MITx: 6.041x Introduction to Probability - The Science of Uncertainty

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Problem 2: Laptop failures

(4/5 points)

Suppose that you have two laptops, both of which you begin using at time 0. Each laptop will eventually fail, and we model each one's lifetime as exponentially distributed with the same parameter λ. The lifetimes of the two laptops are independent. One of the laptops will fail first, followed by the other. Define T_1 as the time of the first failure and T_2 as the time of the second failure.

In parts 1, 2, 4, and 5 below, your answers will be algebraic expressions. Enter 'lambda' for λ and use 'exp()' for exponentials. Follow standard notation .

1. Determine the PDF of T_1.

   For $t > 0,\ f_{T_1}(t) =$ 2*lambda*exp(-2*lambda*t)

   Answer: 2*lambda*exp(-2*lambda*t)

2. Let $X = T_2 - T_1$. Determine the conditional PDF $f_{X|T_1}(x \mid t)$.

   For $x, t > 0,\ f_{X|T_1}(x \mid t) =$ lambda*exp(-lambda*x)

Answer: lambda*exp(-lambda*x)

3. Is X independent of T_1?

Yes, they are independent ▾ ✔ **Answer:** Yes, they are independent

4. Determine the PDF $f_{T_2}(t)$.

For $t > 0,\ f_{T_2}(t) =$ (2*lambda/3)*exp(-(2*lambda/3)*t) ✖

Answer: 2*lambda*(exp(-lambda*t)-exp(-2*lambda*t))

5.

$\mathbf{E}[T_2] =$ 3/(2*lambda) ✔ **Answer:** 1.5/lambda

Answer:

1. Let M_1 be the lifetime of laptop 1 and M_2 the lifetime of laptop 2, where M_1 and M_2 are i.i.d. exponential random variables distributed according to the same CDF $F_M(m) = 1 - e^{-\lambda m}$ for $m \geq 0$. T_1, the time of the first failure, is the minimum of M_1 and M_2. We first find the CDF $F_{T_1}(t)$ and then differentiate to find the PDF $f_{T_1}(t)$. For $t \geq 0$,

$$\begin{aligned} F_{T_1}(t) &= \mathbf{P}(\min(M_1, M_2) \leq t) \\ &= 1 - \mathbf{P}(\min(M_1, M_2) > t) \\ &= 1 - \mathbf{P}(M_1 > t)\mathbf{P}(M_2 > t) \end{aligned}$$

$$= 1 - (1 - F_{M_1}(t))(1 - F_{M_2}(t))$$
$$= 1 - e^{-2\lambda t}.$$

Differentiating $F_{T_1}(t)$ with respect to t yields

$$f_{T_1}(t) = 2\lambda e^{-2\lambda t}, \qquad t \geq 0.$$

Note that this is the PDF of an exponential random variable with parameter 2λ.

For an alternative approach, we consider 2 independent Poisson processes, each with rate λ. We can then interpret M_1 as the first arrival time in process 1 and M_2 as the first arrival time in process 2. If we merge the two processes, the first arrival time in the merged process corresponds precisely to T_1. Since the merged process has rate 2λ, T_1, an interarrival time, is exponentially distributed with parameter 2λ.

2. Conditioned on the time of the first failure, the time remaining until the second failure is an exponential random variable with parameter λ by the memorylessness property. (The memorylessness property tells us that regardless of the elapsed lifetime of the surviving laptop, the time remaining until its failure has the same exponential distribution.) Consequently, for $t > 0$,

$$f_{X|T_1}(x \mid t) = f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

3. By the memorylessness property mentioned in part 2, X and T_1 are independent.

4. The time of the laptop failure T_2 is equal to $T_1 + X$. Since X and T_1 were shown to be independent in part 2, we convolve the densities found in parts 1 and 2 to determine $f_{T_2}(t)$. For $t \geq 0$,

$$\begin{aligned} f_{T_2}(t) &= \int_{-\infty}^{\infty} f_{T_1}(\tau) f_X(t-\tau)\, d\tau \\ &= \int_0^t \left(2\lambda e^{-2\lambda\tau}\right)\left(\lambda e^{-\lambda(t-\tau)}\right)\, d\tau \\ &= 2\lambda e^{-\lambda t}\int_0^t \lambda e^{-\lambda\tau}\, d\tau \\ &= 2\lambda e^{-\lambda t}(1-e^{-\lambda t}). \end{aligned}$$

An alternative method for solving this problem is to note that T_2 is the maximum of M_1 and M_2 and to derive the distribution of T_2 using our standard CDF to PDF method. For $t \geq 0$,

$$\begin{aligned} F_{T_2}(t) &= \mathbf{P}(\max(M_1, M_2) \leq t) \\ &= \mathbf{P}(M_1 \leq t)\mathbf{P}(M_2 \leq t) \\ &= F_{M_1}(t) F_{M_2}(t) \\ &= \left(1-e^{-\lambda t}\right)^2 \\ &= 1 - 2e^{-\lambda t} + e^{-2\lambda t}. \end{aligned}$$

Differentiating $F_{T_2}(t)$ with respect to t yields

$$f_{T_2}(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t},$$

which is equivalent to our solution by convolution above.

5. From part 1, we know that T_1 is exponential with parameter 2λ, and so $\mathbf{E}[T_1] = \frac{1}{2\lambda}$.
From part 2, we know that X is exponential with parameter λ, and so $\mathbf{E}[X] = \frac{1}{\lambda}$.
Hence, by the linearity of expectation, we have that
$\mathbf{E}[T_2] = \mathbf{E}[T_1] + \mathbf{E}[X] = \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$.

You have used 2 of 3 submissions

DISCUSSION

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