Sum of two gamma/Erlang random variables $\Gamma(m, \lambda)$ and $\Gamma(n, \mu)$ with integer numbers $m \neq n, \lambda \neq \mu$

The gamma distribution with parameters $m > 0$ and $\lambda > 0$ (denoted $\Gamma(m, \lambda)$) has density function

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{m-1}}{\Gamma(m)}, x > 0$$

Given two independent gamma random variables $X = \Gamma(m, \lambda)$ and $Y = \Gamma(n, \mu)$ with integer numbers $m \neq n, \lambda \neq \mu$, what is the density function of their sum $X + Y = \Gamma(m, \lambda) + \Gamma(n, \mu)$?

Notice that both X and Y are also Erlang distribution since m, n are positive integers.

My attempt

First, I searched for well-known results about gamma distribution and I got:

(1) If $\lambda = \mu$, the sum random variable is a Gamma distribution $\sim \Gamma(m + n, \lambda)$ (See Math.SE).

(2) $\Gamma(m, \lambda)$ (or $\Gamma(n, \mu)$) is the sum of m (or n) independent exponential random variables each having rate λ (or μ). The hypoexponential distribution is related to the sum of independent exponential random variables. However, it require all the rates *distinct*.

(3) This site is devoted to the problem of **sums of gamma random variables**. In section 3.1, it claims that if m and n are integer numbers (which **is** my case), the density function **can** be expressed in terms of elementary functions (proved in section 3.4). *The answer is likely buried under a haystack of formulas (however, I failed to find it; you are recommended to have a try).*

Then, I try to calculate it:

$$
\begin{aligned}
f_{X+Y}(a) &= \int_0^a f_X(a-y) f_Y(y) dy \\
&= \int_0^a \frac{\lambda e^{-\lambda(a-y)}(\lambda(a-y))^{m-1}}{\Gamma(m)} \frac{\mu e^{-\mu y}(\mu y)^{n-1}}{\Gamma(n)} dy \\
&= e^{-\lambda a} \frac{\lambda^m \mu^n}{\Gamma(m)\Gamma(n)} \int_0^a e^{(\lambda-\mu)y}(a-y)^{m-1} y^{n-1} dy
\end{aligned}
$$

Here, I am stuck with the integral and gain nothing ... Therefore,

> 1. How to compute the density function of $\Gamma(m, \lambda) + \Gamma(n, \mu)$ with integer numbers $m \neq n, \lambda \neq \mu$?
> 2. **Added:** The answers assuming $m = n$ $(\lambda \neq \mu)$ are also appreciated.

(calculus) (probability) (probability-theory) (reference-request) (probability-distributions)

edited May 25 '14 at 1:52

asked May 21 '14 at 9:05

hengxin
892 7 14

I deleted my answer as I had missed (a quite crucial) parenthesis! Will try to have a closer look later instead. – hejseb May 21 '14 at 9:53

@hejseb Thank you all the same. And you are recommended to refer to the material: Sums of Gamma Random Variables mentioned in the post if you want to come back. The answer is likely buried under a haystack of formulas (however, I failed to find it). – hengxin May 21 '14 at 11:37

2 You are almost there. Since m and n are integers, expand $(a-y)^{m-1}y^{n-1}$ via the binomial theorem into a polynomial in y. Then you are left with a sum of integrals of the form $\displaystyle\int_0^a y^i e^{\nu y}\, dy$ each of which can be integrated by parts. – Dilip Sarwate May 21 '14 at 13:14

@DilipSarwate Following your instruction, I get the integrals of the form $\int_0^a e^{(\lambda-\mu)y} y^{n+x-1} dy$. Here x is related to the general term of binomial extension of $(a-y)^{m-1}$. Using Mathematica, I get the Gamma distribution (i.e., $\Gamma(n+x, \mu-\lambda)$) back. In addition, I find it hard to combine the binomial terms together after computing the integrals. Stuck again... – hengxin May 21 '14 at 14:22

1 The Maple code

$$
with(Statistics); X := RandomVariable(GammaDistribution(m, lambda)) : Y := RandomVariable(GammaDistribution(n, mu)) \\
: PDF(X+Y, t)\; assuming\; m :: posint, n :: posint, m <> n;
$$

produces the answer in terms of LaguerreL. – user64494 May 22 '14 at 2:33

2 Answers

A closed form expression is provided in the following paper.

SV Amari, RB Misra, Closed-form expressions for distribution of sum of exponential random variables, IEEE Transactions on Reliability, 46 (4), 519-522.

answered Jul 1 '15 at 19:26

Suprasad Amari
36 2

Update:

We summarize the development in follows:

Step 1: We simplify the case to be $l(\Gamma(m,1)+k\Gamma(n,1))$ by choosing appropriate k,l for a scale transformation.

Step 2. We want to calculate

$$\Gamma(m,1)+k\Gamma(n,1)=\sum_{i=1}^{m}X_i+k\sum_{i=1}^{n}Y_j$$

Step 3. For $m=n$ case, we only need to calculate $X+kY$, where $X,Y\sim\Gamma(1,1)$. In the case of $k=1$, we let

$$Z=X+Y, W=\frac{X}{X+Y}, X=ZW, Y=Z-ZW$$

The Jacobian is Z. Therefore we have

$$f_{Z,W}(z,w)=ze^{-zw}e^{zw-z}dzdw=ze^{-z}dzdw$$

and

$$Z \sim \Gamma(2,1)$$

as desired. In the general case we have

$$Z = X + kY, W = \frac{X}{X+kY}, X = ZW, Y = \frac{1}{k}(Z - ZW)$$

The Jacobian is $\frac{1}{k}Z$. We thus have

$$f_{Z,W}(z,w) = \frac{1}{k} z e^{-zw} e^{\frac{1}{k}(zw-z)} = \frac{1}{k} z e^{-\frac{k-1}{k}zw - \frac{1}{k}z}$$

and I do not have a good way to factorize it.

A reason this technique might not work in general is the moment generating function does not change when we use the scale transformation, and for different β the moment generating function is different. Thus the problem may be better to be attacked numerically.

edited May 25 '14 at 18:12 answered May 22 '14 at 2:03

Bombyx mori
10.8k 2 15 50

As you suggest, the computation of the density function of $X+Y$ can be reduced to that of $X' + kY''$. However, why is $X', Y'' \sim \exp(1)$? In my calculation, it is of form $\sim \Gamma(n,1) = \frac{e^{-x}x^{n-1}}{\Gamma(n)} = \frac{e^{-x}x^{n-1}}{(n-1)!}$ since n is a positive integer. – hengxin May 22 '14 at 6:41

I updated the computation. I think you are right, the factorization now seems rather complicated. – Bombyx mori May 22 '14 at 15:29

I think I found a way out. – Bombyx mori May 22 '14 at 15:50

Some constants may still be off, I need to fix it. – Bombyx mori May 22 '14 at 15:59

Thanks for your efforts. The factorization trick is rather impressive. However, it seems that you have missed the requirement that $m \neq n$. Following your instruction, I get:

$$X + Y = \lambda\Gamma(m,1) + \mu\Gamma(n,1) = \lambda(\Gamma(m,1) + \frac{\mu}{\lambda}\Gamma(n,1)) = \lambda(\sum_{i=1}^{m} \exp(1) + \frac{\mu}{\lambda}\sum_{i=1}^{n} \exp(1))$$

. Unfortunately, we cannot factor $\sum_{i=1}^{i=n}$ out and focus on

$$\exp(1) + \frac{\mu}{\lambda}\exp(1)$$

. What do you think of it? (Now it has been reduced to the *weighted sum* of $\exp(1)$s). – hengxin May 23 '14 at 2:10