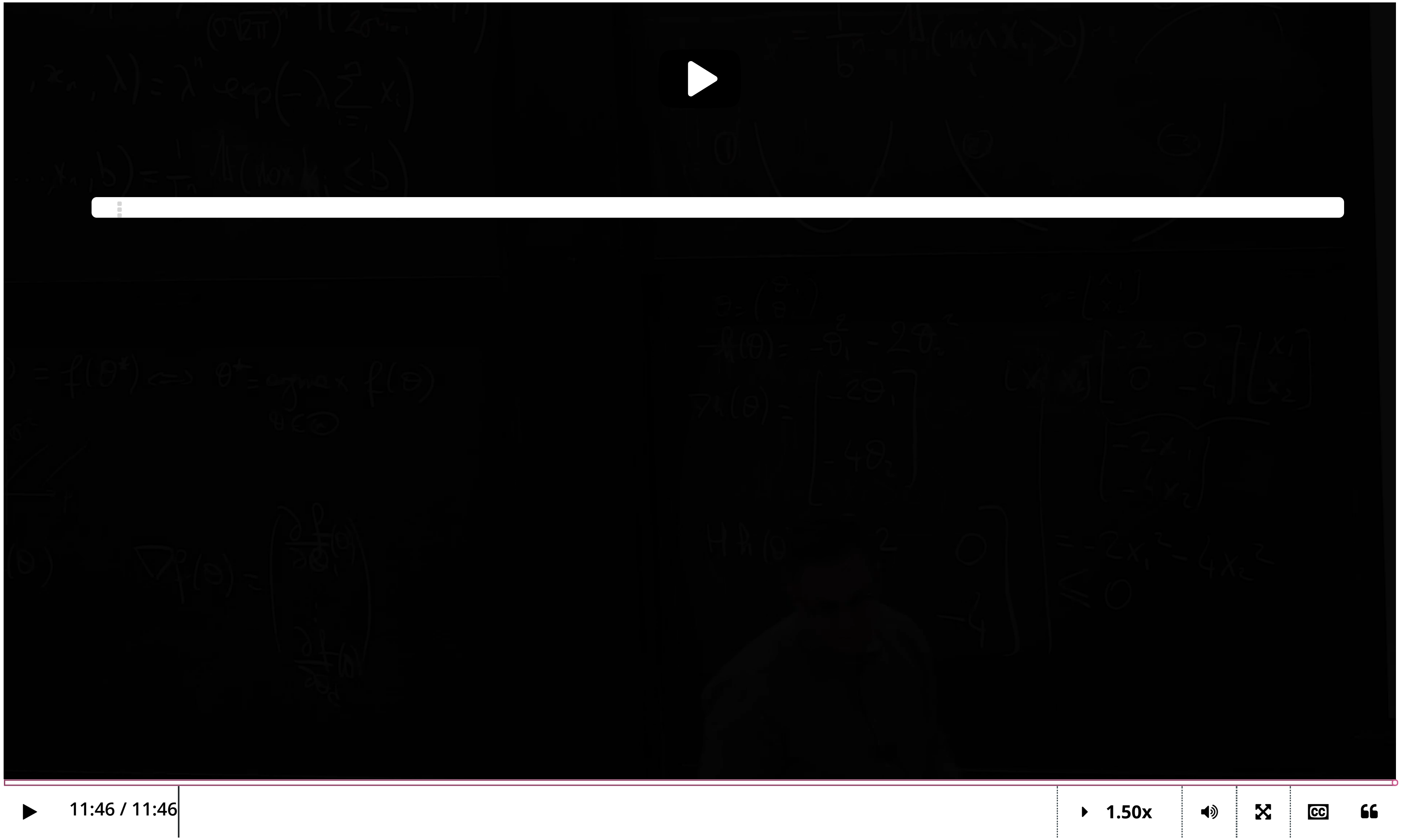
8. Review: Gradients and Hessians; Concavity in Higher dimensions

Concavity in Higher Dimensions: Gradients, Hessians, Semi-Definiteness



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Multivariable Calculus Review: Compute the Gradient

1/1 point (graded)

Let

$$\begin{aligned} f: \quad & \mathbb{R}^d && \to \mathbb{R} \\ & \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \end{pmatrix} && \mapsto f(\theta). \end{aligned}$$

denote a **differentiable** function. The **gradient** of f is the vector-valued function

$$\begin{aligned} \nabla f: \quad & \mathbb{R}^d && \to \mathbb{R}^d \\ & \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \end{pmatrix} && \mapsto \left.\begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \vdots \\ \frac{\partial f}{\partial \theta_d} \end{pmatrix}\right|_{\theta}. \end{aligned}$$

Consider $f(\theta) = -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2$ where $c_1, c_2, c_3 > 0$ are positive real numbers.

Compute the gradient ∇f.

(Enter your answer as a vector, e.g., type **[3,2,x]** for the vector $\begin{pmatrix} 3 \\ 2 \\ x \end{pmatrix}$. Note the square brackets, and commas as separators. Enter **c_i** for c_i, **theta_i** for θ_i.)

$\nabla f =$ [-2*c_1*theta_1,-2*c_2*theta_2,-2*c_3*theta_3] ✔ **Answer:** [-2*c_1*theta_1,-2*c_2*theta_2,-2*c_3*theta_3]

STANDARD NOTATION

Solution:

$$f(\theta) \;=\; -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2$$

$$\nabla f(\theta) \;=\; \left.\begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \frac{\partial f}{\partial \theta_3} \end{pmatrix}\right|_{\theta} \;=\; \begin{pmatrix} -2c_1\theta_1 \\ -2c_2\theta_2 \\ -2c_3\theta_3 \end{pmatrix}.$$

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Answers are displayed within the problem

Multivariable Calulus Review: Compute the Hessian Matrix

1/1 point (graded)

As above, let

$$f: \; \mathbb{R}^d \;\to\; \mathbb{R} \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \end{pmatrix} \;\mapsto\; f(\theta).$$

denote a **twice-differentiable** function.

The **Hessian** of f is the matrix

$$\mathbf{H} f \;:\; \mathbb{R}^d \;\to\; \mathbb{R}^{d\times d}$$

whose entry in the i-th row and j-th column is defined by

$$(\mathbf{H} f)_{ij} \;:=\; \frac{\partial^2}{\partial\theta_i \partial\theta_j} f, \quad 1 \le i,j \le d.$$

The Hessian matrix of f in this context is also denoted by $\nabla^2 f$, the **second derivative** of f. This is not to be confused with the "Laplacian" of f, which is also denoted the same way.

Consider the same function $f(\theta) = -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2$ where $c_1,\, c_2,\, c_3 > 0$ as in the previous problem. Compute the Hessian matrix $\mathbf{H}f$.

(Enter your answer as a matrix, e.g. by typing **[[1,2],[5*x,y-1]]** for the matrix $\begin{pmatrix} 1 & 2 \\ 5x & y-1 \end{pmatrix}$. Note the square brackets, and commas as separaters.)

$\mathbf{H}f =$ [[-2*c_1,0,0],[0,-2*c_2,0],[0,0,-2*c_3]] ✔

Answer: [[-2*c_1,0,0],[0,-2*c_2,0],[0,0,-2*c_3]]

STANDARD NOTATION

Solution:

Recall from the previous problem:

$$\begin{aligned} f(\theta) &= -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2 \\ \nabla f(\theta) &= \left.\begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \frac{\partial f}{\partial \theta_3} \end{pmatrix}\right|_{\theta} = \begin{pmatrix} -2c_1\theta_1 \\ -2c_2\theta_2 \\ -2c_3\theta_3 \end{pmatrix}. \end{aligned}$$

One way to compute the Hessian is to start will in j-th column of the Hessian matrix by the gradient of the j-th component of ∇f. We obtain:

$$\begin{aligned} f(\theta) &= -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2 \\ \mathbf{H}f(\theta) &= \begin{pmatrix} | & | & | \\ \nabla(-2c_1\theta_1) & \nabla(-2c_2\theta_2) & \nabla(-2c_3\theta_3) \\ | & | & | \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -2c_1 & 0 & 0 \\ 0 & -2c_2 & 0 \\ 0 & 0 & -2c_3 \end{pmatrix}.$$

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Semi-Definiteness

3/3 points (graded)

A symmetric (real-valued) $d \times d$ matrix $\mathbf{A}$ is **positive semi-definite** if

$$\mathbf{x}^T \mathbf{A}\,\mathbf{x} \;\geq\; 0 \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d.$$

If the inequality above is strict, i.e. if $\mathbf{x}^T\,\mathbf{A}\,\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x} \in \mathbb{R}^d$, then $\mathbf{A}$ is **positive definite** .

Analogously, a symmetric (real-valued) $d \times d$ matrix $\mathbf{A}$ is **negative semi-definite** (*resp.* **negative definite**) if $\mathbf{x}^T\,\mathbf{A}\,\mathbf{x}$ is **non-positive** (*resp.* **negative**) for all $\mathbf{x} \in \mathbb{R}^d - \{\mathbf{0}\}$.

Note that by definition, positive (or negative) definiteness implies positive (or negative) semi-definiteness.

Consider the same function as in the problems above:

$$f(\theta) \;=\; -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2 \quad \text{where } c_1, c_2, c_3 > 0.$$

Compute $\mathbf{x}^T\,(\mathbf{H}f)\,\mathbf{x}$ where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

$\mathbf{x}^T(\mathbf{H}f)\,\mathbf{x} =$ -2*c_1*x_1^2-2*c_2*x_2^2-2*c_3*x_3^2 ✔ **Answer:** -2*c_1*x_1^2-2*c_2*x_2^2-2*c_3*x_3^2

$-2\cdot c_1\cdot x_1^2 - 2\cdot c_2\cdot x_2^2 - 2\cdot c_3\cdot x_3^2$

The matrix $\mathbf{H}f$ is (Choose all that apply.)

- [ ] positive semi-definite
- [ ] positive definite
- [x] negative semi-definite
- [x] negative definite

✔

Hence, the function f is
(Choose all that apply.)

- [x] concave
- [x] strictly concave
- [ ] convex
- [ ] strictly convex

✔

Solution:

Recall from the previous problem that

$$\mathbf{H}f(\theta) = \begin{pmatrix} -2c_1 & 0 & 0 \\ 0 & 2c_2 & 0 \\ 0 & 0 & -2c_3 \end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{x}^T(\mathbf{H}f)\,\mathbf{x} &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -2c_1 & 0 & 0 & \\ 0 & 2c_2 & 0 & \\ 0 & 0 & & 2c_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= -2c_1x_1^2 - 2c_2x_2^2 - 2c_3x_3^2 < 0.\end{aligned}$$

Since $c_1, c_2, c_3 > 0,$ this means the $\mathbf{H}f$ is negative definite, (also negative semi-definite), and hence f is strictly concave (also concave).

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strictly convex and convex

Since last option allows for multiple answer, I want to be sure the understanding is right. Something that is convex, is not necessarily strictly convex.. but if something is "Strict... 5

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