Well-Orderings

The standard ordering of the natural numbers, $<_{\mathbb{N}}$, is a total ordering:

$$0 <_{\mathbb{N}} 1 <_{\mathbb{N}}, 2 <_{\mathbb{N}} \ldots$$

But it is a total ordering with an important special feature: it is a *well-ordering*.

What this means is that any non-empty set of natural numbers has a $<_{\mathbb{N}}$-smallest element: an element that precedes all others according to $<_{\mathbb{N}}$. (The set of prime numbers, for example, has 2 as its $<_{\mathbb{N}}$-smallest element, and the set of perfect numbers has 6.)

Not every total ordering is a well-ordering.

For example, the standard ordering of the integers, $<_{\mathbb{Z}}$, is not a well-ordering, since there there are non-empty subsets of $\mathbb{Z}$ with no $<_{\mathbb{Z}}$-smallest integer. One example of such a subset is $\mathbb{Z}$ itself:

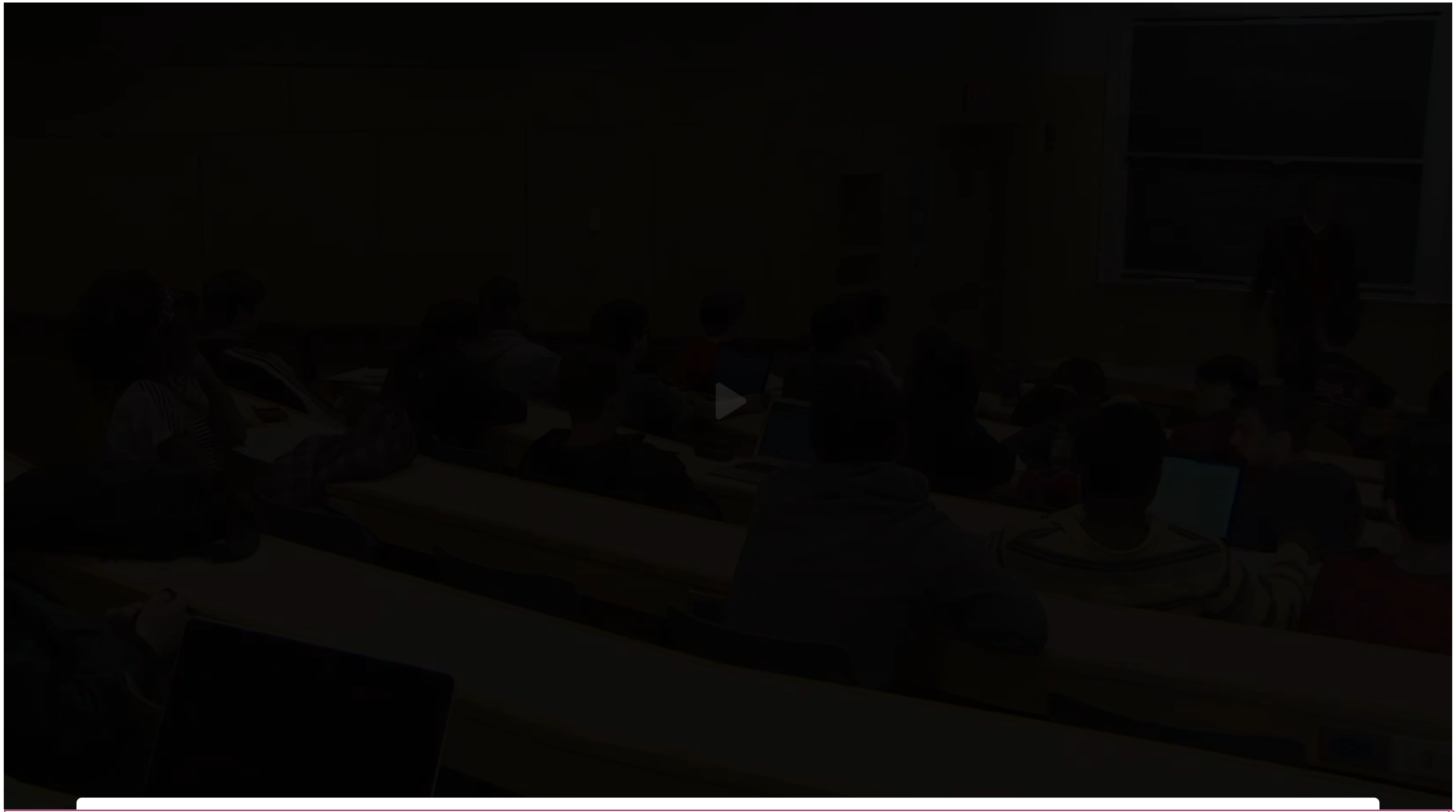
$$\ldots \; <_{\mathbb{Z}} -2 <_{\mathbb{Z}} -1 <_{\mathbb{Z}} 0 <_{\mathbb{Z}} 1 <_{\mathbb{Z}} 2 <_{\mathbb{Z}} 3 <_{\mathbb{Z}} \ldots$$

The set $[0, 1]$ under its standard ordering, $<_{\mathbb{R}}$, also fails to be well-ordered. For even though the entire set $[0, 1]$ has 0 as its $<_{\mathbb{R}}$-smallest element, $[0, 1]$ has subsets with no $<_{\mathbb{R}}$-smallest element. One example of such a subset is the set $(0, 1] = [0, 1] - \{0\}$.

Formally, we shall say that a set A is **well-ordered** by $<$ if A is totally ordered by $<$ and satisfies the following additional condition:

Well-Ordering
Every non-empty subset S of A has a $<$-smallest member (that is, a member x such that $x < y$ for for every y in S other than x).

Video Review: Well-Orderings



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Problem 1

1/1 point (ungraded)
Is the following ordering a well-ordering?

> The positive rational numbers, under the standard ordering $<_{\mathbb{Q}}$, which is such that $a <_{\mathbb{Q}} b$ if and only if $b = q + a$ for q a positive rational number.

- [ ] Yes. It is a well-ordering.
- [x] No. It is not a well-ordering.

Explanation
No, $<_{\mathbb{Q}}$ not a well-ordering of the positive rational numbers. There is no least element in, for instance, the set of all positive rational numbers. For every positive rational number, you can find a smaller one.

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Problem 2

1/1 point (ungraded)
Is the following ordering a well-ordering?

The natural numbers, under an unusual ordering, $<_\star$, which is just like the standard ordering, except that the order of 0 and 1 is reversed:

$$1 <_\star 0 <_\star 2 <_\star 3 <_\star 4 <_\star 5 <_\star \dots$$

- [x] Yes. It is a well-ordering.
- [ ] No. It is not a well-ordering.

✔

Explanation
Yes, the natural numbers are well-ordered by $<_\star$.

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Problem 3

1/1 point (ungraded)
Is the following ordering a well-ordering?

The natural numbers, under an unusual ordering, $<_0$, in which 0 is counted as bigger than every positive number but the remaining numbers are ordered in the standard way:

$$1 <_0 2 <_0 3 <_0 4 <_0 \cdots <_0 0$$

- [x] Yes. It is a well-ordering.
- [ ] No. It is not a well-ordering.

✔

Explanation

Yes, the natural numbers are well-ordered by $<_0$.

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Problem 4

1/1 point (ungraded)

Is the following ordering a well-ordering?

A finite set of real numbers, under the standard ordering $<_{\mathbb{R}}$.

- [x] Yes. It is a well-ordering.
- [ ] No. It is not a well-ordering.

Explanation

Yes, a total ordering of a finite set is always a well-ordering.

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