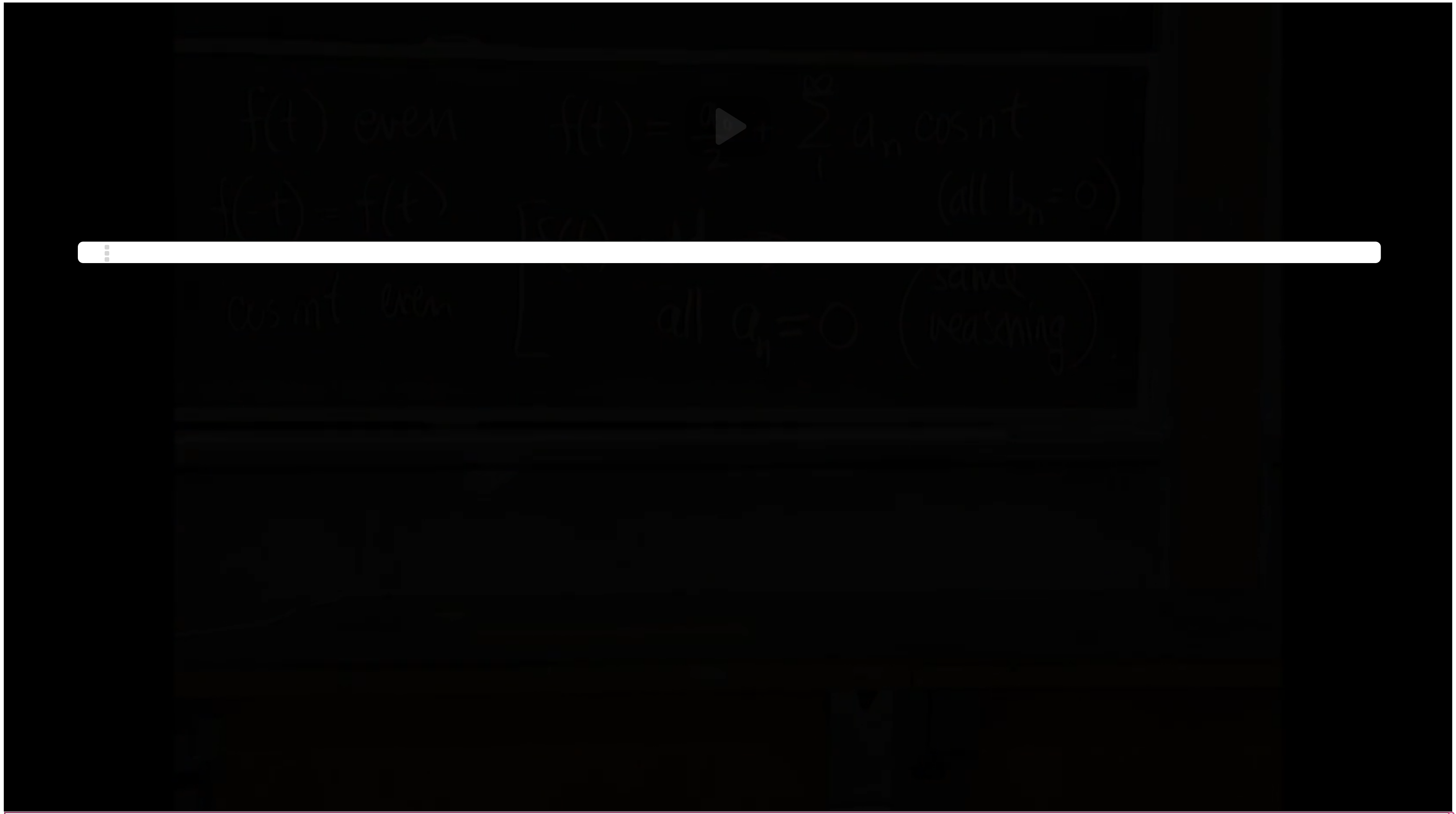
13. Even and odd periodic functions

Fourier series of even and odd periodic functions



6:00 / 6:00 1.50x

Video

Download video file

Transcripts

Download SubRip (.srt) file

Download Text (.txt) file

Even and odd symmetry

- A function $f(t)$ is **even** if $f(-t) = f(t)$ for all t.

- A function $f(t)$ is **odd** if $f(-t) = -f(t)$ for all t.

If

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt,$$

then substituting $-t$ for t gives

$$f(-t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} (-b_n) \sin nt.$$

The right hand sides match if and only if $b_n = 0$ for all n.

Conclusion: The Fourier series of an even function f has only cosine terms (including the constant term):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt.$$

Similarly, the Fourier series of an odd function f has only sine terms:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt.$$

Fourier series of Square wave using oddness

The square wave,

$$\mathrm{Sq}\,(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases},$$

is an odd function, so

$$\mathrm{Sq}\,(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

for some numbers b_n. The Fourier coefficient formula says

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\mathrm{Sq}\,(t) \sin nt \, dt}_{\text{even}} \\ &= \frac{2}{\pi} \int_0^{\pi} \mathrm{Sq}\,(t) \sin nt \, dt \qquad \text{(the two halves of the integral are equal, by symmetry)} \\ &= \frac{2}{\pi} \int_0^{\pi} \sin nt \, dt \qquad \text{(since } \mathrm{Sq}\,(t) = 1 \text{ whenever } 0 < t < \pi\text{)} \\ &= \left. \frac{2\,(-\cos nt)}{\pi n} \right|_0^{\pi} \\ &= \frac{2}{\pi n}(-\cos n\pi + \cos 0) \\ &= \begin{cases} \frac{4}{\pi n}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Thus

$$b_1 = \frac{4}{\pi}, \quad b_3 = \frac{4}{3\pi}, \quad b_5 = \frac{4}{5\pi}, \ldots$$

and all other Fourier coefficients are 0

Conclusion:

$$\mathrm{Sq}\,(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right).$$

Hide

Concept check

1/1 point (graded)

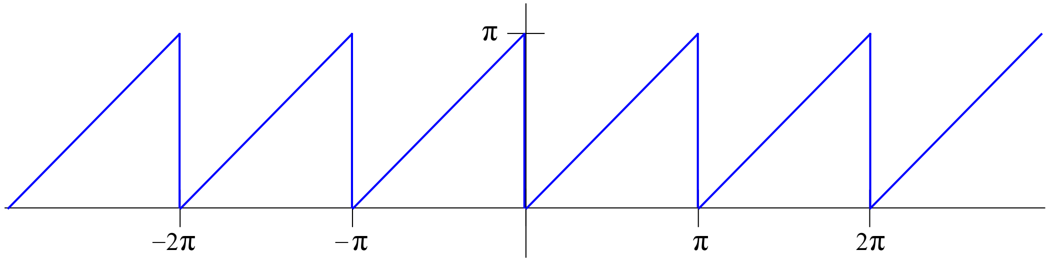
The following functions are 2π-periodic. They are only defined on the interval from $-\pi$ to π. Identify all of the even periodic functions in the list below.

- [ ] $p(x) = \begin{cases} x & : x \in (0,\pi) \\ \pi + x & : x \in (-\pi, 0) \end{cases}$
- [x] $q(x) = \begin{cases} x & : x \in (0,\pi) \\ -x & : x \in (-\pi, 0) \end{cases}$
- [ ] $r(x) = x \quad : x \in (-\pi,\pi)$
- [ ] $s(x) = \begin{cases} x^2 & : x \in (0,\pi) \\ -x^2 & : x \in (-\pi, 0) \end{cases}$
- [x] $t(x) = x^2 \quad : x \in (-\pi,\pi)$

✔

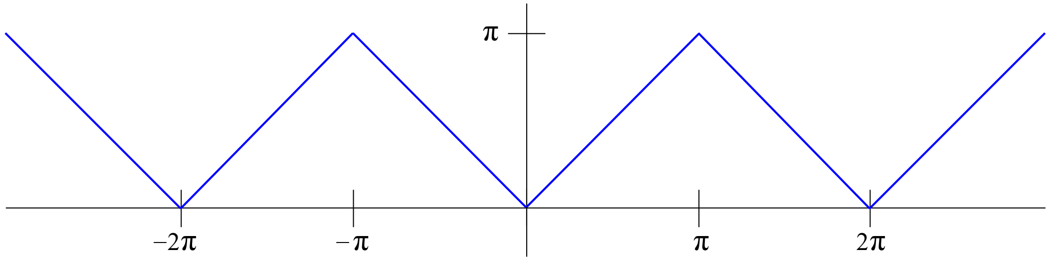
Solution:

- The graph of $p(x)$ is below.



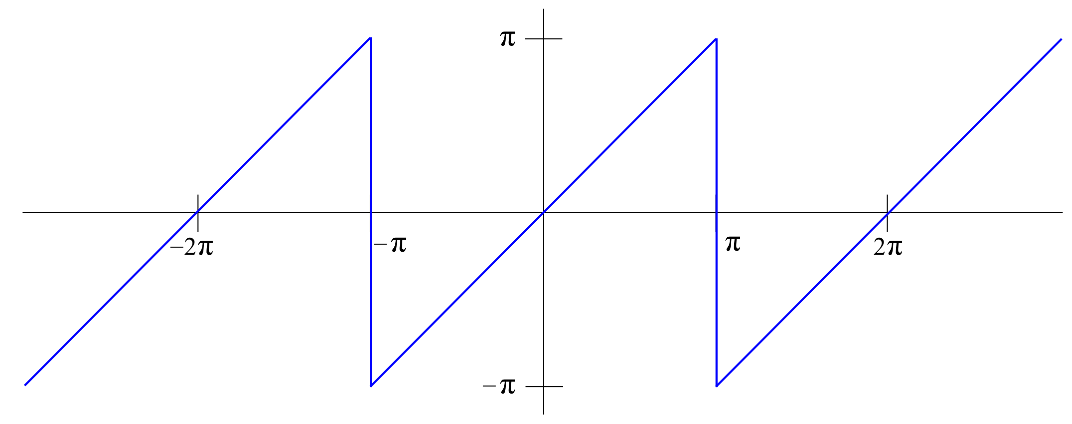
Consider values of x so that $0 < x < \pi$. Note that $p(-x) = \pi - x$, and $p(x) = x$. Since $\pi - x \neq x$ this function is neither even nor odd.

- The graph of $q(x)$ is below.



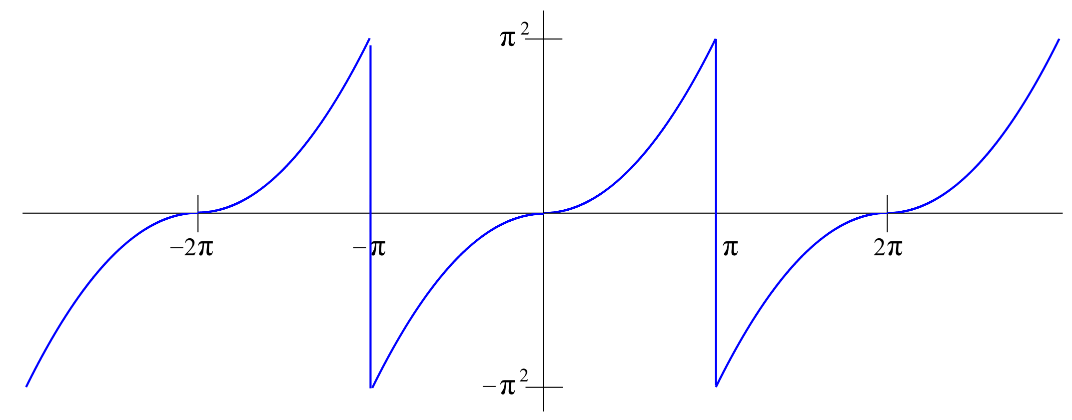
Consider values of x so that $0 < x < \pi$. Note that $q(-x) = x$, and $q(x) = x$. Since $q(-x) = q(x)$ this function is even.

- The graph of $r(x)$ is below.



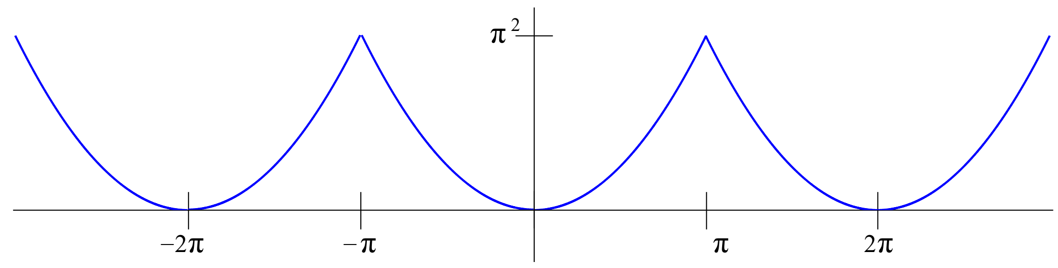
Consider values of x so that $0 < x < \pi$. Note that $r\left(-x\right) = -x$, and $r\left(x\right) = x$. Since $r\left(-x\right) = -r\left(x\right)$ this function is odd.

- The graph of $s\left(x\right)$ is below.



Consider values of x so that $0 < x < \pi$. Note that $s\left(-x\right) = -x^2$, and $s\left(x\right) = x^2$. Since $s\left(-x\right) = -s\left(x\right)$ this function is odd.

- The graph of $t(x)$ is below.



Consider values of x so that $0 < x < \pi$. Note that $t(-x) = x^2$, and $t(x) = x^2$. Since $t(-x) = t(x)$ this function is even.

Submit You have used 1 of 5 attempts

Answers are displayed within the problem

13. Even and odd periodic functions

Hide Discussion

Topic: Unit 1: Fourier Series / 13. Even and odd periodic functions

Add a Post

Show all posts ▼ by recent activity ▼

b_n, n even, terms are zero for square fn
For the square function, since it is odd, its F.S. consists of sines, and we found the even n terms vanish. O.K. All these vanishing terms have a base period pi, while the square f... 2

Learn About Verified Certificates