5.

Wald's Test and Likelihood Ratio test

Consider a sample of i.i.d. random variables $X_1, \ldots, X_n$ and assume their common density is given by

$$f_\theta(x) = \frac{x}{\theta}\exp\left(-\frac{x^2}{2\theta}\right)\mathbf{1}(x \geq 0),$$

where $\theta > 0$ is an unknown parameter

Consider the following set of hypotheses:

$$H_0 : \theta = 1 \quad \text{and} \quad H_1 : \theta \neq 1.$$

You will perform Wald's test and the likelihood ratio test at significance level 7% for these hypotheses.

Maximum Likelihood Estimator

1.0/1 point (graded)

Compute the maximum likelihood estimator $\hat{\theta}$ of θ.

(Enter **barX_n** for $\overline{X_n}$ and **bar(X_n^2)** for $\overline{X_n^2}$. Note “barX_n^2" represents $\left(\overline{X_n}\right)^2$, NOT $\overline{X_n^2}$. Note the extra parentheses needed to enter $\overline{X_n^2}$.)

$\hat{\theta} =$ bar(X_n^2) /2 ✔ **Answer:** bar(X_n^2)/2

STANDARD NOTATION

Solution:

The maximum likelihood estimator $\hat{\theta}$ and the Fisher information $I(\theta)$:

$$\begin{aligned} \hat{\theta} &= \frac{\overline{X_n^2}}{2} \\ I(\theta) &= \frac{1}{\theta^2}. \end{aligned}$$

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Fisher Information

1/1 point (graded)

Compute the Fisher information $I(\theta)$, as a function of θ.

Useful fact: $\int_0^\infty u^3 e^{-u^2/2} du = 2.$

$I(\theta) =$ 1/theta^2 ✔ **Answer:** 1/theta^2

$\frac{1}{\theta^2}$

STANDARD NOTATION

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Wald's Test

2.0/2.0 points (graded)

Write down the test statistic T_n^{Wald} for Wald's test in terms of the maximum likelihood estimator $\hat{\theta}$ of θ, the Fisher information I, and the sample size n. Use the Fisher information $I(\hat{\theta})$ evaluated at $\hat{\theta}$ in the definition of Wald's test.

(To avoid double jeopardy, enter **I** for the $I = I(\hat{\theta})$; or enter your answer directly in terms of $\hat{\theta}$ only. Enter **hattheta** for $\hat{\theta}$.)

$T_n^{\text{Wald}} =$ n*(hattheta-1)^2/(hatthe ✔ **Answer:** n*I*(hattheta-1)^2

When do we reject the null hypothesis in Wald's test

- (selected) When $T_n^{\text{Wald}} > C$ for some $C > 0$
- When $T_n^{\text{Wald}} < C$ for some $C > 0$

✔

Find C such that the Wald's test has asymptotic level 7%.

(Enter a numerical value accurate to at least 2 decimal places.)

$C =$ 3.283020286759533 ✔ **Answer:** 3.283

You have used 2 of 3 attempts

Submit

Answers are displayed within the problem

P-value

3/3 points (graded)

Assume that the sample size is $n = 500$. You observe that

- The sample mean is $\frac{1}{n}\sum_{i=1}^{n} X_i = 0.86$;
- The (biased) sample variance is $\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 = 1.09$.

Enter a numerical value for $\hat{\theta}$. (Enter a value acccurate to at least 2 decimal places.)

$\hat{\theta} =$ 0.9148 ✔ **Answer:** 1.09/2+(0.86)^2/2

Compute the p-value of the Wald's test above. (As above, the Fisher information $I\left(\hat{\theta}\right)$ evaluated at the MLE $\hat{\theta}$ in the definition of Wald's test.)

(Enter a numerical value accurate to at least 3 decimal places.)

p-value: 0.03729094666035515 ✔ **Answer:** 0.037

Does your test reject H_0 at the following asymptotic levels? (Choose all that apply.)

- [ ] reject H_0 at asymptotic level 1%
- [x] reject H_0 at asymptotic level 7%
- [x] reject H_0 at asymptotic level 10%

Fail to reject at asymptotic level 10%.

STANDARD NOTATION

Solution:

THe MLE is

$$\hat{\theta} \;=\; \frac{1}{2}\left(1.09+(0.86)^2\right) \;=\; 0.9148.$$

The p-value is

$$p \;=\; 1-\Phi^{-1}_{\chi^2}\left(\frac{n}{\hat{\theta}^2}(\hat{\theta}-1)^2\right) \;=\; 1-\Phi^{-1}_{\chi^2}(4.337) \;=\; 0.037.$$

Since $1\% < p < 7\% < 10\%$, we reject H_0 at 7% and 10%.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

Likelihood Ratio Test

5/5 points (graded)

You perform the Likelihood Ratio test for the same set of hypotheses:

$$H_0 : \theta = 1 \;\text{ and }\; H_1 : \theta \neq 1.$$

Write down the test statistic T_n^{LR} for the likelihood ratio test in terms of $\hat{\theta}$ and n.

(Enter **hattheta** for $\hat{\theta}^{\mathrm{MLE}}$).

$T_n^{\mathrm{LR}} =$ 2*n*hattheta-2*n*ln(hattheta)-2*n ✔ **Answer:** 2*n*(hattheta-1-ln(hattheta))

When do we reject the Null hypothesis using the likelihood ratio test?

- (●) When $T_n^{\mathrm{LR}} > C$ for some $C>0$
- ( ) When $T_n^{\mathrm{LR}} > C$ for some $C<0$

✔

Find C such that the Likelihood ratio test has asymptotic level 7%.

(Enter a numerical value accurate to at least 2 decimal places.)

$C =$ 3.283020286759533 ✔ **Answer:** 3.283

Compute the p-value of the likelihood ratio test using the same sample as in the previous problem.

(Enter a numerical value accurate to at least 3 decimal places.)

p-value: 0.04975142364884566 ✔ **Answer:** 0.04975

Does your test reject H_0 at the following asymptotic levels? (Choose all that apply.)

- [ ] reject H_0 at asymptotic level 1%
- [x] reject H_0 at asymptotic level 7%

- [x] reject H_0 at asymptotic level 10%
- [ ] Fail to reject at asymptotic level 10%.

Solution:

The likelihood ratio test statistic is

$$\begin{aligned} T_n^{\mathrm{LR}} &= 2\left(l_n\left(\hat{\theta}\right) - l_n\left(1\right)\right) \\ &= n\left(\hat{\theta} - \ln\left(\hat{\theta}\right) - 1\right). \end{aligned}$$

The likelihood ratio test with level 7% is

$$\psi^{\mathrm{LR}} \;=\; \mathbf{1}\left(T_n^{\mathrm{LR}} > C\right) \qquad \text{where } C = q_{0.07,\chi^2} = 3.28302.$$

Since $T_n^{\mathrm{LR}} = 1000\left(0.9148 - \ln\left(0.9148\right) - 1\right) = 3.85$, the p-value is 0.04975. Hence we reject H_0 at levels 7% and 10%.

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Answers are displayed within the problem

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[Polling] What are your answers for this page?

1) bar(X_n^2)/2 2) 1/(theta^2) 3) n*I*(hattheta -1)^2; first choice; 3.28302 4) 0.9148; 0.03729095; reject at 7% and 10% 5) 2*n*(ln(1/hattheta)+hattheta-1); first choice; 3.28302; ... 6 new

Possible resolution
[Of course this could be all wrong. Please correct me if so] # Load the needed Julia packages using Distributions, SymPy ## Maximum Likelihood estimator # specifying the do... 8

Error/Bug:Likelihood Ratio Test
[Edited by staff to remove exam content] 2

[stuff] lack of preview of Deleted by MW-CTA
The input of the **Deleted by MW-CTA** should have a equation preview for us to review the answer since it is not a pure numeric question. 6

bug?
cannot input to the answer of **Deleted by MW-CTA** is it correct or a bug? Please check 4

[Staff] Typo in the answers options
Deleted by MW-CTA. 2