MITx: 6.041x Introduction to Probability - The Science of Uncertainty

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Problem 7: Sum of a random number of r.v.'s

(4/4 points)

A fair coin is flipped independently until the first Heads is observed. Let K be the number of Tails observed **before** the first Heads (note that K is a random variable). For $k = 0, 1, 2, \ldots, K$, let X_k be a continuous random variable that is uniform over the interval $[0, 3]$. The X_k's are independent of one another and of the coin flips. Let the random variable X be defined as the sum of all the X_k's generated before the first Heads. That is, $X = \sum_{k=0}^{K} X_k$. Find the mean and variance of X. You may use the fact that the mean and variance of a geometric random variable with parameter p are $1/p$ and $(1-p)/p^2$, respectively.

$\mathbf{E}[X] =$ 3 **Answer:** 3

$\mathbf{var}(X) =$ 6 **Answer:** 6

Answer:
Since X_k is uniform over $[0, 3]$, we have $\mathbf{E}[X_k] = 3/2$ and $\mathbf{var}(X_k) = 3^2/12 = 3/4$.

As the problem states, K is the number of Tails before the first Heads. Therefore, K can be interpreted as the number of failures until the first success, which is not quite a geometric random variable as we have defined it. However, if we add 1 to K, this accounts for the last trial (which is a success), and $K+1$ can then be interpreted as the number of trials until the first success. Hence, the random variable $N=K+1$ is geometrically distributed with parameter $p=1/2$. Thus, $\mathbf{E}[K+1]=\mathbf{E}[N]=2$ and $\textbf{var}(K+1)=\textbf{var}(N)=2$.

Since $X=\sum_{k=0}^{K} X_k$ is the sum of a random number of independent and identically distributed random variables, we have

$$\mathbf{E}[X]=\mathbf{E}[X_1]\mathbf{E}[K+1]=3,$$

and

$$\textbf{var}(X)=\textbf{var}(X_1)\mathbf{E}[K+1]+(\mathbf{E}[X_1])^2\textbf{var}(K+1)=\frac{3}{4}\cdot 2+\frac{9}{4}\cdot 2=6.$$

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