15. Properties of the matrix exponential
Some properties and computation

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Properties of the matrix exponential

1. $e^{\mathbf{0}} = I$ (here $\mathbf{0}$ is the zero matrix)

 Proof: $e^{\mathbf{0}} = \mathbf{I} + \mathbf{0} + \frac{\mathbf{0}^2}{2!} + \cdots = \mathbf{I}$

2. $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$
   Proof:. Take the derivative of $e^{\mathbf{A}t}$ term by term.

3. If $\mathbf{AB} = \mathbf{BA},$ then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$. Here are a few (but not all) special cases when $\mathbf{AB} = \mathbf{BA}$:

   - $\mathbf{A} = c\mathbf{I}$
   - $\mathbf{B} = -\mathbf{A}$
   - $\mathbf{B} = \mathbf{A}^{-1}$

   Warning: In general, this fails, that is, if $\mathbf{AB} \neq \mathbf{BA}$, then usually $e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}}e^{\mathbf{B}}$, since there are counterexamples ever for real 2x2 matrices.

4. If $\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $e^{\mathbf{A}} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$.
   Proof: $\mathbf{A}^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix}$, and so on. Thus

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots = \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \cdots & 0 \\ 0 & 1 + \lambda_2 + \frac{\lambda_2^2}{2!} + \cdots \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

   (A similar statement holds for diagonal matrices of any size.)

5. Given $\mathbf{A}$ is **diagonalizable**, that is,

$$\mathbf{A} \quad = \quad \mathbf{SDS}^{-1},$$

   where $\mathbf{D}$ is the diagonal matrix of eigenvalues and $\mathbf{S}$ is the invertible matrix whose columns are eigenvectors:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{pmatrix} \qquad \text{where } \mathbf{v}_i \text{ corresponds to } \lambda_i.$$

Then

$$e^{\mathbf{A}} = \mathbf{S}e^{\mathbf{D}}\mathbf{S}^{-1}.$$

Proof: Expand $e^{\mathbf{A}}$ as a power series and use the followin cancellations for each term:

$$\mathbf{A}^n = \mathbf{SD}\underbrace{\mathbf{S}^{-1}\mathbf{S}}_{\text{cancels}}\cdots\underbrace{\mathbf{S}^{-1}\mathbf{S}}_{\text{cancels}}\mathbf{D}\underbrace{\mathbf{S}^{-1}\mathbf{S}}_{\text{cancels}}\cdots\underbrace{\mathbf{S}^{-1}\mathbf{S}}_{\text{cancels}}\mathbf{DS}^{-1} = \mathbf{SD}^n\mathbf{S}^{-1}.$$

6. $e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}(0)^{-1}$ for any fundamental matrix $\mathbf{X}$.
Proof: Both $e^{\mathbf{A}t}$ and $\mathbf{X}(t)\mathbf{X}(0)^{-1}$ satisfy the matrix differential equation $\dot{\mathbf{Y}} = \mathbf{AY}$ and the same initial conditions $\mathbf{Y}(0) = \mathbf{I}$. The uniqueness and existence theorem then guarantees these to be the same. (You can also think of the columns of $e^{\mathbf{A}t}$ separately and apply the existence and uniqueness theorem.)

Problem 15.1 Use the matrix exponential to find the solution to the system

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 2y \end{aligned}$$

satisfying $x(0) = 5$ and $y(0) = 7$.

Solution: This is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} := \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Write $\mathbf{A} = \mathbf{D} + \mathbf{N}$ with $\mathbf{D} := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (diagonal) and $\mathbf{N} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\mathbf{N}^2 = \mathbf{0}$, so

$$e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then $\mathbf{D}t$ and $\mathbf{N}t$ commute (a scalar times $\mathbf{I}$ commutes with any matrix of the same size), so

$$\begin{aligned}
e^{\mathbf{A}t} &= e^{\mathbf{D}t+\mathbf{N}t} \\
&= e^{\mathbf{D}t}e^{\mathbf{N}t} \\
&= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= e^{At}\begin{pmatrix} 5 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 5e^{2t} + 7te^{2t} \\ 7e^{2t} \end{pmatrix}.
\end{aligned}$$

Remark: The matrix exponential is a fundamental matrix even when $\mathbf{A}$ is deficient, as in the example above.

Compute the exponential matrix

0.5/1 point (graded)

Compute the exponential matrix $e^{\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

(Enter **[a,b;c,d]** for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. That is, use commas to separate entries within the same row and **semicolon** to separate rows, and **square bracket** the entire matrix. Similarly for larger matrices.)

$e^{\mathbf{A}} =$ [0.2088333,2.7182818;2.7182818,0.2088333] ✱

Answer: [cosh(1)/e^2, sinh(1)/e^2; sinh(1)/e^2,cosh(1)/e^2]

Solution:

Since $\mathbf{A}$ is symmetric, it is diagonalizable. Observe that

$$\mathbf{A} \;=\; -2\mathbf{I} + \mathbf{B} \qquad \text{where } \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$(-2\mathbf{I})\mathbf{B} = \mathbf{B}(-2\mathbf{I}).$$

This implies that $e^{\mathbf{A}t} \;=\; e^{-2\mathbf{I}}e^{\mathbf{B}}$. Let us compute $e^{\mathbf{B}}$:

$$\begin{aligned} e^{\mathbf{B}} &= \mathbf{I} + \mathbf{B} + \frac{\mathbf{B}^2}{2} + \frac{\mathbf{B}^3}{3!} + \frac{\mathbf{B}^4}{4!} + \cdots && \left(\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\ &= \mathbf{I} + \mathbf{B} + \frac{\mathbf{I}}{2} + \frac{\mathbf{B}}{3!} + \frac{\mathbf{I}}{4!} + \cdots && \text{since } \mathbf{B}^2 = \mathbf{I} \\ &= \mathbf{I}\left(1 + \frac{1}{2} + \frac{1}{4!} + \cdots\right) + \mathbf{B}\left(1 + \frac{1}{3!} + \frac{1}{5!} + \cdots\right) \\ &= \mathbf{I}\cosh(1) + \mathbf{B}\sinh(1) = \begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}. \end{aligned}$$

Therefore,

$$e^{\mathbf{A}t} \;=\; e^{-2\mathbf{I}}\begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix} = e^{-2}\begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}.$$

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