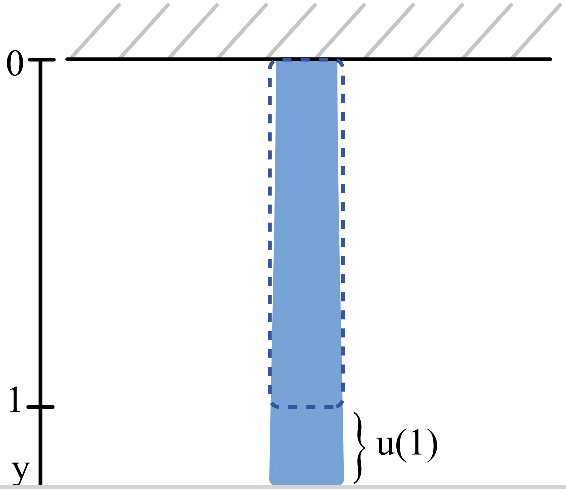
7. Linear elasticity

Boundary conditions are most common when the independent variable is space rather than time. In time, we can typically only assign initial conditions. In space, it makes sense to specify conditions on the end points of a spatial object.

Let's consider some examples that arise from Linear Elasticity.

Linear Elasticity for Vertical Beams

Suppose we have a bar hanging vertically with the top end attached to a solid surface. The y coordinate is the spatial variable running along the length of the bar, so that each point on the non-elongated bar exists on the interval $y \in [0, 1]$.



Under its own weight, or an external force $f\left(y\right)$ pulling downwards, the bar will tend to lengthen. Thus each point on the bar will move to a new location, displaced by some amount u. So that the new position of a point y is $y+u$.

The displacement u is a function of **position,** $u=u\left(y\right)$.

Note the following analog of the situation described here, a simple experiment that you can conduct yourself. Take a slinky, and hang it on one end from the ceiling (or the under-side of a shelf).

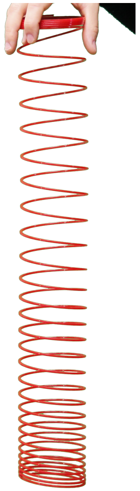


Figure 18: Slinky hanging from a fixed position at top.

Then the rings separate, with the separation between rings decreasing as you go down from the top. The u in our equation is then the distance from where a particular ring is at, to the position it would have if the slinky rings were placed right next to each other as if it were floating in zero gravity.

Figure 19: Slinky with rings placed next to each other in zero gravity.

Then the differential equation describing the displacement in terms of the external stress per coil (of slinky) $f\left(y\right)$ is

$$\frac{d^2u}{dy^2} = \frac{1}{E}f\left(y\right), \tag{3.6}$$

where the constant E is determined by the material elasticity.

When the external force is gravity, this equation becomes

$$\frac{d^2u}{dy^2} = \frac{-\rho g}{E}, \tag{3.7}$$

where the constant ρ is the density of the beam, and g is the constant of gravitational acceleration.

Physics of Linear Elasticity

Click for derivation of differential equation

Note that the following is an abridged explanation of linear elasticity. Please note that the definition of $N\left(y\right)$ in what follows is slightly different than the function $f\left(y\right)$ above.

- above, $f\left(y\right)$ to denotes the external force per volume, in our case $f\left(y\right) = \rho g$.

- below, $N(y)$ is the internal resultant of these external forces, which is defined as $N(y) = \int_y^1 f(w)\, dw$.

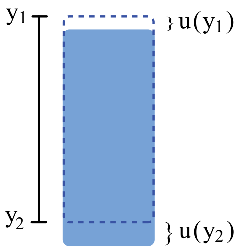
We have suppressed the derivation of this as this is not the focus of this class. To gain a deeper understanding, we recommend taking MITx course 2.01x Elements of Structures.

There is a force per unit area on the bar, which arises from the deformation of the material and its elasticity. There is an internal resultant of this force, $N(y)$, which is felt by the cross-sectional area A through any position y on the vertical axis of the bar, is defined as **stress.**

Linear elasticity tells us that the stress is proportional to the strain. Let E be this constant of proportionality. The **strain** is defined as the ratio of the relative change in length in any small ("infinitesimal") cross-section of the bar relative to its rest state.

In terms of the slinky, the strain is the ratio of the increment in separation between the coils under the stress, and their separation at rest.

To find the strain, choose two points in the non-elongated bar: y_1 and y_2. In the elongated bar, the displacement of y_1 is $u(y_1)$, and the displacement of y_2 is $u(y_2)$. So the total change in length between these two points is $u(y_2) - u(y_1)$. The initial length of the bar between these two points is $y_2 - y_1$.



The ratio is

$$\frac{u(y_2) - u(y_1)}{y_2 - y_1} = \frac{\Delta u}{\Delta y} \qquad (3.8)$$

Taking the limit as y_2 approaches y_1, we get the strain $\left.\frac{du}{dy}\right|_{y=y_1}$.

Let E be this constant of proportionality relating the stress $N(y)/A$ to the strain $\frac{du}{dy}$

$$EA\frac{du}{dy} = N(y). \tag{3.9}$$

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Boundary Conditions

Case 1: Fix one end, other end hanging free

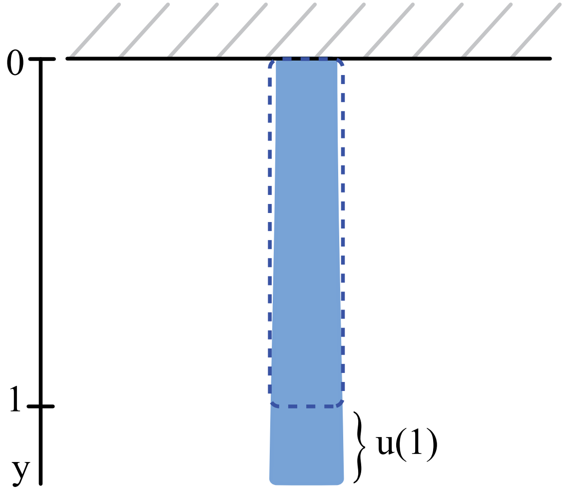

Figure 20: **Boundary conditions:** $u(0) = 0$, $\frac{du}{dy}(1) = 0$

Note that u represents the displacement of a point, and so points on the bar that are not displaced in the stretching will not move. Since the top of the bar is firmly set in place, this gives us our first boundary condition,

$$u(0) = 0. \tag{3.10}$$

At any cross-section along the beam, the force per unit area [the stress] has to be just enough to support the weight of the bar below this point. Thus:

$$EA\frac{du}{dy} = gW\left(y\right) \tag{3.11}$$

where $W\left(y\right)$ is the mass of the bar below y. Suppose the density ρ of the beam and the cross sectional area A are both constant along the beam. Then

$$W\left(y\right) = \int_{y}^{1} \rho A \, dx = \rho A\left(1 - y\right);$$

where 1 is the length of the beam.

Taking derivatives this becomes

$$AE\frac{d^2u}{dy^2} = \frac{d}{dy}gW\left(y\right) = -g\rho A.$$

Note that at the end of our bar, there is no force per unit area, since there is no weight there. So we require the second boundary condition:

$$u'\left(1\right) = 0. \tag{3.12}$$

Question 7.1 What is the formula for the displacement $u\left(y\right)$ if the force is gravity acting along a uniform beam?

See worked solution

The differential equation can be written as

$$\frac{d^2u}{dy^2} = \frac{-g\rho}{E} \tag{3.13}$$

We can just integrate this twice, using the constants of integration c_1 and c_2 the first and second time, respectively, to find

$$u(y) = \frac{1}{2}\frac{-g\rho}{E}y^2 + c_1 y + c_2. \tag{3.14}$$

When we plug these boundary conditions into our differential equation, we get the system

$$u(0) = 0 = c_2, \tag{3.15}$$

$$\frac{du}{dy}(1) = 0 = \frac{-g\rho}{E} + c_1. \tag{3.16}$$

We can see immediately that $c_1 = \frac{g\rho}{E}$ and $c_2 = 0$, which gives us the solution

$$u(y) = -\frac{1}{2}\frac{g\rho}{E}y^2 + \frac{g\rho}{E}y. \tag{3.17}$$

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Find the boundary conditions

1/1 point (graded)

Case 2: Fix both ends

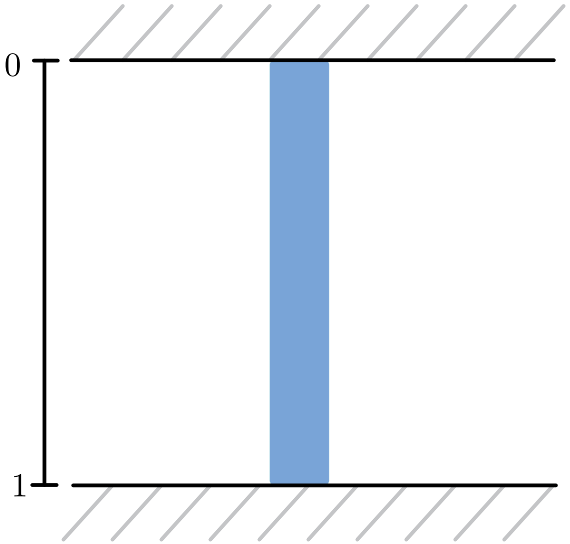

Figure 21: A vertical beam fixed at the top and bottom. The only force acting on it is gravity, and it is at steady state.

What are the boundary conditions?

● $u\left(0\right)=0$ and $u\left(1\right)=0$

○ $u\left(0\right)=0$ and $\dfrac{du}{dy}(1)=0$

○ $\dfrac{du}{dy}(0)=0$ and $u\left(1\right)=0$

○ $\dfrac{du}{dy}(0)=0$ and $\dfrac{du}{dy}(1)=0$

✓

Solution:

Since both the top and bottom are fixed, the boundary conditions are that the top and bottom cross-sections are not displaced at all: $u\left(0\right)=0$ and $u\left(1\right)=0$.

Submit

You have used 1 of 3 attempts

Answers are displayed within the problem

Solve the boundary value problem

1/1 point (graded)

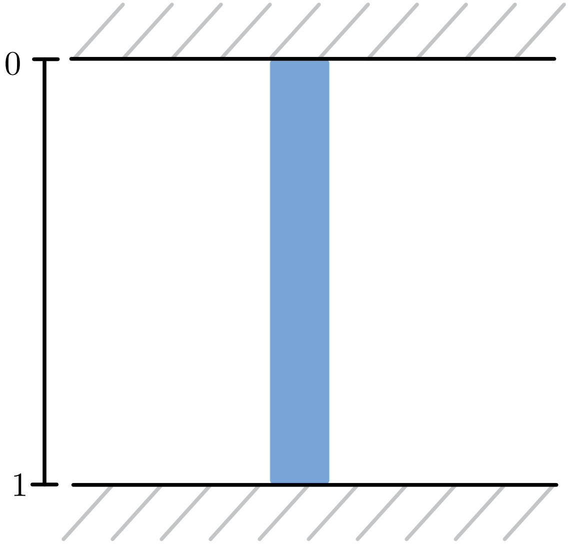

Figure 22: A vertical beam fixed at the top and bottom. The only force acting on it is gravity, and it is at steady state.

In the case of a vertical beam fixed at both ends, what is the formula for the displacement $u\left(y\right)$?

(Enter in terms of ρ (typed as **rho**), g, E, and y.)

$u\left(y\right) =$ -g*rho/2/E*y^2+g*rho/2/E*y ✓ **Answer:** -rho*g*y^2/(2*E)+rho*g*y/(2*E)

$-\frac{g\cdot\rho}{2\cdot E}\cdot y^2+\frac{g\cdot\rho}{2\cdot E}\cdot y$

FORMULA INPUT HELP

Solution:

The differential equation becomes

$$\frac{d^2u}{dy^2} = \frac{-\rho g}{E} \qquad (3.18)$$

As before, we integrate to find the equation with two unknown constants c_1 and c_2:

$$u\left(y\right) = \frac{1}{2}\frac{-\rho g}{E}y^2+c_1y+c_2. \qquad (3.19)$$

When we plug these boundary conditions into our differential equation, we get the system

$$\begin{aligned} u\left(0\right)=0 &= c_2 && (3.20)\\ u\left(1\right)=0 &= \frac{1}{2}\frac{-\rho g}{E}+c_1+c_2 && (3.21)\\ &= \frac{-\rho g}{2E}+c_1 && (3.22) \end{aligned}$$

We can see immediately that $c_1 = \frac{\rho g}{2E}$ and $c_2 = 0$, which gives us the solution

$$u(y) \;=\; \frac{-\rho g}{2E}y^2+\frac{\rho g}{2E}y \qquad (3.23)$$

Note that this solution is similar to the solution for the displacement when one end of the beam is free, but differs by a factor of 2 in the linear term.

Submit You have used 1 of 7 attempts

Answers are displayed within the problem

7. Linear elasticity

Hide Discussion

Topic: Unit 2: Boundary value problems and PDEs / 7. Linear elasticity

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Something in the y that it moves 1

Assumption
Did we assume that cross section of the beam A is also 1? 4

☑ Boundary Condition
For the condition u'(1)=0, does the 1 represents the other end of the rod regardless of where it is or the new part of the rod that would be at 1? For example, if the rod is 1 me... 2