MATHEMATICS

Why is the outer measure of the set of irrational numbers in the interval [0,1] equal to 1?

Asked 5 years, 9 months ago Active 4 months ago Viewed 9k times

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Just learned Lebesgue outer measure from Royden's Real Analysis.

Let me give my proof. First, let A be the set of irrational numbers in [0,1]. So $A \subset [0,1] \Rightarrow m^*(A) \leq m^*([0,1]) = 1$.

Then I want to show $m^*(A) \geq 1$ by using $\sum_{k=1}^{\infty} l(I_k) \leq m^*(A) + \epsilon$. $\{I_k\}_k$ covers A, then add I_0 to this collection. $[0,1] \subset I_0$. So

$l(I_0) + \sum_{k=1}^{\infty} l(I_k) \leq m^*(A) + \epsilon \Rightarrow m^*(A) \geq l(I_0) + \sum_{k=1}^{\infty} l(I_k) - \epsilon \geq 1 + \sum_{k=1}^{\infty} l(I_k) - \epsilon$

We can always choose a small enough $\epsilon > 0$ such that $\sum_{k=1}^{\infty} l(I_k) - \epsilon > 0$. Therefore, $m^*(A) = 1$.

real-analysis analysis measure-theory proof-verification lebesgue-measure

edited Nov 23 '17 at 6:16
reflexive
2,730 8 30

asked Oct 17 '14 at 16:51
Drake Marquis
1,120 8 15

Why is "the" outer measure.... You should say, why is an "outer" measure of.... Unless you are very specific what sort of outer measure you are using, it's NOT obvious – Squirtle Oct 17 '14 at 16:54

3 @Squirtle I disagree, we're talking about the real line, the obvious choice of measure is the Lebesgue measure and the obvious choice of outer measure is the Lebesgue outer measure. – Ian Oct 17 '14 at 17:16

1 If $I_0 \supset [0,1]$ then $l(I_0) \geq 1$ so $l(I_0) + \sum_{i=1}^{\infty} l(I_i) \geq 1 + m^*(A)$, and in fact $1 + m^*(A) = 2$. – DanielWainfleet Nov 23 '17 at 9:44

3 Answers

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The rational numbers has measure zero, so $\mathbb{Q} \in \mathcal{M}(\lambda^*)$. Then

$$1 = \lambda^*([0,1]) = \lambda^*([0,1] \cap \mathbb{Q}) + \lambda^*([0,1] \setminus \mathbb{Q}) = 0 + \lambda^*([0,1] \setminus \mathbb{Q})$$

i.e., $1 = \lambda^*([0,1] \setminus \mathbb{Q})$.

edited Mar 9 at 15:20
Rab
1,034 5 17

answered Oct 17 '14 at 21:09
Jose Antonio
6,376 2 21 34

8

What you know is that $\sum_k l(I_k) \leq m^*(A) + \epsilon$ for **some** sequence of intervals covering A. You've got $l(I_0) \geq 1$ but only add it to the left-hand side of the inequality so your solution is in error.

Do you know that $m^*([0,1]) = 1$ and $m^*(rationals) = 0$? If so use subadditivity and monotonicity:

$$m^*([0,1]) \leq m^*(rationals) + m^*(irrationals) = m^*(irrationals) \leq m^*([0,1])$$

so that

$$m^*(irrationals) = m^*([0,1]) = 1.$$

answered Oct 17 '14 at 16:56

Umberto P.
44.5k 2 37 76

Thanks! Yes! I know $m^*(rationals) = 0$ because the set of rationals is countable. I made the problem more complicated. – Drake Marquis Oct 17 '14 at 17:15

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First we show that if S is a countable subset of $\mathbb{R}$ then the Lebesgue outer measure $m^*(S) = 0$. Second we show that if J is a bounded real interval then $m^*(J) = l(J)$.

Now for $J \subset \mathbb{R}$ and for a countable $S \subset J$ let A be any countable family of open intervals with $\cup A \supset (J \setminus S)$. For $r > 0$ let B_r be a countable family of open intervals with $\cup B_r \supset S$ and $\sum_{b \in B_r} l(b) \leq r$. Then $C = B_r \cup A$ is a countable family of open intervals with $\cup C \supset J$ so

$$m^*(J \setminus S) \leq m^*(J) \leq \sum_{c \in C} l(c) \leq \sum_{b \in B_r} l(b) + \sum_{a \in A} l(a) \leq r + \sum_{a \in A} l(a).$$

Taking the inf of the right-most expression above, over every family A of open intervals that covers $J \setminus S$, we have

$$(\bullet) \quad m^*(J \setminus S) \leq m^*(J) \leq r + m^*(J \setminus S).$$

Since $(\bullet)$ holds for every $r > 0$, we have

$$m^*(J \setminus S) \leq m^*(J) \leq m^*(J \setminus S).$$

Therefore $m^*(J \setminus S) = m^*(J)$.

In particular if J is a bounded interval then $m^*(J \setminus S) = m^*(J) = l(J)$.

edited Nov 23 '17 at 10:52

answered Nov 23 '17 at 10:46

DanielWainfleet
44.5k 3 19 59

You should also learn about inner measure. With m denoting Lebesgue measure, the inner measure of any $T \subset \mathbb{R}$ is $m^i(T) = \sup\{m(U) : U = \overline{U} \subset T\}$..... A set $T \subset \mathbb{R}$ is Lebesgue-measurable iff $m^*(T) = m^i(T)$. – DanielWainfleet Nov 23 '17 at 10:50

This answer will apply verbatim for any $S \subset \mathbb{R}$ such that $m^*(S) = 0$. – DanielWainfleet Nov 23 '17 at 11:03