Homework

Homework due Aug 26, 2020 21:30 IST

The exercises below will count towards your grade. **You have only one chance to answer these questions.** Take your time, and think carefully before answering.

Problem 1

20/20 points (graded)

What do the following sentences of L say?

$\neg\forall x(x = 3)$

- ( ) It is not the case that there is something that is identical to 3.
- (•) There is something that is not identical to 3.
- ( ) There is something that is identical to 3.
- ( ) Nothing is identical to 3.

✔

Explanation

This says, "It is not the case that, for all x, $x = 3$, which is equivalent to: there is something that is not identical to 3.

$\forall x\, \exists y\, (y + 1 = x)$

- ( ) Every number plus 1 equals another number. (That is, every number has a successor.)

- (selected) Every number is such that some number plus 1 equals it. (That is, every number has a predecessor.)
- Every number is unique.
- Every number is such that 1 can be added to it.

✔

Explanation

This says, “For every number x, there exists some number y such that it (y), when added to 1, equals x.” In other words, every number has a predecessor. (Here is one instance of that general claim: 4 is such that some other number, when added to 1, equals 4. Obviously in this case that other number is 3.)

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Problem 2

15/15 points (graded)

Is it possible to construct a Turing Machine M_1 which runs forever outputting sentences of L in such a way that every arithmetical truth is eventually outputted by M_1?

- (selected) Yes
- No

✔

Explanation

Yes. We could program a Turing Machine to output every sentence of L. First it could output all the possible one-symbol sentences of L; then all the two-symbol sentences of L; then all the three-symbol sentences; and so on. So every truth in L would eventually be outputted by this Turing Machine.

Is it possible to construct a Turing Machine M_2 which runs forever outputting sentences of L in such a way that no arithmetical falsehood is ever output by M_2?

- [x] Yes
- [ ] No

Explanation
Yes. We could also program a Turing Machine that could run forever never outputting falsehoods and never outputting the same sentence twice. We could, for example, program a Turing Machine to output the sentences "$1 + 1 = 2$", "$2 + 2 = 4$", "$3 + 3 = 6$", and so on. What Gödel's theorem shows us is that it is impossible to construct a Turing Machine M such that all three of these conditions are met *at once*: M runs forever, outputting arithmetical sentences; every arithmetical truth is eventually outputted by M; and no arithmetical falsehood is ever outputted by M. It is the *conjunction* of those conditions, not any individually, that is impossible.

Is it possible to construct a Turing Machine which satisfies both the conditions of M_1 above and the conditions of M_2 above?

- [ ] Yes
- [x] No

Explanation
No. That's Godel's Theorem.

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Answers are displayed within the problem

Problem 3

12/20 points (graded)
Say that a sentence s of L is *provable* on the basis of a given axiomatization if there is a finite sequence of sentences of L with the following two properties:

(1) the last member of the sequence is s, and

(2) every member of the sequence is either an axiom or something that results from previous members of the sequence by applying a rule of inference.

Now consider an axiomatization $\mathfrak{A}$ that consists of the following axioms:

$(A0)$ $0 = 0$

$(A1)$ $1 = 1$

$(A2)$ $2 = 2$

$\ldots$

and the following rules of inference:

$(R1)$ You may infer "ϕ & ψ" if you have ϕ and ψ.

$(R2)$ You may infer "$\forall x\,(x = x)$" if you have all of the following:

$$0 = 0$$
$$1 = 1$$
$$2 = 2$$
$$\vdots$$

Now, is "$0 = 0$" provable on the basis of $\mathfrak{A}$?

- (●) Yes
- ( ) No

✔

Explanation

Yes. Each axiom of $\mathfrak{A}$ is probable because there is a finite list of sentences such that every member is either an axiom or something that results from previous members of the sequence by applying a rule of inference, namely, the one-sentene proof consisting of the axiom itself. Every member of that sequence is either an axiom or something that results from previous members by applying a rule of inference, because every member is an axiom. And the last member of the sequence (which is also the only member of the sequence) is the sentence we wanted to prove.

Is "$1 = 1\ \&\ 2 = 2\ \&\ 7 = 7$" provable on the basis of $\mathfrak{A}$?

- [x] Yes
- [ ] No

✔

Explanation

Yes. Here is a proof:

$(1)\ \ 1 = 1$ (From $A1$)
$(2)\ \ 2 = 2$ (From $A2$)
$(3)\ \ 7 = 7$ (From $A7$)
$(4)\ \ (1 = 1 \,\&\, 2 = 2)$ (From 1 and 2 by $R1$)
$(5)\ \ ((1 = 1 \,\&\, 2 = 2) \,\&\, 7 = 7)$ (From 3 and 7 by $R1$)

Is the following provable on the basis of $\mathfrak{A}$?

$$0 = 0 \ \ \& \ \ 1 = 1 \ \ \& \ \ 2 = 2 \ \ \& \ \ 3 = 3 \ \ \& \ \ \ldots$$

- [x] Yes
- [ ] No ✔

✖

Explanation

No. Notice, first, that the string above is not a sentence of L. But we wouldn't be able to prove it in $\mathfrak{A}$ even if it were, since that would require infinitely many applications of rule $(R1)$, and therefore an infinitely long proof. But proofs must be finite.

Does rule $(R2)$ license inferring "$\forall x\,(x = x)$" from the axioms of $\mathfrak{A}$?

- [x] Yes
- [ ] No

✔

Explanation

Yes. That's just what $(R2)$ says!

Is "$\forall x\,(x = x)$" provable on the basis of $\mathfrak{A}$?

- ◉ Yes
- ○ No ✔

✖

Explanation

No. Such a sentence could only be proved in $\mathfrak{A}$ by applying rule $(R2)$. But $(R2)$ requires infinitely many premises, and therefore an infinitely long proof. But proofs must be finite.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Problem 4

20/20 points (graded)

Choose the sentence of L (plus abbreviations) that expresses *Goldbach's Conjecture*: the statement that every even number greater than 2 is the sum of two primes.

- ○ $\forall x_0((\mathrm{Even}(x_0)\ \&\ 2 < x_0) \supset \forall x_1 \forall x_2(\mathrm{Prime}(x_1)\ \&\ \mathrm{Prime}(x_2)\ \&\ x_0 = x_1 + x_2))$
- ◉ $\forall x_0((\mathrm{Even}(x_0)\ \&\ 2 < x_0) \supset \exists x_1 \exists x_2(\mathrm{Prime}(x_1)\ \&\ \mathrm{Prime}(x_2)\ \&\ x_0 = x_1 + x_2))$
- ○ $\forall x_0((\mathrm{Even}(x_0)\ \&\ 2 < x_0) \vee \exists x_1 \exists x_2(\mathrm{Prime}(x_1)\ \&\ \mathrm{Prime}(x_2)\ \&\ x_0 = x_1 + x_2))$

✔

Choose the sentence of L (plus abbreviations) that expresses the (incorrect) claim that the natural numbers are dense: between any two natural numbers there is a third.

- ○ $\exists x_0 \exists x_1 \forall x_2((x_0 < x_2\ \&\ x_2 < x_1) \vee (x_1 < x_2\ \&\ x_2 < x_0))$

- ( ) $\forall x_0 \forall x_1 \exists x_2 ((x_0 < x_2 \,\&\, x_2 < x_1) \wedge (x_1 < x_2 \,\&\, x_2 < x_0))$
- (•) $\forall x_0 \forall x_1 \exists x_2 ((x_0 < x_2 \,\&\, x_2 < x_1) \vee (x_1 < x_2 \,\&\, x_2 < x_0))$

✔

Choose the sentence of L (plus abbreviations) that expresses Euclid's Theorem: there are infinitely many primes.

- ( ) $\exists x_0 (\mathrm{Prime}(x_0) \,\&\, \exists x_1 (\mathrm{Prime}(x_1) \,\&\, x_0 < x_1))$
- (•) $\forall x_0 (\mathrm{Prime}(x_0) \supset \exists x_1 (\mathrm{Prime}(x_1) \,\&\, x_0 < x_1))$
- ( ) $\forall x_0 (\mathrm{Prime}(x_0) \supset \forall x_1 (\mathrm{Prime}(x_1) \,\&\, x_0 < x_1))$

✔

Submit You have used 1 of 1 attempt

Answers are displayed within the problem