13. Companion system of the coupled oscillator

Simplifying assumption

Before we find the solutions for this homogeneous system, we will simplify and assume that

- the springs are identical, $k_1 = k_2 = k_3 = k$,
- the masses are equal, so $m_1 = m_2 = m$.

Let $\frac{k}{m} = \omega^2$. This allows us to rewrite this system as follows:

$$\ddot{\mathbf{x}} = \mathbf{B}\mathbf{x} \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{B} = \omega^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Note the second derivatives on the left hand side. This is a **second** order system. How do we solve it? Convert to the first order, companion system!

Companion system

There are different ways to write down a companion matrix. We will follow the one shown in the video on the previous page, so that the resulting companion matrix will be in a form that makes finding eigenvalues easier.

Define a new **vector variable** $\mathbf{y} = \begin{pmatrix} y_1 \\ \ldots \end{pmatrix}$ and set

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$$\begin{aligned} \mathbf{y} &= \dot{\mathbf{x}} \\ \text{or} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}. \end{aligned}$$

Then, the 2×2 system of second order ODEs $\ddot{x} = \mathbf{Bx}$ can be rewritten as

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{Bx}. \\ \text{or} \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \omega^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

As usual for companion matrices, we reorder the equation $\mathbf{y} = \dot{\mathbf{x}}$ and combine the two equations to form a system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{y} \\ \dot{\mathbf{y}} &= \mathbf{Bx}. \end{aligned}$$

Expanded in all components, the system is:

$$\begin{array}{rcll} \dot{x}_1 & = & & y_1 \\ \dot{x}_2 & = & & \quad y_2 \\ \dot{y}_1 & = & -2\omega^2 x_1 + \omega^2 x_2 & \\ \dot{y}_2 & = & \omega^2 x_1 - 2\omega^2 x_2 & \end{array}.$$

This is the companion system, a 4×4 **first order** system of ODEs, of the original 2×2 system of second order ODEs.
Let us now rewrite this in matrix form by combining the two vectors $\mathbf{x}$ and $\mathbf{y}$ into a single

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vector $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ and rewrite this system in a form involving a 4×4 matrix:

$$\frac{d}{dt}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{B} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

$$\text{or} \quad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\omega^2 & \omega^2 & 0 & 0 \\ \omega^2 & -2\omega^2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}.$$

The notation $\begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{B} & 0 \end{pmatrix}$ is called a **block matrix** because the entries of the matrix are themselves matrices.

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