Are the random variables $X+Y$ and $X-Y$ independent if X,Y are distributed normal?

Asked 3 years, 10 months ago Active 10 months ago Viewed 4k times

2

1

Let X,Y be independent random variables such that $X,Y \sim N(\mu,\sigma^2)$, show that $X+Y$ and $X-Y$ are independent using the moment generating function.

I know that the moment generating function of a sum of independent random variables is the product of the MGF.

So, I´m trying to solve that but i don´t know if my process is correct

$M_{X+Y}(t_1,t_2) = M_X(t_1)M_Y(t_2) = M^2_{N(\mu,\sigma^2)}(t)$?

probability statistics moment-generating-functions

edited Jan 25 '16 at 0:46 Zhanxiong 9,512 1 13 34

asked Jan 25 '16 at 0:36 A P 572 4 15

2 What you need to show is the MGF for the random vector $(X+Y, X-Y)$ can be factorized to the product of MGFs of $X+Y$ and $X-Y$. – Zhanxiong Jan 25 '16 at 0:44

3 Answers

6

Recall that for an $\mathcal{N}(\mu,\sigma^2)$ random variable, the moment generating function of it is

$$M(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right). \tag{1}$$

By condition, $X+Y \sim \mathcal{N}(2\mu, 2\sigma^2)$ and $X-Y \sim \mathcal{N}(0, 2\sigma^2)$. Therefore by (1), we have:

$$M_{X+Y}(t) = \exp(2\mu t + \sigma^2 t^2),\ M_{X-Y}(t) = \exp(\sigma^2 t^2).$$

$$
\begin{aligned}
&= E\left\{\exp[(t_1+t_2)X] \times \exp[(t_1-t_2)Y]\right\} \\
&= E\left\{\exp[(t_1+t_2)X]\right\} \times E\left\{\exp[(t_1-t_2)Y]\right\} \quad \text{by independence of } X \text{ and } Y. \\
&= M_X(t_1+t_2)M_Y(t_1-t_2) \\
&= \exp\left(\mu(t_1+t_2)+\frac{1}{2}\sigma^2(t_1+t_2)^2\right)\exp\left(\mu(t_1-t_2)+\frac{1}{2}\sigma^2(t_1-t_2)^2\right) \\
&= \exp\left(2\mu t_1+\sigma^2 t_1^2\right)\exp\left(\sigma^2 t_2^2\right) \\
&= M_{X+Y}(t_1)M_{X-Y}(t_2).
\end{aligned}
$$

Hence $X+Y$ and $X-Y$ are independent.

edited Sep 28 '16 at 13:13

answered Jan 25 '16 at 1:03
Zhanxiong
9,512 1 13 34

I have one question, when you say: $E\left\{\exp[(t_1+t_2)X]\right\} \times E\left\{\exp[(t_1-t_2)Y]\right\}$ by independence of X and Y. it is a property?, I only know that E(XY)=E(X)E(Y) but I didn´t know that property. – A P Jan 25 '16 at 1:24

1 Yes, it is exactly this property, where you treat $(t_1+t_2)X$ and $(t_1-t_2)Y$ are independent random variables. Notice that t_1 and t_2 are all constants. – Zhanxiong Jan 25 '16 at 1:36

so, when I have 2 independent random variables X,Y $E[g(aX)h(bY)] = E[g(aX)] \cdot E[h(bY)]$ is always true? – A P Jan 25 '16 at 1:44

1 Yes, you made a good guess. Of course, rigorous, g and h needs to be measurable, which is usually guaranteed. – Zhanxiong Jan 25 '16 at 1:45

it´s an interesant result that i didn´t know, can you suggest me a site where find the proof to this property? – A P Jan 25 '16 at 1:52

|

-1

No. If X and $2Y = X$, if X is normally distributed so is Y. And $X + Y = 3Y$ and $X - Y = Y$ aren't independent at all.

answered Jan 25 '16 at 0:39
vonbrand
21.2k 6 33 61

OP did mentioned independence of X and Y – Francis Jan 25 '16 at 0:44

$E[(X+Y)(X-Y)] = E[X^2 - Y^2] = \sigma^2 - \sigma^2 = 0 = (2\mu) \cdot 0 = E[X+Y]E[X-Y]$. Thus $X+Y$ and $X-Y$ are independent.

answered Jan 25 at 21:12
nullUser
21k 4 45 110