Problem 2: Calculation with PDFs

(3/3 points)

Let X be a random variable that takes non-zero values in $[1, \infty)$, with a PDF of the form

$$f_X(x) = \begin{cases} \frac{c}{x^3}, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let U be a uniform random variable on $[0, 2]$. Assume that X and U are independent.

1. What is the value of the constant c?

$c =$ 2 ✔ **Answer:** 2

2.

$\mathbf{P}(X \leq U) =$ 1/4 ✔ **Answer:** 0.25

3. Find the PDF of $D = 1/X$. Express your answer in terms of d using standard notation .

For $0 \le d \le 1$, $f_D(d) =$ 2*d ✔ **Answer:** 2*d

Answer:

1. The distribution must integrate to 1. Since

$$\int_1^\infty \frac{c}{x^3}\,dx = -\frac{c}{2x^2}\Big|_1^\infty = \frac{c}{2},$$

we have $c = 2$.

2. Since $X \ge 1$ and $U \le 2$, the event of interest occurs only if $1 \le X \le U \le 2$. Using the law of total probability and the independence of X and U, we have

$$\begin{aligned}\mathbf{P}(X \le U) &= \int_1^2 \mathbf{P}(X \le u)\, f_U(u)\, du \\ &= \int_1^2 \left(\int_1^u f_X(x)\, dx\right) f_U(u)\, du \\ &= \int_1^2 \left(\int_1^u \frac{2}{x^3}\, dx\right) \frac{1}{2}\, du\end{aligned}$$

$$= \int_1^2 \left(1 - \frac{1}{u^2}\right) \cdot \frac{1}{2}\, du$$
$$= \frac{1}{4}.$$

3. Since X takes values in $[1, \infty)$, D takes values in $[0, 1]$. We use the method of derived distributions to find the CDF of D. For $0 \le d \le 1$,

$$\begin{aligned} F_D(d) &= \mathbf{P}(D \le d) \\ &= \mathbf{P}(X \ge 1/d) \\ &= \int_{1/d}^{\infty} \frac{2}{x^3}\, dx \\ &= d^2. \end{aligned}$$

The complete CDF of D is

$$F_D(d) = \begin{cases} 0, & \text{if } d < 0, \\ d^2, & \text{if } 0 \le d \le 1, \\ 1, & \text{if } d > 1. \end{cases}$$

Differentiating the CDF gives the PDF of D:

$$f_D(d) = \begin{cases} 2d, & \text{if } 0 \le d \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, the same result can be achieved without explicitly computing the CDF of D. We express the CDF of D in terms of the CDF of X and use the chain rule of differentiation. For $0 \le d \le 1$,

$$\begin{aligned} F_D(d) &= \mathbf{P}(D \le d) \\ &= \mathbf{P}(X \ge 1/d) \\ &= 1 - F_X(1/d), \\ f_D(d) &= -f_X(1/d) \cdot \frac{-1}{d^2} \\ &= -2d^3 \cdot \frac{-1}{d^2} \\ &= 2d. \end{aligned}$$

Hence, we obtain the same PDF:

$$f_D(d) = \begin{cases} 2d, & \text{if } 0 \le d \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

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