



MITx: 6.041x Introduction to Probability - The Science of Uncertainty



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Problem 5: Covariance for the multinomial

(5/5 points)

Consider n independent rolls of a k -sided fair die with $k \geq 2$: the sides of the die are labelled $1, 2, \dots, k$ and each side has probability $1/k$ of facing up after a roll. Let the random variable X_i denote the number of rolls that result in side i facing up. Thus, the random vector (X_1, \dots, X_k) has a multinomial distribution.


1. Which of the following statements is correct? Try to answer without doing any calculations.

- ☐ X_1 and X_2 are uncorrelated.
- ☐ X_1 and X_2 are positively correlated.
- ☒ X_1 and X_2 are negatively correlated. ✓


Unit 6: Further topics on random variables

Unit overview


Lec. 11: Derived distributions

Exercises 11 due Mar 30, 2016 at 23:59 UTC 

Lec. 12: Sums of independent r.v.'s; Covariance and correlation

Exercises 12 due Mar 30, 2016 at 23:59 UTC 


Lec. 13: Conditional expectation and variance revisited; Sum of a random number of independent r.v.'s

Exercises 13 due Mar 30, 2016 at 23:59 UTC 

Solved problems

Additional theoretical material

Problem Set 6

Problem Set 6 due Mar 30, 2016 at 23:59 UTC 

Unit summary

2. Find the covariance, $\text{cov}(X_1, X_2)$, of X_1 and X_2 . Express your answer as a function of n and k using standard notation. *Hint:* Use indicator variables to encode the result of each roll.

$$\text{cov}(X_1, X_2) = -n/k^2$$

✓ Answer: $-n/(k^2)$

3. Suppose now that the die is biased, with a probability $p_i \neq 0$ that the result of any given die roll is i , for $i = 1, 2, \dots, k$. We still consider n independent tosses of this biased die and define X_i to be the number of rolls that result in side i facing up.

Generalize your answer to part 2: Find $\text{cov}(X_1, X_2)$ for this case of a biased die. Express your answer as a function of n, k, p_1, p_2 using standard notation. Write p_1 and p_2 as p_1 and p_2 , respectively, and wrap them in parentheses in your answer; i.e., enter (p_1) and (p_2) .

$$\text{cov}(X_1, X_2) = -n \cdot p_1 \cdot p_2$$

✓ Answer: $-n \cdot (p_1) \cdot (p_2)$

Answer:

1. The random variables X_1 and X_2 are negatively correlated. There is a fixed number, n , of rolls of the die. Intuitively, a large number of rolls that result in a 1 uses up many of the n total rolls, which leaves fewer remaining rolls that could result in a 2.
2. Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the t th roll resulted in a 1 (respectively, 2). Note that $X_1 = \sum_{t=1}^n A_t$ and $X_2 = \sum_{t=1}^n B_t$, and so

- ▶ Unit 7: Bayesian inference
- ▶ Exam 2
- ▶ Unit 8: Limit theorems and classical statistics
- ▶ Unit 9: Bernoulli and Poisson processes
- ▶ Unit 10: Markov chains
- ▶ Exit Survey
- ▶ Final Exam

$$\mathbf{E}[X_1] = \mathbf{E}[X_2] = \mathbf{E} \left[\sum_{t=1}^n A_t \right] = n\mathbf{E}[A_1] = \frac{n}{k}.$$

Since one roll of the die cannot result in both a 1 and a 2, at least one of A_t and B_t must equal 0. Thus, $\mathbf{E}[A_t B_t] = 0$. Furthermore, since different rolls of the die are independent, A_t and B_s are independent if $t \neq s$. Therefore,

$$\mathbf{E}[A_t B_s] = \mathbf{E}[A_t] \mathbf{E}[B_s] = \frac{1}{k} \cdot \frac{1}{k} = \frac{1}{k^2} \quad \text{for } t \neq s,$$

and so

$$\begin{aligned} \mathbf{E}[X_1 X_2] &= \mathbf{E}[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)] \\ &= \mathbf{E} \left[\sum_{t=1}^n A_t B_t + \sum_{t \neq s} A_t B_s \right] \\ &= n \cdot 0 + n(n-1) \cdot \mathbf{E}[A_1 B_2] \\ &= n(n-1) \cdot \frac{1}{k^2}. \end{aligned}$$

Thus,

$$\text{cov}(X_1, X_2) = \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1] \mathbf{E}[X_2]$$

$$\begin{aligned}
 &= n(n-1) \cdot \frac{1}{k^2} - \frac{n}{k} \cdot \frac{n}{k} \\
 &= -\frac{n}{k^2}.
 \end{aligned}$$

The covariance of X_1 and X_2 is negative as expected.

3. Follow the same reasoning as part 2. Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the t th roll resulted in a 1 (respectively, 2). As in part 2, one roll of the die still cannot result in both a 1 and a 2, so $\mathbf{E}[A_t B_t] = 0$.

Different rolls of the die are still independent, and so

$\mathbf{E}[A_t B_s] = \mathbf{E}[A_t] \mathbf{E}[B_s] = p_1 \cdot p_2$, for $t \neq s$. Thus,

$$\begin{aligned}
 \mathbf{E}[X_1 X_2] &= \mathbf{E}[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)] \\
 &= \mathbf{E} \left[\sum_{t=s} A_t B_t + \sum_{t \neq s} A_t B_s \right] \\
 &= n \cdot 0 + n(n-1) \cdot \mathbf{E}[A_1 B_2] \\
 &= n(n-1)p_1 p_2.
 \end{aligned}$$

Note that $X_1 = \sum_{t=1}^n A_t$ and $X_2 = \sum_{t=1}^n B_t$, and so

$\mathbf{E}[X_1] = \mathbf{E}[\sum_{t=1}^n A_t] = n\mathbf{E}[A_1] = np_1$. Similarly, $\mathbf{E}[X_2] = np_2$.

Therefore,

$$\begin{aligned}
 \text{cov}(X_1, X_2) &= \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1] \mathbf{E}[X_2] \\
 &= n(n-1)p_1 p_2 - (np_1)(np_2)
 \end{aligned}$$

$$= -np_1p_2.$$

The covariance of X_1 and X_2 is again negative, even when the die is no longer fair as it was in parts 1 and 2.

You have used 1 of 2 submissions

DISCUSSION

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