Infinitesimals to the rescue?

One might be tempted to think that there is an easy way out of this difficulty. Perhaps we could reject the assumption that a probability must be a real number between 0 and 1, and instead allow for *infinitesimal* probabilities: probabilities that are greater than zero but smaller than every positive real number.

On the face of it, bringing in infinitesimal values could make all the difference. For suppose we had an infinitesimal value ι with the following property:

$$\underbrace{\iota+\iota+\iota+\iota+\iota+\ldots}_{\text{once for each natural number}}=1$$

Now assign probability ι to each proposition, G_n, that God has selected number n. Doesn't that mean that we will have assigned the same probability to each G_n without having to give up on Countable Additivity?

It is not clear that we have. The problem is not to do with talk of infinitesimals, which were shown to allow for rigorous theorizing by mathematician Abraham Robinson. The problem is to do with what we hope infinitesimals might do for us in the present context. For it is hard to see how they could help us theorize about a uniform probability distributions on a countably infinite set unless infinite sums of infinitesimals are defined using limits, in the usual way:

$$\underbrace{\iota+\iota+\iota+\iota+\iota+\ldots}_{\text{once for each natural number}}=\lim_{n\to\infty} n\cdot\iota$$

But this leads to the conclusion that $\iota+\iota+\iota+\iota+\iota+\ldots$ cannot equal 1, as required in the present context. For on any standard treatment of infinitesimals we have $n\cdot\iota<\frac{1}{m}$ for any positive integers n and m. And from this it follows that $\lim_{n\to\infty} n\cdot\iota$ cannot converge to a positive real number.

What if we were to sidestep standard treatments of infinitesimals altogether? Suppose we introduce a new quantity ι with the double stipulation that (a) ι is greater than zero but smaller than any positive real number, and (b) the result of adding ι to itself countably many times is equal to 1. Even if the details of such a theory could be successfully spelled out, it is not clear that it would put us in a position to assign the same probability to each G_n without giving up on Countable Additivity. For suppose we assume both that $p(G_n) = \iota$, for each n, and that Countable Additivity is in place. Then we are left with the unacceptable conclusion that God's selecting an even number and God's selecting an odd number are both events of probability 1. Here is a proof of one half of that assertion:

$$\begin{aligned} p(G_1 \text{ or } G_3 \text{ or } G_5 \text{ or } \ldots) &= p(G_1) + p(G_3) + p(G_5) \ldots && [\text{Countable Additivity}] \\ &= \underbrace{\iota + \iota + \iota + \iota + \iota + \ldots}_{\text{once for each natural number}} && [p(G_n) = \iota] \\ &= 1 && [\text{stipulation } (b)] \end{aligned}$$

I have not shown that it is impossible to use infinitesimals to construct a probability theory that allows for both Countable Additivity and a uniform probability distribution over a countably infinite set of mutually exclusive propositions. But I hope to have made clear why the prospects of doing so are not as rosy as they might appear.

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