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Eigenspaces

Let λ be an eigenvalue of A. Recall that the eigenvectors of A for λ are the nonzero vectors in the nullspace of $A - \lambda I$. We call the nullspace $A - \lambda I$ the **eigenspace** of A for λ denoted by $\mathcal{E}_A(\lambda)$. In other words, $\mathcal{E}_A(\lambda)$ consists of all the eigenvectors of A for λ and the zero vector.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Note that -1 is an eigenvalue of A. Then $A - (-1)I_2 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. The nullspace of this matrix is spanned by the single vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence, $\mathcal{E}_A(-1)$ is the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Algebraic multiplicity vs geometric multiplicity

The **geometric multiplicity** of an eigenvalue λ of A is the dimension of $\mathcal{E}_A(\lambda)$. In the example above, the geometric multiplicity of -1 is 1 as the eigenspace is spanned by one nonzero vector.

In general, determining the geometric multiplicity of an eigenvalue requires no new technique because one is simply looking for the dimension of the nullspace of $A - \lambda I$.

The **algebraic multiplicity** of an eigenvalue λ of A is the number of times λ appears as a root of p_A. For the example above, one can check that -1 appears only once as a root. Let us now look at an example in which an eigenvalue has multiplicity higher than 1.

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Then $p_A = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$. Note that 1 is a root and it appears twice. Hence, the algebraic multiplicity of 1 is 2. But what is the geometric multiplicity?

Consider $A - \lambda I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. What is the dimension of its nullspace? Clearly, the rank of this matrix is 1. By the rank-nullity formula, we get that the nullspace has dimension 1. Hence, the geometric multiplicity is 1. This is different from the algebraic multiplicity!

In general, **the algebraic multiplicity and geometric multiplicity of an eigenvalue can differ. However, the geometric multiplicity can never exceed the algebraic multiplicity**.

It is a fact that summing up the algebraic multiplicities of all the eigenvalues of an $n \times n$ matrix A gives exactly n. **If for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity, then A is said to be *diagonalizable***. As we will see, it is relatively easy to compute powers of a diagonalizable matrix.

Quick Quiz

Exercises

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

   a. Find all eigenvalues of A. Show answer

b. For each eigenvalue of A, determine its algebraic multiplicity and geometric multiplicity. Show answer

2. Let $A = \begin{bmatrix} 8 & -9 \\ 4 & -4 \end{bmatrix}$.

   a. Find all eigenvalues of A. Show answer

   b. For each eigenvalue of A, determine its algebraic multiplicity and geometric multiplicity. Show answer

3. Is $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ diagonalizable? Show answer

4. Let $A, B \in \mathbb{C}^{n \times n}$. A and B are said to be similar if $B = S^{-1}AS$ for some invertible matrix $S \in \mathbb{C}^{n \times n}$.

   a. Show that similar matrices have the same eigenvalues, including multiple appearances.

   b. Let λ be an eigenvalue of A. Prove that the geometric multiplicity of λ is at most the algebraic multiplicity of λ. (Hint: Let λ be an eigenvalue of A. Let $\{v_1, \ldots, v_k\}$ be a basis for $\mathcal{E}_A(\lambda)$. Extend $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_n\}$ for

$\mathbb{C}^n$. Form the matrix $S = [v_1 \cdots v_n]$. What can you say about the first k columns of the matrix $B = S^{-1}AS$?.)