4. Consistency of Maximum Likelihood Estimator

Review: Definition of MLE

1/1 point (graded)

Let $\left\{E, (\mathbf{P}_\theta)_{\theta\in\Theta}\right\}$ be a statistical model associated with a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n$. Assume that there exists $\theta^* \in \Theta$ such that $X_i \sim \mathbf{P}_{\theta^*}$.

Recall the **Kullback-Leibler (KL) divergence** between two distributions $\mathbf{P}_{\theta^*}$ and $\mathbf{P}_\theta$, with pdfs p_{θ^*} and p_θ respectively, is defined as

$$\mathrm{KL}\left(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta\right) = \mathbb{E}_{\theta^*}\left[\ln\left(\frac{p_{\theta^*}(X)}{p_\theta(X)}\right)\right],$$

and a consistent estimator of $\mathrm{KL}\left(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta\right)$ is

$$\widehat{\mathrm{KL}}_n\left(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta\right) = \text{constant} - \frac{1}{n}\sum_{i=1}^{n} \ln p_\theta\left(X_i\right).$$

Which of the following represents the maximum likelihood estimator of θ^*? (Choose all that apply).

- [x] $\operatorname{argmin}_{\theta\in\Theta}\widehat{\mathrm{KL}}_n\left(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta\right)$
- [x] $\operatorname{argmax}_{\theta\in\Theta}\sum_{i=1}^{n} \ln p_\theta\left(X_i\right)$
- [x] $\operatorname{argmax}_{\theta\in\Theta} \ln\left(\prod_{i=1}^{n} p_\theta\left(X_i\right)\right)$
- [x] $\operatorname{argmax}_{\theta\in\Theta} \ln\left(L_n\left(X_1, X_2, \ldots, X_n; \theta\right)\right)$

✔

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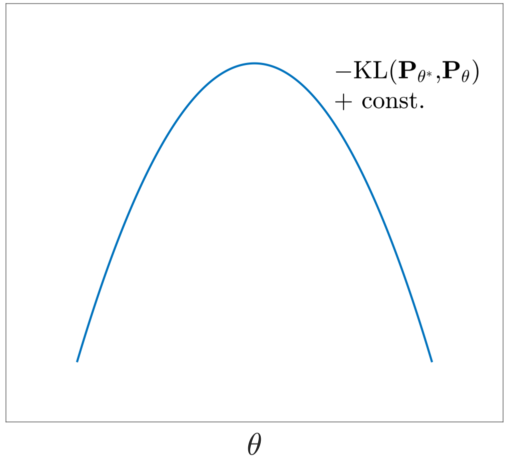
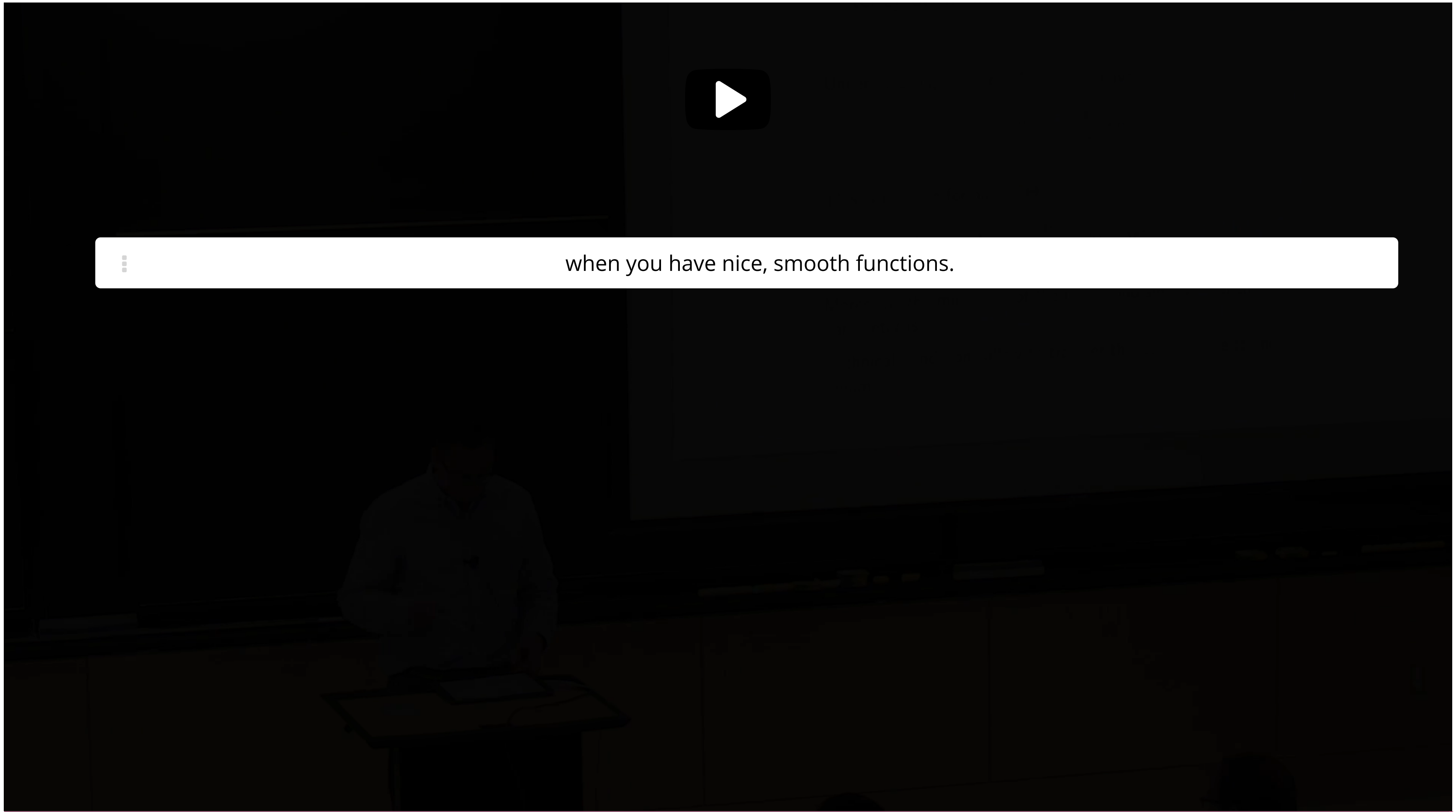
You have used 2 of 3 attempts

Note: In the following video, at around the 3:20 mark, the plot of $-\mathrm{KL}\left(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta\right)$, with θ^* fixed and as a function of θ, is presented incorrectly as a convex curve while it should be concave. This error propagates until the end of the video and we request you to keep the following picture in mind instead:



Consistency of the Maximum Likelihood Estimator



5:24 / 5:24 1.50x

Video

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Consistency of MLE

Given i.i.d samples $X_1, \ldots, X_n \sim \mathbf{P}_{\theta^*}$ and an associated statistical model $\left(E, \{\mathbf{P}_\theta\}_{\theta \in \Theta}\right)$, the maximum likelihood estimator $\hat{\theta}_n^{\text{MLE}}$ of θ^* is a **consistent** estimator under mild regularity conditions (e.g. continuity in θ of the pdf p_θ almost everywhere), i.e.

$$\hat{\theta}_n^{\text{MLE}} \xrightarrow[(p)]{n \to \infty} \theta^*.$$

Note that this is true even if the parameter θ is a vector in a higher dimensional parameter space Θ, and $\hat{\theta}_n^{\text{MLE}}$ is a multivariate random variable, e.g. if $\theta = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \in \mathbb{R}^2$ for a Gaussian statistical model.

Multivariate Random Variables

A **multivariate random variable** , or a **random vector** , is a vector-valued function whose components are (scalar) random variables on the same underlying probability space. More specifically, a random vector $\mathbf{X} = \left(X^{(1)}, \ldots, X^{(d)}\right)^T$ of dimension $d \times 1$ is a vector-valued function from a probability space Ω to $\mathbb{R}^d$:

$$\begin{aligned} \mathbf{X} : \Omega &\to \mathbb{R}^d \\ \omega &\mapsto \begin{pmatrix} X^{(1)}(\omega) \\ X^{(2)}(\omega) \\ \vdots \\ X^{(d)}(\omega) \end{pmatrix} \end{aligned}$$

where each $X^{(k)}$ is a (scalar) random variable on Ω. We will often (but not always) use the bracketed superscript (k) to denote the k-th component of a random vector, especially when the subscript is already used to index the samples.

The **probability distribution** of a random vector $\mathbf{X}$ is the **joint distribution** of its components $X^{(1)}, \ldots, X^{(d)}$.

The **cumulative distribution function (cdf)** of a random vector $\mathbf{X}$ is defined as

$$F : \mathbb{R}^d \to [0, 1]$$

$$\mathbf{x} \quad \mapsto \quad \mathbf{P}\left(X^{(1)} \le x^{(1)}, \ldots, X^{(d)} \le x^{(d)}\right).$$

Convergence in Probability in Higher Dimension

To make sense of the consistency statement $\hat{\theta}_n^{\text{MLE}} \xrightarrow[(p)]{n\to\infty} \theta^*$ where the MLE $\hat{\theta}_n^{\text{MLE}}$ is a random vector, we need to know what convergence in probability means in higher dimensions. But this is no more than convergence in probability for **each component**.

Let $\mathbf{X}_1, \mathbf{X}_2 \ldots$ be a sequence of random vectors of size $d \times 1$, i.e. $\mathbf{X}_i = \begin{pmatrix} X_i^{(1)} \\ \vdots \\ X_i^{(d)} \end{pmatrix}$.

Let $\mathbf{X} = \begin{pmatrix} X^{(1)} \\ \vdots \\ X^{(d)} \end{pmatrix}$ be another vector of size $d \times 1$.

Then

$$\mathbf{X}_n \xrightarrow[n\to\infty]{(p)} \mathbf{X} \quad \iff \quad X_n^{(k)} \xrightarrow[n\to\infty]{(p)} X^{(k)} \text{ for all } 1 \le k \le d.$$

In other words, the sequence $\mathbf{X}_1, \mathbf{X}_2, \ldots$ **converges in probability** to $\mathbf{X}$ if and only if each component sequence $X_1^{(k)}, X_2^{(k)}, \ldots$ converges in probability to $X^{(k)}$.

Hence, for example, in the Gaussian model $\left((-\infty, \infty), \{\mathcal{N}(\mu, \sigma^2)\}_{(\mu,\sigma^2)\in\mathbb{R}\times\mathbb{R}_{>0}}\right)$, consistency of the MLE $\hat{\theta}_n^{\text{MLE}} = \begin{pmatrix} \widehat{\mu} \\ \widehat{\sigma^2} \end{pmatrix}$ means that $\widehat{\mu}$ and $\widehat{\sigma^2}$ are consistent estimators of μ^* and $(\sigma^2)^*$, respectively.

Remark: You can check that this condition is equivalent to the following definition of convergence in probability, which is a straightforward generalization of the 1-dimensional case:

$$P\left(\left\{\omega \in \Omega : \left|\mathbf{X}_n(\omega) - \mathbf{X}(\omega)\right| < \epsilon\right\}\right) \xrightarrow{n\to\infty} 1 \qquad \text{for any } \epsilon > 0.$$

Consistency of the MLE of a Uniform Model

1/1 point (graded)

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Unif}\,[0, \theta^*]$ where θ^* is an unknown parameter. We construct the associated statistical model $(\mathbb{R}_{\geq 0}, \{\mathrm{Unif}\,[0, \theta]\}_{\theta > 0})$

Consider the maximum likelihood estimator $\hat{\theta}_n^{\text{MLE}} = \max_{i=1,\ldots,n} X_i$.

Which of the following are true about $\hat{\theta}_n^{\text{MLE}}$. (Choose all that apply.)

- [x] $\max_{i=1,\ldots,n} X_i$ is a consistent estimator
- [x] For any $0 < \epsilon \leq \theta^*$, $\mathbf{P}\left(|\max_{i=1,\ldots,n} X_i - \theta^*| \geq \epsilon\right) \to 0$ as $n \to \infty$
- [ ] For any $0 < \epsilon \leq \theta^*$, $\mathbf{P}\left(|\max_{i=1,\ldots,n} X_i - \theta^*| \geq \epsilon\right) \to c$ as $n \to \infty$, where $c > 0$ is a constant
- [x] For any $0 < \epsilon \leq \theta^*$, $\mathbf{P}\left(|\max_{i=1,\ldots,n} X_i - \theta^*| \geq \epsilon\right) = \left(\frac{\theta^* - \epsilon}{\theta^*}\right)^n$

Solution:

Choices 1, 2, and 4 are true because of the following proof for consistency of this ML estimator. Let $0 < \epsilon \leq \theta^*$:

$$\begin{aligned}\mathbf{P}\left(\left|\max_{i=1,\ldots,n} X_i - \theta^*\right| \geq \epsilon\right) &= \mathbf{P}\left(\theta^* - \max_{i=1,\ldots,n} X_i \geq \epsilon\right) \\ &= \mathbf{P}\left(\max_{i=1,\ldots,n} X_i \leq \theta^* - \epsilon\right)\end{aligned}$$

$$= \left(\frac{\theta^* - \epsilon}{\theta^*}\right)^n \longrightarrow 0 \text{ as } n \to \infty.$$

Choice 3 is not true because if a sequence (the relevant sequence here is $\left(\frac{\theta^*-\epsilon}{\theta^*}\right)^n$) converges to a limit, then the limit is unique.

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Answers are displayed within the problem

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"Expectation up to a constant.."

question posted 4 days ago by rickytyagi

what does the phrase mean, I've seen it many times before but never understood what it means. For e.g, at around 1:40 the prof says: "...We said we have this scale, and up to constant, it's an expectation of something."

This post is visible to everyone.

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2 responses

Erocha (Community TA)
3 days ago

I believe this is related to the constant which we miss by not knowing the true parameter, but that does not prevent us from finding the argmax of the likelihood.

See slides 12 and 13.

Add a comment

synnfusion
3 days ago

+

...

Up to a constant in general means that you're making a statement that is true if you ignore a constant. So in this context A = B up to a constant means A = B + c for some constant c. In some other contexts it's not always an added constant. sometimes it's a multiplicative constant. For example f(x) = 2x and g(x) = x may be described as equal up to a constant (constant 2 in this case). An example of two functions that are NOT equal up to a constant would be f(x) = x and g(x) =x^2.

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