6. When does an inhomogeneous system have a solution?

Note on video: Professor Strang uses the notation $C(\mathbf{A})$ for the column space of a matrix $\mathbf{A}$. We use $\mathrm{CS}(\mathbf{A})$ in this text.

For which vectors is there at least one solution?

(Caption will be displayed when you start playing the video.)

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We are going to switch gears here to revisit solutions to inhomogeneous linear equations. Our solutions will now involve the column space and the nullspace.

For an inhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$, there are two possibilities:

1. There are no solutions.

2. There exists a solution.

Our first question is: for which vectors $\mathbf{b}$ does $\mathbf{Ax} = \mathbf{b}$ have a solution?

Example 6.1 Consider the matrix equation

$$\mathbf{Ax} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \mathbf{b}.$$

For which vectors $\mathbf{b}$ is there a vector $\mathbf{x}$ so that $\mathbf{Ax} = \mathbf{b}$? We can find some vectors directly.

- $\mathbf{Ax} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$.
- $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ has a solution given by $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ has a solution given by $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

In particular, to figure out which vectors $\mathbf{b}$ are possible, one option is to look at the vectors $\mathbf{Ax}$ for any vector $\mathbf{x}$. Recall that we can think of matrix vector multiplication as a linear combination of the columns of the matrix. Since

$$\mathbf{Ax} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix},$$

the system $\mathbf{Ax} = \mathbf{b}$ can be written as

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \mathbf{b}.$$

Given $\mathbf{b}$, we can find x_1, x_2 and x_3 if and only if $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$. Since the set of all linear combination of the columns of $\mathbf{A}$ is called the column space, $\mathrm{CS}(\mathbf{A})$, this criterion can be rephrased as follows.

Theorem 6.2 The linear system $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $\mathrm{CS}(\mathbf{A})$.

This is why the column space is important!

This leads us to the question, how can we tell if $\mathbf{b}$ is in $\mathrm{CS}(\mathbf{A})$ or not?

When is a system solvable?

(Caption will be displayed when you start playing the video.)

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Interpreting an augmented matrix in row echelon form

Let's consider a linear system describing the intersection of two parallel planes in $\mathbb{R}^3$.

$$x - 3y + 2z \quad = \quad 0$$

$$x - 3y + 2z \quad = \quad 1$$

This system has no solutions. Let's see what happens when we try to solve it by elimination.

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 1 & -3 & 2 & 1 \end{array}\right)$$

Subtracting the first row from the second puts the matrix in row echelon form

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

This row corresponds to the equation

$$0x + 0y + 0z = 1.$$

Such an equation is never satisfied, so we say that the system is **inconsistent.** In general, when an augmented matrix has a row echelon form that has a pivot in the augmented column, the original system has no solutions.

Example 6.3 Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 &= b_3 \\ 4x_1 + 8x_2 + 10x_3 + 12x_4 &= b_4 \end{aligned}$$

(a slightly modified version of the example in the video above).

This system has augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \\ 4 & 8 & 10 & 12 & b_4 \end{array}\right).$$

The first thing that we notice is that the third row is the sum of the first and second rows. This becomes apparent when we do Gaussian elimination since the third row becomes a row of zeros.

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \\ 4 & 8 & 10 & 12 & b_4 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \\ 0 & 0 & 2 & 4 & b_4 - 4b_1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \\ 0 & 0 & 0 & 0 & b_4 - 2b_1 - b_2 \end{array}\right)$$

The last two rows are represented by the equations

$$\begin{aligned} 0x_1 + 0x_2 + 0x_3 + 0x_4 &= b_3 - b_1 - b_2 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= b_4 - 2b_1 - b_2 \end{aligned}$$

In other words, this system is solvable if and only if $b_3 - b_1 - b_2 = 0$ and $b_4 - 2b_1 - b_2 = 0$. This shows us how to use our algorithm for finding solutions to linear systems to determine if a solution is solvable or not.

Cool fact

Show

Algorithm to test if a linear system is consistent (has a solution):

Consider a linear system of m equations in n variables.

1. Construct the $m \times (n+1)$ augmented matrix.

2. Put it in row echelon form. Call this row echelon form $\mathbf{B}$.

3. Look for a row that is all zero except for a nonzero entry in the augmented column. (Alternatively, find all of the pivots of $\mathbf{B}$, and determine if there is a pivot in the augmented column.)

4. If $\mathbf{B}$ has a row that is all zero except for an entry in the augmented column, then one of the new equations has the form

$$0x_1 + \cdots + 0x_n = \underbrace{b.}_{\text{nonzero number}}$$

So the linear system is **inconsistent** . Otherwise, the system is **consistent** .

If the system is consistent, then when solving it by back substitution, the free variables corresponds to the non-pivot columns, excluding the augmented column, so

$$\#\textbf{parameters in general solution} = \underbrace{\#\textbf{non-pivot columns excluding the augmented column.}}_{\#\textbf{free variables}}$$

Is the system consistent?

1/1 point (graded)

Is the system of linear equations represented by the augmented matrix

$$\left(\begin{array}{cccc|c} -7 & 4 & 2 & -1 & 6 \\ 0 & 7 & -3 & 1 & 0 \\ 0 & 0 & -2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

consistent?

- ○ Yes.
- ◉ No. ✔
- ○ It cannot be determined

Solution:

The matrix is already in row echelon form. The last row is zero except for a one in the augmented column. Therefore the system is inconsistent.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

When is the system inconsistent?

1/1 point (graded)

Find the value of a that makes the following system (represented as an augmented matrix) inconsistent.

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & 3 \\ -2 & 4 & a & 5 \end{array}\right)$$

$a =$ -24 ✔ **Answer:** -24

-24

Solution:

We put the matrix in row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & 3 \\ -2 & 4 & a & 5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 7 & 3 \\ 0 & 4 & a-4 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 7 & 3 \\ 0 & 0 & a+24 & 17 \end{array}\right).$$

The system will be inconsistent when the last row is all zeros, except for the entry in the augmented column. The last row is all zeros when $a + 24 = 0,$ so $a = -24.$

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How many parameters?

1/1 point (graded)

How many **free variables** or **parameters** does a solution to the following system, represented as an augmented matrix, have?

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 6 \end{array}\right).$$

(Enter NA if there are no solutions to this system.)

2

Answer: 2

Solution:

This matrix has 3 pivots (in orange), 2 free columns (in blue). Note that the augmented column is neither a pivot column nor a free column.

$$\left(\begin{array}{ccccc|c} \color{orange}{1} & 1 & \color{blue}{0} & 1 & \color{blue}{1} & 2 \\ 0 & \color{orange}{1} & \color{blue}{1} & 0 & \color{blue}{1} & 4 \\ 0 & 0 & \color{blue}{0} & \color{orange}{1} & \color{blue}{0} & 6 \end{array}\right).$$

Therefore the general solution to the system has 2 parameters.

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You have used 1 of 3 attempts

Answers are displayed within the problem

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Typo in "Cool fact"
"Earlier we showed how give a basis for the column space" should be "Earlier we showed how **to** give a bas... 2
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