4.1 Quiz: Solving the Simplified Pendulum Model

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We're using the differential equation

$$\frac{d^2\theta}{dt^2} = -k\theta$$

to model the motion of our pendulum, where $k = \frac{g}{l}$. This is a reasonable model of the motion of a pendulum when angle θ is small.

We can check that the function $\theta(t) = \theta_0 \cos(\sqrt{k}t)$ is one solution to this differential equation. We will explore this solution in the questions below.

(**Note:** Horizontal shifts of this functions, including $\theta_0 \sin(\sqrt{k}t)$, are also solutions, as we'll see later.)

Question 1

1/1 point (graded)

Consider a positive angle θ_0 and the solution $\theta(t) = \theta_0 \cos(\sqrt{k}t)$. Which of the options below best describes the starting position of the pendulum when $t = 0$?

You can add an optional tip or note related to the prompt like this.

- If $t = 0$, then $\theta = 0$ and the pendulum starts with the bob at the lowest point and with the rod hanging straight down (vertical).

- [x] If $t = 0$, then $\theta = \theta_0$ and the pendulum starts with the rod at an angle of θ_0.

- [ ] The differential equation does not specify the initial conditions; the starting position is undetermined.

- [ ] Because θ is very small, $\cos(\sqrt{k}t) \approx \sqrt{k}t$ so the starting position is $\sqrt{k}t\theta_0$.

Explanation

The differential equation $\frac{d^2\theta}{dt^2} = -k\theta$ does not specify the initial conditions, but the solution $\theta(t) = \theta_0 \cos(\sqrt{k}t)$ gives us that information.
In this case, when $t = 0$, the value of $\cos(\sqrt{k}t)$ is 1, so the pendulum starts with the rod at an angle of θ_0 away from vertical.

Note on choice (d): For small values of t, $\sin(\sqrt{k}t) \approx \sqrt{k}t$ and $\cos(\sqrt{k}t) \approx 1$. In addition to approximating the wrong function, a linear approximation like $\sqrt{k}t$ is only valid for a limited range of input values t. We wish to model the motion of a pendulum over a long period of time, so cannot use such simplifications here.

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You have used 1 of 2 attempts

Answers are displayed within the problem

Question 2

1/1 point (graded)

Consider a positive angle θ_0 and the solution $\theta(t) = \theta_0 \cos(\sqrt{k}t)$. Which of the options below best describes the angular velocity $\frac{d\theta}{dt}$ of the pendulum at $t = 0$ and shortly after?

- The starting velocity is 0 after which the velocity becomes positive. The pendulum is not moving at this instant, but after this instant, it will swing outward.
- The starting velocity is 0 after which the velocity becomes negative. The pendulum is not moving at this instant, but after this instant, it will swing toward the vertical.
- The starting velocity is $-\sqrt{k}\theta_0$. The pendulum is swinging toward the vertical.
- The starting velocity is $\sqrt{k}\theta_0$. The pendulum is swinging toward the vertical.
- The starting velocity is $-\sqrt{k}\theta_0$. The pendulum is swinging outward.
- The starting velocity is $\sqrt{k}\theta_0$. The pendulum is swinging outward.

Explanation

The starting velocity is 0. At this instant, the pendulum is not moving, but it will then start to swing toward the vertical. You might have solved this using the graph of $\cos(x)$ and your intuition. If not, you could differentiate $\theta(t)$ to find the answer and let $t = 0$:

$$\begin{aligned}\theta &= \theta_0 \cos(\sqrt{k}t)\\ \frac{d\theta}{dt} &= -\sqrt{k}\theta_0 \sin(\sqrt{k}t)\\ \left.\frac{d\theta}{dt}\right|_0 &= -\sqrt{k}\theta_0 \sin(0)\\ &= 0.\end{aligned}$$

At time $t = 0$, the angular velocity $\frac{d\theta}{dt}$ is 0. But this doesn't mean the pendulum is stopped for all time, since the acceleration is $-k\theta_0 \neq 0$. What it means is that the angle θ is at a maximum of θ_0 at $t = 0$ and the pendulum will start to swing toward the vertical. Note that the derivative $\frac{d\theta}{dt}$ is 0 any time the pendulum stops and changes direction which for this solution is when $\theta = \pm\theta_0$. In summary, this particular solution $\theta(t) = \theta_0 \cos(\sqrt{k}t)$ has θ_0 equal to the maximum value of θ, describing the furthest point in the pendulum's swing.

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Question 3

1/1 point (graded)

Consider a positive angle θ_0 and the solution $\theta(t) = \theta_0 \cos(\sqrt{k}t)$. What is the period of the pendulum?

- ○ k
- ○ $\sqrt{k}$
- ○ $2\pi\sqrt{k}$
- ● $\frac{2\pi}{\sqrt{k}}$ ✔
- ○ $\frac{\pi}{\sqrt{k}}$

- $\frac{\sqrt{k}}{2\pi}$

- None of the above.

Explanation

The solution $\theta(t) = \theta_0 \cos(\sqrt{k}t)$ describes the pendulum angle over time. The period of the pendulum is the length of time it takes to go from maximum to minimum angle and back to its starting position. This is equal to the period of the trigonometric function $\cos(\sqrt{k}t)$ which is $\frac{2\pi}{\sqrt{k}}$.

Note: The maximum value of $\theta(t)$ occurs when $\cos(\sqrt{k}t) = 1$ and the minimum value occurs when $\cos(\sqrt{k}t) = -1$.

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What do our solutions look like? Since we haven't been given a specific length for our pendulum (except for the restriction that θ_0 is small), we can look at the simplest case, when $k = \sqrt{\frac{g}{l}} = 1$. We have the solution for the angle position and we can find the angular velocity by differentiating.

$$\theta(t) = \theta_0 \cos(t),$$
$$\frac{d\theta}{dt}(t) = -\theta_0 \sin(t)$$

Again, we're still considering θ_0 as a positive angle.

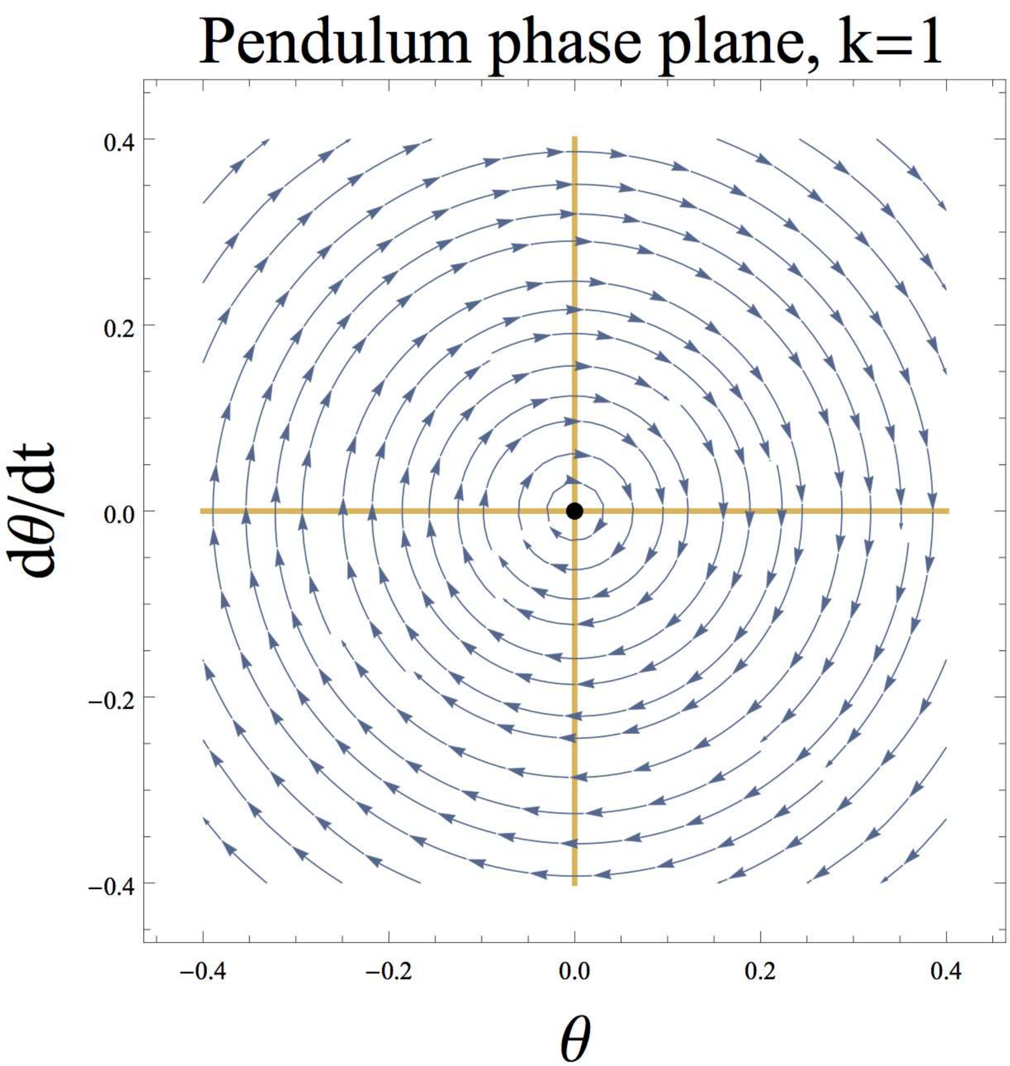
Question 4

1/1 point (graded)

For the case of $k = 1$, consider the solution trajectory in the position and angular velocity phase plane: the curves traced out by $(\theta(t), \frac{d\theta}{dt}(t))$ over time. These solution curves are clockwise circles of radius θ_0. Why is this the case?

They must be circles about the origin because the points $(\theta, \frac{d\theta}{dt})$ satisfy the relationship $\theta^2 + (\frac{d\theta}{dt})^2 = \theta_0^2$. In our qualitative analysis of the phase plane we determined that the trajectories are clockwise.

From these phase plane trajectories, which of the following can we conclude are true?



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- [ ] The pendulum will swing in a full circle about the fixed end of the rod.
- [x] The pendulum will swing back and forth from θ_0 to $-\theta_0$ for all time. ✔

- The pendulum will swing back and forth for all time, but the maximum and minimum angle of the swing will decrease (as we move to smaller and smaller circles).

- None of the above.

Explanation

B) is correct since as the trajectory goes around and around the circle of radius θ_0, the angle $\theta(t)$ will go from θ_0 to $-\theta_0$ and back.

A) is not correct. It's easy to get the position of the point on the phase plane confused with the position of the pendulum in space. The fact that the point $(\theta, \frac{d\theta}{dt})$ moves along a circle in the phase plane does not mean the pendulum swings over the top of the vertical.

C) is not correct. Because the trajectories are closed, the point $(\theta, \frac{d\theta}{dt})$ stays on the same circle for all time.

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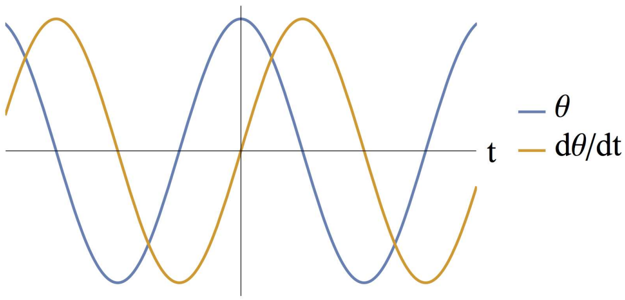
Answers are displayed within the problem

Question 5

1/1 point (graded)

As we did with predator-prey systems, we can graph both θ and $\frac{d\theta}{dt}$ as functions of time. Choose the graph below which shows the functions θ and $\frac{d\theta}{dt}$.

Graph A



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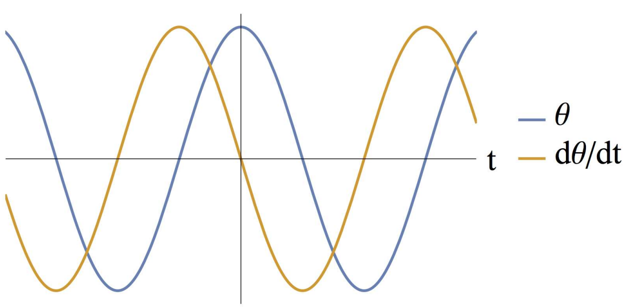
Graph B



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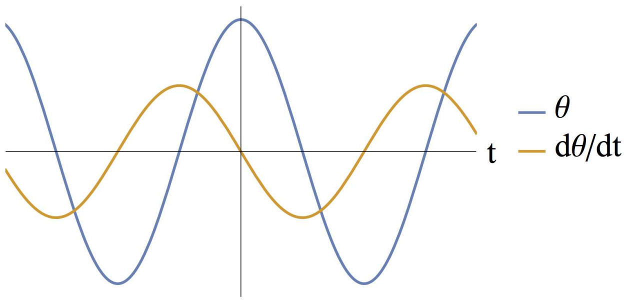
Graph C



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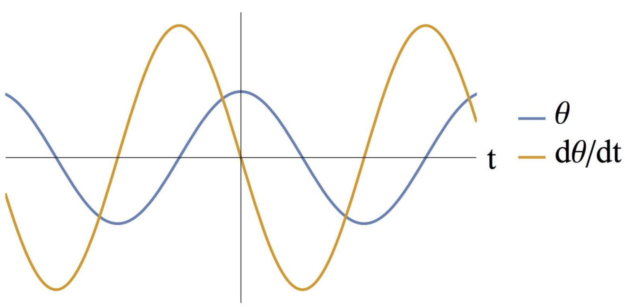
Graph D



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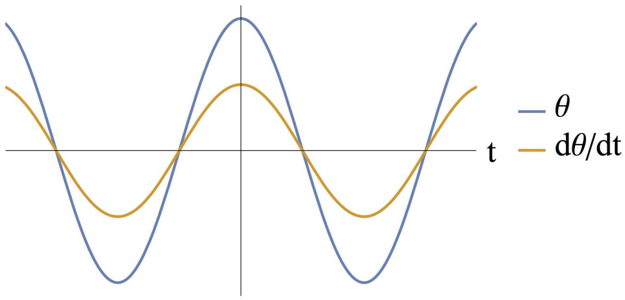
Graph E



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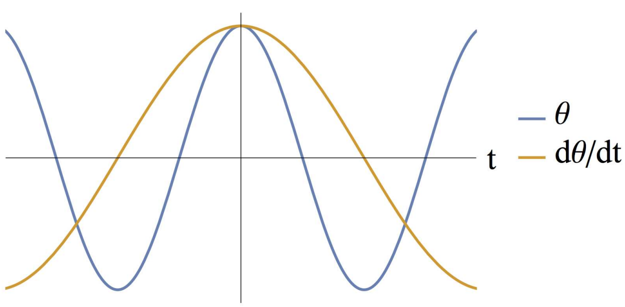
Graph F



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Explanation

For each option the slope of the graph of $\theta(t)$ is 0 at the origin, where it is changing from positive to negative. Three of the graphs proposed for $\frac{d\theta}{dt}$ change from positive to negative at this point. From the equations $\theta = \theta_0 \cos(t)$ and $\frac{d\theta}{dt} = -\theta_0 \sin(t)$ we see that θ and $\frac{d\theta}{dt}$ both have maximum values of θ_0. The answer must be .

Alternately, three of the graphs show a cosine function with positive coefficient and a sine function with negative coefficient. The graph in is the only one in which the two functions have the same amplitude (maximum value).

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