6. Eigenspaces

We now know that the eigenvalues of a matrix are the roots of the characteristic equation $\det(\lambda \mathbf{I} - \mathbf{A}) = 0.$ Let us proceed to find the eigenvectors corresponding to each eigenvalue.

Definition 6.1 For each eigenvalue λ of $\mathbf{A}$, define the **eigenspace** of λ to be all eigenvectors associated to the eigenvalue λ.

In other words,

$$\begin{aligned} \text{the eigenspace of an eigenvalue } \lambda, &= \{\text{all eigenvectors associated to } \lambda\} \\ &= \{\text{all solutions to } (\lambda \mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}\} \\ &= \text{NS}(\, \lambda \mathbf{I} - \mathbf{A}\,). \end{aligned}$$

There is one eigenspace for every eigenvalue. Each eigenspace is a vector space, so it can be described as the span of a basis. To compute the eigenspace of λ, compute $\text{NS}(\lambda \mathbf{I} - \mathbf{A})$ by Gaussian elimination and back-substitution.

Finding eigenvalues and eigenvectors: a 2 by 2 example

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Steps to find all eigenvectors associated to a given eigenvalue λ of a square matrix $\mathbf{A}$:

1. Write down $\boldsymbol{\lambda}\mathbf{I} - \mathbf{A}$.

2. Use Gaussian elimination to find a basis of $\mathrm{NS}(\lambda\mathbf{I}-\mathbf{A})$.

3. The eigenvectors corresponding to the eigenvalue λ are all the linear combinations of these basis vectors.

Example 6.2 Find all the eigenvalues, eigenvectors, and eigenspaces of $\mathbf{A}=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$.

Solution:

The eigenvalues are $0,\ 1,$ and $2.$ (We have found these before, by determining the roots of the $P(\lambda)=\det(\lambda\mathbf{I}-\mathbf{A})$).

Eigenspace of 0:

This is $\mathrm{NS}(0\mathbf{I}-\mathbf{A})=\mathrm{NS}(-\mathbf{A})=\mathrm{NS}(\mathbf{A})$, i.e. the set of all solutions to

$$\begin{aligned}\mathbf{A}\mathbf{v} &= \mathbf{0}\\ \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\mathbf{v} &= \mathbf{0}.\end{aligned}$$

The first and third rows are the same, and so subtracting row 1 from row 3, we get

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\to\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The resulting matrix is in reduced-row-echelon form. The null space of $\mathbf{A}$ is

$$\text{Eigenspace of } 0 = \mathrm{NS}\left(\mathbf{A}\right) = \mathrm{Span}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}.$$

In other words, the eigenspace of the eigenvalue 0 is the 1-dimensional vector space spanned by the eigenvector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and the set of all eigenvectors is all scalar multiples of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Geometrically, the eigenspace consists of all vectors along the line containing $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, which is defined by $x = -z,\ y = 0$.

Terminology: Very often, we say "the eigenvectors of λ" to mean a **basis** of the eigenspace of λ.

Eigenspace of 1:

This is $\mathrm{NS}(\mathbf{I} - \mathbf{A})$, i.e. the set of all solutions to

$$\begin{aligned} (\mathbf{I}-\mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} 1-1 & 0 & -1 \\ 0 & 1-1 & 0 \\ -1 & 0 & 1-1 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\mathbf{v} &= \mathbf{0}. \end{aligned}$$

We can immediately see that the first and third components of $\mathbf{v}$ must be zero, while the second component is a free parameter. Hence, the set of all solutions to $(\mathbf{I}-\mathbf{A})\mathbf{v} = \mathbf{0}$ is

$$\text{Eigenspace of } 1 = \mathrm{NS}\,(\mathbf{I}-\mathbf{A}) = \mathrm{Span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The eigenspace of the eigenvalue 1 is the 1-dimensional vector space spanned by the eigenvector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Geometrically, this consists of all vectors along the y-axis.

Eigenspace of 2:

This is $\mathbf{NS}(2\mathbf{I}-\mathbf{A})$, i.e. the set of all solutions to

$$(2\mathbf{I}-\mathbf{A})\mathbf{v} \quad = \quad \mathbf{0}$$

$$\begin{pmatrix} 2-1 & 0 & -1 \\ 0 & 2-1 & 0 \\ -1 & 0 & 2-1 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\mathbf{v} \quad = \quad \mathbf{0}.$$

The matrix can be reduced to:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives the set of all solutions to $(2\mathbf{I}-\mathbf{A})\mathbf{v}=\mathbf{0}$ to be

$$\text{Eigenspace of } 2 \;=\; \text{NS}\,(2\mathbf{I}-\mathbf{A}) \;=\; \text{Span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenspace of the eigenvalue 2 is the 1-dimensional vector space spanned by the eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Geometrically, this is the line along the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (the line defined by $x=z,\ y=0$).

Conclusion : The eigenvalues and corresponding eigenspaces of the matrix $\mathbf{A}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ are:

$$\text{Eigenvalue} \qquad \text{Corresponding eigenspace}$$

$$\lambda = 0 \quad ; \quad \text{Span}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda = 1 \quad ; \quad \text{Span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 2 \quad ; \quad \text{Span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvectors for each eigenvalue are all vectors in the corresponding eigenspace.

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