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Bayes' Law and Conditionalization

Just like one can speak of conditional *subjective* probabilities, so one can speak of conditional *objective* probabilities. One can speak, for instance, of the objective probability that a pair of coin-tosses yields two Heads, given that it yields at least one Head. In symbols: $p\,(\text{two Heads} \mid \text{at least one Head})$.

How are objective conditional probabilities related to their subjective counterparts?

It is natural to answer this question by setting forth a conditional version of the Objective-Subjective Connection:

> **The Objective-Subjective Connection (Conditional Version)**
> The objective probability of A given H at time t is the subjective conditional probability that a perfectly rational agent would assign to A given H, if she had perfect information about events before t and no information about events after t.

The conditional version of the Objective-Subjective Connection can be used to show that the objective probabilities must evolve in accordance with a version of conditionalization, on the hypothesis that perfectly rational agents update by conditionalization. It can also be used to show that the objective probabilities satisfy Bayes' Law, on the hypothesis that you can't count as a perfectly rational agent unless your conditional and unconditional *subjective* probabilities are related by Bayes' Law.

Problem 0

1/1 point (ungraded)
Use the conditional version of the Objective-Subjective Connection to show that the objective probabilities must evolve in accordance with a version of conditionalization. More specifically, show that for any proposition A and any times $t_0 < t_1$,

$$p_{t_1}(A) = p_{t_0}(A|H_{t_1})$$

where p_{t_i} is the objective probability function at t_i and H_{t_i} consists of perfect information about events before t_i.(You may assume that that perfectly rational agents update by conditionalization.

- [x] done

✔

Explanation

Consider a perfectly rational agent who at each $t_i (i \leq 1)$ has perfect information about events before t_i (and no information about events after t_i). By the Subjective-Objective Connection, our subject's credences at each t_i must line up with the objective probabilities at t_i. Since perfectly rational agents update their credences by conditionalization,

$p^{t_1}(A) = p^{t_0}(A|H_{t_1})$

where p^{t_i} is the subject's credence function at t_i and H_{t_i} consists of perfect information about events before t_i. So the Subjective-Objective Connection yields $p_{t_1}(A) = p_{t_0}(A|H_{t_1})$.

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Answers are displayed within the problem

Problem 1

1/1 point (ungraded)

Bayes' Law allows us to perform all sorts of interesting computations. Suppose, for example, that you have an urn with two red balls and two black balls, and use a random procedure to draw balls from the urn. You draw once, and keep the ball in your pocket. The probability of getting red on your first draw (R_1) is $1/2$. But the probability of getting red on the second draw (R_2) depends on whether you get red on your first draw. If your first draw is red, the urn will be left with two black balls and one red ball. So the probability of red on your second draw is $1/3$: $p(R_2|R_1) = 1/3$. But if your first draw is black (B_1), the urn will be left with one black ball and two red balls. So the probability of red on your second draw is $2/3$: $p(R_2|B_1) = 2/3$. Accordingly:

First Draw	Second Draw
$p(R_1) = 1/2$	$p(R_2|R_1) = 1/3$
$p(B_1) = 1/2$	$p(R_2|B_1) = 2/3$

Use Bayes' Law allows to calculate the unconditional probability, $p(R_2)$, of getting red on your second draw.

1/2

✔ **Answer:** .5

$\frac{1}{2}$

Explanation

$$\begin{aligned}
p(R_2) &= p(R_1R_2 \text{ or } B_1R_2) && R_2 \text{ equivalent to } (R_1R_2 \text{ or } B_1R_2) \\
&= p(R_1R_2) + p(B_1R_2) && \text{[Additivity]} \\
&= p(R_1) \cdot p(R_2|R_1) + R(B_1) \cdot p(R_2|B_1) && \text{[Bayes' Law]} \\
&= 1/2 \cdot 1/3 + 1/2 \cdot 2/3 \\
&= 1/2
\end{aligned}$$

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Problem 2

2/2 points (ungraded)

In Lecture 4 we talked about probabilistic dependence and independence. These notions can be characterized formally, using the notion of conditional probability. Here is one way of doing so:

> If $p(A|B) = p(A|\text{not-}B)$, then A is probabilistically independent of B. Otherwise, each of A and B is probabilistically dependent on the other.

In fact, there are several equivalent ways of defining probabilistic independence. Is the following an equivalent way of defining probabilistic independence?

> $p(A|B) = p(A|\text{not-}B)$ if and only if $p(AB) = p(A) \cdot p(B)$.

- (●) Yes
- ( ) No

✔

Explanation

Bayes' Law gives us each of the following:

$$p(A|B) = \frac{p(AB)}{p(B)} \quad p(A|\text{not-}B) = \frac{p(A\,\text{not-}B)}{p(\text{not-}B)}$$

So the following are equivalent:

$$\begin{aligned}
p(A|B) &= p(A|\text{not-}B)\\
\frac{p(AB)}{p(B)} &= \frac{p(A\,\text{not-}B)}{p(\text{not-}B)}\\
p(\text{not-}B)\cdot p(AB) &= p(B)\cdot p(A\,\text{not-B})\\
(1-p(B))\cdot p(AB) &= p(B)\cdot p(A\,\text{not-B})\\
p(AB)-p(B)\cdot p(AB) &= p(B)\cdot p(A\,\text{not-B})\\
p(AB) &= p(B)\cdot p(A\,\text{not-B})+p(B)\cdot p(AB)\\
p(AB) &= p(B)\cdot(p(A\,\text{not-B})+p(AB))\\
p(AB) &= p(B)\cdot p(A\,\text{not-B or }AB)\\
p(AB) &= p(B)\cdot p(A)\\
p(AB) &= p(A)\cdot p(B)
\end{aligned}$$

How about the following? Is it an equivalent way of defining probabilistic independence?

$p(A|B) = p(A|\text{not-}B)$ if and only if $p(A|B) = p(A)$.

- (•) Yes
- ( ) No

✔

Explanation

Bayes' Law gives us:

$$p(A|B) = \frac{p(AB)}{p(B)}$$

So the following are equivalent:

$$\begin{aligned}
p(AB) &= p(A)\cdot p(B)\\
\frac{p(AB)}{p(B)} &= p(A)\\
p(A|B) &= p(A)
\end{aligned}$$

But, by the previous exercise, we have

$p(A|B) = p(A|\text{not-}B)$ if and only if $p(AB) = p(A)\cdot p(B)$.

So we conclude:

$p(A|B) = p(A|\text{not-}B)$ if and only if $p(A|B) = p(A)$.

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Problem 3

1/1 point (ungraded)

True or false?

$p(A|B) = p(A|\text{not-}B)$ if and only if $p(B|A) = p(B|\text{not-}A)$.

(In other words, A is probabilistically independent of B if and only if B is probabilistically independent of A.)

- (selected) True
- False

✔

Explanation

The previous exercise gives us each of the following:

- $p(A|B) = p(A|\text{not-}B)$ if and only if $p(AB) = p(A) \cdot p(B)$
- $p(B|A) = p(B|\text{not-}A)$ if and only if $p(BA) = p(B) \cdot p(A)$

But since AB is equivalent to BA, we have $p(AB) = p(BA)$. So:

- $p(A|B) = p(A|\text{not-}B)$ if and only if $p(B|A) = p(B|\text{not-}A)$

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Problem 4

1/1 point (ungraded)

Suppose that the objective probability of a fair coin landing Heads is $\frac{1}{2}$. If you toss a fair coin n times, what is the objective probability that it will land Heads on every single toss? (Assume that different tosses are probabilistically independent of one another.)

- ○ $n^{\frac{1}{2}}$
- ○ $n \cdot \frac{1}{2}$
- ◉ $\left(\frac{1}{2}\right)^n$

Explanation

Answer: $1/2^n$

Let H_k be the proposition that the coin lands heads on the kth toss. Since the coin is fair, we know that $p\left(H_k\right) = 1/2$. And since different coin tosses are independent of one another, the previous exercise entails that the following is true whenever $k \neq l$:

$$p\left(H_k H_l\right) = p\left(H_k\right) \cdot p\left(H_l\right)$$

Putting all of this together:

$$p\left(H_1 H_2 \ldots H_n\right) = p\left(H_1\right) \cdot p\left(H_2\right) \cdot \ldots \cdot \left(H_n\right) = \underbrace{1/2 \cdot 1/2 \cdot \ldots \cdot 1/2}_{n} = 1/2^n$$

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Problem 5

1/1 point (ungraded)

Part of what it is for a coin to have a 50% chance of landing Heads is for it to be *possible* that the coin land Heads.

Notice, moreover, that the possibility of a coin's landing Heads does not depend on how many previous coin tosses have landed Heads, since coin tosses are independent of one another. From this it follows that if you toss a fair coin infinitely many times, it is *possible* (though vanishingly unlikely) that the coin lands Heads every single time.

Is the following true or false?

If the probability of such an outcome is a real number between 0 and 1, then it must be 0.

- [x] True
- [ ] False

✔

Explanation

Let $\overset{k\to\infty}{H}$ be the proposition that the coin lands Heads on the kth toss and on every toss thereafter. Let us assume that the probability of $\overset{1\to\infty}{H}$ is a real number between 0 and 1, and show that, on that assumption, the probability must be 0.

The first thing to note is that, for any k, $\overset{1\to\infty}{H}$ is equivalent to $H_1 H_2 \ldots H_{k-1} \overset{k\to\infty}{H}$. But, since different coin tosses are independent of one another, a previous exercise (Problem 2) entails:

$$p\left(\overset{1\to\infty}{H}\right) = p\left(H_1 H_2 \ldots H_k \overset{k+1\to\infty}{H}\right) = p\left(H_1 H_2 \ldots H_k\right) \cdot p\left(\overset{k+1\to\infty}{H}\right)$$

But, by the previous exercise (Problem 4), $p\left(H_1 H_2 \ldots H_k\right) = 1/2^k$. Since $p\left(\overset{k+1\to\infty}{H}\right)$ cannot be greater than 1, this means that

$$p\left(\overset{1\to\infty}{H}\right) \leq 1/2^k$$

for every k. But 0 is the only real number between 0 and 1 which is smaller than or equal to $1/2^k$ for every k.

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Answers are displayed within the problem

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Nothing new in what the expert Susanna Rinard said if you learned and understood propbability 1