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Bayes Estimators for the Continuous Uniform Distributio

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Abstract

Classical estimators for the parameter of a uniform distribution on the interval (0, θ) are often discussed in mathematical ... nts are frequently left wondering how to distinguish which among the variety of classical estimators are better than the others. W ... tors can be derived as Bayes estimators from a family of improper prior distributions. We believe that linking the estimation c ... rk is of value to students in a mathematical statistics course, and we believe that the students benefit from the exposure to Ba ... we compare classical and Bayesian interval estimators for the parameter θ and illustrate the Bayesian analysis with an example.

Keywords: Highest posterior density interval, Improper prior distribution

1 The continuous uniform distribution is widely studied in mathematical statistics textbooks and courses in part because classical estimation criteria produce different estimators for the parameter. Letting X_1, X_2,..., X_n have independent uniform distributions on the interval (0, θ), the likelihood function is for .

2 The maximum likelihood estimator of θ is , while the minimum variance unbiased estimator is . Furthermore, among es which minimizes the mean squared error is . These results can be found in many textbooks on mathematical statistics, ir and Craig (1978), and Larsen and Marx (1986).

3 While we find this example useful for helping students discover that classical estimation criteria can in fact lead to differ less feel a sense of unease when students naturally ask which estimator is "better." At this point we are tempted to turn from iteria of the classical approach to the unifying philosophy and analysis strategy of a Bayesian framework. As we will show, this esian analysis with a simple family of improper prior distributions provides a direct link among several classical estimators.

4 Moreover, we contend that students of mathematical statistics should explore principles of Bayesian inference for a variety of reasons. One is that the development and use of Bayesian methods are on the increase. A growing number of papers appearing in statistical forums such as the *Journal of the American Statistical Association* represent the Bayesian approach, and even some applied statisticians have adopted a Ba *can Statistician* recently presented a collection of papers by Berry (1997), Moore (1997), and Albert (1997), with accompanying ue of a Bayesian perspective in an introductory statistics course.

5 A second reason for encouraging students to study the Bayesian paradigm is that it models the process of science. Berr progresses with scientists altering their opinions as information accumulates, and with scientists trying to persuade other of their opinions." Eliciting opinions, updating after observing data, and quantifying uncertainty using probability distributio tistics.

6 A third motivation for studying Bayesian statistics is that students might better understand classical procedures and est hem in comparison to Bayesian methods.

In this article

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present the Pareto distribution as a conjugate family of prior distributions. One can adopt a simpler form for the prior distribution by considering improper priors which do not integrate to one but still perform the same function as a proper prior distribution. For instance, if one chooses the flat improper prior distribution of the form π(θ) = 1 for θ>0, the posterior distribution is proportional to the likelihood function, $\pi(\theta|\boldsymbol{x}) \propto 1/($ … erior distribution is proper provided that $n > 1$, with the constant of proportionality turning out to be $(n-1)\cdot(\max\{x_i\})^{n-1}$. As … tion, the Bayes estimator equals the posterior mean

$$\mathrm{E}[\theta|\boldsymbol{x}] = \int_{-\infty}^{\infty} \theta\cdot\pi(\theta|\boldsymbol{x})d\theta = \frac{\int_{\max\{x_i\}}^{\infty}\theta^{-n+1}d\theta}{\int_{\max\{x_i\}}^{\infty}\theta^{-n}d\theta} = \frac{n-1}{n-2}\cdot\max\{x_i\},$$

which exists when $n > 2$. This Bayesian analysis produces yet another estimator which equals a constant times the sampl… ant has the form $(n + m) / (n + m - 1)$ for some m and approaches 1 as $n \to \infty$.

8 In fact, one can derive all estimators of this form from a Bayesian perspective. Consider the family of prior distributions … for θ>0. These distributions are improper for any real k. The resulting posterior distribution is $\pi(\theta|\boldsymbol{x}) \propto 1/\theta^{k+n}$ for $\theta \geq \max$ … $+ n > 1$ with the constant of proportionality equaling $(k+n-1)\cdot(\max\{x_i\})^{k+n-1}$. The posterior mean exists when $k + n > 2$, producing a Bayes estimator of

$$\mathrm{E}[\theta|\boldsymbol{x}] = \frac{\int_{\max\{x_i\}}^{\infty}\theta^{-k-n+1}d\theta}{\int_{\max\{x_i\}}^{\infty}\theta^{-k-n}d\theta} = \frac{k+n-1}{k+n-2}\cdot\max\{x_i\}.$$

Notice that this estimator corresponds to the minimum variance unbiased estimator when $k = 2$ and to the minimum me… en $k = 3$. Choosing $k = 1$ yields the estimator $(n/(n-1))\cdot\max\{X_i\}$, which seems to be missing in the sequence of estimators ab… θ that emerge from various classical criteria of estimation can be seen as members of a sequence of Bayes estimators based or … distributions.

9 Positive values of k can be interpreted to represent k unobserved uniform random variables on the interval (0, θ). Large… weight on smaller values of θ and therefore produce lower posterior estimates.

10 One can also compare classical and Bayesian interval estimators of the parameter θ. The classical $100(1-\alpha)\%$ confidence interval for θ is $(\max\{X_i\}, \alpha^{-1/n}\cdot\max\{X_i\})$ since $\Pr(\max\{X_i\} < \theta < \alpha^{-1/n}\cdot\max\{X_i\}) = 1-\alpha$. From the Bayesian perspective, a $100(1$ … nsity …g the family of improper prior distributions described above, turns out to be $(\max\{X_i\}, \alpha^{-1/(k+n-}$ … $^{k+n-1)}\cdot\max\{x_i\}|\boldsymbol{x}) = 1-\alpha$. The classical and Bayesian interval estimators are therefore the sa… when $k = 1$.

11 The choice of $k = 1$ comes highly recommended from the Bayesian literature because it corresponds to the Jeffreys' prior, which is in this case a standard noninformative prior distribution for a scale parameter. The Jeffreys' prior is noninformative because it is invariant to parameter transformations. For example, θ may be transformed to obtain standard deviation σ or variance τ = σ^2. The prior $\pi(\theta) \propto \theta^{-1}$ is equivalent to priors $\pi(\sigma) \propto \sigma^{-1}$ or $\pi(\tau) \propto \tau^{-1}$ on the standard deviation or scale parameters, respectively. Furthermore, $\pi(\theta) \propto \theta^{-1}$ is noninformative on the ratio scale -- t es that all intervals of the form $x < \theta < cx$ are equally likely for any choice of *x*. See, for example, Box and Tiao (1973) for mo s' priors.

12 Larger values of *k* in the prior distribution represent increased prior certainty about the value of the parameter, and th ior HPD intervals.

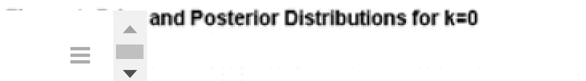
3. Example

13 As an example suppose that *n* = 12 and that the observed data are:

$$x = (2.6, 2.8, 3.6, 4.3, 5.5, 10.3, 12.2, 20.2, 21.8, 28.7, 30.6, 32.2).$$

Starting with a flat improper prior distribution for θ corresponding to *k* = 0 produces the posterior distribution $\pi(\theta|\boldsymbol{x}) \propto$ displayed in Figure 1. Note that the height of the improper prior distribution displayed in Figure 1 is arbitrary. The Bayes e = 35.42, and a 95% posterior HPD interval for θ is (32.2, $(.05)^{-1/11} \cdot 32.2$) = (32.2, 42.28). For the sake of comparison, Table 1 erval estimates of θ for other values of *k* and points out their classical counterparts. Figure 2 graphs Bayes estimates and HPD continuous functions of *k*, and also indicates values that correspond to estimates based on classical criteria.

Figure 1. Prior and Posterior Distributions for *k* = 0.



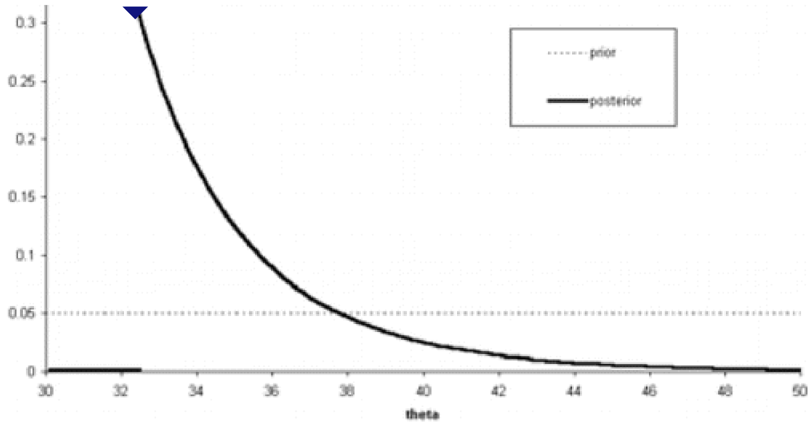


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Table 1. Bayes Estimates for Various Values of *k*

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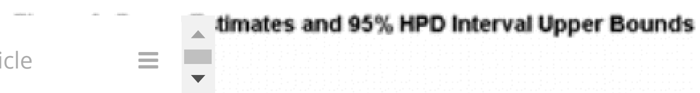
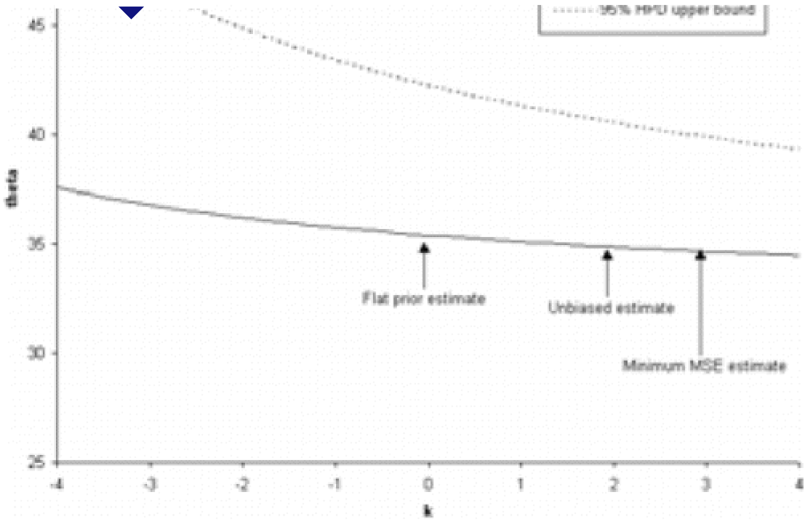
Figure 2. Bayes Estimates and 95% HPD Interval Upper Bounds.





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4. Conclusion

14 We have demonstrated that a Bayesian framework unites the various classical estimators produced by different estim ter of a continuous uniform distribution. The Bayes estimators arise from a family of improper prior distributions and highlight b es of Bayesian and classical analyses.

15 We believe that this comparison can help students of mathematical statistics both to gain valuable experience with Ba understand classical estimation criteria more fully.

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References

1. Albert, J. (1997), "Teaching Bayes' Rule: A Data-Oriented Approach," *The American Statistician*, 51, 247–253. [Taylor & Fr e ®], [Google Scholar]

2. Berry, D. A. (1997), "Teaching Elementary Bayesian Statistics with Real Applications in Science," *The American Statisticia* [Taylor & Francis Online], [Web of Science ®], [Google Scholar]

3. Box, G. E. P., and Tiao, G. C. (1973), *Bayesian Inference in Statistical Analysis*, New York: John Wiley and Sons, Inc. [Goog

4. DeGroot, M. H. (1970), *Optimal Statistical Decisions*, New York: McGraw-Hill, Inc. [Google Scholar]

5. DeGroot, M. H. (1986), *Probability and Statistics* (2nd ed.), Reading, MA: Addison-Wesley Publishing Company. [Google Scholar]

6. Freund, J. E. (1992), *Mathematical Statistics* (5th ed.), Englewood Cliffs, NJ: Prentice Hall. [Google Scholar]

7. Hogg, R. V., and Craig, A. T. (1978), *Introduction to Mathematical Statistics* (4th ed.), New York: Macmillan Publishing Co.

8. Larsen, R. J., and Marx, M. L. (1986), *An Introduction to Mathematical Statistics and Its Applications* (2nd ed.), Englewoo [Google Scholar]

9. Lee, P. M. (1989), *Bayesian Statistics: An Introduction*, New York: Oxford University Press. [Google Scholar]

10. Moore, D. S. (1997), "Bayes for Beginners? Some Reasons to Hesitate," *The American Statistician*, 51, 247–253. [Web of Science ®], [Google Scholar]

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