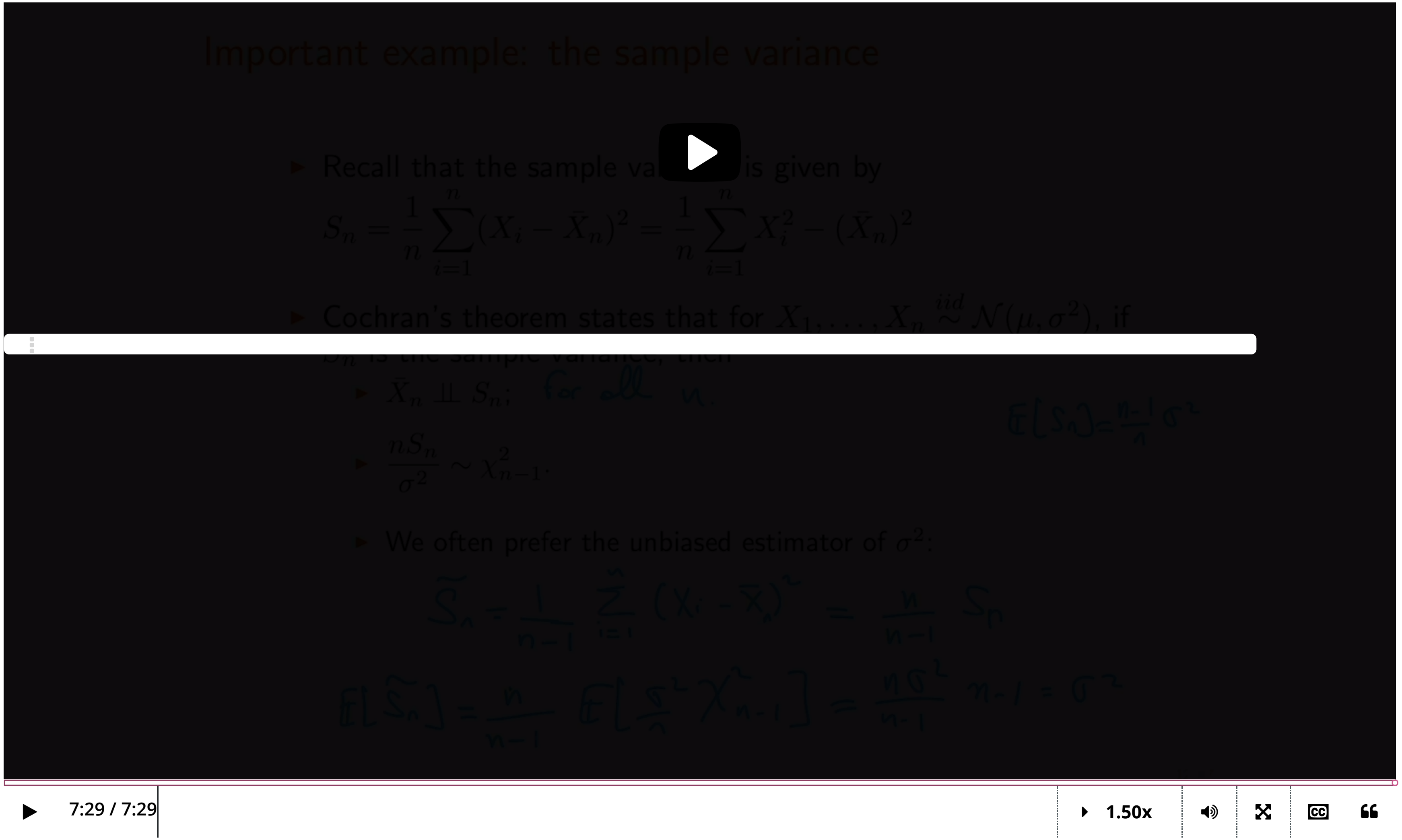
8. Sample Variance and Sample Mean of IID Gaussians: Cochran's Theorem

Cochran's Theorem: Independence of Gaussian Sample Variance and Sample Mean



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A Special Case of Cochran's Theorem I

3/3 points (graded)

Cochran's theorem states that if $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, then the sample variance

$$S_n := \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2\right) - \left(\overline{X}_n\right)^2$$

satisfies:

- $\overline{X}_n$ is independent of S_n, and
- $\frac{nS_n}{\sigma^2} \sim \chi^2_{n-1}$.

In this problem, you will verify that Cochran's theorem holds when $n = 2$. Let $X_1, X_2 \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$.

The expression S_2 can be written in the form A^2 where A is a polynomial in X_1 and X_2.

What is A^2?

Type **X_1** for X_1 and **X_2** for X_2.

$A^2 =$ (X_1-X_2)^2/4 **Answer:** (X_1 - X_2)^2/4

$\frac{(X_1 - X_2)^2}{4}$

The expression A from the previous question is a random variable, and moreover is distributed as $\mathcal{N}\left(\mu^*, (\sigma^*)^2\right)$ for some μ^* and σ^* that can be expressed in terms of the original parameters μ and σ. (Note: A can have two forms, but both would have the same distribution by symmetry).

What is μ^* expressed in terms of μ and σ?

$\mu^* =$ 0 **Answer:** 0.0

0

What is $(\sigma^*)^2$ expressed in terms of μ and σ?

$(\sigma^*)^2 =$ sigma^2/2 ✔ **Answer:** sigma^2/2

$\frac{\sigma^2}{2}$

STANDARD NOTATION

Solution:

Observe that

$$S_n = \frac{X_1^2 + X_2^2}{2} - \left(\frac{X_1 + X_2}{2}\right)^2 = \frac{X_1^2}{4} + \frac{X_2^2}{4} - \frac{1}{2}X_1X_2 = \left(\frac{X_1 - X_2}{2}\right)^2.$$

Hence, we can take $A = \pm\frac{X_1 - X_2}{2}$ (either choice has the same distribution, by symmetry). Next,

$$\mathbb{E}[A] = \frac{1}{2}\mathbb{E}[X_1 - X_2] = \frac{1}{2}(\mu - \mu) = 0,$$

and

$$\mathrm{Var}(A) = \mathrm{Var}\left(\frac{X_1 - X_2}{2}\right) = \frac{1}{4}(\mathrm{Var}(X_1) + \mathrm{Var}(X_2)) = \frac{\sigma^2}{2}.$$

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

A Special Case of Cochran's Theorem II

4/4 points (graded)

As above, let $X_1, X_2 \stackrel{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$.

Recall the random variable A that you found in the previous problem in terms of X_1 and X_2.

Let $\overline{X}_2 = \dfrac{X_1 + X_2}{2}$, i.e. $\overline{X}_n$ when $n = 2$.

What is $\mathbb{E}[A\overline{X}_2]$?

0

✔ **Answer:** 0.0

Using the answer above, which of the following are true? (Choose all that apply.)

- [x] A and $\overline{X}_2$ are independent.
- [ ] A and $\overline{X}_2$ are not independent.
- [ ] $A, \overline{X}_2 \sim \mathcal{N}(0, 2\sigma^2)$.
- [x] $A \sim \mathcal{N}(0, \sigma^2/2)$ and $\overline{X}_2 \sim \mathcal{N}(\mu, \sigma^2/2)$.

✔

For some expression B in terms of σ^2, the random variable $BS_2 \sim \chi^2$. What is B?

$B =$ 2/sigma^2

$\frac{2}{\sigma^2}$

✔ **Answer:** 2/sigma^2

STANDARD NOTATION

How many degrees of freedom does the χ^2 random variable BS_2 have?

1

Answer: 1

Solution:

Recall that $A = \frac{X_1 - X_2}{2}$ and $\overline{X}_2 = \frac{X_1 + X_2}{2}$. Hence,

$$\mathbb{E}\left[A\overline{X}_2\right] = \frac{1}{4}\mathbb{E}\left[\left(X_1 - X_2\right)\left(X_1 + X_2\right)\right] = \frac{1}{4}(\sigma^2 - \sigma^2) = 0.$$

As jointly Gaussian variables (why is it that A and $\overline{X}_2$ are jointly Gaussian?) that are uncorrelated are also independent, A and $\overline{X}_2$ are independent. By the previous problem, we know $A \sim \mathcal{N}\left(0, \sigma^2/2\right)$. A quick calculation shows that $\overline{X}_2 \sim \mathcal{N}\left(\mu, \sigma^2/2\right)$. Hence, the first and last choices are correct in the multiple choice question.

Observe that

$$\frac{2}{\sigma^2} S_2 = \frac{2}{\sigma^2}\left(\frac{X_1 - X_2}{2}\right)^2 = \left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right)^2,$$

and $\frac{X_1 - X_2}{\sqrt{2}\sigma} \sim \mathcal{N}\left(0, 1\right)$. By definition, $\frac{2}{\sigma^2} S_2 \sim \chi_1^2$.

Remark: The last question shows that $\frac{2}{\sigma^2} S_2 \sim \chi_1^2$, which verifies the second claim in Cochran's theorem for this special case. To show the first part of Cochran's theorem, that S_2 and $\overline{X}_2$ are independent, recall that we showed $A = \sqrt{S_2}$ is independent of $\overline{X}_2$. By a standard fact of probability, this also implies that $A^2 = S_2$ is independent of $\overline{X}_2$.

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You have used 1 of 3 attempts

Answers are displayed within the problem

Concept Check: Cochran's Theorem and Unbiased Sample Variance

1/1 point (graded)

Let $X_1, \ldots, X_n$ be i.i.d. and distributed according to $\mathcal{N}(0, \sigma^2)$. Let $\overline{X}_n = \frac{X_1+X_2+\cdots+X_n}{n}$. What is the distribution of $\frac{(n-1)\widetilde{S}_n}{\sigma^2}$, where $\widetilde{S}_n$ is the the unbiased sample variance of $X_1, \ldots, X_n$:

$$\widetilde{S}_n = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2$$

Type **Cn** for chi-squared distribution with n degrees of freedom, **Cn1** for chi-squared distribution with $n-1$ degrees of freedom.

Cn1 ✔ **Answer:** Cn1 + 0*Cn

$Cn1$

STANDARD NOTATION

Solution:

By Cochran's theorem,

$$\frac{nS_n}{\sigma^2} \sim \chi^2_{n-1}$$
$$\iff \frac{(n-1)\,\widetilde{S}_n}{\sigma^2} \sim \chi^2_{n-1}$$

Remark: We will use the random variable $\frac{\widetilde{S}_n}{\sigma^2}$ in the upcoming videos in what is called the Student's T Test. The point of this problem was to show that $\frac{(n-1)\widetilde{S}_n}{\sigma^2}$ is a χ^2_{n-1} random variable, thereby showing that the distribution of $\frac{\widetilde{S}_n}{\sigma^2}$ is the distribution of a χ^2_{n-1} random variable scaled by $\frac{1}{n-1}$.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

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