7. Worked example

Problem 7.1 Find all the eigenvalues, eigenvectors, and eigenspaces of $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$.

Solution:

We found previously that the eigenvalues are $0,\ -3,$ where -3 is of multiplicity 2.

Eigenspace of 0:

This is $\mathrm{NS}(\mathbf{A})$, that is, the set of all solutions to

$$\begin{aligned} \mathbf{Av} &= \mathbf{0} \\ \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\mathbf{v} &= \mathbf{0}. \end{aligned}$$

We reduce the matrix to reduced row-echelon form:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3/2 & 3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, z is a free parameter, and the eigenspace is given by

$$\text{Eigenspace of } 0 = \text{NS}(\mathbf{A}) = \text{Span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Eigenspace of -3:

We use the fact that $\text{NS}(-3\mathbf{I}-\mathbf{A}) = \text{NS}(\mathbf{A}+3\mathbf{I})$ to avoid negative sign errors. The set of all solutions to

$$\begin{aligned} (\mathbf{A}+3\mathbf{I})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -2+3 & 1 & 1 \\ 1 & -2+3 & 1 \\ 1 & 1 & -2+3 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\mathbf{v} &= \mathbf{0}. \end{aligned}$$

The matrix can be reduced to:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, both y and z can be free parameters and $x = -y - z.$ The eigenspace is given by

$$\text{Eigenspace of } -3 = \text{NS}(\mathbf{A}+3\mathbf{I}) = \text{Span}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

This is a 2-dimensional vector space spanned by two independent vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$ Geometrically, this consists of all vectors on the plane $x = -y - z.$

Conclusion : The eigenvalues and corresponding eigenspaces of the matrix $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ are:

Eigenvalue		**Corresponding eigenspace**
$\lambda = 0$	;	$\text{Span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\lambda = -3$	;	$\text{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)$

(The eigenvectors for each eigenvalue are all vectors in the corresponding eigenspace.) Notice that the eigenspace of -3 is 2-dimensional.

Eigenvectors concept check

1/1 point (graded)
A 5×5 matrix $\mathbf{A}$ has eigenvalues $5,\ 1,\ 0.5.$

If $\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ are two eigenvectors associated to the eigenvalue 1. Which of the following must also be eigenvectors with eigenvalue 1?

(Choose all that apply.)

- [ ] $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

- [x] $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ✔
- [x] $\begin{pmatrix} -4 \\ 0 \\ 0 \\ -2 \\ 0 \end{pmatrix}$ ✔
- [x] $\begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ✔
- [x] $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ ✔

✔

Solution:

Any linear combination of two eigenvectors corresponding to the same eigenvalue λ is again an eigenvector corresponding to λ (even if it is the $\mathbf{0}$ vector).

- The vector $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ cannot be a linear combination of $\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, because any linear combinations of these given eigenvectors must have 0 in the second component.
- All other choices of vector are linear combinations of the given eigenvectors and so are eigenvectors themselves.

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Answers are displayed within the problem

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Weird definition of an eigenvector? includes the null vector?
The text and the grader are consistent is asserting that: > "The eigenvectors for each eigenvalue are all v... 5
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