9. Gaussian elimination

Gaussian elimination is an algorithm that specifies a sequence of equation operations guaranteed to convert a linear system into one that we can solve easily. Usually this is described in terms of row operations on an augmented matrix. It converts this matrix into what is called **row echelon form.** Here are the steps. Try reading them together with the example below.

1. Find the left-most nonzero column, and the first nonzero entry in that column (read from the top down). Call this the first **pivot.**

2. If that entry is not already in the first row, interchange its row with the first row.

3. Make all other entries of the column zero by adding suitable multiples of the first row to the others.

4. At this point, the first row is done, so ignore it, and repeat the steps above for the remaining submatrix (with one fewer row). In each iteration, ignore the rows already taken care of. Eventually the whole matrix will be in row echelon form.

With the algorithm, we can program a computer to solve large systems (100 equations with 100 unknowns).

Example 9.1 Let us continue with the example from the previous page, and convert the augmented matrix to row echelon form using Gaussian elimination.

$$\begin{aligned} 8y - 4z &= 0 \\ x - y + 4z &= 1 \\ -x - 5y + 2z &= 0, \end{aligned}$$

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 8 & -4 & 0 \\ 1 & -1 & 4 & 1 \\ -1 & -5 & 2 & 0 \end{array}\right).$$

Step 1. The leftmost nonzero column is the first one, and its first nonzero entry is the 1:

$$\left(\begin{array}{ccc|c} 0 & 8 & -4 & 0 \\ 1 & -1 & 4 & 1 \\ -1 & -5 & 2 & 0 \end{array}\right)$$

Step 2. The 1 is not in the first row, so interchange its row with the first row:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ -1 & -5 & 2 & 0 \end{array}\right)$$

Step 3. To make all other entries of the column zero, we need to add 1 times the first row to the last row (the second row is OK already):

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ 0 & -6 & 6 & 1 \end{array}\right)$$

Step 4. Because entries below the pivot are all zero, the first row (in gray) is done. Start over with the submatrix that remains beneath the first row:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ 0 & -6 & 6 & 1 \end{array}\right)$$

Step 1. The leftmost nonzero column is now the second column, and its first nonzero entry is the 8:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ 0 & -6 & 6 & 1 \end{array}\right)$$

Step 2. The first nonzero entry is 8, which is already in the first row of the submatrix (we are ignoring the first row of the whole matrix), so no interchange is necessary.

Step 3. To make all other entries below the pivot of the column zero, add $6/8$ times the (new) first row to the (new) second row:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

Step 4. Now the first and second row of the original matrix are done (both now in gray). Start over with the submatrix beneath them:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

Step 1. The leftmost nonzero column is now the third column, and its first nonzero entry is the 3 at the bottom:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

There are no columns below this column, so we are done.

The matrix is now in **row echelon** form:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 8 & -4 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

The first nonzero entry in each row is the **pivot.**

Definition 9.2 A matrix is in **row echelon form** if it satisfies the following conditions:

1. All the zero rows (if any) are grouped at the bottom of the matrix.

2. The first nonzero entry in a nonzero row, which we call the **pivot,** lies farther to the right than the pivots of higher rows.

3. All entries in the column below a pivot entry are zero.

Warning: Some books require also that each pivot be a 1. We are not going to require this for row echelon form, but we will require it for **reduced** row echelon form later on.

Example 9.3 The following matrices are all in row echelon form. The pivots are highlighted in **orange.**

$$\begin{pmatrix} 1 & 4 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad \begin{pmatrix} 1 & 4 & 5 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Nonexamples The following matrices are not in row echelon form.

1. The following matrix has a zero row (in blue) that is above a nonzero row.

$$\begin{pmatrix} 1 & 4 & 5 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

2. The following matrix is not in row echelon form because the pivot in the third row is to the left of the pivot in the second row (both in blue).

$$\begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

3. The following matrix is not in row echelon form because the entries below the **pivot** in the 2nd row are not all zero (highlighted in blue).

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 9.4 A row echelon form of a matrix $\mathbf{A}$ is any matrix in row echelon form obtained from $\mathbf{A}$ by a sequence of row operations.

Note: Gaussian elimination is one algorithm for obtaining a row echelon form of a matrix. However, there are other sequences of row operations that will lead to different row echelon forms. However, all of these row echelon forms have the same solution set as the original system.

Identify matrices in row echelon form

1/1 point (graded)

Which of the following matrices are in row echelon form?

- [ ] $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$
- [x] $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$ ✔
- [ ] $\mathbf{C} = \begin{pmatrix} 6 & 2 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{pmatrix}.$
- [x] $\mathbf{D} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$ ✔
- [ ] $\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}.$
- [x] $\mathbf{F} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$ ✔
- [ ] $\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$

✔

Solution:

A matrix is in row echelon form if

1. All the zero rows (if any) are grouped at the bottom of the matrix.
2. The first nonzero entry in a nonzero row, which we call the **pivot,** lies farther to the right than the pivots of higher rows.

3. All entries in the column below a pivot entry are zero.

The only matrices that satisfy all of the conditions in the definition of row echelon form are the matrices:

- $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, which has three pivots (in orange) along the diagonal. Each pivot is to the right of the pivots above it. All entries below a pivot are zero.
- $\mathbf{D} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, which has three pivots (in orange) along the diagonal. Each pivot is to the right of the pivots above it. All entries below a pivot are zero.
- $\mathbf{F} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which has one pivot (in orange) in the first row and first column. All entries below this pivot are zero, and in fact all other rows are zero rows.

Let's see why the other matrices are not in row echelon form:

- $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 \end{pmatrix}$ is not in row echelon form because the pivot in the third row is to the **left** of the pivot in the second row.
- $\mathbf{C} = \begin{pmatrix} 6 & 2 & 7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 5 \end{pmatrix}$ is not in row echelon form because the zero row is not at the bottom of the matrix.
- $\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{2} & 3 & 0 \\ \mathbf{4} & 5 & 6 \end{pmatrix}$ is not in row echelon form because the first entry in the first row is a pivot, but the entries below it in the first column are not zero.

$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ is not in row echelon form because the first entry in the first row is a pivot, but the entries below it in the first column are not zero.

Submit You have used 1 of 15 attempts

Answers are displayed within the problem

Row echelon form concept check I

1/1 point (graded)

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 4 \\ -2 & -3 & 10 \\ 1 & 2 & 1 \end{pmatrix}.$$

You apply Gaussian elimination. The first step is to exchange the first and second rows. After completing the rest of the steps, the resulting row echelon form of $\mathbf{A}$ is

$$\begin{pmatrix} -2 & -3 & 10 \\ 0 & 1 & * \\ 0 & 0 & 4 \end{pmatrix},$$

what is the value of the missing entry (*)?

4

Answer: 4

Solution:

Exchanging the first and the second row converts the matrix

$$\begin{pmatrix} 0 & 1 & 4 \\ -2 & -3 & 10 \\ 1 & 2 & 1 \end{pmatrix} \qquad \text{into} \qquad \begin{pmatrix} -2 & -3 & 10 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix},$$

and we see that we already have the pivots in the first and the second row, so we do not even need to do Gaussian elimination further, because the second row will not be affected by our work towards eliminating the first two entries of the third row. Therefore, the missing entry is equal to 4.

Submit You have used 1 of 7 attempts

Answers are displayed within the problem

Row echelon form concept check II

1/1 point (graded)

Consider the same matrix as above

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 4 \\ -2 & -3 & 10 \\ 1 & 2 & 1 \end{pmatrix}.$$

You want to find a row echelon form of this matrix. Instead of applying Gaussian elimination directly, you first exchange the first and third rows, and then complete the rest of the steps in the algorithm. After completing the steps, the resulting row echelon form of $\mathbf{A}$ is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & * \\ 0 & 0 & -8 \end{pmatrix},$$

what is the value of the missing entry (*)?

12

Answer: 12

Solution:

After exchanging the first and the third row we get the matrix

$$\begin{pmatrix} 0 & 1 & 4 \\ -2 & -3 & 10 \\ 1 & 2 & 1 \end{pmatrix} \quad \textbf{into} \quad \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 10 \\ 0 & 1 & 4 \end{pmatrix},$$

Adding two times the first row to the second row, we get

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 12 \\ 0 & 1 & 4 \end{pmatrix},$$

Subtracting the second row from the third row, we get a matrix in row echelon form:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 12 \\ 0 & 0 & -8 \end{pmatrix},$$

Therefore the missing number is 12.

Submit You have used 1 of 5 attempts

Answers are displayed within the problem

Row echelon form concept check III

1/1 point (graded)

Starting from a single matrix, is it possible to perform two different sequences of row operations leading to two different matrices, both of which are in row echelon form? (Hint: look at the previous two problems.)

- [x] Yes. ✔
- [ ] No.

Solution:

Yes. As we see in the previous two problems, we can get different row echelon forms for the same matrix $\mathbf{A}$.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

9. Gaussian elimination

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- Proper reduced row-echelon form of an augmented matrix. — 1
  if Ax=b has in every column of the row-echelon form H a pivot of numeral 1 in its column, then the syste...
- ? Row echelon form concept check I — 3
- [Staff] Typo in the sentence above Definition 9.2 — 2
  The sentence right above Definition 9.2 reads: > The first nonzero entry in each column is the **pivot**. ...
- There are no columns below this column, so we are done — 3
  in the first example. so you mean there are no rows below this row?
- ? Row echelon form concept check II issue — 5
  I think that the grader is wrong here. Please confirm.

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