Math 230, Fall 2012: HW 8 Solutions

Problem 1 (p.309 #5)**.** SOLUTION. Consider finding the cdf of X^2 first. Let $Y = X^2$. Since $-1 \le X \le 2$, $0 \le Y = X^2 < 4$. The cumulative distribution function of Y, $F_Y(y)$, is defined as

$$\begin{aligned} F_Y(y) &= P(Y < y) \\ &= P(X^2 < y) \\ &= P(-\sqrt{y} < X < \sqrt{y}) \end{aligned}$$

Now, if $-\sqrt{y} > -1$ and $\sqrt{y} < 1$, that is, if $0 \le y \le 1$, then this probability is simply

$$P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \frac{2\sqrt{y}}{3}$$

If $-\sqrt{y} < -1$, i.e. if $1 \le y \le 4$, then

$$P(Y \le y) = P(-1 \le X \le \sqrt{y}) = \frac{\sqrt{y} - (-1)}{3} = \frac{\sqrt{y} + 1}{3}.$$

Differentiating these expressions gives the pdf of $Y = X^2$

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & 0 \le y < 1 \\ \frac{1}{6\sqrt{y}} & 1 \le y < 4 \\ 0 & \text{else} \end{cases}$$

Problem 2 (p. 310 #6)**.** SOLUTION. Again, we can find the density by first finding the cumulative distribution function. Let $F_Y(y)$ be the cdf of the y-coordinate of the intersection between the point and the line $x = 1$. It helps to draw a picture and see what values of θ result in a y-coordinate less than some number y. Observe that

$$\tan\theta = \frac{y}{1}$$

because the intersection of the ray at angle θ and the vertical line $x = 1$ results in a right triangle with three vertices: $(0,0), (1,0), and (1,y)$.

$$\begin{aligned} F_Y(y) &= P(Y < y) \\ &= P\left(-\frac{\pi}{2} \le \theta \le \tan^{-1}(y)\right) \\ &= \frac{\tan^{-1}(y) - \frac{\pi}{2}}{\pi} \end{aligned}$$

Differentiating this with respect to y gives

$$f_Y(y) = \frac{1}{\pi(1+y^2)}.$$

Why? By implicit differentiation: If $g(y) = \tan^{-1}(y)$ then $y = \tan(g(y))$ and differentiating gives $1 = \sec^2(g(y)) \cdot g'(y)$ by the chain rule. Dividing the identity $\cos^2 y + \sin^2 y = 1$ by $\cos^2 y$ gives $1 + \tan^2 y = \sec^2 y$, so

$$g'(y) = \frac{1}{1+\tan^2(g(y))} = \frac{1}{1+\tan^2(\tan^{-1}(y))} = \frac{1}{1+y^2}.$$

Problem 3 (p.310 #9). SOLUTION. Let F_T be the cumulative distribution function for the Weibull random variable T. We don't have to compute it to find the density of T^α. Let $F_{[T^\alpha]}$ be the cumulative distribution function of T^α.

$$\begin{aligned} F_{[T^\alpha]}(t) &= P(T^\alpha < t) \\ &= P(T < t^{\frac{1}{\alpha}}) \\ &= F_T(t^{\frac{1}{\alpha}}) \end{aligned}$$

Therefore, differentiating this with respect to t yields

$$f_{[T^\alpha]}(t) = F_T'(t^{\frac{1}{\alpha}})\frac{1}{\alpha}t^{\frac{1}{\alpha}-1}$$

recalling that F_T' is the Weibull density and making the above substitutions yields an exponential density with parameter λ.

For part (b), note that

$$\begin{aligned} P(T < t) &= P((-\lambda^{-1}\log U)^{\frac{1}{\alpha}} < t) \\ &= P(U < e^{-\lambda t^\alpha}) \\ &= e^{-\lambda t^\alpha} \end{aligned}$$

and differentiating this with respect to α gives the Weibull density.

Problem 4 (p.310 #10). SOLUTION. Throughout, let Φ be the standard normal cdf and let $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}$ be the standard normal pdf. Let $X = |Z|$ and $F_X(x)$ be the cdf for X. Then for $x \geq 0$

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(|Z| \leq x) \\ &= P(-x \leq Z \leq x) \\ &= \Phi(x) - \Phi(-x). \end{aligned}$$

Differentiating this (using the chain rule) gives

$$f_X(x) = \phi(x) + \phi(-x) = \frac{2}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$$

for $x \geq 0$ and $f_X(x) = 0$ for $x < 0$.

Let $Y = Z^2$. Let $F_Y(y)$ be the cdf for Y. Then for $y \geq 0$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(Z^2 \leq y) \\ &= P(-\sqrt{y} \leq Z \leq \sqrt{y}) \\ &= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}). \end{aligned}$$

To find the density of Y, differentiate $F_Y(y)$; the chain rule gives

$$f_Y(y) = F_Y'(y) = \left[\frac{\phi(\sqrt{y})}{2\sqrt{y}} + \frac{\phi(-\sqrt{y})}{2\sqrt{y}}\right]$$

Therefore:

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2} & y \geq 0 \\ 0 & \text{else} \end{cases}$$

Let $W = \frac{1}{Z}$. Then for any $w < 0$

$$\begin{aligned} F_W(w) &= P(\frac{1}{Z} \leq w) \\ &= P(\frac{1}{w} \leq Z < 0) \\ &= \frac{1}{2} - \Phi(\frac{1}{w}), \end{aligned}$$

and for $w > 0$

$$\begin{aligned} F_W(w) &= P(\frac{1}{Z} \leq w) \\ &= P(Z \geq \frac{1}{w}) + P(Z < 0) \\ &= \frac{3}{2} - \Phi(\frac{1}{w}). \end{aligned}$$

Differentiating shows that for any $w \neq 0$

$$f_W(w) = -\phi(1/w)\frac{-1}{w^2} = \frac{1}{w^2\sqrt{2\pi}} e^{-1/(2w^2)}.$$

Let $V = 1/Z^2$. For any $v > 0$

$$\begin{aligned} F_V(v) &= P(\frac{1}{Z^2} \leq v) \\ &= P(Z^2 \geq \frac{1}{v}) \\ &= 1 - F_Y(\frac{1}{v}). \end{aligned}$$

Differentiating and using the second part:

$$f_V(v) = \begin{cases} \frac{1}{v^{3/2}\sqrt{2\pi}} e^{-2/v} & v > 0 \\ 0 & v \leq 0 \end{cases}$$

Problem 5 (p. 323 #6)**.** SOLUTION. For (a) we plug in $P(X \geq \frac{1}{2}) = 1 - P(X < \frac{1}{2}) = 1 - F(\frac{1}{2}-) = 1 - F(\frac{1}{2}) = \frac{7}{8}$. The second to last equality is because F is a continuous function.

For part (b) we differentiate to find $f(x) = F'(x) = 3x^2$ for $x \in (0, 1)$ and 0 otherwise. Then part (c) is computed as

$$EX = \int_0^1 xf(x)dx = \int_0^1 3x^3 dx = \frac{3}{4}.$$

Finally, observe that if G is the c.d.f. of Y_1, then $G(x) = P(Y_1 \leq x) = x$ for $x \in (0,1)$, and is constant (0 or 1) elsewhere. Since Y_1, Y_2, Y_3 are all uniformly distributed on $(0,1)$, they all have c.d.f. G. Then for $x \in (0,1)$

$$\begin{aligned}
P(X \leq x) &= P(Y_1 \leq x, Y_2 \leq x, Y_3 \leq x) \\
&= P(Y_1 \leq x)P(Y_2 \leq x)P(Y_3 \leq x) \\
&= G(x)^3 = x^3.
\end{aligned}$$

So $P(X \leq x) = F(x)$.

Problem 6 (p.323 #7)**.** SOLUTION. Again, we can find the distribution of $Y = \sqrt{T}$ and differentiate it to find the density. Let F_T be the cumulative distribution function for T.

$$\begin{aligned}
F_Y(y) &= P(\sqrt{T} \leq y) \\
&= P(T < y^2) \\
&= F_T(y^2) \\
&= 1 - e^{-\lambda y^2}
\end{aligned}$$

Differentiating this gives the density $f_Y(y) = 2\lambda y e^{-\lambda y^2}$ for $y \geq 0$ and 0 otherwise. To compute the expected value of Y, we can integrate

$$\int_0^\infty 2\lambda y^2 e^{-\lambda y^2} dy$$

or we can relate it to a normal random variable. To do so, we make the change of variables $u^2/2 = \lambda y^2$ so $u = \sqrt{2\lambda} y$, which yields

$$\begin{aligned}
\int_0^\infty 2\lambda y^2 e^{-\lambda y^2} dy &= \frac{1}{\sqrt{2\lambda}} \int_0^\infty u^2 e^{-u^2/2} du \\
&= \frac{1}{2\sqrt{2\lambda}} \int_{-\infty}^\infty u^2 e^{-u^2/2} du \\
&= \frac{\sqrt{2\pi}}{2\sqrt{2\lambda}} \int_{-\infty}^\infty u^2 \phi(u) du \\
&= \frac{\sqrt{\pi}}{2\sqrt{\lambda}} E(X^2) \\
&= \frac{\sqrt{\pi}}{2\sqrt{\lambda}}
\end{aligned}$$

where in the above calculation ϕ is the standard normal density and X is a standard normal random variable. When $\lambda = 3$ this evaluates to $\frac{\sqrt{\pi}}{2\sqrt{3}} = 0.51$.

For part (c), recall that if you can generate uniform $[0,1]$ random variables, you can generate random variables with an arbitrary distribution F if F is invertible. That is, let Y have cumulative distribution function F_Y, and let F_Y^{-1} be the function inverse of F_Y. Recall that if U is a uniform random variable, then

$$\begin{aligned}
P(F_Y^{-1}(U) < y) &= P(U < F(y)) \\
&= F(y)
\end{aligned}$$

and hence $F_Y^{-1}(U)$ is a random variable with the requisite distribution. Now we adapt this to the particular case of F_Y from part (a), where $Y = \sqrt{T}$, so $F^{-1}(U) = \sqrt{-\frac{1}{\lambda}\log(1-U)}$.