Solving $Ax = 0$

Selected Notes — September 21, 2012

Abstract: We demonstrate an organized procedure to find all possible solutions to $Ax = 0$ for a system of m linear equations in n unknowns.

Nullspace of a Matrix

The nullspace $\mathcal{N}(A)$ of a $m \times n$ matrix A is a vector subspace of $\mathbb{R}^n$ defined by:

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

If A is a square matrix ($m = n$) and A^{-1} exists, then the null space $\mathcal{N}(A) = \{0\}$ (HOMEWORK). For general matrices A (including some square matrices), it is possible to have non-zero solutions to $Ax = 0$, i.e. there can exist vectors x which are not the zero vector but which still satisfy the equation $Ax = 0$. For example,

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} \underset{=}{} \underset{0}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} .$$

For this matrix A, the vector $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is not the only non-zero vector such that $Ax = 0$. Another example is

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} -4 \\ 2 \end{bmatrix}} \underset{=}{} \underset{0}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} .$$

We include this example to reiterate, to find the nullspace $\mathcal{N}(A)$ of a matrix A, one needs to find *all solutions* to $Ax = 0$.

How to Solve $Ax = 0$: First Example

Let us start with an example. Suppose we want to find all solutions to $Ax = 0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 11 \end{bmatrix}.$$

A natural instinct is to convert $Ax = 0$ to augmented matrix form and do Gaussian elimination to simplify the system of equations. Let us first take a look at the augmented matrix for $Ax = 0$ in this example.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 6 & 11 & 0 \end{array}\right]$$

Note that the last column of the augmented matrix has a zero in each entry of the column. This means that no matter what row operations we do to the augmented matrix during Gaussian elimination, the last column will always just have zeros in each entry! (To convince yourself, perform the operation Row $2 - 3 \times$ Row 1 to the augmented matrix.) The upside here is that it is not necessary to keep track of the right hand side during Gaussian elimination; that is, we may as well do Gaussian elimination directly to the matrix A rather than to the augmented matrix $[A|0]$.

Let's proceed with Gaussian elimination to transform A into an "upper triangular" U:

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 3 & 6 & 11 \end{bmatrix} \xrightarrow{R2-3R1} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{2} \end{bmatrix} = U.$$

A few comments are in order. The matrix A has two pivots — $\boxed{1}$ in the $(1,1)$ entry of U and $\boxed{2}$ in the $(2,3)$ entry of U. Note that there is not a pivot in the $(2,2)$ entry of U because 0 is never a pivot. To emphasize again, there is a pivot in columns 1 and 3 of U, but there is no pivot in column 2 of U.

> **Definition.** Suppose you have done Gaussian elimination to transform a $m \times n$ matrix A into an $m \times n$ upper triangular matrix U (for rectangular matrices "upper triangular" means that the (i,j) entry of U is zero whenever $i > j$.)
>
> We call column j of A a **pivot column** if there is a pivot in column j of U.
>
> We call column j of A a **free column** if there is not a pivot in column j of U.

Continuing with our example, columns 1 and 3 are pivots columns of A (and U), but column 2 is a free column of A (and U). This is important information for finding all solutions to $Ax = 0$.

So far we have done Gaussian elimination to transform the equations $Ax = 0$ into simpler equations $Ux = 0$. Writing the system $Ux = 0$ long hand, we have

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_3 &= 0 \end{aligned}$$

We now have two equations with three variables; since the number of equations is less than the number of variables, this system of equations will have more than one solution.

> **Definition.** Suppose A is an $m \times n$ matrix and you are trying to solve $Ax = 0$.
>
> We call x_j a **pivot variable** if column j of A is a pivot column.
>
> We call x_j a **free variable** if column j of A is a free column.

So in this example x_1 and x_3 are pivot variables, while x_2 is a free variable.

> **Rule.** When solving $Ax = 0$ (or equivalently $Ux = 0$), you are allowed to choose any values you want for the free variables.

Set the free variable $\boxed{x_2 = 1}$ in $Ux = 0$. With this choice, $Ux = 0$ looks like

$$\begin{aligned} x_1 + 3x_3 &= -2 \\ 2x_3 &= 0 \end{aligned}$$

Now this is an upper triangular system of two equations in two variables — we know how to solve this type of system — use back substitution!

$$\begin{aligned} 2x_3 = 0 &\Longrightarrow \boxed{x_3 = 0} \\ x_1 + 3x_3 = -2 &\Longrightarrow x_1 + 3(0) = -2 \Longrightarrow \boxed{x_1 = -2} \end{aligned}$$

Finally we have found out that one solution to $Ax = 0$ is $\boxed{x_1 = -2,\ x_2 = 1 \text{ and } x_3 = 0}$. But this is not the only solution, because we could have picked a different value for x_2.

Suppose that in the previous paragraph we picked $x_2 = t$ instead of $x_2 = 1$ (where t represents any real number). Substituting this choice into $Ux = 0$, the system of equations becomes

$$\begin{aligned} x_1 + 3x_3 &= -2t \\ 2x_3 &= 0. \end{aligned}$$

Using back substitution we can solve for x_1 and x_3.

$$2x_3 = 0 \Longrightarrow \boxed{x_3 = 0}$$

$$x_1 + 3x_3 = -2t \Longrightarrow x_1 + 3(0) = -2t \Longrightarrow \boxed{x_1 = -2t}$$

Thus another solution to $Ax = 0$ is $\boxed{x_1 = -2t,\ x_2 = t,\ x_3 = 0 \text{ where } t \text{ is any real number.}}$

If we let t range over all real numbers, then we have found *all solutions* to $Ax = 0$. Hence

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Finally note that the nullspace of A is simply the span of the first solution that we found:

$$\mathcal{N}(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

How to Solve $Ax = 0$: Second Example

Let's find all solutions to $Ax = 0$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix}.$$

The first step is to use Gaussian elimination to transform A into an "upper triangular" U:

$$A = \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 4 & -2 & 2 & -2 \end{bmatrix} \xrightarrow{R3-4R1} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & -2 & -2 & -2 \end{bmatrix} \xrightarrow{R3+R2} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

Therefore columns 1 and 2 of A are the pivot columns; columns 3 and 4 of A are the free columns. In particular, to solve $Ux = 0$, we are free to choose the values of the free variables x_3 and x_4.

> **Definition.** Suppose A has f free columns total. Suppose columns $j_1, j_2, \ldots, j_f$ of A are its free columns and $x_{j_1}, x_{j_2}, \ldots, x_{j_f}$ are the associated free variables.
> A **special solution** of $Ax = 0$ is the unique solution x to $Ax = 0$ that is found after setting one of the free variables $x_{j_k} = 1$ and all other free variables $x_{j_l} = 0$.
> A matrix A with f free columns total has exactly f special solutions to $Ax = 0$.

The next step is to find all of the special solutions to $Ax = 0$. Because A has 2 free columns, A has 2 special solutions. (In the previous example, there was only 1 special solution.)

To find the first special solution, set $\boxed{x_3 = 1}$ and $\boxed{x_4 = 0}$. Then $Ux = 0$ is simply

$$x_1 = -1$$
$$2x_2 = -2$$

Hence the first special solution is $\boxed{x_1 = -1,\ x_2 = -1,\ x_3 = 1,\ x_4 = 0}$.

To find the second special solution, set $\boxed{x_3 = 0}$ and $\boxed{x_4 = 1}$. Then $Ux = 0$ becomes

$$x_1 = 0$$
$$2x_2 = -2$$

Hence the second special solution is $\boxed{x_1 = 0,\ x_2 = -1,\ x_3 = 0,\ x_4 = 1}$.

Rule. Every solution to $Ax = 0$ is a linear combination of the special solutions of $Ax = 0$. In other words, the nullspace of A can be described as

$$\mathcal{N}(A) = \text{span}(\text{special solutions of } Ax = 0)$$

Thus for this second example, we have

$$\mathcal{N}(A) = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

How to Solve $Ax = 0$: General Procedure

Here are the steps to find all solutions of $Ax = 0$.

(1) Do Gaussian elimination to transform $Ax = 0$ to $Ux = 0$ where U is upper triangular.
(2) Note which columns are pivots columns of A and which columns are free columns of A.
(3) Find the special solutions of $Ax = 0$. If A has f free columns, A has f special solutions.
  (a) Pick a free variable x_{j_k}. Set $x_{j_k} = 1$ and set all other free variables $x_{j_l} = 0$.
  (b) Substitute these values for the free variables into $Ux = 0$ and solve for the special solution using back substitution.
  (c) Repeat steps (a) and (b) once for each free variable.
(4) A generic solution to $Ax = 0$ is a linear combination of the special solutions of $Ax = 0$. This means that the nullspace of A is given by

$$\mathcal{N}(A) = \text{span}(f \text{ special solutions of } Ax = 0)$$

If A happens to have no free columns ($f = 0$), the only solution to $Ax = 0$ is $x = 0$.