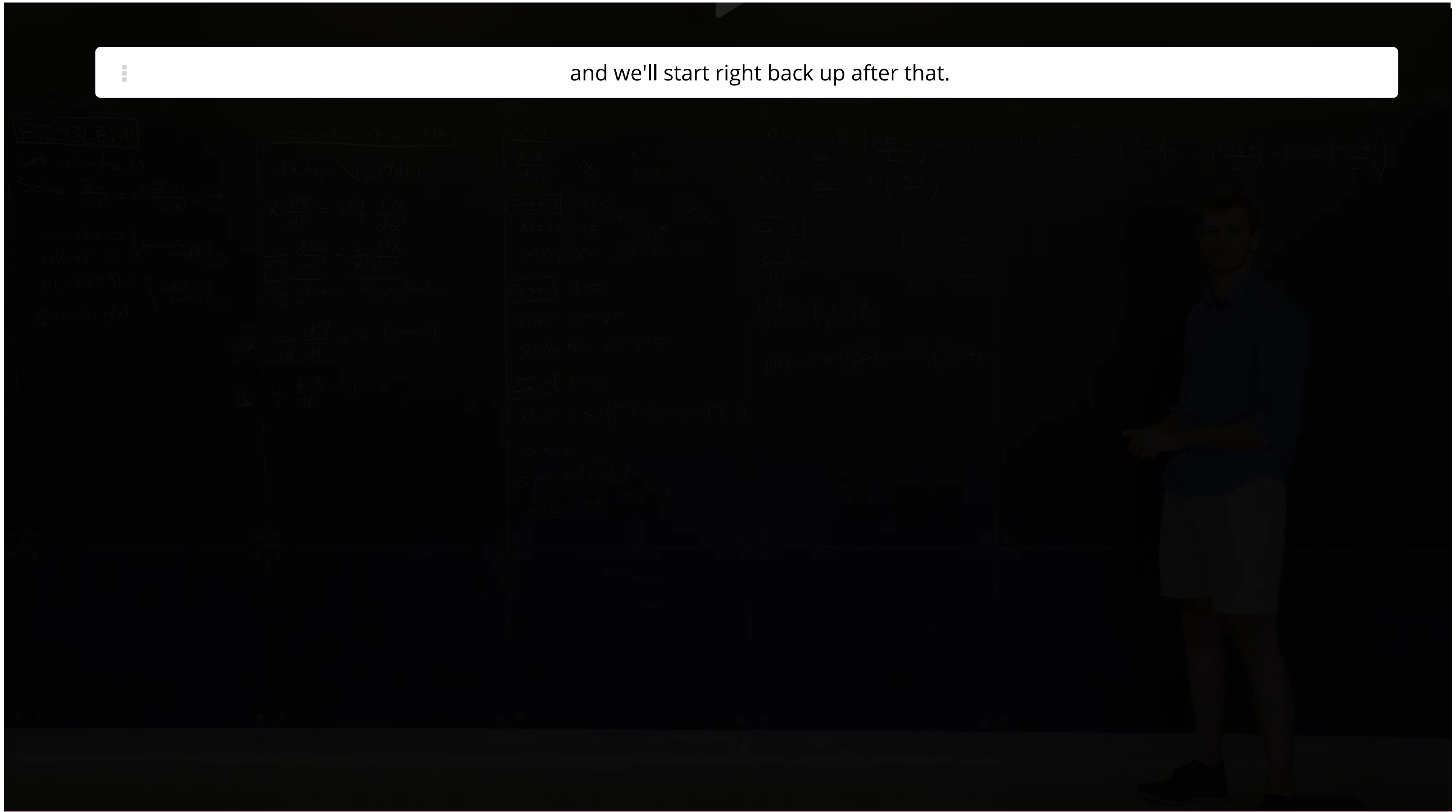
4. Separation of variables in PDEs

Separation of variables for the wave equation



16:08 / 16:08 1.50x

Video

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Video errors note: Note that at 6:25 the equation which reads $X(x) = c_1 e^{\sqrt{\lambda}t} + c_2 e^{-\sqrt{\lambda}t}$ should be $X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$ (a function of x not of t), and at 13:11, $\lambda^2 = -\frac{n^2\pi^2}{L^2}$ should be $\lambda = -\frac{n^2\pi^2}{L^2}$.

For simplicity, suppose that $c = 1$ and $L = \pi$. So now we are solving the PDE with boundary conditions

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi,\ t > 0 \\ u(0,t) &= 0 \\ u(\pi,t) &= 0. \end{aligned}$$

As with the heat equation, we try separation of variables. In other words, try to find normal modes of the form

$$u(x,t) = v(x)\,w(t),$$

for some nonzero functions $v(x)$ and $w(t)$. Substituting this into the PDE gives

$$\begin{aligned} v(x)\,\ddot{w}(t) &= v''(x)\,w(t) \\ \frac{\ddot{w}(t)}{w(t)} &= \frac{v''(x)}{v(x)}. \end{aligned}$$

As usual, a function of t can equal a function of x only if both are equal to the same constant, say λ, so this breaks into two ODEs:

$$\ddot{w}(t) = \lambda\,w(t), \qquad v''(x) = \lambda\,v(x).$$

Moreover, the boundary conditions become $v(0) = 0$ and $v(\pi) = 0$.

We already solved the eigenfunction equation $v''(x) = \lambda\,v(x)$ with the boundary conditions $v(0) = 0$ and $v(\pi) = 0$: nonzero solutions exist only when $\lambda = -n^2$ for some positive integer n, and in this case $v = \sin nx$ (times a scalar). What is different this time is that w satisfies a **second** -order ODE

$$\ddot{w}(t) = -n^2\, w(t)\,.$$

The characteristic polynomial is $r^2 + n^2$, which has roots $\pm in$, so

$$w(t) = \cos nt \quad \text{and} \quad w(t) = \sin nt$$

are possibilities (and all the others are linear combinations). Multiplying each by the $v(x)$ with the matching λ gives the normal modes

$$\cos(nt)\sin(nx)\,, \qquad\qquad \sin(nt)\sin(nx)\,.$$

Any linear combination

$$u(x,t) = \sum_{n\geq 1} a_n \cos(nt)\sin(nx) + \sum_{n\geq 1} b_n \sin(nt)\sin(nx)$$

is a solution to the PDE with boundary conditions, and this turns out to be the general solution.

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