



## MITx: 6.041x Introduction to Probability - The Science of Uncertainty



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**Exam 1**  
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## Problem 2: A binary communication system - Part 2

(1/2 points)

**Note:** The problem statement from part 1 has been repeated here for your convenience.

A binary communication system is used to send one of two messages:

- (i) message A is sent with probability  $2/3$ , and consists of an infinite sequence of zeroes,
- (ii) message B is sent with probability  $1/3$ , and consists of an infinite sequence of ones.

The  $i$ th received bit is "correct" (i.e., the same as the transmitted bit) with probability  $3/4$ , and is "incorrect" (i.e., a transmitted 0 is received as a 1, and vice versa), with probability  $1/4$ . We assume that **conditioned on any specific message sent**, the received bits, denoted by  $Y_1, Y_2, \dots$  are independent.

1. Is  $Y_2 + Y_3$  independent of  $Y_1$ ?

Yes ▼



Answer: No

2. Is  $Y_2 - Y_3$  independent of  $Y_1$ ?

Yes ▼



Answer: Yes

random variables

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Answer:

1. No, they are not independent. Let  $\mathbf{X} = \mathbf{Y}_2 + \mathbf{Y}_3$ . Using the total probability theorem we find:

$$\begin{aligned}\mathbf{P}(\mathbf{X} = 0) &= \mathbf{P}(\mathbf{A})\mathbf{P}(\mathbf{X} = 0 \mid \mathbf{A}) + \mathbf{P}(\mathbf{B})\mathbf{P}(\mathbf{X} = 0 \mid \mathbf{B}) \\ &= (2/3)(3/4)^2 + (1/3)(1/4)^2 \\ &= 19/48 \approx 0.396.\end{aligned}$$

Conditioning on the event  $\mathbf{Y}_1 = 0$ , we now find that:

$$\begin{aligned}\mathbf{P}(\mathbf{X} = 0 \mid \mathbf{Y}_1 = 0) &= \frac{\mathbf{P}(\mathbf{X} = 0 \cap \mathbf{Y}_1 = 0)}{\mathbf{P}(\mathbf{Y}_1 = 0)} \\ &= \frac{\mathbf{P}(\mathbf{A})\mathbf{P}(\mathbf{X} = 0 \cap \mathbf{Y}_1 = 0 \mid \mathbf{A}) + \mathbf{P}(\mathbf{B})\mathbf{P}(\mathbf{X} = 0 \cap \mathbf{Y}_1 = 0 \mid \mathbf{B})}{\mathbf{P}(\mathbf{Y}_1 = 0)} \\ &= \frac{(2/3)(3/4)^3 + (1/3)(1/4)^3}{7/12} \\ &= 55/112 \approx 0.491.\end{aligned}$$

Since we have shown  $\mathbf{P}(\mathbf{X} = 0 \mid \mathbf{Y}_1 = 0) \neq \mathbf{P}(\mathbf{X} = 0)$  with  $\mathbf{P}(\mathbf{Y}_1 = 0) > 0$ , we conclude that  $\mathbf{X}$  and  $\mathbf{Y}_1$  are not independent. Intuitively, knowing  $\mathbf{Y}_1 = 0$  increases the likelihood that message A was transmitted, which increases the likelihood that  $\mathbf{Y}_2 + \mathbf{Y}_3 = 0$ .

2. Yes, they are independent. Let  $\mathbf{Z} = \mathbf{Y}_2 - \mathbf{Y}_3$ . We want to show that  $p_{\mathbf{Y}_1}(\mathbf{y}_1)p_{\mathbf{Z}}(\mathbf{z}) = p_{\mathbf{Y}_1, \mathbf{Z}}(\mathbf{y}_1, \mathbf{z})$ , for all  $(\mathbf{y}_1, \mathbf{z})$ . We have already found the PMF of  $\mathbf{Y}_1$  in part (1):

$$p_{Y_1}(y_1) = \begin{cases} 7/12 & \text{if } y_1 = 0 \\ 5/12 & \text{if } y_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Using the total probability theorem, similar to how we calculated  $\mathbf{P}(\mathbf{X} = 0)$  in part (4), we find the PMF of  $\mathbf{Z}$ :

$$p_Z(z) = \begin{cases} 3/16 & \text{if } z = -1 \text{ or } z = 1 \\ 5/8 & \text{if } z = 0 \\ 0 & \text{otherwise} \end{cases}$$

The joint PMF can also be found using the total probability theorem:

$$p_{Y_1, Z}(y_1, z) = \begin{cases} 7/64 & \text{if } (y_1, z) = \{(0, -1), (0, 1)\} \\ 35/96 & \text{if } (y_1, z) = (0, 0) \\ 5/64 & \text{if } (y_1, z) = \{(1, -1), (1, 1)\} \\ 25/96 & \text{if } (y_1, z) = (1, 0) \\ 0 & \text{otherwise} \end{cases}$$

Therefore,  $p_{Y_1}(y_1)p_Z(z) = p_{Y_1, Z}(y_1, z)$ , for all  $(y_1, z)$ . Intuitively,  $\mathbf{Z}$  is independent of what message was transmitted, and so, even though  $\mathbf{Y}_1$  provides information on which message was transmitted, it ultimately does not provide information on  $\mathbf{Z}$ .

*You have used 1 of 1 submissions*

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