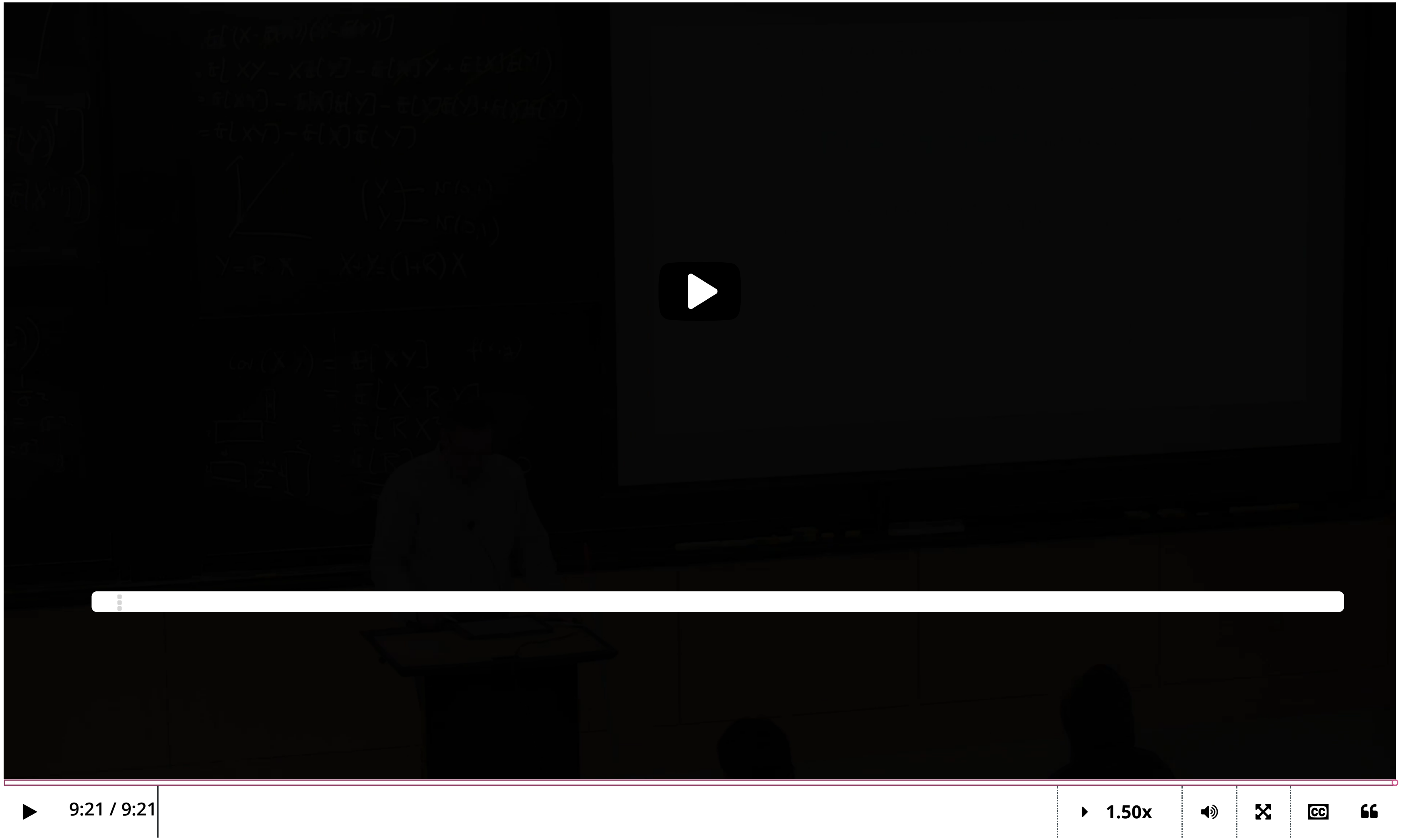
9. Multivariate Gaussian Distribution

Note: Now is a good time to review Gaussian random variables from Lecture 2.

Video Note: In the slide of the video below, there is a typo in the formula of the pdf of the multivariate Gaussian distribution: the exponent d in overall scaling factor should apply only to 2π, rather than $2\pi\mathrm{det}\Sigma$. The correct version is in the note below the video. (The unannotated slides in the resource section have also been corrected).

Multivariate Gaussian Distribution: Definition



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Multivariate Gaussian Random Variable

A random vector $\mathbf{X} = \left(X^{(1)}, \ldots, X^{(d)}\right)^T$ is a **Gaussian vector** , or **multivariate Gaussian or normal variable** , if any linear combination of its components is a (univariate) Gaussian variable or a constant (a "Gaussian" variable with zero variance), i.e., if $\alpha^T\mathbf{X}$ is (univariate) Gaussian or constant for any constant non-zero vector $\alpha \in \mathbb{R}^d$.

The distribution of $\mathbf{X}$, the d**-dimensional Gaussian or normal distribution** , is completely specified by the vector mean $\mu = \mathbb{E}[\mathbf{X}] = \left(\mathbb{E}[X^{(1)}], \ldots, \mathbb{E}[X^{(d)}]\right)^T$ and the $d \times d$ covariance matrix Σ. If Σ is invertible, then the pdf of $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)}, \quad \mathbf{x} \in \mathbb{R}^d$$

where $\det(\Sigma)$ is the determinant of the Σ, which is positive when Σ is invertible.

If $\mu = \mathbf{0}$ and Σ is the identity matrix, then $\mathbf{X}$ is called a **standard normal random vector** .

Note that when the covariant matrix Σ is diagonal, the pdf factors into pdfs of univariate Gaussians, and hence the components are independent.

Linear Transformation of a Multivariate Gaussian Random Vector

1/1 point (graded)

Consider the 2-dimensional Gaussian $\mathbf{X} = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}$ with covariance matrix $\Sigma_X = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and mean $\mu_{\mathbf{X}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Consider the vector $\alpha = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so that $Y = \alpha^T\mathbf{X}$ is a 1-dimensional Gaussian.

What is the variance Var(Y) of Y?

Var$(Y) =$ 2 ✓ **Answer:** 2

Solution:

One way to answer this is to notice that $Y = X^{(1)} - X^{(2)}$, so

$$\text{Var}\,(Y) = \text{Cov}\,(Y, Y) = \text{Var}\,(X^{(1)}) + \text{Var}\,(X^{(2)}) - 2\text{Cov}\,(X^{(1)}, X^{(2)}) = 1 + 5 - 4 = 2.$$

Another way is to define the matrix $M \triangleq \alpha^T = \begin{pmatrix} 1 & -1 \end{pmatrix}$, and apply the formula $\Sigma_Y = M\Sigma_{\mathbf{X}}M^T = 2$.

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You have used 2 of 3 attempts

Answers are displayed within the problem

Singular Covariance Matrices

1/1 point (graded)

Consider again a 2-dimensional Gaussian $\mathbf{X} = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}$. But instead, Σ_X is $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, what is the variance $\text{Var}\,(Y)$ of $Y = \alpha^T\mathbf{X}$?

$\text{Var}\,(Y) =$ 0 ✓ **Answer:** 0

This result tells us that the Gaussian $(X^{(1)}, X^{(2)})^T$ is actually a one-dimensional Gaussian, orthogonal to the direction of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Solution:

Define a matrix $M = \alpha^T$. We have $\Sigma_Y = M\Sigma_X M^T = 0$, since M^T is a column vector in the nullspace of Σ_X.

Such a Gaussian (with a singular covariance matrix) is sometimes referred to as a **degenerate** Gaussian.

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(Optional) Diagonalization of the Covariance Matrix

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(Optional) Gaussian Random Vectors I

0 points possible (ungraded)

Recall from an earlier part of this lecture that the covariance between two random variables being 0 does not necessarily imply that the random variables are independent. However, this is true if the random variables are multivariate Gaussian.

Let $\mathbf{X}$ be a Gaussian random vector with mean μ and covariance Σ. Assume that Σ is positive definite. Determine if the following statement is true or false.

“There exists a vector B and a matrix A such that $A\left(\mathbf{X}+B\right)$ is a Gaussian random vector whose components are independent and each of mean 0".

● True

○ False

✓

Hint: Refer to the note above on diagonalization of the covariance matrix.

Solution:

True. First, in order to remove the effect of μ we can set $B=-\mu$ to make the individual Gaussian random variables be of zero mean. Let $\widehat{\mathbf{X}}=\mathbf{X}-\mu$. From an earlier problem we know that the covariance matrix of $\widehat{\mathbf{X}}$ is the same as Σ.

From the above note on covariance matrices we can see that there exists an orthogonal matrix U such that $D=U\Sigma U^T$.

Consider the following transformation: $\mathbf{Y} = U\widehat{\mathbf{X}}$.

The covariance matrix of $\mathbf{Y}$ is (from an earlier problem)

$$U\Sigma U^T,$$

which is precisely equal to the diagonal matrix D. Therefore, $\mathbf{Y}$ has component Gaussian random variables that are uncorrelated and hence independent.

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