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The Borel Sets

A **Borel Set** is a set that you can get to by starting with the family of line segments and carrying out finitely many applications of the operations of *complementation* and *countable union*:

- The **complementation operation** takes each set A to $\mathbb{R} A$, where \mathbb{R} is the set of real numbers. (I will sometimes refer to the complement of A as \overline{A} .) So, for instance, the result of applying the complement operation to [0,1] is the set $\overline{[0,1]} = \mathbb{R} [0,1]$, which consists of every real number except for the elements of [0,1].
- The **countable union operation** takes each countable (i.e. finite or countably infinite) family of sets A_1, A_2, A_3, \ldots to their union: $\bigcup \{A_1, A_2, A_3, \ldots \}$. So, for instance, the result of applying the countable union operation to the sets $[0, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, \frac{7}{8}], \ldots$ is the set [0, 1), which consists of the real numbers x such that $0 \le x < 1$. (Note that the round bracket on the right-hand side of "[0, 1)" is used to indicate that the end-point 1 is not included in the set.)

Formally, a Borel Set is any member of the smallest class \mathscr{B} such that: (i) every line-segment is in \mathscr{B} , (ii) if a set is in \mathscr{B} , then so is its complement, and (iii) if a countable family of sets is in \mathscr{B} , then so is its union.

To get some practice working with Borel Sets, let us verify that the set of *irrational* real numbers $\overline{\mathbb{Q}}$ is a Borel Set. ($\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$, where \mathbb{Q} is the set of rational numbers.) What we need to show is that the set $\overline{\mathbb{Q}}$ can be generated by starting with a family of line segments [a,b] and applying complementation and countable union as many times as needed. This can be done in three steps:

• Step 1: For every rational number $q \in \mathbb{Q}$, the singleton set $\{q\}$ is identical to the point-sized line-segment [q,q]. Since every line-segment is a Borel Set, it follows that every singleton set $\{q\}$ $(q \in \mathbb{Q})$ is a Borel Set.

- Step 2: Since the set of rational numbers, \mathbb{Q} , is countable, it is the countable union of the family of Borel Sets $\{q\}$, for $q \in \mathbb{Q}$. And since the countable union of Borel Sets is a Borel Set, \mathbb{Q} must be a Borel Set too.
- *Step 3*: Since \mathbb{Q} is a Borel Set, $\overline{\mathbb{Q}}$ is the complement of a Borel Set. So $\overline{\mathbb{Q}}$ must be a Borel Set too.

(Note that a procedure of this kind can be used to show that any countable set is a Borel Set, as is its complement.)

Problem 1

1/1 point (ungraded)

If A_1, A_2, A_3, \ldots is a countable family of Borel Sets, then $\bigcap \{A_1, A_2, A_3, \ldots\}$ is a Borel Set.

(In general, the **intersection** of a set $\{A_1, A_2, A_3, \ldots\}$ (in symbols: $\bigcap \{A_1, A_2, A_3, \ldots\}$) is the set of individuals x such that x is a member of each set A_1, A_2, A_3, \ldots)

True or false?



() False



Explanation

The key observation is that

$$igcap \{A_1,A_2,A_3,\ldots\} = \overline{igcup \{\overline{A_1},\overline{A_2},\overline{A_3},\ldots\}}$$

where $\overline{A} = \mathbb{R} - A$. One can therefore verify that $\bigcap \{A_1, A_2, A_3, \ldots\}$ is a Borel Set as follows. Since each of A_1, A_2, A_3, \ldots is a Borel Set, we can use complementation to show that each of $\overline{A_1}, \overline{A_2}, \overline{A_3}, \ldots$ is a Borel Set. But by countable union, this means that $\bigcup \{\overline{A_1}, \overline{A_2}, \overline{A_3}, \ldots\}$ must also be a Borel Set. So, by complementation, $\bigcap \{A_1, A_2, A_3, \ldots\}$ must also be a Borel Set.

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1 Answers are displayed within the problem

Problem 2

1/1 point (ungraded)

Assume that A and B are Borel Sets. Is it true that A - B (i.e. the set of elements in A which are not in *B*) is a Borel Set?





Explanation

The key observation is that $A - B = A \cap B$. Since the complement of a Borel Set is a Borel Set, and since (as verified in the previous exercise), the intersection of Borel Sets is a Borel Set, this means that A - B is a Borel Set.

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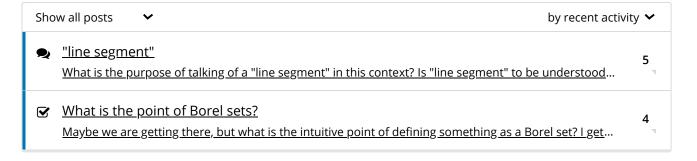
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