7.3 Unit 7 Practice Problems

Unit 7: Markov Chains

Adapted from Blitzstein-Hwang Chapter 11.

Problem 1

1/1 point (graded)

Let $X_0, X_1, X_2, \dots$ be a Markov chain. Is $X_0, X_2, X_4, X_6, \dots$ also a Markov chain?

- (selected) Yes ✔
- No

Solution

Yes. Let $Y_n = X_{2n}$; we need to show $Y_0, Y_1, \dots$ is a Markov chain. By the definition of a Markov chain, we know that $X_{2n+1}, X_{2n+2}, \dots$ ("the future" if we define the "present" to be time $2n$) is conditionally independent of $X_0, X_1, \dots, X_{2n-2}, X_{2n-1}$ ("the past"), given X_{2n}. So given Y_n, we have that $Y_{n+1}, Y_{n+2}, \dots$ is conditionally independent of $Y_0, Y_1, \dots, Y_{n-1}$. Thus,

$$P(Y_{n+1} = y | Y_0 = y_0, \dots, Y_n = y_n) = P(Y_{n+1} = y | Y_n = y_n).$$

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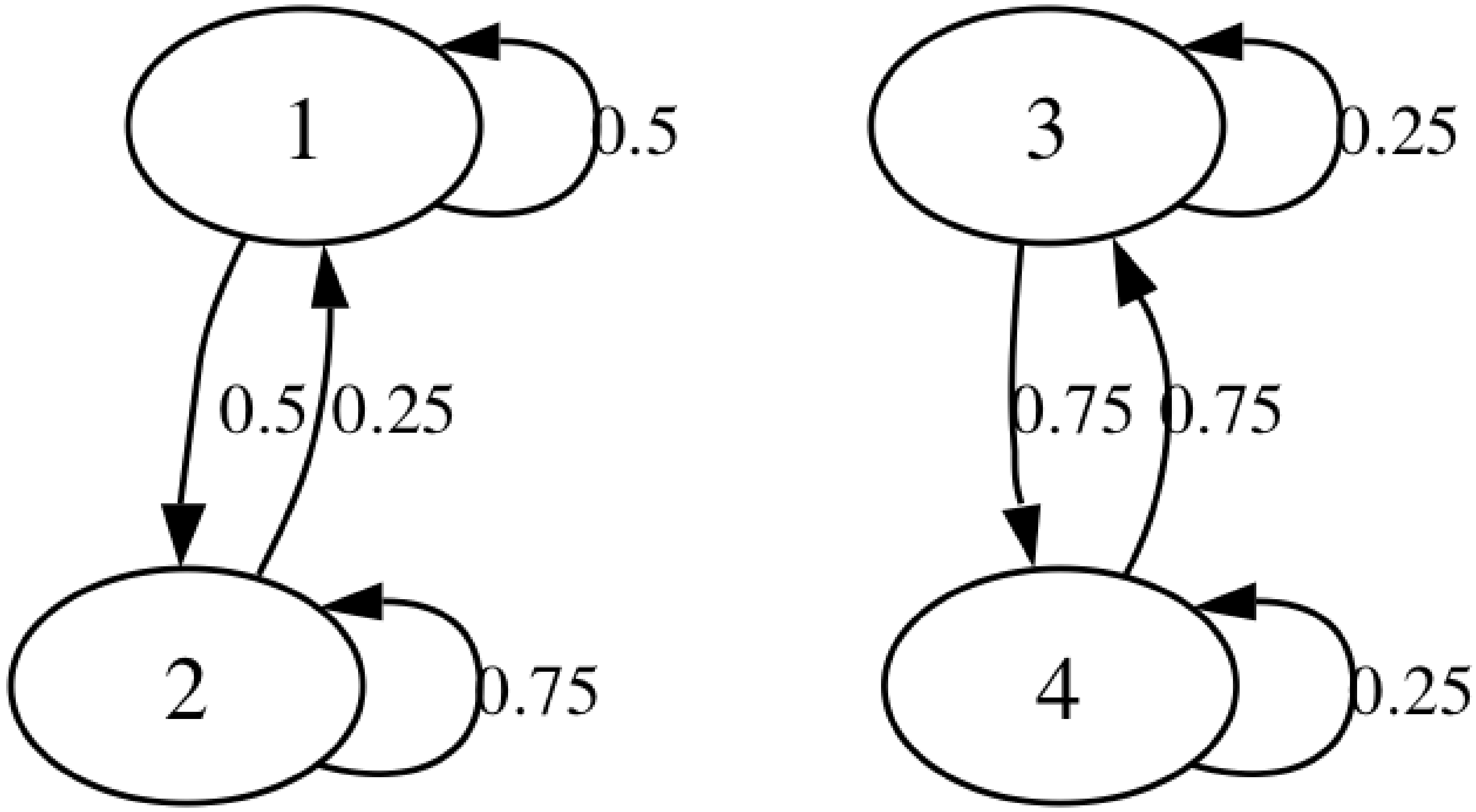
Answers are displayed within the problem

For Problem 2

Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$ and the labels on the arrows indicating transition probabilities.



Markov Chain for Problem 2
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Image Description

Problem 2a

1/1 point (graded)

Which of the following is the transition matrix Q for this chain?

- [x] $$Q = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix}$$

- [ ] $$Q = \begin{pmatrix} 0.25 & 0.75 & 0 & 0 \\ 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

- [ ] $$Q = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{pmatrix}$$

- [ ] $$Q = \begin{pmatrix} 0.75 & 0.25 & 0 & 0 \\ 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix}$$

Solution

The transition matrix is

$$Q = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix}$$

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Problem 2b

1/1 point (graded)
Which states are recurrent?

- [x] State 1 ✔
- [x] State 2 ✔
- [x] State 3 ✔
- [x] State 4 ✔
- [ ] None of the above

✔

Solution
All of the states are recurrent. Starting at state 1, the chain will go back and forth between states 1 and 2 forever (sometimes lingering for a while). Similarly, for any starting state, the probability is 1 of returning to that state.

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Problem 2c

1/1 point (graded)

(c) Which, if any, of the following distributions are stationary distributions for the chain?

- [x] $(1/3, 2/3, 0, 0)$ ✔
- [ ] $(1/3, 0, 1/6, 1/6)$
- [ ] $(0, 1/4, 1/2, 1/4)$
- [x] $(0, 0, 1/2, 1/2)$ ✔
- [ ] None of the above

✔

Solution

Solving

$$(a \quad b)\begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} = (a \quad b)$$

$$(c \quad d)\begin{pmatrix} 0.25 & 0.75 \\ 0.75 & 0.25 \end{pmatrix} = (c \quad d)$$

shows that $(a, b) = (1/3, 2/3)$, and $(c, d) = (1/2, 1/2)$ are stationary distributions on the 1, 2 chain and on the 3, 4 chain respectively, viewed as separate chains. It follows that $(1/3, 2/3, 0, 0)$ and $(0, 0, 1/2, 1/2)$ are both stationary for Q (as is any mixture $p(1/3, 2/3, 0, 0) + (1 - p)(0, 0, 1/2, 1/2)$ with $0 \leq p \leq 1$).

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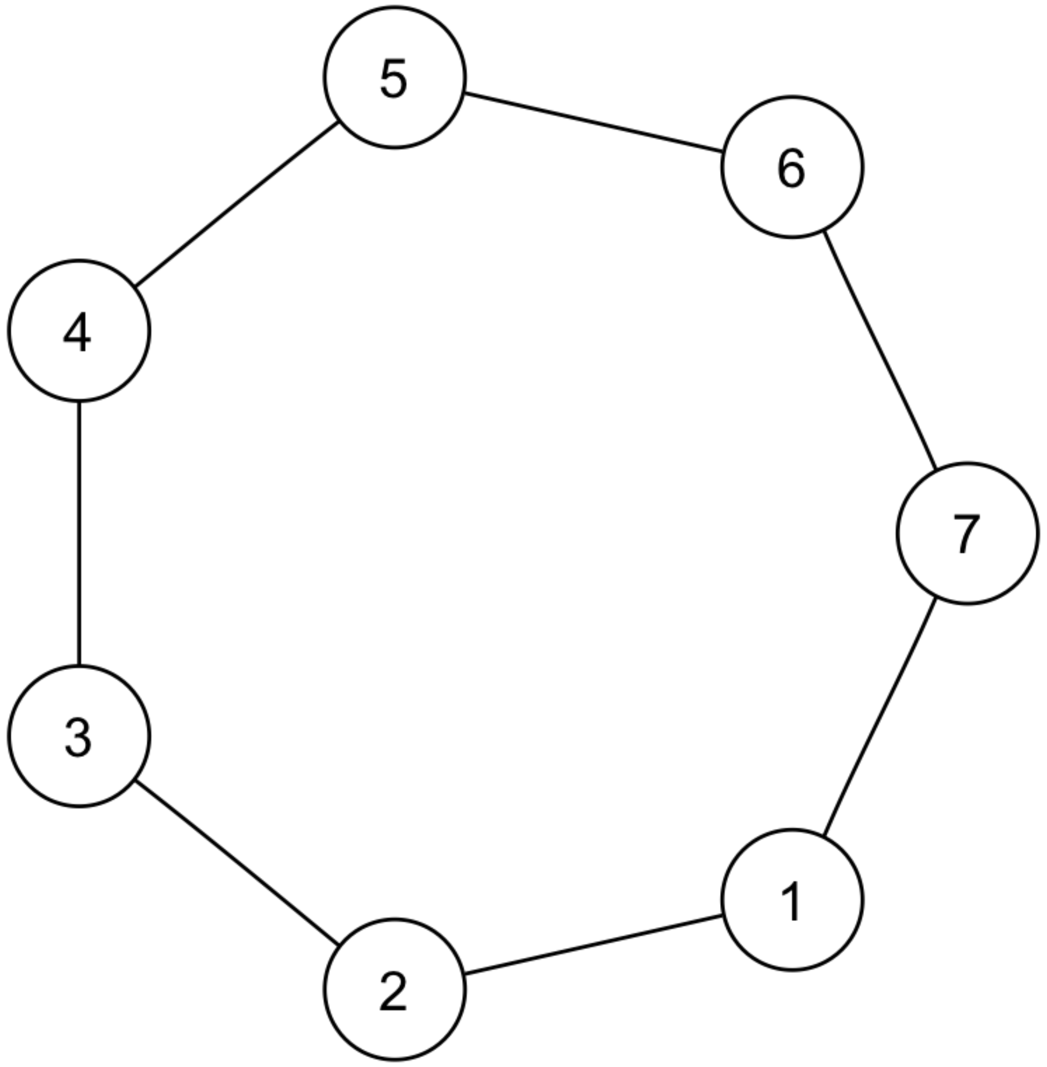
Problem 3a

1/1 point (graded)

(a) Consider a Markov chain on the state space $\{1, 2, \ldots, 7\}$ with the states arranged in a “circle" as shown below, and transitions given by moving one step clockwise or counterclockwise with equal probabilities. For example, from state 6, the chain moves to state 7 or state 5 with probability $1/2$ each; from state 7, the chain moves to state 1 or state 6 with probability $1/2$ each. The chain starts at state 1.



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Find the stationary distribution of this chain.

- $(1, 1, 1, 1, 1, 1, 1)$
- $(1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)$
- $(1/12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/12)$
- $(0, 1/7, 0, 2/7, 0, 4/7, 0)$

Solution

The symmetry of the chain suggests that the stationary distribution should be uniform over all the states. To verify this, note that the reversibility condition is satisfied. So the stationary distribution is $(1/7, 1/7, \ldots, 1/7)$.

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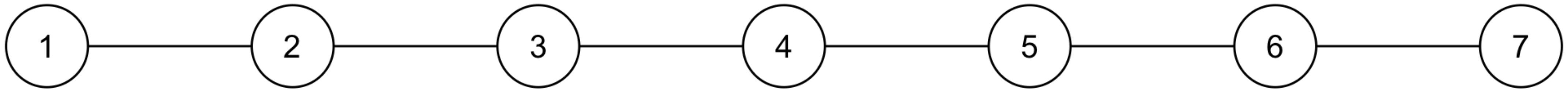
Problem 3b

1/1 point (graded)

(b) Consider a new chain obtained by "unfolding the circle". Now the states are arranged as shown below. From state 1 the chain always goes to state 2, and from state 7 the chain always goes to state 6. Find the new stationary distribution.



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- $(1, 1, 1, 1, 1, 1, 1)$
- $(1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7)$
- $(1/12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/12)$ ✔
- $(0, 1/7, 0, 2/7, 0, 4/7, 0)$

Solution

By the result about random walk on an undirected network, the stationary probabilities are proportional to the degrees. So we just need to normalize $(1, 2, 2, 2, 2, 2, 1)$, obtaining $(1/12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/12)$.

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Answers are displayed within the problem

Problem 4

3/3 points (graded)

In chess, the king can move one square at a time in any direction (horizontally, vertically, or diagonally).

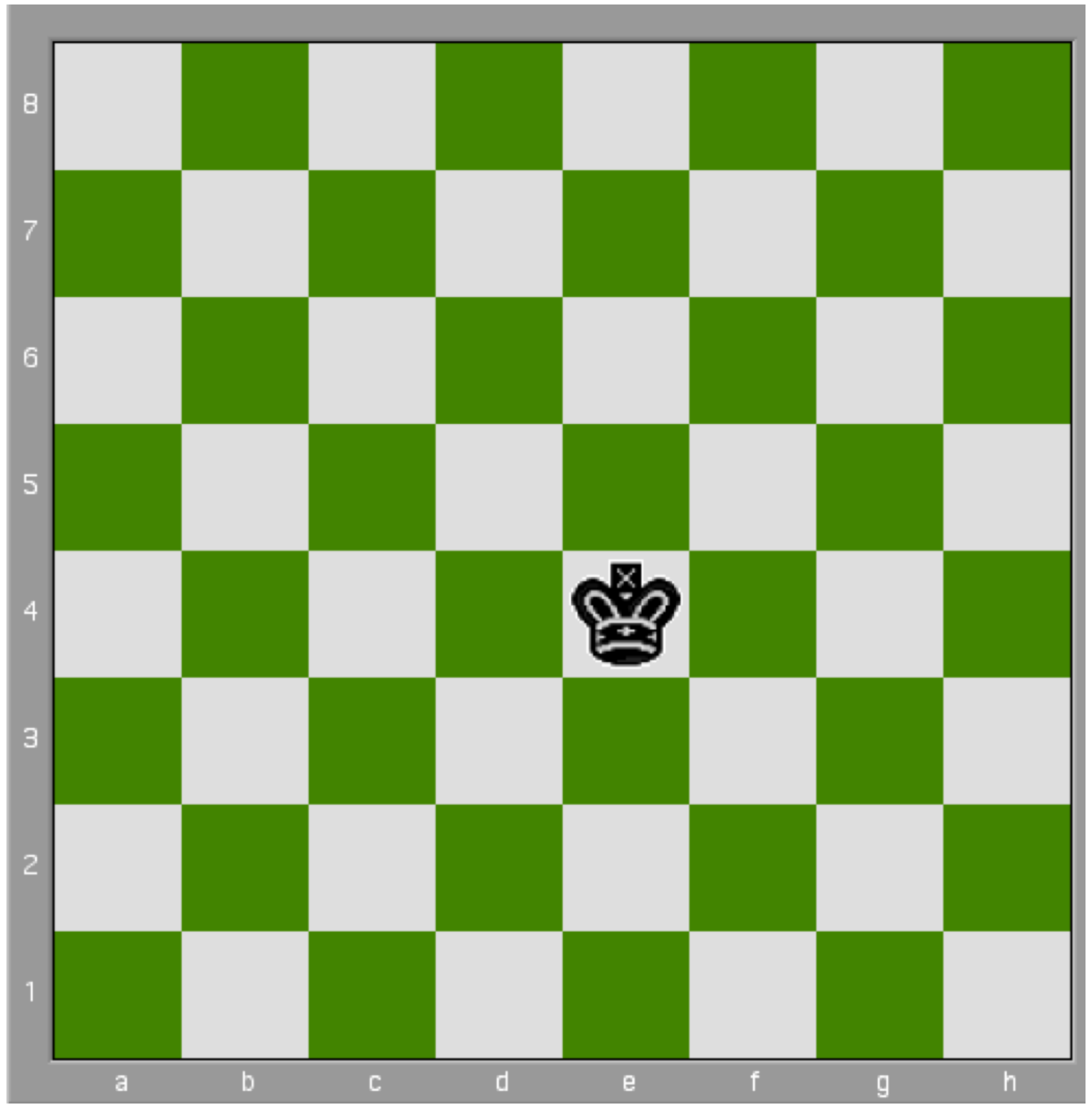

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Image Description

For example, in the diagram, from the current position the king can move to any of 8 possible squares. A king is wandering around on an otherwise empty 8×8 chessboard, where for each move all possibilities are equally likely. Find the stationary distribution of this chain. Note that there are 3 types of square: corner (the 4 corners), edge (first or last row or column, excluding the corners), and normal (any square that is not on the edge or in a corner). All squares of a particular type will turn out to have the same stationary probability, so you just need to specify 3 numbers: the stationary probabilities for a corner square, edge square, and normal square.

Corner square:

3/420 ✔ **Answer:** 3/420

$\frac{3}{420}$

Edge square:

5/420 ✔ **Answer:** 5/420

$\frac{5}{420}$

Normal square:

8/420 ✔ **Answer:** 8/420

$\frac{8}{420}$

Solution

There are 4 corner squares, 24 edge squares, and 36 normal squares, where by "edge" we mean a square in the first or last row or column, excluding the 4 corners, and by "normal" we mean a square that's not on the edge or in a corner. View the chessboard as an undirected network, where there is an edge between two squares if the king can walk from one to the other in one step.

The stationary probabilities are proportional to the degrees. Each corner square has degree 3, each edge square has degree 5, and each normal square has degree 8. The total degree is $420 = 3 \cdot 4 + 24 \cdot 5 + 36 \cdot 8$ (which is also twice the number of edges in the network). Thus, the stationary probability is $\frac{3}{420}$ for a corner square, $\frac{5}{420}$ for an edge square, and $\frac{8}{420}$ for a normal square.

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Answers are displayed within the problem

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