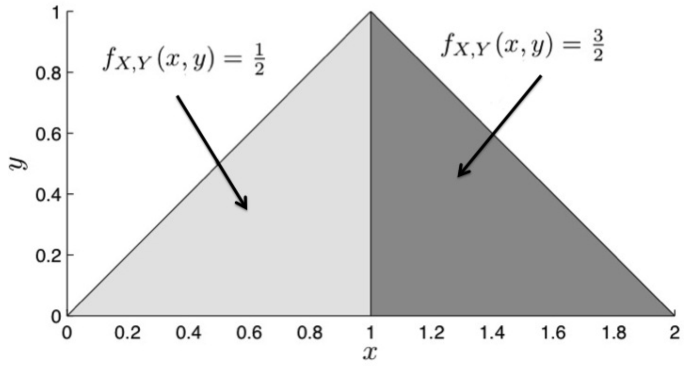
Problem 5: A joint PDF on a triangular region

(7/7 points)

This figure below describes the joint PDF of the random variables X and Y. These random variables take values in $[0, 2]$ and $[0, 1]$, respectively. At $x = 1$, the value of the joint PDF is $1/2$.



1. Are X and Y independent?

Unit overview

Lec. 8: Probability density functions
Exercises 8 due Mar 18, 2016 at 23:59 UTC

Lec. 9: Conditioning on an event; Multiple r.v.'s
Exercises 9 due Mar 18, 2016 at 23:59 UTC

Lec. 10: Conditioning on a random variable; Independence; Bayes' rule
Exercises 10 due Mar 18, 2016 at 23:59 UTC

Standard normal table

Solved problems

Problem Set 5
Problem Set 5 due Mar 18, 2016 at 23:59 UTC

Unit summary

- Unit 6: Further topics on random variables
- Unit 7: Bayesian inference

- Yes
- No ✔

2. Find $f_X(x)$. Express your answers in terms of x using standard notation .

If $0 < x < 1$,

$f_X(x) =$ x/2 ✔ **Answer:** x/2

If $1 < x < 2$,

$f_X(x) =$ 3 - 3*x/2 ✔ **Answer:** (-3/2)*x+3

3. Find $f_{Y|X}(y \mid 0.5)$.

If $0 < y < 1/2$,

$f_{Y|X}(y \mid 0.5) =$ 2 ✔ **Answer:** 2

4. Find $f_{X|Y}(x \mid 0.5)$.

If $1/2 < x < 1$,

$f_{X|Y}(x \mid 0.5) =$ 1/2 **Answer:** 0.5

If $1 < x < 3/2$,

$f_{X|Y}(x \mid 0.5) =$ 3/2 **Answer:** 1.5

5. Let $R = XY$ and let A be the event $\{X < 0.5\}$. Evaluate $\mathbf{E}[R \mid A]$.

$\mathbf{E}[R \mid A] =$ 1/16 **Answer:** 0.0625

Answer:

1. In order for X and Y to be independent, the value of X should not give any information about Y. But if X is smaller than some $\epsilon > 0$, then we can infer that $Y < \epsilon$.

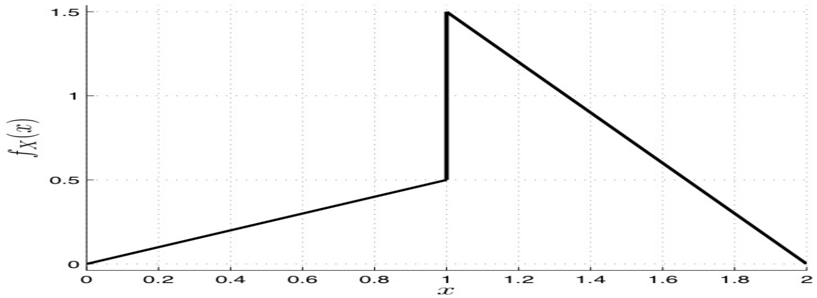
   In other words, $f_{Y|X}(y \mid 0.5) \neq f_Y(y)$. Therefore, X and Y are not independent.

2. Using the formula $f_X(x) = \int f_{X,Y}(x, y)dy$, we have

$$f_X(x) = \begin{cases} \int_0^x \frac{1}{2}\,dy, & \text{if } 0 < x \le 1, \\ \int_0^{2-x} \frac{3}{2}\,dy, & \text{if } 1 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} x/2, & \text{if } 0 < x \le 1, \\ -3x/2 + 3, & \text{if } 1 < x < 2, \\ 0, & \text{otherwise}. \end{cases}$$

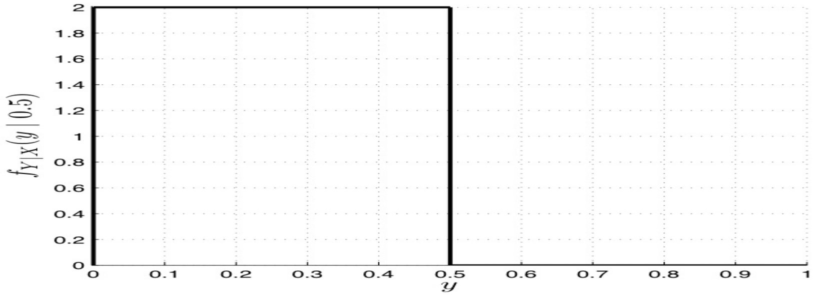
A plot of the PDF is shown below:



3. Given that $X = 0.5$, Y is uniformly distributed between 0 and 1/2. Thus,

$$f_{Y|X}(y \mid 0.5) = \begin{cases} 2, & \text{if } 0 \le y \le 1/2, \\ 0, & \text{otherwise}. \end{cases}$$

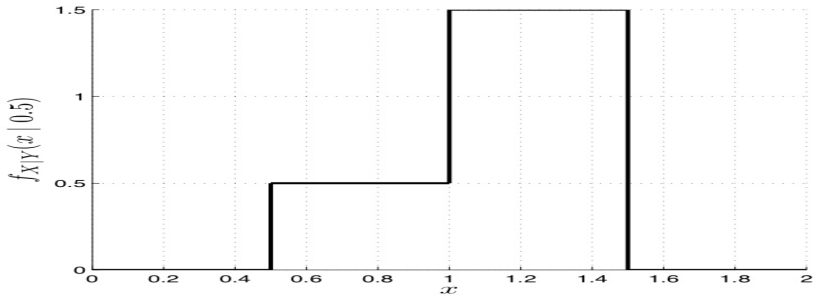
A plot of the conditional PDF is shown below:



4. Given that $Y = 0.5$, the conditional distribution of X is piecewise constant:

$$f_{X|Y}(x \mid 0.5) = \begin{cases} 1/2, & \text{if } 1/2 \le x \le 1, \\ 3/2, & \text{if } 1 < x \le 3/2, \\ 0, & \text{otherwise.} \end{cases}$$

A plot of the conditional PDF is shown below:



5. Under event A, the pair (X, Y) takes values in a triangular region with sides of length $1/2$, and area $1/8$. The conditional point PDF is uniform, so that $f_{X,Y|A}(x, y) = 8$ on that set. The conditional expectation is

$$\begin{aligned}\mathbf{E}[R \mid A] &= \mathbf{E}[XY \mid A] \\ &= \int\int xyf_{X,Y|A}(x,y)\,dx\,dy \\ &= \int_0^{0.5}\int_y^{0.5} 8xy\,dx\,dy \\ &= 1/16.\end{aligned}$$

You have used 3 of 3 submissions

DISCUSSION

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