1.5.2 Optional Exploratory Quiz: An Example of Iterative Reconstruction

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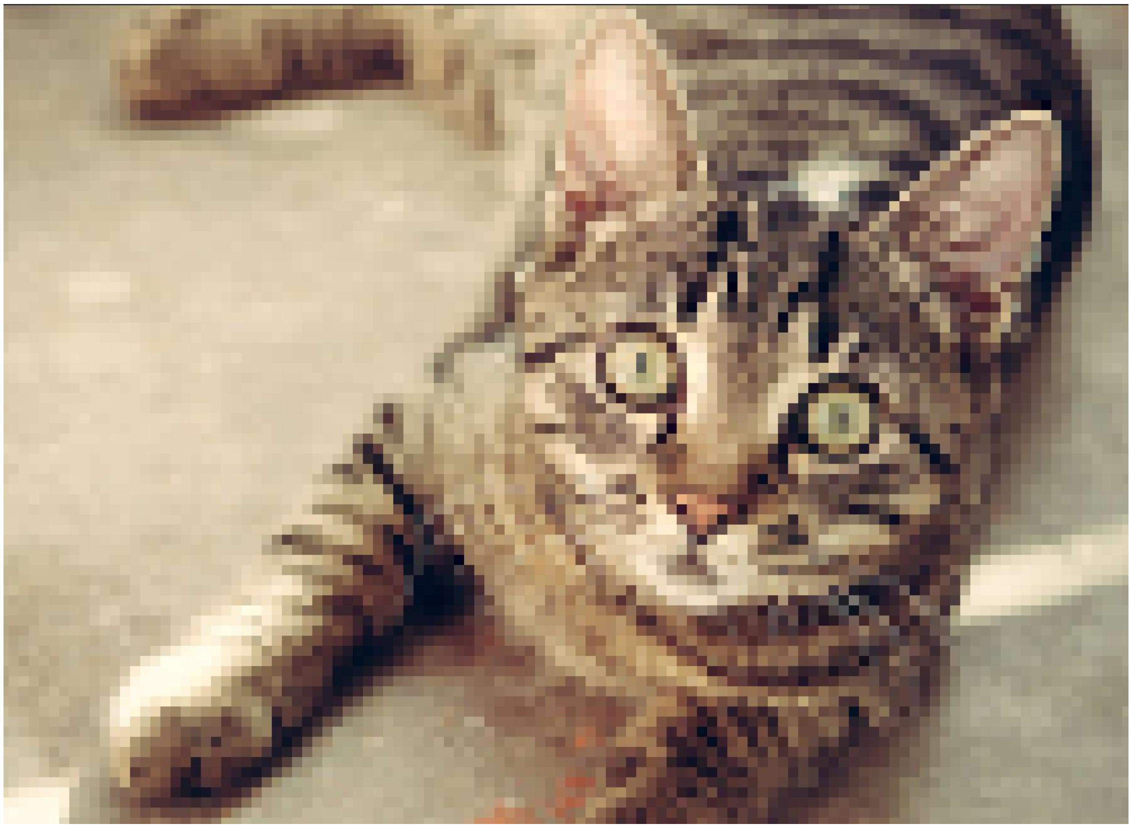
The goal of reconstruction is to recreate a map of the attenuation function μ for a horizontal slice of the body. This is what a CT-scan is.

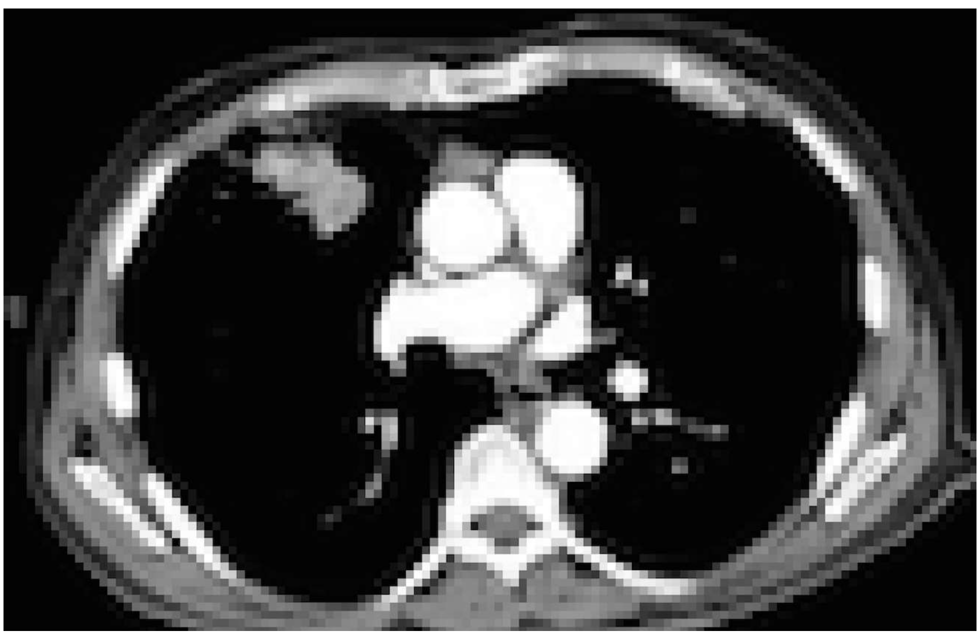
How many variables does this function depend on? As discussed in the previous quiz, the attenuation for the body is a function of 3 variables because the body is 3-dimensional. However, if we restrict ourselves to a horizontal slice of the body (a fixed height from the waist), we can consider μ as a function of two variables, $\mu(x, y)$.

In the iterative reconstruction approach, we focus on finding values of μ at different pixel locations. Why? As Margo mentioned, when we look at a digital picture, we are actually looking at a map of pixels:

Images Have Pixels



A CT-scan is just a map of pixels too.



Instead of trying to find the explicit attenuation function μ, we can find values of μ at different points - these are the pixels. Here is the the example Margo gave, but with some actual numbers.

Suppose we have a mystery 2-dimensional object and we'd like to have a pixelated image of $\mu(x, y)$ for this object, the attenuation function.

We break the image into a 2 by 2 grid, creating 4 squares (pixels). For simplicity, we'll assume each square has side length 1, so $\Delta x = \Delta y = 1$. The goal is to find an approximate value of the $\mu(x, y)$ function for each pixel.

Unknown (Goal): $\mu_1, \mu_2, \mu_3, \mu_4$

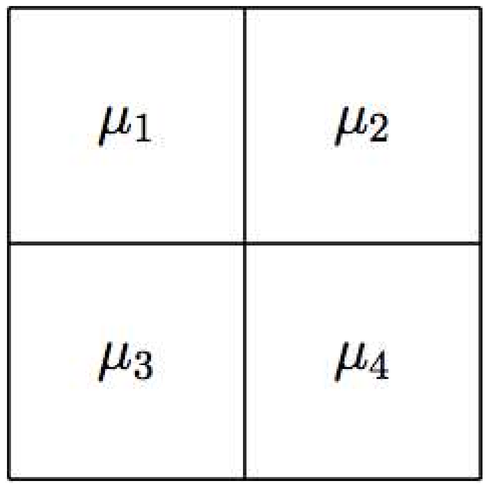


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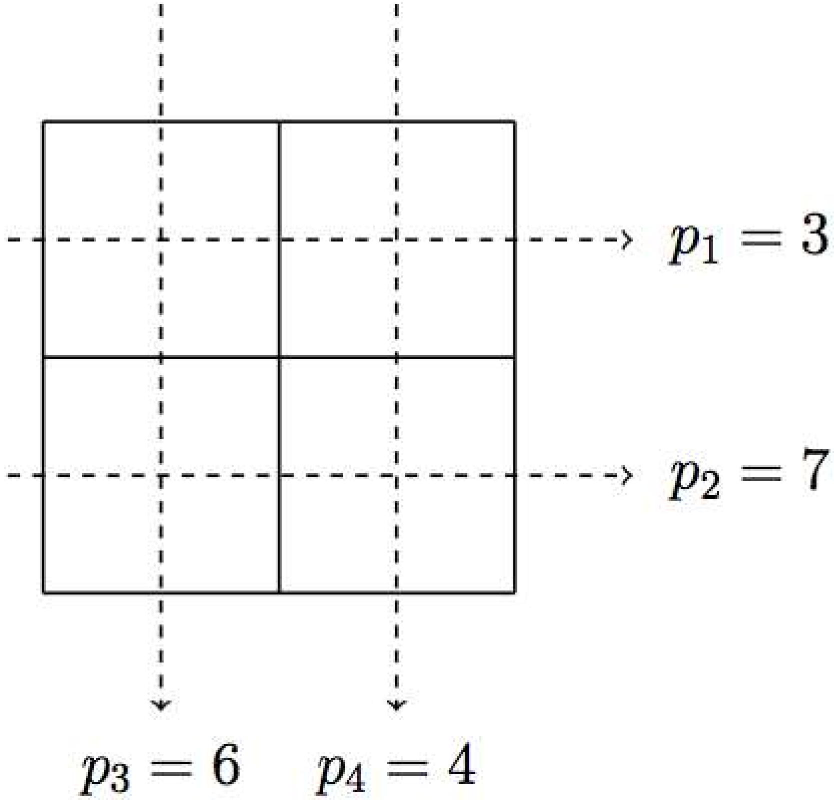
Image Description

Using x-rays, we can find the projection from different views: we take the ratio of output to input intensity and then the negative logarithm of this value. Suppose we've done that and found values p_1, p_2, p_3, p_4 as shown on the diagram.

Known Projections: $p_1 = 3, p_2 = 7, p_3 = 6, p_4 = 4$



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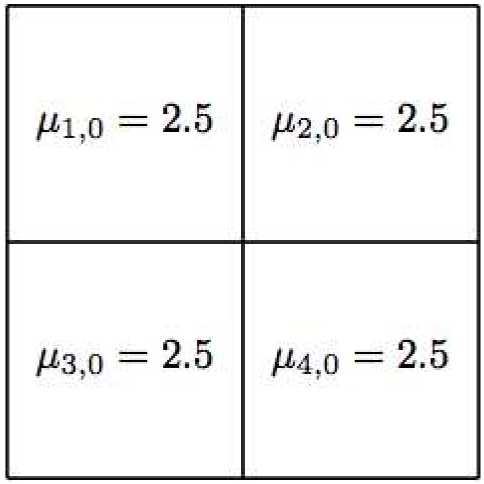
This is the first step of the process:

- Initial Guess Step 0: Assume $\mu(x, y)$ is constant function, so $\mu_1 = \mu_2 = \mu_3 = \mu_4 =$ a constant value.

  What constant value should we choose? We average the measured projections (these are $-\ln(I/I_0)$ where I is the output intensity of the x-ray and I_0 is the input intensity) and assign that as the μ value for each pixel. This gives us $(3 + 7 + 6 + 4)/8 = 2.5$. Why do we divide by 8? We've counted each pixel twice because we used both the vertical and horizontal projections. So we start with $\mu_{1,0} = \mu_{2,0} = \mu_{3,0} = \mu_{4,0} = 2.5$. (The second number in the subscript represents what stage of guessing we are at. We use 0 to represent the original guess.)



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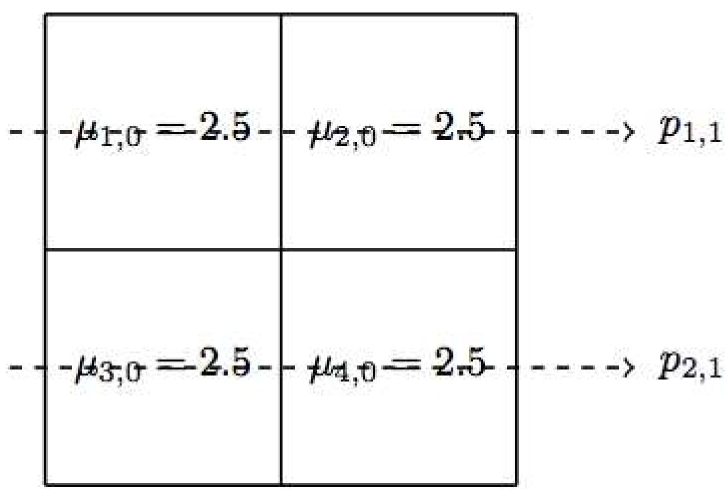
- Compute Projections Step 1: Let’s now compute the horizontal projections based on our guess (we’ll call these estimated projections).

  We do this by adding across: $p_{1,1} = \mu_{1,0} + \mu_{2,0} = 5$ and $p_{2,1} = \mu_{3,0} = +\mu_{4,0} = 5$.

  Why do we add the μ values across to compute the estimated projection? This comes from the fact that the projection is an integral value. So to estimate that integral we do a Riemann sum of the attenuation in each pixel, which is the estimate for μ (the attenuation coefficient) multiplied by Δx. In our simplified example, we’ve assumed $\Delta x = 1$ and $\Delta y = 1$.



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- Compute Error Step 1: $p_1 - p_{1,1} = 5 - 3 = 2$, so we’ve overestimated by 2.

We do the same for $p_2 - p_{2,1} = 5 - 7 = -2$, where we see we underestimate by 2.

- Update Guess Step 1:

  In the first row, because the initial guess led to an overestimate by 2 for the projection, we reduce those two μ-values by 1 each: $\mu_{1,1} = \mu_{1,0} - 1 = 2.5 - 1 = 1.5$ and $\mu_{2,1} = \mu_{2,0} - 1 = 1.5$.

  In the second row, because the initial guess led to an underestimate by 2 for the projection, we increase those two μ-values by 1 each: $\mu_{3,1} = 3.5$ and $\mu_{4,1} = 3.5$.

  Here's a picture of our new guess. Now you try the next step!

$\mu_{1,1} = 1.5$	$\mu_{2,1} = 1.5$
$\mu_{3,1} = 3.5$	$\mu_{4,1} = 3.5$

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Image Description

Question 1 & 2

2/2 points (graded)
Complete the second round of the process, using vertical instead of horizontal projections. Focus first on the left column.

Compute the left vertical projection and the error. Use the error from to update your guess for μ-values in the left column, $\mu_{1,2}$ and $\mu_{3,2}$.

A. $\mu_{1,2}$

2

✔ **Answer:** 2

2

B. $\mu_{3,2}$

4

Answer: 4

4

A. The left vertical projection value is $1.5 + 3.5 = 5$, which is 1 less than the known projection $p_3 = 6$. To adjust, we add half of this error of 1 to each square in the left column, so $\mu_{1,2} = \mu_{1,1} + 0.5 = 1.5 + 0.5 = 1.5 + 0.5 = 2$. Similarly, $\mu_{3,2} = \mu_{3,1} + 0.5 = 3.5 + 0.5 = 4$

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Answers are displayed within the problem

Question 3-4

2/2 points (graded)

Focus now on the right column.

Compute the right vertical projection and the error. Use the error from to update your guess for μ-values in the right column, $\mu_{2,2}$ and $\mu_{4,2}$.

C.$\mu_{2,2}$

1

Answer: 1

1

D.$\mu_{2,4}$

3

Answer: 3

3

Explanation
The right vertical projection value is $1.5 + 3.5 = 5$, which is 1 more than the known projection $p_4 = 4$. To adjust, we subtract half of this error of 1 from each right square, so $\mu_{2,2} = \mu_{2,1} - 0.5 = 1.5 - 0.5 = 1$. Similarly, $\mu_{4,2} = \mu_{4,1} - 0.5 = 3.5 - 0.5 = 3$.

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Answers are displayed within the problem

We repeat this process until it stops because we have zero error. In this example, we now have $\mu_{1,2} = 1, \mu_{2,2} = 2, \mu_{3,2} = 3, \mu_{4,2} = 4$. When we compute projections (vertical and horizontal) we see that there is no error in any of the them, so the process stops.

Clearly, this is a simplified example with only a 2 by 2 grid (4 pixels). In reality, we'd be dealing with images on the order of 128 by 128 pixels. So there are 16384 values of μ to determine, and thus many other x-ray views are used to create better guesses, as well as more sophisticated computing methods.

You might also wonder: Does this process always end? Are we guaranteed to eventually come to the correct values of μ that match the original projections, the original x-rays? These are questions of what mathematicians call **convergence** and **uniqueness** and the answer is 'not necessarily'.

In the example above, you may have noticed that other values of μ_i match the projections, such as $\mu_1 = 1, \mu_2 = 2, \mu_3 = 5, \mu_4 = 2$. How do we know which is correct? The answer is that we don't without more information. In this case, we might compute diagonal projections and use that to rule out multiple solutions. (This non-uniqueness of solutions to a system of linear equations is analogous to when we tried to figure out the function which had a certain value of its integral.) How many projections we need and how to determine a good enough solution are important issues in computed tomography, but beyond the scope of this course. See for example https://en.wikipedia.org/wiki/Iterative_reconstruction.

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- CT Scan, adapted from "The calcified lung nodule: What does it mean?", Fig. 1, by Khan, et.al. Annals of Thoracic Medicine - Vol 5, Issue 2, April-June 2010 ncbi.nlm.nih.gov/pmc/articles/PMC2883201/ (CC BY 2.0, license: creativecommons.org/licenses/by/2.0/)

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