10. Likelihood Ratio Test: Basic Form

Note: Below we introduce the most basic form of the likelihood Ratio test as preparation for the more general form that Prof Rigollet discusses in the next video.

Basic Form of Likelihood Ratio Test

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathbf{P}_{\theta^*}$, and consider the associated statistical model $(E, \{\mathbf{P}_\theta\}_{\theta \in \mathbb{R}^d})$. Suppose that $\mathbf{P}_\theta$ is a discrete probability distribution with pmf given by p_θ.

In its most basic form, the **likelihood ratio test** can be used to decide between two hypotheses of the following form:

$$\begin{aligned} H_0 :& \quad \theta^* = \theta_0 \\ H_1 :& \quad \theta^* = \theta_1. \end{aligned}$$

Recall the likelihood function

$$\begin{aligned} L_n : \mathbb{R}^n \times \mathbb{R}^d &\to \mathbb{R} \\ (x_1, \ldots, x_n; \theta) &\mapsto \prod_{i=1}^{n} p_\theta(x_i). \end{aligned}$$

The **likelihood ratio test** in this set-up is of the form

$$\psi_C = \mathbf{1}\left(\frac{L_n(x_1, \ldots, x_n; \theta_1)}{L_n(x_1, \ldots, x_n; \theta_0)} > C\right).$$

where C is a threshold to be specified.

Basic Likelihood Ratio Test

2/2 points (graded)

In this problem, you have an **unfair** coin, and your friend tells you that the coin comes up heads with probability either 25% or 75%, but doesn't know which is true. You design a test to decide between the two possibilities. First, you model the results of n coin flips as $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Ber}(p^*)$ where Heads is $+1$ and Tails is 0. The associated statistical model is $(\{0,1\}, \{\mathrm{Ber}(p)\}_{p \in (0,1)})$.

You formulate the hypotheses

$$\begin{aligned} H_0 &: \ p^* = 0.25 \\ H_1 &: \ p^* = 0.75. \end{aligned}$$

You decide to use the likelihood-ratio test described above with threshold $C = 1$. Suppose you observe a single coin-flip: $X_1 = 1$.

Should you **reject** or **fail to reject** the null hypothesis?

- (●) Reject
- ( ) Fail to reject

Now suppose you observe the data set

$$\mathbf{X} = \{1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0\}.$$

Using the specified likelihood ratio test, would you **reject** or **fail to reject** the null hypothesis? (**Hint**: You should not have to compute the likelihoods exactly to be able to answer this question.)

- Reject
- Fail to reject

Solution:

If we observe $X_1 = 1$, then

$$L_1(1; 0.25) = 0.25, \quad L_1(1; 0.75) = 0.75.$$

Hence

$$\frac{L_1(1; 0.75)}{L_1(1; 0.25)} > 1,$$

and we would reject the null hypothesis.

Now consider the data set

$$\mathbf{X} = \{1,0,0,0,1,1,1,0,0,1,0,1,0,0,0,0\}.$$

which consists of 6 heads, 10 tails, and 16 total coin flips. Intuitively, since there are more tails than heads, it seems more likely that the parameter p^* is closer to 0 than 1. And indeed, the likelihood ratio test we designed confirms this heuristic.

Note that

$$L_{16}(\mathbf{X}; 0.25) = \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^{10}, \quad L_{16}(\mathbf{X}; 0.75) = \left(\frac{3}{4}\right)^6 \left(\frac{1}{4}\right)^{10}.$$

Since $L_{16}(\mathbf{X}; 0.25) > L_{16}(\mathbf{X}; 0.75)$, we see that $\psi = 0$. For this data set, we fail to reject the null hypothesis.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

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