12.3.4 Properties

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Week 12 due Dec 29, 2023 10:42 IST Completed

12.3.4 Properties

Video 12.3.4 Part 1

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Dr. Robert van de Geijn: In this unit, we're

going to look at some properties of eigenvalues

and eigenvectors of general matrices.

So here's one.

If a matrix A can be partitioned into quadrants

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Reading Assignment

0 points possible (ungraded)

Read Unit 12.3.4 of the notes. [LINK]

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HW 12.3.4.5: Alternative Proof 1

Homework 12.3.4.1

Calculator

1/1 point (graded)

Let $A \in \mathbb{R}^{n \times n}$ and $A = \begin{pmatrix} A_{0,0} & A_{0,1} \\ 0 & A_{1,1} \end{pmatrix}$, where $A_{0,0}$ and $A_{1,1}$ are square matrices.

$\Lambda(A) = \Lambda(A_{0,0}) \cup \Lambda(A_{1,1})$.

Always ✔ Answer: Always

We will show that $\Lambda(A) \subset \Lambda(A_{0,0}) \cup \Lambda(A_{1,1})$ and $\Lambda(A_{0,0}) \cup \Lambda(A_{1,1}) \subset \Lambda(A)$.

$\Lambda(A) \subset \Lambda(A_{0,0}) \cup \Lambda(A_{1,1})$: Let $\lambda \in \Lambda(A)$. Then there exists $x \neq 0$ such that $Ax = \lambda x$. Partition $x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$. Then $\begin{pmatrix} A_{0,0} & A_{0,1} \\ 0 & A_{1,1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$ which implies that $\begin{pmatrix} A_{0,0}x_0 + A_{0,1}x_1 \\ A_{1,1}x_1 \end{pmatrix} = \begin{pmatrix} \lambda x_0 \\ \lambda x_1 \end{pmatrix}$. Now, either $x_1 \neq 0$ (the zero vector), in which case $A_{1,1}x_1 = \lambda x_1$ and hence $\lambda \in \Lambda(A_{1,1})$, or $x_1 = 0$, in which case $A_{0,0}x_0 = \lambda x_0$ and hence $\lambda \in \Lambda(A_{0,0})$ since x_0 and x_1 cannot both equal zero vectors. Hence $\lambda \in \Lambda(A_{0,0})$ or $\lambda \in \Lambda(A_{1,1})$, which means that $\lambda \in \Lambda(A_{0,0}) \cup \Lambda(A_{1,1})$.

$\Lambda(A_{0,0}) \cup \Lambda(A_{1,1}) \subset \Lambda(A)$: Let $\lambda \in \Lambda(A_{0,0}) \cup \Lambda(A_{1,1})$.

Case 1: $\lambda \in \Lambda(A_{0,0})$. Then there exists $x_0 \neq 0$ s.t. that $A_{0,0}x_0 = \lambda x_0$. Observe that

$$\begin{pmatrix} A_{0,0} & A_{0,1} \\ 0 & A_{1,1} \end{pmatrix} \underbrace{\begin{pmatrix} x_0 \\ 0 \end{pmatrix}}_{x} = \begin{pmatrix} \lambda x_0 \\ 0 \end{pmatrix} = \lambda \underbrace{\begin{pmatrix} x_0 \\ 0 \end{pmatrix}}_{x}.$$

Hence we have constructed a nonzero vector x such that $Ax = \lambda x$ and therefore $\lambda \in \Lambda(A)$.

Case 2: $\lambda \notin \Lambda(A_{0,0})$. Then there exists $x_1 \neq 0$ s.t. that $A_{1,1}x_1 = \lambda x_1$ (since $\lambda \in \Lambda(A_{1,1})$) *and* $A_{0,0} - \lambda I$ is nonsingular (and hence its inverse exists). Observe that

$$\underbrace{\begin{pmatrix} A_{0,0} - \lambda I & A_{0,1} \\ 0 & A_{1,1} - \lambda I \end{pmatrix}}_{A - \lambda I} \underbrace{\begin{pmatrix} -(A_{0,0} - \lambda I)^{-1} A_{0,1} x_1 \\ x_1 \end{pmatrix}}_{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence we have constructed a nonzero vector x such that $(A - \lambda I)x = 0$ and therefore $\lambda \in \Lambda(A)$.

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Answers are displayed within the problem

Video 12.3.4 Part 2

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Dr. Robert van de Geijn: So the answer is that this is always the case .

Now how do you prove somethi

sets?

Calculator

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How do you prove the two sets are equal?

Well, we need to show that the eigenvalues of A form a subset

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Homework 12.3.4.2

1/1 point (graded)

Let $A \in \mathbb{R}^{n \times n}$ be symmetric, $\lambda_i \neq \lambda_j$, $Ax_i = \lambda_i x_i$ and $Ax_j = \lambda_j x_j$.

$x_i^T x_j = 0$

Always ✔ Answer: Always

$$\left\{\begin{array}{rcl} Ax_i & = & \lambda_i x_i \\ Ax_j & = & \lambda_j x_j \end{array}\right.$$

$$\text{implies} \quad < \text{Multiplying both sides by transpose of same vector maintains equivalence} >$$

$$\left\{\begin{array}{rcl} x_j^T A x_i & = & x_j^T(\lambda_i x_i) \\ x_i^T A x_j & = & x_i^T(\lambda_j x_j) \end{array}\right.$$

$$\text{implies} \quad < \text{Move scalar to front} >$$

$$\left\{\begin{array}{rcl} x_j^T A x_i & = & \lambda_i x_j^T x_i \\ x_i^T A x_j & = & \lambda_j x_i^T x_j \end{array}\right.$$

$$\text{implies} \quad < \text{Transposing both sides maintains equivalence} >$$

$$\left\{\begin{array}{rcl} (x_j^T A x_i)^T & = & (\lambda_i x_j^T x_i)^T \\ x_i^T A x_j & = & \lambda_j x_i^T x_j \end{array}\right.$$

$$\text{implies} \quad < \text{Property of transposition of product} >$$

$$\left\{\begin{array}{rcl} x_i^T A^T x_j & = & \lambda_i x_i^T x_j \\ x_i^T A x_j & = & \lambda_j x_i^T x_j \end{array}\right.$$

$$\text{implies} \quad < A = A^T >$$

$$\begin{array}{rcl} x_i^T A^T x_j & = & \lambda_i x_i^T x_j \\ \| & & \\ x_i^T A x_j & = & \lambda_j x_i^T x_j \end{array}$$

$$\text{implies} \quad < \text{Transitivity of equivalence} >$$

$$\lambda_i x_i^T x_j = \lambda_j x_i^T x_j$$

$$\text{implies} \quad < \text{Since } \lambda_i \neq \lambda_j >$$

$$x_i^T x_j = 0$$

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Video 12.3.4 Part 3

Calculator

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Dr. Robert van de Geijn: So hopefully you got a chance to do that homework.

And the answer is, that it is the case that these two vectors

are orthogonal to each other.

And how do we prove that?

Well, somehow we need to come up

0:00 / 0:00 2.0x

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Homework 12.3.4.3

1/1 point (graded)

If $Ax = \lambda x$ then $AAx = \lambda^2 x$. (AA is often written as A^2.)

Always ✔ Answer: Always

$$AAx = A(\lambda x) = \lambda Ax = \lambda\lambda x = \lambda^2 x.$$

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Homework 12.3.4.4

1/1 point (graded)

Let $Ax = \lambda x$ and $k \geq 1$. Recall that $A^k = \underbrace{AA\cdots A}_{k\text{~ times}}$.

$A^k x = \lambda^k x$.

Always ✔ Answer: Always

Proof by induction.

Base case: $k = 1$.

$$A^k x = A^1 x = Ax = \lambda x = \lambda^2 x = \lambda^k x.$$

Inductive hypothesis: Assume that $A^k x = \lambda^k x$ for $k = K$ with $K \geq 1$.

Calculator

We will prove that $A^k x = \lambda^k x$ for $k = K + 1$.

$$
\begin{array}{ll}
A^k x & \\
\quad = & < k = K + 1 > \\
A^{K+1} x & \\
\quad = & < \text{Definition of } A^k > \\
(A A^K)\, x & \\
\quad = & < \text{Associativity of matrix multiplication} > \\
A\,(A^K x) & \\
\quad = & < \text{I.H.} > \\
A\,(\lambda^K x) & \\
\quad = & < Ax \text{ is a linear transformation} > \\
\lambda^K A x & \\
\quad = & < Ax = \lambda x > \\
\lambda^K \lambda x & \\
\quad = & < \text{Algebra} > \\
\lambda^{K+1} x & \\
\quad = & < k = K + 1 > \\
\lambda^k x &
\end{array}
$$

We conclude that $A^k x = \lambda^k x$ for $k = K + 1$.

By the Principle of Mathematical Induction the result holds for $k \geq 1$.

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12.3.4.5

1/1 point (graded)

$A \in \mathbb{R}^{n \times n}$ is nonsingular if and only if $0 \notin \Lambda\,(A)$.

TRUE ✔ Answer: TRUE

($\Rightarrow$) Assume A is nonsingular. Then $Ax = 0$ only if $x = 0$. But that means that there is no nonzero vector x such that $Ax = 0x$. Hence $0 \notin \Lambda\,(A)$.

($\Leftarrow$) Assume $0 \notin \Lambda\,(A)$. Then $Ax = 0$ must imply that $x = 0$ since otherwise $Ax = 0x$. Therefore A is nonsingular.

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