3. Minimize and maximize the distance, with scaffolding

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Practice

In the following sequence of problems, we will walk you through the process of using Lagrange multipliers to find the minimum and maximum distance between the unit circle $x^2 + y^2 = 1$ and the point $(3, 4)$.

You may want to try to solve this problem on your own before looking at how we break this problem down into smaller pieces first! Then compare your approach to ours.

First step: Find the formula

1.0/1 point (graded)
Before we can optimize a function, we need a formula.

Find a formula for the distance $D(x, y)$ between the point $(3, 4)$ and any arbitrary point in the plane (x, y).

$D(x, y) =$ sqrt((x-3)^2+(y-4)^2) ✔ **Answer:** sqrt((x-3)^2+(y-4)^2)

Solution:

The formula for the distance between a point (x, y) and a point $(3, 4)$ is the length of the vector connecting them. That is, our formula is

$$D(x, y) = |\langle x - 3, y - 4 \rangle| = \sqrt{(x-3)^2 + (y-4)^2}.$$

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Answers are displayed within the problem

Identify the constraint

1/1 point (graded)
Recall that our goal is to find the minimum and maximum distance between the unit circle $x^2 + y^2 = 1$ and the point $(3, 4)$.

Which of the following equations best describes the constraint against which we are maximizing and minimizing the distance formula above?

- ( ) $(x, y) = (0, 0)$
- ( ) $(x, y) = (3, 4)$
- ( ) $x^2 + y^2 = 0$
- (•) $x^2 + y^2 = 1$
- ( ) $D(x, y) = 0$
- ( ) $D(x, y) - (3, 4)$

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Solution:

The formula for $D(x, y)$ that we found above holds for any point in the plane. We want to find the maximum and minimum constrained to points in the unit circle. Thus the formula $x^2 + y^2 = 1$ is the constraint condition we are looking for!

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A plot twist: change the function!

1/1 point (graded)
Generally, we have our function $D(x, y)$ and our constraint $g(x, y)$, we immediately jump to solving the Lagrange multiplier problem:

$$\nabla D(x, y) = \lambda \nabla g(x, y)$$

You can absolutely jump right in and start solving! However, if the function you are optimizing is complicated, you can often optimize a related function instead, which simplifies the algebra involved but actually won't change your answer.

In this case, which related function should we optimize which will be easier to work with? (Make sure to read the solution for an explanation as to why this works!)

- (selected) $D(x, y)^2$
- $\sqrt{D(x, y)}$
- $D(x, y) - D(x, y)^2$
- We cannot optimize a different function.

Solution:

The function $D(x, y) = \sqrt{(x-3)^2 + (y-4)^2}$ is the positive square root of the sum of two squares. Thus it is greater than or equal to zero everywhere. The function that sends a number to its square $y = x^2$ is monotonically increasing for $x \geq 0$. Therefore, the function $D(x, y)^2 = (x-3)^2 + (y-4)^2$ is the sum of two squares and thus is also greater than or equal to zero, and the maxima and minima are preserved by this transformation. If $D(x, y)^2$ is minimized, then $D(x, y)$ is minimized as well.

Why is that true? Assume that $D(x, y)^2$ is minimized at the point (x_0, y_0). Then if there is a point (x_1, y_1) such that

$$0 \leq D(x_1, y_1) \leq D(x_0, y_0)$$

then we can square both sides to see that

$$0 \leq D(x_1, y_1)^2 \leq D(x_0, y_0)^2 \quad \text{(since squaring is monotonically increasing)}$$

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Thus $D(x_1, y_1) = D(x_0, y_0)$ so $D(x_0, y_0)$ is still minimal.

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Answers are displayed within the problem

The Lagrange multiplier problem

2.0/2 points (graded)
Now let's solve the problem!

What is the minimum distance between the unit circle and the point $(3, 4)$?

4 **Answer:** 4

What is the maximum distance between the unit circle and the point $(3, 4)$?

6 **Answer:** 6

Solution:

Solving our related problem we are looking for a solution to

$$\begin{aligned} \nabla D^2 &= \lambda \nabla g \\ 2(x-3) &= \lambda 2x \\ 2(y-4) &= \lambda 2y \end{aligned}$$

Solving both equations for λ and then setting them equal to each other we get

$$\begin{aligned} \frac{x-3}{x} &= \lambda \\ \frac{y-4}{y} &= \lambda \\ \frac{x-3}{x} &= \frac{y-4}{y} \end{aligned}$$

Cross multiplying and simplifying we get

$$\begin{aligned} x(y-4) &= y(x-3) \\ -4x &= -3y \\ x &= (3/4)\,y \end{aligned}$$

Finally, we apply our constraint equation $x^2 + y^2 = 1$, which gives us

$$\begin{aligned} x^2 + y^2 &= 1 \\ (3/4)^2 y^2 + y^2 &= 1 \\ \left(\frac{9}{16} + \frac{16}{16}\right) y^2 &= 1 \\ y^2 &= \frac{16}{25} \\ y &= \pm\frac{4}{5} \end{aligned}$$

Thus $x = \pm\frac{3}{4}\cdot\frac{4}{5} = \frac{3}{\ldots}$

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Thus $x = \pm\frac{3}{4} \cdot \frac{4}{5} = \frac{3}{5}$.

The critical points are $(3/5, 4/5)$ and $(-3/5, -4/5)$. Plugging them in we find (as we would suspect) that the point $(3/5, 4/5)$ minimizes the distance with a distance of 4, and the point $(-3/5, -4/5)$ maximizes the distance with a distance of 6. (Don't forget to take the distance not the distance squared here!)

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?	May be I don't understand the last question about Min/Max I tried to solve the last question by solving Grad DeltaD(x,y) ^2= Lambda*Grad g(x,y) I found x= +- 3/5 . am I on the right track?	3
	other way of solving this I figured out another of solving this problem. I assumed a function f(x,y) with gradf(x,y) = <3,4>. Since this is a plane with steepest i...	5
	A Plot Twist Interesting. I seem to recall we had a similar problem in an 18.01 recitation, and the simplification was the same as here.	1
?	Second question Is there any mistake with the second question ? What is the constraint we are talking about ?	2
	[Staff] Small typo in the solution Community TA	2

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