MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Problem 2: A binary communication system - Part 2

(1/2 points)

Note: The problem statement from part 1 has been repeated here for your convenience.

A binary communication system is used to send one of two messages:

(i) message A is sent with probability 2/3, and consists of an infinite sequence of zeroes,

(ii) message B is sent with probability 1/3, and consists of an infinite sequence of ones.

The ith received bit is “correct" (i.e., the same as the transmitted bit) with probability 3/4, and is “incorrect" (i.e., a transmitted 0 is received as a 1, and vice versa), with probability 1/4. We assume that **conditioned on any specific message sent,** the received bits, denoted by $Y_1, Y_2, \ldots$ are independent.

1. Is $Y_2 + Y_3$ independent of Y_1?

   Yes ✖ **Answer:** No

2. Is $Y_2 - Y_3$ independent of Y_1?

   Yes ✔ **Answer:** Yes

Answer:

1. No, they are not independent. Let $X = Y_2 + Y_3$. Using the total probability theorem we find:

$$\begin{aligned}\mathbf{P}(X=0) &= \mathbf{P}(A)\mathbf{P}(X=0 \mid A) + \mathbf{P}(B)\mathbf{P}(X=0 \mid B)\\ &= (2/3)(3/4)^2 + (1/3)(1/4)^2\\ &= 19/48 \approx 0.396.\end{aligned}$$

Conditioning on the event $Y_1 = 0$, we now find that:

$$\begin{aligned}\mathbf{P}(X=0 \mid Y_1=0) &= \frac{\mathbf{P}(X=0 \cap Y_1=0)}{\mathbf{P}(Y_1=0)}\\ &= \frac{\mathbf{P}(A)\mathbf{P}(X=0 \cap Y_1=0 \mid A) + \mathbf{P}(B)\mathbf{P}(X=0 \cap Y_1=0 \mid B)}{\mathbf{P}(Y_1=0)}\\ &= \frac{(2/3)(3/4)^3 + (1/3)(1/4)^3}{7/12}\\ &= 55/112 \approx 0.491.\end{aligned}$$

Since we have shown $\mathbf{P}(X=0 \mid Y_1=0) \neq \mathbf{P}(X=0)$ with $\mathbf{P}(Y_1=0) > 0$, we conclude that X and Y_1 are not independent. Intuitively, knowing $Y_1 = 0$ increases the likelihood that message A was transmitted, which increases the likelihood that $Y_2 + Y_3 = 0$.

2. Yes, they are independent. Let $Z = Y_2 - Y_3$. We want to show that $p_{Y_1}(y_1)p_Z(z) = p_{Y_1,Z}(y_1, z)$, for all (y_1, z). We have already found the PMF of Y_1 in part (1):

$$p_{Y_1}(y_1) = \begin{cases} 7/12 & \text{if } y_1 = 0 \\ 5/12 & \text{if } y_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Using the total probability theorem, similar to how we calculated $\mathbf{P}(X = 0)$ in part (4), we find the PMF of Z:

$$p_Z(z) = \begin{cases} 3/16 & \text{if } z = -1 \text{ or } z = 1 \\ 5/8 & \text{if } z = 0 \\ 0 & \text{otherwise} \end{cases}$$

The joint PMF can also be found using the total probability theorem:

$$p_{Y_1,Z}(y_1, z) = \begin{cases} 7/64 & \text{if } (y_1, z) = \{(0,-1),(0,1)\} \\ 35/96 & \text{if } (y_1, z) = (0,0) \\ 5/64 & \text{if } (y_1, z) = \{(1,-1),(1,1)\} \\ 25/96 & \text{if } (y_1, z) = (1,0) \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $p_{Y_1}(y_1)p_Z(z) = p_{Y_1,Z}(y_1, z)$, for all (y_1, z). Intuitively, Z is independent of what message was transmitted, and so, even though Y_1 provides information on which message was transmitted, it ultimately does not provide information on Z.

You have used 1 of 1 submissions