MITx: 6.041x Introduction to Probability - The Science of Uncertainty

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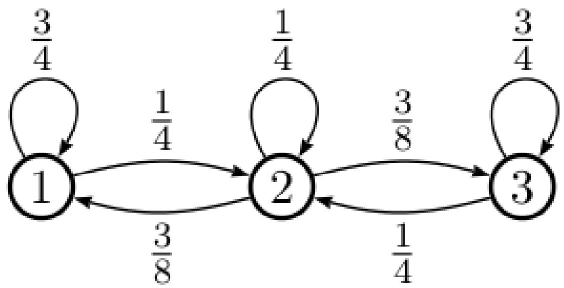
Problem 6: Markov chain

(5/5 points)

Consider a Markov chain $X_0, X_1, X_2, \ldots$ described by the transition probability graph shown below. The chain starts at state 1; that is, $X_0 = 1$.



1. Find the probability that $X_2 = 3$.

$\mathbf{P}(X_2 = 3) =$ 3/32 ✔ **Answer:** 0.09375

2. Find the probability that the process is in state 3 immediately after the second change of state. (A “change of state" is a transition that is not a self-transition.)

1/2 ✔ **Answer:** 0.5

3. Find (approximately) $\mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001})$.

$\mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001}) \approx$ 1/10 ✔ **Answer:** 0.1

4. Let T be the first time that the state is equal to 3.

$\mathbf{E}[T] =$ 32/3 ✔ **Answer:** 10.66667

5. Suppose for this part of the problem that the process starts instead at state 2, i.e., $X_0 = 2$. Let S be the first time by which both states 1 and 3 have been visited.

$\mathbf{E}[S] =$ 12 ✔ **Answer:** 12

Answer:

1. The only path that leads to $X_2 = 3$ is $1 \to 2 \to 3$. Therefore, $\mathbf{P}(X_2 = 3) = \frac{1}{4} \cdot \frac{3}{8} = \frac{3}{32}$.

2. Since the chain starts at state 1, the first change of state will necessarily be from state 1 to state 2. In order for the second change of state to lead to state 3, the chain must transition to state 3 when it changes state from state 2. By symmetry, state 2 is equally likely to change state to state 1 or to state 3. Therefore, the probability of interest is $1/2$.

3. By Bayes' rule, we have

$$\begin{aligned} &\mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001}) \\ = &\ \frac{\mathbf{P}(X_{1000} = X_{1001} = 2)}{\mathbf{P}(X_{1000} = X_{1001})} \\ = &\ \frac{\mathbf{P}(X_{1000} = 2) p_{22}}{\mathbf{P}(X_{1000} = 1) p_{11} + \mathbf{P}(X_{1000} = 2) p_{22} + \mathbf{P}(X_{1000} = 3) p_{33}} \\ \approx &\ \frac{\pi_2 \cdot p_{22}}{\pi_1 \cdot p_{11} + \pi_2 \cdot p_{22} + \pi_3 \cdot p_{33}}, \end{aligned}$$

where the π_i's are the steady-state probabilities.

To compute these steady-state probabilities, we set up the balance equations, noting that the Markov chain represents a birth-death process and that states 1 and 3 are symmetric:

$$\begin{aligned} &\pi_1 \cdot \frac{1}{4} = \pi_2 \cdot \frac{3}{8} \\ &\pi_1 = \pi_3 \\ &\pi_1 + \pi_2 + \pi_3 = 1. \end{aligned}$$

Solving this system of equations yields

$$\pi_1 = \frac{3}{8},$$

$$\begin{aligned} \pi_2 &= \frac{1}{4}, \\ \pi_3 &= \frac{3}{8}. \end{aligned}$$

Therefore,

$$\mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001}) \approx \frac{(1/4)(1/4)}{(3/8)(3/4) + (1/4)(1/4) + (3/8)(3/4)} = \frac{1}{10}.$$

4. We set up the system of equations to find the mean first passage time to reach state 3 starting from 1:

$$\begin{aligned} t_1 &= 1 + \frac{3}{4}t_1 + \frac{1}{4}t_2, \\ t_2 &= 1 + \frac{3}{8}t_1 + \frac{1}{4}t_2. \end{aligned}$$

   Solving this system of equation yields $\mathbf{E}[T] = t_1 = 32/3$.

5. In order to visit both states 1 and 3, the chain must first visit one of the two states. Suppose state 1 is visited first. After this point, the expected remaining time until we visit state 3 (and hence visit both states) is exactly the mean first passage time to state 3 that we calculated in part (4). Similarly, if state 3 is visited first, by symmetry of the chain, the mean first passage time to state 1 is also the same as what was calculated in part (4).

We condition on X_1 using the total expectation theorem to compute $\mathbf{E}[S]$:

$$\begin{aligned}\mathbf{E}[S] &= 1+\mathbf{E}[S \mid X_1=1]\mathbf{P}(X_1=1)+\mathbf{E}[S \mid X_1=2]\mathbf{P}(X_1=2) \\ &\quad +\mathbf{E}[S \mid X_1=3]\mathbf{P}(X_1=3) \\ &= 1+\mathbf{E}[T]\cdot\frac{3}{8}+\mathbf{E}[S]\cdot\frac{1}{4}+\mathbf{E}[T]\cdot\frac{3}{8} \\ &= \frac{4}{3}\left(1+\frac{3}{4}\mathbf{E}[T]\right) \\ &= 12.\end{aligned}$$

An alternative solution is the following. Let G be the number of transitions until the chain first leaves state 2, which is a geometric random variable with $p = 1 - p_{22} = 3/4$. Once the chain leaves state 2, it is either in state 1 or state 3. The subsequent expected time until the chain then reaches state 3 from 1 or state 1 from 3 is $\mathbf{E}[T]$ in both cases. Hence, $\mathbf{E}[S] = \mathbf{E}[G] + \mathbf{E}[T] = \frac{4}{3} + \frac{32}{3} = 12$.

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