The Riemann Hypothesis (1)

$\pi(N)$ = the number of prime numbers P with $P \leq N$

PNT says **$\pi(N)$ increases like N/log(N)**.

But numerical experimentation shows

$\pi(N)$ is **not very close to** N/log(N).

- Can we calculate $\pi(N)$ accurately?
- Which function shall we use instead of N/log(N) ?

The Riemann Hypothesis (2)

- π(N) is approximated by N/log(N).
- But the approximation does not seem very precise.

N	$\pi(N)$	$N/\log(N)$ (approximate value)
10^{10}	455052511	434294482
10^{15}	29844570422669	28952965460217
10^{20}	2220819602560918840	2171472409516259138
10^{25}	176846309399143769411680	173717792761300731060452

The values of π(N) and N/log(N)

https://en.wikipedia.org/wiki/Prime_number_theorem

The Riemann Hypothesis (3)

Conjecture (Unsolved)

π(N) = the number of prime numbers P with P ≤ N

Then the **half the digits** of π(N) and li(N) are equal.

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log(t)}$$

Far-reaching generalization of PNT!

The Riemann Hypothesis (4)

Conjecture (Unsolved)

The **half the digits** of π(N) and li(N) are equal.

N	$\pi(N)$	$\mathrm{li}(N)$ (approximate value)
10^{10}	455052511	455055615
10^{15}	29844570422669	29844571475288
10^{20}	2220819602560918840	2220819602783663484
10^{25}	176846309399143769411680	176846309399198930392619

The values of π(N) and li(N)

https://en.wikipedia.org/wiki/Prime_number_theorem

The Riemann Hypothesis (5)

- The numerical experiments suggest the conjecture might be true.
- Theoretically, it follows from deeper analytic properties of the **Riemann zeta function**.

$$\zeta(s) = \sum_{N=1}^{\infty} \frac{1}{N^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

The Riemann Hypothesis (6)

- A complex number s with

$$\zeta(s) = 0 \quad (0 \leq \mathrm{Re}(s) \leq 1)$$

is called a **non-trivial zero**.

Riemann Hypothesis (Unsolved)
Every non-trivial zero satisfies $\mathrm{Re}(s) = ½$.

- **If RH is true**, the half the digits of $\pi(N)$ and li(N) are equal.
- So far, nobody knows how to prove RH.

The Riemann Hypothesis (7)

- The Riemann Hypothesis is one of the most important open problems in mathematics.
- One of the seven **Millennium Prize Problems** (Award 1 million USD)

Bernhard Riemann
(1826-1866)

https://en.wikipedia.org/wiki/Bernhard_Riemann
http://www.claymath.org/

Summary of Week 1

- Basics on prime numbers:
  - Unique Factorization
  - Infinitude of prime numbers
- Prime Number Thm
- The Riemann zeta function $\zeta(s)$
- The Basel Problem
- The Riemann Hypothesis

Plan of Week 2

We will learn basics on Modular Arithmetic including Fermat's Little Thm, Wilson's Thm, and Fermat's Thm on Sums of Two Squares.

Let's explore beautiful **laws of prime numbers**.

See you next week!

Pierre de Fermat
(1607?-1665)

https://en.wikipedia.org/wiki/Pierre_de_Fermat