5. Total Variation Distance for Discrete Random Variables

Quiz: Probability Mass Functions

1/1 point (graded)

Let X be a discrete random variable whose sample space is $\mathbb{Z}$, the set of integers. Let $p : \mathbb{Z} \to [0,1]$ denote the **probability mass function** (**pmf**) of X. What does $p\,(7) + p\,(10)$ represent?

- The probability that $X = 10$.
- The probability that $X = 7$.
- The probability that $X = 7$ **or** $X = 10$.
- The probability that $X = 7$ **and** $X = 10$.

✔

Solution:

By definition, $p(7) + p(10) = P(X = 10) + P(X = 7)$. The events $X = 10$ and $X = 7$ are disjoint, so in fact $p(7) + p(10) = P(X = 10 \text{ or } X = 7)$.

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Answers are displayed within the problem

Preparation: Probability of Complements

1/1 point (graded)
What is $\mathbf{P}_\theta(A^c) - \mathbf{P}_{\theta'}(A^c)$ in terms of $\mathbf{P}_\theta(A)$ and $\mathbf{P}_{\theta'}(A)$? (Recall A^c is the complement of A in the sample space.)

- (selected) $\mathbf{P}_{\theta'}(A) - \mathbf{P}_\theta(A)$
- $\mathbf{P}_\theta(A) - \mathbf{P}_{\theta'}(A)$

✔

Solution:

$$\mathbf{P}_\theta(A^c) - \mathbf{P}_{\theta'}(A^c) = (1 - \mathbf{P}_\theta(A)) - (1 - \mathbf{P}_{\theta'}(A)) = \mathbf{P}_{\theta'}(A) - \mathbf{P}_\theta(A).$$

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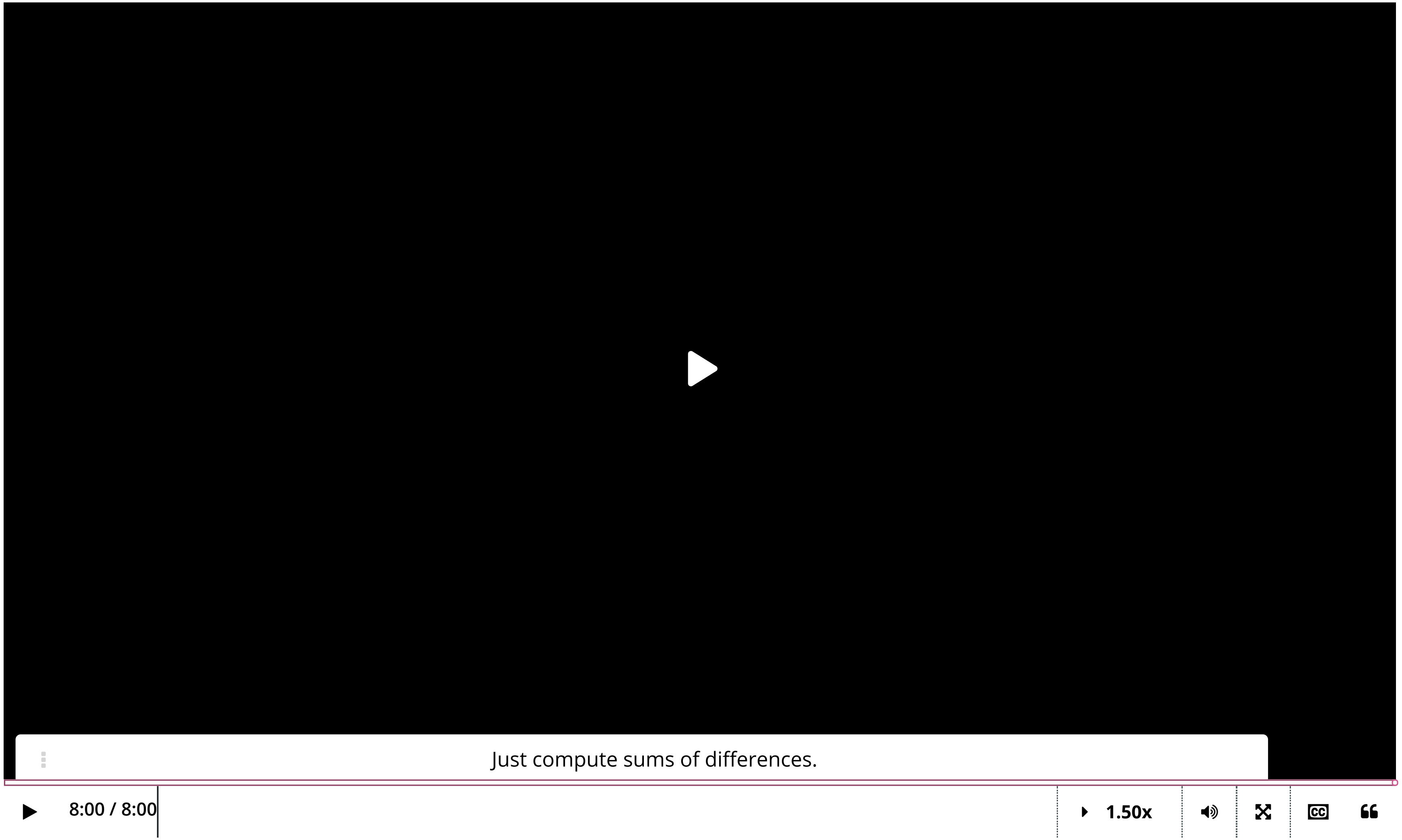
Total Variation Distance for Discrete Distributions



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Let $\mathbf{P}$ and $\mathbf{Q}$ be probability measures with a discrete sample space E and probability mass functions f and g. Then, the total variation distance between $\mathbf{P}$ and $\mathbf{Q}$:

$$]\mathrm{TV}(\mathbf{P},\mathbf{Q}) = \max_{A\subset E}|\mathbf{P}(A) - \mathbf{Q}(A)|,$$

can be computed as

$$\mathrm{TV}(\mathbf{P},\mathbf{Q}) = \frac{1}{2}\sum_{x\in E}|f(x) - g(x)|.$$

Equivalence of Formulas

4/4 points (graded)

Let $E = \{1,2,3,4\}$ be a discrete sample space. Let $\mathbf{P}$ and $\mathbf{Q}$ be probability measures with probability mass functions f and g as follows:

$$f(1) = 1/4,\ f(2) = 1/4,\ f(3) = 1/8,\ f(4) = 3/8$$

$$g(1) = g(2) = g(3) = g(4) = 1/4$$

Find the value of $|\mathbf{P}(A) - \mathbf{Q}(A)|$ for the following choices of A.

For $A = \{3\}$:

$|\mathbf{P}(A) - \mathbf{Q}(A)| =$ 0.125 **Answer:** 1/8

For $A = \{4\}$:

$|\mathbf{P}(A) - \mathbf{Q}(A)| =$ 0.125 **Answer:** 1/8

For $A = \{3,4\}$?

$|\mathbf{P}(A) - \mathbf{Q}(A)| =$ 0 ✔ **Answer:** 0

What is the value of $\max_{A \subset E} |\mathbf{P}(A) - \mathbf{Q}(A)|$?

0.125 ✔ **Answer:** 1/8

STANDARD NOTATION

Solution:

First, compute $|\mathbf{P}(A) - \mathbf{Q}(A)|$ for the different choices of A:

- When $A = \{3\}$, $\mathbf{P}(A) = f(3) = 1/8$ and $\mathbf{Q}(A) = g(3) = 1/4$. Therefore, $|\mathbf{P}(A) - \mathbf{Q}(A)| = 1/8$.
- When $A = \{4\}$, $\mathbf{P}(A) = f(4) = 3/8$ and $\mathbf{Q}(A) = g(4) = 1/4$. Therefore, $|\mathbf{P}(A) - \mathbf{Q}(A)| = 1/8$.
- When $A = \{3,4\}$, $\mathbf{P}(A) = f(3) + f(4) = 1/2$ and $\mathbf{Q}(A) = g(3) + g(4) = 1/2$. Therefore, $|\mathbf{P}(A) - \mathbf{Q}(A)| = 0$.

Now, we find $\max_{A \subset E} |\mathbf{P}(A) - \mathbf{Q}(A)|$. We have already considered $A = \{3\}$, $A = \{4\}$, and $A = \{3,4\}$. For any other non-empty set A, $|\mathbf{P}(A) - \mathbf{Q}(A)|$ takes on one of the values that we have already computed because $f(1) = f(2) = g(1) = g(2) = 1/4$.

In particular, for any set that includes 3 but does not include 4, $|\mathbf{P}(A) - \mathbf{Q}(A)| = |-1/8| = 1/8$. For any set that includes 4 but does not include 3, $|\mathbf{P}(A) - \mathbf{Q}(A)| = |1/8| = 1/8$. And finally, for any set that includes both 3 and 4, $|\mathbf{P}(A) - \mathbf{Q}(A)| = 0$.

Therefore, $\max_{A \subset E} |\mathbf{P}(A) - \mathbf{Q}(A)| = 1/8$, with the maximum achieved with numerous sets as discussed above.

Submit You have used 2 of 2 attempts

Answers are displayed within the problem

Equivalence of Formulas (cont.)

1/1 point (graded)

Setup as above:

Let $E = \{1,2,3,4\}$ be a discrete sample space. Let $\mathbf{P}$ and $\mathbf{Q}$ be probability measures with probability mass functions f and g as follows:

$$f(1) = 1/4,\ f(2) = 1/4,\ f(3) = 1/8,\ f(4) = 3/8$$

$$g(1) = g(2) = g(3) = g(4) = 1/4$$

Question: What is the value of $\frac{1}{2}\sum_{x\in E} |f(x) - g(x)|$?

0.125

Answer: 1/8

Solution:

$$\frac{1}{2}\sum_{x\in E} |f(x) - g(x)| = \frac{1}{2}\left(0+0+\frac{1}{8}+\frac{1}{8}\right) = \frac{1}{8}.$$

This is the same result as in the previous problem.

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You have used 1 of 2 attempts

Answers are displayed within the problem

Computing Total Variation Distance I

1/1 point (graded)

Let $X \sim \mathbf{P} = \mathrm{Ber}\left(1/2\right)$ and $Y \sim \mathbf{Q} = \mathrm{Ber}\left(1/2\right)$. What is $\mathrm{TV}\left(\mathbf{P}, \mathbf{Q}\right)$, the total variation distance between the distributions of the Bernoulli random variables X and Y?

Note that we make no assumptions about X and Y being independent.

0

✔ **Answer:** 0.0

Solution:

Intuitively, since X and Y have the same distribution, we expect the (total variation) distance between their distributions to be 0. And indeed this is the case. Observe that for any event, $\mathbf{P}\left(A\right) = \mathbf{Q}\left(A\right)$ since $\mathbf{P}$ and $\mathbf{Q}$ are both $\mathrm{Ber}\left(1/2\right)$.

$$\mathrm{TV}\left(\mathbf{P}, \mathbf{Q}\right) = \max_{A \subset E} \left|\mathbf{P}\left(A\right) - \mathbf{Q}\left(A\right)\right| = 0.$$

Note that the distance between two distributions only depends on the distributions themselves and *not* their relation to each other (the joint distribution). This is why assuming X and Y are independent (or not) does not affect the total variation distance.

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You have used 1 of 3 attempts

Answers are displayed within the problem

Computing Total Variation II

1/1 point (graded)

Let $X \sim \mathbf{P} = \mathrm{Ber}\left(1/2\right)$ and $Y \sim \mathbf{Q} = \mathrm{Ber}\left(1/3\right)$. What is $\mathrm{TV}\left(\mathbf{P}, \mathbf{Q}\right)$, the total variation distance between the distributions of the Bernoulli random variables X and Y?

1/6 ✔ **Answer:** 1/6

Solution:

For this problem, the sample space of X and Y is $\{0,1\}$. Let f be the pmf of X and let g be the pmf of Y. Note that $f(1) = f(0) = 1/2$ and $g(1) = 1/3, g(0) = 2/3$. Hence, we can apply the given formula:

$$\begin{aligned}\mathrm{TV}(\mathbf{P},\mathbf{Q}) &= \frac{1}{2}\sum_{x\in E}|f(x) - g(x)| \\ &= \frac{1}{2}(|f(0) - g(0)| + |f(1) - g(1)|) \\ &= \frac{1}{2}(1/6 + 1/6) = 1/6 \approx 0.16667.\end{aligned}$$

Remark: In general, we have the formula

$$\mathrm{TV}(\mathrm{Ber}(p), \mathrm{Ber}(q)) = |p - q|.$$

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Answers are displayed within the problem

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? detailed proof of the TV calculation formula for PMF
I just would like to learn more. Could you show me the detailed proof about how to get the TV formula for PMF from the definition TN? 3

10 attempts seem like too many for the last question...
Not that I mind :-) 1