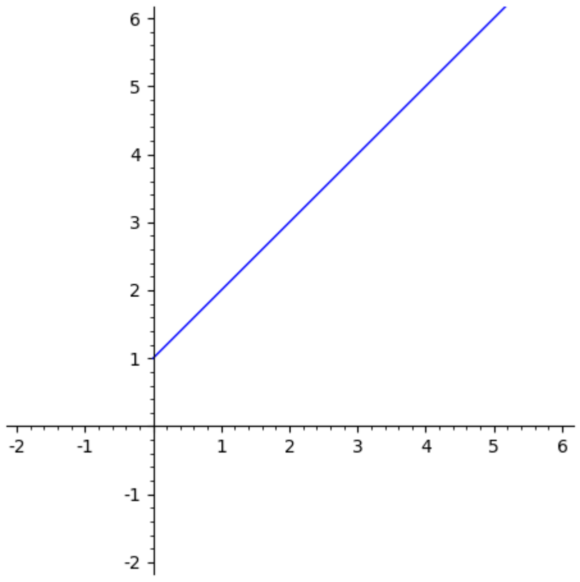
## 2. Example in 2D

Lecture due Oct 5, 2021 20:30 IST

## Explore

Imagine a particle moving through two-dimensional space. We can describe the motion of the particle by specifying the position of the particle at time $t$, where $t$ runs through a set of values. For example, let's imagine a particle whose position at time $t$ is given by the $x, y$-coordinates $(t^2, 1+t^2)$, where $0 \leq t < \infty$.

By plotting a few points, we can see that the particle moves in a straight line from the point $(0,1)$ in the north-eastern direction.




In the image above, we have plotted the particle's **trajectory**, that is, the set of points that the particle goes through. We can imagine letting the tip of a pencil follow the particle around on a piece of paper, which creates this image of the particle's trajectory.

Equations such as $x(t) = t^2$ and $y(t) = 1+t^2$ for $0 \leq t < \infty$ are known as **parametric equations**. The terminology comes from the fact that $x$ and $y$ each depend on the **parameter** $t$.

Here are some questions we will be interested in answering:

1. What is the particle's velocity at time $t$?

2. What is the particle's speed at time $t$?

3. Is there a way to see that $(t^2, 1+t^2)$ describes a straight line without plotting points?

**Straight-line trajectory**

Let's first look at the third question: can we get a better feeling for the motion described by $(t^2, 1+t^2)$? One approach is to use the language of vector arithmetic. We can represent the point $(t^2, 1+t^2)$ by the vector $\begin{pmatrix} t^2 \\ 1+t^2 \end{pmatrix}$. Then, we can separate this vector into the sum of vectors:

$$\begin{pmatrix} t^2 \\ 1+t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.37}$$

This form gives better insight into the motion of the particle. We can see that at $t=0$, the particle will be at the point $(0,1)$, and as $t$ increases, it moves along the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Since the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ doesn't depend on $t$, the trajectory of the particle is indeed a straight line. In fact, the trajectory's line is parallel to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

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**Remark 2.1 (Parameterizing a Line)** In general, the parametric equation $(x(t), y(t)) = \vec{v} + f(t)\vec{w}$ for vectors $\vec{v}, \vec{w}$ and any function $f(t)$, gives a trajectory that is contained within a straight line. The point $\vec{v}$ will be the starting point at $t = 0$, and the vector $\vec{w}$ will be parallel to the trajectory's line.

## Parameterizing a line

1/1 point (graded)
Which of the following parametric equations has a straight-line trajectory?

- [x] $(2+t, 1-t)$
- [ ] $(t^2, 1+t)$
- [x] $(t/3, t/2)$
- [x] $(1+t(t-1), 2t(t-1))$
- [x] $(1-t^2, 1+t^2)$

**Solution:**

In each case, we try to write it in the form $\vec{v} + f(t)\vec{w}$. In this solution, we will highlight $t$ in blue.

First, $(2+t, 1-t)$ can be written as the sum:

$$\begin{pmatrix} 2+t \\ 1-t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.38}$$

Since this matches the form $\vec{v} + f(t)\vec{w}$, this trajectory **is** a straight line.

Second, $(t^2, 1+t)$ cannot be written this way. If we try, the best we can get is:

$$\begin{pmatrix} t^2 \\ 1+t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} t \\ 1 \end{pmatrix} \tag{6.39}$$

The trajectory **is not** a straight line. Over time, the particle will bend towards the horizon.

Third, $(t/3, t/2)$ can be written as:

$$\begin{pmatrix} t/3 \\ t/2 \end{pmatrix} = t\begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix} \tag{6.40}$$

Therefore its trajectory **is** a straight line (here $\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$).

Fourth, $(1+t(t-1), 2t(t-1))$ can be written as:

$$\begin{pmatrix} 1+t(t-1) \\ 2t(t-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t(t-1)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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Therefore its trajectory **is** a straight line. The motion of this particle is a little more interesting: for $t > 0$, it will move along the line, then reverse, and trace out the straight line in the other direction. This happens because $t(t-1)$ switches from decreasing to increasing as $t$ runs through $(0, \infty)$.

Fifth, $(1-t^2, 1+t^2)$ can be written as:

$$\begin{pmatrix} 1-t^2 \\ 1+t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t^2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{6.42}$$

Therefore its trajectory **is** a straight line.

Submit You have used 1 of 4 attempts

Answers are displayed within the problem

**A parametric equation is a vector-valued function**

As in the above example, it is common to think of a pair of parametric equations $(x(t), y(t))$ as a two-dimensional vector that varies with the parameter $t$. It is standard to use the letter $\vec{r}$ to represent this vector.

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{6.43}$$

or sometimes just $\vec{r}$, if the parameter $t$ is clear from context. Technically, $\vec{r}$ is known as a **vector-valued function**, which means it is a function whose output is a vector. The input to the vector-valued function $\vec{r}$ is the parameter $t$.

**Remark 2.2** The notation $\vec{r}$ is used quite often for the position vector at a time $t$, sometimes without explicit comment. It will be important that you can recognize this meaning when you see the letter $\vec{r}$.

A mnemonic: $\vec{r}$ tells you where you **are** ("r").

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