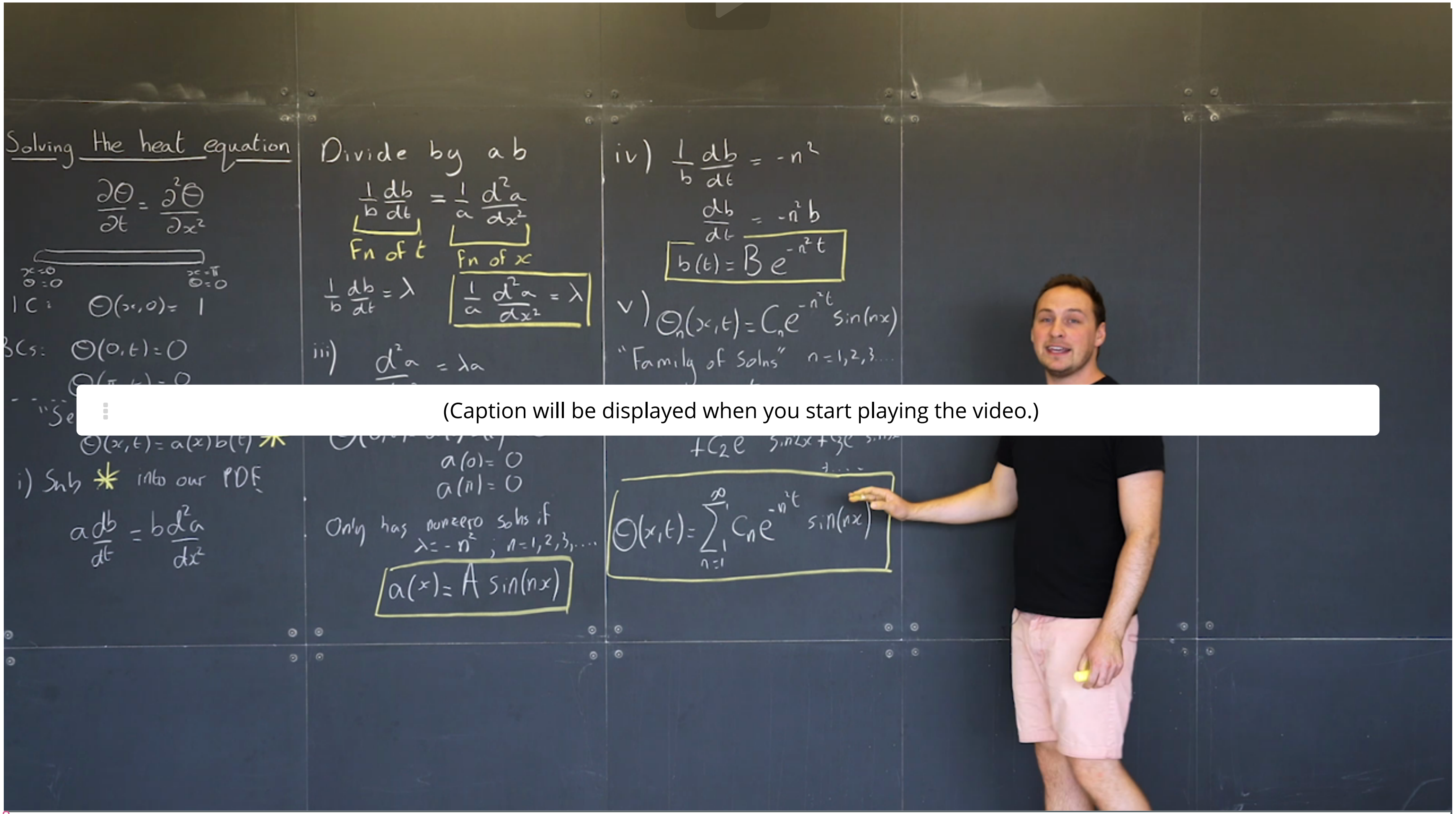
5. Initial conditions

Continued example



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Summary:

- We modeled an insulated metal rod with exposed ends held at $0^\circ\mathrm{C}$.
- Using physics, we found that its temperature $\theta(x,t)$ was governed by the PDE

$$\frac{\partial\theta}{\partial t} = \nu\frac{\partial^2\theta}{\partial x^2}, \quad 0 < x < \pi, \quad \text{(the heat equation)}.$$

  For simplicity, we specialized to the case $\nu = 1$, length π, and initial temperature $\theta(x,0) = 1$.

- Trying $\theta = v(x)\, w(t)$ led to separate ODEs for v and w, leading to solutions $e^{-n^2 t}\sin nx$ for $n = 1, 2, \ldots$ to the PDE with boundary conditions.
- We took linear combinations to get the general solution

$$\boxed{\theta(x,t) = b_1 e^{-t}\sin x + b_2 e^{-4t}\sin 2x + b_3 e^{-9t}\sin 3x + \cdots}$$

  to the PDE with homogeneous boundary conditions $\theta(0,t) = 0$, and $\theta(\pi,t) = 0$.

Initial conditions. As usual, we postponed imposing the initial condition, but now it is time to impose it.

Question 5.1 Which choices of $b_1, b_2, \ldots$ make the general solution above also satisfy the initial condition $\theta(x,0) = 1$ for $x \in (0,\pi)$?

Set $t = 0$ in

$$\text{General solution:} \quad \theta(x,t) = b_1 e^{-t}\sin x + b_2 e^{-4t}\sin 2x + b_3 e^{-9t}\sin 3x + \cdots \tag{3.45}$$

(the general solution to the Heat Equation) and use the initial condition on the left to get

$$1 = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \quad \text{for } \; x \in (0, \pi),$$

which must be solved for $b_1, b_2, \ldots$.

Because the right hand side is odd and of base period 2π, to find such b_i, the left hand side must be extended to an odd period 2π function, namely $\mathrm{Sq}(x)$. So we need to solve

$$\mathrm{Sq}(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \quad \text{for all } x \in \mathbb{R}.$$

We already know the answer:

$$\mathrm{Sq}(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots.$$

In other words $b_n = 0$ for even n, and $b_n = \frac{4}{n\pi}$ for odd n. Substituting these b_n back into the general solution to the heat equation gives

$$\theta(x, t) = \frac{4}{\pi} e^{-t} \sin x + \frac{4}{3\pi} e^{-9t} \sin 3x + \frac{4}{5\pi} e^{-25t} \sin 5x + \cdots.$$

Question 5.2 What does the temperature profile look like when t is large?

Answer: All the Fourier components are decaying, so $\theta(x, t) \to 0$ as $t \to +\infty$ at every position. Thus the temperature profile approaches a horizontal segment, the graph of the zero function. But the Fourier components of higher frequency decay much faster than the first Fourier component, so when t is large, the formula

$$\theta(x, t) \approx \frac{4}{\pi} e^{-t} \sin x$$

$\Theta(x) =$ 0 ✔ **Answer:** 0

0

Solution:

The steady state solution is $\Theta(x) = 0$.

For large times, the solution is dominated by the first term $\theta(x,t) \approx \frac{4}{\pi} e^{-t} \sin x$. But this term tends to zero as t tends to infinity. Thus as you might expect, if you submerge the ends of a metal rod in an ice bath, eventually, the temperature everywhere in the bar will be 0.

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You have used 1 of 3 attempts

Answers are displayed within the problem

Another initial condition

2/2 points (graded)

Suppose that a thin metal bar initially has temperature given by $\theta_0(x) = x$ at time $t = 0$. Then both ends are submerged in an ice bath and held at 0 degrees Celsius.

The general solution can be written as

$$\theta(x,t) = C_1(t) \sin x + C_2(t) \sin 2x + C_3(t) \sin 3x + \cdots.$$

Find the function $C_n(t)$ given that the Fourier series of the 2π-periodic sawtooth wave is given by

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt).$$

(Note that you must find both the constant coefficient, and multiply by the correct function of t.)

$C_n(t) =$ 2*(-1)^(n+1)/n*e^(-n^2*t) ✔ **Answer:** 2*(-1)^(n+1)*e^(-n^2*t)/n

$\frac{2 \cdot (-1)^{n+1}}{n} \cdot e^{-n^2 \cdot t}$

Find the steady state solution $\Theta(x)$.

$\Theta(x) =$ 0 ✔ **Answer:** 0

0

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Solution:

The general solution to the heat equation with homogeneous boundary conditions is always

$$\theta(x,t) = C_1(t)\sin x + C_2(t)\sin 2x + C_3(t)\sin 3x + \cdots.$$

First note that the general solution takes the form

$$\theta(x,t) = b_1 e^{-t}\sin x + b_2 e^{-4t}\sin 2x + b_3 e^{-9t}\sin 3x + \cdots.$$

Thus $C_n(t) = b_n e^{-n^2 t}$.

To find the coefficients b_n, set $t = 0$ and set the general solution to be equal to the Fourier series for the Sawtooth wave, which is the odd, 2π-periodic extension of the function given as the initial condition: $\theta_0(x) = t$, for $0 < t < \pi$.

$$b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = 2 \sin x - \sin(2x) + \frac{2}{3} \sin(3x) + \cdots + 2\frac{(-1)^{n+1}}{n} \sin(nx) + \cdots.$$

Therefore $b_n = \frac{2(-1)^{n+1}}{n}$, and $C_n(t) = \frac{2(-1)^{n+1}}{n} e^{-n^2 t}$.

The general solution therefore is

$$\theta(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 t} \sin nx.$$

Note that as t tends to infinity, every term tends to 0 in this Fourier series due to the exponential decay term in each summand. Therefore the steady state solution $\Theta = 0$ for this initial condition as well.

The steady state solution for the Heat Equation with homogeneous boundary conditions $\theta(0,t) = 0$, and $\theta(L,t) = 0$ will always be the constant zero function.

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Answers are displayed within the problem

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is a very good approximation. Eventually, the temperature profile is indistinguishable from a sinusoid of angular frequency 1 whose amplitude is decaying to 0. This can be observed in the mathlet.

mathlets.org

Steady state

1/1 point (graded)

What is the steady state solution $\Theta(x)$ defined as $\theta(x,t) \to \Theta(x)$ as time $t \to \infty$.