# 12. Basis and dimension

We would like a simple way to describe vector subspaces such as the nullspace. Such a description comes in the form of a minimal set of vectors that span the space. This minimal set is called a basis.

## Basis

**Definition 12.1** A **basis** of a vector space $V$ is a list of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ such that

1. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ are linearly **independent.**
2. They span the space: $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots) = V$.

**Example 12.2** If $S$ is the $xy$-plane in $\mathbb{R}^3$, then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis for $S$.

## Basis concept check I

1/1 point (graded)

Does the pair of vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$?

- (selected) Yes; the pair forms a basis for $\mathbb{R}^2$. ✔
- No; the pair does not form a basis for $\mathbb{R}^2$.

**Solution:**

Yes, they form a basis, because of the following two statements:

1. They are linearly independent, since neither vector is a scalar multiple of the other.

2. We saw earlier that

$$\mathbf{Span}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\right)=\mathbf{Span}\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right),$$

which equals $\mathbb{R}^2$ since each vector $\begin{pmatrix}a\\b\end{pmatrix}$ can be expressed as a linear combination $a\begin{pmatrix}1\\0\end{pmatrix}+b\begin{pmatrix}0\\1\end{pmatrix}$.

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You have used 1 of 2 attempts

Answers are displayed within the problem

## Basis concept check II

1/1 point (graded)

Is the basis of the nonzero vector space $\mathbb{R}^2$ unique?

- ○ Yes.
- ◉ No. ✔

**Solution:**

The answer is no. For example, both pairs of vectors $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}1\\-1\end{pmatrix}$ form a basis for $\mathbb{R}^2$. In fact, any nonzero vector space has infinitely many bases. ("Bases" is the plural of basis.)

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## Example: finding a basis

If I put in 3, 3, 7, it would be the sum of those two.

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## Find the linearly independent vectors

1/1 point (graded)

The vector $\mathbf{v}$ is such that the list $\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\begin{pmatrix}2\\2\\5\end{pmatrix}$, $\mathbf{v}$ is a basis of $\mathbb{R}^3$. Which of the following are possibilities for $\mathbf{v}$?

- [ ] $\begin{pmatrix}3\\3\\7\end{pmatrix}$
- [ ] $\begin{pmatrix}0\\0\\1\end{pmatrix}$
- [ ] $\begin{pmatrix}3\\3\\8\end{pmatrix}$
- [x] $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ✔
- [x] $\begin{pmatrix}2\\3\\7\end{pmatrix}$ ✔

✔

**Solution:**

The vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$ lie in the plane $x - y = 0$, so their span is contained in this plane too. Neither vector is a scalar multiple of the other, so they are independent, so their span is $2$-dimensional and hence must equal the whole plane $x - y = 0$.

If $\mathbf{v}$ lies in this plane, then the three vectors are linearly dependent, so they cannot form a basis. If $\mathbf{v}$ lies outside the plane, then the span of the three vectors is strictly larger than the plane and hence equals $\mathbb{R}^3$; moreover, in this case the span of the three vectors is $3$-dimensional, so the vectors must be linearly independent; thus they form a basis.

Hence the vectors we want are those **not** lying on the plane $x - y = 0$, i.e., the vectors whose first two coordinates are unequal. These are the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$.

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Answers are displayed within the problem

## Finishing the example and defining dimension

Not the same vectors--

there are all sorts of bases--

but the number of vectors is always the same.

And that number is the dimension.

This is a definition now.

This number is the dimension of the space.

OK.

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## Dimension

**Definition 12.3** *Three equivalent definitions.* The **dimension** of a vector space is

1. the largest number of linearly independent vectors one can find in that vector space.
2. the smallest number of vectors needed to span that vector space.
3. the number of vectors in any basis of that vector space.

Every basis for a vector space has the same number of vectors.

**Example 12.4** The line $x + 3y = 0$ in $\mathbb{R}^2$ is a vector space $L$. The vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ by itself is a basis for $L$, so the dimension of $L$ is $1$. (Not a big surprise!)

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## Dimension concept check I

1/1 point (graded)

What is the dimension of $\mathbb{R}^n$?

$\text{Dimension of } \mathbb{R}^n =$

n

**Answer:** n

$n$

**Solution:**

The dimension of $\mathbb{R}^n$ is $n$.

Let's check that the vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

form a basis for $\mathbb{R}^n$.

First, they are linearly independent because any linear combination

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 + \cdots + c_n\mathbf{e}_n = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix}$$

is zero only when $c_1 = c_2 = \cdots = c_n = 0$.

They span $\mathbb{R}^n$ because any vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ in $\mathbb{R}^n$ can be written as a linear combination of these vectors:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 + \cdots + c_n\mathbf{e}_n.$$

Finally, the number of vectors in this basis is $n$, so the dimension of $\mathbb{R}^n$ is $n$.

(The list $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is called the **standard basis** for $\mathbb{R}^n$.)

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Answers are displayed within the problem

## Dimension concept check II

1/1 point (graded)

What is the dimension of the subspace in $\mathbb{R}^3$ given by the equation $x + y + z = 0$?

Dimension of this subspace =

2

✔ **Answer:** 2

$\mathbf{2}$

**Solution:**

The subspace in $\mathbb{R}^3$ given by the equation $x+y+z=0$ is the set of solutions to the homogeneous linear equation

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = 0$$

Thus the answer is the dimension of the nullspace of $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. The dimension is equal to the number of elements in a basis for the nullspace.

To find a basis for the nullspace, we note that the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ is already in reduced row echelon form. The variables $y$ and $z$ are free variables, so we set them equal to parameters:

$$\begin{aligned} y &= c_1, \\ z &= c_2. \end{aligned}$$

Then

$$x = -c_1 - c_2.$$

A general solution is given by

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

and a basis for the nullspace is given by the vectors

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore the dimension is 2.

Alternatively, the dimension is the number of elements in the basis of the subspace, which is the number of free variables, which is 2.

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Answers are displayed within the problem

## Dimension concept check III

1/1 point (graded)

Does the list of the vectors $\begin{pmatrix} 11 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 47 \\ 23 \end{pmatrix}$, $\begin{pmatrix} -13 \\ 8 \end{pmatrix}$ form a basis for $\mathbb{R}^2$?

- Yes.
- No. ✔
- It cannot be determined.

**Solution:**

The dimension of $\mathbb{R}^2$ is 2, so any basis should have $\mathbf{2}$ vectors. But here we have 3 vectors, so they can't be a basis.

Another way to answer the question is to appeal to the definition of a basis, to check if the vectors span and are linearly dependent. To find the linear dependence, find the nullspace of the matrix $\begin{pmatrix} 11 & 47 & -13 \\ -5 & 23 & 8 \end{pmatrix}$. Any nonzero vector in the nullspace gives the coefficients in a linear dependence between the 3 vectors. (This dependence is not easy to guess just from looking at the vectors.)

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Answers are displayed within the problem

## Dimension concept check IV

1/1 point (graded)

What is the dimension of $\mathbf{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix}\right)$?

1 **Answer:** 1

**Solution:**

The second vector is $-2$ times the first vector, so the two vectors are linearly dependent and their span equals the span of the first vector alone. That single vector is a basis for the span. The basis has one element, so the dimension is $1$. (This vector space is a line in $\mathbb{R}^4$.)

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Answers are displayed within the problem

## 12. Basis and dimension

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? Span in a flash 3

? vector spaces, subspaces, span, basis vectors, dimension 5
are these concepts going to be central to solving systems of DEs? I have a feeling they will not.

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