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## 5. Asymptotic Normality of M-

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## 5. Asymptotic Normality of M-estimators

### Asymptotic Normality of M-estimators

## Asymptotic normality



Let  $\mu^* \in \mathcal{M}$  (the *true* parameter). Assume the following:

1.  $\mu^*$  is the only minimizer of the function  $Q$ ;
2.  $J(\mu)$  is invertible for all  $\mu \in \mathcal{M}$ ;

OK.

OK.

Then,  $\hat{\mu}_n$  satisfies:

$$\triangleright \hat{\mu}_n \xrightarrow[n \rightarrow \infty]{\mathbf{P}} \mu^*;$$

$$\triangleright \sqrt{n}(\hat{\mu}_n - \mu^*) \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{N}(0, \mathcal{J}(\mu^*)^{-1} K(\mu^*) \mathcal{J}(\mu^*)^{-1}).$$

▶ 4:38 / 4:38

▶ 1.50x



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## Asymptotic normality of the M-estimators

3/3 points (graded)

Let  $X_1, \dots, X_n \stackrel{iid}{\sim} \mathbf{P}$ . Let  $\rho(x, \mu)$  denote a loss function satisfying

$$\mu^* = \operatorname{argmin}_{\mu \in \mathbb{R}} \mathbb{E} [\rho(X_1, \mu)]$$

where  $\mu^* \in \mathbb{R}$  is some unknown one-dimensional parameter associated with  $\mathbf{P}$  that we would like to estimate. Let

$$J(\mu) = \mathbb{E} \left[ \frac{\partial^2 \rho}{\partial \mu^2}(X_1, \mu) \right]$$
$$K(\mu) = \operatorname{Var} \left[ \frac{\partial \rho}{\partial \mu}(X_1, \mu) \right]$$

You construct the M-estimator  $\hat{\mu}_n$  associated  $\rho$ .

Assuming that the conditions for the asymptotic normality of this M-estimator hold, we have

$$\sqrt{n} \frac{\hat{\mu}_n - \mu^*}{\sqrt{J(\mu^*)^{-2} K(\mu^*)}} \xrightarrow[n \rightarrow \infty]{(d)} Q$$

for some distribution  $Q$ .

What is  $Q$ ?

☐ Poisson with mean 1.

☐ Exponential with mean 1.

☒ Standard normal.

☐  $\mathcal{N}(0, \sigma^2)$  for some unknown parameter  $\sigma^2$ .



Let  $q_\alpha$  denote the  $\alpha$ -quantile of the distribution  $Q$ . For what value of  $q_\alpha$  is it true that

$$\mu^* \in \left[ \hat{\mu}_n - q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}}, \hat{\mu}_n + q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}} \right]$$

with probability 95% as  $n \rightarrow \infty$ ?

$q_\alpha =$

1.96

✓ Answer: 1.96

Let

$$\mathcal{I} := \left[ \hat{\mu}_n - q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}}, \hat{\mu}_n + q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}} \right]$$

denote the interval in the previous question.

Is  $\mathcal{I}$  an asymptotic confidence interval for  $\mu^*$  of confidence level 95%?

☐ Yes, because the previous question solves for  $q_\alpha$  so that this holds.

☐ Yes, because of the asymptotic normality of  $\hat{\mu}_n$ .

☐ No, because we did not define a statistical model for this problem.

☒ No, because the endpoints of  $\mathcal{I}$  depend on the true parameter.



### Solution:

For the first question, the correct response is "Standard normal." Referring to the theorem regarding the asymptotic normality of the M-estimators, we see that the asymptotic variance of  $\widehat{\mu}_n$  is  $J(\mu^*)^{-2}K(\mu^*)$ . Hence,

$$\sqrt{n} \frac{\widehat{\mu}_n - \mu^*}{\sqrt{J(\mu^*)^{-2}K(\mu^*)}} \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{N}(0, 1).$$

For the second question, the correct response is "1.96". By the previous equation,

$$P\left(\sqrt{n} \left| \frac{\widehat{\mu}_n - \mu^*}{\sigma} \right| \geq q_{0.025}\right) = P\left(\mu^* \in \left[ \widehat{\mu}_n - q_{0.025} \sqrt{\frac{J(\mu^*)^{-2}K(\mu^*)}{n}}, \widehat{\mu}_n + q_{0.025} \sqrt{\frac{J(\mu^*)^{-2}K(\mu^*)}{n}} \right]\right) = 0.05$$

where  $q_{0.025} = 1.96$  is the 2.5%-quantile of a standard Gaussian.

For the third question, the correct response is "No, because the endpoints of  $\mathcal{I}$  depend on the true parameter." By definition, the endpoints of a confidence interval should be estimators, and this is not the case for  $\mathcal{I}$  because  $K^{-1}(\mu^*)$  and  $J(\mu^*)$  depend on the true parameter.

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You have used 1 of 2 attempts

**i** Answers are displayed within the problem

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