8 Polya Theorem

8-1 The Plight of Burnside Lemma

组合数学 Combinatorics

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What is Group

群(group)

Set *G* and **binary operation of** *G*, satisfy the following rules is called group.

(a)封闭性(Closure)：

(b)结合律(Associativity)：

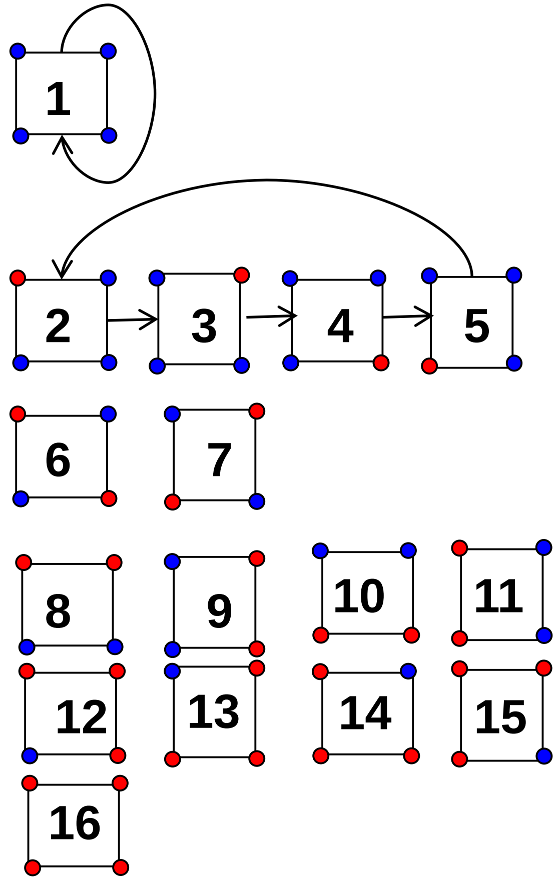
(c)有单位元(Identity)：

(d)有逆元(Invertibility)：

Abstract and corresponding to reality

Rotation Group

- Square vertices of the rotation group
- $G=\{a_1,a_2,a_3,a_4\}$
  - No movement (Rotate 0 degree):
    - a_1=(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)
  - Rotate *90* degree clockwise:
    - a_2= (1)(2 3 4 5)(6 7)(8 9 10 11)(12 13 14 15)(16)
  - Rotate *180* degree clockwise:
    - a_3=(1)(2 4)(3 5)(6)(7)(8 10)(9 11)(12 14)(13 15)(16)
  - Rotate *270* degree clockwise：
    - a_4= (1)(2 5 4 3)(6 7)(8 11 10 9)(12 15 14 13)(16)



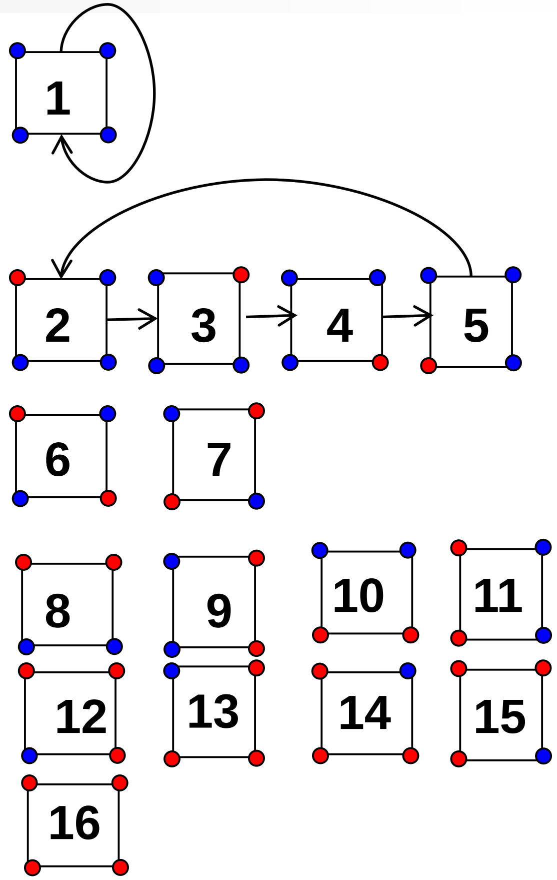
Burnside Lemma

- Burnside Lemma(1897)
  - Set G={a1,a2,…ag} is the permutation group of the target group [1,n]. Each permutation is written into the product of disjointed cycles. $c_1(a_k)$ is number of the fixed points which are not changed by permutation a_k, which is the number of cycles with the length of 1. G decomposed [1, n] into l number of equivalence class. The number of equivalence classes is:

$$l = \frac{1}{|G|}\sum_{j=1}^{g} c_1(a_j)$$

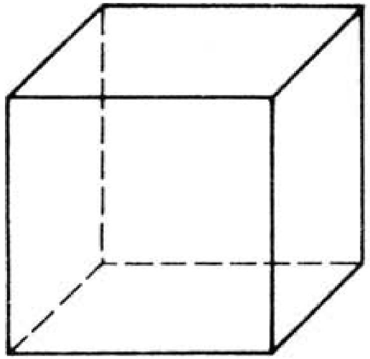
•$l=[c_1(a_1)+c_1(a_2)+\ldots+c_1(a_g)]/|G| = \frac{1}{4}\sum_{f\in G}(16+2+4+2)=6$

- Coloring 2 vertices of square with the consideration of only the rotations, the number of equivalence classes is
- $|G|$: Number of Permutations
  - Consider only rotation: 4
- $c_1(f)$: Number of fixed points
  - No movement: (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)
  - Rotate 90 degree: (1)(2 3 4 5)(6 7)(8 9 10 11)(12 13 14 15)(16)
  - Rotate 180 degree: (1)(2 4)(3 5)(6)(7)(8 10)(9 11)(12 14)(13 15)(16)
  - Rotate 270 degree: (1)(2 5 4 3)(6 7)(8 11 10 9)(12 15 14 13)(16)

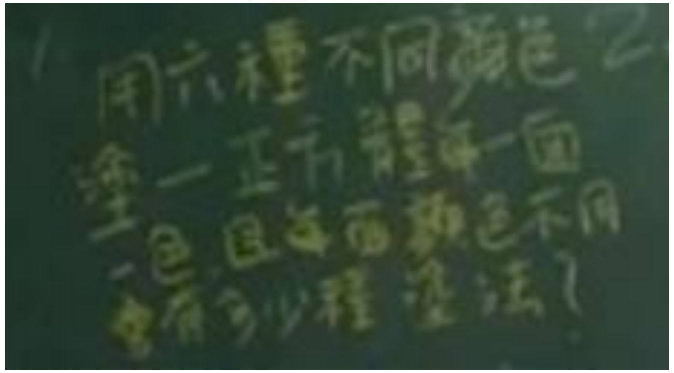


Permutation and Combination on the Blackboard

Use 6 different colors to paint a cube, each side one color and each side is using different color; how many different kinds of coating are there?



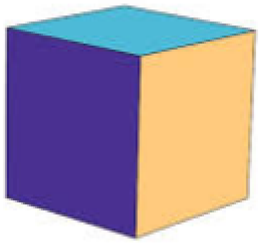
$6*5*P(4,4)/4\ /\ 6 = 30$

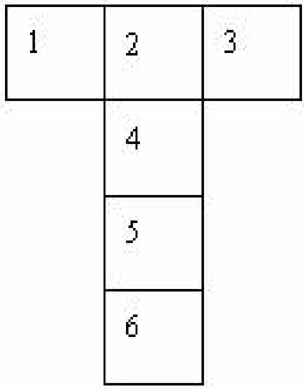


Burnside Lemma?

$$l = \frac{1}{|G|}\sum_{j=1}^{g} c_1(a_j)$$

- Use 6 different colors to paint a cube, each side one color and each side is using different color; how many different kinds of coating are there?
- Rotation group of Color images
- Colored images: 6!
- What are fixed points?





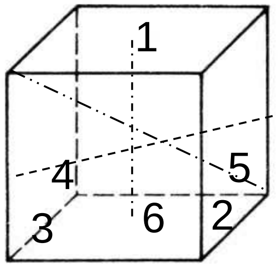
在数学中，函数的**不动点**或**定点**是指被这个函数映射到其自身一个点。例如，定义在实数上的函数f，

$$f(x) = x^2 - 3x + 4，$$

则2是函数f的一个不动点，因为$f(2) = 2$。

  – Color image under rotational permutation unchanged

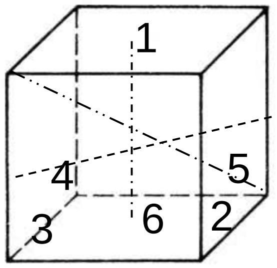
- Hexahedron rotation group: Permutation representation of faces



正六面体

$$l = \frac{1}{|G|}\sum_{j=1}^{g} c_1(a_j)$$

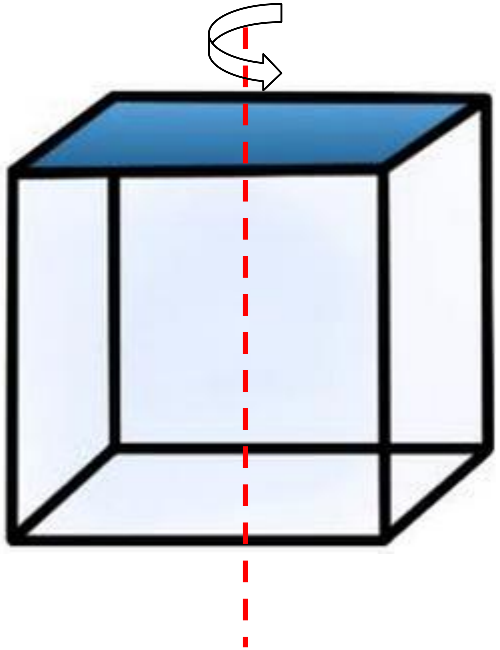
- Use 6 different colors to paint a cube, each side one color and each side is using different color; how many different kinds of coating are there?
- Hexahedron rotation group: Permutation representation of faces
  - No movement: e=(1)(2)(3)…(6!)   1 permutation   **6! fixed points**
  - Centers on two oppositing faces: rotation ±90°   2*3 permutations   no fixed point
  - Centers of two opposite faces rotation180°   3 permutations   no fixed point
  - Middle points on edges 180°   6 permutations   no fixed point
  - Corner to corner (Diagonal axis) rotation ± 120°   2*4 permu
    fixed point
  - The order of Hexahedron rotation group $|G|$=24
- $l=[c_1(e)]/24=6!/24=30$ solutions.



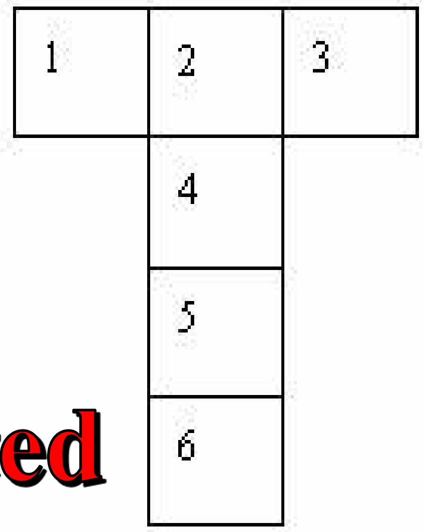
正六面体

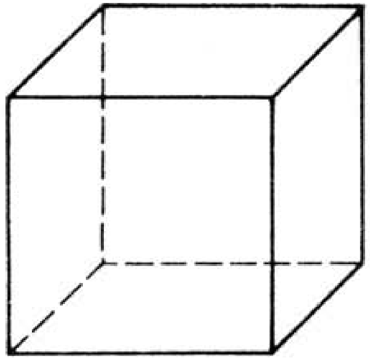
If the color of face allows repetition?

- Use 6 different colors to paint a cube, each side one color and each side is using different color; how many different kinds of coating are there?
- Color image: **6^6=46656**
- Is there fixed points during rotation?



Complicated





Burnside Lemma

$$l = \frac{1}{|G|}\sum_{j=1}^{g} c_1(a_j)$$

- Solve by targeting the rotation group of **colored image set**
- To find the solutions for 2 coloring, may use Burnside Lemma
- But for coloring with multiple colors, and due to its coloring target structure is very complicated, theoretically it can use Burnside for solutions, but it is extremely complicated

Pólya counting Theorem

8 Polya Theorem

8-2 From Burnside to Polya

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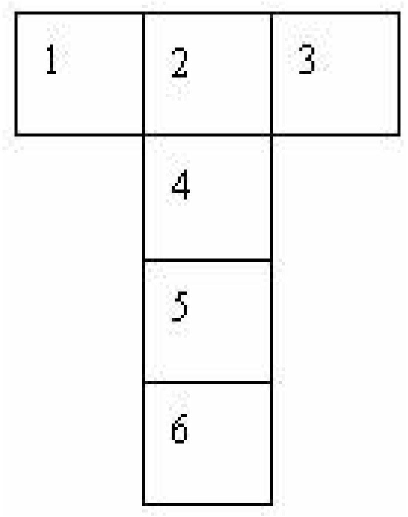
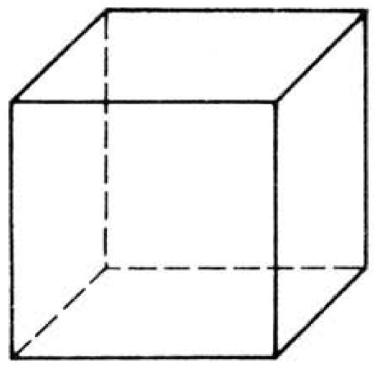
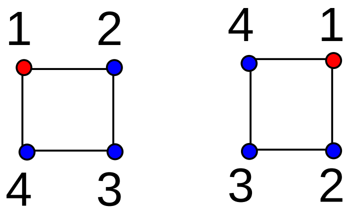
Structure

- Use 6 different colors to paint a cube, each side one color, **different face may use the same color**, how many different kinds of coating are there?
- Color image: **6^6=46656？**
- Is it possible to see the structure of the color target?





Permutation Group Based on Structure Representation

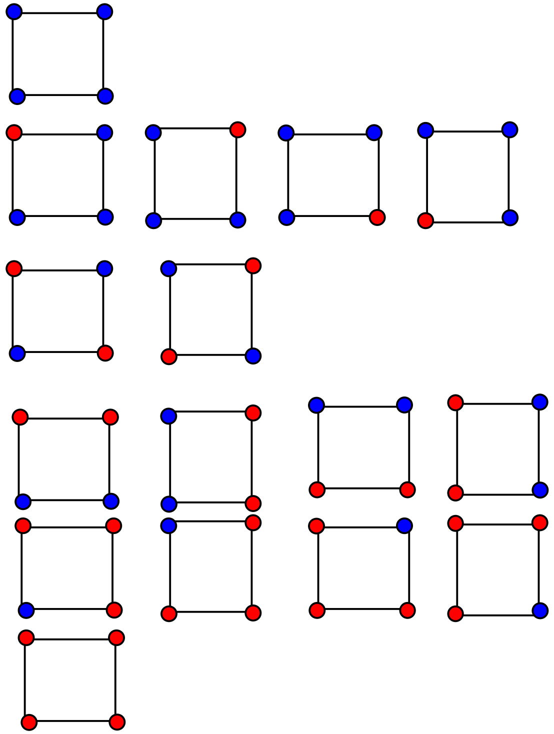


Rotate 90°

$$\begin{pmatrix} 1234 \\ 4123 \end{pmatrix}$$

Cycle Representation of Permutations

（4321）



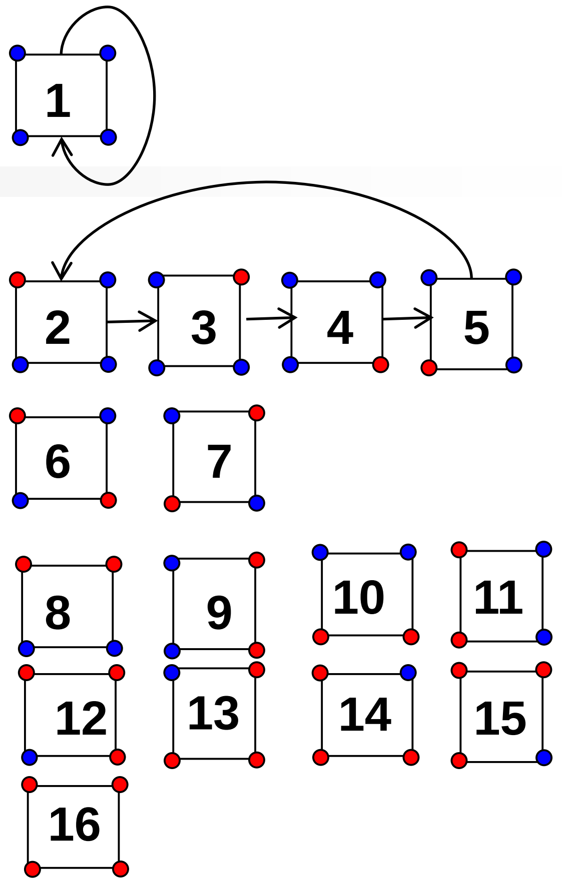
Permutation group of 4 vertices of a square $\bar{G}$

No movement: a_1=(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)

Rotate 90°: a_2= (1)(2 3 4 5)(6 7)(8 9 10 11)(12 13 14 15)(16)

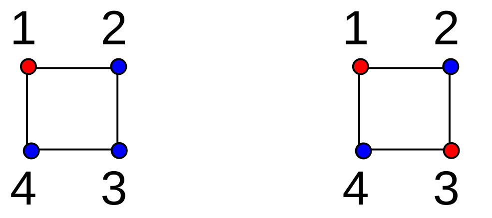
Rotate 180°: a_3= (1)(2 4)(3 5)(6)(7)(8 10)(9 11)(12 14)(13 15)(16)

Rotate 270°: a_4= (1)(2 5 4 3)(6 7)(8 11 10 9)(12 15 14 13)(16)



Permutation group of 4 vertices of a square $\bar{G}$

		$c_1(a_j)$
No movement: $\overline{p_1}$= (1)(2)(3)(4)	2^4	16
Rotate 90°: $\overline{p_2}$= (4 3 2 1)	2^1	2
Rotate 180°: $\overline{p_3}$= (1 2)(3 4)	2^2	4
Rotate 270°: $\overline{p_4}$= (1 2 3 4)	2^1	2



Color with the same color in the same cycle, the image unchanged after rotation.

$$c_1(a_j) = m^{C(\overline{p_j})}$$

Burnside Lemma

Set G={a1,a2,…ag} is the permutation group of the target group [1,n]. Each permutation is written into the product of disjointed cycles. $c_1(a_k)$ is number of the fixed points which are not changed by permutation a_k.

$$l = \frac{1}{|G|}\sum_{j=1}^{g} \boxed{c_1(a_j)}$$

Pólya Theorem Set $\bar{G} = \{\overline{p_1}, \overline{p_2}, \ldots, \overline{p_g}\}$ is the permutation group of n **target object**, **$C(\overline{p_k})$** is the cycle number of permutation p_k, use m type of colors to color n targets, the coloring solution number is

$$l = \frac{1}{|\bar{G}|}\sum_{j=1}^{g} \boxed{m^{C(\overline{Pj})}}$$

Pólya Theorem

- Compare Pólya counting theorem to Burnside Lemma
  - The group in Pólya counting is used on top of the permutation group of original *n* target objects
  - The group in Burnside Lemma is the permutation group on top of the solution group for the colored images of n target objects
  - The relation between these 2 groups: the corresponding coloring solutions generated in group G, have induced permutation *p* which belonged to *G*

$$l=[c_1(a_1)+c_1(a_2)+\ldots+c_1(a_g)]/|G|$$

$$\Downarrow \quad c_1(a) = m^{C(\overline{p})}$$

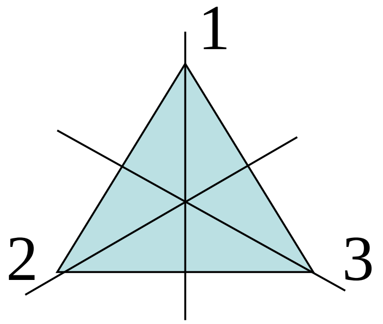
$$l=[m^{C(\overline{P_1})}+m^{C(\overline{P_2})}+\ldots+m^{C(\overline{P_g})}]/|\overline{G}|.$$

$$l = \frac{1}{|\overline{G}|}\sum_{j=1}^{g} m^{C(\overline{P}j)}$$

Example

- **Ex.** Color the 3 vertices of the equilateral triangle with red, blue and green, how many solutions are there?

- **An.** Consider in 3D space, permutation group S3 of 3 vertices.



Rotate $\pm 120^o$ (123); (321)

$(3)^1$: 2;

Symmetry axis flipping (1)(23); (2)(13); (3)(12)

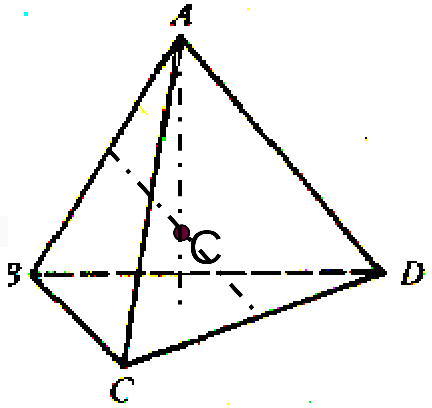
$(1)^1\,(2)^1$: 3;

No movement (1)(2)(3)

$(1)^3$: 1;

- $l = (2 \cdot 3^1 + 3 \cdot 3^2 + 3^3)/6 = 10$

$$l = \frac{1}{|\overline{G}|}\sum_{j=1}^{g} m^{C(\overline{Pj})}$$



Example

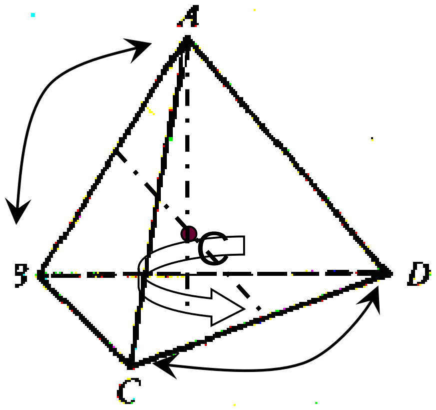
- **Ex.** Random usage of 4 keys in Methane CH_4

  Link H(氢),Cl(氯),CH_3(甲基), C_2H_5(乙基) , how many ways?

- **An.** The structure of CH_4 is a 4-sided space, atom C is located in the center of the 4-sided space. The rotation group of 4-sided object is classified by axis of rotation:
- No movement: (A)(B)(C)(D) Total of 1 cycle $(1)^4$
- Made the top vertice and the opposite center as shaft, rotate ±120°clockwise. Permute (BCD) and (BDC). Permute (ACD) and (ADC), permute (ABD) and (ADB), (ABC) and (ACB) corresponding rotation. Total of 8 $(1)^1(3)^1$cycles.
- Made 3 pairs of edge point line of tetrahedron as shaft rotate 180°: (AB)(CD), (AC)(BD), (AD)(BC), total of 3 $(2)^2$ cycles
- According to Polya Theorem: $(1*4^4+8*4^2+3*4^2)/12=36$

Example

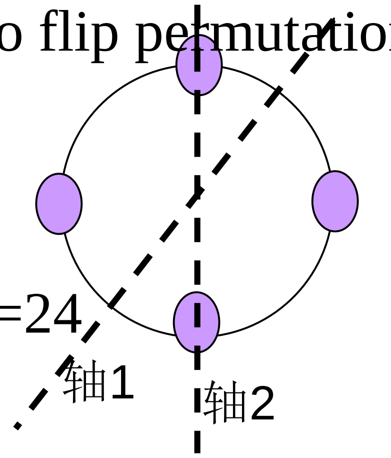
- Is it a must to list all the permutation?
- Vertex-Opposite fece center:
  - $(1)^1(3)^1$ every vertices $\pm 120^\circ$, 4 vertices, total of 8;
- Edge-Edge：
  - $(2)^2$ 3 pairs middle points of edges, total of 3;
- No movement:
  - $(1)^4$ 1;
- Hence, there are 12 group elements of rotation group
- l=1/12[8*4^2+3*4^2 +4^4]=[44+64]/3=36。



Permutation

Example 3 different colors of beads, string as 4-bead necklace, how many ways?

Turnover flipping makes changes to the original image, namely no flip permutation



Rotate around the center ±90 $(4)^1$ 2

Rotate around the center 180 $(2)^2$ 1

No movement $(1)^4$ 1

Total of 4 permutations, solution number $m=(2*3^1+1*3^2+1*3^4)/4=24$

Turnover coincidence regarded as the same

Rotate around the center ±90 $(4)^1$ 2

Rotate around the center 180 $(2)^2$ 1

Flip over axis 1 $(2)^2$ 2

Flip over axis 2 $(1)^2(2)^1$ 2

No movement $(1)^4$ 1

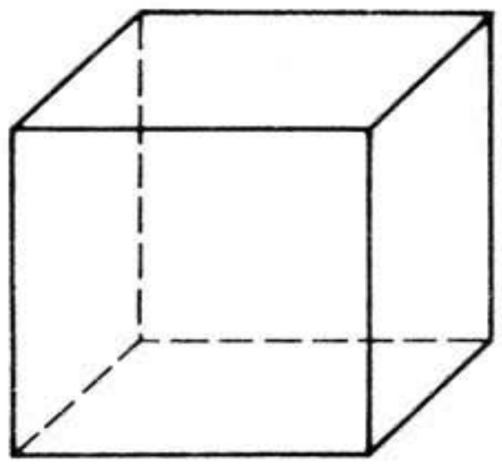
Total of 8 permutations, solution number $m=(2*3^1+1*3^2+2*3^2+2*3^3+1*3^4)/8=21$

Permutation

- Target objects are different, corresponding permutation will be different
  - Flat Surface
    - In 2D space, rotate
    - In 3D space, consider rotation and flip
    - Chess board: Single-sided or double-sided?
  - Cube
    - Rotation
  - Number permutation
  - Dot permutation
  - ……..

Example

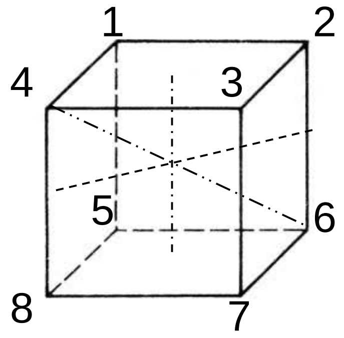
- Rotation group of hexahedron
  - Face Permutation 6 faces
  - Vertex Permutation 8 vertices
  - Edge Permutation 12 edges



正六面体

Example

- Hexahedron rotation group: Permutation representation of vertices
  - No movement: $(1)^8$ 1
  - Centers of two opposite faces $\pm 90^\circ$ $(4)^2$ 2*3
  - Centers of two opposite faces 180° $(2)^4$ 3
  - Middle points of two edges: 180° $(2)^4$ 6
  - Rotate around diagonal shaft $\pm 120^\circ$ $(1)^2(3)^2$ 2*4
  - Order number of hexahedron rotation group is 24



正六面体

Example

- **Ex.** Use 2 colors to paint the 8 vertices of hexahedron, how many possible ways?
- Hexahedron Rotation Group: Permutation representation of vertices
  - No movement: $(1)^8$ 1
  - Rotate around face centers $\pm 90^\circ$ $(4)^2$ 2*3
  - Rotate around face centers 180° $(2)^4$ 3
  - Rotate middle points of two edges 180° $(2)^4$ 6
  - Rotate around diagonal shaft $\pm 120^\circ$ $(1)^2(3)^2$ 2*4
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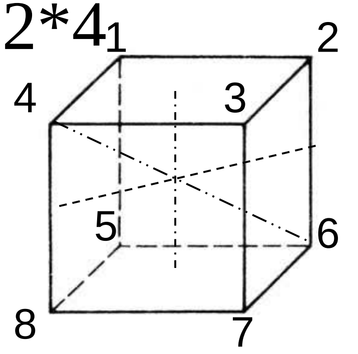
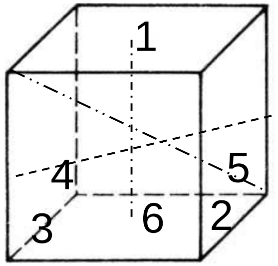


正六面体

- $[17 \cdot 2^4 + 6 \cdot 2^2 + 2^8]/24 = [34+3+32]/3 = 23$

Example

- Hexahedron Rotation Group: Permutation representation of faces
  - No movement: (1)(2)(3)(4)(5)(6) $(1)^6$ 1
  - Rotate around face centers ±90° $(1)^2(4)^1$ 2*3
  - Rotate around face centers 180° $(1)^2(2)^2$ 3
  - Rotate around middle points of two edges 180° $(2)^3$ 6
  - Rotate around diagonal shaft ±120° $(3)^2$ 2*4
  - Order number of hexahedron rotation group is 24



正六面体

Example

- **Ex** Paint 6 faces of hexahedron with red and blue colors, how many possible solutions?
  Answer: Hexahedron rotation group use permutation representation of faces

Hexahedron Rotation Group: Permutation representation of faces

No movement: (1)(2)(3)(4)(5)(6) $(1)^6$ 1

Rotate around face centers ±90° $(1)^2(4)^1$ 2*3

Rotate around face centers 180° $(1)^2(2)^2$ 3

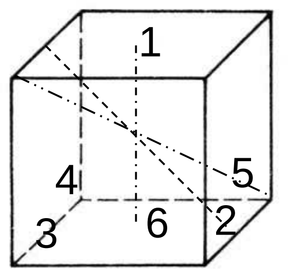
Rotate around middle points of two edges 180°

$(2)^3$ 6

Rotate around diagonal shaft ±120° $(3)^2$ 2*4

Order number of hexahedron rotation group is 24

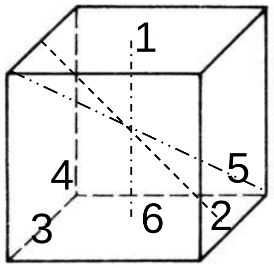
$(2^6+6*2^3+3*2^4+6*2^3+8*2^2)/24 = 10$



正六面体

Use 6 different types of color to paint a cube, each side one color, **different faces may use the same color**, how many ways?

Hexahedron Rotation Group: Permutation Representation of Faces

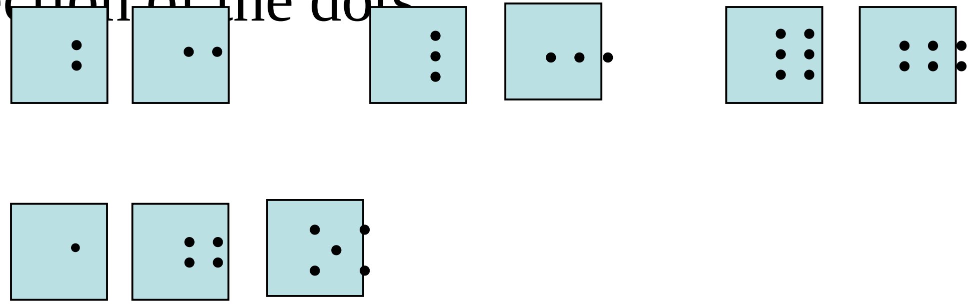


正六面体

- No movement: (1)(2)(3)(4)(5)(6) $(1)^6$ 1
- Rotate around face centers ±90° $(1)^2(4)^1$ 2*3
- Rotate around face centers 180° $(1)^2(2)^2$ 3
- Rotate around middle points of two edges 180° $(2)^3$ 6
- Rotate around diagonal shaft ±120° $(3)^2$ 2*4
- Order number of hexahedron rotation group is 24
- $(6^6+6*6^3+3*6^4+6*6^3+8*6^2)/24$

Example

- **Ex.** 6-sided dice have respectively1,…,6 points each side, how many ways to arrange those points on a dice?
- **An.** Unlike the previous coloring problem, in this question, all 6 colors must all be used
- 1) Hexahedron rotation group contains 24 permutations
- 6! image target group, only the unit possible contains 6! fixed points, the other 23 contains no fixed point.
- From Burnside Lemma, it contains $[c_1(e)]/24=6!/24=30$ solutions. $c_1(p_1)=c_1(p_2)=\ldots=c_1(p_{23})=0$
- 2) Further consideration about the direction of the dots
- 2 pts, 3 pts, 6 pts each have
  2 kinds of orientations
- 1pts, 4pts, 5pts each have
  1 kind of orientation
- Hence, there is $30 \cdot 2^3=240$ solutions



Example

- **Ex.** Draw a diagonal line over the faces of hexahedron, how many ways?
- **An.** There are 2 ways to draw a diagonal line on each face, may reference to the 2 coloring problem.
- But in the permutation of the ±90 rotation of centers of two faces, there is no fixed image. Apart from this, other may follow the 2 coloring problem. The total solution is:

Hexahedron Rotation Group: Permutation Representation

of Face Fixed Image

No movement: (1)(2)(3)(4)(5)(6) $(1)^6$ 1 2^6

Rotate around face centers ±90° $(1)^2(4)^1$ 2*3 0

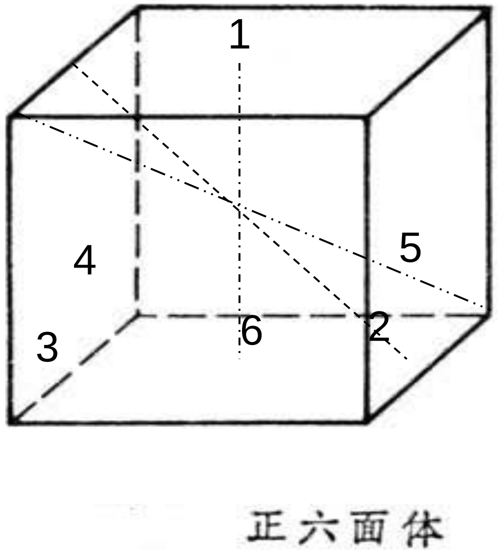
Rotate around face centers 180° $(1)^2(2)^2$ 3 $3*2^4$

Rotate around the middle points of edges: 180° $6*2^3$

$(2)^3$ 6 $8*2^2$

Rotate around diagonal shaft ±120° $(3)^2$ 2*4

Order number of hexahedron rotation group is 24



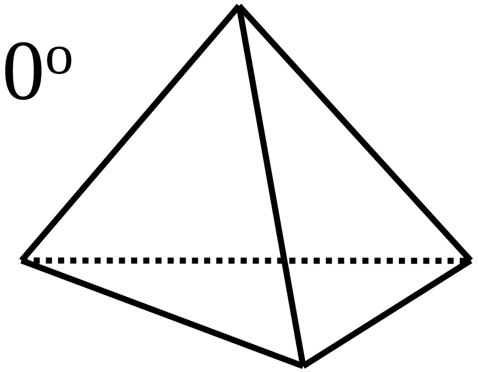
- $[2^6+0+3\cdot 2^4+8\cdot 2^2+6\cdot 2^3]/24=[8+6+4+6]/3=8$

Example

- To solve the calculation problem of polyhedron and some symmetrical polyhedral, the following theorem is introduce.
- **Euler Theorem: The sum number of vertices *v* and faces *f* of any convex polyhedron is always 2 more than the number of edges *e*, which is $v+f-e=2$**
- **The sum inner angle of flat polygon is** $(v\text{-}2)\times 180^o$
- **Definition** The difference between 360^o and convex polyhedron associated sum vertex is known as vertex angle defect.
- **Theorem** The sum of vertex less angle of convex polyhedron is known as 720^o (Proved by Euler Theorem)

Definition The difference between 360^o and angles associated to a vertex is called as the complement angle of this vertex.

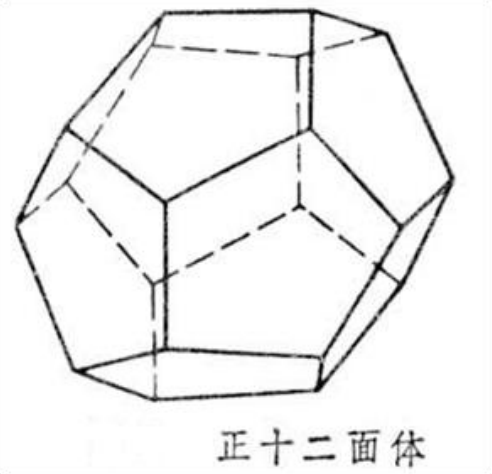
- Tetrahedron
- Each face is triangle-shaped
- Each face angle is 60^o, each vertex has 3 face angle
- Each vertex defect angle =360^o-180^o=180^o
- The sum of 4 vertex defect angle is 180^o*4=720^o



The sum of all inner angle of flat polygon is $(v\text{-}2)\times 180^o$

The difference between 360^o and convex polyhedron associated sum vertex is known as vertex defect angle.

The sum of complement angles for a convex polyhedron is 720^o.



- Use pentagon to form polyhedron:

Inner angle $(5\text{-}2)\cdot 180^o/5=108^o$,

Complement angle $360^o - 3\cdot 108^o = 36^o$。

$720^o / 36^o = 20$(vertices)

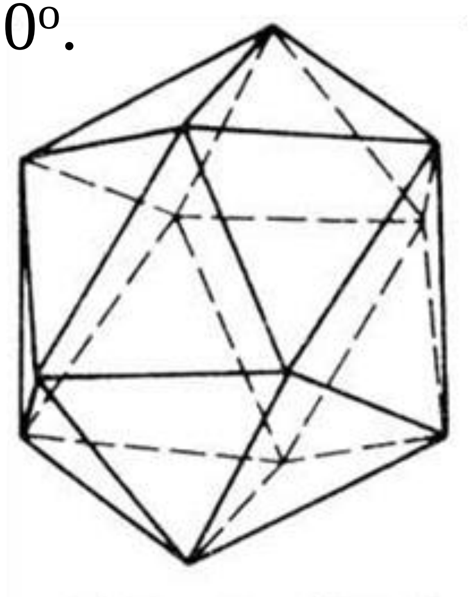
1 vertex 3 edges, the repetition rate 2：$20\cdot 3/2=30$ edges

1 vertex and related 3 faces, repetition rate 5：$20\cdot 3/5=12$ faces

The sum of all inner angle of flat polygon is $(v\text{-}2) \times 180^o$

The difference between 360^o and convex polyhedron associated sum vertex is known as vertex defect angle.

The sum of complement angles for a convex polyhedron is 720^o.



正二十面体

- Use regular triangles to form a regular polyhedron with maximum faces :

Inner angle: 60^o

Complement angle: $360^o - 5 \cdot 60^o = 60^o$ 。

$720^o / 60^o = 12$(vertices)

1 vertex associated with 5 edges, repetition rate 2： $12 \cdot 5/2=30$ edges.

1 vertex associated with 5 faces, repetition rate 3： $12 \cdot 5/3=20$ faces.

Example

- Football：
- Inner angle of pentagon is 108^o, inner angle of hexahedron is 120^o

  Complement angle=360^o –(108^o +2 ·120^o)=12^o

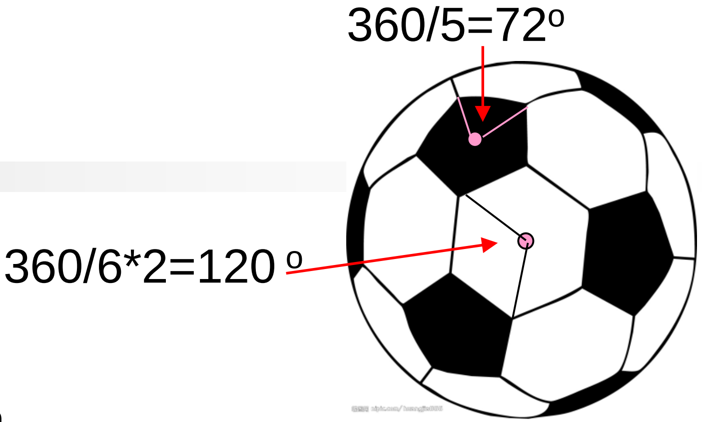
  720 /12 =60 (vertices)

  60 ·3/2=90 (edges)

  60/5=12 (pentagon)

  60 ·2/6=20 (hexagon)

Example



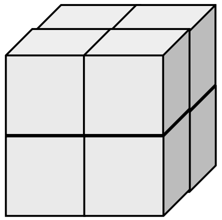
- Use matchstick to build a football, how many ways?
- Refer to 2 coloring of edges,
- Football contains 60 vertices, 90 edges, 12 pentagon, 20 hexagon,
- No rotation $(1)^{90}$ **1**
- Pentagon center to face center rotate n*72º n=1,2,3,4,
  total of 6 pair of face center $(5)^{(90/5)}$, **24**
- Hexagon center to face center rotate n*120º n=1,2,
  total of 10 pairs face center $(3)^{(90/3)}$, **20**
- Made the center point of pentagon to pentagon shape boundary as shaft rotate 180 º, total of 20*3/2/2=15 pairs

  (20 hexagon, each hexagon contains 3 edges, 2 edges contains an axis, 2 pentagons share an edge)

  $(1)^2(2)^{44}$, **15** non-fixed images
- （$2^{90}+24*2^{18}+20*2^{30}$）/60

Example

- Stack with the same dice to form a hexahedron
- Each pentagon contains 24 types of rotation, each dice occupy a vertex, which is equivalence to 24 coloring of vertices
  - No rotation: $(1)^8$ 1
  - Rotate around face center $\pm 90^o$ $(4)^2$ 6
  - Rotate around face center 180^o $(2)^4$ 3
  - Rotate edge to edge 180^o $(2)^4$ 6
  - Rotate around diagonal shaft $\pm 120^o$ $(1)^2(3)^2$ 8
  - $(24^8+6*24^2+3*24^4+6*24^4)/24$



No Fixed Image

组合数学 Combinatorics

8 Polya Theorem

8-3 Generating Function -typed Polya

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Weighted Version of Pólya Theorem

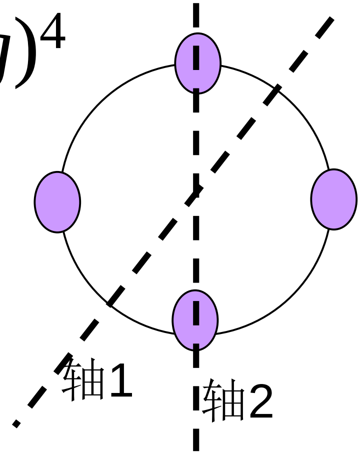
- Pólya Theorem is mainly used for counting
- Generating Function enumerate the status

Weighted version
of Pólya Theorem
(with generating function)

- For example, paint 3 identical balls with 4 colors (Red-r, Yellow-y, Blue-b, Green-g), all the possible color combination is
- $(r+y+b+g)^3=r^3+y^3+b^3+g^3+3r^2y+3r^2g+3ry^2+3y^2b+3y^2g+3rb^2+3b^2g+3rg^2+3yg^2+3bg^2+6ryb+6rbg+6ryg+6ybg$
- All the coefficients is corresponding to the number of coloring schemes.

Weighted Version of Pólya Theorem

- **Ex.** 3 different colors of beads, string into 4 beads of necklace, how many ways?
- **An.** Activity group of tetragon
- Rotate around center ±90 $(4)^1$ 2 $2(r^4+b^4+g^4)$
- Rotate around center 180 $(2)^2$ 1 $(r^2+b^2+g^2)^2$
- Flip around axis 1 $(2)^2$ 2 $2(r^2+b^2+g^2)^2$
- Flip around axis 2 $(1)^2(2)^1$ 2 $2(r+b+g)^2(r^2+b^2+g^2)$
- No movement $(1)^4$ 1 $(r+b+g)^4$
- Total of 8 permutations
- Solution number $m=(2*3^1+1*3^2+2*3^2+2*3^3+1*3^4)/8=21$
- $P(G)=b^4+r^4+g^4+b^3r+br^3+r^3g+bg^3+rg^3+2b^2r^2$
- $+2b^2g^2+2r^2g^2+2b^2rg+2br^2g+2brg^2$
- The solution of 2 black, 1 red, 1 green is exactly 2



Weighted Version of Pólya Theorem

$$l=\frac{1}{|G|}[m^{C(P_1)}+m^{C(P_2)}+\ldots+m^{C(Pg)}].$$

- Set n number of target objects colored with m types of colors： $b_1, b_2, \ldots, b_m$
- $m^{C(pi)}$ use $(b_1+b_2+\ldots+b_m)^{c1(pi)}(b_1^2+b_2^2+\ldots+b_m^2)^{c2(pi)} \ldots (b_1^n+b_2^n+\ldots+b_m^n)^{cn(pi)}$ replace with
- $S_k=(b_1^k+b_2^k+\ldots+b_m^k)$, $k=1,2\ldots, n$ then
- Weighted Version of Pólya Theorem obtains the solution number of:

$$P(G)=\frac{1}{|\overline{G}|}\sum_{j=1}^{g}\prod_{k=1}^{n}S_k^{C_k(\overline{Pj})}$$

Example

- **Ex.** Paint 3 vertices of equilateral triangle with red, blue and green colors, how many possible ways?
- **An.** Consider from 3D space, permutation group S3 of 3 vertices.

$(3)^1$: 2； $2(r^3+b^3+g^3)$

$(1)^1\,(2)^1$: 3； $3(r+b+g)(r^2+b^2+g^2)$

$(1)^3$: 1； $(r+b+g)^3$

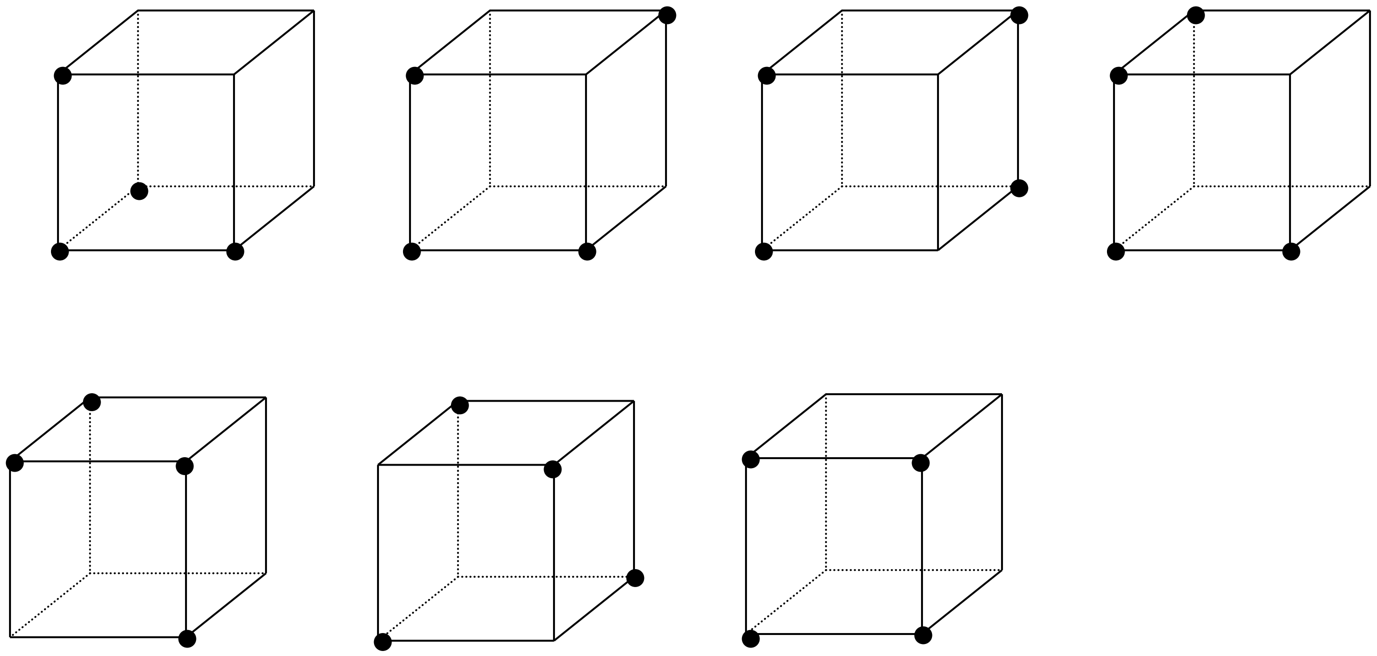
- $l = (2 \cdot 3^1 + 3 \cdot 3^2 + 3^3)/6 = 10$
- The coefficient of r^3 $(2+3+1)/6=1$
- The coefficient of rgb $(3!/(1!1!1!))/6=1$
- The coefficient of r^2b $(3+3!/(2!1!))/6=1$

Weighted Version of Pólya Theorem

- **Ex.** Embed 4 red beads onto the 4 vertices of hexahedron, how many possible way?

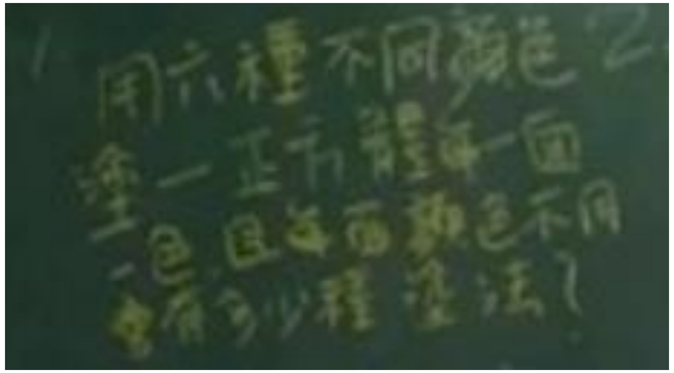
- **An.** Equivalent to paint the vertex with 2 colors: with bead or without beads. Set no bead with b.
- Hexahedron Rotation Group: Permutation Representation of Vertices
  - No movement: $(1)^8$ 1
  - Face center to face center rotate $\pm 90^o$ $(4)^2$ 2*3
  - Face center to face center rotate 180^o $(2)^4$ 3
  - Edge middle point to edge middle point 180^o $(2)^4$ 6
  - Diagonal line as shaft rotate $\pm 120^o$ $(1)^2(3)^2$ 2*4
  - Order number of hexahedron rotation group is 24
  - $p=[(b+r)^8+6(b^4+r^4)^2+9(b^2+r^2)^4+8(b+r)^2(b^3+r^3)^2]/24$
  - Find the coefficient of b^4r^4 $(C(8,4)+12+9*C(4,2)+8*C(2,1)*C(2,1))/24=7$

Weighted Version of Pólya Theorem



Weighted Version of Pólya Theorem

- **Ex.** Use 6 different colors to paint a cube, each side with one color and each face uses different color, how many possible ways?



- **An.** Hexahedron Rotation Group contains 24 permutation
  - No movement: (1)(2)(3)(4)(5)(6) $(1)^6$ 1
  - Face center – face center rotate $\pm 90^o$ $(1)^2(4)^1$ 2*3
  - Face center – face center rotate 180^o $(1)^2(2)^2$ 3
  - Edge center – edge center rotate 180^o $(2)^3$ 6
  - Diagonal line as shaft rotate $\pm 120^o$ $(3)^2$ 2*4
- Written into the generating function format as follow
- $P=[(x_1+x_2+x_3+x_4+x_5+x_6)^6+6(x_1+x_2+x_3+x_4+x_5+x_6)^2(x_1^4+x_2^4+x_3^4+x_4^4+x_5^4+x_6^4)$
  $+3(x_1+x_2+x_3+x_4+x_5+x_6)^2(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)^2$
  $+6(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)^3+8(x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3)^2]/24$

Weighted Version of Pólya Theorem

- **Ex.** Use 6 different colors to paint a cube, each side each color and each face uses different color, how many possible way?
- Written into the generating function format as follow
- $P=[(x_1+x_2+x_3+x_4+x_5+x_6)^6+6(x_1+x_2+x_3+x_4+x_5+x_6)^2(x_1{}^4+x_2{}^4+x_3{}^4+x_4{}^4+x_5{}^4+x_6{}^4)$
  $+3(x_1+x_2+x_3+x_4+x_5+x_6)^2(x_1{}^2+x_2{}^2+x_3{}^2+x_4{}^2+x_5{}^2+x_6{}^2)^2$
  $+6(x_1{}^2+x_2{}^2+x_3{}^2+x_4{}^2+x_5{}^2+x_6{}^2)^3+8(x_1{}^3+x_2{}^3+x_3{}^3+x_4{}^3+x_5{}^3+x_6{}^3)^2]/24$

Find the coefficient of $x_1x_2x_3x_4x_5x_6$

The coefficient which obtained by polynomial expansion of $x_1x_2x_3x_4x_5x_6$ is 6!

Hence, the solution number is 6!/24 =30

Weighted Version of Pólya Theorem

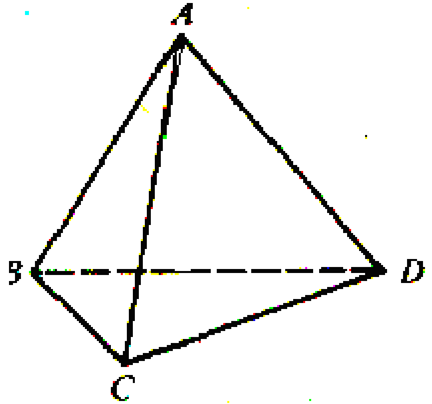
- Ex. Tetrahedron paint with 4 colors, faces with 3 colors, edge with 2 colors, find the solution number

	Point	Face	Edge	Composite
Vertex-Face Center $\pm 120^o$：	$(1)^1(3)^1$	$(1)^1(3)^1$	$(3)^2$	$(x_1)^1(x_3)^1(y_1)^1(y_3)^1(z_3)^2$ 8个
Edge Center - Edge Center：	$(2)^2$	$(2)^2$	$(1)^2(2)^2$	$(x_2)^2(y_2)^2(z_1)^2(z_2)^2$ 3个
No rotation:	$(1)^4$	$(1)^4$	$(1)^6$	$(x_1)^4\ (y_1)^4\ (z_1)^6$ 1个

Hence, the group element of rotation group is 12.

Solution number:

$(8*4^2 3^2 2^2+3*4^2 3^2 2^4+4^4 3^4 2^6)/12$



Weighted Version of Pólya Theorem

- Ex. Put 4 balls *a*, *a*, *b*, *b* into 3 different boxes, find solution number. If empty box is not allowed, how many different arrangement solutions?
- *a*, *a* and *b*, *b* are placed independently
- Find the solution number of placing 2 identical balls into 3 different boxes
- $C(2+3-1,2)=6$
- So, the solution number is 6*6=36
- If does not allow empty box, use inclusions and exclusion theorem
- $C(4,2)^2 - 3*C(3,2)^2+3*C(2,2)^2=36-27+3=12$

Weighted Version of Pólya Theorem

- Put 4 ball *a*, *a*, *b*, *b* into 3 different boxes, find solution number. If does not allow empty box, how many arrangement solutions?
- Answer: Set 4 balls as a_1, a_2, b_1, b_2, put 4 balls into 3 boxes, can image as painting 4 balls with 3 colors
- $G=\{e,(a_1a_2),(b_1b_2),(a_1a_2)(b_1b_2)\}$
- $l=(3^4+2*3^3+3^2)/4=36$
- $P(G)=((r+b+g)^4+2*(r^2+b^2+g^2)(r+b+g)^2+(r^2+b^2+g^2)^2)/4$
- After expansion, get $r^ib^jg^k$ item, $i, j, k>0$
- The coefficient of $r^1b^1g^2$= The coefficient of $r^1b^2g^1$= The coefficient of $r^2b^1g^1$= $(C(4,1)*C(3,1)*C(2,2)+2*C(2,1))/4=4$
- Hence, if does not allow empty box, arrangement solution is 4*3=12

8 Polya Theorem

8-4 Graphical Enumeration

组合数学 Combinatorics

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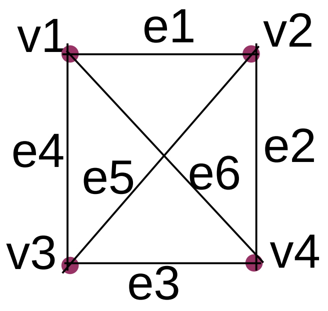
Graphical Enumeration

- The simple graph of n number of vertices contains how many different patterns?
  - Simple graph: there are no more than 1 edge connecting two vertices and there is no ring connecting between vertices
  - The edge of the complete graph of n number of unlabeled vertices perform 2-colors painting, find the number of difference solutions.
  - The number of edge of the complete graph is $n(n-1)/2$

Graphical Enumeration

- The edge of the complete graph of *n* number of unlabeled vertices perform 2-colors painting, find the solution number
- Permutation of Complete Image
  - Edge moves with the vertices
  - Not only rotation or flipping
  - Full permutation of dots, corresponding to symmetry group

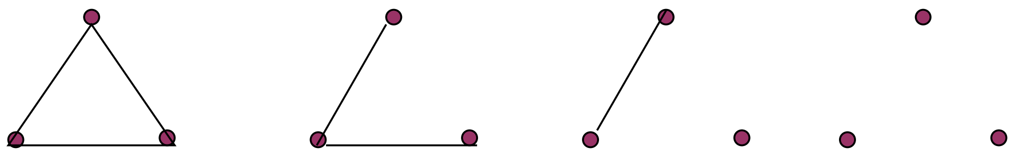


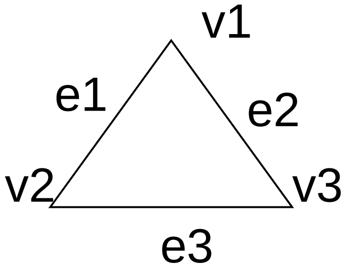
  - S4={(1)(2)(3)(4),(12),(13),(14),(23),(24),(34),(123),(124),(132),(134),(142),(143),(234),(243), (1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (14)(23)}.

Graphical Enumeration

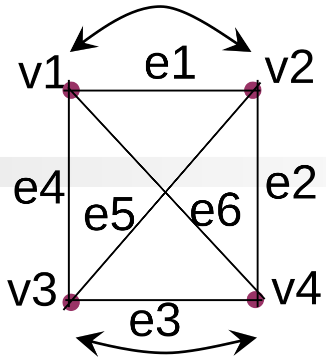
- Use 3 vertices as an example
- Permutation of vertices: $S3=\{e,(v1v2v3), (v3v2v1), (v2v3), (v1v3), (v1v2)\}$
- Permutation of corresponding edge $G=\{e, (e1e2e3), (e3e2e1),(e1e2), (e1e3),(e2e3)\}$
- $P(G)=((x+y)^3+2*(x^3+y^3)+3*(x+y)(x^2+y^2))/6$

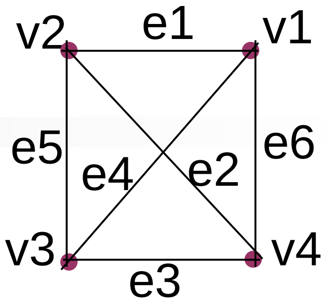
  $= x^3+y^3+xy^2+x^2y$



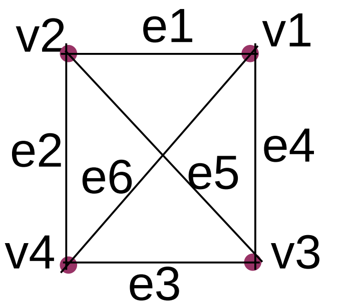


Graphical Enumeration





- **Ex.** Image of 4 vertices
- 2-coloring of edge of complete image
- Each permutation of S4 corresponding to the permutation of the set of 6 edges

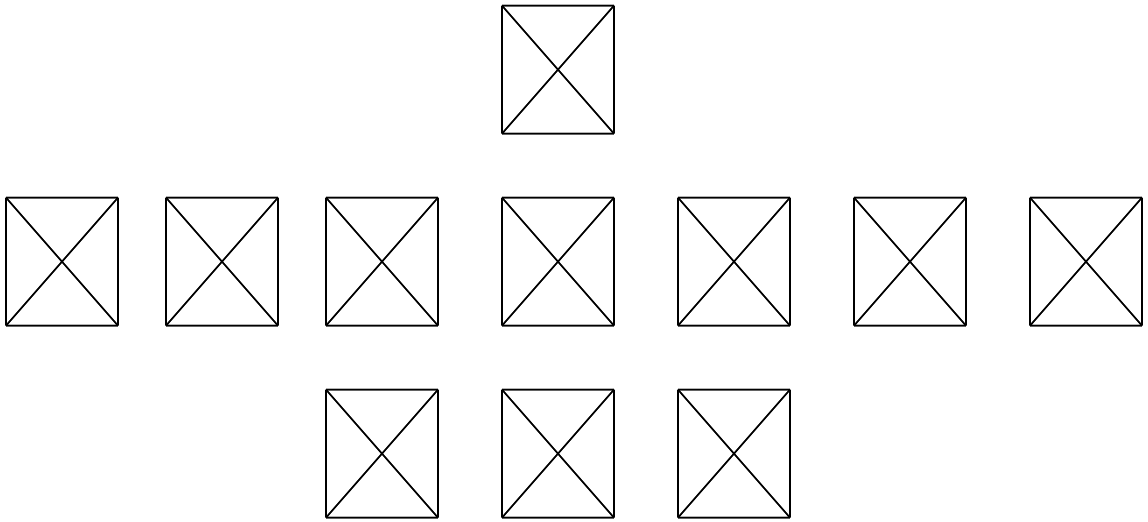


Vertex	Edge	Unit Number
$(1)^4$	$(1)^6$	1
$(1)^2(2)$	$(1)^2(2)^2$	6
$(4)^1$	$(2)^1(4)^1$	6
$(2)^2$	$(1)^2(2)^2$	3
$(1)\ (3)$	$(3)^2$	2*4

$$P(x,y) = [(x+y)^6 + 9(x+y)^2 (x^2+y^2)^2 + 8(x^3+y^3)^2 + 6(x^2+y^2)(x^4+y^4)]/24$$

•S4={(1)(2)(3)(4),(12),(13),(14),(23),(24),(34),(123),(124),(132),(134),(142),(143),(234),(243), (1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (14)(23)}.

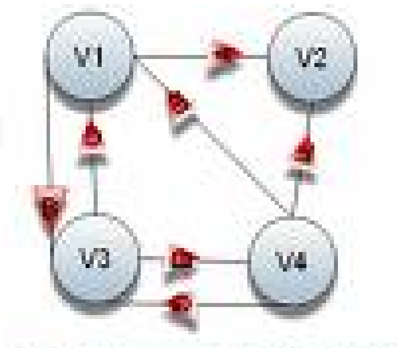
Graphical Enumeration



$$P(x,y) = [(x+y)^6 + 9(x+y)^2 (x^2+y^2)^2 + 8(x^3+y^3)^2 + 6(x^2+y^2)(x^4+y^4)]/24$$

$$= x^6 + x^5y + 2x^4y^2 + 3x^3y^3 + 2x^2y^4 + xy^5 + y^6$$

Graphical Enumeration



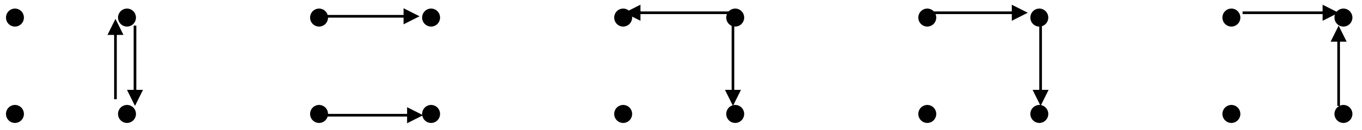
- **Ex.** Find the number heterogeneous undirected graph of 4 vertices
- Edge number is 12, each point contains 3 out-edge, 3 in-edge
- **An.** Vertex Permutation   Directed Graph Permutation
- $(1)^4$ $(1)^{12}$ 1个

$(1)^2(2)$ $(1)^2(2)^5$ 6个

$(4)^1$ $(4)^3$ 6个

$(2)^2$ $(2)^6$ 3个

$(1)\,(3)$ $(3)^4$ 2*4个



$P(x,y) = [(x+y)^{12} + 6(x+y)^2(x^2+y^2)^5 + 3(x^2+y^2)^6 + 8(x^3+y^3)^4 + 6(x^4+y^4)\ ]/24$

x^2y^{10} (coefficient): $\frac{1}{24}[\frac{12!}{2!10!} + 6(1+\frac{5!}{4!}) + 3\frac{6!}{5!} + 0 + 0] = 5$

8 Polya Theorem

8-5 Summary

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组合数学 Combinatorics

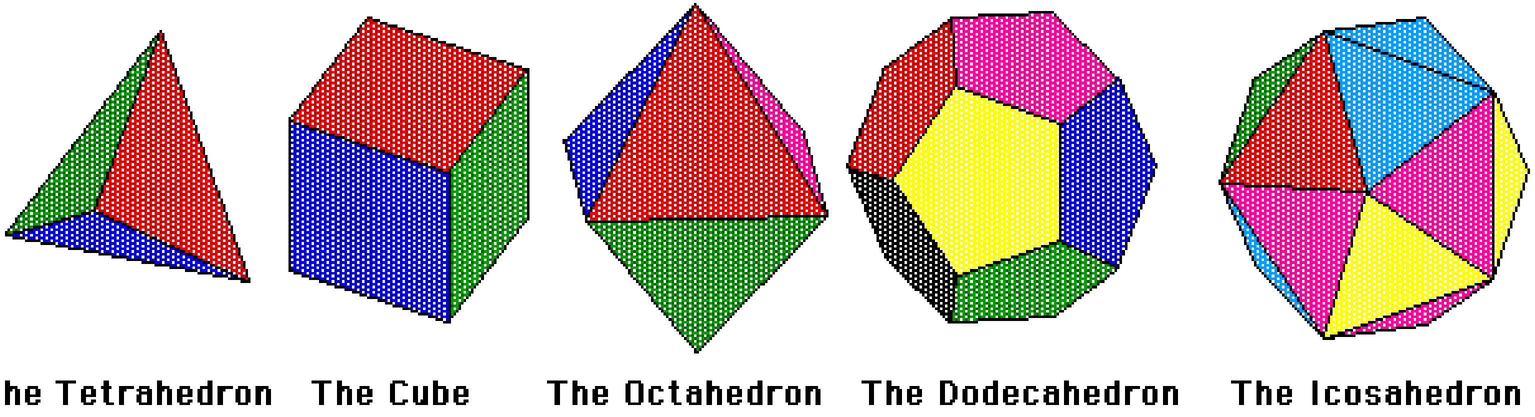
Journey Experience

Paint 4 vertices of a square using red and blue colors, if square rotations is allowed, how many possible way?

Group

Permutation Group

Rotation Group



$$P(G)=\frac{1}{|\overline{G}|}\sum_{j=1}^{g}\prod_{k=1}^{n}S_k^{\,C_k(\overline{Pj})}$$

Problem solving is a skill, it is like swimming, skiing, playing the piano; you can only rely on imitation and practice in order to learn it.

—— 《Mathematical Discovery on Understanding, Learning, and Teaching Problem Solving》 Polya