probability mass function of sum of two independent geometric random variables

How could it be proved that the probability mass function of X + Y, where X and Y are independent random variables each geometrically distributed with parameter p; i.e.

$$p_X(n) = p_Y(n) = \begin{cases} p(1-p)^{n-1} & n = 1, 2, \ldots \\ 0 & otherwise \end{cases}$$

equals to $\mathbf{P_{X+Y}(n) = {\color{red}(n-1)}\, p^2(1-p)^{n-2}}$

Using convolution I get

$\mathbf{P(X+Y=n) = \sum_{n}^{k=0} Pr(X=k) * Pr(Y=n-k) =}$
$\mathbf{\sum_{k=1}^{n} p_X(1-p_x)^{k-1} p_Y(1-p_Y)^{n-k-1}}$

as $p = p_X = p_Y$ it reduces to

$\mathbf{P(X+Y=n) = \sum_{k=1}^{n} p^2(1-p)^{n-2}}$

is this a correct way? I am stuck here, I don't know how to get the final formula.I miss some transition in order to get the (n-1).

(probability) (random-variables) (convolution)

asked Sep 14 '15 at 20:42
Michal
410 2 13

2 Answers

Since $X, Y \geq 1$, the summation should run over $k = 1, 2, \ldots, n-1$. Using this your convolution becomes

$$\begin{aligned} P(X+Y=n) &= \sum_{k=1}^{n-1} p^2(1-p)^{n-2} \\ &= p^2(1-p)^{n-2} \sum_{k=1}^{n-1} 1 \\ &= p^2(1-p)^{n-2}(n-1). \end{aligned}$$

answered Sep 14 '15 at 20:51

Titus
1,653 4 12

Could you please clarify more what happened here with the summation? Based on what was it converted in $\sum_{k=1}^{n-1} 1$ and then further into (n-1). I don't get those steps. – Michal Sep 14 '15 at 21:31

@Michal First move all factors that aren't terms of k to the outside of the sum, as $\sum_{k=1}^{n-1} a = a \sum_{k=1}^{n-1} 1$ Then: $\sum_{k=1}^{n-1} 1 = \underbrace{1+1+\ldots+1}_{\text{how many?}}$ – Graham Kemp Sep 14 '15 at 21:51

ok, clear now. Thanks guys! – Michal Sep 14 '15 at 21:54

A geometric random variable is the count of Bernouli trial *until* a success. We measure the probability of obtaining $n-1$ failures and then 1 success.

$$\mathsf{P}(X=n) = (1-p)^{n-1}p \qquad : n \in \{1, 2, \ldots\}$$

The sum of two such is the count of Bernouli trials *until* the *second* success. We measure the probability of obtaining 1 success and $n-2$ failures, in any arrangement of those $n-1$ trials, followed by the second success.

$$\mathsf{P}(X+Y=n) = (n-1)(1-p)^{n-2}p^2 \qquad : n \in \{2, 3 \ldots\}$$

This may also be counted by summing

$$\begin{aligned}
\mathsf{P}(X+Y=n) &= \sum_{k=1}^{n-1} \mathsf{P}(X=k, Y=n-k) && \text{note the range} \\
&= \sum_{k=1}^{n-1} \mathsf{P}(X=k)\mathsf{P}(Y=n-k) && \text{by independence} \\
&= \sum_{k=1}^{n-1} (1-p)^{k-1} p \cdot (1-p)^{n-k-1} p \\
&= (1-p)^{n-2} p^2 \sum_{k=1}^{n-1} 1 \\
&= (n-1)(1-p)^{n-2} p^2
\end{aligned}$$

Since $X+Y$ must equal n and neither can be less than 1, then neither can be more than $n-1$. Hence this the range of X values we must sum over.

answered Sep 14 '15 at 22:08

Graham Kemp
48.8k 4 17 42

Thank you Graham for the detailed explanation! – Michal Sep 14 '15 at 22:19