5. Asymptotic Normality of M-estimators

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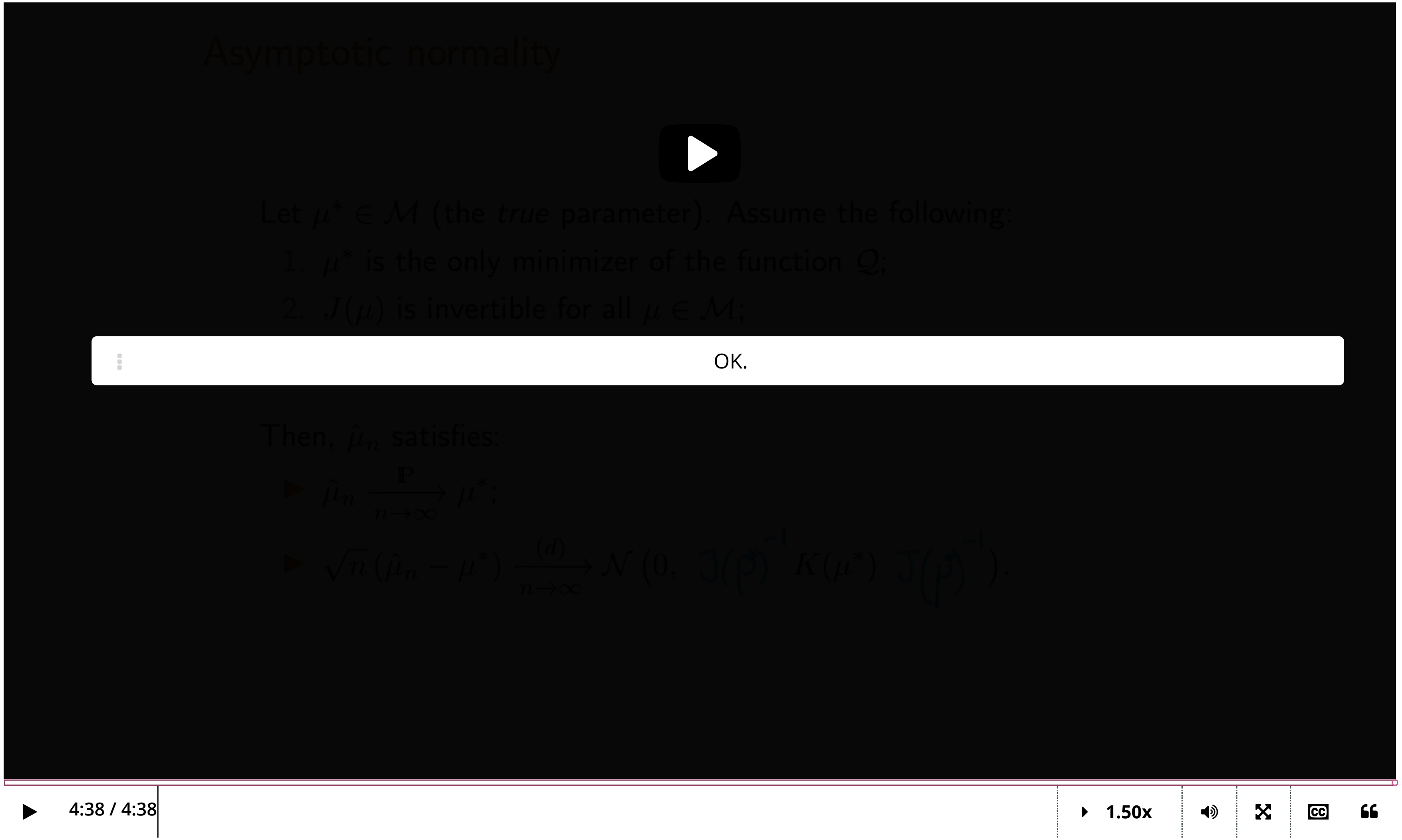

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Asymptotic normality of the M-estimators

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Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathbf{P}$. Let $\rho(x, \mu)$ denote a loss function satisfying

$$\mu^* = \mathrm{argmin}_{\mu \in \mathbb{R}} \mathbb{E}\left[\rho(X_1, \mu)\right]$$

where $\mu^* \in \mathbb{R}$ is some unknown one-dimensional parameter associated with $\mathbf{P}$ that we would like to estimate. Let

$$\begin{aligned} J(\mu) &= \mathbb{E}\left[\frac{\partial^2 \rho}{\partial \mu^2}(X_1, \mu)\right] \\ K(\mu) &= \mathrm{Var}\left[\frac{\partial \rho}{\partial \mu}(X_1, \mu)\right] \end{aligned}$$

You construct the M-estimator $\widehat{\mu}_n$ associated ρ.

Assuming that the conditions for the asymptotic normality of this M-estimator hold, we have

$$\sqrt{n}\frac{\widehat{\mu}_n - \mu^*}{\sqrt{J(\mu^*)^{-2} K(\mu^*)}} \xrightarrow[n \to \infty]{(d)} Q$$

for some distribution Q.

What is Q?

- Poisson with mean 1.
- Exponential with mean 1.
- Standard normal.

- $\mathcal{N}(0, \sigma^2)$ for some unknown parameter σ^2.

✔

Let q_α denote the α-quantile of the distribution Q. For what value of q_α is it true that

$$\mu^* \in \left[\widehat{\mu}_n - q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}}, \widehat{\mu}_n + q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}}\right]$$

with probability 95% as $n \to \infty$?

$q_\alpha =$ 1.96 ✔ **Answer:** 1.96

Let

$$\mathcal{I} := \left[\widehat{\mu}_n - q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}}, \widehat{\mu}_n + q_\alpha \sqrt{\frac{J(\mu^*)^{-2} K(\mu^*)}{n}}\right]$$

denote the interval in the previous question.

Is $\mathcal{I}$ an asymptotic confidence interval for μ^* of confidence level 95%?

- Yes, because the previous question solves for q_α so that this holds.
- Yes, because of the asymptotic normality of $\widehat{\mu}_n$.
- No, because we did not define a statistical model for this problem.

No, because the endpoints of $\mathcal{I}$ depend on the true parameter.

Solution:

For the first question, the correct response is "Standard normal." Referring to the theorem regarding the asymptotic normality of the M-estimators, we see that the asymptotic variance of $\widehat{\mu_n}$ is $J(\mu^*)^{-2}K(\mu^*)$. Hence,

$$\sqrt{n}\frac{\widehat{\mu_n}-\mu^*}{\sqrt{J(\mu^*)^{-2}K(\mu^*)}}\xrightarrow[(d)]{n\to\infty}\mathcal{N}(0,1).$$

For the second question, the correct response is "1.96". By the previous equation,

$$P\left(\sqrt{n}\left|\frac{\widehat{\mu}_n-\mu^*}{\sigma}\right|\geq q_{0.025}\right)=P\left(\mu^*\in\left[\widehat{\mu}_n-q_{0.025}\sqrt{\frac{J(\mu^*)^{-2}K(\mu^*)}{n}},\widehat{\mu}_n+q_{0.025}\sqrt{\frac{J(\mu^*)^{-2}K(\mu^*)}{n}}\right]\right)=0.05$$

where $q_{0.025}=1.96$ is the 2.5%-quantile of a standard Gaussian.

For the third question, the correct response is "No, because the endpoints of $\mathcal{I}$ depend on the true parameter." By definition, the endpoints of a confidence interval should be estimators, and this is not the case for $\mathcal{I}$ because $K^{-1}(\mu^*)$ and $J(\mu^*)$ depend on the true parameter.

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Answers are displayed within the problem

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