6. Wald's Test Continued

Review: Chi-Squared Distribution

2/2 points (graded)
Which of the following random variables follow a χ^2_d distribution?
(Choose all that apply. In the choices, “I" denotes the $d \times d$ identity matrix.)

- [ ] $Z_1 + Z_2 \ldots + Z_d$ where $Z_i \sim \mathcal{N}(\mu, \sigma^2)$
- [ ] $Z_1^2 + Z_2^2 \ldots + Z_d^2$ where $Z_i \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu, \sigma \in \mathbb{R}$
- [ ] $Z_1^2 + Z_2^2 \ldots + Z_d^2$ where $Z_i \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu, \sigma \in \mathbb{R}$ and are independent
- [ ] $Z_1 + Z_2 \ldots + Z_d$ where $Z_i \sim \mathcal{N}(0, 1)$
- [ ] $Z_1^2 + Z_2^2 \ldots + Z_d^2$ where $Z_i \sim \mathcal{N}(0, 1)$
- [x] $Z_1^2 + Z_2^2 \ldots + Z_d^2$ where $Z_i \sim \mathcal{N}(0, 1)$ and are independent

✔

- [ ] $\|\mathbf{Z}\|$ where $\mathbf{Z}$ is multivariate Gaussian $\mathbf{Z} \sim \mathcal{N}_d(\vec{\mu}, \Sigma_{\mathbf{Z}})$ for some $\vec{\mu} \in \mathbb{R}^d$ and $d \times d$ matrix $\Sigma_{\mathbf{Z}}$
- [ ] $\|\mathbf{Z}\|$ where $\mathbf{Z}$ is multivariate Gaussian $\mathbf{Z} \sim \mathcal{N}_d(\vec{\mu}, I)$ for some some $\vec{\mu} \in \mathbb{R}^d$
- [ ] $\|\mathbf{Z}\|$ where $\mathbf{Z}$ is multivariate Gaussian $\mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, I)$
- [ ] $\|\mathbf{Z}\|^2$ where $\mathbf{Z}$ is multivariate Gaussian $\mathbf{Z} \sim \mathcal{N}_d(\vec{\mu}, \Sigma_{\mathbf{Z}})$ for some $\vec{\mu} \in \mathbb{R}^d$ and $d \times d$ matrix $\Sigma_{\mathbf{Z}}$)
- [ ] $\|\mathbf{Z}\|^2$ where $\mathbf{Z}$ is multivariate Gaussian $\mathbf{Z} \sim \mathcal{N}_d(\vec{\mu}, I)$ for some some $\vec{\mu} \in \mathbb{R}^d$
- [x] $\|\mathbf{Z}\|^2$ where $\mathbf{Z}$ is multivariate Gaussian $\mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, I)$

✔

Solution:

The χ^2 distribution with d degrees of freedom is by definition the distribution of

$$Z_1^2 + Z_2^2 \ldots + Z_d^2 \qquad \text{where } Z_i \stackrel{iid}{\sim} \mathcal{N}(0, 1)$$

or equivalently the distribution of

$$\|\mathbf{Z}\|^2 \qquad \text{where } \mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, \mathbf{1}),$$

whose components are independent because the off-diagonal elements of the covariance matrix $\mathbf{1}$ are all 0.

Remark: Recall from a problem on the previous page that the vector $\mathbf{MZ}$, where $\mathbf{M}^T = \mathbf{M}^{-1}$ (or equivalently $\mathbf{M}\mathbf{M}^T = \mathbf{M}^T\mathbf{M} = \mathbf{1}_{d\times d}$,) is also a **standard** multivariate Gaussian vector. Hence $\|\mathbf{MZ}\|^2$ also follows a χ_d^2 distribution.

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Answers are displayed within the problem

Review: Writing the Norm Squared

1/1 point (graded)

Which of the following equals the squared norm $\|\mathbf{A}\mathbf{x}\|^2$ of the vector $\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a **symmetric** $d \times d$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^d$?

(Choose all that apply.)

- [x] $(\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x})$
- [ ] $(\mathbf{A}\mathbf{x})(\mathbf{A}\mathbf{x})^T$
- [x] $\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}$
- [x] $\mathbf{x}^T\mathbf{A}^2\mathbf{x}$

Solution:

$$\begin{aligned}\|\mathbf{A}\mathbf{x}\|^2 &= (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}\\ &= \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} \qquad (\text{since } \mathbf{A}^T = \mathbf{A}) = \mathbf{x}^T\mathbf{A}^2\mathbf{x}\end{aligned}$$

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Wald's Test Continued

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Deriving Wald's Test

1/1 point (graded)

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathbf{P}_{\theta^*}$ for some true parameter $\theta^* \in \mathbb{R}^d$. We construct the associated statistical model $(\mathbb{R}, \{\mathbf{P}_\theta\}_{\theta \in \mathbb{R}^d})$ and the maximum likelihood estimator $\hat{\theta}_n^{MLE}$ for θ^*.

Your goal is to use hypothesis testing to decide between two hypotheses:

$$H_0 : \theta^* = \mathbf{0}$$
$$H_1 : \theta^* \neq \mathbf{0}.$$

Assuming that the null hypothesis is true, the asymptotic normality of the MLE $\hat{\theta}_n^{MLE}$ implies that the following random variable

$$\left\| \sqrt{n}\mathcal{I}(\mathbf{0})^{1/2} (\hat{\theta}_n^{MLE} - \mathbf{0}) \right\|^2$$

converges to a χ_k^2 distribution. What is the degree of freedom k of this χ_k^2 distribution?

$\left\| \sqrt{n}\mathcal{I}(\mathbf{0})^{1/2} (\hat{\theta}_n^{MLE} - \mathbf{0}) \right\|^2 \xrightarrow[n\to\infty]{(d)} \chi_k^2$ for $k =$ d ✔ **Answer:** d

STANDARD NOTATION

Solution:

From the previous problem, we know that under the assumption $X_1, \ldots, X_n \overset{iid}{\sim} P_{\mathbf{0}}$,

$$\sqrt{n}\mathcal{I}(\mathbf{0})^{1/2} (\hat{\theta}_n^{MLE} - \mathbf{0}) \xrightarrow[n\to\infty]{(d)} = \mathcal{N}(\mathbf{0}, I_{d\times d}).$$

Next, if $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, I_{d\times d})$, then $Z_1, \ldots, Z_d \overset{iid}{\sim} \mathcal{N}(0,1)$. Hence,

$$\|\mathbf{Z}\|_2^2 = Z_1^2 + Z_2^2 + \cdots + Z_d^2 \sim \chi_d^2$$

by definition of the χ^2 distribution with d degrees of freedom. Hence by continuity, we have

$$\left\|\sqrt{n}\mathcal{I}(\mathbf{0})^{1/2}\,(\hat{\theta}_n^{MLE} - \mathbf{0})\right\|_2^2 \xrightarrow[n\to\infty]{(d)} \chi_d^2.$$

Remark: The above allows us to derive **Wald's test** . For the given null and alternative hypotheses:

$$\begin{aligned} H_0 &: \ \theta^* = \mathbf{0} \\ H_1 &: \ \theta^* \neq \mathbf{0}, \end{aligned}$$

we define the test statistic

$$W_n := \left\|\sqrt{n}\mathcal{I}(\mathbf{0})^{1/2}\,(\hat{\theta}_n^{MLE} - \mathbf{0})\right\|^2 = n(\hat{\theta}_n^{MLE} - \mathbf{0})^T\,\mathcal{I}\,(\mathbf{0})\,(\hat{\theta}_n^{MLE} - \mathbf{0})\,.$$

Then, then Wald's test of level α is the test

$$\psi_\alpha = \mathbf{1}\,(W_n > q_\alpha\,(\chi_d^2))\,,$$

where $q_\alpha\,(\chi_d^2)$ is the $1-\alpha$-quantile of the (pivotal) distribution χ_d^2.

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? Are we going back to asymptotic distributions?
We needed chi-squared and T test so that we can say something when the sample size is small. Since, we are going back to asymptotic distributions, does that mean there is ... 1

? Review: Chi-Squared Distribution
Quick question, as I am confused about the wording: '$Z21+Z22...+Z2d$ where Zi~N(0,1) and are independent' is a **Deleted by MW-CTA**, but '$Z21+Z22...+Z2d$ where Zi~N(μ,... 3

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