

MITx: 6.041x Introduction to Probability - The Science of Uncertainty

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Problem 4: Maximum likelihood estimation

(3/3 points)

Let heta be an unknown constant. Let W_1,\ldots,W_n be independent exponential random variables each with parameter 1. Let $X_i= heta+W_i$.

1. What is the maximum likelihood estimate of heta based on a single observation $X_1=x_1$? Enter your answer in terms of x_1 (enter as x_1) using standard notation .

$$\hat{ heta}_{ML}(x_1) = \begin{bmatrix} imes_1 \end{bmatrix}$$
 \checkmark Answer: x_1

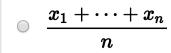
2. What is the maximum likelihood estimate of heta based on a sequence of observations $(X_1,\ldots,X_n)=(x_1,\ldots,x_n)$? $\hat{ heta}_{ML}(x_1,\ldots,x_n)=$

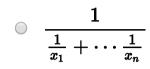
$$(x_1x_2\cdots x_n)^{1/n}$$

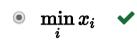
- Unit 6: Further topics on random variables
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Final Exam

Final Exam due May 24, 2016 at 23:59 UTC









None of the above

3. You have been asked to construct a confidence interval of the particular form $[\hat{\Theta}-c,\hat{\Theta}]$, where $\hat{\Theta}=\min_i\{X_i\}$ and c is a constant that we need to choose. For n=10, how should the constant c be chosen so that we have a 95% confidence interval? (Give the smallest possible value of c.) Your answer should be accurate to 3 decimal places.

Answer:

1. To find $\hat{\theta}_{ML}$, we first find $f_{X_1}(x_1;\theta)$. Given $X_1=\theta+W_1$, X_1 is a shifted exponential, where the entire distribution is shifted to the right by θ . Therefore,

$$f_{X_1}(x_1; heta)=egin{cases} e^{-(x_1- heta)},&x_1\geq heta,\ 0,&x_1< heta. \end{cases}$$

This quantity is maximized at $\hat{ heta}_{ML} = x_1$.

2. Let $X=(X_1,\ldots,X_n)$ and $x=(x_1,\ldots,x_n)$. To find $\hat{\theta}_{ML}$, we first find $f_X(x;\theta)$. Since the W_i 's are independent, so are the X_i 's. Hence,

$$egin{aligned} f_X(x; heta) &= \prod_{i=1}^n f_{X_i}(x_i; heta) \ &= egin{cases} \prod_{i=1}^n e^{-(x_i- heta)}, & ext{if } x_i \geq heta \ 0, & ext{otherwise.} \end{cases} \end{aligned}$$

Note that this quantity is nonzero only if θ is no greater than each of the x_i 's. Moreover, $e^{-(x_i-\theta)}$ is greater when θ is closer to x_i . Therefore, this quantity is maximized when we push θ as high as possible while keeping it no greater than each of the x_i 's. This means that $\hat{\theta}_{ML}(x) = \min_i x_i$. (Any larger choice of θ would give $f_X(x;\theta) = 0$.)

3. We wish to find c such that $\mathbf{P}(\hat{\Theta} - c \leq \theta \leq \hat{\Theta}) \geq 0.95$.

Since each of the W_i 's is nonnegative (because they are exponential random variables), we have that $X_i \geq \theta$ for all i. This implies that $\theta \leq \hat{\Theta} = \min_i \{X_i\}$, and so $P(\theta \leq \hat{\Theta}) = 1$. Therefore, we need only $\mathbf{P}(\hat{\Theta} - c \leq \theta) \geq 0.95$.

Since the X_i 's are independent, we have

$$egin{aligned} \mathbf{P}(\hat{\Theta} - c \leq heta) &= \mathbf{P}(\min_i \{X_i\} \leq heta + c) \ &= 1 - \mathbf{P}(\min_i \{X_i\} \geq heta + c) \ &= 1 - \prod_{i=1}^{10} \mathbf{P}(X_i \geq heta + c) \ &= 1 - \prod_{i=1}^{10} \mathbf{P}(W_i \geq c) \ &= 1 - \prod_{i=1}^{10} e^{-c} \ &= 1 - e^{-10c}. \end{aligned}$$

To have a 95% confidence interval, we require $1-e^{-10c} \geq 0.95$, or

$$c \geq rac{-\ln(0.05)}{10} = rac{\ln(20)}{10} pprox 0.29957.$$

You have used 2 of 2 submissions

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