5. Exponential random variables

Sums and products

3/3 points (graded)

Let X be an exponential random variable with parameter $\lambda > 0$ and Y be a Poisson random variable with parameter $\mu > 0$. Assume that X and Y are independent. Compute the following quantities:

$\mathbb{E}[X^2 + Y^2] =$ mu^2+mu+2/lambda^2 ✔ **Answer:** 2/(lambda^2)+ mu + mu^2

$\mu^2 + \mu + \frac{2}{\lambda^2}$

$\mathbb{E}[X^2 Y] =$ 2*mu/lambda^2 ✔ **Answer:** 2 * mu / (lambda^2)

$\frac{2\cdot\mu}{\lambda^2}$

$\text{Var}(2X + 3Y) =$ 4/lambda^2+9*mu ✔ **Answer:** 4/(lambda^2) + 9 * mu

$\frac{4}{\lambda^2} + 9\cdot\mu$

STANDARD NOTATION

Solution:

First, let us review the moments of the Exponential and Poisson distribution:

If $X \sim \text{Exp}(\lambda)$ with $\lambda > 0$, then

$$\mathbb{E}[X] = \lambda, \quad \mathbb{E}[X^2] = \frac{2}{\lambda^2}, \quad \text{Var}(X) = \frac{1}{\lambda^2}. \tag{1.4}$$

If $Y \sim \text{Poi}(\mu)$, again with $\mu > 0$, then

$$\mathbb{E}[Y] = \mu, \quad \mathbb{E}[Y^2] = \mu + \mu^2, \quad \text{Var}(Y) = \mu. \tag{1.5}$$

Now, we can use the rules for expectation and variance to calculate:

$$\begin{aligned} \mathbb{E}[X^2 + Y^2] = & \quad \mathbb{E}[X^2] + \mathbb{E}[Y^2] \qquad (\text{linearity of expectation}) \\ = & \quad \frac{2}{\lambda^2} + \mu + \mu^2 \end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[X^2 Y\right] &= \mathbb{E}\left[X^2\right]\mathbb{E}\left[Y\right] && \text{(multiplicativity of expectation for independent variables)}\\
&= \frac{2\mu}{\lambda^2} &&\\
\mathsf{Var}\left(2X+3Y\right) &= \mathsf{Var}\left(2X\right)+\mathsf{Var}\left(3Y\right) && \text{(additivity of variance for independent variables)}\\
&= 2^2\mathsf{Var}\left(X\right)+3^2\mathsf{Var}\left(Y\right) && \text{(scaling property of variance)}\\
&= \frac{4}{\lambda^2}+9\mu &&
\end{aligned}$$

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Answers are displayed within the problem

Estimators

1/1 point (graded)

Let $X_1,\ldots,X_n$ be i.i.d exponential random variables with parameter λ and let $Z_i = \mathbf{1}\left(X_i \leq 1\right), i=1,\ldots,n$. Recall that $\mathbf{1}\left(X \leq 1\right)$ denotes the **indicator function** that takes the value 1 when $X \leq 1$ and 0 otherwise.

What is the limit in probability, as n goes to infinity, of $\frac{1}{n}\sum_{i=1}^{n} Z_i$?

$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow[n\to\infty]{\mathbf{P}}$ 1-exp(-lambda) ✔ **Answer:** 1 - exp(-lambda)

$1-\exp\left(-\lambda\right)$

STANDARD NOTATION

Solution:

Since the X_i are independent and identically distributed, so are the Z_i . By the Law of Large Numbers, we know that

$$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}\left[Z_i\right],$$

so it is enough to compute that quantity.

For this, note that

$$\mathbb{E}\left[Z_i\right] = \mathbf{P}\left(X_i \leq 1\right) = 1 - \exp\left(-\lambda \times 1\right) = 1 - \exp\left(-\lambda\right),$$

which follows from the formula for the cdf of an Exponential distribution. Hence,

$$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow[n\to\infty]{\mathbf{P}} 1 - \exp\left(-\lambda\right),$$

:

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Answers are displayed within the problem

Properties of the exponential distribution

2/2 points (graded)

Let X be an exponential random variable with parameter 2 that models the lifetime (in years) of a lightbulb. Compute the probability that the lightbulb lasts for at least 2 years. Round your answer to the nearest 10^{-2}.

$\mathbf{P}\left(X \geq 2\right) =$ 0.01831563888873418 ✔ **Answer:** exp(-4)

Given the lightbulb has lasted 2 years, find the probability that it lasts for k more years for any positive integer k.

$\mathbf{P}\left(X \geq k+2 | X \geq 2\right) =$ exp(-2*k) ✔ **Answer:** exp(-2*k)

$\exp\left(-2 \cdot k\right)$

STANDARD NOTATION

Solution:

The exponential distribution with parameter λ has a continuous density on $(0, \infty)$ with cdf

$$F\left(x\right) = 1 - \exp\left(-\lambda x\right).$$

Hence, for $\lambda = 2$,

$$\mathbf{P}\left(X \geq 2\right) = 1 - \mathbf{P}\left(X \leq 2\right) = 1 - \left(1 - \exp\left(-2 \times 2\right)\right) = e^{-4}.$$

For the second part, note that $\{X \geq k+2\} \subseteq \{X \geq 2\}$. Therefore,

$$\mathbf{P}\left(X \geq k+2 | X \geq 2\right) = \frac{\mathbf{P}\left(\{X \geq k+2\} \cap \{X \geq 2\}\right)}{\mathbf{P}\left(X \geq 2\right)} = \frac{\mathbf{P}\left(X \geq k+2\right)}{\mathbf{P}\left(X \geq 2\right)} = \frac{e^{-2(k+2)}}{e^{-4}} = e^{-2k}.$$

Remark: This is an example of the exponential distribution being **memoryless** : The probability of the lightulb lasting k more years given that it already lasted 2 years is exactly the same as the probability of it lasting k years in the first place.

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? Sum and Product Question
Unsure how E(X^2) is not 2*lambda^2. Can someone explain? I feel there is a problem with the solution. 2

? 'submit' button is grey
I can't submit answers now because the 'submit' button is grey and can't be clicked. Any one know why? 3

"with parameter 2"
So I assume this means that lambda = 2. 2

Estimation? 4

Properties of the exponential distribution- Second part
For the probability that given the lightbulb has lasted 2 years... I wrote a completely valid answer and It was considered wrong... so I suppuse that it will be much better if you ... 3