2. Three connected tanks

Three cyclically connected fish tanks

(Caption will be displayed when you start playing the video.)

0:00 / 6:16 2.0x

Video

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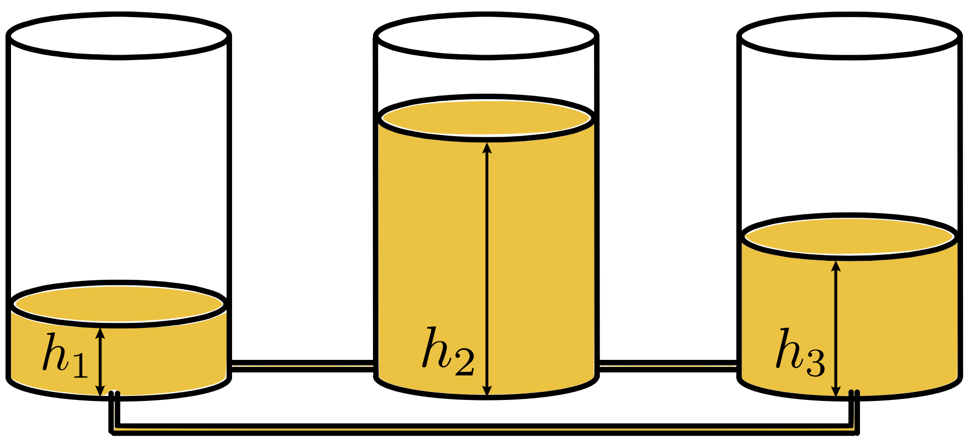
Below we will go through another example which yields exactly the same system of differential equations as in the video.

You've been put in charge of part of a medicine manufacturing plant with three identical cylindrical storage tanks. These tanks are filled with the medicine to different heights and are connected to each other via a pipe at their bottoms, with valves that are initially closed. Our goal is to determine how the fluid will flow once we open the valves. Note that we have solved the analogous problem for two tanks in the course *Differential equations: 2 by 2 systems* previously.

To begin, we name the heights of the fluid in the three tanks h_1, h_2, and h_3 respectively.



Three tanks cyclically connected to each other

Simplifying assumptions

We will make the simplifying assumptions that we can model the flow linearly, and that the flow rate is proportional to the difference in fluid height (as in the case with only 2 tanks). These assumptions are valid when the pipes have a small diameter relative to the overall size of the tanks, the geometry of the pipes and tanks is simple, and the fluid properties of the medicine work out.

Equation for tank 1:

Let us first work out the differential equation describing fluid flow into tank 1. Let V_1 be the volume of fluid in tank 1. The flow rate of fluid into (or out of) tank 1 is by definition $\frac{dV_1}{dt}$.

As mentioned above, we assume that the flow rate is proportional to the difference in heights. The flow rate into tank 1 from tank 2 is $b(h_2 - h_1)$, where b is the constant of proportionality (with dimension $[\text{length}]^2[\text{time}]^{-1}$). Similarly, the flow rate from tank 3 is $c(h_3 - h_1)$, with c the constant of proportionality. The flow rate into tank 1 is therefore:

$$\frac{dV_1}{dt} \quad = \quad b(h_2 - h_1) + c(h_3 - h_1).$$

Note that the constant b is positive because when $h_2 > h_1$, fluid flows into tank 1 so that V_1 is increasing. Similarly, c is positive. We usually set up differential equations so that parameters such as b and c are positive. This way we will be less prone to sign errors.

Since the tank is cylindrical, the rate of change in volume V_1 is given by the cross sectional area A times the rate of change in height h_1 of fluid in tank 1 :

$$\frac{dV_1}{dt} = A\frac{dh_1}{dt}.$$

Therefore, the differential equation in terms of h_1 is

$$\frac{dh_1}{dt} = \frac{1}{A}\frac{dV_1}{dt} = \frac{b}{A}(h_2 - h_1) + \frac{c}{A}(h_3 - h_1).$$

We can combine the constants. Let $a_{21} = \frac{b}{A}$ (with dimension $[\text{time}]^{-1}$), where the subscript 21 indicates that this is the constant governing the flow from tank 2 to tank 1. Similarly, let $a_{31} = \frac{c}{A}$ be the combined constant governing flow from tank 3 and tank 1. The differential equation in terms of the new constants is:

$$\frac{dh_1}{dt} \quad = \quad a_{21}(h_2 - h_1) + a_{31}(h_3 - h_1) \qquad (a_{21},\, a_{31} > 0).$$

Equations for tank 2 and tank 3:

Analogous reasoning as above gives the differential equations for flow rates into tank 2 and tank 3:

$$\frac{dh_2}{dt} = a_{12}(h_1 - h_2) + a_{32}(h_3 - h_2) \qquad (a_{12},\, a_{32} > 0)$$

$$\frac{dh_3}{dt} = a_{13}(h_1 - h_3) + a_{23}(h_2 - h_3) \qquad (a_{13},\, a_{23} > 0).$$

Note that $a_{12} = a_{21}$. For example, if $h_2 > h_1$, then $a_{21}(h_2 - h_1) > 0$ so the flow rate from tank 2 into tank 1 is positive, whereas $a_{12}(h_1 - h_2) < 0$ so that the flow rate from tank 1 to tank 2 is negative. This is consistent because these 2 expressions are equal in magnitude but opposite in sign. Similarly, $a_{13} = a_{31}$, and $a_{23} = a_{32}$.

Matrix form

We can now rewrite this equation in matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} -a_{12} - a_{13} & a_{12} & a_{13} \\ a_{12} & -a_{12} - a_{23} & a_{23} \\ a_{13} & a_{23} & -a_{13} - a_{23} \end{pmatrix}.$$

To make things even nicer, we'll set all constants $a_{ij} = 1$. The resulting equation is

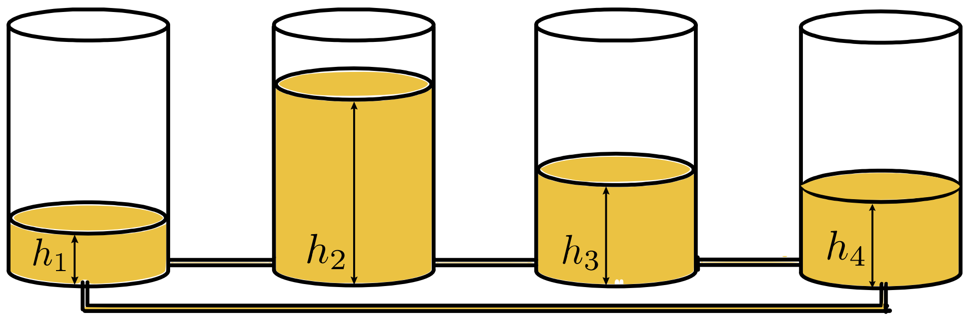
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Symmetry

Show

Four cyclically connected tanks

1/1 point (graded)



Four tanks cyclically connected to each other

Consider the set-up as in the example above but with 4 cyclically-connected tanks instead. As in the example above, assume the flow rate is constant (with dimension $[\text{time}]^{-1}$) along each of the 4 pipes are set to $1,$ and let h_i be the height of fluid in tank i.

Find the matrix $\mathbf{A}$ such that the system of DE describing the fluid flow between the tanks is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}.$$

(Enter **[a,b;c,d]** for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. That is, use commas to separate entries within the same row and **semicolon** to separate rows, and **square bracket** the entire matrix. Similarly for larger matrices.)

$\mathbf{A} =$ [-2,1,0,1;1,-2,1,0;0,1,-2,1;1,0,1,-2]

Answer: [-2,1,0,1;1,-2,1,0;0,1,-2,1;1,0,1,-2]

Solution:

Each tank is connected directly only to the two adjacent tanks. For example, fluid can flow into tank 1 from tanks 2 and 4 only, so the DE for the flow rate into tank 1 is

$$\frac{dh_1}{dt} = a_{21}(h_2 - h_1) + a_{41}(h_4 - h_1) \qquad \text{with } a_{21} = a_{41} = 1.$$

Similarly,

$$\begin{aligned} \frac{dh_2}{dt} &= (h_1 - h_2) + (h_3 - h_2) \\ \frac{dh_3}{dt} &= (h_2 - h_3) + (h_4 - h_3) \\ \frac{dh_4}{dt} &= (h_3 - h_4) + (h_1 - h_4). \end{aligned}$$

Hence, the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}.$$

Put another way, in the 3×3 case:

$$\frac{dh_1}{dt} = \begin{pmatrix} -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

In the 4×4 case, since there is no connection between tanks 1 and 3, $a_{31} = 0,$ and we get

$$\frac{dh_1}{dt} = \begin{pmatrix} -2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}.$$

Similarly, the other equations are determined by the 3×3 case and the fact that $a_{13} = a_{31} = a_{24} = a_{42} = 0$.

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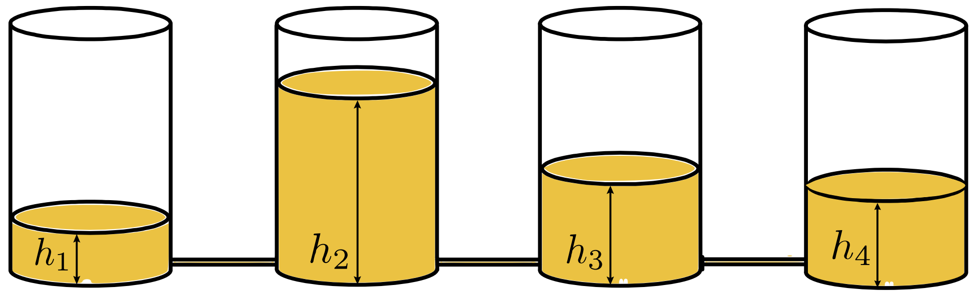
Answers are displayed within the problem

Four linearly connected tanks

1/1 point (graded)



As above, consider four connected fluid tanks. But this time, tank 1 and tank 4 are **not connected**.

As in the example above, assume the flow rate constant (with dimension $[\text{time}]^{-1}$) along each of the 4 pipes are set to 1, and let h_i be the height of fluid in tank i.

Find the matrix $\mathbf{A}$ such that the system of DE describing the fluid flow between the tanks is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}.$$

(Enter **[a,b;c,d]** for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. That is, use commas to separate entries within the same row and **semicolon** to separate rows, and **square bracket** the entire matrix. Similarly for larger matrices.)

$\mathbf{A} =$ [-1,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]

Answer: [-1,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]

Solution:

The only differential equations that are changed are the ones for $\frac{dh_1}{dt}$ and $\frac{dh_4}{dt}$.
In this setup, fluid flows into tank 1 only from tank 2, and hence

$$\frac{dh_1}{dt} = (h_2 - h_1).$$

Similarly, fluid now flows into tank 4 only from tank 3, so

$$\frac{dh_4}{dt} = (h_3 - h_4).$$

The differential equations for $\frac{dh_2}{dt}$ and $\frac{dh_3}{dt}$ remain the same as in the previous problem.
Together these DEs form the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

2. Three connected tanks

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