The Rational Numbers

Recall that the set of **rational numbers**, $\mathbb{Q}$, is the set of numbers equal to some fraction a/b, where a and b are integers and b is distinct from zero. (For instance, $17/3$ is a rational number, and so is $-4 = 4/-1$.) The **non-negative** rational numbers, $\mathbb{Q}^{\geq 0}$, are simply the rational numbers greater than or equal to 0.

In this section we'll prove an astonishing result: there is a bijection between the natural numbers and the non-negative rational numbers.

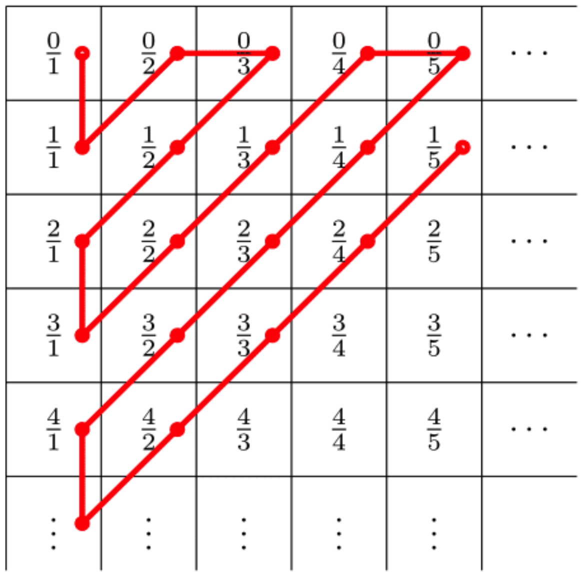
In other words: *there are just as many natural numbers as there are non-negative rational numbers*. (In one of the exercises below, I'll ask you to show that there is also a bijection between the natural numbers and the full set of rational numbers, $\mathbb{Q}$.)

The proof is arrestingly simple. Consider the matrix below, and note that every (non-negative) rational number appears somewhere on the matrix. (In fact, they all appear multiple times, under different labels. For instance, $\frac{1}{2}$ occurs under the labels $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, etc.)

$\frac{0}{1}$	$\frac{0}{2}$	$\frac{0}{3}$	$\frac{0}{4}$	$\frac{0}{5}$	$\cdots$
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\cdots$
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\cdots$
$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The non-negative rational numbers, arranged on an infinite matrix.

Now notice that the red path traverses each cell of the matrix exactly once. We can use this observation to define a bijection from natural numbers to matrix cells: to each natural number n assign the cell at the nth step of the path.



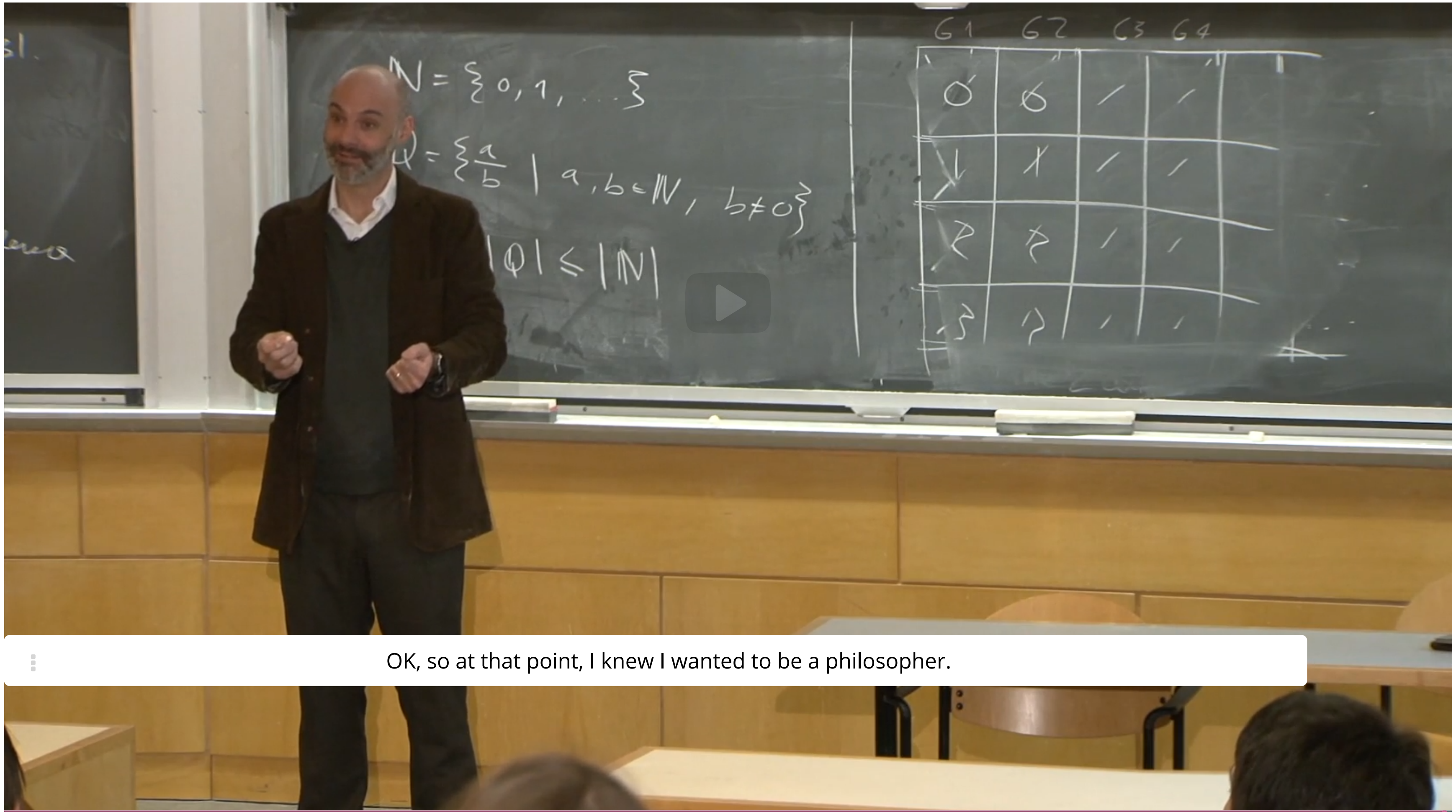
A path that goes through each cell of the matrix exactly once.

If each rational number appeared only once on the matrix, this would immediately yield a bijection from natural numbers to rational numbers.

But we have seen that each rational number appears multiple times, under different labels. Fortunately, it is easy to get around the problem. We can follow the same route as before, except that this time we skip any cells corresponding to rational numbers that had already been counted. The resulting assignment is a bijection from the natural numbers to the (non-negative) rational numbers.

(I was so excited when I first learned about this result that I decided that I wanted to spend the rest of my life thinking about this sort of thing!)

Video Review: There are just as many Rationals as Naturals



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Problem 1

1/1 point (ungraded)

We've seen that there is a bijection between the natural numbers and the non-negative rational numbers.

Is there also a bijection between the natural numbers and the full set of (negative and non-negative) rational numbers?

Yes ✔ **Answer:** Yes

(If so, can you given an example? If not, why not?)

Explanation
We know that there is a bijection f from the natural numbers to the non-negative rational numbers.
A similar construction can be used to show that there is a bijection g from the natural numbers to the negative rational numbers.
We can then show that there is bijection h from the natural numbers to the rational numbers by combining these two results. For we can let:

$$h(n) = \begin{cases} f\left(\frac{n}{2}\right), & \text{if } n \text{ is even} \\ g\left(\frac{(n-1)}{2}\right), & \text{if } n \text{ is odd} \end{cases}$$

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Answers are displayed within the problem

Problem 2

1/1 point (ungraded)
Is there a bijection between the natural numbers and as many copies of the natural numbers as there are natural numbers?

Yes ✔ **Answer:** Yes

(If so, can you given an example? If not, why not?)

Explanation
One way to do so is to follow the route in the figure above, but interpret the matrix differently.
Instead of thinking of cells as corresponding to rational numbers, we think of each column as one copy of the natural numbers. Accordingly, the cell labelled $\frac{a}{b}$ is the ath member of the bth copy of the natural numbers.

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Answers are displayed within the problem

Problem 3

1/1 point (ungraded)

Is there a bijection between the natural numbers and some subset of the rational numbers that includes only rational numbers that are larger than 0 and smaller than or equal to 1?

Yes

Answer: Yes

(If so, can you given an example? If not, why not?)

Explanation

Yes. Here's an example: $f(n) = \frac{1}{n+1}$ is a bijection from $\mathbb{N}$ to a subset of the set of rational numbers larger than 0 and smaller than or equal to 1.

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Answers are displayed within the problem

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Infinity times infinity equals infinity?

Between 0 and 1 there are an infinite number of rational numbers and thus so for 1 and 2 etc. However since there are also an infinite number of natural numbers then the i...

3

What if they want to share?

3

How could we write up that two new guests want to share a room? Is that even possible? Sorry for the dumb question...

Rigor of the Path proof
Why is the proof given by Professor Rayo rigorous? 2

Why Diagonal
Might be a dumb question...but why does the route have to be diagonal? Is there any alternative path that can also achieve the goal of assigning each natural number to each... 8

He assigned 4 two times?
He said that every rational number got assigned a different natural number right? But why does he assign 4 to the rational numbers 3/1 and 1/3? I'm confused... 3

The example given in question 1
The answer looks (to me) like a two bijections involving natural numbers only. If so, I don't see how this is an example of a bijection involving rational numbers, positive or neg... 6

Cartesian Products
Is there a bijection between the natural numbers and the set NXN that is N^2? 3

I am kind of confused about what one of the questions is asking
Can someone please clarify what question two means? 5

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