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STAT 414 Introduction to Probability Theory

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23.1 - Change-of-Variables Technique

Recall, that for the univariate (one random variable) situation: Given X with pdf $f(x)$ and the transformation $Y = u(X)$ with the single-valued inverse $X = v(Y)$, then the pdf of Y is given by

$$g(y) = |v'(y)| f[v(y)].$$

Now, suppose (X_1, X_2) has joint density $f(x_1, x_2)$. and support S_X.

Let (Y_1, Y_2) be some function of (X_1, X_2) defined by $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ with the single-valued inverse given by $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$. Let S_Y be the support of Y_1, Y_2.

Then, we usually find S_Y by considering the image of S_X under the transformation (Y_1, Y_2). Say, given $x_1, x_2 \in S_X$, we can find $(y_1, y_2) \in S_Y$ by

$$x_1 = v_1(y_1, y_2), \qquad x_2 = v_2(y_1, y_2)$$

The joint pdf Y_1 and Y_2 is

$$g(y_1, y_2) = |J| f[v_1(y_1, y_2), v_2(y_1, y_2)]$$

In the above expression, $|J|$ refers to the absolute value of the Jacobian, J. The Jacobian, J, is given by

$$\begin{vmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{vmatrix}$$

i.e. it is the determinant of the matrix

$$\begin{pmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{pmatrix}$$

Example 23-1

Suppose X_1 and X_2 are independent exponential random variables with parameter $\lambda = 1$ so that

$$\begin{array}{ll} f_{X_1}(x_1) = e^{-x_1} & 0 < x_1 < \infty \\ f_{X_2}(x_2) = e^{-x_2} & 0 < x_2 < \infty \end{array}$$

The joint pdf is given by

$$f(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = e^{-x_1 - x_2} \qquad 0 < x_1 < \infty, 0 < x_2 < \infty$$

Consider the transformation: $Y_1 = X_1 - X_2, Y_2 = X_1 + X_2$. We wish to find the joint distribution of Y_1 and Y_2.

We have

$$x_1 = \frac{y_1 + y_2}{2}, x_2 = \frac{y_2 - y_1}{2}$$

OR

$$v_1(y_1, y_2) = \frac{y_1 + y_2}{2}, v_2(y_1, y_2) = \frac{y_2 - y_1}{2}$$

The Jacobian, J is

$$\begin{vmatrix} \frac{\partial\left(\frac{y_1+y_2}{2}\right)}{\partial y_1} & \frac{\partial\left(\frac{y_1+y_2}{2}\right)}{\partial y_2} \\ \frac{\partial\left(\frac{y_2-y_1}{2}\right)}{\partial y_1} & \frac{\partial\left(\frac{y_2-y_1}{2}\right)}{\partial y_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

So,

$$\begin{aligned} g(y_1, y_2) &= e^{-v_1(y_1,y_2)-v_2(y_1,y_2)}\left|\frac{1}{2}\right| \\ &= e^{-\left[\frac{y_1+y_2}{2}\right]-\left[\frac{y_2-y_1}{2}\right]}\left|\frac{1}{2}\right| \\ &= \frac{e^{-y_2}}{2} \end{aligned}$$

Now, we determine the support of (Y_1, Y_2). Since $0 < x_1 < \infty, 0 < x_2 < \infty$, we have $0 < \frac{y_1+y_2}{2} < \infty, 0 < \frac{y_2-y_1}{2} < \infty$ or $0 < y_1 + y_2 < \infty, 0 < y_2 - y_1 < \infty$. This may be rewritten as $-y_2 < y_1 < y_2, 0 < y_2 < \infty$.

Using the joint pdf, we may find the marginal pdf of Y_2 as

$$\begin{aligned} g(y_2) &= \int_{-\infty}^{\infty} g(y_1, y_2) dy_1 \\ &= \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2} dy_1 \\ &= \frac{1}{2}\left[e^{-y_2} y_1 \Big|_{y_1=-y_2}^{y_1=y_2}\right] \\ &= \frac{1}{2} e^{-y_2}(y_2 + y_2) \\ &= y_2 e^{-y_2}, \qquad 0 < y_2 < \infty \end{aligned}$$

Similarly, we may find the marginal pdf of Y_1 as

$$g(y_1) = \begin{cases} \int_{-y_1}^{\infty} \frac{1}{2} e^{-y_2} dy_2 = \frac{1}{2} e^{y_1} & -\infty < y_1 < 0 \\ \int_{y_1}^{\infty} \frac{1}{2} e^{-y_2} dy_2 = \frac{1}{2} e^{-y_1} & 0 < y_1 < \infty \end{cases}$$

Equivalently,

$$g(y_1) = \frac{1}{2} e^{-|y_1|} 0 < y_1 < \infty$$

This pdf is known as the **double exponential** or **Laplace** pdf.