Lebesgue Measure

As it turns out, there is *exactly one* natural way of extending the ordinary notion of length so that it applies to all Borel Sets. More precisely, there is exactly one function λ on the Borel Sets that satisfies these three conditions:

Length on Segments
$\lambda([a, b]) = b - a$.

(This condition is meant to ensure that λ counts as an extension, rather than a modification, of the notion of length.)

Countable Additivity
Let $A_1, A_2, \ldots$ be a countable family of disjoint sets. Whenever $\lambda(A_i)$ is defined for each A_i, we have:

$$\lambda\left(\bigcup\{A_1, A_2, A_3, \ldots\}\right) = \lambda(A_1) + \lambda(A_2) + \lambda(A_3) + \ldots$$

(For $A_1, A_2, \ldots$ to be *disjoint* is for A_i and A_j to have no elements in common whenever $i \neq j$.)

Non-Negativity
For any set A in the domain of λ, $\lambda(A)$ is either a non-negative real number, or the infinite value ∞.

(Note that the length of a line-segment $[a, b]$ is always a non-negative real number. When we transition from measuring line-segments to measuring Borel Sets, however, we allow for sets of “infinite length”, such as $[0, \infty)$, which is the set of non-negative real numbers, or $(-\infty, \infty) = \mathbb{R}$.)

The unique function λ on the Borel Sets that satisfies these three conditions is called the **Lebesgue Measure**, in honor of another great French mathematician: Henry Lebesgue.

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We will not prove that λ exists here, or that it is unique. We will simply assume that λ exists and is well-defined for every Borel Set. (If you'd like to learn how to prove the relevant result, I recommend a measure-theory textbook in Lecture 7.3.)

Problem 1

1/1 point (ungraded)

Identify the value of $\lambda(\emptyset)$, where $\emptyset$ is the empty set.

0

✔ **Answer:** 0

0

Explanation

Since $[0,1]$ and the empty set are both Borel Sets, $\lambda([0,1])$ and $\lambda(\emptyset)$ are both well-defined. So we can use Countable Additivity to get

$$\lambda([0,1]) = \lambda([0,1]) + \lambda(\emptyset)$$

But it follows from Length on Segments that $\lambda([0,1]) = 1$. So $\lambda(\emptyset) = 1 - 1 = 0$.

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Problem 2

1/1 point (ungraded)

Identify the value of $\lambda([a,b))$, where $[a,b) = [a,b] - \{b\}$.

- ○ 0
- ○ $a + b$
- ● $b - a$
- ○ $a - b$

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Explanation

Since $[a, b]$ and $\{b\}$ are Borel Sets, it follows from an exercise from the previous section that $[a, b)$ is a Borel Set. So we know that both $\lambda([a, b))$ and $\lambda(\{b\})$ are defined. We can therefore use Countable Additivity to get the following:

$$\lambda([a, b]) = \lambda([a, b)) + \lambda(\{b\})$$

But since $\{b\} = [b, b]$ and $\lambda([b, b]) = b - b = 0$ (by Length on Segments), $\lambda(\{b\}) = 0$. So $\lambda([a, b)) = \lambda([a, b]) = b - a$.

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Problem 3

1/1 point (ungraded)

Identify the value of $\lambda(\mathbb{R})$.

- ( ) 0
- (•) ∞
- ( ) $-\infty$
- ( ) None of the above

✓

Explanation

Since each set $[n, n + 1)$ is a Borel Set for each integer n, we know that $\lambda([n, n + 1)$ is defined for each n. We can therefore use Countable Additivity to get the following:

$$\lambda(\mathbb{R}) = \ldots \lambda([-2, -1)) + \lambda([-1, 0)) + \lambda([0, 1)) + \lambda([1, 2)) + \ldots$$

But the previous answer entails that $\lambda([a, a + 1)) = 1$ for each a. So we have:

$$\lambda(\mathbb{R}) = \ldots 1 + 1 + 1 + 1 \ldots$$

Since no real number is equal to an infinite sum of ones, Non-Negativity entails that $\lambda(\mathbb{R}) = \infty$.

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Problem 4

1/1 point (ungraded)

True or false?

Every countable (i.e. finite or countably infinite) set has Lebesgue Measure zero.

- [x] True
- [ ] False

Explanation

Let A be the countable set $\{a_0, a_1, a_2, \ldots\}$. Since each $\{a_i\}$ is a Borel Set, we know that each $\lambda(\{a_i\})$ is defined. We can therefore use Countable Additivity to get:

$$\lambda(\{a_0, a_1, a_2, \ldots\}) = \lambda(\{a_0\}) + \lambda(\{a_1\}) + \lambda(\{a_2\}) + \ldots$$

But since $\{a_i\} = [a_i, a_i]$, it follows from [Length on Line-Segments] that $\lambda(\{a_i\}) = 0$ for each i. So we have:

$$\lambda(\{a_0, a_1, a_2, \ldots\}) = 0 + 0 + 0 + \cdots = 0$$

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Problem 5

1/1 point (ungraded)

True or false?

If A and B are both Borel Sets and $B \subseteq A$, then $\lambda(B) \leq \lambda(A)$.

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- [x] True
- [ ] False

Explanation
Since A and B are both Borel Sets, so is $A - B$. So Countable Additivity entails

$$\lambda(A) = \lambda(A - B) + \lambda(B)$$

But Non-Negativity entails that $\lambda(A - B)$ is either a non-negative real number or ∞. In either case, it follows that $\lambda(B) \leq \lambda(A)$.

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Problem 6

1/1 point (ungraded)
Countable Additivity gives us an additivity condition for finite or countably infinite families of disjoint sets. Would it be a good idea to insist on an additivity condition for *uncountable* families of disjoints sets?

- [ ] Yes, it would be a great idea!
- [x] No, it would not be a good idea.

Explanation
Recall that for any $x \in \mathbb{R}$, $\lambda(\{x\}) = 0$. So an uncountable additivity principle would entail that $\lambda([0,1]) = 0$:

$$\lambda([0,1]) = \lambda\left(\bigcup_{x\in[0,1]}(\{x\})\right) = \sum_{x\in[0,1]}(\lambda(\{x\})) = \sum_{x\in[0,1]}(0) = 0$$

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