5. Matrix notation and matrix algebra

We will be using matrices throughout this course. For this reason, it is helpful to ensure that we all describe matrices and think about them in a similar way.

We will start with the novice view of a matrix as a table of numbers:

$$\begin{pmatrix} 1 & 3 & 3 & 0 & 3 \\ 3 & 47 & 3 & 9 & 38 \\ 8 & 3 & 0 & 0 & 0 \\ 0 & 0 & 32 & 4 & 3 \\ 1 & 2 & 1 & 0 & 1 \end{pmatrix}.$$

If we do not know what these numbers are, or wish to describe a generic matrix, we may choose to use variables instead:

$$\begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{pmatrix}.$$

However, as our matrices get larger and larger, we tend to run out of letters pretty quickly. Instead, use a single letter and subscripts to denote its location within the matrix. We say a matrix $\mathbf{A}$ is $m \times n$ if it has m **rows** and n **columns** . We denote the entry in the ith row and jth column by a_{ij} , and write the full matrix as

$$\begin{array}{cc} & \text{column } j \\ \text{row } i & \begin{pmatrix} a_{11} & a_{12} & \cdots & | & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & | & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & | & \cdots & \vdots \\ - & - & - & a_{ij} & - & - \\ \vdots & \vdots & \cdots & | & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & | & \cdots & a_{mn} \end{pmatrix} \end{array}.$$

In this course, we will mostly be concerned with square matrices, which are matrices whose number of rows is equal to the number of columns.

We can think of multiplying vectors and matrices in 2 equivalent ways as follows:

1. To multiply $\mathbf{A}$ and $\mathbf{x}$ to get a vector $\mathbf{b}$, the jth entry of $\mathbf{b}$ is the dot product of the jth row of $\mathbf{A}$ with $\mathbf{x}$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

2. The product $\mathbf{Ax}$ can be expressed as a linear combination of the columns of $\mathbf{A}$:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

where the coefficients are the entries of the vector $\mathbf{x}$.

Properties of matrix vector multiplication

0. For any matrix $\mathbf{A}$, and the zero vector $\mathbf{0}$:

$$\mathbf{A}\mathbf{0} = \mathbf{0}.$$

1. For any matrix $\mathbf{A}$, a c scalar, and a vector $\mathbf{x}$:

$$\mathbf{A}(c\mathbf{x}) = c(\mathbf{A}\mathbf{x}).$$

2. For any matrix $\mathbf{A}$, and vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}.$$

Matrices as functions

Another view of an $m \times n$ matrix that will be useful for us is to think of it as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Example 5.1 Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 7 \\ 8 & 9 \end{pmatrix}$ as a function $\mathbf{f}$ from $\mathbb{R}^2$ to $\mathbb{R}^3$. To evaluate $\mathbf{f}$ at $\mathbf{v} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$, we multiply the vector $\mathbf{v}$ by the matrix $\mathbf{A}$:

$$\begin{pmatrix} 1 & 2 \\ 4 & 7 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} 10 \\ 1 \end{pmatrix} = \begin{pmatrix} 1(10)+2(1) \\ 4(10)+7(1) \\ 8(10)+9(1) \end{pmatrix} = \begin{pmatrix} 12 \\ 47 \\ 89 \end{pmatrix}$$

The result is a vector with 3 entries; a vector in $\mathbb{R}^3$.

Matrices as functions concept check 1

1/1 point (graded)

Which of the following matrices $\mathbf{A}$ represent a function $\mathbf{f} : \mathbb{R}^2$ to $\mathbb{R}^4$, such that $\mathbf{f}(\mathbf{v}) = \mathbf{A}\mathbf{v}$ (i.e. we multiply the matrix by vectors on the right)?

Check all that apply.

- [ ] $\begin{pmatrix} 2 & 5 \\ -4 & 1 \end{pmatrix}$
- [x] $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ✔
- [ ] $\begin{pmatrix} 0 & 1 & 2 & -3 \\ -3 & 0 & 6 & 1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
- [x] $\begin{pmatrix} 3 & -1 \\ 0 & 1 \\ 4 & -4 \\ -2 & -3 \end{pmatrix}$ ✔

✔

Solution:

A matrix represents a function from $\mathbb{R}^2$ to $\mathbb{R}^4$ if it has 4 rows and 2 columns.

Therefore, the correct answers are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & -1 \\ 0 & 1 \\ 4 & -4 \\ -2 & -3 \end{pmatrix}$.

Note that if we were to allow multiplication by vectors on the left, we could also view $\begin{pmatrix} 0 & 1 & 2 & -3 \\ -3 & 0 & 6 & 1 \end{pmatrix}$ as a function from $\mathbb{R}^2$ to $\mathbb{R}^4$ (via multiplication by a vector on the left hand side of the matrix). However, we will not consider such functions formed by left multiplication in this course. It is more natural from our familiarity of functions on $\mathbb{R}$ to consider multiplication by a vector on the right hand side of the matrix.

Submit You have used 2 of 7 attempts

Answers are displayed within the problem

Matrices as functions concept check 2

2/2 points (graded)

Consider a function represented by a 3 by 5 matrix $\mathbf{A}$.

$\mathbf{A}$ is a function from

- ( ) $\mathbb{R}^3$
- (•) $\mathbb{R}^5$ ✓

to

- (•) $\mathbb{R}^3$ ✓
- ( ) $\mathbb{R}^5$

Solution:

The matrix $\mathbf{A}$ takes vectors from $\mathbb{R}^5$ to vectors in $\mathbb{R}^3$. Because $\mathbf{A}$ is 3×5, it can only be multiplied on the right by vectors with 5 entries. Because each column of $\mathbf{A}$ has 3 entries, the result is a linear combination of these columns, which is a vector in $\mathbb{R}^3$.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Properties of matrix vector multiplication check

1/1 point (graded)

A matrix $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$\mathbf{A}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}2\\4\\-1\end{pmatrix}$$

and

$$\mathbf{A}\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

What is $\mathbf{A}\begin{pmatrix}1\\2\\1\end{pmatrix}$?

Hint: $\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}+2\begin{pmatrix}0\\1\\0\end{pmatrix}$.

- $\begin{pmatrix}0\\0\\0\end{pmatrix}$
- $\begin{pmatrix}2\\4\\-1\end{pmatrix}$
- $\begin{pmatrix}0\\1\\-1\end{pmatrix}$

- $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$
- $\begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix}$ ✔

Solution:

First we express the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ it in terms of the other vectors:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Applying the properties of matrix vector multiplication, we get

$$\begin{aligned} \mathbf{A} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} &= \mathbf{A} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\mathbf{A} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix} \end{aligned}$$

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

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About the third question

I am not sure how the final answer is [answer removed by staff]

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