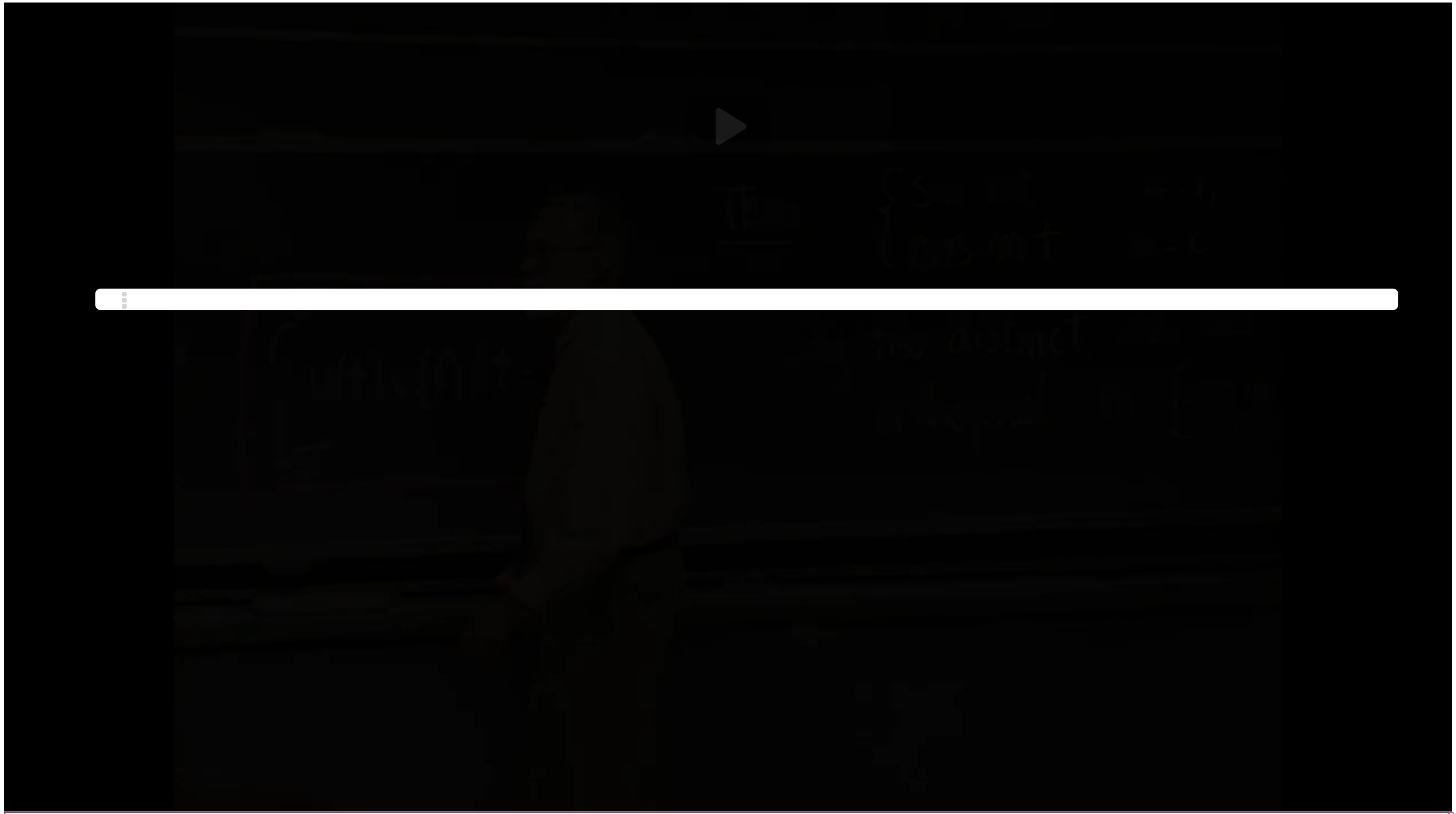
6. Connecting to linear algebra

Orthogonality



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If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then

$$\mathbf{v}\cdot\mathbf{w} := \sum_{k=1}^{n} v_k w_k,$$

where v_k and w_k are the kth components of the vectors $\mathbf{v}$ and $\mathbf{w}$ respectively. Can one define the dot product of two functions? Sort of.

If we think of vectors $\mathbf{v}$ and $\mathbf{w}$ as functions from the finite sets

$$\begin{aligned} \mathbf{v} : \{-\pi + 2\pi/n, -\pi + 4\pi/n, \ldots, -\pi + 2n\pi/n = \pi\} &\quad\rightarrow\quad \mathbb{R} \\ \mathbf{w} : \{-\pi + 2\pi/n, -\pi + 4\pi/n, \ldots, -\pi + 2n\pi/n = \pi\} &\quad\rightarrow\quad \mathbb{R}, \end{aligned}$$

then the inner product of these functions can be written as $\langle \mathbf{v}, \mathbf{w}\rangle$, where

$$\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{k=1}^{n} \mathbf{v}\left(-\pi + 2k\pi/n\right) \mathbf{w}\left(-\pi + 2k\pi/n\right).$$

When we take the continuous limit as $n \to \infty$, $\mathbf{v}$ and $\mathbf{w}$ become continuous, real valued functions defined on $(-\pi, \pi]$, and then the infinite sum becomes an integral.

Definition 6.1 If $u\left(t\right)$ and $v\left(t\right)$ are real-valued periodic functions with period 2π, then we will define their **inner product** as

$$\langle u\left(t\right), v\left(t\right)\rangle := \int_{-\pi}^{\pi} u\left(t\right) v\left(t\right)\ dt$$

This is the continuous analogue of the dot product for vectors. (*Sometimes the inner product is defined on different intervals for functions with different periods. In other definitions, for convenience there is a constant multiplier, which is a normalizing factor.*)

Definition 6.2 We say that two 2π-periodic functions $u\left(t\right)$ and $v\left(t\right)$ are **orthogonal** if

$$\int_{-\pi}^{\pi} u(t)\, v(t)\ dt = 0.$$

Example 6.3 For example,

$$\langle 1, \cos t \rangle = \int_{-\pi}^{\pi} \cos t \, dt = 0.$$

Thus the functions 1 and $\cos t$ are orthogonal.

Compute the inner product, 1

1/1 point (graded)

Compute $\langle 1, \sin t \rangle$

$\langle 1, \sin t \rangle =$ 0 ✔ **Answer:** 0

Solution:

Because $\sin t$ is an odd function,

$$\langle 1, \sin t \rangle = \int_{-\pi}^{\pi} \sin t \, dt = 0.$$

Therefore the functions 1 and $\sin t$ are orthogonal.

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Answers are displayed within the problem

Compute the inner product, 2

0 points possible (ungraded)
Compute $\langle \cos t, \sin t \rangle$.

$\langle \cos t, \sin t \rangle =$ 0 Answer: 0

Solution:

$$\langle \cos t, \sin t \rangle = \int_{-\pi}^{\pi} \cos t \sin t \, dt = \int_{-\pi}^{\pi} \frac{1}{2} \sin 2t \, dt = 0.$$

Because $\sin 2t$ is an odd function on the interval $[-\pi, \pi]$, this integral is zero. Therefore the functions $\cos t$ and $\sin t$ are orthogonal.

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Answers are displayed within the problem

Compute the inner product, 3

0 points possible (ungraded)
Compute $\langle \cos nt, \sin nt \rangle$.

$\langle \cos nt, \sin nt \rangle =$ 0 Answer: 0

Solution:

$$\langle \cos nt, \sin nt \rangle = \int_{-\pi}^{\pi} \cos nt \sin nt \, dt = \int_{-\pi}^{\pi} \frac{1}{2} \sin 2nt \, dt = 0.$$

Because $\sin 2nt$ is an odd function on the interval $[-\pi, \pi]$, this integral is zero. Therefore the functions $\cos nt$ and $\sin nt$ are orthogonal.

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You have used 1 of 4 attempts

Answers are displayed within the problem

Fourier's theorem (from the previous page) tells us that $\{1, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots \}$ is a basis. So "every" 2π-periodic function can be represented as a Fourier series. The other incredibly useful property is that they are **orthogonal** . Orthogonality is what helps us answer our question on finding formulas for the Fourier coefficients.

Question 6.4 Is $\{1, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots \}$ an ortho***normal*** basis for all 2π-periodic functions, which can be represented by a Fourier series?

Answer

No, since $\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \, dt = 2\pi \neq 1.$

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Example 6.5

$$\begin{aligned} \langle \sin t, \sin t \rangle &= \int_{-\pi}^{\pi} \sin^2 t \, dt = \; ? \\ \langle \cos t, \cos t \rangle &= \int_{-\pi}^{\pi} \cos^2 t \, dt = \; ? \end{aligned}$$

Since $\cos t$ is just a shift of $\sin t$, the answers are going to be the same. Also, the two answers add up to

$$\int_{-\pi}^{\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{\text{this is } 1}\, dt = 2\pi,$$

so each is π.

The same idea works to show that

$$\boxed{\langle \cos nt, \cos nt \rangle = \pi} \qquad \text{and} \qquad \boxed{\langle \sin nt, \sin nt \rangle = \pi}$$

for each positive integer n.

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- Why do we consider the function's domain as [-pi,pi]? 8
  Couldn't we also use [0,2*pi] or any other continuous subset of size 2*pi and still end up with the same equations for coefficients? What's special about considering -pi and pi?
- ? Orthogonal basis: are all the elements pairwise orthogonal? 3
- ? About Question 6.4 3
  Please, can you explain how does the question is related to ⟨1,1⟩?
- [typo] missing word 2

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