Proving Stronger Statements May Be Easier!

Recall that the number of moves in the Hanoi Towers puzzle with n discs is given by the recurrent formula

$$T_n = \begin{cases} 2T_{n-1} + 1 & \text{for } n > 1\,; \\ 1 & \text{for } n = 1\,. \end{cases}$$

A simple induction arguments shows that $T_n = 2^n - 1$ for every $n \geq 1$. The base case holds as $T_1 = 1 = 2^1 - 1$. The induction step follows from $T_n = 2T_{n-1} + 1 = 2 \cdot (2^{n-1} - 1) + 1 = 2^n - 1$.

We have an induction proof of the statement $T_n = 2^n - 1$. It must be even easier to prove a weaker statement $T_n \leq 2^n$. Let us try to prove this by induction again. The base case for $n = 1$ trivially holds. However, in the induction step, we can only say that

$$T_n = 2T_{n-1} + 1 \leq 2 \cdot 2^{n-1} + 1 = 2^n + 1\,,$$

which is not sufficient for our goals!

How is it even possible that we can inductively prove that $T_n = 2^n - 1$, but cannot prove a *weaker* statement that $T_n < 2^n$?

While this might seem strange at first, sometimes for an induction proof one needs to strengthen the statement. The trick here is that this also allows one to use a *stronger induction hypothesis*. And, having a stronger hypothesis, one has more tools to prove stronger statements.

Problem:

Prove that
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{99} - \frac{1}{100} = \frac{1}{51} + \frac{1}{52} + \dots + \frac{1}{100}$

Solution:

We will solve a more general problem:

Prove that for every $k \geq 1$:
$1 - \frac{1}{2} + \dots + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k}$

This will imply the initial problem by setting $k = 50$.
The induction base case $k = 1$ is easy to verify:

$$1 - \frac{1}{2} = \frac{1}{2}\,.$$

For the induction step from $k \geq 1$ to $k + 1$, it suffices to show that the sum on the left and the sum on right change by the same amount when going from k to $k + 1$. The sum on the left simply increases by $\frac{1}{2k+1} - \frac{1}{2k+2}$, while the sum on the right increases by
$\frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2(k+1)} - \left(\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k}\right) = \frac{1}{2k+1} + \frac{1}{2(k+1)} - \frac{1}{k+1} = \frac{1}{2k+1} - \frac{1}{2(k+1)} = \frac{1}{2k+1} - \frac{1}{2k+2}$

Thus, the expressions on the left-hand side and on the right-hand side are the same initially for $k = 1$, and each time we increment k these two expressions change by the same value. Therefore, they stay the same for all values of $k \geq 1$
.

Problem:
Prove that for every $n \geq 1$ it holds that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} < 1\,.$$

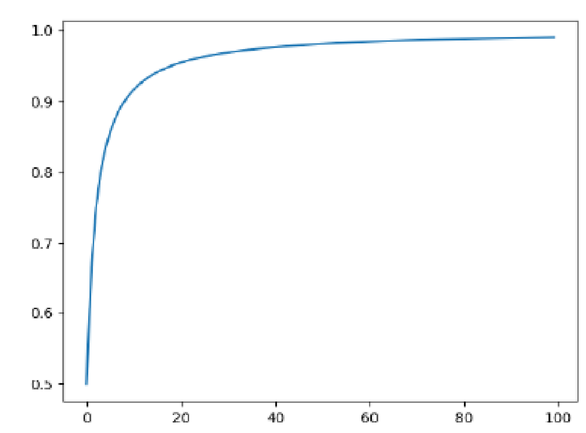
First, let us see whether this sum is less than 1 for small values of n.

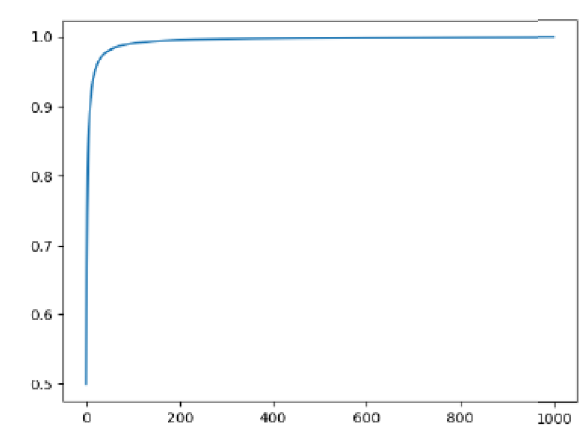
```
from itertools import accumulate
import matplotlib.pyplot as plt

n = 1000
sums = [*accumulate(1 / (i * (i + 1))
                    for i in range(1, n + 1))]

for k in (n // 10, n):
    plt.clf()
    plt.plot(sums[:k])
    plt.savefig(f'sum{k}.png')

```





Now, let us prove an even stronger statement: not only is this sum less than one, but it actually equals $1 - \frac{1}{n+1}$. We will use mathematical induction to prove this stronger statement. When $n = 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{2} = 1 - \frac{1}{n+1}\,.$$

Now if we assume that this holds for some $n \geq 1$, then for $n + 1$ we have

$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)}$

where the last equality is due to induction hypothesis. Now,

$\sum_{i}^{n+1} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} = 1 - \frac{n+2}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = 1 - \frac{n+1}{(n+1)(n+2)} = 1 - \frac{1}{n+2}$

which proves the induction step and thus the statement.