Binary Notation

Our proof will rely on representing real numbers in binary notation. In particular, we will be using **binary expansions** of the form "$0.b_1 b_2 b_3 \ldots$," where each digit b_i is either a zero or a one. (For instance, $0.00100100001\ldots$)

It will be useful start by saying a few worlds about binary expansions. The way to tell which binary expansion corresponds to which number is to apply the following recipe:

$$\text{the binary name “}0.b_1 b_2 b_3 \ldots\text{” represents the number: } \frac{b_1}{2^1} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \ldots$$

Consider, for example, the binary expansion "$0.001\,(0)$" (i.e. "$0.00100000000\ldots$"). It is a name for the number $1/8$, since according to our recipe:

$$\text{the binary name “}0.001\,(0)\text{” represents the number } \frac{0}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \sum_{n=4}^{\infty}\left(\frac{0}{2^n}\right) = \frac{1}{8}$$

As you'll be asked to prove in the exercises below, every real number in $[0,1]$ is named by some binary expansion of the form "$0.b_1 b_2 b_3 \ldots$,".

It is also worth noting that binary notation shares some important features with decimal notation. In particular, the binary expansions of *rational* numbers are always periodic: they end with an infinitely repeating string of digits. And the binary expansions of *irrational* numbers are never periodic. Also, some real numbers are named by more than one binary expansion. For instance, $1/8$ is represented not just by "$0.001\,(0)$", but also by "$0.000\,(1)$", since

$$\text{the binary name "0.000(1)" represents the number } \frac{0}{2^1} + \frac{0}{2^2} + \frac{0}{2^3} + \sum_{n=4}^{\infty}\left(\frac{1}{2^n}\right) = \frac{1}{8}$$

Fortunately, the only real numbers with multiple names are those named by binary expansions that end in an infinite sequence of 1s. Such numbers always have exactly two binary names: one ending in an infinite sequence of 1s and one ending in an infinite sequence of 0s. Since only rational numbers have periodic binary expansions, this means that there are only countably many real numbers with more than one binary name.

Problem 1

1/1 point (ungraded)
Identify the two binary expansions of $11/16$.

- [x] $0.1011\,(0)$
- [ ] $0.1011\,(1)$
- [x] $0.1010\,(1)$
- [ ] $0.1010\,(0)$
- [ ] $0.1000\,(1)$
- [ ] $0.1111\,(0)$

✔

Explanation
The two binary expansions of $11/16$ are $0.1011\,(0)$ and $0.1010\,(1)$.
This can be verified as follows:

$$\begin{aligned} 0.1011\,(0) &= \frac{1}{2^1}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{1}{2^4} \\ &= \frac{8}{16}+\frac{2}{16}+\frac{1}{16} \\ &= \frac{11}{16} \end{aligned}$$

$$\begin{aligned} 0.1010\,(1) &= \frac{1}{2^1}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{0}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\dots \\ &= \frac{1}{2^1}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{0}{2^4}+\frac{1}{2^4} \\ &= \frac{8}{16}+\frac{2}{16}+\frac{1}{16} \\ &= \frac{11}{16} \end{aligned}$$

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Answers are displayed within the problem

Problem 2

1/1 point (ungraded)

Does every real number in $[0,1]$ have a name in binary notation?

Yes ✔ **Answer:** Yes

(If the answer is yes, try to prove it. If the answer is no, find a counterexample.)

Explanation

The answer is yes. Here's the proof.

Let r be a real number in $[0,1]$. We will verify that r is named by the binary expression "$0.b_1b_2b_3\dots$", whose digits are defined as follows

$$\begin{aligned} b_1 &= \begin{cases} 0, & \text{if } r < \frac{1}{2} \\ 1, & \text{otherwise} \end{cases} \\ b_{k+1} &= \begin{cases} 0, & \text{if } r - 0.b_1b_2\dots b_k < \frac{1}{2^{k+1}} \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

These definitions guarantee that $r - 0.b_0\dots b_n < \frac{1}{2^n}$, for each n. From this it follows that the difference between r and $0.b_1b_2b_3\dots$ must be smaller than any positive real number, and therefore equal to 0. So $r = 0.b_1b_2b_3\dots$.

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