MATHEMATICS

You have a biased coin

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3

You have a biased coin, where the probability of flipping a head is 0.70. You flip once, and the coin comes up tails. What is the expected number of flips from that point (so counting that as flip #0) until the number of heads flipped in total equals the number of tails?

My approach is wrong but I don't understand why!!!

$x = 0.7(1) + 0.3(1 + x) \approx 1.43.$

with 0.5 we get a head and we are done in 1 flip. with 0.5 we get an additional tail and so we will have made 1 flip but then we will need to repeat X.

Could you please explain clearly Markov chains. I am not very good at it.

probability markov-chains

Share Cite Edit Follow Flag edited 15 hours ago asked 17 hours ago Marco 133 7 New contributor

4 $\frac{1}{2}$ doesn't come into the picture with this unfair coin. – aschepler 17 hours ago

If the coin were a fair coin, then you would have a 50/50 shot at getting heads on the first toss. This coin is not fair. As a Markov chain, it seems that the states represent the possible discrepancy between heads and tails. You start with 1 extra tails. Each time you flip the coin, a heads moves you to a state with one fewer extra tails, and a tails moves you to a state with one more extra tails. What are the transition probabilities? How many possible states are there? How can you use this to compute the expected number of tosses until you get to the state with no excess tails? – Xander Henderson ♦ 16 hours ago

I'm guessing you can't just do: $1 + 0.3x = 0.7x \implies x = 2.5$. – Adam Rubinson 16 hours ago

Hi guys. Yes, what I meant was actually $x = 0.7(1) + 0.3(1 + x) \approx 1.43$. But from what I know, this is not correct. The correct answer should be the one of @AdamRubinson but I really do not understand why. Could you please explain it in a different way, maybe easier for someday who is not very good with markov chains? – Marco 15 hours ago

@Adam Rubinson: What's wrong with it? After n tosses, the expected number of heads is $0.7n$ and that of tails is $0.3n$. As tails is given a head start of 1, n is determined by $0.7n = 1 + 0.3n$, thus $n = 2.5$. – emacs drives me nuts 13 hours ago

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4 Answers

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The expected number of flip is where the expected number head $(0.7x)$ and the expected number of tail $(1 + 0.3x)$ are equal so that's why it's 2.5.

You're wrong because you forgot to consider the fact that when the tail is flipped, you need to repeat the process twice for the head to catch up, not once. The correct formula is $x = 0.7(1) + 0.3(1 + 2x)$ which give the correct answer.

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edited 13 hours ago
Mohsen Shahriari
4,983 9 24 61

answered 13 hours ago
duck
16

New contributor

1 The result is correct, but I would like to comment that there is a slight difference between "at what time is the expected number of tails and heads equal" and "what is the expected time at which the numbers of heads and tails are equal". The process in this question is simple enough that these two are the same, but there is something slightly non-trivial to prove (and for more complicated processes it actually fails) – Simon 13 hours ago

@duck. Very nice explanation. Just to make sure I understood. Let's say that I am 2 tails down and the game ends when Heads = (Tails + 1). would the equation be $\rightarrow 1 + 07x = 0.3x + 2$? – Marco 13 hours ago

1

I have used an inelegant method for confirmation

- Number of steps taken has to be of the form $(2k + 1)$, i.e. *odd*
- k can range from 0 to ∞
- probabilities will be of the form $p^{k+1}q^k$ with $p = 0.7$

$$\text{Thus the formula} = \sum_{k=0}^{\infty} \binom{2k+1}{k+1} p^{k+1} q^k$$

which confirms the answer $= 2.5$

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edited 12 hours ago

answered 12 hours ago
true blue anil
25.5k 2 22 43

I think you're forgetting the binomial coefficient $\binom{2k+1}{k+1}$ in your formula for the probability. – Matthew Pilling 12 hours ago

@MatthewPilling: Thanks, omission rectified :-} – true blue anil 12 hours ago

This is an interesting problem and all the above solutions I think can be obtained using the law of the total expectation, with an assumption that the expected number of flips gets doubled when we have an additional tail to start with.

Let X be the r.v. denoting the number of flips required. Also let's define a r.v Y denoting the first flip after the initial tail.

Now, we have,

$E[X|Y=H]=1$

$E[X|Y=T]=1+2E[X]$

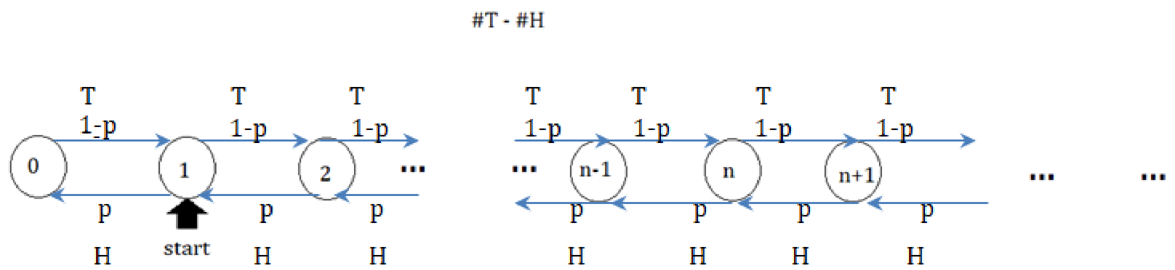
with an assumption that we need to double the number of required flips if we get an additional T (but is the assumption correct?)

Now, by law of total expectation, we have,

$E[X]=E[E[X|Y]]=P(Y=H)E[X|H]+P(Y=T)E[X|T]=p.1+(1-p)(1+2E[X])$

$\implies E[X]=\frac{1}{2p-1}=2.5$, when we have $p=P(Y=H)=0.7$.

Now, let's consider the following **Markov chain** (which looks like the Gambler's ruin or the birth-death process, but it's not finite, can be extended indefinitely to the right and there is no absorbing state).



- A state in the chain is represented as the difference in number of tails and heads (#T - #H) in the coin-tossing experiment.
- We start at state 1 (one single tail) and want to compute the expected time to reach state 0 (equal heads and tails).
- When a head shows up in the coin flip we move to the immediate left state with transition probability p (except for state 0) and if a tail shows up we move to the immediate right state with transition probability $1-p$.
- Note that the chain is not finite, there is no last state to the right.
- In this scenario, the problem gets reduced to computing the **hitting time** from state 1 to state 0.
- In general hitting time x_n from state n to state 0 can be represented as the following recurrence relation:

  $x_n=1+px_{n-1}+(1-p)x_{n+1}$, with $x_0=0$.
- We are interested to compute x_1.

- The above is a non-homogeneous recurrence relation $(1-p)x_{n+1} - x_n + px_{n-1} = -1$ and let's solve the homogeneous version of it first, which has characteristic equation as

  $(1-p)\lambda^2 - \lambda + p = 0$, with roots $1, \frac{p}{1-p}$, so that the homogeneous solution is:

  $x_h = A\left(\frac{p}{1-p}\right)^n + B.1^n$
- Now, let's guess a particular solution $x_p = Cn$, s.t., plugging it into the non-homogeneous equation with some algebra, we get $C = \frac{1}{1-2p}$ and $x_p = \frac{n}{2p-1}$, s.t., the solution is

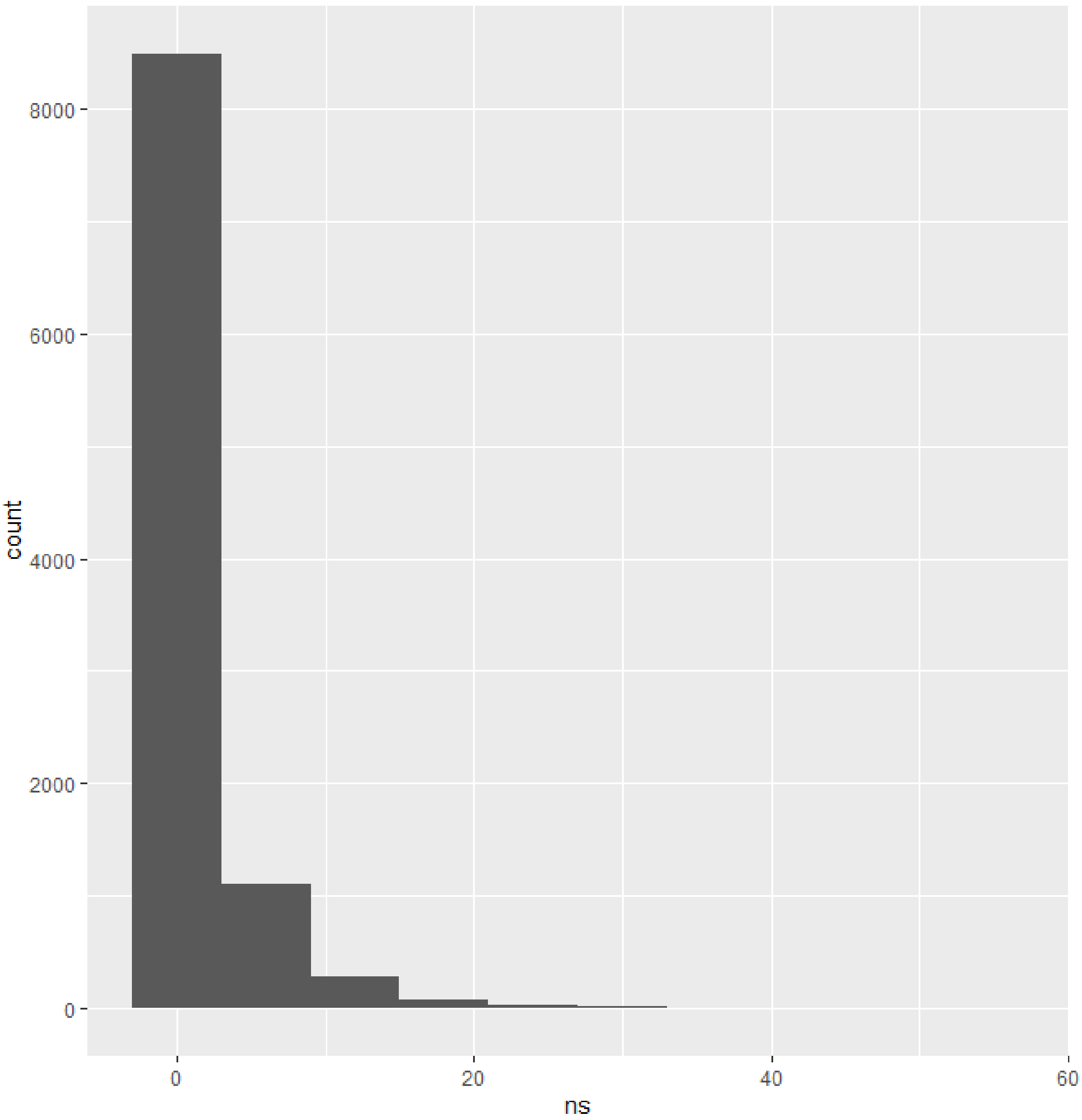
  $x_n = x_h + x_p = A\left(\frac{p}{1-p}\right)^n + B.1^n + \frac{n}{2p-1}$.
- Note that the particular solution $x_p = \frac{5}{2}$, for $n = 1$.
- With boundary condition $x_0 = 0$, we get the final solution as:

  $x_n = x_h + x_p = A\left(\left(\frac{p}{1-p}\right)^n - 1\right) + \frac{n}{2p-1} = A\left(\left(\frac{7}{3}\right)^n - 1\right) + \frac{5n}{2}$, for $p = 0.7$
- $x_1 = A\frac{2p-1}{1-p} + \frac{1}{2p-1}$.
- With $p = 0.7$, we have $x_1 = \frac{4A}{3} + \frac{5}{2}$, which depends on the arbitrary constant A.
- Letting $A = 0$, we get back our earlier solution $x_1 = \frac{5}{2} = 2.5$

The following simulation in `R` supports the above fact:

```
seed(123)
N <- 10000
ns <- replicate(N,
          {
            n <- 0
            nT <- 1
            nH <- 0
            while (nT != nH) {
              toss <- sample(0:1, 1, prob = c(0.3, 0.7))
              if (toss == 1) {
                nH <- nH + 1
              } else {
                nT <- nT + 1
              }
              n <- n + 1
            }
            n
          })
mean(ns)
#[1] 2.4696
```

with the following histogram of the hitting time.



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edited 17 mins ago

answered 9 hours ago

Sandipan Dey

1,174 6 9

1 I think you have the role of heads and tails in the problem mixed up; here you start out with a bias in favor of tails and the coin itself is biased in favor of heads, so the process tends to terminate. – Ian 7 hours ago

@Ian 16 I see your point, but using the Markov chain hitting time we should be able to obtain a similar solution, that simulation results should also support. Can you see anything I am missing there? My point is that there is always a chance that the random walk will go towards the right of the chain, which is kind of unbounded, absence of an absorbing state and also lack of any other initial condition will add to it, is not it the case? – Sandipan Dey 7 hours ago

1 I think your simulation is incorrect because sample() is not taking the bias into account. And no, the expected hitting time is still finite because there is a net drift to the left. The heavy-handed theorem

to back this claim up is called the Foster-Lyapunov theorem, which basically says that if you have a "Lyapunov function" (which must have certain properties) which decreases on average in each step then the chain is positive recurrent. – Ian 7 hours ago

@Ian thank you for mentioning the theorem, I am not aware of it and I shall check, still I shall be interested in having a solution with Markov hitting time, is it at all possible? Is so, any idea on what additional initial condition can be used? Thanks in advance. – Sandipan Dey 7 hours ago

@Ian regarding simulation you are right, I am using equally likely head-tail outcomes mistakenly, that I can change – Sandipan Dey 7 hours ago

0

The accepted answer gets the correct result largely by coincidence. There is not a priori any relationship between the time when the expected number of heads and tails are equal vs. the expected time when the number of heads and tails are equal, and the latter is what is asked for.

To work this out, you can consider a Markov chain for the cumulative number of tails minus the cumulative number of heads. Then this jumps by $+1$ with probability 0.3, jumps by -1 with probability 0.7, and it starts at 1. You want the expected time to hit 0. One way to do this is to consider starting at any $n \geq 0$ and compute the expected times $T(n)$ to hit 0 starting from all such n together. To do this you apply the total expectation formula to get the recursive relation

$$T(n) = 1 + 0.3T(n+1) + 0.7T(n-1).$$

You can find the general homogeneous solution to this recurrence which is given by $T(n) = c_1\lambda_1^n + c_2\lambda_2^n$ where $\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{1-4 \cdot 0.3 \cdot 0.7}}{0.6} = \frac{1 \pm 0.4}{0.6} = 1, \frac{7}{3}$.

A particular solution can be obtained by the method of undetermined coefficients; you can guess $T(n) = cn$ which yields $cn = 1 + 0.3c(n+1) + 0.7c(n-1) = 1 + cn - 0.4c$ yielding $c = 5/2$. So the general solution is $T(n) = \frac{5}{2}n + c_1 + c_2\left(\frac{7}{3}\right)^n$.

Now clearly $T(0) = 0$, which implies that we have $T(n) = \frac{5}{2}n + c\left(\left(\frac{7}{3}\right)^n - 1\right)$ for some c.

Now we want $T(1)$, and $T(1)$ actually depends on c, so we need to figure out what c is. This is a bit subtle, since it's not obvious that we have any other boundary condition in the problem. One technique for circumventing this is to consider artificially stopping the process when the state $N > 0$ is reached and then send $N \to \infty$ to "suppress" the artificial termination scenario. Thus we consider setting $T(N) = 0$ also.

Proceeding in this manner you obtain the equation $\frac{5}{2}N + c\left(\left(\frac{7}{3}\right)^N - 1\right) = 0$, and so $T(n) = \frac{5}{2}n - \frac{\frac{5}{2}N}{(7/3)^N - 1}((7/3)^n - 1)$. Then $T(1) = \frac{5}{2} - \frac{\frac{5}{2}N}{(7/3)^N - 1}\frac{4}{3} \to \frac{5}{2}$ as $N \to \infty$.

Share Cite Edit Follow Flag edited 7 hours ago answered 7 hours ago

Ian

88.1k 2 66 130