5. Introduction to Wald's Test

Introduction to Wald's Test

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Review: Manipulating Multivariate Gaussians

1/1 point (graded)

Recall that a **multivariate Gaussian** $\mathcal{N}(\vec{\mu}, \Sigma)$ is a random vector $\mathbf{Z} = \left[Z^{(1)}, \ldots, Z^{(n)}\right]^T$ where $Z^{(1)}, \ldots, Z^{(n)}$ are **jointly Gaussian** , meaning that the density of $\mathbf{Z}$ is given by the joint pdf

$$\begin{aligned} f : \mathbb{R}^n &\rightarrow \mathbb{R} \\ \mathbf{Z} &\mapsto \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{Z}-\vec{\mu})^T \Sigma^{-1}(\mathbf{Z}-\vec{\mu})\right) \end{aligned}$$

where

$$\begin{aligned} \vec{\mu}_i &= \mathbb{E}\left[Z^{(i)}\right], \qquad \text{(vector mean)}. \\ \Sigma_{ij} &= \mathsf{Cov}\left(Z^{(i)}, Z^{(j)}\right) \qquad \text{(positive definite covariance matrix)}. \end{aligned}$$

Suppose that $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \Sigma)$. Let $\mathbf{M}$ denote an $n \times n$ matrix.

What is the distribution of $\mathbf{MZ}$?

- ○ $\mathcal{N}(\mathbf{0}, \Sigma)$
- ○ $\mathcal{N}(\mathbf{0}, \mathbf{M}\Sigma)$
- ○ $\mathcal{N}(\mathbf{0}, \Sigma\mathbf{M})$
- ◉ $\mathcal{N}(\mathbf{0}, \mathbf{M}\Sigma\mathbf{M}^T)$

✔

Solution:

Linear transformations, e.g. $\mathbf{MZ}$, of Gaussian vectors are still Gaussian vectors. Hence, we only need to figure out the mean and covariance matrix of $\mathbf{MZ}$. By linearity of expectation:

$$\mathbb{E}\left[(\mathbf{MZ})_i\right] = \mathbb{E}\left[\sum_{j=1}^{n} \mathbf{M}_{ij} Z^{(j)}\right] = 0$$

for all i, so $\mathbb{E}\left[\mathbf{MZ}\right] = 0$. Or equivalently, in vector notation, (which is still correct, by linearity of expectation):

$$\mathbb{E}\left[\mathbf{MZ}\right] = \mathbf{M}\mathbb{E}\left[\mathbf{Z}\right] = \mathbf{M0} = \mathbf{0}.$$

Next we compute the covariance. We will use the vector notation. Observe that

$$\Sigma = \mathbb{E}\left[\mathbf{ZZ}^T\right].$$

The covariance matrix of $\mathbf{MZ}$ is given by

$$\mathbb{E}\left[(\mathbf{MZ})(\mathbf{MZ})^T\right] = \mathbb{E}\left[\mathbf{MZZ}^T\mathbf{M}^T\right] = \mathbf{M}\cdot\mathbb{E}\left[\mathbf{ZZ}^T\right]\mathbf{M}^T = \mathbf{M}\Sigma\mathbf{M}^T,$$

where we applied linearity of expectation for the third equality and the definition of Σ in the final equality.

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Answers are displayed within the problem

Aside: Rotation of Standard Gaussian

4/4 points (graded)

Suppose

$$\mathbf{M} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Is it true that $\mathbf{M}^T\mathbf{M} = \mathbf{1}_{2\times 2}$?

(Here $\mathbf{1}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix in 2 dimensions.)

- (selected) True
- False

✔

Now, let $\mathbf{Z} \sim \mathcal{N}_2\left(\mathbf{0}, \mathbf{1}_{2\times 2}\right)$, i.e. $\mathbf{Z}$ is a standard Gaussian in 2 dimensions. Is it true that $\mathbf{MZ} \sim \mathcal{N}_2\left(\vec{\mu}, \Sigma_{\mathbf{MZ}}\right)$, for some $\vec{\mu}, \Sigma_{\mathbf{MZ}}$?

- (selected) True
- False

✔

Find the mean $\vec{\mu} = \mathbb{E}\left[\mathbf{MZ}\right]$ and covariance matrix $\Sigma_{\mathbf{MZ}}$ of $\mathbf{MZ}$.

(Enter your answer as a vector or matrix. For example, type **[1,3]** for the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$; type **[[1,2],[5,1]]** for the matrix $\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$. Note the square brackets, and the commas as separators.)

$\vec{\mu} = \mathbb{E}\left[\mathbf{MZ}\right] =$ [0,0] ✔ **Answer:** [0,0]

$\Sigma_{\mathbf{MZ}} =$ [[1,0],[0,1]] ✔ **Answer:** [[1,0],[0,1]]

STANDARD NOTATION

Solution:

- $$\begin{aligned}\mathbf{M}\mathbf{M}^T &= \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\\ &= \begin{pmatrix}\cos^2\phi+\sin^2\phi & \cos\phi\sin\phi-\cos\phi\sin\phi\\ \cos\phi\sin\phi-\cos\phi\sin\phi & \sin^2\phi+\cos^2\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

  Hence $\mathbf{M}^T\mathbf{M} = \mathbf{1}_{2\times 2}$ or equivalenly $\mathbf{M}^T = \mathbf{M}^{-1}$
  Remark: Geometrically, $\mathbf{M}$ rotates a vector $\mathbf{z}$ by an angle ϕ counterclockwise. Hence $\|\mathbf{M}\mathbf{z}\| = \|\mathbf{z}\|$ for any nonzero $\mathbf{z}$.

- Recall a main property of (multivariate) Gaussian variables is that any linear transformation of them remain (multivariate) Gaussian.

- Compute the mean and covariance of $\mathbf{MZ}$:

  $$\begin{aligned}\mathbb{E}[\mathbf{MZ}] &= \mathbf{M0} = \mathbf{0}\\ \Sigma_{\mathbf{MZ}} &= \mathbf{M}\Sigma_{\mathbf{Z}}\mathbf{M}^T = \mathbf{M}\mathbf{1}_{2\times 2}\mathbf{M}^T = \mathbf{M}\mathbf{M}^{-1} = \mathbf{1}_{2\times 2}.\end{aligned}$$

  Hence, $\mathbf{MZ} \sim \mathcal{N}_d\left(\mathbf{0}, \mathbf{1}_{2\times 2}\right)$, , i.e. a **standard** Gaussian vector.

Remark: Real matrices satisfying $\mathbf{M}^T = \mathbf{M}^{-1}$ (or equivalently $\mathbf{M}\mathbf{M}^T = \mathbf{M}^T\mathbf{M} = \mathbf{1}_{d\times d}$,) are called **orthogonal** matrices. In general, in d dimensions and for any orthogonal matrix $\mathbf{M}$, $\mathbf{MZ}$ is also a **standard** multivariate Gaussian vector if $\mathbf{Z}$ is a standard multivariate Gaussian.

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Review: Asymptotic Normality of the MLE

1/1 point (graded)

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathbf{P}_{\theta^*}$ for some true parameter $\theta^* \in \mathbb{R}^d$. We construct the associated statistical model $(\mathbb{R}, \{\mathbf{P}_\theta\}_{\theta\in\mathbb{R}^d})$ and the maximum likelihood estimator $\hat{\theta}_n^{MLE}$ for θ^*.

Recall that, under some technical conditions,

$$\sqrt{n}\,(\hat{\theta}_n^{MLE}-\theta^*)\xrightarrow[n\to\infty]{(d)}\mathcal{N}\left(0,\mathcal{I}(\theta^*)^{-1}\right)$$

where $\mathcal{I}(\theta^*)$ denotes the Fisher information. That is, the MLE $\hat{\theta}_n^{MLE}$ is asymptotically normal with asymptotic covariance matrix $\mathcal{I}(\theta^*)^{-1}$.

Standardize the statement of asymptotic normality above. Answer by finding the power a of the Fisher information $\mathcal{I}(\theta^*)$ such that the following is true:

$$\sqrt{n}\mathcal{I}(\theta^*)^a\,(\hat{\theta}_n^{MLE}-\theta^*)\xrightarrow[n\to\infty]{(d)}\mathcal{N}\left(0,I_{d\times d}\right)$$

where $I_{d\times d}$ denotes the $d\times d$ identity matrix.

Hint: Use the result of the previous problem.

$a=$ 1/2 ✔ **Answer:** 1/2

STANDARD NOTATION

Solution:

By the result of the previous problem, if $\mathbf{X}\sim\mathcal{N}\left(\mathbf{0},\mathcal{I}(\theta^*)^{-1}\right)$, then $\mathcal{I}(\theta^*)^{1/2}\mathbf{X}$ is mean 0 and has covariance matrix

$$\mathcal{I}(\theta^*)^{1/2}\mathcal{I}(\theta^*)^{-1}\left(\mathcal{I}(\theta^*)^{1/2}\right)^T=\mathcal{I}(\theta^*)^{1/2}\mathcal{I}(\theta^*)^{-1}\mathcal{I}(\theta^*)^{1/2}=I_{d\times d}.$$

Indeed, $\mathcal{I}(\theta^*)^{1/2}\mathbf{X}\sim\mathcal{N}\left(\mathbf{0},I_{d\times d}\right)$.

By the asymptotic normality of the MLE,

$$\sqrt{n}\,(\hat{\theta}_n^{MLE} - \theta^*) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, \mathcal{I}(\theta^*)^{-1}\right)$$

so that, by continuity,

$$\sqrt{n}\mathcal{I}(\theta^*)^{1/2}\,(\hat{\theta}_n^{MLE} - \theta^*) \xrightarrow[n\to\infty]{(d)} \mathcal{I}(\theta^*)^{1/2}\mathbf{N}\left(\mathbf{0}, \mathcal{I}(\theta^*)^{-1}\right) = \mathbf{N}\left(0, I_{d\times d}\right).$$

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