Problem 5: Office hours

(6/6 points)

A dedicated professor has been holding infinitely long office hours. Undergraduate students arrive according to a Poisson process at a rate of $\lambda_u = 3$ per hour, while graduate students arrive according to a second, independent Poisson process at a rate of $\lambda_g = 5$ per hour. An arriving student receives immediate attention (the previous student's stay is immediately terminated) and stays with the professor until the next student arrives. (Thus, the professor is always busy, meeting with the most recently arrived student.)

(1) What is the probability that exactly three undergraduates arrive between 10:00 pm and 10:30 pm?

0.1255107

Answer: 0.12551

(2) What is the expected length of time in hours that the 10th arriving student (undergraduate or graduate) will stay with the professor?

1/8

Answer: 0.125

(3) Given that the professor is currently talking with an undergraduate, what is the expected number of subsequent student arrivals up to and including the next graduate student arrival?

8/5

Answer: 1.6

(4) Given that the professor is currently talking with an undergraduate, what is the probability that 5 of the next 7 students to arrive will be undergraduates?

0.0608325

Answer: 0.06083

As rumors spread around campus, a worried department head drops in at midnight and begins observing the professor.

(5) Beginning at midnight, what is the expected length of time until the next student arrives, conditioned on the event that the next student will be an undergraduate?

1/8

Answer: 0.125

(6) What is the expected time that the department head will have to wait until the set of students he/she has observed meeting with the professor (including the student who was meeting the professor when the deparment head arrived) include both an undergraduate and a graduate student?

17/60

Answer: 0.28333

Answer:

1. This is the probability of exactly 3 arrivals in the undegraduate Poisson process during 0.5 hours, which can be calculated from the Poisson PMF with $k = 3$, $\lambda = 3$, and $\tau = 0.5$:

$$P(3, 0.5) = e^{-3 \cdot 0.5} \frac{(3 \cdot 0.5)^3}{3!} \approx 0.12551.$$

2. We can merge the two independent processes for the arrivals of undergraduate and graduate students into a single merged Poisson process with $\lambda = 8$. Therefore, the expected length of time that any given student will stay with the professor is $\frac{1}{\lambda} = \frac{1}{8}$ hours or 7.5 minutes.

3. Given an arrival in the merged process, the probability that the student is a graduate student is $\frac{5}{8}$. The expected number of new arrivals is then the expected value of a geometric random variable with $p = \frac{5}{8}$, which is $\frac{1}{p} = \frac{8}{5}$.

4. This is a binomial probability with $n = 7$, $k = 5$, and $p = \frac{3}{8}$. The probability is therefore

$$\binom{7}{5} \left(\frac{3}{8}\right)^5 \left(\frac{5}{8}\right)^2 \approx 0.06083.$$

5. In the merged Poisson process with rate $\lambda_u + \lambda_g = 8$, whenever there is an arrival, we flip a coin, and with probability $\lambda_u / (\lambda_u + \lambda_g) = 3/8$ the next student is an undergraduate. The outcome of this coin flip has no bearing on the amount of time until the arrival in the merged process occurred. Hence, the answer is simply $1/(\lambda_u + \lambda_g) = \frac{1}{8}$.

A more formal approach involves the use of Bayes' rule. Let T be the length of time until the next student arrives starting from midngith, when the department head arrived. Let A be the random variable that takes on the value 1 if the next student is an undergraduate and 0 otherwise.

Using Bayes' rule,

$$f_{T|A}(t \mid 1) = \frac{p_{A|T}(1 \mid t) f_T(t)}{p_A(1)}.$$

We have that $p_{A|T}(1 \mid t) = p_A(1)$ since the time of the arrival is independent of the type of arrival in the merged process. Therefore, $f_{T|A}(t \mid 1) = f_T(t)$, which is the distribution of the interarrival time of the merged process. Hence, the expected value is $1/(\lambda_u + \lambda_g) = 1/8$ as argued above.

6. Let B be the time until the department head observes both an undergraduate and a graduate student. Let $U = 1$ if there is an undergraduate student in the professor's office when the department heads arrives, and let $U = 0$ if there is a graduate student. Using the previously derived probability of an arrival being of each type, we have $\mathbf{P}(U = 1) = 3/8$ and $\mathbf{P}(U = 0) = 5/8$. Therefore, by the total expectation theorem:

$$\begin{aligned} \mathbf{E}[B] &= \mathbf{E}[B \mid U = 1]\mathbf{P}(U = 1) + \mathbf{E}[B \mid U = 0]\mathbf{P}(U = 0) \\ &= \frac{1}{5} \cdot \frac{3}{8} + \frac{1}{3} \cdot \frac{5}{8} \end{aligned}$$

$$= \frac{17}{60} \approx 0.28333.$$

The second equality holds as $\mathbf{E}[B \mid U = 1]$ is the expected value of an exponential random variable with parameter $\lambda_g = 5$ since the department head must wait until a graduate student arrives. The second term follows from a similar argument for the expected time until the next undergraduate student arrives.

You have used 2 of 3 submissions