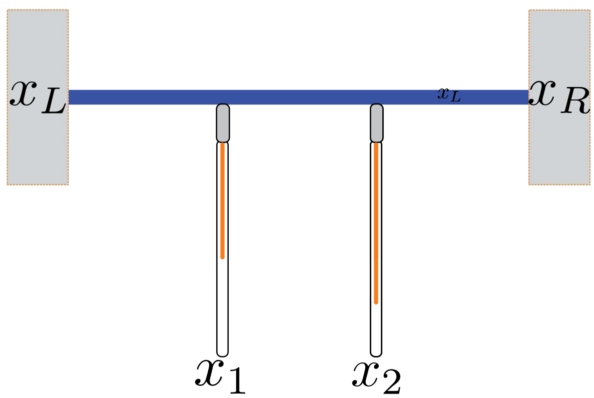
# 9. Worked Example: heated rod with fixed temperature at the ends




Recall the inhomogeneous system modelling the insulated metal rod with the temperatures at its two ends fixed at $x_L$ and $x_R$, and with two thermometers placed at equal distance along it:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_L \\ x_R \end{pmatrix}$$

**Homogeneous solution**

Let us first consider the case where the end points are held fixed at $0$ degree Celsius, so $x_L = x_R = 0.$ This makes the system **homogeneous**.

The eigenvalues of the system are $\lambda_1 = -1$ and $\lambda_2 = -3$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Hence, a fundamental matrix of the homogeneous system is given by the product:

$$\mathbf{X}(t) \quad = \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix}.$$

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## Review: phase portraits

1/1 point (graded)

As above, consider the homogeneous system describing the rates of change of temperatures $x_1$ and $x_2$ along a metal bar whose two ends are fixed at $0$ degree:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad = \quad \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

What is the phase portrait of this system?

- ( ) Stable spiral
- (•) Stable node ✔
- ( ) Saddle
- ( ) Unstable spiral
- ( ) Unstable node
- ( ) None of the above

**Solution:**

Since the eigenvalues are distinct, negative, and real, the phase portrait is a stable node. In particular, the homogenous system exhibits no oscillations, and both temperatures $x_1$, $x_2$ will tend to zero over time, as expected for a rod held at 0 degrees at both ends.

You have used 1 of 2 attempts

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Answers are displayed within the problem

## Variation of parameters practice

4/4 points (graded)

We now consider the case in which the left end of the rod is held at 0 degrees, so $x_L = 0$, but the right end is held at 100 degrees, so $x_R = 100$. The inhomogeneous system is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{r} \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 0 \\ 100 \end{pmatrix}.$$

Variation of parameters says that the general inhomogeneous solution is $\mathbf{X}\mathbf{v}$, where $\mathbf{v} = \int \mathbf{X}^{-1}\mathbf{r}\,dt,$ and $\mathbf{X}$ is a fundamental matrix of the associated homogeneous system. In this problem, we will carry out the necessary computations.

Compute $\mathbf{X}^{-1}\mathbf{r}$ using the fundamental matrix we previously found,

$$\mathbf{X}(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix}.$$

(Enter a single vector as your answer. Enter **[a;c]** for the vector $\begin{pmatrix} a \\ c \end{pmatrix}$. That is, use **semicolon** to separate the entries, and **square bracket** the entire matrix. )

$\mathbf{X}^{-1}\mathbf{r} =$ [50*e^t;-50*e^(3*t)] **Answer:** [50*e^t;-50*e(3*t)]

Integrate each component to find $\mathbf{v} = \int \mathbf{X}^{-1}\mathbf{r}\,dt.$ Use $c_1$ and $c_2$ as the constants of integration for the first and second components respectively.

$\mathbf{v} = \int \mathbf{X}^{-1}\mathbf{r}\,dt =$ [50*e^t+c_1;-(50/3)*e^(3

**Answer:** [50*e^t+c_1;-50/3*e(3*t)+c_2]

Find the general inhomogeneous solution by computing $\mathbf{Xv},$ with the fundamental matrix $\mathbf{X}$ given above, and the $\mathbf{v}$ you just computed. Enter the two components $x_1(t)$ and $x_2(t)$ separately below. (Your answer should involve the constants of integration $c_1$ and $c_2$ from the previous step.)

$x_1(t) =$ 100/3+c_1*e^(-t)+c_2*e^ **Answer:** c_1*e^(-t)+ c_2 *e^(-3*t) + 100/3

$\frac{100}{3} + c_1 \cdot e^{-t} + c_2 \cdot e^{-3\cdot t}$

$x_2(t) =$ 200/3+c_1*e^(-t)-c_2*e^( **Answer:** c_1*e^(-t)- c_2*e^(-3*t) + 200/3

$\frac{200}{3} + c_1 \cdot e^{-t} - c_2 \cdot e^{-3\cdot t}$

FORMULA INPUT HELP

**Solution:**

$$\begin{aligned} X^{-1}\mathbf{r} &= \frac{1}{2}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 100 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 100 \\ -100 \end{pmatrix} \\ &= \begin{pmatrix} 50e^t \\ -50e^{3t} \end{pmatrix} \end{aligned}$$

Integrating component wise, we get

$$\begin{aligned} \mathbf{v} = \int X^{-1}\mathbf{r}\,dt &= \begin{pmatrix} 50e^t + c_1 \\ -\frac{50}{3}e^{3t} + c_2 \end{pmatrix} \\ &= \begin{pmatrix} 50e^t \\ -\frac{50}{3}e^{3t} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

Now we multiply by $\mathbf{X}$ to find the inhomogeneous solution. The general homogeneous solution equals the product of $\mathbf{X}$ with the second term in $\mathbf{v}$:

$$\begin{aligned}
\mathbf{X}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \left[\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\right]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
&= c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}.
\end{aligned}$$

On the other hand, a particular solution arises from $\mathbf{X}$ multiplied by the the first term in $\mathbf{v}$:

$$\begin{aligned}
\mathbf{x}_p &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} 50e^{t} \\ -\frac{50}{3}e^{3t} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 50 \\ -50/3 \end{pmatrix} \\
&= \begin{pmatrix} 100/3 \\ 200/3 \end{pmatrix}
\end{aligned}$$

Therefore, the general solutions for the two temperatures $x_1$ and $x_2$ are

$$\begin{aligned}
x_1(t) &= \frac{100}{3} + \left(c_1 e^{-t} + c_2 e^{-3t}\right) \\
x_2(t) &= \frac{200}{3} + \left(c_1 e^{-t} - c_2 e^{-3t}\right)
\end{aligned}$$

**Remark:** The particular solution above is the steady state solution of the inhomogeneous system, and it is exactly as we could have predicted. The temperature diffuses no matter what the initial condition until it is linearly distributed across the bar from 0 to 100.

Submit You have used 2 of 7 attempts

Answers are displayed within the problem

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