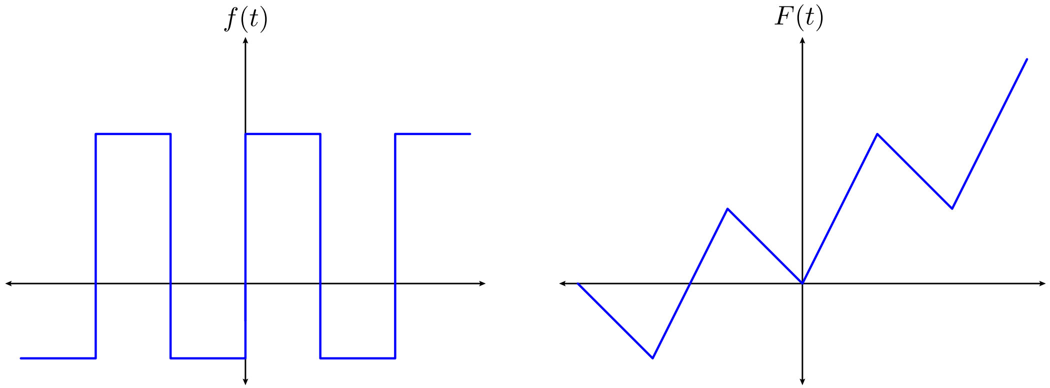
11. Antiderivative of a Fourier series

Suppose that f is a piecewise differentiable periodic function, and that F is an antiderivative of f. (If f has jump discontinuities, one can still define $F(t) := \int_0^t f(\tau)\,d\tau + C$, but at the jump discontinuities F will be only continuous, not differentiable.)

The function F might not be periodic. For example, if f is a function of period 2 such that

$$f(t) := \begin{cases} 2, & \text{if } 0 < t < 1, \\ -1 & \text{if } -1 < t < 0, \end{cases}$$

then $F(t)$ creeps upward over time.



An even easier example: if $f(t) = 1$, then $F(t) = t + C$ for some C, so $F(t)$ is not periodic.

But if the constant term $a_0/2$ in the Fourier series of f is 0, then F **is** periodic, and its Fourier series can be obtained by taking the simplest antiderivative of each cosine and sine term, and adding an overall $+C$, where C is the average value of F.

Problem 11.1 Let $T(t)$ be the triangle wave of period 2 and amplitude 1: so that $T(t) = |t|$ for $-1 \leq t \leq 1$. Find the Fourier series of $T(t)$.

Solution: We could use the Fourier coefficient formula. But instead, notice that $T(t)$ has slope -1 on $(-1, 0)$ and slope 1 on $(0, 1)$, so $T(t)$ is an antiderivative of the period 2 square wave

$$\mathrm{Sq}(\pi t) = \sum_{n \geq 1 \mathrm{odd}} \frac{4}{n\pi} \sin n\pi t.$$

Taking an antiderivative termwise (and using that the average value of $T(t)$ is $1/2$) gives

$$\begin{aligned} T(t) &= \frac{1}{2} + \sum_{n \geq 1 \mathrm{odd}} \frac{4}{n\pi} \left(\frac{-\cos n\pi t}{n\pi} \right) \\ &= \frac{1}{2} - \sum_{n \geq 1 \mathrm{odd}} \frac{4}{n^2 \pi^2} \cos n\pi t. \end{aligned}$$

Warning: If a periodic function $f(t)$ is not continuous, it will not be an antiderivative of any piecewise differentiable function, so you cannot find the Fourier series of $f(t)$ by integration.

Remark 11.2 The Fourier series for a function with discontinuities can (formally) be differentiated term by term, but **the result will not converge.** For example, the term-wise derivative of the Fourier series for $Sq(t)$ is

$$\frac{4}{\pi}\sum_{n \text{ odd}} \cos(nt).$$

This does not converge anywhere (since the nth term does not even vanish as $n \to \infty$). However, note that it is possible to make sense of this series, and of the anti-derivative of the square wave function, in terms of Dirac's delta functions and the theory of distributions — seen in more advanced courses than this one.

Integrate to find the Fourier series (*)

2/2 points (graded)

Find the Fourier series of the function $f(t) = t^2$ defined on $[-1, 1]$.

First find the constant term.

$\frac{a_0}{2} =$ 1/3 ✔ **Answer:** 1/3

$\frac{1}{3}$

Next, find the remaining terms of the Fourier series in terms of n:

$\frac{a_0}{2} + \sum_{n=1}^{\infty}$ 4*(-1)^n/(n^2*pi^2)*cos(n*pi*t) ✔ **Answer:** 4*(-1)^n*cos(n*pi*t)/(n^2*pi^2)

Solution:

The function $f(t) = t^2, \quad -1 < t < 1$ is even of period 2. This function is continuous, so it could be the antiderivative of another function. In particular, note that $f(t)$ is the antiderivative of the period 2 sawtooth wave $g(t) = 2t, \quad -1 < t < 1$.

The Fourier series for the 2π-periodic sawtooth wave $W(u)$ is

$$W(u) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nu).$$

Therefore, scaling by $u = \pi t$ so that when $u = \pi$, we have $t = 1$ we get that

$$g(t) = \frac{2}{\pi} W(u) = \frac{2}{\pi} W(\pi t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi t).$$

Finally, to find the Fourier series for $f(t)$, we integrate the Fourier series for $g(t)$ term-by-term.

$$\begin{aligned} \int g(t)\,dt &= \frac{4}{\pi} \sum_{n=1}^{\infty} \int \frac{(-1)^{n+1}}{n} \sin(n\pi t)\,dt \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{-(-1)^{n+1}}{n^2\pi} \cos(n\pi t) \\ &= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t). \end{aligned}$$

To find the constant term, we find the average value of $f(t)$ on the interval $-1 < t < 1$.

$$\frac{a_0}{2} = \frac{\int_{-1}^{1} t^2\,dt}{1-(-1)} = \frac{1}{2}\left(\frac{1-(-1)}{3}\right) = \frac{1}{3}.$$

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Nice!

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1 response

jfrench (Staff)
23 days ago

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