7. More variables

Lecture due Oct 5, 2021 20:30 IST

Explore

Chain Rule More Variables

7:00 / 7:00 2.0x

So dx is going to be--

well, I can use the differential again, dx.

Well, x is a function of u and v.

That will be x sub u times du plus x sub v times dv.

That's, again, taking the differential

of a function of two variables.

Does that make sense?

Then, we have the other guy f sub y times--

what is dy?

Well, similarly, dy is y sub u du plus y sub v dv.

And now we have a relation between dw and du and dv.

We are expressing how w reacts to changes in u and v,

which was our goal.

Now, let's actually collect terms

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Chain rule with more variables

The multivariable chain rule also comes up if the inputs to the function depend on more than one parameter. Imagine a quantity w is given by a function of x and y as $w = f(x, y)$. Now suppose we cannot control x and y directly, but they each depend on two variables u and v. **How can we write $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ in terms of the rate of change of w (derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$) and the rate of change of x and y (derivatives x_u, x_v and y_u, y_v)?**

We will give a specific example later on this page. First let's solve the problem in general.

Using differentials

Differentials give us an indirect way to keep track of everything. First we will write the total differential of w, then substitute in the total differentials of x and y, and rewrite the equation to elicit the coefficients on du and dv.

We start with the total differential of w:

$$dw = f_x\,dx + f_y\,dy \qquad (6.154)$$

Next we can replace the differentials dx and dy by their total differentials:

$$dw = f_x\left(\underbrace{x_u\,du + x_v\,dv}_{dx}\right) + f_y\left(\underbrace{y_u\,du + y_v\,dv}_{dy}\right)$$

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Now recollecting terms:

$$dw = (f_x x_u + f_y y_v)\ du + (f_x x_v + f_y y_v)\ dv \tag{6.156}$$

It follows that the coefficients on du and dv are the unknown partial derivatives:

$$dw = \underbrace{(f_x x_u + f_y y_u)}_{\partial w/\partial u}\ du + \underbrace{(f_x x_v + f_y y_v)}_{\partial w/\partial v}\ dv \tag{6.157}$$

Thus we have

$$\frac{\partial w}{\partial u} = f_x x_u + f_y y_u \tag{6.158}$$

$$\frac{\partial w}{\partial v} = f_x x_v + f_y y_v \tag{6.159}$$

Understanding the Chain Rule Formula

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Well, it's the sum of the two effects.

Does that make sense?

Good.

Of course, if f depends on more variables,

then you just have-- you just have more terms in here.

OK, here's another thing that may be a little bit confusing.

So what is tempting?

Well, what is tempting here would be to simplify these

formulas by removing these partial x's.

So let's simplify by partial x.

Let's simplify by partial y.

We get partial f, partial u equals partial f, partial u

plus partial f, partial u.

Something is not working properly

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We might get a better understanding by rewriting the above results using ∂ notation:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \tag{6.160}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \tag{6.161}$$

Although these formulas look complicated, it is possible to make sense of them, with some practice. Let's look at a more concrete example to get the idea.

Example 7.1

Imagine a box with height y and square base of width x. The volume is $V = x^2 y$.

Now suppose we can't control x and y directly, but they both depend on two parameters a and b.

In this example, let's imagine $x = (a+3b)^2$ and $y = 2a$. Now changing either of the variables a or b will cause the volume V to change, and we would like to know the corresponding rates of change, that is, the values of $\frac{\partial V}{\partial a}$ and $\frac{\partial V}{\partial b}$. In this example, we will do out $\frac{\partial V}{\partial a}$. There is a space below for you to work out $\frac{\partial V}{\partial b}$.

There are three ways of tackling this question.

Approach 1: Direct substitution

Since we have formulas for x and y, we can substitute them in to $V = x^2 y$, and then take normal partial derivatives. We get $V = (a+3b)^4\,(2a)$. After applying one product rule, we have our desired partial derivative:

$$\frac{\partial V}{\partial a} = 4(a+3b)^3\,(2a) + (a+3b)^4\,(2) \qquad (6.162)$$

Approach 2: Substitute differentials one-at-a-time.

One way to solve this problem is to try to write dV in terms of da and db. The coefficients will have to be the unknown $\frac{\partial V}{\partial a}$ and $\frac{\partial V}{\partial b}$, so this gives us a way of solving for our unknowns. First, we have the total differential of V:

$$dV = V_x\,dx + V_y\,dy \qquad (6.163)$$

$$= 2xy\,dx + x^2\,dy \qquad (6.164)$$

This is not our answer, because we have dx and dy instead of da and db. But since $x = (a+3b)^2$ and $y = 2a$, we can write each of dx and dy in terms of da and db.

$$dx = 2\,(a+3b)\;da + 6\,(a+3b)\;db \qquad (6.165)$$

$$dy = 2\,da + 0\,db \qquad (6.166)$$

This works, and the rest of this calculation involves just doing out the algebra and collecting the terms:

By substitution we get

$$dV = 2xy\cdot(2\,(a+3b)\;da + 6\,(a+3b)\;db) + x^2\cdot(2\,da + 0\,db) \qquad (6.167)$$

Rearranging terms, we can extract the coefficients on da and db.

$$dV = \left(2xy\cdot 2\,(a+3b) + x^2\cdot 2\right)\;da + (\ldots)\;db \qquad (\text{focussing only on } da) \qquad (6.168)$$

This equation tells us that changing a by Δa causes V to change by

$$\left(2xy\cdot 2\,(a+3b) + x^2\cdot 2\right)\;\Delta a \qquad (6.169)$$

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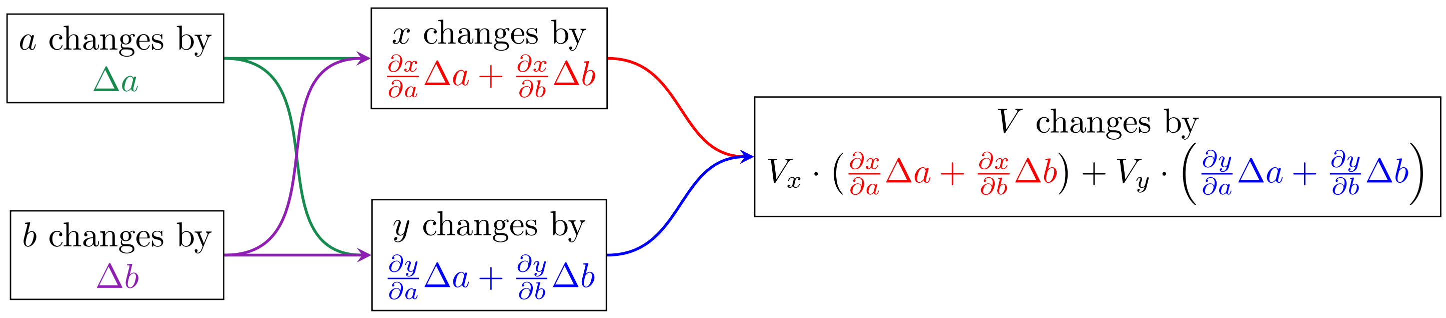
Therefore we have

$$\frac{\partial V}{\partial a} = 2xy \cdot 2\,(a+3b) + x^2 \cdot 2. \tag{6.170}$$

For our final answer, we should express everything in terms of a and b.

$$\frac{\partial V}{\partial a} = 2\left((a+3b)^2\right)(2a) \cdot 2\,(a+3b) + \left((a+3b)^4\right) \cdot 2. \tag{6.171}$$

The following “chain" diagram captures the various relationships that we have made use of:



Approach 3: Ask ourselves “how does V depend on a?"

Another way of finding $\frac{\partial V}{\partial a}$ is to try to remind ourselves what the chain rule formula should be based on the question “how does V depend on a?" In this example, there are two ways in which V depends on a, and each of these ways contributes a term in the summation. We have

1. V depends on x and x depends on a.

2. V depends on y and y depends on a.

The first of these gives us the term $\frac{\partial V}{\partial x}\frac{\partial x}{\partial a}$, because the connection from V to a goes through x. Similarly, the second gives us the term $\frac{\partial V}{\partial y}\frac{\partial y}{\partial a}$. In total, we have:

$$\frac{\partial V}{\partial a} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial a}. \tag{6.172}$$

With the formula in hand, it is simply a matter of computing each term and substituting. We find $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ from $V = x^2 y$. We find $\frac{\partial x}{\partial a}$ from $x = (a+3b)^2$ and we find $\frac{\partial y}{\partial a}$ from $y = 2a$.

Check your understanding

1/1 point (graded)

Suppose $V = x^2 y$ and $x = (a+3b)^2$ and $y = 2a$. What is $\frac{\partial V}{\partial b}$?

Try to solve this problem using at least two different methods, and check that you ge

$\frac{\partial V}{\partial b} =$ 24*a*(a+3*b)^3 ✔ **Answer:** 24*a*(a+3*b)^3

? INPUT HELP

Solution:

Approach 1: Direct Substitution

In terms of a and b, we have $V = (a+3b)^4 2a$. Taking $\frac{\partial V}{\partial b}$ we obtain $24a(a+3b)^3$.

Approach 2: Multivariable Chain Rule

The multivariable chain rule in this setting says:

$$\frac{\partial V}{\partial b} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial V}{\partial y}\frac{\partial x}{\partial b} \qquad (6.173)$$

We have:

$$\frac{\partial V}{\partial x} = 2xy = 2(a+3b)^2\,(2a) \qquad (6.174)$$

$$\frac{\partial V}{\partial y} = x^2 = (a+3b)^4 \qquad (6.175)$$

$$\frac{\partial x}{\partial a} = 6\,(a+3b) \qquad (6.176)$$

$$\frac{\partial y}{\partial b} = 0 \qquad (6.177)$$

Thus, in total we get

$$\frac{\partial V}{\partial b} = 24a(a+3b)^3 + 0 \qquad (6.178)$$

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Answers are displayed within the problem

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