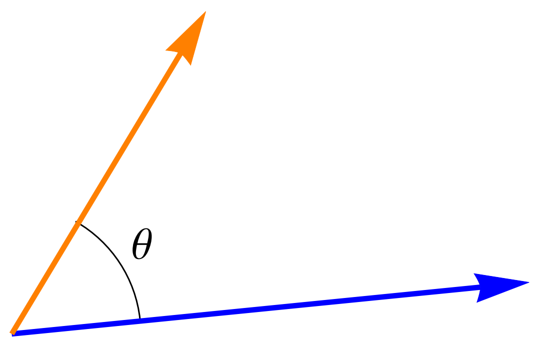
12. Dot product: geometric formula

Reflect

There is a more geometric formula way to understand the dot product. Let's consider two vectors $\vec{v}$ and $\vec{w}$ where the angle between them is θ.



POLL
The geometric formula for the dot product is given by which one of the following?

RESULTS

- **[mathjaxinline]\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\sin\theta[/mathjaxinline]** **4%**
- **[mathjaxinline]\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta[/mathjaxinline]** **81%**
- **[mathjaxinline]\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\tan\theta[/mathjaxinline]** **7%**
- **I do not know how to think about this yet** **9%**

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FEEDBACK
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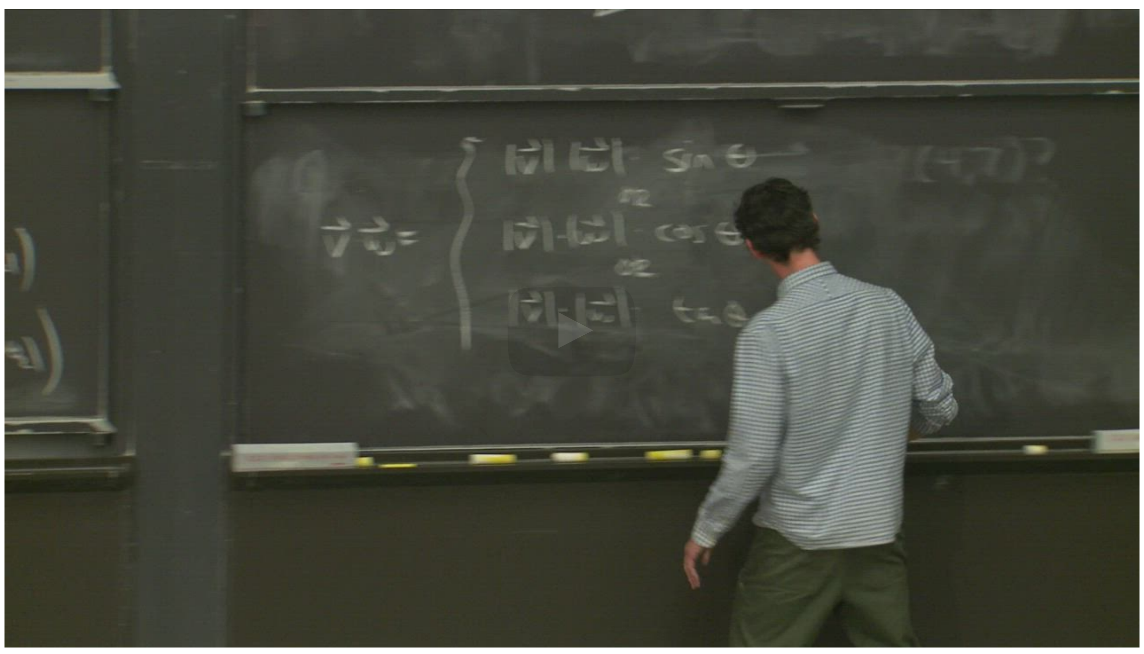
Geometric formula of the dot product



Yeah?

STUDENT: Maybe compare it to that first formula you gave us?

PROFESSOR: Yeah.

Yeah.

Yeah, that's good.

That's good.

So something I like to do is to have

in mind a few examples, a few simple examples, where

we can work out these things.

And then I can test whatever I need

to remember about dot products by looking

at the simple examples.

So can people think of an example

where it

out wha

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0:00 / 0:00 2.0x

Yeah.

STUDENT: [INAUDIBLE]

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To help us remember the geometric definition, we'll try some simple examples using the formula for the dot product

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2$$

and compare it to the above formulas.

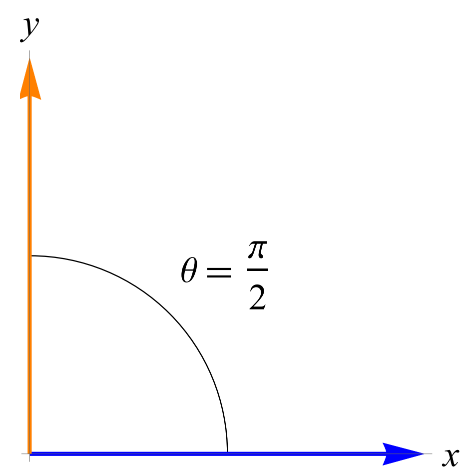
A first example: One good case to consider is

$$\vec{v} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{w} = \langle 0, 1 \rangle.$$

Equation for the dot products gives us

$$\vec{v} \cdot \vec{w} = \langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 0.$$

Drawing the two vectors, we see that $\theta = \pi/2$.



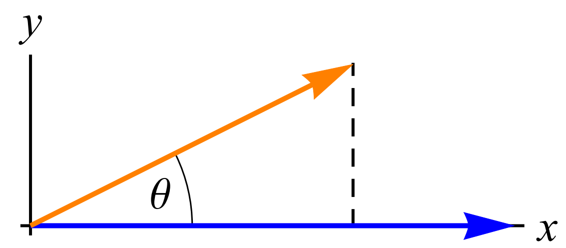
This matches with the second choice since

$$|\vec{v}||\vec{w}| \cos\left(\frac{\pi}{2}\right) = 0.$$

A second example: Consider the two vectors

$$\vec{v} = \langle v_1, 0 \rangle \quad \text{and} \quad \vec{w} = \langle w_1, w_2 \rangle,$$

which are sketched below.



Then

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$$\vec{v} \cdot \vec{w} = v_1 w_1$$

Since $v_2 = 0$, we know that

$$v_1 = |\vec{v}|.$$

We also know that

$$w_1 = |\vec{w}| \cos\theta.$$

Putting this all together gives

$$\vec{v} \cdot \vec{w} = v_1 w_1 = |\vec{v}||\vec{w}| \cos\theta.$$

Where does the relationship between the algebraic definition and geometric interpretation come from?

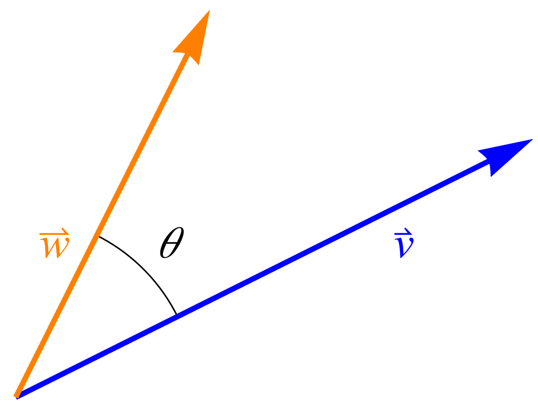
(Optional) proof of equivalence

images need alt text

Proof that

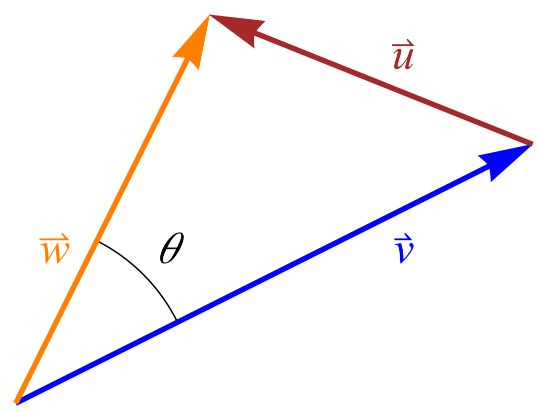
$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta \tag{3.24}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$ as shown in the figure below.



To see this, let's use the vectors in the figure above to form a triangle with sides $\vec{v} = \langle v_1, v_2 \rangle$, $\vec{w} = \langle w_1, w_2 \rangle$, and $\vec{u}$ where

$$\vec{u} = \vec{w} - \vec{v} = \langle w_1 - v_1, w_2 - v_2 \rangle. \tag{3.25}$$



The law of cosines tells us that

$$|\vec{u}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| \cos\theta.$$

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Note: When $\theta = \pi/2$, the triangle is a right triangle and the law of cosines simplifies to the Pythagorean theorem. In our case, θ is not specified.

However, we also know from the properties of dot products that

$$\begin{aligned} |\vec{u}|^2 &= \vec{u}\cdot\vec{u} && (3.27)\\ &= (\vec{w}-\vec{v})\cdot(\vec{w}-\vec{v}) && (3.28)\\ &= \vec{w}\cdot\vec{w}+\vec{v}\cdot\vec{v}-\vec{w}\cdot\vec{v}-\vec{v}\cdot\vec{w} && (3.29)\\ &= |\vec{w}|^2+|\vec{v}|^2-2\vec{v}\cdot\vec{w}. && (3.30) \end{aligned}$$

using our previous definition of the dot product.

In summary, we just showed that the quantity $|\vec{u}|^2$ satisfies two equalities. Namely,

$$\begin{aligned} |\vec{u}|^2 &= |\vec{v}|^2+|\vec{w}|^2-2|\vec{v}||\vec{w}|\cos\theta && (3.31)\\ |\vec{u}|^2 &= |\vec{w}|^2+|\vec{v}|^2-2\vec{v}\cdot\vec{w}. && (3.32) \end{aligned}$$

Setting both expressions for $|\vec{u}|^2$ equal to each other gives

$$|\vec{v}|^2+|\vec{w}|^2-2|\vec{v}||\vec{w}|\cos\theta = |\vec{w}|^2+|\vec{v}|^2-2\vec{v}\cdot\vec{w}. \qquad (3.33)$$

Solving for $\vec{v}\cdot\vec{w}$ then gives

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta. \qquad (3.34)$$

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12. Dot product: geometric formula

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