4. Eigenvalues of symmetric matrices using MATLAB

Eigenvectors of a symmetric matrix (External resource)

(1.0 points possible)

There is another useful result in linear algebra which says that the eigenvectors of any symmetric matrix are orthogonal. MATLAB gives us a simple way to test whether or not this is a reasonable claim. MATLAB can be a useful tool for testing out hypotheses, however, you must remember that numerical experiments are never a substitute for a full proof.

In order to understand the experiment below, it is useful to note that when you use the eig() command in MATLAB, the eigenvectors it gives you are of unit length. Therefore all of the dot products given will have magnitude less than 1.

Your Script

Save   Reset   MATLAB Documentation (https://www.mathworks.com/help/)

```
% Firstly generate a random 100x100 matrix A of numbers randomly chosen between -1
% Note that rand() generates random numbers between 0 and 1, so we've modified the
% by first subtracting 0.5 from each entry to get randomly generated entries betwe
% then multiplying by 20 so that the numbers lie between -10 and 10.

A = (rand(100)-0.5)*20;
% Now calculate the symmetric part of the matrix A. Call this matrix B
B = (A + A') / 2;
% Now calculate the eigenvalues and eigenvectors of A using eig(). Store them in t
[S1, D1] = eig(A);
% Now calculate the eigenvalues and eigenvectors of B using eig(). Store them in t
[S2, D2] = eig(B);

% We will now create a scatter plot of the first eigenvector of A and B dotted wit
% of A and B using the script you have just completed.
% If you have filled it in correctly, then you should see that the dot product of
% of A are scattered across the complex plane,
% while the dot product of one eigenvector of B with any other eigenvector of B is
figure(1)
hold on
for m=2:100
    plot(real(S1(:,m)'*S1(:,1)),imag(S1(:,m)'*S1(:,1)),'b*')
end
xlabel('Real part')
ylabel('Imaginary part')
xlim([-0.5, 0.5])
ylim([-0.5, 0.5])
title('Dot products of eigenvectors of A')
figure(2)
hold on
for m=2:100
    plot(real(S2(:,m)'*S2(:,1)),imag(S2(:,m)'*S2(:,1)),'r*')
end
xlabel('Real part')
ylabel('Imaginary part')
xlim([-0.5, 0.5])
ylim([-0.5, 0.5])
```

```
title('Dot products of eigenvectors of B')
```

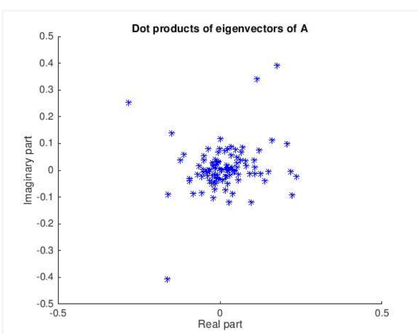
▶ Run Script ? ()

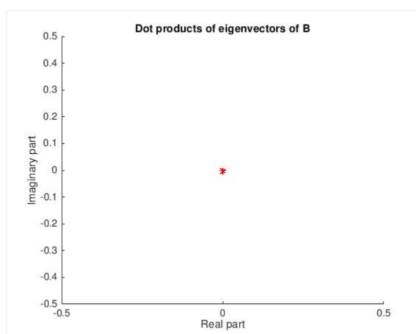
Assessment: Correct

Submit ? ()

- ✔ **Symmetric part correct**
- ✔ **S1 correct**
- ✔ **D1 correct**
- ✔ **S2 correct**
- ✔ **D2 correct**

Output





For your entertainment, here are proofs that the eigenvalues of a symmetric matrix are real, and that any two eigenvectors of distinct eigenvalues are distinct. These proofs rely on two facts about how matrix multiplication interacts with complex conjugation and transposes. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices, possibly with complex entries.

- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
- $\overline{\mathbf{AB}} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$
- $(\overline{\mathbf{AB}})^T = \overline{\mathbf{B}}^T\overline{\mathbf{A}}^T$

Proof that eigenvalues of a symmetric matrix are real

Let $\mathbf{M}$ be a symmetric real valued matrix. Then in particular $\overline{\mathbf{M}} = \mathbf{M}$ and $\mathbf{M}^T = \mathbf{M}$.

Let $\mathbf{v}$ be a nonzero eigenvector of $\mathbf{M}$ with eigenvalue λ. Note that the length of any vector is positive, so in particular $|\mathbf{Mv}| > 0$. Writing this in terms of matrix products we get

$$\begin{aligned} |\mathbf{Mv}|^2 &= ((\overline{\mathbf{Mv}})^T)(\mathbf{Mv}) \\ &= (\overline{\mathbf{v}}^T\overline{\mathbf{M}}^T)(\mathbf{Mv}) \\ &= \overline{\mathbf{v}}^T\mathbf{M}^2\mathbf{v} \\ &= \overline{\mathbf{v}}^T\lambda^2\mathbf{v} = \lambda^2|\mathbf{v}|^2 \end{aligned}$$

The last line is true since if $\mathbf{Mv} = \lambda\mathbf{v}$, then $\mathbf{M}^2\mathbf{v} = \mathbf{M}(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$.

Note that we now know that $|\mathbf{Mv}|^2 = \lambda^2|\mathbf{v}|^2 > 0$. Since $|\mathbf{v}|^2 > 0$, and $|\mathbf{Mv}|^2 > 0$, it follows that $\lambda^2 > 0$ as well. Therefore λ is real.

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Proof that eigenvectors corresponding to different eigenvalues of a symmetric matrix are orthogonal

Let $\mathbf{M}$ be a symmetric real valued matrix. (Then in particular $\overline{\mathbf{M}} = \mathbf{M}$ and $\mathbf{M}^T = \mathbf{M}$.)

Let $\mathbf{v_1}$ be an eigenvector of $\mathbf{M}$ with eigenvalue λ_1, and $\mathbf{v_2}$ be an eigenvector of $\mathbf{M}$ with eigenvalue λ_2 for distinct eigenvalues λ_1 and λ_2.

To show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, we will express $\overline{\mathbf{v}}_2^T\mathbf{M}\mathbf{v}_1$ in two ways.

$$\begin{aligned} \overline{\mathbf{v}}_2^T(\mathbf{M}\mathbf{v}_1) &= \overline{\mathbf{v}}_2^T(\lambda_1\mathbf{v}_1) \\ &= \lambda_1\overline{\mathbf{v}}_2^T\mathbf{v}_1 \end{aligned}$$

Using associativity to move the parentheses on the opposite side we can also write this expression as

$$\begin{aligned} (\overline{\mathbf{v}}_2^T\mathbf{M})\mathbf{v}_1 &= (\overline{\mathbf{v}}_2^T\overline{\mathbf{M}}^T)\mathbf{v}_1 \\ &= (\overline{\mathbf{M}\mathbf{v}_2})^T\mathbf{v}_1 \\ &= (\overline{\lambda_2\mathbf{v}_2})^T\mathbf{v}_1 \\ &= \lambda_2\overline{\mathbf{v}_2}^T\mathbf{v}_1 \qquad \text{since the eigenvalues are real} \end{aligned}$$

Therefore we have

$$\lambda_1\overline{\mathbf{v}}_2^T\mathbf{v}_1 = \lambda_2\overline{\mathbf{v}}_2^T\mathbf{v}_1.$$

Since $\lambda_1 \neq \lambda_2$, this equality can only hold if $\overline{\mathbf{v}}_2^T\mathbf{v}_1 = 0$. Therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

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