5. Variation of parameters

Variation of parameters

need to put it in.

Just find one particular solution is good enough.

12:58 / 12:58

2.0x HD

End of transcript. Skip to the start.

Video

Download video file

Transcripts

Download SubRip (.srt) file

Download Text (.txt) file

Long ago, we used **variation of parameters** to solve single first order inhomogeneous linear ODEs

$$\dot{x} + p(t)x = r(t).$$

We first find a solution x_h to the associated homogeneous equation, seek a particular solution of the form $x_p(t) = u(t)x_h(t),$ and use the original inhomogeneous ODE to solve for the unknown function $u(t)$.

Now, we are going to use the same idea to solve an inhomogeneous linear $n \times n$ **system** of ODEs:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{r},$$

where $\mathbf{r}$ is a vector-valued function of t.

First, find a basis of solutions to the corresponding homogeneous system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

Call the basis solutions $\mathbf{x}_1(t),\ \mathbf{x}_2(t),\ \cdots,\ \mathbf{x}_n(t).$ The general homogeneous solution is any linear combination of these:

$$\begin{aligned}\mathbf{x}_h(t) &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n \\ &= \mathbf{X}\mathbf{c} \qquad \text{where } \mathbf{X} = \begin{pmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & & | \end{pmatrix},\ \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.\end{aligned}$$

Notice that in 1 dimension, $x_h(t) = cx_1(t) = x_1(t)c,$ but in higher dimensions, $\mathbf{x}_h(t) = \mathbf{X}\mathbf{c}$ where $\mathbf{c}$ is a column vector and must be placed to the right of the fundamental matrix $\mathbf{X}$.

To find a particular solution, we let the coefficients c_i vary with time. In other words, replace the constant vector $\mathbf{c}$ by the vector **function**

$$\mathbf{v}(t) \quad = \quad \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{pmatrix}.$$

Now, substitute $\mathbf{x} = \mathbf{X}\mathbf{v}(t)$ in the original system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{r} \\ \dot{\mathbf{X}}\mathbf{v} + \mathbf{X}\dot{\mathbf{v}} &= \mathbf{A}\mathbf{X}\mathbf{v} + \mathbf{r} \qquad \text{(product rule of differentiation)} \\ \mathbf{A}\mathbf{X}\mathbf{v} + \mathbf{X}\dot{\mathbf{v}} &= \mathbf{A}\mathbf{X}\mathbf{v} + \mathbf{r} \qquad (\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}) \\ \mathbf{X}\dot{\mathbf{v}} &= \mathbf{r} \\ \dot{\mathbf{v}} &= \mathbf{X}^{-1}\mathbf{r} \qquad (\mathbf{X}\text{ invertible}). \end{aligned}$$

This means

$$\mathbf{v}(t) \quad = \quad \int \mathbf{X}^{-1}\mathbf{r}\, dt.$$

and the general solution to the inhomogeneous system is

\mathbf{x}(t)& =& \mathbf{X}\mathbf v(t)\, =\, \mathbf{X}\left(\int \mathbf{X}^{-1} \mathbf{r}\, dt\right),

for any fundamental matrix $\mathbf{X}$ of the associated homogeneous system.

This is a family of solutions because the indefinite integral on the right hand side will result in a constant of integration. Note that the constant of integration is a column vector.

5. Variation of parameters

Hide Discussion

Topic: Unit 3: Solving systems of first order ODEs using matrix methods / 5. Variation of parameters

Add a Post

Show all posts ▼ by recent activity ▼

Not rendering... 3

Learn About Verified Certificates