Problem 6: Two measurement instruments

(4/4 points)

Let Θ be an unknown random variable that we wish to estimate. It has a prior distribution with mean 1 and variance 2. Let W be a noise term, another unknown random variable with mean 3 and variance 5. Assume that Θ and W are independent.

We have two different instruments that we can use to measure Θ. The first instrument yields a measurement of the form $X_1 = \Theta + W$, and the second instrument yields a measurement of the form $X_2 = 2\Theta + 3W$. We pick an instrument at random, with each instrument having probability $1/2$ of being chosen. Assume that this choice of instrument is independent of everything else. Let X be the measurement that we observe, without knowing which instrument was used.

Give numerical answers for all parts below.

1.

$\mathbf{E}[X] =$ 7.5

Answer: 7.5

2.

$\mathbf{E}[X^2] =$ 98.5

Answer: 98.5

3. The LLMS estimator of Θ given X is of the form $aX + b$. Give the numerical values of a and b.

$a =$ 12/169

Answer: 0.07101

$b =$ 79/169

Answer: 0.46746

Answer:

1. Let I be the instrument used to perform the measurement, with $\mathbf{P}(I = 1) = \mathbf{P}(I = 2) = 1/2$.

   Using the total expectation theorem, we have

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{E}[X \mid I = 1]\mathbf{P}(I = 1) + \mathbf{E}[X \mid I = 2]\mathbf{P}(I = 2) \\ &= \frac{1}{2} \cdot (\mathbf{E}[\Theta + W] + \mathbf{E}[2\Theta + 3W]). \\ &= \frac{1}{2} \cdot (4 + 11) \end{aligned}$$

$$= \frac{15}{2}.$$

2. Note that

$$\begin{aligned} \mathbf{E}[\Theta^2] &= \text{var}(\Theta) + (\mathbf{E}[\Theta])^2 = 3, \\ \mathbf{E}[W^2] &= \text{var}(W) + (\mathbf{E}[W])^2 = 14. \end{aligned}$$

Also, $\mathbf{E}[\Theta W] = \mathbf{E}[\Theta]\mathbf{E}[W]$ by independence.

Using the total expectation theorem again, we obtain

$$\begin{aligned} \mathbf{E}[X^2] &= \mathbf{E}[X^2 \mid I = 1]\mathbf{P}(I = 1) + \mathbf{E}[X^2 \mid I = 2]\mathbf{P}(I = 2) \\ &= \frac{1}{2} \cdot (\mathbf{E}[(\Theta + W)^2] + \mathbf{E}[(2\Theta + 3W)^2]). \\ &= \frac{1}{2} \cdot (5\mathbf{E}[\Theta^2] + 14\mathbf{E}[\Theta W] + 10\mathbf{E}[W^2]) \\ &= \frac{1}{2} \cdot (15 + 42 + 140) \\ &= \frac{197}{2}. \end{aligned}$$

3. The LLMS estimator of Θ given X is

$$\widehat{\Theta}_{LLMS} = \mathbf{E}[\Theta] + \frac{\textbf{cov}(X,\Theta)}{\textbf{var}(X)}(X - \mathbf{E}[X]).$$

First, $\textbf{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = 197/2 - (15/2)^2 = 169/4$. Next, the covariance term is computed by first calculating $\mathbf{E}[X\Theta]$:

$$\begin{aligned}
\mathbf{E}[X\Theta] &= \mathbf{E}[X\Theta \mid I = 1]\mathbf{P}(I = 1) + \mathbf{E}[X\Theta \mid I = 2]\mathbf{P}(I = 2) \\
&= \mathbf{E}[(\Theta + W)\Theta]\mathbf{P}(I = 1) + \mathbf{E}[(2\Theta + 3W)\Theta]\mathbf{P}(I = 2) \\
&= \frac{1}{2} \cdot (3\mathbf{E}[\Theta^2] + 4\mathbf{E}[\Theta W]) \\
&= \frac{1}{2} \cdot (9 + 12) \\
&= \frac{21}{2}.
\end{aligned}$$

Hence, $\textbf{cov}(X,\Theta) = 21/2 - (15/2) \cdot 1 = 3$. Therefore,

$$\begin{aligned}
\widehat{\Theta}_{LLMS} &= 1 + \frac{3}{169/4}\left(X - \frac{15}{2}\right) \\
&= \frac{12}{169}X + \frac{79}{169}.
\end{aligned}$$

You have used 1 of 2 submissions