

MITx: 6.041x Introduction to Probability - The Science of Uncertainty

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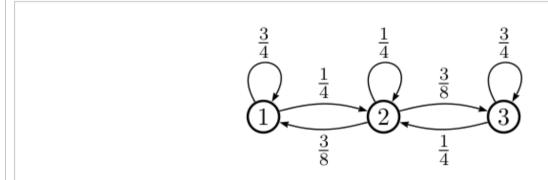
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Problem 6: Markov chain

(5/5 points)

Consider a Markov chain X_0, X_1, X_2, \ldots described by the transition probability graph shown below. The chain starts at state 1; that is, $X_0 = 1$.



1. Find the probability that $X_2=3$.

2. Find the probability that the process is in state 3 immediately after the second change of state. (A "change of state" is a transition that is not a self-transition.)

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Final Exam

Final Exam due May 24, 2016 at 23:59 UTC



3. Find (approximately) $\mathbf{P}(X_{1000}=2\mid X_{1000}=X_{1001})$.

$$\mathbf{P}(X_{1000}=2\mid X_{1000}=X_{1001})pprox 1/10$$
 $ightharpoons$ Answer: 0.1

4. Let T be the first time that the state is equal to 3.

5. Suppose for this part of the problem that the process starts instead at state 2, i.e., $X_0=2$. Let S be the first time by which both states 1 and 3 have been visited.

$$\mathbf{E}[S] = \boxed{12}$$
 Answer: 12

Answer:

- 1. The only path that leads to $X_2=3$ is 1 o 2 o 3. Therefore, ${f P}(X_2=3)=rac{1}{4}\cdot rac{3}{8}=rac{3}{32}.$
- 2. Since the chain starts at state 1, the first change of state will necessarily be from state 1 to state 2. In order for the second change of state to lead to state 3, the chain must transition to state 3 when it changes state from state 2. By symmetry, state 2 is equally likely to change state to state 1 or to state 3. Therefore, the probability of interest is 1/2.

3. By Bayes' rule, we have

$$egin{aligned} \mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001}) \ &= rac{\mathbf{P}(X_{1000} = X_{1001} = 2)}{\mathbf{P}(X_{1000} = X_{1001})} \ &= rac{\mathbf{P}(X_{1000} = 2)p_{22}}{\mathbf{P}(X_{1000} = 1)p_{11} + \mathbf{P}(X_{1000} = 2)p_{22} + \mathbf{P}(X_{1000} = 3)p_{33}} \ &pprox rac{\pi_2 \cdot p_{22}}{\pi_1 \cdot p_{11} + \pi_2 \cdot p_{22} + \pi_3 \cdot p_{33}}, \end{aligned}$$

where the π_i 's are the steady-state probabilities.

To compute these steady-state probabilities, we set up the balance equations, noting that the Markov chain represents a birth-death process and that states 1 and 3 are symmetric:

$$egin{aligned} \pi_1 \cdot rac{1}{4} &= \pi_2 \cdot rac{3}{8} \ \pi_1 &= \pi_3 \ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

Solving this system of equations yields

$$\pi_1 = \frac{3}{8},$$

$$egin{array}{ll} \pi_2 &= rac{1}{4}, \ &\pi_3 &= rac{3}{8}. \end{array}$$

Therefore,

$$\mathbf{P}(X_{1000}=2 \mid X_{1000}=X_{1001}) pprox rac{(1/4)(1/4)}{(3/8)(3/4)+(1/4)(1/4)+(3/8)(3/4)} = rac{1}{10}.$$

4. We set up the system of equations to find the mean first passage time to reach state 3 starting from 1:

$$egin{array}{ll} t_1 &= 1 + rac{3}{4}t_1 + rac{1}{4}t_2, \ t_2 &= 1 + rac{3}{8}t_1 + rac{1}{4}t_2. \end{array}$$

Solving this system of equation yields $\mathbf{E}[T] = t_1 = 32/3$.

5. In order to visit both states 1 and 3, the chain must first visit one of the two states. Suppose state 1 is visited first. After this point, the expected remaining time until we visit state 3 (and hence visit both states) is exactly the mean first passage time to state 3 that we calculated in part (4). Similarly, if state 3 is visited first, by symmetry of the chain, the mean first passage time to state 1 is also the same as what was calculated in part (4).

We condition on X_1 using the total expectation theorem to compute $\mathbf{E}[S]$:

$$\begin{aligned}
\mathbf{E}[S] &= 1 + \mathbf{E}[S \mid X_1 = 1]\mathbf{P}(X_1 = 1) + \mathbf{E}[S \mid X_1 = 2]\mathbf{P}(X_1 = 2) \\
&+ \mathbf{E}[S \mid X_1 = 3]\mathbf{P}(X_1 = 3) \\
&= 1 + \mathbf{E}[T] \cdot \frac{3}{8} + \mathbf{E}[S] \cdot \frac{1}{4} + \mathbf{E}[T] \cdot \frac{3}{8} \\
&= \frac{4}{3} \left(1 + \frac{3}{4} \mathbf{E}[T] \right) \\
&= 12.
\end{aligned}$$

An alternative solution is the following. Let G be the number of transitions until the chain first leaves state 2, which is a geometric random variable with $p=1-p_{22}=3/4$. Once the chain leaves state 2, it is either in state 1 or state 3. The subsequent expected time until the chain then reaches state 3 from 1 or state 1 from 3 is $\mathbf{E}[T]$ in both cases. Hence, $\mathbf{E}[S]=\mathbf{E}[G]+\mathbf{E}[T]=\frac{4}{3}+\frac{32}{3}=12$.

You have used 1 of 2 submissions

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