MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Bookmarks

- Unit 0: Overview
- Entrance Survey
- Unit 1: Probability models and axioms
- Unit 2: Conditioning and independence
- **Unit 3: Counting**

Lec. 4: Counting
Exercises 4 due Feb 24, 2016 at 23:59 UTC

Solved problems

Problem Set 3
Problem Set 3 due Feb 24, 2016 at 23:59 UTC

Unit 3: Counting > Problem Set 3 > Problem 5 Vertical: Hats in a box

Bookmark

Problem 5: Hats in a box

(5/5 points)

Each one of n persons, indexed by $1, 2, \ldots, n$, has a clean hat and throws it into a box. The persons then pick hats from the box, at random. Every assignment of the hats to the persons is equally likely. In an equivalent model, each person picks a hat, one at a time, in the order of their index, with each one of the remaining hats being equally likely to be picked. Find the probability of the following events.

1. Every person gets his or her own hat back.

● $\frac{1}{n!}$ ✓

○ $\frac{1}{(n+1)!}$

- Unit 4: Discrete random variables
- Exam 1
- Unit 5: Continuous random variables
- Unit 6: Further topics on random variables
- Unit 7: Bayesian inference
- Exam 2
- Unit 8: Limit theorems and classical statistics
- Unit 9: Bernoulli and Poisson processes
- Unit 10: Markov chains

○ $\frac{1}{n}$

○ $\frac{1}{n+1}$

2. Each one of persons $1, \ldots, m$ gets his or her own hat back, where $1 \leq m \leq n$.

○ $\frac{(n+m)!}{n!}$

● $\frac{(n-m)!}{n!}$ ✓

○ $\frac{n!}{(n+m)!}$

- Exit Survey
- Final Exam

○ $\dfrac{m!}{n!}$

3. Each one of persons $1, \ldots, m$ gets back a hat belonging to one of the last m persons (persons $n-m+1, \ldots, n$), where $1 \leq m \leq n$.

● $\dfrac{1}{\binom{n}{m}}$ ✓

○ $\dfrac{m}{\binom{n}{m}}$

○ $\dfrac{n-m}{\binom{n}{m}}$

○ $\dfrac{n}{\binom{n}{m}}$

Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). Find the probability that:

4. Persons $1, \ldots, m$ will pick up clean hats.

○ $(1-p)^{n-m}$

○ $m(1-p)^m$

● $(1-p)^m$ ✓

○ $m(1-p)^{n-m}$

5. Exactly m persons will pick up clean hats.

○ $\frac{\binom{n}{m}}{n!}(1-p)^m p^{n-m}$

○ $(1-p)^m p^{n-m}$

○ $\binom{n}{m}(1-p)^{n-m} p^m$

● $\binom{n}{m}(1-p)^m p^{n-m}$ ✓

Answer:

1. Consider the sample space of all possible hat assignments. It has $n!$ elements (n hat selections for the first person, after that $n-1$ for the second, etc.), with every assignment equally likely (hence each assignment has probability $1/n!$). The event that everyone gets his or her own hat back corresponds to exactly one of these $n!$ assignments. Therefore, the answer is $1/n!$.

2.

Consider the same sample space and probabilities as in the solution of part 1. The event of interest assigns the first m people to their own hats and allows for an arbitrary assignment of hats to the remaining $n-m$ persons, so that there are $(n-m)!$ possible assignments. The probability of an event with $(n-m)!$ elements is $(n-m)!/n!$.

3. Consider the m hats belonging to the last m persons. There are $m!$ ways to distribute these m hats among the first m persons. Then, there are $(n-m)!$ ways to distribute the remaining $n-m$ hats to everyone else. The probability of an event with $m!(n-m)!$ elements is $m!(n-m)!/n!$, which is equal to $1/\binom{n}{m}$.

4. The probability of a given person picking up a clean hat is $1-p$. By the independence assumption, the probability of m specific persons picking up clean hats is $(1-p)^m$.

5. Think of picking a clean hat as an independent Bernoulli trial with success probability $1-p$. The probability of m successes out of n trials is $\binom{n}{m}(1-p)^m p^{n-m}$.

You have used 1 of 2 submissions

DISCUSSION

Click “Show Discussion" below to see discussions on this problem.