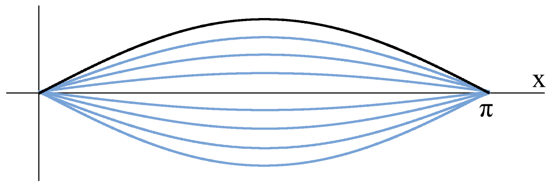
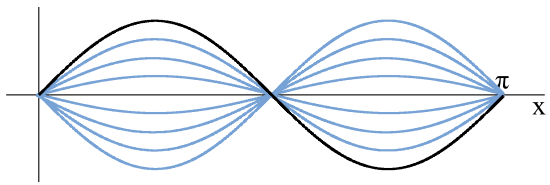
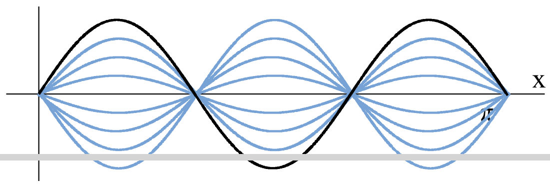
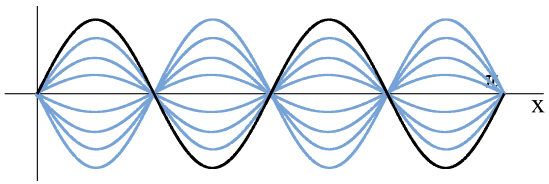
6. Normal modes: standing waves

In the previous example of transverse waves in a string with fixed ends, the eigenfunctions $c_n v_n(x)$ correspond to standing waves. These standing waves are the normal modes for this system, and are depicted below.

n	**Eigenfunction**	**Plot of standing wave**
1	$\sin(x)$ on $0 < x < \pi$	
2	$\sin(2x)$ on $0 < x < \pi$	
3	$\sin(3x)$ on $0 < x < \pi$	

4 $\sin(4x)$ on $0 < x < \pi$



Watch the demo of standing transverse waves in a different physical setup: the ends of the string are free. In this case the same partial differential equation models the situation but the boundary condition is determined by there being no tension force at the end of the string, and the boundary conditions reduce to $\dfrac{\partial u}{\partial x}(0,t) = 0$ and $\dfrac{\partial u}{\partial x}(L,t) = 0$.

Observe how the normal modes with free ends differ from the case of fixed ends.

The following video was created by TSG@MIT Physics.

Standing waves with free ends

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \pi \\ \frac{\partial u}{\partial x}(0,t) &= 0 \\ \frac{\partial u}{\partial x}(\pi,t) &= 0. \end{aligned}$$

Separation of variables leads to

$$\ddot{w}(t) = \lambda w(t), \qquad v''(x) = \lambda v(x).$$

The boundary conditions become $v'(0) = 0$ and $v'(\pi) = 0$.

Find λ_n and $v_n(x)$. Observe that the normal modes exactly correspond to the standing waves in the demo video.

For $n = 0, 1, 2, \ldots,\ \ \lambda_n =$ -n^2 ✔ **Answer:** -n^2

$-n^2$

For $n = 0, 1, 2, \ldots,\ \ v_n(x) =$ cos(n*x) ✔ **Answer:** cos(n*x)

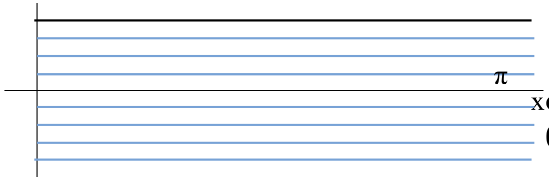
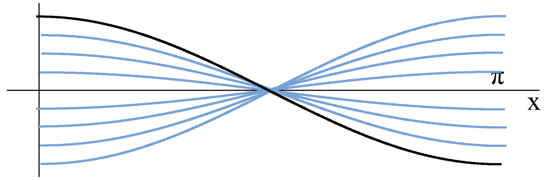
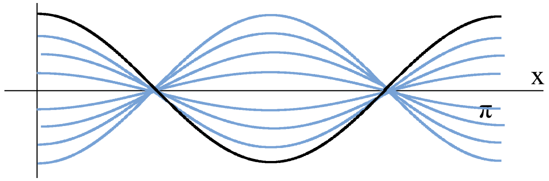
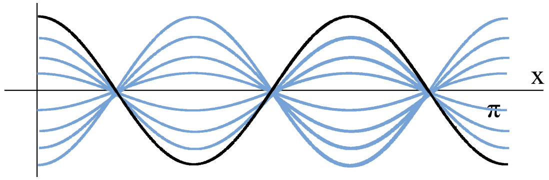
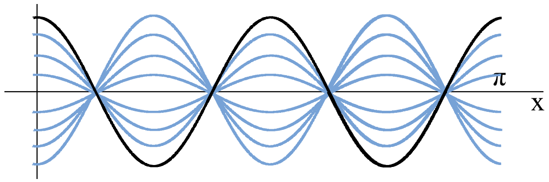
$\cos(n \cdot x)$

FORMULA INPUT HELP

Solution:

We already solved the eigenfunction equation $v''(x) = \lambda v(x)$ with the boundary conditions $v'(0) = 0$ and $v'(\pi) = 0$: nonzero solutions exist only when $\lambda = -n^2$ for some nonnegative integer n, and in this case $v = \cos nx$ (times a scalar).

Note that as in the previous example, $w_n(t) = a_n \cos(nt) + b_n \sin(nt)$. The standing waves are given by the eigenfunctions below.

n	**Eigenfunction**	**Plot of standing wave**
0	1 on $0 < x < \pi$	
1	$\cos(x)$ on $0 < x < \pi$	
2	$\cos(2x)$ on $0 < x < \pi$	
3	$\cos(3x)$ on $0 < x < \pi$	
4	$\cos(4x)$ on $0 < x < \pi$	

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Bell Labs Wave Machine: Standing Waves

(Caption will be displayed when you start playing the video.)

1:16 / 1:16 2.0x HD

Find the normal modes in the case of free ends

2/2 points (graded)

For simplicity, suppose that $c = 1$ and $L = \pi$. So now we are solving the PDE with boundary conditions