Unit 6: Quiz

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The problems in the quiz will be automatically graded.

You are given only **ONE** attempt for each problem. You will be able to see the solution after you submit your answers.

Problem 1

1/2 points (graded)

1a. Suppose X and Y are independent Poisson random variables, each with expected value 2. Define $Z = X + Y$. Find $P(Z \leq 3)$.

0.4334701

✔ **Answer:** 0.4335

1b. Consider a Poisson random variable X with parameter $\lambda = 5.3$, and its probability mass function, $p_X(x)$. Where does $p_X(x)$ have its peak value?

5.3

✖ **Answer:** 5

Explanation

1a. Since X and Y are independent Poisson random variables, then Z is a Poisson random variable too. We have $\mathbb{E}(Z) = \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 2 + 2 = 4$. We have
$P(Z \le 3) = p_Z(0) + p_Z(1) + p_Z(2) + p_Z(3) = e^{-4} + 4e^{-4} + 8e^{-4} + \frac{32}{3}e^{-4}$
$= 0.0183 + 0.0733 + 0.1465 + 0.1954 = 0.4335$.

1b. We calculate a few values of the probability mass function of X, and we find that $p_X(x)$ attains its maximum when $x = 5$; indeed, we have $p_X(5) = e^{-5.3}(5.3)^5/5! = 0.1740$.

Submit You have used 1 of 1 attempt

Problem 2

2/2 points (graded)

2a. If X is a Poisson random variable with expected value 2.2, find the conditional probability that $X > 4$, given that $X > 2$.

0.192152 ✔ **Answer:** 0.1922

2b. If X is a Poisson random variable with expected value 2.2, find the conditional probability that $X \le 1$, given that $X \le 3$.

0.4327443 ✔ **Answer:** 0.433

Explanation

2a. We have

$$P(X > 4 \mid X > 2) = \frac{P(X>4\ \&\ X>2)}{P(X>2)} = \frac{P(X>4)}{P(X>2)}$$

$$= \frac{1-P(X\le 4)}{1-P(X\le 2)} = \frac{1-0.1108-0.2438-0.2681-0.1966-0.1082}{1-0.1108-0.2438-0.2681} = 0.1922$$

2b. We have

$$P(X \le 1 \mid X \le 3) = \frac{P(X\le 1\ \&\ X\le 3)}{P(X\le 3)} = \frac{P(X\le 1)}{P(X\le 3)}$$

$$= \frac{0.1108+0.2438}{0.1108+0.2438+0.2681+0.1966} = 0.433$$

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Problem 3

2/2 points (graded)

3a. Suppose that, during a given week, $5{,}000{,}000$ people play a lottery game. If their chances to win the lottery are independent, and if each person has probably $1/2{,}000{,}000$ of winning the lottery, write an *exact expression* for the probability that there are exactly 4 winners of the lottery that week. (This actual probability corresponds to a particular value of the probability mass function of a Binomial random variable.)

- (selected, correct) $\binom{5000000}{4}\left(\frac{1}{2000000}\right)^4\left(\frac{1999999}{2000000}\right)^{5000000-4}$
- $\binom{2000000}{4}4^{\left(\frac{4}{5000000}\right)}(2000000-4)^{\left(\frac{1999999}{2000000}\right)}$

- $\left(\frac{1}{2000000}\right)^4 \left(\frac{1999999}{2000000}\right)^{5000000-4}$

- $4^{\left(\frac{4}{5000000}\right)} (2000000-4)^{\left(\frac{1999999}{2000000}\right)}$

3b. Briefly explain how you can *approximate* the value in part (3a) using a Poisson random variable. Then give an approximate value for the probability that there are exactly 4 winners.

P(there are exactly 4 winners)≈ 0.1336019 ✔ **Answer:** 0.1336

Explanation

3a. The exact expression is $\binom{5000000}{4}\left(\frac{1}{2000000}\right)^4\left(\frac{1999999}{2000000}\right)^{5000000-4} = 0.133601909\ldots$
(You do not need this last number in your answer; it takes a computer to approximate the answer.)
3b. The actual number of winners is a Binomial random variable with $n = 5000000$ and $p = 1/2000000$. So n is large and $np(1-p)$ is roughly $5/2$ which is a moderate size number, i.e., not too far from 1. So the number of winners is approximately Poisson with $\lambda = np = 5/2$. So the probability of 4 winners is approximately $e^{-5/2}(5/2)^4/4! = 0.133601886\ldots$.

Submit You have used 1 of 1 attempt

Problem 4

1/1 point (graded)

4. Suppose that X is a Poisson random variable with $\mathbb{E}(X)=\lambda$. Find $\mathbb{E}((X)(X-1)(X-2))$.

- $e^{-\lambda}$
- λ^3 ✔
- $\lambda^3 e^{-\lambda}$
- $\lambda^3 e^{-2\lambda}$

Explanation

4. We have

$$E((X)(X-1)(X-2)) = \sum_{x=0}^{\infty}(x)(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!}$$
$$= \sum_{x=3}^{\infty}(x)(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{because } x=0,1,2 \text{ terms are themselves } 0$$
$$= \sum_{x=3}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x-3)!} \qquad \text{divide out by } x \text{ and } x-1 \text{ and } x-2$$
$$= \lambda^3 e^{-\lambda}\sum_{x=3}^{\infty}\frac{\lambda^{x-3}}{(x-3)!} \qquad \text{factor out } e^{-\lambda} \text{ and } \lambda^3$$
$$= \lambda^3 e^{-\lambda}\left(\frac{\lambda^0}{0!}+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\dots\right)$$
$$= \lambda^3 e^{-\lambda}e^{\lambda}$$
$$= \lambda^3$$

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Problem 5

4/4 points (graded)

5. At a lunch buffet there are 13 burgers without guacamole and 7 burgers with guacamole. Isabella, Rodrigo, and their two children each blindly reach for a burger.

5a. If they independently pick at once and (chaotically!) reach for their burger-and all selections are equally likely-this is just like choosing with replacement. Let X be the number of the people that reach for burgers with guacamole. What are the expected number and variance of X?

$\mathbb{E}(X) =$ 7/5 **Answer:** 1.4

$\text{Var}(X) =$ 0.91 **Answer:** 0.91

5b. More realistically, if they take turns, without replacement, and each person draws blindly from the remaining burgers, this is choosing without replacement. Let Y be the number of the people that get burgers with guacamole. What are the expected number and variance of Y?

$\mathbb{E}(Y) =$ 7/5 **Answer:** 1.4

$\text{Var}(Y) =$ 0.7663158 **Answer:** 0.7663

Explanation

5a. Since X is Binomial with $n = 4$ and $p = 7/20$, then $\mathbb{E}(X) = np = (4)(7/20) = 7/5$, and
$\text{Var}\,(X) = np(1-p) = (4)(7/20)(13/20) = 91/100$.
5b. Since Y is Hypergeometric with $N = 20$, $M = 7$, and $n = 4$, then we get
$\mathbb{E}(Y) = n(M/N) = (4)(7/20) = 7/5$, and

$$\begin{aligned}\text{Var}\,(Y) &= n(M/N)(1 - M/N)(N-n)/(N-1) \\ &= (4)(7/20)(1-7/20)(20-4)/(20-1) = 364/475 = 0.7663.\end{aligned}$$

Submit You have used 1 of 1 attempt

Problem 6

1/1 point (graded)

6. Suppose that X and Y are independent Hypergeometric random variables that each have parameters $N = 6$, $M = 3$, and $n = 2$. What is the probability that X and Y are equal, i.e., what is $P(X = Y)$?

11/25 ✔ **Answer:** 0.44

Explanation

6. We have

$$\begin{aligned}P(X = Y) &= P(X = Y = 0) + P(X = Y = 1) + P(X = Y = 2) \\ &= (1/5)^2 + (3/5)^2 + (1/5)^2 = 11/25.\end{aligned}$$

Submit You have used 1 of 1 attempt

Problem 7

2/2 points (graded)

7a. Suppose that X is a Hypergeometric random variable with parameters $N = 50{,}000$, $M = 15{,}000$, and $n = 10$. Write an exact expression for $P(X = 4)$. You do not need to evaluate the expression.

- $\binom{15000}{6}\binom{35000}{4}/\binom{50000}{10}$
- $\binom{15000}{4}\binom{50000}{6}/\binom{75000}{10}$
- $\binom{15000}{4}\binom{35000}{6}/\binom{50000}{10}$ ✔
- $\binom{15000}{6}\binom{50000}{4}/\binom{75000}{10}$

7b. Now approximate the expression from part **7a.**

0.2001209 ✔ **Answer:** 0.2001

Explanation

7a. The exact expression is $P(X=4)=\binom{15000}{4}\binom{35000}{6}/\binom{50000}{10}=0.20013524\ldots$ (You did not have to put the decimal value, of course; it is probably way too large for your calculator.)
7b. Since X is approximately Binomial with $n=10$ and $p=M/N=35000/50000=7/10$, then $P(X=4)$ is approximately equal to $\binom{10}{4}(3/10)^4(7/10)^6=0.20012095\ldots$

Submit You have used 1 of 1 attempt

Problem 8

1/1 point (graded)
8. Consider a Binomial random variable X with parameters n and p, and consider a Hypergeometric random variable Y with parameters N, M, n (the same value of n as for the Binomial), and suppose that p and M/N happen to be the same value.

8a. If $n=1$, convince yourself that $P(X=1)$ and $P(Y=1)$ are always the same. Why? Is there an intuitive reason for this?

8b. If $n\geq 2$, which is larger, $P(X=n)$ or $P(Y=n)$? Why?

- (selected) $P(X=n)>P(Y=n)$ ✔
- $P(X=n)<P(Y=n)$

Explanation

8a. If $n = 1$, then $P(X = 1) = \binom{1}{1} p^1 (1-p)^{1-1} = p$ and $P(Y = 1) = M/N$, so these are the same value. The intuitive reason is that X corresponds to a sampling of one item with replacement, to see if it is a success, and Y corresponds to a sampling of one item without replacement, to see if it a success, but we don't worry about whether or not we are replacing after picking, because we only pick one item to test.

8b. We have $P(X = n) = \binom{n}{n} p^n (1-p)^{n-n} = p^n$, which is equal to $(M/N)^n$. In contrast, $P(Y = n) = (\frac{M}{N})(\frac{M-1}{N-1})(\frac{M-2}{N-2}) \cdots (\frac{M-n+1}{N-n+1}) < (M/N)^n$, so $P(Y = n) < P(X = n)$.

Submit

You have used 1 of 1 attempt

Problem 9

2/2 points (graded)

9. A playlist contains 10 rock songs, 3 country songs, 5 R&B songs, and 2 blues songs. In shuffle mode, each song is played exactly once, and all possible equal orderings are equally likely. Suppose that a person starts this playlist in shuffle mode and continues until a country music song plays, and then stops. Let X denote the number of songs played *before* the country music song (but not including the country music song itself). [[Hint: Write $X = X_1 + \cdots + X_{17}$, where $X_j = 1$ if the j th non-country song is played before all of the country songs, or $X_j = 0$ otherwise.]]

9a. Find $\mathbb{E}(X)$.

17/4

Answer: 4.25

9b. Find $\mathrm{Var}(X)$.

13.3875

✔ **Answer:** 13.3875

Explanation

9a. We have $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{17}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{17})$. Also $\mathbb{E}(X_j) = 1/4$, so it follows that $\mathbb{E}(X) = (17)(1/4) = 17/4 = \mathbf{4.25}$.

9b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{17})^2)$, which has 17 terms of the form $\mathbb{E}(X_j^2)$ and $17^2 - 17 = 272$ terms of the form $\mathbb{E}(X_i X_j)$. Also $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 1/4$ and $\mathbb{E}(X_i X_j) = (2)(1/5)(1/4) = 1/10$. Thus $\mathbb{E}(X^2) = (17)(1/4) + (272)(1/10) = 31.45$. So altogether we have $\mathbf{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 31.45 - (4.25)^2 = \mathbf{13.3875}$.

Submit You have used 1 of 1 attempt

Problem 10

4/4 points (graded)

10. In question (9a), suppose that we randomly pick 4 songs (without repetitions) to play.

10a. What is the probability that we get 1 song from each of the 4 genres?

20/323

✔ **Answer:** 0.06192

10b. What is the probability that all 4 songs are selected from the same 1 genre?

0.04437564

✔ **Answer:** 0.04438

10c. What is the probability that 3 of the 4 genres appear during the 4 songs?

0.4726522 ✔ **Answer:** 0.4727

10d. Knowing the answers to a, b, c, you can use the complement to find the probability that 2 of the 4 genres appear during the 4 songs. (To test your strength, are you able to also calculate this probability directly?)

$P($2 of the 4 genres appear during the 4 songs$)=$ 0.4210526 ✔ **Answer:** 0.4211

Explanation

10a. The probability is $\binom{10}{1}\binom{3}{1}\binom{5}{1}\binom{2}{1}/\binom{20}{4} = 20/323 = 0.06192$.

10b. The probability is $(\binom{10}{4} + \binom{5}{4})/\binom{20}{4} = 43/969 = 0.04438$.

10c. The probability is

$$\begin{aligned}&(\binom{10}{2}\binom{3}{1}\binom{5}{1} + \binom{10}{2}\binom{3}{1}\binom{2}{1} + \binom{10}{2}\binom{5}{1}\binom{2}{1} + \binom{3}{2}\binom{10}{1}\binom{5}{1} + \binom{3}{2}\binom{10}{1}\binom{2}{1} + \binom{3}{2}\binom{5}{1}\binom{2}{1}\\ &\quad +\binom{5}{2}\binom{10}{1}\binom{3}{1} + \binom{5}{2}\binom{10}{1}\binom{2}{1} + \binom{5}{2}\binom{3}{1}\binom{2}{1} + \binom{2}{2}\binom{10}{1}\binom{3}{1} + \binom{2}{2}\binom{10}{1}\binom{5}{1} + \binom{2}{2}\binom{3}{1}\binom{5}{1})/\binom{20}{4}\\ &\quad = 458/969 = 0.4727.\end{aligned}$$

10d. The probability is

$$\begin{aligned}&(\binom{10}{2}\binom{3}{2} + \binom{10}{2}\binom{5}{2} + \binom{10}{2}\binom{2}{2} + \binom{3}{2}\binom{5}{2} + \binom{3}{2}\binom{2}{2} + \binom{5}{2}\binom{2}{2} + \binom{10}{3}\binom{3}{1} + \binom{10}{3}\binom{5}{1}\\ &\quad +\binom{10}{3}\binom{2}{1} + \binom{3}{3}\binom{5}{1} + \binom{3}{3}\binom{2}{1} + \binom{5}{3}\binom{2}{1} + \binom{10}{1}\binom{3}{3} + \binom{10}{1}\binom{5}{3} + \binom{3}{1}\binom{5}{3})/\binom{20}{4}\\ &\quad = 8/19 = 0.4211.\end{aligned}$$

[[Indeed, the four probabilities above do sum to exactly 1.]]

Submit You have used 1 of 1 attempt

Problem 11

2/2 points (graded)

11. A bag of candy contains 10 green M&M's and 10 red M&M's. Suppose that 10 students pick 2 candies each, without replacement. Let X denote the number of students who get one red and one green candy.

11a. Find $\mathbb{E}(X)$.

5.26 ✔ **Answer:** 5.2632

11b. Find $\mathbf{Var}(X)$.

2.63 ✔ **Answer:** 2.64

Explanation

11a. We can write $X = X_1 + \cdots + X_{10}$ where $X_j = 1$ if the jth pair has 1 red and 1 green, or $X_j = 0$ otherwise. Then $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10})$. Also $\mathbb{E}(X_j) = 10/19$, so it follows that $\mathbb{E}(X) = (10)(10/19) = 100/19 = \mathbf{5.2632}$.

11b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{10})^2)$, which has 10 terms of the form $\mathbb{E}(X_j^2)$ and $10^2 - 10 = 90$ terms of the form $\mathbb{E}(X_i X_j)$. Also $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 10/19$ and $\mathbb{E}(X_i X_j) = (10/19)(9/17) = 90/323$. Thus

$\mathbb{E}(X^2) = (10)(10/19) + (90)(90/323) = 9800/323 = 30.3406$. So altogether we have $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 30.3406 - (5.2632)^2 = 2.64$.

Submit You have used 1 of 1 attempt

✔ Correct (2/2 points)

Problem 12

2/2 points (graded)

12. Consider the even positive integers $2, 4, 6, \ldots, 100$. Let X be one of these integers, with all selections equally likely.

12a. Find $\mathbb{E}(X)$.

51 ✔ **Answer:** 51

12b. Find $\text{Var}(X)$.

833 ✔ **Answer:** 833

Explanation

12a. We have $\mathbb{E}(X) = \mathbb{E}(2Y) = 2\mathbb{E}(Y) = 2(50+1)/2 = 51$.

12b. We have $\text{Var}\,(X) = \text{Var}\,(2Y) = 4\,\text{Var}\,(Y) = 4(50^2 - 1)/12 = 833$.