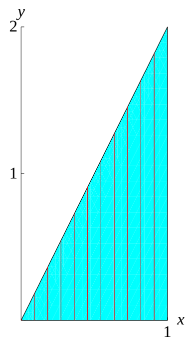
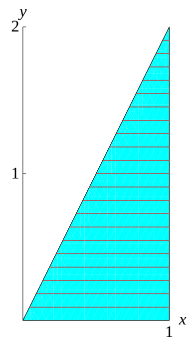
Double Integrals over General Regions

1. *Let $\mathcal{R}$ be the region in the plane bounded by the lines $y = 0$, $x = 1$, and $y = 2x$. Evaluate the double integral* $\displaystyle\iint_{\mathcal{R}} 2xy \; dx\, dy$.

Solution. We can either slice the region $\mathcal{R}$ vertically or horizontally.[(1)]





- **Slicing vertically**:

  Slicing vertically amounts to slicing the interval $[0, 1]$ on the x-axis, so our outer integral will be $\displaystyle\int_0^1$ something dx. To figure out the inner integral, we look at a general slice. Remember that, on a single slice, x is (roughly) constant, and we want to describe what y does. The bottom of each slice is on the line $y = 0$, and the top is on the line $y = 2x$, so the inner integral has endpoints of integration 0 and $2x$. Therefore, our iterated integral is

$$\begin{aligned}
\boxed{\int_0^1 \int_0^{2x} 2xy \; dy \; dx} &= \int_0^1 \left(xy^2 \Big|_{y=0}^{y=2x} \right) \; dx \\
&= \int_0^1 4x^3 \; dx \\
&= x^4 \Big|_{x=0}^{x=1} \\
&= \boxed{1}
\end{aligned}$$

- **Slicing horizontally**:

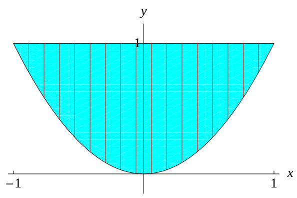
  Slicing horizontally amounts to slicing the interval $[0, 2]$ on the y-axis, so our outer integral will be $\displaystyle\int_0^2$ something dy. To figure out the inner integral, we look at a general slice. The left end of each slice is on the line $y = 2x$, and the right end is on the line $x = 1$. Since we are describing

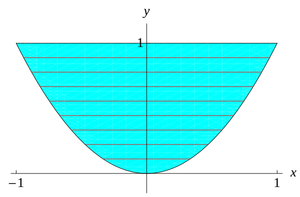
(1) Remember that this is a streamlined version of the real process. Really, to get a Riemann sum approximation, we chop the region $\mathcal{R}$ into lots of small rectangles, each of width Δx and height Δy. The area of each piece is then $\Delta A = \Delta x \Delta y$. We have one product "$f(x, y)\Delta x\Delta y$" per little rectangle, and we need to add these all up to get a Riemann sum. (See #2 of the worksheet "Double Integrals" for more details.) When converting to an iterated integral, we're really deciding whether we want to add up in rows or columns first. If we add up in rows, we visualize adding up in a horizontal slice first and getting one sum per horizontal slice (then we add up all of those sums, one per slice). Similarly, if we add up in columns, we visualize adding up in a vertical slice first and then adding up all those sums, one per vertical slice. So, when we say "slice horizontally," we *really* mean we're going to add up in rows first.

a horizontal slice, we want to describe how x varies, so x goes from $\frac{y}{2}$ to 1. Thus, the iterated integral is $\boxed{\int_0^2 \int_{y/2}^1 2xy \; dx \; dy}$, which is of course also equal to 1.

2. *Let $\mathcal{R}$ be the region bounded by $y = x^2$ and $y = 1$. Write the double integral $\iint_{\mathcal{R}} f(x,y)\; dx\, dy$ as an iterated integral in both possible orders.*

Solution. Again, we think of slicing either vertically or horizontally.





- **Slicing vertically**:

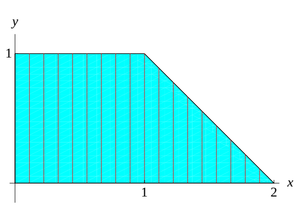
  Slicing vertically amounts to slicing the interval $[-1, 1]$ on the x-axis, so the outer integral will be $\int_{-1}^{1}$ something dx. To write the inner integral, we want to describe what y does within a single slice (thinking of x as being constant). The bottom of each slice lies on $y = x^2$, and the top lies on $y = 1$, so the iterated integral is $\boxed{\int_{-1}^1 \int_{x^2}^1 f(x,y) \; dy \; dx}$.

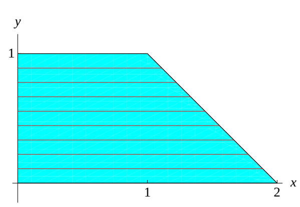
- **Slicing horizontally**:

  Slicing horizontally amounts to slicing the interval $[0, 1]$ on the y-axis, so the outer integral will be $\int_0^1$ something dy. The left side of each slice lies on $y = x^2$, and the right side of each slice also lies on $y = x^2$. Remember, though, that we are trying to describe how x varies in a slice (and we think of y as being constant), so x goes from the left half of $y = x^2$, where $x = -\sqrt{y}$, to the right half, where $x = \sqrt{y}$. Thus, the iterated integral is $\boxed{\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y) \; dx \; dy}$.

3. *Let $\mathcal{R}$ be the trapezoid with vertices $(0,0)$, $(2,0)$, $(1,1)$, and $(0,1)$. Write the double integral $\iint_{\mathcal{R}} f(x,y)\; dx\, dy$ as an iterated integral.*

Solution. Let's compare slicing vertically with slicing horizontally:





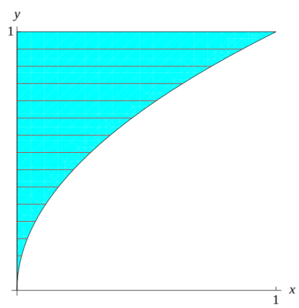
Notice that, if we slice vertically, there are two "types" of slices. The slices to the left of $x = 1$ go from $y = 0$ to $y = 1$, whereas the slices to the right go from $y = 0$ to the diagonal side of the trapezoid.

In contrast, if we slice horizontally, all of the slices have the same description: they go from $x = 0$ to the diagonal side. This seems simpler, so let's go with this method. When we slice horizontally, we are slicing the interval $[0, 1]$ on the y-axis, so our outer integral will be $\int_0^1$ something dy. Each slice goes from $x = 0$ to the diagonal side. The diagonal side is $y = 2 - x$ (we know it's a line containing the points $(2, 0)$ and $(1, 1)$). We want to describe how x varies in each slice, so x goes from 0 to $2 - y$.

So, the iterated integral is $\boxed{\int_0^1 \int_0^{2-y} f(x, y)\ dx\ dy}$.[2]

4. *Evaluate the double integral* $\iint_{\mathcal{R}} \sqrt{y^3 + 1}\, dx\, dy$ *where* $\mathcal{R}$ *is the region in the first quadrant bounded by* $x = 0$, $y = 1$, *and* $y = \sqrt{x}$. *(To decide the order of integration, first think about whether it's easier to integrate the integrand with respect to* x *or with respect to* y*.)*

   Solution. The integrand is much easier to integrate with respect to x than with respect to y. Therefore, we should try to rewrite the double integral as an iterated integral where the inner integral is with respect to x. This means our outer integral will be with respect to y, which corresponds in our strategy to slicing the region horizontally.



   This amounts to slicing the interval $[0, 1]$ on the y-axis, so the outer integral will be $\int_0^1$ something dy. Each slice has its left end on $x = 0$ and its right end on $y = \sqrt{x}$. We want to describe how x varies within a slice, so we rewrite $y = \sqrt{x}$ as $x = y^2$. This gives the iterated integral

$$\begin{aligned} \int_0^1 \int_0^{y^2} \sqrt{y^3 + 1}\ dx\ dy &= \int_0^1 \left(x\sqrt{y^3 + 1} \Big|_{x=0}^{x=y^2} \right)\ dy \\ &= \int_0^1 y^2 \sqrt{y^3 + 1}\ dy \end{aligned}$$

   We can evaluate this integral using substitution: if we let $u = y^3 + 1$, then $du = 3y^2\ dy$, and we can rewrite the integral as

$$\begin{aligned} \int_1^2 \frac{1}{3}\sqrt{u}\ du &= \frac{2}{9} u^{3/2} \Big|_{u=1}^{u=2} \\ &= \boxed{\frac{2}{9}\left(2^{3/2} - 1\right)} \end{aligned}$$

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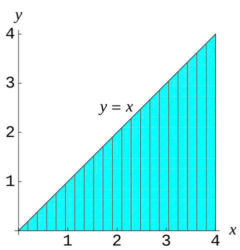
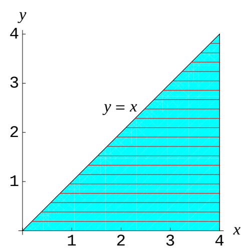
(2) If you used the other order of integration, you should have a sum of iterated integrals $\int_0^1 \int_0^1 f(x, y)\ dy\ dx + \int_1^2 \int_0^{2-x} f(x, y)\ dy\ dx$.

5. In each part, you are given an iterated integral. Sketch the region of integration, and then change the order of integration.

   (a) $\displaystyle\int_0^4 \int_0^x f(x,y)\; dy\; dx.$

   Solution. Let's just think of our strategy in reverse. The fact that the outer integral is $\displaystyle\int_0^4$ something dx tells us that we are slicing the interval $[0,4]$ on the x-axis, so we are making vertical slices from $x=0$ to $x=4$. The inner integral tells us that the bottom of each slice is on $y=0$, and the top of each slice is on $y=x$. So, the region of integration (with vertical slices) looks like the picture on the left:

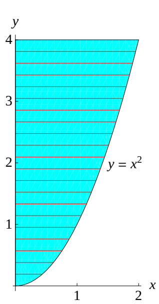


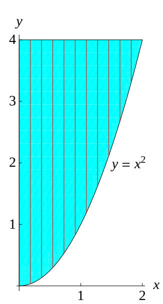


   To change the order of integration, we want to instead use horizontal slices (the picture on the right). Now, we are slicing the interval $[0,4]$ on the y-axis, so the outer integral is $\displaystyle\int_0^4$ something dy. Each slice has its left edge on $y=x$ (or $x=y$, since we really want to describe x in terms of y) and its right edge on $x=4$, so we can rewrite the iterated integral as $\boxed{\displaystyle\int_0^4 \int_y^4 f(x,y)\; dx\; dy}$.

   (b) $\displaystyle\int_0^4 \int_0^{\sqrt{y}} f(x,y)\; dx\; dy.$

   Solution. The fact that the outer integral is $\displaystyle\int_0^4$ something dy tells us that we are slicing the interval $[0,4]$ on the y-axis, so we are making horizontal slices from $y=0$ to $y=4$. The inner integral tells us that the left side of each slice is on $x=0$ and the right side is on $x=\sqrt{y}$ (or $y=x^2$). So, the region of integration looks like this:

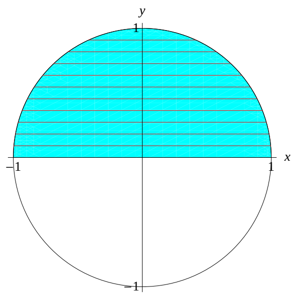


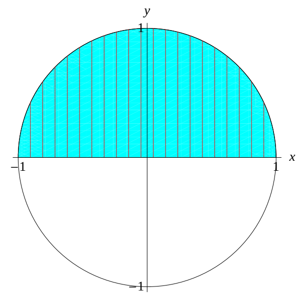


   To change the order of integration, we use vertical slices. Now, we are slicing the interval $[0,2]$ on the x-axis. The bottom of each slice is on $y=x^2$, and the top of each slice is on $y=4$, so we can rewrite the integral as $\boxed{\displaystyle\int_0^2 \int_{x^2}^4 f(x,y)\; dy\; dx}$.

(c) $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\ dx\ dy.$

Solution. The fact that the outer integral is $\displaystyle\int_0^1$ something dy tells us that we are slicing the interval $[0,1]$ on the y-axis, so we are making horizontal slices from $y=0$ to $y=1$. The inner integral tells us that the left side of each slice is on $x=-\sqrt{1-y^2}$ and the right side of each slice is on $x=\sqrt{1-y^2}$. $x=-\sqrt{1-y^2}$ describes the left half of the circle $x^2+y^2=1$, and $x=\sqrt{1-y^2}$ describes the right half, so the region of integration looks like this:

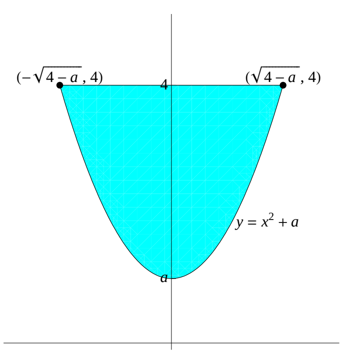




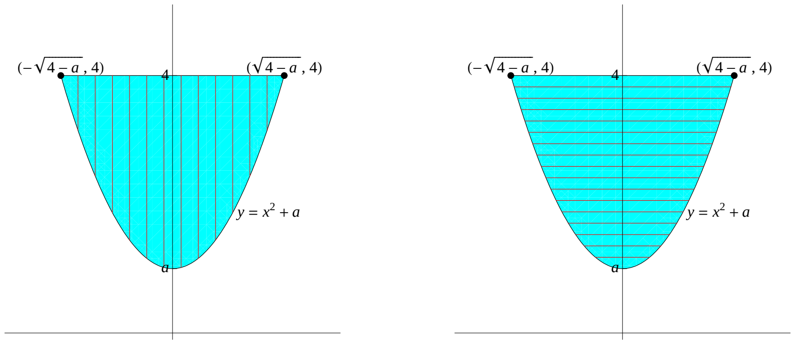
To change the order of integration, we use vertical slices. Now, we are slicing the interval $[-1,1]$ on the x-axis, so the outer integral is $\displaystyle\int_{-1}^1$ something dx. Each slice has its bottom edge on $y=0$ and its top edge on the top half of the circle $x^2+y^2=1$ (or $y=\sqrt{1-x^2}$), so we can rewrite the iterated integral as $\boxed{\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)\ dy\ dx}$.

6. *Let a be a constant between 0 and 4. Let $\mathcal{R}$ be the region bounded by $y=x^2+a$ and $y=4$. Write the double integral $\displaystyle\iint_{\mathcal{R}} f(x,y)\ dx\,dy$ as an iterated integral in both possible orders.*

Solution. The curves $y=x^2+a$ and $y=4$ intersect where $x^2=4-a$, so $x=\pm\sqrt{4-a}$. So, the region $\mathcal{R}$ looks like this:



To write the double integral as an iterated integral, we think of slicing either vertically or horizontally.



- **Slicing vertically**:

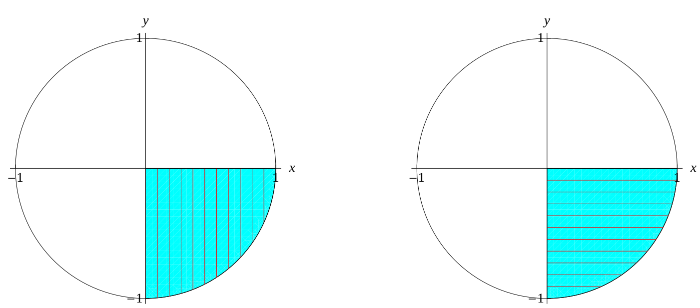
  Slicing vertically corresponds to slicing the interval $[-\sqrt{4-a}, \sqrt{4-a}]$ on the x-axis, so the outer integral will be $\int_{-\sqrt{4-a}}^{\sqrt{4-a}}$ something dx. Each slice has its bottom edge on $y = x^2 + a$ and its top edge on $y = 4$, so the iterated integral is $\boxed{\int_{-\sqrt{4-a}}^{\sqrt{4-a}} \int_{x^2+a}^{4} f(x, y)\ dy\ dx}$. Remember that a is a constant, so it's fine to have it in the outer integral.

- **Slicing horizontally**:

  Slicing horizontally corresponds to slicing the interval $[a, 4]$ on the y-axis, so the outer integral will be $\int_a^4$ something dy. Each slice has its left edge on $y = x^2 + a$ (so $x = -\sqrt{y-a}$) and its right edge on $y = x^2 + a$ (so $x = \sqrt{y-a}$). Thus, the iterated integral is $\boxed{\int_a^4 \int_{-\sqrt{y-a}}^{\sqrt{y-a}} f(x, y)\ dx\ dy}$.

7. *Evaluate the iterated integral* $\int_0^1 \int_{-\sqrt{1-x^2}}^{0} 2x \cos\left(y - \frac{y^3}{3}\right)\ dy\ dx.$

Solution. We don't know how to integrate the integrand with respect to y, but we can integrate it with respect to x. This suggests that we should change the order of integration, as in **??**. First, let's figure out what the region looks like. The fact that the outer integral is $\int_0^1$ something dx tells us that we are slicing the interval $[0, 1]$ on the x-axis, so we are making vertical slices from $x = 0$ to $x = 1$. The inner integral tells us that the bottom of each slice is on $y = -\sqrt{1-x^2}$ (the bottom half of the circle $x^2 + y^2 = 1$) and the top of each slice is on $y = 0$. So, the region of integration looks like this:



To change the order of integration, we switch to using horizontal slices. Now, we are slicing the interval $[-1,0]$ on the y-axis, so our outer integral will be $\int_{-1}^{0}$ something dy. Each slice has its left edge on $x=0$ and its right edge on the right half of the circle $x^2+y^2=1$ (so $x=\sqrt{1-y^2}$). Therefore, we can rewrite the given integral as

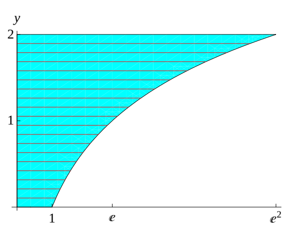
$$\begin{aligned}\int_{-1}^{0}\int_{0}^{\sqrt{1-y^2}} 2x\cos\left(y-\frac{y^3}{3}\right)\,dx\,dy &= \int_{-1}^{0}\left[x^2\cos\left(y-\frac{y^3}{3}\right)\Big|_0^{\sqrt{1-y^2}}\right]dy\\ &= \int_{-1}^{0}(1-y^2)\cos\left(y-\frac{y^3}{3}\right)dy\end{aligned}$$

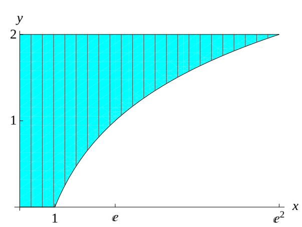
We can use substitution to evaluate this integral: let $u=y-\frac{y^3}{3}$; then, $du=(1-y^2)dy$, so the integral becomes

$$\begin{aligned}\int_{-2/3}^{0}\cos u\,du &= \sin u\Big|_{u=-2/3}^{u=0}\\ &= \boxed{-\sin\left(-\frac{2}{3}\right)}\end{aligned}$$

8. *A flat plate is in the shape of the region in the first quadrant bounded by $x=0$, $y=0$, $y=\ln x$ and $y=2$. If the density of the plate at point (x,y) is xe^y grams per cm^2, find the mass of the plate. (Suppose the x- and y-axes are marked in cm.)*

   Solution. As we learned in #2(b) of the worksheet "Double Integrals", we can find the mass of the plate by taking the double integral of the density, where the region of integration is the plate. In this case, the integrand xe^y is easy to integrate with respect to x and with respect to y, so we will pick an order of integration based on the shape of the region. We can either slice horizontally or vertically:





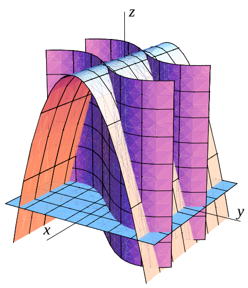
   As in **??**, this region is simpler to describe using horizontal slices: with vertical slices, there are two "types" of slices, but with horizontal slices, there is only one.

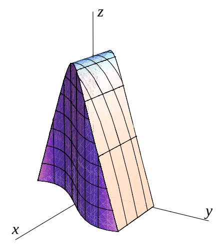
   If we use horizontal slices, we are slicing the interval $[0,2]$ on the y-axis. Each slice goes from $x=0$ to $y=\ln x$ (or $x=e^y$), so the iterated integral is

$$\begin{aligned}\int_0^2\int_0^{e^y} xe^y\,dx\,dy &= \int_0^2\left(\frac{1}{2}x^2e^y\Big|_{x=0}^{x=e^y}\right)dy\\ &= \int_0^2\frac{1}{2}e^{3y}\,dy\\ &= \frac{1}{6}e^{3y}\Big|_{y=0}^{y=2}\\ &= \boxed{\frac{1}{6}\left(e^6-1\right)}\end{aligned}$$

9. *Let $\mathcal{U}$ be the solid above $z = 0$, below $z = 4 - y^2$, and between the surfaces $x = \sin y - 1$ and $x = \sin y + 1$. Find the volume of $\mathcal{U}$.*

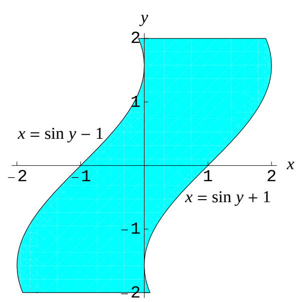
Solution. The picture on the left shows the four surfaces $z = 0$, $z = 4 - y^2$, $x = \sin y - 1$, and $x = \sin y + 1$. The picture on the right shows just the solid $\mathcal{U}$.



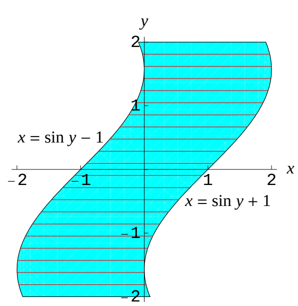


This solid can be described as the solid under $z = 4 - y^2$ over the region $\mathcal{R}$, where $\mathcal{R}$ is where the solid meets the xy-plane. So, its volume will just be $\iint_{\mathcal{R}} (4 - y^2)\; dx\, dy$.

To calculate this double integral, we need to describe $\mathcal{R}$ and convert the double integral to an iterated integral. The surface $z = 4 - y^2$ intersects the xy-plane $z = 0$ where $4 - y^2 = 0$, or $y = \pm 2$, so $y = 2$ and $y = -2$ are 2 boundx dyries of the region $\mathcal{R}$. The other two are $x = \sin y - 1$ and $x = \sin y + 1$. So, $\mathcal{R}$ looks like this:



It's easier to slice this region horizontally:



This amounts to slicing the interval $[-2, 2]$ on the y-axis, so the outer integral will be $\int_{-2}^{2}$ something dy.

The left side of each slice is on $x = \sin y - 1$, and the right side is on $x = \sin y + 1$, so we can rewrite the double integral as an iterated integral

$$\begin{aligned}
\int_{-2}^{2} \int_{\sin y - 1}^{\sin y + 1} (4 - y^2)\ dx\ dy &= \int_{-2}^{2} \left[x(4 - y^2) \Big|_{x=\sin y - 1}^{x=\sin y + 1} \right] dy \\
&= \int_{-2}^{2} 2(4 - y^2) dy \\
&= 8y - \frac{2y^3}{3} \Big|_{y=-2}^{y=2} \\
&= \boxed{\frac{64}{3}}
\end{aligned}$$