

Why is the outer measure of the set of irrational numbers in the interval [0,1] equal to 1?

Asked 5 years, 9 months ago Active 4 months ago Viewed 9k times



Just learned Lebesgue outer measure from Royden's Real Analysis.



Let me give my proof. First, let A be the set of irrational numbers in [0,1]. So $A \subset [0,1] \Rightarrow m^*(A) \leq m^*([0,1]) = 1$.



Then I want to show $m^*(A) \geq 1$ by using $\sum_{k=1}^{\infty} l(I_k) \leq m^*(A) + \epsilon$. $\{I_k\}_k$ covers A, then add I_0 to this collection. $[0,1] \subset I_0$. So



A)

 $l(I_0) + \sum_{k=1}^\infty l(I_k) \leq m^*(A) + \epsilon \Rightarrow m^*(A) \geq l(I_0) + \sum_{k=1}^\infty l(I_k) - \epsilon \geq 1 + \sum_{k=1}^\infty l(I_k) - \epsilon$

We can always choose a small enough $\epsilon>0$ such that $\sum_{k=1}^{\infty}l(I_k)-\epsilon>0$. Therefore, $m^*(A)=1$.

real-analysis

analysis

measure-theory

proof-verification

lebesgue-measure

edited Nov 23 '17 at 6:16



reflexive

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asked Oct 17 '14 at 16:51



Drake Marquis

- Why is "the" outer measure.... You should say, why is an "outer" measure of.... Unless you are very specific what sort of outer measure you are using, it's NOT obvious Squirtle Oct 17 '14 at 16:54
- @Squirtle I disagree, we're talking about the real line, the obvious choice of measure is the Lebesgue measure and the obvious choice of outer measure is the Lebesgue outer measure. Ian Oct 17 '14 at 17:16
- 1 $rightharpoonup If <math>I_0\supset [0,1]$ then $l(I_0)\geq 1$ so $l(I_0)+\sum_{i=1}^\infty l(I_i)\geq 1+m^*(A),$ and in fact $1+m^*(A)=2.$ DanielWainfleet Nov 23 '17 at 9:44

3 Answers

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The rational numbers has measure zero, so $\mathbb{Q} \in \mathcal{M}(\lambda^*).$ Then



$$1=\lambda^*([0,1])=\lambda^*([0,1]\cap\mathbb{Q})+\lambda^*([0,1]\setminus\mathbb{Q})=0+\lambda^*([0,1]\setminus\mathbb{Q})$$



i.e.,
$$1 = \lambda^*([0,1] \setminus \mathbb{Q})$$
.



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edited Mar 9 at 15:20

answered Oct 17 '14 at 21:09



Jose Antonio

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What you know is that $\sum_k l(I_k) \leq m^*(A) + \epsilon$ for **some** sequence of intervals covering A. You've got $l(I_0) \geq 1$ but only add it to the left-hand side of the inequality so your solution is

8

in error.



1

Do you know that $m^*([0,1]) = 1$ and $m^*(rationals) = 0$? If so use subadditivity and monotonicity:

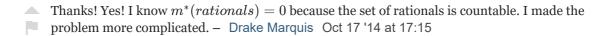
$$m^*([0,1]) \leq m^*(rationals) + m^*(irrationals) = m^*(irrationals) \leq m^*([0,1])$$

so that

$$m^*(irrationals) = m^*([0,1]) = 1.$$

answered Oct 17 '14 at 16:56



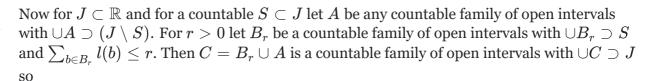




First we show that if S is a countable subset of \mathbb{R} then the Lebesgue outer measure $m^*(S) = 0$. Second we show that if J is a bounded real interval then $m^*(J) = l(J)$.



1



$$m^*(J\setminus S) \leq m^*(J) \leq \sum_{c\in C} l(c) \leq \sum_{b\in B_r} l(b) + \sum_{a\in A} l(a) \leq r + \sum_{a\in A} l(a).$$

Taking the inf of the right-most expression above, over every family A of open intervals that covers $J \setminus S$, we have

$$(\bullet) \quad m^*(J\setminus S) \leq m^*(J) \leq r + m^*(J\setminus S).$$

Since (\bullet) holds for every r > 0, we have

$$m^*(J \setminus S) \le m^*(J) \le m^*(J \setminus S).$$

Therefore $m^*(J \setminus S) = m^*(J)$.

In particular if J is a bounded interval then $m^*(J \setminus S) = m^*(J) = l(J)$.

edited Nov 23 '17 at 10:52

answered Nov 23 '17 at 10:46



You should also learn about inner measure. With m denoting Lebesgue measure, the inner measure of any $T \subset \mathbb{R}$ is $m^i(T) = \sup\{m(U) : U = \overline{U} \subset T\}$ A set $T \subset \mathbb{R}$ is Lebesgue-measurable iff $m^*(T) = m^i(T)$. – DanielWainfleet Nov 23 '17 at 10:50

riangle This answer will apply verbatim for any $S \subset \mathbb{R}$ such that $m^*(S) = 0$. – DanielWainfleet Nov 23 '17

7/25/2020	real analysis - Why is the outer measure of the set of irrational numbers in the interval [0,1] equal to 1? - Mathematics Stack Exchange
	at 11:03