MATHEMATICS

What is the maximum number of warriors one can put on a chess board so that no two warriors attack each other?

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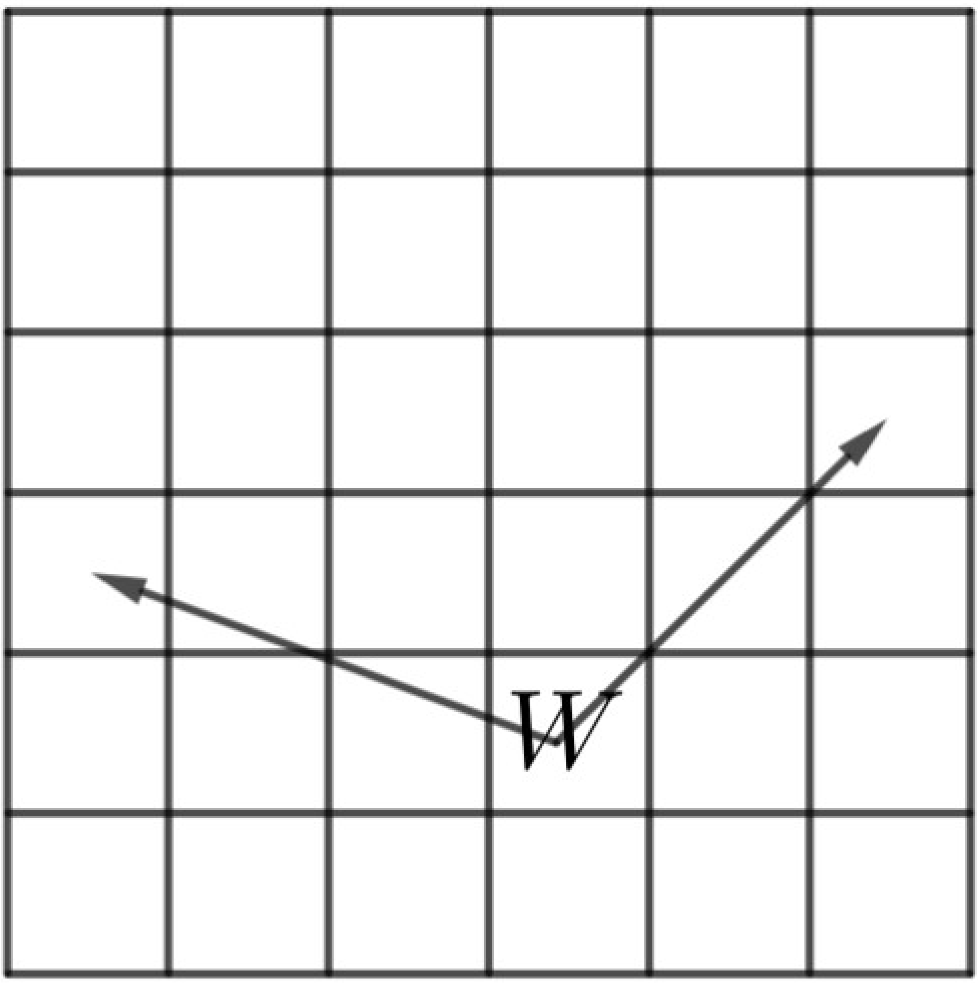
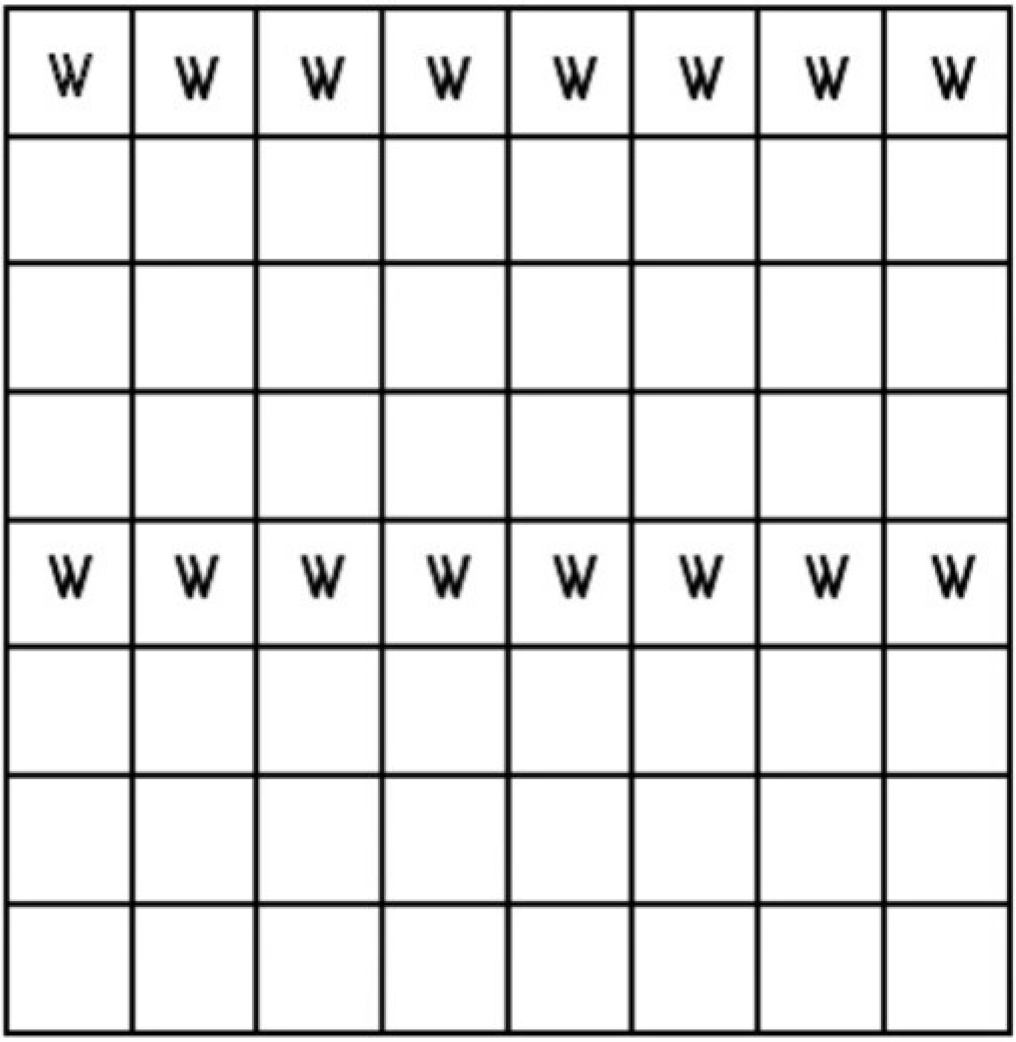
> In chess, a normal knight goes two steps forward and one step to the side, in some orientation. Thanic thought that he should spice the game up a bit, so he introduced a new kind of piece called *a warrior*. A *warrior* can either go three steps forward and one step to the side, or two steps forward and two steps to the side in some orientation.
> Given a 2020×2020 chess board. Find, with proof, the maximum number of warriors one can put on its cells such that no two warriors attack each other.

The question is a modified version of a problem from Bangladesh Mathematical Olympiad 2019. For more clarity, here is a picture that shows example moves of a *warrior*:



This is my first time solving this kind of problem. I've made the following progress in solving the question:

We place the warriors in each cell of n-th column where $n \equiv 1 \pmod 4$. The following picture shows this strategy in an 8×8 board:



It can be seen that no two *warriors* can attack each other. Hence, the answer to our original problem should be 2020×505.

Though this result matches with the original answer, I have still some confusions. Firstly, the optimal strategy is that in the 2020×2020 board, we place a warrior in each cell of n-th column. But what if we don't place them with that strategy or we just randomly place the *warriors* so that they cannot attack each other? How will I know other strategies would not give a result greater than 2020×505? More specifically, how do I write a formal proof for this kind of problems?

combinatorics optimization proof-writing contest-math chessboard

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edited 2 hours ago

asked 6 hours ago

Unknown
3,359 1 6 30

1 Regarding your confusion, just because you have the numerical value, doesn't mean that your proof is correct. Like you realized, for a complete solution, you also need to show that "no other strategies will give a larger number". Similar to when we want to show something is a maximum, we need to show A) It can be achieved (which you have done) and B) No larger value can be achieved. – Calvin Lin 3 hours ago

1 @CalvinLin Yes, my question is regarding B. – Unknown 3 hours ago

1 Note that the actual BDMO question doesn't ask to find the exact maximum, but asks to show that the maximum is $\leq \frac{2}{5} \times 2020 \times 2020$ Source - BdMO 2019 Higher Secondary Qn 10. This greatly changes how one can approach the question. – Calvin Lin 3 hours ago

In your post, you stated "This question is from Bangladesh Mathematical Olympiad 2020.", but not that you modified it (so strictly speaking, it isn't from there). While I haven't seen the solution, I don't think this is "a bit modified". I suspect that to find the actual maximum, it will involve a lot more machinery. Please go to the actual source when you can. Even your link states the actual problem in the second post. – Calvin Lin 3 hours ago

@CalvinLin The question is from BDMO 2019 (That was a mistake, I edited the source). And yes, I am aware of the actual problem. But as I said, I am rather interested in finding the exact maximum. – Unknown 2 hours ago

2 Answers

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$505 \times 2020 = 1{,}020{,}100$ is certainly not optimal. By tiling a 2019×2020 board with a rectangular pattern of the following 3×5 rectangle, you can fit

$$4 \times \frac{2019}{3} \times \frac{2020}{5} = 1{,}087{,}568$$

warriors onto a 2019×2020 board alone.

		W		
	W		W	
		W		

You strategy achieves a density of $1/4$ warriors/square, while mine has a density of $4/15$, so the latter should always be better for large enough boards. I do not know if this can be improved at all.

Let D be the optimal packing density for warriors. In addition to the lower bound of $D \geq 4/15$, I can prove the upper bound $D \leq 1/3$.

For each warrior, imagine placing a token on the 12 squares that the warrior can attack. Some squares will have multiple tokens. However, you can show that every square will have at most 6 tokens. Indeed, for any unoccupied square X, if we partition the 12 squares that can attack X into 6 attacking pairs as shown in this table, (pairs are labeled A through F), then we see that X can be attacked from at most one square in each pair.

		C		D		
	B				C	
A						D
			X			
B						E
	A				F	
		F		E		

This means that each warrior effectively occupies $1 + 12 \times \frac{1}{6} = 3$ squares, so you can have no more than $1/3$ warriors per square.

This is only a "long-run" result, since warriors at the boundary of a grid will place fewer than 12 tokens. However, this effect is negligible in the long run.

Share Cite Edit Follow Flag edited 34 mins ago answered 2 hours ago

Mike Earnest
49.2k 5 44 91

With your idea, we can show that the inner 2014×2014 grid has at most $1/3 \times 2020^2$ warriors. Adding on at most $2020^2 - 2014^2$ warriors on the remaining border, that's still less than $2/5 \times 2020^2$. – Calvin Lin 44 mins ago

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If we can find 3 squares that attack each other, then we can place at most 1 warrior on these 3 squares. This is possible, like with:
$(a, b), (a + 1, b + 3), (a + 3, b + 1)$ and $(a + 5, b), (a + 4, b + 3), (a + 2, b + 1)$.

Now, with $a = 6k + 1, k \in [1, 336]$ and $b \in [1, 2017]$, we can cover $3 \times 2 \times 336 \times 2017$ of the 2020^2 squares. (Verify that these squares are distinct, and lie in our grid.)

These squares contain at most $2 \times 336 \times 2017$ warriors.
Addining on the remaining $2020^2 - 3 \times 2 \times 336 \times 2017$ squares, we get at most 1369552 squares for warriors to be on. This gives us a density of $33.6\% < 40\%$.

Notes

- I originally was caught up looking at 5-cycles (because of the ratio $\frac{2}{5}$), till I saw Mike's density bound of $\frac{1}{3}$. This led me to focus on 3-cycles, hence the above solution.
- We just need to make up for the boundary cases (which is minimal in a 2020 grid), of which there are several approaches.
- The upper bound on density is $\frac{1}{3}$, which is easily obtained from the above approach of finding 3 cycles (in an densely packed manner) and accounting for the number of leftover cells (~3 in each row and 1-3 empty columns -> hence density of 0).

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edited 21 mins ago

answered 33 mins ago

Calvin Lin

58.3k 5 69 145