10. (Optional) D'Alembert on half-infinite intervals

Example 10.1 Consider an infinitely long string that is fixed at one end.

Model this by the wave equation on the half-line $x > 0$,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x > 0,\ t > 0,$$

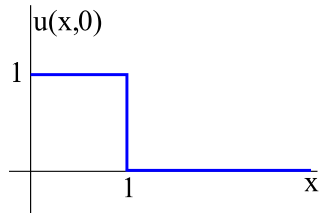
where the fixed end is at $x = 0$.

The boundary condition at $x = 0$ is $u(0,t) = 0$.

Suppose the initial conditions are

$$u(x,0) = s(x-1) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0,$$

a step function pulse centered at $x = 1$ with zero initial velocity, defined for $x > 0$.



Initial position defined for $x > 0$

If we let $f(x) = s(x-1)$ for $x > 0$, then the **incoming wave** (left traveling wave) $f(x+ct)$ obeys the PDE and the initial condtion ($t = 0$), but does not obey the boundary condition ($x = 0$).

Determine the proper **reflected wave** (right traveling wave) $g(x-ct)$ that needs to be added to the incoming wave in order to satisfy the Dirichlet boundary condition.

Draw the corresponding wavefronts in the (x, t) plane.

Worked solution

Let $u(x,t) = f(x+ct) + g(x-ct)$. At $x = 0$, the boundary condition gives $0 = f(ct) + g(-ct)$, hence we need to take $g(z) = -f(-z)$. Therefore $g(x) = -s(-x-1)$ for $x < 0$.

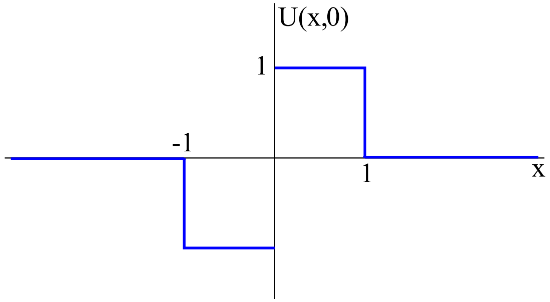
Thus in order to solve this boundary value problem, we must extend the initial condition $U(x,0)$ to be defined for all x. The first step is to extend f and g to functions F and G defined for all x.

Let $F(x) = s(x-1)\,s(-x)$. This function is defined for all x and is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases},$$

which is the behavior we want. Similarly, the $G(x) = -s(-x-1)\,s(x)$ has the correct properties so that

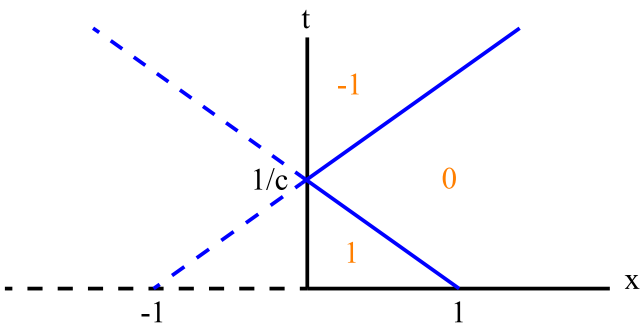
$U(x,0) = F(x) + G(x) = s(x-1)\,s(-x) - s(-x-1)\,s(x)$ is defined for all x, and is an odd extension of $s(x-1)$ defined for $x > 0$.



Odd extension of initial condition defined for all $-\infty < x < \infty$

The space-time diagram includes

- a first line with slope $-1/c$ passing through $(1,0)$ (the incident wave)
- a second line with slope $1/c$ passing through $(-1,0)$ (the reflected wave)



The two lines meet at $(0, 1/c)$, where the 2 waves cancel out exactly. The physical wave first travels as a left-going step function along the first line toward the origin, reflects because of the boundary condition, and then re-emerges as a right-going step function with amplitude -1

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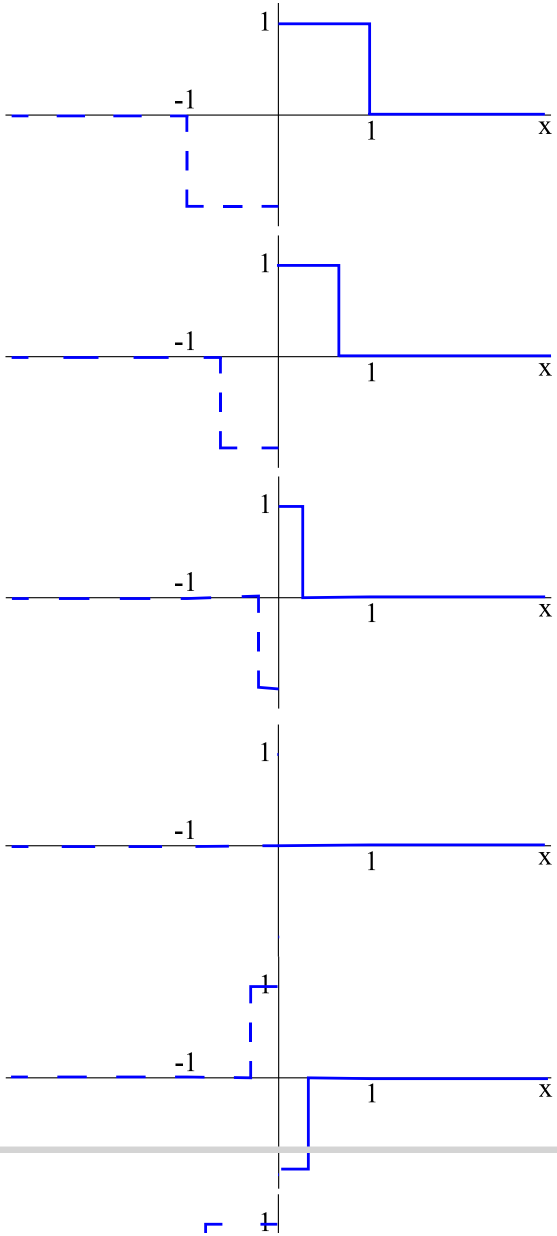
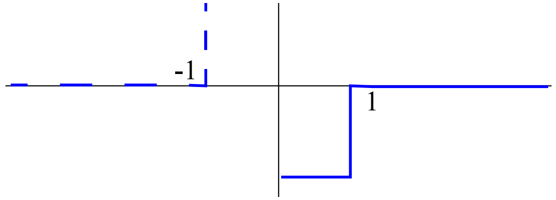
To help understand the process, the image below shows a series of photographs of the wave in the string for $x > 0$ with the extended odd wave shown as a dotted line for a sequence of times starting from the initial position.





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In the next problem, we explore how to handle the case of a half-line $x > 0$ with the end at $x = 0$ free. To help you anticipate the mathematics, it may be helpful to watch the following demo video again and observe what is happening at the end points in the two boundary conditions shown: free and fixed.

This demo video was created by TSG@MIT Physics.

Boundary conditions and effect on reflected wave

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x > 0,\ t > 0,$$

where the end of the string is free at $x = 0$.

The boundary condition at $x = 0$ is $\dfrac{\partial u}{\partial x}(0, t) = 0$.

Suppose the initial conditions are

$$u(x, 0) = s(x - 1) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

a step function pulse centered at $x = 1$ with zero initial velocity.

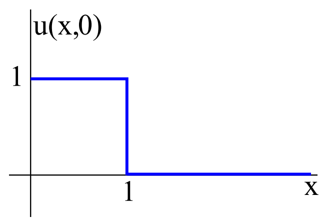


Figure 28: Initial position defined for $x > 0$

If we let $f(x) = s(x - 1)$ for $x > 0$, then the **incoming wave** (left traveling wave) $f(x + ct)$ obeys the PDE and the initial condtion ($t = 0$), but does not obey the boundary condition ($x = 0$).

Determine the proper **reflected wave** (right traveling wave) $g(x - ct)$ that needs to be added to the incoming wave in order to satisfy the Dirichlet boundary condition.

Draw the corresponding wavefronts in the (x, t) plane.

See worked solution

Let $u(x,t) = f(x+ct) + g(x-ct)$. Then $\frac{du}{dx}(x,t) = f'(x+ct) + g'(x-ct)$. At $x=0$, we get $0 = f'(ct) + g'(-ct)$. Thus $g'(x) = -f'(-x)$, which can be integrated to get $g(x) = f(-x)$. Thus $g(x) = s(-x-1)$. In order to combine f and g to a function defined for all x, we must extend the functions to both be defined for all real numbers x. The easiest way to do this is to define $f(x)$ to be 0 for $x<0$, and define $g(x)$ to be 0 for $x>0$. We can do this by the formulas

$$\begin{aligned} F(x) &= f(x)\,s(-x) \\ G(x) &= g(x)\,s(x) \end{aligned}$$

By summing these two functions, we get an initial condition to be an even function defined everywhere $U(x,0) = s(x-1)\,s(-x) + s(-x-1)\,s(x)$.

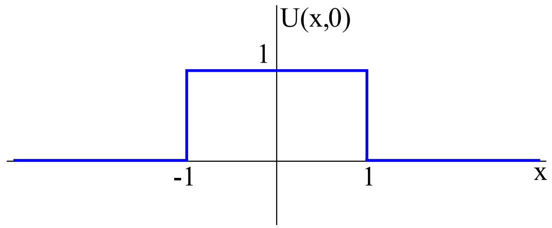
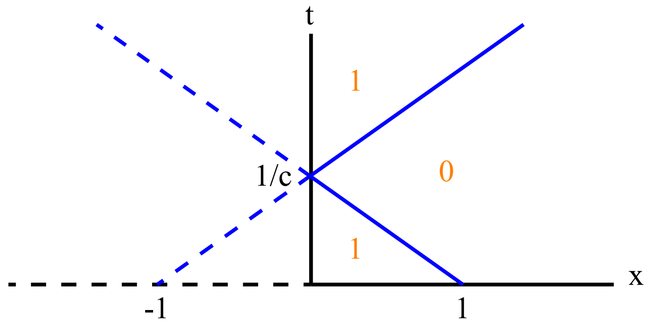


Figure 29: Even extension of initial condition defined for all $-\infty < x < \infty$

This tells us that the reflected wave bounced back at the free end with the same direction.

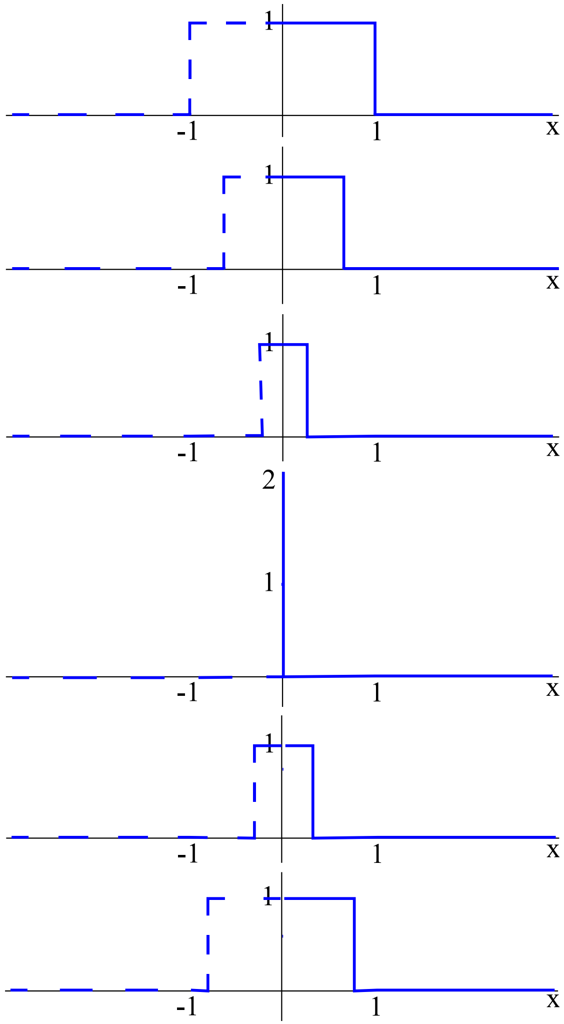
The space-time diagram includes

- a first line with slope $-1/c$ passing through $(1,0)$ (the incident wave)
- a second line with slope $1/c$ passing through $(-1,0)$ (the reflected wave)



The two lines meet at $(0, 1/c)$, where the 2 waves superpose and add to briefly have amplitude 2. The physical wave first travels as a left-going step function along the first line toward the origin, reflects because of the boundary condition, and then re-emerges as a right-going step function with amplitude 1.

To help understand the process, the image below shows a series of photographs of the wave in the string for $x > 0$ with the extended odd wave shown as a dotted line for a sequence of times starting from the initial position.



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Extending d'Alembert's solution to finite intervals

It turns out that you can use the d'Alembert idea to solve the wave equation on finite intervals $[0, L]$ as well. The main idea is to take the initial conditions for the initial position and velocity, which are only defined for $0 < x < L$ and extend these to either even or odd $\mathbf{2L}$**-periodic** functions defined for all $-\infty < x < \infty$ that are consistent with the specified boundary conditions. You will recover exactly the functions we found using Fourier's method (but expressed in a slightly different form).

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10. (Optional) D'Alembert on half-infinite intervals

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Topic: Unit 2: Boundary value problems and PDEs / 10. (Optional) D'Alembert on half-infinite intervals

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Example 10.2 Soluition 2

Representation of superposition? 1
The representations of the superpositions of the square pulses seem incorrect. For the first representation (when the one pulse is oriented in the positive direction and the ot...

Bell Labs Wave Machine: Reflection

(Caption will be displayed when you start playing the video.)

0:00 / 1:00 2.0x

Example 10.2

(Try working this problem out on your own before looking at the solution, and compare to the previous problem solution.)

Consider an infinitely long string that is free at one end and modeled this by the wave equation on the half-line $x > 0$,