14. Solving the companion system of the coupled oscillator

Let us now solve the companion system of the unforced coupled oscillator:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \begin{pmatrix}0 & \mathbf{I}\\ \mathbf{B} & 0\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \qquad \text{where} \quad \mathbf{B} = \omega^2\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix}.$$

The first step is to find the eigenvalues and eigenvectors of the companion matrix $\begin{pmatrix}0 & \mathbf{I}\\ \mathbf{B} & 0\end{pmatrix}$. We could work out the characteristic polynomial directly, but this involves computing the determinant of a 4×4 matrix. But in fact, because this is a companion matrix, its eigenvalues and eigenvectors can be written in terms of those of the smaller matrix $\mathbf{B}$ within it. Let us inspect the following eigenvalue-eigenvector equation:

$$\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{B} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \lambda\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}.$$

To calculate the left hand side, we can use multiplication of block matrices, which works like usual matrix multiplication:

$$\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{B} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \quad = \quad \begin{pmatrix}\mathbf{y}\\ \mathbf{Bx}\end{pmatrix}$$

Hence, the eigenvalue-eigenvector equation becomes

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{Bx} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \iff \begin{cases} \mathbf{y} & = \lambda \mathbf{x} \\ \mathbf{Bx} & = \lambda \mathbf{y}. \end{cases}$$

Now we perform a trick: we plug the first equation into the second to eliminate $\mathbf{y}$:

$$\mathbf{Bx} = \lambda^2 \mathbf{x}.$$

This says that $\mathbf{x}$ is an eigenvector of $\mathbf{B}$ with eigenvalue λ^2. Now we can solve for $\mathbf{y}$, namely $\mathbf{y} = \lambda \mathbf{x}$ and $\begin{pmatrix} \mathbf{x} \\ \lambda \mathbf{x} \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$.

The good news is that this procedure gives all the eigenvalues and eigenvectors of the 4×4 matrix.

The matrix $\mathbf{B}$ has two eigenvalues and eigenvectors. Let the eigenvalues be called λ_1^2, λ_2^2, and the associated eigenvectors be $\mathbf{v}_1$, $\mathbf{v}_2$. As long as λ_1 and λ_2 are distinct and non-zero, the matrix $\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$ will have four eigenvalues:

$$\lambda_1, \quad -\lambda_1, \quad \lambda_2, \quad -\lambda_2$$

with corresponding eigenvectors

$$\begin{pmatrix} \mathbf{v}_1 \\ \lambda \mathbf{v}_1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{v}_1 \\ -\lambda \mathbf{v}_1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{v}_2 \\ \lambda \mathbf{v}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{v}_2 \\ -\lambda \mathbf{v}_2 \end{pmatrix}.$$

On the next page, we will carry out the computations and see that in this example, $\mathbf{B}$ does have distinct non-zero eigenvalues.

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