Uniform and Non-Uniform Measures

So far we have been thinking of the notion of measure as a generalization of the notion of *length*. But, as I suggested at the beginning of this chapter, the notion of measure can also be thought of as a generalization of the notion of *probability*. Suppose, for example, that we have a random procedure for selecting individual points from the line-segment $[0,1]$. For A a subset of $[0,1]$, let $p(A)$ be the probability that the next point selected from $[0,1]$ by our random procedure is a member of A. As long as $p(\ldots)$ satisfies Countable Additivity, it will count as a measure on the Borel Sets in [0,1]. But whether or not $p(\ldots)$ turns out to be the Lebesgue Measure will depend on the details of our random selection procedure. The best way to see this is to consider two different kinds of selection procedures:

> **Standard Coin-Toss Procedure**
> You toss a fair coin once for each natural number. Each time the coin lands Heads you write down a zero, and each time it lands Tails you write down a one. This gives you an infinite sequence $\langle d_1, d_2, d_3, \ldots \rangle$, The selection procedure is then as follows: pick whichever number in $[0,1]$ has $0.d_1d_2d_3\ldots$ as its binary expansion.

(If you'd like a refresher on binary expansions, have a look at Lecture 1.5.1. As noted in that discussion, the rational numbers within [0,1] have two different binary expansions: one ending in 0s and the other ending in 1s. To simplify the present discussion, I assume that the output of the Coin-Toss Procedure is declared void if it corresponds to a binary expansion ending in 1s.)

> **Square Root Coin-Toss Procedure**
> As before, but this time you pick the *square root* of the number represented by $0.d_1d_2d_3\ldots$. (Suppose, for instance, that sequence of coin-tosses yields 0.0100(0), which is the binary expansion of $\frac{1}{4}$. Then you should select $\frac{1}{2}$, since $\frac{1}{2} = \sqrt{\frac{1}{4}}$.)

The Standard Coin-Toss Procedure delivers a uniform probability distribution. In fact, when one makes suitable assumptions about the probabilities of sequences of coin tosses, the Standard Coin Toss Procedure generates a Lebesgue Measure over $[0,1]$. The Square Root Coin Toss Procedure, in contrast, does not satisfy Uniformity. It can be used to define a measure over $[0,1]$, but not the Lebesgue Measure.

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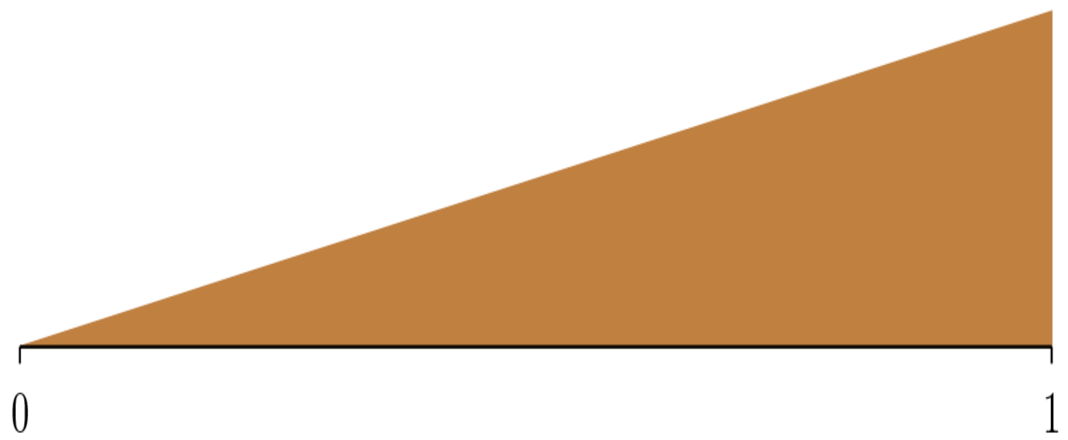
Here is an intuitive way of visualizing the difference between the two measures. Suppose we have 1kg of mud, and that we are told to pour it on the line segment $[0, 1]$ in such a way that the amount of mud above an interval within $[0, 1]$ is proportional to the probability that the next point selected by one of our coin-toss procedures will fall within that interval. In the case of the Standard Coin-Toss Procedure, our mud distribution should look like this:

In other words: the probability of getting a number within a given interval does not depend on where the interval is located within $[0, 1]$: it depends only on the size of the interval. When it comes to the Square Root Coin-Toss Procedure, in contrast, our mud distribution should look like this:



In other words: the probability of getting a number within a given interval depends not just on the size of the interval, but also on the *location* of the interval within $[0, 1]$: the closer to 1 the interval is, the higher the probability of getting a number within that interval.

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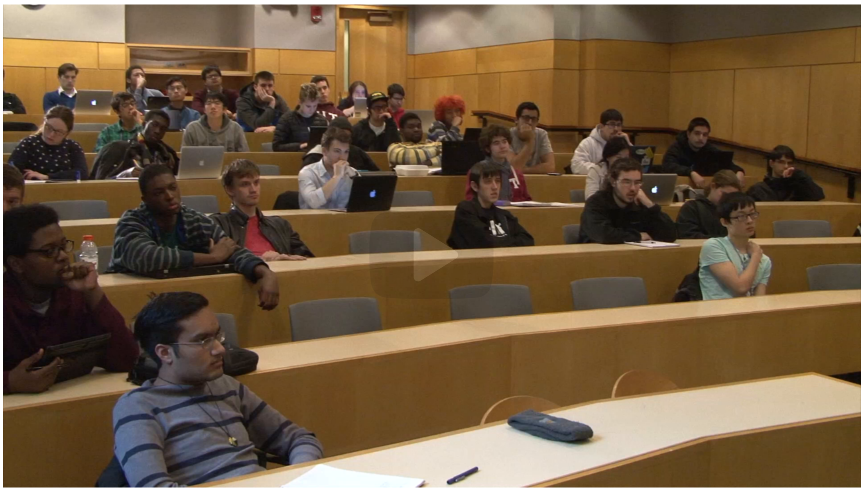
Picking Points on a Line



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So suppose that we're doing this.

Suppose that we have the segment [0, 1].

So the set of real numbers greater or equal to 0

and smaller or equal to 1.

And we want to describe a mechanism for picking

one of those points at random.

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Problem 1

10/10 points (ungraded)

Answer the following questions with respect to each of our two coin-toss procedures:

(1) What is the probability that the next selected point will be in $[\frac{1}{2}, 1]$?

Standard Coin-Toss Procedure:

1/2 ✔ **Answer:** .5

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Square Root Coin-Toss Procedure:

3/4

✔ **Answer:** .75

$\frac{3}{4}$

Explanation

Let us start with the ordinary Coin-Toss Procedure. The first thing to note is that the ordinary Coin-Toss Procedures yield a number in $[\frac{1}{2}, 1]$ if and only if the first toss lands Tails.

To verify this, note that the real number $\frac{1}{2}$ is represented by the binary expansion $0.1\,(0)$. (It is also represented by the binary expansion $0.0\,(1)$, but this latter expansion is stipulated not to count as a valid output of our Coin-Toss Procedures.) Accordingly, each valid binary sequence which is not of the form $0.1d_2d_3\ldots$ represents a real number smaller than $\frac{1}{2}$, and each valid binary sequence which is of the form $0.1d_2d_3\ldots$ represents a real number equal to or larger than $\frac{1}{2}$.

With this observation in place, the answer is straightforward. Since (on standard assumptions about the probabilities of coin tosses) probability that the first toss lands Tails is $\frac{1}{2}$, the probability that one gets a number in $[\frac{1}{2}, 1]$ as output is $\frac{1}{2}$.

On the Square Root Coin-Toss Procedure, in contrast, one gets a number in $[\frac{1}{2}, 1]$ if and only if at least one of the first two tosses lands Tails.

To verify this, consider two different cases: (1) the first toss can land Tails, and (2) the first toss lands Heads and the second toss lands Tails. If the first toss lands Tails, then the sequence of coin-tosses will yield a number of the form $0.1d_2d_3\ldots$, which represents a real number greater than or equal to $\frac{1}{2}$. Since $(\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$, this means that the Square Root Coin-Toss Procedure will yield a number greater than or equal to $\frac{1}{\sqrt{2}}$. If, on the other hand, the first toss lands Heads and the second toss lands Tails, then the sequence of coin-tosses will yield a number of the form $0.01d_2d_3\ldots$, which represents a real number greater than or equal to $\frac{1}{4}$ but smaller than $\frac{1}{2}$. Since $(\frac{1}{2})^2 = \frac{1}{4}$ and $(\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$, this means that the Square Root Coin-Toss Procedure will yield a number greater than or equal to $\frac{1}{2}$ but smaller than $\frac{1}{\sqrt{2}}$. Putting these two cases together yields the desired result: one gets a number in $[\frac{1}{2}, 1]$ if and only if at least one of the first two tosses lands Tails.

With this observation in place, the answer is straightforward. Since (on standard assumptions about the probabilities of coin tosses) the probability of getting Tails in at least one of the first two tosses is $\frac{3}{4}$, the probability that one gets a number in $[\frac{1}{2}, 1]$ as output is $\frac{3}{4}$.

(2) What is the probability that the next selected point will be precisely $\frac{1}{2}$? (Assume that the relevant probability is defined and is a real number between 0 and 1.)

Standard Coin-Toss Procedure:

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0

✔ **Answer:** 0

0

Square Root Coin-Toss Procedure:

0

✔ **Answer:** 0

0

Explanation
Start with the ordinary Coin-Toss Procedure. The unique binary expansion of $\frac{1}{2}$ that is a valid output of our coin-toss procedures is 0.1(0). So the only way for $\frac{1}{2}$ to be selected is for our sequence of coin tosses to result in exactly the following sequence:

T H H H H H H H H H H H H H H H H H H ...

And what is the probability that this will happen? Zero! One can verify this by noting that each of the following must be true:

- The probability must be smaller than 1, since it requires our first toss to land Tails, which is an event of probability $\frac{1}{2}$.
- The probability must be smaller than $\frac{1}{2}$, since it requires our first two tosses to land Tails-Heads, which is an event of probability $\frac{1}{4}$.
- The probability must be smaller than $\frac{1}{4}$, since it requires our first two tosses to land Tails-Heads-Heads, which is an event of probability $\frac{1}{8}$.
- And so forth: the probability must be smaller than $\frac{1}{2^n}$ for each $n \in \mathbb{N}$.

Since we are assuming that the relevant probability exists, since probabilities are real number between 0 and 1, and since the only such number smaller than $\frac{1}{2^n}$ for each $n \in \mathbb{N}$ 0, the probability of selecting $\frac{1}{2}$ must be 0.
Notice, moreover, that $\frac{1}{2}$ is not a special case. For any x in $[0, 1]$, the probability that our Coin-Toss Procedures yield precisely x as output is always zero. (It is worth keeping in mind, however, that, as we saw in Lecture 4, saying that an event has probability zero is *not* the same

Generating Speech Output t can't happen. To see this, note our procedure always yields *some* number in $[0, 1]$ as output, and *whatever* number we get, the probability of getting that number was zero. But it happened anyway. For an event with well-defined probability to have probability zero is for the event to be so vanishingly unlikely that the probability of its occurrence is too small to be measured by a positive real number, no matter how small.)

Let us now consider the Square Root Coin-Toss Procedure. Since $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, the only way for the Square Root Coin-Toss Procedure to yield $\frac{1}{2}$ is for our sequence of coin tosses to yield a sequence that represents $\frac{1}{\sqrt{4}}$ in binary notation. But an analogue of the argument above shows that the probability of getting such a sequence is zero.

(3) What is the probability that the next selected point will be in $[0, \frac{1}{2}]$? (*Hint:* use your answer to the previous question, and assume that the relevant probabilities satisfy Additivity.)

Standard Coin-Toss Procedure:

1/2 ✔ **Answer:** .5

$\frac{1}{2}$

Square Root Coin-Toss Procedure:

1/4 ✔ **Answer:** .25

$\frac{1}{4}$

Explanation

We first consider the ordinary Coin-Toss Procedure. There are only two ways for a number in $[0, \frac{1}{2}]$ to be selected. The first is for the first toss to land Heads, in which case the procedure yields a number in $[0, \frac{1}{2})$ (i.e. a number x such that $0 \leq x < \frac{1}{2}$). The second is to get exactly the following sequence:

T H H H H H H H H H H H H H H H H H H ...

in which case the procedure yields the number $\frac{1}{2}$.

The probability of the first outcome is $\frac{1}{2}$, and we know from the previous exercise that the probability of the second is 0. Since $[0, \frac{1}{2}] = [0, \frac{1}{2}) \cup \{\frac{1}{2}\}$, and since $[0, \frac{1}{2})$ and $\{\frac{1}{2}\}$ have no

Generating Speech Output Additivity yields

$$p\left(\left[0, \frac{1}{2}\right]\right) = p\left(\left[0, \frac{1}{2}\right)\right) + p\left(\left\{\frac{1}{2}\right\}\right) = \frac{1}{2} + 0 = \frac{1}{2}$$

Let us now consider the Square Root Coin-Toss Procedure. Since $(\frac{1}{2})^2 = \frac{1}{4}$, there are two ways for a number in $[0, \frac{1}{2}]$ to be selected. The first is for the first two tosses to land Heads, in which case the sequence of coin-tosses corresponds to a number in $[0, \frac{1}{4})$ and the Square Root Coin-Toss Procedure yields a number in $[0, \frac{1}{2})$. The second is to get exactly the following sequence:

H T H H H H H H H H H H H H H H H H H ...

in which case our coin-toss sequence represents the number $\frac{1}{4}$ and the Square Coin-Toss Procedure yields the number $\frac{1}{2}$.
The probability of the first outcome is $\frac{1}{4}$, and we know from the previous exercise that the probability of the second is 0. Since $[0, \frac{1}{2}] = [0, \frac{1}{2}) \cup \{\frac{1}{2}\}$, and since $[0, \frac{1}{2})$ and $\{\frac{1}{2}\}$ have no elements in common, Additivity yields

$$p\left(\left[0, \frac{1}{2}\right]\right) = p\left(\left[0, \frac{1}{2}\right)\right) + p\left(\left\{\frac{1}{2}\right\}\right) = \frac{1}{4} + 0 = \frac{1}{4}$$

(4) Suppose $\{x_0, x_1, x_2, \ldots\}$ is a countable set of numbers in $[0, 1]$. What is the probability that the next selected point will be in $\{x_0, x_1, x_2, \ldots\}$? (Assume that the relevant probabilities satisfy Countable Additivity.) *Hint:* Use your answer to the questions above.

Standard Coin-Toss Procedure:

0 ✔ **Answer:** 0

0

Square Root Coin-Toss Procedure:

0 ✔ **Answer:** 0

0

Explanation

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The following argument works for either version of the Coin-Toss Procedure. We know from an earlier question that $p(\{x_i\}) = 0$ for each individual point x_i. But by Countable Additivity,

$$p(\{x_0, x_1, x_2, \ldots\}) = p(\{x_0\}) + p(\{x_1\}) + p(\{x_2\}) + \ldots$$

So $p(\{x_0, x_1, x_2, \ldots\})$ must be 0.

(5) What is the probability that the next point selected will be a member of $[0,1] - \{\frac{1}{2}\}$? What is the probability that it will be a member of $[0,1] - \{x_0, x_1, x_2, \ldots\}$? (Assume that the relevant probabilities are all well-defined, and that they satisfy Countable Additivity.)

Standard Coin-Toss Procedure:

1

Answer: 1

1

Square Root Coin-Toss Procedure:

1

Answer: 1

1

Explanation

The following arguments work for either version of the Coin-Toss Procedure.

Notice, first, that $p([0,1]) = 1$, since the next valid output of either procedure is guaranteed to be a number in $[0,1]$. We know, moreover, from question (2) above that $p(\{\frac{1}{2}\}) = 0$. Since we are assuming that $p([0,1] - \{\frac{1}{2}\})$ is well defined, Additivity yields

$$p([0,1]) = p\left([0,1] - \left\{\frac{1}{2}\right\}\right) + p\left(\left\{\frac{1}{2}\right\}\right)$$

So we know that

$$1 = p\left([0,1] - \left\{\frac{1}{2}\right\}\right) + 0$$

and therefore that

$$p\left([0,1] - \left\{\frac{1}{2}\right\}\right) = 1$$

A similar argument yields the result that $p([0,1] - \{x_0, x_1, x_2, \ldots\}) = 1$, since we know from the previous question that $p(\{x_0, x_1, x_2, \ldots\}) = 0$.

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Square Root Coin-Toss Procedure: Computing the pdf

discussion posted 3 minutes ago by sandipan_dey

Standard Coin-Toss Procedure

We have $X \sim \mathcal{U}(0, 1)$, pdf $f_X(x) = 1,\ x \in [0, 1]$

Square Root Coin-Toss Procedure

We have $Y = \sqrt{X}$

$$\Rightarrow y = \sqrt{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dx}{dy} = 2\sqrt{x} = 2y$$

By change of variables, we have

$$f_Y(y) = f_X(x) \,.\left|\frac{dx}{dy}\right| = 1.2y = 2y$$

That's why the pdf for Y has shape of a triangle and the probability is linearly increasing with slope 2.

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