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Problem 4: Trajectory estimation

(5/5 points)

The vertical coordinate ("height") of an object in free fall is described by an equation of the form

$$x(t) = \theta_0 + \theta_1 t + \theta_2 t^2,$$

where θ_0, θ_1, and θ_2 are some parameters and t stands for time. At certain times $t_1, \dots, t_n$, we make noisy observations $Y_1, \dots, Y_n$, respectively, of the height of the object. Based on these observations, we would like to estimate the object's vertical trajectory.

We consider the special case where there is only one unknown parameter. We assume that θ_0 (the height of the object at time zero) is a known constant. We also assume that θ_2 (which is related to the acceleration of the object) is known. We view θ_1 as the realized value of a continuous random variable Θ_1. The observed height at time t_i is $Y_i = \theta_0 + \Theta_1 t_i + \theta_2 t_i^2 + W_i$, $i = 1, \dots, n$, where W_i models the observation noise. We assume that $\Theta_1 \sim N(0,1)$, $W_1, \dots, W_n \sim N(0, \sigma^2)$, and all these random variables are independent.

In this case, finding the MAP estimate of Θ_1 involves the minimization of

$$\theta_1^2 + \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \theta_0 - \theta_1 t_i - \theta_2 t_i^2)^2$$

with respect to θ_1.

1. Carry out this minimization and choose the correct formula for the MAP estimate, $\hat{\theta}_1$, from the options below.

- $\hat{\theta}_1 = \frac{\sum_{i=1}^{n} t_i (y_i - \theta_0 - \theta_2 t_i^2)}{\sigma^2}$

- (selected) $\hat{\theta}_1 = \frac{\sum_{i=1}^{n} t_i(y_i - \theta_0 - \theta_2 t_i^2)}{\sigma^2 + \sum_{i=1}^{n} t_i^2}$ ✔
- $\hat{\theta}_1 = \frac{\sum_{i=1}^{n} t_i(y_i - \theta_0 - \theta_2 t_i^2)}{\sigma^2 + \sum_{i=1}^{n} \theta_2 t_i^2}$
- none of the above

2. The formula for $\hat{\theta}_1$ can be used to define the MAP estimator, $\hat{\Theta}_1$ (a random variable), as a function of $t_1, \ldots, t_n$ and the random variables $Y_1, \ldots, Y_n$. Identify whether the following statement is true.

   The MAP estimator $\hat{\Theta}_1$ has a normal distribution.

   True ✔

3. Let $\sigma = 1$ and consider the special case of only two observations ($n = 2$). Write down a formula for the mean squared error $\mathbf{E}[(\hat{\Theta}_1 - \Theta_1)^2]$, as a function of t_1 and t_2. Enter 't1' for t_1 and 't2' for t_2.

   $\mathbf{E}[(\hat{\Theta}_1 - \Theta_1)^2] =$ 1/(1+(t1)^2+(t2)^2) ✔

4. Consider the "experimental design" problem of choosing when to make measurements. Under the assumptions of part (3), and under the constraints $0 \le t_1, t_2 \le 10$, find the values of t_1 and t_2 that minimize the mean squared error associated with the MAP estimator.

   $t_1 =$ 10 ✔

   $t_2 =$ 10 ✔

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