# 8. Main result and example with a nonlinear function

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Lecture due Aug 18, 2021 20:30 IST Completed

## Explore

We have just seen how to choose the direction in which a linear function has the steepest increase. This exact same procedure works for nonlinear functions too, because near a point, all functions can be approximated by linear functions!

The big result of this lecture is the following theorem.

**Theorem**

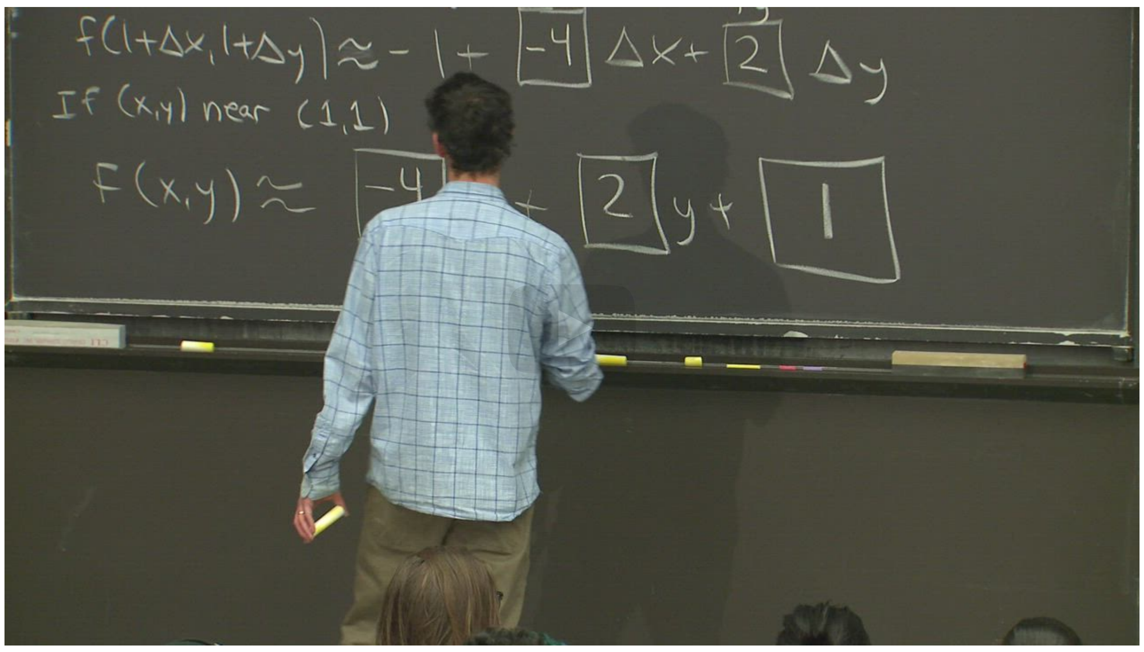
1. The gradient of $f$ is normal to the level curves of $f$.
2. $\nabla f$ points in the direction of steepest increase.
3. $|\nabla f|$ is the "slope" of the tangent plane to $f$ in the direction of the gradient.

Note that by slope we mean the approximate change in $f$ that results from moving by $\nabla f$, up to linear approximation.

Suppose you are looking at a hiking map, and you want to know how much the elevation changes if you move a small distance in the direction of steepest increase along the map. The ratio of the resulting change in elevation over the change in horizontal distance is exactly the quantity measured by the magnitude of the gradient.

### Nonlinear example

Start of transcript. Skip to the end.




0:00 / 0:00 2.0x

PROFESSOR: So that was a warm-up for understanding.

So with each of these questions, we have a warm-up up

where we understand the linear function

and then we can use those ideas to understand

a non-linear function because it's very well approximated

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Let's revisit the nonlinear function from the beginning of lecture

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$$f(x,y) = y^2 - x^3 - x.$$

We saw that the linear approximation could be written in two ways:

$$f(1+\Delta x, 1+\Delta y) \approx -1 - 4\Delta x + 2\Delta y$$
$$f(x,y) \approx -4x + 2y + 1 \text{ near } (1,1).$$

**Question:** Start at $(1,1)$ and move in any direction by a distance $0.1$. How can we choose the direction that maximizes $f(x,y)$?

**Answer:**

First, let's note that we only move by $0.1$ in this case because the linear approximation is only valid for a very small change in $(x,y)$. We have

$$f(1+\Delta x, 1+\Delta y) \approx -1 \underbrace{-4\Delta x + 2\Delta y}_{\text{change in } f}.$$

We want the change in $f$ to be as large as possible. But we can write the change in $f$ as a dot product

$$\begin{aligned} -4\Delta x + 2\Delta y &= \underbrace{\langle -4, 2\rangle}_{\nabla f(1,1)} \cdot \langle \Delta x, \Delta y\rangle \\ &= |\nabla f(1,1)| \underbrace{|\langle \Delta x, \Delta y\rangle|}_{=0.1} \cos\theta \\ &= 0.1 |\nabla f| \cos\theta \end{aligned}$$

where we note that $\langle -4, 2\rangle = \nabla f(1,1)$, $|\langle \Delta x, \Delta y\rangle| = 0.1$ since that's how far we move from $(1,1)$, and $\theta$ is the angle between the vectors $\nabla f$ and $\langle \Delta x, \Delta y\rangle$. This value is largest when $\cos\theta = 1$ which means $\theta = 0$. So we maximize $f$ when $\langle \Delta x, \Delta y\rangle$ is in the same direction as $\nabla f$.

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## Enter the vector

3.0/3 points (graded)

How can we move from $(1,1)$ by a distance of 0.1 while increasing $f$ as much as possible? Let $\langle \Delta x, \Delta y\rangle$ be the corresponding change in $x$ and $y$. What is $\langle \Delta x, \Delta y\rangle$?

(Enter the vector $\langle \Delta x, \Delta y\rangle$ surrounded by square brackets, and can be multiplied by scalars: e.g. `(1/sqrt(2))*[1,1]`.)

[-0.2/sqrt(5),0.1/sqrt(5)] ✔ **Answer:** [-4,2]*(0.1/sqrt(20))

Find the approximate value of the function at $f(1+\Delta x, 1+\Delta y)$ to 2 decimal places.

-0.55 ✔ **Answer:** -0.55

Find the exact value of the function at $f(1+\Delta x, 1+\Delta y)$ to 2 decimal places.

-0.57 ✔ **Answer:** -0.57

? INPUT HELP

**Solution:**

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We saw above that $\langle \Delta x, \Delta y\rangle$ should be parallel to the gradient, which is $\langle -4, 2\rangle$. Therefore $\langle \Delta x, \Delta y\rangle = \lambda\langle -4, 2\rangle$. To find the value of $\lambda$, we use the constraint $|\langle \Delta x, \Delta y\rangle| = 0.1$.

The vector pointing in the direction of $\langle -4, 2\rangle$ with magnitude $0.1$ is

$$0.1/\sqrt{20}\langle -4, 2\rangle$$

The approximate value of $f(1+\Delta x, 1+\Delta y)$ is

$$-1 + |\nabla f(1,1)|\underbrace{|(\Delta x, \Delta y)|}_{=0.1} = -1 + 0.1\sqrt{20} \approx -0.55$$

The exact value is $\approx -0.57$.

Note that this is an error of about $4\%$.

Submit You have used 2 of 5 attempts

Answers are displayed within the problem

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- [Staff] Clarification
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- I am stumped.
  Do we use some kind of ratio to find the values of delta x and delta y? 5
- inverse of a vector?
  If exists an inverse of a vector: 1/[4,-2] could solve this problem using the approximation formula. 2
- [Staff] grader precision is incorrect
  Hi, if one enters 4 decimal places of accuracy in the answers, they are marked as incorrect. Even though the problem clearly states ... 14
- Exact value
  The explanation of exact value of function in answer is not given. Could anyone include in answer please? Or give hint how the exact... 2

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