7. The Decomposition Theorem for Symmetric Matrices

Concept Check: Orthogonal Matrices

4/4 points (ungraded)

A matrix $P \in \mathbb{R}^{d \times d}$ is **orthogonal** (sometimes referred to as a **rotation matrix**) if $PP^T = P^TP = I_d$. Suppose that

$$P = \begin{pmatrix} v_1 & v_2 & \cdots & v_d \end{pmatrix}$$

where $v_1, v_2, \ldots, v_d \in \mathbb{R}^d$ are column vectors.

Is the identity matrix I_d an orthogonal matrix?

- (•) Yes
- ( ) No

✔

What is $\sum_{i=1}^{d} \left(v_1^i\right)^2$?

1

✔ **Answer:** 1

What is $v_1 \cdot v_2$?

0

Answer: 0

Are the rows of P unit vectors?

- (selected) Yes
- No

Solution:

The identity matrix is an orthogonal matrix because

$$I_d I_d^T = I_d, \quad I_d^T I_d = I_d.$$

For the second question, we know that $P^T P = I_d$. The i-th row of P^T is v_i^T, and the j-th row of P is v_j^T. By matrix multiplication and the orthogonal property,

$$(P^T P)_{ij} = v_i \cdot v_j = (I_d)_{ij}$$

.

In particular, if $i = j$, then

$$\left(P^T P\right)_{ii} = v_i \cdot v_i = \sum_{j=1}^{d} \left(v_i^j\right)^2 = \left(I_d\right)_{ii} = 1.$$

Therefore, $\sum_{i=1}^{d} \left(v_1^i\right)^2 = 1$.

If $i \neq j$, then

$$\left(P^T P\right)_{ij} = v_i \cdot v_j = \sum_{k=1}^{d} v_i^k v_j^k = \left(I_d\right)_{ij} = 0.$$

Therefore $v_1 \cdot v_2 = 0$. This gives the answer to the third question.

We have just shown that the columns of P are mutually orthogonal unit vectors. The same holds true for the rows of P. To see this, we use the fact that $PP^T = I_d$ and follow the same procedure as above. Namely, let $w_1, \ldots, w_d$ denote the columns of P^T, and use the fact that

$$\left(PP^T\right)_{ij} = w_i \cdot w_j = \left(I_d\right)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Therefore, the answer to the fourth question is "Yes."

Submit

You have used 1 of 3 attempts

Answers are displayed within the problem

Eigenvectors and Eigenvalues of a Decomposition of a Symmetric Matrix

3/3 points (ungraded)

Let $A \in \mathbb{R}^{d\times d}$ be a symmetric matrix, and suppose that

$$A = PDP^T$$

where

- P is a $d \times d$ orthogonal matrix, and
- D is a diagonal matrix.

Suppose that the diagonal entries of D are all equal to $\lambda > 0$ (*i.e.*, for $1 \leq i \leq d$, we have $D_{ii} = \lambda > 0$).

For $1 \leq i \leq d$, let $\mathbf{v}_i$ denote the i-th column of P.

Using the decomposition above, express $A\mathbf{v}_1$ in terms of λ and $\mathbf{v}_1$.

Input your answer for $i = 1$. Use **v_1** for $\mathbf{v}_1$.

lambda*v_1

Answer: lambda*v_1

$\lambda \cdot v_1$

Which of the following properties does $\mathbf{v}_1$ have? (Choose all that apply.)

- [x] $\mathbf{v}_1$ is a vector.
- [ ] $\mathbf{v}_1$ is a square matrix.
- [ ] $\mathbf{v}_1$ is an eigenvalue of A.
- [x] $\mathbf{v}_1$ is an eigenvector of A.

Which of the following properties does A have? (Choose all that apply.)

- [x] A is symmetric.
- [x] A is positive semidefinite.
- [x] A is positive definite.

STANDARD NOTATION

Solution:

For the first question, we compute $A\mathbf{v}_1 = PDP^T$ using the decomposition. Observe first that

$$P^T\mathbf{v}_1 = \begin{pmatrix} \longleftarrow & \mathbf{v}_1^T & \longrightarrow \\ \longleftarrow & \mathbf{v}_2^T & \longrightarrow \\ \vdots & \vdots & \vdots \\ \longleftarrow & \mathbf{v}_d^T & \longrightarrow \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^1 \\ \mathbf{v}_1^2 \\ \vdots \\ \mathbf{v}_1^d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

by the property that $P^TP = I_d$.

Next, observe that

$$D\left(P^T\mathbf{v}_1\right) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Finally, we left-multiply by the matrix P:

$$P\left(DP^T\mathbf{v}_1\right) = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_d \end{pmatrix} \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \mathbf{v}_1^1 \\ \lambda \mathbf{v}_1^2 \\ \vdots \\ \lambda \mathbf{v}_1^d \end{pmatrix} = \lambda \mathbf{v}_1.$$

We conclude that $A\mathbf{v}_1 = PDP^T\mathbf{v}_1 = \lambda\mathbf{v}_1$.

For the second part, we examine the choices in order.

- The first choice is correct. By definition, $\mathbf{v}_1 \in \mathbb{R}^d$, so $\mathbf{v}_1$ is a vector.
- The second choice is incorrect. Since $\mathbf{v}_1$ is a vector, it cannot be a square matrix.

  Remark: However, sometimes it is useful to think of $\mathbf{v}_1$ as a $d \times 1$ matrix for the sake of computations.
- The third choice is incorrect. Eigenvalues are numbers, and so since $\mathbf{v}_1$ is a vector, it cannot be an eigenvalue of A.
- The fourth choice is correct. By definition, $\mathbf{v}_1$ is an eigenvector of A is $A\mathbf{v}_1 = \lambda\mathbf{v}_1$ for some real number λ. We showed in the first part of this question that $A\mathbf{v}_1 = \lambda\mathbf{v}_1$

For the third and final part, we examine the choices in order. Note that all of the given choices are correct.

- The first choice "A is symmetric." is correct. It is given in the problem statement that A is symmetric.

  Remark: Note that the decomposition theorem stated above only applied to symmetric matrices.
- The second choice "A is positive semidefinite." is correct. We are given that $A = PDP^T$ where D is a diagonal matrix with all diagonal entries equal to λ. We are also given that $\lambda > 0$, so all of the eigenvalues of λ are positive. Note that

$$\mathbf{u}^T A\mathbf{u} \geq 0 \text{ for all } \mathbf{u} \Leftrightarrow \text{all eigenvalues of } A \text{ are nonnegative,}$$

so indeed A is positive semidefinite.

- The third choice "A is positive definite" is correct. It is also true that

$$\mathbf{u}^T A\mathbf{u} > 0 \text{ for all } \mathbf{u} \Leftrightarrow \text{all eigenvalues of } A \text{ are positive.}$$

Since $\lambda > 0$, and all eigenvalues of A are equal to λ, it follows that A is positive definite.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Concept Check: The Decomposition Theorem for Symmetric Matrices

2/2 points (ungraded)

Suppose that $A \in \mathbb{R}^{d\times d}$ is a symmetric matrix. The **decomposition theorem** states that for all **symmetric** matrices A with **real** entries

$$A = PDP^T$$

where

- $P \in \mathbb{R}^{d\times d}$ is an orthogonal matrix (*i.e.*, $P^TP = PP^T = I_d$), and
- $D \in \mathbb{R}^{d\times d}$ is a diagonal matrix.

Which of the following properties hold for the matrix P ? (Choose all that apply.)

- [x] The columns of P are unit vectors.
- [x] The columns of P are eigenvectors of A.
- [x] The dot product of any two **different** columns of P is 0.
- [x] The rows of P are unit vectors.

✔

Which of the following properties hold for the matrix D ? (Choose all that apply.)

- [x] The diagonal entries of D are the eigenvalues of A.
- [x] The first diagonal element of D (*i.e.* in the top left corner) is the eigenvalue corresponding to the eigenvector which is in the first (*i.e.* leftmost) column of P.
- [ ] It is not possible for any of the diagonal entries of D to be zero.
- [ ] It is not possible for all of the diagonal entries of D to be the same.

✔

Solution:

We handle the first part, examining the choices in order. Note that all of the given options are correct.

- The first choice "The columns of P are unit vectors." is correct. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ denote the columns of P, labeled from left to right. We know that $P^T P = I_d$. For $1 \leq i \leq d$, we have

$$(P^T P)_{ii} = \mathbf{v}_i \cdot \mathbf{v}_i = \|\mathbf{v}_i\|_2^2 = (I_d)_{ii} = 1.$$

Indeed, for $1 \leq i \leq d$, the vector $\mathbf{v}_i$ is a unit vector.

- The second choice "The columns of P are eigenvectors of A." is correct. Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ denote the diagonal entries of D, *i.e.*,

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_d \end{pmatrix}.$$

  Recall that $\mathbf{v}_1, \ldots, \mathbf{v}_d$ denote the columns of P. Using the computation from the previous question, we may verify that

$$A\mathbf{v}_i = PDP^T\mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

  Indeed, the columns of P are eigenvectors.

- The third choice "The dot product of any two **different** columns of P is 0." is correct. Note that $P^TP = I_d$ and recall that $\mathbf{v}_1, \ldots, \mathbf{v}_d$ denote the columns of P. By matrix multiplication, if $i \neq j$,

$$(P^TP)_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j = (I_d)_{ij} = 0.$$

  Therefore, the column vectors of P are mutually orthogonal.

- The fourth choice "The rows of P are unit vectors." is correct. Using that $PP^T = I_d$ and letting $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$ denote the **rows** of P, we can use the same strategy as in the first bullet (look at the entry $(PP^T)_{ii}$) to see that the rows of P are indeed unit vectors.

We now handle the second part, examining the choices in order.

- The first choice "The diagonal entries of D are the eigenvalues of A." is correct. See the explanation in the second bullet.

- The second choice "The first diagonal element of D (*i.e.* in the top left corner) is the eigenvalue corresponding to the eigenvector which is in the first (*i.e.* leftmost) column of P" is correct. Let's write

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_d \end{pmatrix}, \quad \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_d \end{pmatrix}$$

Using matrix multiplication and the same strategy as in the previous problem, we see that

$$A\mathbf{v}_i = PDP^T\mathbf{v}_i = \lambda_i\mathbf{v}_i.$$

Indeed, the diagonal entries of A are eigenvalues of A, and in particular, the eigenvalue λ_i corresponds to the eigenvector $\mathbf{v}_i$.

- The third choice "It is not possible for any of the diagonal entries of D to be zero." is incorrect. Consider the (symmetric) matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

all of whose entries are 0. Observe that $A_d = I_d A I_d^T$ and I_d is orthogonal, so this is indeed a decomposition as guaranteed by the theorem. However, all of the diagonal entries of A are equal to 0, contradicting the claim.

- The fourth choice "It is not possible for all of the diagonal entries of D to be the same." is incorrect. A counterexample is given by the matrix consisting of all zeros given in the previous problem. Another counterexample is given by the identity matrix I_d, which has a decomposition $I_d I_d I_d^T$ and has all of its eigenvalues/diagonal entries equal to 1.

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Answers are displayed within the problem

Concept Check: Properties of Covariance Matrices

1/1 point (ungraded)

Let Σ denote a covariance matrix for some random vector $\mathbf{X} \in \mathbb{R}^d$. (Assume that $\mathbb{E}\left[\|\mathbf{X}\|_2^2\right] < \infty$.)

Which of the following properties does Σ necessarily have? (Choose all that apply.)

- [x] Symmetric
- [x] Positive Semidefinite
- [ ] Positive Definite
- [ ] Orthogonal

✔

Solution:

We examine the choices in order.

- The first choice is correct. The covariance matrix of a random vector is symmetric because the i, j-th entry is given by $\text{Cov}\left(\mathbf{X}^i, \mathbf{X}^j\right)$ and it is true that $\text{Cov}\left(\mathbf{X}^i, \mathbf{X}^j\right) = \text{Cov}\left(\mathbf{X}^j, \mathbf{X}^i\right)$.

- The second choice is also correct. The covariance matrix of a random vector is positive semidefinite. Recall that in the previous page we learned that for all $\mathbf{u} \in \mathbb{R}^d$ it is true that $\mathbf{u}^T \Sigma \mathbf{u} = \text{Var}\left(\mathbf{u}^T \mathbf{X}\right) \geq 0$. The inequality follows because the variance of a random variable is always non-negative. Since $\mathbf{u}^T \Sigma \mathbf{u} \geq 0$ for all $\mathbf{u} \in \mathbb{R}^d$, by definition Σ is positive semidefinite.

- The third choice is incorrect. While it is true that $\mathbf{u}^T \Sigma \mathbf{u} \geq 0$ for all $\mathbf{u} \in \mathbb{R}^d$, there is no reason that a strict inequality should hold. For example, if $\mathbf{X}$ is non-random (*i.e.*, $\mathbf{X} = \mathbf{x}$ with probability 1 where $\mathbf{x}$ is fixed), then the covariance matrix of $\mathbf{X}$ has all of its entries equal to 0. In this case, $\mathbf{u}^T \Sigma \mathbf{u} = 0$ for *all* $\mathbf{u} \in \mathbb{R}^d$.

- The fourth choice is incorrect. Using the counterexample from the previous bullet, the square matrix with all entries equal to 0 is a covariance matrix, but it is not orthogonal because its columns are not unit vectors.

Remark 1: It is good to keep in mind that the positive semidefinite matrices and orthogonal matrices are mathematically significantly different properties. In particular, most positive semidefinite matrices are not orthogonal, and most orthogonal matrices are not positive semidefinite.

Remark 2: When we get into principal component analysis (PCA) in the next lecture, our general strategy will be to diagonalize the (empirical covariance) matrix, and select the eigenvectors whose eigenvalues are the largest as axes on which to visualize our data set or point cloud.

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Answers are displayed within the problem

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