# 8. Linear combinations and span

A **linear combination** of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a vector of the form $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for some real (or complex) scalars $c_1, \dots, c_n$.

## Linear combinations concept check

1/1 point (graded)

Can $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ be written as a linear combination of the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$?

- (•) Yes. ✔
- ( ) No.

**Solution:**

The answer is yes:

$$\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

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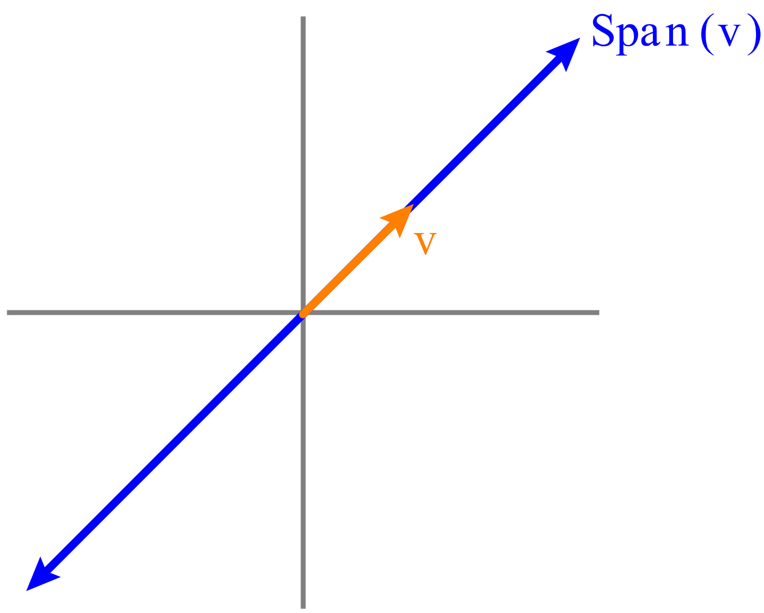
Answers are displayed within the problem

**Definition 8.1** Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the set of **all** linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{\text{all vectors } c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n, \text{ where } c_1, \ldots, c_n \text{ are scalars}\}.$$

**Example 8.2** If $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $\text{Span}(\mathbf{v}) = \left\{ c \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ where $c$ is a scalar. This is the line $y = x$ in $\mathbb{R}^2$.




Notice that $\text{Span}(\mathbf{v})$ is an infinite set of vectors because it contains $\begin{pmatrix} c \\ c \end{pmatrix}$ for every real number $c$.

**Example 8.3** The subspace $\text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right)$ is the same subspace defined in the previous example. The vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is in the span $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, since

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore there are no new vectors in this subspace,

$$\mathbf{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \mathbf{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

This is the line $y = x$ in $\mathbb{R}^2$.

**Remark:** Sometimes we are lazy, and use a single parenthesis for the span of a single vector: $\mathrm{Span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ rather than $\mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.

**Example 8.4** If $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $\mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2)$ is the set of all vectors of the form

$$c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}.$$

These form the $xy$-plane in $\mathbb{R}^3$, whose equation is $z = 0$.

Note that in all of the examples above, the span is a subspace. In fact the span is always a vector space. (You can check this.) The reason we talk about span is because it gives a compact notation for describing an infinite set of vectors that form a subspace. For example, the set of solutions to a homogeneous linear equation.

**Example 8.5** In our second example of solving the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ on the third page, we found that the solutions were of the form

$$c_1 \begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -11/6 \\ 0 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 19/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

That is, set of solutions to the homogeneous linear system (or nullspace) is the span

$$\mathrm{NS}(\mathbf{A}) = \{\text{the set of solutions to } \mathbf{A}\mathbf{x} = \mathbf{0}\} = \mathrm{Span}\left(\begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -11/6 \\ 0 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 19/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right).$$

**Definition 8.6** Given a matrix $\mathbf{A}$, whose columns are the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, the span of these columns is a vector space which we call the **column space of $\mathbf{A}$**, and denote using the notation $\mathrm{CS}(\mathbf{A})$.

$$\mathrm{CS}(\mathbf{A}) = \mathrm{Span}\,(\text{columns of } \mathbf{A}) = \mathrm{Span}\,(\mathbf{v}_1, \dots \mathbf{v}_n)$$

## Span concept check

1/1 point (graded)

Are $\mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ and $\mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$ equal?

- (selected) Yes. ✔
- No.

**Solution:**

The answer is yes. They are both all of $\mathbb{R}^2$.

What does it mean for the span to be all of $\mathbb{R}^2$? It means that every vector $\begin{pmatrix} a \\ b \end{pmatrix}$ can be written as a linear combination of the two vectors in each spanning set. To do this, we express the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ as a linear combination of both sets of vectors.

First notice that any vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If $\mathbf{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$ is all of $\mathbb{R}^2$, then all vectors can be written as a linear combination of the two vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. In particular, we must be able to express the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a linear combination of these. We see that

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned}$$

So every vector that is a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is also a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= a\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right] + b\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right] \end{aligned}$$

$$= \frac{1}{2}(a+b)\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}(-a+b)\begin{pmatrix}-1\\1\end{pmatrix}$$

Therefore we've shown that every vector in $\mathbb{R}^2$ is in both spans, therefore the two are equal, and are equal to $\mathbb{R}^2$.

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

## Span concept check II

1/1 point (graded)

Which of the following sets of vectors is a plane passing through the origin in $\mathbb{R}^3$? (Check all that apply.)

- [x] $\text{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$ ✔
- [x] $\text{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right)$ ✔
- [ ] $\text{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}\right)$

✔

**Solution:**

The answer is that both the first and the second sets of vectors span a plane through the origin in $\mathbb{R}^3$.

- The pair of vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ spans the $xy-$plane in $\mathbb{R}^3$, so they do span a plane passing through the origin in $\mathbb{R}^3$.
- The three vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ also span the same plane because the last one is the sum of the first two vectors, so it does not add anything new to their span, since it is contained in it.
- The two vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ span a line in $\mathbb{R}^3$, not a plane.

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Answers are displayed within the problem

An optional video on span and linear independence: Linear combinations, span, and basis vectors

## 8. Linear combinations and span

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