MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Problem 7: Sampling families

(3/3 points)

We are given the following statistics about the number of children in the families of a small village.

There are 100 families: 10 families have no children, 40 families have 1 child each, 30 families have 2 children each, 10 families have 3 each, and 10 families have 4 each.

1. If you pick a family at random (each family in the village being equally likely to be picked), what is the expected number of children in that family?

   1.7

   Answer: 1.7

2. If you pick a child at random (each child in the village being equally likely to be picked), what is the expected number of children in that child's family (including the picked child)?

   41/17

   Answer: 2.41176

3.

Generalize your approach from part 2: Suppose that a fraction p_k of the families have k children each. Let K be the number of children in a randomly selected family, and let $a = \mathbf{E}[K]$ and $b = \mathbf{E}[K^2]$. Let W be the number of children in a randomly chosen child's family. Express $\mathbf{E}[W]$ in terms of a and b using standard notation .

$\mathbf{E}[W] =$ b/a ✔ **Answer:** b/a

Answer:

1. The PMF describing K, the number of children in a randomly selected family, is

$$p_K(k) = \begin{cases} 1/10, & k = 0, \\ 4/10, & k = 1, \\ 3/10, & k = 2, \\ 1/10, & k = 3, \\ 1/10, & k = 4, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[K] = 0 \cdot \frac{1}{10} + 1 \cdot \frac{4}{10} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{10} + 4 \cdot \frac{1}{10} = \frac{17}{10}.$$

2.

Note that there are a total of 170 children in the village; 40 of them come from a family with only one child, 60 of them from a family with two children, 30 of them from a family with three children and 40 of them from a family of four children. Each child is equally likely to be picked. Thus, the PMF of W, the number of children in the family of a randomly selected *child*, is

$$p_W(w) = \begin{cases} 4/17, & w = 1, \\ 6/17, & w = 2, \\ 3/17, & w = 3, \\ 4/17, & w = 4, \\ 0, & \text{otherwise}. \end{cases}$$

Hence,

$$\mathbf{E}[W] = 1 \cdot \frac{4}{17} + 2 \cdot \frac{6}{17} + 3 \cdot \frac{3}{17} + 4 \cdot \frac{4}{17} = \frac{41}{17}.$$

3. Parts 1 and 2 both deal with a random variable that describes the number of children in a particular family; the distinction is, of course, in the manner in which that particular family is selected. By selecting a child at random, we immediately remove the possibility of selecting a family with no children and in general induce a bias towards families with many children. It is a clear illustration of the random incidence paradox; it is only when we appreciate the differences in the underlying experiments that the paradox is resolved.

There is a neat relationship between K, the number of members in a randomly selected set, and W, the number of members in the set associated with a randomly selected member. Generalizing the logic in part 2, the PMF of W is merely the PMF of K, but weighted in proportion to the number of members, k, of each set. Mathematically, letting c denote a normalizing constant,

$$p_W(k) = c \cdot k p_K(k) \quad \Rightarrow \quad c = \frac{1}{\mathbf{E}[K]} \quad \Rightarrow \quad p_W(k) = \frac{k p_K(k)}{\mathbf{E}[K]}, k = 0, 1, \ldots.$$

From this, it follows that

$$\mathbf{E}[W] = \sum_k k p_W(k) = \sum_k \frac{k^2 p_K(k)}{\mathbf{E}[K]} = \frac{\mathbf{E}[K^2]}{\mathbf{E}[K]}.$$

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DISCUSSION

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