8. Beam bending equation

In this section, we derive the fourth order differential equation that describes the static bending of a slender, horizontal beam due to a distributed load. The main simplifying assumption we are making is that we are only modeling the steady state, static bending. That is our model will not change in time, so there will be no time derivatives. The other simplifying assumption is that the beam does not bend very much.

We start the modeling process by drawing a picture, and labeling the relevant variables.

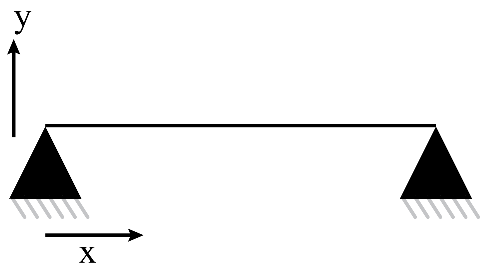
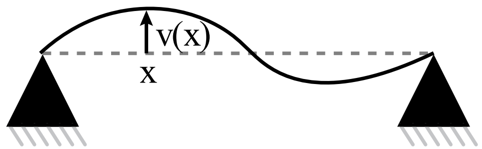


Figure 23: Undeformed beam

We will start with a beam that is pinned at both ends, which we draw by placing it on little triangles that are stuck into the ground. We label the vertical direction as the y-axis, and the horizontal direction along the undeformed beam is the x-axis. The quantity we wish to describe is the vertical displacement, or deflection, $v\left(x\right)$ of the beam at every point x along the beam.



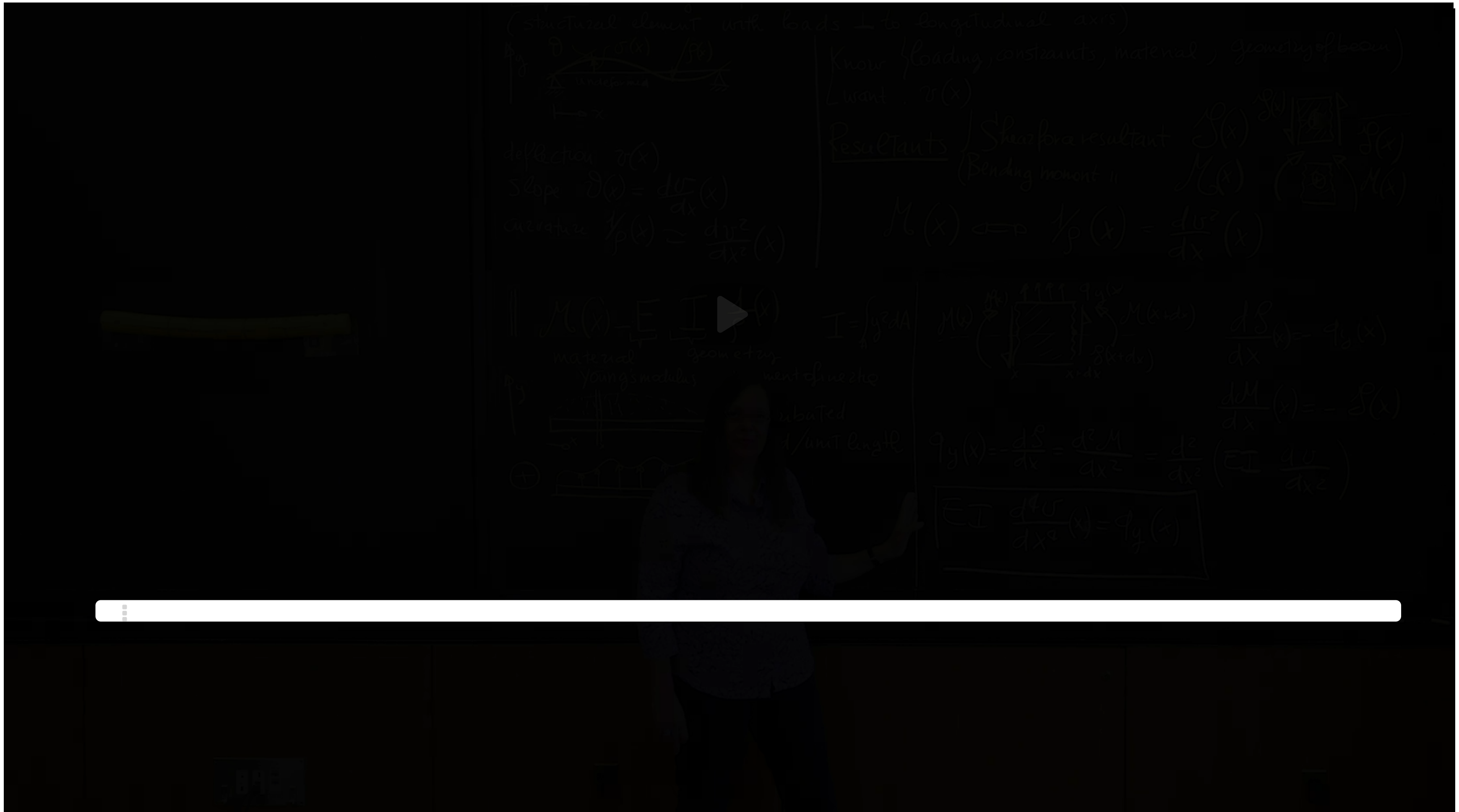
Deformed beam

Geometric quantities of interest	**Variable name**	
Vertical deflection of the beam at each point	$v\left(x\right)$	
Slope of the beam at each point	$\frac{dv}{dx}\left(x\right)$	
Curvature of the beam at each point	$\approx\frac{d^2v}{dx^2}\left(x\right)$	(assuming that the deflection $v\left(x\right)$ is small)

Our goal is to determine $v\left(x\right)$ given

- the loading on the beam,
- the boundary conditions (constraints),
- the material of the beam, and
- the geometry of the beam.

(Optional) Derivation of the beam equation



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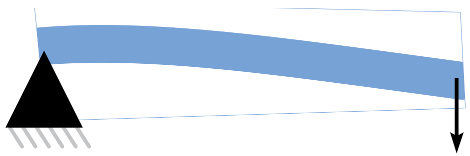
In order to determine how the load will affect beam bending, we need to understand all of the resultant forces inside of the beam that occur due to external loading.

There are two main resultants that will be important to us.

1. The **shear force** resultant, $S(x)$
2. The **bending moment** resultant, $M(x)$ (equivalent to torque)

To gain some intuition for these forces, we will look at a toy example of a long slender beam, where we have hung a heavy object off the one end.



If you were to cut the beam near the weight, the beam would fall. So what this tells us is that there is a force internal to the beam that is preventing the force of the mass from causing the beam to fall off. That is, internal to the beam, there is an equal and opposite resultant force. This is the **shear force** resultant.

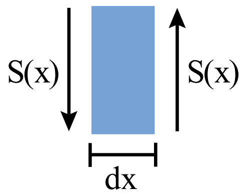


Figure 24: Shear force S with positive orientation on infinitesimal slice of beam

On an infinitesimal slice of beam, we draw a positive shear force resultant so that there is a shear force in the upward direction on the right, and the corresponding downward facing shear force on the left.

In our toy example, the shear force resultant is constant along the beam, exactly canceling the effect of the weight in order to hold it up. However, that is not the only thing happening. Because if we hold the beam further and further from the point where the mass is hung, we not only have to apply a vertical shear force, but the further away our point of contact, the force also creates a torque, and the beam must supply a resultant torque (or **bending moment** as it is called in the mechanics of beams) to prevent the beam from rotating.

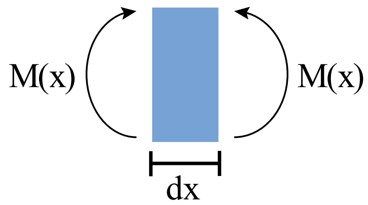


Figure 25: Bending moment M with positive orientation on an infinitesimal slice of beam

Positive bending moments are drawn so that they bend the beam in the direction that causes it to look like a smile.

The more the beam is trying to curve, the more bending moment must be applied to counter the rotation. In our toy example, we see that the further we are from the applied shear force, the more moment we need to counteract the effect. Thus we have a relationship between the bending moment and the shear force

$$\frac{d}{dx} M\left(x\right) = -S\left(x\right).$$

Additionally, the bending moment is exactly what causes the beam to bend. Thus it is proportional to the curvature, and the relationship between the bending moment and the curvature is given by

$$M\left(x\right) = EI \frac{d^2 v}{dx^2},$$

where E depends on the material of the beam, and I is the moment of inertia, which depends on the geometry of the beam. Now that we have all of the forces, we can finally write down our model.

How are resultant forces affected by a distributed load?

Suppose that along our beam, we have some distributed load $q_y\left(x\right)$ (units of [Force]/[Length]), where the subscript y is used to denote the fact that the positive direction points in the positive y direction.

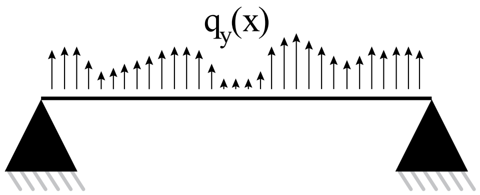


Figure 26: Beam with distributed load $q_y\left(x\right)$ in positive orientation (pointing up).

To see how this load affects the resultant forces inside the beam, we need to do a force and torque balance on an infinitesimal segment of our beam.

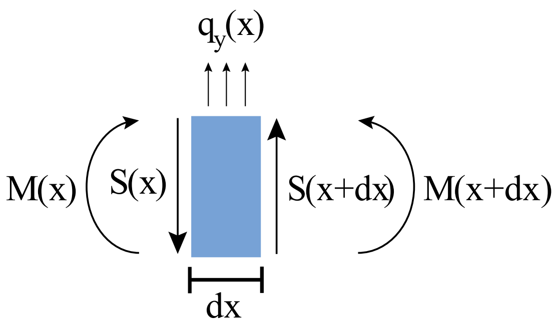


Figure 27: Load, shear forces, and bending moments on a slice of beam of width dx.

On an infinitesimal segment of beam of width dx, we can assume that the load is the constant value $q_y\left(x\right)$. Thus the total force is $q_y\left(x\right)\ dx$. Force balance tells us that

$$S\left(x+dx\right)-S\left(x\right)+q_y\left(x\right)\ dx=0.$$

Dividing by dx and taking the limits as $dx\rightarrow 0$, this gives us

$$\frac{d}{dx}S\left(x\right)=-q_y\left(x\right).$$

Therefore, using the fact that $\frac{d}{dx}M(x) = -S(x)$, we can rewrite this as

$$q_y(x) = -\frac{d}{dx}S(x) = \frac{d^2}{dx^2}M(x) = \frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right).$$

In the case that the material and geometry of the beam are constant throughout, this reduces to the forth order differential equation relating the deflection of the beam to the external loading

$$\boxed{EI\frac{d^4v(x)}{dx^4} = q_y(x).}$$

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8. Beam bending equation

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dM(x)/dx = - S(x)?

Was looking though the literature and found proofs for dM(x)/dx=S(x) Not sure if there are different sign conventions ...

4