

14.310x Lecture 13

Statistics

Where were we anyhow?

Summary to date

Probability basics

Introduced concept and talked about simple sample spaces, independent events, conditional probabilities, Bayes Rule

Random variables

Defined a random variable, discussed ways to represent distributions (PF, PDF, CDF), covered random variable versions of concepts above

Functions of random variables

Saw some basic strategies and several important examples

Summary to date

Moments

Defined moments of distributions and learned many techniques and properties to help compute moments of functions of random variables

Special distributions

Binomial, hypergeometric, geometric, negative binomial, Poisson, exponential, uniform, normal

Estimation

CLT, had general discussion and discussion about sample mean, criteria for assessing, frameworks for deriving

Statistics---quantifying reliability

We have seen various estimators, made observations about their distributions, discussed how we might derive them, and also discussed criteria that we might use to choose among them.

That's all well and good, but when we actually have to report estimates, people will want to have some objective measure of how good, or reliable, or precise, our estimates are. One way we quantify this is by reporting the variance (or estimated variance) of the estimator along with the estimate.

Statistics---quantifying reliability

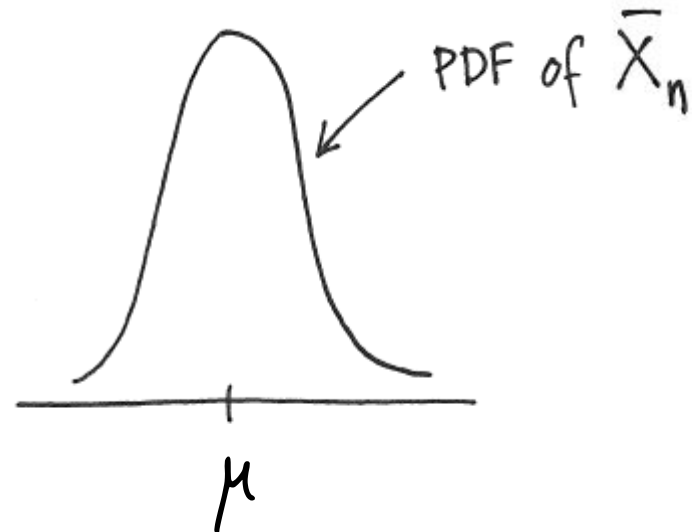
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So we often report an estimate along with its standard error. Sometimes we put the standard error in parentheses right after the estimate.

Statistics---quantifying reliability

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In other words, we could construct an interval using information about the distribution of the estimator. That interval will be narrow when the estimator has a tight distribution and wide when it has a dispersed distribution. We'll call it a confidence interval.

Statistics---confidence intervals

We want to find functions of the random sample $A(X_1, \dots, X_n)$ and $B(X_1, \dots, X_n)$ such that

$$P(A(X_1, \dots, X_n) < \theta < B(X_1, \dots, X_n)) = 1 - \alpha$$

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random functions

true parameter---
fixed but unknown

desired "degree of
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Statistics---confidence intervals

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(Compare this with point estimation: want a function such that $E(\hat{\theta}(X_1, \dots, X_n)) = \theta$, for instance.)

Statistics---confidence intervals

If you can find such functions A & B , then

$$[A(x_1, \dots, x_n), B(x_1, \dots, x_n)]$$

is said to be a $1-\alpha$ confidence interval for θ .

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After we plug in the realizations, these are just numbers now.

Statistics---confidence intervals

If you can find such functions $A \leq B$, then

$$[A(x_1, \dots, x_n), B(x_1, \dots, x_n)]$$

is said to be a $1-\alpha$ confidence interval for θ .

Notes: ---These functions are not unique. So how do we choose them? Typically, we choose $A \leq B$ such that $\alpha/2$ of the probability falls on each side of the interval.

---Keep in mind that $A(x_1, \dots, x_n)$ and $B(x_1, \dots, x_n)$ are just numbers, so probability statements involving those quantities and θ don't make sense.

Statistics---confidence intervals

Where do we find those functions? You can find them "from scratch." In most cases that you will encounter in everyday data analysis, though, others have found those functions and a resulting formula for a "95% confidence interval for the mean of an unknown distribution with sample size greater than 30" (or whatever).

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Important to note: in order to find those functions and derive the formulae, one needs to know how an estimator for the unknown parameter is distributed (typically not just the mean and variance).

Statistics--- χ^2 , t , and F distributions

This is precisely where our friends, the χ^2 , t , and F distributions, come in. These are all distributions that, unlike other special distributions we've encountered, don't really appear in nature, don't really describe stochastic phenomena we observe. Rather, they were "invented" because estimators or functions of estimators had distributions that needed to be described and tabulated.

Statistics--- χ^2 , t , and F distributions

χ^2

Recall that we briefly mentioned an estimator called the sample variance, s^2 .

$$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x}_n)^2$$

We said that it was an unbiased estimator for the variance of a distribution.

Well, $(n-1)s^2/\sigma^2$ has one of these distributions, in particular,

$$(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$$

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This is a parameter of the χ^2 distribution---it's called "degrees of freedom"

Statistics--- χ^2 , t , and F distributions

t

If $X \sim N(0,1)$ and $Z \sim \chi^2_n$ and they're independent, then

$$X/(Z/n)^{1/2} \sim t_n$$

Why is that a useful fact? Suppose we are sampling from a $N(\mu, \sigma^2)$ distribution. We know that $(\bar{X} - \mu)/(\sigma^2/n)^{1/2} \sim N(0,1)$ and that $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$. (We do not know that they're independent but, in fact, they are.)

We form $\frac{\sqrt{n}(\bar{X} - \mu)/\sigma}{\sqrt{\frac{\sum ((X_i - \bar{X})/\sigma)^2}{(n-1)}}}$ cancel a few things, and get $\frac{\sqrt{n}(\bar{X} - \mu)}{s}$

Statistics--- χ^2 , t , and F distributions

t

So we have that $\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t_{n-1}$ for X_i i.i.d. $N(\mu, \sigma^2)$.

Statistics--- χ^2 , t , and F distributions

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again, "degrees of freedom"



Statistics--- χ^2 , t, and F distributions

The t distribution was formulated by William Sealy Gosset in his job as Chief Brewer in the Guinness Brewery in Dublin. He derived and tabulated this distribution to aid in his analysis of data for quality control across batches of beer. He published it under the pseudonym "Student" in 1908.



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took photo during my recent visit to the brewery

Statistics--- χ^2 , t , and F distributions

E

If $X \sim \chi^2_n$ and $Z \sim \chi^2_m$ and they're independent, then
 $(X/n)/(Z/m) \sim F_{n,m}$

Why is that a useful fact? Suppose we have samples from two different populations. We might want to know whether the distributions in the two populations were, in fact, the same. If they are, we can form the ratio of the sample variances divided by their degrees of freedom (true variances canceling because they're the same) and the ratio then has the above distribution.

Statistics---confidence intervals

So let's construct some confidence intervals. We will focus initially on two cases. These are not the only cases you will ever encounter, but they are, by far, the most important.

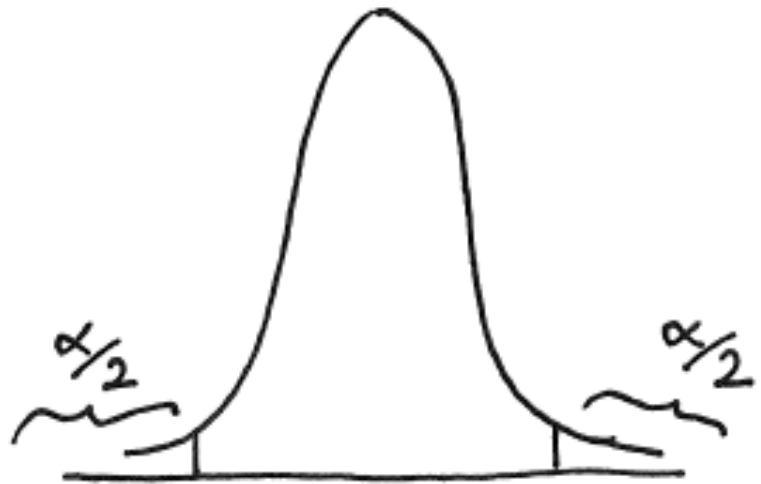
Case 1: We are sampling from a normal distribution with a known variance and we want a confidence interval for the mean.

Case 2: We are sampling from a normal distribution with an unknown variance and we want a confidence interval for the mean.

Statistics---confidence intervals, case 1

We have an estimator for the mean, \bar{X} , which has a normal distribution with mean μ and variance σ^2/n .

$$\text{So } P\left\{\Phi^{-1}(\alpha/2) < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < -\Phi^{-1}(\alpha/2)\right\} = 1 - \alpha$$



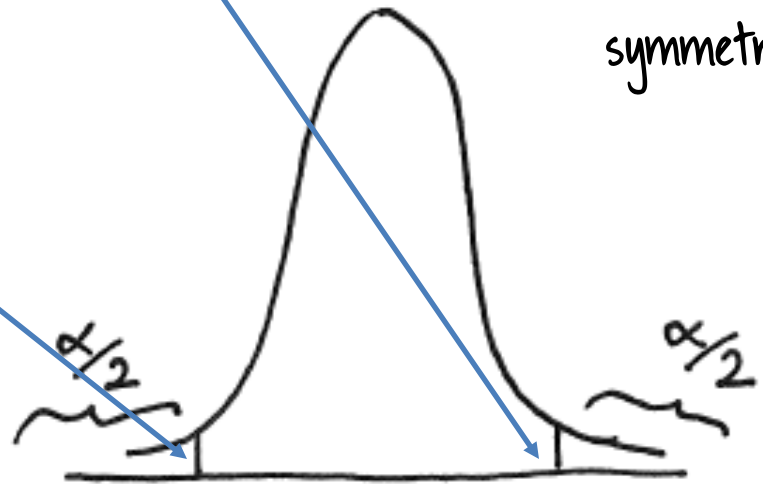
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This is the inverse CDF of the standard normal evaluated at $\alpha/2$.

We just put a negative sign in front of it to get the value for the other tail due to symmetry.



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$$\text{So, rearranging we have, } P\left\{\bar{X} + \Phi^{-1}(\alpha/2) \sigma/\sqrt{n} < \mu < \bar{X} - \Phi^{-1}(\alpha/2) \sigma/\sqrt{n}\right\} = 1 - \alpha.$$

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Our $1 - \alpha$ CI is $[\bar{X} + \Phi^{-1}(\alpha/2) \sigma/\sqrt{n}, \bar{X} - \Phi^{-1}(\alpha/2) \sigma/\sqrt{n}]$

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We plug realizations in and our CI is just two numbers.

$$\text{Our } 1 - \alpha \text{ CI is } [\bar{X} + \Phi^{-1}(\alpha/2) \sigma/\sqrt{n}, \bar{X} - \Phi^{-1}(\alpha/2) \sigma/\sqrt{n}]$$

Statistics---confidence intervals, case 2

We have an estimator for the mean, \bar{X} , which has a normal distribution with mean μ and variance σ^2/n , but we do not know σ^2 . We do know, though, that $\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t_{n-1}$.

$$\text{So } P\left\{t_{n-1}^{-1}(\alpha/2) < \frac{\sqrt{n}(\bar{X}-\mu)}{S} < -t_{n-1}^{-1}(\alpha/2)\right\} = 1-\alpha$$

↑
This is the inverse
of the t_{n-1} CDF
evaluated at $\alpha/2$.

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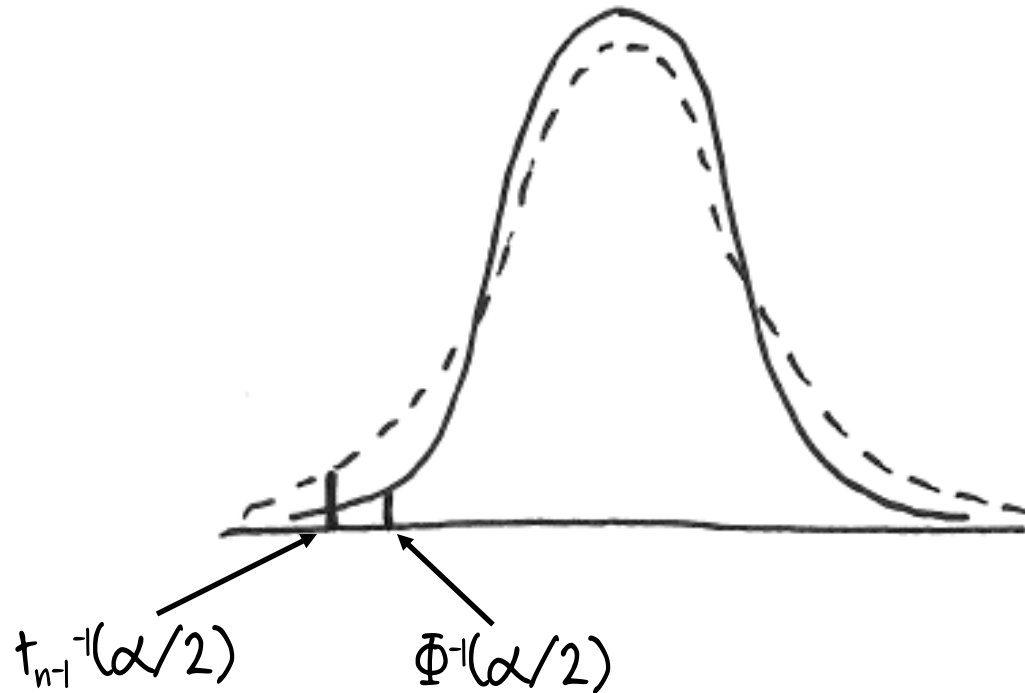
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Statistics---confidence intervals

Comparison of cases 1 and 2:



The t distribution is similar to the normal but has "fatter tails." It converges to the normal as $n \rightarrow \infty$.

Statistics---confidence intervals

Comparison of cases 1 and 2:

$$-t_{n-1}^{-1}(\alpha/2) \rightarrow \Phi^{-1}(\alpha/2) \text{ as } n \rightarrow \infty$$

The t gives you a wider interval than the normal for finite n . The intuition is that you are less sure of the distribution of your estimator because you don't know $\text{Var}(\bar{X})$ and must estimate it. The t "penalizes" your confidence interval by making it wider, reflecting your greater uncertainty. As n goes to infinity, your uncertainty becomes relatively less important.

Statistics---confidence intervals

We don't always fall into case 1 or case 2. What do we do then?

Using facts that you know about how functions of random variables are distributed, you can construct a confidence interval "from scratch" on your own.

In practice, we usually appeal to CLT-like results to argue that the estimator has an approximate normal distribution, and then just use the t confidence interval formula with an estimated variance. (For large n , the t and normal confidence intervals are the same.)

Statistics---hypothesis testing

Well, now we know what an estimator is, how to estimate unknown parameters, and a couple of different ways to express how confident we are in our estimates. That gets us a long way and gives us a very good foundation for studying all kinds of estimation going forward.

One more foundational bit is quite important: hypothesis testing.

Statistics---hypothesis testing

In social science (as well as other settings for data analysis), we often encounter questions that we want to answer. (Some of you have started formulating them for your empirical project.) Do the lifespans of popes follow a lognormal distribution? Does the income tax rate affect the number of hours employees are willing to work? Do used books cost more on the internet than they do in brick and mortar stores? Has NAFTA hurt US manufacturing workers?

Statistics---hypothesis testing

The tool that statisticians have invented to help answer such questions (and quantify how confident we are in the answers) is the hypothesis test.

Purpose: Given a random sample from a population, is there enough evidence to contradict some assertion about the population?

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Let's build the structure underlying the hypothesis test.

Statistics---hypothesis testing

First, we'll need a bunch of definitions:

An hypothesis is an assumption about the distribution of a random variable in a population.

A maintained hypothesis is one that cannot or will not be tested.

A testable hypothesis is one that can be tested using evidence from a random sample.

The null hypothesis, H_0 , is the one that will be tested.

The alternative hypothesis, H_A , is a possibility (or series of possibilities) other than the null.

Statistics---hypothesis testing

For instance, we might want to perform a test concerning unknown parameter θ where $X_i \sim f(x|\theta)$.

$$H_0: \theta \in \Theta_0$$

$$H_A: \theta \in \Theta_A, \text{ where } \Theta_0 \text{ and } \Theta_A \text{ disjoint.}$$

More definitions:

A simple hypothesis is one characterized by a single point, i.e., $\Theta_0 = \theta_0$.

A composite hypothesis is one characterized by multiple points, i.e., Θ_0 is multiple values or a range of values.

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usual set-up: simple null and composite alternative

Statistics---example set-up

X_i i.i.d. $N(\mu, \sigma^2)$, where σ^2 known.

Interested in testing whether $\mu = 0$.

$$H_0: \mu = 0$$

$$H_A: \mu = 1$$

Statistics---example set-up

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testable hypothesis

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null hypothesis, simple

$$H_A: \mu = 1$$

alternative hypothesis, simple

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$H_A: \mu \neq 0$

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$H_0: \mu = 0$ null hypothesis, simple

$H_A: \mu \neq 0$ alternative hypothesis, composite, two-sided

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$H_A: \mu > 0$

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We then either "accept" or "reject" the null hypothesis based on evidence from our sample.

Statistics---hypothesis testing

Obviously, mistakes can be made---we can "reject" a null that is true or "accept" a null that is false. We want to set up our hypothesis test to analyze and control these errors. We first need a taxonomy.

	H_0 true	H_0 false
accept H_0	No error	Type II error
reject H_0	Type I error	No error

Statistics---hypothesis testing

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The significance level of the test, α , is the probability of type I error.

The operating characteristic of the test, β , is the probability of type II error.

We call $1-\alpha$ the confidence level. We call $1-\beta$ the power.

Statistics---hypothesis testing

Finally, we define the critical region of the test, C or C_X , as the region of the support of the random sample for which we reject the null.

Statistics---example

X_i i.i.d. $N(\mu, \sigma^2)$, where σ^2 known.

$$H_0: \mu = 0$$

$$H_A: \mu = 1$$

Suppose, first, that $n = 2$.

Think about what kind of sample would lead you to believe the null or doubt the null in favor of the alternative.

Statistics---example

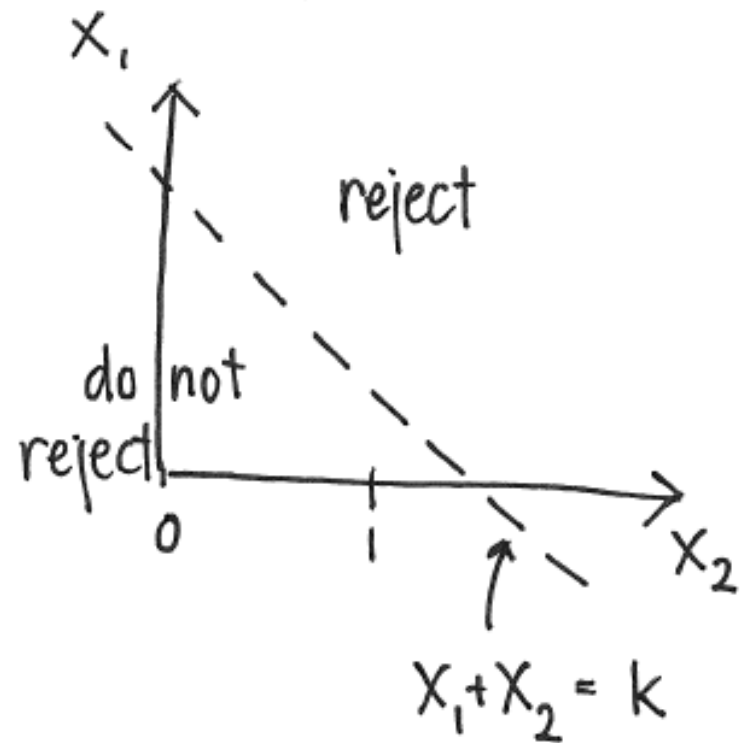
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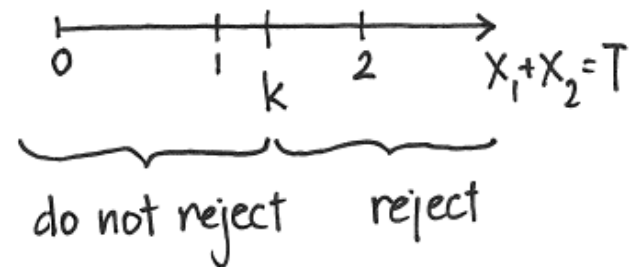
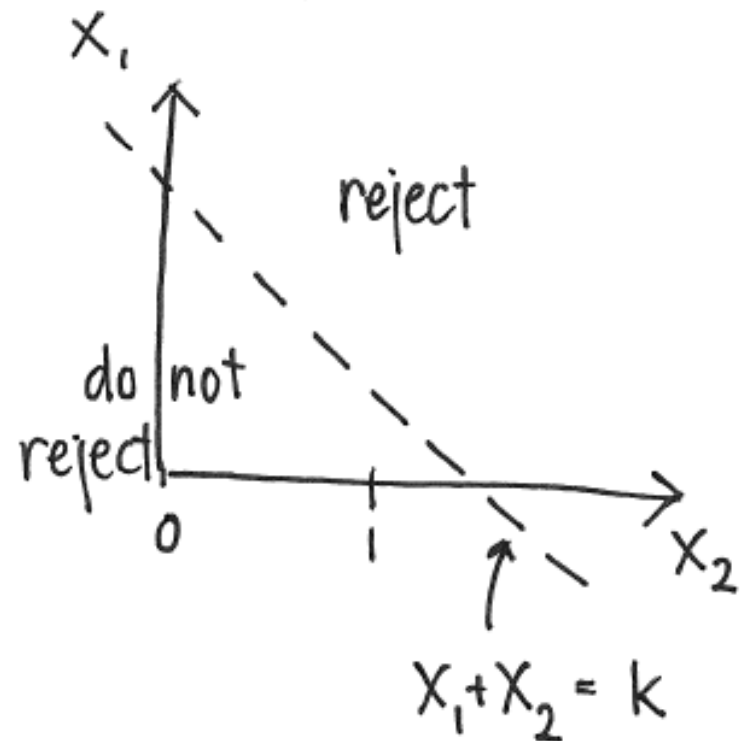
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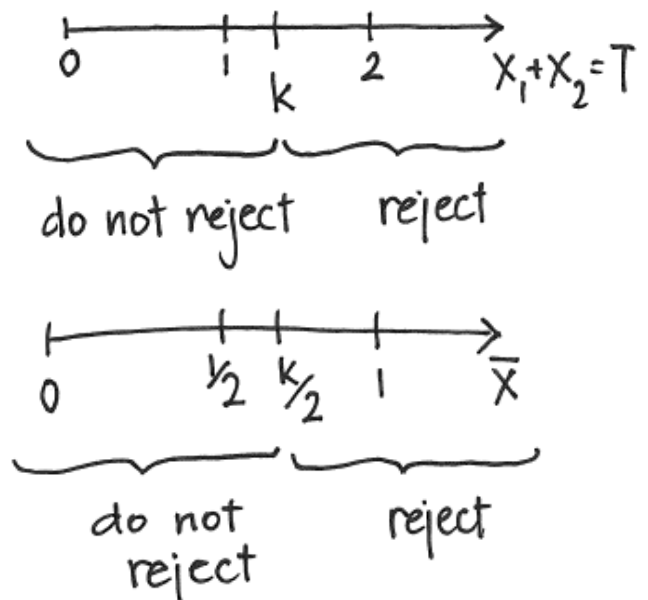
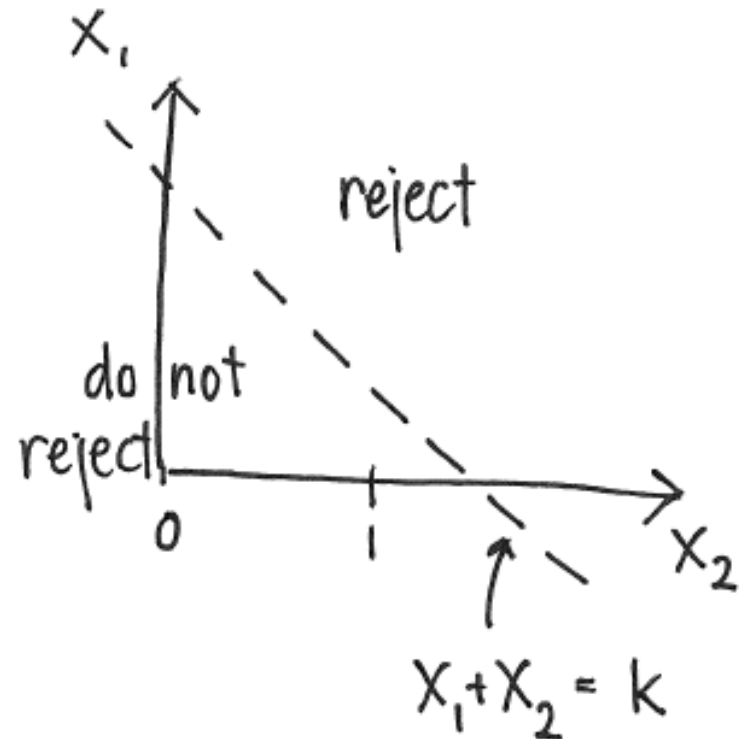
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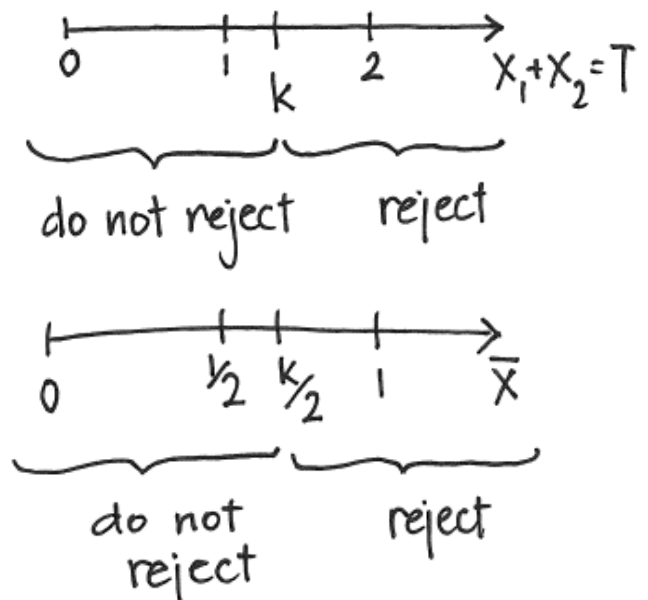
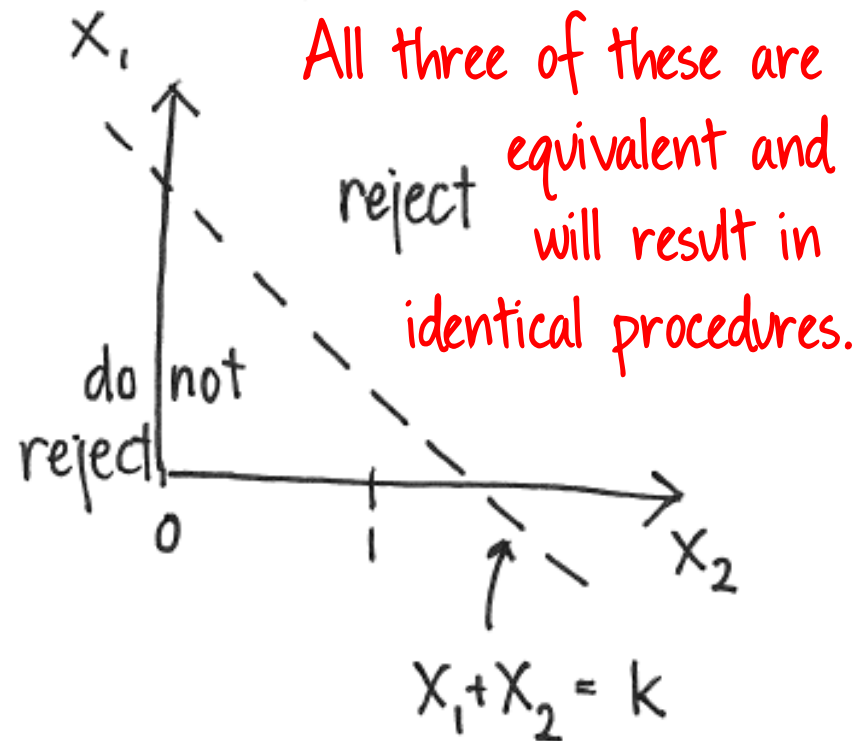
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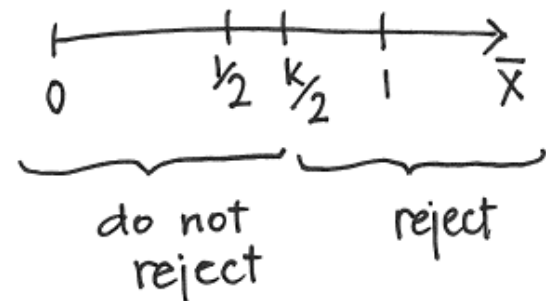
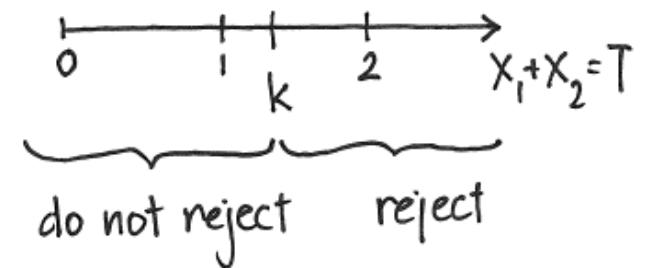
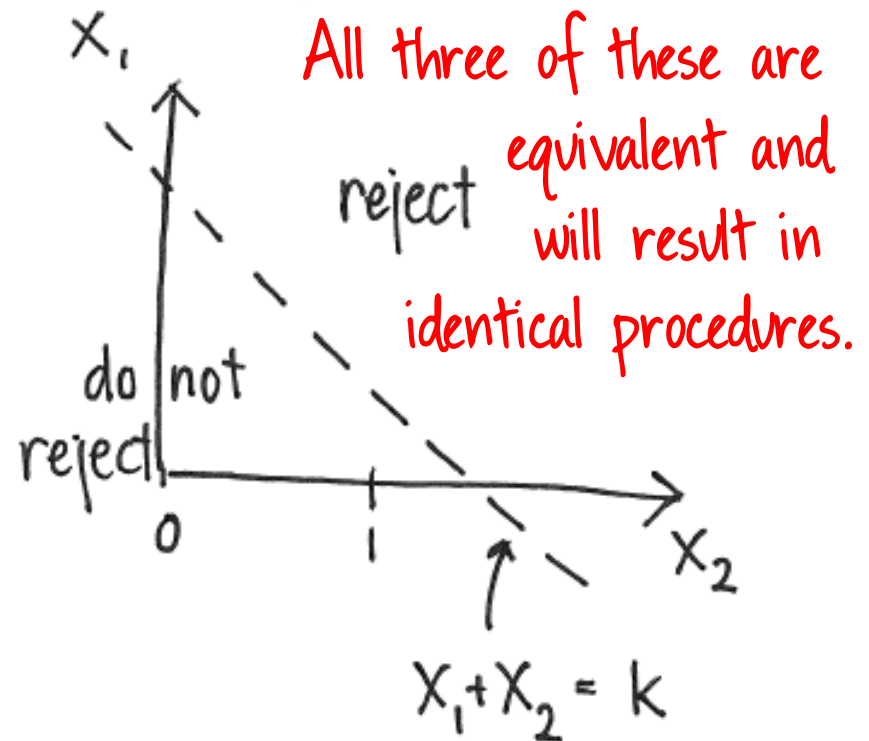
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Statistics---example

So do we prefer one over the others? Not really, but when n gets big, we don't want to have to worry about n -dimensional spaces, so we would just as soon base the test on the sum or the sample mean.



Statistics---example

So we'll base our testing procedure on the test statistic

$T = \bar{X}$ and reject for "large" values.

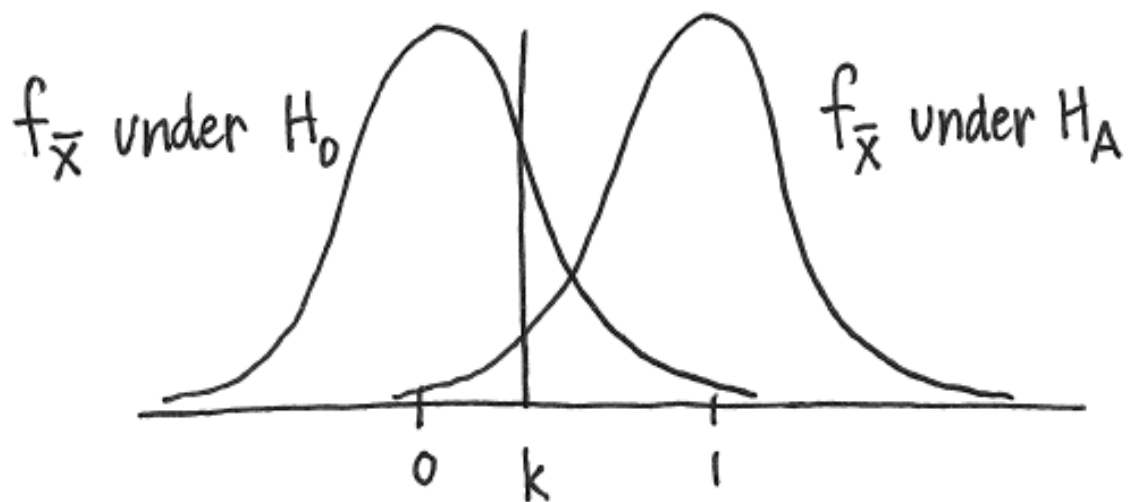
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How do we choose k ? Trade off two types of error.

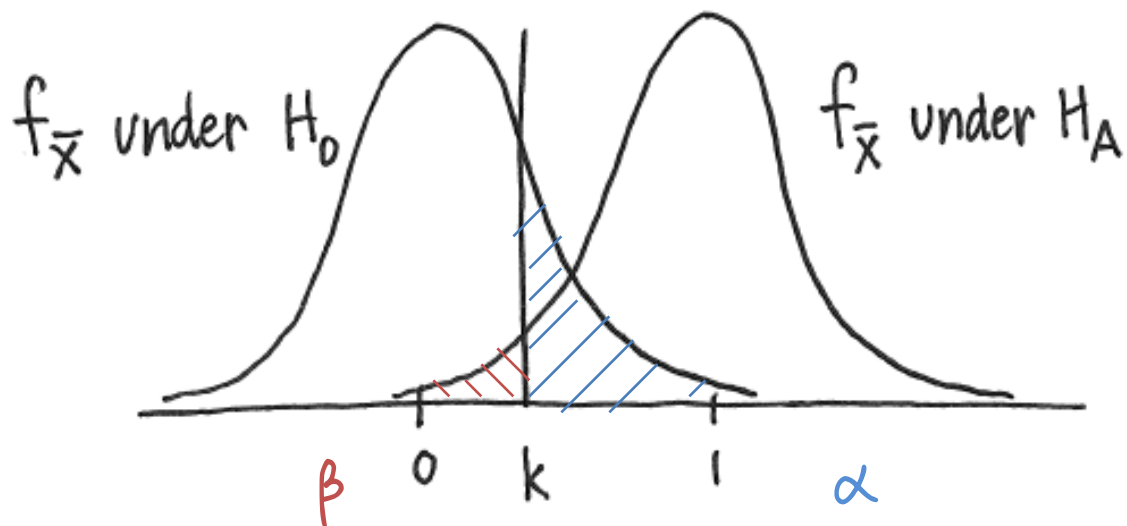


Statistics---example

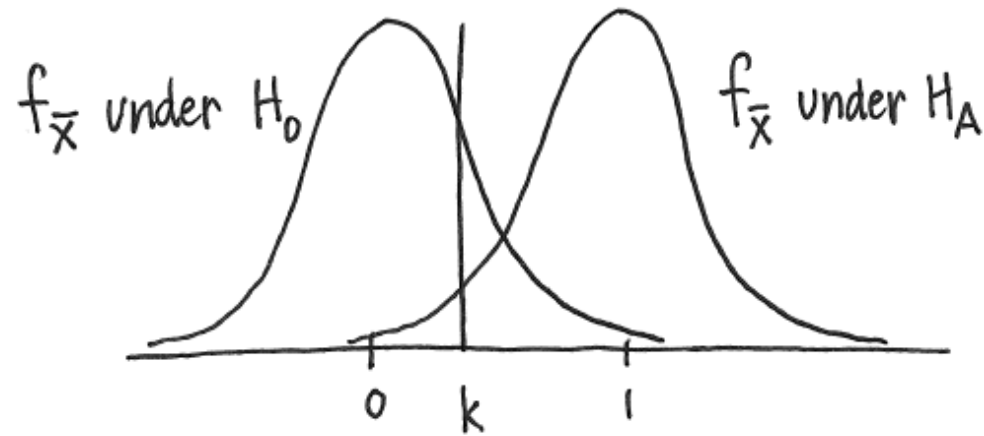
So we'll base our testing procedure on the test statistic
 $T = \bar{X}$ and reject for "large" values.

In other words, critical region C will take the form $\bar{X} > k$
for some k yet to be determined.

How do we choose k ? Trade off two types of error.



Statistics---example

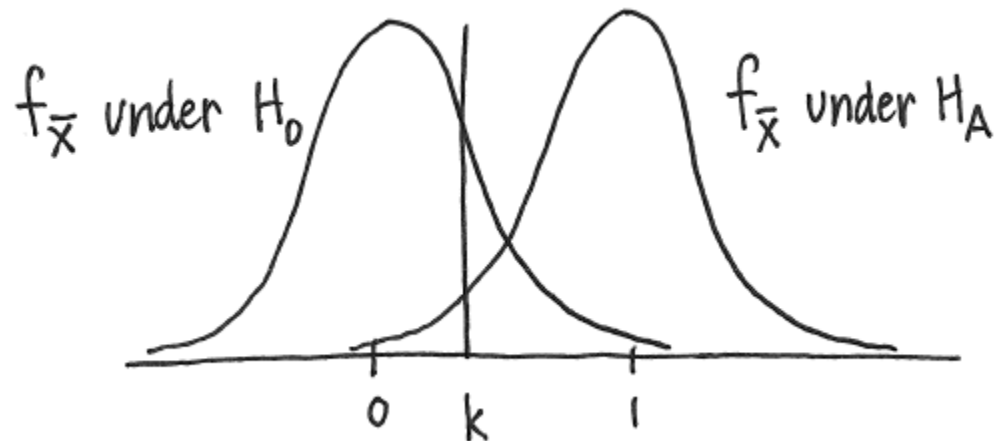


Choice of any one of α , β , or k determines the other two.
Furthermore, choosing them involves an explicit trade-off between the probability of type I and type II errors.

increasing k means $\alpha \downarrow$ and $\beta \uparrow$

decreasing k means $\alpha \uparrow$ and $\beta \downarrow$

Statistics---example



Let's compute α and β using some specific numbers. If $\sigma^2 = 4$ and $n = 25$, then $\bar{T} \sim N(0, 4/25)$ under H_0 and $N(1, 4/25)$ under H_A .

$$\alpha = P(T > k | \mu = 0) = 1 - \Phi((k - 0)/(2/5))$$

$$\beta = P(T < k | \mu = 1) = \Phi((k - 1)/(2/5))$$

Statistics---example

$$\alpha = P(T > k | \mu = 0) = 1 - \Phi((k-0)/(2/5))$$

$$\beta = P(T < k | \mu = 1) = \Phi((k-1)/(2/5))$$

If you plugged in different values of k , you would get a graph in α - β space that looked like this:

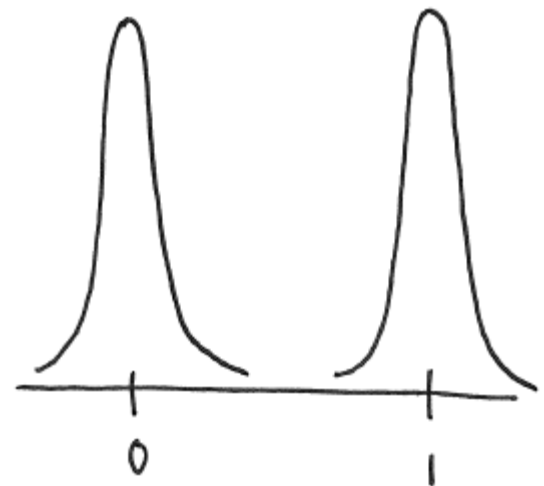
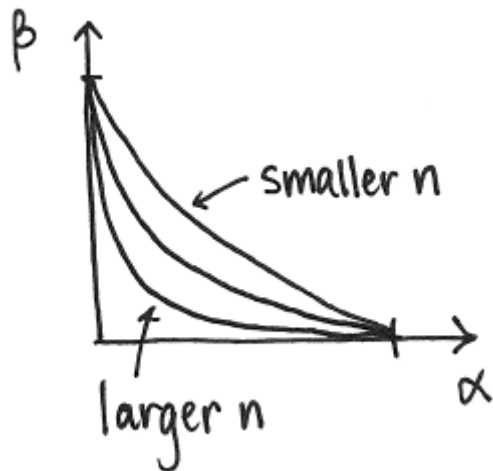


Statistics---example

$$\alpha = P(T > k | \mu = 0) = 1 - \Phi((k-0)/(2/5))$$

$$\beta = P(T < k | \mu = 1) = \Phi((k-1)/(2/5))$$

What happens as n increases or decreases?



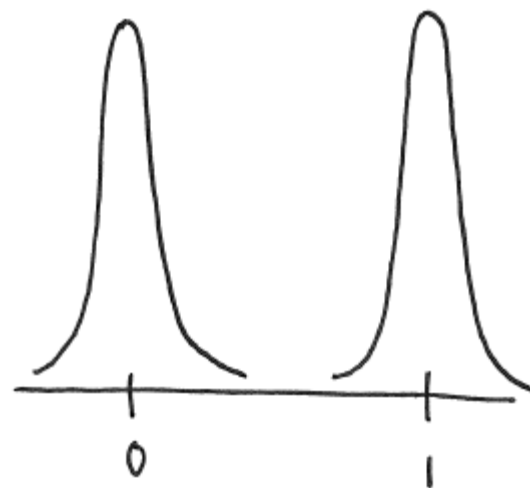
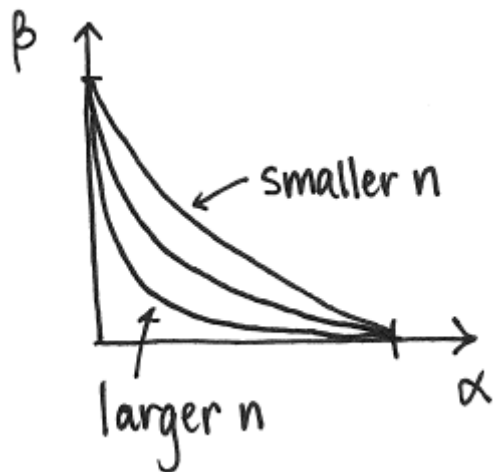
Statistics---example

$$\alpha = P(T > k | \mu = 0) = 1 - \Phi((k-0)/(2/5))$$

$$\beta = P(T < k | \mu = 1) = \Phi((k-1)/(2/5))$$

this changes

What happens as n increases or decreases?



Statistics---example

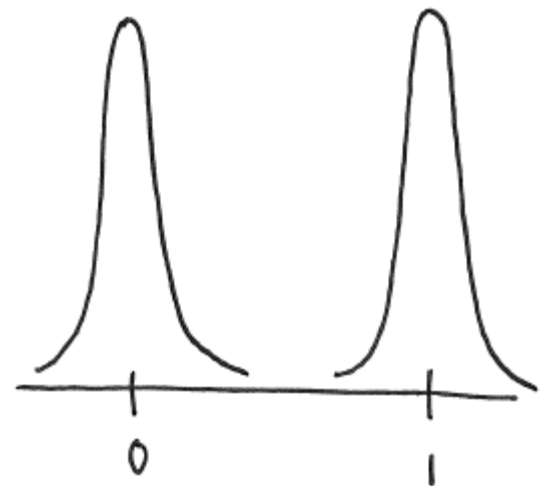
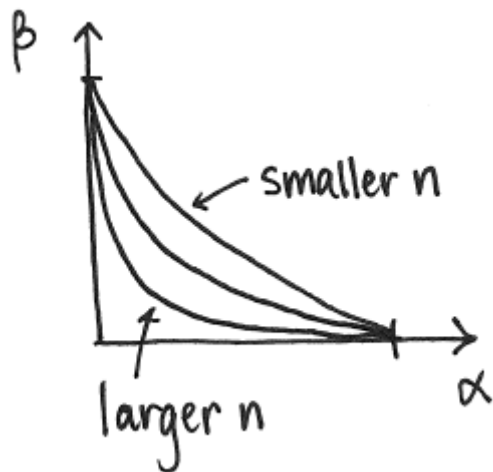
$$\alpha = P(T > k | \mu = 0) = 1 - \Phi((k - 0) / (\sqrt{2/5}))$$

$$\beta = P(T < k | \mu = 1) = \Phi((k - 1) / (\sqrt{2/5}))$$

As n increases, the two distributions get tighter around their means---can do better on both α and β .

this changes

What happens as n increases or decreases?



Statistics---hypothesis testing

Notes:

How do we know which hypothesis should be the null and which should be the alternative? Well, we are free to choose. We often choose so that the type I error is the more serious of the errors. Then we will choose k so that α is at an acceptably low level, such as .05 or .01. (Of course, if n is really large and we keep α fixed, then β might be very small, which might not be what we want.)

Statistics---hypothesis testing

Notes:

What if μ is not either 0 or 1? Well, much of the time we set up a hypothesis test so that $\Theta_0 \cup \Theta_A$ is the entire parameter space. So either the null or the alternative must be true. That means that one or both of the hypotheses are composite. When we have composite hypotheses, the test becomes more difficult to analyze. In particular, α and β may no longer be values but could be functions of the unknown parameter(s) θ .

Statistics---hypothesis testing

Notes:

What if our hypotheses were

$$H_0: \mu = 0$$

$$H_A: \mu \neq 0?$$

We could use the same test statistic \bar{X} , but what should the critical region look like?