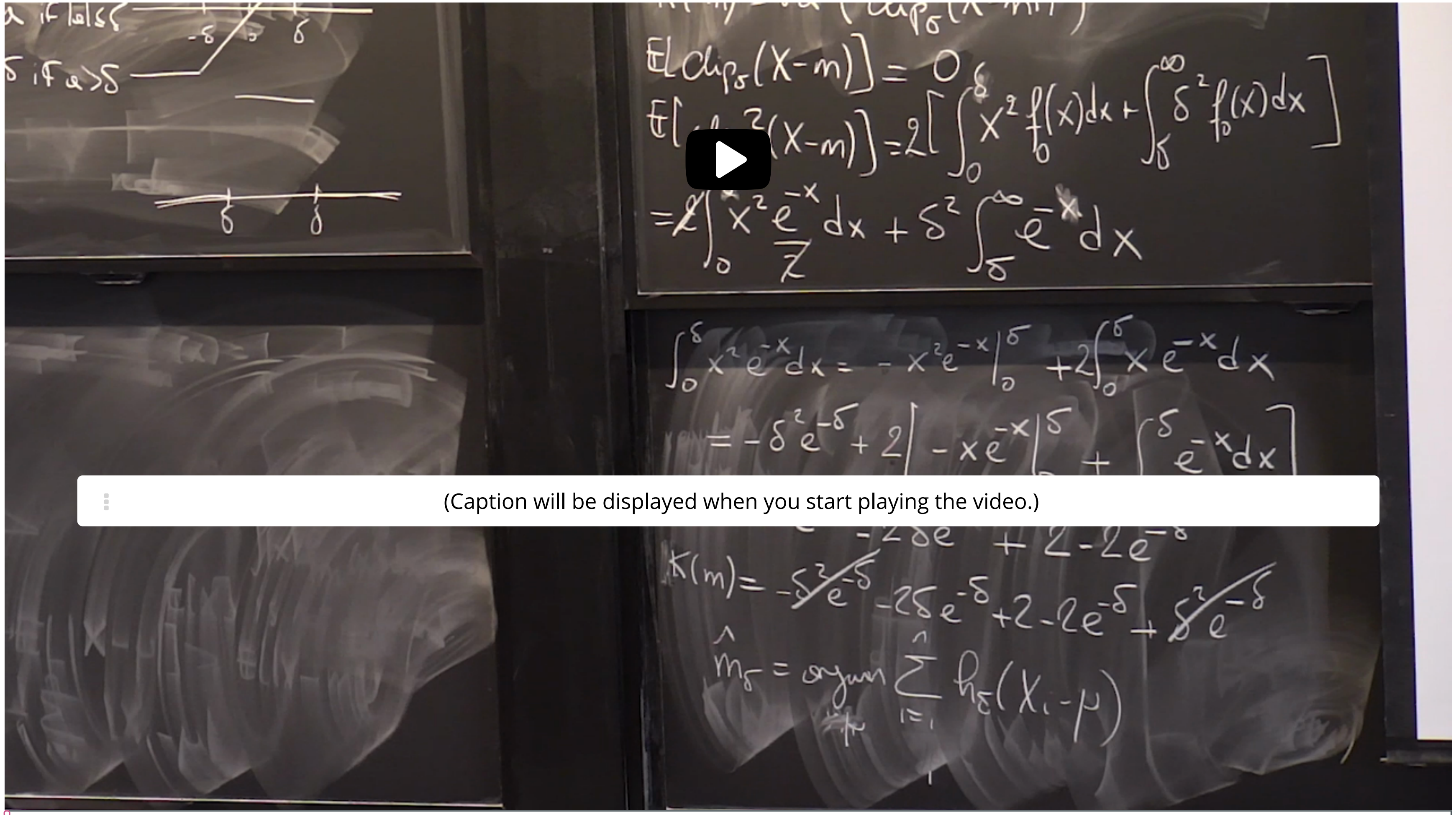
# 9. Applying Huber's loss to the Laplace distribution (Continued)

## Applying Huber's Loss to the Laplace distribution (Continued)

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## Asymptotic Variance of the M-estimator for a Laplace distribution

2/2 points (graded)

We use the same statistical set-up from the previous three questions. As before, $m^*$ denotes the location parameter for a Laplace distribution, and $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Lap}\,(m^*)$. Recall the M-estimator

$$\widehat{m}\,(\delta) = \text{argmin}_{m\in\mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} h_\delta\,(X_i - m)\,,$$

where now we emphasize the dependence on the parameter $\delta \in (0, \infty)$.

In lecture, we showed that

$$\sqrt{n}\,(\widehat{m}\,(\delta) - m^*) \xrightarrow[n\to\infty]{(d)} N\,(0\,,\,g\,(\delta))\,.$$

where

$$g\,(\delta) = \frac{2\,(1 - \delta e^{-\delta} - e^{-\delta})}{(1 - e^{-\delta})^2}.$$

We can extend $g$ to be a continuous function with domain $[0, \infty]$ by setting $g\,(0) = 1$ and $g\,(\infty) = 2$.

Where is the minimum of $g$ attained on $[0, \infty]$? (You may use computational software.)

(If applicable type **inf** for $\infty$.)

0

✔ **Answer:** 0

$0$

Where is the maximum of $g$ attained on $[0, \infty]$? (You may use computational software.)

(If applicable type **inf** for $\infty$.)

inf

✔ **Answer:** inf

$inf$

STANDARD NOTATION

**Solution:**

One can see by graphing that $g(\delta)$ is an increasing function on $[0, \infty]$. Hence, the minimum is attained at $\delta = 0$, and the maximum is attained at $\delta = \infty$. Therefore, the correct response to the first question is "0", and the correct response to the second question is "I". Below we justify this rigorously.

If we are able to show that

$$g'(\delta) \geq 0,$$

for $\delta \in [0, \infty)$, then the result follows. By the quotient rule for derivatives,

$$g'(\delta) = 2 \cdot \left( \frac{\delta e^{-\delta}}{(1-e^{-\delta})^2} - \frac{2(1-\delta e^{-\delta} - e^{-\delta})e^{-\delta}}{(1-e^{-\delta})^3} \right) = 2 \cdot \frac{\delta e^{-\delta} - 2e^{-\delta} + \delta e^{-2\delta} + 2e^{-2\delta}}{(1-e^{-\delta})^3}.$$

The denominator is positive for $\delta \in [0, \infty]$, so it suffices to show that the numerator is nonnegative. Let $\tilde{g}(\delta) = \delta - 2 + \delta e^{-\delta} + 2e^{-\delta}$ denote the numerator of the above divided by $e^{-\delta}$. Observe that $\tilde{g}(\delta) \geq 0$ if and only if

$$h(\delta) := e^{\delta}(\delta - 2) + \delta + 2 \geq 0.$$

Since $h(0) = 0$, if we can show that $h'(\delta) \geq 0$ for $\delta \in [0, \infty)$, then this implies $h(\delta)$ is increasing, and hence, $h(\delta) \geq 0$ for $\delta \in [0, \infty)$. Therefore $\tilde{g} \geq 0$ as well, which would suffice to prove what we want.

Observe that

$$h'(\delta) = e^x(x-2) + e^x + 1 = xe^x - e^x + 1.$$

Since $h'(0) = 0$, we would be done if we can show that $h''(\delta) \geq 0$ because

$$\begin{aligned} h''(\delta) &\geq 0 \Rightarrow \\ h'(\delta) &\geq 0 \Rightarrow \\ h(\delta) &\geq 0 \Rightarrow \\ \tilde{g}(\delta) &\geq 0 \Rightarrow \\ g'(\delta) &\geq 0 \end{aligned}$$

on the interval $[0, \infty)$. Finally, $h''(\delta) = \delta e^{-\delta} \geq 0$, so we have shown analytically that $g(\delta)$ is an increasing function on $[0, \infty)$, as desired.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

## Extreme Values of Huber's loss I

1/1 point (graded)

If $\delta = \infty$, it makes sense to extend the definition of Huber's loss to be

$$h_\infty(x) = \frac{x^2}{2}.$$

Setting $\delta = \infty$, we have

$$\widehat{m}(\infty) = \mathrm{argmin}_{m \in \mathbb{R}} \frac{1}{2n} \sum_{i=1}^{n} (X_i - m)^2.$$

What is another name for $\widehat{m}(\infty)$?
*Hint:* You may use the fact that the objective function is strictly convex.

- (selected) The sample average.
- The sample median.
- The sample average divided by $2$.
- The sample median divided by $2$.

✔

**Solution:**

The correct response is "The sample average.". We will show this analytically. Let us differentiate and find the value of $m$ that is a critical point of the function

$$F(m) := \frac{1}{2n} \sum_{i=1}^{n} (X_i - m)^2.$$

Observe that

$$F'(m) = -\frac{1}{n}\sum_{i=1}^{n}(X_i - m).$$

Setting $m = \frac{1}{n}\sum_{i=1}^{n} X_i$, we see that $F'(m) = 0$. By strict convexity, this implies that the sample average is the unique global minimizer of $F(m)$.

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

## Extreme values of Huber's loss II

1/1 point (graded)

Note that for all $\delta > 0$,

$$\begin{aligned}\widehat{m}(\delta) &= \mathrm{argmin}_{m\in\mathbb{R}}\frac{1}{n}\sum_{i=1}^{n} h_\delta(X_i - m)\\ &= \mathrm{argmin}_{m\in\mathbb{R}}\frac{1}{n}\sum_{i=1}^{n}\frac{h_\delta(X_i - m)}{\delta}\end{aligned}$$

Moreover, for all $x \in \mathbb{R}$,

$$\lim_{\delta\to 0^+}\frac{h_\delta(x)}{\delta} = |x|.$$

Therefore, it makes sense to define

$$\widehat{m}(0) = \mathrm{argmin}_{m \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |X_i - m|.$$

What is another name for $\widehat{m}(0)$?

- The sample average.
- The sample median.
- The true mean.
- The true median.

**Solution:**

The correct response is "The sample median." This is a direct consequence of the definition of the sample median from the problem "The Sample Median" on the page "Applying Huber's Loss to the Laplace Distribution."

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

## Discussion

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**Topic:** Unit 3 Methods of Estimation:Lecture 12: M-Estimation / 9. Applying Huber's loss to the Laplace distribution (Continued)

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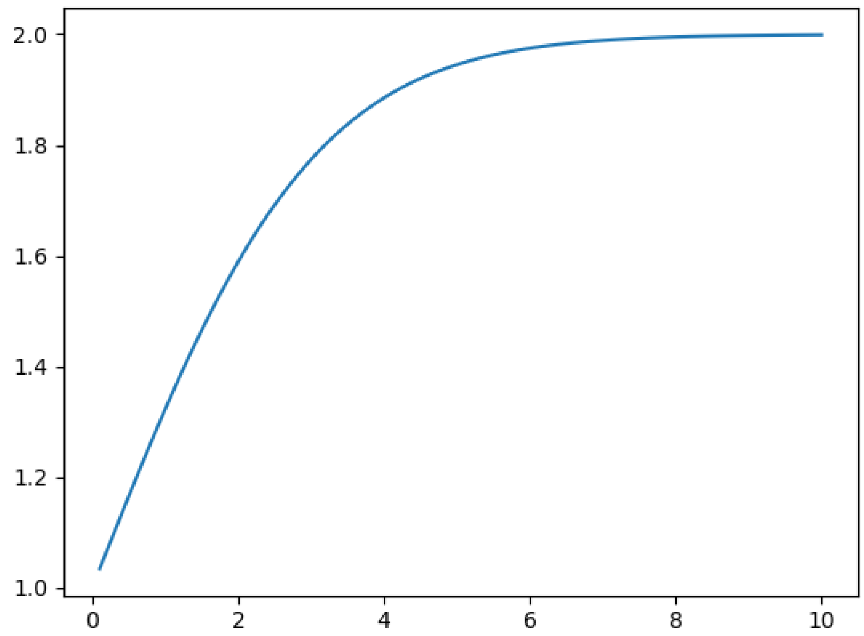
# Asymptotic Variance of the M-estimator for a Laplace distribution, minimum/maximum of g(.)

question posted about 4 hours ago by **sandipan_dey**

I thought $g(0) = \frac{0}{0}$ is undefined (not $\infty$) and the limiting value of $g(.)$ at 0 exists (using L'Hospital's rule e.g.,) and it's used to make the function continuous (remove the discontinuity at 0). So, min and max of $g(.)$ in $[0, \infty)$ (is the interval closed $[0, \infty]$ or half-open $[0, \infty)$?) Also the function looks like the following:




and the derivative $g'(x) = \frac{(1-e^{-x}).e^{-x}.(x(1-3e^{-x})+2(1-e^{-x}))}{(1-e^{-x})^4} = 0 \Rightarrow x = 0, \infty, 0$ where $g(0) = 1, g(\infty) = 2$ (limiting values), but grader does not accept, any clue? thanks in advance.

This post is visible to everyone.

**Jang Park**
about 2 hours ago

Note they are asking for **where** the maximum or the minimum is attained not what the maximum or the minimum values are. This confused me as well until I read the question more carefully.

Thank you very much @Jang_Park, I got it wrong, the question asks $argmin$ and $argmax$ instead.

posted less than a minute ago by **sandipan_dey**

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