3. Method

Consider a first-order homogeneous linear system of ODEs:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $n \times n$ matrix with constant, real entries.

Recall (from the course *Differential equations: 2 by 2 systems*) that $\mathbf{v}e^{\lambda t}$ is a solution if and only if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ.

Reason

If $\mathbf{x} = \mathbf{v}e^{\lambda t}$, then $\dot{\mathbf{x}} = \lambda \mathbf{v}e^{\lambda t}$ and $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{v}e^{\lambda t}$. Therefore,

$$\begin{array}{rrcll} & \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x} & \\ \iff & \lambda \mathbf{v}e^{\lambda t} & = & \mathbf{A}\mathbf{v}e^{\lambda t} & (\text{for all } t). \\ \iff & \lambda \mathbf{v} & = & \mathbf{A}\mathbf{v} & (\text{cancel } e^{\lambda t} \text{ from both sides}). \end{array}$$

Note that we are following the convention that in expressions like $\lambda \mathbf{v}e^{\lambda t}$, scalar functions such as $e^{\lambda t}$ are placed to the right, while constant scalars and constant vectors are placed to the left.

Conclusion: $\mathbf{v}e^{\lambda t}$ is a solution if and only if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ.

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Steps to find a basis of solutions to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, given an $n \times n$ matrix $\mathbf{A}$:

1. Find the eigenvalues of $\mathbf{A}$. These are the roots of the characteristic polynomial $\det(\lambda\mathbf{I} - \mathbf{A})$.

2. For each eigenvalue λ:

   - Find a basis for the corresponding eigenspace $\mathrm{NS}(\lambda\mathbf{I} - \mathbf{A})$. Call these basis vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$.

   - Each vector-valued function $\mathbf{v}_i e^{\lambda t}$ is a solution. A solution of this form is called a **normal mode.**

3. If n such solutions were found (i.e., the sum of the dimensions of the eigenspaces is n), then these n solutions are enough to form a basis of all solutions.

Remark 3.1 The solutions of this type will automatically be linearly independent, since their values at $t = 0$ are linearly independent. (The chosen eigenvectors within each eigenspace are independent, and there is no linear dependence between eigenvectors with different eigenvalues.)

Remark 3.2 Note that λ and $\mathbf{v}$ may be complex, which means that eventually you will want to find a basis of real solutions.

Remark 3.3 The only thing that could go wrong is this: if there is a repeated eigenvalue λ, and the dimension of the eigenspace of λ is less than the multiplicity of λ, then the method above does not produce enough solutions. We will not deal with this case until the next lecture.

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