8. Linear Independence, Subspaces and Dimension

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_n$ such that (1) not all c_i's are zero and (2) $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$.

Otherwise, they are said to be **linearly independent** : the only scalars $c_1, \ldots, c_n$ that satisfy $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$ are $c_1 = \cdots = c_n = 0$.

The collection of non-zero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ determines a **subspace** of $\mathbb{R}^m$, which is the set of all linear combinations $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ over different choices of $c_1, \ldots, c_n \in \mathbb{R}$. The **dimension** of this subspace is the size of the **largest possible, linearly independent** sub-collection of the (non-zero) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Row and Column Rank

2/2 points (graded)

Suppose $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. The rows of the matrix, $(1,3)$ and $(2,6)$, span a subspace of dimension

1

✔ **Answer:** 1 . This is the **row rank** of $\mathbf{A}$.

The columns of the matrix, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ span a subspace of dimension

1

✔ **Answer:** 1 . This is the **column rank** of $\mathbf{A}$.

We will be using these ideas when studying **Linear Regression**, where we will work with larger, possibly rectangular matrices.

Solution:

In both cases, the two vectors are linearly dependent.

$$2 \cdot (1,3) - (2,6) = (0,0)$$

$$3\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}3\\6\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

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Answers are displayed within the problem

The rank of a matrix

3/3 points (graded)

In general, row rank is always equal to the column rank, so we simply refer to this common value as the **rank** of a matrix.

What is the largest possible rank of a 2×2 matrix?

2 ✔ **Answer:** 2

What is the largest possible rank of a 5×2 matrix?

2 ✔ **Answer:** 2

In general, what is the largest possible rank of an $m \times n$ matrix?

- m
- n
- $\min(m, n)$ ✔
- $\max(m, n)$
- None of the above

Solution:

In general, the rank of any $m \times n$ matrix can be at most $\min(m, n)$, since rank = column rank = row rank. For example, if there are five columns and three rows, the column rank cannot be larger than the largest possible row rank – the largest possible row rank for three rows is, unsurprisingly, 3. The opposite is also true if there are more rows than columns. If a matrix has two columns and six rows, then the row rank cannot exceed the column rank, which is at most 2.
In general, a matrix $\mathbf{A}$ is said to have **full rank** if $\text{rank}(\mathbf{A}) = \min(m, n)$. (note the $=$, instead of $\leq$).

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Examples of rank

5/5 points (graded)

What is the rank of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

1

✔ **Answer:** 1

What is the rank of $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$?

2

✔ **Answer:** 2

What is the rank of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$?

0

✔ **Answer:** 0

What is the rank of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$?

2

✔ **Answer:** 2

What is the rank of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$?

3

✔ **Answer:** 3

Solution:

1. The set of rows describe a subspace of dimension 1, spanned by $(1, 1)$.
2. This matrix has rank 2, since $(1, -1)$ and $(1, 0)$ are linearly independent.
3. This matrix has rank zero. By definition, the rank is equal to the number of nonzero linearly independent vectors.
4. The second and third rows are independent. However, the sum of the second and third rows are equal to the first: $(1, 0, 1) + (0, 1, 0) = (1, 1, 1)$. So this matrix has rank 2.
5. All three rows are independent. An easy way to check is to notice that this matrix is **upper triangular** , with nonzero entries along the diagonal.

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Answers are displayed within the problem

The rank of a matrix continued

2/2 points (graded)

This question is meant to serve as an answer to the following: *If you sum two rank-1 matrices, do you get a rank-2 matrix? What about products? More generally, what rank is the sum of a rank-r_1 and a rank-r_2 matrix?*"

Let $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Observe that all four of these matrices are rank 1.

There are many ways to determine rank. Here is one useful fact that you could use for this problem:

> **"Every rank-1 matrix can be written as an outer product. Conversely, every outer product $\mathbf{u}\mathbf{v}^T$ is a rank-1 matrix."**

For example, $\mathbf{A} = \mathbf{u}\mathbf{v}^T$, $\mathbf{B} = \mathbf{v}\mathbf{v}^T$, $\mathbf{C} = \mathbf{w}\mathbf{w}^T$ and $\mathbf{D} = \mathbf{x}\mathbf{x}^T$, where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Which combination of these matrices has rank 2? Choose all that apply.

- [ ] $\mathbf{A} + \mathbf{A}$
- [ ] $\mathbf{A} + \mathbf{B}$

- [x] $\mathbf{A}+\mathbf{C}$ ✔
- [ ] $\mathbf{AB}$
- [ ] $\mathbf{AC}$
- [ ] $\mathbf{BD}$

✔

Which combination of these matrices has rank 1? Choose all that apply.

- [x] $\mathbf{A}+\mathbf{A}$ ✔
- [x] $\mathbf{A}+\mathbf{B}$ ✔
- [ ] $\mathbf{A}+\mathbf{C}$
- [x] $\mathbf{AB}$ ✔
- [x] $\mathbf{AC}$ ✔
- [ ] $\mathbf{BD}$

✔

Solution:

The choices are of two general types: sums of matrices, and products of matrices.

- $\mathbf{A}+\mathbf{A}=2\mathbf{A}$, which has rank 1.
- $\mathbf{A}+\mathbf{B}=\mathbf{u}\mathbf{v}^T+\mathbf{v}\mathbf{v}^T=(\mathbf{u}+\mathbf{v})\,\mathbf{v}^T$, which has rank 1.
- $\mathbf{A}+\mathbf{C}=\begin{pmatrix}-1 & 1\\ -3 & 4\end{pmatrix}$. This has two linearly independent rows, hence its rank is 2.

The last three choices $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{BD}$ cannot have rank 2 since they are products of rank-1 matrices.

- $\mathbf{AB}=\mathbf{u}\mathbf{v}^T\mathbf{v}\mathbf{v}^T=\mathbf{u}\langle\mathbf{v},\mathbf{v}\rangle\mathbf{v}^T=\langle\mathbf{v},\mathbf{v}\rangle\mathbf{u}\mathbf{v}^T$. Note that the inner product $\mathbf{v}^T\mathbf{v}=\langle\mathbf{v},\mathbf{v}\rangle$ “floats" to the front because it is a scalar. This is an outer product of two vectors, which has rank 1.
- $\mathbf{AC}=\mathbf{u}\mathbf{v}^T\mathbf{w}\mathbf{w}^T=\langle\mathbf{v},\mathbf{w}\rangle\mathbf{u}\mathbf{w}^T$, which again has rank 1.
- $\mathbf{BD}=\mathbf{v}\mathbf{v}^T\mathbf{x}\mathbf{x}^T=\langle\mathbf{v},\mathbf{x}\rangle\mathbf{v}\mathbf{x}^T$. Notice that $\mathbf{v}$ is orthogonal to $\mathbf{x}$, so $\mathbf{BD}=0\mathbf{v}\mathbf{x}^T$ is the zero matrix. Its rank is zero.

In general, the sum of two matrices can have a varying range of ranks, and they can be greater **or** less than the ranks of matrices that are being summed up. On the other hand, it is a general fact that if $\mathbf{A}$ and $\mathbf{B}$ are arbitrary (possibly rectangular) matrices, $\mathrm{rank}\,(\mathbf{AB})\leq\min\left(\mathrm{rank}\,(\mathbf{A})\,,\mathrm{rank}\,(\mathbf{B})\right)$. It is possible to use **determinants** to reason about rank. For choices such as $\mathbf{A}+\mathbf{B}=\begin{pmatrix}0 & 0\\ -2 & 2\end{pmatrix}$, the rank is obviously 1. Sometimes, it is easier if you know how to factor matrices – in this problem, we gave you the factorizations of rank-1 matrices into outer products of vectors. Other times, one may resort to using Gaussian Elimination – the rank of any upper triangular matrix is **at least** the number of non-zero entries along the diagonal.

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Answers are displayed within the problem

Invertibility of a matrix

0 points possible (ungraded)

An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ has full rank, i.e. $\mathrm{rank}\,(\mathbf{A}) = n$.

Which of the following matrices are invertible? Choose all that apply.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

- [ ] $\mathbf{A}$
- [x] $\mathbf{B}$ ✔
- [x] $\mathbf{C}$ ✔
- [ ] $\mathbf{D}$

✔

Solution:

We saw in a previous exercise that the rank of $\mathbf{A}$ is 1. The rank of $\mathbf{B}$ is 2, since $(1,2)$ and $(2,1)$ are linearly independent, since e.g. by Gaussian Elimination one obtains the reduced upper triangular matrix $\begin{pmatrix} 1 & 2 \\ 0 & 3/2 \end{pmatrix}$. In general, an upper triangular matrix with nonzero entries along the diagonal has full rank.

By the same reasoning, $\mathbf{C}$ also has full rank. Finally, $\mathbf{D}$ does not have full rank, since $(\text{row } 1) + (\text{row } 2) + (\text{row } 3) = \vec{0}$.

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? The rank of a matrix continued

Puzzled: if a given sum or product of these matrices is not rank 2, must it not be rank 1? (Assuming it's not a null matrix). Having got the first part cor... 3

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