5.4 Unit 5 Homework Problems

Unit 5: Averages

Adapted from Blitzstein-Hwang Chapters 4 and 5.

Problem 1

2/2 points (graded)

Bobo, the amoeba from Unit 2, currently lives alone in a pond. After one minute Bobo will either die, split into two amoebas, or stay the same, with equal probability. Find the expectation of the number of amoebas in the pond after one minute.

1

Answer: 1

1

Find the variance of the number of amoebas in the pond after one minute.

2/3

Answer: 2/3

$\frac{2}{3}$

Solution:

Let X be the number of amoebas in the pond after 1 minute, so $P(X=0)=P(X=1)=P(X=2)=1/3$. Then

$$E(X)=\frac{1}{3}\cdot 0+\frac{1}{3}\cdot 1+\frac{1}{3}\cdot 2=1,$$

$$E(X^2)=\frac{1}{3}\cdot 0^2+\frac{1}{3}\cdot 1^2+\frac{1}{3}\cdot 2^2=\frac{5}{3},$$

$$\mathrm{Var}(X)=E(X^2)-(E(X))^2=\frac{5}{3}-1^2=\frac{2}{3}.$$

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Answers are displayed within the problem

Problem 2

1/1 point (graded)

Two researchers independently select simple random samples from a population of size N, with sample sizes m and n (for each researcher, the sampling is done without replacement, with all samples of the prescribed size equally likely). Find the expected size of the overlap of the two samples as a function of N, n and m. For $m = 20, n = 30, N = 100,$ find the expected size of the overlap of the two samples.

6 **Answer:** 6

6

Solution

Label the elements of the population $1, 2, \ldots, N$, and let I_j be the indicator of element j being in both samples. Let X be the size of the overlap. By symmetry, linearity, and the fundamental bridge,

$$E(X) = N\left(\frac{m}{N} \cdot \frac{n}{N}\right) = \frac{mn}{N}.$$

Alternatively, note that $X \sim \text{HGeom}(m, N - m, n)$ by the story of the Hypergeometric (imagine two sets of tags, one for each researcher). So again we have

$$E(X) = n\frac{m}{N} = \frac{mn}{N}.$$

For $m = 20, n = 30, N = 100,$ this evaluates to 6.

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Answers are displayed within the problem

Problem 3

1/1 point (graded)

For $X \sim \text{Pois}(3)$, find $E(2^X)$, if it is finite.

20.08554 **Answer:** 20.086

20.08554

Solution:
Let $\lambda = 3$. By LOTUS,

$$E(2^X) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(2\lambda)^k}{k!} = e^{-\lambda} e^{2\lambda} = e^{\lambda} = e^3 \approx 20.086.$$

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Problem 4

1/1 point (graded)
Three students are working independently on their probability homework. They all start at the same time. Each takes an Exponential time with mean 6 hours to complete the homework. How many hours will it take until all 3 students have completed the homework, on average?

11 **Answer:** 11

11

Solution:
Label the students as $1, 2, 3$, and let X_j be how long it takes student j to finish the homework. Let T be the time it takes for all 3 students to complete the homework, so $T = T_1 + T_2 + T_3$ where $T_1 = \min(X_1, X_2, X_3)$ is how long it takes for one student to complete the homework, T_2 is the additional time it takes for a second student to complete the homework, and T_3 is the additional time until all 3 have completed the homework. Then $T_1 \sim \textbf{Expo}(\frac{3}{6})$ since the minimum of independent Exponentials is Exponential with rate the sum of the rates. By the memoryless property, at the first time when a student completes the homework the other two students are starting from fresh, so $T_2 \sim \textbf{Expo}(\frac{2}{6})$. Again by the memoryless property, $T_3 \sim \textbf{Expo}(\frac{1}{6})$. Thus,

$$E(T) = 2 + 3 + 6 = 11.$$

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Answers are displayed within the problem

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