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# What is the difference between a consistent estimator and an unbiased estimator?

Asked 7 years, 2 months ago Active 10 months ago Viewed 130k times

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I'm really surprised that nobody appears to have asked this already...

When discussing estimators, two terms frequently used are "consistent" and "unbiased". My question is simple: what's the difference?

The precise technical definitions of these terms are fairly complicated, and it's difficult to get an intuitive feel for what they *mean*. I can imagine a good estimator, and a bad estimator, but I'm having trouble seeing how any estimator could satisfy one condition and not the other.

unbiased-estimator estimators consistency

edited Oct 31 '18 at 11:56
Ferdi
4,011 5 28 56

asked Jun 24 '12 at 16:41
MathematicalOrchid
1,070 2 9 14

13 Have you looked at the very first figure in the Wikipedia article on consistent estimators, which specifically explains this distinction? – whuber ♦ Jun 24 '12 at 16:45

4 I've read the articles for both consistency and bias, but I still don't really understand the distinction. (The figure you refer to claims that the estimator is consistent but biased, but doesn't explain *why*.) – MathematicalOrchid Jun 24 '12 at 16:47

1 Which part of the explanation do you need help with? The caption points out that each of the estimators in the sequence is biased and it also explains why the sequence is consistent. Do you need an explanation of how the bias in these estimators is apparent from the figure? – whuber ♦ Jun 24 '12 at 16:50

5 +1 The comment thread following one of these answers is very illuminating, both for what it reveals about the subject matter and as an interesting example of how an online community can work to expose and rectify misconceptions. – whuber ♦ Jan 12 '13 at 17:51

Related: stats.stackexchange.com/questions/173152/… – kjetil b halvorsen Oct 1 '15 at 16:26

## 3 Answers

125

To define the two terms without using too much technical language:

- An estimator is **consistent** if, as the sample size increases, the estimates (produced by the estimator) "converge" to the true value of the parameter being estimated. To be slightly more precise - consistency means that, as the sample size increases, the sampling distribution of the estimator becomes increasingly concentrated at the true parameter value.
- An estimator is **unbiased** if, on average, it hits the true parameter value. That is, the mean of the sampling distribution of the estimator is equal to the true parameter value.
- The two are not equivalent: **Unbiasedness** is a statement about the expected value of the sampling distribution of the estimator. **Consistency** is a statement about "where the sampling distribution of the estimator is going" as the sample size increases.

It certainly is possible for one condition to be satisfied but not the other - I will give two examples. For both examples consider a sample $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ population.

- **Unbiased but not consistent:** Suppose you're estimating $\mu$. Then $X_1$ is an unbiased estimator of $\mu$ since $E(X_1) = \mu$. But, $X_1$ is not consistent since its distribution does not become more concentrated around $\mu$ as the sample size increases - it's always $N(\mu, \sigma^2)$!
- **Consistent but not unbiased:** Suppose you're estimating $\sigma^2$. The maximum likelihood estimator is

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

where $\overline{X}$ is the sample mean. It is a fact that

$$E(\hat{\sigma}^2) = \frac{n-1}{n} \sigma^2$$

herefore, $\hat{\sigma}^2$ which can be derived using the information here. Therefore $\hat{\sigma}^2$ is biased for any finite sample size. We can also easily derive that

$$\text{var}(\hat{\sigma}^2) = \frac{2\sigma^4(n-1)}{n^2}$$

From these facts we can informally see that the distribution of $\hat{\sigma}^2$ is becoming more and more concentrated at $\sigma^2$ as the sample size increases since the mean is converging to $\sigma^2$ and the variance is converging to $0$. (**Note:** This does constitute a proof of consistency, using the same argument as the one used in the answer here)

edited Apr 13 '17 at 12:44
Community ♦
1

answered Jun 25 '12 at 2:06
Macro
33.9k 6 128 142

9 (+1) Not all MLEs are consistent though: the general result is that there exists a consistent subsequence in the sequence of MLEs. For proper consistency a few additional requirements, e.g. identifiability, are needed. Examples of MLEs that aren't consistent are found in certain errors-in-variables models (where the "maximum" turns out to be a saddle-point). – MånsT Jun 25 '12 at 6:42

2 Well, the EIV MLEs that I mentioned are perhaps not good examples, since the likelihood function is unbounded and no maximum exists. They're good examples of how the ML approach can fail though :) I'm sorry that I can't give a relevant link right now - I'm on vacation. – MånsT Jun 25 '12 at 6:59

Thank you @MånsT. The necessary conditions were outlined in the link but that wasn't clear from the wording. – Macro Jun 25 '12 at 11:12

2 Just a side note: The parameter space is certainly not compact in this case, in contrast to the conditions at that link, nor is the log likelihood concave wrt $\sigma^2$ itself. The stated consistency result still holds, of course. – cardinal Jun 25 '12 at 12:43

2 You're right, @cardinal, I'll delete that reference. It's clear enough that $E(\hat{\sigma}^2) \to \sigma^2$ and $\mathrm{var}(\hat{\sigma}^2) \to 0$ but I don't want to stray from the point by turning this into an exercise of proving the consistency of $\hat{\sigma}^2$. – Macro Jun 25 '12 at 12:54

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Consistency of an estimator means that as the sample size gets large the estimate gets closer and closer to the true value of the parameter. Unbiasedness is a finite sample property that is not affected by increasing sample size. An estimate is unbiased if its expected value equals the true parameter value. This will be true for all sample sizes and is exact whereas consistency is asymptotic and only is approximately equal and not exact.

To say that an estimator is unbiased means that if you took many samples of size $n$ and computed the estimate each time the average of all these estimates would be close to the true parameter value and will get closer as the number of times you do this increases. The sample mean is both consistent and unbiased. The sample estimate of standard deviation is biased but consistent.

**Update following the discussion in the comments with @cardinal and @Macro:** As described below there are apparently pathological cases where the variance does not have to go to 0 for the estimator to be strongly consistent and the bias doesn't even have to go to 0 either.

edited May 30 '16 at 19:55
amoeba
**66.2k** 18 226 278

answered Jun 24 '12 at 17:20
Michael Chernick
**35.8k** 8 63 134

9 @MichaelChernick +1 for your answer but, regarding your comment, the variance of a consistent estimator does not necessarily goes to $0$. For example if $(X_1, \ldots, X_n)$ is a sample from $\mathrm{Normal}(\mu, 1)$, $\mu \neq 0$, then $1/\bar{X}$ is a (strong) consistent estimator of $1/\mu$, but $\mathrm{var}(1/\bar{X}) = \infty$, for all $n$. – user10525 Jun 24 '12 at 20:38

6 @Procrastinator: (+2) The bias need not shrink to zero, either, even when the mean exists for each $n$ – cardinal Jun 24 '12 at 20:46

6 Michael, the body of your answer is pretty good; I think the confusion was introduced by your first comment, which leads with two statements that are plainly false and potential points of confusion. (Indeed, many students walk away from an introductory graduate statistics class with precisely these misconceptions due to poor delineation between the different modes of convergence and their meaning. Your last comment could be taken to be a little on the harsh side.) – cardinal Jun 25 '12 at 13:33

9 Unfortunately, the first two sentences in your first comment and the entire second comment are false. But, I fear it is not fruitful to further try to convince you of these facts. – cardinal Jun 25 '12 at 13:48

11 Here is an admittedly absurd, but **simple** example. The idea is to **illustrate** exactly what can go wrong and why. It *does* have practical applications. **Example**: Consider the typical iid model with finite second moment. Let $\hat{\theta}_n = \bar{X}_n + Z_n$ where $Z_n$ is independent of $\bar{X}_n$ and $Z_n = \pm an$ each with probability $1/n^2$ and is zero otherwise,

with $a > 0$ arbitrary. Then $\hat{\theta}_n$ is unbiased, has variance bounded *below* by $a^2$, and $\hat{\theta}_n \to \mu$ almost surely (it's strongly consistent). I leave as an exercise the case regarding the bias. – cardinal Jun 25 '12 at 14:48

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Consistency : very well explained before [as the sample size increases, the estimates (produced by the estimator) "converge" to the true value of the parameter being estimated]

Unbiasedness : It satisfies the 1-5 MLR assumptions known as the Gauss-Markov Theorem

1. linearity,
2. random sampling
3. zero conditional mean error expectation
4. no perfect collinearity
5. homoskedasticity

Then the estimator is said to be BLUE (best linear unbiased estimator

edited Oct 31 '18 at 12:03
Ferdi
4,011 5 28 56

answered Oct 31 '18 at 11:43
Nikolina Langura
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