4. Gibbs' Phenomenon

You might be wondering how we can use Fourier series, series composed of continuous functions, to approximate a discontinuous function. While it's true that we can get a Fourier series that converges to a function with discontinuities, the convergence fails at the discontinuities, and near points of discontinuity **the convergence is very slow** and very peculiar. Namely, the **partial sums** of the Fourier series overshoot and undershoot the values of the function near these points of disconinuity. This is known as the **Gibbs' Phenomenon** . All discontinuous functions exhibit Gibbs' phenomenon.

We illustrate this phenomenon using the example of the square wave of height $\pi/4$ — which we denote by $S(t) = \frac{\pi}{4}\mathrm{Sq}(t)$. In this case the discontinuities are at the integer multiples of π. The Fourier series for $S(t)$ is

$$\begin{aligned} S(t) &= \frac{\pi}{4}\mathrm{Sq}(t) = \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots \\ &= \sum_{n\ \mathrm{odd}}^{\infty} \frac{1}{n}\sin nt. \end{aligned}$$

For an odd positive integer N, write $S_N(t)$ for the Nth **partial sum**

$$S_N(t) := \sum_{n \text{ odd}}^{N} \frac{1}{n} \sin nt.$$

Then $S(t)$ is the limit of the functions $S_N(t)$ as $N \to \infty$. It turns out that it is possible to do a calculation that's a little bit beyond the scope of this class that for N very large, S_N overshoots $\pi/4$ (or undershoots $-\pi/4$) at a point of discontinuity by about 9%. This overshooting and undershooting can easily be seen in the figures below.

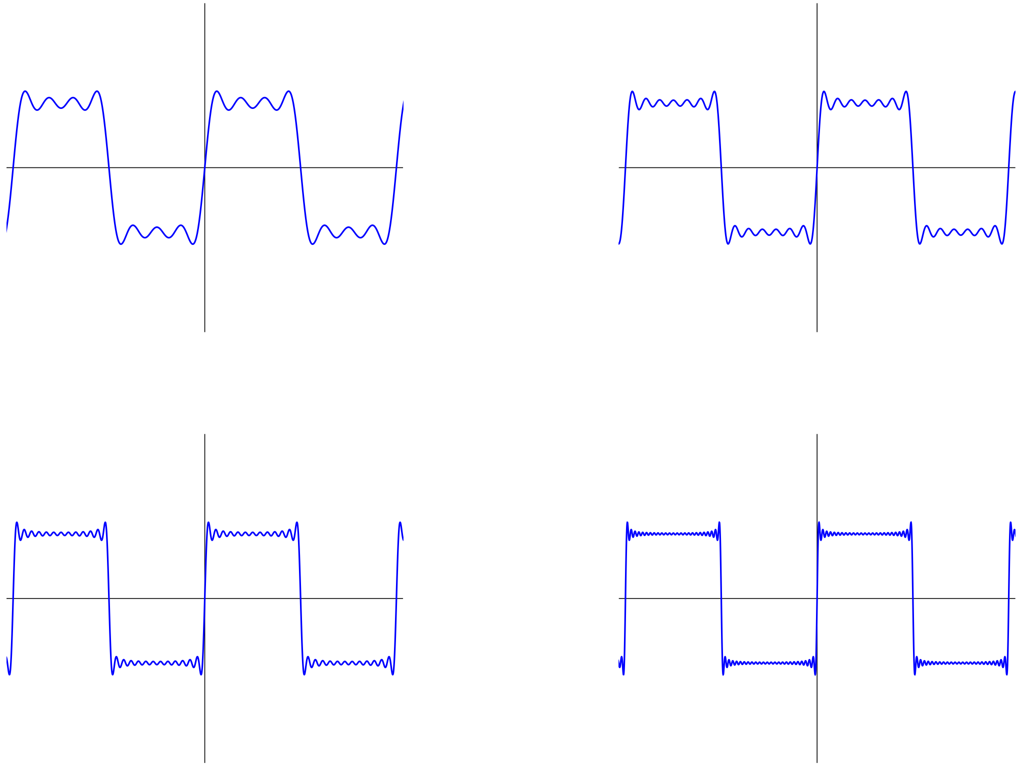


Figure 7: The partial sum functions $S_N(t)$ for $N = 7, 13, 25, 49$.

This overshoot/undershoot by about 9% happens **always** , not just for the case of $S(t)$, but for any function with discontinuities. In practice, one can only generate approximate Fourier series (with only a finite number of Fourier coefficients, known with some error). Thus the poor convergence of Fourier series for functions with discontinuities — and particularly the Gibb's phenomenon — creates (serious) difficulties in areas such as signal processing.

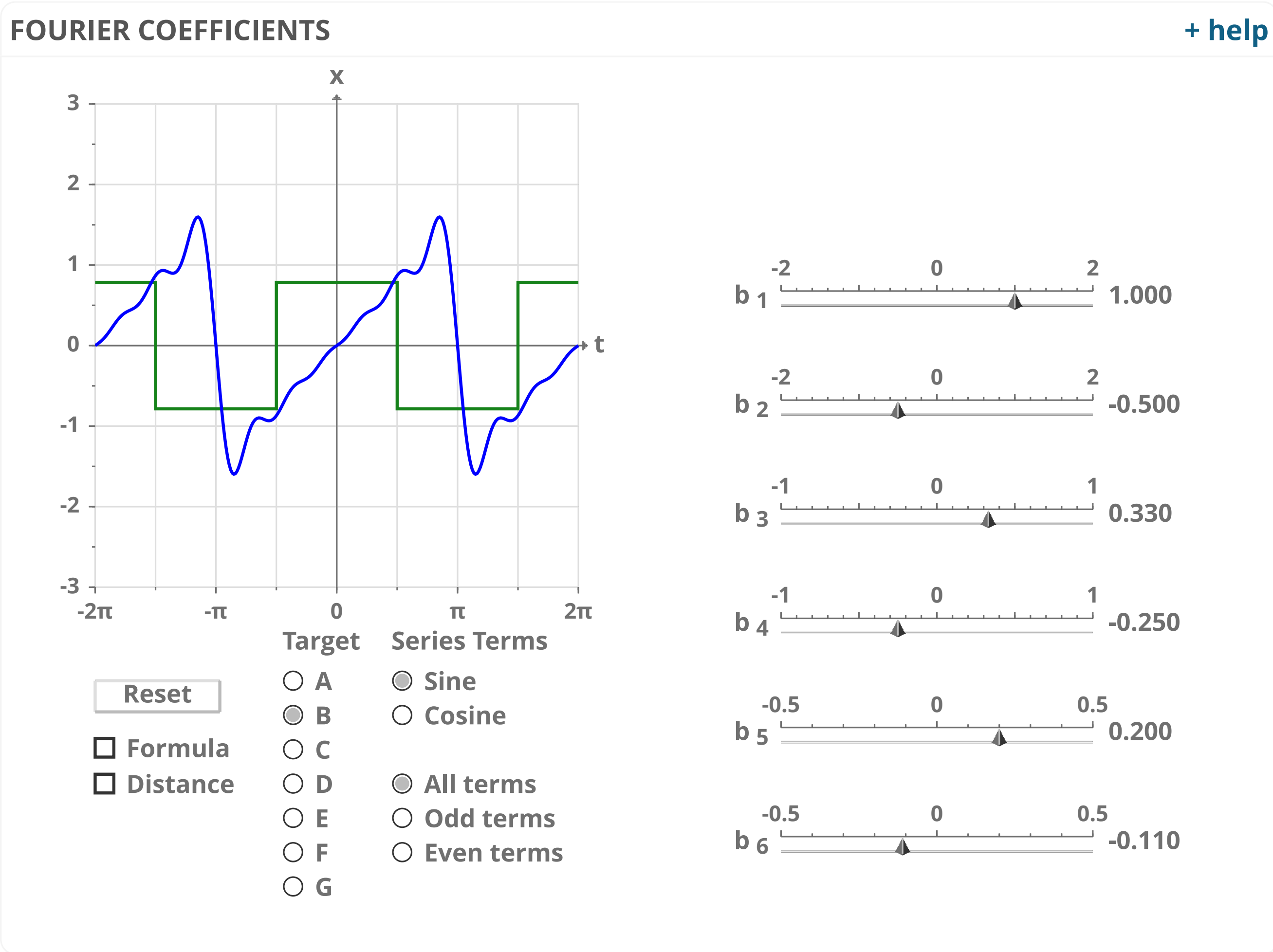
The intuitive reason that the approximation S_N behaves so poorly near discontinuities is that S_N is continuous (it is a finite sum of continuous functions). A continuous function cannot perfectly approximate a discontinuous one. A confusing point, is that even though for N arbitrarily large, the function S_N acts poorly at the discontinuity, the series $\sum_{n \text{odd}}^{\infty} \frac{1}{n} \sin nt$ doesn't. This illustrates how limits can act strangely sometimes, as well as the fact that the convergence of $S_N(t)$ to $S(t)$ “doesn't happen in the best possible way". If you're interested in questions like this, you might find 18.311 or 18.100 interesting.

Mathlet exploration

In the mathlet below, choose the Target function D, which is the 2π-periodic function which is defined by $D(t) = t/2$ for $-\pi < t < \pi$. This function has Fourier series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt).$$

Select “sine terms" move the sliders to match the Fourier coefficients. Observe how the first few terms approximate the function near the discontinuity.



mathlets.org

Note that you can find the Fourier series of more functions and observe the Gibbs' phenomenon (or not if the target is continuous) by selecting different target functions. Click on the **Distance** checkbox. Find the coefficients by using the sliders to minimize the root mean squared **Distance** between the sum of the first few terms in the series and the target function.

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Fun with the Mathlet
I didn't really appreciate the utility of these "mathlets" until the above example. As you play with the sliders and minimize the distance function I see that the values for the b ... 3

Please provide a reference
In the above discussion the statement: "If you're interested in questions like this, you might find 18.311 or 18.100 interesting," is not very helpful. Can you provide a reference... 6

Very cool!
It's as if at points of discontinuities, the function is confused, and learning how to behave. Maybe "machine learning" isn't all that new... 2

Aren't Fourier series the exact representation of a function?
As I understand from reading this topic, the truncated Fourier series always overshoots or undershoots at the point of discontinuity. But when we take infinitely many terms i... 6

? 9% of what?
Sorry for dumb question but is the overshoot 9% of the discontinuity (i.e. 9% peak to peak for square wave)? When I plotted for square wave of amplitude 1, the blip was at 1.... 1

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