Problem 3: Sums of a random number of random variables

(4/4 points)

Let $N, X_1, Y_1, X_2, Y_2, \ldots$ be independent random variables. The random variable N takes positive integer values and has mean a and variance r. The random variables X_i are independent and identically distributed with mean b and variance s, and the random variables Y_i are independent and identically distributed with mean c and variance t. Let

$$A = \sum_{i=1}^{N} X_i \quad \text{and} \quad B = \sum_{i=1}^{N} Y_i.$$

1. Find $\mathbf{cov}(A, B)$. Express your answer in terms of the given means and variances using standard notation .

$\mathbf{cov}(A, B) =$ b*c*r

Answer: b*c*r

2.

Find $\mathbf{var}(A+B)$. Express your answer in terms of the given means and variances using standard notation .

$\mathbf{var}(A+B) =$ a*s+a*t+b^2*r+c^2*r+2*b*c*r

Answer: a*(s+t) + r*(b+c)^2

Answer:

1. The covariance of A and B is given by $\mathbf{cov}(A,B) = \mathbf{E}[AB] - \mathbf{E}[A]\mathbf{E}[B]$. Using the law of iterated expectations,

$$\begin{aligned}
\mathbf{E}[AB] &= \mathbf{E}[\mathbf{E}[AB \mid N]] \\
&= \mathbf{E}\left[\mathbf{E}\left[(X_1 + X_2 + \cdots + X_N)(Y_1 + Y_2 + \cdots + Y_N) \mid N\right]\right] \\
&= \mathbf{E}[N^2 \mathbf{E}[X_1 Y_1 \mid N]] \\
&= \mathbf{E}[N^2 \mathbf{E}[X_1]\mathbf{E}[Y_1]] \\
&= bc\mathbf{E}[N^2] \\
&= bc(r + a^2),
\end{aligned}$$

where the third equality holds because there are N^2 terms after expanding the product, each of which have the same expected value since the X_i's and Y_i's are both identically distributed. The fourth equality holds by independence.

Also using the law of iterated expectations,

$$\begin{aligned}
\mathbf{E}[A] &= \mathbf{E}[\mathbf{E}[X_1 + X_2 + \cdots + X_N \mid N]] \\
&= \mathbf{E}[Nb] \\
&= ab, \\
\mathbf{E}[B] &= \mathbf{E}[\mathbf{E}[Y_1 + Y_2 + \cdots + Y_N \mid N]] \\
&= \mathbf{E}[Nc] \\
&= ac.
\end{aligned}$$

Hence, the covariance of A and B is

$$\begin{aligned}
\mathrm{cov}(A, B) &= bc(r + a^2) - a^2 bc \\
&= bcr.
\end{aligned}$$

2. We are dealing with a sum of a random number of independent random variables. Let us define $Z_i = X_i + Y_i$ so that $A + B = \sum_{i=1}^{N} Z_i$. Since the X_i's are i.i.d. and the Y_i's are also i.i.d., the Z_i's are i.i.d. as well, with mean $b + c$ and variance $s + t$.

   Using the formula for the variance of the sum of a random number of i.i.d. random variables, we have

$$\begin{aligned}
\mathrm{var}(A + B) &= \mathbf{E}[N]\mathrm{var}(X_1 + Y_1) + \mathrm{var}(N)(\mathbf{E}[X_1 + Y_1])^2 \\
&= a(s + t) + r(b + c)^2.
\end{aligned}$$

   Alternatively, we can also use the formula

$$\mathbf{var}(A+B) = \mathbf{var}(A) + \mathbf{var}(B) + 2 \cdot \mathbf{cov}(A,B).$$

We can calculate $\mathbf{var}(A)$ and $\mathbf{var}(B)$ in a similar manner as above and apply the result from part (1) for $\mathbf{cov}(A,B)$.

You have used 2 of 3 submissions