11. Complicated example with free variables

Goal: After writing the system as an augmented matrix, apply Gaussian elimination to put the augmented matrix into row echelon form, and then use back substitution to solve the system.

$$\begin{aligned} 6z+2u-4v-8w &= 8 \\ 3z+u-2v-4w &= 4 \\ 2x-3y+z+4u-7v+w &= 2 \\ 6x-9y+11u-19v+3w &= 0 \end{aligned}$$

Solution: Start by writing the system as an augmented matrix.

$$\left(\begin{array}{cccccc|c} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 0 \end{array}\right).$$

Next we apply the steps of Gaussian elimination. **Step 1.** The leftmost nonzero column is the first one, and its first nonzero entry is the 2:

$$\left(\begin{array}{cccccc|c} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 0 \end{array}\right).$$

Step 2. The 2 is not in the first row, so interchange its row with the first row:

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 0 \end{array}\right).$$

Step 3. To make all other entries of the column zero, we need to subtract 3 times the first row from the last row (the second and third rows are OK already):

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -6 \end{array}\right).$$

Step 4. Because entries below the pivot are all zero, the first row (in gray) is done. Start over with the submatrix that remains beneath the first row:

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -6 \end{array}\right).$$

Step 1. The leftmost nonzero column is now the third column, and its first nonzero entry is the 3:

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -6 \end{array}\right).$$

Step 2. The first nonzero entry is 3, which is already in the first row of the submatrix (we are ignoring the first row of the whole matrix), so no interchange is necessary.

Step 3. To make all other entries below the pivot of the column zero, subtract 2 times the (new) first row to the (new) second row, and add the (new) first row to the (new) third row:

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -2 \end{array}\right).$$

Step 4. Now the first and second row of the original matrix are done (both now in gray). Start over with the submatrix beneath them:

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -2 \end{array}\right).$$

Step 1. The leftmost nonzero column is now the sixth column, and its first nonzero entry is the -4 at the bottom:

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-4} & \mathbf{-2} \end{array}\right).$$

Step 2. The -4 is not in the first row of the submatrix, so interchange its row with the first row of the submatrix:

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -4 & \mathbf{-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} \end{array}\right).$$

Step 3. The other entry in this column of the submatrix is already 0, so this step is not necessary.

The matrix is now in **row echelon** form. The first nonzero entry in each column is the **pivot.**

$$\left(\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This is the augmented matrix for the system

$$\begin{aligned} 2x-3y+z+4u-7v+w &= 2 \\ 3z+u-2v-4w &= 4 \\ -4w &= -2. \end{aligned}$$

Suppose that a matrix is in row echelon form. Then any column that contains a pivot is called a **pivot column** . A variable whose corresponding column is a pivot column is called a **dependent variable** or **pivot variable** . The other variables are called **free variables** . (The augmented column does not correspond to any variable.)

In the problem above, x, z, w are dependent variables, and y, u, v are free variables.

Back substitution

Now we are ready to back-substitute. Solve for w in

$$-4w = -2,$$

to get

$$w = 1/2.$$

There is no equation for v in terms of the later variable w, so v can be any number; set

$$v \quad = \quad c_1 \quad \text{for a parameter } c_1.$$

There is no equation for u in terms of v and w, so set

$$u \quad = \quad c_2 \quad \text{for a parameter } c_2.$$

Solving the second equation for z we find

$$\begin{aligned} 3z + c_2 - 2c_1 - 4\frac{1}{2} &= 4 \\ z &= 2 - \frac{1}{3}c_2 + \frac{2}{3}c_1, \qquad \text{where } c_1 \text{ and } c_2 \text{ are free parameters.} \end{aligned}$$

There is no equation for y in terms of the later variables z,u,v and w, so set

$$y \quad = \quad c_3 \quad \text{for a parameter } c_3.$$

Solving the first equation for x we find

$$\begin{aligned} 2x - 3c_3 + z + 4c_2 - 7c_1 + w &= 2 \\ 2x - 3c_3 + (2 - \frac{1}{3}c_2 + \frac{2}{3}c_1) + 4c_2 - 7c_1 + \frac{1}{2} &= 2 \\ 2x - 3c_3 + \frac{11}{3}c_2 - \frac{19}{3}c_1 &= -\frac{1}{2} \\ x &= -1/4 + \frac{19}{6}c_1 - \frac{11}{6}c_2 + \frac{3}{2}c_3, \qquad \text{where } c_1, c_2 \text{ and } c_3 \text{ are free parameters.} \end{aligned}$$

Therefore the general solution is given by

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -1/4 + (19/6)c_1 - (11/6)c_2 + (3/2)c_3 \\ c_3 \\ 2 + (2/3)c_1 - (1/3)c_2 \\ c_2 \\ c_1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} + c_1 \begin{pmatrix} 19/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -11/6 \\ 0 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that the three parameters correspond to the three free variables.

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