5. Fourier series

A linear combination like $2\sin 3t - 4\sin 7t$ is another periodic function of period 2π.

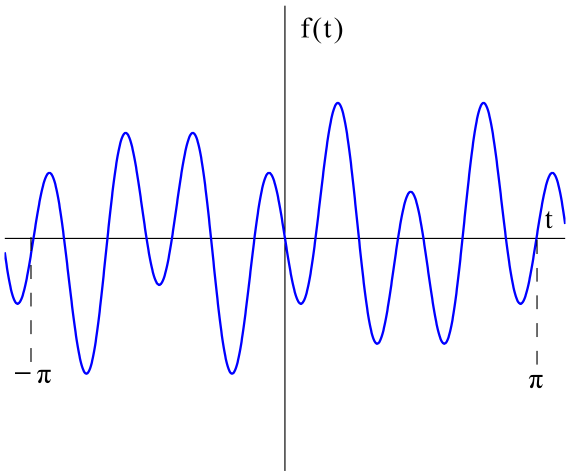

Figure 2: $2\sin 3t - 4\sin 7t$

Definition 5.1 A **Fourier series** is a linear combination of the infinitely many functions $\cos nt$ and $\sin nt$ as n ranges over integers:

$$\begin{aligned} f(t) &= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \cdots \\ & \quad + b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots \end{aligned}$$

(Terms like $\cos(-2t)$ are redundant since $\cos(-2t) = \cos 2t$. Also $\sin 0t = 0$ produces nothing new. But $\cos 0t = 1$ is included. We'll explain later why we write the first coefficient as a_0 divided by 2 instead of a_0.)

Written using sigma-notation:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt.$$

Example 5.2 Recall that, for example, $g(t) = \sum_{n=1}^{\infty} b_n \sin nt$ means the sum of the series whose n^{th} term is obtained by plugging in the positive integer n into the expression $b_n \sin nt$, so

$$g(t) = \sum_{n \geq 1} b_n \sin nt = b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots.$$

In particular, at $t = 0$,

$$g(0) = \sum_{n \geq 1} b_n \sin(n0) = \sum_{n \geq 1} b_n \cdot 0 = 0.$$

Example 5.3 If $f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos(nt)$, then

$$
\begin{aligned}
f(\pi) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n \\
&= \frac{a_0}{2} - a_1 + a_2 - a_3 + \cdots .
\end{aligned}
$$

Any Fourier series as above is periodic of period 2π. *(Later we'll extend to the definition of Fourier series to include functions of other periods.)*

- The numbers a_n and b_n are called the **Fourier coefficients** of f.
- Each summand ($a_0/2$, $a_n \cos nt$, or $b_n \sin nt$) is called a **Fourier component** of f.

Example 5.4 Define

$$
\mathrm{Sq}(t) := \begin{cases} 1, & \text{if } 0 < t < \pi, \\ -1 & \text{if } -\pi < t < 0. \end{cases}
$$

and extend it to a periodic function of period 2π, called a **square wave** .

The function $\mathrm{Sq}(t)$ has jump discontinuities, for example at $t = 0$. The graph is usually drawn with vertical segments at the jumps (even though this violates the vertical line test).

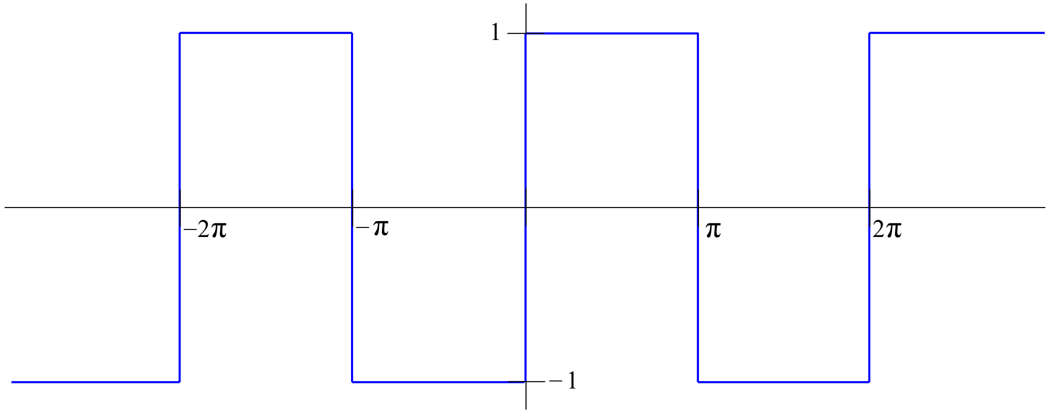

Figure 3: The 2π-periodic square wave.

It turns out that

$$\mathrm{Sq}\,(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right).$$

We'll explain later today where this comes from.

Fourier's Theorem. “Every" periodic function f of period 2π “is" a Fourier series, and the Fourier coefficients are uniquely determined by f.

(The word “Every" has to be taken with a grain of salt: The function has to be “reasonable". Piecewise differentiable functions with jump discontinuities are reasonable, as are virtually all other functions that arise in physical applications. The word “is" has to be taken with a grain of salt: If f has a jump discontinuity at τ, then the Fourier series might disagree with f there; the value of the Fourier series at τ is always the average of the left limit $f\,(\tau^-)$ and the right limit $f\,(\tau^+)$, regardless of the actual value of $f\,(\tau)$.)

In other words, the functions

$$1, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots$$

form a basis for the vector space of "all" periodic functions of period 2π.

Question 5.5 Given f, how do you find the Fourier coefficients a_n and b_n?

In other words, how do you find the coordinates of f with respect to the basis of cosines and sines?

5. Fourier series

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- ? What is a non reasonable function? — 5
  When you depict a reasonable function, you say that "Piecewise differentiable functions with jump discontinuities are reasonable". So, a non reasonable the function should b...
- ? Fourier vs Taylor — 6
  So are Fourier series a bit like Taylor series? Whereas Taylor series uses an infinite sum of power terms with coefficients (polynomial) to approximate a function, a Fourier seri...
- Question 5.5 Discussion — 2
  I assume we are supposed to discuss this question since I do not see any input fields. So... Given some function f(t) that is periodic, with period 2pi we want to determine a_n ...
- ? About basis function — 8
  From definition Fourier series consist only of linear combination of sine and cosine function, but what about the other trigonometric functions?
- Math Renderer setting? — 8
  For proper rendering of formula, what is the proper setting? Math Settings > Math renderer >(?) What should we choose? SVG? I'm asking because I'm having to reload pages t...
- cos(-2t) is redundant what about sin(-2t) ? — 2
- Why Fourier series fails with nonperiodic functions? — 3
  Why does Fourier series represent only periodic functions? In contrast, this condition is relaxed by Tayor series?

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