Does the variance of a sum equal the sum of the variances?

Is it (always) true that

$$\mathrm{Var}\left(\sum_{i=1}^m X_i\right) = \sum_{i=1}^m \mathrm{Var}(X_i) \ ?$$

variance

edited Jun 27 '12 at 2:30
Macro
22.1k 3 89 123

asked Jun 26 '12 at 22:44
Abe
998 2 12 31

The answers below provide the proof. The intuition can be seen in the simple case var(x+y): if x and y are positively correlated, both will tend to be large/small together, increasing total variation. If they are negatively correlated, they will tend to cancel each other, decreasing total variation. – Assad Ebrahim Jul 17 '15 at 10:51

3 Answers

The answer to your question is "Sometimes, but not in general".

To see this let $X_1, \ldots, X_n$ be random variables (with finite variances). Then,

$$\mathrm{var}\left(\sum_{i=1}^n X_i\right) = E\left(\left[\sum_{i=1}^n X_i\right]^2\right) - \left[E\left(\sum_{i=1}^n X_i\right)\right]^2$$

Now note that $(\sum_{i=1}^n a_i)^2 = \sum_{i=1}^n \sum_{j=1}^n a_i a_j$, which is clear if you think about what you're doing when you calculate $(a_1 + \ldots + a_n) \cdot (a_1 + \ldots + a_n)$ by hand. Therefore,

$$E\left(\left[\sum_{i=1}^n X_i\right]^2\right) = E\left(\sum_{i=1}^n \sum_{j=1}^n X_i X_j\right) = \sum_{i=1}^n \sum_{j=1}^n E(X_i X_j)$$

similarly,

$$\left[E\left(\sum_{i=1}^n X_i\right)\right]^2 = \left[\sum_{i=1}^n E(X_i)\right]^2 = \sum_{i=1}^n \sum_{j=1}^n E(X_i)E(X_j)$$

so

$$\mathrm{var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \sum_{j=1}^n \left(E(X_i X_j) - E(X_i)E(X_j)\right) = \sum_{i=1}^n \sum_{j=1}^n \mathrm{cov}(X_i, X_j)$$

by the definition of covariance.

Now regarding *Does the variance of a sum equal the sum of the variances?*:

- **If the variables are uncorrelated, yes**: that is, $\mathrm{cov}(X_i, X_j) = 0$ for $i \neq j$, then

$$\mathrm{var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \sum_{j=1}^n \mathrm{cov}(X_i, X_j) = \sum_{i=1}^n \mathrm{cov}(X_i, X_i) = \sum_{i=1}^n \mathrm{var}(X_i)$$

- **If the variables are correlated, no, not in general**: For example, suppose X_1, X_2 are two random variables each with variance σ^2 and $\mathrm{cov}(X_1, X_2) = \rho$ where $0 < \rho < \sigma^2$. Then $\mathrm{var}(X_1 + X_2) = 2(\sigma^2 + \rho) \neq 2\sigma^2$, so the identity fails.
- **but it is possible for certain examples**: Suppose X_1, X_2, X_3 have covariance matrix

$$\begin{pmatrix} 1 & 0.4 & -0.6 \\ 0.4 & 1 & 0.2 \\ -0.6 & 0.2 & 1 \end{pmatrix}$$

then $\mathrm{var}(X_1 + X_2 + X_3) = 3 = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \mathrm{var}(X_3)$

Therefore **if the variables are uncorrelated** then the variance of the sum is the sum of the variances, but converse is **not** true in general.

edited Jun 28 '12 at 1:12

answered Jun 26 '12 at 22:51


Macro
22.1k 3 89 123

3 This is very cool. – Kris Harper Jun 27 '12 at 12:34

1 Thank you for providing such a detailed and clear explanation. – Abe Jun 27 '12 at 17:44

Regarding the example covariance matrix, is the following correct: the symmetry between the upper right and lower left triangles reflects the fact that $\mathrm{cov}(X_i, X_j) = \mathrm{cov}(X_j, X_i)$, but the symmetry between the upper left and the lower right (in this case that $\mathrm{cov}(X_1, X_2) = \mathrm{cov}(X_2, X_3) = 0.3$ is just part of the example, but could be replaced with two different numbers that sum to 0.6 e.g., $\mathrm{cov}(X_1, X_2) = a$ and $\mathrm{cov}(X_2, X, 3) = 0.6 - a$? Thanks again. – Abe Jun 27 '12 at 17:56

1 thanks. I proposed an edit to make it more 'textbook' like / dummy resistant. – Abe Jun 27 '12 at 18:06

$$\mathrm{Var}\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} \mathrm{Var}(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j).$$

So, if the covariances average to 0, which would be a consequence if the variables are pairwise uncorrelated or if they are independent, then the variance of the sum is the sum of the variances.

An example where this is not true: Let $\mathrm{Var}(X_1) = 1$. Let $X_2 = X_1$. Then $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(2X_1) = 4$.

answered Jun 26 '12 at 22:59


Douglas Zare
6,410 1 16 35

(+1) For completeness. – cardinal ♦ Jun 26 '12 at 23:27

It will rarely be true for sample variances. – DWin Jun 27 '12 at 2:35

@DWin, "rare" is an understatement - if the Xs have a continuous distribution, the probability that the sample variance of the sum is equal to the sum of the sample variances in exactly 0 :) – Macro Jun 27 '12 at 13:41

Yes, if each pair of the X_i's are uncorrelated, this is true.

See the explanation on Wikipedia

edited Jun 27 '12 at 11:21

community wiki
3 revs, 3 users 67%
Abe

I agree. You also find a simple(r) explanation on Insight Things. – Jan 16 hours ago