2. Review of the nullspace and an introduction to the column space, a geometric example

Let $\mathbf{f}$ be the function from $\mathbb{R}^3$ to $\mathbb{R}^3$ that projects all of $\mathbb{R}^3$ onto the xy-plane:

$$\mathbf{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

1. What is the matrix $\mathbf{A}$ that represents $\mathbf{f}$?
2. Which vectors from the input vector space does $\mathbf{f}$ map to $\mathbf{0}$?
3. Describe the range (or image) of $\mathbf{f}$ geometrically in the output space. Recall that the range of $\mathbf{f}$ is the set of all vectors $\mathbf{b} = \mathbf{f}(\mathbf{x})$ for all vectors $\mathbf{x}$ in the input space.

Solution:

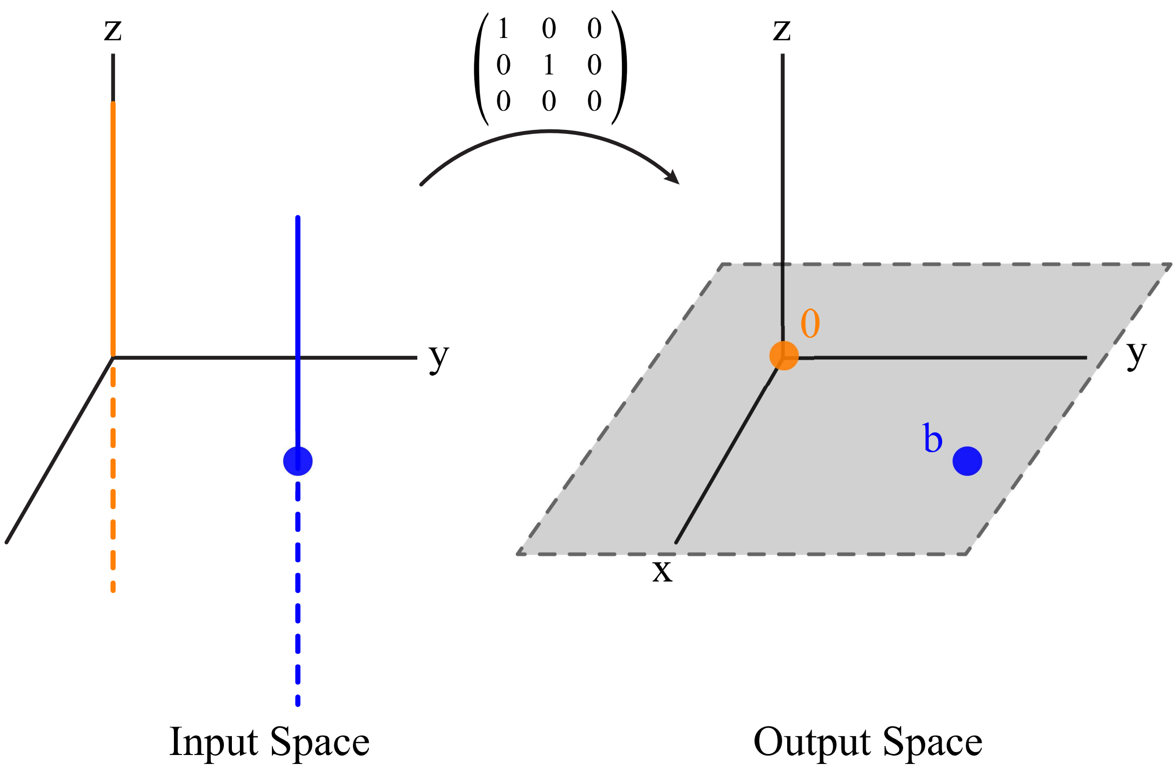
1. $\mathbf{A}$ is a $\mathbf{3} \times \mathbf{3}$ matrix such that

$$\begin{aligned} (\text{first column of } \mathbf{A}) &= \mathbf{f}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ (\text{second column of } \mathbf{A}) &= \mathbf{f}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

$$\text{(third column of } \mathbf{A}) \quad = \mathbf{f}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Thus $\mathbf{A} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$.

2. Note that $\mathbf{f}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\0\end{pmatrix}$ is the zero vector if and only if $x = 0$ and $y = 0$. Thus the projection takes any vector of the form $\begin{pmatrix}0\\0\\z\end{pmatrix}$ to the zero vector. Geometrically, this says that the entire z-axis is sent to the zero vector.



The set of vectors mapped to $\mathbf{0}$ by $\mathbf{f}$ is the same as the set of solutions $\mathbf{Ax} = \mathbf{0}$. Using our algorithm from the last lecture, we find the set of solutions $\mathbf{Ax} = \mathbf{0}$. This set of solutions forms a vector space called the nullspace. The nullspace $\mathrm{NS}(\mathbf{A})$ is a subspace of the **input** space; in this example:

$$\begin{aligned}\mathrm{NS}(\mathbf{A}) &= \{\text{solutions to } \mathbf{A}\mathbf{x} = \mathbf{0}\} \\ &= \text{solutions to } \mathbf{f}(x,y,z) = \mathbf{0} \\ &= \{(0,0,z) : z \in \mathbb{R}\} \\ &= \text{the } z\text{-axis in the input space } \mathbb{R}^3.\end{aligned}$$

3. The set of all vectors $\mathbf{b}$ in the image (also called the range) of $\mathbf{f}$ are all vectors $\mathbf{f}(\mathbf{x})$; in our example:

$$\begin{aligned}\text{Image of } \mathbf{f}(x,y,z) &= \{(x,y,0) : x,y \in \mathbb{R}\} \\ &= \text{the } xy\text{-plane in the output space } \mathbb{R}^3.\end{aligned}$$

Another way of thinking of this is as the set of all vectors $\mathbf{b}$ that can be written as $\mathbf{A}\mathbf{x}$ for some vector $\mathbf{x}$; in our example:

$$\{\text{All vectors } \mathbf{A}\mathbf{x}\} = \text{all linear combinations of the columns of } \mathbf{A}.$$

Therefore the set of all vectors $\mathbf{b} = \mathbf{A}\mathbf{x}$ is the same as the span of the columns of $\mathbf{A}$. The span of the columns of $\mathbf{A}$ is called the **column space** $\mathrm{CS}(\mathbf{A})$; it is a subspace of the **output space** .

---

2. Review of the nullspace and an introduction to the column space, a geometric example

Hide Discussion

Topic: Unit 2: Linear Algebra, Part 2 / 2. Review of the nullspace and an introduction to the column space, a geometric example

Add a Post

Show all posts ▼ by recent activity ▼

There are no posts in this topic yet.

✖

Learn About Verified Certificates