MATHEMATICS

Is a convex function always continuous?

Asked 2 years, 10 months ago Active 2 years, 10 months ago Viewed 5k times

13

11

It is well known that a convex function defined on $\mathbb{R}$ is continuous (it is even left and right differentiable. Now you can define a convex function for any normed vector space E : $f : E \mapsto \mathbb{R}$ is convex iff

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y)$$

I know that such a function is not necessarily continuous if E has infinite dimension: f can be a discontinuous linear form. For instance, if $E = \ell^2(\mathbb{N})$ the space of square summable sequences (endowed with the supremum norm $||\cdot||_\infty$ instead of its natural norm), and $f(u) = \sum\limits_{i\geq 1} \frac{u_i}{i}$, then f is linear, thus convex, yet it is well-known that f is not continuous.

Now my question is: **what about finite dimensions? Does there exist a convex function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not continuous?**

I know that there are discontinuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ that have derivatives in every direction (that's a good start since this is a necessary condition !) but I don't know any that is convex.

functional-analysis convex-analysis

Share Cite Edit Follow Flag edited Oct 20 '18 at 23:47 asked Oct 19 '18 at 8:29

charmd
4,701 1 14 40

4 Answers

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4

No: all convex functions $f : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

Here's a slightly more general statement. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $\mathbf{x}^* \in \mathbb{R}^n$. We show that f is continuous at $\mathbf{x}^*$.

Let $S = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x}^* - \mathbf{y}\| = 1\}$. Our first goal is to show that there's some $M \in \mathbb{R}$ such that $f(\mathbf{y}) \leq M$ for all $\mathbf{y} \in S$.

To prove that M exists: by Jensen's inequality, if $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are arbitrary points in $\mathbb{R}^n$, and $\mathbf{x}$ is a point in their convex hull, then $f(\mathbf{x})$ is a weighted average of $f(\mathbf{x}^{(1)}), \dots, f(\mathbf{x}^{(m)})$, so it is bounded above by $\max\{f(\mathbf{x}^{(1)}), \dots, f(\mathbf{x}^{(m)})\}$. From there, it's enough to find finitely many points whose convex hull contains S: for example, the vertices of a hypercube circumscribed about S.

Now suppose we take some $\mathbf{x}$ close to $\mathbf{x}^*$. Let $r = \|\mathbf{x}^* - \mathbf{x}\|$; we may assume $r < 1$, since ultimately we want to consider $\|\mathbf{x}^* - \mathbf{x}\|$ arbitrarily small.

On the line through $\mathbf{x}$ and $\mathbf{x}^*$, we can pick points $\mathbf{y}^-, \mathbf{y}^+ \in S$ such that they appear in the order $\mathbf{y}^-, \mathbf{x}^*, \mathbf{x}, \mathbf{y}^+$ on that line. They can be defined by:

$$\mathbf{y}^- = \mathbf{x}^* - \frac{\mathbf{x} - \mathbf{x}^*}{r} \text{ and } \mathbf{y}^+ = \mathbf{x}^* + \frac{\mathbf{x} - \mathbf{x}^*}{r}.$$

From this, we have

- $\mathbf{x}^* = \frac{r}{r+1}\mathbf{y}^- + \frac{1}{r+1}\mathbf{x}$, so $f(\mathbf{x}^*) \le \frac{r}{r+1} f(\mathbf{y}^-) + \frac{1}{r+1} f(\mathbf{x})$, which gives us the lower bound

$$f(\mathbf{x}) - f(\mathbf{x}^*) \ge r f(\mathbf{x}^*) - r f(\mathbf{y}^-) \ge r(f(\mathbf{x}^*) - M).$$

- $\mathbf{x} = r\mathbf{y}^+ + (1-r)\mathbf{x}^*$, so $f(\mathbf{x}) \le r f(\mathbf{y}^+) + (1-r) f(\mathbf{x}^*)$, which gives us the upper bound

$$f(\mathbf{x}) - f(\mathbf{x}^*) \le r f(\mathbf{y}^+) - r f(\mathbf{x}^*) \le r(M - f(\mathbf{x}^*)).$$

Putting these together, we get

$$-r(M - f(\mathbf{x}^*)) \le f(\mathbf{x}) - f(\mathbf{x}^*) \le r(M - f(\mathbf{x}^*))$$

which is the statement we need to prove continuity. (In the usual ϵ-δ form: given $\epsilon > 0$, take $\delta = \frac{\epsilon}{M - f(\mathbf{x}^*)}$. Then if $\|\mathbf{x}^* - \mathbf{x}\| < \delta$, the inequalities above tell us that $|f(\mathbf{x}^*) - f(\mathbf{x})| < \epsilon$.)

Share Cite Edit Follow Flag edited Oct 21 '18 at 22:38 answered Oct 19 '18 at 15:53

Misha Lavrov
96.3k 10 90 160

I don't really see how you can prove that M exists – charmd Oct 21 '18 at 19:03

Is there something specific you don't understand about my proof that M exists? – Misha Lavrov Oct 21 '18 at 20:20

When do you first define M ? When reading your first bullet point (btw there is a typo: it is $f(\mathbf{x}^*) \le \frac{r}{r+1} \ldots$ and not $\ge$), I had the feeling that you assumed that M was defined as $\sup_{\|y - x^*\| = 1} |f(y)|$. Is that it ? – charmd Oct 21 '18 at 22:26

1 Yes. That is the definition of M. Proof of existence is in the third paragraph, which I've now signposted more carefully. – Misha Lavrov Oct 21 '18 at 22:37

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Corollary 10.1.1 of Convex Analysis by Rockafellar says all convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ are continuous. The proof is very long and it is not worth reproducing the complete proof here. In the infinite dimensional case there are are discontinuous linear functionals.

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edited Oct 22 '18 at 5:29

answered Oct 19 '18 at 8:32

Kavi Rama Murthy

275k 18 70 145

19 I think to deserve a bunch of upvotes, an answer should also add at least some explanation rather than just stating a result. Basically, this is little more than a link-only answer. – leftaroundabout Oct 19 '18 at 13:31

Indeed. I accepted it since it was the only answer, but would have preferred to have a complete solution – charmd Oct 20 '18 at 14:29

I did not expect 6 upvotes for my answer. But it is not at all uncommon to find strange voting patterns, more so with downvoting. – Kavi Rama Murthy Oct 20 '18 at 23:16

@CharlesMadeline What extra information are you looking for? I will try to include more information if you tell me what is missing in my answer. – Kavi Rama Murthy Oct 21 '18 at 4:33

@Kavi Rama Murthy, thanks for your continued interest. I just would have likes to see 3-4 main ppins in the proof (e.g. a set of points with a special property whuch is introduced, or if the proof shows that the restrictions of f to segments are uniformly Lipschitz on compact sets, or something like that). That would be enough for me to accept your answer oc – charmd Oct 21 '18 at 9:36

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Yes, if E is an infinite-dimensional real Banach space then a discontinuous linear functional is a discontinuous convex function. But the map f defined by $f(u) = \sum u_i/i$ is certainly continuous on ℓ_2.

You're not going to be able to write down a formula for a discontinuous linear functional on a Banach space - it takes the Axiom of Choice to show such a thing exists.

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answered Oct 19 '18 at 16:58

David C. Ullrich

81.1k 5 58 124

Indeed, I've changed it a bit: same space and linear form, but with the supremum norm $||\cdot||_\infty$. – charmd Oct 21 '18 at 0:15

Another example, taken from math.stackexchange.com/questions/99206/... : on $E = \mathcal{C}^1([0,1],\mathbb{R})$, with the supremum norm $||f|| := \sup_{x\in[0,1]} |f(x)|$. Then $L : f \mapsto f'(0)$ is discontinuous – charmd Oct 21 '18 at 9:40

2 @CharlesMadeline We should note of course that the domain of those explicit unbounded linear functionials is not a Banach space... – David C. Ullrich Oct 21 '18 at 13:25

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The answer to the question in your title is "no". Consider any convex function f defined on (0,1), and extend f to [0,1) by taking $f(0)$ as $1 + \sup_{(0,1)} f(x)$. So we do need the condition that the domain of f be open.

As for the question in the body, it is sufficient to show that f is continuous iff given any function g that takes $\mathbb{R}$ to a line in $\mathbb{R}^n$, the function $t \to f(g(t))$ is continuous.

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answered Oct 19 '18 at 17:12

Acccumulation

9,978 3 13 24

1 Unfortunately, the sufficient statement you give is false. For one example (taken from here), let $f(x,y) = \frac{y}{x^2}(1 - \frac{y}{x^2})$ if $0 < y < x^2$ and $f(x,y) = 0$ otherwise. This is continuous away from $(0,0)$, and continuous along every line through $(0,0)$. So it is continuous along all lines everywhere. But it is not continuous at $(0,0)$. – Misha Lavrov Oct 19 '18 at 19:12