3. GPS

GPS

3/3 points (graded)

The Global Positioning System (GPS) works by sending signals from satellites to your smartphone. Satellites know their exactly position $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ with respect to the center of the Earth (which we declare to be the origin). Suppose your phone knows its distance r from each satellite. Knowing the distance r of your phone from one satellite, tells you that your phone lies on the sphere of radius r about that satellite, which is given by the equation

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2.$$

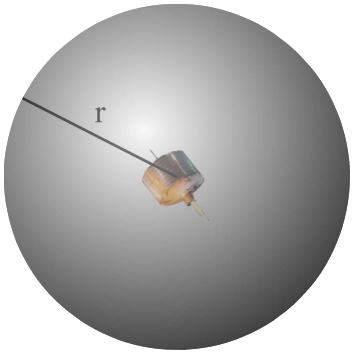

Image depicts a GPS satellite at the center of the sphere. The cellphone may be at any location on the surface of the sphere.

If you know the locations of two satellites, one of distance r_1 and one of distance r_2 to your phone, your phone must lie on the intersection of two spheres of radius r_1 and r_2 about each satellite respectively.

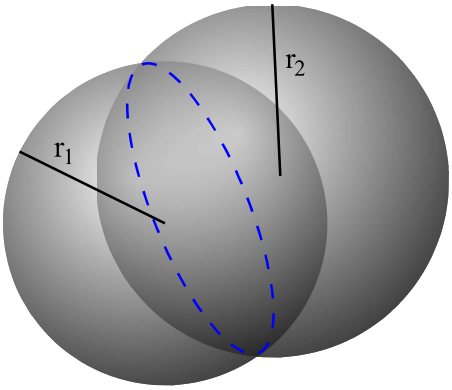

Image depicts two GPS satellites, each at the center a sphere. The overlap describes the solution to
two simultaneous nonlinear equations described by the equations of the two spheres. The cellphone
may be at any location along the intersection of these spheres (designated by the dotted blue circle).

That is it must satisfy both of the following equations simultaneously.

$$\begin{aligned}(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 &= r_1^2\\ (x-x_2)^2+(y-y_2)^2+(z-z_2)^2 &= r_2^2\end{aligned}$$

(Notice that this problem is nonlinear!)

Once your phone knows the distance between itself and four satellites, your phone can calculate its position. Let's see how:

Assume you are at an unknown position $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where x, y and z are calculated relative to the center of the Earth, which is the origin. The constants in this problem are scaled so that the radius of Earth is 1.

Your phone receives signals from 4 satellites, which give you their location and how far they are from you. This gives you the following system of equations that needs to be satisfied:

$$\begin{aligned} x^2+y^2+(z-4)^2 &= 9 \\ (x-0.2)^2+(y+0.6)^2+(z-4)^2 &= 9.4 \\ (x-0.2)^2+(y-1.7)^2+(z-3.5)^2 &= 9.18 \\ x^2+(y-1.7)^2+(z-3.8)^2 &= 10.73 \end{aligned}$$

This a nonlinear system of 4 equations in 3 unknowns. Use a little creativity first, then some standard linear algebra, to solve this system and find the unique solution that gives your position on the Earth. Check that your solution lies exactly on the surface of the Earth!

(You can use MATLAB Online to help find the solution.)

$x =$ 0 ✔ Answer: 0

$y =$ 0 ✔ Answer: 0

$z =$ 1 ✔ Answer: 1

Solution:

First we expand each equation

$$\begin{aligned} x^2+y^2+(z^2-8z+16) &= 9 \\ (x^2-0.4x+0.04)+(y^2+1.2y+0.36)+(z^2-8z+16) &= 9.4 \\ (x^2-0.4x+0.04)+(y^2-3.4y+2.89)+(z^2-7z+12.25) &= 9.18 \\ x^2+(y^2-3.4y+2.89)+(z^2-7.6z+14.44) &= 10.73 \end{aligned}$$

Next subtract the first equation from each of the other equations to eliminate the nonlinear square terms:

$$\begin{aligned} x^2+y^2+(z^2-8z+16) &= 9 \\ -0.4x+0.04+1.2y+0.36 &= 0.4 \\ -0.4x+0.04-3.4y+2.89+z-3.75 &= 0.18 \end{aligned}$$

$$-3.4y + 2.89 + 0.4z - 1.56 = 1.73.$$

Move all constant terms to the right hand side:

$$\begin{aligned} x^2 + y^2 + (z^2 - 8z + 16) &= 9 \\ -0.4x + 1.2y &= 0 \\ -0.4x - 3.4y + z &= 1.0 \\ -3.4y + 0.4z &= 0.4. \end{aligned}$$

Note that now we have 3 linear equations in 3 variables, which we have methods for solving.

$$\begin{pmatrix} -0.4 & 1.2 & 0 \\ -0.4 & -3.4 & 1 \\ 0 & -3.4 & 0.4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1.0 \\ 0.4 \end{pmatrix}$$

At this point, we can plug this system into MATLAB and solve to find the solution $x = 0$, $y = 0$, and $z = 1$. Note that this is on the surface of the Earth, defined by the equation $x^2 + y^2 + z^2 = 1$. As a check, one should verify that this point satisfies the 4th equation, which we did not use directly in our solution. (Hint: it does.)

Alternative solution: look up how to solve nonlinear systems using MATLAB in the documentation!

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Answers are displayed within the problem

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Another alternative solution

discussion posted 7 days ago by **yves-M**

There is a 5th equation that can be added to the system and would make your calculation much easier :)

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1 response

jfrench (Staff)
6 days ago

Too true. :)

The suspense is killing me ... Oh, I see :-)

Wan-Teh

posted 6 days ago by **Wan-TehChang**

Cool.

posted 6 days ago by **hchar**

Love it

posted about 16 hours ago by **DavidAOConnor**

keeping in mind that "your solution lies exactly on the surface of the Earth!", right? :)

posted about a minute ago by **sandipan_dey**

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