6. Review: Conditional Likelihood and Bayes' Rule

Note: The following two problems will be quick review of the concepts of **conditional probability and Bayes' rule** , which were covered in the previous course 6.431x. **If understanding these problems or their solutions pose a significant difficultly, please review these concepts before proceeding.**

An Observation Model

3/3 points (graded)

Let $\theta \sim \pi(\theta)$ be a parameter supported on $\mathbb{Z}$, the integers. Suppose that we observe random variables

$$Y_i = \theta X_i$$

for $i = 1, \ldots, n$. The outcomes of $X_1, \ldots, X_n$ are unknown to you, but you do know that they are i.i.d. and uniformly distributed on the set $\{-1, 0, 1\}$. Assume that X_i is independent of θ for all i.

Compute each of the probabilities below:

- $\mathbb{P}(Y_1 = 6 \mathrm{and} Y_2 = 0 | \theta = 3) =$

0

✔ **Answer:** 0

- $\mathbb{P}\left(Y_1 = 7 \text{and} Y_2 = -7 \text{and} Y_3 \in \{0,7\} | \theta = -7\right) =.$

2/27 ✔ **Answer:** 2/27

- $\mathbb{P}\left(Y_1 = Y_2 + Y_3 | \theta = 5\right).$

7/27 ✔ **Answer:** 7/27

Solution:

- Note that, conditional on $\theta = 3$,

$$Y_1 = 6, Y_2 = 0 \implies X_1 = 2, X_2 = 0.$$

Since $X_1 \in \{-1, 0, 1\}$, this probability is 0, as X_1 cannot be 2.

- Similar to the item above, we have, conditional on $\theta = -7$,

$$Y_1 = 7, Y_2 = -7, Y_3 \in \{0,7\} \implies X_1 = -1, X_2 = 1 \quad \text{and} \quad X_3 \in \{0,1\}.$$

In particular,

$$\mathbb{P}\left(Y_1 = 7 \text{and} Y_2 = -7 \text{and} Y_3 \in \{0,7\} | \theta = -7\right) = \mathbb{P}\left(X_1 = -1 \text{and} X_2 = 1 \text{and} X_3 \in \{0,1\}\right),$$

which, by using independence, is equal to,

$$\mathbb{P}\left(X_1 = -1 \text{and} X_2 = 1 \text{and} X_3 \in \{0,1\}\right) = \mathbb{P}\left(X_1 = -1\right)\mathbb{P}\left(X_2 = 1\right)\mathbb{P}\left(X_3 \in \{0,1\}\right) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{27}.$$

- Note that,

$$Y_1 = Y_2 + Y_3 \iff X_1 = X_2 + X_3.$$

In particular, using the law of total probability,

$$\mathbb{P}(X_1 = X_2 + X_3) = \sum_{i=-1}^{1} \mathbb{P}(X_1 = X_2 + i | X_3 = i)\, \mathbb{P}(X_3 = i) = \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{9} \cdot \frac{1}{3} + \frac{2}{9} \cdot \frac{1}{3} = \frac{7}{27}.$$

since, $\mathbb{P}(X_3 = i) = 1/3$ for each $i \in \{-1, 0, 1\}$ and

- if $i = -1$, then $X_1 = X_2 - 1$ iff $(X_1, X_2) = (0, 1)$ or $(X_1, X_2) = (-1, 0)$;
- if $i = 0$, then $X_1 = X_2$ in three possible ways: $X_1 = X_2 = j$ for $j \in \{-1, 0, 1\}$; and
- if $i = 1$, then $X_1 = X_2 + 1$ iff $(X_1, X_2) = (1, 0)$ or $(X_1, X_2) = (0, -1)$.

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You have used 2 of 3 attempts

Answers are displayed within the problem

Probability Review: Bayes' Rule

2/2 points (graded)

Assume that, each person is Republican or Democrat with probability $1/2$ for each; independent of any other person. If two persons are of same political view, they become friends with probability a; and if they are of opposite political view, they become friends with probability b.

What is the probability that Amy and Ben are friends?

(a+b)/2

Answer: (a+b)/2

$\frac{a+b}{2}$

Given that Amy and Ben are two friends, what is the probability that they have the same political views?

a/(a+b)

Answer: a/(a+b)

$\frac{a}{a+b}$

STANDARD NOTATION

Solution:

- Let E be the event that Amy and Ben are friends, and let σ_A denote the view of Amy; and σ_B denote the view of Ben. Observe that,

$$\begin{aligned}\mathbb{P}(\sigma_A = \sigma_B) &= \mathbb{P}(\sigma_A = \sigma_B = \text{Republican}) + \mathbb{P}(\sigma_A = \sigma_B = \text{Democrat}) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} \\ &= 1/2,\end{aligned}$$

  where, the first line uses the definition (namely, Amy and Ben have the same political view, if and only if, either both are Democrat; or both are Republican), and the second line uses the independence, and uniformity of the distribution. Similarly, $\mathbb{P}(\sigma_A \neq \sigma_B) = 1/2$. With this,

$$\mathbb{P}(E) = \mathbb{P}(E|\sigma_A = \sigma_B)\,\mathbb{P}(\sigma_A = \sigma_B) + \mathbb{P}(E|\sigma_A \neq \sigma_B)\,\mathbb{P}(\sigma_A \neq \sigma_B) = \frac{a+b}{2},$$

  using the law of total probability.

- Our goal is to compute,

$$\mathbb{P}\left(\sigma_A = \sigma_B | E\right),$$

which, by Bayes' rule;

$$\begin{aligned}\mathbb{P}\left(\sigma_A = \sigma_B | E\right) &= \frac{\mathbb{P}\left(E | \sigma_A = \sigma_B\right)\mathbb{P}\left(\sigma_A = \sigma_B\right)}{\mathbb{P}\left(E\right)} \\ &= \frac{a \cdot (1/2)}{\left(\left(a+b\right)/2\right)} \\ &= \frac{a}{a+b}.\end{aligned}$$

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Discussion

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Topic: Unit 5 Bayesian statistics:Lecture 17: Introduction to Bayesian Statistics / 6. Review: Conditional Likelihood and Bayes' Rule

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First problem - Xi
Xi takes only discrete values, right? (from {-1,0,1}) 1