2. The eigenvalue-eigenvector problem

In the course *Differential equations: 2 by 2 systems*, we learned that the first step in solving a linear 2×2 system of differential equations, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$ is to find the eigenvalues and eigenvectors of the 2×2 matrix $\mathbf{A}$. The procedure for solving linear $n \times n$ systems of DEs is the same, and starts with finding eigenvalues and eigenvectors. And even outside the context of differential equations, the eigenvalues and eigenvectors of a matrix tell us a lot about what the matrix does as a function on $\mathbb{R}^n$.

Definition 2.1 Let $\mathbf{A}$ be an $n \times n$ matrix.

- An **eigenvalue of $\mathbf{A}$** is a **scalar** λ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some **nonzero** vector $\mathbf{v}$.
- An **eigenvector of $\mathbf{A}$ associated with an eigenvalue λ** is a vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$

(We also say that an eigenvector $\mathbf{v}$ “corresponds to," or “belongs to" an eigenvalue λ.)

The eigenvalue-eigenvector problem is to find all possible scalars $\lambda,$ and for each $\lambda,$ all vectors $\mathbf{v}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

Warning: Eigenvalues and eigenvectors are defined only for **square** matrices.
Warning: Some authors require an eigenvector to be nonzero, but we allow $\mathbf{0}$ as an eigenvector. However, there must be at least one nonzero eigenvector for each eigenvalue, or it isn't an eigenvalue.
Note: Everyone allows that $\lambda = 0$ can be an eigenvalue.

Example 2.2 The matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ satisfies

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathbf{v} \quad = \quad 5\mathbf{v} \qquad \textbf{for all } \mathbf{v} \textbf{ in } \mathbb{R}^3.$$

Therefore, the number 5 is an eigenvalue and all vectors in $\mathbb{R}^3$ are eigenvectors associated to the eigenvalue 5.

Example 2.3 The diagonal matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ satisfies the following:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad = \quad 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad = \quad -1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The first equation above shows that the scalar 2 is an eigenvalue with associated eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Similarly, the second and third equations show that 0 and -1 are both eigenvalues, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors associated to 0 and -1 respectively.

Example 2.4 The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the scalar 1 is an eigenvalue with an associated eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and -1 is an eigenvalue with an associated eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Eigenvalue and eigenvector concept check

1/1 point (graded)

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 2 & 1 & 0 \end{pmatrix}$. Given that $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ is an eigenvector, find the eigenvalue λ that $\mathbf{v}$ is associated with.

$\lambda =$ 3 ✔ **Answer:** 3

3

Solution:

The calculation

$$\mathbf{Av} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = 3\mathbf{v},$$

shows that $\mathbf{v}$ is an eigenvector associated with eigenvalue 3.

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Scalar multiples of eigenvectors

1/1 point (graded)

As above, let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 2 & 1 & 0 \end{pmatrix}$. We know that $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ is an eigenvector associated to an eigenvalue λ (which you found in the previous problem).

Which of the following is true about the vector $\mathbf{w} = 2\mathbf{v}$?

- $\mathbf{w}$ is not an eigenvector.
- $\mathbf{w}$ is an eigenvector corresponding to the **same** eigenvalue λ. ✔
- $\mathbf{w}$ is an eigenvector corresponding to an eigenvalue **different** from λ.

Solution:

We have

$$\mathbf{Aw} = \mathbf{A}(2\mathbf{v}) = 2(\mathbf{Av}) = 2(3\mathbf{v}) = 6\mathbf{v} = 3\mathbf{w},$$

so $\mathbf{w}$ is an eigenvector associated to the same eigenvalue, 3.

(Alternatively, we could have multiplied out $\mathbf{Aw}$ explicitly to find out how it compared to $\mathbf{w}$, but that would have been more work.)

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