8. A new boundary condition: insulated ends

Warm up problem

1/1 point (graded)
Solve the differential equation with boundary conditions:

$$v''(x) = \lambda v(x)\,,\ \text{for } 0 < x < \pi, \qquad v'(0) = 0,\ \ v'(\pi) = 0.$$

The nonzero solutions are:

- ( ) $c\sin(nx)$ for $\lambda = -n^2$, where $n = 1, 2, 3, \ldots$.
- ( ) $c\cos(nx)$ for $\lambda = -n^2$, where $n = 1, 2, 3, \ldots$.
- (•) $c\cos(nx)$ for $\lambda = -n^2$, where $n = 0, 1, 2, 3, \ldots$.
- ( ) $a\cos(nx) + b\sin(nx)$ for $\lambda = -n^2$, where $n = 1, 2, 3, \ldots$.
- ( ) $a\cos(nx) + b\sin(nx)$ for $\lambda = -n^2$, where $n = 0, 1, 2, 3, \ldots$.

✔

Solution:

There are three cases to consider.

1. If $\lambda > 0$, then the general solution is $v(x) = c_1 e^{nx} + c_2 e^{-nx}$ where $n^2 = \lambda$. The boundary conditions give

$$\begin{aligned} nc_1 - nc_2 &= 0 \qquad \longrightarrow c_1 = c_2 \\ nc_1 e^{n\pi} - nc_1 e^{-n\pi} &= 0 \end{aligned}$$

   The last equation cannot hold for any $n > 0$, therefore there are no solutions in this case.

2. If $\lambda = 0$, then the general solution is $v(x) = c_1 x + c_2$. The first boundary condition tells us that $c_1 = 0$. The second boundary condition tells us that $c_1 = 0$ too. Therefore there are infinitely many solutions of the form $v(x) = c_2$, where c_2 is any constant.

3. If $\lambda < 0$, then the general solution is $v(x) = c_1 \cos(nx) + c_2 \sin(nx)$ where $\lambda = -n^2$. Applying the boundary conditions we find

$$\begin{aligned} -c_1 n \sin(0) + c_2 n \cos(0) &= 0, \qquad \longrightarrow c_2 = 0 \\ -c_1 n \sin(n\pi) &= 0, \qquad \longrightarrow c_1 = 0 \text{ or } \sin(n\pi) = 0. \end{aligned}$$

   The nontrivial solution occurs when n is an integer, in other words for $\lambda = -n^2$ and $n = 0, 1, 2, 3, \ldots$. Therefore the general solution is $v(x) = c\cos(nx)$ for $\lambda = -n^2$. (Note that by allowing $n = 0$ we are covering the case $\lambda = 0$ of constant solutions.)

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You have used 1 of 2 attempts

Answers are displayed within the problem

Problem 8.1 Consider the same insulated uniform metal rod as before ($\nu = 1$, length π) but now assume that the ends are insulated too (instead of exposed and held in ice), and that the initial temperature is given by $\theta(x, 0) = x$ for $x \in (0, \pi)$. Now what is $\theta(x, t)$?

Solution: As usual, we temporarily forget the initial condition, and use it only at the end.

“Insulated ends" means that there is zero heat flow through the ends, so the heat flux density function $q \propto -\frac{\partial\theta}{\partial x}$ is 0 when $x = 0$ or $x = \pi$. In other words, “insulated ends" means that the boundary conditions are

$$\text{insulated ends}: \qquad \frac{\partial \theta}{\partial x}(0,t) = 0, \qquad \frac{\partial \theta}{\partial x}(\pi,t) = 0 \qquad \text{for all } t > 0, \tag{3.49}$$

instead of $\theta\left(0,t\right)=0$ and $\theta\left(\pi,t\right)=0$. So we need to solve the Heat Equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$$

with the boundary conditions for insulated ends. Separation of variables $\theta\left(x,t\right)=a\left(x\right)\,b\left(t\right)$ leads to

$$a''\left(x\right) = \lambda\, a\left(x\right) \qquad \text{with } a'\left(0\right)=0 \text{ and } a'\left(\pi\right)=0 \tag{3.50}$$

$$\dot{b}\left(t\right) = \lambda\, b\left(t\right). \tag{3.51}$$

for a constant λ. Looking at the cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$, we find that

$$\lambda = -n^2 \qquad \text{and} \qquad a\left(x\right) = \cos nx \text{ (times a scalar)}$$

where n is one of $0, 1, 2, \ldots$. (This time n starts at 0 since $\cos 0x$ is a nonzero function.) For each such $a\left(x\right)$, the corresponding b is $b\left(t\right)=e^{-n^2 t}$ (times a scalar), and the normal mode is

$$\theta_n\left(x,t\right) = e^{-n^2 t}\cos nx.$$

The case $n=0$ is the constant function 1, so the general solution is

$$\theta\left(x,t\right) = \frac{a_0}{2} + a_1 e^{-t}\cos x + a_2 e^{-4t}\cos 2x + a_3 e^{-9t}\cos 3x + \cdots.$$

Finally, we bring back the initial condition: substitute $t = 0$ and use the initial condition on the left to get

$$\theta(x, 0) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$

for all $x \in (0, \pi)$.

Even or odd

1/1 point (graded)

Given the form of the general solution (right hand side), what is the appropriate periodic extension of the intitial condition to use?

- [ ] The 2π-periodic Square wave, defined by $\mathrm{Sq}(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$.
- [x] The 2π-periodic Triangle wave, defined by $T(x) = |x|$ on $-\pi < x < \pi$.
- [ ] The 2π-periodic Sawtooth wave, defined by $W(x) = x$ on $-\pi < x < \pi$.

✔

Solution:

The right hand side is a period 2π even function, so extend the left hand side to a period 2π even function gives $T(x)$, a triangle wave.

Recall that the Fourier series for the Triangle wave is

$$T(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right)$$

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You have used 1 of 2 attempts

Answers are displayed within the problem

The solution that satisfies the initial condition is

$$\theta(x,t) \quad = \quad \frac{\pi}{2} - \frac{4}{\pi}\left(e^{-t}\cos x + e^{-9t}\frac{\cos 3x}{9} + e^{-25t}\frac{\cos 5x}{25} + \cdots\right).$$

This answer makes physical sense: when the entire bar is insulated, its temperature tends to a constant equal to the average of the initial temperature.

Another setup

4/4 points (graded)

Suppose that a metal bar of length π initially has a temperature distribution given by $\theta(x,0) = x$ for $0 \le x \le \pi$. The right end is then put into an ice resevoir (temperature 0 degrees Celsius), and the left end is insulated.

Identify the boundary conditions. (If it is determined by the setup, enter the formula into the answer box. If it is unknown, type UNK into the answer box.)

For $t > 0,\ \ \theta(0,t) =$ UNK ✔

Answer: UNK

For $t > 0,\ \ \theta(\pi,t) =$ 0 ✔ **Answer:** 0

For $t > 0,\ \ \dfrac{\partial}{\partial x}\theta(0,t) =$ 0 ✔ **Answer:** 0

For $t > 0,\ \ \dfrac{\partial}{\partial x}\theta(\pi,t) =$ UNK ✔

Answer: UNK

FORMULA INPUT HELP

Solution:

The right end temperature being held at 0 degrees Celsius gives us the boundary condition $\theta(\pi, t) = 0$. The left end being insulated gives the boundary condition $\frac{\partial}{\partial x}\theta(0, t) = 0$. The other conditions must be determined from the partial differential equation and these boundary conditions.

Notice that the boundary conditions are not consistent with the initial conditions, because $\theta(x, t)$ is discontinuous at $t = 0$.

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Answers are displayed within the problem

One of the homework/recitation problems will lead you through the method to solve the Heat Equation with mixed boundary conditions of this type.

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TYPO 2

One solution of "Another setup" is questionable. 4

Why a_0/2?
Why not just a_0 given the function at t=0 is equal to 1? 2

What does the heat flux density function mean? 2