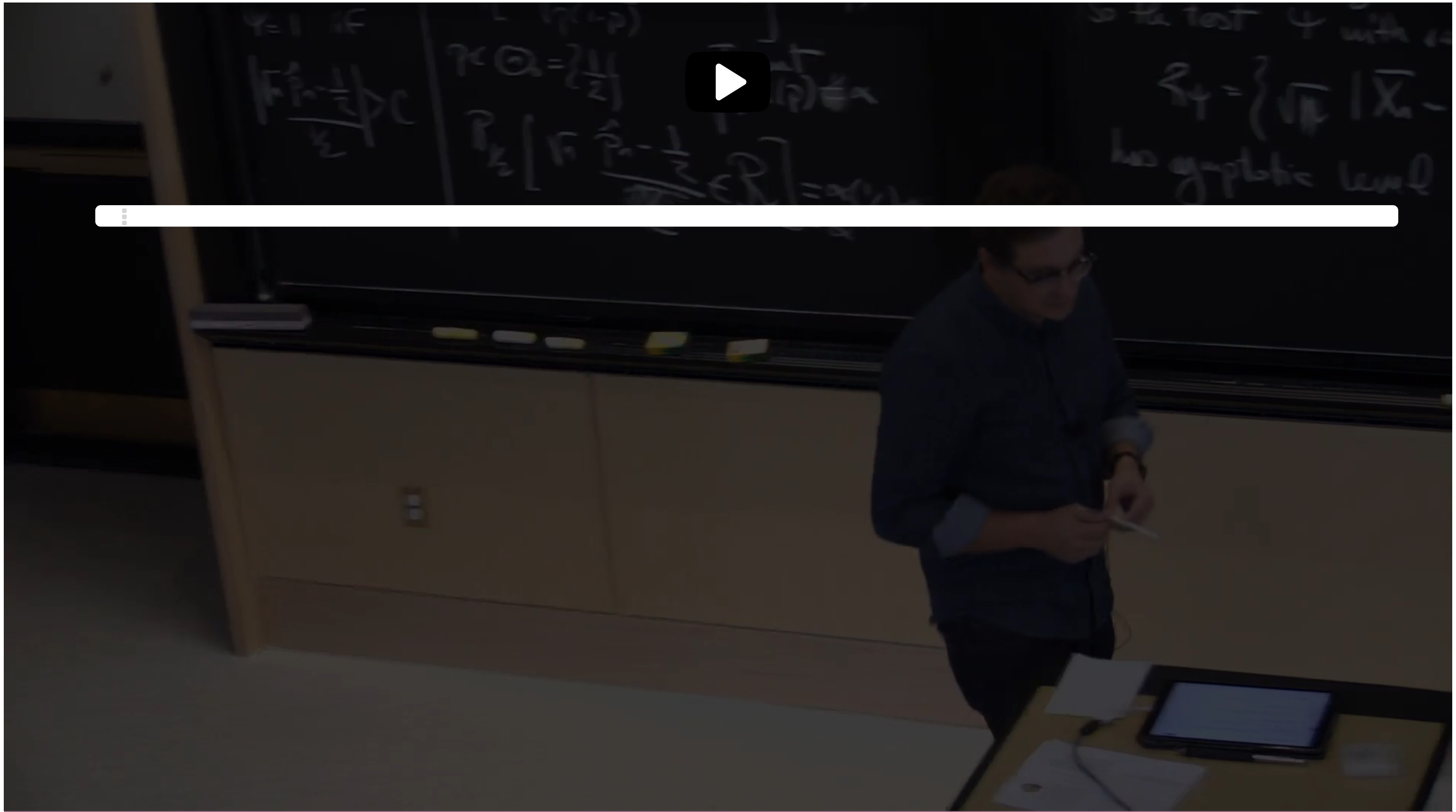
3. Review of Parametric Hypothesis Testing

Worked Example: A Two-Sided Test Associated to a Bernoulli Experiment



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Review: Interpreting the Level

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1/1 point (graded)

Which of the following is a correct interpretation of the (smallest) **level** of a test? (Choose all that apply.)

- [x] The level of a test is an upper bound on the type 1 error.
- [ ] The level of a test is an upper bound on the type 2 error.
- [ ] The level of a test is a random variable that depends on the sample.
- [x] The level of a test gives an upper bound on the worst-case probability of making an error under the null hypothesis.

✔

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Concept Check: Test Statistics

1/1 point (graded)

Setup:

Recall the **statistical experiment** in which you flip a coin n times to decide the coin is fair.

You model the coin flips as $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Ber}(p)$ where p is an unknown parameter, and formulate the hypothesis:

$$\begin{aligned} H_0 &: \ p = 0.5 \\ H_1 &: \ p \neq 0.5, \end{aligned}$$

and design the test ψ using the statistic T_n:

$$\begin{aligned} \psi_n &= \mathbf{1}(T_n > C) \\ \text{where} \quad T_n &= \sqrt{n}\frac{\left|\overline{X}_n - 0.5\right|}{\sqrt{0.5(1-0.5)}} \end{aligned}$$

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where the number C is the threshold. Note the absolute value in T_n for this two sided test.

Question:
If it is true that $p = 1/2$, which of the following are true about T_n?
(Choose all that apply.)

- [ ] T_n is a consistent estimator of the true parameter $p = 1/2$.
- [x] $\lim_{n\to\infty} T_n \xrightarrow[n\to\infty]{(d)} |Z|$ where $Z \sim N(0,1)$ is a standard Gaussian.
- [x] T_n involves a shift and rescaling of the sample average so that as $n \to \infty$, this random variable will converge in distribution.
- [x] The limiting distribution of T_n can be understood using computational software or tables.

✔

Solution:

We examine the choices in order.

- The first choice is incorrect. The statistic T_n does **not** converge to a real number as $n \to \infty$. By the CLT, T_n converges in *distribution*, meaning that asymptotically, it is a random variable.

- The remaining choices are correct. To construct T_n we have shifted the sample mean $\overline{X}_n$ by $1/2$, rescaled by $\sqrt{\frac{n}{0.5(1-0.5)}}$. The CLT guarantees that T_n converges in distribution to a random variable $|Z|$ where $Z \sim N(0,1)$. Since the density of Z is given explicitly, we can work with the limiting distribution using computational software. Alternatively, there are also tables available containing the quantiles of a standard Gaussian.

Remark: This example illustrates one of the main strategies involved in hypothesis testing. Namely, we want to work with a test statistic, that, asymptotically, tends to a distribution that we can easily work with. In many cases, this will involve shifting and rescaling the sample mean so that the CLT applies and we can just work with a standard Gaussian $\mathcal{N}(0,1)$.

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You have used 2 of 2 attempts

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Answers are displayed within the problem

Designing a Test to have a Given Asymptotic Level

4/4 points (graded)

In this problem, we will see the condition for a threshold of a hypothesis test graphically.

Setup as above:

You observe $X_1, \ldots, X_n \stackrel{i.i.d.}{\sim} \text{Ber}(p^*)$ (each X_i models a coin flip) and want to decide if $p^* = 1/2$. Let the null and alternative hypotheses be

- $H_0 : p^* = 0.5$
- $H_1 : p^* \neq 0.5.$

You construct the statistical test:

$$\begin{aligned} \psi_n &= \mathbf{1}(T_n > C) \\ \text{where} \quad T_n &= \sqrt{n}\frac{\left|\overline{X}_n - 0.5\right|}{\sqrt{0.5(1-0.5)}} \end{aligned}$$

where the number C is the threshold to be determined. Note the absolute value in T_n; this is a two-sided test.

Recall that the test ψ has **asymptotic level** α if

$$\lim_{n\to\infty} P_{1/2}(\psi = 1) \leq \alpha.$$

Questions:

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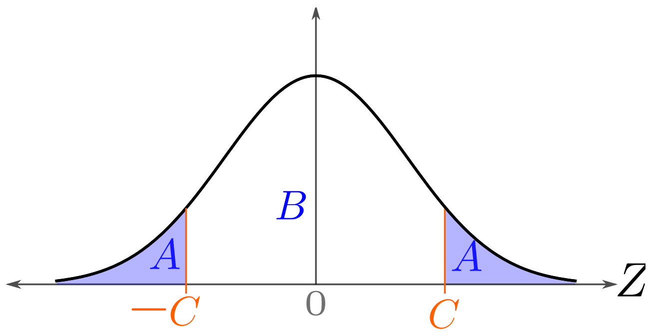

The graph of the standard normal distribution $\mathcal{N}(0,1)$, along with the lines $Z = \pm C$. The letters A, B denote the areas of the corresponding shaded regions:

$$\begin{aligned} A &= \mathbf{P}(Z < -C) = \mathbf{P}(Z > C) \qquad (\text{recall that } \mathbf{P}(Z < -C) = \mathbf{P}(Z > C) \text{ by symmetry}), \\ B &= \mathbf{P}(-C \le Z \le C) \end{aligned}$$

where $\mathbf{P}$ is the probability distribution of $\mathcal{N}(0,1)$.

What is the smallest C such that the test $\psi(T_n > C)$ has asymptotic level α? (The level is often given as a specification for the test.)

Answer not by giving the value of C, but by **giving the condition** that C must satisfy, i.e.refer to the figure above, the smallest C such that $\psi(T_n > C)$ has asymptotic level α must be chosen such that, in terms of A and B in the figure above, α equals...

$\alpha =$ 1-B ✔ **Answer:** 2*A

Hence, as a function of α, what is C_α? (To enter the quantiles of the standard Gaussian, for instance q_α, type **q(alpha)**. Recall q_α denotes the $1-\alpha$-quantile of a standard Gaussian, i.e. the value such that $P(Z \ge q_\alpha) = \alpha$ for $Z \sim \mathcal{N}(0,1)$.)
Denote by C_α the smallest C such that the test $\psi(T_n > C)$ has asymptotic level α.

$C_\alpha =$ q(alpha/2) ✔ **Answer:** q(alpha/2)

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Let the rejection region for the test $\psi(T_n > C_\alpha)$ be

$$R_\alpha = \left\{ (X_1, \ldots, X_n) \in \{0,1\}^n : \overline{X}_n < L \cup \overline{X}_n > R \right\}.$$

What are L and R?

(Your answers will depend on α and n.)
(To enter quantiles, for instance q_α, type **q(alpha)**.)

$L =$ 1/2-1/(2*sqrt(n))*q(alpha ✔ **Answer:** 0.5-q(alpha/2)*sqrt(0.5*(1 - 0.5))/(sqrt(n))

$R =$ 1/2+1/(2*sqrt(n))*q(alph ✔ **Answer:** 0.5+q(alpha/2)*sqrt(0.5*(1 - 0.5))/(sqrt(n))

STANDARD NOTATION

Solution:

- By the central limit theorem, if $\mathbb{E}[X] = p^* = 0.5$, then

$$\sqrt{n}\frac{\overline{X}_n - 0.5}{\sqrt{0.5\,(1-0.5)}} \xrightarrow[n\to\infty]{(d)} N(0,1).$$

Let $\mathbf{P}_{1/2} = \mathrm{Ber}(1/2)$ for notational convenience. Then for the test statistics

$$T_n = \left| \sqrt{n}\frac{\overline{X}_n - 0.5}{\sqrt{0.5\,(1-0.5)}} \right|,$$

we have

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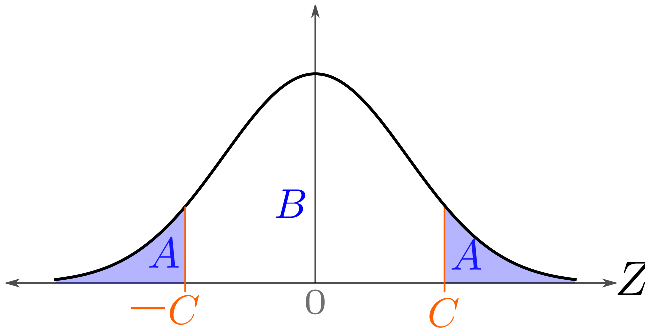
$$\mathbf{P}_{1/2}(T_n > C) \xrightarrow[n\to\infty]{} A + A = 2A$$

where $2A$ are the total area of the shaded regions under the graph of the normal distribution:



The graph of the standard normal distribution $\mathcal{N}(0,1)$, along with the lines $Z = \pm C$. The letters A, B denote the areas of the corresponding shaded regions; hence:

$$\begin{aligned} A &= P(Z < -C) \\ B &= P(Z \leq C) \\ A &= P(Z > C) \end{aligned}$$

where P is the probability distribution of $\mathcal{N}(0,1)$

Since H_0 is defined by a single value $p = 1/2$, the asymptotic level is equal to the asymptotical type 1 error at $p = 1/2$, which is $\mathbf{P}_{1/2}(T_n > C)$. Therefore, given a desired asymptotic level α, choosing a threshold C_α such that

$$\alpha \;=\; P(Z < -C_\alpha) + P(Z > C_\alpha) \;=\; A + A = 2A \qquad Z \sim \mathcal{N}(0,1)$$

will result in a test $\psi = \mathbf{1}(T_n > C_\alpha)$ that has asymptotic level α. Furthermore, for any threshold $C < C_\alpha$ will yield a larger asymptotic type 1 error, as shown in the figure below

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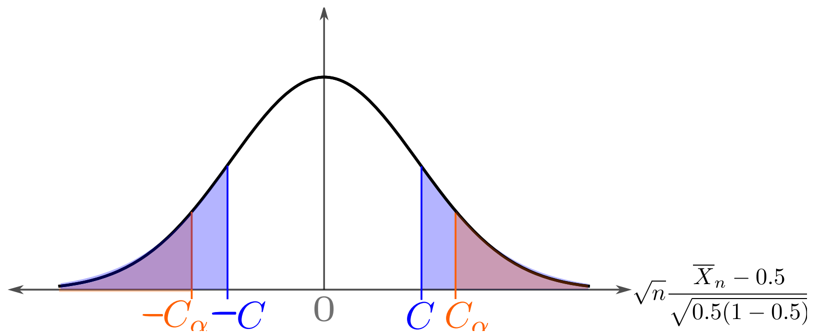

The graph of the standard normal distribution $\mathcal{N}(0,1)$;
For $C < C_\alpha$, the **type 1 error for** $\psi = \mathbf{1}(T_n > C)$ (shaded **blue**) is larger than the **type 1 error for** $\psi = \mathbf{1}(T_n > C_\alpha)$ (shaded **orange**).

This means that C_α is the smallest choice of threshold C such that the test $\psi(T_n > C)$ has asymptotic level α.

- Since $\alpha = P(Z < -C_\alpha) + P(Z > C_\alpha) = 2P(Z > C_\alpha)$ by symmetry, we have $C_\alpha = q_{\alpha/2}$.
- The rejection region of $\psi = \mathbf{1}(T_n > q_{\alpha/2})$ is defined by

$$T_n = \left|\sqrt{n}\frac{\overline{X}_n - 0.5}{\sqrt{0.5(1-0.5)}}\right| > q_{\alpha/2}$$
$$\Longrightarrow \overline{X}_n < 0.5 - q_{\alpha/2}\frac{\sqrt{0.5(1-0.5)}}{\sqrt{n}} \cup \overline{X}_n > 0.5 + q_{\alpha/2}\frac{\sqrt{0.5(1-0.5)}}{\sqrt{n}}.$$

Remark: We have done similar manipulations when looking for two-sided confidence interval of level $1 = \alpha$. But here, we look for a range of $\overline{X}_n$ in terms of the assumed value of the parameter p under the null hypothesis.

Remark: Since the limiting distribution of our test statistic is well-known (the absolute value of a standard Gaussian), it is straightforward to specify the asymptotic level of our test using computational tools or tables. Later in this course, we will also encounter tests where for fixed n we can compute the (non-asymptotic) level of ψ_n using computational tools or tables.

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Answers are displayed within the problem

Concept Check: Rejection Region

2/2 points (graded)

You observe $X_1, \ldots, X_n \stackrel{i.i.d.}{\sim} \mathbf{P}_{\theta^*}$ and design a test ψ to test between a null hypothesis and an alternative hypothesis.

True or False: The rejection region of ψ depends on the value of the true unknown parameter θ^*.

- True
- False (selected)

✔

True or False: To define a statistical test ψ, it is enough to define the rejection region R_ψ.

- True (selected)
- False

✔

Solution:

- The rejection region does not depend on the true parameter. It is fixed when a test is designed, as in the example in the problem above.
- As pointed out above, a test is by definition an indicator function of its rejection region:

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$$\psi = \mathbf{1}\left((X_1, \ldots, X_n) \in R_\psi\right)$$

Hence, yes, to define a test, all that is needed is to define its rejection region.

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Answers are displayed within the problem

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Hi, I was working on third excercise but after submit I find the message: "Could not format HTML for problem. Contact course staff in the discussion forum for assistance." Ple... 3

in terms of A and B in the figure above, α equals..
I found the expression quite confusing. A can be referred to both A regions combined, it can also mean just one of them... 2

? Review: Interpreting the Level P(Type 2 error)
Isn't there the possibility of making a Type 2 error? And could not this probability be higher than alpha? I.e. wouldn't there be another, possibly greater, upper bound for the o... 3

? Question from Bernoulli Experiment Slide 4

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