Paradox

We are before a paradox.

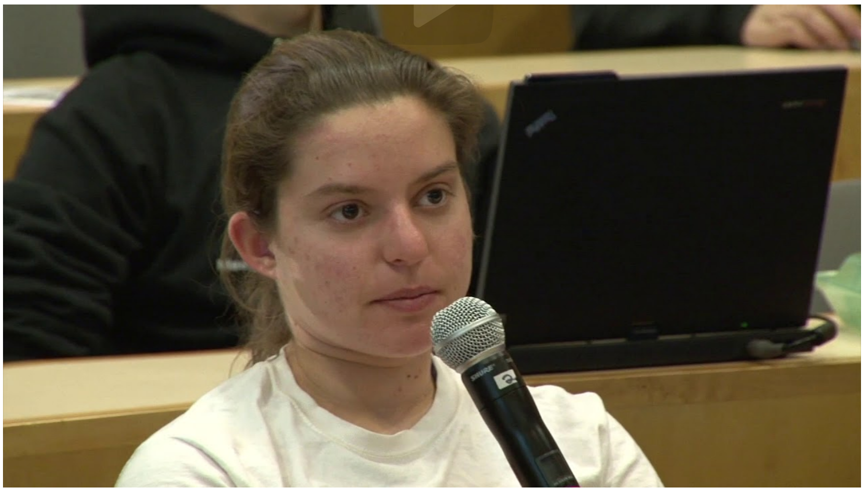
We have an apparently valid argument telling us that you should switch *regardless of which of the envelopes you've selected.* And that can't be the right conclusion, since it would lead you to switch envelopes indefinitely.

Where have we gone wrong?

The first time I heard about the Two-Envelope Paradox, I was unimpressed. It seemed to me that the setup relied on a problematic assumption: the assumption that the man in the white suit is able to pick a natural number at random, and do so in such a way that different numbers always have the same probability of being selected. (We relied on this assumption above in assigning equal probability to the outcome in which you get $2k$ dollars by switching and that outcome in which you get $k/2$ dollars by switching.) As we saw in Lecture 6.4, however, there can be no such thing as a *uniform* probability distribution over the natural numbers, on pain of violating Countable Additivity—a principle which I very much hoped to keep in place. No uniform probability distribution, no paradox! Or so I thought. . .

Video Review: Countable Additivity

Start of transcript. Skip to the end.



So let me tell you guys what I think

is wrong with this version of the problem.

I think that this assumption that God can just

pick a number at random is problematic,

because what we tacitly assume-- so when I just assigned half

0:00 / 0:00 1.50x

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I have since come to see that I was wrong. It is a mistake to think that the paradox requires a uniform probability distribution over the natural numbers. As it turns out, there are non-uniform probability distributions which yield the result that switching envelopes always has a higher expected value than staying put.

Here is an example, due to Oxford philosopher John Broome.

Take a die, and toss it until it lands One or Two. If the die first lands One or Two on the kth toss, place 2^{k-1} in the first envelope, and twice that amount in the second. As you'll be asked to prove below, this setup entails that the expected value of switching is always greater than the expected value of staying put.

Video Review: Revenge of the Paradox

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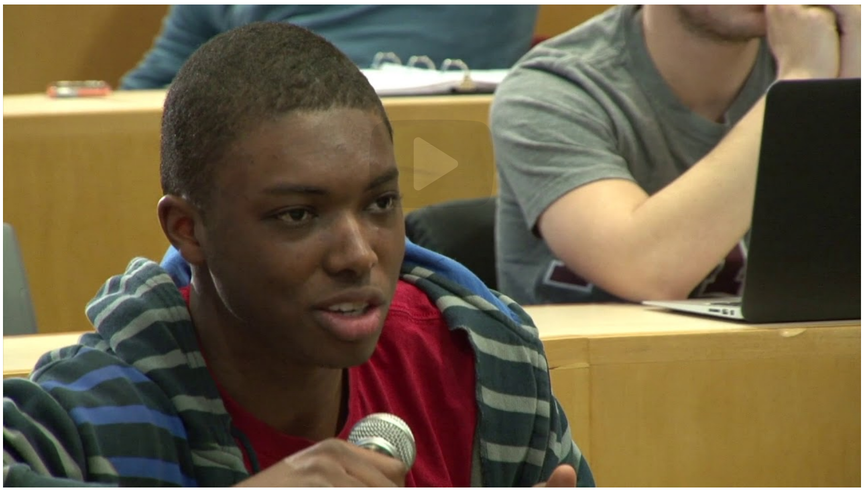

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When I first understood this, so when I first said, well,

if it's uniform, you get the advice

that you ought to switch.

If it's not, then you get a different situation.

Then I thought, well, surely it's

all to do with the uniformity assumption.

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Problem 1

1/1 point (ungraded)

What is the probability that the man fills the first envelope with 2^n, and the second envelope with twice that amount?

- ○ $\frac{1}{2^n}$
- ● $\frac{2^n}{3^{n+1}}$
- ○ $\frac{2^{n+1}}{3^n}$

✔

Explanation

For each k, let S_k be the proposition that the number 2^k is selected. Then:

$$\begin{array}{rcccc} p(S_0) & = & 1/3 & & \\ p(S_1) & = & 2/3 \cdot 1/3 & = & 2/9 \\ p(S_2) & = & 2/3 \cdot 2/3 \cdot 1/3 & = & 4/27 \\ & & \vdots & & \\ p(S_n) & = & (2/3)^n \cdot 1/3 & = & 2^n/3^{n+1} \end{array}$$

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Answers are displayed within the problem

Problem 2

1/1 point (ungraded)

If your envelope contains \$1, you should definitely switch. Suppose your envelope contains 2^k dollars for some $k > 0$. Then the expected value of not switching is $\$2^k$. What is the expected value of switching?

- ○ 2^k
- ◉ $2^k \cdot \frac{11}{10}$
- ○ $2^k + \frac{11}{10}$

✔

Explanation

Since your envelope contains 2^k dollars ($k > 0$), the number originally selected by the man must have been either 2^{k-1} or 2^k. So there are two relevant states of the world. In the first scenario, the man selected 2^{k-1}, and the envelope you did not pick contains 2^{k-1} dollars. (I call this scenario $O_{2^{k-1}}$.) In the second scenario, the man selected 2^k, and the envelope you did not pick contains 2^{k+1} dollars. (I call this scenario $O_{2^{k+1}}$.)

On Broome's method, the probability of selecting 2^{k-1} is $2^{k-1}/3^k$ and the probability of selecting 2^k is $2^k/3^{k+1}$. So, by Additivity, the probability of selecting one or the other is:

$$\frac{2^{k-1}}{3^k} + \frac{2^k}{3^{k+1}}$$

This allows us to calculate $p(O_{2^{k-1}})$, which is the probability that the man selects 2^{k-1}, given that he selects 2^{k-1} or 2^k:

$$p\left(O_{2^{k-1}}\right)=\frac{\frac{2^{k-1}}{3^k}}{\frac{2^{k-1}}{3^k}+\frac{2^k}{3^{k+1}}}=\frac{2^{k-1}}{2^{k-1}+\frac{2^k}{3}}=\frac{1}{1+\frac{2}{3}}=\frac{3}{5}$$

We can also calculate $p\left(O_{2^{k+1}}\right)$, which is the probability that the man selects 2^k, given that he selects 2^{k-1} or 2^k:

$$p\left(O_{2^{k+1}}\right)=1-\frac{3}{5}=\frac{2}{5}$$

Accordingly, the expected value of switching (S) can be computed as follows:

$$\begin{aligned} EV(S)= &\quad 2^{k-1}\cdot p\left(O_{2^{k-1}}\right)+2^{k+1}\cdot p\left(O_{2^{k+1}}\right)\\ = &\quad 2^{k-1}\cdot\frac{3}{5}+2^{k+1}\cdot\frac{2}{5}\\ = &\quad 2^k\cdot\frac{11}{10}\end{aligned}$$

So the expected value of switching ($2^k\cdot 11/10$ dollars) is larger than the expected utility of not switching (2^k dollars).

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Answers are displayed within the problem

Discussion

Topic: Week 6 / Paradox

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I refuse to "submit" an incorrect answer to problem 2.
As explained in my comment on the previous page, the expected value of switching is 0. This is clearl... 3

Lewin ball experiment as a copy of what Feynman did in his lectures way before.
Done at Caltech before it was done at MIT 1