11. Likelihood Ratio Test

Likelihood Ratio Test

I can actually perform tests at level alpha.

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Concept Check: The Constrained Maximum Likelihood Estimator

1/1 point (graded)

In the general form of the likelihood ratio test, we have an unknown parameter $\theta^* \in \mathbb{R}^d$, and we are deciding between two hypotheses of the form

$$\begin{aligned} H_0 : (\theta^*_{r+1}, \ldots, \theta^*_d) &= (\theta^{(0)}_{r+1}, \ldots, \theta^{(0)}_d) \\ H_1 : (\theta^*_{r+1}, \ldots, \theta^*_d) &\neq (\theta^{(0)}_{r+1}, \ldots, \theta^{(0)}_d). \end{aligned}$$

for some $r \geq 0$.

Thus Θ_0, the region defined by the null hypothesis, is

$$\Theta_0 := \{\mathbf{v} \in \mathbb{R}^d : (v_{r+1}, \ldots, v_d) = (\theta^{(0)}_{r+1}, \ldots, \theta^{(0)}_d)\}$$

where $(\theta^{(0)}_{r+1}, \ldots, \theta^{(0)}_d)$ consists of *known* values.

The likelihood ratio test involves the test-statistic

$$T_n = 2\left(\ell_n\left(\widehat{\theta_n}^{MLE}\right) - \ell_n\left(\widehat{\theta_n}^{c}\right)\right)$$

where ℓ_n is the log-likelihood.

The estimator $\widehat{\theta_n^c}$ is the **constrained MLE** , and it is defined to be

$$\widehat{\theta_n^c} = \mathrm{argmax}_{\theta \in \Theta_0} \ell_n\left(X_1, \ldots, X_n; \theta\right).$$

Which of the following are possible? (Choose all that apply.)

- [ ] $\ell_n\left(\widehat{\theta_n}^{MLE}\right) < \ell_n\left(\widehat{\theta_n}^{c}\right)$

- [x] $\ell_n\left(\widehat{\theta_n}^{MLE}\right) = \ell_n\left(\widehat{\theta_n}^{c}\right)$
- [x] $\ell_n\left(\widehat{\theta_n}^{MLE}\right) > \ell_n\left(\widehat{\theta_n}^{c}\right)$

✔

Solution:

Recall that the MLE is defined by the optimization problem

$$\widehat{\theta_n^{MLE}} = \mathrm{argmax}_{\theta\in\Theta}\ell_n\left(X_1,\ldots,X_n;\theta\right)$$

In particular, we find the maximizer over the *entire* parameter space Θ. The constrained MLE

$$\widehat{\theta_n^c} = \mathrm{argmax}_{\theta\in\Theta_0}\ell_n\left(X_1,\ldots,X_n;\theta\right)$$

finds the maximum over a subset of Θ, so it is not possible that $\ell_n\left(\widehat{\theta_n}^{MLE}\right) < \ell_n\left(\widehat{\theta_n}^{c}\right)$. However, it may be the case that $\ell_n\left(\widehat{\theta_n}^{MLE}\right) = \ell_n\left(\widehat{\theta_n}^{c}\right)$ or $\ell_n\left(\widehat{\theta_n}^{MLE}\right) > \ell_n\left(\widehat{\theta_n}^{c}\right)$. In general, we will have that $\ell_n\left(\widehat{\theta_n}^{MLE}\right) \geq \ell_n\left(\widehat{\theta_n}^{c}\right)$.

Remark: The likelihood ratio test is a natural test in a situation where we only care about *some* (e.g., the last $d-r$ coordinates) of the unknowns involved in the parameter $\theta^* \in \mathbb{R}^d$.

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Answers are displayed within the problem

Concept Check: Test-Statistic for the Likelihood Ratio Test

1/1 point (graded)

Suppose we are hypothesis testing between a null and alternative of the form

$$\begin{aligned} H_0 : (\theta^*_{r+1}, \ldots, \theta^*_d) &= (\theta^{(0)}_{r+1}, \ldots, \theta^{(0)}_d) \\ H_1 : (\theta^*_{r+1}, \ldots, \theta^*_d) &\neq (\theta^{(0)}_{r+1}, \ldots, \theta^{(0)}_d) \,. \end{aligned}$$

Above, $\theta^* \in \mathbb{R}^d$ is an unknown parameter while the values $\theta^{(0)}_{r+1}, \ldots, \theta^{(0)}_d$ are known. To perform the likelihood ratio test, we define the test statistic

$$T_n = 2\left(\ell_n\left(\widehat{\theta_n^{MLE}}\right) - \ell_n\left(\widehat{\theta_n^c}\right)\right)$$

Assume that the technical conditions needed for the MLE to be a consistent estimator are satisfied, and assume that the null-hypothesis is true.

Which of the following are true about the above test statistic T_n? (Choose all that apply. Refer to the slides.)

- [x] T_n is a pivotal statistic; *i.e.*, it converges to a pivotal distribution.
- [ ] T_n is asymptotically normal.
- [x] $T_n \xrightarrow[n\to\infty]{(d)} \chi^2_{d-r}$
- [ ] $T_n \xrightarrow[n\to\infty]{(d)} \chi^2_r$

✔

Solution:

We examine the choices in order.

- The first answer choice is correct. Under the null hypothesis,

$$T_n \xrightarrow[n\to\infty]{(d)} \chi^2_{d-r}.$$

The distribution χ^2_r is pivotal because it does not depend on the specific value of the true parameter θ^*. Hence T_n is also a pivotal statistic.

- The second answer choice is incorrect. T_n is not asymptotically normal; rather it is asymptotically a χ^2 random variable, as stated in the previous bullet. Note that the normal distribution and χ distribution are very different from each other (e.g., χ^2 has significantly heavier tails).
- The third answer choice is correct. As stated in the first bullet, $T_n \xrightarrow[n\to\infty]{(d)} \chi^2_{d-r}$ assuming the null hypothesis and the technical conditions mentioned in the problem statement.
- The fourth answer choice is incorrect. It is true that T_n converges to a χ^2 random variable, but this choice gives the wrong number of degrees of freedom.

Remark: Be careful not to be confused about the following point. While the parameter space corresponding to H_0 is $\Theta_0 = \mathbb{R}^r$ which, intuitively, has r free variables, the test statistic T_n converges to a χ^2 distribution with $d-r$ degrees of freedom. This convergence fact follows from a technical result of Wilks, and we do not discuss aspects of its proof here.

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