Unit 5: Quiz

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The problems in the quiz will be automatically graded.

You are given only **ONE** attempt for each problem. You will be able to see the solution after you submit your answers.

Problem 1

3/3 points (graded)

1. Suppose X and Y are independent Binomial random variables, each with $n = 3$ and $p = 9/10$.

1a. Find the probability that X and Y are equal, i.e., find $P(X = Y)$.

0.59122 ✔ **Answer:** 0.59122

1b. Find the probability that X is strictly larger than Y, i.e., find $P(X > Y)$.

0.20439 ✔ **Answer:** 0.20439

1c. Find the probability that Y is strictly larger than X, i.e., find $P(Y > X)$.

0.20439

Answer: 0.20439

[Hint: Once you find the value in 1a, you can very easily find the values in 1b and 1c, with almost no effort at all.]

Explanation

1a. We compute that

$$P(X = Y) = P(X = Y = 0) + P(X = Y = 1) + P(X = Y = 2) + P(X = Y = 3)$$
$$= \binom{3}{0}(9/10)^0(1/10)^3\binom{3}{0}(9/10)^0(1/10)^3 + \binom{3}{1}(9/10)^1(1/10)^2\binom{3}{1}(9/10)^1(1/10)^2$$
$$+\binom{3}{2}(9/10)^2(1/10)^1\binom{3}{2}(9/10)^2(1/10)^1 + \binom{3}{3}(9/10)^3(1/10)^0\binom{3}{3}(9/10)^3(1/10)^0$$

which simplifies to

$$P(X = Y) = 1/1000000 + 729/1000000 + 59049/1000000 + 531441/1000000$$
$$= 29561/50000 = 0.59122.$$

1b. We have $P(X \neq Y) = 1 - P(X = Y) = 1 - 29561/50000 = 20439/50000$, or written with decimals, this is $P(X \neq Y) = 1 - 0.59122 = 0.40878$. By the symmetry of X and Y, we have $P(X > Y) = P(Y > X)$ and thus

$$P(X > Y) = P(X \neq Y)/2 = (20439/50000)(1/2) = 20439/100000 = 0.20439.$$

1c. Similarly to 1b, we have $P(Y > X) = 20439/100000 = 0.20439$.

Submit You have used 1 of 1 attempt

Problem 2

3/3 points (graded)

2. Consider a deck of 15 cards containing 5 blue cards, 5 red cards, and 5 green cards. Shuffle the cards and deal all 15 of the cards out, around a circular table, with one card per seat.

2a. A card is called “isolated" if its color does not agree with either of the nearby cards (i.e., if it has a different color than the card to its right and a different color than the card to its left). Let X denote the number of isolated cards. Find $\mathbb{E}(X)$.

7.4

✔ **Answer:** 7.4176

2b. A card is called “semi-happy" if its color agrees with exactly one (but not both) of the nearby cards (i.e., if its color agrees with the color of the card on its left or on its right, but not both). Let Y denote the number of semi-happy cards. Find $\mathbb{E}(Y)$.

6.6

✔ **Answer:** 6.5934

2c. A card is called “joyous" if its color agrees with both of the nearby cards (i.e., if its color agrees with the color of the card on its left and on its right). Let Z denote the number of joyous cards. Find $\mathbb{E}(Z)$. [Hint: You answers to a, b, c, must add up to 15.]

1

✔ **Answer:** 0.9890

Explanation

2a. Let $X_i = 1$ if the ith card is isolated, and $X_i = 0$ otherwise. Then $\mathbb{E}(X_i) = P(X_i = 1) = (10/14)(9/13) = 45/91 = 0.4945$. So $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{15}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{15}) = (15)(45/91) = 675/91 = 7.4176$.

2b. Let $Y_i = 1$ if the ith card is semi-happy, and $Y_i = 0$ otherwise. Then $\mathbb{E}(Y_i) = P(Y_i = 1) = (10/14)(4/13) + (4/14)(10/13) = 40/91 = 0.43956$. So $\mathbb{E}(Y) = \mathbb{E}(Y_1 + \cdots + Y_{15}) = \mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_{15}) = (15)(40/91) = 600/91 = 6.5934$.

2c. Let $Z_i = 1$ if the ith card is joyous, and $Z_i = 0$ otherwise. Then $\mathbb{E}(Z_i) = P(Z_i = 1) = (4/14)(3/13) = 6/91 = 0.06593$. So $\mathbb{E}(Z) = \mathbb{E}(Z_1 + \cdots + Z_{15}) = \mathbb{E}(Z_1) + \cdots + \mathbb{E}(Z_{15}) = (15)(6/91) = 90/91 = 0.9890$.
[We verify, by the way, that $\mathbb{E}(X) + \mathbb{E}(Y) + \mathbb{E}(Z) = 7.4176 + 6.5934 + 0.9890 = 15$.]

Submit You have used 1 of 1 attempt

✔ Correct (3/3 points)

Problem 3

6/6 points (graded)

3. You randomly choose cookies from a very large container. Assume that 35% of the cookies are chocolate chip and 65% of the cookies are not chocolate. Assume that your selections of cookies are independent, and assume that the container is so large that these percentages do not change with each subsequent draw. (If you prefer, you can just sample the cookies with replacement, but nobody likes to put cookies back!) Let X denote the number of chocolate chip cookies that you get, when you choose 5 cookies from the cookie jar.

3a. Suppose that your brother also chooses 5 cookies, and let Y denote the number of chocolate chip cookies that he gets. Assume that X and Y are independent. Define $W = X + Y$. Is W a Binomial random variable? (and Why?)

◉ Yes ✔

- No

If you choose “Yes", what are the parameters n and p for the random variable W?
(Put 0 if you don't think W is a random variable.)

$n =$ 10 **Answer:** 10

$p =$ 0.35 **Answer:** 0.35

3b. Is $U = 2X - Y$ a Binomial random variable? (and why?)

- Yes
- No

If you choose “Yes", what are the parameters n and p for the random variable U?
(Put 0 if you don't think U is a random variable.)

$n =$ 0 **Answer:** 0

$p =$ 0 **Answer:** 0

Explanation

3a. Yes, $W = X + Y$ is a Binomial random variable. To see this, notice that $X = X_1 + \cdots + X_5$ and $Y = Y_1 + \cdots + Y_5$ where the 10 Bernoulli random variables $X_1, \ldots, X_5, Y_1, \ldots, Y_5$ are independent and each have $p = 0.35$. So W is the sum of 10 independent Bernoulli random variables with $p = 0.35$, so W is a Binomial random variable with $n = 10$ and $p = 0.35$.
3b. No, U is not a Binomial random variable. An easy way to see this is to note, for instance, that if $X = 0$ and $Y = 3$, then $U = -3$, so U can take on negative values. Binomial random variables only take on values $0, \ldots, n$ for some n, and so U cannot be a Binomial random variable.

Submit You have used 1 of 1 attempt

Problem 4

2/2 points (graded)

4. Consider a die with 2 red sides, 2 green sides, and 2 blue sides. Roll the die 5 times, and let X denote the number of times that the die has a red result.

Flip a coin 5 times, and let Y denote the number of times that the coin shows “heads."

4a. Find $\mathbb{E}(X - Y)$.

-5/6 ✔ **Answer:** -0.8333

4b. Find $\text{Var}(X - Y)$.

85/36 ✔ **Answer:** 2.3611

Explanation

4a. Since X is a Binomial random variable with $n = 5$ and $p = 2/6 = 1/3$, then $\mathbb{E}(X) = np = 5/3$. Since Y is a Binomial random variable with $n = 5$ and $p = 1/2$, then $\mathbb{E}(Y) = np = 5/2$. Thus $\mathbb{E}(X - Y) = \mathbb{E}(X) - \mathbb{E}(Y) = 5/3 - 5/2 = -5/6$.

4b. For this part (but not for the last part), we need to use the fact that X and Y are independent. So we have

$$\text{Var}(X - Y) = \text{Var}(X) + (-1)^2\,\text{Var}(Y) = (5)(1/3)(2/3) + (5)(1/2)(1/2) = 85/36 = 2.3611$$

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Submit You have used 1 of 1 attempt

Problem 5

2/2 points (graded)

5. Suppose X and Y are independent Geometric random variables, with $\mathbb{E}(X) = 4$ and $\mathbb{E}(Y) = 3/2$

.

5a. Find the probability that X and Y are equal, i.e., find $P(X = Y)$.

2/9 ✔ **Answer:** 0.2222

5b. Find the probability that X is strictly larger than Y, i.e., find $P(X > Y)$. [Hint: Unlike Problem 1b, we do *not* have symmetry between X and Y here, so you must calculate.]

2/3 ✔ **Answer:** 0.6667

Explanation

5a. We observe that $P(X=Y)=\sum_{j=1}^{\infty}P(X=Y=j)=\sum_{j=1}^{\infty}(3/4)^{j-1}(1/4)(1/3)^{j-1}(2/3)$, where the last equality holds since X and Y are independent, so their joint probability mass function is a product of their probability mass functions. We simplify to obtain $P(X=Y)=(1/4)(2/3)\sum_{j=1}^{\infty}((3/4)(1/3))^{j-1}=\frac{(1/4)(2/3)}{1-(3/4)(1/3)}=2/(12-3)=2/9$.

5b. We have

$$P(X>Y)=\sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty}P(X=x,Y=y)=\sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty}(3/4)^{x-1}(1/4)(1/3)^{y-1}(2/3).$$

We can factor the $(1/4)(1/3)^{y-1}(2/3)$ out of the inner sum, to obtain

$$P(X>Y)=\sum_{y=1}^{\infty}(1/4)(1/3)^{y-1}(2/3)\sum_{x=y+1}^{\infty}(3/4)^{x-1}=\sum_{y=1}^{\infty}(1/4)(1/3)^{y-1}(2/3)\frac{(3/4)^{y}}{1-3/4}.$$

This simplifies to $P(X>Y)=(2/3)(3/4)\sum_{y=1}^{\infty}(1/4)^{y-1}=(2/3)(3/4)/(1-1/4)=2/3$.

Alternative (easier) method. Let us treat the outcomes that yield the values of X and Y as two sequences of independent trials that are performed at the same time, and see which sequence of trials succeeds first. We can effectively ignore all of the trials at the beginning in which both trials fail, and we focus on the first trial in which one or the other trial succeeds. The probability that one or the other (or both) trials succeed is $(3/4)(2/3)+(1/4)(1/3)+(1/4)(2/3)$. Thus, $P(X=Y)=\frac{(1/4)(2/3)}{(3/4)(2/3)+(1/4)(1/3)+(1/4)(2/3)}=2/9$. Similarly, $P(X>Y)=\frac{(3/4)(2/3)}{(3/4)(2/3)+(1/4)(1/3)+(1/4)(2/3)}=2/3$.

Submit You have used 1 of 1 attempt

Problem 6

2/2 points (graded)

6. Suppose X and Y are independent Geometric random variables, each with expected value $5/4$.

6a. Find the probability that $X + Y = 5$.

64/3125 ✔ **Answer:** 0.02048

6b. What is the variance of $2Y - 3X$?

65/16 ✔ **Answer:** 4.0625

Explanation

6a. We note

$P(X + Y = 5) = P(X = 4, Y = 1) + P(X = 3, Y = 2) + P(X = 2, Y = 3) + P(X = 1, Y = 4$

but each of these four terms is equal to $(1/5)^3(4/5)(4/5) = 16/3125$, because each term corresponds to a sequence of trials that is performed until the second success, i.e., which has 3 failures and 1 success within the first 4th trials, and then a success on the 5th trial. So

$P(X + Y = 5) = (4)(16/3125) = 64/3125$.

6b. Each of X and Y has variance $(1/5)/(4/5)^2 = 5/16$. Since X and Y are independent, this yields

$\mathrm{Var}(2Y - 3X) = 2^2\,\mathrm{Var}(Y) + (-3)^2\,\mathrm{Var}(X) = 4(5/16) + 9(5/16) = 65/16 = 4.0625$.

Submit You have used 1 of 1 attempt

Problem 7

2/2 points (graded)

7. Let X be a Geometric random variable with $\mathbb{E}(X) = \mathbf{3}$. Let A denote the event that X is even, i.e., is a multiple of 2.

7a. Find $P(A)$.

2/5 ✔ **Answer:** 0.4

7b. Let B denote the event that X is a multiple of 4. Are A and B independent events?

- Yes
- No ✔

Explanation

7a. We have

$$\begin{aligned} P(A) &= P(X = 2) + P(X = 4) + P(X = 6) + P(X = 8) \\ &= (2/3)(1/3) + (2/3)^3(1/3) + (2/3)^5(1/3) + (2/3)^7(1/3) + \cdots \end{aligned}$$

$$= (2/3)(1/3)(1 + (2/3)^2 + (2/3)^4 + (2/3)^6 + \cdots)$$
$$= (2/3)(1/3)(1 + 4/9 + (4/9)^2 + (4/9)^3 + \cdots)$$
$$= (2/3)(1/3)\frac{1}{1-4/9} = 2/5.$$

7b. No, A and B are not independent, because B is a subset of A. (If one event is a subset of the other, they are dependent, unless one of them is $\emptyset$ or S.)

Submit You have used 1 of 1 attempt

Problem 8

3/3 points (graded)

8. Let X, Y, and Z be independent Geometric random variables that each have expected value $5/3$.

8a. Find $P(X > 10)$. Give your answer as a decimal to at least 6 decimal places.

0.0001048576 ✔ **Answer:** 0.0001048576

8b. Find $P(X + Y > 10)$. Give your answer as a decimal to at least 4 decimal places.

0.001677722 ✔ **Answer:** 0.0016777216

8c. Find $P(X + Y + Z > 10)$. Give your answer as a decimal to at least 4 decimal places.

0.01229455 ✔ **Answer:** 0.0122945536

Explanation

8a. We have $P(X > 10) = (2/5)^{10}$.

8b. We have $P(X + Y > 10) = (2/5)^{10} + \binom{10}{1}(2/5)^9(3/5)$.

8c. We have $P(X + Y + Z > 10) = (2/5)^{10} + \binom{10}{1}(2/5)^9(3/5) + \binom{10}{2}(2/5)^8(3/5)^2$.

Submit You have used 1 of 2 attempts

Problem 9

3/3 points (graded)

9. Suppose Jessica picks homework problems at random to practice for her midterm exam. She practices until she has solved 5 worthwhile questions, and then she quits after that. Her selections of problems are independent, each with a probability of 0.90 of being worthwhile.

9a. Find the probability that she solves 8 or fewer questions.

0.9949757 ✔ **Answer:** 0.995

9b. Find the conditional probability that she solves 6 or fewer questions, given that she solves 8 or fewer questions.

0.8902077 ✔ **Answer:** 0.890

9c. Find the variance of the total number of questions that she solves.

0.617284

✔ **Answer:** 0.6173

Explanation

9a. The probability that she solves 8 or fewer questions is

$$\binom{4}{4}(1/10)^0(9/10)^5+\binom{5}{4}(1/10)^1(9/10)^5+\binom{6}{4}(1/10)^2(9/10)^5+\binom{7}{4}(1/10)^3(9/10)^5$$

which simplifies to $0.5905+0.2952+0.08857+0.02067=0.995$.

9b. The probability that she solves 6 or fewer questions is

$$\binom{4}{4}(1/10)^0(9/10)^5+\binom{5}{4}(1/10)^1(9/10)^5=0.5905+0.2952=0.886.$$

So the desired conditional probability is $0.886/0.995=0.890$.

9c. The variance is $rq/p^2=(5)(1/10)/(9/10)^2=50/81=0.6173$.

Submit You have used 1 of 1 attempt

Problem 10

2/2 points (graded)

10. Suppose that 60% of people in Chicago are fans of da Bears. Assume that the fans' preferences are independent. We interview fans until we find the 3rd person who is a fan of da Bears. (This is different than the setup from question #1 of Problem Set 12. Here we need to interview at least 3 people, but in that former question, we interviewed exactly 3 people.) Let X denote the number of people we interview altogether.

10a. Find the probability that $X > 6$.

0.1792

✔ **Answer:** 0.1792

10b. Find the conditional probability that $X > 6$, given that $X > 4$.

0.3414634

✔ **Answer:** 0.3415

Explanation

10a. We have $P(X > 6) = 1 - P(X \le 6)$, and
$P(X \le 6) = \binom{2}{2}(4/10)^0(6/10)^3 + \binom{3}{2}(4/10)^1(6/10)^3 + \binom{4}{2}(4/10)^2(6/10)^3 + \binom{5}{2}(4/10)^3(6/10)^3$
$= 0.216 + 0.2592 + 0.20736 + 0.13824 = 0.8208$.
$P(X \le 6) = \binom{2}{2}(4/10)^0(6/10)^3$
So $P(X > 6) = 1 - 0.8208 = 0.1792$.
10b. We have $P(X > 4) = 1 - P(X \le 4)$, and

$$P(X \le 4) = \binom{2}{2}(4/10)^0(6/10)^3 + \binom{3}{2}(4/10)^1(6/10)^3 = 0.216 + 0.2592 = 0.4752.$$

So $P(X > 4) = 1 - 0.4752 = 0.5248$. We conclude that $P(X > 6 \mid X > 4) = \frac{P(X>6 \ \& \ X>4)}{P(X>4)} = \frac{P(X>6)}{P(X>4)} = 0.1792/0.5248 = 0.3415$.

Submit You have used 1 of 1 attempt

Problem 11

3/3 points (graded)

11. Let X_1, X_2, X_3 be independent Geometric random variables, each with expected value $10/7$. Let Y be a Negative Binomial random variable, with parameters $r = 3$ and $p = 7/10$. Let $Z = 3X_1$.

11a. Do $X_1 + X_2 + X_3$ and Y have the same distribution? Why or why not?

- (selected) Yes ✔
- No

11b. Do $X_1 + X_2 + X_3$ and Z have the same distribution? Why or why not?

- Yes
- (selected) No ✔

11c. Do Y and Z have the same distribution? Why or why not?

- Yes
- No ✔

Explanation

11a. Yes, $X_1 + X_2 + X_3$ is a Negative Binomial random variable with $r = 3$ and $p = 1/(10/7) = 7/10$. So $X_1 + X_2 + X_3$ has the same distribution as Y.

11b. No, $X_1 + X_2 + X_3$ and Z do not have the same distribution, because $X_1 + X_2 + X_3$ can take on any positive integer values of 3 or larger, but Z can only take on values that are positive integer multiples of 3.

11c. No, Y and Z do not have the same distribution, because Y can take on any positive integer values of 3 or larger, but Z can only take on values that are positive integer multiples of 3.

Submit You have used 1 of 1 attempt

Problem 12

2/2 points (graded)

12. Roll a 6 sided die until you have seen all of the sides as a result. Let X denote the number of rolls required.

12a. Is X a negative binomial random variable? If so, what are the parameters? If not, then why not?

- Yes
- No ✔

12b. Find $\mathbb{E}(X)$.
Hint: Let X_i denote the number of additional rolls needed until the ith new result appears. So $X = X_1 + \cdots + X_6$.

14.7

✔ **Answer:** 14.7

Explanation
12a. No, X is not a Negative Binomial random variable. Instead, X is the sum of 6 independent random variables, each of which has a different value of p.
12b. We have

$$\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_6) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_6)$$
$$= \frac{1}{6/6} + \frac{1}{5/6} + \frac{1}{4/6} + \frac{1}{3/6} + \frac{1}{2/6} + \frac{1}{1/6} = 147/10 = 14.7.$$

Submit You have used 1 of 1 attempt