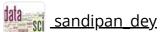


<u>Help</u>



<u>Unit 6: Joint Distributions and</u>
Course > Conditional Expectation

6.6 Conditional expectation given a

> <u>6.1 Reading</u> > random variable

## 6.6 Conditional expectation given a random variable Unit 6: Joint Distributions and Conditional Expectation

Adapted from Blitzstein-Hwang Chapters 7 and 9.

In this section we introduce <u>conditional expectation</u> given a <u>random variable</u>. That is, we want to understand what it means to write E(Y|X) for an r.v. X. We will see that E(Y|X) is a *random variable* that is, in a certain sense, our best prediction of Y, assuming we get to know X.

The key to understanding E(Y|X) is first to understand E(Y|X=x). Since X=x is an event, E(Y|X=x) is just the conditional expectation of Y given this event, and it can be computed using the conditional distribution of Y given X=x.

If Y is discrete, we use the conditional PMF P(Y=y|X=x) in place of the unconditional PMF P(Y=y):

$$E(Y|X=x) = \sum_{y} y P(Y=y|X=x).$$

Analogously, if Y is continuous, we use the conditional PDF  $f_{Y|X}(y|x)$  in place of the unconditional PDF:

$$E(Y|X=x)=\int_{-\infty}^{\infty}yf_{Y|X}(y|x)dy.$$

Notice that because we sum or integrate over y, E(Y|X=x) is a function of x only. We can give this function a name, like g: let g(x) = E(Y|X=x). We define E(Y|X) as the random variable obtained by finding the form of the function g(x), then plugging in X for x.

DEFINITION 6.6.1 (CONDITIONAL EXPECTATION GIVEN AN R.V.).

Let g(x) = E(Y|X = x). Then the conditional expectation of Y given X, denoted E(Y|X), is defined to be the random variable g(X). In other words, if after doing the experiment X crystallizes into x, then E(Y|X) crystallizes into g(x).

♥ WARNING 6.6.2.

The notation in this definition sometimes causes confusion. It does *not* say "g(x) = E(Y|X=x), so g(X) = E(Y|X=X), which equals E(Y) because X=X is always true". Rather, we should first compute the function g(x), then plug in X for x. For example, if  $g(x) = x^2$ , then  $g(X) = X^2$ .

**₩**WARNING 6.6.3.

By definition, E(Y|X) is a function of X, so it is a random variable. Thus it makes sense to compute quantities like E(E(Y|X)) and  $\operatorname{Var}(E(Y|X))$ , the mean and variance of the r.v. E(Y|X). It is easy to be ensnared by category errors when working with conditional expectation, so we should always keep in mind that conditional expectations of the form E(Y|X) are random variables.

## Example 6.6.4.

Suppose we have a stick of length 1 and break the stick at a point X chosen uniformly at random. Given that X = x, we then choose another breakpoint Y uniformly on the interval [0, x]. Find E(Y|X), and its mean and variance.

## Solution

From the description of the experiment,  $X \sim \mathrm{Unif}(0,1)$  and  $Y|X=x \sim \mathrm{Unif}(0,x)$ . Then E(Y|X=x)=x/2, so by plugging in X for x, we have

$$E(Y|X) = X/2.$$

The expected value of E(Y|X) is

$$E(E(Y|X)) = E(X/2) = 1/4.$$

(We will show in the next section that a general property of conditional expectation is that E(E(Y|X)) = E(Y), so it also follows that E(Y) = 1/4.) The variance of E(Y|X) is

$$\operatorname{Var}(E(Y|X)) = \operatorname{Var}(X/2) = 1/48.$$

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