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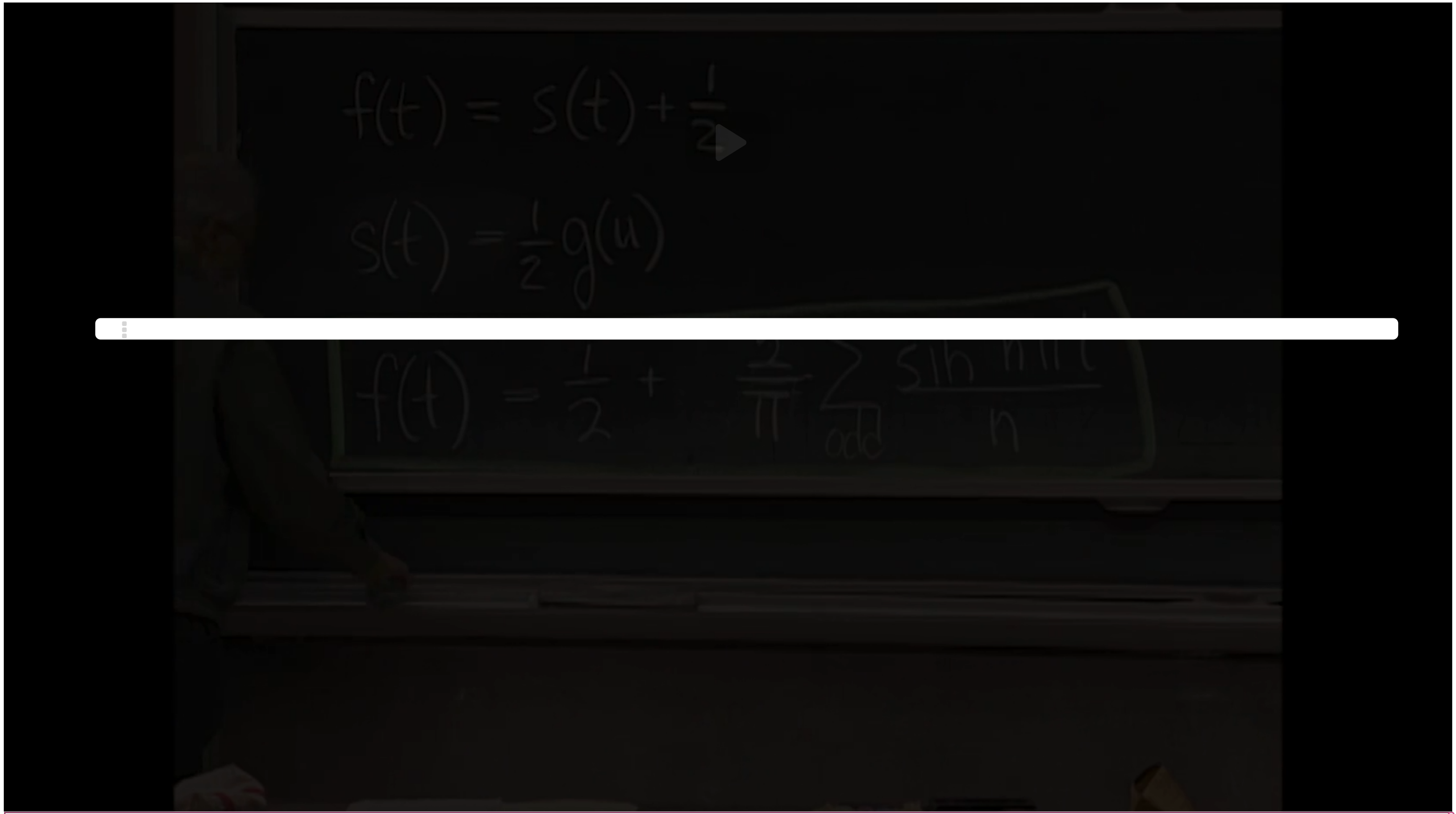
6. Functions of arbitrary period

Manipulating Fourier series



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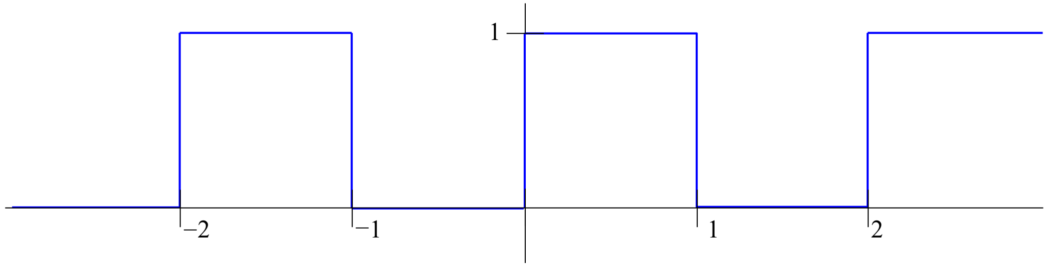
Everything we did with periodic functions of period 2π can be generalized to periodic functions of other periods.

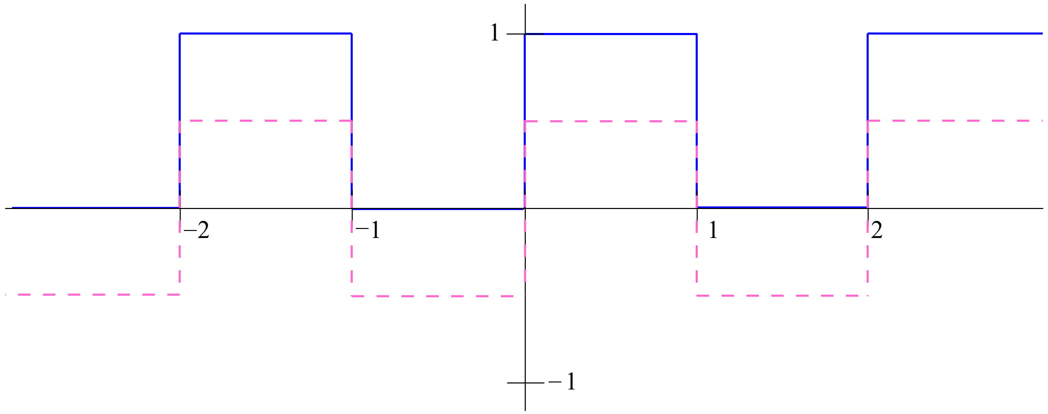
Problem 6.1 Define

$$f(t) := \begin{cases} 1 & \text{if } 0 < t < 1, \\ 0 & \text{if } -1 < t < 0. \end{cases}$$



and extend it to a periodic function of period 2. Express this new square wave $f(t)$ in terms of Sq. Then use the known Fourier series of Sq to find the Fourier series for $f(t)$.

Solution: This function is neither even nor odd. However, if we shift the function downwards by $1/2$, it is an odd function.

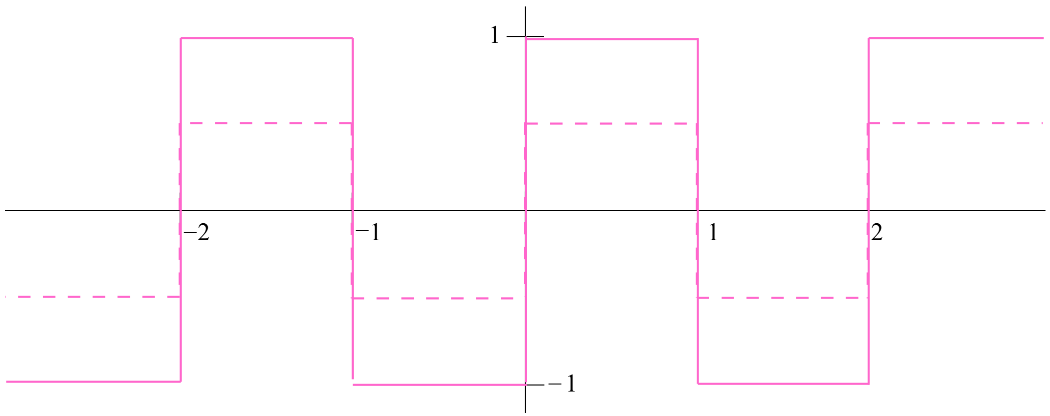


Write the shifted function

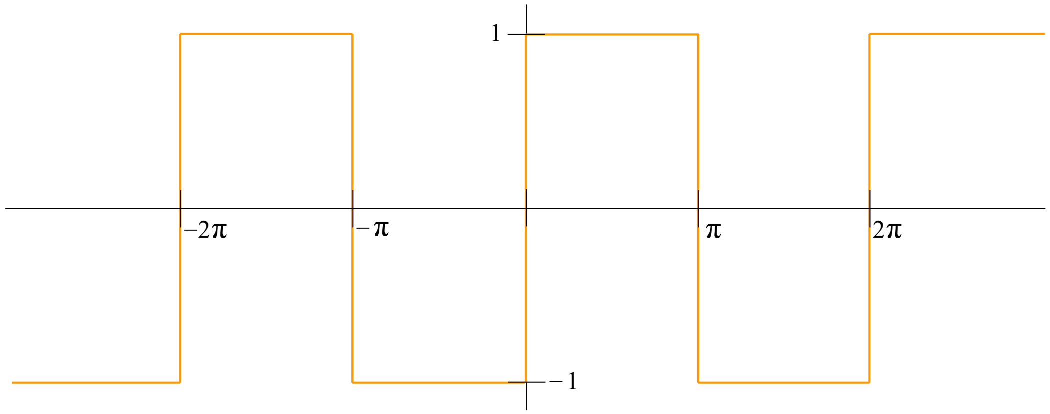
$$s(t) = f(t) - 1/2 = \begin{cases} 1/2 & 0 < t < 1 \\ -1/2 & -1 < t < 0 \end{cases}.$$

If we multiply this function by 2,

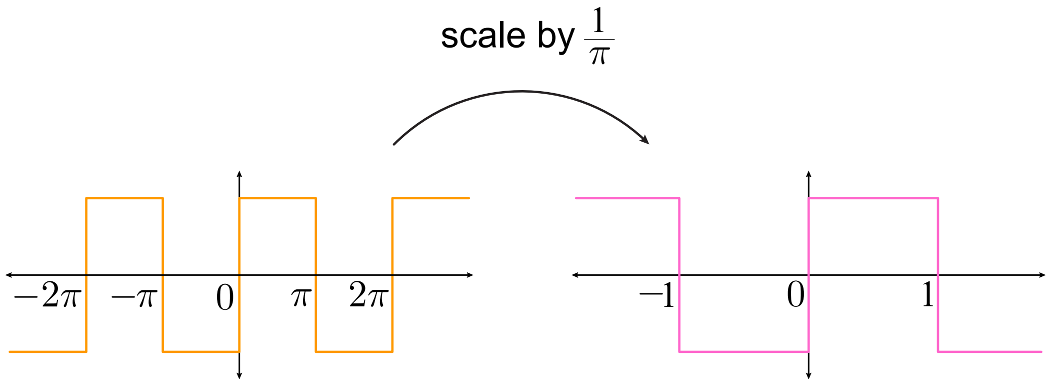
$$2s(t) = 2f(t) - 1 = \begin{cases} 1 & 0 < t < 1 \\ -1 & -1 < t < 0 \end{cases}.$$



what we get is very similar to the square wave of period 2π.



To avoid confusion, let's use u as the variable for Sq. Scaling the horizontal axis by factor of $1/\pi$ produces the graph of $2s\left(t\right)$.



In other words, if t and u are related by $u=\pi t$ (so that $u=\pi$ corresponds to $t=1$), then $2s\left(t\right)=\mathrm{Sq}\left(u\right)$. In other words,

$$2s\left(t\right)=\mathrm{Sq}\left(\pi t\right).$$

Therefore,

$$f(t) = \frac{1}{2} + s(t) = \frac{1}{2} + \frac{1}{2}\mathrm{Sq}(\pi t).$$

Since we know the Fourier series

$$\mathrm{Sq}(u) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nu)}{n},$$

the Fourier series for $f(t)$ is

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}.$$

Period $2L$ functions

Similarly we can scale the horizontal axis of any function of period 2π to get a function of different period. Let L be a positive real number. Start with “any" periodic function

$$g(u) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos nu + \sum_{n\geq 1} b_n \sin nu,$$

of period 2π. Stretching horizontally by a factor L/π gives a periodic function $f(t)$ of period $2L$, and “every" f of period $2L$ arises this way. By the same calculation as above,

$$\begin{aligned} f(t) &= g\left(\frac{\pi t}{L}\right) \\ &= \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos \frac{n\pi t}{L} + \sum_{n\geq 1} b_n \sin \frac{n\pi t}{L} \end{aligned}$$

The substitution $u = \frac{\pi t}{L}$ (and $du = \frac{\pi}{L}\,dt$) also leads to Fourier coefficient formulas for period $2L$:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g\left(u\right)\cos nu\,du \\ &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)\frac{\pi}{L}\,dt \\ &= \frac{1}{L}\int_{-L}^{L} f\left(t\right)\cos\left(\frac{n\pi t}{L}\right)\,dt. \end{aligned}$$

A similar formula gives b_n in terms of f.

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