6.4 Unit 6 Homework Problems

Unit 6: Joint Distributions and Conditional Expectation

Adapted from Blitzstein-Hwang Chapters 7 and 9.

For Problem 1

Alice, Bob, and Carl arrange to meet for lunch on a certain day. They arrive independently at uniformly distributed times between 1 pm and 1:30 pm on that day.

Problem 1a

1/1 point (graded)
(a) What is the probability that Carl arrives first?

0.33333 ✔ **Answer:** 1/3

0.33333

Solution
By symmetry, the probability that Carl arrives first is $1/3$.

Submit You have used 1 of 5 attempts

Answers are displayed within the problem

For Problem 1

For the rest of this problem, assume that Carl arrives first at 1:10 pm, and condition on this fact.

Problem 1b

1/1 point (graded)

(b) What is the probability that Carl will be waiting alone for more than 10 minutes?

0.25 ✔ **Answer:** 1/4

0.25

Solution

There is a 50% chance that Alice will arrive within the next 10 minutes and a 50% chance that Bob will arrive within the next 10 minutes. So by independence, the probability is $1/4$ that neither Alice nor Bob will arrive within the next 10 minutes.

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Answers are displayed within the problem

Problem 1c

1/1 point (graded)

(c) What is the probability that Carl will have to wait more than 10 minutes until his party is complete?

0.75 ✔ **Answer:** 3/4

0.75

Solution

The probability is $1/4$ that both Alice and Bob will arrive within the next 10 minutes, so the probability is $3/4$ that Carl will have to wait more than 10 minutes in order for both Alice and Bob to have arrived.

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Answers are displayed within the problem

Problem 1d

1/1 point (graded)

(d) What is the probability that the person who arrives second will have to wait more than 5 minutes for the third person to show up?

0.562 **Answer:** 9/16

0.562

Solution

We need to find $P(|A-B|>5)$, where A and B are i.i.d. $\mathrm{Unif}(0,20)$ r.v.s. Letting $X=A/20$ and $Y=B/20$, we need to find $P(|X-Y|>0.25)$, where X and Y are i.i.d. $\mathrm{Unif}(0,1)$. This can be done geometrically, interpreting probability as area. The desired area consists of two disjoint triangles, each with area $\frac{1}{2}\left(\frac{3}{4}\right)^2$. Therefore,

$$P(|X-Y|>0.25)=\frac{9}{16}.$$

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Answers are displayed within the problem

Problem 2

2/2 points (graded)

Each of $n\geq 2$ people puts his or her name on a slip of paper (no two have the same name). The slips of paper are shuffled in a hat, and then each person draws one (uniformly at random at each stage, without replacement). Find the mean and standard deviation of the number of people who draw their own names.

1 **Answer:** 1

1

mean

1 **Answer:** 1

1

standard deviation

Solution

Label the people as $1, 2, \dots, n$, let I_j be the indicator of person j getting his or her own name, and let $X = I_1 + \dots + I_n$. By symmetry and linearity,

$$E(X) = nE(I_1) = n \cdot \frac{1}{n} = 1.$$

To find the variance of X, we can expand in terms of covariances:

$$\begin{aligned}\mathrm{Var}(X) &= n\mathrm{Var}(I_1) + 2\binom{n}{2}\mathrm{Cov}(I_1, I_2) \\ &= \frac{n}{n}\left(1 - \frac{1}{n}\right) + n(n-1)(E(I_1 I_2) - E(I_1)E(I_2)) \\ &= 1 - \frac{1}{n} + n(n-1)\left(\frac{1}{n(n-1)} - \frac{1}{n^2}\right) \\ &= 1 - \frac{1}{n} + 1 - \frac{n-1}{n} \\ &= 1.\end{aligned}$$

Thus, the mean and standard deviation of X are both 1.

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Answers are displayed within the problem

For Problem 3

Let $N \sim \mathrm{Pois}(\lambda_1)$ be the number of movies that will be released next year. Suppose that for each movie the number of tickets sold is $\mathrm{Pois}(\lambda_2)$, independently.

Problem 3a

1/1 point (graded)
(a) Find the mean of the number of movie tickets that will be sold next year.

- ○ $N + \lambda_2$
- ◉ $\lambda_1 \cdot \lambda_2$ ✔
- ○ $\lambda_1 + \lambda_2$
- ○ $N \cdot \lambda_2$

Solution
Let $X_j \sim \text{Pois}(\lambda_2)$ be the number of tickets sold for the jth movie released next year, and $X = X_1 + \cdots + X_N$ be the total number of tickets sold for movies released next year. By Adam's law,

$$E(X) = E(E(X|N)) = E(N\lambda_2) = \lambda_1\lambda_2.$$

Submit

You have used 1 of 2 attempts

Answers are displayed within the problem

Problem 3b

1/1 point (graded)
(b) Find the variance of the number of movie tickets that will be sold next year.

- ○ $\lambda_1 \cdot \lambda_2$
- ○ $\lambda_1 \cdot \lambda_2 + \lambda_1^2 \cdot \lambda_2$
- ○ $\lambda_1 \cdot \lambda_2 + \lambda_1^2 \cdot \lambda_2^2$
- ◉ $\lambda_1 \cdot \lambda_2 + \lambda_1 \cdot \lambda_2^2$ ✔

Solution

By Eve's law,

$$\mathrm{Var}(X) = E(\mathrm{Var}(X|N)) + \mathrm{Var}(E(X|N)) = E(N\lambda_2) + \mathrm{Var}(N\lambda_2) = \lambda_1\lambda_2 + \lambda_1\lambda_2^2.$$

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Answers are displayed within the problem

Problem 4

1/1 point (graded)

Jimmy's computer will last an $\mathrm{Expo}(\lambda)$ amount of time until it has a malfunction. When that happens, he will try to get it fixed. With probability p, he will be able to get it fixed. If he is able to get it fixed, the computer is good as new again and will last an additional, independent $\mathrm{Expo}(\lambda)$ amount of time until the next malfunction (when again he is able to get it fixed with probability p, and so on). If after any malfunction Jimmy is unable to get it fixed, he will buy a new computer. Find the expected amount of time (in years) until Jimmy buys a new computer, for $1/\lambda = 2$ years and $p = 0.4$. (Assume that the time spent on computer diagnosis, repair, and shopping is negligible.)

10/3 ✔ **Answer:** 3.33

$\frac{10}{3}$

Solution

Let $N \sim \mathrm{FS}(1-p)$ be the number of malfunctions of the computer until Jimmy can no longer get it fixed (including the last malfunction). Let T_1 be the time until the first malfunction, T_2 be the additional time until the second malfunction, etc. By Adam's law, the expected time until Jimmy buys a new computer is

$$E(T_1 + T_2 + \cdots + T_N) = E(E(T_1 + \cdots + T_N|N)) = E(N/\lambda) = \frac{1}{\lambda(1-p)} = 3.33 \text{ years.}$$

Submit You have used 1 of 5 attempts

Answers are displayed within the problem