Linearity of the Inner Product

Any function $f(\underline{u})$ of a vector $\underline{u}\in\mathbb{C}^N$ (which we may call an *operator* on $\mathbb{C}^N$) is said to be *linear* if for all $\underline{u}\in\mathbb{C}^N$ and $\underline{v}\in\mathbb{C}^N$, and for all scalars α and β in $\mathbb{C}$,

$$f(\alpha\underline{u}+\beta\underline{v}) = \alpha f(\underline{u}) + \beta f(\underline{v}).$$

A linear operator thus ``commutes with mixing.'' Linearity consists of two component properties:

- *additivity*: $f(\underline{u}+\underline{v}) = f(\underline{u}) + f(\underline{v})$
- *homogeneity*: $f(\alpha\underline{u}) = \alpha f(\underline{u})$

A function of multiple vectors, *e.g.*, $f(\underline{u},\underline{v},\underline{w})$ can be linear or not with respect to each of its arguments.

The inner product $\langle\underline{u},\underline{v}\rangle$ is *linear in its first argument*, *i.e.*, for all $\underline{u},\underline{v},\underline{w}\in\mathbb{C}^N$, and for all $\alpha,\beta\in\mathbb{C}^N$,

$$\langle\alpha\underline{u}+\beta\underline{v},\underline{w}\rangle = \alpha\left<\underline{u},\underline{w}\right> + \beta\left<\underline{v},\underline{w}\right>.$$

This is easy to show from the definition:

$$\begin{aligned}
\langle\alpha\underline{u}+\beta\underline{v},\underline{w}\rangle &\triangleq \sum_{n=0}^{N-1}\left[\alpha u(n)+\beta v(n)\right]\overline{w(n)}\\
&= \sum_{n=0}^{N-1}\alpha u(n)\overline{w(n)} + \sum_{n=0}^{N-1}\beta v(n)\overline{w(n)}\\
&= \alpha\sum_{n=0}^{N-1}u(n)\overline{w(n)} + \beta\sum_{n=0}^{N-1}v(n)\overline{w(n)}\\
&\triangleq \alpha\left<\underline{u},\underline{w}\right> + \beta\left<\underline{v},\underline{w}\right>
\end{aligned}$$

The inner product is also *additive* in its second argument, *i.e.*,

$$\langle\underline{u},\underline{v}+\underline{w}\rangle = \langle\underline{u},\underline{v}\rangle + \langle\underline{u},\underline{w}\rangle,$$

but it is only *conjugate homogeneous* (or *antilinear*) in its second argument, since

$$\langle \underline{u}, \alpha \underline{v} \rangle = \overline{\alpha} \langle \underline{u}, \underline{v} \rangle \neq \alpha \langle \underline{u}, \underline{v} \rangle .$$

The inner product *is* strictly linear in its second argument with respect to *real* scalars a and b :

$$\langle \underline{u}, a\underline{v} + b\underline{w} \rangle = a \langle \underline{u}, \underline{v} \rangle + b \langle \underline{u}, \underline{w} \rangle , \quad a, b \in \mathbb{R}$$

where $\underline{u}, \underline{v}, \underline{w} \in \mathbb{C}^N$.

Since the inner product is linear in both of its arguments for real scalars, it may be called a *bilinear operator* in that context.

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