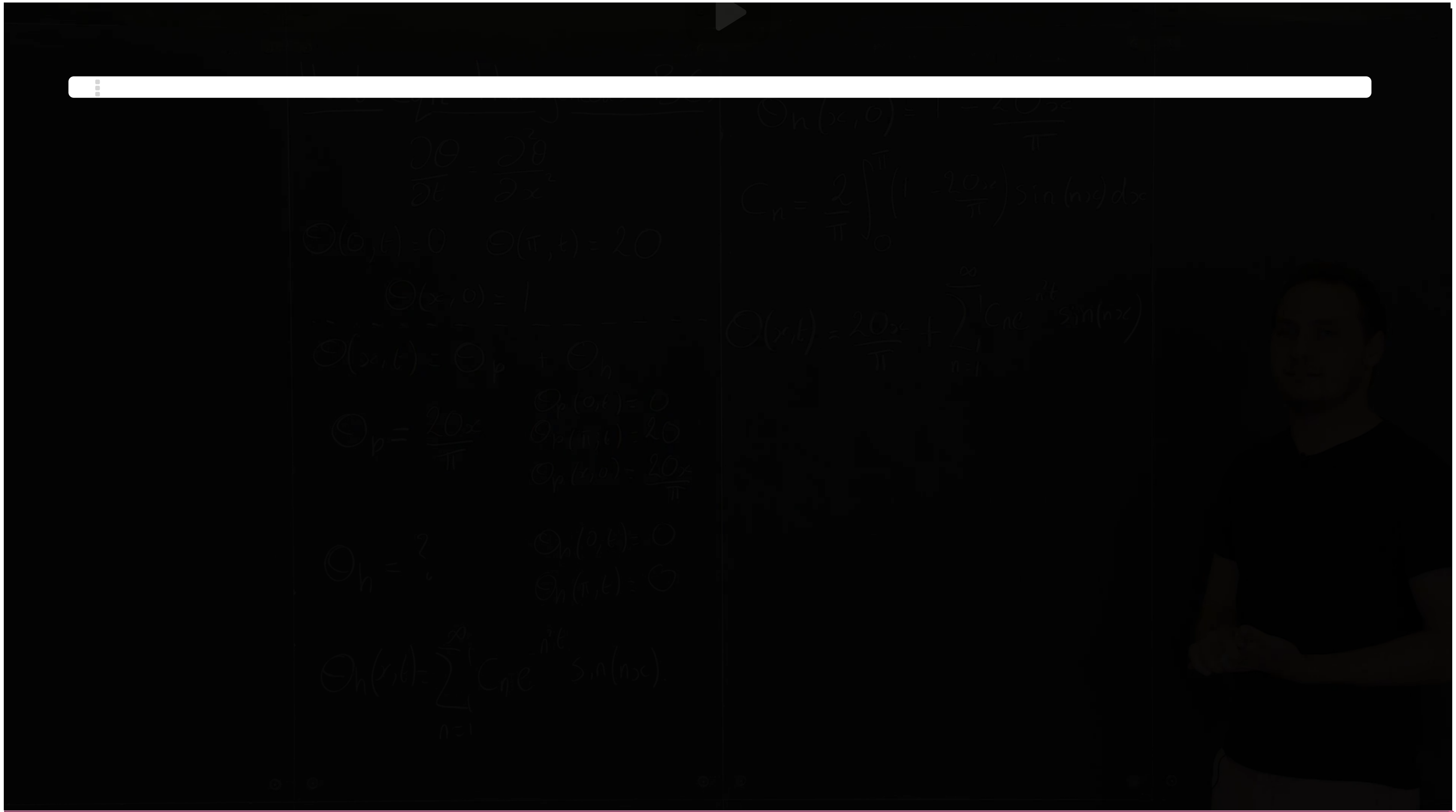
7. Solving the PDE with inhomogeneous boundary conditions

Worked example: inhomogeneous boundary conditions



6:10 / 6:10 1.50x

Video

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Boundary conditions that are not all zero are called **inhomogeneous boundary conditions.**

Steps to solve a linear PDE with inhomogeneous boundary conditions:

1. Find a particular solution θ_p to the PDE with the inhomogeneous boundary conditions (but without initial conditions). If the boundary conditions do not depend on t, try to find the steady-state solution $\theta_p\left(x\right)$, i.e., the solution that does not depend on t.

2. Then $\theta := \theta_p + \theta_h$ is the general solution to the PDE with the inhomogeneous boundary conditions, where θ_h is the general solution to the PDE with the homogeneous boundary conditions.

3. If initial conditions $\theta\left(x,0\right)$ are given, use the initial condition $\theta\left(x,0\right) - \theta_p$ to find the specific solution to the PDE with the inhomogeneous boundary conditions. (This often involves finding Fourier coefficients.)

Problem 7.1 Consider the same insulated uniform metal rod as before ($\nu = 1$, length π, initial temperature $1^\circ\mathrm{C}$), but now suppose that the left end is held at $0^\circ\mathrm{C}$ while the right end is held at $20^\circ\mathrm{C}$. Now what is $\theta\left(x,t\right)$?

Solution:

1. Forget the initial condition for now and look for a solution $\theta_p = \theta_p\left(x\right)$ that does not depend on t. Plugging this into the Heat Equation PDE gives $0 = \frac{\partial^2\theta}{\partial x^2}$. The general solution to this simplified DE is $\theta_p\left(x\right) = ax + b$ Imposing the boundary conditions $\theta_p\left(0\right) = 0$ and $\theta_p\left(\pi\right) = 20$ leads to $b = 0$ and $a = 20/\pi$, so $\theta_p = \frac{20}{\pi}x$.

2. Write $\theta\left(x,t\right) = \theta_p\left(x\right) + \theta_h\left(x,t\right)$. Because our PDE is linear, and both $\theta\left(x,t\right)$ and $\theta_p\left(x\right)$ satisfy the heat equation, it follows that $\theta_h\left(x,t\right)$ also satisfies the heat equation

$$\frac{\partial}{\partial t}\theta_h\left(x,t\right) = \frac{\partial^2}{\partial x^2}\theta_h\left(x,t\right) \qquad 0 < x < \pi.$$

   Moreover, $\theta_h\left(x,t\right)$ has homogeneous boundary conditions

$$\theta_h\left(0,t\right) = 0, \quad \text{and} \quad \theta_h\left(\pi,t\right) = 0, \qquad \text{for } t > 0.$$

3. The PDE with the homogeneous boundary conditions is what we solved earlier; therefore the general solution for θ_h is

$$\theta_h = b_1 e^{-t} \sin x + b_2 e^{-4t} \sin 2x + b_3 e^{-9t} \sin 3x + \cdots.$$

4. The general solution to the PDE with inhomogeneous boundary conditions is

$$\theta(x,t) = \theta_p + \theta_h = \frac{20}{\pi} x + b_1 e^{-t} \sin x + b_2 e^{-4t} \sin 2x + b_3 e^{-9t} \sin 3x + \cdots. \tag{3.46}$$

5. To find the b_n, set $t = 0$ and use the initial condition on the left:

$$1 = \frac{20}{\pi} x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \quad \text{for all } x \in (0,\pi). \tag{3.47}$$

$$1 - \frac{20}{\pi} x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \quad \text{for all } x \in (0,\pi). \tag{3.48}$$

Extend $1 - \frac{20}{\pi} x$ on $(0, \pi)$ to an odd periodic function $f(x)$ of period 2π. Then use the Fourier coefficient formulas to find the b_n such that

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots;$$

alternatively, find the Fourier series for the odd periodic extensions of 1 and x separately, and take a linear combination to get $1 - \frac{20}{\pi} x$. Once the b_n are found, plug them back into the general solution for the heat equation with inhomogeneous boundary conditions.

Find the steady state solution

1/1 point (graded)

Consider the same insulated uniform metal rod as before ($\nu = 1$, length π) but with initial temperature $\theta(x,0) = x^2$

Suppose that the left end is held at $20^\circ\mathrm{C}$ while the right end is held at $20^\circ\mathrm{C}$.

Find the steady state solution $\Theta(x)$.

$\Theta(x) =$ 20 ✔ **Answer:** 20

20

FORMULA INPUT HELP

Solution:

The steady state solution occurs when the entire bar has temperature $20^\circ\mathrm{C}$.

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Answers are displayed within the problem

Find the initial condition and boundary conditions

3/3 points (graded)
The solution is $\theta(x,t) = \Theta(x) + \theta_h(x,t)$ where $\Theta(x)$ is the steady state solution you found in the previous problem. The function $\theta_h(x,t)$ satisfies

$$\frac{\partial}{\partial t}\theta_h(x,t) = \frac{\partial^2}{\partial x}\theta_h(x,t) \qquad 0 < x < \pi.$$

What initial conditions and boundary conditions must $\theta_h(x,t)$ satisfy?

Initial condition:

For $0 < x < \pi,\ \ \theta_h(x,0) =$ x^2-20 ✔ **Answer:** x^2-20

$x^2 - 20$

Boundary conditions:

For $t > 0$, $\theta_h(0,t) =$ 0 ✔ **Answer:** 0

0

For $t > 0$, $\theta_h(\pi,t) =$ 0 ✔ **Answer:** 0

0

FORMULA INPUT HELP

Solution:

First we find the initial condition. We know that

$$\theta(x,0) = \Theta(x) + \theta_h(x,0),$$

therefore

$$x^2 = 20 + \theta_h(x,0), \qquad \text{or} \qquad \theta_h(x,0) = x^2 - 20.$$

Next we solve for the boundary conditions. Since $\theta(0,t) = \theta(\pi,t) = 20$, and $\Theta(x) = 20$

$$\begin{aligned} \theta(0,t) &= \Theta(0) + \theta_h(0,t) \\ 20 &= 20 + \theta_h(0,t) \\ \theta_h(0,t) &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} \theta(\pi,t) &= \Theta(\pi)+\theta_h(\pi,t) \\ 20 &= 20+\theta_h(\pi,t) \\ \theta_h(\pi,t) &= 0 \end{aligned}$$

Thus $\theta_h(x,t)$ must satisfy the homogeneous boundary conditions, which is what we expect.

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TYPO in last problem. 2

Discontinuity at boundaries? 4
When they say theta(x,0) =1, and that the temperature is thus 1 for all x, do they really mean all x except precisely at x = 0 and x=pi, where in the above example the temp. mu...

Why is the separation of variables method not applicable ? 2
At 0:52 of the video, we learn that the method "separation of variables" won't work because the equation has inhomogeneous boundary conditions. I don't remember seeing ...