How to compute the sum of random variables of geometric distribution

Let $X_i, i = 1, 2, \ldots, n$, be independent random variables of geometric distribution, that is, $P(X_i = m) = p(1-p)^{m-1}$. How to compute the PDF of their sum $\sum_{i=1}^n X_i$?

I know intuitively it's a negative binomial distribution

$$P\left(\sum_{i=1}^n X_i = m\right) = \binom{m-1}{n-1} p^n (1-p)^{m-n}$$

but how to do this deduction?

(probability) (probability-distributions) (random-variables)

edited Mar 30 at 15:39 Math1000 14.7k 3 11 33

asked Nov 2 '13 at 0:38 TonyLic 151 1 1 4

I think the probabilistic interpretation leads quite naturally to the desired formula. One could do an induction on n and use convolution, but that is less informative. – André Nicolas Nov 2 '13 at 1:28

I think the language interpretation cannot be treated as math deduction. I know I should use convolution, but could anyone teach me that? – TonyLic Nov 2 '13 at 1:52

2 Typo: $P\left(\sum_{i=1}^n X_i = n\right)$ should be replaced by: $P\left(\sum_{i=1}^n X_i = m\right)$ – drhab Nov 2 '13 at 12:12

1 Answer

Let $X_1, X_2, \ldots$ be independent rvs having the geometric distribution with parameter p, i.e. $P\left[X_i = m\right] = pq^{m-1}$ for $m = 1, 2. \ldots$ (here $p + q = 1$).

Define $S_n := X_1 + \cdots + X_n$.

With induction on n it can be shown that S_n has a negative binomial distribution with parameters p and n, i.e. $P\left\{S_n = m\right\} = \binom{m-1}{n-1} p^n q^{m-n}$ for $m = n, n+1, \ldots$.

It is obvious that this is true for $n = 1$ and for S_{n+1} we find for $m = n+1, n+2, \ldots$:

> $P\left[S_{n+1} = m\right] = \sum_{k=n}^{m-1} P\left[S_n = k \wedge X_{n+1} = m-k\right] = \sum_{k=n}^{m-1} P\left[S_n = k\right]$
> $\times P\left[X_{n+1} = m-k\right]$

Working this out leads to $P\left[S_{n+1} = m\right] = p^{n+1} q^{m-n-1} \sum_{k=n}^{m-1} \binom{k-1}{n-1}$ so it remains to be shown that $\sum_{k=n}^{m-1} \binom{k-1}{n-1} = \binom{m-1}{n}$.

This can be done with induction on m:

> $\sum_{k=n}^{m} \binom{k-1}{n-1} = \sum_{k=n}^{m-1} \binom{k-1}{n-1} + \binom{m-1}{n-1} = \binom{m-1}{n} + \binom{m-1}{n-1} = \binom{m}{n}$

answered Nov 2 '13 at 12:08

drhab
54k 4 26 76

Thank you very much. This really helps me a lot!!! – TonyLic Nov 2 '13 at 16:28