MATHEMATICS

Differentiating an Inner Product

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If $(V, \langle\cdot,\cdot\rangle)$ is a finite-dimensional inner product space and $f, g : \mathbb{R} \longrightarrow V$ are differentiable functions, a straightforward calculation with components shows that

$$\frac{d}{dt}\langle f, g\rangle = \langle f(t), g'(t)\rangle + \langle f'(t), g(t)\rangle$$

This approach is not very satisfying. However, attempting to apply the definition of the derivative directly doesn't seem to work for me. Is there a slick, perhaps intrinsic way, to prove this that doesn't involve working in coordinates?

real-analysis functional-analysis inner-product-space derivatives

edited Jan 4 '12 at 4:48
t.b.
66.3k 7 221 303

asked Jan 4 '12 at 3:21
ItsNotObvious
8,341 6 41 80

2 Answers

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Observe that

$$\begin{aligned}
&\frac{1}{h}[\langle f(t+h),\, g(t+h)\rangle - \langle f(t),\, g(t)\rangle] \\
&\quad= \frac{1}{h}[\langle f(t+h),\, g(t+h)\rangle - \langle f(t),\, g(t+h)\rangle] + \frac{1}{h}[\langle f(t),\, g(t+h)\rangle - \langle f(t),\, g(t)\rangle] \\
&\quad= \left\langle \frac{1}{h}[f(t+h) - f(t)]\,,\, g(t+h) \right\rangle + \left\langle f(t),\, \frac{1}{h}[g(t+h) - g(t)] \right\rangle.
\end{aligned}$$

As $h \to 0$ the first expression converges to

$$\frac{d}{dt}\langle f(t), g(t)\rangle$$

and the last expression converges to

$$\langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle$$

by definition of the derivative, by continuity of g and by continuity of the scalar product. Hence the desired equality follows.

Note that this doesn't use finite-dimensionality and that the argument is the exact same as the one for the ordinary product rule from calculus.

edited Jan 4 '12 at 18:27 answered Jan 4 '12 at 3:36
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This answer may be needlessly complicated if you don't want such generality, taking the approach of first finding the Fréchet derivative of a bilinear operator.

If V, W, and Z are normed spaces, and if $T : V \times W \to Z$ is a continuous (real) bilinear operator, meaning that there exists $C \geq 0$ such that $\|T(v,w)\| \leq C\|v\|\|w\|$ for all $v \in V$ and $w \in W$, then the derivative of T at (v_0, w_0) is $DT|_{(v_0,w_0)}(v,w) = T(v, w_0) + T(v_0, w)$. (I am assuming that $V \times W$ is given a norm equivalent with $\|(v,w)\| = \sqrt{\|v\|^2 + \|w\|^2}$.) This follows from the straightforward computation

$$\frac{\|T(v_0+v, w_0+w) - T(v_0,w_0) - (T(v,w_0) + T(v_0,w))\|}{\|(v,w)\|} = \frac{\|T(v,w)\|}{\|(v,w)\|} \leq C\frac{\|v\|\|w\|}{\|(v,w)\|} \to 0$$

as $(v,w) \to 0$.

With $V = W$, $Z = \mathbb{R}$ or $Z = \mathbb{C}$, and $T : V \times V \to Z$ the inner product, this gives $DT_{(v_0,w_0)}(v,w) = \langle v, w_0\rangle + \langle v_0, w\rangle$. Now if $f, g : \mathbb{R} \to V$ are differentiable, then $F : \mathbb{R} \to V \times V$ defined by $F(t) = (f(t), g(t))$ is differentiable with $DF|_t(h) = h(f'(t), g'(t))$. By the chain rule,

$$D(T \circ F)|_t(h) = DT|_{F(t)} \circ DF|_t(h) = h(\langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle),$$

which means $\frac{d}{dt}\langle f, g\rangle = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle$.

answered Jan 4 '12 at 4:31
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2 I notice that $\langle v, w_0\rangle + \langle v_0, w\rangle$ is not linear in (v,w). (I.e. it's conjugate linear in w). So how can this be the derivative? I stumbled upon this page with exactly this question in mind. – Eric Auld Jan 16 '15 at 1:02

3 @Eric: It is real linear. That it what is used here, as noted above, "if T... is a continuous (real) bilinear operator...". – Jonas Meyer Jan 16 '15 at 2:33