8. (Optional) D'Alembert's solution

Observation 1: Wave forms in a string appear to travel along the string.

Observe the way that a pulse appears to travel along the "string" in the demo video below created by TSG@MIT Physics.

Observe a traveling wave

$$u(x,t) := f(x-ct)$$

is a solution to the PDE, as shown by the following calculations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= (-c)\, f'(x-ct) & \frac{\partial u}{\partial x} &= f'(x-ct) \\ \frac{\partial^2 u}{\partial t^2} &= (-c)^2 f''(x-ct) & \frac{\partial^2 u}{\partial x^2} &= f''(x-ct)\,, \end{aligned}$$

so

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

What is the physical meaning of this solution? At $t=0$, we have $u(x,0) = f(x)$, so $f(x)$ is the initial position. For any number t, the position of the wave at time t is the graph of $f(x-ct)$, which is the graph of f shifted ct units to the right. Thus the wave travels at **constant speed** c to the right, maintaining its shape.

The function $u(x,t) := g(x+ct)$ (for any reasonable function $g(x)$) is a solution too, a wave moving to the left. It turns out that the general solution is a superposition

$$u(x,t) = f(x-ct) + g(x+ct)\,.$$

There is a tiny bit of redundancy: one can add a constant to f and subtract the same constant from g without changing u.

It is important to note that these solutions assume that the wave is defined for all $-\infty < x < \infty$. However, there are ways to extend these general solutions to the case of finite intervals with boundary conditions. This is why we are able to show demo videos of finite strings to see some of the phenomena described here.

Remark 8.1 Note that the d'Alembert solution is what allows us to easily understand the coefficient c in the PDE as being the **wave speed** .

Example problem Suppose that $c = 1$, that the initial position is $I(x)$, and that the initial velocity is 0. What does the wave look like?

Solution: The initial conditions $u(x,0) = I(x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ become

$$\begin{aligned} f(x) + g(x) &= I(x) \\ -f'(x) + g'(x) &= 0. \end{aligned}$$

The second equation says that $g(x) = f(x) + C$ for some constant C, and we can adjust f and g by constants to assume that $C = 0$. Then $f(x) = I(x)/2$ and $g(x) = I(x)/2$. So the wave

$$u(x,t) = I(x-t)/2 + I(x+t)/2$$

consists of two equal waveforms, one traveling to the right and one traveling to the left.

You can observe the wave an initial pulse of zero velocity splits into two half amplitude pulses traveling in opposite directions in the following demo video created by TSG@MIT Physics.

Observe this split of traveling waves

Bell Labs Wave Machine: Reflection

(Caption will be displayed when you start playing the video.)

0:59 / 0:59 2.0x HD

Traveling waves

D'Alembert figured out another way to write down solutions, in the case when $u\left(x,t\right)$ is defined for all real numbers x instead of a finite interval $0 \leq x \leq L$. Then, for any reasonable function f,

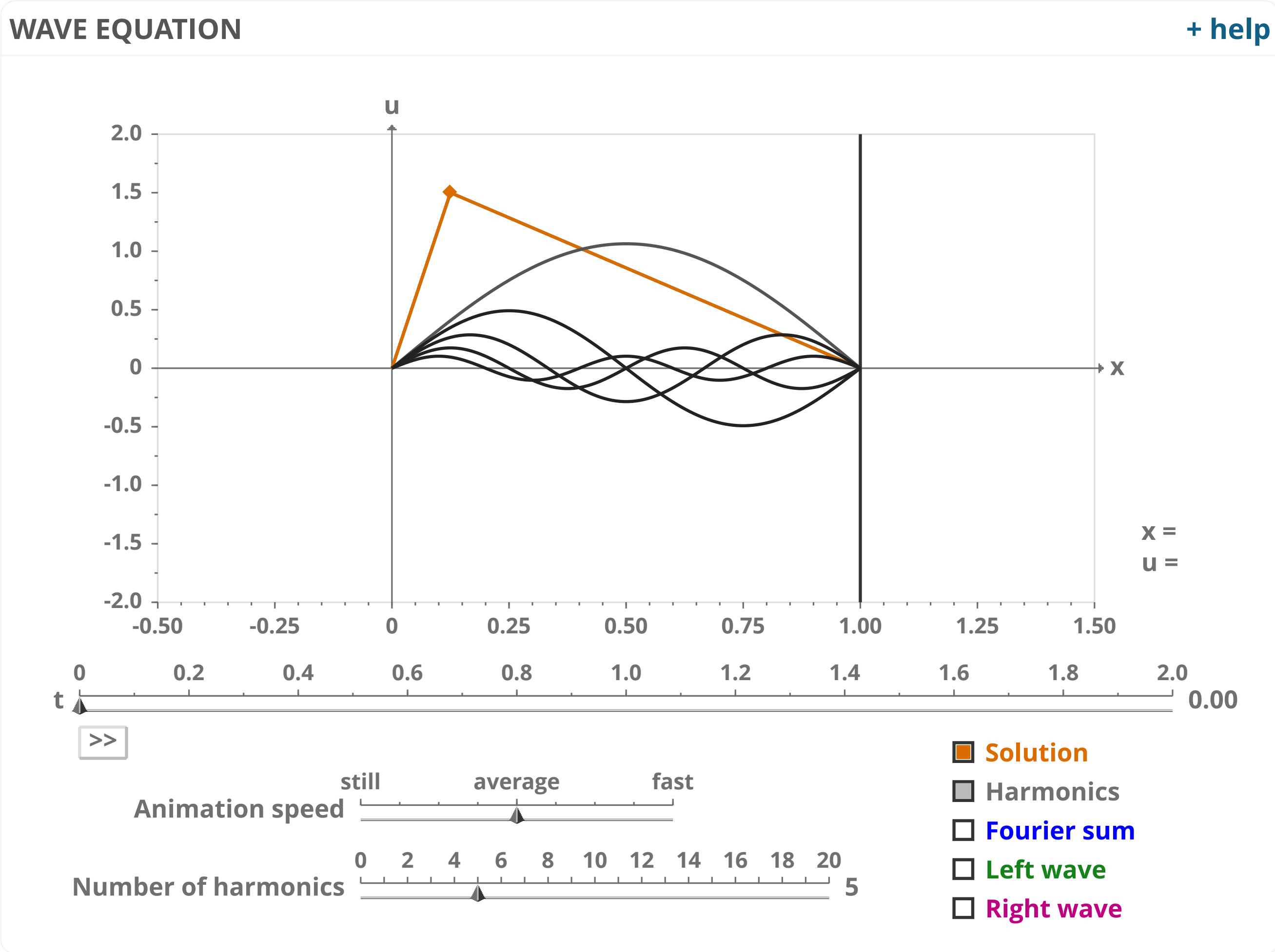
Bell Labs Wave Machine: Superposition

(Caption will be displayed when you start playing the video.)

0:34 / 0:34 2.0x HD

Wave Equation mathlet

Check out the way an initial condition splits into right and left traveling waves with the Wave Equation mathlet. Un-click the “Harmonics", and click the “Right" and “Left" waves buttons.



mathlets.org

8. (Optional) D'Alembert's solution

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