1.2.3 Qualitative Analysis of Differential Equations

Bookmark this page

Qualitative analysis of differential equations is a strategy to get information about solution behavior without explicitly solving the equation(s), usually by creating a picture to represent the solution behavior. How do we create this picture? We use what we know about the derivative and regions of increase and decrease and no change.

For example, in the previous quiz, we determined for which population values the population will increase and decrease, as well as the solutions where the population remains constant, where $\frac{dP}{dt} = 0$. These equilibrium solutions were $P = 0$ (zero fish) and $P = 100$ (the carrying capacity).

Using this information, we determined the most likely graph of solution curves for different initial conditions, such as $P_0 = 25$ or $P_0 = 125$.

We can now ask what do solution curves look like for other starting values of the population? Using similar reasoning, we can see that the general shapes should look roughly like one of those two. If $P_0 < 100$, the graph will increase but level off as P approaches 100. If $P_0 > 100$, the graph will decrease toward 100, leveling off as P gets close to 100.

Notice that time t does not play a role here in whether the population increases or decreases. The only deciding factor is the current population level, P, because the differential equation

$$\frac{dP}{dt} = 0.4P(1 - \frac{P}{100})$$

does not explicitly depend on time t.

A differential equation with independent variable t is called **autonomous** if the differential equation does not explicitly depend on t. When t represents time, we call this **time-independent** or **time-invariant**. (See Autonomous system (mathematics) from Wikipedia.) This makes the qualitative analysis especially straightforward.

For other types of differential equations, there are other qualitative analysis strategies. For general first-order differential equations, **slope fields** works. See Slope Field from Wikipedia. For a systems of two first-order differential equations, we can make **phase portrait** which we'll see later in this section.

> **Note:** First-order means the differential equation involves only the first derivative of P (not second derivatives or higher.)

Strategy for Qualitative Analysis of an Autonomous Differential Equation

This strategy is for any first-order differential equation which is autonomous and continuously differentiable:

$$\frac{dP}{dt} = f(P).$$

Here $f(P)$ represents the fact that the differential equation is a function only of P not t.

In particular, it is the strategy we used in our example,

$$\frac{dP}{dt} = 0.4P(1 - \frac{P}{100}).$$

- Find where $\frac{dP}{dt} = 0$. If $\frac{dP}{dt} = 0$ for some population value P_0, this means the population does not change: the population is P_0 for all time. We say that $P(t) = P_0$ is an **equilibrium solution** of the differential equation.

- Sketch the equilibrium solutions on a plot of population versus time (with time on the horizontal axis). The equilibrium solutions divide the plane into horizontal strips.

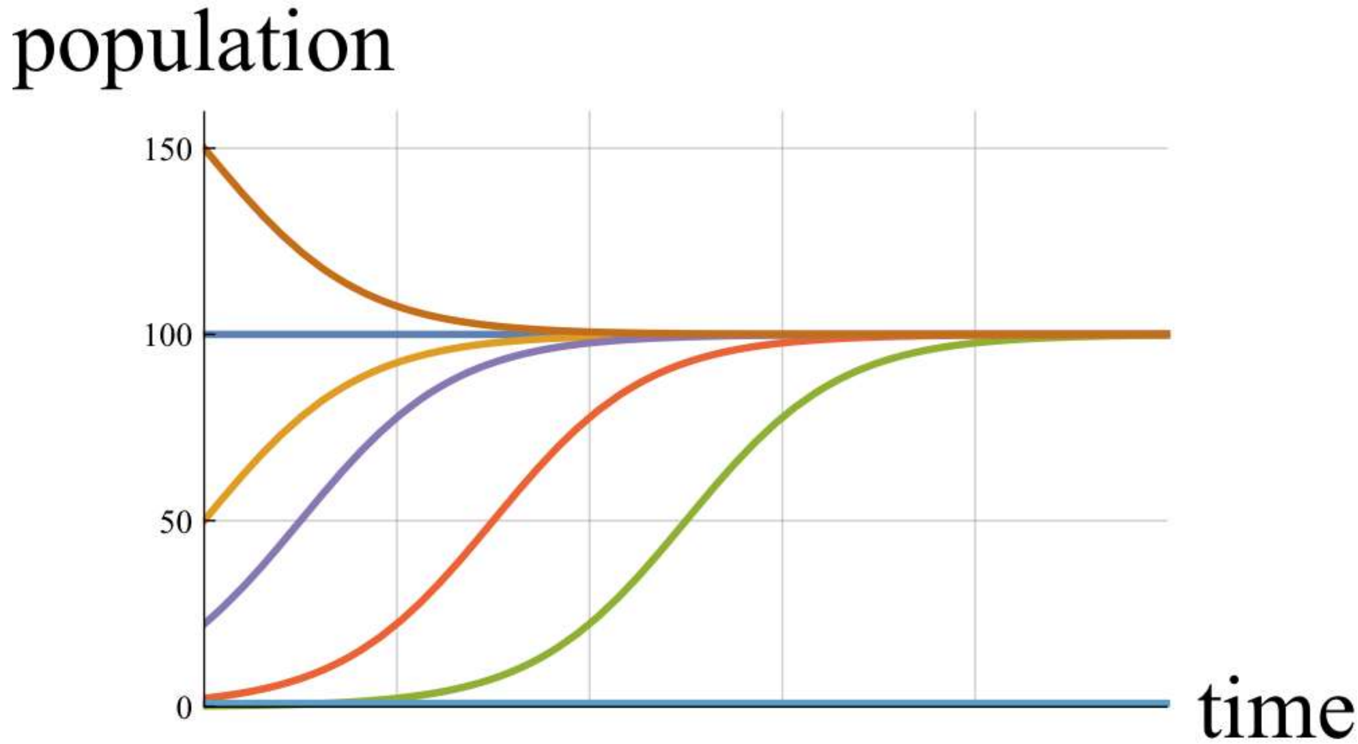
- The theory of differential equations states that for continuously differentiable differential equations *solution curves to differential equations cannot cross or merge.* This means the non-equilibrium solution curves must stay in the horizontal strip they start in and can only increase for all time or decrease for all time within those strips. (Otherwise the solution would have $\frac{dP}{dt} = 0$ at some time, meaning they'd merge into an equilibrium solution, which can't happen.)

  So in each strip, determine whether $\frac{dP}{dt}$ is positive or negative, and sketch a sample curve accordingly.

- All other solution curves are horizontal shifts of these sample curves.

We can put all of this information together in a single picture. Each curve is a solution curve corresponding to some starting population conditions. Together they describe different possible outcomes for the population. While we don't have explicit formulas for these solution curves, we have a *qualitative* sense of what the population will do over time.

> **Note:**In the image below, it may appear that solution curves merge at the equilibrium solution $P(t) = 100$. However, this is just because of the thickness of the graphs. The solution curves remain distinct - they do not cross or merge.



View Larger Image
Image Description

Learn About Verified Certificates

English ▼

POWERED BY