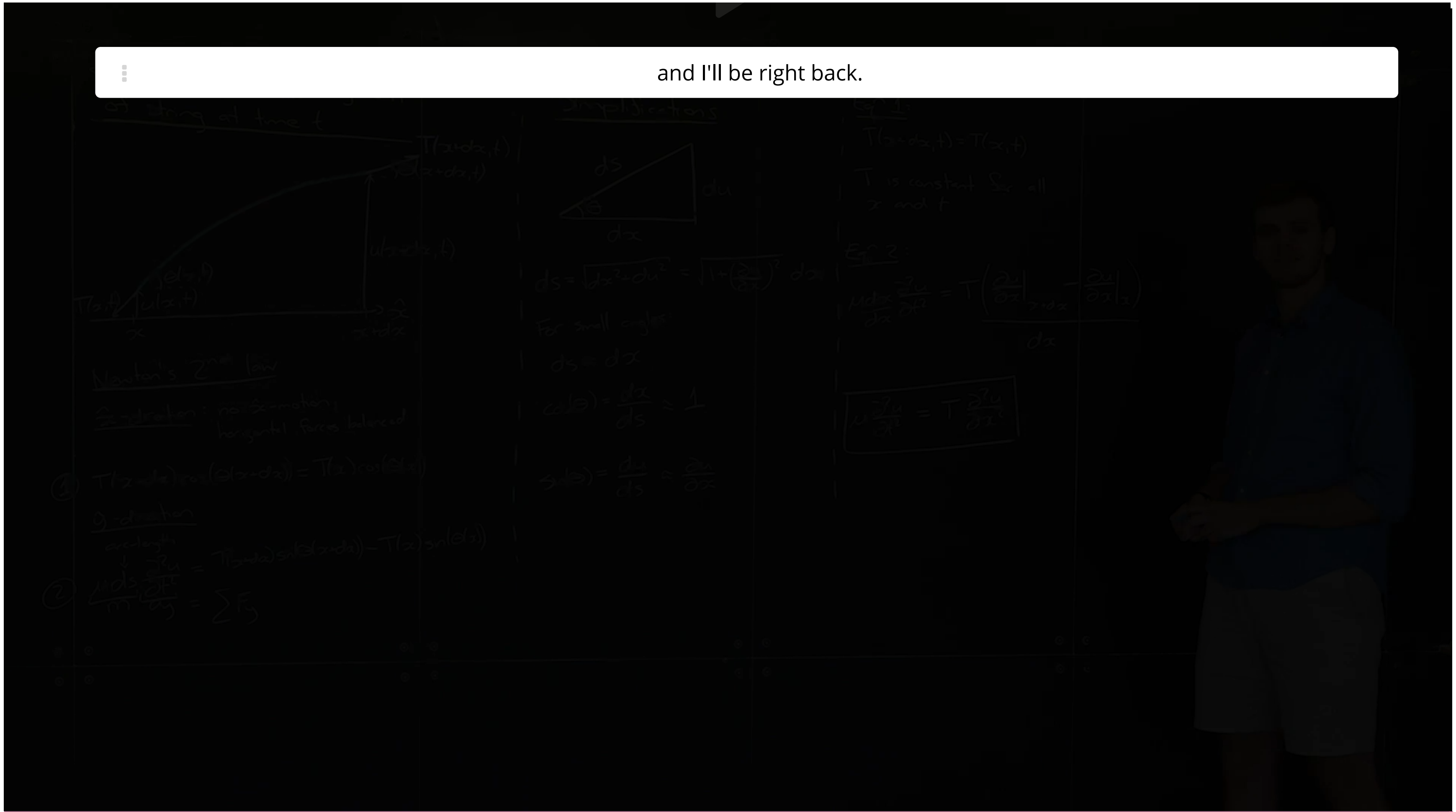
2. Modeling transverse waves

Modeling the wave equation



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The wave equation is a PDE that models light waves, sound waves, waves along a string, etc.

Problem 2.1 Model a vibrating guitar string.

Variables and functions: Define

L	length of the string
μ	mass per unit length
T	**magnitude** of the tension force
t	time
x	position along the string (from 0 to L)
u	vertical displacement of a point on the string

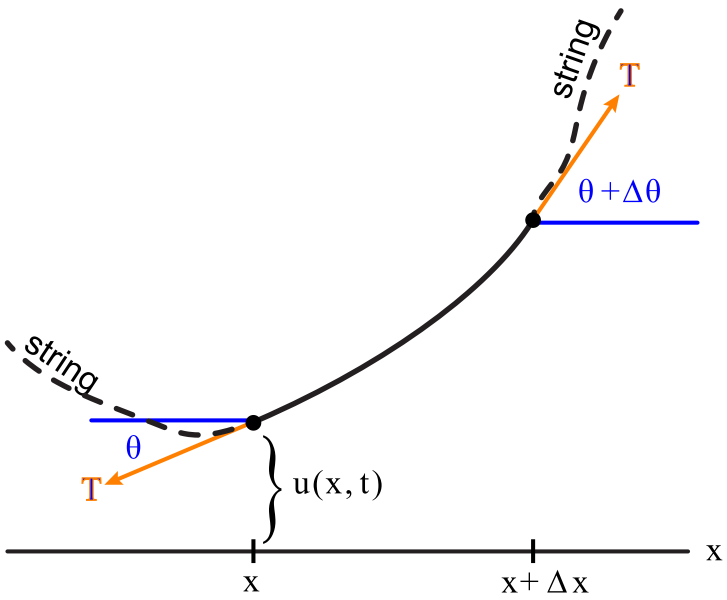
Here

- L, μ, T are constants;
- t, x are independent variables; and
- $u = u(x, t)$ is a function defined for $x \in [0, L]$ and $t \geq 0$. The vertical displacement is measured relative to the equilibrium position in which the string makes a straight line.

At any given time t, the string is in the shape of the graph of $u(x, t)$ as a function of x.

Assumption: The string is taut, so the vertical displacement of the string is small, and the slope of the string at any point is small.

Consider the piece of string between positions x and $x + \Delta x$. Let θ be the (small) angle formed by the string and the horizontal line at position x, and let $\theta + \Delta\theta$ be the same angle at position $x + \Delta x$.



Derivation of the Wave Equation

Newton's second law says that $m\alpha = \mathbf{F}$. Taking the vertical component of each side gives

$$\begin{aligned} \underbrace{\mu\, dx}_{\text{mass}} \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} &= \underbrace{T \sin(\theta + d\theta) - T \sin\theta}_{\text{vertical component of force}} \\ &= T\, d(\sin\theta)\,. \end{aligned}$$

Side calculation:

$$\begin{aligned} d(\sin\theta) &= \cos\theta\, d\theta \\ d(\tan\theta) &= \frac{1}{\cos^2\theta}\, d\theta, \end{aligned}$$

but $\cos\theta = 1 - \frac{\theta^2}{2!} + \cdots \approx 1$, so up to a factor that is very close to 1 (because we are assuming θ is small), we get

$$d(\sin\theta) \approx d\underbrace{(\tan\theta)}_{\text{slope of string}} = d\left(\frac{\partial u}{\partial x}\right).$$

Substituting this in gives

$$\mu\, dx\, \frac{\partial^2 u}{\partial t^2} \approx T\, d\left(\frac{\partial u}{\partial x}\right).$$

Divide by $\mu\, dx$ to get

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &\approx T\mu^{-1}\frac{d\left(\frac{\partial u}{\partial x}\right)}{dx}\\ &\approx T\mu^{-1}\frac{\partial^2 u}{\partial x^2}.\end{aligned}$$

If we define a new constant $c := \sqrt{T\mu^{-1}}$, then this becomes the

$$\text{wave equation:} \quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Note about dimensions: Observe that T is a force, so it has dimension of mass times length over time squared $[mL/t^2]$. And μ is a mass per unit length $[m/L]$. So the constant $c = \sqrt{T/\mu}$ has dimension of $[L/t]$ velocity.

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The resulting equation describing the evolution of the wave over time is given by the following PDE:

$$\text{wave equation:} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This makes sense intuitively, since at places where the graph of the string is concave up $\left(\frac{\partial^2 u}{\partial x^2} > 0\right)$ the tension pulling on both sides should combine to produce an upward force, and hence an upward acceleration.

Comparing units of both sides of the wave equation shows that the units for c are $\mathrm{m/s}$. The physical meaning of c as a velocity will be explained later.

The ends of a guitar string are fixed, so we have boundary conditions

$$\begin{aligned} u(0,t) &= 0 \quad \text{for all } t \geq 0 \\ u(L,t) &= 0 \quad \text{for all } t \geq 0. \end{aligned}$$

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