6.3 Unit 6 Practice Problems

Unit 6: Joint Distributions and Conditional Expectation

Adapted from Blitzstein-Hwang Chapters 7 and 9.

Problem 1a

1/1 point (graded)

(a) A stick is broken into three pieces by picking two points independently and uniformly along the stick, and breaking the stick at those two points. What is the probability that the three pieces can be assembled into a triangle?

0.25

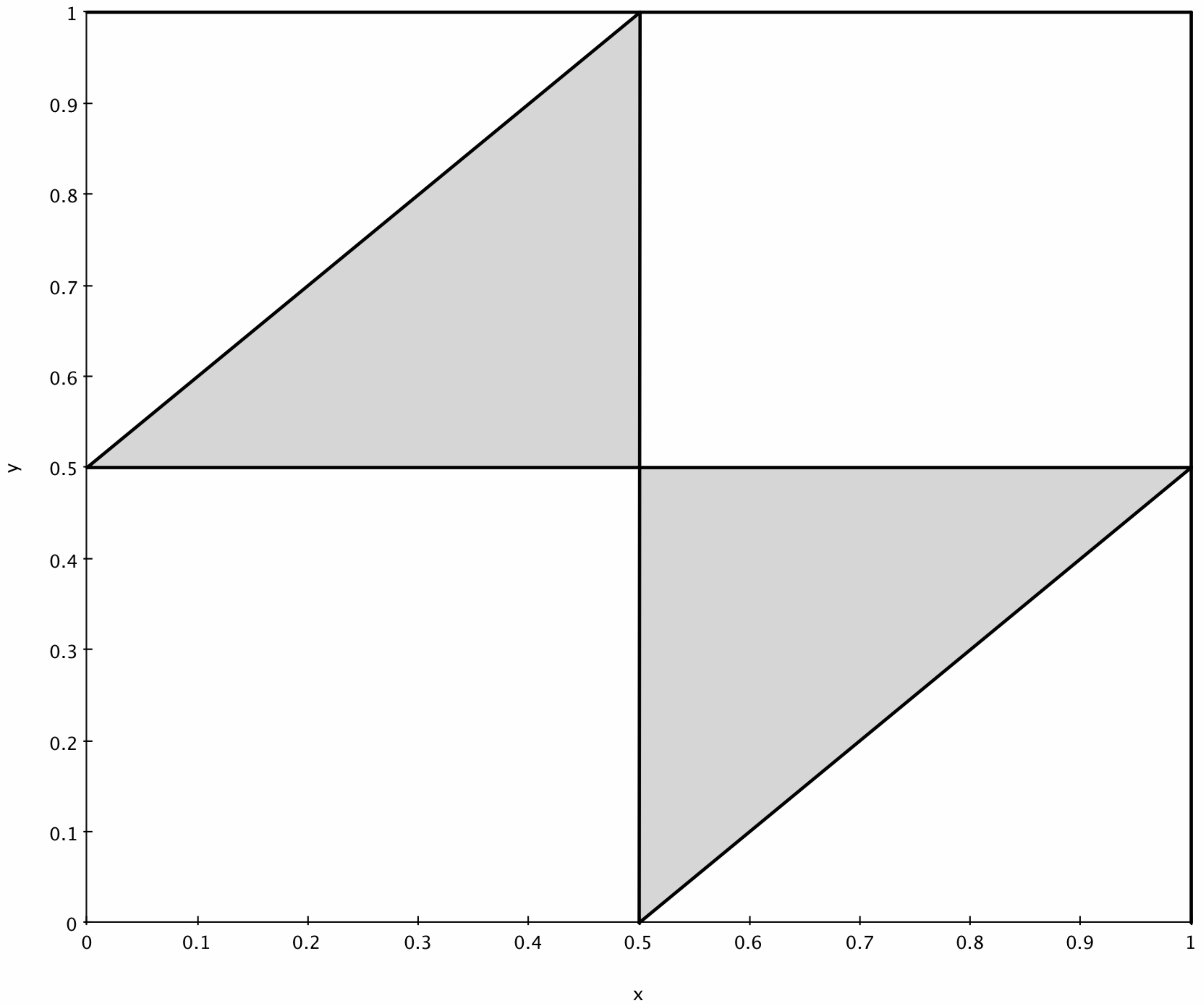
✔ **Answer:** 1/4

0.25

Solution

We can assume the length is 1 (in some choice of units, the length will be 1, and the choice of units for length does not affect whether a triangle can be formed). So let X, Y be i.i.d. $\textbf{Unif}(0,1)$ random variables. Let x and y be the observed values of X and Y respectively. If $x < y$, then the side lengths are $x, y - x$, and $1 - y$, and a triangle can be formed if and only if $y > \frac{1}{2}, y < x + \frac{1}{2}, x < \frac{1}{2}$. Similarly, if $x > y$, then a triangle can be formed if and only if $x > \frac{1}{2}, x < y + \frac{1}{2}, y < \frac{1}{2}$.

Since (X, Y) is Uniform over the square $0 \le x \le 1, 0 \le y \le 1$, the probability of a subregion is proportional to its area. The region given by $y > 1/2, y < x + 1/2, x < 1/2$ is a triangle with area $1/8$, as is the region given by $x > 1/2, x < y + 1/2, y < 1/2$, as illustrated in the picture below. Thus, the probability that a triangle can be formed is $1/8 + 1/8 = 1/4$.



Note that the idea of interpreting probabilities as areas works here because (X, Y) is *Uniform* on the square. For other distributions, in general we would need to find the joint PDF of X, Y and integrate over the appropriate region.

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Answers are displayed within the problem

Problem 1b

1/1 point (graded)

(b) Three legs are positioned uniformly and independently on the perimeter of a round table. What is the probability that the table will stand?

0.25

✔ **Answer:** 1/4

0.25

Solution

Think of the legs as points on a circle, chosen randomly one at a time, and choose units so that the circumference of the circle is 1. Let A, B, C be the arc lengths from one point to the next (clockwise, starting with the first point chosen). Then

$$\begin{aligned} P(\textbf{table falls}) &= P(\textbf{the 3 legs are all contained in some semicircle}) \\ &= P(\textbf{at least one of } A, B, C \textbf{ is greater than } 1/2) = 3/4, \end{aligned}$$

by Part (a). So the probability that the table will stand is $1/4$.

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Answers are displayed within the problem

For Problem 2

Two fair six-sided dice are rolled (one green and one orange), with outcomes X and Y respectively for the green and the orange.

Problem 2a

1/1 point (graded)

Compute the covariance of $X + Y$ and $X - Y$.

0

✔ **Answer:** 0

0

Solution

We have

$$\mathrm{Cov}(X+Y, X-Y) = \mathrm{Cov}(X, X) - \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) - \mathrm{Cov}(Y, Y) = 0.$$

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Problem 2b

1/1 point (graded)

(b) Are $X+Y$ and $X-Y$ independent?

- Yes
- No ✔

Solution

They are not independent: information about $X+Y$ may give information about $X-Y$, as shown by considering an *extreme example*. Note that if $X+Y=12$, then $X=Y=6$, so $X-Y=0$. Therefore, $P(X-Y=0|X+Y=12)=1 \neq P(X-Y=0)$, which shows that $X+Y$ and $X-Y$ are not independent. Alternatively, note that $X+Y$ and $X-Y$ are both even or both odd, since the sum $(X+Y)+(X-Y)=2X$ is even.

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For Problem 3

A coin with probability $p=1/3$ of Heads is flipped repeatedly.

Problem 3a

1/1 point (graded)

(a) What is the expected number of flips until the pattern *HT* is observed?

9/2

Answer: 4.5

$\frac{9}{2}$

Solution

This can be thought of as “Wait for Heads, then wait for the first Tails after the first Heads," so the expected value is $\frac{1}{p} + \frac{1}{q}$, with $q = 1 - p$. For $p = 1/3$, this evaluates to $3 + 1.5 = 4.5$.

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Answers are displayed within the problem

Problem 3b

1/1 point (graded)

(b) What is the expected number of flips until the pattern *HH* is observed?

12

Answer: 12

12

Solution

Let X be the waiting time for HH and condition on the first toss, writing H for the event that the first toss is Heads and T for the complement of H:

$$E(X) = E(X|H)p + E(X|T)q = E(X|H)p + (1 + EX)q.$$

To find $E(X|H)$, condition on the second toss:

$$E(X|H) = E(X|HH)p + E(X|HT)q = 2p + (2 + EX)q.$$

Solving for $E(X)$, we have

$$E(X) = \frac{1}{p} + \frac{1}{p^2}.$$

For $p = 1/3$, this evaluates to $3 + 9 = 12$.

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Answers are displayed within the problem

Problem 4

2/2 points (graded)
Consider a group of n roommate pairs at a college (so there are $2n$ students). Each of these $2n$ students decides randomly whether to take a certain course, with probability p of taking it. The students' choices of whether to take the course are independent.

Let N be the number of students among these $2n$ who take the course, and let X be the number of roommate pairs where both roommates in the pair take the course. For $n = 20$ and $p = 0.5$, find $E(X)$ and $E(X|N = 23)$.

Historical note: An equivalent problem was first solved in the 1760s by Daniel Bernoulli, a nephew of Jacob Bernoulli. (The Bernoulli distribution is named after Jacob Bernoulli.)

$E(X) =$

5 ✔ **Answer:** 5

5

$E(X|N = 23) =$

6.49 ✔ **Answer:** 6.49

6.49

Solution
Create an indicator r.v. I_j for the jth roommate pair, equal to 1 if both take the course. The expected value of such an indicator r.v. is p^2, so $E(X) = np^2$ by symmetry and linearity. For $n = 20$ and $p = 0.5$, this evaluates to 5.
Similarly, $E(X|N) = nE(I_1|N)$. We have

$$E(I_1|N) = \frac{N}{2n}\frac{N-1}{2n-1}$$

since given that N of the $2n$ students take the course, the probability is $\frac{N}{2n}$ that any particular student takes Stat 110 (the p no longer matters), and given that one particular student in a roommate pair takes the course, the probability that the other roommate does is $\frac{N-1}{2n-1}$. Thus,

$$E(X|N) = nE(I_1|N) = \frac{N(N-1)}{2(2n-1)}.$$

For $n = 20$, we have

$$E(X|N=23) = \frac{23 \cdot 22}{2(2 \cdot 20 - 1)} \approx 6.49.$$

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Answers are displayed within the problem