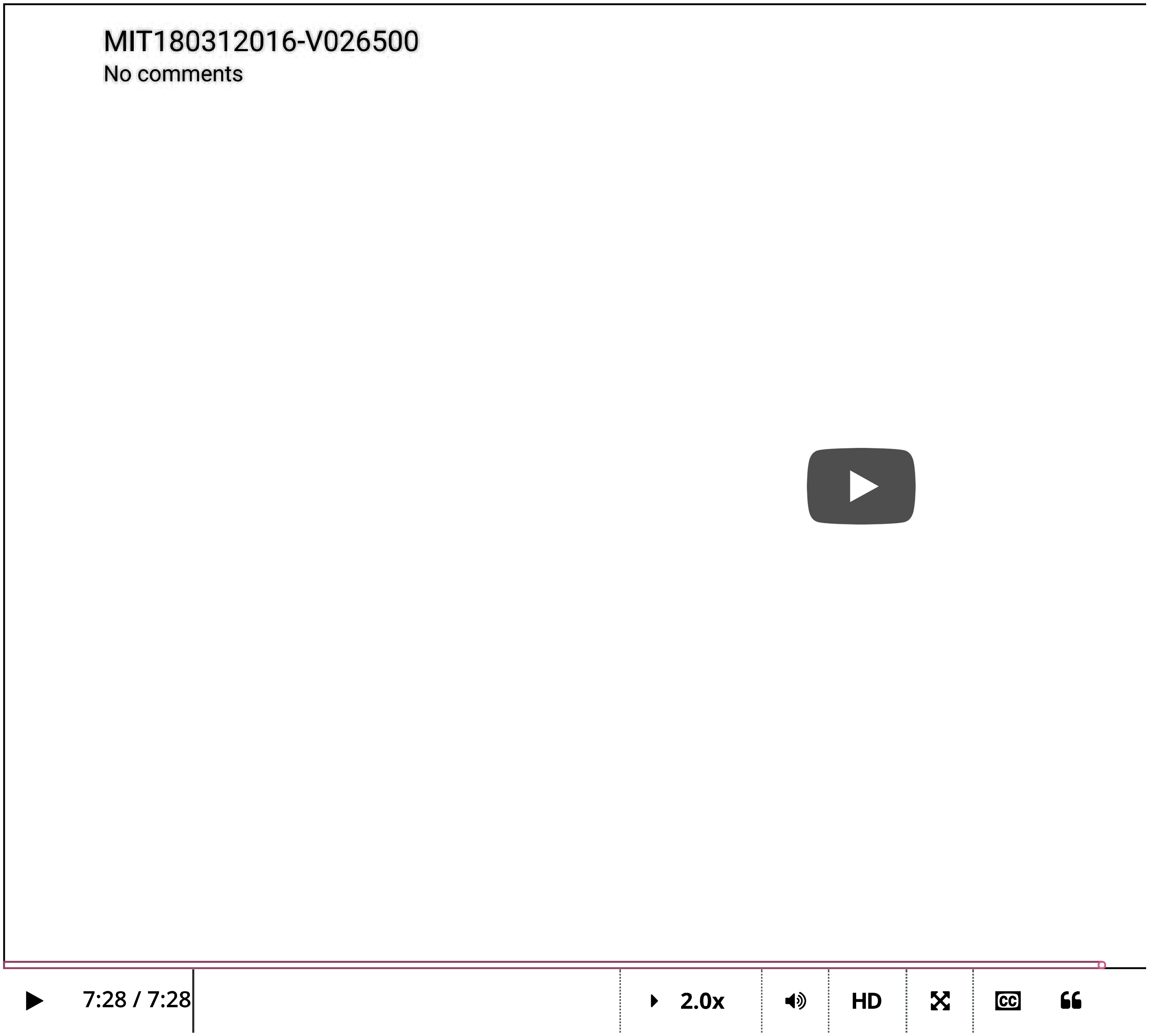
4. A mixing problem

Modeling a mixing problem



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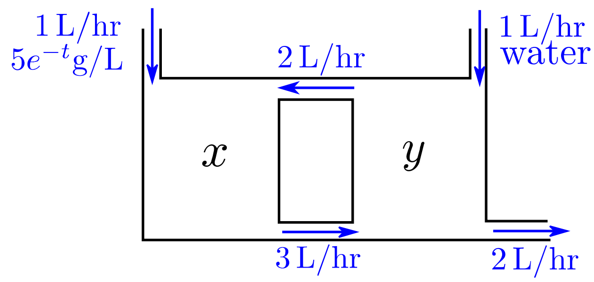
Review: solve the associated homogeneous system

1/1 point (graded)

Here is the same system as in the video above.



Two connected 1-liter tanks with salt solution.

Let x (in grams) and y (in grams) be the amounts of salt in the left and right tanks respectively.

A salt solution of concentration $5e^{-t}$ grams/liter is mixed into the system through a pipe connected to the left tank at a flow rate of 1 liters/hour. In addition, water flows into the right tank through a top pipe at the rate of 1 liters/hour.

The salt solution in the left tank then flows through the bottom pipe to right tank at 3 liters/hour, and the solution in the right tank flows through the top pipe to the left tank at 2 liters/hour.

The salt solution inside the right tank also flows out through a bottom pipe at the rate of 2 liters/hour.

Assume that the salt is instantaneously mixed.

In general, let V_1 be the volume of the left tank (in liters), and V_2 be the volume of the right tank (in liters). The equation describing the rate of change of amounts of salt in the left tank is

$$\dot{x}[\mathrm{g/hr}] \;=\; -3[\mathrm{L/hr}]\frac{x[\mathrm{g}]}{V_1[\mathrm{L}]} + 2[\mathrm{L/hr}]\frac{y[\mathrm{g}]}{V_2[\mathrm{L}]} + (1[\mathrm{L/hr}])\, 5e^{-t}[\mathrm{g/L}]$$

For our system, $V_1 = V_2 = 1$, and hence the system of equations that describes the rate of change of the amount of salt in the tanks is

$$\begin{aligned} \dot{x} &= -3x + 2y + 5e^{-t} \\ \dot{y} &= 3x - 4y. \end{aligned}$$

Which of the following matrix products is a fundamental matrix for the associated homogeneous system?
(Choose all that apply.)

- [ ] $\begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}$
- [ ] $\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{pmatrix}$
- [ ] $\begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{pmatrix}$
- [x] $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{pmatrix}$ ✔
- [ ] $\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$
- [x] $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ ✔
- [ ] $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

✔

Solution:

The associated homogeneous system in matrix form is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}$$

(The homogeneous system corresponds to the case when only water, no salt, is flowing through both incoming pipes.) The eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial:

$$(\lambda+3)(\lambda+4)-6 \;=\; \lambda^2+7\lambda+6 \;=\; (\lambda+6)(\lambda+1) \Longrightarrow \lambda_1=-1,\, \lambda_2=\,-6.$$

By inspection, an eigenvector of -1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and an eigenvalue of -6 is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. These give the two normal modes:

$$e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad e^{-6t}\begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Hence, a fundamental matrix, with the two normal modes as columns, is

$$\mathbf{X}(t) \;=\; \begin{pmatrix} e^{-t} & -2e^{-6t} \\ e^{-t} & 3e^{-6t} \end{pmatrix}.$$

This factors into

$$\mathbf{X}(t) \;=\; \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{pmatrix} \qquad \text{(the 4th choice).}$$

where the first matrix is the matrix whose columns are eigenvectors, and the second is the diagonal matrix with $e^{\lambda t}$ in the corresponding diagonal entries.

All other fundamental matrices can be written in the form $\mathbf{XC}$ where $\mathbf{C}$ is any 2×2 invertible, constant matrix. So among the last 2 choices the first one is also a fundamental matrix, but the second one isn't, it is not invertible.

All other choices are not fundamental matrices because some of their columns are not solutions to the system. You can check by multiplying the matrices.

Remark: Recall another way to describe the system is to use a phase portrait. Since the two eigenvalues are distinct, real and negative, the phase portrait of the homogeneous system is a stable node.

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You have used 1 of 3 attempts

Answers are displayed within the problem

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? what happened to the 1L/h that enters tank y?
The equation for the y tank seems to be missing the water intake. Am I reading this wrong?
4

A minor typo in "Review: solve the associated homogeneous system"
"The salt solution in the left **tanks** then flows" should be "The salt solution in the left **tank** then ...
2
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