Numbers

We are used to numbers and we use them constantly in our life. But why do we actually need them? The obvious reason is to count objects. Indeed, this is the simplest task we encounter that requires numbers. For this we need natural numbers: $1, 2, 3, \ldots$.

Once we have numbers we need some operations on them. We can add and multiply them, and these operations are also used frequently in real life.

Now, can we reverse these basic operations? Yes, we can. If we add b to a, we then can subtract b back and obtain the original number a. If we multiply number a by number b, we can divide the result by b and obtain the original number a. Subtraction is an inverse operation for addition and division is an inverse operation for multiplication. Clearly, these operations are also useful in every day life. But are they always applicable to the numbers $1, 2, 3, \ldots$?

Clearly not. For example, we cannot subtract 3 from 2 if we operate only with numbers $1, 2, 3, \ldots$. But there is a solution to this problem: we can introduce negative numbers (and zero). If we consider *integer* numbers

$$\ldots, -2, -1, 0, 1, 2, \ldots,$$

then the subtraction operation becomes valid for all numbers. Note also that addition and multiplication operations can be defined for negative numbers as well. This is the main reason to introduce the negative numbers: to make subtraction operation well defined.

Once we introduce negative numbers, we have that addition, subtraction and multiplication are well defined. There is still an issue with division: we cannot say what is $3/2$ if we know only integer numbers. The standard solution here, is to introduce *rational numbers* (fractions). This approach is standard in mathematics and works in many settings. But sometimes it is important to stay within integer numbers and still consider division operation (recall the Judgement of Solomon).

Number theory is the area of mathematics dedicated to the studies of integers and operations on them. The basics of number theory has natural applications related to real life problems. However, as number theory developed it turned out that advanced topics in this field could not find any applications or connections to real life. This was explicitly stated (and actually praised) by prominent number theorists in the first half of the 20th century, such as Godfrey Hardy and Leonard Dickson.

But the situation changed dramatically soon after that! First, as Donald Knuth said "... virtually every theorem in elementary number theory arises in a natural, motivated way in connection with the problem of making computers do high-speed numerical calculations". But maybe even more importantly, number theory (including its highly involved parts) is vital for the modern cryptography. And cryptography, in its turn, dramatically affects our life. Technologies underlying email services, messengers, online transactions, Internet as a whole, heavily rely on cryptographic protocols.

Thus, number theory is an important area of mathematics with crucial applications in computer science.