Homework

Homework due Jul 1, 2020 21:30 IST

The exercises below will count towards your grade. **You have only one chance to answer these questions.** Take your time, and think carefully before answering.

Problem 1

25/25 points (graded)

Is $\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\emptyset\right)\right)\right)$ totally ordered by $\in$?

- ( ) Yes
- (•) No

✔

Explanation

No, it is not. Note that

$$\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\emptyset\right)\right)\right) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

and neither $\emptyset \in \{\{\emptyset\}\}$ nor $\{\{\emptyset\}\} \in \emptyset$.

Is $\mathcal{P}\left(\mathcal{P}\left(\mathcal{P}\left(\emptyset\right)\right)\right) - \{\{\{\emptyset\}\}\}$ totally ordered by $\in$?

- (•) Yes

No

Explanation

Yes, it is. Since $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \{\{\{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, which is the ordinal $0'''$ and is therefore well-ordered by $\in$.
Note that $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \{\{\{\emptyset\}\}\}$ is not the result of removing $\{\{\{\emptyset\}\}\}$ from amongst the elements of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$, but the result of removing $\{\{\emptyset\}\}$ from amongst the elements of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$. (This is because $A - B$ is not the result of of eliminating set B from amongst the elements of A but the result of removing any elements of B from A.)

Some definitions:

- $\emptyset$ is the empty set.
- if A and B are sets, $A - B$ is the set of objects in A but not in B. (For instance, $\{1, 2\} - \{1\} = \{2\}$.)
- $\mathcal{P}$ is the powerset operation.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Problem 2

25/25 points (graded)
Are the following statements true or false?

- $\alpha + 0' = \alpha \cup \{\alpha\}$

True

False

Explanation

It follows from the Construction Principle that $\alpha \cup \{\alpha\} = \alpha'$, and it follows from the definition of addition that $\alpha + 0' = (\alpha + 0)' = \alpha'$. Putting the two together, $\alpha \cup \{\alpha\} = \alpha'$.

- $0' + \omega = \omega + 0'$

( ) True

(•) False

✔

Explanation
$0' + \omega = \omega$, but $\omega + 0'$ is greater than ω.

- $0' + 0''' = 0''' + 0'$

(•) True

( ) False

✔

Explanation
$0' + 0''' = 0'''' = 0''' + 0'$.

- $(\omega + 0'') + \omega <_o (\omega + \omega) + 0''$

(•) True

( ) False

✔

Explanation
$(\omega + 0'') + \omega = \omega + \omega$, which is smaller than $(\omega + \omega) + 0''$.

- $0' \times 0''' = 0''' \times 0'$

- [x] True
- [ ] False

✓

Explanation
$0' \times 0''' = 0''' = 0''' \times 0'$.

- $0''' \times \omega = (\omega + \omega) + \omega$

- [ ] True
- [x] False

✓

Explanation
$0''' \times \omega = \omega$, which is smaller than $(\omega + \omega) + \omega$.

- $\omega \times 0''' = \omega + (\omega + \omega)$

- [x] True
- [ ] False

✓

Explanation
$\omega \times 0''' = \omega + (\omega + \omega)$.

- $(\omega \times 0'') + \omega <_o (\omega \times \omega) + 0''$

- [x] True
- [ ] False

✓

Explanation
$(\omega \times 0'') + \omega = ((\omega + \omega) + \omega)$, which is much smaller than $\omega \times \omega$ (and therefore much smaller than $(\omega \times \omega) + 0''$.)

- $\omega \times \omega <_o \omega \times (0'' \times \omega)$

( ) True

(•) False

✔

Explanation
$\omega \times (0'' \times \omega) = \omega \times \omega$.

- $\omega \times (\omega + \omega) = (\omega \times \omega) + (\omega \times \omega)$

(•) True

( ) False

✔

Explanation
$\omega \times (\omega + \omega)$ is the result of using $(\omega + \omega)$ as a template, and filling each position with an ω-sequence. So we get an ω-sequence of ω-sequences followed by an ω-sequence of ω-sequences.
$(\omega \times \omega) + (\omega \times \omega)$ is the result of starting with a sequence of type $(\omega \times \omega)$ (i.e. an ω-sequence of ω-sequences), and appending another sequence of type $(\omega \times \omega)$ (i.e. another ω-sequence of ω-sequences) to the right. So, again, we get an ω-sequence of ω-sequences followed by an ω-sequence of ω-sequences.

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Answers are displayed within the problem

Problem 3

25/25 points (graded)

Let U be the following set:

$$\cup\{\mathbb{N}, \mathcal{P}^1(\mathbb{N}), \mathcal{P}^2(\mathbb{N}), \ldots\}$$

Does U contain the set $\underbrace{\{\{\ldots\{}_{n \text{ times}}\{17\underbrace{\}\}\ldots\}\}}_{n \text{ times}}$ for each $n > 0$?

- (selected) Yes
- No

✔

Explanation

Yes, it does. Let n be an arbitrary natural number greater than 0. Then $\mathcal{P}^n(\mathbb{N})$ contains the set $\underbrace{\{\{\ldots\{}_{n \text{ times}}\{17\underbrace{\}\}\ldots\}\}}_{n \text{ times}}$.
Since U contains every member of $\mathcal{P}^n(\mathbb{N})$, U must also contain $\underbrace{\{\{\ldots\{}_{n \text{ times}}\{17\underbrace{\}\}\ldots\}\}}_{n \text{ times}}$.

Does U contain the set $\underbrace{\{\{\ldots\{}_{\infty \text{ times}}\{17\underbrace{\}\}\ldots\}\}}_{\infty \text{ times}}$?

- Yes
- (selected) No

✔

Explanation

No, it doesn't. For the only members of U are members of $\mathcal{P}^n(\mathbb{N})$ for *some* n, and no $\mathcal{P}^n(\mathbb{N})$ contains $\underbrace{\{\{\ldots\{}_{\infty \text{ times}}\{17\underbrace{\}\}\ldots\}\}}_{\infty \text{ times}}$.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Problem 4

10/10 points (graded)

Is there an ordinal α such that $\alpha <_o \emptyset$?

- Yes
- No

Explanation

By definition, $\alpha <_o \emptyset$ if and only if $\alpha \in \emptyset$. But $\emptyset$ has no members. So it cannot be the case that $\alpha <_o \emptyset$.

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Answers are displayed within the problem

Problem 5

5/5 points (graded)

Recall the Ordinal Construction Principle:

> *Construction Principle*: At each stage, introduce a new ordinal: the set of all ordinals that have been introduced at previous stages.

Keeping this in mind, state whether the following is true or false: If α is an ordinal with infinitely many members, then either $\alpha = \omega$ or $\omega <_o \alpha$.

- True
- False

Explanation

Ordinals are introduced in stages. The first stage at which the process yields an ordinal with infinitly many members is the stage at which ω is introduced. So any ordinal distinct from ω with infinitely many members must be introduced at some later stage of the process. But, by the Construction Principle, every ordinal that is introduced after ω must contain ω. And, by definition, $\omega <_o \alpha$ if and only if $\omega \in \alpha$.

You have used 1 of 1 attempt

Submit

Answers are displayed within the problem

Problem 6

10/10 points (graded)

Which of the following sets are totally ordered by their respective relation, $<$?

Select all correct answers.

- [x] The set of natural numbers, where $1 < 2 < 3 < \ldots < 0$
- [x] The set of rational numbers, where $a < b$ if and only if $b = a + q$ for q a positive rational number
- [ ] The set of people on Earth, where $a < b$ if and only if a's year of birth is before b's year of birth
- [ ] The set $\mathcal{P}(\{0, 1, 2, 3\})$, where $a < b$ if and only if $a \subsetneq b$
- [ ] The set of Olympic athletes, where $a < b$ if and only if a has won fewer medals than b

✔

Explanation

The first two sets are totally ordered by their respective sets; the latter three are not:

- The set of natural numbers is totally ordered by the proposed ordering because every number turns out to be either greater or smaller than any other number.
- The set of rational numbers is totally ordered by the proposed ordering because for any distinct rationals a and b, there is a positive rational q such that either $b = a + q$ or $a = b + q$.
- The set of people on Earth is not totally ordered by the proposed ordering because there are people on Earth who share the same birthday. In other words, there is someone whose birthday is neither before nor after someone else's.

- The set $\mathcal{P}(\{0, 1, 2, 3\})$ is not totally ordered by the proposed ordering because, for example, we have neither $\{0\} < \{1\}$ nor $\{1\} < \{0\}$.
- The set of Olympic athletes is not totally ordered by the proposed ordering because there are Olympic athletes who have won the same number of medals.

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Answers are displayed within the problem