Homework

The exercises below will count towards your grade. **You have only one chance to answer these questions.** Take your time, and think carefully before answering.

Problem 1a

15.0/15.0 points (graded)

Three properties a relation might have are *reflexivity*, *symmetry*, and *transitivity.*

If A is a set and R is a relation that holds amongst members of that set:

- R is reflexive on A if and only if for any $a \in A$, aRa.
- R is symmetric on A if and only if for any $a, b \in A$, if aRb then bRa.
- R is transitive on A if and only if for any $a, b, c \in A$, if aRb and bRc then aRc.

Here is an example. Let A be the set of people, and let R be the relation "having the same birthday as". R is reflexive because everyone has the same birthday as herself. R is symmetric because whenever a has the same birthday as b, b will have the same birthday as a. And R is transitive because for any people a, b, and c, if a has the same birthday as b, and b has the same birthday as c, then a will have the same birthday as c.

Now consider the relation "is less than or equal to".

Is this relation reflexive, symmetric, and transitive on the set of natural numbers?

Reflexive?

- [x] yes
- [ ] no

✔

Explanation
"Less than or equal to" is reflexive on the natural numbers, since every number is less than or equal to itself.

Symmetric?

- [ ] yes
- [x] no

✔

Explanation
"Less than or equal to" is transitive on the natural numbers, since if x is less than or equal to y, and y is less than or equal to z, x is less than or equal to z.

Transitive?

- [x] yes
- [ ] no

✔

Explanation
“Less than or equal to” is not symmetric on the natural numbers, since if x is less than or equal to y, there is no guarantee that y is less than or equal to x. That is true only in the special case where x is equal to y.

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Answers are displayed within the problem

Problem 1b

15.0/15.0 points (graded)
There are ten guests at a dinner party, sitting around a large table. Consider the relation R such that guest a bears R to guest b just in case a's seat is immediately adjacent to b's seat. (Note that a seat is not adjacent to itself.)

Is R reflexive, symmetric, and transitive on the set of guests?

Reflexive?

- [ ] yes
- [x] no

✔

Explanation
R fails to be reflexive because nobody is in a seat immediately adjacent to her own seat (your seat is only immediately adjacent to your neighbor's seats).

Symmetric?

- [x] yes

no

Explanation

R is not transitive; to see this, suppose that a is seated immediately to the right of b and that b is seated immediately to the right of c; then a bears R to b and b bears R to c, but a doesn't bear R to c.

Transitive?

yes

no

Explanation

R is symmetric, since the only way for a to be in a seat immediately adjacent to (b\) is for b to be in a seat immediately adjacent to a.

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Answers are displayed within the problem

Problem 2

15.0/15.0 points (graded)

Recall that a function from set A to set B is an assignment of each member of A to some member of B. For a function f from A to B to be an *injection* is for f never to assign the same element of B to two different elements of A. For a function f from A to B to be a *surjection* is for there to be no element of B to which f fails to assign some element of A. For a function to be a *bijection* is for it to be both an injection and a surjection.

Describe each of the following functions f from A to B:

$A = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$B = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$f(x) = x + 2$

- ○ f is an injection (but not a surjection)
- ○ f is a surjection (but not an injection)
- ◉ f is a bijection
- ○ f is neither a bijection nor a surjection

✔

Explanation
The function is bijective, since every element of set A is paired with exactly one element of set B, and every element of set B is paired with exactly one element of set A. 0 is paired with 2, -5 is paired with -3, -1000 is paired with -998, and so on.

$A = \{1, 2, 3, \ldots, 10\}$

$B = \{1, 2, 3, \ldots, 100\}$

$f(x) = x^2$

- ◉ f is an injection (but not a surjection)
- ○ f is a surjection (but not an injection)

- f is a bijection
- f is neither a bijection nor a surjection ✔

✔

Explanation

The function is injective, since no two elements of A are mapped to the same element of B. It is not surjective, since not every element of B is hit by an element of A. E.g., there is no number in set A whose square is 70, although 70 is an element of set B.

$$A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

$$B = \{0, 1, 4, 9, 16, 25\}$$

$$f(x) = x^2$$

- f is an injection (but not a surjection)
- (selected) f is a surjection (but not an injection)
- f is a bijection
- f is neither a bijection nor a surjection

✔

Explanation

The function is surjective, since every element of set B is mapped to by at least one element of set A. It is not injective, since there are elements of set B (most of them, in fact) that are mapped to by more than one element of A. E.g., both -5 and 5 are mapped to 25.

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Answers are displayed within the problem

Problem 3

0.0/5.0 points (graded)
A set is *dense* (relative to a given ordering) if and only if there is a member of the set between any two members of the set (according to that ordering).

Does the fact that a set is dense (relative to some ordering) entail that the set is bigger than the set of the natural numbers?

- (selected) yes
- no ✔

✖

Explanation
No. The set of rational numbers is dense (relative to the standard ordering), but there are as many rational numbers as natural numbers.

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Answers are displayed within the problem

Problem 4

20.0/20.0 points (graded)
Between which of these infinite sets is it possible to construct a bijection:

The set of natural numbers and the set of integers?

- (selected) Yes

No

The set of prime numbers and the set of real numbers?

Yes

No

The set of rational numbers and the set of real numbers between 0 and 1?

Yes

No

The set of real numbers and the power set of the real numbers?

Yes

No

Explanation

The set of natural numbers, the set of integers and the set of prime numbers are all of the same size. The set of real numbers between 0 and 1 has the same cardinality as the set of real numbers (and therefore greater cardinality than the set of natural numbers), so there can be no bijection between the set of rational numbers and the set of reals between 0 and 1 or between the set of prime numbers and the set of real numbers. Since the power set of a set is strictly larger than set itslef, the power set of the real numbers is bigger than the set of real numbers.

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Answers are displayed within the problem

Problem 5

5.0/5.0 points (graded)

Suppose that Hilbert's Hotel is completely full. We showed that when as many new guests as there are natural numbers show up, the new guests can be accommodated.

Now suppose that as many new guests as there are *real* numbers show up. Can they be accommodated? (In other words: can we fit *uncountably* new guests in Hilbert's Hotel?)

- Yes
- No

✔

Explanation

No. There are only countably many hotel rooms. If we could fit uncountably many guests in the hotel, then there would be a bijection between a countable set (the set of hotel rooms) and an uncountable set (the set of guests). But there is no such bijection. So an uncountable number of guests will not fit.

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Answers are displayed within the problem

Problem 6

5.0/5.0 points (graded)

There is a bijection between the set of natural numbers and the set of prime numbers. To see this, note that one can assign the smallest prime number (i.e. the zeroth-smallest prime number) to 0, the next smallest prime number (i.e. the first-smallest prime number) to 1, and, in general, assigning the nth smallest prime to n.

Is there also a bijection between the set of prime numbers and the set of integers?

- (selected) Yes
- No

✔

Explanation

Yes. Here is one example of a bijection: assign the kth-smallest prime number to $k/2$ if k is even and to $-(k+1)/2$ if k is odd.

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Answers are displayed within the problem

Problem 7

10/10 points (graded)

Is there a bijection between the set of natural numbers, $\mathbb{N}$, and the set of all pairs of natural numbers, $\{\langle n, m\rangle : n, m \in \mathbb{N}\}$?

- (selected) Yes
- No

✔

Explanation

Yes. This can be verifed using the same construction that was used to show that the natural numbers are in one-one correspondence with the rational numbers. So put the pairs $\langle n, m\rangle$ in a matrix, where n increases rightward on the horizontal axis and m increases downward on the veritcal axis, and then count diagonally across the table just as we do to count the rationals.

Is there a bijection between the set of natural numbers, $\mathbb{N}$, and the set $\{x : x$ is a function that assigns each natural number to a member of $\{1,2,3,4,5\}\}$?

Hint: Can you derive a contradiction from the assumption that there is such a bijection? How about from the assumption that there isn't?

- [ ] Yes
- [x] No

✔

Explanation

No. We can verify this using a version of the construction that we used to show that there are more real numbers than natural numbers. Assume for *reductio* that there is a bijection f from the natural numbers to the relevant set of functions. Let $f(n) = g_n$, where g_n is a function from natural numbers to $\{1,2,3,4,5\}$. Then we know that every function from the natural numbers to $\{1,2,3,4,5\}$ is g_m for some natural number n. Now consider the 'diagonal' sequence $\langle g_0(0), g_1(1), \ldots\rangle$, and construct its evil twin $\langle t(g_0(0)), t(g_1(1)), \ldots\rangle$, where $t(x) = 1$ if $x \neq 1$, and $t(x) = 2$ if $x = 1$. The evil twin funcction $h(n) = t(g_n(n))$ is a function from natural numbers to $\{1,2,3,4,5\}$. But $h \neq g_m$ for every m, since $g_m(m) \neq t(g_m(m)) = h(m)$. This contradicts our earlier claim that every function from natural numbers to $\{1,2,3,4,5\}$ is g_m for some natural number m.

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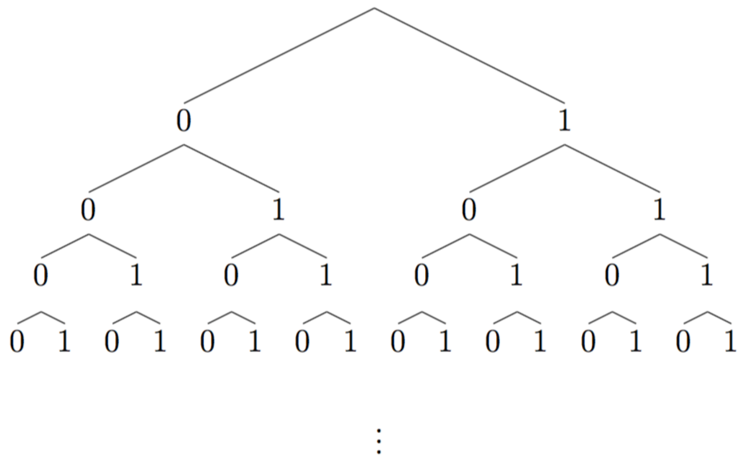
Answers are displayed within the problem

Problem 8

10/10 points (graded)

Consider the infinite tree in the image below. (When fully spelled out, the tree contains one row for each natural number. The zero-th row contains one node, the first row contains two nodes, the second row contains four nodes, and, in general, the nth row contains 2^n nodes.)



Is there a bijection between the set of nodes of the tree and the set of natural numbers?

- Yes
- No

Explanation
There are as many nodes as natural numbers. One way to see this is to note that each node can be arrived at as a result of taking a finite number of $0-1$ decisions as one travels down on the tree. Accordingly, each node corresponds to a unique finite sequence of zeroes and ones. To see that there are no more finite sequences of zeroes and ones than there are natural numbers, consider the function that assigns to each sequence the number whose decimal notation that results from writing down a '5' followed by each number in the sequence. (For instance, to the sequence $\langle 0,0,1,0,1,1\rangle$ we assign the number 5001011.) This gives us an injection from the set of finite sequences of zeroes and ones into the set of natural numbers.

Is there a bijection between the set of paths of this tree and set of natural numbers?
A path is an infinite sequence of nodes which starts at the top of the tree and contains a node at every row, with each node connected to its successor by an edge. Paths can be represented as infinite sequences of zeroes and ones.

- Yes

No

Explanation

We can show that there are more paths than natural numbers. One way to see this is by noting that each path can be arrived at as a the result of taking infinitely many $0-1$ decisions as one travels down the tree. So each node corresponds to an infinite sequence of zeroes and ones, or, more precisely, to a function from the natural numbers to the set $\{0,1\}$. But we can show that there are more such functions than natural numbers, using a variant of the proof discussed in "The Proof" subsection of "The Power Set of Natural Numbers" section. In fact, there are as many paths as there are real numbers.

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Answers are displayed within the problem