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Norm inequalities for sums and differences of positive operators

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Abstract

It is shown that if A and B are positive operators on a separable complex Hilbert space, and if $|||\cdot|||$ is any unitarily invariant norm, then

$$2|||A \oplus B \oplus 0 \oplus 0||| \leqslant |||(A - B) \oplus (A - B) \oplus 0 \oplus 0||| + |||A \oplus A \oplus B \oplus B||| + |||A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}|||.$$

When specialized to the usual operator norm $\|\cdot\|$, this inequality reduces to

$$\max(\|A\|, \|B\|) - \|A^{1/2}B^{1/2}\| \leqslant \|A - B\|.$$

Related inequalities for sums and differences of positive operators are obtained, and applications of these inequalities to norms of self-commutators are also considered.

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1. Introduction

The self-commutator of a Hilbert space operator A is the self-adjoint operator $A^*A - AA^*$. It is known that

$$\|A\|^2 - \|A^2\| \leqslant \|A^*A - AA^*\| \leqslant \|A\|^2, \tag{1}$$

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where $\|\cdot\|$ denotes the usual operator (or the spectral) norm. The first inequality in (1), which is a comparison between two measures of non-normality, has been recently proved in [11], and the second inequality in (1) has been proved earlier in [6]. A generalization, in the finite-dimensional setting, of the first inequality in (1) to the case involving two matrices has been recently given in [15]. Though this generalization has been confined to matrices regarded as operators on a finite-dimensional Hilbert space, by a slight modification it can be extended to operators on an infinite-dimensional Hilbert space. Generalizations of the second inequality in (1) can be found in [2] (see, also, [1, p. 280]).

Using ideas similar to those employed in [11] to prove the first inequality in (1), it can be shown that if A and B are positive operators on a Hilbert space, then

$$\max(\|A\|, \|B\|) - \|A^{1/2}B^{1/2}\| \leqslant \|A - B\|. \tag{2}$$

The inequality (2) yields Theorem 4 in [15], which generalizes the first inequality in (1) for matrices. It also supplements an inequality given in [10] concerning sums of positive operators, which asserts that if A and B are positive operators on a Hilbert space, then

$$\|A + B\| \leqslant \max(\|A\|, \|B\|) + \|A^{1/2}B^{1/2}\|. \tag{3}$$

A weaker version of the inequality (3), where $\|A^{1/2}B^{1/2}\|$ is replaced by $\|AB\|^{1/2}$, has been useful in the theory of best approximation in C*-algebras given in [5]. The inequality (3) is a special case of a general norm inequality involving 2×2 operator matrices, which asserts that if A and B are positive operators in a norm ideal of operators on a Hilbert space, then

$$|||(A + B) \oplus 0||| \leqslant |||A \oplus B||| + |||A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}|||, \tag{4}$$

where $|||\cdot|||$ is an associated unitarily invariant norm.

In this paper we establish a general norm inequality involving 4×4 operator matrices, from which the inequality (2) follows as a special case. We also obtain several norm inequalities for sums and differences of positive operators, which enable us to give refinements of the inequalities (1)–(3). Our inequalities seem natural enough and applicable to be widely useful. For a host of norm inequalities concerning sums and differences of positive operators, and for diverse applications of these inequalities, we refer to [1,3,4,9,14], and references therein.

2. Main result

Let $B(H)$ denote the C*-algebra of all bounded linear operators on a separable complex Hilbert space H. In addition to the usual operator norm, which is defined on all of $B(H)$, we consider unitarily invariant (or symmetric) norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$, and for the sake of brevity, we will make no explicit mention of this ideal. Thus, when we talk of $|||T|||$, we are assuming that the operator T belongs to the norm ideal associated with $|||\cdot|||$. Moreover, each unitarily invariant norm $|||\cdot|||$ enjoys the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $B(H)$. For the theory of unitarily invariant norms, we refer to [1,8], or [13].

If S and T are operators in $B(H)$, we write the direct sum $S \oplus T$ for the 2×2 operator matrix $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$, regarded as an operator on $H \oplus H$. Thus, for the usual operator norm,

$$\|S \oplus T\| = \max(\|S\|, \|T\|). \tag{5}$$

It follows easily from the basic properties of unitarily invariant norms that

$$|||S||| = |||\,|S|\,||| = |||S^*|||, \tag{6}$$

$$|||S^*S||| = |||SS^*|||, \tag{7}$$

$$|||\,|S|\,|T|\,||| = |||ST^*|||, \tag{8}$$

$$|||S \oplus T||| = \left|\left|\left|\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right|\right|\right|, \tag{9}$$

$$|||S \oplus S^*||| = |||S \oplus S|||, \tag{10}$$

where $|S| = (S^*S)^{1/2}$ is the absolute value of S.

Our main result can be stated as follows.

Theorem 1. *If A and B are positive operators in $B(H)$, then*

$$\begin{aligned} 2|||A \oplus B \oplus 0 \oplus 0||| \leqslant\ & |||(A - B) \oplus (A - B) \oplus 0 \oplus 0||| \\ & + |||A \oplus A \oplus B \oplus B||| \\ & + |||A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}|||. \end{aligned} \tag{11}$$

Proof. First, observe that

$$\begin{aligned} 2A \oplus 2B \oplus 0 \oplus 0 =\ & ((A - B) \oplus (B - A) \oplus 0 \oplus 0) \\ & + ((A + B) \oplus (A + B) \oplus 0 \oplus 0). \end{aligned}$$

Hence, by the basic properties of unitarily invariant norms and the triangle inequality, we have

$$\begin{aligned} 2|||A \oplus B \oplus 0 \oplus 0||| \leqslant\ & |||(A - B) \oplus (A - B) \oplus 0 \oplus 0||| \\ & + |||(A + B) \oplus 0 \oplus (A + B) \oplus 0|||. \end{aligned} \tag{12}$$

If $S = \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}$, then $S^*S = (A+B) \oplus 0$ and $SS^* = \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix}$. Applying the property (7) to the operator $S \oplus S$, we have

$$
\begin{aligned}
&|||(A+B) \oplus 0 \oplus (A+B) \oplus 0||| \\
&\quad = \left|\left|\left| \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} \oplus \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} \right|\right|\right|. \qquad (13)
\end{aligned}
$$

Now, by the inequality (12), the identity (13), the triangle inequality, and the properties (6), (9), (10), we conclude that

$$
\begin{aligned}
&2|||A \oplus B \oplus 0 \oplus 0||| \\
&\quad \leqslant |||(A-B) \oplus (A-B) \oplus 0 \oplus 0||| \\
&\quad\quad + \left|\left|\left| \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} \oplus \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} \right|\right|\right| \\
&\quad \leqslant |||(A-B) \oplus (A-B) \oplus 0 \oplus 0||| + |||A \oplus B \oplus A \oplus B||| \\
&\quad\quad + \left|\left|\left| \begin{bmatrix} 0 & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & 0 \end{bmatrix} \right|\right|\right| \\
&\quad = |||(A-B) \oplus (A-B) \oplus 0 \oplus 0||| + |||A \oplus A \oplus B \oplus B||| \\
&\quad\quad + |||A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}|||,
\end{aligned}
$$

as required. □

Specializing Theorem 1 to the usual operator norm, we obtain the inequality (2), which when combined with the inequality (3), together with some obvious inequalities, furnishes the following chain of norm inequalities for sums and differences of positive operators.

Corollary 1. *If A and B are positive operators in $B(H)$, then*

$$
\begin{aligned}
\max(\|A\|, \|B\|) - \|A^{1/2}B^{1/2}\| &\leqslant \|A - B\| \\
&\leqslant \max(\|A\|, \|B\|) \\
&\leqslant \|A + B\| \\
&\leqslant \max(\|A\|, \|B\|) + \|A^{1/2}B^{1/2}\|. \qquad (14)
\end{aligned}
$$

It is easy to see that for positive operators A and B in $B(H)$, $AB = 0$ if and only if $A^{1/2}B^{1/2} = 0$. The "if" part is obvious, and the "only if" part is an immediate consequence of the Löwner–Heinz type inequality

$$\|A^{1/2}B^{1/2}\| \leqslant \|AB\|^{1/2} \qquad (15)$$

(see, e.g., [7] or [10]). Thus, it is evident that equality holds through the chain of the inequalities (14) if and only if $AB = 0$, i.e., if and only if A and B have orthogonal ranges.

3. Remarks

1. For general (i.e., not necessarily positive) operators A and B in $B(H)$, applying the inequalities (2) and (3) to the positive operators A^*A and BB^*, using the fact that $\|T^*T\| = \|T\|^2$ for every operator T in $B(H)$, and invoking the property (8), we obtain the inequalities

$$\max(\|A\|^2, \|B\|^2) - \|AB\| \leqslant \|A^*A - BB^*\| \tag{16}$$

and

$$\|A^*A + BB^*\| \leqslant \max(\|A\|^2, \|B\|^2) + \|AB\|. \tag{17}$$

The inequality (16), which is a generalization of the first inequality in (1), has been proved in [15] for matrices using an analysis similar to that used in [11] to prove the first inequality in (1). Letting $A = B$ in the inequality (17), we have

$$\|A^*A + AA^*\| \leqslant \|A\|^2 + \|A^2\|, \tag{18}$$

which has been observed in [11].

2. In a recent paper [12], using a certain norm inequality for 2×2 operator matrices, it has been proved that if A and B are positive operators in $B(H)$, then

$$\|A + B\| \leqslant \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2}\right). \tag{19}$$

This is sharper than both the inequality (3) and the triangle inequality. In fact, it has been observed in [12] that

$$\begin{aligned}&\frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2}\right)\\ &\quad\leqslant \max(\|A\|, \|B\|) + \|A^{1/2}B^{1/2}\|\end{aligned} \tag{20}$$

and

$$\frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2}\right) \leqslant \|A\| + \|B\|. \tag{21}$$

3. If A and B are operators in $B(H)$, then, by the triangle inequality, we have

$$2\max(\|A\|, \|B\|) \leqslant \|A - B\| + \|A + B\|. \tag{22}$$

While the inequalities (19) and (20), when combined, give a refinement of the inequality (3), the inequalities (19), (20), and (22) can be utilized to show that if A and B are positive operators in $B(H)$, then

$$\begin{aligned}&\max(\|A\|, \|B\|) - \|A^{1/2}B^{1/2}\|\\ &\quad\leqslant 2\max(\|A\|, \|B\|) - \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2}\right)\\ &\quad\leqslant \|A - B\|,\end{aligned} \tag{23}$$

which is a refinement of the inequality (2).

It should be mentioned here that refinements of the inequalities (16) and (17) can be obtained from those of the inequalities (2) and (3) by considering the positive operators A^*A and BB^*.

4. For a positive operator A in $B(H)$, let $m(A) = \inf\{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$, and recall that $\|A\| = \sup\{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$. It is easy to see that $m(A) \geqslant 0$, with equality if and only if A is not invertible. Moreover, if A is invertible, then $m(A) = \|A^{-1}\|^{-1}$. It can be easily shown that if A and B are positive operators in $B(H)$, then

$$\|A - B\| \leqslant \max(\|A\|, \|B\|) - \min(m(A), m(B)) \tag{24}$$

and

$$\max(\|A\|, \|B\|) + \min(m(A), m(B)) \leqslant \|A + B\|. \tag{25}$$

In particular, if A and B are invertible, then

$$\|A - B\| \leqslant \max(\|A\|, \|B\|) - \min(\|A^{-1}\|^{-1}, \|B^{-1}\|^{-1}) \tag{26}$$

and

$$\max(\|A\|, \|B\|) + \min(\|A^{-1}\|^{-1}, \|B^{-1}\|^{-1}) \leqslant \|A + B\|. \tag{27}$$

The inequalities (26) and (27) improve the second and the third inequalities in Corollary 1, respectively.

5. For an operator A in $B(H)$, in spite of the property (7), if H is infinite-dimensional, then $m(A^*A)$ and $m(AA^*)$ may be different. To see this, consider the unilateral shift operator. But, if H is finite-dimensional, then $m(A^*A) = m(AA^*)$, which is the square of the smallest singular value of A. However, if A is invertible, then regardless of the dimension of H, $m(A^*A) = m(AA^*) = \|A^{-1}\|^{-2}$.

For general operators A and B in $B(H)$, applying the inequalities (24) and (25) to the positive operators A^*A and BB^*, we see that

$$\|A^*A - BB^*\| \leqslant \max(\|A\|^2, \|B\|^2) - \min(m(A^*A), m(BB^*)) \tag{28}$$

and

$$\max(\|A\|^2, \|B\|^2) + \min(m(A^*A), m(BB^*)) \leqslant \|A^*A + BB^*\|. \tag{29}$$

In particular, if A and B are invertible, then

$$\|A^*A - BB^*\| \leqslant \max(\|A\|^2, \|B\|^2) - \min(\|A^{-1}\|^{-2}, \|B^{-1}\|^{-2}) \tag{30}$$

and

$$\max(\|A\|^2, \|B\|^2) + \min(\|A^{-1}\|^{-2}, \|B^{-1}\|^{-2}) \leqslant \|A^*A + BB^*\|. \tag{31}$$

6. Letting $A = B$ in the inequalities (30) and (31), we have

$$\|A^*A - AA^*\| \leqslant \|A\|^2 - \min(m(A^*A), m(AA^*)) \tag{32}$$

and

$$\|A\|^2 + \min(m(A^*A), m(AA^*)) \leqslant \|A^*A + AA^*\|. \tag{33}$$

In particular, if A is invertible, then

$$\|A^*A - AA^*\| \leqslant \|A\|^2 - \|A^{-1}\|^{-2}, \tag{34}$$

which improves the second inequality in (1), and

$$\|A\|^2 + \|A^{-1}\|^{-2} \leqslant \|A^*A + AA^*\|. \tag{35}$$

The inequalities (34) and (35) complement the first inequality in (1) and the inequality (18), respectively.

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