Gamma density

Consider the distribution of the sum of two independent Exponential(λ) random variables. I showed that it has a density of the form:

$$f_S(s) = \int_0^s \lambda^2 e^{-\lambda s} dy = \lambda^2 s e^{-\lambda s}$$

This density is called the $Gamma(2, \lambda)$ density. In general the gamma density is defined with 2 parameters (t, λ) (both positive reals, most often *t* is actually integer) as being non zero on the positive reals and defined as:

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{t-1}}{\Gamma(t)}$$

where $\Gamma(t)$ is the constant that makes the integral of the density sum to one:

$$\Gamma(t) = \int_0^{+\infty} e^{-y} y^{t-1} dy$$

By integration by parts we showed the important recurrence relation:

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

Because $\Gamma(1) = \int_0^{+\infty} e^{-y} dy = 1$, we have for integer *t*=*m*

$$\Gamma(m) = (m-1)(m-2)\cdots\Gamma(1) = (m-1)!$$

The particular case of the integer *t* can be compared to the sum of *n* independent exponentials, it is the waiting time to the *nth* event, it is the *twin* of the negative binomial.

From this we can guess what the expected value and the variance are going to be: If all the X_i's are independent $Exponential(\lambda)$, then if we sum *n* of them we have $E(\sum_1^n X_i) = \frac{n}{\lambda}$ and if they are independent: $var(\sum_1^n X_i) = \frac{n}{\lambda^2}$

This generalizes to the non integer *t* case:

$$E(X) = \frac{1}{\Gamma(t)} \int_0^{+\infty} e^{-\lambda x} (\lambda x)^{t-1} dx = \frac{1}{\lambda\Gamma(t)} \int_0^{+\infty} \lambda e^{-\lambda x} (\lambda x)^t = \frac{\Gamma(t+1)}{\lambda\Gamma(t)} = \frac{t}{\lambda}$$

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