8. General solution in terms of fundamental matrix

Video note: The new content starts at 3:55, but we include the introductory remarks as review. (Be careful not to confuse the constants c_1 and c_2 with the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ in the video below.)

General solution in terms of fundamental matrix

(Caption will be displayed when you start playing the video.)

9:40 / 9:40 2.0x HD

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Suppose $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are n linearly independent solutions to the $n \times n$ system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, then the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n = \begin{pmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Conclusion: If $\mathbf{X}(t)$ is a fundamental matrix, then the general solution is the product

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c},$$

where $\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ ranges over all constant vectors.

Question 8.1 What do all fundamental matrices look like?

Recall that

1. each column of a fundamental matrix is a solution,
2. given a fundamental matrix $\mathbf{X},$ any solution is of the form $\mathbf{Xc}$ for some constant vector $\mathbf{c}.$

This means that all other fundamental matrices must be of the following form:

$$\begin{pmatrix} | & | & & | \\ \mathbf{Xc}_1 & \mathbf{Xc}_2 & \cdots & \mathbf{Xc}_n \\ | & | & & | \end{pmatrix} = \mathbf{XC}$$

where $\mathbf{C}$ is the $n \times n$ matrix whose columns are the vectors c_i:

$$\mathbf{C} = \begin{pmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \\ | & | & & | \end{pmatrix}$$

and to ensure that the columns $\mathbf{Xc}_i$ are linearly independent, we need $\mathbf{C}$ to be invertible, i.e. $|\mathbf{C}| \neq \mathbf{0}.$

Conclusion: If $\mathbf{X}(t)$ is a fundamental matrix, then all other fundamental matrices are of the form

$$\mathbf{XC} \qquad \text{where } \mathbf{C} \text{ is } n \times n, \text{ and } |\mathbf{C}| \neq 0.$$

Remark 8.2 For the system on the previous page, $\dot{\mathbf{x}} = \mathbf{Ax}$ where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, the general solution is

$$\begin{aligned} \mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \qquad \text{where} \quad \mathbf{x}_1 &= e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \mathbf{x}_2 &= e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ are the normal modes.} \end{aligned}$$

This can be written in terms of the fundamental matrix $\mathbf{X} = (\mathbf{x}_1 \quad \mathbf{x}_2)$, as

$$\mathbf{x}(t) \quad = \quad \mathbf{Xc} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Most other fundamental matrices are “ugly." For example, here is another one: $(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2)$. We will use the simplest one, the one built from the normal modes, as much as we can.

Initial conditions

Once we have a fundamental matrix $\mathbf{X}(t)$, finding the solution $\mathbf{x}(t)$ with initial value $\mathbf{x}(a)$ at $t = a$ means finding the vector $\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ satisfying

$$\mathbf{X}(a)\,\mathbf{c} \quad = \quad \mathbf{x}(a).$$

Since $\mathbf{X}(a)$ is invertible for any a, we have

$$\mathbf{c} \quad = \quad \mathbf{X}(a)^{-1}\mathbf{x}(a).$$

Example 8.3 For the same system as above, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, use the simplest fundamental matrix $\mathbf{X}$

$$\mathbf{X}(t) = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}.$$

to find the solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Solution:

The solution to the initial value problem is $\mathbf{X}(t)\,\mathbf{c}$ for some constant vector $\mathbf{c}$. Thus

$$\mathbf{x} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

for some c_1, c_2 to be determined. Set $t = 0$ and use the initial condition to get

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Solving leads to $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$. Therefore,

$$\mathbf{x} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} \begin{pmatrix} -1 \\ 6 \end{pmatrix} = (-1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + 6\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Initial condition practice

2/2 points (graded)

In the example above, we found the general solution in terms of the fundamental matrix

$$\mathbf{X} = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Find coefficients c_1 and c_2 for the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$c_1 =$ 1 ✔ **Answer:** 1

$c_2 =$ -1 ✔ **Answer:** -1

Solution:

The initial condition gives the equation

$$\begin{aligned} \mathbf{x}(0) &= \mathbf{X}(0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

We could solve using $\mathbf{c} = X(0)^{-1}\mathbf{x}(0)$, but in this case, inspection shows us that $c_1 = 1$ and $c_2 = -1$.

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Answers are displayed within the problem

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