4. Linearize the system at critical points

In general, given n functions $f_1(\mathbf{x}), f_2(\mathbf{x}) \ldots, f_n(\mathbf{x})$, defined for vectors $\mathbf{x}$ in $\mathbb{R}^n$, the best linear approximation to this vector valued function near the point (n-vector) $\mathbf{a}$ is

$$\begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix} \approx \underbrace{\begin{pmatrix} f_1(\mathbf{a}) \\ f_2(\mathbf{a}) \\ \vdots \\ f_n(\mathbf{a}) \end{pmatrix}}_{\text{value at } \mathbf{a}} + \underbrace{\left.\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & \vdots & \vdots & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{pmatrix}\right|_{\mathbf{a}}}_{\text{derivative at } \mathbf{a}} (\mathbf{x} - \mathbf{a}).$$

The matrix $\mathbf{J} = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & \vdots & \vdots & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{pmatrix}$ is called the **Jacobian matrix** .

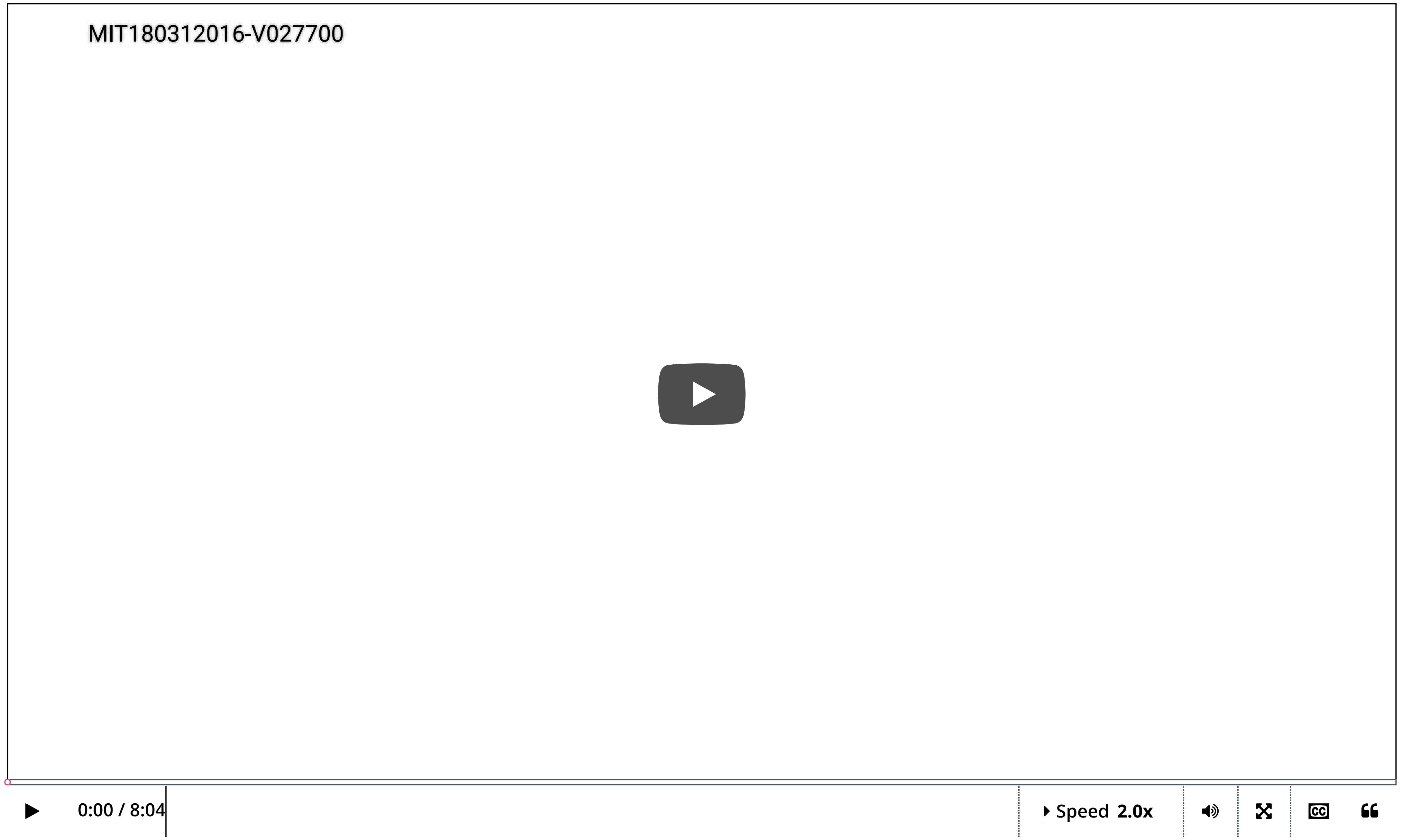
We can rewrite this linearization using more compact notation as

$$\mathbf{f} \approx \mathbf{f}(\mathbf{a}) + \mathbf{J}(\mathbf{a})(\mathbf{x} - \mathbf{a}), \qquad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Note that this is exactly the same notation we had in the 2×2 case.

To determine the stability of the system at the critical points, you evaluate the Jacobian at each critical point. If all of the eigenvalues of the Jacobian at a point $\mathbf{a}$ have negative real part, the critical point is stable. If there is an eigenvalue with positive real part, the critical point is unstable. (However, if the highest real part among eigenvalues is exactly 0, one cannot determine the stability just from the linear approximation of the system.)

3x3 linearization



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Determine conditions for stability

1/1 point (graded)

Determine conditions on a and b so that the critical point with all three coordinates nonzero is stable.

(Give this as a condition on b in terms of a. We suggest using MATLAB or other computer software for help.)

$b <$ 2-a ✔

$2 - a$

FORMULA INPUT HELP

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Behavior in the unstable region

0/1 point (graded)
(Note you will be able to see the answer to this question after the due date.)

Let's focus on the region of parameter values a and b such that the critical point with all three populations positive is unstable and the critical points with only one nonzero population are all unstable (as well as the critical point with all three populations zero). Verify for yourself that in this case the remaining critical points (with two nonzero populations) always lie outside of the physically meaningful region $x, y, z > 0$.

When all relevant critical points are unstable, something interesting should happen to the solutions. Try plotting numerical solutions to determine the behavior of the system.

Recall from course *Differential equations: 2x2 systems* that a **limit cycle** is a closed trajectory, which is

- isolated (there are no other closed trajectories near by), and
- stable.

Note that limit cycles are only possible in nonlinear system, and since the limit cycle is a trajectory, it cannot pass through critical points.

What happens to the majority of the solutions when all of the critical points $x, y, z > 0$ are unstable?

(Choose the best answer from the options below based on your MATLAB exploration.)

- They tend to a limit cycle. ✔
- They tend to a limit cycle formed by trajectories that connect the three critical points with only one nonzero population.
- They tend to the critical points that are outside of the meaningful range $x, y, z > 0$ ✖

They escape to infinity.

Solution:

You do not have the tools to prove this from this course, but if you are curious, you can find the solution explained in this paper.

Submit You have used 2 of 2 attempts

Answers are displayed within the problem

4. Linearize the system at critical points

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Project 2 3d plots and screencast using Maple 2015 for unstable and stable solutions 4

Behavior in the unstable region.
Hi. I can't understand behavior of a system. On my 3-d plot (I can attach graph and matlab code for ode, (a,b) and IC if needed) trace approaching critical point in spiral mann... 9

Determine conditions for stability (hint(s) needed)
After correctly solving for the 8th critical point, I've been working on this for a couple of days and I'm at a wall. Must be a lack of mathematical maturity somewhere. I have tw... 10

How can I plot a 3x3 system phase portrait in matlab?
How can I plot a 3x3 system phase portrait in Matlab? Can we use the same script you provided us before? 4 new_ 7

I solved "Determine conditions for stability" using Routh–Hurwitz stability criterion 3

Relationship between the stability of a critical point and sign of the real part of eigenvalues of Jacobian at that Critical point
Sorry if this is already discussed in the previous course that I unfortunately did not take, I missed the intuition (if any) behind the relationship between the stability of a critical... 7
Following

Definition of "to connect"? 3
Community TA

Phase Types of 3x3 Linear Diff Eq Systems
Is there a list of the phase types that can be produced with a 3x3 system (similar to the 2x2 phase saddle, node, star, etc..)? 1