13. Gauss–Jordan elimination

Jordan recognized the fact that Gaussian elimination can be continued to algorithmically perform the back substitution step as well. The result of this algorithm is called the **reduced row echelon form** .

Definition 13.1 A matrix is in **reduced row echelon form (`rref`)** if it satisfies all of the following conditions:

1. It is in row echelon form.

2. Each pivot is a 1.

3. In each pivot column, all the entries are 0 except for the pivot itself.

Example problem 1

3/3 points (graded)

Find the solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the system represented by the following augmented matrix in reduced row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$x =$ 4 ✔ **Answer:** 4

$y =$ 1/3 ✔ **Answer:** 1/3

$z =$ -2 ✔ **Answer:** -2

FORMULA INPUT HELP

Solution:

For a matrix in row echelon form, you can essentially read off the answers without back-solving! The process of getting the matrix to reduced row echelon form reduces the augmented matrix so that each row is an equation in terms of only one pivot variable.

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Answers are displayed within the problem

Example problem 2

3/3 points (graded)

Find the solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the system represented by the following augmented matrix in reduced row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(Type c for the parameter introduced by the free variable z.)

$x =$ 4-2*c ✔ **Answer:** 4-2*c

$4 - 2 \cdot c$

$y =$ 1/3-c ✔ **Answer:** 1/3-c

$\frac{1}{3} - c$

$z =$ c ✔ **Answer:** c

c

FORMULA INPUT HELP

Solution:

In this case, we cannot solve for the variable z as it corresponds to a nonpivot column. Therefore we let $z = c$, an arbitrary real parameter. Then we can use back substitution to solve for y and x in terms of c. We find that $y = 1/3 - c$, and $x = 4 - 2c$.

Submit You have used 1 of 7 attempts

Answers are displayed within the problem

The algorithm:

Gauss–Jordan elimination is the name of the algorithm for converting any matrix into **reduced** row echelon form by performing row operations. Here are the steps:

1. Use Gaussian elimination to convert the matrix to row echelon form.

2. Divide the last nonzero row by its pivot, to make the pivot 1.
3. Make all entries in that pivot's column 0 by adding suitable multiples of the pivot's row to the rows above.
4. At this point, the row in question (and all rows below it) are done. Ignore them, and go back to Step 2, but now with the remaining submatrix, above the row just completed.

Eventually the whole matrix will be in reduced row echelon form.

Problem 13.2 Convert the 4×7 matrix

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

to reduced row echelon form.

Solution:

Step 1. The matrix is already in row echelon form.

Step 2. The last nonzero row is the third row, and its pivot is the -4, so divide the third row by -4:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 3. To make all other entries of that pivot's column 0, add -1 times the third row to the first row, and add 4 times the third row to the second row:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 0 & 3/2 \\ 0 & 0 & 3 & 1 & -2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4. Now the last two rows are done:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 0 & 3/2 \\ 0 & 0 & 3 & 1 & -2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Go back to Step 2, but with the 2×7 submatrix above them.

Step 2. The last nonzero row of the new matrix (ignoring the bottom two rows of the original matrix) is the second row, and its pivot is the 3, so we divide the second row by 3:

$$\begin{pmatrix} 2 & -3 & 1 & 4 & -7 & 0 & 3/2 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 3. To make the other entries of that pivot's column 0, add -1 times the second row to the first row:

$$\begin{pmatrix} 2 & -3 & 0 & 11/3 & -19/3 & 0 & -1/2 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4. Now the last three rows are done:

$$\begin{pmatrix} 2 & -3 & 0 & 11/3 & -19/3 & 0 & -1/2 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Go back to Step 2, but with the 1×7 submatrix above them.

Step 2. The last nonzero row of the new matrix is the only remaining row (the first row), and its pivot is the initial 2, so we divide the first row by 2:

$$\begin{pmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 0 & -1/4 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix is now in reduced row echelon form.

Remark 13.3 Performing row operations on $\mathbf{A}$ in a different order than specified by Gaussian elimination and Gauss-Jordan elimination can lead to different row echelon forms. But it turns out that row operations leading to **reduced row echelon form** always give the same result, a matrix that we will write as $\mathtt{rref}(\mathbf{A})$.

Identify matrices in reduced row echelon form

1/1 point (graded)

Which of the following matrices are in reduced row echelon form?

- [ ] $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$
- [ ] $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$
- [ ] $\mathbf{C} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$
- [ ] $\mathbf{D} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$
- [ ] $\mathbf{E} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/4 \\ 0 & 0 & 1 \end{pmatrix}.$
- [x] $\mathbf{F} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$ ✔
- [x] $\mathbf{G} = \begin{pmatrix} 1 & 2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$ ✔

✔

Solution:

The matrices in reduced row echelon form are $\mathbf{F}$ and $\mathbf{G}$.

- $\mathbf{F} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ has pivots in the first and second rows (highlighted in orange). All other entries in the columns which have a pivot are zero.
- $\mathbf{G} = \begin{pmatrix} 1 & 2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has one pivot in the first row (highlighted in orange). All other entries in the first column are zero. All other rows are zero rows.

Let's see why the other matrices are not in reduced row echelon form.

- $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{1} & 1 \\ 0 & \mathbf{1} & 0 \end{pmatrix}$ is not in reduced row echelon form because it is not even in row echelon form. The entry below the pivot in the second row is not zero.

- $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & \mathbf{3} \end{pmatrix}$ is not in reduced row echelon form because the pivots in the second and third rows are not equal to one.

- $\mathbf{C} = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{1} \\ 0 & 1 & \mathbf{1} \\ 0 & 0 & 1 \end{pmatrix}$ is not in reduced row echelon form because there are pivots in each row along the diagonal, but the entries above the pivots in the second and third rows are not zero.

- $\mathbf{D} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & \mathbf{4} & 5 \\ 0 & 0 & \mathbf{6} \end{pmatrix}$ is not in reduced row echelon form because the pivots in the second and third rows are not one. Additionally, the entries above these pivots are not zero.

- $\mathbf{E} = \begin{pmatrix} 1 & \mathbf{2} & \mathbf{3} \\ 0 & 1 & \mathbf{5/4} \\ 0 & 0 & 1 \end{pmatrix}$ is not in reduced row echelon form because there are pivots in each row along the diagonal, but the entries above the pivots in the second and third rows are not zero.

Submit You have used 1 of 5 attempts

Answers are displayed within the problem

Reduced row echelon form concept check

1/1 point (graded)

Consider the matrix which we had in some previous problems

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 4 \\ -2 & -3 & 10 \\ 1 & 2 & 1 \end{pmatrix}.$$

What is the reduced row echelon form of $\mathbf{A}$?

- $\begin{pmatrix} -2 & -3 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

- $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 12 \\ 0 & 0 & -8 \end{pmatrix}$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ✔

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution:

To get to the reduced row echelon form we first get the row echelon form of $\mathbf{A}$ and then continue until we get to RREF($\mathbf{A}$).

$$\begin{pmatrix} -2 & -3 & 10 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & 10 \\ 0 & 1 & 4 \\ 0 & 1/2 & 6 \end{pmatrix} \begin{pmatrix} -2 & -3 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so

$$\text{RREF}(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Submit

You have used 1 of 4 attempts

Answers are displayed within the problem

Number of pivots concept check

0/1 point (graded)

Let $\mathbf{A}$ be a matrix in row echelon form. Let $\mathbf{B}$ be the reduced row echelon matrix form of $\mathbf{A}$; in other words, $\mathbf{B} = \mathtt{rref}(\mathbf{A})$. Which of the following must be true?

- The number of pivots of $\mathbf{B}$ is strictly greater than the number of pivots of $\mathbf{A}$.
- The number of pivots of $\mathbf{B}$ is greater than or equal to the number of pivots of $\mathbf{A}$. ✘

- The number of pivots of $\mathbf{B}$ is equal to the number of pivots of $\mathbf{A}$. ✔
- The number of pivots of $\mathbf{B}$ is less than or equal to the number of pivots of $\mathbf{A}$.
- The number of pivots of $\mathbf{B}$ is strictly less than the number of pivots of $\mathbf{A}$.
- The number of pivots of $\mathbf{B}$ may be greater than, less than, or equal to the number of pivots of $\mathbf{A}$.

Solution:

When we go from the row echelon form to the reduced row echelon form of a matrix, we eliminate the entries above the pivot positions, but the pivots all remain in the same positions and, therefore, their number stays the same.

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Answers are displayed within the problem

13. Gauss–Jordan elimination

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? Identify matrices in reduced row echelon form, issue 8
I don't think the grader is right about this. Please confirm. By definition, rref() requires a 1 in the pivot position with zeroes n the column undern...

Typo in the solution to "Example problem 1" 3
"For a matrix in row echelon form," should be "For a matrix in **reduced** row echelon form," or even better, "**From** a matrix in **reduced...

? Number of pivots concept check I - pivots of A 6

Minor typo in solution of Identify matrices in reduced row echelon form 2
The reasoning of D in the solution has a minor typo in the first sentence; should replace "zero." at the end of that sentence with "one.".

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