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10. Composition and matrix multiplication

Suppose that we have two functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. The first rotates all points 90 degrees counter clockwise about the origin. The second reflects across the x-axis. Both of these functions can be represented by a matrix.

90 degree rotation counterclockwise:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

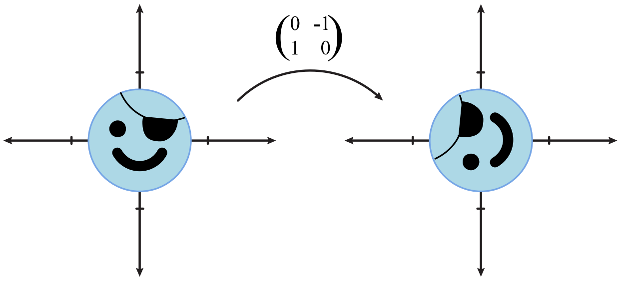
$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

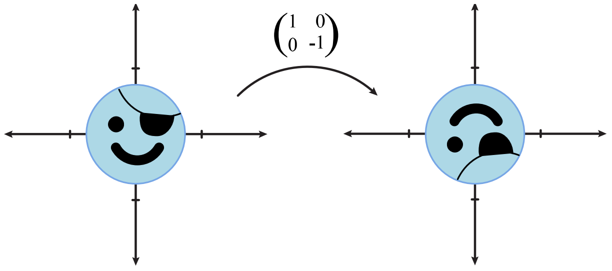
Reflection across x-axis:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

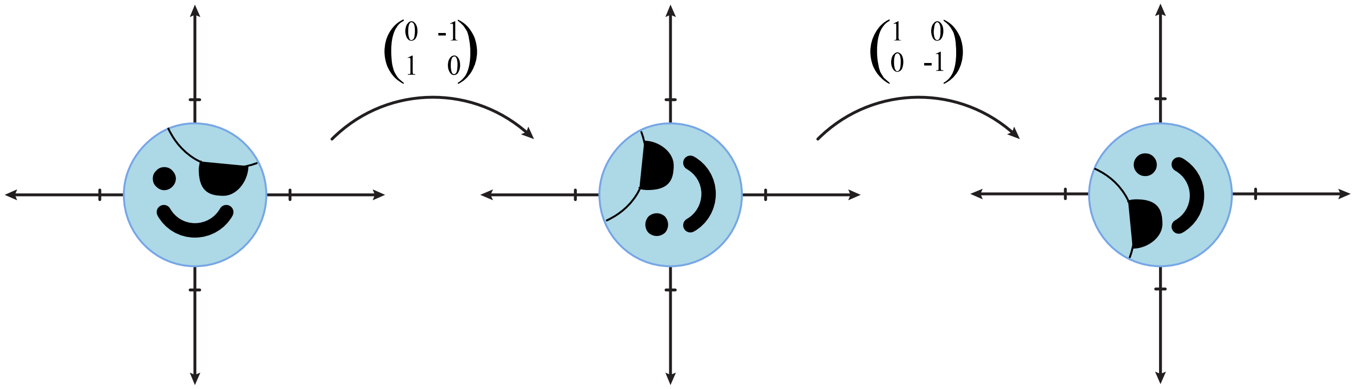
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$





We can compose these two functions, first rotating 90 degrees counterclockwise, and then reflecting across the x-axis.



But what matrix represents this composed function? One way is to figure it out by seeing where each of the standard basis vectors is sent by the composition.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Therefore the matrix representing the composed matrix is

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

There is an easier way to figure out what the composed matrix is in terms of the original two matrices. And that's the matrix product.

Defining the matrix product

Suppose $\mathbf{A}$ is an $m \times n$ matrix, and $\mathbf{B}$ is an $n \times p$ matrix. Starting with a vector $\mathbf{v}$ in $\mathbb{R}^p$, applying left multiplication by the matrix $\mathbf{B}$ gives a vector in $\mathbb{R}^n$. Then applying left multiplication by $\mathbf{A}$ gives a vector in $\mathbb{R}^m$. This composition of functions defines a new function sending the vector $\mathbf{v}$ to $\mathbf{A}(\mathbf{B}\mathbf{v})$.

$$\begin{tikzcd}[column sep=huge]
\mathbb{R}^p \arrow[r, bend left, "\text{left multiplication by } \mathbf{B}"] \arrow[rr, dashed, bend right, "\text{left multiplication by ?}"'] & \mathbb{R}^n \arrow[r, bend left, "\text{left multiplication by } \mathbf{A}"] & \mathbb{R}^m
\end{tikzcd}$$

This composition function itself can be viewed as left multiplication by some matrix. What is it? The answer is called the matrix product $\mathbf{AB}$. The matrix product $\mathbf{AB}$ is characterized by the property

$$(\mathbf{AB})\mathbf{v} = \mathbf{A}(\mathbf{B}\mathbf{v}) \qquad \textbf{for all vectors } \mathbf{v} \textbf{ in } \mathbb{R}^p.$$

The matrix product $\mathbf{AB}$ is defined if and only if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

Quick question

1/1 point (graded)

Suppose $\mathbf{A}$ is an $m \times n$ matrix, and $\mathbf{B}$ is an $n \times p$ matrix. What is the size of the product matrix $\mathbf{AB}$?

○ $m \times n$

○ $n \times p$

● $m \times p$ ✓

○ $p \times m$

○ $p \times n$

○ $n \times m$

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✓ Correct (1/1 point)

Matrix multiplication

Let us illustrate matrix multiplication with an example.

Example 10.1 Multiply a 2×3 matrix by a 3×2 a matrix:

$$\begin{pmatrix} 2 & 4 & 1 \\ -3 & 5 & -1 \end{pmatrix} \begin{pmatrix} x & u \\ y & v \\ z & w \end{pmatrix} = \begin{pmatrix} 2x + 4y + 1z & 2u + 4v + 1w \\ -3x + 5y - 1z & -3u + 5v - 1w \end{pmatrix}.$$

The product is a 2×2 matrix. The **i, j-entry** of a matrix is the entry in the ith row and jth column of the matrix.

Consider two matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & & \vdots \\ & & & a_{ij} & & \\ \vdots & \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & & \cdots & a_{mn} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & & & \vdots \\ & & & b_{ij} & & \\ \vdots & \vdots & & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & & \cdots & b_{np} \end{pmatrix}$$

The i, j-entry of the product $\mathbf{AB}$ is the dot product of the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$. For example, the entry in the third row and second column of $\mathbf{C} = \mathbf{AB}$ is the product of the third row of $\mathbf{A}$ and the second column of $\mathbf{B}$:

$$c_{32} = \begin{pmatrix} a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix} = a_{31}b_{12} + a_{32}b_{22} + \cdots + a_{3n}b_{n2}.$$

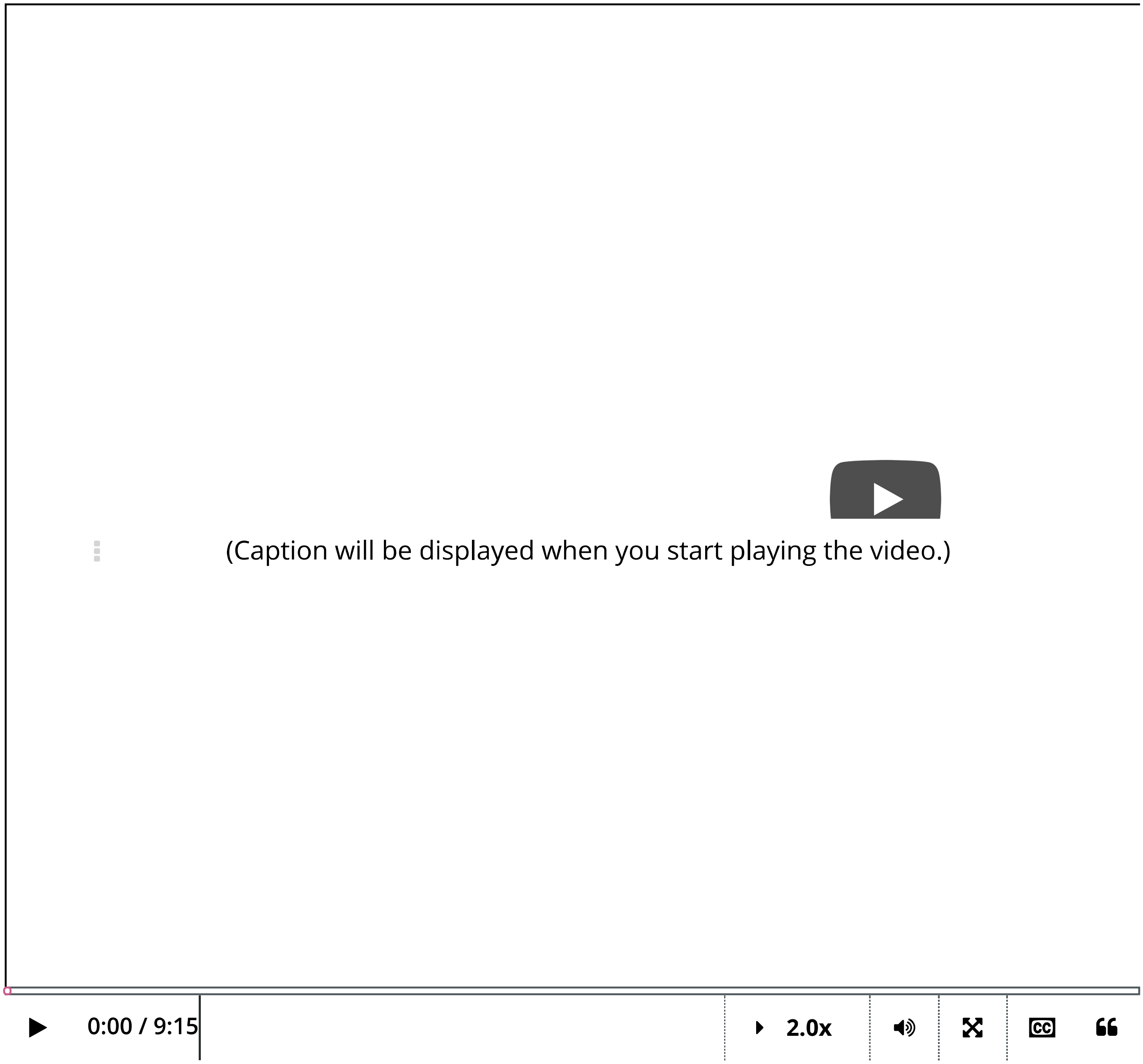
Another perspective:

We can also compute the matrix product one column at a time. Multiplying $\mathbf{A}$ by (the jth column of $\mathbf{B}$) gives (the jth column of $\mathbf{C} = \mathbf{AB}$).

$$\mathbf{A}(\textbf{the } j\textbf{th column of } \mathbf{B}) = \textbf{the } j\textbf{th column of } \mathbf{C}$$

This shows that each column of $\mathbf{C}$ is a linear combination of the columns of $\mathbf{A}$. The coefficients of that linear combination are the entries in the the corresponding column of $\mathbf{B}$.

How to multiply matrices



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The identity matrix

Let's consider the do-nothing function from $\mathbb{R}^3$ to $\mathbb{R}^3$, called the identity function. This is the function $\mathbf{I}$ that sends every vector $\mathbf{x}$ back to itself:

$$\mathbf{I}(\mathbf{x}) = \mathbf{x}.$$

The matrix that represents this function is called the **identity matrix** $\mathbf{I}$.

The first column of $\mathbf{I}$ is $\mathbf{I}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

The second column of $\mathbf{I}$ is $\mathbf{I}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

The third column of $\mathbf{I}$ is $\mathbf{I}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Therefore $\mathbf{I} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

For any n, the $n \times n$ identity matrix is the matrix with ones along the diagonal and zeros elsewhere:

$$\mathbf{I} = \begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&&\ddots&\vdots\\0&\cdots&0&1\end{pmatrix}.$$

For any $n \times n$ matrix $\mathbf{A}$,

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}.$$

More generally, we have $\mathbf{IA} = \mathbf{A}$ for any matrix $\mathbf{A}$ for which the product is defined (i.e., any matrix with n rows); this is because applying the function associated to $\mathbf{A}$ and then doing nothing is the same as applying the function associated to $\mathbf{A}$. Similarly, $\mathbf{BI} = \mathbf{B}$ for any matrix $\mathbf{B}$ for which the product is defined (i.e., any matrix with n columns).

What is the size of the product matrix?

1/1 point (graded)

Take the product of the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to form a new matrix $\mathbf{AB}$.

What is the size of the product matrix $\mathbf{AB}$?

- ● 1×4 ✓
- ○ 5×1
- ○ 4×5
- ○ 5×4
- ○ These matrices cannot be multiplied.

Solution:

The matrix $\mathbf{A}$ has one row and five columns, 1×5. The matrix $\mathbf{B}$ is 5×4. The product is a 1×4 matrix.

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You have used 1 of 5 attempts

ℹ Answers are displayed within the problem

Find the size again

1/1 point (graded)

Take the product of the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

to form a new matrix $\mathbf{AB}$.

What is the size of the product matrix $\mathbf{AB}$?

○ 4×4

○ 4×5

○ 5×4

○ 5×5

● These matrices cannot be multiplied. ✓

Solution:

To multiply matrices $\mathbf{A}$ and $\mathbf{B}$, the number of columns of $\mathbf{A}$ must be equal the number of rows of $\mathbf{B}$. Since $\mathbf{A}$ has 5 columns and $\mathbf{B}$ has 4 rows, these matrices cannot be multiplied.

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Which matrices multiply?

1/1 point (graded)

Suppose a matrix product $\mathbf{AB}$ is defined and

$$\mathbf{B} = \begin{pmatrix} 1 & 4 & 5 & 3 & 5 & 7 \\ 2 & 0 & 7 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 4 & 5 \end{pmatrix}.$$

Which of the following are candidates for the matrix $\mathbf{A}$? (Choose all that are possible.)

☑ $\begin{pmatrix} 1 & 4 & 5 \end{pmatrix}$ ✓

☐ $\begin{pmatrix} 1 & 4 & 5 & 1 & 0 & 1 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 4 \\ 5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

☑ $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 7 \end{pmatrix}$ ✓

☐ $\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 & 1 & 1 \end{pmatrix}$

☑ $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 7 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 4 & 0 \end{pmatrix}$ ✓

☐ $\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

✓

Solution:

The matrix $\mathbf{B}$ is 3×6. Therefore $\mathbf{A}$ must be $? \times 3$. This means that $\mathbf{A}$ must have 3 columns. Any matrix with 3 columns works. The options that are correct are $\begin{pmatrix} 1 & 4 & 5 \end{pmatrix}$,

$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 7 \end{pmatrix}$, and $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 7 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 4 & 0 \end{pmatrix}$

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Answers are displayed within the problem

More properties of matrix products

The following identities hold whenever the matrix products are defined.

1. $\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}$ (associativity)
2. $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}+\mathbf{B}\mathbf{C}$ and $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{A}\mathbf{B}+\mathbf{A}\mathbf{C}$ (distributivity)

The proofs can be found in any linear algebra textbook.

10. Composition and matrix multiplication

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