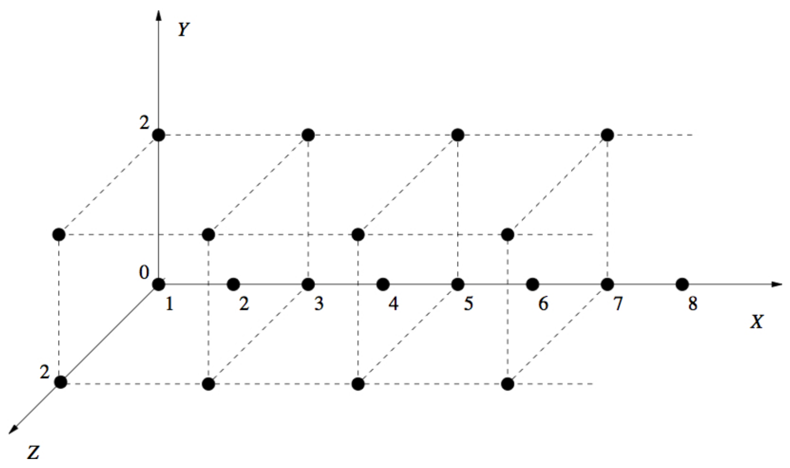
MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Problem 5: Joint PMF calculations - Part 2

(5/5 points)

Note: The problem statement from part 1 has been repeated here for your convenience.

Consider three random variables X, Y, and Z, associated with the same experiment. The random variable X is geometric with parameter $p \in (0,1)$. If X is even, then Y and Z are equal to zero. If X is odd, (Y,Z) is uniformly distributed on the set $S=\{(0,0),(0,2),(2,0),(2,2)\}$. The figure below shows all the possible values for the triple (X,Y,Z) that have $X \leq 8$. (Note that the X axis starts at 1 and that a complete figure would extend indefinitely to the right.)



1. Find the joint PMF $p_{X,Y,Z}(x,y,z)$. Express your answers in terms of x and p using standard notation .

   If x is odd and $(y,z) \in \{(0,0),(0,2),(2,0),(2,2)\}$,

   $p_{X,Y,Z}(x,y,z) =$ (1-p)^(x-1)*p/4 ✔ **Answer:** (1/4)*p*(1-p)^(x-1)

   If x is even and $(y,z) = (0,0)$,

   $p_{X,Y,Z}(x,y,z) =$ (1-p)^(x-1)*p ✔ **Answer:** p*(1-p)^(x-1)

2. Find $p_{X,Y}(x,2)$, for when x is odd. Express your answer in terms of x and p using standard notation .
   If x is odd,

   $p_{X,Y}(x,2) =$ (1-p)^(x-1)*p/2 ✔ **Answer:** (1/2)*p*(1-p)^(x-1)

3. Find $p_Y(2)$. Express your answer in terms of p using standard notation .

   $p_Y(2) =$ 1/(4-2*p) ✔ **Answer:** 1/(2*(2-p))

4. Find $\mathbf{var}(Y+Z \mid X=5)$.

   2 ✔ **Answer:** 2

Answer:

1. An easy way to derive $p_{X,Y,Z}(x,y,z)$ uses the multiplication rule: $p_X(x) \cdot p_{Y,Z|X}(y,z|x)$. Note that X is geometric with parameter p. Conditioned on X even, $(Y,Z)=(0,0)$ with probability 1. Conditioned on X odd, $p_{Y,Z|X}(y,z) = \frac{1}{4}$ for $(y,z) \in \{(0,0),(0,2),(2,0),(2,2)\}$.

$$p_{X,Y,Z}(x,y,z) = \begin{cases} \frac{1}{4}p(1-p)^{x-1}, & \textbf{if } x \textbf{ is odd and } (y,z) \in \{(0,0),(0,2),(2,0),(2,2)\} \\ p(1-p)^{x-1}, & \textbf{if } x \textbf{ is even and } (y,z)=(0,0) \\ 0, & \textbf{otherwise.} \end{cases}$$

2. $p_{X,Y}(x,2) = \sum_z p_{X,Y,Z}(x,2,z)$. From part 1, we know that when x is odd and $(y,z) \in \{(0,0),(0,2),(2,0),(2,2)\}$, $p_{X,Y,Z}(x,y,z) = \frac{1}{4}p(1-p)^{x-1}$ and so:

$$\begin{aligned} p_{X,Y}(x,2) &= p_{X,Y,Z}(x,2,0) + p_{X,Y,Z}(x,2,2) \\ &= \frac{1}{2}p(1-p)^{x-1}, \end{aligned}$$

when x is odd.

3. $p_Y(2) = \sum_x p_{X,Y}(x,2)$. Since $p_{X,Y}(x,2)$ is non-zero only when x is odd, we can use the result from the previous question to find:

$$p_Y(2) = \sum_{x \text{ is odd}} p_{X,Y}(x,2)$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{x \text{ is odd}} p(1-p)^{x-1} \\
&= \frac{1}{2}(p(1-p)^0 + p(1-p)^2 + p(1-p)^4 + p(1-p)^6 + \cdots) \\
&= \frac{p}{2}((1-p)^0 + (1-p)^2 + (1-p)^4 + (1-p)^6 + \cdots) \\
&= \frac{p}{2}(\frac{1}{1-(1-p)^2}) \\
&= \frac{1}{2(2-p)}
\end{aligned}
$$

4. If $X = 5$, then Y and Z are uniformly distributed on the set S specified in the problem statement, so $Y + Z$ takes the values 0 and 4 with probability $\frac{1}{4}$, and takes the value 2 with probability $\frac{1}{2}$. This PMF is symmetric about 2, so the mean value of $Y + Z$ is evidently 2. Hence the variance is

$$(0-2)^2 \frac{1}{4} + (4-2)^2 \frac{1}{4} = 2.$$

You have used 1 of 2 submissions