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Abstract

The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y , when X and Y are two independent random variables following Pareto and Exponential distributions, respectively.

Keywords

Product Distribution, Ratio Distribution, Pareto Distribution, Exponential Distribution, probability density function, Moment of order r, Survival function, Hazard function.

1 Introduction

For given random variables X and Y, the distributions of the product XY and the ratio X/Y are of interest in many areas of the sciences. In traditional portfolio selection models certain cases involve random products and The best examples of this are in the case of investment in a number of different overseas markets. In portfolio diversification models (see, e.g., Grubel, 1968) not only are prices of shares in local markets uncertain, but also the exchange rates are uncertain, and so the value of the portfolio in domestic currency is related to a product of random variables. Similarly in models of diversified production by multinationals (see, e.g., Rugman, 1979) there are local production uncertainty and exchange rate uncertainty, and so profits in home currency are again related to a product of random variables. An entirely different example is drawn from the econometric literature.In making a forecast from an estimated equation Feldstein (1971) pointed out that both the parameter and the value of the exogenous variable in the forecast period could be considered random variables. Hence the forecast was proportional to a product of random variables.

An important example of ratios of random variables is the stress-strength model in the context of reliability. It describes the life of a component that has a random strength Y and is subjected to random stress X The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $Y > X$. Thus, $\Pr(X < Y)$ is a measure of component reliability. It has many applications, especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels.

The distributions of XY and X/Y have been studied by several authors, especially when X and Y are independent random variables and come from the same family. With respect to products of random variables, see Sakamoto, 1943, [25] for uniform family, Harter, 1951, [5] and Wallgren, 1980, [22] for Student's t family, Stuart, 1962, [17] and Podolski, [12] 1972 for gamma family, Steece, 1976, [16], Bhargava and Khatri, 1981, [2], and Tang and Gupta 1984, [23], for beta family, AbuSalih 1983, [3], for power function family, and Malik and Trudel 1986, [8], for exponential family (for a comprehensive review of known results, see also Rathie and Rohrer, 1987, [15]. With respect to ratios of random variables, see Therar, Khaled, Seifedine, 2017, [21], for hyper-erlang family, Marsaglia, 1965, [24] and Korhonen and Narula, 1989, [7] for normal family, Press, 1969, [13] for Student's t family, Basu and Lochner, 1971, [1] for Weibull family, Shcolnick, 1985, [19] for stable family, Hawkins and Han, 1986, [5] for noncentral chisquared family, Provost, 1989, [14] for gamma family, and PhamGia, 2000, [11] for beta family. In latest years, the study of product and ratio when X and Y belong to different families has been a great interest in many areas of the sciences. For example,The distributions of the product and ratio, when X and Y are independent random variables with Pareto and Rayleigh independent random variables have been studied by Noura and Seifedine, 2019, [20]. Gamma and Weibull independent random variables respectively, have been studied by Nadarajah and Kotz, 2006, [9]. The distributions of the product and ratio , when X and Y are independent random variables with Pareto and Gamma distributions respectively, have been studied by Nadarajah, Saralees, 2010, [10]. The Ratio of Pareto and KUMARASWAMY Random Variables have been studied by IDRIZI, 2014, [6]. In the applications mentioned before, it is quite possible that X and Y could arise from different but similar distributions. In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y , when X and Y are two independent random variables following Pareto and Rayleigh distributions respectively. with probability density functions (p.d.f.s)

$$f_X(x) = \frac{ca^c}{x^{c+1}} \tag{1}$$

$$f_Y(y) = \lambda e^{-\lambda y} \tag{2}$$

respectively, for $a \leq x < \infty$, $a > 0$, $c > 0$, $y > 0$, $\lambda > 0$.

Notations and Preliminaries

Recall some special mathematical functions, these will be used repeatedly throughout this article.

- The upper incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty \exp(-t) t^{a-1} dt \tag{3}$$

- The lower incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x \exp(-t) t^{a-1} dt \tag{4}$$

- The generalized hypergeometric function is denoted by

$${}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k ... (a_p)_k}{(b_1)_k (b_2)_k ... (b_p)_k} \frac{z^k}{k!} \tag{5}$$

  where $(a)_k$, $(b)_k$ represent Pochhammer's symbol given by

$$(a)_k = a(a+1)...(a+k-1) = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}. \tag{6}$$

- The Exponential integral is generalized to,
  for $n = 0, 1, 2, ..., x > 0$

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt \tag{7}$$

  where n is then the order of the integral.

The calculations of this paper involve several Lemmas

Lemma 1 For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$I(\alpha, r, b) = \int_{\alpha}^{\infty} x^r e^{-bx} dx = \frac{1}{b^{r+1}} \Gamma(r+1, b\alpha) \tag{8}$$

Proof Let $u = bx$, then

$$I(\alpha, r, b) = \int_{b\alpha}^{+\infty} \frac{u^r}{b^{r+1}} e^{-u} du = \frac{1}{b^{r+1}} \Gamma(r+1, b\alpha) \tag{9}$$

Lemma 2 For $t \in \mathbb{R}$,

$$\frac{d}{dx} \Gamma(t, v(x)) = -v(x)^{t-1} e^{-v(x)} \frac{d}{dx} v(x) \tag{10}$$

Proof

$$\frac{d}{dx} \Gamma(t, v) = \frac{d}{dv} \Gamma(t, v) \frac{dv}{dx} \tag{11}$$

Note that

$$\frac{d}{dv} \Gamma(t, v) = -v^{t-1} e^{-v}$$

Lemma 3 For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$\int_0^{\alpha} x^r e^{-bx} dx = \frac{1}{b^{r+1}} \gamma(r+1, b\alpha) \tag{12}$$

Proof For $u = bx$

$$\int_0^{b\alpha} \frac{u^r}{b^{r+1}} e^{-u} du = \frac{1}{b^{r+1}} \gamma(r+1, b\alpha)$$

Lemma 4 The Exponential integral is generalized to,
for $n = 0, 1, 2, ..., x > 0$

$$E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n} dt \tag{13}$$

where n is then the order of the integral. This expression (13) is closely related to the incomplete gamma function as follows, for $n = 0, 1, 2, ..., x > 0$

$$E_n(x) = x^{n-1} \Gamma(1-n, x) \tag{14}$$

Proof For $u = xt$

$$E_n(x) = x^{n-1} \int_x^\infty e^{-u} u^{-n} du = x^{n-1} \Gamma(-n+1)$$

2 Distribution of the Product XY

Theorem 2.1. Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the cumulative distribution function *c.d.f.* of $Z = XY$ can be expressed as:

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1 - \frac{a^c}{z^c \lambda^c} \Gamma(c+1) + \frac{a^c}{z^c \lambda^c} \Gamma(c+1, \frac{\lambda z}{a}) - e^{-\frac{\lambda z}{a}} & \text{if } z > 0 \end{cases} \tag{15}$$

Proof the c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$ Thus, one can write the c.d.f. of XY as

$$\begin{aligned} Pr(XY \leq z) &= \int_0^\infty F_X(\frac{z}{y}) f_Y(y) dy \\ &= \int_0^{z/a} (1 - (\frac{ay}{z})^c) f_Y(y) dy \end{aligned} \tag{16}$$

We can write $F_Z(z)$ as:

$$F_Z(z) = \int_0^\infty (1 - (\frac{ay}{z})^c) f_Y(y) dy - \int_{z/a}^\infty (1 - (\frac{ay}{z})^c) f_Y(y) dy \tag{17}$$

Let

$$I_1 = \int_0^\infty (1 - (\frac{ay}{z})^c) f_Y(y) dy \tag{18}$$

And

$$I_2 = \int_{z/a}^\infty (1 - (\frac{ay}{z})^c) f_Y(y) dy \tag{19}$$

Then

$$F_Z(z) = I_1 - I_2$$

.

Calculus of I_1

$$\begin{aligned} I_1 &= \int_0^\infty (1 - (\frac{ay}{z})^c) f_Y(y) dy \\ &= 1 - \frac{a^c}{z^c} \int_0^\infty y^c \lambda e^{-\lambda y} dy \\ &= 1 - \frac{\lambda a^c}{z^c} \int_0^\infty y^c e^{-\lambda y} dy \end{aligned} \tag{20}$$

Using **Lemma 1** (8) in the integral above

$$I_1 = 1 - \frac{a^c}{z^c\lambda^c}\Gamma(c+1) \tag{21}$$

Calculus of I_2

$$\begin{aligned} I_2 &= \int_{z/a}^{\infty} (1-(\frac{ay}{z})^c) f_Y(y)dy \\ &= \int_{z/a}^{\infty} \lambda e^{-\lambda y} dy - \frac{a^c}{z^c}\int_{z/a}^{\infty} y^c \lambda e^{-\lambda y} dy \\ &= e^{-\lambda z/a} - \frac{a^c}{z^c}\int_{z/a}^{\infty} y^c \lambda e^{-\lambda y} dy \end{aligned} \tag{22}$$

Using **Lemma 1** (8) in the integral above

$$I_2 = e^{-\lambda z/a} - \frac{a^c}{z^c\lambda^c}\Gamma(c+1, \frac{\lambda z}{a})$$

And finally

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1 - \frac{a^c}{z^c\lambda^c}\Gamma(c+1) + \frac{a^c}{z^c\lambda^c}\Gamma(c+1, \frac{\lambda z}{a}) - e^{-\frac{\lambda z}{a}} & \text{if } z > 0 \end{cases} \tag{23}$$

Corollary 2.2. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$, the probability density function *p.d.f.* of $Z = XY$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{ca^c}{\lambda^c z^{c+1}} \Bigg[\Gamma(c+1) - \Gamma(c+1, \frac{\lambda z}{a})\Bigg] & \text{if } z > 0 \end{cases} \tag{24}$$

Proof The probability density function $f_Z(z)$ in (24) follows by differentiation using **Lemma 2**

Corollary 2.3. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > r$, $z > \alpha$, $\alpha > 0$. the moment of order r of $Z = XY$ can be expressed as:

$$E[Z^r] = \left[\frac{ca^c\Big[\Gamma(c+1) - \Gamma(c+1, \lambda\alpha/a)\Big]}{\alpha^{c-r}\lambda^c(c-r)} + \frac{ca^c\Gamma(r+1, \lambda\alpha/a)}{\lambda^c(c-r)}\right] \tag{25}$$

Proof

$$\begin{aligned} E[Z^r] &= \int_0^{+\infty} z^r f_Z(z)dz \\ &= \lim_{\alpha\to 0^+} \int_\alpha^{+\infty} z^r \frac{ca^c}{\lambda^c z^{c+1}} \Big[\Gamma(c+1) - \Gamma(c+1, \lambda z/a)\Big] dz \\ &= \lim_{\alpha\to 0^+} \left[\int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c-r+1}} \Gamma(c+1)dz - \int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c+1-r}} \Gamma(c+1, \lambda z/a)dz\right] \end{aligned} \tag{26}$$

Let

$$I_1 = \int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c-r+1}} \Gamma(c+1)dz$$

and

$$I_2 = \int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c+1-r}} \Gamma(c+1, \lambda z/a)dz$$

$$E[Z^r] = \Big[I_1 - I_2\Big] \tag{27}$$

Calculus of I_1

$$I_1 = \int_\alpha^{\infty} \frac{ca^c}{\lambda^c z^{c-r+1}} \Gamma(c+1)dz = \frac{ca^c \Gamma(c+1)}{\lambda^c \alpha^{c-r}(c-r)} \tag{28}$$

Calculus of I_2

$$I_2 = \frac{ca^c}{\lambda^c} \int_\alpha^{\infty} z^{r-1-c} \Gamma(c+1, \lambda z/a)dz \tag{29}$$

Integration by parts implies of I_2:

$$I_2 = -\frac{\alpha^{r-c}}{r-c} \Gamma(c+1, \lambda\alpha/a) + \frac{\lambda^{c+1}}{a^{c+1}(r-c)} \int_\alpha^{\infty} z^r e^{-\lambda z/a} dz \tag{30}$$

Using **Lemma 1** (8) in the integral above then

$$I_2 = \frac{ca^c \Gamma(c+1, \lambda\alpha/a)}{\lambda^c (c-r)\alpha^{c-r}} - \frac{ca^c \Gamma(r+1, \lambda\alpha/a)}{\lambda^c (c-r)} \tag{31}$$

And finally

$$E[Z^r] = \left[\frac{ca^c \Big[\Gamma(c+1) - \Gamma(c+1, \lambda\alpha/a)\Big]}{\alpha^{c-r} \lambda^c (c-r)} + \frac{ca^c \Gamma(r+1, \lambda\alpha/a)}{\lambda^c (c-r)}\right] \tag{32}$$

Corollary 2.4. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 1$. the Expected value of $Z = XY$ can be expressed as:

$$E[Z] = \left[\frac{ca^c\Big[\Gamma(c+1) - \Gamma(c+1, \lambda\alpha/a)\Big]}{\alpha^{c-1}\lambda^c(c-1)} + \frac{ca^c\Gamma(2, \lambda\alpha/a)}{\lambda^c(c-1)}\right] \tag{33}$$

Corollary 2.5. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 2$. the Variance of $Z = XY$ can be expressed as:

$$\begin{aligned} \sigma^2 =& \frac{ca^c\Big[\Gamma(c+1) - \Gamma(c+1, \lambda\alpha/a)\Big]}{\alpha^{c-2}\lambda^c(c-2)} + \frac{ca^c\Gamma(3, \lambda\alpha/a)}{\lambda^c(c-2)} \\ &- \left[\frac{ca^c\Big[\Gamma(c+1) - \Gamma(c+1, \lambda\alpha/a)\Big]}{\alpha^{c-1}\lambda^c(c-1)} + \frac{ca^c\Gamma(2, \lambda\alpha/a)}{\lambda^c(c-1)}\right]^2 \end{aligned} \tag{34}$$

Proof By definition of variance :

$$\sigma^2 = E[Z^2] - E[Z]^2$$

Corollary 2.6. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. the Survival function of $Z = XY$ can be expressed as:

$$S_Z(z) = \begin{cases} 1 & \text{if } z \le 0 \\ \frac{a^c}{z^c\lambda^c}\Gamma(c+1) - \frac{a^c}{z^c\lambda^c}\Gamma(c+1, \frac{\lambda z}{a}) + e^{-\frac{\lambda z}{a}} & \text{if } z > 0 \end{cases} \tag{35}$$

Proof By definition of the Survival function $S_Z(z) = 1 - F_Z(z)$

Corollary 2.7. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. the Hazard function of $Z = XY$ can be expressed as:

$$h_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{ca^c\Big[\Gamma(c+1) - \Gamma(c+1, \lambda z/a)\Big]}{za^c\Big[\Gamma(c+1) - \Gamma(c+1, \lambda z/a)\Big] + z^{c+1}\lambda^c e^{-\lambda z/a}} & \text{if } z > 0 \end{cases} \tag{36}$$

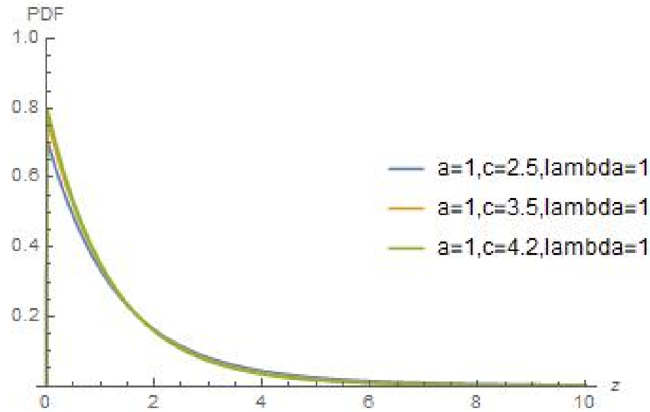

Figure 1: Plots of the pdf (24) for $a = 1, c = 2.5, 3.5, 4.2, \lambda = 1$

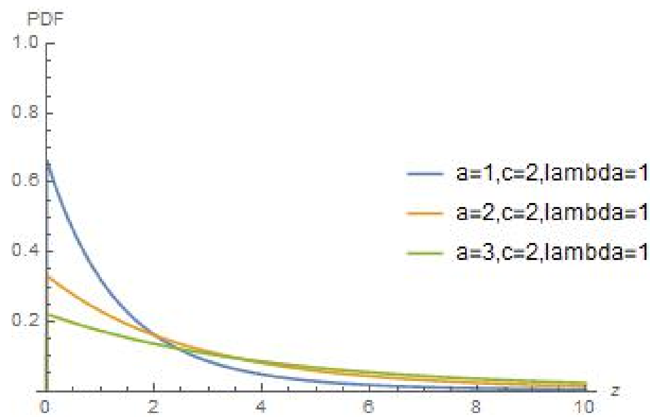

Figure 2: Plots of the pdf (24) for $a = 1, 2, 3, c = 2, \lambda = 1$

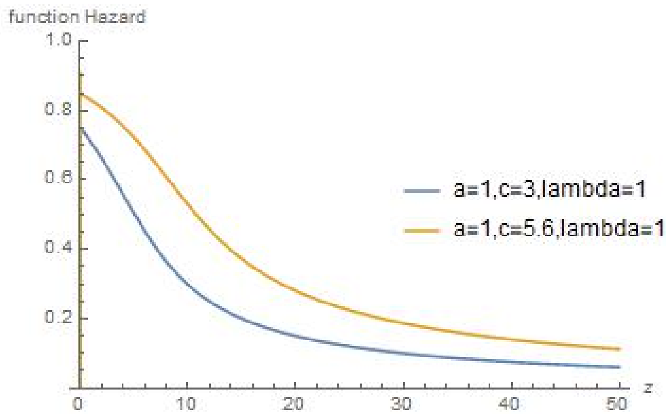

Figure 3: Plots of the Hazard function (36) for $a = 1, c = 3, 5.6, \lambda = 1$

3 Distribution of the Ratio X/Y

Theorem 3.1. Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the cumulative distribution function *c.d.f.* of $Z = X/Y$ can be expressed as:

$$F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ e^{-\lambda a/z} - \frac{a^c \lambda^c}{z^c} \Gamma(-c+1, \lambda a/z) & \text{if } z > 0, c < 1 \\ e^{-\lambda a/z} - \frac{a\lambda}{z} E_c(\lambda a/z) & \text{if } z > 0, c = 1, 2, 3, ... \\ e^{-\lambda a/z} & \\ -\dfrac{a^c \lambda^c}{z^c} \left[\displaystyle\int_0^\infty t^{-c} \Big[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^i}{i!} \Big] dt \right. & \text{if } z > 0, -n < 1 - c < -n + 1 \\ \left. - \displaystyle\int_0^{\lambda a/z} t^{-c} \Big[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^i}{i!} \Big] dt - \sum_{i=0}^{n-1} (-1)^i \frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!} \right] & \end{cases} \tag{37}$$

Proof the c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$

$$\begin{aligned}
F_Z(z) &= pr(X/Y \leq z) \\
&= \int_0^\infty F_X(zy) f_Y(y) dy \\
&= \int_{a/z}^\infty (1 - (\frac{a}{yz})^c) f_Y(y) dy \\
&= \int_{a/z}^\infty f_Y(y) dy - \frac{a^c}{z^c} \int_{a/z}^\infty \frac{f_Y(y)}{y^c} dy
\end{aligned} \tag{38}$$

Let

$$I_1 = \int_{a/z}^\infty \lambda e^{-\lambda y} dy = e^{-\lambda a/z}$$

And

$$I_2 = \int_{a/z}^\infty \frac{\lambda e^{-\lambda y}}{y^c} dy$$

Then we can write

$$F_Z(z) = I_1 - I_2$$

Calculus of I_2

$$\begin{aligned}
I_2 &= \lambda \int_{a/z}^\infty \frac{e^{-\lambda y}}{y^c} dy \\
&= \lambda \int_{a/z}^\infty y^{-c} e^{-\lambda y} dy
\end{aligned}$$

if we substitute $u = \lambda y$, Then

$$\begin{aligned}
I_2 &= \lambda \int_{\lambda a/z}^\infty e^{-u} (\frac{u}{\lambda})^{-c} \frac{du}{\lambda} \\
&= \lambda^c \int_{\lambda a/z}^\infty u^{-c} e^{-u} du \\
&= \lambda^c \Gamma(1 - c, \lambda a/z)
\end{aligned} \tag{39}$$

Finally we get

$$F_Z(z) = -\frac{a^c \lambda^c}{z^c} \Gamma(-c + 1, \frac{\lambda a}{z}) + e^{-\lambda \frac{a}{z}} \tag{40}$$

For $c < 1$.

For $c = 1, 2, 3, ...$ using **Lemma 4** we have

$$\Gamma(1 - c, \lambda a/z) = (\frac{\lambda a}{z})^{1-c} E_c(\lambda a/z)$$

, and

$$F_Z(z) = e^{-\lambda a/z} - \frac{a\lambda}{z} E_c(\lambda a/z) \tag{41}$$

Where c is the order of the integral.

For $-n < 1-c < -n+1$, $1-c \neq -1, -2, ...$ we have

$$\begin{aligned}\Gamma(1-c, \lambda a/z) = &\int_0^\infty t^{-c}[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^i}{i!}]dt \\ &- \int_0^{\lambda a/z} t^{-c}[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^i}{i!}]dt - \sum_{i=0}^{n-1} (-1)^i \frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!}\end{aligned} \tag{42}$$

Corollary 3.2. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$, the probability density function $p.d.f.$ of $Z = X/Y$ can be expressed as:

$$f_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ c(a\lambda)^c \frac{\Gamma(1-c, a\lambda/z)}{z^{c+1}} & \text{if } z > 0, c < 1 \\ \frac{\lambda a}{z^2} e^{-\lambda a/z} - \frac{(a\lambda)^2}{z^3} E_{c-1}(\lambda a/z) + \frac{a\lambda}{z^2} E_c(\lambda a/z) & \text{if } z > 0, c = 1, 2, 3, ... \end{cases} \tag{43}$$

For $z > 0, -n < 1-c < -n+1, (1-c) \neq -1, -2, -3, ...$

$$\begin{aligned} f_Z(z) = &\frac{ca^c\lambda^c}{z^{c+1}} \left[\int_0^\infty t^{-c} \left[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^i}{i!} \right] dt \right] + \frac{a\lambda e^{-a\lambda/z}}{z^2} Q(n, -a\lambda/z) \\ &+ (-1)^{n+1} c\Gamma(1-c+n) \frac{(a\lambda)^{n+1}}{z^{n+2}} \frac{{}_2F_2(1, 1-c+n, 1+n, 2-c+n, -a\lambda/z)}{\Gamma(1+n)\Gamma(2-c+n)} \\ &- \frac{(a\lambda)^c}{z^{c+1}} c\Gamma(1-c, 0, a\lambda/z) - \frac{a\lambda}{z^2} e^{-a\lambda/z} \frac{\Gamma(n, -a\lambda/z)}{(n-1)!} \\ &+ (-1)^{n+1} (a\lambda)^{n+1} \frac{1}{n! z^{n+2}} (\frac{c}{-1+c-n})_2F_2(1, 1-c+n, 1+n, 2-c+n, -\frac{a\lambda}{z}) \end{aligned} \tag{44}$$

Where $Q(n, -a\lambda/z)$ is the Gamma regularized function, and ${}_2F_2(1, 1-c+n, 1+n, 2-c+n, -a\lambda/z)$ is the generalized hypergeometric function.

Proof The probability density function $f_Z(z)$ in (43) follows by differentiation using **Lemma 2**(10), and

$$\frac{d}{dz} E_n(z) = -E_{n-1}(z) \tag{45}$$

For $n = 1, 2, 3, ...$

Corollary 3.3. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$.the Survival function of $Z = X/Y$ can be expressed as:

$$S_Z(z) = \begin{cases} 1 & \text{if } z \le 0 \\ 1 - e^{-\lambda a/z} + \frac{a^c\lambda^c}{z^c}\Gamma(-c+1, \lambda a/z) & \text{if } z > 0, c < 1 \\ 1 - e^{-\lambda a/z} + \frac{a\lambda}{z}E_c(\lambda a/z) & \text{if } z > 0, c = 1, 2, 3, ... \\ 1 - e^{-\lambda a/z} & \\ \quad + \dfrac{a^c\lambda^c}{z^c}\Bigg[\displaystyle\int_0^\infty t^{-c}\Big[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\Big]dt & \text{if } z > 0, -n < 1 - c < -n + 1 \\ \quad - \displaystyle\int_0^{\lambda a/z} t^{-c}\Big[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\Big]dt - \sum_{i=0}^{n-1}(-1)^i\frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!}\Bigg] & \end{cases} \tag{46}$$

Proof By definition of the Survival function $S_Z(z) = 1 - F_Z(z)$

Corollary 3.4. Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. the Hazard function of $Z = X/Y$ can be expressed as:

$$h_Z(z) = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{c(a\lambda)^c\Gamma(1-c, a\lambda/z)}{z(a\lambda)^c\Gamma(1-c, a\lambda/z) + z^{c+1}(1-e^{-\lambda a/z})} & \text{if } z > 0, c < 1 \\ \frac{\frac{\lambda a}{z^2}e^{-\lambda a/z} - \frac{(a\lambda)^2}{z^3}E_{c-1}(\lambda a/z) + \frac{a\lambda}{z^2}E_c(\lambda a/z)}{1 - e^{-\lambda a/z} + \frac{a\lambda}{z}E_c(\lambda a/z)} & \text{if } z > 0, c = 1, 2, ... \end{cases} \tag{47}$$

For $z > 0, -n < 1-c < -n+1, (1-c) \neq -1, -2, -3, ...$

$$h_Z(z) = \frac{\frac{ca^c\lambda^c}{z^{c+1}}\left[\int_0^\infty t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt\right]}{1 - e^{-\lambda a/z} + \frac{a^c\lambda^c}{z^c}\left[\int_0^\infty t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \int_0^{\lambda a/z} t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \sum_{i=0}^{n-1}(-1)^i\frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!}\right]}$$
$$+ \frac{\frac{a\lambda e^{-a\lambda/z}}{z^2}Q(n, -a\lambda/z) + (-1)^{n+1}c\Gamma(1-c+n)\frac{(a\lambda)^{n+1}}{z^{n+2}}\frac{{}_2F_2(1,1-c+n,1+n,2-c+n,-a\lambda/z)}{\Gamma(1+n)\Gamma(2-c+n)}}{1 - e^{-\lambda a/z} + \frac{a^c\lambda^c}{z^c}\left[\int_0^\infty t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \int_0^{\lambda a/z} t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \sum_{i=0}^{n-1}(-1)^i\frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!}\right]}$$
$$- \frac{\frac{(a\lambda)^c}{z^{c+1}}c\Gamma(1-c, 0, a\lambda/z) - \frac{a\lambda}{z^2}e^{-a\lambda/z}\frac{\Gamma(n, -a\lambda/z)}{(n-1)!}}{1 - e^{-\lambda a/z} + \frac{a^c\lambda^c}{z^c}\left[\int_0^\infty t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \int_0^{\lambda a/z} t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \sum_{i=0}^{n-1}(-1)^i\frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!}\right]}$$
$$+ \frac{(-1)^{n+1}(a\lambda)^{n+1}\frac{1}{n!z^{n+2}}(\frac{c}{-1+c-n})_2F_2(1, 1-c+n, 1+n, 2-c+n, -\frac{a\lambda}{z})}{1 - e^{-\lambda a/z} + \frac{a^c\lambda^c}{z^c}\left[\int_0^\infty t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \int_0^{\lambda a/z} t^{-c}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt - \sum_{i=0}^{n-1}(-1)^i\frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!}\right]} \quad (48)$$

Proof By definition of the hazard function $h_Z(z) = \frac{f_Z(z)}{S_Z(z)}$

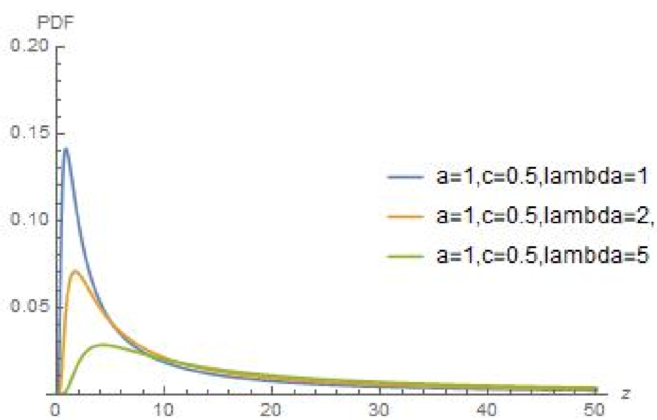

Figure 4: Plots of the probability density function (43) for $a = 1, c = 0.5, \lambda = 1, 2, 5$

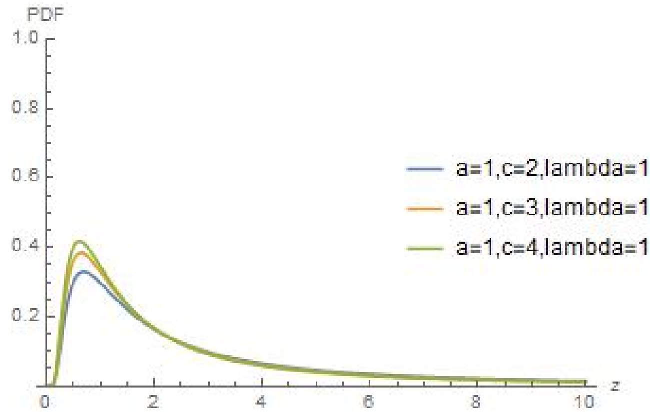

Figure 5: Plots of the probability density function (43) for $a = 1, c = 2, 3, 4, \lambda = 1$

4 Hazard function

It is useful to think about real phenomena and how their hazard functions might be shaped. For example, if T denotes the age of a car when it first has a serious engine problem, then one might expect the corresponding hazard function $h(t)$ to be increasing in t; that is, the conditional probability of a serious engine problem in the next month, given no problem so far, will increase with the life of the car. In contrast, if one were studying infant mortality in a region of the world where there was poor nutrition, one might expect $h(t)$ to be decreasing during the first year of life. This is known to be due to selection during the first year of life. Finally, in some applications (such as when T is the lifetime of a light bulb or the time to which you won a BIG lottery), the hazard function will be approximately constant in t. This means that the chances of failure in the next short time interval, given that failure hasn't yet occurred, does not change with t; e.g., a 1-month old bulb has the same probability of burning out in the next week as does a 5-year old bulb. As we will see below, this 'lack of aging' or 'memory less' property uniquely defines the exponential distribution, which plays a central role in survival analysis. The hazard function may assume more a complex form. For example, if T denote the age of death, then the hazard function $h(t)$ is expected to be decreasing at first and then gradually increasing in the end, reflecting higher hazard of infants and elderly.

Let take an example, for $Z = X/Y$ quotient of Pareto and Exponential random variables, For $z > 0$

$$h_Z(z) = \frac{c(a\lambda)^c\Gamma(1-c, a\lambda/z)}{z(a\lambda)^c\Gamma(1-c, a\lambda/z) + z^{c+1}(1-e^{-\lambda a/z})} \tag{49}$$

1. **Continuity of $h_Z(z)$:**

for $z > 0$

$$h_Z(z) = \frac{u(z)}{v(z)}$$

Where

$$u(z) = c(a\lambda)^c\Gamma(1-c, a\lambda/z) \tag{50}$$

And

$$v(z) = z(a\lambda)^c\Gamma(1-c, a\lambda/z) + z^{c+1}(1-e^{-\lambda a/z}) \tag{51}$$

We have $\Gamma(1-c, a\lambda/z)$ is continuous, then $u(z)$ is a continuous function , and for $z > 0$, $(1-e^{-\lambda a/z}) > 0$ and continuous, then the function $v(z) > 0$ is positive and continuous. finally we get that, $h_Z(z) = \frac{u(z)}{v(z)}$ is a continuous function.

2. **Derivative of $h_Z(z)$**

the derivative of $h_Z(z)$ is given as

$$\begin{aligned}\frac{d}{dz}h_Z(z) = &\frac{c(1+c)(1-e^{-a\lambda/z})z^c(a\lambda)^c\Gamma(1-c, a\lambda/z) + (a\lambda)^{2c}c(\Gamma(1-c, a\lambda/z))^2}{\left[z(a\lambda)^c\Gamma(1-c, a\lambda/z) + z^{c+1}(1-e^{-\lambda a/z})\right]^2} \\ &+ \frac{(a\lambda)ce^{-a\lambda/z}}{z^{-c+2}\left[(1-e^{-a\lambda/z})z^{1+c} + (a\lambda)^c z\Gamma(1-c, a\lambda/z)\right]}\end{aligned} \tag{52}$$

Proof

$$\frac{d}{dz}\left(\frac{u(z)}{v(z)}\right) = \frac{\frac{du(z)}{dz}v(z) - u(z)\frac{dv(z)}{dz}}{v(z)^2} \tag{53}$$

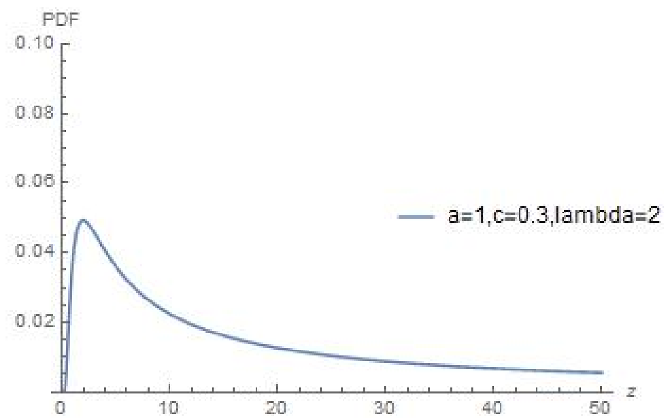

Figure 6: Plot of the Hazard function (47) for $a = 1, c = 0.3$ *and* $\lambda = 2$.

5 Applications and simulations

5.1 Replenishment of a Life Support system

With the advent of long-duration manned space flights, estimating the changes that will occur in the quantities of certain substances in the desired ecological system has become more complex.
Ideally, a mathematical model for general system analysis of an ecological system for.lung-duration flights will provide for:

1. A formulation of the control problem from which optimal control functions car be determined.
2. A preliminary design for the ecological system in respect to system stability and control considerations.
3. A method for determining the time required to restore the Işystem to a suitable balance in the case of a mishap and the time required initially to put the system into operation.
4. The determination of resupply requirements.

One aspect of a preliminary model designed to include the above considerations has resulted in the requirement for evaluating a product of random variables. It may be stated in the following manner:

Consider the amount of oxygen in a cabin atmosphere. The amount is affected by leakage, crew consumption, and resupply from a storage capacity. Let:

$X(t)$=amount of oxygen (in moles) in the cabin atmosphere at time t.

L= proportion of rate of loss of oxygen due to leakage from the cabin atmosphere per time period.

Y_1=rate of increase in oxygen content in the cabin atmosphere from storage per time period.

K=rate of decrease in the oxygen content in the cabin atmosphere due to crew consumption per time period.

The estimated amount of oxygen at time t is:

$$X(t) = \exp(Lt)(Y_1 - K) + X(0) \tag{54}$$

In this study, $W = exp(Lt)$ and $Z = (Y_1 - K)$ represent random variables which are functions of time. A knowledge of combining random Variables in product forms is required for solving this problem. Suppose Z is a random variable follows Pareto distribution with parameter $c = 2$ and $a = 1$, and W is a random variable follows Exponential distribution with parameter $\lambda = 1$, then The estimated amount of oxygen at time t is:

$$X(t) = \begin{cases} 0 & \text{if } t < 0 \\ X(0) & \text{if } t = 0 \\ \frac{2[\Gamma(3) - \Gamma(3,t)]}{t^3} & \text{if } t > 0 \end{cases} \tag{55}$$

5.2 Measurement or Radiation by Electronic Counters

Proportional, Geiger, and scintillation counters are often used to ' detect X and γ radiation, as well as other charged particles such as electrons and α particles. Design of these counters and their associated circuits depends to some extent on what is to be detected. A device common to all counters is a scaler. This electronic device counts pulses produced by the counter. Once the number of pulses over a measured period of time is known, the average counting rate is obtained by simple division. If the rate of pulses is too high for a mechanical device, it is necessary to scale down the pulses by a known factor before feeding them to a mechanical counter. There are two kinds of scalers: the binary scaler in which the scaler factor is some power of 2, and the decade scaler in which the scaling factor is some power of 10.

A typical binary scaler has several scaling factors ranging from 2^0 to 2^{14}. The scaling circuit is made up of a number of identical "stages" connected in series, the number of stages being equal to n, where 2^n is the desired scaling factor, Each stage is composed of a number of vacuum tubes, capacitors, and resistors, connected so that only one pulse of current is transmitted for every two pulses received, Since the output of one stage is connected to the input of another, this division by two is repeated as many times as there are stages. The output of the last stage is connected to a mechanical counter that will register one count for every pulse transmitted to it by the last stage. Thus, if N pulses from a counter are passed through a circuit of n stages, only $\frac{N}{2^n}$ will register on the mechanical counter. Because arrival of X-ray quanta in the counter is random in time, the accuracy of a counting rate measurement is governed by the laws of probability. Two counts of the same X-ray beam for identical periods of time will not be precisely the same because of the random spacing between pulses, even though the counter and scaler are functioning perfectly. Clearly, the accuracy of a rate measurement of this kind improves as the time of counting is prolonged. It is therefore; important to know how long to count in order to attain a specified degree of accuracy. This problem is complicated when additional background causes contamination in the counting process. This unavoidable background is due to cosmic rays and may be augmented, particularly in some laboratories, by nearby radioactive materials. Suppose we want to estimate the diffraction background in the presence of a fairly large unavoidable background. Let N be the number of pulses ! counted in a given time from a radiation source; Let N_b be the number counted in the same time with radiation source removed. The N_b counts are due to unavoidable background and $(N - N_b)$ to the diffract able background being measured. The relative probable error in $(N - N_b)$ is

$$E_{N-N_b} = \frac{67\sqrt{N + N_b}}{N - N_b}$$

percent.

since N and N_b are random variables, the desirability of obtaining the density function of the above quotient from of random variable is apparent.
For instance, if $\sqrt{N + N_b}$ is a random variable follows Pareto distribution with parameter $c = 0.5$ and $a = 1$, and $N - N_b$ is a random variable follows Exponential distribution with parameter $\lambda = 1$, then by using our result The relative probable error in $(N - N_b)$ is

$$E_{N-N_b} = \begin{cases} 0 & \text{if } z \leq 0 \\ \frac{33.5\Gamma(0.5,1/z)}{z^{1.5}} & \text{if } z > 0 \end{cases} \tag{56}$$

6 Conclusion

This paper has derived The analytical expressions of the PDF, CDF,the moment of order r, the survival function, and the hazard function, for the distributions of XY and X/Y when X and Y are Pareto and Exponential random variables distributed independently of each other. we illustrate our results in some graphics of the distributions of product and ratio.
Finally we have discussed two examples of engineering applications for the distribution of the product and ratio.

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