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15. Eigenvectors computation: coupled oscillators

Let us continue with finding the eigenvalues and eigenvectors of the companion matrix of the unforced coupled oscillator:

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{B} & 0 \end{pmatrix} \qquad \text{where } \mathbf{B} = \omega^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Eigenvalues

Let us compute the eigenvalues and eigenvectors of $\mathbf{B}$. The characteristic polynomial of $\mathbf{B}$ is

$$\det(\lambda \mathbf{I} - \mathbf{B}) \;=\; \lambda^2 + 4\omega^2\lambda + 3\omega^4 \;=\; (\lambda + \omega^2)(\lambda + 3\omega^2),$$

so the eigenvalues of $\mathbf{B}$ are $-\omega^2$ and $-3\omega^2$.

We know that the eigenvalues of $\mathbf{A}$ are the square roots of the eigenvalues of $\mathbf{B}$. Therefore, the eigenvalues of $\mathbf{A}$ are

$$\pm i\omega \qquad \pm\sqrt{3}i\omega.$$

Eigenvectors

Let us find the eigenvectors for $\mathbf{B}$. (We are almost there!)

For the eigenvalue $-\omega^2$, we need to find a basis for the null space of

$$\mathbf{B} - (-\omega^2 \mathbf{I}) = \omega^2 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will do.

Similarly, an eigenvector for the eigenvalue $-3\omega^2$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Recall from the previous page that the eigenvectors of $\mathbf{A}$ with eigenvalue λ are $\begin{pmatrix} \mathbf{x} \\ \lambda \mathbf{x} \end{pmatrix}$ where $\mathbf{x}$ is an eigenvector of $\mathbf{B}$ with eigenvalue λ^2. Therefore, the four pairs of eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\text{for } \lambda = \pm i\omega : \begin{pmatrix} 1 \\ 1 \\ \pm i\omega \\ \pm i\omega \end{pmatrix} \quad , \quad \text{for } \lambda = \pm i\sqrt{3}\omega : \begin{pmatrix} 1 \\ -1 \\ \pm i\sqrt{3}\omega \\ \mp i\sqrt{3}\omega \end{pmatrix}.$$

Note that as in the 2×2 case, the complex eigenvalues come in pairs of complex conjugates, and the eigenvectors of the conjugate eigenvalue $\overline{\lambda}$ are the conjugates of the eigenvectors of λ.

Solutions to the companion system

The four pairs of eigenvalues and eigenvectors of $\mathbf{A}$ give the four exponential solutions:

$$e^{i\omega t} \begin{pmatrix} 1 \\ 1 \\ i\omega \\ i\omega \end{pmatrix} \quad , \quad e^{i\sqrt{3}\omega t} \begin{pmatrix} 1 \\ -1 \\ i\sqrt{3}\omega \\ -i\sqrt{3}\omega \end{pmatrix}, \qquad \text{and their complex conjugates.}$$

The general (complex) solution is therefore linear combinations of these.

The real solutions and physical interpretation

But we are interested in the real solutions because the components of $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ represent the displacements in real physical space of the two masses.

As usual, we find a basis for the real solutions by taking the real and imaginary parts of the exponential solutions.

Since the bottom half $\mathbf{y}$ of $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ is given by $\mathbf{y} = \dot{\mathbf{x}},$ we only need to write down the top halves of the real solutions:

$$\begin{pmatrix} \cos(\omega t) \\ \cos(\omega t) \end{pmatrix} \quad , \quad \begin{pmatrix} \sin(\omega t) \\ \sin(\omega t) \end{pmatrix} \quad , \quad \begin{pmatrix} \cos(\sqrt{3}\omega t) \\ -\cos(\sqrt{3}\omega t) \end{pmatrix} \quad , \quad \begin{pmatrix} \sin(\sqrt{3}\omega t) \\ -\sin(\sqrt{3}\omega t) \end{pmatrix}.$$

The first two solutions combine to give a general sinusoid of angular frequency ω for x_1 **and** $x_2 = x_1$. In this mode, the masses are moving together; the spring between them is relaxed.

The general solution is

$$\begin{aligned} x_1 &= A_1 \cos(\omega t + \phi_1) + A_2 \cos(\sqrt{3}\omega t + \phi_2) \\ x_2 &= A_1 \cos(\omega t + \phi_1) - A_2 \cos(\sqrt{3}\omega t + \phi_2). \end{aligned}$$

There are two purely sinusoidal modes:

- When $A_2 = 0,\ x_1 = x_2 = A_1 \cos(\omega t + \phi_1),$ and the two masses oscillate together with the spring in between at equilibrium,
- When $A_1 = 0,\ x_1 = -x_2 = A_2 \cos(\sqrt{3}\omega t + \phi_2),$ and the oscillations are synchronous but in opposite directions.

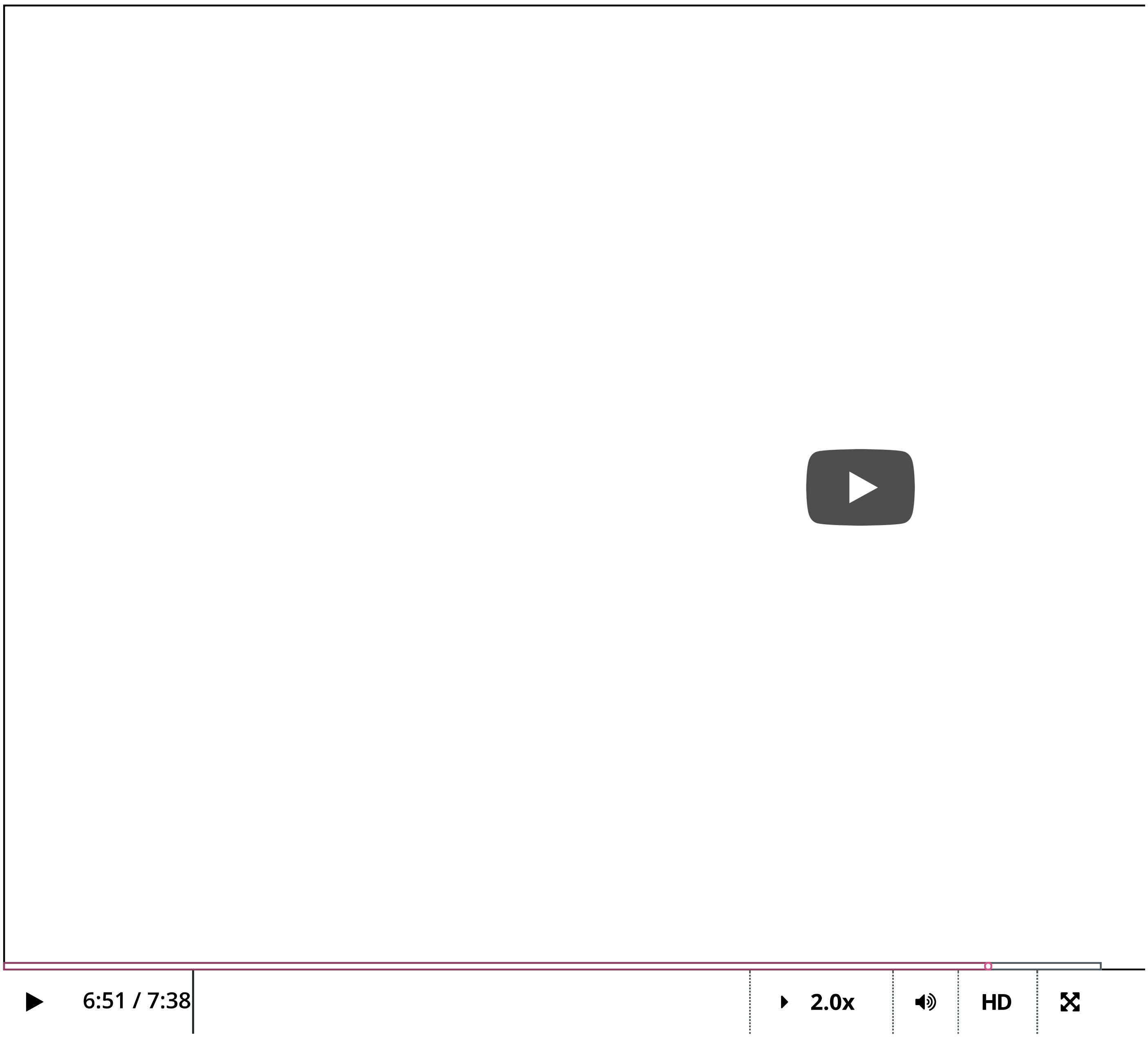
Demonstration of coupled oscillator using air carts

The Technical Services Group in the physics department has graciously provided some video footage of the couple oscillator in real action using carts on an air track.

The springs are difficult to discern in the video, and at the outset the carts have brakes on. When the technician turns on the switch, the carts become elevated on an air track and move. This air track makes the assumption of zero drag quite close to reality.

Video note: In the video below, the segment relevant to this example ends at 1:07, but we include the rest of the demonstration for fun.

Carts on air tracks



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fundamental matrix

1/1 point (graded)

Consider the coupled oscillator as above. Which of the following is a fundamental matrix of the companion system describing the coupled oscillator? (Choose all that apply.)

- [ ] $$\begin{pmatrix} \cos(\omega t) & \sin(\omega t) & \cos(\sqrt{3}\omega t) & \sin(\sqrt{3}\omega t) \\ \cos(\omega t) & \sin(\omega t) & \cos(\sqrt{3}\omega t) & -\sin(\sqrt{3}\omega t) \end{pmatrix}$$

- [ ] $$\begin{pmatrix} \cos(\omega t) & \cos(\sqrt{3}\omega t) \\ \cos(\omega t) & -\cos(\sqrt{3}\omega t) \end{pmatrix}$$

- [x] $$\begin{pmatrix} \cos(\omega t) & \sin(\omega t) & \cos(\sqrt{3}\omega t) & \sin(\sqrt{3}\omega t) \\ \cos(\omega t) & \sin(\omega t) & -\cos(\sqrt{3}\omega t) & -\sin(\sqrt{3}\omega t) \\ -\omega\sin(\omega t) & \omega\cos(\omega t) & -\sqrt{3}\omega\sin(\sqrt{3}\omega t) & \sqrt{3}\omega\cos(\sqrt{3}\omega t) \\ -\omega\sin(\omega t) & \omega\cos(\omega t) & \sqrt{3}\omega\sin(\sqrt{3}\omega t) & -\sqrt{3}\omega\cos(\sqrt{3}\omega t) \end{pmatrix}$$ ✔

- [x] $$\begin{pmatrix} e^{i\omega t} & e^{-i\omega t} & e^{i\sqrt{3}\omega t} & e^{-i\sqrt{3}\omega t} \\ e^{i\omega t} & e^{-i\omega t} & -e^{i\sqrt{3}\omega t} & -e^{-i\sqrt{3}\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} & \sqrt{3}i\omega e^{i\sqrt{3}\omega t} & -\sqrt{3}ie^{-i\sqrt{3}\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} & -\sqrt{3}i\omega e^{i\sqrt{3}\omega t} & \sqrt{3}ie^{-i\sqrt{3}\omega t} \end{pmatrix}$$ ✔

✔

Solution:

The matrices in the first two choices are of the wrong sizes.
The last two choices are fundamental matrices of the system since in each case, the four columns in the matrix are four independent solutions of the system.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

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