

Q and A:

(1) When making superpositions of independent solutions to differential equations of complex functions, should we use complex coefficient or real coefficient?

We should always use complex coefficient to make sure that we get the most general form of solution, although sometimes it does not matter.

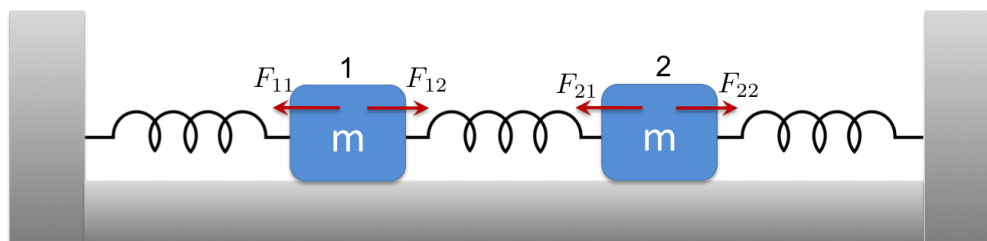
(2) In driven oscillation, what if the driving force is not of the sinusoidal form?

Any force can be decomposed into superposition of sinusoidal forms. The solution can be decomposed correspondingly.

3 Coupled Harmonic Oscillators, Normal Modes

3.1 Two masses coupled by spring

Having studied the oscillation of a single degree of freedom in detail, now let's move on to the discussion of the oscillation of several coupled degree of freedom. As the simplest example, consider two blocks as arranged in the following figure.



Suppose that the two blocks are of the same mass m , the three springs have the same Hooke's constant k . Initially, all three springs are relaxed and the two masses are at equilibrium position x_{10} and x_{20} . Assume no friction or external driving force. If we offset the two masses to a non-equilibrium position, the system is going to start to oscillate and the oscillation of the two masses are going to be coupled. To understand this coupled oscillation, let's write down the EOM.

For mass 1, the EOM is

$$m \frac{d^2 x_1}{dt^2} = -k(x_1 - x_{10}) + k[(x_2 - x_{20}) - (x_1 - x_{10})] \quad (1)$$

For mass 2, the EOM is

$$m \frac{d^2 x_2}{dt^2} = -k(x_2 - x_{10}) - k[(x_2 - x_{20}) - (x_1 - x_{10})] \quad (2)$$

Let's change variables to $\tilde{x}_1 = x_1 - x_{10}$, $\tilde{x}_2 = x_2 - x_{20}$ and write everything in matrix form

$$X = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \quad M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \quad (3)$$

and the EOM becomes

$$M \frac{d^2 X}{dt^2} = -KX \quad (4)$$

which looks very close in form to the EOM for a single mass on a spring, but we should keep in mind that everything is now a matrix. Moving M to the right hand side of the equation (by taking a matrix inverse), we get

$$\frac{d^2 X}{dt^2} = -M^{-1}KX = -\begin{pmatrix} 2k/m & -k/m \\ -k/m & 2k/m \end{pmatrix} X \quad (5)$$

To solve this set of equation, we put it again into complex form

$$\frac{d^2 Z}{dt^2} = -M^{-1}KZ \quad (6)$$

where Z is a two component complex vector. Suppose that $Z(t)$ takes the form of $Z(t) = Ae^{-i\omega t}$, where A is a vector $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ of complex numbers that is independent of time. Substituting this form of solution into the EOM, we get

$$\frac{d^2 Z}{dt^2} = -\omega^2 Ae^{-i\omega t} = -M^{-1}KAe^{-i\omega t} \quad (7)$$

Note that while we can cancel the two $e^{-i\omega t}$ factors on the two sides, we cannot cancel the two A s because they are vectors. The EOM becomes

$$M^{-1}KA = \omega^2 A \quad (8)$$

which is the eigenvalue equation of $M^{-1}K$ with eigenvector A and eigenvalue ω^2 .

Solving this eigenvalue equation, we get

$$\omega^{(1)} = \sqrt{\frac{k}{m}}, \quad \omega^{(2)} = \sqrt{\frac{3k}{m}} \quad (9)$$

and correspondingly

$$A^{(1)} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A^{(2)} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (10)$$

Putting this together, we find two solutions, or two modes, of the EOM. In mode 1, we have

$$Z^{(1)}(t) = a^{(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega^{(1)}t} \quad (11)$$

Taking the real part we get

$$X^{(1)}(t) = \left(\text{Re}(a^{(1)}) \cos(\omega^{(1)}t) - \text{Im}(a^{(1)}) \sin(\omega^{(1)}t) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (12)$$

More specifically,

$$\hat{x}_1^{(1)} = \hat{x}_2^{(1)} = \text{Re}(a^{(1)}) \cos(\omega^{(1)}t) - \text{Im}(a^{(1)}) \sin(\omega^{(1)}t) = |a^{(1)}| \cos(\omega^{(1)}t + \varphi^{(1)}) \quad (13)$$

That is, the two masses oscillate in the same way: the same frequency, the same amplitude, and the same phase. Their motion is the same and the middle spring is not stretched or compressed at all.

In mode 2, we have

$$Z^{(2)}(t) = a^{(2)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega^{(2)}t} \quad (14)$$

Taking the real part we get

$$X^{(2)}(t) = \left(Re(a^{(2)}) \cos(\omega^{(2)}t) - Im(a^{(2)}) \sin(\omega^{(2)}t) \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (15)$$

More specifically,

$$\tilde{x}_1^{(2)} = -\tilde{x}_2^{(2)} = Re(a^{(2)}) \cos(\omega^{(2)}t) - Im(a^{(2)}) \sin(\omega^{(2)}t) = |a^{(2)}| \cos(\omega^{(2)}t + \varphi^{(2)}) \quad (16)$$

That is, the two masses oscillate in the opposite way: the same frequency, the same amplitude, and with a phase difference of π .

The general motion of the two masses is the superposition of these two modes

$$X(t) = |a^{(1)}| \cos(\omega^{(1)}t + \varphi^{(1)}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + |a^{(2)}| \cos(\omega^{(2)}t + \varphi^{(2)}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (17)$$

There are four free parameters in this solution, $Re(a^{(1)})$, $Im(a^{(1)})$, $Re(a^{(2)})$ and $Im(a^{(2)})$ which are determined from the initial conditions $\tilde{x}_1(0)$, $\tilde{x}_1'(0)$, $\tilde{x}_2(0)$, $\tilde{x}_2'(0)$.

Let's consider some special cases of initial conditions and see what kind of motion it starts.

$$(1) \ x_1(0) = x_2(0) = x_0, \ x_1'(0) = x_2'(0) = 0$$

The two blocks are displaced by the same amount and then released. In this case, only mode 1 is excited and we have

$$X(t) = x_0 \cos(\omega^{(1)}t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (18)$$

$$(2) \ x_1(0) = x_2(0) = 0, \ x_1'(0) = x_2'(0) = v_0$$

The two blocks are set to move at the same initial speed, but with no initial displacement. Again, only mode 1 is excited and we have

$$X(t) = \frac{v_0}{\omega^{(1)}} \sin(\omega^{(1)}t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (19)$$

$$(3) \ x_1(0) = -x_2(0) = x_0, \ x_1'(0) = x_2'(0) = 0$$

The two blocks are displaced by the opposite amount and then released. In this case, only mode 2 is excited and we have

$$X(t) = x_0 \cos(\omega^{(2)}t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (20)$$

$$(4) \ x_1(0) = x_0, \ x_2(0) = x'_1(0) = x'_2(0) = 0$$

Even though only mass 1 is displaced, both blocks will move. The whole motion is given by

$$X(t) = \frac{1}{2}x_0 \cos(\omega^{(1)}t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}x_0 \cos(\omega^{(2)}t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (21)$$

That is, both modes are excited, with the same amplitude. The motion of the two blocks is the superposition of two SHO with different frequencies.

From these examples, we see that in a coupled Harmonic oscillator, even though the original degree of freedoms are coupled and move in a complicated way, some transformed degree of freedom ($x_1 + x_2$ and $x_1 - x_2$ in this case) have independent motion and each undergoes a SHO.

3.2 Coupled oscillator with many DOF

Now let's look at how a coupled oscillator with many DOF oscillates. The intuitive picture is basically given by the two DOF example, but we want to be more general about the mathematical setup.

Consider a set of n EOM with n DOF involved.

$$\begin{aligned} m_1 \frac{d^2}{dt^2} x_1 &= F_{11} + F_{12} + F_{1n} \\ &\dots\dots \\ m_n \frac{d^2}{dt^2} x_n &= F_{n1} + F_{n2} + F_{nn} \end{aligned} \quad (22)$$

where F_{ij} is the force acting on the i th particle due to small displacement of the j th particle away from its equilibrium position.

Assume that at equilibrium, $x_1 = x_2 = \dots = x_n = 0$. F_{ij} depends linearly on x_j as

$$F_{ij} = k_{ij}x_j \quad (23)$$

Note that $k_{ij} = k_{ji}$. This is because, suppose that the total potential energy in the system which depends on x_1, \dots, x_n is V . Then we have

$$k_{ij} = -\frac{\partial F_i}{\partial x_j} = -\frac{\partial}{\partial x_j} \left(-\frac{\partial V}{\partial x_i} \right) = \frac{\partial^2 V}{\partial x_j \partial x_i} \quad (24)$$

At the same time, we have

$$k_{ji} = -\frac{\partial F_j}{\partial x_i} = -\frac{\partial}{\partial x_i} \left(-\frac{\partial V}{\partial x_j} \right) = \frac{\partial^2 V}{\partial x_j \partial x_i} \quad (25)$$

Therefore, $k_{ij} = k_{ji}$.

We can write the set of EOM in matrix form as

$$M \frac{d^2}{dt^2} X = KX \quad (26)$$

where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix}$, $K = \begin{pmatrix} k_{11} & \dots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \dots & k_{nn} \end{pmatrix}$.

As long as M is invertible, we can move it to the right hand side and get

$$\frac{d^2}{dt^2}X = M^{-1}KX \quad (27)$$

To solve this equation, we first put it into the complex form

$$\frac{d^2}{dt^2}Z = M^{-1}KZ \quad (28)$$

and then try solutions of the form

$$Z^{(j)} = A^{(j)}e^{i\omega^{(j)}t} \quad (29)$$

$A^{(j)}$ then satisfies the eigen equation

$$M^{-1}KA^{(j)} = (\omega^{(j)})^2 A^{(j)} \quad (30)$$

Is it guaranteed that there are always solutions? If so, how do they look like? To answer this question, we make use of a math theorem:

A real symmetric $n \times n$ matrix has n real eigenvalues (two or more of them could be the same) and correspondingly n real eigenvectors which are orthogonal to each other.

In order to apply this theorem, we need to modify the eigen equation by multiplying $M^{1/2} = \begin{pmatrix} \sqrt{m_1} & & \\ & \ddots & \\ & & \sqrt{m_n} \end{pmatrix}$ from the left on both sides

$$M^{-1/2}KM^{-1/2} \left(M^{1/2}A^{(j)} \right) = (\omega^{(j)})^2 \left(M^{1/2}A^{(j)} \right) \quad (31)$$

As both $M^{1/2}$ and K are symmetric real matrices, $M^{-1/2}KM^{-1/2}$ is real symmetric and there are n solutions with orthogonal eigenvectors. That is, there are n eigenmodes, characterized by eigenfrequency $\omega^{(j)}$ and eigenvector $A^{(j)}$. Note that the $A^{(j)}$ s may not be orthogonal to each other but the $M^{1/2}A^{(j)}$ s are.

$$\left(A^{(k)} \right)^T M A^{(j)} = \delta_{jk} \quad (32)$$

The eigenvalue $(\omega^{(j)})^2$ is real, but it can be positive, zero or negative, corresponding to $\omega^{(j)}$ being real, zero or imaginary. What does it mean?

(1) If $(\omega^{(j)})^2 > 0$, $\omega_\omega^{(j)} = \pm\sqrt{(\omega^{(j)})^2}$ are real numbers. The general solution of the motion is

$$Z(t) = \left[a_+^{(j)} e^{i\omega_+^{(j)}t} + a_-^{(j)} e^{i\omega_-^{(j)}t} \right] A^{(j)} \quad (33)$$

Taking the real part, we get

$$X(t) = \left[\left(\text{Re}(a_+^{(j)}) + \text{Re}(a_-^{(j)}) \right) \cos(\sqrt{(\omega^{(j)})^2}t) + \left(-\text{Im}(a_+^{(j)}) + \text{Im}(a_-^{(j)}) \right) \sin(\sqrt{(\omega^{(j)})^2}t) \right] A^{(j)} \quad (34)$$

which describes oscillatory motion.

(2) If $(\omega^{(j)})^2 = 0$, $\omega^{(j)} = 0$. In this case, we have $Z(t) = A^{(j)}$, which is stationary. Actually, we are missing a solution $Z(t) = tV^{(j)}$, which describes constant velocity motion. So the general solution is given by

$$Z(t) = A^{(j)} + tV^{(j)} \quad (35)$$

(3) If $(\omega^{(j)})^2 < 0$, $\omega_\omega^{(j)} = \pm i\sqrt{-(\omega^{(j)})^2}$ are pure imaginary numbers. The general solution of the motion is

$$Z(t) = \left[a_+^{(j)} e^{-\sqrt{-(\omega^{(j)})^2}t} + a_-^{(j)} e^{\sqrt{-(\omega^{(j)})^2}t} \right] A^{(j)} \quad (36)$$

without loss of generality, we can take a to be real. Then $X(t) = Z(t)$. The first part of the solution describes an exponentially decaying motion while the second part describes an exponentially growing motion, so that after a little while the whole motion is exponentially growing. This corresponds to motion away from an unstable equilibrium point.

Now let's focus on the case with $(\omega^{(j)})^2 > 0$ and take a more careful look at the form of the solution.

$$Z^{(j)} = a^{(j)} A^{(j)} e^{i\omega^{(j)}t} = |a^{(j)}| e^{i\omega^{(j)}t + \varphi^{(j)}} A^{(j)} \quad (37)$$

Here $a^{(j)}$ is a complex number with phase $\varphi^{(j)}$, $A^{(j)}$ is a real vector, and $\omega^{(j)}$ is a real number.

Taking the real part of the solution, we get

$$X^{(j)} = |a^{(j)}| \cos(\omega^{(j)}t + \varphi^{(j)}) A^{(j)} = |a^{(j)}| \cos(\omega^{(j)}t + \varphi^{(j)}) \begin{pmatrix} A_1^{(j)} \\ \vdots \\ A_n^{(j)} \end{pmatrix} \quad (38)$$

That is, in an eigenmode, all the DOF oscillates with the same frequency, the same phase, and with a fixed ratio of amplitude (note that the ratio can be negative corresponding to a π phase shift).

Combining all the eigenmodes, we get the total motion

$$X(t) = \sum_j |a^{(j)}| \cos(\omega^{(j)}t + \varphi^{(j)}) A^{(j)} \quad (39)$$

The free parameters $|a^{(j)}|$ and $\varphi^{(j)}$ are to be determined from the initial condition $x_i(0)$ and $x'_i(0)$. The n normal modes form a linearly independent (not necessarily orthogonal), complete set of basis for the full motion.