7.4 Unit 7 Homework Problems

Unit 7: Markov Chains

Adapted from Blitzstein-Hwang Chapter 11.

Problem 1

1/1 point (graded)

Consider the following Markov chain with $52! \approx 8 \times 10^{67}$ states. The states are the possible orderings of a standard 52-card deck. To run one step of the chain, pick 2 different cards from the deck, with all pairs equally likely, and swap the 2 cards. Is the stationary distribution of the chain uniform over the $52!$ states, i.e., is the stationary distribution $\frac{1}{52!}(1, 1, \ldots, 1)$?

● Yes ✓

○ No

Solution

Yes. The transition probability from state x to state y is $\frac{1}{\binom{52}{2}}$ if x and y differ by a single swap, and 0 otherwise. So the transition matrix is symmetric. Thus, the stationary distribution $\mathbf{s}$ is uniform over the $52!$ states, i.e., $\mathbf{s} = \frac{1}{52!}(1, 1, \ldots, 1)$.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Problem 2

1/1 point (graded)

Alice and Bob are wandering around randomly, independently of each other, in a house with M rooms, labeled $1, 2, \ldots, M$. Let d be the

Alice and Bob are wandering around randomly, independently of each other, in a house with M rooms, labeled $1, 2, \dots, M$. Let d_i be the number of doors in room i (leading to other rooms, not leading outside). At each step, Alice moves to another room by choosing randomly which door to go through (with equal probabilities). Bob does the same, independently. The Markov chain that each of them follows is irreducible and aperiodic. Find the limit of the probability that Alice is in room i and Bob is in room j, as time goes to ∞.

○ $$\frac{d_i d_j}{\sum_k d_k^2}$$

● $$\frac{d_i d_j}{(\sum_k d_k)^2}$$

✓

○ $$\frac{d_i d_j}{(\sum_k d_k)^4}$$

○ $$\frac{d_i + d_j}{\sum_k d_k}$$

Solution

Alice is following a random walk on an undirected network, so the stationary probability of being at i is proportional to d_i. Likewise for Bob. Their walks are independent, so the desired probability is

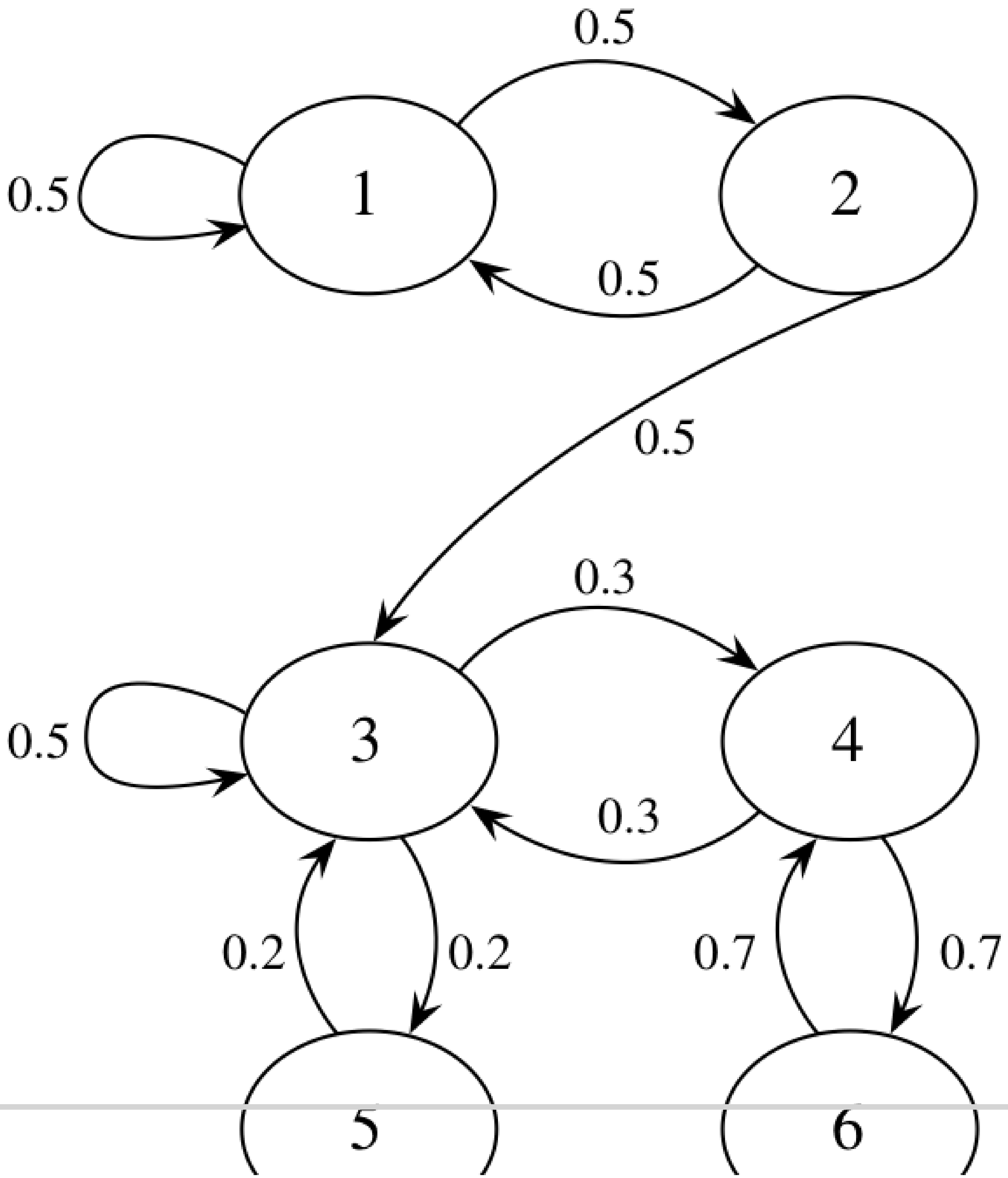
$$\frac{d_i d_j}{(\sum_k d_k)^2}.$$

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

For Problem 3

Consider the following Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$.



Markov Chain for Problem 3
View Larger Image
Image Description

Problem 3a

2/2 points (graded)
(a) Suppose the chain starts at state 1. The distribution of the number of times that the chain returns to state 1 is:

Distribution:

Geometric ▾ ✓ **Answer:** Geometric

Parameter:

1/4 ▾ ✓ **Answer:** 1/4

Solution
From state 1, the probability of not returning to state 1 is 0.25, since the chain will not return if and only if the next 2 transitions are state 1 to state 2, followed by state 2 to state 3. Considering this sequence of transitions a "success" and returning to state 1 as a "failure", we have that the number of returns is $\text{Geom}(1/4)$.

Submit You have used 1 of 4 attempts

Answers are displayed within the problem

Problem 3b

1/1 point (graded)

(b) In the long run, what fraction of the time does the chain spend in state 3?

0.25906736 ✓ **Answer:** 1/4

0.25906736

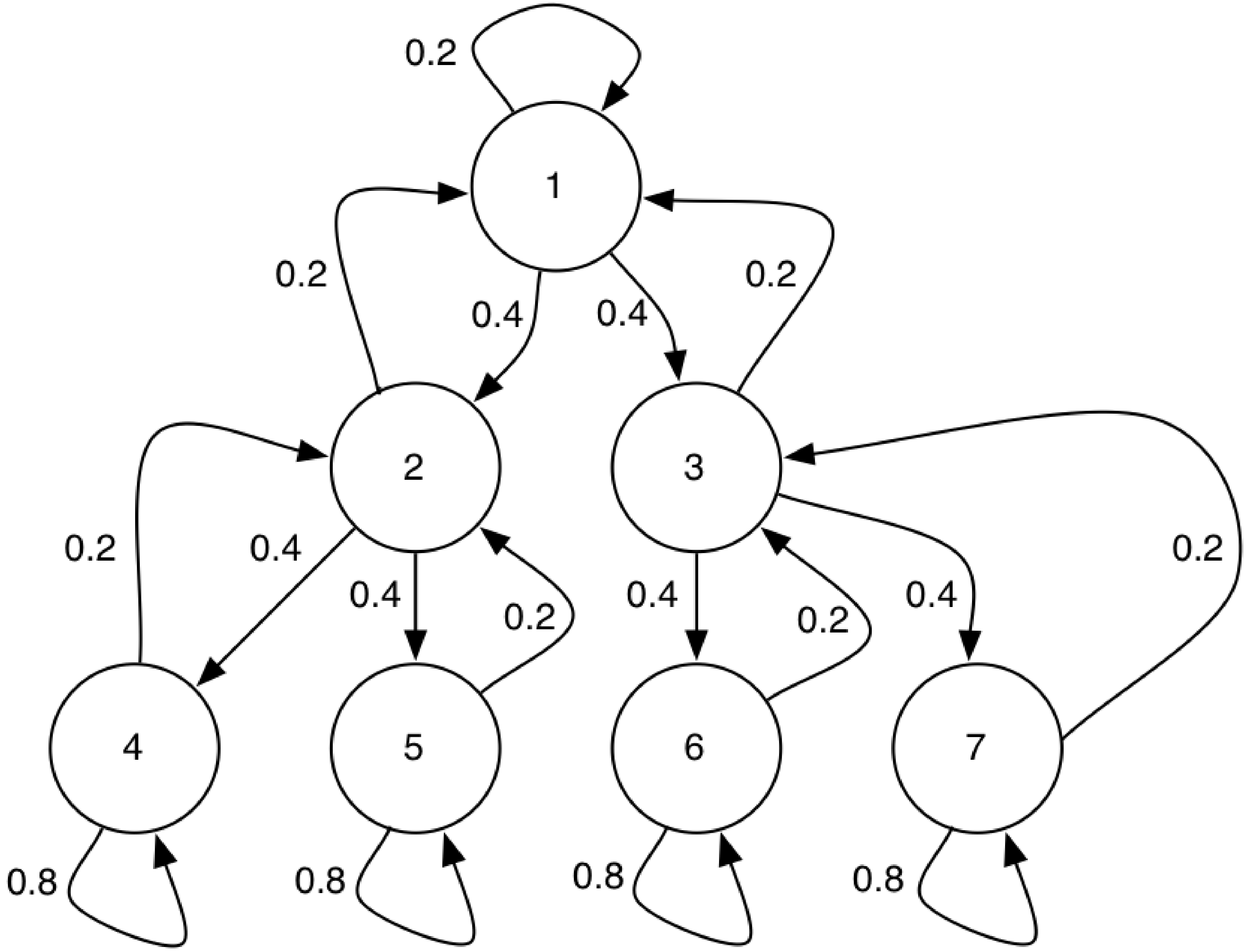
Solution
States 1 and 2 are transient, so in the long run the chain will spend all its time in states 3, 4, 5, and 6, never again returning to state 1 or state 2. So we can restrict attention to the 4-state chain with states 3, 4, 5, and 6. The transition matrix for these states is symmetric, so the stationary distribution is uniform on these 4 states. So in the long run, the original chain spends $1/4$ of its time in state 3.

Submit You have used 1 of 5 attempts

ℹ Answers are displayed within the problem

Problem 4

3/3 points (graded)
Find the probabilities of states 1, 2, and 4 in the stationary distribution of the Markov chain s shown below. The label to the left of an arrow gives the corresponding transition probability.



Markov Chain for Problem 4

View Larger Image

Image Description

$s_1 =$ 0.04761905

✓ **Answer:** 1/21

0.04761905

$s_2 =$ 0.09523810

✓ **Answer:** 2/21

0.09523810

$s_4 =$ 0.19047619

✓ **Answer:** 4/21

0.19047619

Solution

Let q_{ij} be the transition probability from i to j. Let's try to solve for $\mathbf{s}$ such that $s_i q_{ij} = s_j q_{ji}$ for all i, j.

- $s_1 q_{12} = s_2 q_{21}$ gives $s_2 = s_1 q_{12} / q_{21} = 2 s_1$. Similarly, $s_3 = 2 s_1$.
- $s_2 q_{24} = s_4 q_{42}$ gives $s_4 = 2 s_2 = 4 s_1$. Similarly, $s_5 = s_6 = s_7 = 4 s_1$.

Then $s_i q_{ij} = s_j q_{ji}$ for all i, j, so the chain is reversible. Normalizing, the stationary distribution is

$$\mathbf{s} = \frac{1}{21}(1, 2, 2, 4, 4, 4, 4).$$

Submit You have used 1 of 5 attempts

ℹ Answers are displayed within the problem