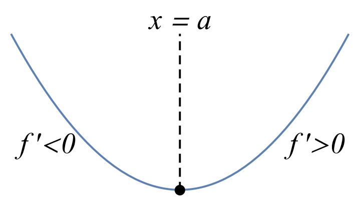
11. Graphical methods

Synthesize

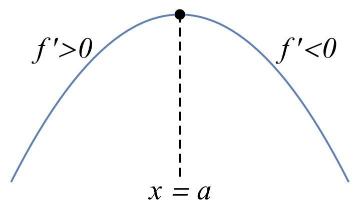
In single variable calculus, we used the **first derivative test** to get information about the shape of the graph of a function $f(x)$ near critical points.

Suppose the function $f(x)$ is continuous at $x = a$ and has a critical point at $x = a$ with $f'(a) = 0$.

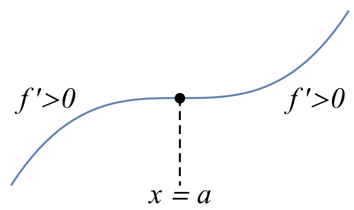
Case 1: If $f'(x) < 0$ just to the left of a and $f'(x) > 0$ just to the right of a, then $f(x)$ has a local minimum at $x = a$.

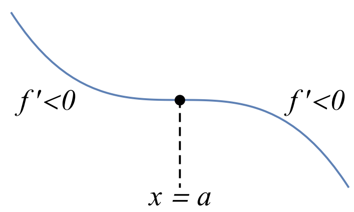


Case 2: If $f'(x) > 0$ just to the left of a and $f'(x) < 0$ just to the right of a, then $f(x)$ has a local maximum at $x = a$.



Case 3: If $f'(x)$ has the same sign just to the left of a and just to the right of a, then $x = a$ is neither a local maximum nor a local minimum of $f(x)$.





Just to the left or right

When we use the phrase "$f'(x) > 0$ just to the left of a," we mean that there is some open interval $a - c < x < a$ of positive width c on which f' is positive. This interval does not have to be very big, as long as it has some size!

Similarly, "$f'(x) > 0$ just to the right of a" means that there is some open interval $a < x < a + d$ of positive width d on which f' is positive.

Hide

Now consider a function of two variables $f(x, y)$. There are two **graphical methods** that we can use to determine the type of critical point.

- If we have a plot of the level curves. We can read off the type of critical point based on the geometry of the level curves.
- If we do not have the function, but have its derivative, the gradient, then we can use calculus to connect the gradient to its level curves and identify types of critical points.

Suppose (a, b) is a critical point of $f(x, y)$ (meaning $\nabla f(a, b) = \langle 0, 0 \rangle$).

Case 1: If the vectors representing $\nabla f(x, y)$ surrounding (a, b) are pointing away from (a, b), then $f(x, y)$ is decreasing as we approach (a, b) from every direction.

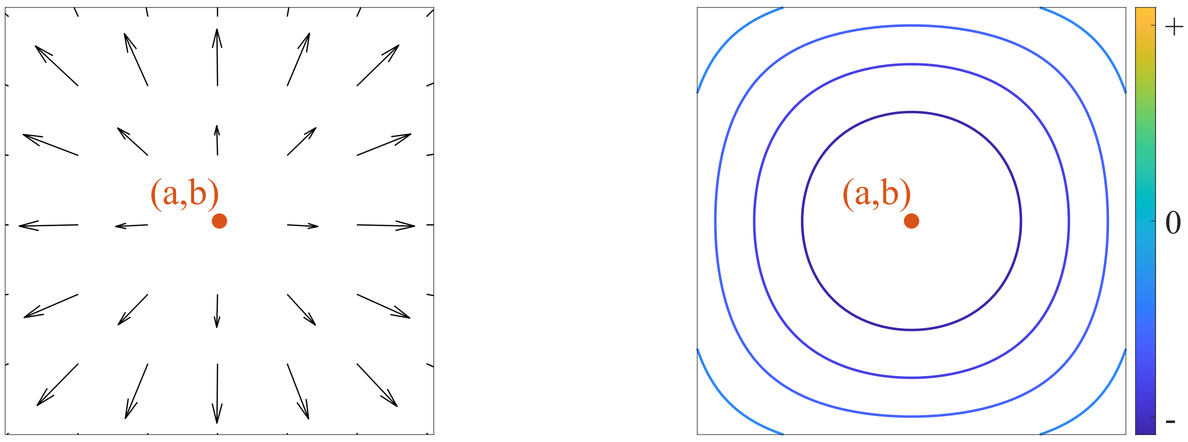
Mathematically, this means that for any point (x, y), the angle between the gradient $\nabla f(x, y)$ and the vector $\langle a - x, b - y \rangle$ is larger than $\pi/2$. That is, for all points (x, y) near (a, b), the dot pro

Calculator Hide Notes

$$\langle a-x, b-y\rangle \cdot \nabla f(x,y) < 0.$$

This means (a,b) is a local minimum of $f(x,y)$.

The figure below on the left shows the gradient field near a local minimum. The figure below on the right shows the corresponding level curves.



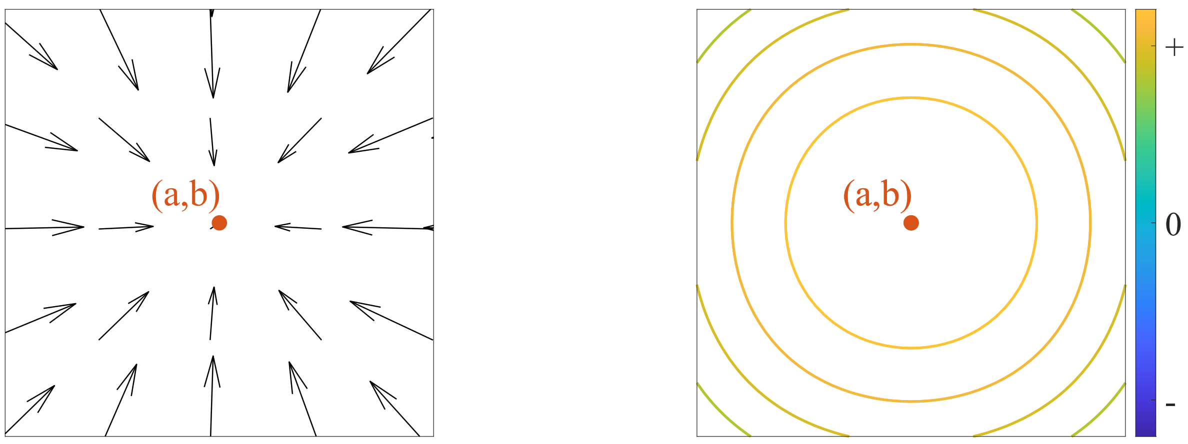
Case 2: If the vectors representing $\nabla f(x,y)$ surrounding (a,b) are pointing towards (a,b), then $f(x,y)$ is increasing as we approach (a,b) from every direction.

Mathematically, this means that for any point (x,y), the angle between the gradient $\nabla f(x,y)$ and the vector $\langle a-x, b-y\rangle$ is less than $\pi/2$. That is, for all points (x,y) near (a,b), the dot product

$$\langle a-x, b-y\rangle \cdot \nabla f(x,y) > 0.$$

This means (a,b) is a local maximum of $f(x,y)$.

The figure below on the left shows the gradient field near a local maximum. The figure below on the right shows the corresponding level curves.



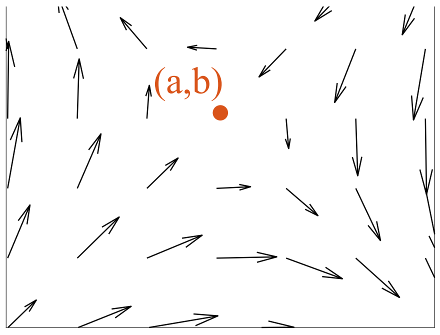
Case 3: If some vectors representing $\nabla f(x,y)$ near (a,b) point towards (a,b) and some point away from (a,b), then $f(x,y)$ is increasing as we approach (a,b) from some directions and decreasing as we approach (a,b) from other directions.

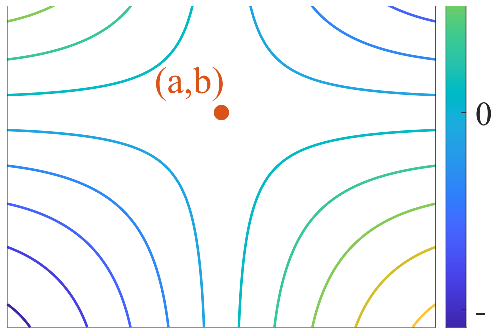
Mathematically, this means that there are some points (x,y) such that the angle between the gradient $\nabla f(x,y)$ and the vector $\langle a-x, b-y\rangle$ is larger than $\pi/2$, and some where this angle is less than $\pi/2$. That is, there are points (x,y) near (a,b), such that the dot product $\langle a-x, b-y\rangle \cdot \nabla f(x,y)$ is positive and negative.

This means (a,b) is a saddle point of $f(x,y)$.

The figure below on the left shows the gradient field near a saddle point. The figure below on the right shows the corresponding level curves.

Calculator Hide Notes





Remark 11.1 The level curves give you the value of the maximum or minimum directly from the contour plot. However, the gradient doesn't tell you the value of the function, only how the derivative changes. More information is needed to determine the value of the function at the critical point.

---

11. Graphical methods

Hide Discussion

Topic: Unit 3: Optimization / 11. Graphical methods

Add a Post

Show all posts by recent activity

should it be other way around 4

Dot product 3
Dot product should be between (a-x, b-x) and gradient of f why is different?

[staff] typo in explanation? 2
Hello, I think there's a typo in a phrase "If have a plot of the level curves", probably it was meant to be "If **we** have a plot of the le...

[Staff] Typos (a) in case 1 and case 2 , and (b) in case 3? 5
Community TA

Previous Next

edX

About
Affiliates
edX for Business
Open edX
Careers

Calculator Hide Notes

News

Legal

Terms of Service & Honor Code
Privacy Policy
Accessibility Policy
Trademark Policy
Sitemap

Connect

Blog
Contact Us
Help Center
Media Kit
Donate

GET IT ON Google play
Download on the App Store

Calculator
Hide Notes