15. (Optional) Generalizing Fourier series: The Fourier transform

So far, we have only worked with periodic functions and signals. In the real world, signals may not be periodic.

Example 15.1 The function $f(t) = e^{-t^2}$ is a non periodic function that is important in probability and differential equations.

Example 15.2 Sound signals are not truly periodic as the signal exists for finite time.

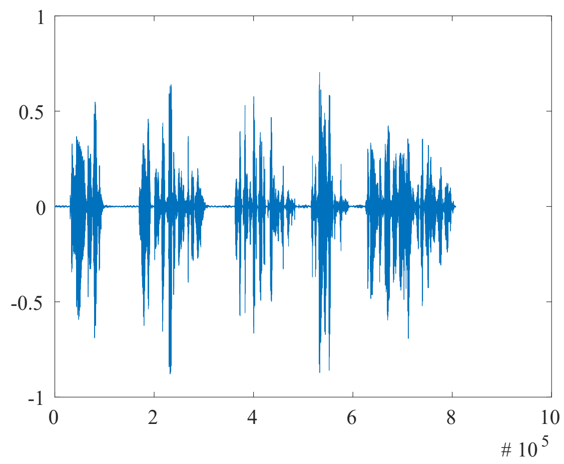

Figure 12: A sound wave coming from a narration of text.

Below let's generalize the method of Fourier series to non-periodic signals. This is known as the Fourier transform.

Suppose you have a radio signal, sound wave, or other signal $f(t)$ with period $2L$. Then we can write the signal in the form

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{(ik\pi/L)t},$$

where the c_k are the coefficients determined by

$$c_k = \frac{1}{2L} \int_{-L}^{L} f(t)\, e^{-(ik\pi/L)t}\, dt.$$

We can think of a non-periodic signal as the limit as L goes to infinity of a periodic signal of period $2L$. As L increases, the spacing between the frequencies in our sum are approaching zero. This turns the sum into an integral in the limit, and we have the equations:

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{ikt}\, dk$$

$$\widehat{f}\,(k) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} f\,(t)\, e^{-ikt}\, dt$$

We call $\widehat{f}\,(k)$ the **Fourier transform** of $f\,(t)$.

Note that the continuous function $\widehat{f}\,(k)$ replaces the discrete coefficients c_k. So now $f\,(t)$ can be composed of a continuous infinite sum (an integral) of complex sinusoids e^{ikt} with the weights being given by the $\widehat{f}\,(k)$ function.

A derivation of Fourier transform

The following argument does not constitute proof, but does show you how the formula for the Fourier transform arises. We leave out some technical conditions related to the convergence of the finite Fourier series and improper integral of $f\,(t)$ (the function $f\,(x)$ satisfies the Dirichlet conditions on every finite interval and $\int_{-\infty}^{\infty} |f\,(x)|\, dx$ is finite).

Start with

$$f\,(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}t}.$$

and make a change of variable so that $\omega_k = k\pi/L$, and $\omega_{k+1} - \omega_k = \pi/L = \Delta\omega$. The Fourier series can be rewritten as the following, where the dummy variable t has been replaced by u in the equation for c_k to avoid confusion.

$$\begin{aligned} f\,(t) \quad &= \quad \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k t}, \\ c_k \quad &= \quad \frac{1}{2L}\int_{-L}^{L} f\,(t)\, e^{-i\omega_k t}\, dt = \frac{1}{2L}\int_{-L}^{L} f\,(u)\, e^{-i\omega_k u}\, du. \end{aligned}$$

Substituting the expression for c_k into the formula for $f\,(t)$ and using the fact that $\frac{1}{2L} = \frac{\Delta\omega}{2\pi}$, we get

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(u)\, e^{-i\omega_k u}\, du \right] e^{i\omega_k t} \\
&= \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(u)\, e^{-i\omega_k u} e^{i\omega_k t}\, du
\end{aligned}
$$

Now take the limit as L tends to ∞. We obtain

$$
\begin{aligned}
\lim_{L\to\infty} \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(u)\, e^{-i\omega_k u} e^{i\omega_k t}\, du &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u} e^{i\omega t}\, du \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u} e^{i\omega t}\, du\, d\omega \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du \right) e^{i\omega t}\, d\omega
\end{aligned}
$$

Define

$$
\widehat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt,
$$

then it follows that

$$
f(t) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega t}\, d\omega.
$$

In practice, the above Fourier transform is often a starting point for analysis of signals. However, discrete Fourier series are sufficient approximations in many cases. You'll likely encounter a Fourier transform in other courses. See for example 18.103, 18.303, 18.311, 18.353, and 18.354.

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Wow
The derivation is so fxxking cool.
2

Non-periodic signal as the limit as L goes to infinity of a periodic signal of period $2L$
2

? Problem with seeing the equations in the derivation
For some reason, most of the equations (but not all) in the "A derivation of Fourier transform" window do not display correctly in my viewer. Could you please post an image ...
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