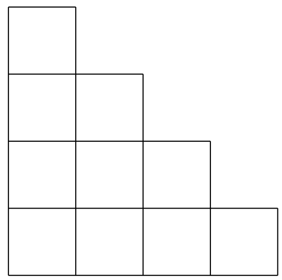
Parity Arguments

Can you cover the following diagram with 5 dominoes?



Hint: Think of the diagram as lying on a checkerboard and count the number of black and white squares.

Even versus odd permutations

Theorem. Every permutation can be written uniquely (up to order) as a product of disjoint cycles.

Example.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 4 & 8 & 2 & 9 & 1 & 7 & 6 & 10 & 3 \end{pmatrix} = (1,5,9,10,3,8,6)\,(2,4)\,(7),$$

a product of three cycles of lengths 7, 2, and 1 respectively.

For a permutation π, let $N(\pi)$ denote the number of distinct cycles of π, where we always include those cycles of length 1.

Theorem. Every cycle can be written as a product of transpositions.

Proof.

$$(a_1, a_2, a_3, \ldots, a_n) = (a_1, a_n)\,(a_1, a_{n-1}) \ldots (a_1, a_4)\,(a_1, a_3)\,(a_1, a_2).$$

Example.

$$(1,5,9,10,3,8,6) = (1,6)\,(1,8)\,(1,3)\,(1,10)\,(1,9)\,(1,5).$$

Note that $N(id) = n$.

Theorem. Let $\tau = (a, b)$ be a transposition and π be a permutation. Then

$$N((a,b)\pi) = \begin{cases} N(\pi) + 1 & \text{if } a \text{ and } b \text{ belong to the same cycle of } \pi \\ N(\pi) - 1 & \text{if } a \text{ and } b \text{ belong to different cycles of } \pi. \end{cases}$$

Proof. If a and b belong to one cycle, then

$$(a,b)\,(a,c_1,c_2,c_3,\ldots,c_k,b,d_1,d_2,\ldots,d_\ell) = (a,c_1,c_2,c_3,\ldots,c_k)\,(b,d_1,d_2,\ldots,d_\ell),$$

thus increasing the number of disjoint cycles by 1. If a and b belong to different cycles, then

$$(a,b)\,(a,c_1,c_2,c_3,\ldots,c_k)\,(b,d_1,d_2,\ldots,d_\ell) = (a,c_1,c_2,c_3,\ldots,c_k,b,d_1,d_2,\ldots,d_\ell)$$

thus decreasing the number of disjoint cycles by 1.

It follows from the previous theorem that

Theorem. Let $\tau_1, \tau_2, \tau_3, \ldots, \tau_m$ be m transpositions and let π be a permutation. Then

$$N(\tau_1\tau_2\cdots\tau_m\pi) \equiv N(\pi) + m \pmod{2}.$$

There are many ways to write a permutation as a product of transpositions. In S_4, for example, the permutation $(1,2,3,4)$ can be written as $(1,4)\,(1,3)\,(1,2)$ or as $(3,4)\,(2,3)\,(1,2)\,(2,4)\,(1,3)$. In general, suppose a permutation π can be written as a product of transpositions in two different ways:

$$\pi = \tau_1\tau_2\tau_3\cdots\tau_m$$

and

$$\pi = \sigma_1\sigma_2\sigma_3\cdots\sigma_\ell.$$

Since $N(id) = n$, it follows from the previous theorem that

$$N(\pi) \equiv m + n \pmod{2} \quad \text{and} \quad N(\pi) \equiv \ell + n \pmod{2}.$$

Consequently $m \equiv \ell \pmod{2}$. This proves

Theorem. A given permutation is either a product of an even number of transpositions or a product of an odd number of transpositions, but never both.

A permutation is **even** if it is expressible as a product of an even number of transpositions, and **odd** if it is a product of an odd number of transpositions.

Example: In S_3, the even permutations are

$$id \qquad (1,2,3) = (1,3)\,(1,2) \qquad (1,3,2) = (1,2)\,(2,3)$$

and the odd permutations are

$$(1,2) \qquad (1,3) \qquad (2,3).$$

The Fifteen Puzzle

Consider the following 4 by 4 square, with the numbers 1–15 placed in the boxes:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

The 15 blocks are free to slide within the large square, but they cannot be removed. By a sequence of such moves, can you ever obtain the pattern

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	