The Shapes of Well-Orderings

There are many different ways of well-ordering the natural numbers. There is, of course, the standard ordering, $<_{\mathbb{N}}$. But one can also well-order the natural numbers using an ordering, $<_{\backsim}$, which is like the standard ordering except that it reverses the position of each even number and its successor:

$$1 <_{\backsim} 0 <_{\backsim} 3 <_{\backsim} 2 <_{\backsim} 5 <_{\backsim} 4 <_{\backsim} \ldots$$

Although $<_{\mathbb{N}}$ and $<_{\backsim}$ correspond to different ways of ordering the natural numbers, they have exactly the same structure. The way to see this is to think of each of the two orderings as consisting of a sequence of "positions", each of which is occupied by a particular number:

$$\begin{array}{l} <_{\mathbb{N}}: \ \boxed{0}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \ldots \\ <_{\backsim}: \ \boxed{1}\ \boxed{0}\ \boxed{3}\ \boxed{2}\ \boxed{5}\ \boxed{4}\ \ldots \end{array}$$

When we abstract away from the numbers that occupy the positions and consider only the positions themselves, we get the exact same structure in both cases:

$$\begin{array}{l} <_{\mathbb{N}}: \ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \ldots \\ <_{\backsim}: \ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \ldots \end{array}$$

Accordingly, we can say that $<_{\mathbb{N}}$ and $<_{\backsim}$ are well-orderings with the same "shape". I will sometimes represent this shape schematically, by shrinking the width of squares, until they look like lines:

$$||||||\ldots$$

Formally, we say that the well-orderings $<_1$ and $<_2$ have the same structure—or the same "shape"—if they are **isomorphic** to one another, in the following sense:

> there is a bijection f from the domain of $<_1$ to the domain of $<_2$ such that, for every x and y in the domain of $<_1$, $x <_1 y$ if and only if $f(x) <_2 f(y)$

Not all well-orderings of $\mathbb{N}$ are isomorphic to one another.

To see this, consider the well-ordering $<_0$, which figures as an exercise below. It is just like the standard ordering, except that it takes every positive number to precede zero:

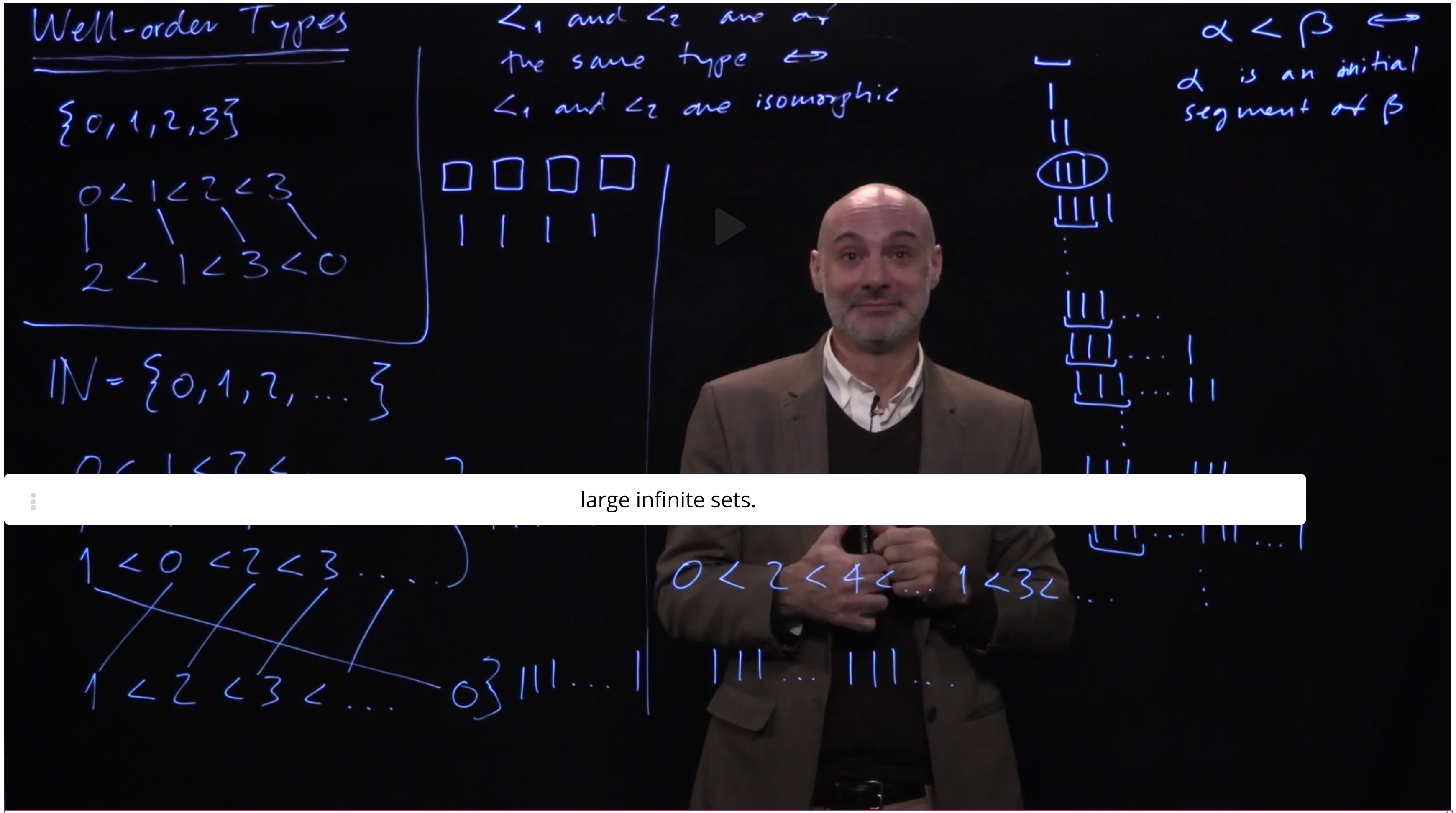
$$1 <_0 2 <_0 3 <_0 4 <_0 \ldots <_0 0$$

Note that $<_0$ is not isomorphic to $<_{\mathbb{N}}$, since its shape is "$||||\ldots|$", which is different from the shape of of $<_{\mathbb{N}}$, "$||||\ldots$".

In this lecture we won't be interested in individual well-orderings, like $<_{\mathbb{N}}$ or $<_0$. Instead, we will be focusing on *types* of well-orderings, or **well-order types**. Intuitively, each well-order type corresponds to a particular a shape of well-ordering.

Formally, a well-order type is an isomorphism-class of well-orderings: a class that consists of a well-ordering, and every well-ordering it is isomorphic to. For instance, $<_{\mathbb{N}}$ and $<_{\backsim}$ fall under the same well-order type, because they are isomorphic to one another. But $<_{\mathbb{N}}$ and $<_0$ do not fall under the same well-order type, because they are not isomorphic to one another.

Video



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Problems 1 and 2

2/2 points (ungraded)

Let $<_{\mathbb{N}}$ be the standard ordering of the natural numbers.

Of each of the following pairs of sets, determine whether they correspond to the same well-order type when ordered by $<_{\mathbb{N}}$:

$\{7, 2, 13, 12\}$	$\{412, 708, 20081\}$

different well-order type ✔ **Answer:** different well-order type

Explanation
These ordering do not have the same well-order type. One way to see this is to note that whereas the shape of $\{7, 2, 13, 12\}$ is

$$||||$$

the shape of $\{412, 708, 20081\}$ is

$$|||$$

Generally speaking, two well-orderings can only be of the same well-order type if they order sets of the same cardinality, because the orderings can only be isomorphic to one another if there is a bijection between the sets they order.

the set of natural numbers	the set of natural numbers greater than 17

same well-order type ✔ **Answer:** same well-order type

Explanation
They are of the same well-order type. Notice, in particular, that they both have the shape: "$||||| \ldots$". Formally, the orderings are isomorphic to one another because the function $x + 18$ is an order-preserving bijection from the set of natural numbers to the set of natural numbers greater than 17.

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Answers are displayed within the problem

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? Is it "..." a comparable tool to define orderings and shapes of sets with infinite elements
So, what I mean is that when we are comparing orderings and shapes we used the ,..., at the end of a set and then start again for another set I feel misleaded by thinking that ...
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