4.2 Uniform

Unit 4: Continuous Random Variables

Adapted from Blitzstein-Hwang Chapter 5.

Intuitively, a Uniform r.v. on the interval (a, b) is a completely random number between a and b. We formalize the notion of ''completely random'' on an interval by specifying that the PDF should be *constant* over the interval.

Definition 4.2.1 (Uniform distribution).

A continuous r.v. U is said to have the *Uniform distribution* on the interval (a, b) if its PDF is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this by $U \sim \text{Unif}(a, b)$.

This is a valid PDF because the area under the curve is just the area of a rectangle with width $b - a$ and height $1/(b - a)$. The CDF is the accumulated area under the PDF:

$$F(x) = \begin{cases} 0 & \text{if } x \leq a, \\ \frac{x-a}{b-a} & \text{if } a < x < b, \\ 1 & \text{if } x \geq b. \end{cases}$$

The Uniform distribution that we will most frequently use is the $\text{Unif}(0, 1)$ distribution, also called the standard Uniform. The $\text{Unif}(0, 1)$ PDF and CDF are particularly simple: $f(x) = 1$ and $F(x) = x$ for $0 < x < 1$. Figure 4.2.2 shows the $\text{Unif}(0, 1)$ PDF and CDF side by side.

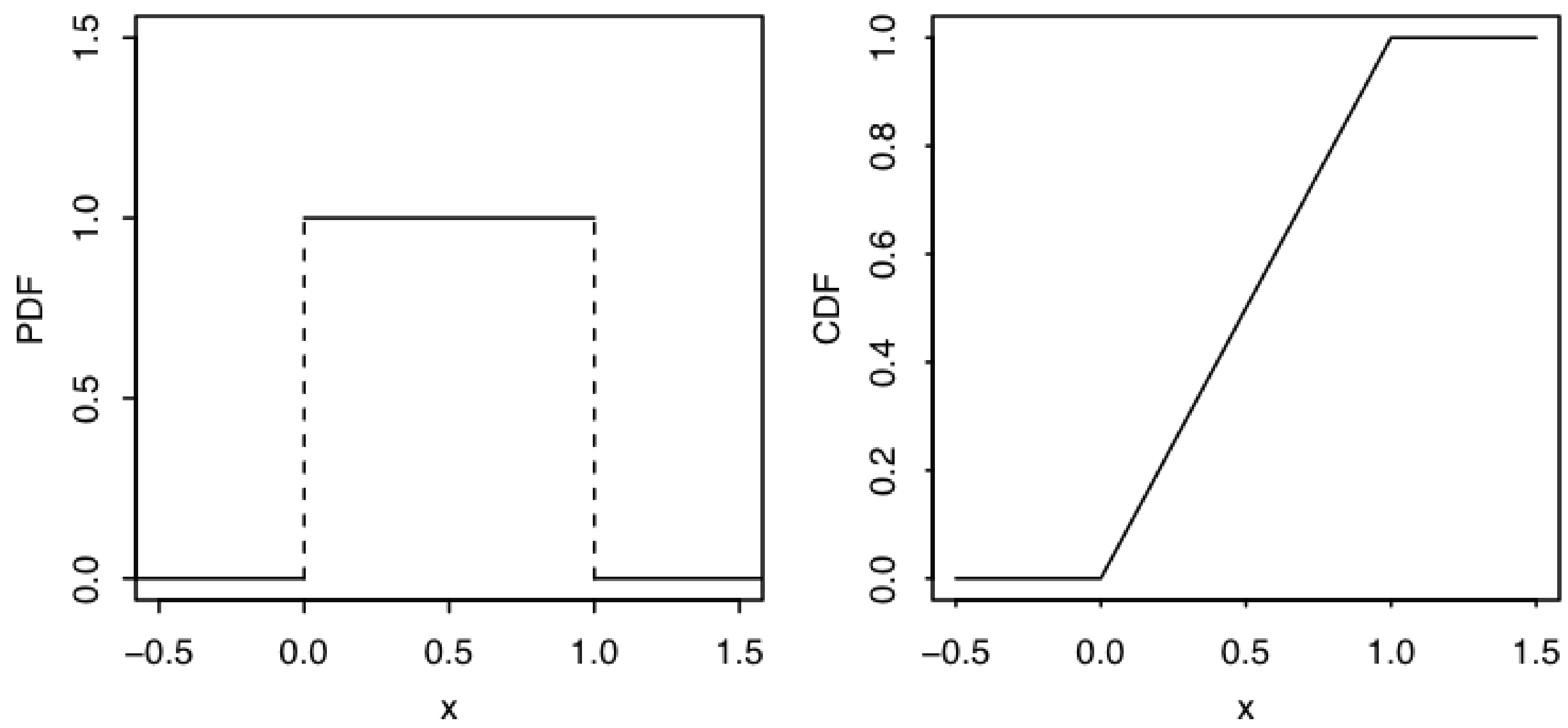

Figure 4.2.2: $\text{Unif}(0, 1)$ PDF and CDF.

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Image Description

For a general $\textbf{Unif}(a, b)$ distribution, the PDF is constant on (a, b), and the CDF is ramp-shaped, increasing linearly from 0 to 1 as x ranges from a to b.

For Uniform distributions, *probability is proportional to length*.

Proposition 4.2.3.

Let $U \sim \textbf{Unif}(a, b)$, and let (c, d) be a subinterval of (a, b), of length l (so $l = d - c$). Then the probability of U being in (c, d) is proportional to l. For example, a subinterval that is twice as long has twice the probability of containing U, and a subinterval of the same length has the same probability.

Proof

Since the PDF of U is the constant $\frac{1}{b-a}$ on (a, b), the area under the PDF from c to d is $\frac{l}{b-a}$, which is a constant times l.

There is a simple way to convert a Uniform random variable on some interval to a Uniform random variable on another interval, by shifting and scaling. For example, if X is Uniform on the interval $(1, 2)$, then $X + 5$ is Uniform on the interval $(6, 7)$, $2X$ is Uniform on the interval $(2, 4)$, and $2X + 5$ is Uniform on $(7, 9)$.

Definition 4.2.4 (Location-scale transformation).

Let X be an r.v. and $Y = \sigma X + \mu$, where σ and μ are constants with $\sigma > 0$. Then we say that Y has been obtained as a *location-scale transformation* of X. Here μ controls how the location is changed and σ controls how the scale is changed.

☣ **Warning 4.2.5.**

In a location-scale transformation, starting with $X \sim \text{Unif}(a, b)$ and transforming it to $Y = cX + d$ where c and d are constants with $c > 0$, Y is a *linear* function of X and Uniformity is preserved: $Y \sim \text{Unif}(ca + d, cb + d)$. But if Y is defined as a *nonlinear* transformation of X, then Y will *not* be linear in general. For example, for $X \sim \text{Unif}(a, b)$ with $0 \le a < b$, the transformed r.v. $Y = X^2$ has support (a^2, b^2) but is *not* Uniform on that interval.

In studying Uniform distributions, a useful strategy is to start with an r.v. that has the simplest Uniform distribution, figure things out in the friendly simple case, and then use a location-scale transformation to handle the general case.

☣ **Warning 4.2.6.**

When using location-scale transformations, the shifting and scaling should be applied to the *random variables* themselves, not to their PDFs. For example, let $U \sim \text{Unif}(0, 1)$, so the PDF f has $f(x) = 1$ on $(0, 1)$ (and $f(x) = 0$ elsewhere). Then $3U + 1 \sim \text{Unif}(1, 4)$, but $3f + 1$ is the function that equals 4 on $(0, 1)$ and 1 elsewhere, which is not a valid PDF since it does not integrate to 1.