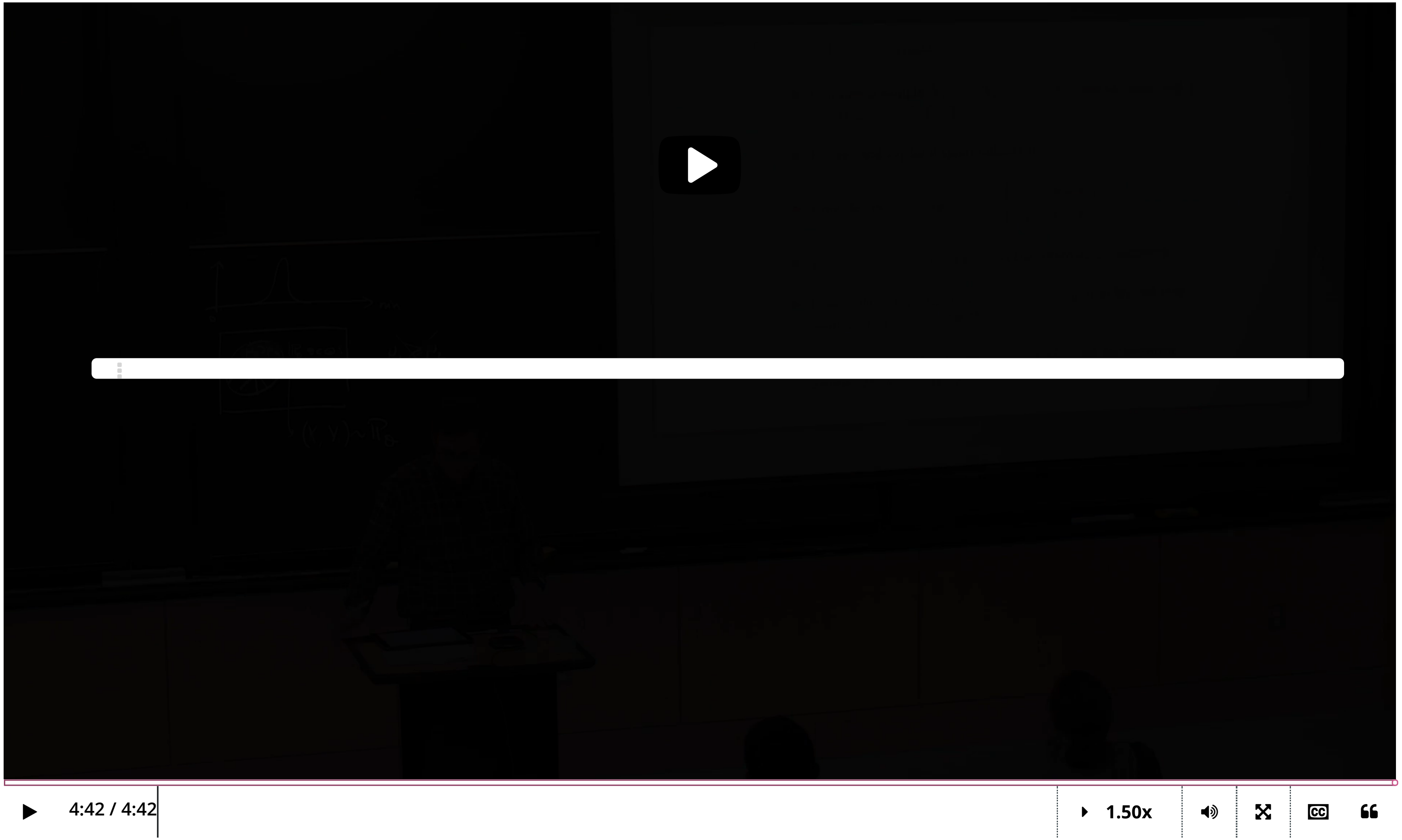
9. Example: Is this Coin Fair?

Video Note: In the video below, at 3:24, and on the last line of the annotated slide on the conclusion, there is an important misprint: The conclusion should be **It is not unlikely that the coin is fair .**

Testing Fairness of a Coin



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CLT Concept Check

1/1 point (graded)

In the next few questions, we will flip a coin 200 times in order to try and answer the hypothesis testing **question of interest**:

"**Is this coin fair?**."

As in lecture, we model the i'th flip as X_i where $X_i = 1$ for a heads and $X_i = 0$ for a tails. Since the flips should not interact with each other and we always flip the same coin, we make the familiar modeling assumption $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Ber}(p)$ where p is an unknown parameter. Then our original question of interest can be rephrased:

"**Does $\mathbf{p = 0.5}$ or does $\mathbf{p \neq 0.5}$?**".

Note that this is a very specific question. In particular, we do not care so much about the particular value of p– we just want to test whether or not it is equal to 0.5.

To answer this question, we consider the statistic

$$\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5\,(1-0.5)}}$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ denotes the sample mean.

Recall that we do not know the true value of p. Assume that n is very large. Can we conclude that the distribution of $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ is very close to the distribution of a standard Gaussian $\mathcal{N}(0,1)$?

Choose the correct response that also has the correct explanation.

- Yes, because $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ is a shift and rescaling of a binomial distribution. We know that for n large enough, the binomial distribution $\mathrm{Bin}(n,p)$ provides a good approximation to the distribution of a standard Gaussian $\mathcal{N}(0,1)$.

- Yes, because the central limit theorem (CLT) guarantees that for n sufficiently large, $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}} \approx \mathcal{N}\left(0,1\right)$ (in distribution).

- [x] No. Since we do not know for sure that $p=0.5$, we cannot conclude that $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}} \to \mathcal{N}\left(0,1\right)$ in distribution. (e.g. If $p=0.6$, then this estimator will not converge to $\mathcal{N}\left(0,1\right)$.)

- No. Even if $p=0.5$, it is not true that $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}} \to \mathcal{N}\left(0,1\right)$ in distribution. Hence, even in the case of a fair coin, we do not expect this estimator be close in distribution to $\mathcal{N}\left(0,1\right)$.

✔

Solution:

We examine the choices in order.

- "Yes, because $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}}$ is a shift and rescaling of a binomial distribution. We know that for n large enough, the binomial distribution $\text{Bin}\left(n,p\right)$ provides a good approximation to the distribution of a standard Gaussian $\mathcal{N}\left(0,1\right)$." is incorrect. The explanation is wrong: $\text{Bin}\left(n,p\right)$ does **not** provide a good approximation for the distribution $\mathcal{N}\left(0,1\right)$.
  Remark: However, by the CLT, if $X \sim \text{Bin}\left(n,p\right)$, then for as $n \to \infty$,

  $$\sqrt{n}\left(\frac{\frac{X}{n}-p}{\sqrt{p\left(1-p\right)}}\right) \to \mathcal{N}\left(0,1\right)$$

  in distribution.

- "Yes, because the central limit theorem (CLT) guarantees that for n sufficiently large, $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}} \approx \mathcal{N}\left(0,1\right)$ (in distribution)." is incorrect. We can only apply the CLT to the given estimator if the mean is 0.5 and the variance is $0.5\left(1-0.5\right)$. This is only the case if the coin is fair, *i.e.*, $p=0.5$.

- "No. Since we do not know for sure that $p = 0.5$, we cannot conclude that $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}} \to \mathcal{N}(0,1)$ in distribution. (e.g. If $p = 0.6$, then this estimator will not converge to $\mathcal{N}(0,1)$.)" is the correct response. We can only apply the CLT to conclude $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}} \to \mathcal{N}(0,1)$ in distribution if $p = 0.5$, as discussed in the previous bullet.

- "No. Even if $p = 0.5$, it is not true that $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}} \to \mathcal{N}(0,1)$ in distribution. Hence, even in the case of a fair coin, we do not expect this estimator be close in distribution to $\mathcal{N}(0,1)$." is incorrect. Though the answer is "No", the explanation is incorrect: the case where $p = 0.5$ is the **only** situation in which we can apply the CLT to say that $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}} \to \mathcal{N}(0,1)$ in distribution.

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

In the next two problems, we will illustrate some of the basic steps behind hypothesis testing.

The set up is the same as in the problem above:

Let $X_1, \ldots, X_{200} \overset{iid}{\sim} \text{Ber}(p)$, and we are interested in determining from the sample whether or not $p = 0.5$. The hypothesis testing question of interest is then

$$\textbf{Does p} = \mathbf{0.5} \qquad \textbf{or} \qquad \textbf{does p} \neq \mathbf{0.5}\,\textbf{?}$$

To answer this question, we introduced the statistic, which is also an estimator:

$$\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5\left(1-0.5\right)}}.$$

The reason for considering this estimator is that, **if $\mathbf{p} = \mathbf{0.5}$, then the CLT applies** (check this!), so that for n very large we may assume

$$\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5\left(1-0.5\right)}} \approx \mathcal{N}\left(0,1\right).$$

In other words, **if $\mathbf{p} = \mathbf{0.5}$**, then the above estimator distributed approximately as a standard Gaussian when n is large enough.

Our strategy will be to evaluate this estimator on the data set. Supposing that $p = 0.5$, then the value of our statistic should resemble the typical value of a single observation of a standard Gaussian random variable. Hence, if the value $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}}$ lies deep in the tails of the standard normal distribution, we would logically conclude that it is **unlikely** that $p = 0.5$. Otherwise, we will not be able to refute that $p = 0.5$.

Hypothesis Testing: A Sample Data Set of Coin Flips I

3/3 points (graded)

We use the statistical set-up from the previous problem. Consider a statistical experiment where you flip the coin 200 times. In one run of this experiment, you observe **80 heads**. We will use this data and the estimator $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}}$ (as in the previous problem) to provide an answer to the hypothesis testing **question of interest**: "**Does $\mathbf{p} = \mathbf{0.5}$ or does $\mathbf{p} \neq \mathbf{0.5}$?**".

Let D_1 denote the value of the realization of the statistic $\sqrt{n}\frac{\left(\overline{X}_n-0.5\right)}{\sqrt{0.5(1-0.5)}}$ on the given data set. (Here $n = 200$, the number of flips.) What is D_1?

$D_1 =$ -2.8284271247461894 ✔ **Answer:** -2.82842

Let $Z \sim \mathcal{N}(0,1)$. What is $\mathbf{P}(Z < D_1)$?
(You are welcome to use table or any computational tools e.g. R, or this online normal table calculator.)

$P(Z < D_1) =$ 0.002338867490523637 ✔ **Answer:** 0.00234

Since $n = 200$ is fairly large, we may assume that if $p = 0.5$ that $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}} \sim \mathcal{N}(0,1)$.

Suppose that $p = 0.5$ and you ran the experiment above (consisting of 200 coin flips) a total of 1000 times (i.e. a total 200×1000 coin flips). What is the expected number of experiments such that the estimator $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ is smaller than the value D_1 attained in the first experiment? (Round your answer to the nearest integer.)

2 ✔ **Answer:** 2

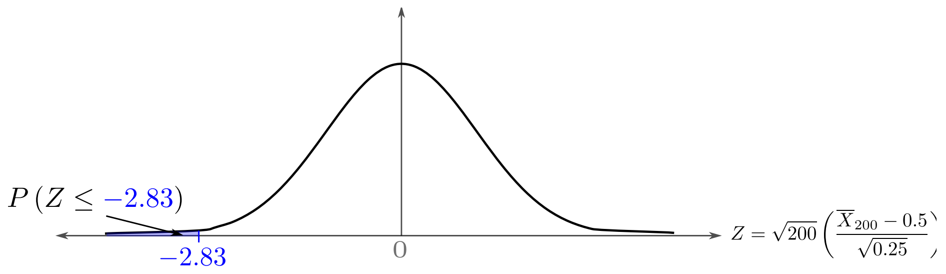
Solution:

First,

$$D_1 = \sqrt{200}\left(\frac{\frac{80}{200} - 0.5}{\sqrt{0.25}}\right) \approx -2.82842.$$

Using a table or computational software, we can also compute that if $Z \sim \mathcal{N}(0,1)$,

$$P(Z < D_1) = \int_{-\infty}^{-2.82842} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \approx .00234$$



Hence, for a single experiment, if $p = 0.5$, then there is (approximately) a 0.23% chance of seeing an observation smaller than $D_1 \approx -2.82842$. Thus if we run 1000 experiments, we would expect to see

$$1000 * (.00234) \approx 2.33907$$

experiments where $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ is smaller than $D_1 \approx -2.82842$.

Remark: By the previous result, it seems reasonable to conclude, for our first experiment, that it is **unlikely** that $p = 0.5$. Indeed, if $p = 0.5$, observing the value $D_1 \approx -2.82842$ would be a very rare event, intuitively speaking. In practice, one has to set the threshold of what determines a "very rare" event, and this will be studied later in this lecture.

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You have used 1 of 3 attempts

Answers are displayed within the problem

Hypothesis Testing: Another Sample Data Set of Coin Flips

3/3 points (graded)

We repeat the above exercise with a different data set.

As above, consider a **statistical experiment** where you flip the coin 200 times. However, in this run of the experiment, you observe $\mathbf{106}$ **heads**.

We will use this data and the statistic $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ from the previous problem to provide an answer to the hypothesis testing question of interest:

"**Does $\mathbf{p} = \mathbf{0.5}$ or does $\mathbf{p} \neq \mathbf{0.5}$?**"

Let D_2 denote the value of the realization of the estimator $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ on the given data set. (Here $n = 200$, the number of flips.) What is D_2?

$D_2 =$ 0.8485281374238578 ✔ **Answer:** 0.8485

Let $Z \sim \mathcal{N}(0,1)$. What is $\mathbf{P}(Z > D_2)$?
(You are welcome to use any tables or any computational tools e.g. R or this online normal table calculator.)

$\mathbf{P}(Z > D_2) =$ 0.19807195457603688 ✔ **Answer:** 0.19808

Since $n = 200$ is fairly large, we may assume if $p = 0.5$ that $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}} \sim \mathcal{N}(0,1)$.

Suppose that $p = 0.5$ and you ran the experiment above (consisting of 200 coin flips) a total of 1000 times. What is the expected number of experiments such that the estimator $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ is larger than the value D_2 attained in the first experiment? (Round your answer to the nearest integer.)

198 ✔ **Answer:** 198

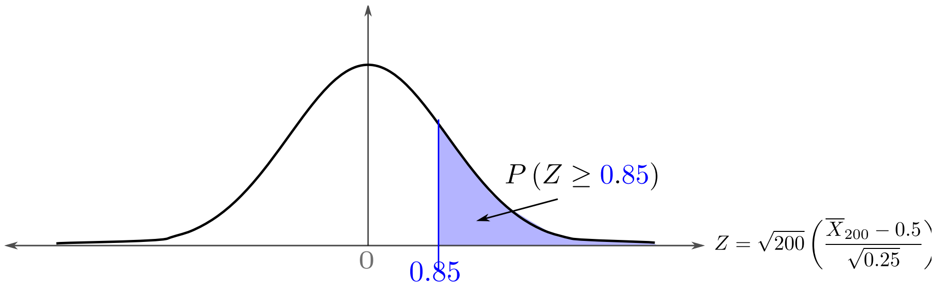
Solution:

First,

$$D_2 = \sqrt{200}\left(\frac{\frac{106}{200} - 0.5}{\sqrt{0.25}}\right) \approx 0.8485.$$

Using a table or computational software, we can also compute that if $Z \sim \mathcal{N}(0,1)$,

$$\mathbf{P}(Z > D_2) = \int_{0.8485}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \approx 0.19808.$$



Hence, for a single experiment, if $p = 0.5$, then there is (approximately) a 19.8% chance of seeing an observation larger than $D \approx 0.8485$. Thus if we run 1000 experiments, we would expect to see

$$1000 * (.00234) \approx 198.08$$

experiments where $\sqrt{n}\frac{\left(\overline{X}_n - 0.5\right)}{\sqrt{0.5(1-0.5)}}$ is larger than $D_2 \approx 0.8485$.

Remark 1: By the previous result, from a heuristic perspective, we would be unable to refute the hypothesis that $p = 0.5$ (Note that this is a **different** conclusion than saying "We may conclude that $p = 0.5$"). Indeed, if $p = 0.5$, observing a value larger than $D_2 \approx 0.8485$ would be **not** be a rare event, intuitively speaking. In practice, one has to set the threshold of what determines a "rare" event, and this will be studied later in this lecture.

Remark 2: Though we are considering a very specific example and applying a very specific test, the steps taken in this problem and the previous one are illustrative of the general principles of hypothesis testing. In general, we will transform our data into a given statistic whose distribution we know well that does **not** depend on the true parameter (e.g., as in this problem, the standard Gaussian). Such a distribution is known as **pivotal** . Then we can reduce our hypothesis testing question to a problem of deciding whether or not a given observation is likely (or not) for this pivotal distribution.

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Link to the online normal table not functional
Hello, I test it many times. Online normal table provided is not functional. Please replace it. Best, 3

The slides and the conclusion was ok
"It is **UNlikely** that the coin is **UNfair**", this is the conclusion. Isn't it equal to "It is not unlikely that the coin is fair"? 3