Stochastic Processes Spring 2018: Homework 1

Exercise 1 For positive numbers a and b, we say that the RV $X \sim Par(a, b)$ (Pareto distribution) if X has the p.d.f $f(x) = ab^a x^{-a-1}$ for $x \geq b$ and $f(x) = 0$ for $x < b$. Use the inversion method to generate X.

Exercise 2 A standardized logistic RV X has the p.d.f $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}$ with $-\infty < x < \infty$.

1. Use the inversion method to generate the random variable X.
2. Write down a computer program which simulate from X and use it to reconstruct the graph of the pdf of X.

Exercise 3 In class to generate a normal random variable $X \sim \mathcal{N}(0, 1)$ we used the rejection method with a exponential random variable with parameter $\lambda = 1$. Can you devise a more efficient algorithms (i.e. with less rejections) if you use an exponential random variable with another parameter $\lambda \neq 1$?

Exercise 4 Consider a random variable X with the semicircular density

$$f(x) = \frac{2}{\pi}\sqrt{1 - x^2}\,, -1 \leq x \leq 1\,.$$

1. Use the rejection method with $Y \sim \mathcal{U}([-1, 1])$ (i.e. uniform on [-1,1]) to generate X.
2. Write down a computer program which simulate from X and use it to reconstruct the graph of the pdf of X.

Exercise 5 (Composition method) Suppose that you can simulate from the random variables $X_1, \cdots, X_n$ with respective CDF $F_1(x), F_2(X), \cdots F_n(X)$.

1. How can you simulate from the random variable with CDF

$$\sum_{i=1}^{N} q_i F_i(x) \quad q_i \geq 0 \quad \sum_{i=1}^{n} q_i = 1\,?$$

2. Give an explicit algorithm to simulate the RV with CDF

$$F(x) = \begin{cases} 0 & -\infty < x \leq 0 \\ \frac{1-e^{-2x}+2x}{3} & 0 \leq x < 1 \\ \frac{3-e^{-2x}}{3} & 1 \leq x < \infty \end{cases}\,.$$

Exercise 6 Suppose that you can simulate from the random variables $X_1, \cdots, X_n$ with respective CDF $F_1(x), F_2(X), \cdots F_n(X)$.

1. How can we simulate from $F(x) = \prod_{i=1}^{n} F_i(x)$?
2. How can we simulate from $F(x) = 1 - \prod_{i=1}^{n}(1 - F_i(x))$?
3. Give two methods to simulate a RV with CDF $F(x) = x^n$ for $0 \leq x \leq 1$.

Exercise 7 In class we have seen that we can generate a $\Gamma(n, \lambda)$ random variable using n exponential random variable which could be expensive if n is large. Consider the alternative technique of generating a $\Gamma(n, \lambda)$ random variable by using the rejection method with $g(x)$ being the p.d.f of an exponential with parameter λ/n.

1. Show that the average number of iterations of the algorithm is $n^n e^{1-n}/(n-1)!$.
2. Use Stirling formula to show that for large n the answer in 1. is approximately $e\sqrt{(n-1)/2\pi}$, that is it is quite a bit cheaper than the alternative for large n.
3. Show that the rejection method is equivalent to the following
   - **Step 1:** Generate Y_1 and Y_2 independent exponentials with parameters 1.
   - **Step 2:** If $Y_1 < (n-1)[Y_2 - \log(Y_2) - 1]$ return to step 1.
   - **Step 3:** Set $X = nY_2/\lambda$.

   Hint: If T is exponential with parameter λ, what is αT?.

Exercise 8 (Generating a random permutation) In many instances it is useful to generate a random permutations of n symbols (think for example of mixing a deck of cards).

Consider the following algorithm to generate a random permutation of n elements $1, 2, 3, \cdots, n$. We will denote by S(i) the element in position i. For example for the permutation $(2, 4, 3, 1, 5)$ of 5 elements we have $S(1) = 2$, $S(2) = 4$, ...

1. Set $k = 1$
2. Set $S(1) = 1$
3. If $k = n$ stop. Otherwise let $k = k + 1$.
4. Generate a random number U, and let
$$S(k) = S(\lfloor kU \rfloor + 1)\,,$$
$$S(\lfloor kU \rfloor + 1) = k\,.$$
   Go to step 3.

Explain, in words, what the algorithm is doing and show that at iteration k, – i.e. when the value of $S(k)$ is initially set– $S(1), S(2), \cdots, S(k)$ is a random permutation of $1, 2, \cdots, k$.
Hint: Relate the probability P_k obtained at iteration k with the probability P_{k-1} obtained at iteration $k-1$.