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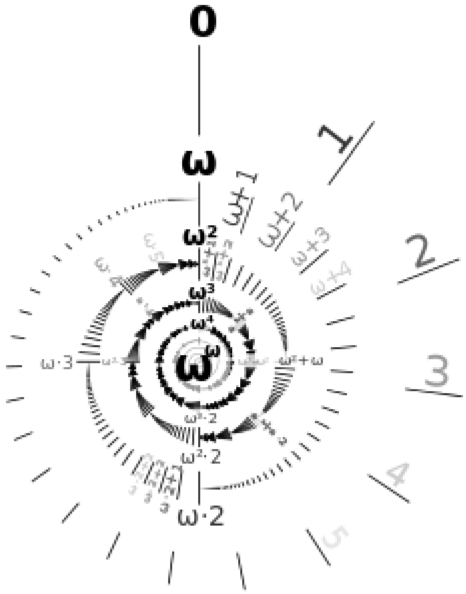
Limit ordinal

In set theory, a **limit ordinal** is an ordinal number that is neither zero nor a successor ordinal. Alternatively, an ordinal λ is a limit ordinal if there is an ordinal less than λ, and whenever β is an ordinal less than λ, then there exists an ordinal γ such that $\beta < \gamma < \lambda$. Every ordinal number is either zero, or a successor ordinal, or a limit ordinal.

For example, ω, the smallest ordinal greater than every natural number is a limit ordinal because for any smaller ordinal (i.e., for any natural number) n we can find another natural number larger than it (e.g. n+1), but still less than ω.

Using the Von Neumann definition of ordinals, every ordinal is the well-ordered set of all smaller ordinals. The union of a nonempty set of ordinals that has no greatest element is then always a limit ordinal. Using Von Neumann cardinal assignment, every infinite cardinal number is also a limit ordinal.



Representation of the ordinal numbers up to ω^ω. Each turn of the spiral represents one power of ω. Limit ordinals are those that are non-zero and have no predecessor, such as ω or ω^2

Contents

Alternative definitions

Various other ways to define limit ordinals are:

- It is equal to the supremum of all the ordinals below it, but is not zero. (Compare with a successor ordinal: the set of ordinals below it has a maximum, so the supremum is this maximum, the previous ordinal.)
- It is not zero and has no maximum element.
- It can be written in the form $\omega\alpha$ for $\alpha > 0$. That is, in the Cantor normal form there is no finite number as last term, and the ordinal is nonzero.
- It is a limit point of the class of ordinal numbers, with respect to the order topology. (The other ordinals are isolated points.)

Some contention exists on whether or not 0 should be classified as a limit ordinal, as it does not have an immediate predecessor; some textbooks include 0 in the class of limit ordinals[1] while others exclude it.[2]

Examples

Because the class of ordinal numbers is well-ordered, there is a smallest infinite limit ordinal; denoted by ω (omega). The ordinal ω is also the smallest infinite ordinal (disregarding *limit*), as it is the least upper bound of the natural numbers. Hence ω represents the order type of the natural numbers. The next limit ordinal above the first is ω + ω = ω·2, which generalizes to ω·n for any natural number n. Taking the union (the supremum operation on any set of ordinals) of all the ω·n, we get ω·ω = ω^2, which generalizes to ω^n for any natural number n. This process can be further iterated as follows to produce:

$$\omega^3, \omega^4, \ldots, \omega^\omega, \omega^{\omega^\omega}, \ldots, \epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}, \ldots$$

In general, all of these recursive definitions via multiplication, exponentiation, repeated exponentiation, etc. yield limit ordinals. All of the ordinals discussed so far are still countable ordinals. However, there is no recursively enumerable scheme for systematically naming all ordinals less than the Church–Kleene ordinal, which is a countable ordinal.

Beyond the countable, the first uncountable ordinal is usually denoted ω_1. It is also a limit ordinal.

Continuing, one can obtain the following (all of which are now increasing in cardinality):

$$\omega_2, \omega_3, \ldots, \omega_\omega, \omega_{\omega+1}, \ldots, \omega_{\omega_\omega}, \ldots$$

In general, we always get a limit ordinal when taking the union of a nonempty set of ordinals that has no maximum element.

The ordinals of the form $\omega^2\alpha$, for $\alpha > 0$, are limits of limits, etc.

Properties

The classes of successor ordinals and limit ordinals (of various cofinalities) as well as zero exhaust the entire class of ordinals, so these cases are often used in proofs by transfinite induction or definitions by transfinite recursion. Limit ordinals represent a sort of "turning point" in such procedures, in which one must use limiting operations such as taking the union over all preceding ordinals. In principle, one could do anything at limit ordinals, but taking the union is continuous in the order topology and this is usually desirable.

If we use the Von Neumann cardinal assignment, every infinite cardinal number is also a limit ordinal (and this is a fitting observation, as *cardinal* derives from the Latin *cardo* meaning *hinge* or *turning point*): the proof of this fact is done by simply showing that every infinite successor ordinal is equinumerous to a limit ordinal via the Hotel Infinity argument.

Cardinal numbers have their own notion of successorship and limit (everything getting upgraded to a higher level).

See also

- Ordinal arithmetic
- Limit cardinal
- Fundamental sequence (ordinals)

References

1. for example, Thomas Jech, *Set Theory*. Third Millennium edition. Springer.
2. for example, Kenneth Kunen, *Set Theory. An introduction to independence proofs*. North-Holland.

Further reading

- Cantor, G., (1897), *Beitrage zur Begrundung der transfiniten Mengenlehre. II* (tr.: Contributions to the Founding of the Theory of Transfinite Numbers II), Mathematische Annalen 49, 207-246 English translation (https://archive.org/details/117770262).
- Conway, J. H. and Guy, R. K. "Cantor's Ordinal Numbers." In *The Book of Numbers*. New York: Springer-Verlag, pp. 266–267 and 274, 1996.
- Sierpiński, W. (1965). *Cardinal and Ordinal Numbers* (2nd ed.). Warszawa: Państwowe Wydawnictwo Naukowe. Also defines ordinal operations in terms of the Cantor Normal Form.