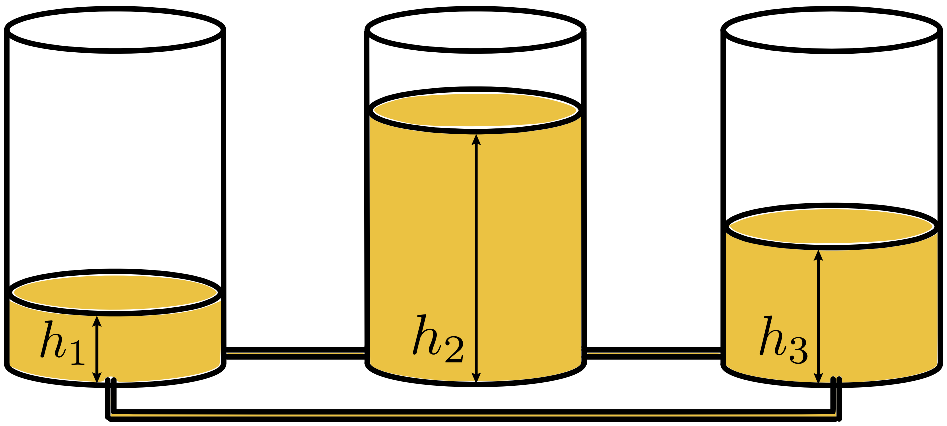
9. Back to 3 tank example



Recall that the fluid flow between three cyclically connected tanks can be described by the system of DE.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

The variables h_1, h_2, h_3 are the fluid heights in tank 1, 2, and 3 respectively.

Problem 9.1 Write the general solution (that we have found previously) in terms of a fundamental matrix.

Solution :

We previously found the normal modes to be

$$\mathbf{v}_1 e^{(0)t}, \qquad \mathbf{v}_2 e^{-3t}, \qquad \mathbf{v}_3 e^{-3t} \qquad \text{where } \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Place each of these into the column of matrix:

$$\mathbf{X} = \begin{pmatrix} | & | & | \\ \mathbf{v}_1 e^{(0)t} & \mathbf{v}_2 e^{-3t} & \mathbf{v}_3 e^{-3t} \\ | & | & | \end{pmatrix}.$$

Then $\mathbf{X}$ is a fundamental matrix of the system because its columns form a basis of the space of all solutions. The general solution is

$$\mathbf{x} \quad = \quad \mathbf{Xc}, \qquad \text{where } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Note that this fundamental matrix can be written as a product of two matrices:

$$\mathbf{X} = \mathbf{SD} \qquad \text{where} \qquad \mathbf{S} \quad = \quad \begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{D} \quad = \quad \begin{pmatrix} e^{(0)t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}.$$

In general, a fundamental matrix of a system $\dot{\mathbf{x}} = \mathbf{Ax}$ for a complete matrix $\mathbf{A}$ is $\mathbf{X} = \mathbf{SD}$ where

$$\mathbf{S} \quad = \quad \begin{pmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{pmatrix}$$

$$\mathbf{D} \quad = \quad \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & e^{\lambda_3 t} & \\ & & & e^{\lambda_n t} \end{pmatrix}.$$

Here, $\mathbf{S}$ is the constant matrix whose columns are the eigenvectors $\mathbf{v}_i$ of $\mathbf{A}$, and $\mathbf{D}$ is the diagonal matrix whose diagonal entries are $e^{\lambda_i t}$, and the eigenvector $\mathbf{v}_i$ corresponds to the eigenvalue λ_i.

Problem 9.2

Before opening the valves, we measure the heights of fluid in the 3 tanks. The starting heights, measured in meters, are

$$\mathbf{x}(0) = \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix}.$$

What are the coefficients c_i that correspond to this initial condition?

Solution

In terms of the fundamental matrix $\mathbf{X}$ (whose columns are the normal nodes), the initial condition is

$$\mathbf{X}(0)\,\mathbf{c} \quad = \quad \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 e^{-3(0)} & \mathbf{v}_3 e^{-3(0)} \\ | & | & | \end{pmatrix} \mathbf{c} \quad = \quad \begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{pmatrix} \mathbf{c} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix}.$$

Hence, $\mathbf{c}$ is the (unique) solution to this linear equation. We can use Gaussian elimination or a computer to find $\mathbf{c}$:

Hence,

$$\mathbf{c} \quad = \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}.$$

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