11. Summary

Summarize

Big Picture

When we are considering a function of 2 (or more) variables, it is natural to wonder what the slope of the graph of this function is. However, the function has a slope that may be different depending on the direction you move away from this point. This geometric notion of the slope of the graph along a particular direction is the **directional derivative** , and can be computed with a dot product with the gradient of the function.

Mechanics

Directional derivatives definition

Definition 11.1

The **directional derivative** of a function $f(x, y)$ in the direction of the unit vector $\hat{u}$ at the point (x, y) is given by

$$D_{\hat{u}} f(x, y) = \nabla f \cdot \hat{u}.$$

For the directional derivative along any non-zero vector $\vec{v}$, we use $D_{\vec{v}} = D_{\vec{v}/|\vec{v}|}$.

Extension to higher dimension: Directional derivatives

In n dimensions, the definition of the directional derivative is the same. We would have an n-dimensional unit vector $\hat{u} = \langle u_1, u_2, \ldots, u_n \rangle$. Then

$$\begin{aligned} D_{\hat{u}} f(x_1, x_2, \ldots, x_n) &= \nabla f(x_1, x_2, \ldots, x_n) \cdot \hat{u} \\ &= \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle \cdot \langle u_1, u_2, \ldots, u_n \rangle \\ &= f_{x_1}(x_1, x_2, \ldots, x_n) u_1 + f_{x_2}(x_1, x_2, \ldots, x_n) u_2 + \cdots + f_{x_n}(x_1, x_2, \ldots, x_n) u_n. \end{aligned}$$

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Directional derivatives given an angle

Let θ be an angle measured from the positive x-axis.The rate of change of $f(x, y)$ in the direction of the angle θ is given by

$$D_{\hat{u}} f(x, y) = f_x \cos\theta + f_y \sin\theta$$

This is the same as the directional derivative of f in the direction of $\hat{u} = \langle \cos\theta, \sin\theta \rangle$.

Directional derivatives with non-unit vectors

Given a vector $\vec{v}$ whose magnitude is not 1, we can obtain a unit vector in the directio

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$$\hat{u} = \frac{1}{|\vec{v}|}\vec{v}.$$

Then the directional derivative of f in the direction of $\vec{v}$ is

$$D_{\vec{u}} f(x, y) = \frac{\nabla f \cdot \vec{v}}{|\vec{v}|}.$$

Direction of maximal change in f

We can write the directional derivative as the dot product

$$D_{\hat{u}} f(x, y) = \nabla f(x, y) \cdot \hat{u} = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $\hat{u}$. This quantity is maximized when $\theta = 0$, which implies that **the gradient is the direction of the maximum rate of change of f**.

Ask Yourself

Is the directional derivative a vector or a scalar?

It's a scalar.

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Why is the directional derivative useful?

We have seen that, when we move in the horizontal direction by a small amount Δx, linear approximation can tell us how f will change. In fact, f will change by $\frac{\partial f}{\partial x} \cdot \Delta x$. But there is nothing special about moving horizontally – sometimes we wish to move in an arbitrary direction. Directional derivatives give us a compact expression for the resulting change in f. If we move from the point (x_0, y_0) in the direction $\vec{u}$ by a distance of Δs, then the change in f will be $D_{\hat{u}} f(x_0, y_0) \cdot \Delta s$.

In summary: if we move the input to f along the direction of $\vec{v}$, then $D_{\vec{v}} f$ tells us how the function changes up to linear approximation.

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What does it mean if the directional derivative in a certain direction is zero?

This means that, up to linear approximation, moving in that direction doesn't change the value of f. For example, if $f(x, y) = x^2 + y^2$, then at the point $(3, 4)$, if we use $\hat{u} = (-4/5, 3/5)$ then $D_{\hat{u}} f(3, 4) = 0$. This means that moving from $(3, 4)$ in the direction $(-4/5, 3/5)$ will keep the value of x^2 + y^2 approximately constant.

Concretely, $f(3, 4) = 25$ and $f(3 - 4/5, 4 + 3/5) = 26$, not much of a change! In contrast, if we move in the direction $(1, 0)$ then the new value of f would be $f(3 + 1, 4) = 32$, a big change!

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extension to higher dimensions -- small typo
Hello. Probably edit out the "c" from "derivativecs." Best wishes.
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