13. Worked example: Finding a basis for solutions to homogeneous linear systems

Problem 13.1 The homogeneous linear system

$$\begin{aligned} x + 2y \quad\quad + 2v + 3w &= 0 \\ -y + 2z + 3v + w &= 0 \\ 2w &= 0 \end{aligned}$$

has matrix

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Find the general solution to the system.

Solution: This matrix is already in row echelon form. So we are ready for back-substitution. The variables x, y, w are the **pivot variables** or **dependent variables** , and the variables z, v are the **free variables** or **independent variables** . Start with

$$w = 0.$$

There is no equation for v in terms of the later variable w as it is a free variable, so set

$$v = c_1 \quad \text{for a parameter } c_1.$$

There is no equation for z in terms of v, w as z is also a free variable, so set

$$\begin{aligned} z &= c_2 \quad \text{for a parameter } c_2. \\ -y+2c_2+3c_1 &= 0 \\ y &= 3c_1+2c_2 \\ x+2(3c_1+2c_2)+2c_1 &= 0 \\ x &= -8c_1-4c_2. \end{aligned}$$

General solution:

$$\begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} = \begin{pmatrix} -8c_1-4c_2 \\ 3c_1+2c_2 \\ c_2 \\ c_1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where c_1, c_2 are parameters.

Set of all solutions:

$$\text{Span}\left(\begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

Since the set of solutions is a span, it is a vector space. Moreover, the two vectors are linearly **independent,** since the only c_1, c_2 such that

$$c_1 \begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

are $c_1 = 0$, $c_2 = 0$ (this is obvious if you look at the blue rows, the row corresponding to the free variables).

Conclusion: The set of solutions to this homogeneous linear system is a 2-dimensional vector space with basis

$$\begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Remark 13.2 Whenever you solve a homogeneous linear system by Gaussian elimination and back substitution, the vectors you find will always be linearly independent for the same reason as in the example above.

Suppose we didn't care about the basis, but all we wanted was to know the dimension of the nullspace. That is the number of vectors in the basis, which is the same as the number of parameters. There is one parameter for each free variable (free variables correspond to non-pivot columns). Therefore the dimension of the nullspace is the number of non-pivot columns.

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