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3. Directional derivatives definition

Explore

Directional derivatives from linear approximation

Linear approximation in the y direction

Recall that if we start at a point (x, y) and move by a small amount Δy in the positive y-direction, we can approximate the value of f by

$$f(x, y + \Delta y) \approx f(x, y) + f_y(x, y)\,\Delta y. \tag{3.98}$$

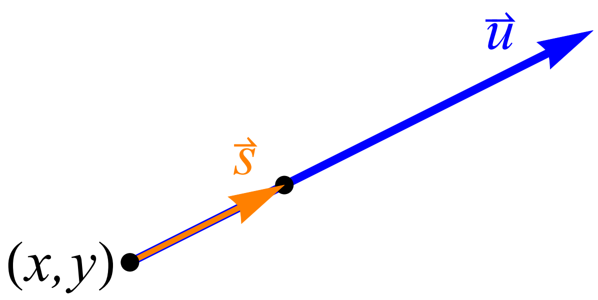
Linear approximation in the $\hat{u}$ direction

This is the same as approximating the function f in the direction the vector $\hat{u} = \langle 0, 1\rangle$. Using our notation for directional derivatives, this becomes

$$f(x, y + \Delta y) \approx f(x, y) + D_{\langle 0,1\rangle} f(x, y)\,\Delta y. \tag{3.99}$$

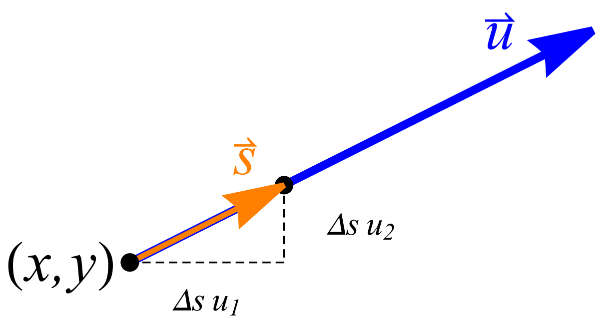
Linear approximation in the $\Delta s \hat{u}$ direction

Now suppose we are at the point (x, y) and we want to move a small amount Δs in the direction of an arbitrary unit vector $\hat{u}$. We will denote this move by the vector $\vec{s}$ which lies along the unit vector $\hat{u} = \langle u_1, u_2\rangle$ and has magnitude Δs.



Because $\vec{s}$ is parallel to $\hat{u}$, we know that $\vec{s}$ is a scalar multiple of $\hat{u}$. Since $|s| = \Delta s$, we can say $\vec{s} = (\Delta s)\,\hat{u}$. Then

$$\begin{aligned} \langle x, y\rangle + \vec{s} &= \langle x, y\rangle + (\Delta s)\,\hat{u} && (3.100)\\ &= \langle x, y\rangle + \langle u_1 \Delta s, u_2 \Delta s\rangle && (3.101)\\ &= \langle x + u_1 \Delta s, y + u_2 \Delta s\rangle. && (3.102)\end{aligned}$$



Now we want to figure out how $f(x, y)$ changes when we move from (x, y) to a point that is Δs units along $\hat{u}$. This puts us at the end of the vector

$$\langle x, y\rangle + \vec{s} = \langle x + u_1 \Delta s, y + u_2 \Delta s\rangle.$$

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So the quantity we want to approximate is

$$f\left(x+u_1\Delta s, y+u_2\Delta s\right). \tag{3.104}$$

Let's apply what we know about linear approximations to this quantity. This gives

$$f\left(x+u_1\Delta s, y+u_2\Delta s\right) \approx f\left(x,y\right)+f_x u_1\Delta s+f_y u_2\Delta s \tag{3.105}$$

$$= f\left(x,y\right)+\underbrace{\left(f_x u_1+f_y u_2\right)}_{D_{\hat{u}}f}\Delta s. \tag{3.106}$$

Directional derivative in the $\hat{u}$ direction

By thinking of the directional derivative as the rate of change of f when we move a distance Δs in the direction of $\hat{u}$, we have

$$D_{\hat{u}}f\left(x,y\right)=f_x u_1+f_y u_2. \tag{3.107}$$

Notice that we can also write this as the dot product

$$D_{\hat{u}}f\left(x,y\right)=\nabla f\cdot\hat{u}. \tag{3.108}$$

Definition 3.1

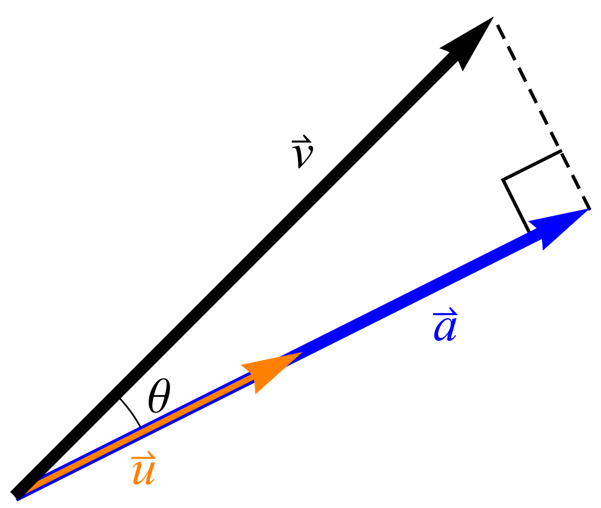
The **directional derivative** of a function $f\left(x,y\right)$ in the direction of the unit vector $\hat{u}$ at the point $\left(x,y\right)$ is given by

$$D_{\hat{u}}f\left(x,y\right)=\nabla f\cdot\hat{u}.$$

Directional derivatives from vector components

Another way to see this definition is to think about the directional derivative as the component of the gradient pointing in the direction $\hat{u}$. In other words, the directional derivative measures how much ∇f points in the direction given by $\hat{u}$. We saw in Lecture 5 that given a vector $\vec{v}$ and a direction $\vec{u}$, we can define

$$\vec{a}=\text{the component of }\vec{v}\text{ in the }\vec{u}\text{ direction.}$$



We saw that $\vec{a}$ is given by

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$$\vec{a} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right) \vec{u}.$$

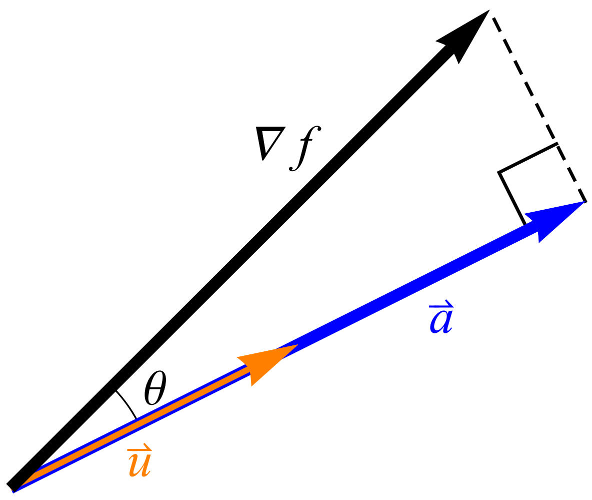
When $\hat{u}$ is a unit vector, this reduces to

$$\vec{a} = (\hat{u} \cdot \vec{v})\, \hat{u}.$$

If we replace $\vec{v}$ by ∇f, we see that the component of the gradient pointing in the direction of a unit vector $\hat{u}$ is

$$\vec{a} = \underbrace{(\hat{u} \cdot \nabla f)}_{D_{\hat{u}} f}\hat{u}.$$

The term inside the parentheses is the directional derivative $D_{\hat{u}} f$.



Remark 3.2 Notice that $D_{\hat{u}} f\,(x, y)$ is a function of x and y. When we evaluate that function at a specific point (x_0, y_0), the quantity $D_{\hat{u}} f\,(x_0, y_0)$ is a scalar.

Extension to higher dimension: Directional derivatives

In n dimensions, the definition of the directional derivative is the same. We would have an n-dimensional unit vector $\hat{u} = \langle u_1, u_2, \ldots, u_n \rangle$. Then

$$\begin{aligned} D_{\hat{u}} f\,(x_1, x_2, \ldots, x_n) &= \nabla f\,(x_1, x_2, \ldots, x_n) \cdot \hat{u} \\ &= \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle \cdot \langle u_1, u_2, \ldots, u_n \rangle \\ &= f_{x_1}\,(x_1, x_2, \ldots, x_n)\, u_1 + f_{x_2}\,(x_1, x_2, \ldots, x_n)\, u_2 + \cdots + f_{x_n}\,(x_1, x_2, \ldots, x_n)\, u_n. \end{aligned}$$

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