# 9. Trace

Let us take a quick detour to discuss the relationship between eigenvalues and the trace and determinant of a matrix.

**Definition 9.1** The **trace** of a square matrix $\mathbf{A}$ is the sum of the entries along the main diagonal. That is, for

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}.$$

the trace is

$$\mathbf{tr}\mathbf{A} = a_{11} + a_{22} + \cdots + a_{nn}.$$

**Example 9.2** If $A = \begin{pmatrix} 4 & 6 & 9 \\ 1 & 7 & 8 \\ 2 & 3 & 5 \end{pmatrix}$, then $\mathbf{tr}A = 4 + 7 + 5 = 16$.

**Warning:** Recall that trace and determinant make sense only for **square** matrices.

**Problem 9.3** For an $n \times n$ matrix $A$, how do $\mathbf{tr}(-\mathbf{A})$ and $\det(-\mathbf{A})$ relate to $\mathbf{tr}\mathbf{A}$ and $\det\mathbf{A}$?

**Solution**

Negating $A$ negates in particular all diagonal entries of $\mathbf{A}$, so $\mathbf{tr}(-\mathbf{A}) = -\mathbf{tr}\mathbf{A}$.

On the other hand, negating $\mathbf{A}$ amounts to multiplying every row by $-1$, which multiplies $\det\mathbf{A}$ by $(-1)^n$ because there is one factor of $-1$ for each row. Thus

- If $n$ is even, then $\det(-\mathbf{A}) = \det\mathbf{A}$.
- If $n$ is odd, then $\det(-\mathbf{A}) = -\det\mathbf{A}$.

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## Trace of sums

1/1 point (graded)
If $\mathbf{A}$ and $\mathbf{B}$ are both $5\times 5$ matrices and $\mathbf{tr}\mathbf{A} = 7,\ \mathbf{tr}\mathbf{B} = 16,$ what is the value of $\mathbf{tr}(\mathbf{A}+\mathbf{B})$?

$\mathbf{tr}(\mathbf{A}+\mathbf{B}) =$ 23 ✔ **Answer:** 23

$23$

**Solution:**

Denote the entries of $\mathbf{A}$ and $\mathbf{B}$ at the $i^{\text{th}}$ row and $j^{\text{th}}$ column by $a_{ij}$ and $b_{ij}$ respectively. Then

$$\begin{aligned}\mathbf{tr}(\mathbf{A}+\mathbf{B}) &= (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{55}+b_{55}) \\ &= (a_{11}+\cdots+a_{55}) + (b_{11}+\cdots+b_{55}) \\ &= \mathbf{tr}\mathbf{A} + \mathbf{tr}\mathbf{B} \ = \ 7+16 \ = \ 23.\end{aligned}$$

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Answers are displayed within the problem

## Trace of scalar multiple

1/1 point (graded)
Let $\mathbf{A}$ be an $5\times 5$ matrix. If $\mathbf{tr}\mathbf{A} = -2,$ what is the trace of $4\mathbf{A}$?

$\mathrm{tr}(4\mathbf{A}) =$ -8 ✔ **Answer:** -8

$-8$

**Solution:**

Again denote the entries of $\mathbf{A}$ and $\mathbf{B}$ at the $i^{\text{th}}$ row and $j^{\text{th}}$ column by $a_{ij}$ and $b_{ij}$ respectively. Then

$$\mathrm{tr}(4\mathbf{A}) \;=\; 4a_{11} + \cdots + 4a_{55} \;=\; 4\left(a_{11} + \cdots + a_{55}\right) \;=\; 4\mathrm{tr}(\mathbf{A}) \;=\; -8.$$

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## Trace of identity matrix

1/1 point (graded)

Let $\mathbf{I}$ be the $n \times n$ **identity matrix**. Compute $\mathrm{tr}(\mathbf{I})$.

$\mathrm{tr}(\mathbf{I}) =$ n ✔

$n$

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✔ Correct (1/1 point)

## Trace and determinant of products of matrices

2/2 points (graded)

Let $\mathbf{A}, \mathbf{B}$ be $n \times n$ matrices.

**True or False?** The trace of $\mathbf{AB}$ is the product of the individual traces:

$$\mathbf{tr}\left(\mathbf{AB}\right)=\left(\mathbf{tr}\mathbf{A}\right)\left(\mathbf{tr}\mathbf{B}\right)$$

- True
- False ✔

**True or False?** The determinant of $\mathbf{AB}$ is the product of the individual determinants:

$$\det\left(\mathbf{AB}\right)=\left(\det\mathbf{A}\right)\left(\det\mathbf{B}\right).$$

- True ✔
- False

**Solution:**

- The trace of $\mathbf{AB}$ is **NOT** the product of the individual traces. Consider the matrix $\mathbf{J}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. We see that $\mathbf{J}^2=-\mathbf{I}$, so $\mathbf{tr}(\mathbf{J}^2)=-\mathbf{2}$ even though $\mathbf{tr}(\mathbf{J})=0$.
- We told you in the previous lecture that $\det\left(\mathbf{AB}\right)=\left(\det\mathbf{A}\right)\left(\det\mathbf{B}\right)$.

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