- Unit 0: Overview
- Entrance Survey
- Unit 1: Probability models and axioms
- Unit 2: Conditioning and independence
- Unit 3: Counting
- Unit 4: Discrete random variables
- Exam 1
- Unit 5: Continuous random variables

Problem 4: Maximum likelihood estimation

(3/3 points)

Let θ be an unknown constant. Let $W_1, \dots, W_n$ be independent exponential random variables each with parameter 1. Let $X_i = \theta + W_i$.

1. What is the maximum likelihood estimate of θ based on a single observation $X_1 = x_1$? Enter your answer in terms of x_1 (enter as x_1) using standard notation .

   $\hat{\theta}_{ML}(x_1) =$ x_1 ✔ **Answer:** x_1

2. What is the maximum likelihood estimate of θ based on a sequence of observations $(X_1, \dots, X_n) = (x_1, \dots, x_n)$?
   $\hat{\theta}_{ML}(x_1, \dots, x_n) =$

   - ○ $(x_1 x_2 \cdots x_n)^{1/n}$

- Unit 6: Further topics on random variables
- Unit 7: Bayesian inference
- Exam 2
- Unit 8: Limit theorems and classical statistics
- Unit 9: Bernoulli and Poisson processes
- Unit 10: Markov chains
- Exit Survey
- Final Exam

Final Exam
Final Exam due May 24, 2016 at 23:59 UTC

- $\dfrac{x_1 + \cdots + x_n}{n}$
- $\dfrac{1}{\frac{1}{x_1} + \cdots \frac{1}{x_n}}$
- $\min_i x_i$ ✔
- $\max_i x_i$
- None of the above

3. You have been asked to construct a confidence interval of the particular form $[\hat{\Theta} - c, \hat{\Theta}]$, where $\hat{\Theta} = \min_i\{X_i\}$ and c is a constant that we need to choose. For $n = 10$, how should the constant c be chosen so that we have a 95% confidence interval? (Give the smallest possible value of c.) Your answer should be accurate to 3 decimal places.

$c =$ 0.2995732 ✔ **Answer:** 0.29957

Answer:

1. To find $\hat{\theta}_{ML}$, we first find $f_{X_1}(x_1;\theta)$. Given $X_1 = \theta + W_1$, X_1 is a shifted exponential, where the entire distribution is shifted to the right by θ. Therefore,

$$f_{X_1}(x_1;\theta) = \begin{cases} e^{-(x_1-\theta)}, & x_1 \geq \theta, \\ 0, & x_1 < \theta. \end{cases}$$

   This quantity is maximized at $\hat{\theta}_{ML} = x_1$.

2. Let $X = (X_1, \ldots, X_n)$ and $x = (x_1, \ldots, x_n)$. To find $\hat{\theta}_{ML}$, we first find $f_X(x;\theta)$. Since the W_i's are independent, so are the X_i's. Hence,

$$\begin{aligned} f_X(x;\theta) &= \prod_{i=1}^{n} f_{X_i}(x_i;\theta) \\ &= \begin{cases} \prod_{i=1}^{n} e^{-(x_i-\theta)}, & \text{if } x_i \geq \theta \ \forall i, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that this quantity is nonzero only if θ is no greater than each of the x_i's. Moreover, $e^{-(x_i-\theta)}$ is greater when θ is closer to x_i. Therefore, this quantity is maximized when we push θ as high as possible while keeping it no greater than each of the x_i's. This means that $\hat{\theta}_{ML}(x) = \min_i x_i$. (Any larger choice of θ would give $f_X(x;\theta) = 0$.)

3. We wish to find c such that $\mathbf{P}(\hat{\Theta} - c \leq \theta \leq \hat{\Theta}) \geq 0.95$.

Since each of the W_i's is nonnegative (because they are exponential random variables), we have that $X_i \geq \theta$ for all i. This implies that $\theta \leq \hat{\Theta} = \min_i \{X_i\}$, and so $P(\theta \leq \hat{\Theta}) = 1$. Therefore, we need only $\mathbf{P}(\hat{\Theta} - c \leq \theta) \geq 0.95$.

Since the X_i's are independent, we have

$$\begin{aligned}
\mathbf{P}(\hat{\Theta} - c \leq \theta) &= \mathbf{P}(\min_i \{X_i\} \leq \theta + c) \\
&= 1 - \mathbf{P}(\min_i \{X_i\} \geq \theta + c) \\
&= 1 - \prod_{i=1}^{10} \mathbf{P}(X_i \geq \theta + c) \\
&= 1 - \prod_{i=1}^{10} \mathbf{P}(W_i \geq c) \\
&= 1 - \prod_{i=1}^{10} e^{-c} \\
&= 1 - e^{-10c}.
\end{aligned}$$

To have a 95% confidence interval, we require $1 - e^{-10c} \geq 0.95$, or

$$c \geq \frac{-\ln(0.05)}{10} = \frac{\ln(20)}{10} \approx 0.29957.$$

You have used 2 of 2 submissions