Problem 2: A binary communication system - Part 1

(4/5 points)

A binary communication system is used to send one of two messages:

(i) message A is sent with probability 2/3, and consists of an infinite sequence of zeroes,

(ii) message B is sent with probability 1/3, and consists of an infinite sequence of ones.

The ith received bit is “correct" (i.e., the same as the transmitted bit) with probability 3/4, and is “incorrect" (i.e., a transmitted 0 is received as a 1, and vice versa), with probability 1/4. We assume that **conditioned on any specific message sent,** the received bits, denoted by $Y_1, Y_2, \ldots$ are independent.

Note: Enter numerical answers; do not enter '!' or combinations.

1. Find $\mathbf{P}(Y_1 = 0)$, the probability that the first bit received is 0.

   7/12 ✔ **Answer:** 0.58333

2.

Given that message A was transmitted, what is the probability that exactly 6 of the first 10 received bits are ones? (Answer with at least 3 decimal digits.)

0.016222 ✔ **Answer:** 0.01622

3. Find the probability that the first and second received bits are the same.

5/8 ✔ **Answer:** 0.625

4. Given that $Y_1, \ldots, Y_5$ were all equal to 0, what is the probability that Y_6 is also zero?

0.7489733 ✔ **Answer:** 0.74897

5. Find the mean of K, where $K = \min\{i : Y_i = 1\}$ is the index of the first bit that is 1.

1 ✖ **Answer:** 3.11111

Answer:

1. Let event A be the case where message A is transmitted. Let event B be the case where message B is transmitted. Using the total probability theorem we find:

$$\begin{aligned}\mathbf{P}(Y_1 = 0) &= \mathbf{P}(A)\mathbf{P}(Y_1 = 0 \mid A) + \mathbf{P}(B)\mathbf{P}(Y_1 = 0 \mid B) \\ &= (2/3)(3/4) + (1/3)(1/4)\end{aligned}$$

$$= 7/12.$$

2. Given A, we can consider each bit sent as an independent Bernoulli trial with the probability of getting a 1 equal to $1/4$.

   $$\mathbf{P}(Y_1 + Y_2 + \ldots + Y_{10} = 6 \mid A) = \binom{10}{6}(1/4)^6(3/4)^4 \approx 0.01622.$$

3. Let event C be the event that the first and second received bits are the same (i.e., $C = \{(Y_1, Y_2) = (0,0)\} \cup \{(Y_1, Y_2) = (1,1)\}$). Using the total probability theorem we find:

   $$\begin{aligned} \mathbf{P}(C) &= \mathbf{P}(A)\mathbf{P}(C \mid A) + \mathbf{P}(B)\mathbf{P}(C \mid B) \\ &= (2/3)(9/16 + 1/16) + (1/3)(1/16 + 9/16) \\ &= 5/8. \end{aligned}$$

4. Using the total probability theorem:
   $\mathbf{P}(Y_1 = 0, \ldots, Y_6 = 0) = (2/3)(3/4)^6 + (1/3)(1/4)^6$
   $\mathbf{P}(Y_1 = 0, \ldots, Y_5 = 0) = (2/3)(3/4)^5 + (1/3)(1/4)^5$
   $\mathbf{P}(Y_6 = 0 \mid Y_1 = 0, \ldots, Y_5 = 0) = \frac{\mathbf{P}(Y_1=0,\ldots,Y_6=0)}{\mathbf{P}(Y_1=0,\ldots,Y_5=0)} \approx 0.74897.$

5. If message A (respectively, B) is transmitted, then K is geometric with parameter $1/4$ (respectively, $3/4$). Therefore, using the total expectation theorem:

   $$\begin{aligned} \mathbf{E}[K] &= \mathbf{P}(A)\mathbf{E}[K \mid A] + \mathbf{P}(B)\mathbf{E}[K \mid B] \\ &= \frac{2}{3} \cdot \frac{1}{1/4} + \frac{1}{3} \cdot \frac{1}{3/4} \end{aligned}$$

$$= 28/9$$

You have used 2 of 2 submissions