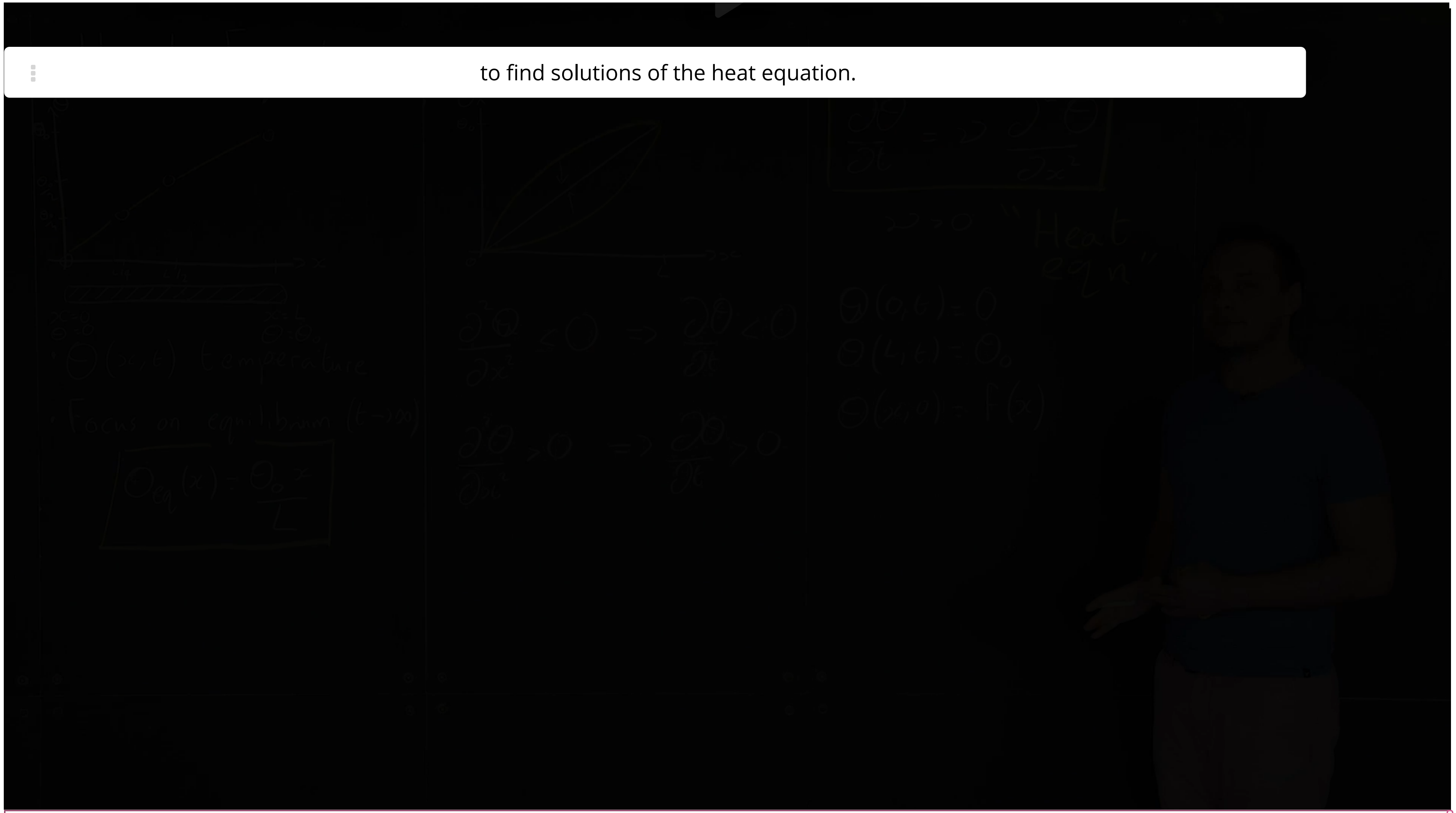
3. Deriving the heat equation
Derivation



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To explain the Heat Equation, we start with a thought experiment. If we fix the temperature at the ends, $\theta(0,t)=0$ and $\theta(L,t)=T$, what will happen in the long term as $t\to\infty$? Assume for the moment that there is a steady state. That is, that $\theta(x,t)$ converges to some temperature profile $\Theta(x)$:

$$\theta(x,t)\to\Theta(x), \quad t\to\infty.$$

The answer is that

$$\Theta(x)=\frac{T}{L}x \quad (\text{linear}).$$

The temperature $\theta(L/2,t)$ at the midpoint $L/2$ tends to the average of 0 and T, namely $T/2$. At the point $L/4$, halfway between 0 and $L/2$, the temperature tends to the average of the temperature at 0 and $T/2$, and so forth.

At a very small scale, this same mechanism, the tendency of the temperature profile toward a straight line equilibrium means that if θ is concave down then the temperature in the middle should decrease (so the profile becomes closer to being straight). If θ is concave up, then the temperature in the middle should increase (so that, once again, the profile becomes closer to being straight). We write this as

$$\frac{\partial^2\theta}{\partial x^2}<0 \quad \Longrightarrow \quad \frac{\partial\theta}{\partial t}<0,$$

$$\frac{\partial^2\theta}{\partial x^2}>0 \quad \Longrightarrow \quad \frac{\partial\theta}{\partial t}>0.$$

The simplest relationship that reflects this is a linear (proportional) relationship,

$$\frac{\partial\theta}{\partial t}=\nu\frac{\partial^2\theta}{\partial x^2}, \quad \nu>0.$$

Fourier introduced the Heat Equation, solved it, and confirmed in many cases that it predicts correctly the behavior of temperature in experiments like the one with the metal bar.

Actually, Fourier crushed the problem, figuring out the whole formula for $\theta(x,t)$ and not just when the initial value is $\theta(x,0)=1$, but also when the initial temperature varies with x. His formula even predicts accurately what happens when $0<t<\tau$.

If you are interested you can click below to see a scale model derivation of the Heat Equation.

Another derivation (optional)

We are going to look at one way to derive a partial differential equation describing the evolution of temperature in an insulated, uniform bar of length L. We first looked at this problem in the course *Linear Algebra and NxN Systems of Differential equations*, when we were studying $n\times n$ systems. Initially, we considered a rod of length L, with two thermometers placed evenly along its length, at the points $x_1=L/3$ and $x_2=2L/3$. The left end of the rod was held at the temperature θ_L and the right end was held at θ_R. We used *Newton's law of cooling*, which said that the rate of change of temperature at x_1, $\frac{d\theta_1}{dt}$ is affected by the adjacent temperatures θ_L and θ_2.The left contribution is proportional to the difference in temperature $\theta_L-\theta_1$, while the right contribution is proportional to $\theta_2-\theta_1$. This meant that we could write down a system of two equations for θ_1 and θ_2:

$$\frac{d\theta_1}{dt}=k(\theta_L-\theta_1)+k(\theta_2-\theta_1)\quad=k(\theta_L-2\theta_1+\theta_2) \tag{3.39}$$

$$\frac{d\theta_2}{dt}=k(\theta_1-\theta_2)+k(\theta_R-\theta_2)\quad=k(\theta_1-2\theta_2+\theta_R)\,. \tag{3.40}$$

We were a little careless when we first looked at this system. Looking more carefully, we can see that if we want to place many thermometers on the bar and track the interaction of temperatures at many points, then we should adjust the constant k for the distance between points. Let's suppose that we place N thermometers at points $x_n=n\Delta x, n=1,\ldots,N$, equally spaced along the bar, with $\Delta x=\frac{L}{N+1}$. If $\theta_n(t)$ is the temperature at x_n, then the correct scaling for the influence of θ_{n+1} and θ_{n-1} on θ_n is the following version of Newton's law:

$$\frac{d\theta_n}{dt}\quad=\nu\frac{(\theta_{n-1}-\theta_n)}{(\Delta x)^2}-\nu\frac{(\theta_n-\theta_{n+1})}{(\Delta x)^2}. \tag{3.41}$$

In other words, $k = \nu/(\Delta x)^2$ is the appropriate scaling for k, with ν a property of the material. (This inverse square law is what Fourier discovered by testing bars of metal of various lengths. He then applied the insight that calculus brings, namely that the same rule should work at infinitesimal scales.) We can now simplify the right hand side to obtain

$$\frac{d\theta_n}{dt} = \nu \frac{(\theta_{n+1} - 2\theta_n + \theta_{n-1})}{(\Delta x)^2}. \tag{3.42}$$

We want to consider the limit $N \to \infty$ so that $\Delta x \to 0$. We are interested in deriving a partial differential equation for the temperature $\theta(x,t)$ at any position x along the bar. We therefore need to choose n so that $x_n \to x$ as $N \to \infty$. A possible choice is $n = \lfloor x/\Delta x \rfloor = \lfloor x(N+1)/L \rfloor$. As $N \to \infty$, the right hand side of the equation for $\frac{d\theta_n}{dt}$ becomes $\frac{\partial^2\theta}{\partial x^2}(x)$. In this limit, we have the following partial differential equation describing the continuous evolution of temperature in the bar:

$$\frac{\partial\theta}{\partial t} = \nu \frac{\partial^2\theta}{\partial x^2}. \tag{3.43}$$

This is the heat equation.

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The heat equation does not determine the solution completely by itself. We also need to specify what are known as **boundary conditions** and **initial conditions** .

Boundary conditions are the temperatures at the two ends of the rod at each time. For example, we can impose the boundary conditions that temperature is fixed for all time on the left and right sides at given temperatures:

$$\theta(0,t) = \theta_L, \quad \text{and} \quad \theta(L,t) = \theta_R.$$

The initial condition can be any smooth function $\theta_0(x)$ so that

$$\theta(x, t=0) = \theta_0(x).$$

As we shall see later, the boundary conditions and initial conditions together with the heat equation determine a **unique** temperature profile $\theta(x, t)$ for all future times $t > 0$.

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