3. Wave equation in MATLAB

Simple numerical method to solve the wave equation

We wish to numerically solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.67}$$

with the boundary conditions $u\left(0,t\right) = f\left(t\right)$ and $u\left(L,t\right) = g\left(t\right)$ and initial conditions $u\left(x,0\right) = q\left(x\right)$ and $\frac{\partial u}{\partial t}\left(x,0\right) = s\left(x\right)$.

We will use a **centered time and space** numerical scheme. Let u_j^i denote the solution at time $i\Delta t$ and position $j\Delta x$. Then the discrete (centered) second time derivative is

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{\Delta t^2} + \text{terms of order } \left(\Delta t^2\right) \text{ and higher,} \tag{3.68}$$

and the discrete (centered) second space derivative is

$$\frac{\partial^2 u}{\partial x^2} = \frac{u^i_{j+1} - 2u^i_j + u^i_{j-1}}{\Delta x^2} + \text{terms of order } \left(\Delta x^2\right) \text{ and higher.} \tag{3.69}$$

Substituting the discrete time and space derivatives into the wave equation gives

$$\begin{aligned}
\frac{u^{i+1}_j - 2u^i_j + u^{i-1}_j}{\Delta t^2} &= c^2 \frac{u^i_{j+1} - 2u^i_j + u^i_{j-1}}{\Delta x^2} \\
u^{i+1}_j &= \frac{c^2 \Delta t^2}{\Delta x^2}\left(u^i_{j+1} - 2u^i_j + u^i_{j-1}\right) + 2u^i_j - u^{i-1}_j \\
u^{i+1}_j &= r^2 u^i_{j+1} + 2\left(1 - r^2\right) u^i_j + r^2 u^i_{j-1} - u^{i-1}_j, \quad r = \frac{c\Delta t}{\Delta x}.
\end{aligned}$$

In matrix notation, this is

$$\begin{pmatrix} u^{i+1}_1 \\ u^{i+1}_2 \\ \vdots \\ u^{i+1}_{N-1} \\ u^{i+1}_N \end{pmatrix} = \begin{pmatrix} 2\left(1-r^2\right) & r^2 & & & \\ r^2 & 2\left(1-r^2\right) & r^2 & & \\ & \ddots & \ddots & \ddots & \\ & & r^2 & 2\left(1-r^2\right) & r^2 \\ & & & r^2 & 2\left(1-r^2\right) \end{pmatrix} \begin{pmatrix} u^i_1 \\ u^i_2 \\ \vdots \\ u^i_{N-1} \\ u^i_N \end{pmatrix} - \begin{pmatrix} u^{i-1}_1 \\ u^{i-1}_2 \\ \vdots \\ u^{i-1}_{N-1} \\ u^{i-1}_N \end{pmatrix}.$$

where at each time step i we impose the boundary conditions $u^i_1 = f\left(i\Delta t\right)$ and $u^i_N = g\left(i\Delta t\right)$. Note that to compute u^{i+1}, we need information about u^i and u^{i-1}. So, how do we start the method (ensuring that our error has order less than Δt^2)? We have that

$$\begin{aligned}
\frac{u\left(x,\Delta t\right) - u\left(x,0\right)}{\Delta t} &= \frac{\partial u}{\partial t}\left(x,0\right) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \mathcal{O}\left(\Delta t^2\right) \\
u\left(x,\Delta t\right) &= u\left(x,0\right) + \Delta t\frac{\partial u}{\partial t}\left(x,0\right) + \frac{c^2\Delta t^2}{2}\frac{\partial^2 u}{\partial x^2}\left(x,0\right) + \mathcal{O}\left(\Delta t^2\right)
\end{aligned} \tag{3.70}$$

Discretizing gives

$$u_j^1 = q_j + \Delta t\, s_j + \frac{r^2}{2}(q_{j+1} - 2q_j + q_{j-1})\,.$$

Condition for numerical stability

$$\boxed{\frac{c\Delta t}{\Delta x} \leq 1.}$$

Download the example script

Download the following script to see how to solve the wave equation using MATLAB Online. This example has a dynamic input (it varies in time).

(Use short cut commands for copy and paste: ctrl-c and ctrl-v on windows, and cmd-c, cmd-v on a MAC.)

```
url = 'https://courses.edx.org/asset-v1:MITx+18.03Fx+3T2018+type@asset+block@waveEqn.m';
websave('waveEqn.m',url)
```

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