subgroups of the group of pentagon symmetries

Question

The pentagon has 5 line symmetries and therefore we will have 10 symmetries. So, we let the group G with order 10 denote the symmetry group of a pentagon.

A subset H of G is a subgroup $(H, *)$ to the group $(G, *)$ if and only if $(H, *)$ is a group. To determine how many subgroups there are, I can use the Lagrange's theorem which tells us that the subgroups of a group with order n have a order m such that $m|n$.

By this theorem we get know that the group G has 5 subgroups since the divisors m of 10 is $m = 1, 2, 5, 10$.

The question is, how shall I sketch the lattice of subgroups? Shall I check every possible subset and then check if they are subgroups or is there a faster way?

I know that the 10 symmetries are the identity element, 4 rotation and 5 reflections. We can see these operation of the transformation as permutation.

The first transformation that rotates 72 degrees are the permutation (ABCDE), 144 degrees are the permutation (ABCDE)^2 etc. This permutation have the order 5 since if we rotate 5 times we will get the identity ("the original pentagon") .

The reflection transformations are permutations with 2 cycles with length 2.

But how do I know that the subgroups with order 5 are the ones with identity and 4 rotations?

EDIT

The order 2 is easy to determine. The subsets $\{i, g\}$ there $g \in G - \{reflections\}$ are not groups because the inverse of g is not in the subset. BUT if g is one of the reflections then we have a group, since the reflections have the order 2 which means that the element g is the inverse of it self. This means that the subset is a subgroup because (i) it is closed and (ii) the inverse is in the subset.

EDIT 2

Can I think like this? We let a subset be

$$H = \{\text{id}, r, r^2, r^3, z\}$$

there r are the rotation and z is the reflection. By checking the properties for a group, we can see that the properties about that every element in H has an inverse does not hold. the element r has the inverse r^4 which is not in H and therefore the subset H cannot be a subgroup. So, by removing one of the rotations and putting one of the reflections, we see that we are removing some inverses. So, the only subgroup of order 5 are

$$K = \{\text{id}, r, r^2, r^3, r^4\}$$

Am I thinking correctly? Or is it better to write down the group table to find the subsets which are subgroups?

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Answer

Your thinking is more or less correct. Generally, when we wish to think about the subgroups of a group G, we take different subsets of G and look at what groups those subsets generate.

The obviously thing to do is start with one element. Let's say I start with a rotation r^k for some $k = 1, 2, 3, 4$. Since 5 is a prime, the numbers $1, 2, 3, 4$ have multiplicative inverses mod 5. This implies that $\langle r^k \rangle$ actually contains r. Hence, $\langle r^k \rangle = \langle r \rangle = \{1, r, r^2, r^3, r^4\}$ for each k. Now, let's say I start with a reflection z instead. It's easy to see that $\langle z \rangle = \{1, z\}$. Thus, we have 5 different subgroups of order 2, one for each reflection. The only remaining case is $\langle 1 \rangle = \{1\}$.

Now let's consider a generating set with two elements. If we take $\langle r^k, r^l \rangle$ for some k, l, this is clearly just $\langle r \rangle$ again. Suppose instead we take $\langle r^k, z \rangle$. Then this subgroup contains r and z, hence it contains $r^l z$ for each $l = 0, 1, 2, 3, 4$, i.e. it contains every reflection. It also contains the powers of r, which are the rotations, hence $\langle r^k, z \rangle = G$. Finally, we could take $\langle z_1, z_2 \rangle$ for two reflections z_1, z_2. But then $z_1 z_2$ is a rotation which is in $\langle z_1, z_2 \rangle$, hence by the previous case we have $\langle z_1, z_2 \rangle = G$.

From here it's easy to see that we've found every subgroup of G. Obviously this took a lot less time than checking all 2^{10} subsets of G.

If you find this answer useful please share it with other students.

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