Twin prime

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A **twin prime** is a prime number that has a prime gap of two. In other words, to qualify as a twin prime, the prime number must be either 2 less or 2 more than another prime number (which by definition would mean that it, too, is a twin prime)—for example, the twin prime pair (41, 43). Two is not considered a twin prime with the number three, since it violates the aforementioned rule.[1] Sometimes the term *twin prime* is used for a pair of twin primes; an alternative name for this is **prime twin** or **prime pair**. Twin primes appear despite the general tendency of gaps between adjacent primes to become larger as the numbers themselves get larger due to the prime number theorem (the "average gap" between primes less than n is $\log(n)$).

Unsolved problem in mathematics:

Are there infinitely many twin primes?
(more unsolved problems in mathematics)

Contents

History

The question of whether there exist infinitely many twin primes has been one of the great open questions in number theory for many years. This is the content of the **twin prime conjecture**, which states: *There are infinitely many primes p such that p* + 2 *is also prime.* In 1849, de Polignac made the more general conjecture that for every natural number k, there are infinitely many prime pairs p and p' such that $p' - p = 2k$. The case $k = 1$ is the twin prime conjecture.

A stronger form of the twin prime conjecture, the Hardy–Littlewood conjecture (see below), postulates a distribution law for twin primes akin to the prime number theorem.

On April 17, 2013, Yitang Zhang announced a proof that for some integer N that is less than 70 million, there are infinitely many pairs of primes that differ by N.[2][3] Zhang's paper was accepted by *Annals of Mathematics* in early May 2013.[4] Terence Tao subsequently proposed a Polymath Project collaborative effort to optimize Zhang's bound.[5] As of April 14, 2014, one year after Zhang's announcement, according to the Polymath project wiki, the bound has been reduced to 246.[6] Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively.[7] These improved bounds were discovered using a different approach that was simpler than Zhang's and was discovered independently by James Maynard and Terence Tao. This second approach also gave bounds for the smallest $f(m)$ needed to guarantee that infinitely many intervals of width $f(m)$ contain at least m primes.

Brun's theorem

In 1915, Viggo Brun showed that the sum of reciprocals of the twin primes was convergent.[8] This famous result, called Brun's theorem, was the first use of the Brun sieve and helped initiate the development of modern sieve theory. The modern version of Brun's argument can be used to show that the number of twin primes less than N does not exceed

$$\frac{CN}{(\log N)^2}$$

for some absolute constant $C > 0$.[9]

Other theorems weaker than the twin-prime conjecture

In 1940, Paul Erdős showed that there is a constant $c < 1$ and infinitely many primes p such that $(p' - p) < (c \ln p)$ where p' denotes the next prime after p. This result was successively improved; in 1986 Helmut Maier showed that a constant $c < 0.25$ can be used. In 2004 Daniel Goldston and Cem Yıldırım showed that the constant could be improved further to $c = 0.085786\ldots$ In 2005, Goldston, János Pintz and Yıldırım established that c can be chosen to be arbitrarily small[10][11]

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.$$

By assuming the Elliott–Halberstam conjecture or a slightly weaker version, they were able to show that there are infinitely many n such that at least two of n, $n + 2$, $n + 6$, $n + 8$, $n + 12$, $n + 18$, or $n + 20$ are prime. Under a stronger hypothesis they showed that for infinitely many n, at least two of n, $n + 2$, $n + 4$, and $n + 6$ are prime.

The result of Zhang,

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < N \quad \text{with} \quad N = 7 \times 10^7,$$

is a major improvement on the Goldston–Graham–Pintz–Yıldırım result. The Polymath project optimization of Zhang's bound and Maynard claims to have reduced the bound to N = 246 are further improvements.

Every twin prime pair except (3, 5) is of the form ($6n - 1$, $6n + 1$) for some natural number n, and with the exception of $n = 1$, n must end in 0, 2, 3, 5, 7, or 8. A002822

It has been proven that the pair (m, $m + 2$) is a twin prime if and only if

$$4((m-1)!+1) \equiv -m \pmod{m(m+2)}.$$

If $m - 4$ or $m + 6$ is also prime then the three primes are called a prime triplet.

Largest known twin prime pair

On January 15, 2007, two distributed computing projects, Twin Prime Search and PrimeGrid found the largest known twin primes, $2003663613 \cdot 2^{195000} \pm 1$. The numbers have 58711 decimal digits. Their discoverer was Eric Vautier of France.

On August 6, 2009, those same two projects announced that a new record twin prime had been found.[12] It is $65516468355 \cdot 2^{333333} \pm 1$.[13] The numbers have 100355 decimal digits.

On December 25, 2011, PrimeGrid announced that yet another record twin prime had been found.[14] It is $3756801695685 \cdot 2^{666669} \pm 1$.[15] The numbers have 200700 decimal digits.

An empirical analysis of all prime pairs up to $4.35 \cdot 10^{15}$ shows that if the number of such pairs less than x is $f(x) \cdot x/(\log x)^2$ then $f(x)$ is about 1.7 for small x and decreases towards about 1.3 as x tends to infinity.

There are 808,675,888,577,436 twin prime pairs below 10^{18}.[16]

The limiting value of f(x) is conjectured to equal twice the twin prime constant (not to be confused with Brun's constant)

$$2 \prod_{\substack{p \text{ prime} \\ p \geq 3}} \left(1 - \frac{1}{(p-1)^2}\right) = 1.3203236\ldots;$$

A114907 this conjecture would imply the twin prime conjecture, but remains unresolved.

The twin prime conjecture would give a better approximation, as with the prime counting function, by

$$\pi_2(x) \approx 2C_2 \ \mathrm{li}_2(x) = 2C_2 \int_2^x \frac{dt}{(\log_e t)^2}.$$

Properties

The first few twin prime pairs are:

(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), (107, 109), (137, 139), … A077800.

The only even prime is 2; except for the pair (2, 3), twin primes are as closely spaced as possible for two primes.

Every third odd number is divisible by 3, which requires that no three successive odd numbers can be prime unless one of them is 3. Five is therefore the only prime that is part of two pairs. Along the same lines, other than the first pair, the number centered between the twin primes must always be divisible by 6. The lower member of a pair is by definition a Chen prime.

First Hardy–Littlewood conjecture

The **Hardy–Littlewood conjecture** (after G. H. Hardy and John Littlewood) is a generalization of the twin prime conjecture. It is concerned with the distribution of prime constellations, including twin primes, in analogy to the prime number theorem. Let $\pi_2(x)$ denote the number of primes $p \leq x$ such that $p + 2$ is also prime. Define the **twin prime constant** C_2 as[17]

$$C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.66016181584686957392781211001 4\ldots$$

A005597 (here the product extends over all prime numbers $p \geq 3$). Then the conjecture is that

$$\pi_2(n) \sim 2C_2 \frac{n}{(\ln n)^2} \sim 2C_2 \int_2^n \frac{dt}{(\ln t)^2}$$

in the sense that the quotient of the two expressions tends to 1 as n approaches infinity.[18] (The second ~ is not part of the conjecture and is proved by integration by parts.)

This conjecture can be justified (but not proven) by assuming that 1 / ln t describes the density function of the prime distribution, an assumption suggested by the prime number theorem.

Polignac's conjecture

Polignac's conjecture from 1849 states that for every positive even natural number k, there are infinitely many consecutive prime pairs p and p' such that $p' - p = k$ (i.e. there are infinitely many prime gaps of size k). The case $k = 2$ is the twin prime conjecture. The conjecture has not yet been proven or disproven for any specific value of k, but Zhang's result proves that it is true for at least one (currently unknown) value of k. Indeed, if such a k did not exist, then for any positive even natural number N there are at most finitely many n such that $p_{n+1} - p_n = m$ for all $m<N$ and so for n large enough we have $p_{n+1} - p_n > N$, which would contradict Zhang's result.

Isolated prime

An **isolated prime** is a prime number p such that neither $p - 2$ nor $p + 2$ is prime. In other words, p is not part of a twin prime pair. For example, 23 is an isolated prime, since 21 and 25 are both composite.

The first few isolated primes are

2, 23, 37, 47, 53, 67, 79, 83, 89, 97, … A007510.

See also

- Cousin prime
- Prime gap
- Prime *k*-tuple
- Prime quadruplet
- Prime triplet
- Sexy prime

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