Linear programming duality

Primal LP (P)

$$\min \mathbf{7x_1 + x_2 + 5x_3} :$$
$$\mathbf{x_1 - x_2 + 3x_3 \geq 10} \quad (1)$$
$$\mathbf{5x_1 + 2x_2 - x_3 \geq 6} \quad (2)$$
$$\mathbf{x_1, x_2, x_3 \geq 0} \quad (3, 4, 5)$$

Dual LP (D)

$$\max \mathbf{10y_1 + 6y_2} :$$
$$\mathbf{y_1 + 5y_2 \leq 7} \quad (1')$$
$$\mathbf{-y_1 + 2y_2 \leq 1} \quad (2')$$
$$\mathbf{3y_1 - y_2 \leq 5} \quad (3')$$
$$\mathbf{y_1, y_2 \geq 0} \quad (4', 5')$$

Lemma

$$\min\{\mathbf{7x_1 + x_2 + 5x_3 : x \ \text{feasible}}\} \geq \max\{\mathbf{10y_1 + 6y_2 : y \ \text{feasible}}\}$$

Applying the same ideas to a maximization problem

$$
\begin{aligned}
&\max \mathbf{10y_1 + 6y_2} : \\
&\quad \mathbf{y_1 + 5y_2 \le 7} && (1') \\
&-\mathbf{y_1 + 2y_2 \le 1} && (2') \\
&\mathbf{3y_1 - y_2 \le 5} && (3') \\
&\mathbf{y_1, y_2 \ge 0} && (4', 5')
\end{aligned}
$$

How do we prove a lower bound?

$$\max \mathbf{10y_1 + 6y_2} :$$

$$\mathbf{y_1 + 5y_2 \le 7} \qquad (\mathbf{1'})$$

$$\mathbf{-y_1 + 2y_2 \le 1} \qquad (\mathbf{2'})$$

$$\mathbf{3y_1 - y_2 \le 5} \qquad (\mathbf{3'})$$

$$\mathbf{y_1, y_2 \ge 0} \qquad (\mathbf{4', 5'})$$

Exhibit a feasible (y1,y2)
its value is a lower bound

How do we prove an upper bound?

$$\max \mathbf{10y_1 + 6y_2} :$$

$$\mathbf{y_1 + 5y_2 \leq 7} \qquad (1')$$

$$\mathbf{-y_1 + 2y_2 \leq 1} \qquad (2')$$

$$\mathbf{3y_1 - y_2 \leq 5} \qquad (3')$$

$$\mathbf{y_1, y_2 \geq 0} \qquad (4', 5')$$

Convex combination of (1'),(2'),(3') s.t. coeff of y1 is at least 10 and coeff of y2 is at least 6

Upper bound, formally:

$$\begin{aligned} &\max \mathbf{10y_1 + 6y_2} : \\ &\mathbf{y_1 + 5y_2 \le 7} \quad (1') \\ &\mathbf{-y_1 + 2y_2 \le 1} \quad (2') \\ &\mathbf{3y_1 - y_2 \le 5} \quad (3') \\ &\mathbf{y_1, y_2 \ge 0} \quad (4', 5') \end{aligned}$$

$$\mathbf{z_1 \times (1') + z_2 \times (2') + z_3 \times (3')}$$

$$\textbf{If } \mathbf{z_1 - z_2 + 3z_3 \ge 10} \textbf{ and } \mathbf{5z_1 + 2z_2 - z_3 \ge 6}$$

Then, upper bound for OPT:

$$\mathbf{7z_1 + z_2 + 5z_3}$$

Best upper bound:

$$\begin{aligned} &\max \mathbf{10y_1 + 6y_2} : \\ &\mathbf{y_1 + 5y_2 \leq 7} && (1') \\ &\mathbf{-y_1 + 2y_2 \leq 1} && (2') \\ &\mathbf{3y_1 - y_2 \leq 5} && (3') \\ &\mathbf{y_1, y_2 \geq 0} && (4', 5') \end{aligned}$$

$$\begin{aligned} &\min \mathbf{7z_1 + z_2 + 5z_3} : \\ &\mathbf{z_1 - z_2 + 3z_3 \geq 10} \\ &\mathbf{5z_1 + 2z_2 - z_3 \geq 6} \\ &\mathbf{z_1, z_2, z_3 \geq 0} \end{aligned}$$

Maximization LP

$$\max \mathbf{10y_1 + 6y_2} :$$

$$\mathbf{y_1 + 5y_2 \leq 7} \qquad \mathbf{(1')}$$

$$\mathbf{-y_1 + 2y_2 \leq 1} \qquad \mathbf{(2')}$$

$$\mathbf{-3y_1 - y_2 \leq 5} \qquad \mathbf{(3')}$$

$$\mathbf{y_1, y_2 \geq 0} \qquad \mathbf{(4', 5')}$$

Minimization LP

$$\min \mathbf{7z_1 + z_2 + 5z_3} :$$

$$\mathbf{z_1 - z_2 + 3z_3 \geq 10}$$

$$\mathbf{5z_1 + 2z_2 - z_3 \geq 6}$$

$$\mathbf{z_1, z_2, z_3 \geq 0}$$

Surprise!

Primal	Dual	Dual of dual
$\min 7\mathbf{x_1} + \mathbf{x_2} + 5\mathbf{x_3}$:	$\max 10\mathbf{y_1} + 6\mathbf{y_2}$:	$\min 7\mathbf{z_1} + \mathbf{z_2} + 5\mathbf{z_3}$:
$\mathbf{x_1} - \mathbf{x_2} + 3\mathbf{x_3} \geq 10$	$\mathbf{y_1} + 5\mathbf{y_2} \leq 7 \quad (1')$	$\mathbf{z_1} - \mathbf{z_2} + 3\mathbf{z_3} \geq 10$
$5\mathbf{x_1} + 2\mathbf{x_2} - \mathbf{x_3} \geq 6$	$-\mathbf{y_1} + 2\mathbf{y_2} \leq 1 \quad (2')$	$5\mathbf{z_1} + 2\mathbf{z_2} - \mathbf{z_3} \geq 6$
$\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3} \geq 0$	$-3\mathbf{y_1} - \mathbf{y_2} \leq 5 \quad (3')$	$\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3} \geq 0$
	$\mathbf{y_1}, \mathbf{y_2} \geq 0 \quad (4', 5')$	

The dual of the dual is the primal!