Expectation of Minimum of n i.i.d. uniform random variables.

$X_1, X_2, \ldots, X_n$ are n i.i.d. uniform random variables. Let $Y = \min(X_1, X_2, \ldots, X_n)$. Then, what's the expectation of Y (i.e., $E(Y)$)?

I have conducted some simulations by Matlab, and the results show that $E(Y)$ may equal to $\frac{1}{n+1}$. Can anyone give a rigorous proof or some hints? Thanks!

(probability-theory) (expectation)

asked May 8 '14 at 13:07

jet
68 1 5

Have you found the density function for Y? – Alex G. May 8 '14 at 13:13

2 Answers

To calculate the expected value, we're going to need the density function for Y. To get that, we're going to need the distribution function for Y. Let's start there.

By definition, $F(y) = P(Y \leq y) = 1 - P(Y > y) = 1 - P(\min(X_1, \ldots, X_n) > y)$. Of course, $\min(X_1, \ldots X_n) > y$ exactly when $X_i > y$ for all i. Since these variables are i.i.d., we have $F(y) = 1 - P(X_1 > y)P(X_2 > y) \ldots P(X_n > y) = 1 - P(X_1 > y)^n$. Assuming the X_i are uniformly distributed on (a, b), this yields

$$F(y) = \begin{cases} 1 - \left(\frac{b-y}{b-a}\right)^n & : y \in (a, b) \\ 0 & : y < a \\ 1 & : y > b \end{cases}$$

We take the derivative to get the density function.

$$f(y) = \begin{cases} \frac{n}{b-a}\left(\frac{b-y}{b-a}\right)^{n-1} & : y \in (a, b) \\ 0 & : \text{otherwise} \end{cases}$$

Now $E(Y) = \int_a^b y f(y) dy$. The integral is straightforward; I'll leave the details to you. I calculate $E(Y) = \frac{b+na}{n+1}$.

edited May 8 at 18:57

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answered May 8 '14 at 13:38

Alex G.
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Thanks for your answer. You extend it to a more general setting, which is very helpful. – jet May 8 '14 at 13:42

Yes. Assuming a $U(0, 1)$, note that

$$\Pr\left(\min_i X_i \leq x\right) = 1 - \Pr\left(\min_i X_i \geq x\right) = 1 - (1 - x)^n.$$

So the density function is

$$f(x) = n(1 - x)^{n-1}.$$

Then

$$\int_0^1 xf(x)dx = n\int_0^1 x(1-x)^{n-1}dx = n\int_0^1 (1-t)t^{n-1}dt = \frac{1}{n+1}.$$

answered May 8 '14 at 13:35

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1 Thanks for your answer, which is very helpful. – jet May 8 '14 at 13:43