12. Dimension of an eigenspace

Theorem 12.1 Let λ be an eigenvalue of an $n \times n$ matrix $\mathbf{A}$. Suppose that the multiplicity of λ (as a root of the characteristic polynomial) is m. Then

$$1 \quad \leq \quad (\textbf{dimension of eigenspace of } \lambda) \quad \leq \quad m.$$

Given an eigenvalue λ, the dimension of the eigenspace of λ is the maximum number of linearly independent eigenvectors corresponding to λ. This dimension is at least $1,$ since there is at least one eigenvector of eigenvalue λ (otherwise λ would not have been an eigenvalue). That this dimension is at most m requires more work to prove and we will not prove this in this course. One of the main ingredients is that eigenvectors corresponding to different eigenvalues must be linearly independent, which we will show below.

Example 12.2 A 9×9 matrix has characteristic polynomial $(\lambda - 2)^3(\lambda - 5)^6$. What are the possibilities for the dimension of the eigenspace of 2?

Solution: In this case, $m = 3$, so the dimension is 1, 2, or 3.

Definition 12.3

- The eigenvalue λ and its eigenspace are called **complete** (or **perfect**) if the dimension of the eigenspace equals the multiplicity m of λ.
- The eigenvalue λ and its eigenspace are called **deficient** (**defective** , **degenerate** , or **incomplete**) if the dimension of the eigenspace is less than m.

- A **matrix** is **complete** if **all** of its eigenspaces are complete; it is **deficient** if any **one** of its eigenspaces is deficient.

Warning : Different authors use different terminology here.

Example 12.4 If the multiplicity of an eigenvalue is 1, then the dimension of its eigenspace is sandwiched between 1 and 1, and so must be 1. In other words, the eigenspace of any eigenvalue of multiplicity 1 is complete.

Theorem 12.5 Fix an $n \times n$ matrix $\mathbf{A}$. The eigenvectors corresponding to distinct eigenvalues are linearly independent.

proof

Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that there were a linear relation

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}.$$

Apply $\lambda_1\mathbf{I} - \mathbf{A}$ to both sides; this gets rid of the first summand on the left. Next apply $\lambda_2\mathbf{I} - \mathbf{A}$, and so on, up to $\lambda_{n-1}\mathbf{I} - \mathbf{A}$. This shows that some nonzero number times $c_n\mathbf{v}_n$ equals $\mathbf{0}$. But $\mathbf{v}_n \neq \mathbf{0}$, so $c_n = 0$. Similarly each c_i must be 0. Thus only the trivial relation between $\mathbf{v}_1, \ldots, \mathbf{v}_n$ exists, so they are linearly independent.

Hide

As a consequence,

Fact: If all the eigenspaces of a $n \times n$ matrix $\mathbf{A}$ are complete, then there is a set of n linearly independent eigenvectors of $\mathbf{A}$.

proof

Show

Fact: If instead the matrix $\mathbf{A}$ is deficient, then the maximum number of linearly independent eigenvectors is less than n.

Example 12.6 Recall for the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, the eigenvalues and corresponding eigenvectors are :

$$\begin{array}{lcl} \textbf{Eigenvalue} & & \textbf{Eigenvectors (basis of eigenspace)} \\ \lambda = 0 & ; & \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ \lambda = 1 & ; & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \lambda = 2 & ; & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \end{array}$$

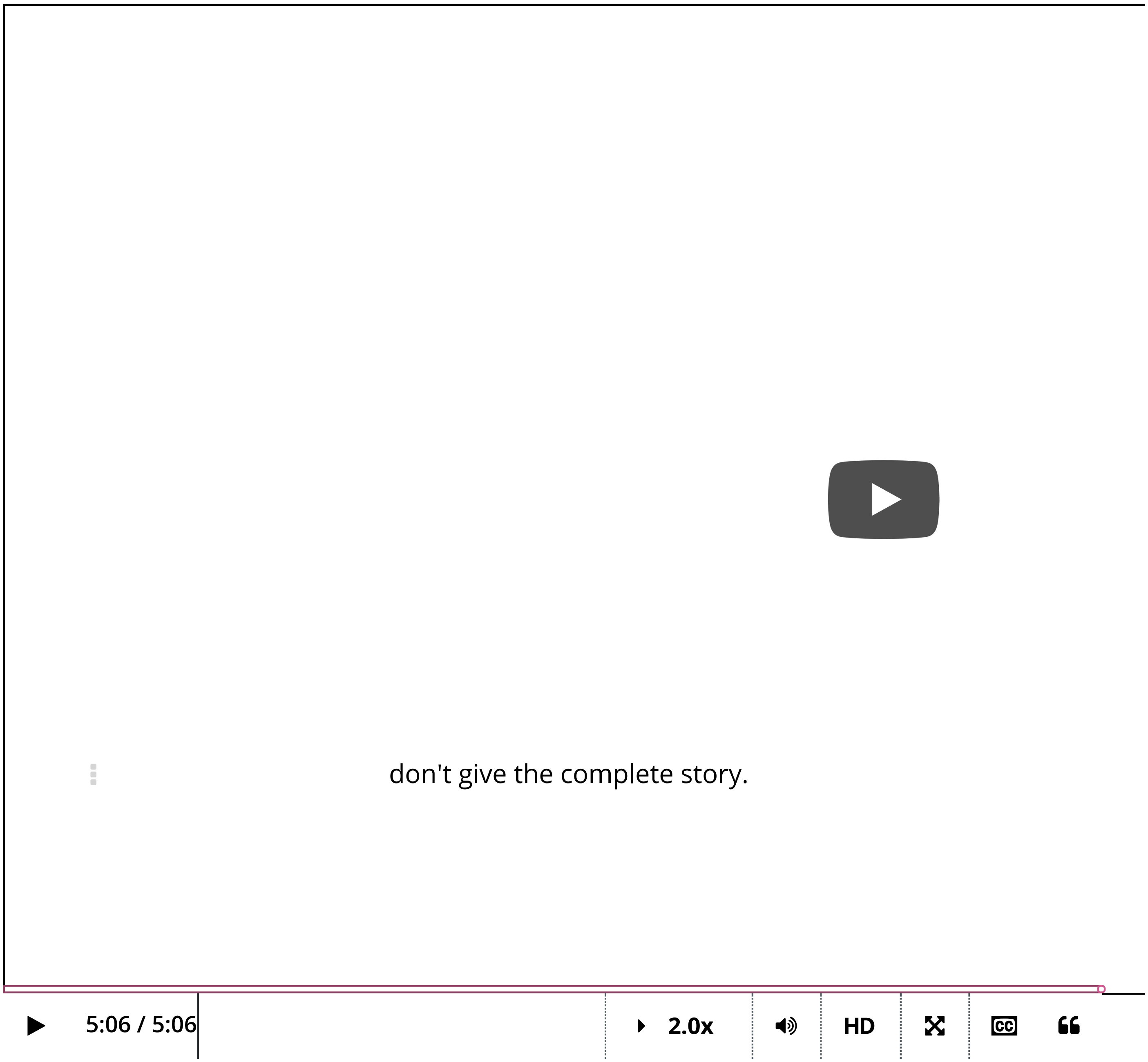
There are 3 distinct eigenvalues (each of multiplicity 1). So the corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent and form a basis of $\mathbb{R}^3$ (or $\mathbb{C}^3$).

Example 12.7 For $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, the eigenvalues and corresponding eigenvectors are

$$\begin{array}{lcl} \textbf{Eigenvalue} & & \textbf{Eigenvectors (basis of eigenspace)} \\ \lambda = 0 & ; & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \lambda = -3 & ; & \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

The eigenvalue -3 has multiplicity 2 (as a root of the characteristic polynomial) and its eigenspace is of dimension 2 and has a basis $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. These two eigenvectors of -3 together with any one eigenvector of 0 form a basis of $\mathbb{R}^3$ since the 3 vectors must be linearly independent.

A deficient example



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A deficient matrix

In this course, we will only deal with linear systems of DEs described by complete (not deficient) matrices. Nonetheless, here is an example of a deficient matrix.

Example 12.8 The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial is $(\lambda - 1)^2$, and so its only eigenvalue is $1,$ with multiplicity 2. However, the eigenspace of 1 is $\mathrm{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is only 1-dimensional. Hence, this matrix is **deficient**.

Optional: the spectral theorem

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The **transpose** of a matrix $\mathbf{A}$ is defined to be the matrix $\mathbf{A}^T$ formed by turning the rows of $\mathbf{A}$ into the columns of $\mathbf{A}^T$. In other words, the entry in the ith row and jth column of $\mathbf{A}^T$ is the entry in the jth row and ith column of $\mathbf{A}$:

$$(\mathbf{A}^T)_{ij} = \mathbf{A}_{ji}.$$

Example 12.9 The transpose of $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is the matrix

$$\mathbf{A}^T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A **symmetric matrix** is a square $n \times n$ matrix $\mathbf{A}$ that is equal to its transpose.

$$\mathbf{A}^T = \mathbf{A}.$$

In other words, the entry in the ith row and jth column is equal to the entry in the jth row and ith column

$$a_{ij} = a_{ji}.$$

Example 12.10 The matrix $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ is symmetric.

Spectral Theorem: If $\mathbf{A}$ is a real, symmetric, $n \times n$ matrix, then all of its eigenvalues are complete.

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