MATHEMATICS

Calculation of the n-th central moment of the normal distribution $\mathcal{N}(\mu, \sigma^2)$

Asked 7 years, 9 months ago Active 3 years, 9 months ago Viewed 57k times

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Since integration is not my strong suit I need some feedback on this, please:

Let Y be $\mathcal{N}(\mu, \sigma^2)$, the *normal distrubution* with parameters μ and σ^2. I know μ is the expectation value and σ is the variance of Y.

I want to calculate the n-th central moments of Y.

The *density function* of Y is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

The n-th *central moment* of Y is

$$E[(Y - E(Y))^n]$$

The n-th *moment* of Y is

$$E(Y^n) = \psi^{(n)}(0)$$

where ψ is the *Moment-generating function*

$$\psi(t) = E(e^{tX})$$

So I started calculating:

$$\begin{aligned}
E[(Y - E(Y))^n] &= \int_{\mathbb{R}} \left(f(x) - \int_{\mathbb{R}} f(x)dx\right)^n dx \\
&= \int_{\mathbb{R}} \sum_{k=0}^{n} \left[\binom{n}{k} (f(x))^k \left(-\int_{\mathbb{R}} f(x)dx\right)^{n-k}\right] dx \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\int_{\mathbb{R}} \left[(f(x))^k \left(-\int_{\mathbb{R}} f(x)dx\right)^{n-k}\right] dx\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\int_{\mathbb{R}} \left[(f(x))^k (-\mu)^{n-k}\right] dx\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left((-\mu)^{n-k} \int_{\mathbb{R}} (f(x))^k \, dx\right)
\end{aligned}$$

$$= \sum_{k=0} \binom{n}{k} \left((-\mu)^{n-k} E\left(Y^k\right)\right)$$

Am I on the right track or completely misguided? If I have made no mistakes so far, I would be glad to get some inspiration because I am stuck here. Thanks!

probability normal-distribution

edited Dec 22 '11 at 1:04 André Caldas 3,521 14 27

asked Dec 19 '11 at 0:27 Aufwind 1,100 2 11 26

1 Since $Y - E(Y)$ has mean 0 and in this case is normally distributed $N(0, \sigma^2)$, the n-th central moment should not be affected by the original mean μ. – Henry Dec 19 '11 at 0:42

What you have so far is correct, but as @Henry points out, the central moments are invariant under a shift. So you may as well simplify things by taking $\mu = 0$ from the start. In any case, you still need to find $E[Y^n]$ for the normal distribution with mean 0. – mjqxxxx Dec 19 '11 at 1:02

1 Your question has a typo in the normal density: there should be a square in the exponent. Also, I disagree with @mjqxxxx's statement that what you have so far is correct." The first step

$$E[(Y - E(Y))^n] = \int_{\mathbb{R}} \left(f(x) - \int_{\mathbb{R}} f(x)dx\right)^n dx$$

is wrong: it should read

$$E[(Y - E(Y))^n] = \int_{\mathbb{R}} \left(x - \int_{\mathbb{R}} x f(x)dx\right)^n f(x)\,dx = \int_{\mathbb{R}} (x-\mu)^n f(x)\,dx$$

and the last step follows immediately upon expanding $(x-\mu)^n$ via the binomial theorem, separating into a sum of integrals, and identifying $\int_{\mathbb{R}} x^k f(x)\,dx = E[Y^k]$. – Dilip Sarwate Dec 19 '11 at 1:41

2 "Am I on the right track.....?" It depends on where you want to go! As Sasha shows in his answer,

$$E[(Y-\mu)^n] = \hat{m}_n = \begin{cases} 0, & n \text{ odd}, \\ \sigma^n (n-1)(n-3)\cdots 3\cdot 1, & n \text{ even}, \end{cases}$$

can be evaluated in straightforward fashion. On the other hand, your approach succeeded in expressing the central $\hat{m}_n$ in terms of the standard (non-central) moments $m_k = E[Y^k]$ and so now you have the task of evaluating n different integrals to find the m_k's. So your approach does not seem too promising to say the least. – Dilip Sarwate Dec 19 '11 at 3:28

@Dilip: Thank you for pointing that out. – Aufwind Dec 19 '11 at 4:11

1 Answer

The n-th central moment $\hat{m}_n = \mathbb{E}\left(\left(X - \mathbb{E}(X)\right)^n\right)$. Notice that for the normal distribution $\mathbb{E}(X) = \mu$, and that $Y = X - \mu$ also follows a normal

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The n-th central moment $\hat{m}_n = \mathbb{E}\left((X - \mathbb{E}(X))^n\right)$. Notice that for the normal distribution $\mathbb{E}(X) = \mu$, and that $Y = X - \mu$ also follows a normal distribution, with zero mean and the same variance σ^2 as X.

Therefore, finding the central moment of X is equivalent to finding the raw moment of Y.

In other words,

$$\begin{aligned}
\hat{m}_n &= \mathbb{E}\left((X - \mathbb{E}(X))^n\right) = \mathbb{E}\left((X-\mu)^n\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} (x-\mu)^n \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x \\
&\overset{y=x-\mu}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} y^n \mathrm{e}^{-\frac{y^2}{2\sigma^2}} \mathrm{d}y \overset{y=\sigma u}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \sigma^n u^n \mathrm{e}^{-\frac{u^2}{2}} \sigma \mathrm{d}u \\
&= \sigma^n \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} u^n \mathrm{e}^{-\frac{u^2}{2}} \mathrm{d}u
\end{aligned}$$

The latter integral is zero for odd n as it is the integral of an odd function over a real line. So consider

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} u^{2n} \mathrm{e}^{-\frac{u^2}{2}} \mathrm{d}u = 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} u^{2n} \mathrm{e}^{-\frac{u^2}{2}} \mathrm{d}u \\
&\overset{u=\sqrt{2w}}{=} \frac{2}{\sqrt{2\pi}} \int_0^{\infty} (2w)^n \mathrm{e}^{-w} \frac{\mathrm{d}w}{\sqrt{2w}} = \frac{2^n}{\sqrt{\pi}} \int_0^{\infty} w^{n-1/2} \mathrm{e}^{-w} \mathrm{d}w = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)
\end{aligned}$$

where $\Gamma(x)$ stands for the Euler's [Gamma function](). Using its [properties]() we get

$$\hat{m}_{2n} = \sigma^{2n}(2n-1)!! \qquad \hat{m}_{2n+1} = 0$$

edited Oct 20 '15 at 20:38
alick
30 8

answered Dec 19 '11 at 0:55
Sasha
61.9k 5 112 190

1 Would you be so kind to explain this a little further, please: *The latter integral is zero for odd n as an integral of even and odd functions over a real line.* – Aufwind Dec 19 '11 at 2:02