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1.3.4 (Optional) Qualitative Analysis of Differential Equations: Visualizing the Long-Term Behavior

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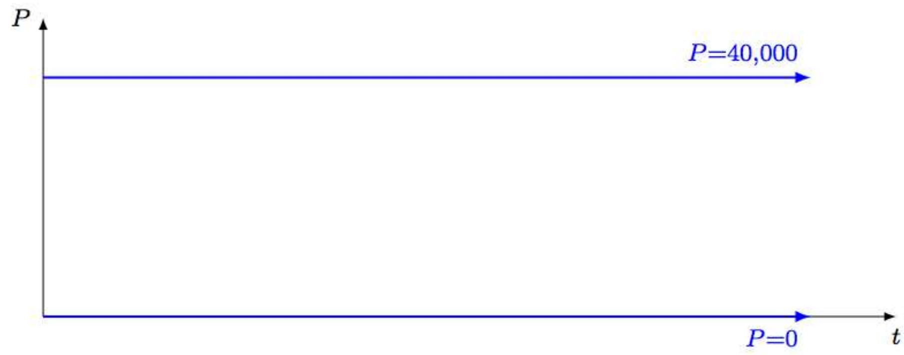
Population vs. Time

So far we've seen graphs of dP/dt versus P, but if we're discussing population, another natural graph to talk about it population versus time. We can actually say quite a bit about this graph, and the point of this subsection is to figure out what we can say and what that means for long-term outcomes for our population.

Let's start with the initial graph, when there is no fishing (so $\alpha = 0$). How can we tell what is happening to the population? We know the population is governed by the differential equation:

$$\frac{dP}{dt} = \frac{1}{10}P\left(1 - \frac{P}{40,000}\right).$$

How is the population changing when $P(t) = 40,000$? We substitute in $P = 40,000$ and see that $dP/dt = 0$. So the answer is, it isn't changing at all! This means that the population will stay constant at that level forever, or in other words $P(t) = 40,000$ is a constant solution, which we'll call an *equilibrium solution*. There's one other equilibrium solution – that is, one other solution where $dP/dt = 0$ for some particular value of the population P – namely, $P(t) = 0$. In terms of the graph of population versus time, this means we have two horizontal solutions:



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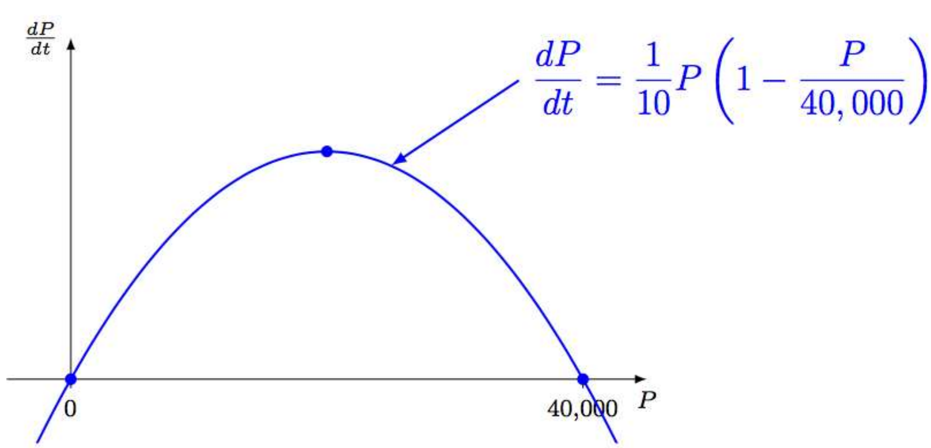
What happens when the population is between $P = 0$ and $P = 40,000$? Let's plug in an example value, say, $P = 20,000$. When we replace P by $20,000$ in our differential equation, we find that dP/dt is $1/10$ times $20,000$ times $1 - 1/2$ which is positive. (We're trying to get a sense of the graph, not actually plot it precisely. This is called a *qualitative analysis*.) That the derivative dP/dt is positive means that the population P is increasing when $P = 20,000$.

What we've actually done is plot three points on the graph of dP/dt, which is simply a parabola:



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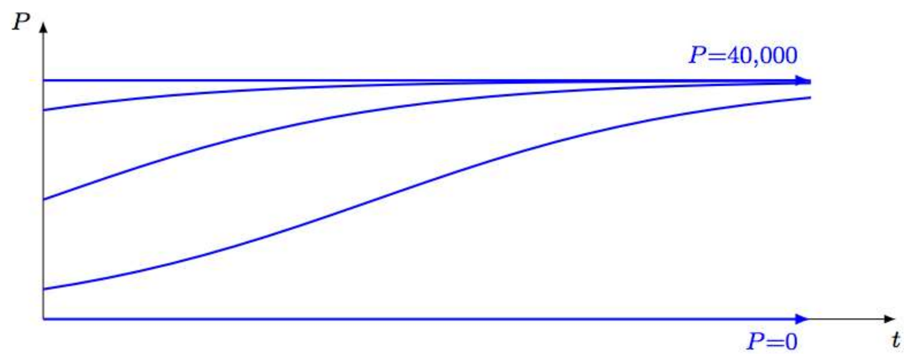
We've noted the points we plotted as dots on the curve. Notice that for *every* value of the population between $P = 0$ and $P = 40,000$ we have that dP/dt is positive. Thus the population is always increasing when the population is between 0 and $40,000$.

This means that solutions that have a starting population between 0 and $40,000$ are increasing functions. You also might notice that as the population increases toward $40,000$, dP/dt decreases toward zero. So the rate of increase decreases, creating a concave down portion of the curve. Now there's a fact you might know from the theory of solutions to differential equations: two different solution curves can't cross or meet. So these increasing solutions are bounded above by the horizontal solution $P = 40,000$. We get a number of (again, non-intersecting) curves:



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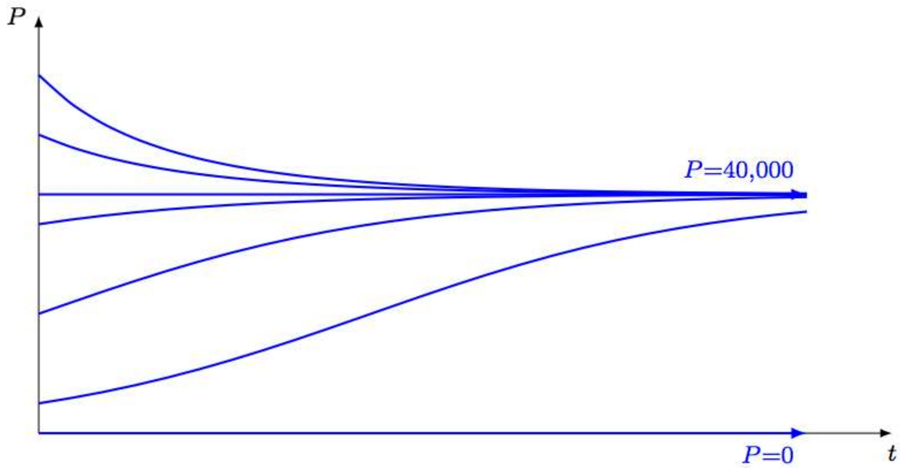
This curve is generally called a *logistic growth curve* and the original differential equation a logistic growth equation.

If you've completed the section on Item Response Theory, you'll recognize this curve, though it doesn't arise directly from a differential equation in that case.

What if the initial population is greater than $40,000$? Then the differential equation tells us that dP/dt is negative (just plug in a sample value, like $P = 60,000$, to check – or look at the graph of dP/dt for P greater than $40,000$). Then we get decreasing solutions above the horizontal line at $P = 40,000$:



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That's a pretty comprehensive understanding of the graph of population versus time – we can simply specify the population at time zero which determines the appropriate graph.

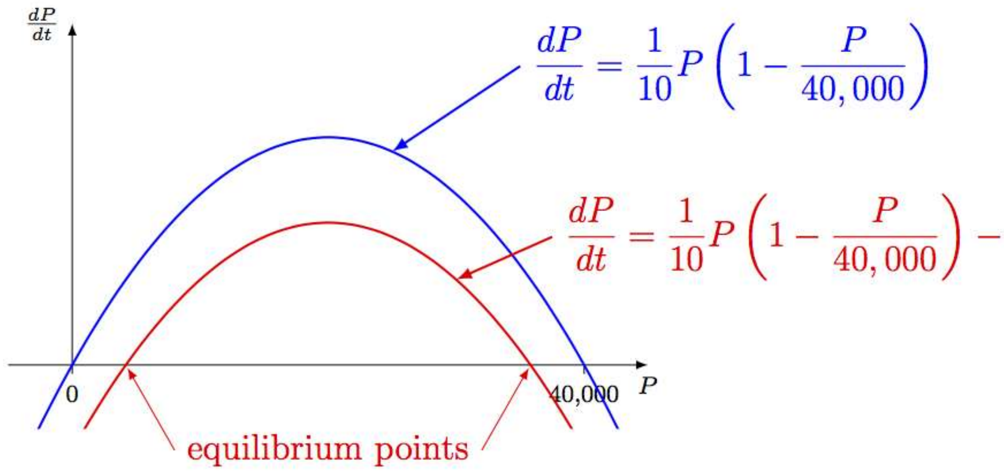
Notice we don't actually solve for the equation for each solution graph (except when P is an equilibrium solution); we just gave a rough sketch of them. This is why this type of analysis is called qualitative analysis of differential equations.

Now what if we add fishing? To find the equilibrium points, we'd need to solve the equation

$$\frac{dP}{dt} = \frac{1}{10}P\left(1 - \frac{P}{40,000}\right) - \alpha = 0.$$

where α is some constant value.

This is do-able, but it sounds like a lot of work just to get a rough picture. Can we tell instead roughly where the equilibrium solutions would be? Remember when we graph this, we can see where $dP/dt = 0$:



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So while we can't tell exactly what the equilibrium values are without doing some computation, we *can* see that these two values are "pinching in" from 0 and $40,000$ and are centered around a population of $20,000$.

So what does this mean for the population-time graph? We can draw in the equilibrium solutions right away:

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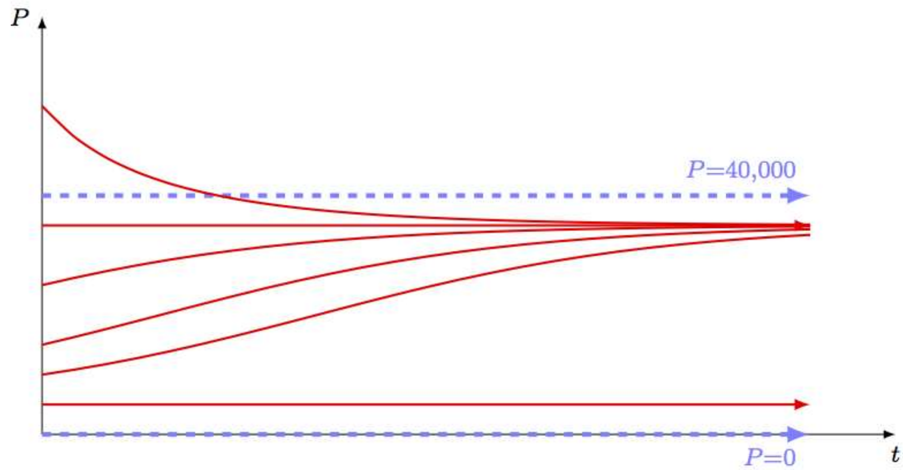
Note that they're closer together than the original two equilibrium points, which are shown in light-blue dotted lines. We can also notice that the work we did before – about the solutions between the equilibrium solutions and above the top one – still apply, so most of the solution curves are similar to what we had before:



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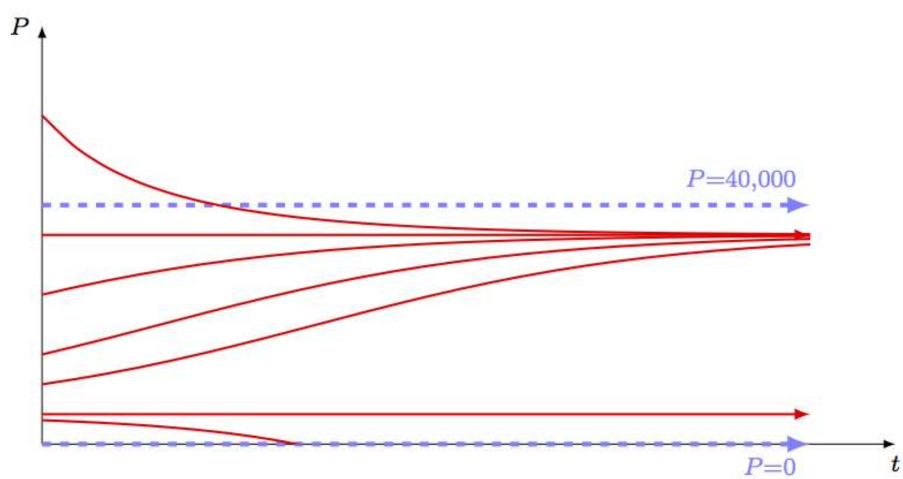
What about solutions that start with a population between zero and the lower equilibrium point? We can see from the graph of dP/dt versus P that population is decreasing, so our final set of solutions will look like this:



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Notice that this curve is cut-off at $P = 0$, because we don't want to worry about the model's non-physically reasonable results (like negative populations). What does this mean? It means extinction for the fish population: the population will decrease to zero.

In the exercises, you'll try another example and see what happens for large α and for the critical α. The big takeaway is that the graph of dP/dt versus P also allows us to get a qualitative understanding of the population versus time graph.