edX

Course › Probab... › The Pri... › An awk...

# An awkward consequence of rejecting Countable Additivity

Wouldn't it be better to give up on Countable Additivity altogether?

By giving up Countable Additivity, one would be free to assign credence 0 to each of a countable set of mutually exclusive propositions.

For example, one could assign credence 0 to each of our propositions $G_n$. (Keep in mind that saying that assigning an event credence 0 is not the same as taking the event to be impossible. It is just to say that your degree of belief is so low that no positive real number is small enough to measure it.)

Unfortunately, giving up Countable Additivity can lead to a theory with undesirable mathematical properties.

Here is an example, which I borrow from Kenny Easwaran.

Let $X$ and $Y$ be sets of positive integers. Let $p(X)$ be the probability that God selects a number in $X$, and let $p(X|Y)$ be the probability that God selects a number in $X$ given that She selects a number in $Y$. One might think that the following is an attractive way of characterizing $p(X)$ and $p(X|Y)$:

$$p(X|Y) =_{df} \lim_{n\to\infty} \frac{|X \cap Y \cap \{1,2,\ldots,n\}|}{|Y \cap \{1,2,\ldots,n\}|}$$

$$p(X) =_{df} p(X|\mathbb{Z}^+)$$

(Notice that $p(X)$ is finitely additive but not countably additive, since $p(\mathbb{Z}^+) = 1$ but $p(\{k\}) = 0$ for each $k \in \mathbb{Z}^+$. Notice also that $p$ is not well-defined for arbitrary sets of integers, since $\lim_{n\to\infty} \frac{|X \cap \{1,2,\ldots,n\}|}{|\{1,2,\ldots,n\}|}$ is not generally well-defined; for instance, it is not well-defined when $X$ consists of the integers $k$ such that $2^m \leq k < 2^{m+1}$, for some even $m$.)

As it turns out, $p(X)$ has the following awkward property: there is a set $S$ and a partition $E_i$ of $\mathbb{Z}^+$ such that $p(S) = 0$ even though $p(S|E_i) \geq 1/2$ for each $E_i$.

For example, let $S$ be the set of perfect squares, and for each $i$ which is not a power of any other positive integer, let $E_i$ be the set of powers of $i$. In other words:

$$\begin{array}{ccc} S & = & \{1, 4, 9, 16, 25, \ldots\} \\ E_1 & = & \{1\} \\ E_2 & = & \{2, 4, 8, 16, 32, \ldots\} \\ E_3 & = & \{3, 9, 27, 81, 243, \ldots\} \\ [\text{No } E_4, \text{ since } 4 = 2^2] & & \\ E_5 & = & \{5, 25, 125, 625, 3125, \ldots\} \\ & \vdots & \end{array}$$

It is easy to verify that $p(S|E_1) = 1$ and $p(S|E_n) = 1/2$ for each $n > 1$, even though $p(S) = 0$.

To see why this is awkward, imagine that Susan's credences are given by $p$. Then Susan can be put in the following situation: there is a sequence of bets such that Susan thinks she ought to take each of the bets, but such that she believes to degree 1 that she will lose money if she takes them all.

Here is one way of spelling out the details. For each $E_i$ Susan is offered the following bet:

> $\mathbf{B_{E_i}}$ **:**
> Suppose God selects a number in $E_i$. Then you'll receive \$2 if the selected number is in $S$, and you'll be forced to pay \$1 if the selected number is not in $S$. (If the selected number is not in $E_i$, then the bet is called off, and no money exchanges hands.)

We know that $p(S|E_i) \geq 1/2$ for each $E_i$. So the expected value of $B_{E_i}$ on the assumption that the selected number is in $E_i$ will always be positive. (It is at least $\$2 \cdot 1/2 - \$1 \cdot 1/2 = \$0.5$.) So Susan should be willing to accept $B_{E_i}$ for each $E_i$. Notice, however, that Susan believes to degree 1 that if she does accept all such bets, she'll loose money. For Susan believes to degree 1 that God will select a number $x$ outside $S$ ( $p(S) = 0$). And if $x$ is outside $S$ then Susan looses \$1 on bet $B_{E^x}$ (where $E^x$ is the $E_i$ such that $x \in E_i$), with no money exchanging hands on any other bet.

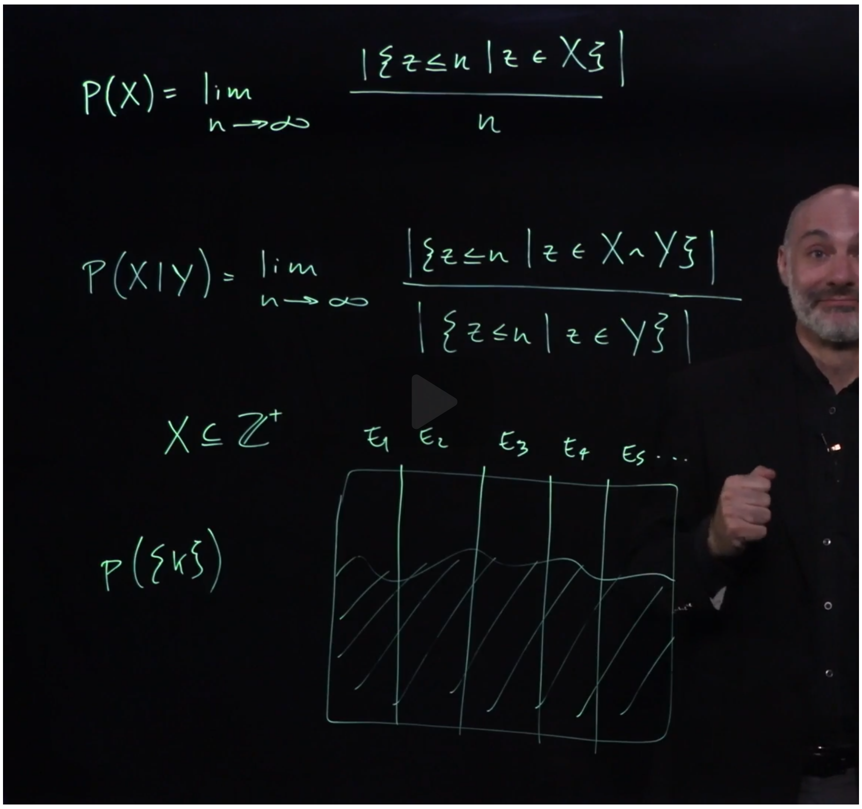
As it turns out, problems of this general form are inescapable: they will occur whenever a probability function on a countable set of possibilities fails to be countably additive. (See Section 6.6 for further details.) This might lead one to think that rejecting Countable

Additivity is less attractive than one might have thought: it is true that doing so allows one to distribute probability uniformly across a countable set of mutually exclusive propositions, but the sorts of distributions one gets are not especially attractive.

My own view is that, on balance, it is best to accept Countable Additivity. But it is important to be clear that the issues here are far from straightforward.

---

## Video Review: Rejecting Countable Additivity




14:25 / 14:25 1.50x

countable additivity is even worse than the cost of having

it.

So my own view is that we should just accept countable identity

and live with the fact that we cannot have uniform

**distributions of probability over countably infinite sets.**

End of transcript. Skip to the start.

### Video

Download video file

### Transcripts

**Download SubRip (.srt) file**

**Download Text (.txt) file**

# Discussion

Hide Discussion

**Topic:** Week 6 / An awkward consequence of rejecting Countable Additivity

Add a Post

All Posts

## Problem is with "infinite number of bets"

discussion posted 2 days ago by **denisos**

For any finite k, it is trivially true that over the group of integers up to k, P(S) = sum (P(Ei) * P(SIEi) if the Ei are independent. For example, if k = 10, then, this sum = 0.1*1 + 0.3*0.3333 + 0.2*0.5 + 0 + 0.1 * 0 + 0.1*0 + 0.1*0 + 0.1*0 + 0 + 0 + 0.1*0 = 0.3, which is correct (1, 4, 9 are 0.3 of the integers between 1-10). Interestingly, P(SIEi) is = 0.5 for only 1 of the 10 i's, and mostly it = 0.

Something goes wrong as k -> infinity. Here is my best guess at what it is:

As k (the set of all the integers from which God has to select one) gets very large, it becomes increasingly likely that the element God will select, let's say m, will belong to Em. (for example, 5 is in E5, 6 is in E6, 7 is in E7, ... ) and so the probability that the number chosen, m, is in Em for any m approaches 1.

So now, if you make an infinite number of bets, all the bets but one are trivially null and void - for any n not = m, then m is not in En, and so the bet is off.

There is always one case in which m is in Ei - but as k goes to infinity, it is increasingly likely that this m is just Em.

So if you say that you can claim that P(S) = 0 for any m, then logically P(Em) is also = 0. So by exactly the same logic that you used to claim P(S) = 0, you could argue that the probability that you actually make the bet Em is zero.

To put it in simple words:

1. P(S) = 0, but if you painstakingly count through all the infinite numbers, you will find numbers which are squares, so for those, individually, the payoff would be 1. It is only that the proportion of these relative to non-squares gets infinitesimally small as the total number of integers in the set approaches infinity.

2. If you painstakingly go through all the infinite Ei's and make the bet for each Ei, you will find a case where the chosen number is in Ei and for which, therefore P(SIEi) = 0.5, but the probability that you will actually make this bet is 0, because as the total number of integers in the set approaches infinity, the chances that a chosen number is in any Ei approaches zero.

I could probably write this a lot more eloquently and coherently if I thought about it. But I am convinced that, once again, the paradox arises from considering infinity inconsistently in the two cases. If you cannot sum up all the infinite probabilities one by one, then you cannot presume to make all the infinite number of bets one by one.

This post is visible to everyone.

Add a Response

1 response

**tschrans**
2 days ago

What if I told you that I can guarantee you that you can go in a casino and play roulette and walk out with a net gain of 1 dollar.

I presume you mean the scheme where you bet 1 dollar at even money. If you win, you've made a dollar. If not, you bet two dollars next time. If you win, you've made a dollar, if not, you bet 4 dollars next time ... Ignoring the 0's, the odds that you win 1 dollar are 1 + 1/2 + 1/4 ... which sums up to 1 - so you're certain to win 1 dollar.

And we could even imagine an simulated roulette wheel where you'd input your strategy and it would keep running until you finally won, and pay you out one dollar.

The flaw in the logic, though, is that you potentially have to bet an infinite amount of money to gain this one dollar. And if at some point before infinity the simulation were to stop, or the casino were to close, there is a small but finite chance that you could owe a humungous amount of money.

As the number of games goes to infinity, your chances of not winning go to 0, but the amount you owe if you don't win also goes to infinity. It's another case where people use the convenient infinity in one sense (the number of times you can play) but ignore it in the other sense (the amount of money you have to risk).

posted a day ago by **denisos**

That's the reason for casinos having a limit to the amount you can bet on roulette.

posted a day ago by **Jimbof**

You got it. Those are the drawbacks. Assuming casinos are open 24/7 and they have no limits. You can calculate the probabilities of how large your bank account needs to be and what the EV of your bank account needs to be.

Note that this is valid no matter how many zeros there are, so technically you don't even need even money. Even money is just better to get to your 1 dollar win sooner. Look at it this way

First bet I play 1 and lose 1

2nd bet I play 2 and lose so I've lost so far 3

3rd bet I play 4 and lose, so I've lost 7

....

nth bet I play $2^{n-1}$ and if I lose I've lost $\sum_{k=1}^{n} 2^{k-1} = \sum_{k=0}^{n-1} 2^{k-2} = \frac{2^n-1}{2-1} = 2^n - 1$

n+1 beth you beth $2^n$ or 1 more than what you've lost so far.

It doesn't matter what the probabilities are at some point you win because the probability of winning is not zero.

You should try it by betting on zero. You'll win much more than 1. Hopefully there is enough money available in the universe to survive this.

posted a day ago by **tschrans**

interesting to ponder all this stuff.

unrelated story: there was a guy in my PhD engineering class who was very passionate about horse-racing. He developed an algorithm which pretty much guaranteed a profit. It wasn't based on any clever trick, or even on infinite series, but more on human psychology. People go to the racetrack to have fun. If you have a race with a very strong favourite, say 5-to-1 on favourite, then it's not much fun to bet on that horse. If you bet $10, you only win 2. Where's the fun in that. So, statistically, he showed that horses that are strong favourites are actually under-valued. Say that horse that is 5-to-1 on (should win 5 times out of 6) is realistically probably likely to win 10 times out of 11). So if you keep backing the strong favourites, you have a net positive income. However, even he, going to the track, wouldn't follow his strategy, because it's more fun to just read the form and try to pick an outsider, and try (and usually fail) to outsmart the wisdom of the crowd.

posted about 11 hours ago by **denisos**

He let his emotional thinking take over his rational thinking. He'll die poor and happy.

posted about 4 hours ago by **tschrans**

Add a comment

Showing all responses

## Add a response:

Preview

Submit