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Central Limit Theorem

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Let $X_1, X_2, \ldots, X_N$ be a set of N independent random variates and each X_i have an *arbitrary* probability distribution $P(x_1, \ldots, x_N)$ with mean μ_i and a finite variance σ_i^2. Then the normal form variate

$$X_{\text{norm}} \equiv \frac{\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \mu_i}{\sqrt{\sum_{i=1}^{N} \sigma_i^2}} \tag{1}$$

has a limiting cumulative distribution function which approaches a normal distribution.

Under additional conditions on the distribution of the addend, the probability density itself is also normal (Feller 1971) with mean $\mu = 0$ and variance $\sigma^2 = 1$. If conversion to normal form is not performed, then the variate

$$X \equiv \frac{1}{N} \sum_{i=1}^{N} x_i \tag{2}$$

is normally distributed with $\mu_X = \mu_x$ and $\sigma_X = \sigma_x / \sqrt{N}$.

Kallenberg (1997) gives a six-line proof of the central limit theorem. For an elementary, but slightly more cumbersome proof of the central limit theorem, consider the inverse Fourier transform of $P_X(f)$.

$$\begin{aligned}
\mathcal{F}_f^{-1}[P_X(f)](x) &\equiv \int_{-\infty}^{\infty} e^{2\pi i f X} P(X)\,dX & (3)\\
&= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(2\pi i f X)^n}{n!} P(X)\,dX & (4)\\
&= \sum_{n=0}^{\infty} \frac{(2\pi i f)^n}{n!} \int_{-\infty}^{\infty} X^n P(X)\,dX & (5)\\
&= \sum_{n=0}^{\infty} \frac{(2\pi i f)^n}{n!} \langle X^n \rangle. & (6)
\end{aligned}$$

Now write

$$\begin{aligned}
\langle X^n \rangle &= \langle N^{-n} (x_1 + x_2 + \ldots + x_N)^n \rangle \\
&= \int_{-\infty}^{\infty} N^{-n} (x_1 + \ldots + x_N)^n P(x_1) \cdots P(x_N)\,dx_1 \cdots dx_N,
\end{aligned} \tag{7}$$

so we have

$$\mathcal{F}_f^{-1}[P_X(f)](x) = \tag{8}$$

THINGS TO TRY:
- binomial distribution
- normal distribution
- ack(2,ack(2,1))

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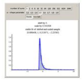
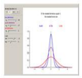
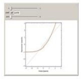


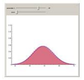
Generalized Central Limit Theorem
Roger J. Brown



Comparing Standard Errors of the Means
Scott R. Colwell



Illustrating the Central Limit Theorem Using the Quantile Plot for Sums of Unit Exponential Random Variables
Ian McLeod



Normal Approximation to a Poisson Random Variable
Chris Boucher

$$\sum_{n=0}^{\infty} \frac{(2\pi i f)^n}{n!} \langle X^n \rangle$$
$$= \sum_{n=0}^{\infty} \frac{(2\pi i f)^n}{n!} \int_{-\infty}^{\infty} N^{-n} (x_1 + \ldots + x_N)^n \times P(x_1) \cdots P(x_N)\, d x_1 \cdots d x_N \quad (9)$$
$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \left[\frac{2\pi i f (x_1 + \ldots + x_N)}{N}\right]^n \frac{1}{n!} P(x_1) \cdots P(x_N)\, d x_1 \cdots d x_N \quad (10)$$
$$= \int_{-\infty}^{\infty} e^{2\pi i f (x_1 + \ldots + x_N)/N} P(x_1) \cdots P(x_N)\, d x_1 \cdots d x_N \quad (11)$$
$$= \left[\int_{-\infty}^{\infty} e^{2\pi i f x_1/N} P(x_1)\, d x_1\right] \times \cdots \times \left[\int_{-\infty}^{\infty} e^{2\pi i f x_N/N} P(x_N)\, d x_N\right] \quad (12)$$
$$= \left[\int_{-\infty}^{\infty} e^{2\pi i f x/N} P(x)\, d x\right]^N \quad (13)$$
$$= \left\{\int_{-\infty}^{\infty} \left[1 + \left(\frac{2\pi i f}{N}\right) x + \frac{1}{2}\left(\frac{2\pi i f}{N}\right)^2 x^2 + \ldots\right] P(x)\, d x\right\}^N \quad (14)$$
$$= \left[1 + \frac{2\pi i f}{N} \langle x \rangle - \frac{(2\pi f)^2}{2 N^2} \langle x^2 \rangle + O(N^{-3})\right]^N \quad (15)$$
$$= \exp\left\{N \ln\left[1 + \frac{2\pi i f}{N} \langle x \rangle - \frac{(2\pi f)^2}{2 N^2} \langle x^2 \rangle + O(N^{-3})\right]\right\}. \quad (16)$$

Now expand

$$\ln(1 + x) = x - \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 + \ldots, \quad (17)$$

so

$$\mathcal{F}_f^{-1}[P_X(f)](x) \approx \exp\left\{N\left[\frac{2\pi i f}{N} \langle x \rangle - \frac{(2\pi f)^2}{2 N^2} \langle x^2 \rangle + \frac{1}{2} \frac{(2\pi i f)^2}{N^2} \langle x \rangle^2 + O(N^{-3})\right]\right\} \quad (18)$$
$$= \exp\left[2\pi i f \langle x \rangle - \frac{(2\pi f)^2 \left(\langle x^2 \rangle - \langle x \rangle^2\right)}{2 N} + O(N^{-2})\right] \quad (19)$$
$$\approx \exp\left[2\pi i f \mu_x - \frac{(2\pi f)^2 {\sigma_x}^2}{2 N}\right], \quad (20)$$

since

$$\mu_x \equiv \langle x \rangle \quad (21)$$
$$\sigma_x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2. \quad (22)$$

Taking the Fourier transform,

$$P_X \equiv \int_{-\infty}^{\infty} e^{-2\pi i f x} \mathcal{F}^{-1}[P_X(f)]\, d f \quad (23)$$
$$= \int_{-\infty}^{\infty} e^{2\pi i f (\mu_x - x) - (2\pi f)^2 {\sigma_x}^2/2N}\, d f. \quad (24)$$

This is of the form

$$\int_{-\infty}^{\infty} e^{i a f - b f^2}\, d f, \quad (25)$$

where $a \equiv 2\pi(\mu_x - x)$ and $b \equiv (2\pi\sigma_x)^2/2N$. But this is a Fourier transform of a Gaussian function, so

$$\int_{-\infty}^{\infty} e^{i a f - b f^2} \, d f = e^{-a^2/4b} \sqrt{\frac{\pi}{b}} \qquad (26)$$

(e.g., Abramowitz and Stegun 1972, p. 302, equation 7.4.6). Therefore,

$$P_X = \sqrt{\frac{\pi}{\frac{(2\pi\sigma_x)^2}{2N}}} \exp\left\{\frac{-[2\pi(\mu_x - x)]^2}{4\,\frac{(2\pi\sigma_x)^2}{2N}}\right\} \qquad (27)$$

$= \qquad (28)$

(29)

But and , so

(30)

The "fuzzy" central limit theorem says that data which are influenced by many small and unrelated random effects are approximately normally distributed.

SEE ALSO:
Berry-Esséen Theorem, Fourier Transform--Gaussian, Lindeberg Condition, Lindeberg-Feller Central Limit Theorem, Lyapunov Condition

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