6 Solving the wave equation for the infinite string

In this lecture I assume that my string (or rod) are so long that it is reasonable to disregard the boundary conditions, i.e., I consider an infinite space. In this case I get the initial value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \quad t > 0, \quad -\infty < x < \infty, \tag{6.1}$$

with the initial conditions

$$u(0, x) = f(x), \quad u_t(0, x) = g(x), \quad -\infty < x < \infty. \tag{6.2}$$

Arguably the best way to get an intuitive understanding what is modeled with this equation is to imagine an infinite guitar string, where $u(t, x)$ represent a transverse displacement at time t at position x.

Very surprisingly (do not get used to it, this is a very rare case for PDE), problem (6.1)–(6.2) can be solved explicitly.

6.1 The general solution to the wave equation

First I will find the *general solution* to (6.1), i.e., the formula that includes *all* possible solutions to the wave equation. To do this I note that I can rewrite (6.1) as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0.$$

Denoting

$$v = \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u,$$

I find that my wave equation (6.1) is equivalent to two first order linear PDE:

$$v_t - c v_x = 0,$$
$$u_t + c u_x = v.$$

From the previous lectures we know immediately that the first one has the general solution $v(t, x) = F^*(x + ct)$, for some arbitrary F^*, and the second one has the solution

$$u(t, x) = \int_0^t v(s, x - c(t - s)) \mathrm{d}s + G^*(x - ct) = \int_0^t F^*(x - ct + 2cs) \mathrm{d}s + G^*(x - ct).$$

Making the change of the variables $\tau = x - ct + 2cs$ in the integral, I have

$$u(t, x) = \frac{1}{2c} \int_{x-ct}^{x+ct} F^*(\tau) \mathrm{d}\tau + G^*(x - ct). \tag{6.3}$$

Finally, since F^*, G^* are arbitrary, by denoting

$$F(x + ct) = \frac{1}{2c} \int_0^{x+ct} F^*(\tau) \mathrm{d}\tau, \quad G(x - ct) = \frac{1}{2c} \int_{x-ct}^0 F^*(\tau) \mathrm{d}\tau + G^*(x - ct),$$

I get the final result that
$$u(t, x) = F(x + ct) + G(x - ct),$$
for arbitrary $\mathcal{C}^{(2)}$ functions F and G. This expression, and the analysis from previous lectures, tell me that the general solution to the wave equation is a sum of two linear traveling waves, one of which moving to the left and another one moving to the right.

6.2 d'Alembert's formula

Here I will solve problem (6.1)-(6.2) and reproduce a famous formula, first obtained by Jean-Baptiste le Rond d'Alembert in 1747.

From the first condition in (6.2) I immediately get that $G^* = f$. To use the second condition I calculate
$$u_t(0, x) = F^*(x) - cf'(x) = g(x) \implies F^*(x) = g(x) + cf'(x).$$
This yields
$$u(t, x) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\mathrm{d}s + \frac{1}{2}f(x + ct) - \frac{1}{2}f(x - ct) + f(x - ct),$$
and finally
$$u(t, x) = \frac{1}{2}\Big(f(x + ct) + f(x - ct)\Big) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\mathrm{d}s, \tag{6.4}$$
which is the d'Alembert's formula.

Remark 6.1. All the results above are obtained in a different way in the textbook. Namely, the change of variables
$$\eta = x + ct, \quad \xi = x - ct$$
reduces the wave equation to its canonical form
$$v_{\eta\xi} = 0.$$
I invite the students to perform these calculations on their own (or read the textbook).

6.3 Two examples

To get an intuitive understanding how formula (6.4) works consider two examples. I assume that $c = 1$ below.

Example 6.2. Let my initial condition be such that the initial velocity is zero. Then my formula simplifies to
$$u(t, x) = \frac{1}{2}\Big(f(x + ct) + f(x - ct)\Big),$$
and hence my solution is a sum of two identical linear waves, each of which is exactly half of the initial displacement. Therefore I can envision the behavior of solutions by first dividing in half the initial displacement, then shifting one half to the left and the other one to the right, and then adding them.

Let, e.g, f be defined as
$$f(x) = \begin{cases} 0, & x < -0.5, \\ 1, & -0.5 \le x \le 0.5, \\ 0, & x > 0.5. \end{cases}$$

The form of the solution at different time moments is shown in Fig. 1.

Now it should be clear how the three dimensional surface of the solution naturally appears (see Fig. 2, left panel).

Finally, the behavior of solution should be also clear from looking at the plane (t, x) and several straight lines of the form

$$x = ct + \xi, \quad x = -ct + \eta,$$

which are also called the *characteristics* of the wave equation. Note that the signal spreads along the characteristics (Fig. 2, right panel).

Example 6.3. For the second example I take $f(x) = 0$ and

$$g(x) = \begin{cases} 0, & x < -0.5, \\ 1, & -0.5 \le x \le 0.5, \\ 0, & x > 0.5. \end{cases}$$

Now the solution takes the form

$$u(t, x) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\mathrm{d}s = \frac{1}{2c}\Big(G(x + ct) - G(x - ct)\Big),$$

where G is any antiderivative of g, e.g., I can always take

$$G(x) = \int_{-\infty}^{x} g(s)\mathrm{d}s.$$

Therefore the solution now is a difference of two traveling waves, each of which is exactly half of G. In my case

$$G(x) = \begin{cases} 0, & x < -0.5, \\ x + 0.5, & -0.5 \le x \le 0.5, \\ 1, & x > 0.5, \end{cases}$$

and my solution is given in Fig. 3. See also Fig. 4 for the three dimensional picture. Again the overall picture can be figured out from the plane (t, x) and characteristics on it.

These two examples give a general idea how actually solutions to the wave equation behave. Note that in both cases I used initial conditions that are not continuously differential (they have "corners") and hence my solutions are not the classical solutions. However, the notion of the solution to the wave equation can be extended in a way to include these, nondifferential, solutions. They are usually called *weak solutions.*

Another point to note that characteristics of the wave equation allows immediately to see which initial conditions contribute to the solution at a given point (t, x) (this is called *domain of dependence*) and also how the given point ξ on the initial condition spreads the signal with time (*range of influence*), see Fig. 5.
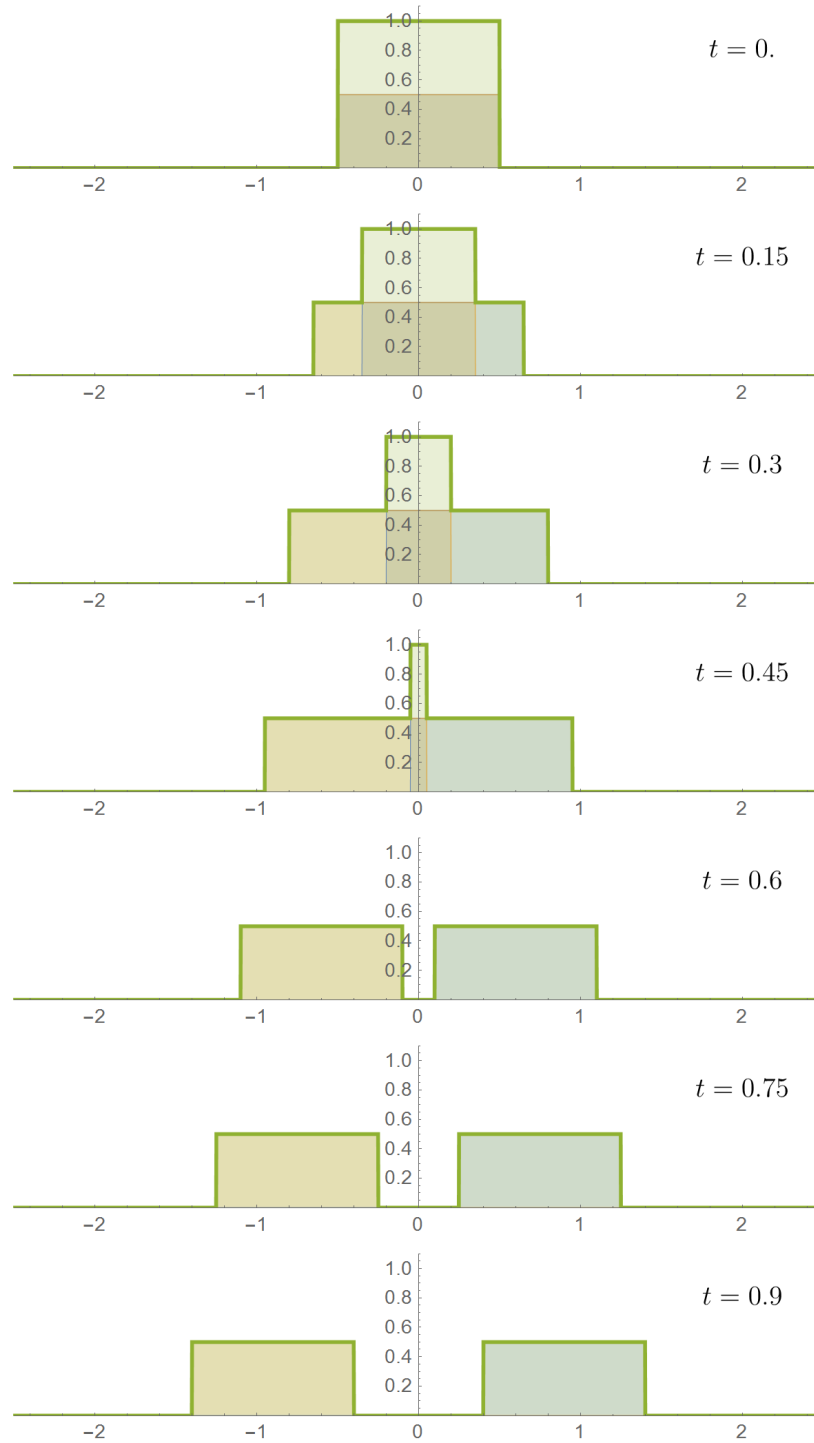
Figure 1: Solution to the initial value problem for the wave equation in case when the initial velocity is zero for different time moments.
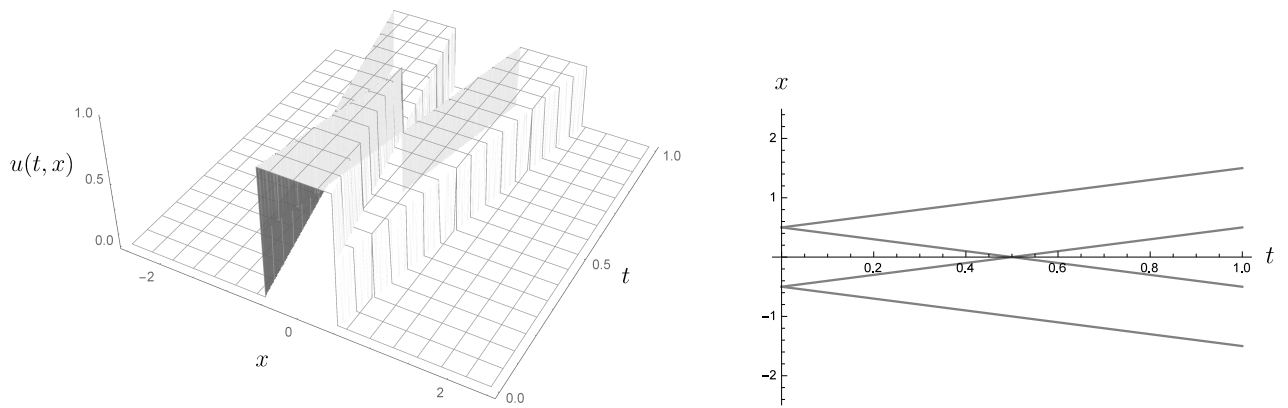
Figure 2: Left: Solution in the coordinates (x, t) to the initial value problem for the wave equation in case when the initial velocity is zero. Right: Characteristics of the wave equation.
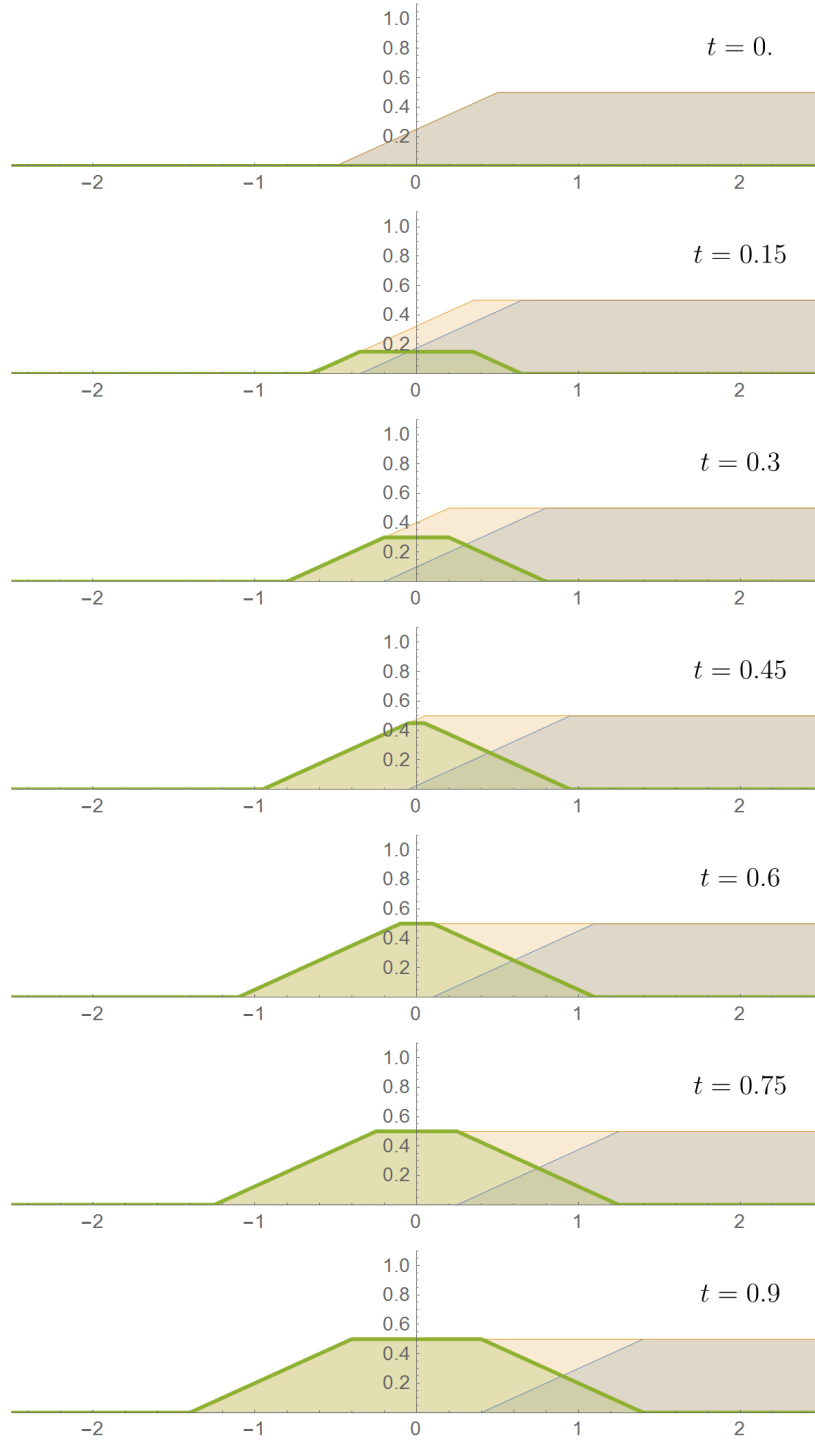
Figure 3: Solution for different time moments to the initial value problem for the wave equation in case when the initial displacement is zero. The actual solution is shown with the green border.
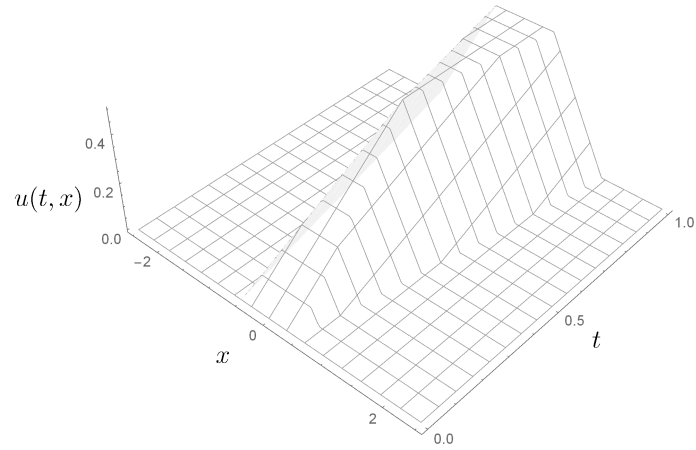
Figure 4: Solution in the coordinates (x, t) to the initial value problem for the wave equation in case when the initial displacement is zero
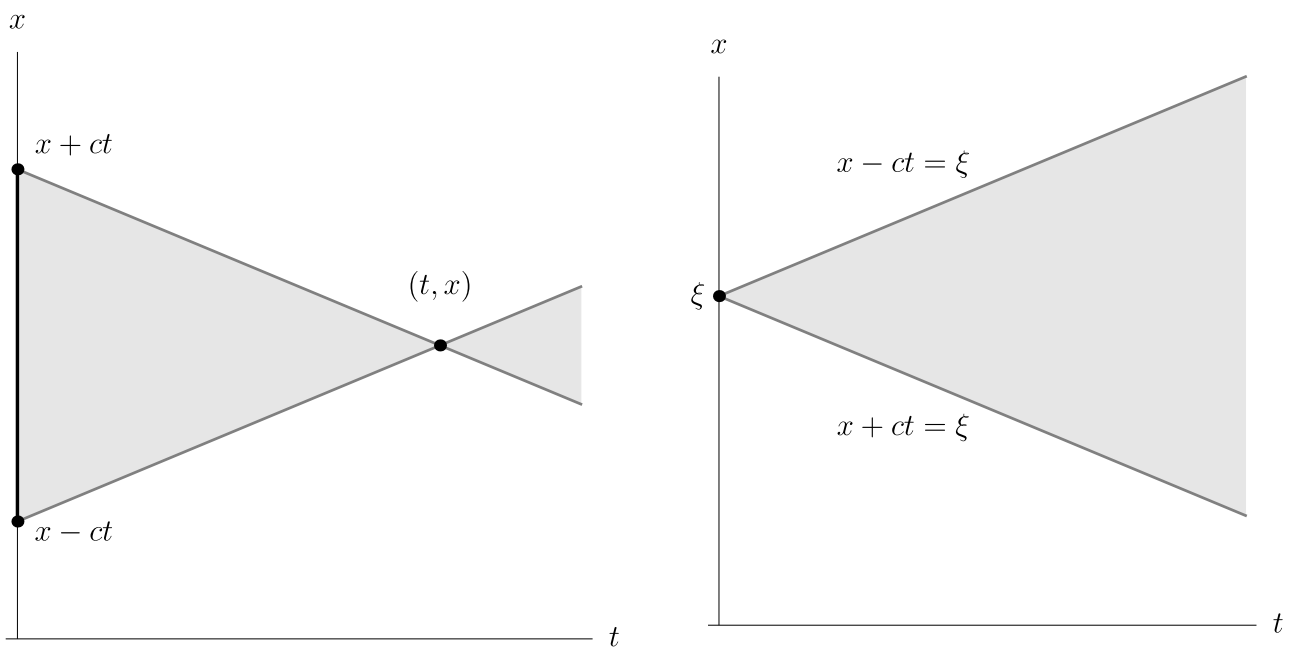

Figure 5: Domain of dependence (left, the bold line on the x axis) and the range of influence (right, the shaded area).

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