[Optional: Extending Lebesgue Measure Beyond the Borel Sets]

As it turns out, there is a natural way of extending the notion of Lebesgue Measure beyond the Borel Sets while preserving the crucial properties of Countable Additivity and Non-Negativity. (The property of Length on Segments is preserved automatically by any extension of λ.)

The key idea is that any subset of a Borel Set of measure zero should be treated as having measure zero. Formally speaking, one extends the notion of Lebesgue Measure in two steps. The first step is to introduce a notion of "Lebesgue measurability". We say that A is **Lebesgue Measurable** if and only if:

1. A is a Borel Set; or
2. $A^+ = A^B \cup A^0$ where A^B is a Borel Set and A^0 is a subset of some Borel Set of Lebesgue Measure zero.

The second step is to extend the function λ by stipulating that $\lambda(A^+) = \lambda(A^B)$, whenever A^+ and A^B are as above.

This procedure doesn't affect the value of λ on Borel Sets. And, as you'll verify in the exercises below, it yields the result that Countable Additivity and Non-Negativity hold for any Lebesgue Measurable sets, whether or not they are Borel Sets.

It is worth noting that our definition of Lebesgue measurability wouldn't get us beyond the Borel Sets if the only Borel Sets with measure zero were countable sets, since every subset of a countable set is countable (and therefore a Borel Set), and since the union of two Borel Sets is always a Borel Set. As it turns out, however, there are *uncountable* Borel Sets of Lebesgue Measure zero which have subsets that are not Borel Sets. The most famous example of an uncountable set of Lebesgue Measure zero is the *Cantor Set*, which is named in honor of our old friend from Lecture 1, Georg Cantor. We won't get into it here, but you can find plenty of information about it on the Web.

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Problem 1

1/1 point (ungraded)

Suppose that that A^0 is a subset of a Borel Set with Lebesgue Measure zero. Is it then true that $\lambda\left(A^0\right) = 0$?

● Yes

○ No

○ None of the above

✓

Explanation

We are assuming that A^0 is a subset of a Borel Set of measure zero. Since $\emptyset$ is a Borel Set, and since $A^0 = \emptyset \cup A^0$ and since, the stipulation we used to extend λ yields $\lambda\left(A^0\right) = \lambda\left(\emptyset\right)$. But we showed in a previous exercise that $\lambda\left(\emptyset\right) = 0$.

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ℹ Answers are displayed within the problem

Problem 2

1/1 point (ungraded)

Is Lebesgue measurability closed under complements?

● Yes

○ No

✓

Explanation

Since A is Lebesgue Measurable for each i, we know that $A = A^B \cup A^0$ for each i, where A^B is a Borel Set and A^0 is a subset of a Borel Set B of measure zero.

Let us now show that $\overline{A}$ is Lebesgue Measurable. Notice, first, that $\overline{A^B \cup B}$ is a Borel Set (since A^B and B are Borel Sets, and Borel Sets are closed under complements and countable unions).

Generating Speech Output $\overline{A^B} \cap \overline{A^0})$ is a subset of a Borel Set of measure zero (since B is a Borel

Set of measure zero). This means that one can show that $\overline{A}$ is Lebesgue Measurable by verifying the following equation:

$$\overline{A} = \overline{A^B \cup B} \cup (B \cap \overline{A^B} \cap \overline{A^0})$$

which can be done in the following steps:

$$\begin{aligned}
A &= A^B \cup A^0 \\
A &= ((A^B \cup B) \cap \overline{B}) \cup A^B \cup A^0 \\
A &= ((A^B \cup B) \cap \overline{B}) \cup ((A^B \cup B) \cap A^B) \cup A^0 \\
A &= ((A^B \cup B) \cap \overline{B}) \cup ((A^B \cup B) \cap A^B) \cup ((A^B \cup B) \cap A^0) \\
A &= (A^B \cup B) \cap (\overline{B} \cup A^B \cup A^0) \\
A &= (A^B \cup B) \cap \overline{(B \cap \overline{A^B} \cap \overline{A^0})} \\
\overline{A} &= \overline{A^B \cup B} \cup (B \cap \overline{A^B} \cap \overline{A^0})
\end{aligned}$$

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Problem 3

1/1 point (ungraded)

Is Lebesgue measurability closed under countable unions?

● Yes

○ No

✓

Explanation

Suppose that $A_1, A_2, A_3, \ldots$ is a countable family of Lebesgue Measurable sets. We will verify that $A_1 \cup A_2 \cup A_3 \cup \ldots$ is Lebesgue Measurable.

Since A_i is Lebesgue Measurable for each i, we know that $A_i = A_i^B \cup A_i^0$ for each i, where A_i^B is a Borel Set and A_i^0 is a subset of a Borel Set B_i of measure zero.

We also know that

$$A_1 \cup A_2 \cup A_3 \cup \cdots = (A_1^B \cup A_2^B \cup A_3^B \cup \ldots) \cup (A_1^0 \cup A_2^0 \cup A_3^0 \cup \ldots)$$

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This means that all we need to do to verify that $A_1 \cup A_2 \cup A_3 \cup \ldots$ is Lebesgue Measurable is check that $A_1^B \cup A_2^B \cup A_3^B \cup \ldots$ is a Borel Set and that $A_1^0 \cup A_2^0 \cup A_3^0 \cup \ldots$ is a subset of a Borel Set of measure zero. But the former follows from the fact that each of $A_1^B, A_2^B, A_3^B \ldots$ is a Borel Set and the fact that Borel Sets are closed under countable unions, and the latter follows from these five observations: (i) $A_1^0 \cup A_2^0 \cup A_3^0 \cup \cdots \subseteq B_1 \cup B_2 \cup B_3 \cup \ldots$, (ii) $B_1 \cup B_2 \cup B_3 \cup \ldots$ is a Borel Set, since each of the $B_1, B_2, B_3, \ldots$ is a Borel Set and Borel Sets are closed under countable unions, and (iii) $B_1 \cup B_2 \cup B_3 \cup \cdots = B_1 \cup (B_2 - B_1) \cup (B_3 - (B_1 \cup B_2)) \cup \ldots$, and (iv) since each of the $B_1, B_2, B_3, \ldots$ has measure zero, each $B_k - (B_1 \cup \cdots \cup B_{k-1})$ $(k > 1)$ has measure zero (by problem 5 of the section on Lebesgue Measure), and (v) $B_1 \cup (B_2 - B_1) \cup (B_3 - (B_1 \cup B_2)) \cup \ldots$ has measure zero, since Countable Additivity holds for disjoint Borel Sets.

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Problem 4

1/1 point (ungraded)

Whenever A is Lebesgue Measurable, $\lambda(A)$ is either a non-negative real number or ∞, and therefore Non-Negativity is preserved when λ is extended to all Lebesgue Measurable sets.

True or false?

● True

○ False

✓

Explanation

Since A is Lebesgue Measurable, $A = A^B \cup A^0$, where A^B is a Borel Set and A^0 is a subset of a Borel Set of measure zero. We know, moreover, that $\lambda(A) = \lambda(A^B)$. Since Non-Negativity applies to Borel Sets, this means that $\lambda(A^B)$ is either a non-negative real number or ∞, so the same must be true of $\lambda(A)$.

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Problem 5

1/1 point (ungraded)

Our extension of λ to the Lebesgue Measurable sets preserves Countable Additivity.

In other words: assume that $A_1, A_2, \ldots$ are a finite or countably infinite family of disjoint sets. Whenever A_i is Lebesgue Measurable for each i, we have:

$$\lambda\left(A_1 \cup A_2 \cup A_3 \cup \ldots\right) = \lambda\left(A_1\right) + \lambda\left(A_2\right) + \lambda\left(A_3\right) + \ldots$$

True or false?

● True

○ False

✓

Explanation

Since each of the A_i is Lebesgue Measurable, we know that $A_i = A_i^B \cup A_i^0$, where A_i^B is a Borel Set and A_i^0 is a subset of a Borel Set B_i of measure zero.
We verified in the exercise above that

$$A_1 \cup A_2 \cup A_3 \cup \cdots = \left(A_1^B \cup A_2^B \cup A_3^B \cup \ldots\right) \cup \left(A_1^0 \cup A_2^0 \cup A_3^0 \cup \ldots\right)$$

where $A_1^B \cup A_2^B \cup A_3^B \cup \ldots$ is a Borel Set and $A_1^0 \cup A_2^0 \cup A_3^0 \cup \ldots$ is a subset of a Borel Set of measure zero.
So the stipulation we used to extend λ therefore entails that

$$\lambda\left(A_1 \cup A_2 \cup A_3 \cup \ldots\right) = \lambda\left(A_1^B \cup A_2^B \cup A_3^B \cup \ldots\right)$$

Since $A_1 \cup A_2 \cup A_3 \cup \ldots$ are pairwise disjoint, $A_1^B \cup A_2^B \cup A_3^B \cup \ldots$ are pairwise disjoint. So, by Countable Additivity on Borel Sets, we have:

$$\lambda\left(A_1^B \cup A_2^B \cup A_3^B \cup \ldots\right) = \lambda\left(A_1^B\right) + \lambda\left(A_2^B\right) + \lambda\left(A_3^B\right) + \ldots$$

Putting all of this together gives us

$$\lambda\left(A_1 \cup A_2 \cup A_3 \cup \ldots\right) = \lambda\left(A_1^B\right) + \lambda\left(A_2^B\right) + \lambda\left(A_3^B\right) + \ldots$$

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But the stipulation we used to extend λ entails that $\lambda\left(A_i\right) = \lambda\left(A_i^B\right)$ for each i. So we have

$$\lambda\left(A_1 \cup A_2 \cup A_3 \cup \ldots\right) = \lambda\left(A_1\right) + \lambda\left(A_2\right) + \lambda\left(A_3\right) + \ldots$$

which is what we wanted.

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