MATHEMATICS

Computing a limit of a sum mixed with product.

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The bounty expires in 4 hours. Answers to this question are eligible for a +50 reputation bounty. Guy Fsone is looking for an **answer from a reputable source**.

Given $d \geq 2$ integer and $m_0 > 0$ define

$$m_k = m_0 \left(\frac{d}{d-1}\right)^k \quad \text{and} \quad \sigma_n = \frac{1}{m_n + d}$$

I would like to compute

$$\lim_{n\to\infty} \prod_{j=1}^{n} (1-\sigma_j)$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n-2} \sigma_k \prod_{j=k+1}^{n-1} (1-\sigma_j)$$

I expect the following results $\frac{m_0}{m_0+d}$ for the first and $\frac{1}{m_0+d}$ for the second.

real-analysis sequences-and-series algebra-precalculus limits infinite-product

Share Cite Edit Follow Flag edited Sep 17 at 8:01 asked Sep 13 at 12:16

Guy Fsone
21.4k 4 44 89

1 Were perhaps d and g confused in the question? We are given g but never use it, and we are not given d but use it. – Jukka Kohonen Sep 16 at 0:14

sorry typo, it is d and not g – Guy Fsone Sep 16 at 9:46

d must be greater than 1, otherwise, zero appears in the denominator. – Danny Pak-Keung Chan Sep 16 at 20:14

2 Answers

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1

Let $a=\frac{d}{d-1}$ and we need to compute

$$P_n=\prod_{k=1}^n \left(1-\frac{1}{m_0\, a^k+d}\right)$$

Using Pochhamer symbols

$$P_n=\frac{d+m_0}{d+m_0-1} \left(\frac{d-1}{d}\right)^{n+1} \frac{\left(-\frac{m_0}{d-1};a\right)_{n+1}}{\left(-\frac{m_0}{d};a\right)_{n+1}}$$

Replace a by its value and simplify to obtain

$$P_n=\frac{1}{m_0+d-1}\left(\frac{d-1}{d}\right)^n\left(m_0\left(\frac{d}{d-1}\right)^n+d-1\right)=\frac{m_0+(d-1)\left(1-\frac{1}{d}\right)^n}{m_0+d-1}$$

Therefore

$$P_\infty=\frac{m_0}{m_0+d-1}$$

Checking for $m=10$ and $d=3$, the P_n make the sequence

$$\left\{\frac{17}{18},\frac{49}{54},\frac{143}{162},\frac{421}{486},\frac{1247}{1458},\frac{3709}{4374},\frac{11063}{13122},\frac{33061}{39366},\frac{98927}{118098},\frac{296269}{354294},\frac{887783}{1062882},\frac{2661301}{3188646}\right.$$

The last value

$$P_{12}=\frac{2661301}{3188646}=\frac{5}{6}+\frac{2048}{1594323}=\frac{5}{6}+0.00129$$

$$P_{24}=\frac{1412164459621}{1694577218886}=\frac{5}{6}+\frac{8388608}{847288609443}=\frac{5}{6}+0.00001$$

I cannot do the second one.

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answered Sep 17 at 11:45

Claude Leibovici

198k 51 84 184

1

We have, $\frac{m_k+d-1}{m_{k+1}+d}=\frac{d-1}{d}, \forall k$

$$\lim_{n\to\infty}\prod_{j=1}^n(1-\sigma_j)=\lim_{n\to\infty}\prod_{j=1}^n\frac{m_j+d-1}{m_j+d}=\lim_{n\to\infty}\left(\frac{1}{m_1+d}\right)\cdot\prod_{j=1}^{n-1}\frac{m_j+d-1}{m_{j+1}+d}.(m_n+d-1)$$

$$= \left(\frac{1}{m_1+d}\right)\cdot \lim_{n\to\infty}\left(\frac{d-1}{d}\right)^{n-1}.\left(m_0.\left(\frac{d}{d-1}\right)^n+d-1\right) = \frac{m_0\cdot\dfrac{d}{d-1}}{m_0\cdot\dfrac{d}{d-1}+d}$$

$$= \frac{m_0}{m_0+d-1}$$

Also, we have,

$$t_k = \sigma_k \prod_{j=k+1}^{n-1}(1-\sigma_j) = \sigma_k \prod_{j=k+1}^{n-1}\frac{m_j+d-1}{m_j+d}$$

$$= \sigma_k \frac{1}{m_{k+1}+d}\prod_{j=k+1}^{n-2}\frac{m_j+d-1}{m_{j+1}+d}.(m_{n-1}+d-1)$$

$$= \sigma_k\sigma_{k+1}\left(\frac{d-1}{d}\right)^{n-k-2}.\left(m_0.\left(\frac{d}{d-1}\right)^{n-1}+d-1\right)$$

$$= \sigma_k\sigma_{k+1}\left(m_0.\left(\frac{d}{d-1}\right)^{k+1}+\frac{1}{d}\left(\frac{d-1}{d}\right)^{n-k-1}\right)$$

$$= \sigma_k\sigma_{k+1}\left(m_{k+1}+\frac{1}{d}\left(\frac{d-1}{d}\right)^{n-k-1}\right)$$

$$\Rightarrow \lim_{n\to\infty} t_k = \sigma_k\sigma_{k+1}m_{k+1}$$

Hence,

$$\lim_{n\to\infty}\sum_{k=1}^{n-2}\sigma_k\prod_{j=k+1}^{n-1}(1-\sigma_j) = \lim_{n\to\infty}\sum_{k=1}^{n-2}\lim_{n\to\infty}t_k = \lim_{n\to\infty}\sum_{k=1}^{n-2}\sigma_k\sigma_{k+1}m_{k+1}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n-2}\frac{m_k}{(m_k+d)(m_k+d-1)} \qquad \text{since } m_{k+1} = \left(\frac{d}{d-1}\right)m_k, \forall k$$

Now it boils down to computing sum of this series, where $T_k = \dfrac{m_k}{(m_k+d)(m_k+d-1)}$. Also, notice that $T_1 > T_2 > \ldots$, i.e., $\{T_k\}$ is monotonically decreasing and $\lim_{n\to\infty} T_n = \lim_{n\to\infty}\dfrac{1/m_n}{(1+d/m_n)(1+(d-1)/m_n)} = 0$.

Another way, as pointed out by @Wiley,

since we have

$$\frac{1}{\sigma_{k+1}} - \frac{1}{\sigma_k} = (m_{k+1}-d)-(m_k+d) = m_{k+1}-m_k = \left(1-\frac{d-1}{d}\right)m_{k+1} = \frac{m_{k+1}}{d}$$

$$\implies \frac{\sigma_k-\sigma_{k+1}}{\sigma_k\sigma_{k+1}} = \frac{m_{k+1}}{d}$$

$$\implies \sigma_k\sigma_{k+1}m_{k+1} = d(\sigma_k-\sigma_{k+1})$$

$$\implies \sum_{k=1}^{n-2} \sigma_k \sigma_{k+1} m_{k+1} = d \sum_{k=1}^{n-2} (\sigma_k - \sigma_{k+1}) = d(\sigma_1 - \sigma_{n-1})$$

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n-2} \sigma_k \prod_{j=k+1}^{n-1} (1-\sigma_j) = \lim_{n\to\infty} \sum_{k=1}^{n-2} \lim_{n\to\infty} t_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n-2} \sigma_k \sigma_{k+1} m_{k+1}$$

$$= d \lim_{n\to\infty} (\sigma_1 - \sigma_{n-1}) = d\sigma_1 \text{, since } \lim_{n\to\infty} \sigma_{n-1} = 0$$

$$= \frac{d}{m_1 + d} = \frac{d-1}{m_0 + d - 1}$$

Share Cite Edit Delete Flag edited 7 hours ago answered Sep 17 at 20:14

Sandipan Dey
1,201 7 9

3 We have

$$\sum_{k=1}^{n} a_k \prod_{j=k+1}^{n} (1-a_j) = 1 - \prod_{j=1}^{n}(1-a_j),$$

it's the inclusion-exclusion formula in disguise. – Maxim Sep 18 at 3:36

@Maxim How do you get this identity? – Guy Fsone 2 days ago

2 To finish it off, instead of taking the limit for t_k, one should keep the m_{n-1} term and note $\sigma_k - \sigma_{k+1} = \sigma_k \sigma_{k+1} \frac{m_0 q^k}{d-1}$, where $q = \frac{d}{d-1}$. Therefore $\sum_{k=1}^{n-2} t_k = \frac{(m_{n-1}+d-1)(d-1)}{m_0 q^{n-2}} \sum_{k=1}^{n-2} (\sigma_k - \sigma_{k+1}) = \frac{(m_0 q^{n-1}+d-1)(d-1)}{m_0 q^{n-2}} (\sigma_1 - \sigma_{n-1})$. Taking the limit $= \frac{(m_0 q^{n-1}+d-1)(d-1) m_0 q (q^{n-2}-1)}{m_0 q^{n-2} (m_0 q + d)(m_0 q^{n-1}+d)}$ when $n \to \infty$ gives $\frac{(d-1)q}{m_0 q + d} = \frac{d-1}{m_0 + d - 1}$ – Wiley yesterday

Thanks for the hint @Wiley, I was wondering whether we can compute the sum of $\{T_k\}$ too, since numerical simulation shows that the sum converges. – Sandipan Dey 7 hours ago

I doubt the second result is correct because it is a sum of positive numbers. – Guy Fsone 2 hours ago