MATHEMATICS

Mean square law of large numbers

Asked 3 years, 9 months ago Active 3 years, 9 months ago Viewed 514 times

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I have the following line in my notes (which I believe is flawed):

$$E\left[(\hat{\mu}(N)-\mu)^2\right]=E\left[\left(\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)\right)^2\right]=\frac{\sigma^2}{N}$$

I think the error is in brackets, $\frac{1}{N}$ us not multiplied by $\sum_{i=1}^{N}(x_i-\mu)$, but only: $\sum_{i=1}^{N}x_i$, since $\hat{\mu}$ is defined as the sample mean: $\frac{1}{N}\sum_{i=1}^{N}x_i$.

But even given that (and supposing that I am correct), I cannot arrive at the required result of σ^2/N.

This is what I get:

$$E\left[\left(\frac{1}{N}\sum_{i=1}^{N}x_i-\mu\right)^2\right]=\frac{1}{N^2}E\left[\sum_{i=1}^{N}x_i^2\right]-\frac{2\mu}{N}E\left[\sum_{i=1}^{N}x_i\right]+\mu^2$$

EDIT:

If I have i.i.d RV's with finite mean, then can I develop the above by saying the following:

$E[X]=< X >=\frac{1}{N}\sum_{i=1}^{N}x_i$ and $E[X^2]=< X^2 >=\frac{1}{N}\sum_{i=1}^{N}x_i^2$, therefore I have

$$E\left[\left(\frac{1}{N}\sum_{i=1}^{N}x_i-\mu\right)^2\right]=\frac{1}{N^2}E\left[\sum_{i=1}^{N}x_i^2\right]-\frac{2\mu}{N}E\left[\sum_{i=1}^{N}x_i\right]+\mu^2=\frac{1}{N}< X^2 > - < X >^2$$
$$=\frac{1}{N}(< X^2 > - < X >^2)=\frac{\sigma^2}{N}$$

probability

edited Jan 13 '16 at 15:16

asked Jan 13 '16 at 14:40

i squared - Keep it Real
1,681 1 13 35

Your edit uses some notation that is new to me. May I ask you what $<\cdot>$ means in your example? – N. Wouda Jan 13 '16 at 15:20

It is the physics notation for mean – i squared - Keep it Real Jan 13 '16 at 15:21

1 Yes, got it. I was a bit confused, since you seem use μ and $< X >$ interchangeably. I should note that $E[X] = \frac{1}{N}\sum_{i=1}^{N} x_i$ is an asymptotic result (that is, $E[X] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i$). – N. Wouda Jan 13 '16 at 15:31

2 Answers

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The trick is a clever grouping of terms:

$$\begin{aligned} E\left[\left(\frac{1}{N}\sum_{i=1}^{N}(x_i) - \mu\right)^2\right] &= E\left[\left(\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)\right)^2\right] \\ &= \frac{1}{N^2}E\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \sum_{i=1}^{N}\sum_{j=1,\ j\neq i}^{N}(x_i-\mu)(x_j-\mu)\right] \\ &= \frac{\sigma^2}{N} + \frac{1}{N^2}E\left[\sum_{i=1}^{N}\sum_{j=1,\ j\neq i}^{N}(x_i-\mu)(x_j-\mu)\right] \end{aligned}$$

Now, why is the last term zero?

What we have really shown here is two basic facts:

- The variance of a sum of uncorrelated variables is the sum of the variances.
- The variance of αX is $\alpha^2 \mathrm{Var}(X)$, if $\alpha \in \mathbb{R}$.

This technique can also be used to prove that the expected value of $\sum_{i=1}^{N}(x_i - \hat{\mu}(N))^2$ is $(N-1)\sigma^2$ (which explains the somewhat mysterious $N-1$ in the standard formula for the sample variance).

edited Jan 13 '16 at 15:14
JnxF
1,101 1 8 22

answered Jan 13 '16 at 14:50
Ian
72k 2 57 105

can you quickly check my edit please – i squared - Keep it Real Jan 13 '16 at 15:17

@isquared-KeepitReal There are some notational problems: for instance you've replaced expected value with sample mean, when they are different. What you really mean to say is that $E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i] = NE[X]$ and similarly $E\left[\sum_{i=1}^{N} X_i^2\right] = \sum_{i=1}^{N} E[X_i^2] = NE[X^2]$. This same point was made in a comment on the OP. – Ian Jan 13 '16 at 16:27

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The secret to the middle expression is simple: if you add together N copies of the exact same quantity, you get N times the original quantity.

$$\sum_{i=1}^{N} \mu = N\mu.$$

So if we take $\frac{1}{N}$ of the sum, we get back the original quantity:

$$\frac{1}{N}\sum_{i=1}^{N} \mu = \frac{1}{N}(N\mu) = \mu.$$

Now combine this with the already-known formula for $\hat{\mu}(N)$:

$$\begin{aligned}
\hat{\mu}(N) &= \frac{1}{N}\sum_{i=1}^{N} x_i, \\
\hat{\mu}(N) - \mu &= \frac{1}{N}\sum_{i=1}^{N} x_i - \mu \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i - \frac{1}{N}\sum_{i=1}^{N} \mu \\
&= \frac{1}{N}\left(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \mu\right).
\end{aligned}$$

Now apply the well-known fact that $\sum_{i=1}^{N} a_n - \sum_{i=1}^{N} b_n = \sum_{i=1}^{N}(a_n - b_n)$. That is, instead of adding up all the x_is and then subtracting off the sum of all the μs in that order, pair off each x_i with one of the μs that we are going to subtract:

$$\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \mu = \sum_{i=1}^{N}(x_i - \mu).$$

Therefore

$$\hat{\mu}(N) - \mu = \frac{1}{N}\left(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \mu\right) = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu).$$

The first equals sign in your question is simply taking the expectation of the square of the quantity on both sides of an equation.

This is a somewhat long-winded way of showing how the first equals sign in Ian's answer works. Follow that answer for the rest of the derivation.

edited Jan 13 '16 at 15:29

answered Jan 13 '16 at 15:21

David K
60.8k 4 48 137