15. Conditions for invertibility

There are two types of square matrices A:

- those with $\det \mathbf{A} \neq 0$ (called **nonsingular** or **invertible**), and
- those with $\det \mathbf{A} = 0$ (called **singular**).

Nonsingular matrices

Theorem 15.1 For a square $n \times n$ matrix $\mathbf{A}$, the following are equivalent:

1. $\det \mathbf{A} \neq 0$
2. $\mathrm{NS}(\mathbf{A}) = \{\mathbf{0}\}$ (the only solution to $\mathbf{A}\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$)
3. $\mathbf{rank}(\mathbf{A}) = n$ (image is n-dimensional)
4. $\mathrm{CS}(\mathbf{A}) = \mathbb{R}^n$ (image is the whole space $\mathbb{R}^n$)
5. For each vector $\mathbf{b}$, the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has exactly one solution.
6. $\mathbf{A}^{-1}$ exists.
7. $\mathtt{rref}(\mathbf{A}) = \mathbf{I}$

So if you have a matrix $\mathbf{A}$ for which one of these conditions holds, then **all** of the conditions hold for $\mathbf{A}$.

Consequences of a nonzero determinant

Let's explain the consequences of $\det \mathbf{A} \neq 0$.

1. The input space $\mathbb{R}^n$ is not flattened by $\mathbf{A}$.
2. Intuitively, there are no “crushed dimensions", so $\mathrm{NS}(\mathbf{A}) = \{\mathbf{0}\}$. Since no dimensions were crushed, the image $\mathrm{CS}(\mathbf{A})$ has the same dimension as the input space, namely n.
3. By definition, $\mathbf{rank}(\mathbf{A}) = \mathbf{dim}\,\mathrm{CS}(\mathbf{A}) = n$. (Alternatively, this follows from $\mathbf{dim}\,\mathrm{NS}(\mathbf{A}) + \mathbf{rank}(\mathbf{A}) = n$.)
4. The only n-dimensional subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself, so $\mathrm{CS}(\mathbf{A}) = \mathbb{R}^n$. Thus every $\mathbf{b}$ is in $\mathrm{CS}(\mathbf{A})$, so $\mathbf{Ax} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
5. The system $\mathbf{Ax} = \mathbf{b}$ has the same number of solutions as $\mathbf{Ax} = \mathbf{0}$ (they are just shifted by adding a particular solution $\mathbf{x}_p$); that number is 1 (the only solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{0}$).
6. To say that $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for each $\mathbf{b}$ means that the associated linear transformation $\mathbf{f}$ is a 1-to-1 correspondence, so $\mathbf{f}^{-1}$ exists, so $\mathbf{A}^{-1}$ exists. (Moreover, we showed how to find $\mathbf{A}^{-1}$ by Gauss–Jordan elimination.)
7. So we have $\mathtt{rref}(\mathbf{A}) = \mathbf{I}$ as explained earlier, since $\mathbf{I}$ is the only RREF square matrix with nonzero determinant.

Invertibility concept check

1/1 point (graded)

Suppose $\mathbf{A}$ is a nonsingular matrix, and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are three $n \times n$ matrices. **True or False?** If $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

● True ✓

○ False

Solution:

True. If $\mathbf{A}$ is nonsingular, then it is invertible. Multiplying both sides of the equation $\mathbf{AB} = \mathbf{AC}$ by $\mathbf{A}^{-1}$ we see that

$$\begin{aligned}\mathbf{A}^{-1}(\mathbf{AB}) &= \mathbf{A}^{-1}(\mathbf{AC})\\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} &= (\mathbf{A}^{-1}\mathbf{A})\mathbf{C}\\ \mathbf{IB} &= \mathbf{IC}\\ \mathbf{B} &= \mathbf{C}.\end{aligned}$$

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Answers are displayed within the problem

Invertibility concept check II

1/1 point (graded)

If $\mathbf{A}$ is an $n \times n$ matrix with $n \geq 3$, and $\mathbf{column\ 1} + \mathbf{column\ 2} = \mathbf{column\ 3}$, is $\mathbf{A}$ invertible?

○ Yes; $\mathbf{A}$ is invertible.

● No; $\mathbf{A}$ is not invertible. ✓

Solution:

No, $\mathbf{A}$ is not invertible. If $\mathbf{column\ 1} + \mathbf{column\ 2} = \mathbf{column\ 3}$, then

$$A\begin{pmatrix}1\\1\\-1\\0\\\vdots\\0\end{pmatrix} = \mathbf{column\ 1} + \mathbf{column\ 2} - \mathbf{column\ 3} = \mathbf{0},$$

so Condition 5 for invertibility fails since $\mathbf{Ax} = \mathbf{0}$ has more than one solution.

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