$N(\theta,\theta)$: MLE for a Normal where mean=variance

Asked 6 years, 6 months ago Active 1 year, 5 months ago Viewed 6k times

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For an n-sample following a Normal$(\mu=\theta, \sigma^2=\theta)$, how do we find the mle?

I can find the root of the score function

$$\theta = \frac{1 \pm \sqrt{1 - 4\frac{s}{n}}}{2}, s = \sum x_i^2,$$

but I don't see which one is the maximum.
I tried to substitute in the second derivative of the log-likelihood, without success.

For the likelihood, with $x = (x_1, x_2, \ldots, x_n)$,

$$f(x) = (2\pi)^{-n/2}\theta^{-n/2}\exp\left(-\frac{1}{2\theta}\sum(x_i-\theta)^2\right),$$

then, with $s = \sum x_i^2$ and $t = \sum x_i$,

$$\ln f(x) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\theta - \frac{s}{2\theta} - t + \frac{n}{2}\theta,$$

so that

$$\partial_\theta \ln f(x) = -\frac{n}{2}\frac{1}{\theta} + \frac{s}{2\theta^2} + \frac{n}{2},$$

and the roots are given by

$$\theta^2 - \theta + \frac{s}{n} = 0.$$

Also,

$$\partial_{\theta,\theta} \ln f(x) = \frac{n}{2}\frac{1}{\theta^2} - \frac{s}{\theta^3}.$$

$2 \quad \theta^2 \quad \theta^3$

maximum-likelihood

asked Apr 16 '13 at 19:39
user21186

2 It looks to me like there might be an error in your calculation of $\log f(x)$. I think it should be $\text{const} - \frac{n}{2}\log(\theta) - \frac{s}{2\theta} + t - \frac{n\theta}{2}$. As is, there is a positive probability chance that $1 - 4\frac{s}{n} < 0$ which is a problem. – guy Apr 16 '13 at 19:57

2 Answers

6

There are some typos (or algebraical mistakes) in the signs of the log-likelihood, followed by the corresponding unpleasant consequences.

Since this is a well-known problem, I will only point out a reference with the solution:

Asymptotic Theory of Statistics and Probability pp. 53, by Anirban DasGupta.

answered Apr 16 '13 at 19:59
Gustav
76 1

3

Recall that the normal distribution $N(\mu, \sigma^2)$ has pdf $f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, Note here that $\mu = \theta$ and $\sigma^2 = \theta$ and therefore $\sigma = \sqrt{\theta}$

$$\begin{aligned} L(x_1, x_2, \ldots, x_n|\theta) &= \prod_{i=1}^{n} f(x_i|\theta) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \ \exp\Big\{ -\frac{1}{2\theta}(x_i-\theta)^2 \Big\} \\ &= (2\pi)^{-n/2}(\theta)^{-n/2} \prod_{i=1}^{n} \ \exp\Big\{ -\frac{1}{2\theta}(x_i-\theta)^2 \Big\} \\ &= (2\pi)^{-n/2}(\theta)^{-n/2} \ \exp\Big\{ -\frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)^2 \Big\} \\ \log L &= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)^2 \end{aligned}$$

Consider the term $\frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)^2$ which can be expanded and simplified

$$\begin{aligned} \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)^2 &= \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\theta)(x_i-\theta) \\ &= \frac{1}{2\theta}\sum_{i=1}^{n}\left(x_i^2 - 2\theta x_i + \theta^2\right) \\ &= \frac{1}{2\theta}\left(\sum_{i=1}^{n}(x_i^2) - 2\theta\sum_{i=1}^{n}(x_i) + n\theta^2\right) \\ &= \frac{1}{2\theta}\sum_{i=1}^{n}(x_i^2) - \sum_{i=1}(x_i) + \frac{n\theta}{2} \end{aligned}$$

We can now compute the derivative with respect to θ, equate to zero and solve for θ

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\theta) - \left(\frac{1}{2\theta}\sum_{i=1}^{n}(x_i^2) - \sum_{i=1}(x_i) + \frac{n\theta}{2}\right)$$
$$\frac{d}{d\theta}\log L = \frac{-n}{2\theta} - \left(\frac{-1}{2\theta^2}\sum_{i=1}^{n}(x_i^2) + \frac{n}{2}\right) = 0$$
$$= \frac{-n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i^2) - \frac{n}{2}$$
$$= -\theta^2 - \theta + \frac{1}{n}\sum_{i=1}^{n}(x_i^2)$$
$$\text{let } s = \frac{1}{n}\sum_{i=1}^{n}(x_i^2)$$
$$0 = -\theta^2 - \theta + s$$
$$\hat{\theta} = \frac{\sqrt{1+4s} - 1}{2}$$

edited Apr 23 '18 at 14:30

answered Apr 21 '18 at 17:21

Xavier Bourret Sicotte
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