MATHEMATICS

Independent chances of 3 events $\frac{a}{a+x}$, $\frac{b}{b+x}$, $\frac{c}{c+x}$

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Here's a problem from my probability textbook:

> Of three independent events the chance that the first *only* should happen is a; the chance of the second *only* is b; the chance of the third *only* is c. Show that the independent chances of the three events are respectively
>
> $$\frac{a}{a+x}, \quad \frac{b}{b+x}, \quad \frac{c}{c+x},$$
>
> where x is a root of the equation
>
> $$(a+x)(b+x)(c+x) = x^2.$$

Here's what I did. We have

$$a = p_1(1-p_2)(1-p_3), \quad b = (1-p_1)p_2(1-p_3), \quad c = (1-p_1)(1-p_2)p_3.$$

Without loss of generality let's consider a. We have

$$p_1 = \frac{a}{(1-p_2)(1-p_3)}.$$

If we assume the result we want to show, then this equals

$$p_1 = \frac{a}{\left(1-\frac{b}{b+x}\right)\left(1-\frac{c}{c+x}\right)} = \frac{a}{\frac{x^2}{(b+x)(c+x)}} = \frac{a}{a+x}.$$

However, we assumed in part what we wanted to show, which possibly makes this circular.

Another observation I noticed is that

$$p_1p_2p_3 + p_1p_2(1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2p_3 + (1-p_1)(1-p_2)(1-p_3) + a + + c = 1.$$

However, despite what I've tried, I'm stuck and do not know how to proceed further. Could anybody help me? Is there a way to turn my circular approach into a noncircular one?

probability combinatorics algebra-precalculus polynomials

Share Cite Edit Follow Flag edited 50 mins ago asked 1 hour ago

Emperor Concerto
1,081 1 13

2 Hint: think about what x should be as an expression of p_1, p_2, p_3. – user3257842 39 mins ago

3 Answers

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Note that you want $p_1 = \frac{a}{a+x}$. Since $a = p_1(1-p_2)(1-p_3)$, then $a + x = (1-p_2)(1-p_3)$. You can further simplify $x = (1-p_1)(1-p_2)(1-p_3)$. This satisfies the equation in the question.

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answered 32 mins ago
sankhya
11 2
New contributor

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$$\begin{aligned} a &= p(1-q)(1-r) \\ b &= (1-p)q(1-r) \\ c &= (1-p)(1-q)r \end{aligned}$$

(where p, q, r are the probabilities of those events. I prefer to use p, q, r rather than p_1, p_2, p_3 - for the ease of typing!)

Now, let $x = (1-p)(1-q)(1-r)$ - the probability that *none* of the events happened. Then, obviously:

$$\begin{aligned} \frac{a}{a+x} &= \frac{p(1-q)(1-r)}{p(1-q)(1-r)+(1-p)(1-q)(1-r)} \\ &= \frac{p(1-q)(1-r)}{(1-q)(1-r)} \\ &= p \end{aligned}$$

and similarly for q and r. Moreover:

$$\begin{aligned} (a+x)(b+x)(c+x) &= (1-q)(1-r)\cdot(1-p)(1-r)\cdot(1-p)(1-q) \\ &= (1-p)^2(1-q)^2(1-r)^2 \\ &= x^2 \end{aligned}$$

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answered 35 mins ago
Stinking Bishop
12.7k 14 29

To get corner cases out of the way, if $p_1 = 1$ then $b = c = 0$, and $x = 1 - a$ will satisfy all the wanted properties. If $p_1 = 0$, then $a = 0$ and the problem reduces to a similar one with only

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two events. The same goes for p_2 and p_3, so from here on assume $0 < p_1 < 1, 0 < p_2 < 1$, and $0 < p_3 < 1$.

From your observation

$$a = p_1(1-p_2)(1-p_3), \quad b = (1-p_1)p_2(1-p_3), \quad c = (1-p_1)(1-p_2)p_3$$

we can get

$$(1-p_1)(1-p_2)(1-p_3) = \frac{1-p_1}{p_1}a = \frac{1-p_2}{p_2}b = \frac{1-p_3}{p_3}c$$

If we call this value $x = (1-p_1)(1-p_2)(1-p_3)$, then $x > 0$ and

$$x = \frac{1-p_1}{p_1}a$$

$$\frac{x}{a} = \frac{1}{p_1} - 1$$

$$\frac{1}{p_1} = \frac{a+x}{a}$$

$$p_1 = \frac{a}{a+x}$$

Solving for p_2 and p_3 works just the same.

To show that $(a+x)(b+x)(c+x) = x^2$, it will be easier to look at the left side divided by x^3:

$$\begin{aligned}\frac{(a+x)(b+x)(c+x)}{x^3} &= \left(1+\frac{a}{x}\right)\left(1+\frac{b}{x}\right)\left(1+\frac{c}{x}\right)\\ &= \left(1+\frac{p_1}{1-p_1}\right)\left(1+\frac{p_2}{1-p_2}\right)\left(1+\frac{p_3}{1-p_3}\right)\\ &= \frac{1}{(1-p_1)(1-p_2)(1-p_3)} = \frac{1}{x}\end{aligned}$$

Now just multiply both sides by x^3 to get the polynomial equation in the question.

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answered 30 mins ago

aschepler

3,088 13 16