3. Geometric meaning

Geometric meaning of real eigenvalues and eigenvectors

Recall that any $n \times n$ matrix $\mathbf{A}$ represents a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Therefore, an eigenvector $\mathbf{v}$ of $\mathbf{A}$, which satisfies the equation

$$\mathbf{Av} = \lambda\mathbf{v} \qquad \text{for some scalar } \lambda,$$

is a vector whose image under $\mathbf{A}$ is a scalar multiple of itself. When the eigenvalue λ is real, this means that an eigenvector is a vector $\mathbf{v}$ whose image lies on the line in $\mathbb{R}^n$ through $\mathbf{0}$ and $\mathbf{v}$, with the eigenvalue λ as the scaling factor.

Let us revisit the examples above.

Example 3.1 The function represented by $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ stretches every vector in $\mathbb{R}^3$ to 5 times its length but does not change its direction; hence, every vector is an eigenvector associated to the eigenvalue 5.

Example 3.2 The function represented by the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- stretches the eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to 2 times (2 is the corresponding eigenvalue) its length but does not change its direction;
- collapses the eigenvector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to the $\mathbf{0}$ vector, since 0 is the corresponding eigenvalue;
- flips the eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ across the origin to the other side of the z-axis without changing its length, since -1 is the corresponding eigenvalue.

Example 3.3 The (function represented by the) matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

- does not change the direction or length of the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ since the corresponding eigenvalue is 1;
- flips the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ across the origin to the other side of the line $y = -x$, since the eigenvalue is -1.

In each case, the image of the eigenvector lies on the same line as the eigenvector itself.

Matrix vector mathlet

The mathlet below shows the input and output vectors of a 2×2 matrix $\mathbf{A}$ and their relationship with the eigenlines. Recall from the course *Differential equations: 2 by 2 systems* that when the eigenvectors of an eigenvalue are all scalar mulitiples of just one eigenvector, then the line consisting of all the eigenvectors is called an **eigenline** .

To see the action of $\mathbf{A}$ on its eigenvector:

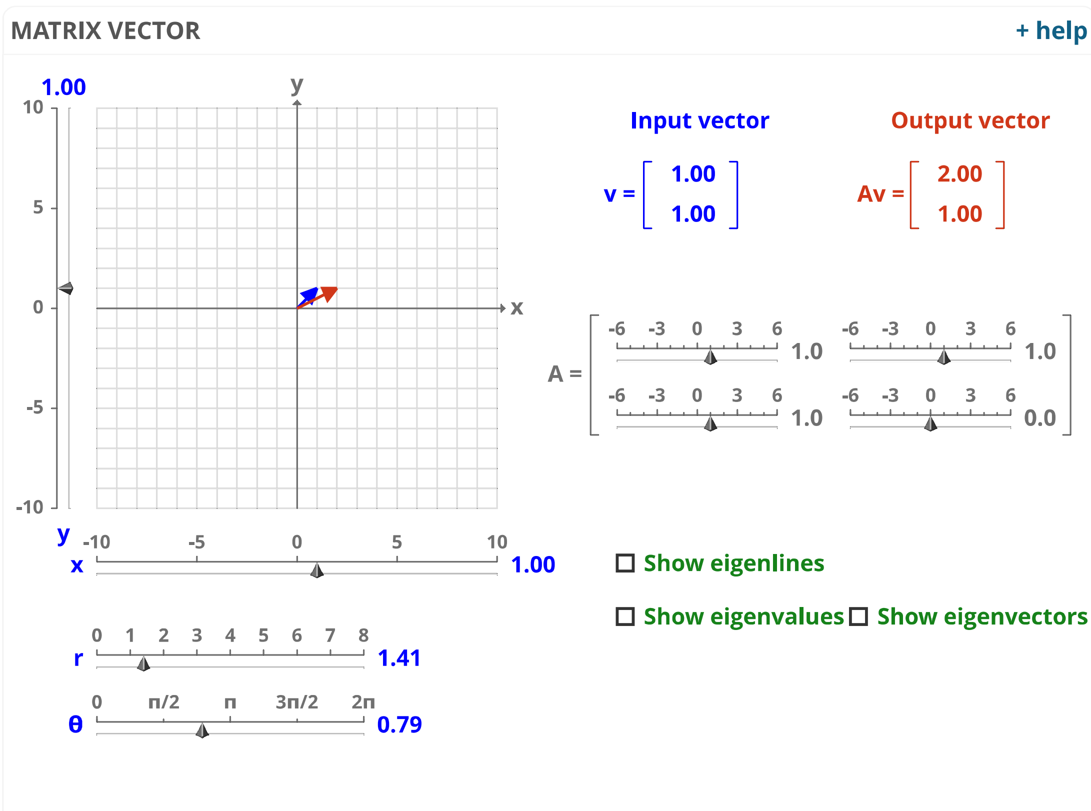
1. click on all three boxes “Show eigenlines," “Show eigenvalues," and “Show eigenvectors";

2. choose values of $\mathbf{A}$ (on the right) so that two eigenlines (green) are shown on the graph (on the left);

3. click on a point along the eigenlines (green on the graph) to select an eigenvector $\mathbf{v}$ (blue) as an input to $\mathbf{A}$;

4. observe that the output $\mathbf{A}\mathbf{v}$ (red) lies along the same line as $\mathbf{v},$ with the corresponding eigenvalue, λ_1 or $\lambda_2,$ (bottom right) as scalar factors.

Observe that when the input vector $\mathbf{v}$ (blue) is **not** an eigenvector (i.e. not on an eigenline), the output $\mathbf{A}\mathbf{v}$ (red) is not on the same line as $\mathbf{v}$.



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Projections

2/2 points (graded)

There are two eigenvalues for the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, namely 0 and 1.

Which of the following are eigenvectors associated to

the eigenvalue 1:
(Choose all that apply.)

the eigenvalue 0:
(Choose all that apply.)

☑ $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ ✓	☐ $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$
☑ $\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$ ✓	☐ $\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$
☐ $\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$	☑ $\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$ ✓
☑ $\begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix}$ ✓	☐ $\begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix}$
☐ $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$	☐ $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$
✓	✓

Solution:

Since $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{v}$ for the choices $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix}$, these are the eigenvectors for the eigenvalue 1.

Note: Any vector of the form $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ is an eigenvector of 1. Geometrically, any vector lying on the xy-plane is unchanged by the matrix.

On the other hand, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix} = \mathbf{0}$, so $\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$ is an eigenvector of 0.

Note: $\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ for any number c is an eigenvector of 0. Geometrically, any vector lying on the z-axis is collapsed to the origin by the matrix.

Finally,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

which is not a scalar multiple of $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. So $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is not an eigenvector associated to any eigenvalue. Therefore, the matrix represents the projection function that sends any vector in $\mathbb{R}^3$ to its “shadow" on the xy-plane.

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Answers are displayed within the problem

Introduction to eigenvectors

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3. Geometric meaning

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