Really hard two-dimensial recurrence relations of form p[n,m]=p[n,m-1]+p[n-1,m]+p[n-1,m-1]*(n-1)

Asked 3 years, 6 months ago Active 3 years, 6 months ago Viewed 430 times

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I have a two-dimensional recurrence equation, help me solve this:

```
p[n,m]=p[n,m-1]+p[n-1,m]+p[n-1,m-1]*(n-1)
p[n,0]=1
p[0,m]=0
p[0,0]=0
```

I generated these numbers for 1<=n,m<=6:

n row, m column

1 1 1 1 1 1

3 5 7 9 11 13

6 17 34 57 86 121

10 45 130 289 546 925

15 100 410 1219 2921 6030

21 196 1106 4375 13391 34026

Firstly I saw, that p[n,1] = n*(n+1)/2

Next, fix n = 2, look for the differences between p[n,i] and p[n,i-1].

They are all equals 2 = 2! (remember that)

Now, fix n = 3, also look for the differences between p[n,i] and p[n,i-1]

We have 11, 16, 23, 29. Okay so now look for the differences between differences :)

They are all equals 6 = 3!

Now, fix n = 4, also (hah) look for the differences between p[n,i] and p[n,i-1]

We have 35, 85, 159, 257. Look for the differences between differences.

We have 50, 74, 98. Also look for the differences between differences.

They are all equals 24 = 4!

Now, fix n = 5, also (hah) look for the differences between p[n,i] and p[n,i-1]
85, 310, 809, 1702 ->

225, 499, 893 ->

274, 394 ->

120 = 5!

And so on...

That's all for now :(

updated: I found oeis sequence which is very similar to mine!

math recursion wolfram-mathematica mathematica-8 dimensional Edit tags

Share Edit Follow Close Flag edited Mar 11 '18 at 12:59 asked Mar 9 '18 at 21:28
Андрей Орап
21 3

What is `p[0,m]` equal to? What are the ranges of `n` and `m` ? – meowgoesthedog Mar 10 '18 at 0:14

1 <= n, m <= 5*10^5 – Андрей Орап Mar 10 '18 at 10:30

1 This looks useful: stackoverflow.com/a/4942124/879601 – Chris Degnen Mar 10 '18 at 12:44

Also stackoverflow.com/a/6293838/879601 – Chris Degnen Mar 10 '18 at 17:29

1 Answer

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I suspect the solution to this difference equation is very strongly (factorially?) divergent, so calculating values for n,m as large as 10^5 is going to be challenging.

One can obviously (and inefficiently!) compute p(n,m) by simple recurrence as follows (in Python):

```
import numpy
n_max, m_max = 10, 10
p = numpy.zeros((n_max+1, m_max+1), dtype='int64')
p[:,0] = 1
p[0,:] = 0
p[0,0] = 0
for n in range(1, n_max+1):
    for m in range(1, m_max+1):
        p[n,m] = p[n, m-1] + p[n-1, m] + p[n-1, m-1] * (n-1)
```

This gives the following result for p(n,m):

```
[[         0          0          0          0          0          0
           0          0          0          0          0]
 [         1          1          1          1          1          1
           1          1          1          1          1]
 [         1          3          5          7          9         11
          13         15         17         19         21]
 [         1          6         17         34         57         86
         121        162        209        262        321]
 [         1         10         45        130        289        546
         925       1450       2145       3034       4141]
 [         1         15        100        410       1219       2921
        6030      11180      19125      30739      47016]
 [         1         21        196       1106       4375      13391
       34026      75356     150381     276745     477456]
 [         1         28        350       2632      13643      53284
      167656     447168    1049685    2228716    4366642]
 [         1         36        582       5664      37731     186516
      727160    2347920    6527781   16104292   36071946]
 [         1         45        915      11235      94278     582642
     2801930   10967130   36278271  104604811  269511093]
 [         1         55       1375      20845     216238    1647382
     9693090   45877590  180860031  611969281 1822923673]]
```

which already contains some sizeable values even for n=m=10. Extending that computation to n=m=100, and using floating-point arithmetic, indicates that p(100,100) may be as large as 5x10^172.

Using a generating function

$$\tilde{p}(\alpha, \beta) = \sum_{n,m} p(n,m)\alpha^n \beta^m$$,

I believe you can convert your 2D difference equation into something like

$$\tilde{p} = \beta\tilde{p} + \alpha\tilde{p} + \alpha^2\beta\frac{\partial}{\partial\alpha}\tilde{p},$$

which perhaps might help your analysis. However, as an illustrative comparison, one could consider a difference equation of the form

$$p(n) = p(n-1) + p(n-1).(n-1)$$

which can be converted to the following differential equation for the generating function:

$$\frac{d\tilde{p}}{d\alpha} = \frac{(1-\alpha)\tilde{p}}{\alpha^2}$$

which has a solution of the form:

$$\tilde{p} \propto \alpha^{-1}\exp(-\alpha^{-1})$$

Clearly, such a generating function would have a very poorly behaved Taylor expansion near $\alpha = 0$.

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edited Mar 11 '18 at 9:41

answered Mar 11 '18 at 9:08

rwp
1,516 2 10 21

you should print answer by modulo 10^9+7 (prime number), it is not a problem if you know the formulas. can you give me formula, please? – Андрей Орап Mar 11 '18 at 12:38