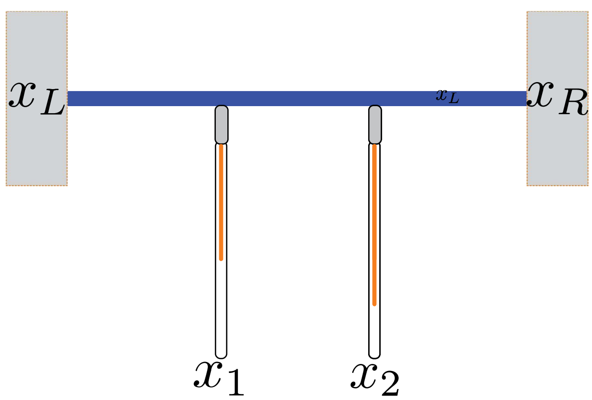
8. Modeling an insulated metal bar heated at the ends

Consider a thermally insulated metal bar with the left and right ends fixed at the temperatures x_L and x_R (in degrees Celsius) respectively. (To maintain the temperatures at the end points, one has to connect the ends to heating or cooling sources.) We are interested in how heat flows through the rod.



To start, assume that the bar is 30 centimeters long and install a thermometer every 10 centimeters. Hence two temperatures $x_1 = x_1(t)$, $x_2 = x_2(t)$ (in degrees Celsius) are recorded as time passes.

What we have done is simplify the model for the temperature along the bar to a discrete model that tracks the temperatures at only two locations.

Newton's law of cooling says that the rate of change of temperature is proportional to the difference in temperatures. The rate of change of the temperature x_1 is affected by the temperatures x_L and x_2 at the adjacent sites. The contribution to $\frac{dx_1}{dt}$ from the left is $k(x_L - x_1)$ where k is a positive constant (with dimension time^{-1}). Note that if $x_L > x_1$,

then $k(x_L - x_1) > 0,$ and heat flows from the left end of the bar towards the measuring site of x_1.
Similarly, the contribution to $\frac{dx_1}{dt}$ from the right is $k(x_2 - x_1)$. Note that we have used the same constant k because we assume the metal bar has the same conductivity and that the location of x_1 is equidistant from x_L and x_2. The equation for $\frac{dx_1}{dt}$ is therefore

$$\frac{dx_1}{dt} = k(x_L - x_1) + k(x_2 - x_1) = k(x_L - 2x_1 + x_2).$$

By the same reasoning, the equation for $\frac{dx_2}{dt}$ is

$$\frac{dx_2}{dt} = k(x_1 - x_2) + k(x_R - x_2) = k(x_1 - 2x_2 + x_R).$$

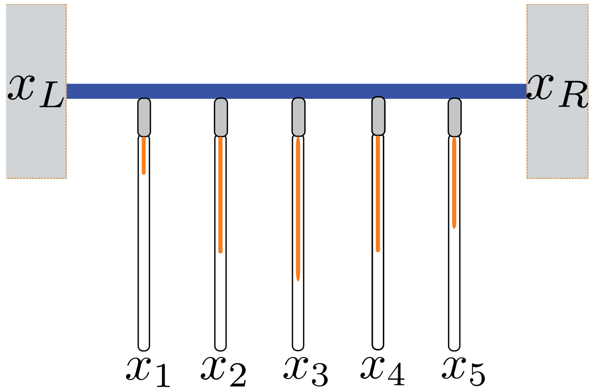
For simplicity, let $k = 1$. In matrix form, the 2×2 inhomogeneous system is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{r} \quad \text{where} \quad & \mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ && \mathbf{A} &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \\ && \mathbf{r} &= \begin{pmatrix} x_L \\ x_R \end{pmatrix}. \end{aligned}$$

More measurements

2/2 points (graded)
To get closer to the continuum, we subdivide the metal bar and instead place 5 thermometers along it at equal distances (5 centimeters) apart.

As above, the two end points are held at fixed temperatures: x_L on the left, and x_R on the right.



For example, the contribution to $\frac{dx_3}{dt}$ from the influence of x_4 is $k(x_4 - x_3)$. (The value of k is different from the example above because the distance between the thermometers are 5 cm instead of 10 cm. In fact, it turns out that this new k is $(10/5)^2 = 4$ times the old k. We will discuss this in the course *Fourier series and partial differential equations*.)

The system of ODEs that describes the rates of change of temperatures is of the form:

$$\dot{\mathbf{x}} = k\left(\mathbf{A}\mathbf{x} + \mathbf{r}\right) \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

Find the matrix $\mathbf{A}$ and the vector $\mathbf{r}$ in the equation above.

(Enter **[a,b;c,d]** for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. That is, use commas to separate entries within the same row and **semicolon** to separate rows, and **square bracket** the entire matrix. Similarly for larger matrices.)

$\mathbf{A} =$ [-2,1,0,0,0;1,-2,1,0,0;0,1,-2,1,0;0,0,1,-2,1;0,0,0,1,-2]

Answer: [-2,1,0,0,0;1,-2,1,0,0;0,1,-2,1,0;0,0,1,-2,1;0,0,0,1,-2]

$\mathbf{r} =$ [x_L;0;0;0;x_R]

Answer: [x_L;0;0;0;x_R]

Solution:

By Newton's cooling, the rate of change of the temperature x_i is affected by the two temperatures to the left and to the right. Therefore, following the reasoning as in the example above, the system is

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = k\left[\underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}}_{\mathbf{A}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \underbrace{\begin{pmatrix} x_L \\ 0 \\ 0 \\ 0 \\ x_R \end{pmatrix}}_{\mathbf{r}}\right].$$

Remark: Note that $\mathbf{r} = \mathbf{r}(t) = \begin{pmatrix} x_L \\ 0 \\ 0 \\ 0 \\ x_R \end{pmatrix}$ can vary with time, and the same model works as in this case where the end points are held at fixed temperatures!

Remark: The symmetric matrix is called the second-order difference matrix because it is a discrete analogue of the 2nd derivative. We will discuss this more in the next course.

Submit You have used 1 of 4 attempts

Answers are displayed within the problem

8. Modeling an insulated metal bar heated at the ends

Hide Discussion

Topic: Unit 3: Solving systems of first order ODEs using matrix methods
/ 8. Modeling an insulated metal bar heated at the ends

Add a Post

All Posts

Answer entry

discussion posted 3 days ago by DavidAOConnor

Are we doing this in terms of k? I assume we can't put the k (Numerical or symbolic) in front based on an error message I receive. x_L vs X_l etc. Running into an issue of the grader seeing 4 terms as well.

This post is visible to everyone.

Add a Response

1 response

jfrench (Staff)
3 days ago

You are correct. That is why we show the k outside the matrix!

It should be x_L and x_R, not X_l and X_r also.

Thanks; got it to work by using the x_L format, and removing k entirely; it should not be present in numerical or symbolic form.

posted 3 days ago by DavidAOConnor

Add a comment

Showing all responses

Add a response:

Preview

Submit

Learn About Verified Certificates