DelftX: OT.1x Observation theory: Estimating the Unknown

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Exercises: error propagation (2)

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Error Propagation

2/2 points (ungraded)

Apply the propagation laws to find the mean and variance of the difference $\underline{y}_d = \underline{y}_2 - \underline{y}_1$, when given that:

$$E\{\begin{bmatrix}\underline{y}_1\\ \underline{y}_2\end{bmatrix}\} = \begin{bmatrix}3\\ 7\end{bmatrix}; \quad D\{\begin{bmatrix}\underline{y}_1\\ \underline{y}_2\end{bmatrix}\} = Q_{yy} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

The mean of $\underline{y}_d$ is equal to:

4

✓ **Answer:** 4

4

Answer

Correct: Since $E\{\underline{y}_d\} = [-1 \quad 1]\, E\{\begin{bmatrix}\underline{y}_1\\ \underline{y}_2\end{bmatrix}\} = E\{\underline{y}_2\} - E\{\underline{y}_1\} = 7 - 3$

The variance is equal to:

Assessment
Graded Assignment due Feb 8, 2017 17:30 IST

Q&A Forum

Feedback

- 6. Does the estimate make sense?
- Pre-knowledge Mathematics
- MATLAB Learning Content

2 ✓ **Answer:** 2

2

Answer

$$\sigma^2_{y_d}=\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}=2$$

In general for difference:

Correct:

$$\sigma^2_{y_d}=\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}\sigma^2_1 & \sigma^2_{12}\\ \sigma^2_{21} & \sigma^2_2\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}$$
$$=\sigma^2_1+\sigma^2_2-2\sigma^2_{12}$$

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Error propagation of the sum

2/2 points (ungraded)

We take the sum of three independent observations $\underline{y}_i$ ($i=1,2,3$):

$$\underline{y}=\underline{y}_1+\underline{y}_2+\underline{y}_3$$

Each observation has a systematic error of $+10\,\mathrm{cm}$, and a variance $\sigma^2_{y_i}=0.25\,\mathrm{cm}^2$

Hence,

$$\underline{y}_i \sim N(x_i + 10\,\mathrm{cm}, 0.25\,\mathrm{cm}^2)$$

where x_i is the true value.

We are interested to see how the systematic and random errors will propagate to the mean and variance of $\underline{y}$?

The expectation of $\underline{y}$ is:

○ $30\,\mathrm{cm}$

● $x_1 + x_2 + x_3 + 30\,\mathrm{cm}$ ✓

○ $x_1 + x_2 + x_3$

○ None of the above.

Answer

Correct: $E\{\underline{y}\} = [1\;1\;1]\begin{bmatrix} x_1 + 10 \\ x_2 + 10 \\ x_3 + 10 \end{bmatrix}$

The standard deviation of $\underline{y}$ is (in cm):

Give your answer to 2 decimal places

0.87 ✓ **Answer:** 0.87

0.87

Answer

Correct:

$$\sigma_y^2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot (0.25 I_3) \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.25 + 0.25 + 0.25 = 0.75\,\mathrm{cm}^2$$

Standard deviation $\sigma_y = \sqrt{0.75} = 0.87\,\mathrm{cm}$.

The systematic error is the same for each measurement, and therefore is propagated in the mean, in this case it becomes 30cm. The random error will be different for each measurement, its standard deviation is 0.87cm and is not affected by the systematic bias.

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Mean propagation law

2/2 points (ungraded)

Use the mean propagation law

$$\underline{u} = L \cdot \underline{v} \quad \rightarrow \quad E\{\underline{u}\} = L \cdot E\{\underline{v}\}$$

to answer the following questions.

Which is correct for the expectation $E\{\underline{\hat{y}}\}$?

○ $E\{\underline{\hat{y}}\} = y$

○ $E\{\underline{\hat{y}}\} = 0$

● $E\{\underline{\hat{y}}\} = Ax$ ✓

○ $E\{\underline{\hat{y}}\} = A\hat{x}$

Which is correct for the expectation $E\{\underline{\hat{e}}\}$?

○ $E\{\underline{\hat{e}}\} = y - \hat{y}$

● $E\{\underline{\hat{e}}\} = 0$ ✓

○ $E\{\underline{\hat{e}}\} = Ax$

○ $E\{\underline{\hat{e}}\} = Ax - A\hat{x}$

Explanation

$\underline{\hat{x}} = (A^TQ_{yy}^{-1}A)^{-1}A^TQ_{yy}^{-1}\underline{y}$ and therefore:
$E\{\underline{\hat{x}}\} = (A^TQ_{yy}^{-1}A)^{-1}A^TQ_{yy}^{-1}E\{\underline{y}\} = (A^TQ_{yy}^{-1}A)^{-1}A^TQ_{yy}^{-1}Ax = x$
$\underline{\hat{y}} = A\underline{\hat{x}}$ and therefore: $E\{\underline{\hat{y}}\} = AE\{\underline{\hat{x}}\} = Ax$
$\underline{e} = \underline{y} - \underline{\hat{y}}$ and therefore: $E\{\underline{\hat{e}}\} = E\{\underline{y}\} - E\{\underline{\hat{y}}\} = Ax - Ax = 0$

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Deriving the covariance matrix of the BLUE (1)

1/1 point (ungraded)

You will now be asked to derive that $Q_{\hat{x}\hat{x}} = (A^TQ_{yy}^{-1}A)^{-1}$ yourself, by applying the covariance propagation law. Thereby we need the following properties of the matrix transpose:

1. $(A^T)^T = A$
2. $(A \cdot B \cdot C)^T = C^T \cdot B^T \cdot A^T$
3. $Q^T = Q$ if Q is symmetric
4. $(Q^{-1})^T = Q^{-1}$ if Q is symmetric

Using these properties, select which of the following is correct for the normal matrix $N = A^TQ_{yy}^{-1}A$.

☑ N is symmetric

☐ N is not symmetric

☑ $((A^T Q_{yy}^{-1} A)^{-1})^T = (A^T Q_{yy}^{-1} A)^{-1}$

✓

Explanation

It follows that $(A^T \cdot Q_{yy}^{-1} \cdot A)^T = A^T \cdot (Q_{yy}^{-1})^T \cdot (A^T)^T = A^T Q_{yy}^{-1} A$. Hence, N is symmetric since $N^T = N$, therefore also $(N^{-1})^T = N^{-1}$.

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Deriving the covariance matrix of the BLUE (2)

1/1 point (ungraded)

The BLUE is given by $\hat{\underline{x}} = (A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} \underline{y}$. Let's call $L = (A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}$. Select the correct answer:

● $L^T = Q_{yy}^{-1} A (A^T Q_{yy}^{-1} A)^{-1}$ ✓

○ $L^T = (A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1}$

○ Both expression are incorrect.

Explanation

We can also write $L = N^{-1} \cdot A^T \cdot Q_{yy}^{-1}$. Now we can derive that $L^T = (Q_{yy}^{-1})^T \cdot (A^T)^T \cdot (N^{-1})^T = Q_{yy}^{-1} \cdot A \cdot N^{-1} = (Q_{yy}^{-1} A(A^T Q_{yy}^{-1} A)^{-1}$. Here we used the properties of the matrix transpose.

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Deriving the covariance matrix of the BLUE (3)

1/1 point (ungraded)

Finally we can apply the covariance propagation law to find the covariance matrix of $\underline{\hat{x}}$. Which of the following is correct:

○ $Q_{\hat{x}\hat{x}} = L^T Q_{yy} L$

● $Q_{\hat{x}\hat{x}} = L Q_{yy} L^T$ ✓

○ $Q_{\hat{x}\hat{x}} = L^T Q_{yy}^{-1} L$

○ $Q_{\hat{x}\hat{x}} = L Q_{yy}^{-1} L^T$

Explanation

We have that $\underline{\hat{x}} = L\underline{y}$ and therefore the second equation is correct.

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For completeness we show again the derivation of the final result:

$$\begin{aligned} Q_{\hat{x}\hat{x}} &= LQ_{yy}L^T \\ &= (A^TQ_{yy}^{-1}A)^{-1}A^TQ_{yy}^{-1} \cdot Q_{yy} \cdot Q_{yy}^{-1}A(A^TQ_{yy}^{-1}A)^{-1} \\ &= (A^TQ_{yy}^{-1}A)^{-1}A^TQ_{yy}^{-1}A(A^TQ_{yy}^{-1}A)^{-1} \\ &= (A^TQ_{yy}^{-1}A)^{-1} \end{aligned}$$

Deriving the covariance matrix of the BLUE (5)

1/1 point (ungraded)

You are now ready to derive the covariance matrix of $\underline{\hat{y}} = A\underline{\hat{x}}$. Select the correct answer below.

$Q_{\hat{y}\hat{y}} =$

● $A(A^TQ_{yy}^{-1}A)^{-1}A^T$ ✓

○ $A^T(A^TQ_{yy}^{-1}A)^{-1}A$

○ $A^TQ_{yy}^{-1}A$

○ $AQ_{yy}^{-1}A^T$

Explanation

The covariance propagation law gives: $Q_{\hat{y}\hat{y}} = AQ_{\hat{x}\hat{x}}A^T = A(A^TQ_{yy}^{-1}A)^{-1}A^T$. Note: the derivation of the covariance matrix of $\underline{\hat{e}}$ is in the Summary.

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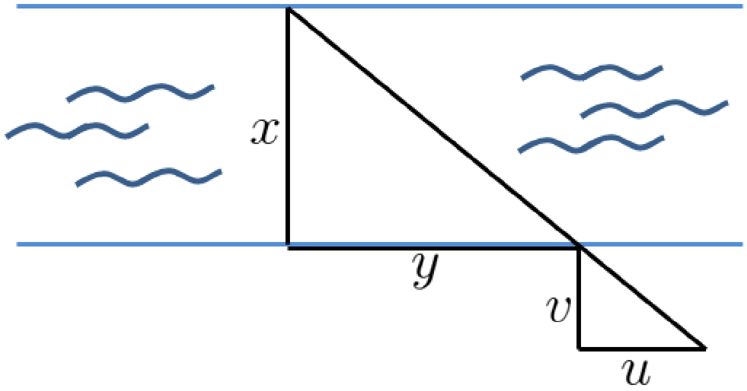
Precision of the nerd teams' estimate

1/1 point (ungraded)

Remember the nerd team who determined the canal width using the approach visualized below, where u and v are known, and distance y is measured.



Using their approach with a single observation would give the following functional and stochastic model:

$$E\{\underline{y}\} = \frac{u}{v}x; \quad D\{\underline{y}\} = \sigma^2 = 4\text{cm}^2$$

The precision of the BLUE $\hat{\underline{x}}$ will be:

● better if u is larger than v. ✓

○ worse if u is larger than v.

○ independent of u and v.

Answer

Correct:

$$Q_{\hat{x}\hat{x}} = (A^T Q_{yy}^{-1} A)^{-1}$$

$$\sigma_{\hat{x}}^2 = (\frac{u}{v} \cdot \frac{1}{\sigma^2} \cdot \frac{u}{v})^{-1}$$

$$= \sigma^2 (\frac{v}{u})^2$$

Note: From this example you can see that the A-matrix $A = \frac{u}{v}$ is driven by the 'design' of the measurement set-up, and this influences the precision. Changing the design, in this case increasing the ratio $\frac{u}{v}$, helps to improve the precision.

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Precision of the nerd teams' estimate (2)

1/1 point (ungraded)

With the same setup as in the previous question, now assume $\frac{v}{u} = 0.7$ and $\sigma^2 = 4\,\mathrm{cm}^2$.

What is the variance of the best linear unbiased estimator $\underline{\hat{x}}$?

Give your answer to 2 decimal places

1.96 ✓ **Answer:** 1.96

1.96

Answer

Correct: $\sigma_{\hat{x}}^2 = \sigma^2 (\frac{v}{u})^2 = 4 \cdot 0.7^2 = 1.96$

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Precision of the nerd teams' estimate (3)

1/1 point (ungraded)

With the settings, $\frac{u}{v} = 0.7$ and $\sigma^2 = 4\,\mathrm{cm}^2$, the team will collect m measurements. The requirement is that $\sigma_{\hat{x}}^2$ is smaller or equal to 0.5.

How many times ($= m$) does the team need to repeat the measurement to fulfil the requirement?

17

✓ **Answer:** 17

17

Answer

$$\sigma_{\hat{x}}^2 = (\frac{m \cdot 0.7^2}{4})^{-1} = \frac{4}{0.49m}$$

Correct: Our requirement is that $\sigma_{\hat{x}}^2 \leq 0.5$. Solving $\frac{4}{0.49m} \leq 0.5$, yields $m \geq 17$

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