



## MITx: 6.041x Introduction to Probability - The Science of Uncertainty




Bookmarks

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Unit overview

Lec. 5: Probability mass functions and expectations

Exercises 5 due Mar 02, 2016 at 23:59 UTC 

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## Problem 5: Indicator variables

(6/6 points)

Consider a sequence of independent tosses of a biased coin at times  $k = 0, 1, 2, \dots, n$ . On each toss, the probability of Heads is  $p$ , and the probability of Tails is  $1 - p$ .

A reward of one unit is given at time  $k$ , for  $k \in \{1, 2, \dots, n\}$ , if the toss at time  $k$  resulted in Tails and the toss at time  $k - 1$  resulted in Heads. Otherwise, no reward is given at time  $k$ .

Let  $R$  be the sum of the rewards collected at times  $1, 2, \dots, n$ .

We will find  $\mathbf{E}[R]$  and  $\mathbf{var}(R)$  by carrying out a sequence of steps. Express your answers below in terms of  $p$  and/or  $n$  using standard notation. Remember to write '\*' for all multiplications and to include parentheses where necessary.


We first work towards finding  $\mathbf{E}[R]$ .

1. Let  $I_k$  denote the reward (possibly 0) given at time  $k$ , for  $k \in \{1, 2, \dots, n\}$ . Find  $\mathbf{E}[I_k]$ .


$$\mathbf{E}[I_k] = (1-p)*p$$

Answer:  $p*(1-p)$

**Lec. 6: Variance;  
Conditioning on an event;  
Multiple r.v.'s**

Exercises 6 due Mar 02, 2016 at 23:59 UTC 


**Lec. 7: Conditioning on a random variable;  
Independence of r.v.'s**

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**Solved problems**

**Additional theoretical material**

**Problem Set 4**

Problem Set 4 due Mar 02, 2016 at 23:59 UTC 

**Unit summary**

- ▶ Exam 1
- ▶ Unit 5: Continuous random variables
- ▶ Unit 6: Further topics on random variables
- ▶ Unit 7: Bayesian

2. Using the answer to part 1, find  $\mathbf{E}[R]$ .

$$\mathbf{E}[R] = \boxed{n \cdot (1-p) \cdot p} \quad \checkmark \quad \text{Answer: } n \cdot p \cdot (1-p)$$

The variance calculation is more involved because the random variables  $I_1, I_2, \dots, I_n$  are not independent. We begin by computing the following values.

3. If  $k \in \{1, 2, \dots, n\}$ , then

$$\mathbf{E}[I_k^2] = \boxed{(1-p) \cdot p} \quad \checkmark \quad \text{Answer: } p \cdot (1-p)$$

4. If  $k \in \{1, 2, \dots, n-1\}$ , then

$$\mathbf{E}[I_k I_{k+1}] = \boxed{0} \quad \checkmark \quad \text{Answer: } 0$$

5. If  $k \geq 1$ ,  $\ell \geq 2$ , and  $k + \ell \leq n$ , then

$$\mathbf{E}[I_k I_{k+\ell}] = \boxed{(1-p)^2 \cdot p^2} \quad \checkmark \quad \text{Answer: } p^2 \cdot (1-p)^2$$

6. Using the results above, calculate the numerical value of  $\text{var}(R)$  assuming that  $p = 3/4$ ,  $n = 10$ .

$$\text{var}(R) = \boxed{0.890625} \quad \checkmark \quad \text{Answer: } 0.890625$$

## inference

- ▶ Exam 2
- ▶ Unit 8: Limit theorems and classical statistics
- ▶ Unit 9: Bernoulli and Poisson processes
- ▶ Unit 10: Markov chains
- ▶ Exit Survey
- ▶ Final Exam

Answer:

1. Since  $I_k$  is a Bernoulli indicator variable and the tosses are independent, we have

$$\mathbf{E}[I_k] = \mathbf{P}(I_k = 1) = \mathbf{P}(\text{Tails at time } k \text{ and Heads at time } k - 1) = p(1 - p).$$

2. The total reward over all the tosses,  $R$ , is the sum of all the  $I_k$ 's, for  $k = 1, 2, \dots, n$ . By linearity of expectations, we have

$$\mathbf{E}[R] = \mathbf{E}\left[\sum_{k=1}^n I_k\right] = \sum_{k=1}^n \mathbf{E}[I_k] = np(1 - p).$$

3. Since  $I_k$  can be only 0 or 1,  $\mathbf{E}[I_k^2] = \mathbf{E}[I_k] = p(1 - p)$ .

4.  $I_k I_{k+1}$  equals 1 if  $I_k = 1$  and  $I_{k+1} = 1$ , i.e., if a reward was given at time  $k$  and at time  $k + 1$ . Otherwise,  $I_k I_{k+1}$  equals 0. But  $I_k$  and  $I_{k+1}$  cannot both equal 1:  $I_k = 1$  means that the toss at time  $k$  resulted in Tails, while  $I_{k+1} = 1$  means that the toss at time  $k$  resulted in Heads. Hence, it is not possible to obtain a reward at consecutive times  $k$  and  $k + 1$ . Therefore,  $\mathbf{E}[I_k I_{k+1}] = 0$ .

5. Part 4 above considered the rewards at two consecutive times. We now consider the rewards at two times that are at least 2 periods apart. Since the reward at time  $k$  depends only on the tosses at times  $k$  and  $k - 1$ , the rewards at times that are at

least 2 periods apart depend on different, non-overlapping pairs of coin tosses, and hence  $I_k$  and  $I_{k+\ell}$  are independent for  $\ell \geq 2$ . Therefore,  
 $\mathbf{E}[I_k I_{k+\ell}] = \mathbf{E}[I_k] \mathbf{E}[I_{k+\ell}] = p^2(1-p)^2$  for the values of  $k$  and  $\ell$  specified in the problem statement for this part.

6. From part 2, we have already calculated  $\mathbf{E}[R]$ . We now find  $\mathbf{E}[R^2]$  and use the identity  $\text{var}(R) = \mathbf{E}[R^2] - (\mathbf{E}[R])^2$ .

$$\mathbf{E}[R^2] = \mathbf{E} \left[ \left( \sum_{k=1}^n I_k \right) \left( \sum_{m=1}^n I_m \right) \right] = \mathbf{E} \left[ \sum_{k=1}^n \sum_{m=1}^n I_k I_m \right] = \sum_{k=1}^n \sum_{m=1}^n \mathbf{E}[I_k I_m]$$

There are  $n^2$  terms in this double summation. We can divide them into three groups:

1. There are  $n$  terms where  $k = m$ . From part 3, we know that  $\mathbf{E}[I_k I_m] = p(1-p)$  for this case.
2. There are  $n-1$  terms where  $k = m+1$  and another  $n-1$  terms where  $m = k+1$ . From part 4, we know that  $\mathbf{E}[I_k I_m] = 0$  for these cases.
3. The remaining  $n^2 - n - 2(n-1) = n^2 - 3n + 2$  terms are those where  $k$  and  $m$  differ by at least 2. From part 5, we know that  $\mathbf{E}[I_k I_m] = p^2(1-p)^2$  for these cases.

Putting these cases together, we have

$$\mathbf{E}[R^2] = n \cdot p(1-p) + 2(n-1) \cdot 0 + (n^2 - 3n + 2) \cdot p^2(1-p)^2.$$

Therefore,

$$\begin{aligned}\text{var}(R) &= \mathbf{E}[R^2] - (\mathbf{E}[R])^2 \\ &= np(1-p) + (n^2 - 3n + 2)p^2(1-p)^2 - n^2p^2(1-p)^2 \\ &= np(1-p) - (3n-2)p^2(1-p)^2.\end{aligned}$$

When  $p = 3/4$  and  $n = 10$ , we have that  $\text{var}(R) = 57/64 = 0.890625$ .

*You have used 1 of 3 submissions*

## DISCUSSION

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