15. Worked example

Example 15.1 Balance the combustion reaction that turns octane and oxygen into carbon dioxide and water

$$a\mathrm{C}_8\mathrm{H}_{18} + b\mathrm{O}_2 \longrightarrow c\mathrm{CO}_2 + d\mathrm{H}_2\mathrm{O}.$$

That is, find the smallest positive integers a, b, c, and d that make the number of each atom on both sides of the reaction equal.

Worked solution

As we've seen before, we can use the constraint that the number of atoms on each side of the reaction must be equal to write a system of linear equations in a, b, c, and d. The system can be written in the form $\mathbf{Ax} = \mathbf{0}$. So this problem is equivalent to finding the vector with the smallest integer coefficients in the **nullspace** of the matrix

$$\mathbf{A} = \begin{pmatrix} 8 & 0 & -1 & 0 \\ 18 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{pmatrix}$$

where the first row is an equation for C, second row for H, and the last for O.

We can use Gauss-Jordan Elimination to put $\mathbf{A}$ into reduced row echelon form:

$$S = \begin{pmatrix} 8 & 0 & -1 & 0 \\ 18 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 18 & 0 & 0 & -2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 1 & -1 & -1/2 \\ 18 & 0 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 9/4 & -2 \end{pmatrix}$$

We see there are 3 pivot columns and one free column, therefore the dimension of the solution space is 1. We can either use back substitution or go all the way to reduced echelon form. We'll do the latter.

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & -8/9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 1 & 0 & -25/18 \\ 0 & 0 & 1 & -8/9 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -25/18 \\ 0 & 0 & 1 & -8/9 \end{pmatrix} = \mathbf{rref}(\mathbf{A}).$$

Normally we set the free variable d equal to some parameter c_1. Because we already have c in this problem, instead we set $d = t$ a parameter. Now we can read off the other variables in terms of t:

$$\begin{aligned} a &= \frac{1}{9}t \\ b &= \frac{25}{18}t \\ c &= \frac{8}{9}t. \end{aligned}$$

Recall that we are looking for the smallest integer solutions. In this case, we cannot set $t = 1$. Instead, we need to choose a value of t which will clear the denominators, and check that the remaining numbers have no common factors. In this case, choosing $t = 18$ works to clear the denominators and we are left with

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = t \begin{pmatrix} 2 \\ 25 \\ 16 \\ 18 \end{pmatrix}.$$

These integers have no common factors (seen quickly by noting that the first number is 2, and the second number is odd), so they are the smallest possible integer solutions to this linear system.

In particular, the nullspace of $\mathbf{A}$ is

$$\mathrm{NS}\left(\mathbf{A}\right) = \mathrm{Span}\begin{pmatrix} 2 \\ 25 \\ 16 \\ 18 \end{pmatrix},$$

and the balanced reaction is

$$2\mathrm{C}_8\mathrm{H}_{18} + 25\mathrm{O}_2 \longrightarrow 16\mathrm{CO}_2 + 18\mathrm{H}_2\mathrm{O}.$$

Now we have a systematic way of solving these problems!

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This worked example reminds me of this quote: > *The ultimate goal of mathematics is to eliminate ... 2

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