Bijections Between Infinite Sets

When the Bijection Principle is in place, many different infinite sets can be shown to have the same size. Four prominent examples are the natural numbers, the integers, the rational numbers and the finite sequences of natural numbers:

Set	Symbol	Members
Natural numbers	$\mathbb{N\ }$	$0, 1, 2, 3, \dots$
Integers	$\mathbb{Z\ }$	$\dots -2, -1, 0, 1, 2, \dots$
Rational numbers	$\mathbb{Q\ }$	a/b (for $a,b \in \mathbb{Z}$ and $b \neq 0$)
Finite sequences of natural numbers	$\mathfrak{F}$	$\langle n_1,\dots, n_k\rangle$ for $k \geq 0$ and $n_1,\dots,n_k \in \mathbb{N}$

Let us verify that the set $\mathbb{N}$ of natural numbers has the same size as the set $\mathbb{Z}$ of integers. All we need to so is define a bijection f from the natural numbers to the integers. One such bijection assigns the even natural numbers to the non-negative integers and the odd natural numbers to the negative integers:

$$\begin{array}{cccccccl} 0 & 1 & 2 & 3 & 4 & 5 &\dots\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & -1 & 1 & -2 & 2 & -3 & \dots \end{array}$$

Formally: \[f(n)=\begin{cases} \frac{n}{2}, \hspace{2mm} \text{if n is even}\\ -\frac{(n+1)}{2}, \hspace{2mm} \text{if n is odd} \end{cases}\]In the next section we will verify that the set \(\mathbb{N}\) of natural numbers has the same size as the set \(\mathbb{Q}\) of rational numbers. After that we'll verify that the set \(\mathbb{N}\) of natural numbers has the same size as the set \(\mathfrak{F}\) of finite sequences of natural numbers.

Reminder

The problems below are exercises and will **not** count towards your grade. The only problems that will count towards your grade are those in the subsections labelled "Homework", which appear at the end of each lecture (except for Lecture 8, which has no homework).

Problem 1

1/1 point (ungraded)

Which of the following, if any, is a bijection from the natural numbers to the even natural numbers?

- [x] $f(n) = 2n$
- [ ] $f(n) = n + 1$
- [ ] $f(n) = \frac{2(2n+1)}{3}$
- [ ] $f(n) = 2$

Explanation

$f(n) = 2n$ is a bijection from the natural numbers to the even natural numbers: $f(1) = 2$, $f(2) = 4$, $f(3) = 6$, $f(4) = 8, \ldots$ And so on and so forth. None of the other options are bijections from the natural numbers to the even natural numbers. For each, try to see why by producing a counterexample.

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Answers are displayed within the problem

Problem 2

1/1 point (ungraded)

Does there exist a bijection from the natural numbers to the integers that are multiples of 7?

Yes

Answer: Yes

(If so, can you give an example? If not, why not?)

Explanation

There does exist a bijection from the natural numbers to the integers that are multiples of 7.

One way to define such a bijection is to combine the technique used to match natural numbers with the integers and the technique used to match natural numbers with even natural numbers. If n is an even natural number, let $f(n) = \frac{7n}{2}$, and if n is an odd natural number, let $f(n) = -\frac{7(n+1)}{2}$.

Have a go at verifying that this is indeed a bijection from the natural numbers to the integers that are multiples of 7.

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Answers are displayed within the problem

Problem 3

3/3 points (ungraded)

Which of the following, if any, are true of bijections?

- [x] **Reflexivity**
  For any set A, there is a bijection from A to A.

Symmetry
For any sets A and B, if there is a bijection from A to B, then there is a bijection from B to A.

Transitivity
For any sets A, B, and C, if there is a bijection from A to B, and there is a bijection from B to C, then there is a bijection from A to C.

Explanation

1. *Reflexivity*: We want to show that for any set A there is a bijection from A to A. Let f be the identity function, that is, $f(x) = x$ for any x. f is obviously a bijection from A to itself.

2. *Symmetry*: Suppose that f is a bijection from A to B, and consider the *inverse* function f^{-1} (i.e. the function g such that for each $b \in B$, $g(b)$ is the $a \in A$ such that $f(a) = b$). We verify that f^{-1} is a bijection from B to A:

   - f^{-1} is defined for all $b \in B$ because f is surjective.
   - f^{-1} is a function because f is injective.
   - f^{-1} is injective because f is a function.
   - f^{-1} is surjective because f is defined for all $a \in A$.

3. *Transitivity*: Suppose that f is a bijection from A to B and that g is a bijection from B to C. Now consider the *composite* function $g \circ f$ (i.e., the function $g(f(x))$). It is straightforward to verify that $g \circ f$ is a function, and that it is defined for every member of A. Let us now verify that it is a *bijection* from A to C.

   - $g \circ f$ is *injective*: For $a_1, a_2 \in A$, let $a_1 \neq a_2$. Since f is injective, $f(a_1) \neq f(a_2)$, and since g is injective, $g(f(a_1)) \neq g(f(a_2))$. So $g \circ f(a_1) \neq g \circ f(a_2)$. So $g \circ f$ is injective.
   - $g \circ f$ is *surjective*: Let $c \in C$. Since g is surjective, there is some $b \in B$ such that $g(b) = c$, and since f is surjective, there is some $a \in A$ such that $f(a) = b$. So $g \circ f(a) = c$, for arbitrary $c \in C$. So $g \circ f$ is surjective.

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Answers are displayed within the problem

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There are no new guests!!!! Send them back to their room as they are just lying
As both guests and rooms are countable, there is a bijection between both sets. Therefore every possible guest already has a room assigned. So if they come ask for a room, j... 12

problem 2
Does there exist a bijection from the natural numbers to the integers that are multiples of 7? isn't f(n) ---> 7n sufficient? 3

Math processing error
Where anything mathematical is supposed to be, it just says Math processing error. Is this a bug? Even in the exercise under 'bijection between infinite sets', all the options on... 4

Can anyone have 2 sets of infinite elements which presents no bijection?
I think there's no set of infinite numbers where you can't prove bijection. is there? 6

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