MITx 18.02.1x

Multivariable Calculus 1: Vectors and Derivatives

Help sandipan_dey

Course Progress Dates Calendar Discussion Notes

Course / Unit 2: Geometry of Derivatives / Lecture 6: Gradients

You are taking "Exam (Timed, No Correctness Feedback)" as a timed exam. Show more

End My Exam 43:58:32

Previous ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ Next

7. Maximize a linear function

Bookmark this page

Calculator Hide Notes

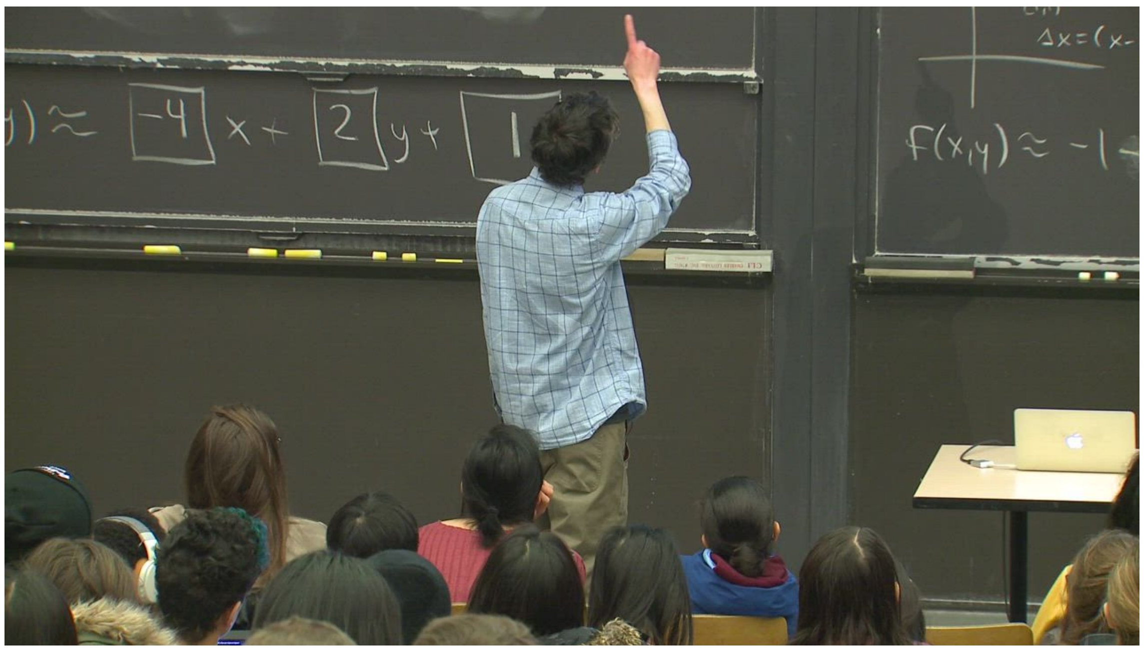
Lecture due Aug 18, 2021 20:30 IST Completed

Explore

Gradients are important because they tell us how quickly a function changes. In data science, machine learning algorithms refine models by finding the direction that minimizes an error function of the model's predictive capabilities in the fastest way possible. The key element to finding the direction which minimizes the error function fastest is the gradient.

Our goal here is to find the direction that maximizes (or minimizes) a general function $f(x, y)$. To understand this problem, we do what we always do in calculus: first we explore this question in a simpler context. The context that is easier to understand is the case of a linear function $L(x, y)$.

The problem



▶ 0:00 / 0:00 ▸ 2.0x

Start of transcript. Skip to the end.

PROFESSOR: So that is part of the significance

of the gradient.

It points perpendicular to the level curves.

But the gradient, like any vector, it has a direction

and it has a magnitude.

And the magnitude of the gradient is also important.

The next thing we're going to talk

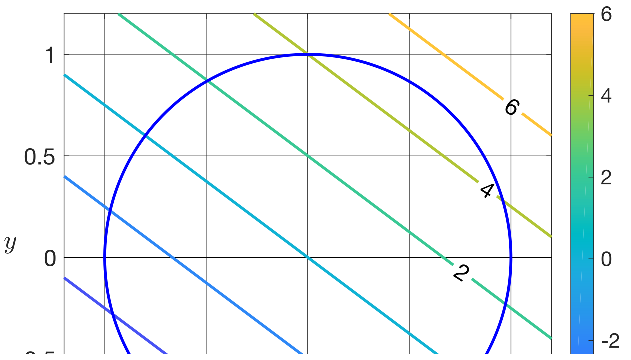
Video
Download video file

Transcripts
Download SubRip (.srt) file
Download Text (.txt) file

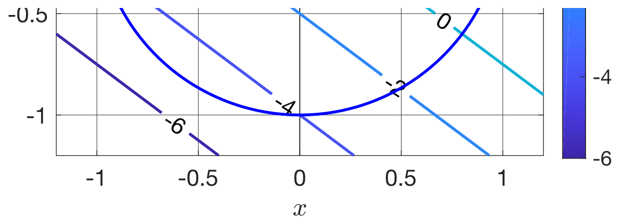
Warmup problem

1/1 point (graded)

Consider the linear function $L(x, y) = 3x + 4y$. We know that this function is a plane, and that the level curves are parallel lines. Let's start at the point $(0, 0)$. And move in a unit direction from the origin. The points a unit distance from the origin are indicated by the blue circle.



Calculator Hide Notes



If you move a unit distance from the origin in the direction indicated by the vector, which vector gives the largest increase in the value of $L(x,y)$?

○ $\langle 1, 0\rangle$

○ $\langle 0, 1\rangle$

○ $\langle 0, -1\rangle$

● $\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$

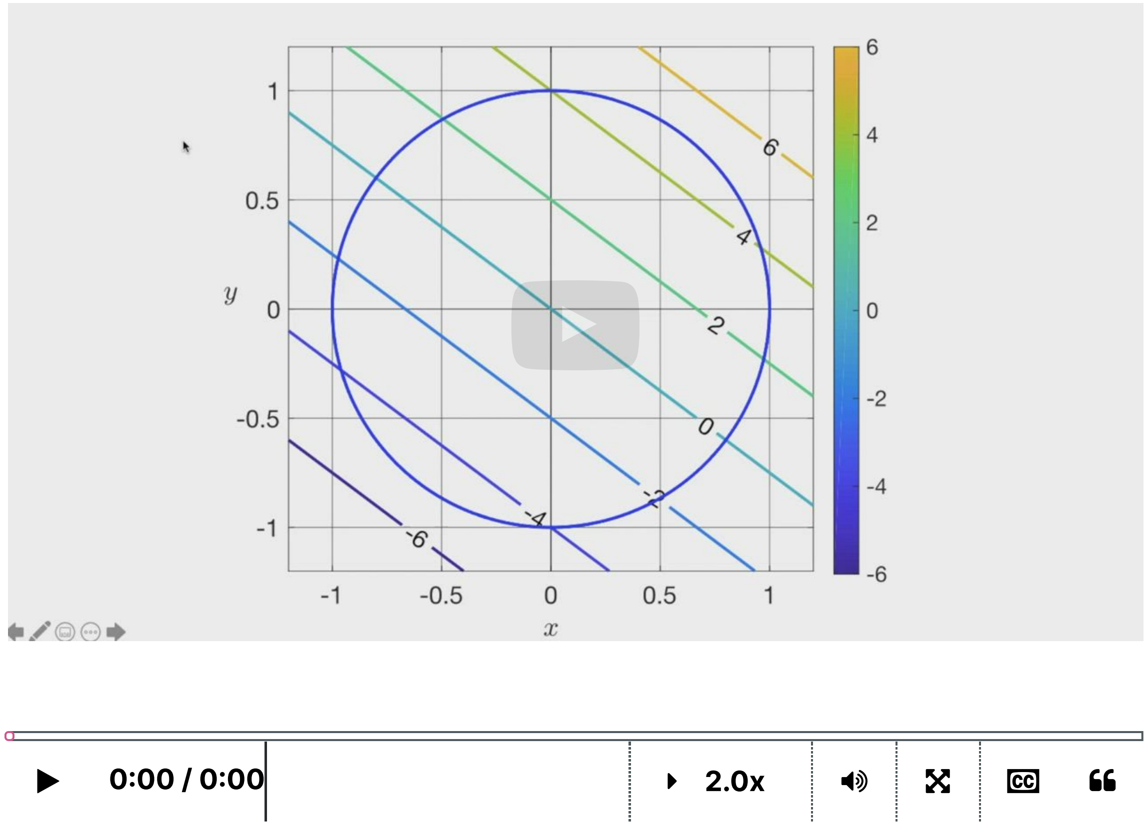
○ $\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$

✓

Submit You have used 1 of 2 attempts

Goal: Our goal is to find the direction $\langle x, y\rangle$ that gives the greatest possible increase in the linear function $L(x,y)$.

Which direction?



Start of transcript. Skip to the end.

PROFESSOR: So we can go anywhere in this blue circle,

and we want to make the function as big as possible.

You can see from the picture over here, the function is 4.

Here, it's bigger than 4.

The maximum is somewhere in that region over there

Video
Download video file

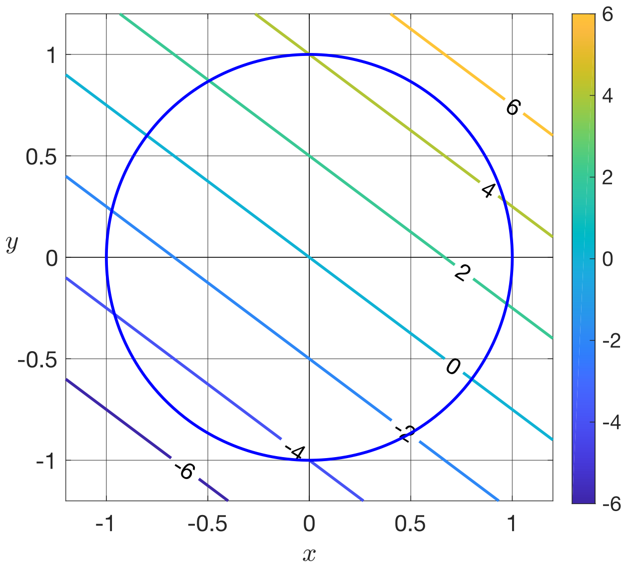
Transcripts
Download SubRip (.srt) file
Download Text (.txt) file

Consider the function

Calculator Hide Notes

$$L(x,y) = 3x + 4y.$$

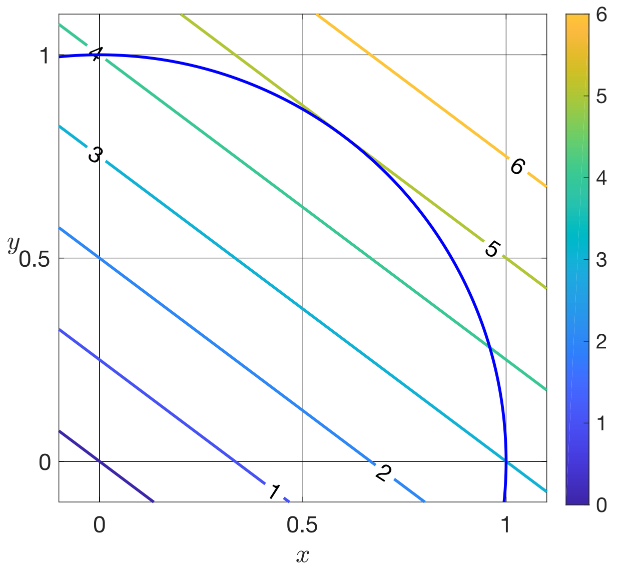
If we start at $(0,0)$ and can move in any direction by a distance 1, in which direction should we move to maximize L?



This figure shows the level curves of $L(x,y)$. The blue circle depicts all of the points that are a distance of 1 from the origin, in other words, the unit circle.

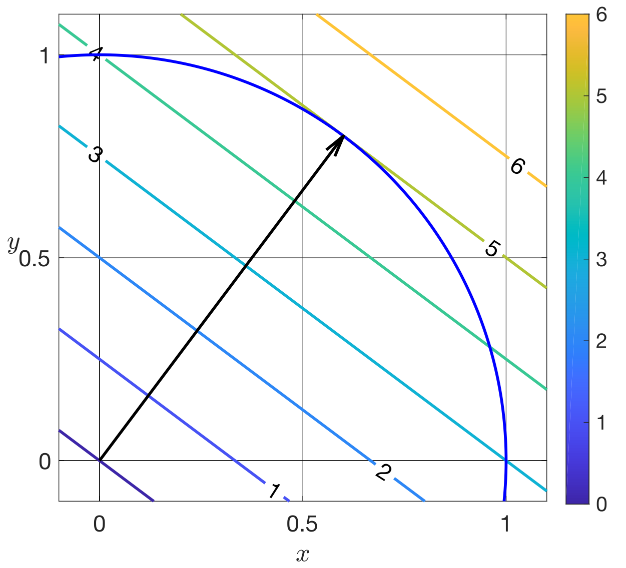
Answer 1 (Visual intuition): One way we might think about answering this is starting at some point on the circle and seeing if the function L gets bigger or smaller as we move along the circle. Remember that the level curves indicate heights of L.

If we start at, say, $(0,1)$, we are at the height $L=4$. If we move clockwise around the circle from this point, we get closer to the level curve of height $L=5$, so the function is increasing. We can see this more clearly in the zoomed in figure below.



At some point, we reach the maximum value because after that, moving along the circle causes us to move towards lower values of the level curves.

If we draw an arrow to that maximum point, it looks like it is the vector that is perpendicular to the level curves.



So our intuition is that the vector $\langle x, y\rangle$ that is perpendicular to the level curves of L will be in the direction we want. We know that $\langle 3,4\rangle$ is perpendicular to the level curves, i.e.,

Calculator Hide Notes

$$\langle x, y\rangle = \lambda\langle 3, 4\rangle \text{ for some } \lambda.$$

We also know that the point (x, y) is on the unit circle. Thus thinking of this point as a vector, $\langle x, y\rangle$ has magnitude 1, i.e.,

$$|\langle x, y\rangle| = 1.$$

With this information, we can find lambda and hence find $\langle x, y\rangle$. So we would do the computation

$$\begin{aligned} 1 = |\langle x, y\rangle| &= |\lambda\langle 3, 4\rangle| \\ &= \lambda\sqrt{9+16} \\ &= 5\lambda \implies \lambda = \frac{1}{5} \end{aligned}$$

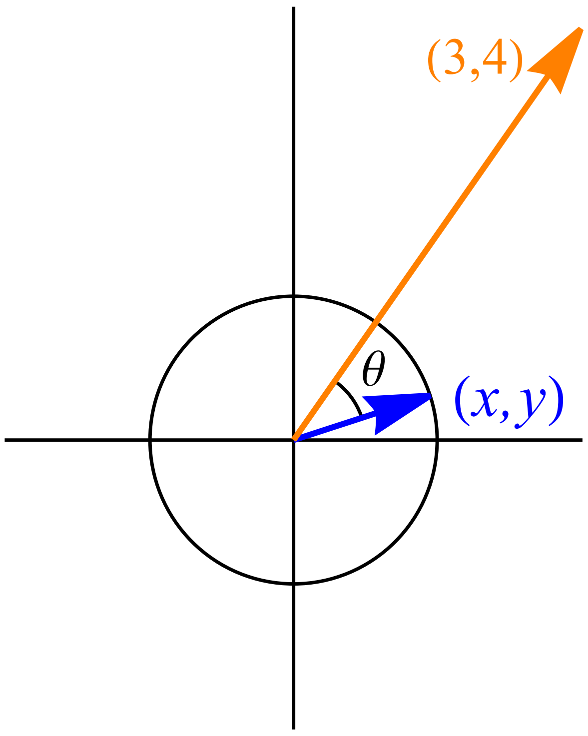
and so

$$\langle x, y\rangle = \frac{1}{5}\langle 3, 4\rangle = \langle \frac{3}{5}, \frac{4}{5}\rangle.$$

Answer 2 (Hidden dot product): We can rewrite $L(x, y)$ as the following dot product

$$3x + 4y = \langle 3, 4\rangle \cdot \langle x, y\rangle = |\langle 3, 4\rangle|\underbrace{|\langle x, y\rangle|}_{=1}\cos\theta = |\,(3, 4)\,|\cos\theta = 5\cos\theta,$$

where θ is the angle between $\langle 3, 4\rangle$ and $\langle x, y\rangle$. Notice that $|\langle x, y\rangle| = 1$ since the point (x, y) lies on the unit circle. To see what this dot product looks like, we'll draw the following picture.



In order to make $3x + 4y$ as large as possible, we need to choose θ so that $5\cos\theta$ is as large as possible. Based on this, how should we choose θ to maximize $L(x, y)$? In the poll below, choose one of the following.

1. $\theta = 0$ ($\langle x, y\rangle$ in the same direction as $\langle 3, 4\rangle$)
2. $\theta = \pi/2$ ($\langle x, y\rangle$ perpendicular to $\langle 3, 4\rangle$)
3. None of the above

Calculator Hide Notes

POLL

The numbers below refer to the options in the text above. Choose one.

RESULTS

● **1.**	**91%**
○ **2.**	**8%**
○ **3.**	**1%**

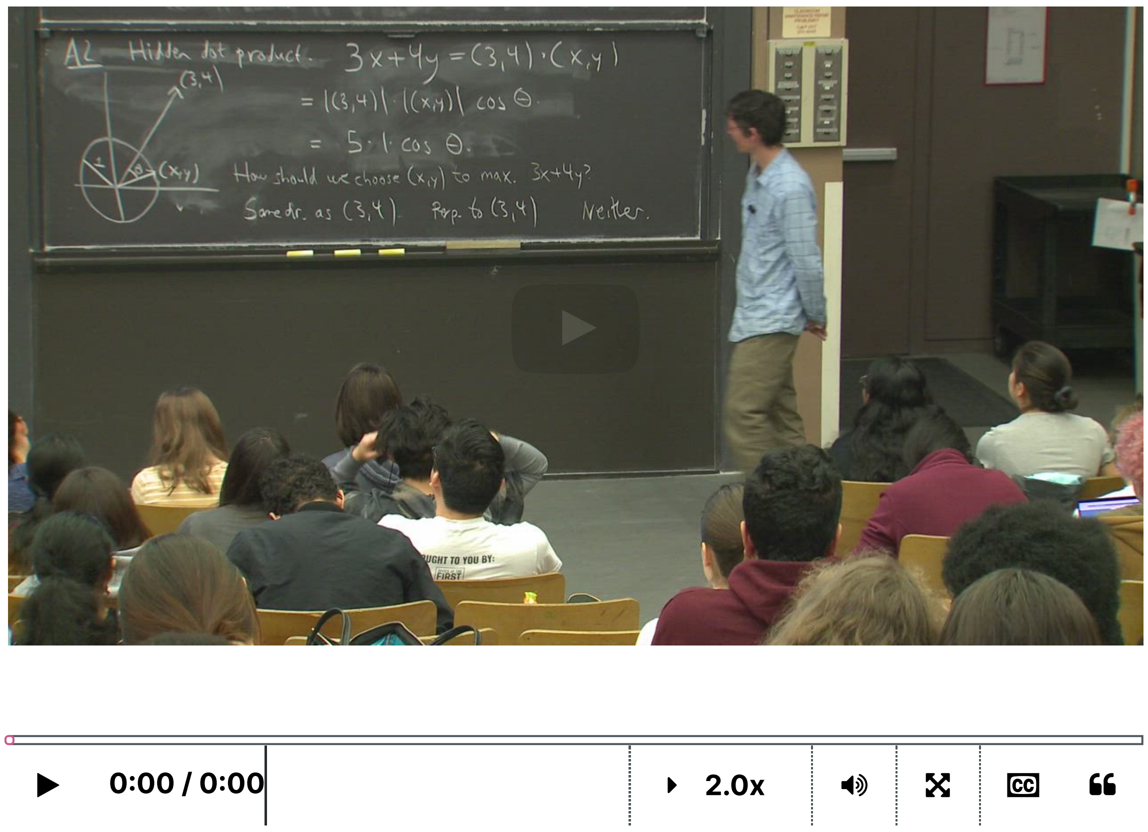
Submit

Results gathered from 514 respondents.

FEEDBACK

Your response has been recorded

What we learned



Start of transcript. Skip to the end.

PROFESSOR: OK.

Let's see what people are thinking.

So here are the choices.

Should choose (x, y) in the same direction as (3, 4),

or perpendicular to (3, 4), or neither?

So if it looks to you like it's the same direction as (3, 4),

thumbs up, perpendicular to (3, 4), and neither.

Video

Download video file

Transcripts

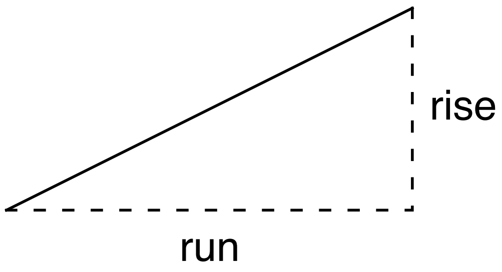
Download SubRip (.srt) file

Download Text (.txt) file

Magnitude of the gradient

The vector $\langle 3,4\rangle$ is in the direction where $L\left(x,y\right)$ increases most steeply. Moreover, $5=\left|\langle 3,4\rangle\right|$ is the slope of the plane.

For single variable functions, we think of the slope as the rise over the run.



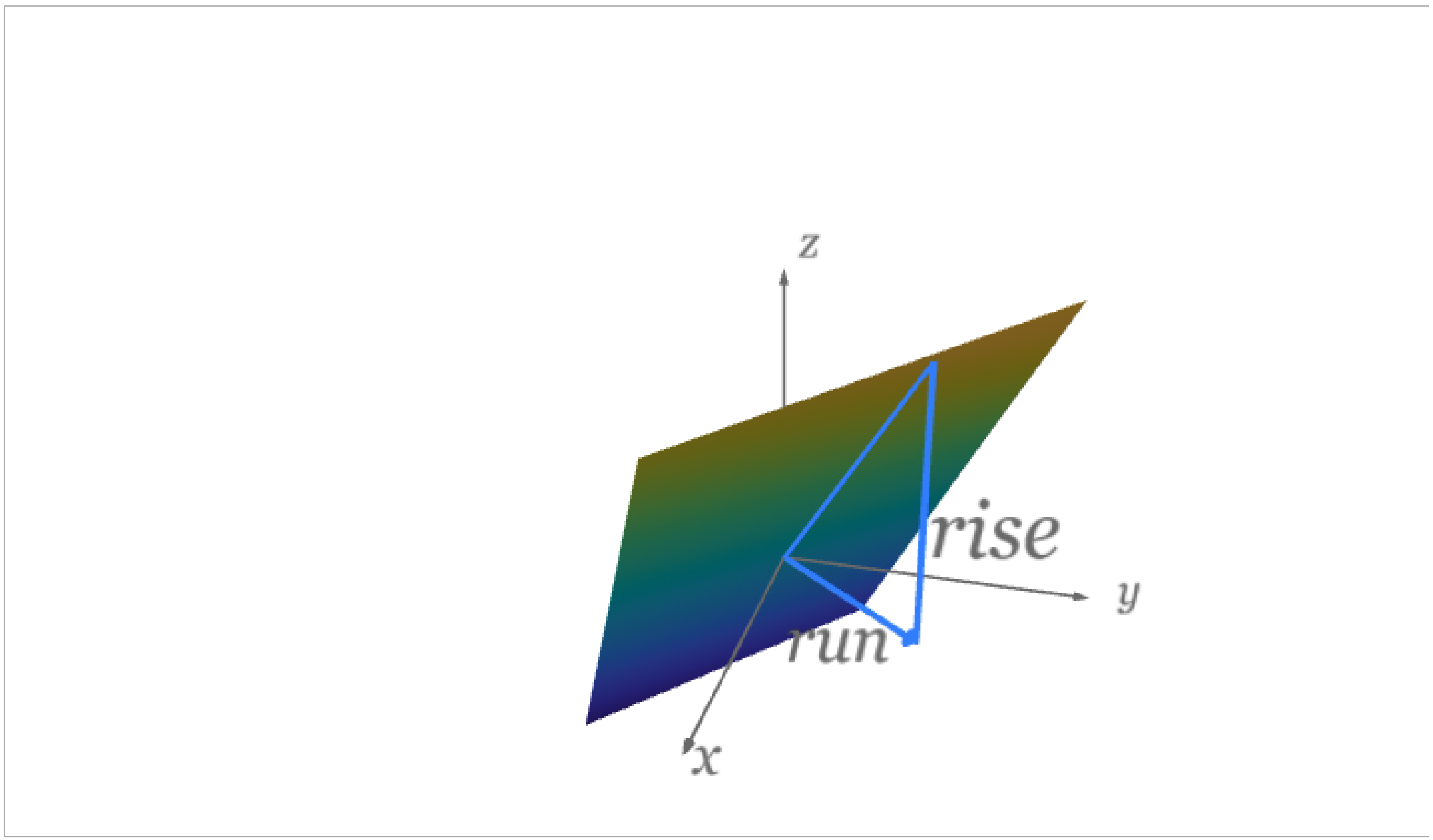
The idea of the slope is similar for multivariable functions. If we imagine the plane defined by $z=L\left(x,y\right)$ as a hill, the steepest direction is directly up the hill, which is the direction that lies above t

dimensions.

Calculator

Hide Notes

▶ **Slope of a plane in the direction of the gradient**



Question: If we move in the direction $\langle 3,4\rangle$ by a horizontal distance d, how much does L increase by?

Answer:

$$\frac{\text{Change in } L}{d} = \text{slope} = |\nabla L| = 5$$

The change in the value of L divided by d is a measure of the slope of the line connecting the initial and ending value on the graph of the plane $z = L(x,y)$. The main result is that this ratio of "rise" over "run" is also equal to $|\nabla L| = 5$. Therefore the amount L increases by is $5d$.

---

7. Maximize a linear function

Hide Discussion

Topic: Unit 2: Geometry of Derivatives / 7. Maximize a linear function

Add a Post

Show all posts ⌄ by recent activity ⌄

? Any chance there will be another extension for this module? 2
I've been preparing for a job interview and fell behind a bit on this one

The direction of the gradient as the maximum increase 3

‹ Previous Next ›

Calculator Hide Notes

edX

[About](#)
[Affiliates](#)
[edX for Business](#)
[Open edX](#)
[Careers](#)
[News](#)

Legal

[Terms of Service & Honor Code](#)
[Privacy Policy](#)
[Accessibility Policy](#)
[Trademark Policy](#)
[Sitemap](#)

Connect

[Blog](#)
[Contact Us](#)
[Help Center](#)
[Media Kit](#)
[Donate](#)

GET IT ON Google play
Download on the App Store

Calculator Hide Notes