Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Linear Algebra and its Applications 428 (2008) 2730–2749
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

# The expected hitting times for finite Markov chains ☆

Haiyan Chen [a,*], Fuji Zhang [b]

[a] Department of Mathematics, School of Sciences, Jimei University, Xiamen 361021, China
[b] Institute of Mathematics, Xiamen University, Xiamen 361005, China

Received 18 May 2006; accepted 4 January 2008
Available online 20 February 2008
Submitted by R.A. Brualdi

**Abstract**

In this paper, using the theory of matrix algebra, we obtain a new expression for the expected hitting times of irreducible aperiodic Markov chains. Then, using it, we calculate the expected hitting times of random walks on several kinds of graphs. These examples show that in many cases our approach is better than the others.


*AMS classification:* 60J20

*Keywords:* Markov chains; The expected hitting time; Graph; Random walks

## 1. Introduction

Let $G = (V, D, \omega)$ be a weighted directed graph (with loops), where $V$ is a finite vertex set; $D$ is an arc set of ordered pairs of the elements of $V$, and $\omega$ is a weight function on $D$, that is, to each $(i, j) \in D$, we assign a positive weight $\omega_{ij}$. Let $\omega_i = \sum_{(i,j)\in D} \omega_{ij}$, define

$$p_{ij} = \begin{cases} \frac{\omega_{ij}}{\omega_i} & \text{if } (i, j) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Then, **a random walk** on $G$ is a walk $X_0, X_1, \ldots$ obtained in a random fashion, that is, if $X_k = i$, then the probability of $X_{k+1} = j$ is $p_{ij}$. Apparently, $X_0, X_1, \ldots$ is a Markov chain on

☆ This work is supported by NSFC10501018 and NSFC10771086.
* Corresponding author.
*E-mail address:* chey6@vip.sohu.com (H. Chen).

0024-3795/$ - see front matter 
doi:10.1016/j.laa.2008.01.003

$V = V(G)$ with transition probability matrix $P = (p_{ij})_{i,j\in V}$. Conversely, for a finite Markov chain with state space $V$ and transition probability matrix $P$, we can obtain a weighted directed graph $G$: the vertices are the states of the chain, $(i, j) \in D$ (with weight $\omega_{ij} = p_{ij}$) whenever $p_{ij} > 0$. Then, the chain can be visualized as the random walk on $G$. So we see that finite Markov chains are just random walks on weighted directed graphs. For convenience, in the following we will tend to use the two terms at the same time. Given a Markov chain (finite or infinite), the classical theory of Markov chains is concerned with closed subsets of it; states which communicate; the period of a state, recurrent and transient states; ergodicity [8,9]; but now motivated by computer science, especially the design of efficient algorithm, some important parameters of finite Markov chains, such as the expected hitting times, the expected cover times and the mixing times, have become of main concerns [2,4]. In this paper, using matrix algebra, we derive a new expression for the expected hitting times of a finite irreducible aperiodic Markov chain. Although the computing complexity of this new method is as complex as the other known methods, under some circumstances, our method is more advantageous than the others. Now, we give the definitions and notations we need in this paper.

Let $X_0, X_1, \ldots$ be a Markov chain on state space $V$. Then for any $i, j \in V$, write

$$T_j = \min\{t \geqslant 0 : X_0 = i, X_t = j\}.$$

Then, the expected value of $T_j$ is called **the expected hitting time** from $i$ to $j$, denoted by $E_iT_j$, that is

$$E_iT_j = \sum_{k=1}^{\infty} kP\{T_j = k\}.$$

In the following, we will identify $V$ with a set $\{1, \ldots, N\}$. Then, **the transition probability matrix** $P = (p_{ij})_{i,j\in V}$ is a $N \times N$ stochastic matrix. If the matrix $P$ is irreducible and aperiodic, then the corresponding Markov chain is called **an irreducible and aperiodic Markov chain.** A probability distribution $\pi = \{\pi_1, \pi_2, \ldots, \pi_N\}$ is called a stationary distribution provided that $\pi P = \pi$; in addition, if the matrix $P$ and the stationary distribution $\pi$ satisfy the detailed balance equations

$$\pi_i p_{ij} = \pi_j p_{ji}$$

for all $i, j \in V$, then the chain is called **a reversible Markov chain.**

Corresponding to the random walk on $G = (V, D, \omega)$, it is easy to see that (for the classical notions and notations from graph theory, we refer to [7]):

(i) if $G$ is a strongly connected directed graph, then the Markov chain is irreducible;
(ii) if $G$ is an undirected graph (i.e., $D$ is an edge set of unordered pairs of elements of $V$, denoted by $E$ in the following), then the chain is reversible; furthermore if $G$ is a connected undirected non-partite graph, then the chain is reversible, irreducible and aperiodic.

If $\forall(i, j) \in D, \omega_{ij} = 1$, then $\omega_i = d(i)$ (the degree of vertex $i$), $p_{ij} = \frac{1}{d(i)}$, $(i, j) \in D$ (in this condition, we call it **the simple random walk** on $G$).

For the purposes of comparison, we give the four known expressions for the expected hitting time of Markov chain:

**Theorem 1.1** [2]. *For a finite irreducible and aperiodic Markov chain on V* $\forall i, j \in V$, *we have*

$$\pi_j E_iT_j = \sum_{t=0}^{\infty}(p_{jj}^t - p_{ij}^t), \tag{1.1}$$

*where $\pi = \{\pi_1, \pi_2, \ldots, \pi_N\}$ is the unique stationary distribution, $p_{ij}^t$ denotes the $ij$-entry of $P^t$.*

**Theorem 1.2** [6,10]. *For a finite irreducible Markov chain on $V$ $\forall i, j \in V$, let $Q$ denote the matrix obtained by deleting the $j$th row and the $j$th column from $P$, then*

$$E_i T_j = \sum_{l \neq j} (I - Q)_{il}^{-1}. \tag{1.2}$$

**Theorem 1.3** [12]. *For a simple random walk on an undirected simple connected graph $G = (V, E)$ $\forall i, j \in V$, we have*

$$E_i T_j = 2m \sum_{k=2}^{n} \frac{1}{1 - \lambda_k} \left( \frac{\upsilon_{kj}^2}{d(j)} - \frac{\upsilon_{ki}\upsilon_{kj}}{\sqrt{d(i)d(j)}} \right), \tag{1.3}$$

*where $\lambda_1 = 1 > \lambda_2 \geqslant \cdots \geqslant \lambda_N$ are the eigenvalues of the transition probability matrix $P$, $\upsilon_k$ is the eigenvector with length 1 corresponding to $\lambda_k$, $m$ is the edge number of $G$.*

**Theorem 1.4** [16]. *For a simple random walk on an undirected simple connected graph $G = (V, E)$ $\forall i, j \in V$, we have*

$$E_i T_j = \frac{1}{2} \sum_{k \in V(G)} d(k)(R_{ik} - R_{jk} + R_{ij}), \tag{1.4}$$

*where $R_{ij}$ denotes the effective resistance between i and $j$, that is, the graph $G = (V, E)$ can be viewed as an electrical network with unit resistance on every edge.*

The first formula is suitable for any finite irreducible and aperiodic Markov chain, but clearly it cannot be applied directly to calculate the explicit value of $E_i T_j$ of general chains. Up to now, the other three are the main methods to calculate the value of $E_i T_j$. The second one (the so called "absorbing chain technique") can be applied to any irreducible chain. The last two are only suitable for reversible Markov chains. In the following, using the theory of matrix algebra, we directly translate the sum of infinite terms in (1.1) into the sum of finite terms. So, we get a new method to calculate the expected hitting times.

The remainder of the paper is organized as follows: In Section 2, we obtain the main results of this paper by matrix algebra. In Sections 3 and 4, using the results of Section 2, we calculate the expected hitting times of random walks on several kinds of digraphs and graphs, respectively. Through these examples, we show that under some circumstances calculating the expected hitting times is easy by using our method.

## 2. The expected hitting times of Markov chains

As above, let $P = (p_{ij})_{i,j \in V}$ denote the probability transition matrix of an irreducible aperiodic Markov chain, and $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ be the stationary distribution. Then 1 is the maximum eigenvalue of $P$ with multiplicity 1, so the minimum polynomial of $P$ is of the form $q(x) = (x - 1)f(x)$. Let

$$f(x) = a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_{k-1} x + a_k.$$

Clearly $a_0 = 1$. Then, we have the main result of this paper as follows:

**Theorem 2.1.** *For notations as above, then for any $i, j \in V$, we have*

$$\pi_j E_i T_j = \frac{1}{f(1)} \sum_{t=0}^{k-1} \sum_{l=0}^{k-t-1} a_l (p_{jj}^t - p_{ij}^t).$$

Before proceeding with the proof, we first give three lemmas:

**Lemma 2.2.** *The row vectors of matrix $f(P)$ are similar.*

**Proof.** Since $q(x) = (x-1)f(x)$ is the minimum polynomial of $P$, we obtain $Pf(P) = f(P)$; that is, each column of $f(P)$ is an eigenvector of $P$ corresponding to the eigenvalue 1. By the knowledge of matrix theory, it follows that each column of $f(P)$ is a multiple of $u$, $u$ is a column vector with each entry 1. Then, the result is obtained. □

**Lemma 2.3.** *Let $f(x) = x^k + a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_{k-1}x + a_k$, $\alpha_0 = (-a_1, -a_2, \ldots, -a_k)$, and $\beta = (x^{k-1}, x^{k-2}, \ldots, x, 1)^{\mathrm{T}}$. Then, for any $m \geqslant 0$, there exists a polynomial $q_m(x)$ of degree $m$ and a row vector $\alpha_m = (\alpha_{m,1}, \alpha_{m,2}, \ldots, \alpha_{m,k})$ such that*

$$x^{k+m} = q_m(x) f(x) + \alpha_m \beta,$$

*where*

$$\alpha_m = \alpha_0 M^m$$

*and*

$$M = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

**Proof.** We will apply induction on $m$ to prove this lemma.

For $m = 0$ the assertion is trivial, so suppose the lemma holds for $m = n$.

Now suppose $m = n + 1$, then

$$\begin{aligned} x^{k+n+1} &= x x^{k+n} = x q_n(x) f(x) + x \alpha_n \beta \\ &= x q_n(x) f(x) + \alpha_{n,1} x^k + \alpha_{n,2} x^{k-1} + \cdots + \alpha_{n,k-1} x^2 + \alpha_{n,k} x \\ &= x q_n(x) f(x) + \alpha_{n,1}(f(x) - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - a_k) \\ &\quad + \alpha_{n,2} x^{k-1} + \cdots + \alpha_{n,k-1} x^2 + \alpha_{n,k} x \\ &= (x q_n(x) + \alpha_{n,1}) f(x) + (-\alpha_{n,1} a_1 + \alpha_{n,2}) x^{k-1} + (-\alpha_{n,1} a_2 + \alpha_{n,3}) x^{k-2} \\ &\quad + \cdots + (-\alpha_{n,1} a_{k-1} + \alpha_{n,k}) x - \alpha_{n,1} a_k \\ &= (x q_n(x) + \alpha_{n,1}) f(x) + \alpha_n M \beta \\ &= q_{n+1}(x) f(x) + \alpha_0 M^{n+1} \beta. \end{aligned}$$

The last equality is derived by letting $x q_n(x) + \alpha_{n,1} = q_{n+1}(x)$ and induction, so the lemma holds for all $m \geqslant 0$. □

**Lemma 2.4.** *Let $P = (p_{ij})_{i,j\in V}$ denote the probability transition matrix of an irreducible aperiodic Markov chain, $(x-1)f(x)$ be the minimum polynomial of $P$ and $f(x) = a_0x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots + a_{k-1}x + a_k$. Then,*

$$\sum_{m=0}^{\infty} P^{k+m} = \sum_{t=0}^{k-1} b_t P^t + B,$$

*where $B$ is a $N \times N$ matrix with the same row vectors and $b_t = \frac{-\sum_{l=k-t}^{k} a_l}{f(1)}$.*

**Proof.** From Lemma 2.3, we have

$$\sum_{m=0}^{\infty} x^{k+m} = \left(\sum_{m=0}^{\infty} q_m(x)\right) f(x) + \alpha_0 \left(\sum_{m=0}^{\infty} M^m\right)\beta,$$

where $\alpha_0, M, \beta$ are the same as in Lemma 2.3. Then,

$$\sum_{m=0}^{\infty} P^{k+m} = \left(\sum_{m=0}^{\infty} q_m(P)\right) f(P) + \alpha_0 \left(\sum_{m=0}^{\infty} M^m\right)(P^{k-1}, P^{k-2}, \ldots, P, I)^{\mathrm{T}}.$$

On the one hand, by Lemma 2.2 $Pf(P) = f(P)P = f(P)$, so $B = \left(\sum_{m=0}^{\infty} q_m(P)\right) f(P)$ is a $N \times N$ matrix with the same row vectors. On the other hand, since $|\lambda I - M| = f(\lambda)$, we derive that 1 is not the eigenvalue of $M$, that is, $I - M$ is invertible. Furthermore,

$$(I-M)^{-1} = \begin{pmatrix} 1+a_1 & a_2 & \cdots & a_{k-1} & a_k \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}^{-1}$$
$$= \frac{1}{f(1)} \begin{pmatrix} 1 & -\sum_{t=2}^{k} a_t & -\sum_{t=3}^{k} a_t & \cdots & -\sum_{t=k}^{k} a_t \\ 1 & 1+a_1 & -\sum_{t=3}^{k} a_t & \cdots & -\sum_{t=k}^{k} a_t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1+a_1 & 1+a_1+a_2 & \cdots & \sum_{t=0}^{k-1} a_t \end{pmatrix}.$$

So

$$\begin{aligned} \alpha_0\left(\sum_{m=0}^{\infty} M^m\right) &= \alpha_0(I-M)^{-1} \\ &= \frac{1}{f(1)}(-a_1, -a_2, \ldots, -a_k) \\ &\quad \times \begin{pmatrix} 1 & -\sum_{t=2}^{k} a_t & -\sum_{t=3}^{k} a_t & \cdots & -\sum_{t=k}^{k} a_t \\ 1 & 1+a_1 & -\sum_{t=3}^{k} a_t & \cdots & -\sum_{t=k}^{k} a_t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1+a_1 & 1+a_1+a_2 & \cdots & \sum_{t=0}^{k-1} a_t \end{pmatrix} \\ &= \frac{1}{f(1)}\left(-\sum_{t=1}^{k} a_t, -\sum_{t=2}^{k} a_t, \ldots, -\sum_{t=k-1}^{k} a_t, -a_k\right). \end{aligned}$$

From the above, the lemma is derived. $\square$

**Proof of Theorem 2.1.** Let $e_i$ be the $1 \times N$ matrix with 1 in the $i$th leftmost position and 0's elsewhere, then $p_{ij}^t = e_i P^t e_j^{\mathrm{T}}$. From Theorem 1.1 and Lemma 2.4, we have

$$\begin{aligned}
\pi_j E_i T_j &= \sum_{t=0}^{\infty}(p_{jj}^t - p_{ij}^t) \\
&= e_j\left(\sum_{t=0}^{\infty} P^t\right)e_j^{\mathrm{T}} - e_i\left(\sum_{t=0}^{\infty} P^t\right)e_j^{\mathrm{T}} \\
&= e_j\left(\sum_{t=0}^{k-1} P^t\right)e_j^{\mathrm{T}} - e_i\left(\sum_{t=0}^{k-1} P^t\right)e_j^{\mathrm{T}} \\
&\quad + e_j\left(\sum_{m=0}^{\infty} P^{k+m}\right)e_j^{\mathrm{T}} - e_i\left(\sum_{m=0}^{\infty} P^{k+m}\right)e_j^{\mathrm{T}} \\
&= e_j\left(\sum_{t=0}^{k-1} P^t\right)e_j^{\mathrm{T}} - e_i\left(\sum_{t=0}^{k-1} P^t\right)e_j^{\mathrm{T}} \\
&\quad + e_j B e_j^{\mathrm{T}} - e_i B e_j^{\mathrm{T}} + e_j\left(\sum_{t=0}^{k-1} b_t P^t\right)e_j^{\mathrm{T}} - e_i\left(\sum_{t=0}^{k-1} b_t P^t\right)e_j^{\mathrm{T}} \\
&= e_j\left(\sum_{t=0}^{k-1}(1+b_t) P^t\right)e_j^{\mathrm{T}} - e_i\left(\sum_{t=0}^{k-1}(1+b_t) P^t\right)e_j^{\mathrm{T}} \\
&= e_j\left(\sum_{t=0}^{k-1} \frac{\sum_{l=0}^{k-t-1} a_l}{f(1)} P^t\right)e_j^{\mathrm{T}} - e_i\left(\sum_{t=0}^{k-1} \frac{\sum_{l=0}^{k-t-1} a_l}{f(1)} P^t\right)e_j^{\mathrm{T}} \\
&= \frac{1}{f(1)}\sum_{t=0}^{k-1}\sum_{l=0}^{k-t-1} a_l(p_{jj}^t - p_{ij}^t).
\end{aligned}$$

So, the theorem is proved. □

From the procedure of proving Theorem 2.1, we can derive the more general result as follows:

**Theorem 2.1′.** *For any polynomial $g(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$, if $g(1) \neq 0$ and $g(P)$ is a matrix with the same row vectors. Then, for $i, j \in V$*

$$\pi_j E_i T_j = \frac{1}{g(1)}\sum_{t=0}^{n-1}\sum_{l=0}^{n-t-1} a_l(p_{jj}^t - p_{ij}^t).$$

So for any irreducible and aperiodic matrix $P$, if we can find a polynomial with the above properties, then we can use Theorem 2.1′ to compute the expected hitting times. Furthermore, it is easy to see that if the minimum polynomial and the characteristic polynomial of $P$ are $(x-1)f_1(x)$ and $(x-1)f_2(x)$, respectively, then $f_1(x)$ and $f_2(x)$ all satisfy the conditions of Theorem 2.1′. And it is worth noting that there exist efficient methods to calculate the characteristic polynomial, for example, the Faddeev–Leverrier method.

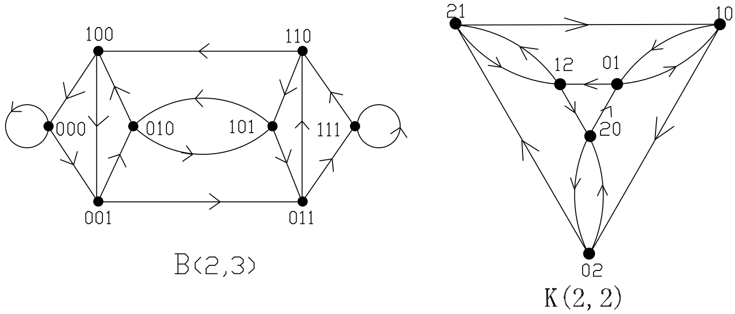


Fig. 1. B(2,3) and K(2,2).

It is easy to see that to find hitting times, any known method requires a matrix inversion, either finding $(I - Q)^{-1}$ (Theorem 1.2), or finding eigenvalues and eigenvectors (Theorem 1.3), or finding the matrix of effective resistances from the transition probability matrix (Theorem 1.4). All these methods, therefore, have a computing complexity roughly of order $N^3$. As to our method, we know that finding the stationary distribution and a polynomial with the property of Theorem 2.1′ can be based on the matrix multiplication (the Faddeev–Leverrier method). Besides that the dominating computing part in the double sum of Theorem 2.1′ is also the matrix multiplication. And the complexity of matrix multiplication is not more than $N^3$. So the computing complexity of our method is also about $N^3$. Although the computing complexity of those methods are about the same, under different circumstances, some perform better than others: for instance, the effective resistance method uses all the physical insight of electric networks that is not available for the absorbing chain technique; sometimes for vertex transitive graph, we can avoid the computation of stationary distributions or inverses. Especially, if both the stationary distribution and the polynomial are easy to be obtained, then our method may be the most suitable choice. In Sections 3 and 4, we will give some explanation examples.

## 3. The random walks on several kinds of digraphs

**Example 1.** The simple random walks on $n$-dimensional de Bruijn digraphs and Kautz digraphs.

Let $Z_k = \{0, 1, \ldots, k-1\}$ $(k \geqslant 2)$. The $n$-dimensional **de Bruijn digraph** $B(k, n)$ is defined as $V(B(k, n)) = \{i_1 i_2 \cdots i_n : i_l \in Z_k, 1 \leqslant l \leqslant n\}$; vertex $i$ is adjacent to vertex $j$ if and only if $i_l = j_{l-1}, 1 < l \leqslant n$. The **Kautz digraph** $K(k, n)$ is defined as $V(K(k, n)) = \{i_1 i_2 \cdots i_n : i_l \in Z_{k+1}, 1 \leqslant l \leqslant n$ and $i_t \neq i_{t+1}$ for $1 \leqslant t < n\}$; vertex $i$ is adjacent to vertex $j$ if and only if $i_l = j_{l-1}, 1 < l \leqslant n$ (see Fig. 1).

Obviously, $B(k, n)$ and $K(k, n)$ are strongly connected $k$-regular digraphs with $k^n$ and $(k+1)k^{n-1}$ vertices, respectively. For the simple random walks on them, we can easily see the following facts:

(i) the transition probability matrices and the adjacency matrices of them all satisfy the relation $P = \frac{1}{k}A$, where $A$ denotes the adjacency matrix;
(ii) $\left(\frac{1}{k^n}, \frac{1}{k^n}, \ldots, \frac{1}{k^n}\right)$ and $\left(\frac{1}{(k+1)(k)^{n-1}}, \frac{1}{(k+1)k^{n-1}}, \ldots, \frac{1}{(k+1)k^{n-1}}\right)$ are their stationary distributions, respectively.

For clearness, for any $i = i_1 \cdots i_n$, $j = j_1 \cdots j_n \in V(B(k,n))$ or $V(K(k,n))$, $0 \leqslant t < n$, define the indicator functions as follows:

$$I(i,j,t) = \begin{cases} 1 & \text{if } j_l = i_{t+l},\ 1 \leqslant l \leqslant n-t, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have the following two lemmas:

**Lemma 3.1** [17]. *For any $i, j \in V(B(k,n))$ or $V(K(k,n))$, $0 \leqslant t < n$. If $I(i,j,t) = 1$, then there exists a unique directed path between $i$ and $j$ with length $t$.*

**Lemma 3.2** [17]. *If $A$ is the adjacent matrix of $B(k,n)$, then $A^n = J$; and if $A$ is the adjacent matrix of $K(k,n)$, then $A^n + A^{n-1} = J$, where $J$ is the matrix each of whose entries is* 1.

From the above two lemmas and Theorem 2.1′, we can easily derive the following theorems:

**Theorem 3.3.** *For simple random walks on $B(k,n)$ $\forall i, j \in V(B(k,n))$, the expected hitting time is*

$$E_i T_j = \sum_{t=0}^{n-1} k^{n-t}(I(j,j,t) - I(i,j,t)).$$

**Proof.** Since $P = \frac{1}{k}A$, by Lemma 3.1, for any $0 \leqslant t \leqslant n-1$, we have

$$p_{ij}^t = \frac{1}{k^t} I(i,j,t),$$

let $f(x) = x^n$, then by Lemma 3.2, $f(P) = \frac{1}{k^n}A^n = \frac{1}{k^n}J$, so by Theorem 2.1′, we have

$$\begin{aligned} E_i T_j &= \frac{1}{\pi_j}\sum_{t=0}^{n-1}(P_{jj}^t - P_{ij}^t) \\ &= \sum_{t=0}^{n-1} k^{n-t}(I(j,j,t) - I(i,j,t)). \qquad \square \end{aligned}$$

**Theorem 3.4.** *For simple random walks on $K(k,n)$ $\forall i, j \in V(K(k,n))$, the expected hitting time*

$$E_i T_j = \sum_{t=0}^{n-1}(k+1)k^{n-t-1}(I(j,j,t) - I(i,j,t)) + I(i,j,n-1) - I(j,j,n-1).$$

**Proof.** Also by Lemma 3.1, for any $0 \leqslant t \leqslant n-1$, we have

$$p_{ij}^t = \frac{1}{k^t} I(i,j,t),$$

let $f(x) = x^{n-1}(x + \frac{1}{k})$, then by Lemma 3.2, $f(P) = \frac{1}{k^n}(A^n + A^{n-1}) = \frac{1}{k^n}J$, so by Theorem 2.1′, we have

$$\begin{aligned} E_i T_j &= \frac{1}{\pi_j}\left(\sum_{t=0}^{n-2}(P_{jj}^t - P_{ij}^t) + \frac{k}{k+1}(p_{jj}^{n-1} - p_{ij}^{n-1})\right) \\ &= \sum_{t=0}^{n-1}(k+1)k^{n-t-1}(I(j,j,t) - I(i,j,t)) + I(i,j,n-1) - I(j,j,n-1). \qquad \square \end{aligned}$$

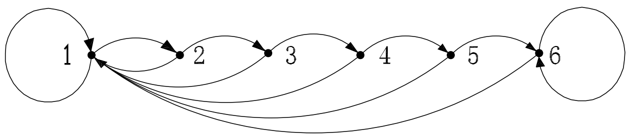


Fig. 2. The winning streak chain for $n = 6$.

From these two theorems, we see, for any $i, j \in V(B(k,n))$ or $V(K(k,n))$, $E_iT_j$ is easy to be obtained. Not only that, we can immediately get the maximum and minimum expected hitting times of simple random walks on them.

For $B(k,n)$,

$$\max_{i,j} E_iT_j = \frac{k^{n+1}-k}{k-1}, \qquad \min_{i,j} E_iT_j = \frac{k^{n+1}-2k^n+k}{k-1}.$$

For $K(k,n)$,

$$\max_{i,j} E_iT_j = \frac{k(k^n-1)}{k-1}, \qquad \min_{i,j} E_iT_j = \frac{k(k^n-k^{n-1}-k^{n-2}+1)}{k-1}.$$

**Example 2.** The winning streak chain.

The winning streak chain is defined as $V = \{1, 2, \ldots, n\}$, and the transition probability matrix is $P = (p_{ij})_{i,j\in V}$, where

$$p_{ij} = \begin{cases} 1/2 & \text{if } j = i+1, \\ 1/2 & \text{if } j = 1 \text{ and } 1 \leqslant i \leqslant n, \\ 1/2 & \text{if } j = i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, this is an irreducible and aperiodic Markov chain (see Fig. 2). Furthermore, it is easy to check that the stationary distribution is

$$\pi_i = \begin{cases} 2^{-i} & \text{if } 1 \leqslant i \leqslant n-1, \\ 2^{-(n-1)} & \text{if } i = n. \end{cases}$$

and $P^{n-1}$ is a matrix with the same row vector $\pi$.

So from Theorem 2.1′, we derive the following theorem immediately:

**Theorem 3.5.** *For the winning streak chain, any $i, j \in V$, the expected hitting time from $i$ to $j$ is*

$$E_iT_j = \begin{cases} 2^j - 2^i & if\ i < j, \\ 2^j & if\ i > j. \end{cases}$$

**Proof.** By Theorem 2.1′, for any $i, j \in V$

$$E_iT_j = \frac{1}{\pi_j}\sum_{t=0}^{n-2}(p_{jj}^t - p_{ij}^t) = \frac{1}{\pi_j}\sum_{t=0}^{j-1}(p_{jj}^t - p_{ij}^t),$$

the second equality is because of $p^t_{jj} = p^t_{ij}$ when $t \geqslant j$. Then,

(i) when $j = n$,

$$E_i T_n = \frac{1}{\pi_n} \sum_{t=0}^{n-i-1} p^t_{nn} = 2^{n-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-i-1}}\right) = 2^n - 2^i,$$

(ii) when $j > i$ and $j \neq n$,

$$E_i T_j = \frac{1}{\pi_j} \sum_{t=0}^{j-1} (p^t_{jj} - p^t_{ij}) = 2^j(1 - 2^{i-j}) = 2^j - 2^i,$$

(iii) when $j < i$,

$$E_i T_j = \frac{1}{\pi_j} \sum_{t=0}^{j-1} (p^t_{jj} - p^t_{ij}) = 2^j(1 - 0) = 2^j,$$

so the result is obvious. □

**Example 3.** The random walks on weighted de Bruijn graph $B(2, n)$.

In Example 1, we have derived the expected hitting times for simple random walks on $B(2, n)$, now we equip weights to its edges. Let $p_{00} = 1 - a$, $p_{01} = a$; $p_{10} = b$, $p_{11} = 1 - b$ $(0 < a \leqslant b \leqslant 1)$, for any $i = i_1 i_2 \cdots i_n$, $j = j_1 j_2 \cdots j_n$, define the weight function as follows:

$$\omega_{ij} = \begin{cases} p_{i_n j_n} & \text{if } (i, j) \in D(B(2, n)), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the random walk on $B(2, n)$ defined above is an irreducible and aperiodic Markov chain with transition probability matrix $P = (\omega_{ij})$, we can also easily get the following two lemmas:

**Lemma 3.6.** *Let $\pi = \{\pi_1, \pi_2, \ldots, \pi_{2^n}\}$ denote the stationary distribution of the random walk defined above. Then,*

$$\pi_i = \frac{c}{a + b} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n},$$

*where* $c = \begin{cases} a & if\ i_1 = 1, \\ b & if\ i_1 = 0. \end{cases}$

**Proof.** We only need to check that for any $j \in V(B(2, n))$, $\pi_j = \sum_{i \in V(B(2,n))} \pi_i p_{ij}$. □

**Lemma 3.7.** *Let $f(x) = x^{n-1}(x + a + b - 1)$, then $f(P)$ has the same row vectors.*

**Proof.** We only need to prove for any $j \in V(B(2, n))$, $p^n_{ij} + (a + b - 1) p^{n-1}_{ij}$ is independent of $i$. □

From these two lemmas and Theorem 2.1′, we derive the following result:

**Theorem 3.8.** *For the random walks on $B(2, n)$ $\forall i, j \in V(B(2, n))$, the expected hitting time from $i$ to $j$ is*

$$E_iT_j = \frac{1}{\pi_j}\sum_{t=0}^{n-2}(p^t_{jj} - p^t_{ij}) + \frac{1}{\pi_j(a+b)}(p^{n-1}_{jj} - p^{n-1}_{ij}),$$

*where* $\pi_j$ *is given by Lemma* 3.6, *and*

$$p^t_{ij} = \begin{cases} p_{j_{n-t}j_{n-t+1}}p_{j_{n-t+1}j_{n-t+2}}\cdots p_{j_{n-1}j_n} & if\ I(i,j,t)=1,\\ 0 & if\ I(i,j,t)=0.\end{cases}$$

*Here* $I(i, j, t)$ *is the indicator function in Example* 1.

In this example, if $a = b = \frac{1}{2}$, then it is just the simple random walk on $B(2, n)$. If $n = 1$, it is the two states Markov chain [2,13].

**Remark.** For the above three examples, the simple random walks on them are not reversible, so Theorems 1.3 and 1.4 cannot be applied to them. And except the winning streak chain, the absorbing chain technique is more complex than ours.

## 4. The simple random walks on several kinds of undirected graphs

In this section, we consider the simple random walks on several kinds of regular undirected graphs. Under those conditions, the random walk is reversible, and the stationary distribution is trivial to be obtained (that is, the uniform distribution). Furthermore, the transition probability matrix $P = \frac{1}{k}A$, where $k$ and $A$ are regular degree and adjacent matrix of the corresponding graph, respectively. So if the eigenvalues of $A$ are known, the minimum polynomial of $P$ is easy to be obtained. In theory, we can use any known method to calculate the expected hitting times. But by using our method, the only thing to be done is to take the corresponding values into 2.1 and compute. Up to now, there have been many kinds of graphs whose eigenvalues are known, for example, the circulant graphs, the strong regular graphs, the Prisms, and the cubes [3]. As explanation examples, in the following, we consider several kinds of graphs with a few distinct eigenvalues. In general, graphs with a few distinct eigenvalues have nice combinatorial properties and a rich structure. So we will see that, by using our method, the computation of the expected hitting times on those graphs can be completed only by pen and paper. Naturally, we will first consider strongly regular graphs (the regular graphs with only three distinct eigenvalues). For clearness, we give its definition.

$G = (V, E)$ is called a $(d, k, \lambda, \mu)$ **strongly regular graph**, if $G$ is $k$-regular with $d$ vertices $(0 < k < d - 1)$, and any pair of adjacent vertices has $\lambda$ common neighbors and any two non-adjacent vertices have $\mu (\mu > 1)$ common neighbors.

It is well known that a strongly regular graph with parameters $(d, k, \lambda, \mu)$ has only three distinct eigenvalues [1]: $k, r, s$, where $r + s = \lambda - \mu$, $rs = \mu - k$, so its minimum polynomial is

$$(x - k)(x^2 + (\mu - \lambda)x + (\mu - k)). \tag{4.1}$$

And the following relation is very useful for our calculation:

$$k^2 + (\mu - \lambda)k + (\mu - k) = d\mu. \tag{4.2}$$

For simplicity, in the following, when we say a strongly regular graph, we mean a non-partite strongly regular graph with parameters $(d, k, \lambda, \mu)$, and $i \sim j$ denotes that $i$ is adjacent to $j$. From (4.1), we can easily prove the following result by Theorem 2.1.

**Theorem 4.1.** *For the simple random walk on a strongly regular graph $G = (V, E)$ $\forall i, j \in V$, we have*

$$E_i T_j = \begin{cases} d - 1 & if\ i \sim j, \\ d - 1 + \frac{k}{\mu} & otherwise. \end{cases}$$

**Proof.** By (4.1), the minimum polynomial of $P = \frac{1}{k}A$ is

$$(x-1)\left(x^2 + \frac{(\mu-\lambda)}{k}x + \frac{(\mu-k)}{k^2}\right).$$

Then by Theorem 2.1, we have

$$\begin{aligned} E_i T_j &= \frac{1}{\pi_j}\left(\frac{k^2 + (\mu-\lambda)k}{k^2 + (\mu-\lambda)k + (\mu-k)}(p^0_{jj} - p^0_{ij}) \right. \\ &\quad \left. + \frac{k^2}{k^2 + (\mu-\lambda)k + (\mu-k)}(p_{jj} - p_{ij})\right) \\ &= \left(d - 1 + \frac{k}{\mu}\right)\left(A^0_{jj} - A^0_{ij}) + \frac{k}{\mu}(A_{jj} - A_{ij}\right) \\ &= d - 1 + \frac{k}{\mu} - \frac{k}{\mu}A_{ij}. \end{aligned}$$

The second equality is derived by (4.2), then the result is obvious. □

The strongly regular graph is a well-investigated family of graphs. Because of its nice properties, the expected hitting times of simple random walks on them had been obtained many years ago by other methods [11,15]. Here we go a step further, considering several kinds of graphs constructed from them.

**Example 1.** The line graphs of strongly regular graphs.

In **the line graph** $L(G)$ of a graph $G = (V, E)$, each vertex represents an edge of $G$, that is $V(L(G)) = E(G)$; and the vertex $i = i_1 i_2$ is adjacent to $j = j_1 j_2$ if and only if the edge $i_1 i_2$ is adjacent to the edge $j_1 j_2$ in $G$.

From definition, if $G$ is a $k$-regular connected graph with $n$ vertices, then $L(G)$ is a $2k-2$-regular connected graph with $\frac{1}{2}nk$ vertices. Furthermore, the characteristic polynomials $\chi(G, x)$, $\chi(L(G), x)$ have the following relation [14]:

$$\chi(L(G), x) = (x + 2)^{m-n}\chi(G, x + 2 - k), \tag{4.3}$$

where $m = \frac{1}{2}nk$ denotes the edge number of $G$. Then, we have the following theorem:

**Theorem 4.2.** *Suppose $G = (V, E)$ is a strongly regular graph, then for the simple random walk on $L(G)$ $\forall i = i_1 i_2, j = j_1 j_2 \in V(L(G))$, we have*

$$E_i T_j = \begin{cases} \frac{2k-2}{4\mu}(d\mu + (\mu-\lambda) + k - 1) & if\ |V(H)| = |E(H)| = 3, \\ \frac{2k-2}{4\mu}(d\mu + (\mu-\lambda) + k) & if\ |V(H)| = 3, |E(H)| = 2, \\ \frac{2k-2}{4\mu}(d\mu + 2(\mu-\lambda) + 2k + 4 - |E(H)|) & if\ |V(H)| = 4, \end{cases}$$

*where $H$ denotes the subgraph of $G$ induced by $i_1, i_2, j_1, j_2$.*

**Proof.** From (4.1) and (4.3), the minimum polynomial of $L(G)$ is

$$(x-2k+2)(x+2)((x+2-k)^2+(\mu-\lambda)(x+2-k)+\mu-k).$$

So the minimum polynomial of $P=\frac{1}{2k-2}A$ is

$$(x-1)\left(x+\frac{2}{2k-2}\right)\\ \times\left(x^2+\frac{\mu-\lambda-2k+4}{2k-2}x+\frac{(2-k)^2+(\mu-\lambda)(2-k)+(\mu-k)}{(2k-2)^2}\right).$$

Let

$$\begin{aligned} f(x)&=\left(x+\frac{2}{2k-2}\right)\\ &\quad\times\left(x^2+\frac{\mu-\lambda-2k+4}{2k-2}x+\frac{(2-k)^2+(\mu-\lambda)(2-k)+(\mu-k)}{(2k-2)^2}\right)\\ &=x^3+\frac{\mu-\lambda-2k+6}{2k-2}x^2+\frac{d\mu+(\mu-\lambda)(4-2k)+12-8k}{(2k-2)^2}x\\ &\quad+\frac{2(d\mu+(\mu-\lambda)(2-2k)+4-4k)}{(2k-2)^3}, \end{aligned}$$

then

$$\begin{aligned} f(1)&=a_0+a_1+a_2+a_3=\frac{2kd\mu}{(2k-2)^3},\\ a_0+a_1&=\frac{\mu-\lambda+4}{2k-2},\\ a_0+a_1+a_2&=\frac{d\mu+2(\mu-\lambda+2)}{(2k-2)^2}. \end{aligned}$$

So by Theorem 2.1

$$\begin{aligned} E_iT_j&=\frac{1}{\pi_j f(1)}((a_0+a_1+a_2)(p_{jj}^0-p_{ij}^0)+(a_0+a_1)(p_{jj}-p_{ij})+a_0(p_{jj}^2-p_{ij}^2))\\ &=\frac{(2k-2)^3}{4\mu}\left(\frac{d\mu+2(\mu-\lambda+2)}{(2k-2)^2}-\frac{\mu-\lambda+4}{(2k-2)^2}A_{ij}+\frac{1}{(2k-2)^2}(2k-2-A_{ij}^2)\right)\\ &=\frac{2k-2}{4\mu}(d\mu+2(\mu-\lambda+2)-(\mu-\lambda+4)A_{ij}+2k-2-A_{ij}^2). \end{aligned}$$

Let $H$ denote the subgraph of $G$ induced by $i_1, i_2, j_1, j_2$, now the arguments shall be divided into the following three different cases:

(i) if $|V(H)|=|E(H)|=3$, then $A_{ij}=1$, $A_{ij}^2=k-1$, so

$$E_iT_j=\frac{2k-2}{4\mu}(d\mu+(\mu-\lambda)+k-1),$$

(ii) if $|V(H)|=3$, $|E(H)|=2$, then $A_{ij}=1$, $A_{ij}^2=k-2$, so

$$E_iT_j = \frac{2k-2}{4\mu}(d\mu + (\mu-\lambda) + k),$$

(iii) if $|V(H)| = 4$, then $A_{ij} = 0$, $A_{ij}^2 = |E(H)| - 2$, so

$$E_iT_j = \frac{2k-2}{4\mu}(d\mu + 2(\mu-\lambda) + 2k + 4 - |E(H)|). \qquad \square$$

**Example 2.** The join graphs of strongly regular graphs.

Suppose $G_1$ and $G_2$ are two disjoint graphs, let $G_1 \cup G_2$ denote **the union** of $G_1$ and $G_2$, that is, $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, then **the join** $G_1 + G_2$ is obtained from $G_1 \cup G_2$ by adding all edges between $G_1$ and $G_2$. If $tG$ denotes the join of $t$ disjoint copies of $G$, then we have the following theorem:

**Theorem 4.3.** *Suppose $G$ is a strongly regular graph, then for simple random walk on $tG(t > 1)$ $\forall i, j \in V(tG)$, we have*

(i) *if $i, j$ lie in the same copy of $G$ and $i \sim j$, then*

$$E_iT_j = \frac{(td+k-d)(t^2d^2 + (\mu-\lambda+k-d-1)td)}{d((t-1)^2d + (2k+\mu-\lambda)(t-1)+\mu)},$$

(ii) *if $i, j$ lie in the same copy of $G$ and $i \nsim j$, then*

$$E_iT_j = \frac{(td+k-d)(t^2d^2 + (\mu-\lambda+k-d)td)}{d((t-1)^2d + (2k+\mu-\lambda)(t-1)+\mu)},$$

(iii) *if $i, j$ lie in the different copies of $G$, then*

$$E_iT_j = \frac{(td+k-d)(t^2d^2 + (\mu-\lambda+k-d-1)td - 2k + \lambda + d)}{d((t-1)^2d + (2k+\mu-\lambda)(t-1)+\mu)}.$$

**Proof.** By definition, $tG$ is a $td+k-d$-regular and connected graph, furthermore we know that it only has four distinct eigenvalues [5]: $td+k-d$, $k-d$, $r$, $s$. So its minimum polynomial is

$$(x - td + d - k)(x + d - k)(x^2 + (\mu-\lambda)x + \mu - k).$$

Then, the minimum polynomial of $P = \frac{1}{td+k-d}A$ is

$$(x-1)\left(x + \frac{d-k}{td+k-d}\right)\left(x^2 + \frac{\mu-\lambda}{td+k-d}x + \frac{\mu-k}{(td+k-d)^2}\right).$$

Let

$$\begin{aligned} f(x) &= \left(x + \frac{d-k}{td+k-d}\right)\left(x^2 + \frac{\mu-\lambda}{td+k-d}x + \frac{\mu-k}{(td+k-d)^2}\right) \\ &= x^3 + \frac{\mu-\lambda+d-k}{td+k-d}x^2 \\ &\quad + \frac{\mu-k+(\mu-\lambda)(d-k)}{(td+k-d)^2}x + \frac{(\mu-k)(d-k)}{(td+k-d)^3}. \end{aligned}$$

Then

$$
\begin{aligned}
a_0 + a_1 &= \frac{td + \mu - \lambda}{td + k - d}, \\
a_0 + a_1 + a_2 &= \frac{t^2d^2 + (\mu - \lambda + k - d)td + \mu - k}{(td + k - d)^2}, \\
f(1) &= a_0 + a_1 + a_2 + a_3 \\
&= \frac{td^2((t-1)^2 d + (2k + \mu - \lambda)(t-1) + \mu)}{(td + k - d)^3}.
\end{aligned}
$$

By Theorem 2.1, we have

$$
\begin{aligned}
E_i T_j &= \frac{1}{\pi_j f(1)}((a_0 + a_1 + a_2)(p^0_{jj} - p^0_{ij}) + (a_0 + a_1)(p_{jj} - p_{ij}) + a_0(p^2_{jj} - p^2_{ij})) \\
&= \frac{(td + k - d)^3}{d((t-1)^2 d + (2k + \mu - \lambda)(t-1) + \mu)} \left( \frac{t^2d^2 + (\mu - \lambda + k - d)td + \mu - k}{(td + k - d)^2} \right. \\
&\quad \left. - \frac{td + \mu - \lambda}{(td + k - d)^2} A_{ij} + \frac{1}{(td + k - d)^2}(td + k - d - A^2_{ij}) \right) \\
&= \frac{td + k - d}{d((t-1)^2 d + (2k + \mu - \lambda)(t-1) + \mu)} (t^2d^2 + (\mu - \lambda + k - d)td + td \\
&\quad + \mu - d - (td + \mu - \lambda)A_{ij} - A^2_{ij}).
\end{aligned}
$$

From here, we distinguish three cases to discuss

(i) if $i, j$ lie in the same copy of $G$ and $i \sim j$, then
$A_{ij} = 1, \quad A^2_{ij} = td + \lambda - d,$
so
$$E_i T_j = \frac{(td + k - d)(t^2d^2 + (\mu - \lambda + k - d - 1)td)}{d((t-1)^2 d + (2k + \mu - \lambda)(t-1) + \mu)},$$
(ii) if $i, j$ lie in the same copy of $G$ and $i \nsim j$, then
$A_{ij} = 0, \quad A^2_{ij} = td + \mu - d,$
so
$$E_i T_j = \frac{(td + k - d)(t^2d^2 + (\mu - \lambda + k - d)td)}{d((t-1)^2 d + (2k + \mu - \lambda)(t-1) + \mu)},$$
(iii) if $i, j$ lie in the different copies of $G$, then
$A_{ij} = 1, \quad A^2_{ij} = td + 2k - 2d,$
so
$$E_i T_j = \frac{(td + k - d)(t^2d^2 + (\mu - \lambda + k - d - 1)td - 2k + \lambda + d)}{d((t-1)^2 d + (2k + \mu - \lambda)(t-1) + \mu)}. \qquad \square$$

**Example 3.** The twisted double of conference graphs.

Suppose $G'$ is a conference graph [1], that is, a strongly regular graph which has parameters $(4\mu + 1, 2\mu, \mu - 1, \mu)$, then **the twisted double** of $G'$ (denoted by $G$) is obtained from $G' \cup \overline{G}'$

by adding edges between the corresponding vertices of $G'$ and $\overline{G}'$, where $\overline{G}'$ denotes the complement of $G'$. For clearness, let $V(G) = \{uv |\ u \in V(G'), v = 1, 2\}$, here $v = 1$ corresponding to the vertices of $G'$, $v = 2$, the vertices of $\overline{G}'$. Apparently, $G$ is a $2\mu + 1$-regular and connected graph, and has four distinct eigenvalues [5]:

$$2\mu + 1, 2\mu - 1, -\frac{1}{2} + \frac{1}{2}\sqrt{4\mu + 5}, -\frac{1}{2} - \frac{1}{2}\sqrt{4\mu + 5},$$

so its minimum polynomial is

$$(x - 2\mu - 1)(x - 2\mu + 1)(x^2 + x - \mu - 1).$$

Then, we derive the following theorem:

**Theorem 4.4.** *For the simple random walk on G* $\forall i = u_1v_1, j = u_2v_2 \in V(G)$, *we have*

(i) *if* $v_1 = v_2 = 1$ *and* $u_1 \sim u_2$ *in* $G'$, *or* $v_1 = v_2 = 2$ *and* $u_1 \sim u_2$ *in* $\overline{G}'$, *then*

$$E_iT_j = \frac{2(2\mu + 1)^2}{\mu + 1},$$

(ii) *if* $v_1 = v_2 = 1$ *and* $u_1 \nsim u_2$ *in* $G'$, *or* $v_1 = v_2 = 2$ *and* $u_1 \nsim u_2$ *in* $\overline{G}'$, *then*

$$E_iT_j = 4(2\mu + 1),$$

(iii) *if* $u_1 = u_2, v_1 \neq v_2$, *then*

$$E_iT_j = \frac{(2\mu + 1)(5\mu + 1)}{\mu + 1},$$

(iv) *if* $v_1 \neq v_2, u_1 \neq u_2$, *then*

$$E_iT_j = \frac{(2\mu + 1)(5\mu + 3)}{\mu + 1}.$$

**Proof.** Let

$$f(x) = \left(x - \frac{2\mu - 1}{2\mu + 1}\right)\left(x^2 + \frac{1}{2\mu + 1}x - \frac{\mu + 1}{(2\mu + 1)^2}\right),$$

then

$$f(1) = \frac{2(\mu + 1)(4\mu + 1)}{(2\mu + 1)^3},$$
$$a_0 + a_1 = \frac{3}{2\mu + 1},$$
$$a_0 + a_1 + a_2 = \frac{3(\mu + 1)}{(2\mu + 1)^2}.$$

So by Theorem 2.1

$$E_iT_j = \frac{2\mu + 1}{\mu + 1}(5\mu + 4 - 3A_{ij} - A_{ij}^2).$$

Now, we distinguish four cases to discuss

(i) if $v_1 = v_2 = 1$ and $u_1 \sim u_2$ in $G'$, or $v_1 = v_2 = 2$ and $u_1 \sim u_2$ in $\overline{G}'$, then

$$A_{ij} = 1, \quad A^2_{ij} = \mu - 1,$$

so

$$E_i T_j = \frac{2(2\mu+1)^2}{\mu+1},$$

(ii) if $v_1 = v_2 = 1$ and $u_1 \nsim u_2$ in $G'$, or $v_1 = v_2 = 2$ and $u_1 \nsim u_2$ in $\overline{G}'$, then

$$A_{ij} = 0, \quad A^2_{ij} = \mu,$$

so

$$E_i T_j = 4(2\mu+1),$$

(iii) if $u_1 = u_2$, $v_1 \neq v_2$, then

$$A_{ij} = 1, \quad A^2_{ij} = 0,$$

so

$$E_i T_j = \frac{(2\mu+1)(5\mu+1)}{\mu+1};$$

(iv) if $v_1 \neq v_2$, $u_1 \neq u_2$ then

$$A_{ij} = 0, \quad A^2_{ij} = 1$$

so

$$E_i T_j = \frac{(2\mu+1)(5\mu+3)}{\mu+1}. \qquad \square$$

**Example 4.** The product graphs constructed from strongly regular graphs.

If $G$ is a strongly regular graph with adjacency matrix $A'$, then we denote by $G \otimes J_m$ the graph with adjacency matrix $A' \otimes J$, that is, $V(G \otimes J_m) = \{uv|\ u \in V(G), v = 1, 2, \ldots, m\}$, $u_1v_1 \sim u_2v_2$ if and only if $u_1 \sim u_2$; and by $G * J_m$ the graph with adjacency matrix $(A' + I) \otimes J_m - I$, that is, $V(G * J_m) = \{uv|\ u \in V(G), v = 1, 2, \ldots, m\}, u_1v_1 \sim u_2v_2$ if and only if $u_1 \sim u_2$ or $u_1 = u_2$ and $v_1 \neq v_2$. From the definition, $G \otimes J_m$ is a $km$-regular connected graph, and $G * J_m$ is a $m(k+1) - 1$-regular connected graph. Furthermore, both of them have only four distinct eigenvalues [5]: $mk, mr, ms, 0$ for the first one; $m(k+1) - 1, m(r+1) - 1, m(s+1) - 1, -1$ for the second one. So, the minimum polynomials of them are

$$x(x - km)(x^2 + m(\mu - \lambda)x + m^2(\mu - k)), \tag{4.4}$$

$$(x+1)(x - mk - m + 1)(x^2 - (m(\lambda - \mu + 2) - 2)x + m^2(\lambda - k + 1) - m(\lambda - \mu + 2) + 1). \tag{4.5}$$

Once more by Theorem 2.1, we derive the following two theorems:

**Theorem 4.5.** *For the simple random walk on $G \otimes J_m$ $\forall i, j \in V(G \otimes J_m)$, we have*

(i) *if $i \sim j$, then*

$$E_i T_j = dm - 1,$$

(ii) *suppose $i = u_1 v_1$, $j = u_2 v_2$, if $v_1 \neq v_2$ and $u_1 \nsim u_2$, then*

$$E_i T_j = dm + \frac{k}{\mu} - 1,$$

(iii) *suppose $i = u_1 v_1$, $j = u_2 v_2$, if $v_1 = v_2$ and $u_1 \nsim u_2$, then*

$$E_i T_j = dm.$$

**Proof.** By (4.4), the minimum polynomial of transition probability matrix $P$ is

$$x(x-1)\left(x^2 + \frac{\mu - \lambda}{k}x + \frac{\mu - k}{k^2}\right).$$

Let $f(x) = x\left(x^2 + \frac{\mu-\lambda}{k}x + \frac{\mu-k}{k^2}\right)$, then $\forall i, j \in V(G \otimes J_m)$,

$$\begin{aligned} E_i T_j &= \frac{1}{\pi_j}(P^0_{jj} - P^0_{ij}) \\ &\quad + \frac{k^2 + (\mu - \lambda)k}{k^2 + (\mu - \lambda)k + (\mu - k)}(P_{jj} - P_{ij}) \\ &\quad + \frac{k^2}{k^2 + (\mu - \lambda)k + (\mu - k)}(P^2_{jj} - P^2_{ij}) \\ &= dm - \left(\frac{d-1}{k} + \frac{1}{\mu}\right) A_{ij} + \frac{1}{m\mu}(km - A^2_{ij}). \end{aligned}$$

Here, we only have three different cases

(i) if $i \sim j$, then $A_{ij} = 1$, $A^2_{ij} = m\lambda$, so

$$\begin{aligned} E_i T_j &= dm - \left(\frac{d-1}{k} + \frac{1}{\mu}\right) + \frac{k - \lambda}{\mu} \\ &= dm + \frac{k^2 - \lambda k + \mu - k - d\mu}{k\mu} = dm - 1, \end{aligned}$$

the last equality is derived from (4.2).

(ii) suppose $i = u_1 v_1 \nsim j = u_2 v_2$, if $v_1 \neq v_2$ and $u_1 \nsim u_2$, then $A_{ij} = 0$, $A^2_{ij} = m\mu$, so

$$E_i T_j = dm + \frac{k}{\mu} - 1,$$

(iii) suppose $i = u_1 v_1 \nsim j = u_2 v_2$, if $v_1 = v_2$ and $u_1 \nsim u_2$, then $A_{ij} = 0$, $A^2_{ij} = km$, so

$$E_i T_j = dm. \qquad \square$$

**Theorem 4.6.** *For the simple random walk on $G * J_m$ $\forall i = u_1v_1$, $j = u_2v_2 \in V(G * J_m)$, we have*

(i) *if $i \sim j$ and $u_1 = u_2$, then*
$$E_iT_j = \frac{d(m(k+1)-1)}{k+1},$$
(ii) *if $i \sim j$ and $u_1 \neq u_2$, then*
$$E_iT_j = \frac{(m(k+1)-1)(md\mu + k - \lambda - 1)}{m\mu(k+1)},$$
(iii) *if $i \nsim j$, then*
$$E_iT_j = \frac{(m(k+1)-1)(md\mu + 2k - \lambda)}{m\mu(k+1)}.$$

**Proof.** From (4.5), we know the minimum polynomial of $P$ is
$$(x-1)\left(x - \frac{m(r+1)-1}{m(k+1)-1}\right)\left(x - \frac{m(s+1)-1}{m(k+1)-1}\right)\left(x + \frac{1}{m(k+1)-1}\right).$$
Let
$$f(x) = \left(x - \frac{m(r+1)-1}{m(k+1)-1}\right)\left(x - \frac{m(s+1)-1}{m(k+1)-1}\right)\left(x + \frac{1}{m(k+1)-1}\right),$$
then
$$a_0 + a_1 = \frac{m(\mu-\lambda) + m(k-1) + 2}{m(k+1)-1},$$
$$a_0 + a_1 + a_2 = \frac{m^2d\mu + m(\mu-\lambda) + m(k-1) + 1}{(m(k+1)-1)^2},$$
$$f(1) = a_0 + a_1 + a_2 + a_3 = \frac{m^3d\mu(k+1)}{(m(k+1)-1)^3},$$
by Theorem 2.1 $\forall i, j \in V(G)$,
$$\begin{aligned} E_iT_j &= \frac{1}{\pi_j f(1)}((a_0 + a_1 + a_2)(P_{jj}^0 - P_{ij}^0) \\ &\quad + (a_0 + a_1)(P_{jj} - P_{ij}) + (P_{jj}^2 - P_{ij}^2)) \\ &= \frac{m(k+1)-1}{m^2\mu(k+1)}(m^2d\mu + m(\mu-\lambda) + m(k-1) + 1 \\ &\quad - (m(\mu-\lambda) + m(k-1) + 2)A_{ij} + m(k+1) - 1 - A_{ij}^2). \end{aligned}$$

Now we divide $i = u_1v_1$, $j = u_2v_2$ into the following three cases:

(i) if $u_1 = u_2$, $v_1 \neq v_2$, then $i \sim j$, and
$$A_{ij} = 1, \quad A_{ij}^2 = km + m - 2,$$
so
$$E_iT_j = \frac{d(m(k+1)-1)}{k+1},$$

(ii) if $u_1 \sim u_2$, then $i \sim j$, and

$$A_{ij} = 1, \quad A^2_{ij} = m\lambda + 2(m-1)$$

so

$$E_i T_j = \frac{(m(k+1)-1)(md\mu + k - \lambda - 1)}{m\mu(k+1)},$$

(iii) if $u_1 \neq u_2$ and $u_1 \nsim u_2$, then

$$A_{ij} = 0, \quad A^2_{ij} = m\mu,$$

so

$$E_i T_j = \frac{(m(k+1)-1)(md\mu + 2k - \lambda)}{m\mu(k+1)}. \qquad \square$$

## Acknowledgments

We would like to thank the anonymous referees for their valuable comments and suggestions. These comments helped us to improve the contents and the presentation of the paper dramatically.

## References

[1] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, New York, 1993.
[2] D. Aldous, J. Fill, Reversible Markov Chains and Random Walks on Graphs, book draft. http://www.stat.berkely.edu/users/aldous/book.html.
[3] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs-Theory and Application, Deutscher Verlag der Wissenschaften–Academic Press, Berlin, 1982.
[4] E. Behrends, Introduction to Markov Chains: with Special Emphasis on Rapid Mixing, Vieweg & Sohn, Braunchweig, Germany, 2000.
[5] E.R. Van dam, Regular graphs with four eigenvalues, Linear Algebra Appl. 226–228 (1995) 139–162.
[6] E. Seneta, Non-negative Matrices and Markov Chains, second ed., Springer, New York, 1981.
[7] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, American Elsevier, New York, 1976.
[8] J.G. Kemeny, J.L. Snell, Finite Markov Chains, D.Van Nostrand, Princeton, NJ, 1960.
[9] J.G. Kemeny, J.L. Snell, A.W. Knapp, Denumerable Markov Chains, Springer-Verlag, New York, 1976.
[10] J.L. Palacios, Bounds on expected hitting times for random walk on a connected graph, Linear Algebra Appl. 141 (1990) 241–252.
[11] L. Devroye, A. Sbihi, Random walks on highly symmetric graphs, J. Theoret. Probab. 4 (1990) 497–514.
[12] L. Lovász, Random walks on graphs: a survey, in combinatorics, Paul Erdös is eighty, Bolyai Soc. Math. Stud. 2 (1993) 1–46.
[13] L. Lovász, P. Winkler, Mixing times, in: D. Aldous, J. Propp (Eds.), Microsurveys in Discrete Probability, DIMACS Ser. 41 (1998) 85–134.
[14] N.L. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, 1993.
[15] N.L. Biggs, Potential theory on distance–regular graphs, in: Cambridge Conference in Honor of Paul Erdös, 1993.
[16] P. Tetali, Random walks and effective resistance of networks, J. Theoret. Probab. 4 (1991) 101–109.
[17] Junming Xu, Topological Structure and Analysis of Interconnection Network, Kluwer Academic Publisher, 2002.