Living Large

There is an important strategy for building infinite sets of bigger and bigger cardinalities. It is based on deploying two basic resources: the power set operation, and the union operation.

We'll begin by reviewing these resources.

The power set operation

In Lecture 1, we proved Cantor's Theorem, which shows that a set's power set always has more members than the set itself. This result entails that there is an infinite hierarchy of bigger and bigger infinities. More specifically, it entails:

$$|\mathbb{N}| < |\mathscr{P}^1(\mathbb{N})| < |\mathscr{P}^2(\mathbb{N})| < |\mathscr{P}^3(\mathbb{N})| < \dots$$

where:

$$\mathscr{P}^n(S) = \underbrace{\mathscr{P}(\mathscr{P}(\dots \mathscr{P}(S) \dots))}_{n \text{ times}}$$

The sequence above includes some very big sets. For example, the set $\mathscr{P}^{10^{10^{10}}}(\mathbb{N})$ is fantastically bigger than any set we considered in Lecture 1.

But could we go bigger still? Could we characterize a set that is bigger than $\mathscr{P}^k(\mathbb{N})$ for *every* natural number k?

One might be tempted to introduce a set by way of the following definition:

$$\mathscr{P}^\infty(\mathbb{N}) = \underbrace{\mathscr{P}(\mathscr{P}(\ldots \mathscr{P}(\mathbb{N})\ldots))}_{\infty \text{ times}}$$

But it is not clear that such a definition makes sense.

To see the problem, note that $\mathscr{P}^\infty(\mathbb{N})$ is supposed to be defined by iterating the ordinary power set operation, $\mathscr{P}$. But each application of $\mathscr{P}$ requires a definite input. When k is a positive integer, the input of $\mathscr{P}^k(\mathbb{N})$ is $\mathscr{P}^{k-1}(\mathbb{N})$. But what input should one use at the supposed "infinite-th" stage of the process?

There is not a clear answer to this question. (Notice, in particular, that "$\mathscr{P}^{\infty-1}(\mathbb{N})$" won't do as an answer, since it is not clear what $\mathscr{P}^{\infty-1}(\mathbb{N})$ is supposed to be.)

The union operation will help us get around this problem.

The Union Operation

In Lecture 1, we encountered a version of the union operation that takes finitely many sets as input, and delivers a single set as output. More specifically, we took $A_1 \cup A_2 \cup \cdots \cup A_n$ to be the set of individuals x such that x is in at least one of $A_1, A_2 \ldots, A_n$.

We will now consider a variant of the union operation that that takes as input a set A of arbitrarily many sets. Let A be a set of sets; as it might be,

$$A = \{S_1, S_2, S_3, \ldots\}$$

Then the union of A (in symbols: $\bigcup A$) is the result of pooling together the elements of each of the sets in A. In other words: $\bigcup A = S_1 \cup S_2 \cup S_3 \cup \ldots$. (Formally, we define $\bigcup A$ as the set of individuals x such that x is a member of some member of A.)

The key advantage of this new version of the union operation is that $\bigcup A$ is well-defined even if A has infinitely many sets as members. This makes the union operation incredibly powerful.

To illustrate this point, consider the set $\{\mathbb{N}, \mathscr{P}^1(\mathbb{N}), \mathscr{P}^2(\mathbb{N}), \ldots\}$. Even though it only has as many elements as there are natural numbers, its union

$$\bigcup\{\mathbb{N}, \mathscr{P}^1(\mathbb{N}), \mathscr{P}^2(\mathbb{N}), \ldots\}$$

is far bigger than the set of natural numbers. In fact, it is bigger than $\mathscr{P}^k(\mathbb{N})$ for each natural number k, since it includes everything in $\mathscr{P}^{k+1}(\mathbb{N})$.

Video Review: Unions

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Problem 1

1 point possible (ungraded)

Which of the following, if any, is a set A such that $|\bigcup A| < |A|$?

- $A = \{\mathbb{N}\}$
- $A = \mathscr{P}(\mathbb{N})$
- $A = \{\mathbb{N}, \mathscr{P}^1(\mathbb{N}), \mathscr{P}^2(\mathbb{N}), \ldots\}$
- None of the above.

Submit

Bringing the two operations together

This is what we've learned so far:

$$|\mathbb{N}| < |\mathscr{P}^1(\mathbb{N})| < |\mathscr{P}^2(\mathbb{N})| < \cdots < |\bigcup\{\mathbb{N}, \mathscr{P}^1(\mathbb{N}), \ldots\}|$$

Can we construct sets that are bigger still?

Of course we can!

Let us use "$\mathscr{U}$" to abbreviate "$\bigcup\{\mathbb{N}, \mathscr{P}^1(\mathbb{N}), \ldots\}$". Then Cantor's Theorem tells us that if we apply the powerset operation to $\mathscr{U}$, we will get something even bigger. And each successive application of the powerset operation gives us something bigger still:

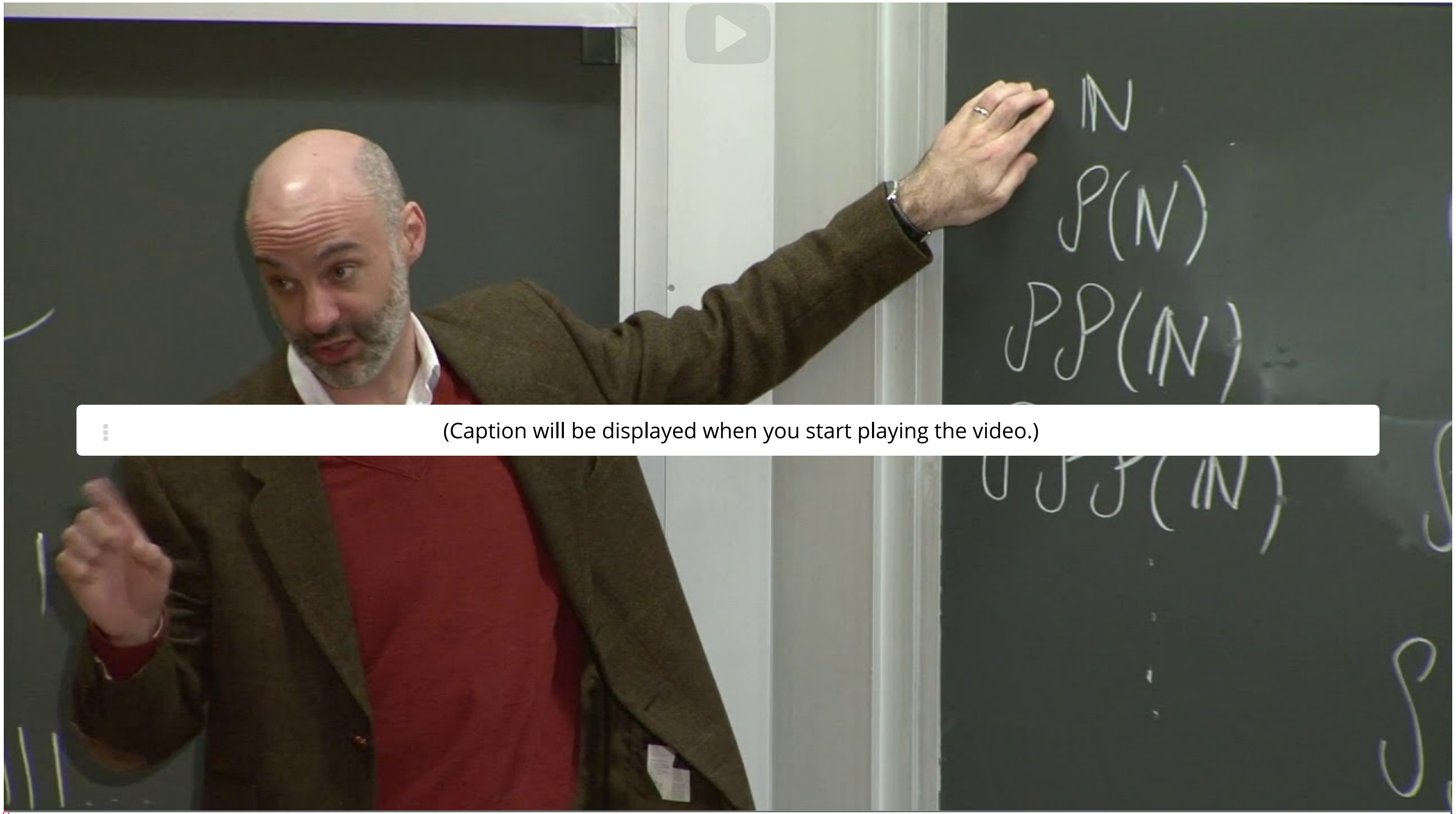
$$|\mathscr{U}| < \ldots |\mathscr{P}^1(\mathscr{U})| < |\mathscr{P}^2(\mathscr{U})| < \ldots$$

And we can keep going. We can apply the union operation to the set of everything we’ve built so far. And then apply further iterations of the powerset operation. And then apply the union operation to everything we’ve built so far. And then apply further iterations of the power sets. And so forth.

We now have a procedure for constructing sets of greater and greater cardinality. In rough outline the procedure is very simple: after applying the powerset operation countably many times, apply the union operation. And repeat.

The main objective of this chapter is to show you how to develop the idea properly, by introducing you to one of the most beautiful tools in the whole of mathematics: the notion of an ordinal.

Video Question: Clarifying the Union Operation



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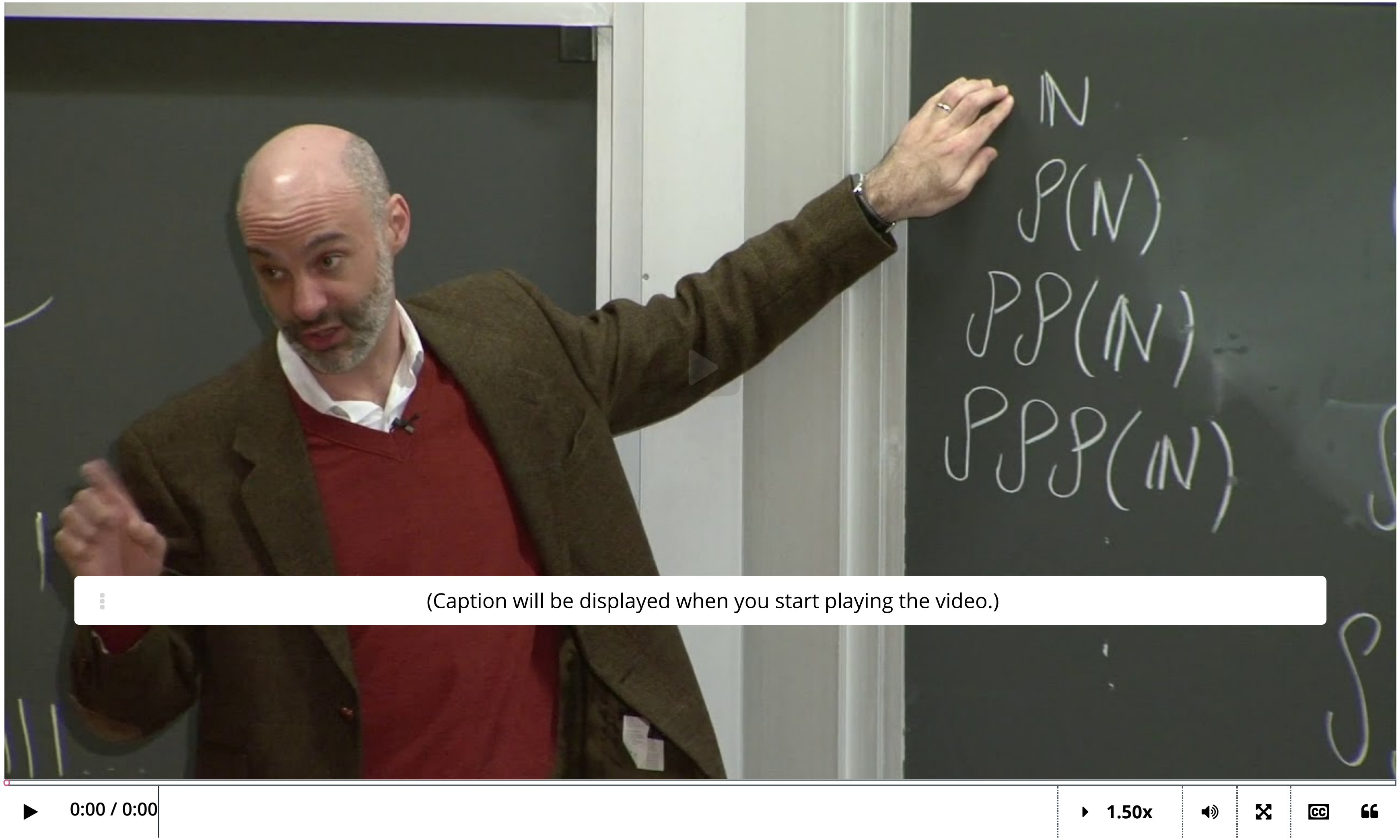
Duplicate of 'Video Question: Clarifying the Powerset Operation'



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Note: There's an important omission on 4:07 of the video. If I had written out a description of $\mathscr{P}^2(\mathbb{N})$ on the board, it might have looked like this:

$$\mathscr{P}^2(\mathbb{N}) = \big\{\{\}, \{\{\}\}, \{\{0\}\}, \{\{1\}\}, \ldots, \{\{\}, \{0\}\}, \{\{\}, \{1\}\}, \{\{0\}, \{1\}\}, \ldots\big\}$$

So although it's true that most members of $\mathscr{P}^2(\mathbb{N})$ have two levels of nested brackets, there is one exception, since the empty set $\{\}$ is a member of $\mathscr{P}^2(\mathbb{N})$ (and of $\mathscr{P}^n(\mathbb{N})$ for each n) but has only one level of nested brackets.

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