MITx: 6.041x Introduction to Probability - The Science of Uncertainty

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Problem 3: A Joint PDF given by a simple formula

(4/4 points)

Random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the constant a.

$a =$ 3/7 **Answer:** 0.42857

2. Determine the marginal PDF $f_Y(y)$. (Your answer can be either numerical or algebraic functions of y).

For $0 \le y \le 1$,

$f_Y(y) =$ 9/14 **Answer:** 0.64286

For $1 < y \leq 2$,

$f_Y(y) =$ (3/14)*(4-y^2) **Answer:** (3/14)*(4-y^2)

3. Determine the conditional expectation of $1/X$ given that $Y = 3/2$.

$\mathbf{E}[1/X \mid Y = 3/2] =$ 4/7 **Answer:** 0.57143

Answer:
Let us draw the region where $f_{X,Y}(x,y)$ is nonzero:

- Exam 2
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- Unit 10: Markov chains
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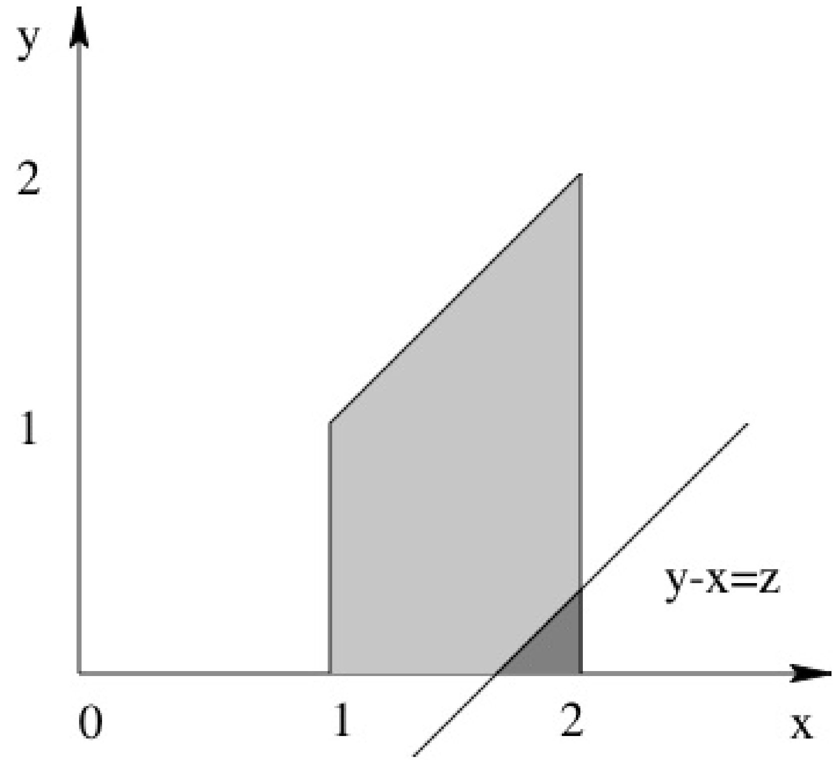

1. The joint PDF has to integrate to 1. From

$$\int_1^2 \int_0^x ax\, dy\, dx = \frac{7}{3}a = 1,$$

we get $a = \frac{3}{7}$.

2. To find the marginal PDF of Y, we integrate the joint PDF over x:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx \\ &= \begin{cases} \int_1^2 \frac{3}{7}x\,dx, & \text{if } 0 \le y \le 1, \\ \int_y^2 \frac{3}{7}x\,dx, & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} \frac{9}{14}, & \text{if } 0 \le y \le 1, \\ \frac{3}{14}(4-y^2), & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

3. We first find the conditional PDF of X given $Y = 3/2$.

$$f_{X|Y}\left(x \,\middle|\, \frac{3}{2}\right) = \frac{f_{X,Y}(x, \frac{3}{2})}{f_Y(\frac{3}{2})} = \frac{\frac{3}{7}x}{\frac{3}{14}\left(4 - \left(\frac{3}{2}\right)^2\right)} = \frac{8}{7}x, \text{ for } \frac{3}{2} \le x \le 2,$$

and equals 0 otherwise.

Then,

$$\mathbf{E}\left[\frac{1}{X}\,\Big|\,Y=\frac{3}{2}\right]=\int_{-\infty}^{\infty}\frac{1}{x}\cdot f_{X|Y}\left(x\,\Big|\,\frac{3}{2}\right)\,dx=\int_{3/2}^{2}\frac{1}{x}\cdot\frac{8}{7}x\,dx=\frac{4}{7}.$$

You have used 2 of 3 submissions

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