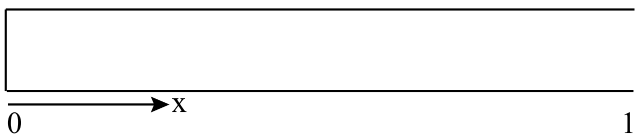
2. Modeling a clarinet

The clarinet

2/2 points (graded)

A clarinet is a curious instrument because its shape is almost exactly a cylinder. You blow air through a reed at one end, and this creates a sound wave inside of the cylindrical clarinet. (Note that clarinets are actually very close to being exactly cylindrical, which is why the following model works.)

We model the geometry of the clarinet as a cylinder of length 1 with one (essentially) closed end (where the mouth piece is) and one open end.



Inside the clarinet, the sound wave satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}, \qquad 0 < x < 1,\ t > 0,$$

where $u(x,t)$ is the horizontal displacement of air particles, $p(x,t)$ is the difference from ambient air pressure, and the speed of sound c is set

Generating Speech Output nputational convenience.

At the closed end, air molecules cannot move horizontally, and the pressure difference is maximized.

Which boundary conditions are zero? (Each column is graded independently. The left hand conditions are all at $x = 0$, the right hand conditions are at $x = 1$.)

☑ $u(0,t) = 0$	☐ $u(1,t) = 0$
☐ $\frac{\partial u}{\partial x}(0,t) = 0$	☑ $\frac{\partial u}{\partial x}(1,t) = 0$
☐ $p(0,t) = 0$	☑ $p(1,t) = 0$
☑ $\frac{\partial p}{\partial x}(0,t) = 0$	☐ $\frac{\partial p}{\partial x}(1,t) = 0$
✔	✔

Solution:

You have seen in recitation, that at the open end of the clarinet ($x = 1$) the pressure difference $p(1,t) = 0$, and thus the derivative $\frac{\partial u}{\partial x}(1,t) = 0$.

At the closed end, we are told that the displacement of air molecules is 0, thus $u(0,t) = 0$. We are also told that the pressure difference is maximized. This tells us that the derivative of the pressure difference at $x = 0$ must be zero.

Submit You have used 2 of 2 attempts

Answers are displayed within the problem

Find the eigenfunctions

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4/4 points (graded)

Use separation of variables $u(x,t) = v(x)\,w(t)$ and $p(x,t) = q(x)\,r(t)$ to find the eigenvalues λ and eigenfunctions $v_k(x)$ and $q_k(x)$ in terms of $k = 0, 1, 2, 3, \ldots$.

For $k = 0, 1, 2, 3, \ldots,\ \lambda_k =$ -(2*k+1)^2*pi^2/4 ✔ **Answer:** -((2*k+1)/2)^2*pi^2

$$-\frac{(2\cdot k+1)^2\cdot\pi^2}{4}$$

For $k = 0, 1, 2, 3, \ldots,\ v_k(x) =$ sin((2*k+1)*pi/2*x) ✔ **Answer:** sin((2*k+1)*pi*x/2)

$$\sin\left(\frac{(2\cdot k+1)\cdot\pi}{2}\cdot x\right)$$

For $k = 0, 1, 2, 3, \ldots,\ q_k(x) =$ cos((2*k+1)*pi/2*x) ✔ **Answer:** cos((2*k+1)*pi*x/2)

$$\cos\left(\frac{(2\cdot k+1)\cdot\pi}{2}\cdot x\right)$$

(To find the functions of t, which are generic, use a for the arbitrary constant in front of the cosine term, and use b for the arbitrary constant in front of the sine term. Do not express in amplitude/phase lag form.)

For $k = 0, 1, 2, 3, \ldots,\ w_k(t) = r_k(t) =$ a*cos((2*k+1)*pi/2*t)+b* ✔ **Answer:** a*cos((2*k+1)*pi*t/2)+b*sin((2*k+1)*pi*t/2)

$$a\cdot\cos\left(\frac{(2\cdot k+1)\cdot\pi}{2}\cdot t\right) + b\cdot\sin\left(\frac{(2\cdot k+1)\cdot\pi}{2}\cdot t\right)$$

FORMULA INPUT HELP

Solution:

Note that to solve for $v_k(x)$, this is the same problem solved on the previous page. Thus

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$$-\left(\frac{(2k+1)\,\pi}{2}\right)^2, \qquad v_k\,(x) = \sin\left(\frac{(2k+1)\,\pi}{2}x\right) \qquad k = 0, 1, 2, \ldots.$$

To solve for $q_k\,(x)$ we use the opposite boundary conditions and end up with cosines instead of sines. Thus

$$q_k\,(x) = \cos\left(\frac{(2k+1)\,\pi}{2}x\right) \qquad k = 0, 1, 2, \ldots.$$

The main difference here is that the wave equation is second order in t as well as in x, therefore

$$w_k\,(t) = r_k\,(t) = a\cos\left(\frac{(2k+1)\,\pi c}{2}t\right) + b\sin\left(\frac{(2k+1)\,\pi c}{2}t\right).$$

Submit You have used 1 of 7 attempts

Answers are displayed within the problem

Food for thought

How does what you have found here compare with the problem you did in Recitation 2 comparing the sound signals of a clarinet, guitar, and human voice. Does this boundary value problem explain the unique feature of the clarinet that you discovered in the frequency spectrum?

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? [Grader error for w(t) and r(t)?] 2

The grader does not allow the variable c in the answer for w(t) and r(t). I removed c and got a green check mark.

☑ [Progress chart working?] 6

Hi, I didn't know where else to post this question, but anyway, I looked at my progress chart and according to the little "total" bar at the far right it says I have a total of 44% so...

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