# Well-Ordering Ordinals

We have been considering a picture on which the ordinals are introduced in stages, and the stages never run out.

What is the shape of the resulting hierarchy of stages? As you'll be asked to verify below, the answer is that *the stages of the hierarchy are well-ordered*. More precisely:

> **Stage Well-Ordering**
> Every set of stages is well-ordered by the relation of occurring-earlier-than.

An immediate consequence of this result is that *every set of ordinals is well-ordered*. To see this, let us start by introducing a relation $<_o$ for ordering ordinals:

$$\alpha <_o \beta \leftrightarrow \alpha \in \beta$$

Since $\alpha \in \beta$ if and only if $\alpha$ was introduced at an earlier stage than $\beta$, $<_o$ is, in fact, the relation of being-introduced-at-an-earlier-stage-than. Since every set of stages is well-ordered by the relation of occurring-earlier-than, this means that *every set of ordinals is well-ordered by* $<_o$. And since every ordinal is a set of ordinals, this yields:

> **Ordinal Well-Ordering**
> Every ordinal is well-ordered by $<_o$.

Ordinal Well-Ordering is one of the key properties of orderings. One reason it is so important is that it allows us to use ordinals to represent well-order types. More specifically: each ordinal can be used as a representative for the well-order type that it itself instantiates under $<_o$. For example:

| Ordinal | Ordering under $<_o$ | Well-order type represented |
|---|---|---|
| $0$ | | |
| $0'$ | $0$ | $\vert$ |
| $0''$ | $0 <_o 0'$ | $\vert\vert$ |
| $0'''$ | $0 <_o 0' <_o 0''$ | $\vert\vert\vert$ |
| $\vdots$ | $\vdots$ | |
| $\omega$ | $0 <_o 0' <_o 0'' \dots$ | $\vert\vert\vert \dots$ |
| $\omega'$ | $0 <_o 0' <_o 0'' \dots <_o \omega$ | $\vert\vert\vert\vert\vert \dots \vert$ |
| $\omega''$ | $0 <_o 0' <_o 0'' \dots <_o \omega <_o \omega'$ | $\vert\vert\vert\vert\vert \dots \vert\vert$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Ordinals are such natural representatives for well-order types that I will sometimes blur the distinction between an ordinal and the well-order type it represents, by speaking of ordinals as if they were themselves well-order types.

## Problem 1

1 point possible (ungraded)

Use the intuitive picture of ordinals we have developed so far to justify Stage Well-Ordering.

☐ Done

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