MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Problem 5: Fire alarm

(4/4 points)

Consider a fire alarm that senses the environment constantly to figure out if there is smoke in the air and hence to conclude whether there is a fire or not. Consider a simple model for this phenomenon. Let Θ be the unknown true state of the environment: $\Theta = 1$ means that there is a fire and $\Theta = 0$ means that there is no fire. The signal observed by the alarm at time n is $X_n = \Theta + W_n$, where the random variable W_n represents noise. Assume that W_n is Gaussian with mean 0 and variance 1 and is independent of Θ. Furthermore, assume that for $i \neq j$, W_i and W_j are independent. Suppose that Θ is 1 with probability 0.1 and 0 with probability 0.9.

Give numerical answers for all parts below.

1. Given the observation $X_1 = 0.5$, calculate the posterior distribution of Θ. That is, find the conditional distribution of Θ given $X_1 = 0.5$.

$\mathbf{P}(\Theta = 0 \mid X_1 = 0.5) =$ 0.9 **Answer:** 0.9

$\mathbf{P}(\Theta = 1 \mid X_1 = 0.5) =$ 0.1 **Answer:** 0.1

- Unit 6: Further topics on random variables
- Unit 7: Bayesian inference
- Exam 2

Exam 2
Exam 2 due Apr 20, 2016 at 23:59 UTC

- Unit 8: Limit theorems and classical statistics
- Unit 9: Bernoulli and Poisson processes
- Unit 10: Markov chains
- Exit Survey
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2. What is the LMS estimate of Θ given $X_1 = 0.5$?

$\hat{\theta}_{LMS} =$ 0.1 ✔ **Answer:** 0.1

3. What is the resulting conditional mean squared error of the LMS estimator given $X_1 = 0.5$?

0.09 ✔ **Answer:** 0.09

Answer:

1. By symmetry, $f_{W_1}(w) = f_{W_1}(-w)$. Using Bayes' rule,

$$\begin{aligned}
p_{\Theta|X_1}(0 \mid 0.5) &= \frac{p_\Theta(0) f_{X_1|\Theta}(0.5 \mid 0)}{f_{X_1}(0.5)} \\
&= \frac{p_\Theta(0) f_{X_1|\Theta}(0.5 \mid 0)}{p_\Theta(0) f_{X_1|\Theta}(0.5 \mid 0) + p_\Theta(1) f_{X_1|\Theta}(0.5 \mid 1)} \\
&= \frac{0.9 \cdot f_{W_1}(0.5)}{0.9 \cdot f_{W_1}(0.5) + 0.1 \cdot f_{W_1}(-0.5)} \\
&= \frac{0.9 \cdot f_{W_1}(0.5)}{f_{W_1}(0.5)} \\
&= 0.9.
\end{aligned}$$

Hence, $p_{\Theta|X_1}(1 \mid 0.5) = 0.1$.

2. From the posterior distribution found in part (1), we calculate

$$\hat{\theta}_{LMS} = \mathbf{E}[\Theta \mid X_1 = 0.5] = 0.1.$$

3. The conditional mean squared error of the LMS estimator is the conditional variance:

$$\mathrm{var}(\Theta \mid X_1 = 0.5) = 0.9 \cdot 0.1 = 0.09.$$

You have used 2 of 2 submissions