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Problem 2: Hypothesis test between two coins

(5/6 points)

Alice has two coins. The probability of Heads for the first coin is $1/3$, and the probability of Heads for the second is $2/3$. Other than this difference, the coins are indistinguishable. Alice chooses one of the coins at random and sends it to Bob. The random selection used by Alice to pick the coin to send to Bob is such that the first coin has a probability p of being selected. Assume that $0 < p < 1$ and that Bob knows the value of p. Bob tries to guess which of the two coins he received by tossing it 3 times in a row and observing the outcome. Assume that for any particular coin, all tosses of that coin are independent.

1. Given that Bob observed k Heads out of the 3 tosses (where $k = 0, 1, 2, 3$), what is the conditional probability that he received the first coin?

- $\frac{1}{2^{3-k}}$
- $\frac{2^{3-k}p}{2^{3-k}p + 2^k(1-p)}$ ✔
- $\frac{p}{2^{3-k}}$

- $\dfrac{1}{1+\frac{p}{1-p}2^{3-2k}}$

2. We define an error to have occurred if Bob decides that he received one coin from Alice, but he actually received the other coin. He decides that he received the first coin when the number of Heads, k, that he observes on the 3 tosses satisfies a certain condition. When one of the following conditions is used, Bob will minimize the probability of error. Choose the correct threshold condition.

- $k \geq \dfrac{3}{2}+\dfrac{1}{2}\log_2\dfrac{p}{1-p}$
- (selected) $k \leq \dfrac{3}{2}+\dfrac{1}{2}\log_2\dfrac{p}{1-p}$ ✔
- $k \leq \dfrac{1}{2}\log_2\dfrac{p}{1-p}$
- $k \geq \dfrac{1}{2}\log_2\dfrac{p}{1-p}$
- none of the above

3. For this part, assume that $p = 2/3$.

(a) What is the probability that Bob will guess the coin correctly using the decision rule from part 2?

0.6714597 ✖ **Answer:** 0.74074

(b) Suppose instead that Bob tries to guess which coin he received without tossing it. He still guesses the coin in order to minimize the probability of error. What is the probability that Bob will guess the coin correctly under this scenario?

2/3 ✔ **Answer:** 0.66667

4. Suppose that we increase p. Then does the number of different values of k for which Bob decides that he received the first coin increase, decrease, or stay the same?

It increases or stays the same. ▾ ✔ **Answer:** It increases or stays the same.

5. Find the values of p for which Bob will never decide that he received the first coin, regardless of the outcome of the 3 tosses.

p is less than 1/9 ✔ **Answer:** 0.11111

Answer:

1. Let Y be the number of Heads Bob observed in the three tosses. Let C denote the coin that Bob received, so that $C = 1$ if Bob received the first coin, and $C = 2$ if Bob received the second coin. Then $\mathbf{P}(C = 1) = p$ and $\mathbf{P}(C = 2) = 1 - p$. Given the value of C, Y is a binomial random variable.

We can find the conditional probability that Bob received the first coin given that he observed k Heads using Bayes' rule.

$$\begin{aligned}\mathbf{P}(C=1 \mid Y=k) &= \frac{\mathbf{P}(Y=k \mid C=1)\cdot\mathbf{P}(C=1)}{\mathbf{P}(Y=k \mid C=1)\cdot\mathbf{P}(C=1)+\mathbf{P}(Y=k \mid C=2)\cdot\mathbf{P}(C=2)}\\ &= \frac{\binom{3}{k}(1/3)^k(2/3)^{3-k}\cdot p}{\binom{3}{k}(1/3)^k(2/3)^{3-k}\cdot p+\binom{3}{k}(2/3)^k(1/3)^{3-k}\cdot(1-p)}\\ &= \frac{2^{3-k}p}{2^{3-k}p+2^k(1-p)}.\end{aligned}$$

2. Given that Bob observes k Heads, he is to decide whether the first or second coin was used. To minimize the probability of error, he should use the MAP rule, which in this case is to decide on the first coin when $\mathbf{P}(C=1 \mid Y=k) \geq \mathbf{P}(C=2 \mid Y=k)$. We can calculate $\mathbf{P}(C=2 \mid Y=k)$ using Bayes' rule in a similar fashion as in part 1. We then have the following equivalent versions of this decision rule:

$$\begin{aligned}\mathbf{P}(C=1 \mid Y=k) &\geq \mathbf{P}(C=2 \mid Y=k),\\ \frac{2^{3-k}p}{2^{3-k}p+2^k(1-p)} &\geq \frac{2^k(1-p)}{2^{3-k}p+2^k(1-p)},\\ 2^{3-k}p &\geq 2^k(1-p),\\ 2^{2k-3} &\leq \frac{p}{1-p},\\ k &\leq \frac{3}{2}+\frac{1}{2}\log_2\frac{p}{1-p}.\end{aligned}$$

3.

(a) If $p = 2/3$, the threshold in the rule above is equal to $\frac{3+\log_2 2}{2} = 2$. Therefore, Bob will decide that he received the first coin when he observes 0, 1, or 2 Heads, and will decide that he received the second coin when he observes 3 Heads.

We find the probability of a correct decision using the total probability theorem:

$$\begin{aligned}
\mathbf{P}(\text{Correct}) &= \mathbf{P}(\text{Correct} \mid C=1) \cdot p + \mathbf{P}(\text{Correct} \mid C=2) \cdot (1-p) \\
&= \mathbf{P}(Y<3 \mid C=1) \cdot p + \mathbf{P}(Y=3 \mid C=2) \cdot (1-p) \\
&= (1-\mathbf{P}(Y=3 \mid C=1)) \cdot p + \mathbf{P}(Y=3 \mid C=2) \cdot (1-p) \\
&= (1-(1/3)^3)(2/3) + (2/3)^3(1/3) = 20/27.
\end{aligned}$$

(b) In the absence of any data, Bob should simply guess that he received whichever coin Alice was more likely to choose, which in this case is the first coin. His decision will be correct if he indeed receives the first coin, which happens with probability $p = 2/3$.

Note that observing 3 coin tosses increases the probability of making a correct decision from $2/3$ to $20/27$, a difference of approximately 0.07407.

4. If p is increased, the threshold in the decision rule in part 3 increases, and therefore the number of values of k for which Bob decides he received the first coin will either increase if the threshold passes a new integer boundary (e.g., goes from 1.9 to 2.1) or stay the same if the threshold increases but does not pass a new integer boundary (e.g., goes from 1.9 to 1.95). The number of values of k can never decrease.

5. Bob will never decide that he received the first coin if the threshold in the decision rule in part 3 is negative, i.e., when

$$\frac{3}{2} + \frac{1}{2}\log_2 \frac{p}{1-p} \; < \; 0,$$

$$\begin{aligned}\log_2 \frac{p}{1-p} &< -3,\\ \frac{p}{1-p} &< \frac{1}{8},\\ p &< \frac{1}{9}.\end{aligned}$$

If $p < 1/9$, the prior probability of receiving the first coin is so low that no amount of evidence from 3 tosses of the coin will make Bob decide he received the first coin.

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