Fun with Prime Numbers (3)

Invitation to the Mysterious World of Mathematics

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Mystery of Triangles (6)

<u>Conjecture</u>

For a prime number P, the following are equivalent:

(A) There exists a P-triangle.

(B) The number of (X,Y,Z) with

$$P = 2X^2 + Y^2 + 8Z^2$$

is equal to twice the number of (A,B,C) with

$$P = 2A^2 + B^2 + 32C^2.$$

Mystery of Triangles (7)

<u>Example</u> (P=5)

$5 = 2X^2 + Y^2 + 8Z^2$: no such triples.

$5 = 2A^2 + B^2 + 32C^2$: no such triples.

$0 = 0 \times 2$ ➡ A 5-triangle exists.

<u>Example</u> (P=11)

$11 = 2X^2 + Y^2 + 8Z^2$: $(\pm 1, \pm 3, 0)$, $(\pm 1, \pm 1, \pm 1)$

$11 = 2A^2 + B^2 + 32C^2$: $(\pm 1, \pm 3, 0)$

$12 \neq 4 \times 2$ ➡ A 11-triangle does not exist.

Mystery of Triangles (8)

<u>Example</u> (P=41)

$41 = 2X^2 + Y^2 + 8Z^2$: 32 such triples.

$41 = 2A^2 + B^2 + 32C^2$: 16 such triples.

$32 = 16 \times 2$ ➡ A 41-triangle exists.

(Hint: $X=40/3$, $Y=123/20$)

<u>Example</u> (P=157)

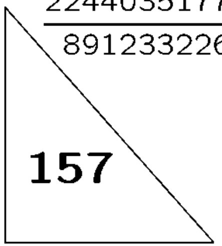
$157 = 2X^2 + Y^2 + 8Z^2$: no such triples.

$157 = 2A^2 + B^2 + 32C^2$: no such triples.

$0 = 0 \times 2$ ➡ A 157-triangle exists.

Mystery of Triangles (9)

The simplest 157-triangle is computed by Zagier.



$$\frac{224403517704336969924557513090674863160948472041}{8912332268928859588025535178967163570016480830}$$

$$\frac{411340519227716149383203}{21666555693714761309610}$$

$$\frac{6803298487826435051217540}{411340519227716149383203}$$

Don Zagier (1951-)

Reference https://en.wikipedia.org/wiki/Don_Zagier

Mystery of Triangles (10)

<u>Theorem</u> (Tunnell, 1983)

- If there exists a P-triangle, the number of (X,Y,Z) with

$$P = 2X^2 + Y^2 + 8Z^2$$

is equal to twice the number of (A,B,C) with

$$P = 2A^2 + B^2 + 32C^2.$$

- The converse holds if the Birch and Swinnerton-Dyer conjecture is true.

Reference http://www.claymath.org/

This week

- There are many reciprocity laws generalizing Fermat's theorem on sums of two squares:

  Quadratic Reciprocity Laws, Class Field Theory
- The Langlands Program is expected to be one of the most general Reciprocity Laws. It is still mostly conjectural.
- Congruent Number Problem is an old problem on the area of triangles, which is open for more than 1000 years.

Plan of the next week

- Prime numbers are also expected to satisfy many laws other than Reciprocity Laws.
- We will study the ABC Conjecture and its polynomial analogues. See you next week!

Joseph Oesterlé
(1954-)

David Masser
(1948-)

Shinichi Mochizuki
(1969-)

Reference
https://en.wikipedia.org/wiki/Joseph_Oesterl%C3%A9
https://en.wikipedia.org/wiki/David_Masser
http://www.kurims.kyoto-u.ac.jp/~motizuki/