Math Counterexamples

ANALYSIS

A DISCONTINUOUS REAL CONVEX FUNCTION

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Consider a function f defined on a real interval $I \subset \mathbb{R}$. f is called convex if:

$$\forall x, y \in I \; \forall \lambda \in [0,1] : \; f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y)$$

Suppose that I is a closed interval: $I = [a, b]$ with $a < b$. For $a < s < t < u < b$ one can prove that:

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t}.$$

It follows from those relations that f has left-hand and right-hand derivatives at each point of the interior of I. And therefore that f is continuous at each point of the interior of I.
Is a convex function defined on an interval I continuous at all points of the interval? That might not be the case and a simple example is the function:

$$f: \left| \begin{array}{rcl} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 0 \text{ for } x \in (0,1) \\ x & \longmapsto & 1 \text{ else} \end{array} \right.$$

It can be easily verified that f is convex. However, f is not continuous at 0 and 1.

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