^2 + \alpha x + \frac{1}{24};$$

writing $\alpha = -1/(2x+1)$ in the first equation and substituting into the second, we have

$$x = -\frac{x^2 + x}{2x+1} + \frac{1}{24},$$

or $x = (-23 \pm \sqrt{601})/72$. This yields $\alpha = -1/(2x+1) = (13 \pm \sqrt{601})/12$.
If instead the tangents to C_1 at P_1, P_2 are not perpendicular to L, then we claim there cannot be any point where C_1 and C_2 are tangent. Indeed, if we count intersections of C_1 and C_2 (by using C_1 to substitute for y in C_2, then solving for y), we get at most four solutions counting multiplicity. Two of these are P_1 and P_2, and any point of tangency counts for two more. However, off of L, any point of tangency would have a mirror image which is also a point of tangency, and there cannot be six solutions. Hence we have now found all possible α.

15. The problem fails if f is allowed to be constant, e.g., take $f(n) = 1$. We thus assume that f is nonconstant. Write $f(n) = \sum_{i=0}^{d} a_i n^i$ with $a_i > 0$. Then

$$\begin{aligned} f(f(n)+1) &= \sum_{i=0}^{d} a_i (f(n)+1)^i \\ &\equiv f(1) \pmod{f(n)}. \end{aligned}$$

If $n = 1$, then this implies that $f(f(n)+1)$ is divisible by $f(n)$. Otherwise, $0 < f(1) < f(n)$ since f is nonconstant and has positive coefficients, so $f(f(n)+1)$ cannot be divisible by $f(n)$.

16. We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2+y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \le 6r,$$

or

$$(r-3)^2 + z^2 \le 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x-3)^2 + z^2 \le 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

17. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $i = 0, \dots, n$, put $s_i = x_1 + \cdots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \dots, \{s_n\}$ into ascending order, and call the result $t_0, \dots, t_n$. Since $0 = t_0 \le \cdots \le t_n < 1$, the differences

$$t_1 - t_0, \dots, t_n - t_{n-1}, 1 - t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n+1)$; if it is anything other than $1 - t_n$, it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \le i < j \le n$. Put $S = \{x_{i+1}, \dots, x_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$; then

$$\begin{aligned} \left| m + \sum_{s \in S} s \right| &= |m + s_j - s_i| \\ &= |\{s_j\} - \{s_i\}| \\ &\le \frac{1}{n+1}, \end{aligned}$$

as desired. In case $1 - t_n \le 1/(n+1)$, we take $S = \{x_1, \dots, x_n\}$ and $m = -\lceil s_n \rceil$, and again obtain the desired conclusion.

18. We proceed by induction, with base case $1 = 2^0 3^0$. Suppose all integers less than $n-1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n. If n is odd, put $m = \lfloor \log_3 n \rfloor$, so that $3^m \le n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n - 3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then

$$n = 3^m + 2s_1 + \cdots + 2s_k,$$

and clearly none of the $2s_i$ divide each other or 3^m. Moreover, since $2s_i \le n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

19. Take $P(x, y) = (y-2x)(y-2x-1)$. To see that this works, first note that if $m = \lfloor a \rfloor$, then $2m$ is an integer less than or equal to $2a$, so $2m \leq \lfloor 2a \rfloor$. On the other hand, $m+1$ is an integer strictly greater than a, so $2m+2$ is an integer strictly greater than $2a$, so $\lfloor 2a \rfloor \leq 2m+1$.

20. Yes. Suppose otherwise. Then there would be an N such that $S(N) < 80\%$ and $S(N+1) > 80\%$; that is, O'Keal's free throw percentage is under 80% at some point, and after one subsequent free throw (necessarily made), her percentage is over 80%. If she makes m of her first N free throws, then $m/N < 4/5$ and $(m+1)/(N+1) > 4/5$. This means that $5m < 4n < 5m+1$, which is impossible since then $4n$ is an integer between the consecutive integers $5m$ and $5m+1$.

21. We have

$$(m+n)^{m+n} > \binom{m+n}{m} m^m n^n$$

because the binomial expansion of $(m+n)^{m+n}$ includes the term on the right as well as some others. Rearranging this inequality yields the claim.

22. There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k = 1, \dots, n$ (and clearly no others). To see this, note that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$, then

$$\begin{aligned} ka_1 &= a_1 + a_1 + \cdots + a_1 \\ &\leq n \leq a_1 + (a_1+1) + \cdots + (a_1+1) \\ &= ka_1 + k - 1. \end{aligned}$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1+a_2+\cdots+a_k$: if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i-1)$. The possible values of i are $1, \dots, k$, and exactly one of these sums comes out equal to n, proving our claim.

23. By differentiating $P_n(x)/(x^k-1)^{n+1}$, we find that

$$P_{n+1}(x) = (x^k-1)P_n'(x) - (n+1)kx^{k-1}P_n(x)\,.$$

Substituting $x = 1$ yields $P_{n+1}(1) = -(n+1)kP_n(1)$. Since $P_0(1) = 1$, an easy induction gives $P_n(1) = (-k)^n n!$ for all $n \geq 0$.

Note: one can also argue by expanding in Taylor series around 1. Namely, we have

$$\frac{1}{x^k-1} = \frac{1}{k(x-1)+\cdots} = \frac{1}{k}(x-1)^{-1} + \cdots\,,$$

so

$$\frac{d^n}{dx^n}\frac{1}{x^k-1} = \frac{(-1)^n n!}{k(x-1)^{-n-1}}$$

and

$$\begin{aligned}P_n(x) &= (x^k-1)^{n+1}\frac{d^n}{dx^n}\frac{1}{x^k-1}\\ &= (k(x-1)+\cdots)^{n+1}\\ &\quad\left(\frac{(-1)^n n!}{k}(x-1)^{-n-1}+\cdots\right)\\ &= (-k)^n n! + \cdots .\end{aligned}$$

24. Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.

25. The hypothesis implies $((b*a)*b)*(b*a) = b$ for all $a, b \in S$ (by replacing a by $b*a$), and hence $a*(b*a) = b$ for all $a, b \in S$ (using $(b*a)*b = a$).

26. First solution: Let a be an even integer such that a^2+1 is not prime. (For example, choose $a \equiv 2 \pmod 5$, so that a^2+1 is divisible by 5.) Then we can write a^2+1 as a difference of squares x^2-b^2, by factoring a^2+1 as rs with $r \geq s > 1$, and setting $x = (r+s)/2$, $b = (r-s)/2$. Finally, put $n = x^2-1$, so that $n = a^2+b^2$, $n+1 = x^2$, $n+2 = x^2+1$.

Second solution: It is well-known that the equation $x^2-2y^2 = 1$ has infinitely many solutions (the so-called "Pell" equation). Thus setting $n = 2y^2$ (so that $n = y^2+y^2$, $n+1 = x^2+0^2$, $n+2 = x^2+1^2$) yields infinitely many n with the desired property.

Third solution: As in the first solution, it suffices to exhibit x such that x^2-1 is the sum of two squares. We will take $x = 3^{2^n}$, and show that x^2-1 is the sum of two squares by induction on n: if $3^{2^n}-1 = a^2+b^2$, then

$$\begin{aligned}(3^{2^{n+1}}-1) &= (3^{2^n}-1)(3^{2^n}+1)\\ &= (3^{2^{n-1}}a+b)^2 + (a-3^{2^{n-1}}b)^2.\end{aligned}$$

27. Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r,s) = (r+s+|r-s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned}F(x) &= \max\{-3x-3, 0\} - \max\{5x, 0\} + 3x + 2\\ &= (-3x-3+|3x-3|)/2\\ &\quad - (5x+|5x|)/2 + 3x + 2\\ &= |(3x-3)/2| - |5x/2| - x + \frac{1}{2},\end{aligned}$$

so we may set $f(x) = (3x-3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

28. Consider the plane containing both the axis of the cone and two opposite vertices of the cube's bottom face. The cross section of the cone and the cube in this plane consists of a rectangle of sides s and $s\sqrt{2}$ inscribed in an isosceles triangle of base 2 and height 3, where s is the side-length of the cube. (The $s\sqrt{2}$ side of the rectangle lies on the base of the triangle.) Similar triangles yield $s/3 = (1 - s\sqrt{2}/2)/1$, or $s = (9\sqrt{2} - 6)/7$.

29. We may discard any solutions for which $a_1 \neq a_2$, since those come in pairs; so assume $a_1 = a_2$. Similarly, we may assume that $a_3 = a_4$, $a_5 = a_6$, $a_7 = a_8$, $a_9 = a_{10}$. Thus we get the equation

$$2/a_1 + 2/a_3 + 2/a_5 + 2/a_7 + 2/a_9 = 1.$$

Again, we may assume $a_1 = a_3$ and $a_5 = a_7$, so we get $4/a_1 + 4/a_5 + 2/a_9 = 1$; and $a_1 = a_5$, so $8/a_1 + 2/a_9 = 1$. This implies that $(a_1 - 8)(a_9 - 2) = 16$, which by counting has 5 solutions. Thus N_{10} is odd.

Problem Solving in Math (Math 43900) Fall 2013

Week six (October 1) solutions

Instructor: David Galvin

A non-Putnam warm-up exercise

Using the trick of repeatedly differentiating the identity

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots,$$

find a nice expression for the coefficients of the power series (about 0) of $1/(1-x)^k$. Use this to derive, via generating functions, the identity

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

and if you are feeling masochistic, go on to find a nice closed-form for

$$\sum_{i=0}^{n} i^3$$

using the same idea.

Solution: Differentiating the left-hand side k times, we get

$$\frac{k!}{(1-x)^{k+1}},$$

and differentiating the right-hand side k times, we get an power series where the coefficient of x^{n-k} is $n(n-1)\ldots(n-(k-1))$, so the coefficient of x^n is $(n+k)(n+k-1)\ldots(n+1)$. Dividing through by $k!$, the coefficient of x^n in $1/(1-x)^{k+1}$ is

$$\frac{(n+k)(n+k-1)\ldots(n+1)}{k!} = \binom{n+k}{k}.$$

Let $a_n = \sum_{i=0}^{n} i^2$; a_n satisfies the recurrence $a_0 = 0$ and $a_n = a_{n-1} + n^2$ for $n > 0$. Letting

$A(x) = a_0 + a_1x + a_2x^2 \ldots$ be the generating function of the a_n's, we get

$$\begin{aligned}
A(x) &= 0 + (a_0 + 1^2)x + (a_1 + 2^2)x^2 + \ldots \\
&= xA(x) + (1^2x + 2^2x^2 + 3^2x^3 + \ldots) \\
&= xA(x) + [1.0 + 1]x + [2.1 + 2]x^2 + [3.2 + 3]x^3 + \ldots \\
&= xA(x) + (1.0x + 2.1x^2 + 3.2x^3 + \ldots) + (1x + 2x^2 + 3x^3 + \ldots) \\
&= xA(x) + x^2(2.1 + 3.2x + \ldots) + x(1 + 2x + 3x^2 + \ldots) \\
&= xA(x) + x^2\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) + x\frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= xA(x) + \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= xA(x) + \frac{x^2 + x}{(1-x)^3}
\end{aligned}$$

so

$$A(x) = \frac{x^2 + x}{(1-x)^4} = \frac{x^2}{(1-x)^4} + \frac{x}{(1-x)^4}.$$

This means that a_n consists of two parts — the coefficient of x^{n-2} in $1/(1-x)^4$ and the coefficient of x^{n-1} in $1/(1-x)^4$. By what we established earlier, this is

$$\binom{n+1}{3} + \binom{n+2}{3} = \frac{n(n+1)(2n-1)}{6}.$$

If you were feeling masochistic, you might have used the same method to discover

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

The problems

For these, time got away from me and I was unable to write up full solutions. Instead I've given the source of the problem (all but one are Putnam problems), so you can find the solution either (for pre-2000) in the appropriate book that's on reserve in the math library, or (for post-2000) by following the links on the course website.

1. Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

   Source: Putnam 1996, B1. Notes: the anser is the nth Fibonacci number!

2. Define a sequence $(p_n)_{n\geq 1}$ recursively by $p_1 = 3$, $p_2 = 7$ and, for $n \geq 3$,

   $$p_n = 4 + p_{n-1} + 2p_{n-2} + \ldots + 2p_1$$

   (so, for example, $p_3 = 4 + p_2 + 2p_1 = 17$ and $p_4 = 4 + p_3 + 2p_2 + 2p_1 = 41$). Find a closed-form expression for p_n for general n.

Solution: The first few values are $p_1 = 3$, $p_2 = 7$, $p_3 = 17$, $p_4 = 41$, $p_3 = 17$, $p_4 = 41$. A pattern seems to be emerging: $p_n = 2p_{n-1} + p_{n-2}$, with $p_1 = 3$, $p_2 = 7$. We verify this by induction on n. It's certainly true for $n = 3$. For $n > 3$,

$$\begin{aligned}
p_n &= 4 + p_{n-1} + 2p_{n-2} + 2p_{n-3} + \ldots + 2p_3 + 2p_2 + 2p_1 \\
&= (p_{n-1} + p_{n-2}) + 4 + p_{n-2} + 2p_{n-3} + \ldots + 2p_3 + 2p_2 + 2p_1 \\
&= (p_{n-1} + p_{n-2}) + p_{n-2} \quad \text{(induction)} \\
&= 2p_{n-1} + p_{n-2},
\end{aligned}$$

as required. With this new recurrence, it is easy to apply the method of generating functions, as described in the introduction, to get

$$p_n = \frac{(1+\sqrt{2})^{n+1}}{2} + \frac{(1-\sqrt{2})^{n+1}}{2}.$$

Source: This problem arose in my research. A *graph* is a collection of points, some pairs of which are joined by edges. A *Widom-Rowlinson* coloring of a graph is a coloring of the points using 3 colors, red, white and blue, in such a way that no point colored red is joined by an edge that is colored blue. I was looking at how many Widom-Rowlinson colorings there are of the graph P_n that consists of n points, numbered 1 up to n, with edges from 1 to 2, from 2 to 3, etc., up to from $n-1$ to n. It turns out that there are p_n such colorings, where p_n satisfies the first recurrence. In trying to find a closed form for p_n, I realized that p_n satisfies the Fibonacci-like recurrence described in the solution above, and so was able to solve for p_n explicitly using generating functions.

3. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

   Source: Putnam 1993, A2.

4. Define a sequence by $a_k = k$ for $k = 1, 2, \ldots, 2006$ and

   $$a_{k+1} = a_k + a_{k-2005}$$

   for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

   Source: Putnam 2006, A3. Notes: the key points are 1) any recursive sequence of this kind can be extended both forward and backwards, and the resulting doubly infinite sequence, reduced to any modulus, is periodic.

5. The last question was clearly written with the years 2005 and 2006 in mind. Does the conclusion remain true for an arbitrary year? That is, fix $m \geq 1$. Define a sequence by $a_k = k$ for $k = 1, 2, \ldots, m+1$ and

   $$a_{k+1} = a_k + a_{k-m}$$

   for $k \geq m+1$. For which m is it true that the sequence has m consecutive terms each divisible by $m+1$?

   Solution: The solution to the previous problem goes through fine with general positive m.

   Source: An idle thought.

6. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

(Here $\lfloor a \rfloor$ denote the round-down of a to the nearest integer at or below a; so for example $\lfloor 3.4 \rfloor = 3$, $\lfloor 2.999 \rfloor = 2$ and $\lfloor 5 \rfloor = 5$.)

Source: Putnam 1997, B4.

7. Define $(a_n)_{n \geq 0}$ by

$$\frac{1}{1-2x-x^2} = \sum_{n \geq 0} a_n x^n.$$

Show that for each $n \geq 0$, there is an $m = m(n)$ such that $a_m = a_n^2 + a_{n+1}^2$.

Source: Putnam 1999, A3.