# 3. Fibonacci sequence

The **Fibonacci sequence** $1, 1, 2, 3, 5, 8, 13, \ldots$ is defined recursively by:

$$F_0 = 1, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1.$$

In this problem, we'll use eigenvalues to compute an explicit formula for and describe the growth rate of $F_n$.

## Fibonacci part (a)

1.0/1 point (graded)

Find a $2 \times 2$ matrix $\mathbf{M}$ such that, for any $k \geq 1$,

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \mathbf{M} \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix}.$$

(Enter a matrix in square brackets, entries in each row separated by commas, rows separated by semicolons: e.g. type **[a, b; c, d]** for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.)

$\mathbf{M} =$ [1,1;1,0] ✔ **Answer:** [1, 1; 1, 0]

**Solution:**

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \begin{pmatrix} F_k + F_{k-1} \\ F_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix},$$

so $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

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We can compute $F_n$ using a simple matrix expression:

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \mathbf{M}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} = \cdots = \mathbf{M}^{n-2} \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = \mathbf{M}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \mathbf{M}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The goal of the rest of the problem will be to compute $\mathbf{M}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and thus $F_n$.

## Part (b)

2/2 points (graded)

Find the eigenvalues $\lambda_1 > \lambda_2$. (Enter the larger eigenvalue in the first answer box. Give an analytic expression rather than a numerical one.)

$\lambda_1 =$ (1+sqrt(5))/2 **Answer:** (1+sqrt(5))/2

$\frac{1+\sqrt{5}}{2}$

$\lambda_2 =$ (1-sqrt(5))/2 **Answer:** (1-sqrt(5))/2

$\frac{1-\sqrt{5}}{2}$

**Solution:**

The eigenvalues are the roots of the characteristic polynomial

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{M}) &= \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right) = \det\begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{pmatrix} \\ &= (\lambda - 1)\lambda - (-1)(-1) = \lambda^2 - \lambda - 1. \end{aligned}$$

Its roots are given by

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

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## Part (c)

2/2 points (graded)

Find the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of $\mathbf{M}$ corresponding to eigenvalues $\lambda_1 > \lambda_2$. In particular, find the numbers $u$ and $w$ so that

$$\mathbf{v}_1 = \begin{pmatrix} u \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} w \\ 1 \end{pmatrix}.$$

*Hint: Use the fact that* $\lambda^2 - \lambda - 1 = 0$ *can be rearranged to show* $\lambda - 1 = 1/\lambda$. *Use this in your computations to simplify algebra.*

(Note that $\mathbf{v}_1$ is the eigenvector corresponding to the larger eigenvalue.)

$u =$ (1+sqrt(5))/2 **Answer:** (1+sqrt(5))/2

$\frac{1+\sqrt{5}}{2}$

$w =$ (1-sqrt(5))/2 **Answer:** (1-sqrt(5))/2

$\frac{1-\sqrt{5}}{2}$

**Solution:**

With $\lambda$ equal to either $\lambda_1$ or $\lambda_2$ we have

$$\lambda - 1 = 1/\lambda.$$

The eigenvectors are the elements in the nullspace of the matrix $\lambda\mathbf{I} - \mathbf{M}$.

$$\mathrm{NS}(\lambda\mathbf{I} - \mathbf{M}) = \mathrm{NS}\begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{pmatrix} = \mathrm{NS}\begin{pmatrix} 1/\lambda & -1 \\ -1 & \lambda \end{pmatrix}.$$

Subtracting a multiple of row 1 from row 2 does not change the nullspace, therefore

$$\mathrm{NS}\begin{pmatrix} 1/\lambda & -1 \\ -1 & \lambda \end{pmatrix} = \mathrm{NS}\begin{pmatrix} 1/\lambda_i & -1 \\ 0 & 0 \end{pmatrix}.$$

Therefore $\begin{pmatrix} \lambda \\ 1 \end{pmatrix}$ is an eigenvector.

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## Part (d)

2/2 points (graded)

Write the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination $a\mathbf{v}_1 + b\mathbf{v}_2$ of the eigenvectors.

$a =$ (sqrt(5)+1)/(2*sqrt(5)) **Answer:** (1+sqrt(5))/(2*sqrt(5))

$\frac{\sqrt{5}+1}{2\cdot\sqrt{5}}$

$b =$ (sqrt(5)-1)/(2*sqrt(5)) **Answer:** (-1+sqrt(5))/(2*sqrt(5))

$\frac{\sqrt{5}-1}{2\cdot\sqrt{5}}$

**Solution:**

We want

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + b \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix},$$

so $a$ and $b$ satisfy the system of linear equations

$$\begin{aligned} a\lambda_1 + b\lambda_2 &= 1 \\ a + b &= 1. \end{aligned}$$

Solving this gives

$$a = \frac{1-\lambda_2}{\lambda_1 - \lambda_2} = \frac{1+\sqrt{5}}{2\sqrt{5}} \quad \text{and} \quad b = 1 - a = \frac{-1+\sqrt{5}}{2\sqrt{5}}.$$

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## Part (e)

1/1 point (graded)

Use your answers from parts (a)-(d) to find a formula for $F_n$. The formula should be something you can compute directly, not something that writes $F_n$ in terms of other $F_j$.

$F_n =$

(1/sqrt(5))*(((1+sqrt(5))/2)^(n+1) - ((1-sqrt(5))/2)^(n+1))

**Answer:** (1/sqrt(5))*( ((1+sqrt(5))/2)^(n+1) - ((1-sqrt(5))/2)^(n+1) )

$$\left(\frac{1}{\sqrt{5}}\right) \cdot \left( \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right)$$

**Solution:**

We have

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2,$$

and

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \mathbf{M}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Expanding this out and using the fact that $\mathbf{M}^k \mathbf{v}_i = \lambda_i^k \mathbf{v}_i$, we get

$$\begin{aligned} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \mathbf{M}^{n-1}(a\mathbf{v}_1 + b\mathbf{v}_2) \\ &= a\lambda_1^{n-1}\mathbf{v}_1 + b\lambda_2^{n-1}\mathbf{v}_2 \\ &= a\lambda_1^{n-1}\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + b\lambda_2^{n-1}\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \\ &= \frac{\frac{1+\sqrt{5}}{2}}{\sqrt{5}}\begin{pmatrix} \lambda_1^n \\ \lambda_1^{n-1} \end{pmatrix} + \frac{\frac{-1+\sqrt{5}}{2}}{\sqrt{5}}\begin{pmatrix} \lambda_2^n \\ \lambda_2^{n-1} \end{pmatrix} \\ &= \frac{\lambda_1}{\sqrt{5}}\begin{pmatrix} \lambda_1^n \\ \lambda_1^{n-1} \end{pmatrix} - \frac{\lambda_2}{\sqrt{5}}\begin{pmatrix} \lambda_2^n \\ \lambda_2^{n-1} \end{pmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}. \end{aligned}$$

Taking the first coordinate gives our formula for $F_n$:

$$F_n = \frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1}) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

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## Part (f)

1/1 point (graded)

Find $\ln F_{1,000,000}$ to 10 decimal places.

(Use MATLAB online. Note that in MATLAB, typing `log` gives the natural logarithm, not the base 10 logarithm. Observe that MATLAB typically only displays four decimal places, but stores a larger number. To see the larger number of decimal digits type `format long` before you display numbers of interest.)

481211.5015524723

✔ **Answer:** 481211.5015524723

**Solution:**

In the previous problem you found that

$$F_n = \frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1}).$$

The term $-\lambda_2^{n+1}$ is negligible for large $n$ since $|\lambda_2| < 1$; therefore

$$\ln F_n \approx \ln \frac{1}{\sqrt{5}} + (n+1)\ln(\lambda_1).$$

We compute this directly using MATLAB for $n = 10^6$.

Note that if you try to compute $\ln F_{10^6}$ directly, MATLAB will output `inf`. Instead, you must first observe that only the term involving the larger eigenvalue will contribute most. If you omit the $\ln(1/\sqrt{5})$ term your answer will not be correct up to the desired number of decimal digits.

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## Part (g)

2/2 points (graded)

Consider a generalized fibonacci $G$ sequence $1, 1, 1, 3, 5, 9, 17, \ldots$ is created by summing the last 3 entries in the sequence together:

$$\begin{aligned} G_0 &= 1, \quad G_1 = 1, \quad G_2 = 1, \quad \text{and} \\ G_{n+1} &= G_n + G_{n-1} + G_{n-2} \text{ for } n \geq 2. \end{aligned}$$

Find a $3 \times 3$ matrix $\mathbf{M}$ such that, for any $k \geq 2$,

$$\begin{pmatrix} G_{k+1} \\ G_k \\ G_{k-1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} G_k \\ G_{k-1} \\ G_{k-2} \end{pmatrix}.$$

Use MATLAB online to find a numerical value for $G_{25}$.

$G_{25} =$ 1800281 **Answer:** 1800281

Find $\lim_{n\to\infty} \frac{\ln G_n}{n}$ to 10 decimal places.

$\lim_{n\to\infty} \frac{\ln G_n}{n} =$ 0.609377863436006 **Answer:** 0.609377863436006

**Solution:**

In this case $\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Since

$$\begin{pmatrix} G_3 \\ G_2 \\ G_1 \end{pmatrix} = \mathbf{M} \begin{pmatrix} G_2 \\ G_1 \\ G_0 \end{pmatrix},$$

$$\begin{pmatrix} G_4 \\ G_3 \\ G_2 \end{pmatrix} = \mathbf{M}^2 \begin{pmatrix} G_2 \\ G_1 \\ G_0 \end{pmatrix},$$

it follows that $G_{25}$ is the first entry in the vector given by $\mathbf{M}^{23} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Plugging into MATLAB for use as a calculator gives the answer directly.

To find $\lim_{n\to\infty} \frac{\ln G_n}{n}$, we use an approach analogous to what we did with the Fibonacci sequence. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be eigenvectors of $\mathbf{M}$ so that $\lambda_1$ is the largest eigenvalue.

We start by expressing the vector representing the first three numbers as a linear combination of the eigenvectors

$$\begin{pmatrix} G_2 \\ G_1 \\ G_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3.$$

We write

$$\begin{aligned} \begin{pmatrix} G_n \\ G_{n-1} \\ G_{n-2} \end{pmatrix} &= \mathbf{M}^{n-2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \mathbf{M}^{n-2}(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \\ &= a_1\lambda_1^{n-2}\mathbf{v}_1 + a_2\lambda_2^{n-2}\mathbf{v}_2 + a_3\lambda_3^{n-2}\mathbf{v}_3 \\ &\approx a_1\lambda_1^{n-2}\mathbf{v}_1 \end{aligned}$$

The number $G_n$ is the first entry of this vector, which is

$$G_n = b_1\lambda_1^{n-2},$$

where $b_1 = a_1$ times the first entry of $\mathbf{v}_1$. Taking the natural logarithm and dividing by $n$ we get

$$\frac{\ln G_n}{n} = \frac{\ln b_1 + (n+2)\ln(\lambda_1)}{n}.$$

As $n$ tends to infinity, this tends to $\ln(\lambda_1)$, which we compute using MATLAB. (In this case the constant term $\ln b_1/n$ is zero in the limit.)

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