Solving Every Sudoku Puzzle

by Peter Norvig

In this essay I tackle the problem of solving every Sudoku puzzle. It turns out to be quite easy (about one page of code for the main idea and two pages for embellishments) using two ideas: **constraint propagation** and **search**.

Sudoku Notation and Preliminary Notions

First we have to agree on some notation. A Sudoku puzzle is a *grid* of 81 squares; the majority of enthusiasts label the columns 1-9, the rows A-I, and call a collection of nine squares (column, row, or box) a *unit* and the squares that share a unit the *peers*. A puzzle leaves some squares blank and fills others with digits, and the whole idea is:

> *A puzzle is solved if the squares in each unit are filled with a permutation of the digits 1 to 9.*

That is, no digit can appear twice in a unit, and every digit must appear once. This implies that each square must have a different value from any of its peers. Here are the names of the squares, a typical puzzle, and the solution to the puzzle:

```
 A1 A2 A3| A4 A5 A6| A7 A8 A9    4 . . |. . . |8 . 5     4 1 7 |3 6 9 |8 2 5
 B1 B2 B3| B4 B5 B6| B7 B8 B9    . 3 . |. . . |. . .     6 3 2 |1 5 8 |9 4 7
 C1 C2 C3| C4 C5 C6| C7 C8 C9    . . . |7 . . |. . .     9 5 8 |7 2 4 |3 1 6
---------+---------+---------    ------+------+------    ------+------+------
 D1 D2 D3| D4 D5 D6| D7 D8 D9    . 2 . |. . . |. 6 .     8 2 5 |4 3 7 |1 6 9
 E1 E2 E3| E4 E5 E6| E7 E8 E9    . . . |. 8 . |4 . .     7 9 1 |5 8 6 |4 3 2
 F1 F2 F3| F4 F5 F6| F7 F8 F9    . . . |. 1 . |. . .     3 4 6 |9 1 2 |7 5 8
---------+---------+---------    ------+------+------    ------+------+------
 G1 G2 G3| G4 G5 G6| G7 G8 G9    . . . |6 . 3 |. 7 .     2 8 9 |6 4 3 |5 7 1
 H1 H2 H3| H4 H5 H6| H7 H8 H9    5 . . |2 . . |. . .     5 7 3 |2 9 1 |6 8 4
 I1 I2 I3| I4 I5 I6| I7 I8 I9    1 . 4 |. . . |. . .     1 6 4 |8 7 5 |2 9 3
```

Every square has exactly 3 units and 20 peers. For example, here are the units and peers for the square `C2`:

```
    A2   |         |                    |         |            A1 A2 A3|         |
    B2   |         |                    |         |            B1 B2 B3|         |
    C2   |         |            C1 C2 C3| C4 C5 C6| C7 C8 C9   C1 C2 C3|         |
---------+---------+---------  ---------+---------+---------  ---------+---------+---------
    D2   |         |                    |         |                    |         |
    E2   |         |                    |         |                    |         |
    F2   |         |                    |         |                    |         |
---------+---------+---------  ---------+---------+---------  ---------+---------+---------
    G2   |         |                    |         |                    |         |
    H2   |         |                    |         |                    |         |
    I2   |         |                    |         |                    |         |
```

We can implement the notions of units, peers, and squares in the programming language Python (2.5 or later) as follows:

```
def cross(A, B):
    "Cross product of elements in A and elements in B."
    return [a+b for a in A for b in B]

digits   = '123456789'
rows     = 'ABCDEFGHI'
cols     = digits
squares  = cross(rows, cols)
unitlist = ([cross(rows, c) for c in cols] +
            [cross(r, cols) for r in rows] +
            [cross(rs, cs) for rs in ('ABC','DEF','GHI') for cs in ('123','456','789')])
units = dict((s, [u for u in unitlist if s in u])
             for s in squares)
peers = dict((s, set(sum(units[s],[]))-set([s]))
             for s in squares)
```

If you are not familiar with some of the features of Python, note that a `dict` or dictionary is Python's name for a hash table that maps each key to a value; that these are specified as a sequence of (key, value) tuples; that `dict((s, [...]) for s in squares)` creates a dictionary which maps each square `s` to a value that is the list `[...]`; and that the expression `[u for u in unitlist if s in u]` means that this value is the list of units `u` such that the square `s` is a member of `u`. So read this assignment statement as "`units` is a dictionary where each square maps to the list of units that contain the square". Similarly, read the next assignment statement as "`peers` is a dictionary where each square `s` maps to the set of squares formed by the union of the squares in the units of `s`, but not `s` itself".

It can't hurt to throw in some tests (they all pass):

```
def test():
    "A set of unit tests."
    assert len(squares) == 81
    assert len(unitlist) == 27
    assert all(len(units[s]) == 3 for s in squares)
    assert all(len(peers[s]) == 20 for s in squares)
    assert units['C2'] == [['A2', 'B2', 'C2', 'D2', 'E2', 'F2', 'G2', 'H2', 'I2'],
                           ['C1', 'C2', 'C3', 'C4', 'C5', 'C6', 'C7', 'C8', 'C9'],
                           ['A1', 'A2', 'A3', 'B1', 'B2', 'B3', 'C1', 'C2', 'C3']]
    assert peers['C2'] == set(['A2', 'B2', 'D2', 'E2', 'F2', 'G2', 'H2', 'I2',
                               'C1', 'C3', 'C4', 'C5', 'C6', 'C7', 'C8', 'C9',
                               'A1', 'A3', 'B1', 'B3'])
    print 'All tests pass.'
```

Now that we have squares, units, and peers, the next step is to define the Sudoku playing grid. Actually we need two representations: First, a textual format used to specify the initial state of a puzzle; we will reserve the name *grid* for this. Second, an internal representation of any state of a puzzle, partially solved or complete; this we will call a *values* collection because it will give all the remaining possible values for each square. For the textual format (*grid*) we'll allow a string of characters with 1-9 indicating a digit, and a 0 or period specifying an empty square. All other characters are ignored (including spaces, newlines, dashes, and bars). So each of the following three grid strings represent the same puzzle:

```
"4.....8.5.3..........7......2.....6.....8.4......1.......6.3.7.5..2.....1.4......"

"""
400000805
030000000
000700000
020000060
000080400
000010000
000603070
500200000
104000000"""

"""
4 . . |. . . |8 . 5
. 3 . |. . . |. . .
. . . |7 . . |. . .
------+------+------
. 2 . |. . . |. 6 .
. . . |. 8 . |4 . .
. . . |. 1 . |. . .
------+------+------
. . . |6 . 3 |. 7 .
5 . . |2 . . |. . .
1 . 4 |. . . |. . .
"""
```

Now for *values*. One might think that a 9 x 9 array would be the obvious data structure. But squares have names like `'A1'`, not `(0,0)`. Therefore, *values* will be a dict with squares as keys. The value of each key will be the possible digits for that square: a single digit if it was given as part of the puzzle definition or if we have figured out what it must be, and a collection of several digits if we are still uncertain. This collection of digits could be represented by a Python `set` or `list`, but I chose instead to use a string of digits (we'll see why later). So a grid where `A1` is `7` and `C7` is empty would be represented as `{'A1': '7', 'C7': '123456789', ...}`.

Here is the code to parse a grid into a *values* dict:

```
def parse_grid(grid):
    """Convert grid to a dict of possible values, {square: digits}, or
    return False if a contradiction is detected."""
    ## To start, every square can be any digit; then assign values from the grid.
    values = dict((s, digits) for s in squares)
    for s,d in grid_values(grid).items():
        if d in digits and not assign(values, s, d):
            return False ## (Fail if we can't assign d to square s.)
    return values

def grid_values(grid):
    "Convert grid into a dict of {square: char} with '0' or '.' for empties."
    chars = [c for c in grid if c in digits or c in '0.']
    assert len(chars) == 81
    return dict(zip(squares, chars))
```

Constraint Propagation

The function `parse_grid` calls `assign(values, s, d)`. We could implement this as `values[s] = d`, but we can do more than just that. Those with experience solving Sudoku puzzles know that there are two important strategies that we can use to make progress towards filling in all the squares:

> *(1) If a square has only one possible value, then eliminate that value from the square's peers.*
> *(2) If a unit has only one possible place for a value, then put the value there.*

As an example of strategy (1) if we assign 7 to `A1`, yielding `{'A1': '7', 'A2':'123456789', ...}`, we see that `A1` has only one value, and thus the 7 can be removed from its peer `A2` (and all other peers), giving us `{'A1': '7', 'A2': '12345689', ...}`. As an example of strategy (2), if it turns out that none of `A3` through `A9` has a 3 as a possible value, then the 3 must belong in `A2`, and we can update to `{'A1': '7', 'A2':'3', ...}`. These updates to `A2` may in turn cause further updates to its peers, and the peers of those peers, and so on. This process is called *constraint propagation*.

The function `assign(values, s, d)` will return the updated *values* (including the updates from constraint propagation), but if there is a contradiction--if the assignment cannot be made consistently--then `assign` returns `False`. For example, if a grid starts with the digits `'77...'` then when we try to assign the 7 to `A2`, `assign` would notice that 7 is not a possibility for `A2`, because it was eliminated by the peer, `A1`.

It turns out that the fundamental operation is not assigning a value, but rather eliminating one of the possible values for a square, which we implement with `eliminate(values, s, d)`. Once we have `eliminate`, then `assign(values, s, d)` can be defined as "eliminate all the values from `s` except `d`".

```
def assign(values, s, d):
    """Eliminate all the other values (except d) from values[s] and propagate.
    Return values, except return False if a contradiction is detected."""
    other_values = values[s].replace(d, '')
    if all(eliminate(values, s, d2) for d2 in other_values):
        return values
    else:
        return False

def eliminate(values, s, d):
    """Eliminate d from values[s]; propagate when values or places <= 2.
    Return values, except return False if a contradiction is detected."""
    if d not in values[s]:
        return values ## Already eliminated
    values[s] = values[s].replace(d,'')
    ## (1) If a square s is reduced to one value d2, then eliminate d2 from the peers.
    if len(values[s]) == 0:
        return False ## Contradiction: removed last value
    elif len(values[s]) == 1:
        d2 = values[s]
        if not all(eliminate(values, s2, d2) for s2 in peers[s]):
            return False
    ## (2) If a unit u is reduced to only one place for a value d, then put it there.
    for u in units[s]:
        dplaces = [s for s in u if d in values[s]]
        if len(dplaces) == 0:
            return False ## Contradiction: no place for this value
        elif len(dplaces) == 1:
            # d can only be in one place in unit; assign it there
            if not assign(values, dplaces[0], d):
                return False
    return values
```

Now before we can go much further, we will need to display a puzzle:

```
def display(values):
    "Display these values as a 2-D grid."
    width = 1+max(len(values[s]) for s in squares)
    line = '+'.join(['-'*(width*3)]*3)
    for r in rows:
        print ''.join(values[r+c].center(width)+('|' if c in '36' else '')
                      for c in cols)
        if r in 'CF': print line
    print
```

Now we're ready to go. I picked the first example from a list of [easy puzzles] from the fine [Project Euler] [Sudoku problem] and tried it:

```
>>> grid1 = '003020600900305001001806400008102900700000008006708200002609500800203009005010300'

>>> display(parse_grid(grid1))
4 8 3 |9 2 1 |6 5 7
9 6 7 |3 4 5 |8 2 1
```

```
2 5 1 |8 7 6 |4 9 3
------+------+------
5 4 8 |1 3 2 |9 7 6
7 2 9 |5 6 4 |1 3 8
1 3 6 |7 9 8 |2 4 5
------+------+------
3 7 2 |6 8 9 |5 1 4
8 1 4 |2 5 3 |7 6 9
6 9 5 |4 1 7 |3 8 2
```

In this case, the puzzle was completely solved by rote application of strategies (1) and (2)! Unfortunately, that will not always be the case. Here is the first example from a list of hard puzzles:

```
>>> grid2 = '4.....8.5.3..........7......2.....6.....8.4......1.......6.3.7.5..2.....1.4......'

>>> display(parse_grid(grid2))
   4      1679   12679  |  139     2369    269   |   8      1239     5    
 26789     3    1256789 | 14589   24569  245689  | 12679    1249   124679 
  2689   15689   125689 |   7     234569  245689 | 12369   12349   123469 
------------------------+------------------------+------------------------
  3789     2     15789  |  3459   34579    4579  | 13579     6     13789  
  3679   15679   15679  |  359      8     25679  |   4     12359   12379  
 36789     4     56789  |  359      1     25679  | 23579   23589   23789  
------------------------+------------------------+------------------------
  289      89     289   |   6      459      3    |  1259     7     12489  
   5      6789     3    |   2      479      1    |   69     489     4689  
   1      6789     4    |  589     579     5789  | 23569   23589   23689  
```

In this case, we are still a long way from solving the puzzle--61 squares remain uncertain. What next? We could try to code more sophisticated strategies. For example, the *naked twins* strategy looks for two squares in the same unit that both have the same two possible digits. Given `{'A5': '26', 'A6':'26', ...}`, we can conclude that 2 and 6 must be in `A5` and `A6` (although we don't know which is where), and we can therefore eliminate 2 and 6 from every other square in the `A` row unit. We could code that strategy in a few lines by adding an `elif len(values[s]) == 2` test to `eliminate`.

Coding up strategies like this is a possible route, but would require hundreds of lines of code (there are dozens of these strategies), and we'd never be sure if we could solve *every* puzzle.

Search

The other route is to *search* for a solution: to systematically try all possibilities until we hit one that works. The code for this is less than a dozen lines, but we run another risk: that it might take forever to run. Consider that in the `grid2` above, `A2` has 4 possibilities (`1679`) and `A3` has 5 possibilities (`12679`); together that's 20, and if we keep multiplying, we get $4.62838344192 \times 10^{38}$ possibilities for the whole puzzle. How can we cope with that? There are two choices.

First, we could try a brute force approach. Suppose we have a very efficient program that takes only one instruction to evaluate a position, and that we have access to the next-generation computing technology, let's say a 10GHz processor with 1024 cores, and let's say we could afford a million of them, and while we're shopping, let's say we also pick up a time machine and go back 13 billion years to the origin of the universe and start our program running. We can then compute that we'd be almost 1% done with this one puzzle by now.

The second choice is to somehow process much more than one possibility per machine instruction. That seems impossible, but fortunately it is exactly what constraint propagation does for us. We don't have to try all 4×10^{38} possibilities because as soon as we try one we immediately eliminate many other possibilities. For example, square H7 of this puzzle has two possibilities, 6 and 9. We can try 9 and quickly see that there is a contradiction. That means we've eliminated not just one possibility, but fully *half* of the 4×10^{38} choices.

In fact, it turns out that to solve this particular puzzle we need to look at only 25 possibilities and we only have to explicitly search through 9 of the 61 unfilled squares; constraint propagation does the rest. For the list of 95 hard puzzles, on average we need to consider 64 possibilities per puzzle, and in no case do we have to search more than 16 squares.

What is the search algorithm? Simple: first make sure we haven't already found a solution or a contradiction, and if not, choose one unfilled square and consider all its possible values. One at a time, try assigning the square each value, and searching from the resulting position. In other words, we search for a value `d` such that we can successfully search for a solution from the result of assigning square `s` to `d`. If the search leads to an failed position, go back and consider another value of `d`. This is a *recursive* search, and we call it a *depth-first* search because we (recursively) consider all possibilities under `values[s] = d` before we consider a different value for `s`.

To avoid bookkeeping complications, we create a new copy of `values` for each recursive call to `search`. This way each branch of the search tree is independent, and doesn't confuse another branch. (This is why I chose to implement the set of possible values for a square as a string: I can copy `values` with `values.copy()` which is simple and efficient. If I implemented a possibility as a

Python `set` or `list` I would need to use `copy.deepcopy(values)`, which is less efficient.) The alternative is to keep track of each change to `values` and undo the change when we hit a dead end. This is known as *backtracking search*. It makes sense when each step in the search is a single change to a large data structure, but is complicated when each assignment can lead to many other changes via constraint propagation.

There are two choices we have to make in implementing the search: *variable ordering* (which square do we try first?) and *value ordering* (which digit do we try first for the square?). For variable ordering, we will use a common heuristic called *minimum remaining values*, which means that we choose the (or one of the) square with the minimum number of possible values. Why? Consider `grid2` above. Suppose we chose `B3` first. It has 7 possibilities (`1256789`), so we'd expect to guess wrong with probability 6/7. If instead we chose `G2`, which only has 2 possibilities (`89`), we'd expect to be wrong with probability only 1/2. Thus we choose the square with the fewest possibilities and the best chance of guessing right. For value ordering we won't do anything special; we'll consider the digits in numeric order.

Now we're ready to define the `solve` function in terms of the `search` function:

```
def solve(grid): return search(parse_grid(grid))

def search(values):
    "Using depth-first search and propagation, try all possible values."
    if values is False:
        return False ## Failed earlier
    if all(len(values[s]) == 1 for s in squares):
        return values ## Solved!
    ## Chose the unfilled square s with the fewest possibilities
    n,s = min((len(values[s]), s) for s in squares if len(values[s]) > 1)
    return some(search(assign(values.copy(), s, d))
                for d in values[s])

def some(seq):
    "Return some element of seq that is true."
    for e in seq:
        if e: return e
    return False
```

That's it! We're done; it only took one page of code, and we can now solve any Sudoku puzzle.

Results

You can view the complete program. Below is the output from running the program at the command line; it solves the two files of 50 easy and 95 hard puzzles (see also the 95 solutions), eleven puzzles I found under a search for [hardest sudoku], and a selection of random puzzles:

```
% python sudo.py
All tests pass.
Solved 50 of 50 easy puzzles (avg 0.01 secs (86 Hz), max 0.03 secs).
Solved 95 of 95 hard puzzles (avg 0.04 secs (24 Hz), max 0.18 secs).
Solved 11 of 11 hardest puzzles (avg 0.01 secs (71 Hz), max 0.02 secs).
Solved 99 of 99 random puzzles (avg 0.01 secs (85 Hz), max 0.02 secs).
```

Analysis

Each of the puzzles above was solved in less than a fifth of a second. What about really hard puzzles? Finnish mathematician Arto Inkala described his 2006 puzzle as "the most difficult sudoku-puzzle known so far" and his 2010 puzzle as "the most difficult puzzle I've ever created." My program solves them in 0.01 seconds each (`solve_all` will be defined below):

```
>>> solve_all(from_file("hardest.txt")[0:2], 'Inkala')
8 5 . |. . 2 |4 . .
7 2 . |. . . |. . 9
. . 4 |. . . |. . .
------+------+------
. . . |1 . 7 |. . 2
3 . 5 |. . . |9 . .
. 4 . |. . . |. . .
------+------+------
. . . |. 8 . |. 7 .
. 1 7 |. . . |. . .
. . . |. 3 6 |. 4 .

8 5 9 |6 1 2 |4 3 7
7 2 3 |8 5 4 |1 6 9
1 6 4 |3 7 9 |5 2 8
------+------+------
9 8 6 |1 4 7 |3 5 2
```

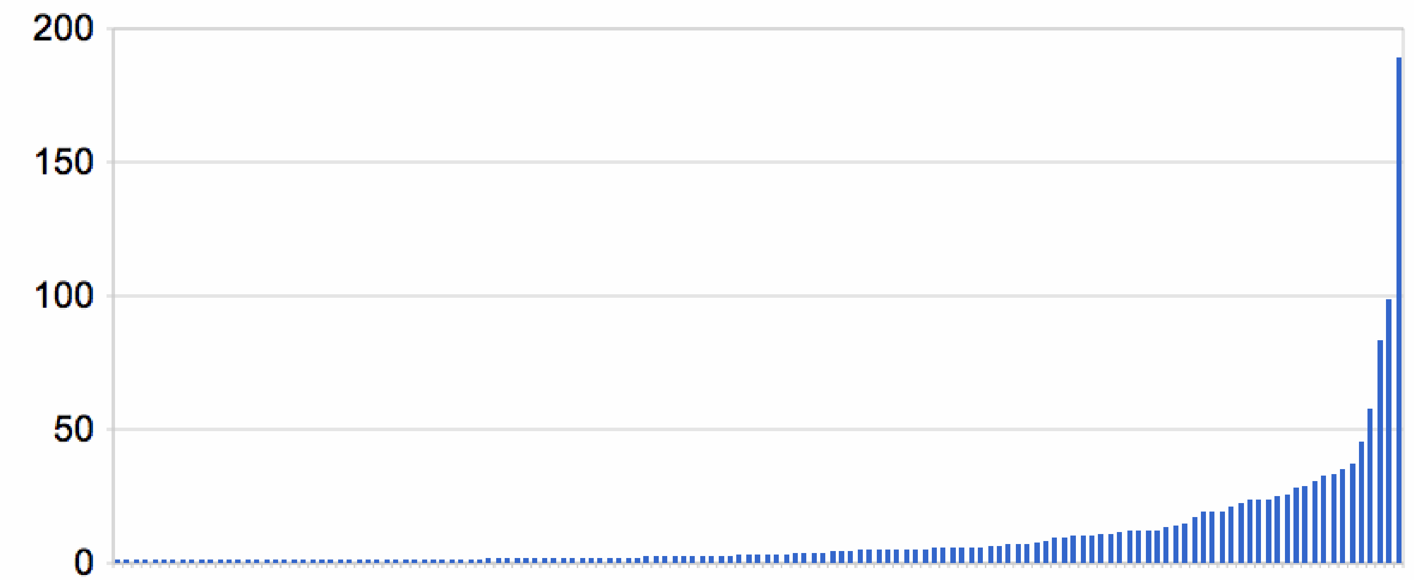
```
3 7 5 |2 6 8 |9 1 4
2 4 1 |5 9 3 |7 8 6
------+------+------
4 3 2 |9 8 1 |6 7 5
6 1 7 |4 2 5 |8 9 3
5 9 8 |7 3 6 |2 4 1

(0.01 seconds)

. . 5 |3 . . |. . .
8 . . |. . . |. 2 .
. 7 . |. 1 . |5 . .
------+------+------
4 . . |. . 5 |3 . .
. 1 . |. 7 . |. . 6
. . 3 |2 . . |. 8 .
------+------+------
. 6 . |5 . . |. . 9
. . 4 |. . . |. 3 .
. . . |. . 9 |7 . .

1 4 5 |3 2 7 |6 9 8
8 3 9 |6 5 4 |1 2 7
6 7 2 |9 1 8 |5 4 3
------+------+------
4 9 6 |1 8 5 |3 7 2
2 1 8 |4 7 3 |9 5 6
7 5 3 |2 9 6 |4 8 1
------+------+------
3 6 7 |5 4 2 |8 1 9
9 8 4 |7 6 1 |2 3 5
5 2 1 |8 3 9 |7 6 4

(0.01 seconds)

Solved 2 of 2 Inkala puzzles (avg 0.01 secs (99 Hz), max 0.01 secs).
```

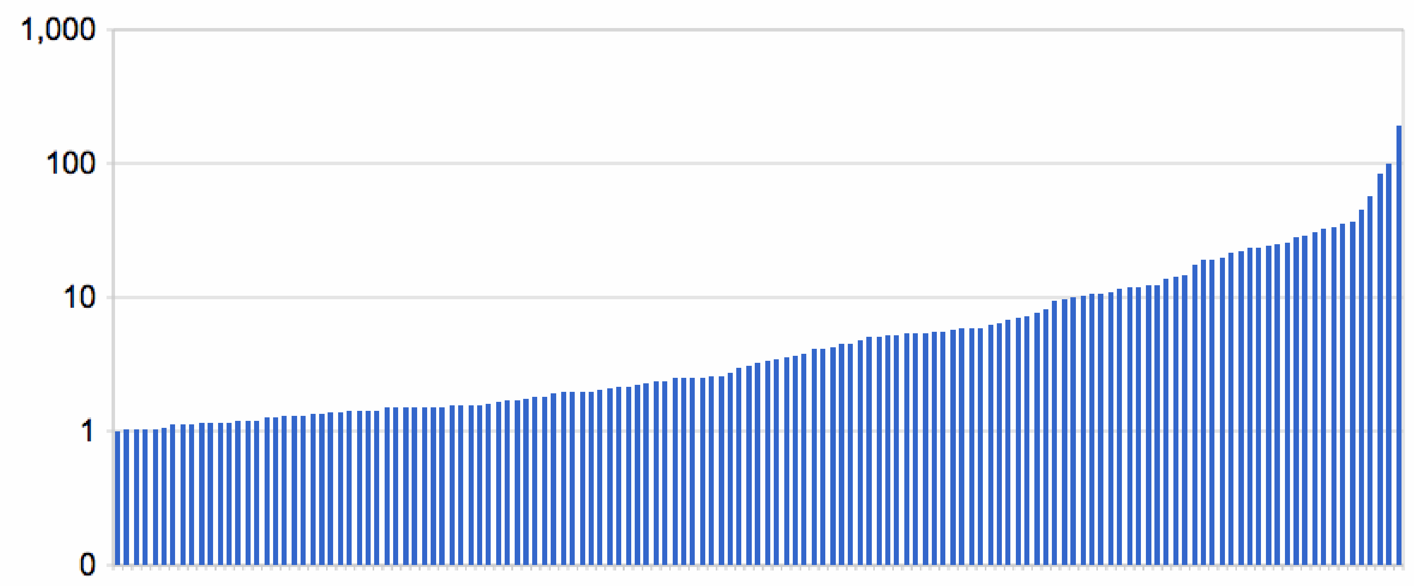
I guess if I want a really hard puzzle I'll have to make it myself. I don't know how to make hard puzzles, so I generated a million random puzzles. My algorithm for making a random puzzle is simple: first, randomly shuffle the order of the squares. One by one, fill in each square with a random digit, respecting the possible digit choices. If a contradiction is reached, start over. If we fill at least 17 squares with at least 8 different digits then we are done. (Note: with less than 17 squares filled in or less than 8 different digits it is known that there will be duplicate solutions. Thanks to Olivier Grégoire for the fine suggestion about 8 different digits.) Even with these checks, my random puzzles are not guaranteed to have one unique solution. Many have multiple solutions, and a few (about 0.2%) have no solution. Puzzles that appear in books and newspapers always have one unique solution.

The average time to solve a random puzzle is 0.01 seconds, and more than 99.95% took less than 0.1 seconds, but a few took much longer:

0.032%	(1 in 3,000) took more than 0.1 seconds
0.014%	(1 in 7,000) took more than 1 second
0.003%	(1 in 30,000) took more than 10 seconds
0.0001%	(1 in 1,000,000) took more than 100 seconds

Here are the times in seconds for the 139 out of a million puzzles that took more than a second, sorted, on linear and log scales:





It is hard to draw conclusions from this. Is the uptick in the last few values significant? If I generated 10 million puzzles, would one take 1000 seconds? Here's the hardest (for my program) of the million random puzzles:

```
>>> hard1  = '.....6....59.....82....8....45........3........6..3.54...325..6..................'
>>> solve_all([hard1])
. . . |. . 6 |. . .
. 5 9 |. . . |. . 8
2 . . |. . 8 |. . .
------+------+------
. 4 5 |. . . |. . .
. . 3 |. . . |. . .
. . 6 |. . 3 |. 5 4
------+------+------
. . . |3 2 5 |. . 6
. . . |. . . |. . .
. . . |. . . |. . .

4 3 8 |7 9 6 |2 1 5
6 5 9 |1 3 2 |4 7 8
2 7 1 |4 5 8 |6 9 3
------+------+------
8 4 5 |2 1 9 |3 6 7
7 1 3 |5 6 4 |8 2 9
9 2 6 |8 7 3 |1 5 4
------+------+------
1 9 4 |3 2 5 |7 8 6
3 6 2 |9 8 7 |5 4 1
5 8 7 |6 4 1 |9 3 2

(188.79 seconds)
```

Unfortunately, this is not a true Sudoku puzzle because it has multiple solutions. (It was generated before I incorporated Olivier Grégoire's suggestion about checking for 8 digits, so note that any solution to this puzzle leads to another solution where the 1s and 7s are swapped.) But is this an intrinsicly hard puzzle? Or is the difficulty an artifact of the particular variable- and value-ordering scheme used by my `search` routine? To test I randomized the value ordering (I changed `for d in values[s]` in the last line of `search` to be `for d in shuffled(values[s])` and implemented `shuffled` using `random.shuffle`). The results were starkly bimodal: in 27 out of 30 trials the puzzle took less than 0.02 seconds, while each of the other 3 trials took just about 190 seconds (about 10,000 times longer). There are multiple solutions to this puzzle, and the randomized `search` found 13 different solutions. My guess is that somewhere early in the search there is a sequence of squares (probably two) such that if we choose the exact wrong combination of values to fill the squares, it takes about 190 seconds to discover that there is a contradiction. But if we make any other choice, we very quickly either find a solution or find a contradiction and move on to another choice. So the speed of the algorithm is determined by whether it can avoid the deadly combination of value choices.

Randomization works most of the time (27 out of 30), but perhaps we could do even better by considering a better value ordering (one popular heuristic is *least-constraining value*, which chooses first the value that imposes the fewest constraints on peers), or by trying a smarter variable ordering.

More experimentation would be needed before I could give a good characterization of the hard puzzles. I decided to experiment on another million random puzzles, this time keeping statistics on the mean, 50th (median), 90th and 99th percentiles, maximum and standard deviation of run times. The results were similar, except this time I got two puzzles that took over 100 seconds, and one took quite a bit longer: 1439 seconds. It turns out this puzzle is one of the 0.2% that has no solution, so maybe it doesn't count. But the main message is that the mean and median stay about the same even as we sample more, but the maximum keeps going up--dramatically. The standard deviation edges up too, but mostly because of the very few very long times that are way out beyond the 99th percentile. This is a *heavy-tailed* distribution, not a normal one.

For comparison, the tables below give the statistics for puzzle-solving run times on the left, and for samples from a normal (Gaussian) distribution with mean 0.014 and standard deviation 1.4794 on the right. Note that with a million samples, the max of the Gaussian is 5 standard deviations above the mean (roughly what you'd expect from a Gaussian), while the maximum puzzle run time is 1000 standard deviations above the mean.

Samples of Puzzle Run Time

N	mean	50%	90%	99%	max	std. dev
10	0.012	0.01	0.01	0.01	0.02	0.0034
100	0.011	0.01	0.01	0.02	0.02	0.0029
1,000	0.011	0.01	0.01	0.02	0.02	0.0025
10,000	0.011	0.01	0.01	0.02	0.68	0.0093
100,000	0.012	0.01	0.01	0.02	29.07	0.1336
1,000,000	0.014	0.01	0.01	0.02	1439.81	1.4794

Samples of *N*(0.014, 1.4794)

N	mean	50%	90%	99%	max	std. dev
10	0.312	1.24	1.62	1.62	1.62	1.4061
100	0.278	0.18	2.33	4.15	4.15	1.4985
1,000	0.072	0.10	1.94	3.38	6.18	1.4973
10,000	0.025	0.05	1.94	3.45	6.18	1.4983
100,000	0.017	0.02	1.91	3.47	7.07	1.4820
1,000,000	0.014	0.01	1.91	3.46	7.80	1.4802

Here is the impossible puzzle that took 1439 seconds:

```
. . . |. . 5 |. 8 .
. . . |6 . 1 |. 4 3
. . . |. . . |. . .
------+------+------
. 1 . |5 . . |. . .
. . . |1 . 6 |. . .
3 . . |. . . |. . 5
------+------+------
5 3 . |. . . |. 6 1
. . . |. . . |. . 4
. . . |. . . |. . .
```

Here is the code that defines `solve_all` and uses it to verify puzzles from a file as well as random puzzles:

```
import time, random

def solve_all(grids, name='', showif=0.0):
    """Attempt to solve a sequence of grids. Report results.
    When showif is a number of seconds, display puzzles that take longer.
    When showif is None, don't display any puzzles."""
    def time_solve(grid):
        start = time.clock()
        values = solve(grid)
        t = time.clock()-start
        ## Display puzzles that take long enough
        if showif is not None and t > showif:
            display(grid_values(grid))
```

```
        if values: display(values)
        print '(%.2f seconds)\n' % t
    return (t, solved(values))
    times, results = zip(*[time_solve(grid) for grid in grids])
    N = len(grids)
    if N > 1:
        print "Solved %d of %d %s puzzles (avg %.2f secs (%d Hz), max %.2f secs)." % (
            sum(results), N, name, sum(times)/N, N/sum(times), max(times))

def solved(values):
    "A puzzle is solved if each unit is a permutation of the digits 1 to 9."
    def unitsolved(unit): return set(values[s] for s in unit) == set(digits)
    return values is not False and all(unitsolved(unit) for unit in unitlist)

def from_file(filename, sep='\n'):
    "Parse a file into a list of strings, separated by sep."
    return file(filename).read().strip().split(sep)

def random_puzzle(N=17):
    """Make a random puzzle with N or more assignments. Restart on contradictions.
    Note the resulting puzzle is not guaranteed to be solvable, but empirically
    about 99.8% of them are solvable. Some have multiple solutions."""
    values = dict((s, digits) for s in squares)
    for s in shuffled(squares):
        if not assign(values, s, random.choice(values[s])):
            break
        ds = [values[s] for s in squares if len(values[s]) == 1]
        if len(ds) >= N and len(set(ds)) >= 8:
            return ''.join(values[s] if len(values[s])==1 else '.' for s in squares)
    return random_puzzle(N) ## Give up and make a new puzzle

def shuffled(seq):
    "Return a randomly shuffled copy of the input sequence."
    seq = list(seq)
    random.shuffle(seq)
    return seq

grid1  = '003020600900305001001806400008102900700000008006708200002609500800203009005010300'
grid2  = '4.....8.5.3..........7......2.....6.....8.4......1.......6.3.7.5..2.....1.4......'
hard1  = '.....6....59.....82....8....45........3........6..3.54...325..6..................'

if __name__ == '__main__':
    test()
    solve_all(from_file("easy50.txt", '========'), "easy", None)
    solve_all(from_file("top95.txt"), "hard", None)
    solve_all(from_file("hardest.txt"), "hardest", None)
    solve_all([random_puzzle() for _ in range(99)], "random", 100.0)
```

Why?

Why did I do this? As computer security expert Ben Laurie has stated, Sudoku is "a denial of service attack on human intellect". Several people I know (including my wife) were infected by the virus, and I thought maybe this would demonstrate that they didn't need to spend any more time on Sudoku. It didn't work for my friends (although my wife has since independently kicked the habit without my help), but at least one stranger wrote and said this page worked for him, so I've made the world more productive. And perhaps along the way I've taught something about Python, constraint propagation, and search.

Translations

This code has been reimplemented by several people in several languages:

- C++ by Pau Fernandez
- C# with LINQ version by Richard Birkby
- Clojure version by Justin Kramer
- Erlang version by Andreas Pauley
- Haskell version by Emmanuel Delaborde
- Java version by Johannes Brodwall
- Javascript version by Brendan Eich
- Javascript version by Pankaj Kumar
- Ruby version by Luddite Geek
- Ruby version by Martin-Louis Bright

You can see a Korean translation of this article by JongMan Koo, or use the translation widget below:

[Peter Norvig](http://norvig.com)

131 Comments norvig.com Login

Recommend 96 Share Sort by Best

Join the discussion…

naeg • 6 years ago

This article should be called "how to win sudoku".

24 • Reply • Share ›

Michel Lespinasse • 6 years ago

I like your writeup & your concise implementation.

When doing my own sudoku solver, I used a similar approach but with a twist in the constraint propagation. I had constraints not just on cell contents like in yours, but also in values that MAY or MUST be seen within blocks of 3 consecutive cells (horizontally or vertically). I found that this allowed my algorithm to make quick progress in cases where the search method would not - for example, I can find out that your 1439 seconds puzzle has no solution without having to invoke search even once.

As an example of how this works:
If the possible values for 3 consecutive cells don't include a given value, one can note that the corresponding cell block may not include that value.
if two aligned cell blocks may not include a given value, then the third one must include that value (even if we don't know which of the 3 cells will have the value in).
Since we know the value will be present in that third cell block, it may not be present in any of the other 6 cells in the corresponding 3x3 block.
This is a good heuristic to have because it's very complementary with the search - it works well in cases where the search does not.

17 • Reply • Share ›

Ravi Annaswamy • 6 years ago

Genius is fluency. Genius is going to a distance where no one has gone and then going further and further away.

Peter Norvig teases me. As though if you give him an hour, he can take any issue and make a giant step. Wow! I memorized the first version of your sudoku.py to learn Python idioms. I spent a couple of months with that code and it led me to functional programming, lisp and so many beautiful things. Proof that if you go deep, you encompass wide. Thank you.
Ravi

15 • Reply • Share ›

Anonymous • 5 years ago

Ahh, but there is an advantage to playing sudoku (as opposed to just completing the puzzle with a computer program). That advantage is making your brain think in a logical fashion, not just work from rote. And it is this logical thought process that gets the blood flowing to sometimes under-utilized portions of the brain, and can help stave off both boredom and possibly even dementia.

7 • Reply • Share ›

Harald Schilly → Anonymous • 5 years ago

this is right, i think, but i also read that it only helps against dementia as long as you are "learning" the rules. once you managed to learn sudoku to the point where you are into it and can solve it ... means you know some patterns and it feels nice to live in this world of rules ... you are already beyond the point where it helps. at this point, you need to search for a new puzzle and don't get sucked into this one too much :-)

6 ^ | v • Reply • Share ›

Thomas Lambrecht • 5 years ago

working on the C implementation right now (i have got to make this for an exam)
It's really helpfull to have some guidlines to follow, thank you for the guide.

Step (1) and (2) work already... starting on the search part

5 ^ | v • Reply • Share ›

Dave_Bernazani • 3 years ago

Mr. Norvig,
I am neither a mathematician nor a programmer, but I am an avid sudoku player, and your article was so well written it was understandable and interesting even to me! Thanks for sharing this fascinating info.
David Bernazani, Lafayette, CA

4 ^ | v • Reply • Share ›

Abel Mengistu • 4 years ago

A simple suggestion:

This last weekend I solved this puzzle on Project Euler, after which I read a comment in the forum that led me here.
I had used the same constraint propagation strategies (1) and (2), and did the search in a similar way starting with squares that have the fewest options. However, with the same reasoning for choosing the squares with fewest possibilities, I was ordering the options themselves based on their frequency in the whole grid. I had bookkeeping code, that kept the frequencies. I tested it on the **hard1** puzzle from above, and it ran in less than 0.06 seconds. :)

4 ^ | v • Reply • Share ›

Idk ➦ Abel Mengistu • 2 years ago

Could you share your code for this approach?
I'm currently learning python an am thinking about how one could speed this thing up.Thanks!

^ | v • Reply • Share ›

Bill Zwicky • 6 years ago

Your link for "easy 50" has the wrong file name and wrong format, and the program no longer works with them.
To fix, use this from_file which supports regular expressions:

```
def from_file(filename, sep='\n'):
"Parse a file into a list of strings, separated by sep."
s = file(filename).read().strip()
l = re.split(sep, s)
l = filter(lambda z: len(z)>0, l)
return l
```

and this test line:

```
solve_all(from_file("easy50.txt", "Grid [0-9]+"), "easy", None)
```

4 ^ | v • Reply • Share ›

Peter Norvig Mod ➦ Bill Zwicky • 6 years ago

Bill, you are exactly right: I tried to give more credit to project Euler by linking directly to their file, when I should have linked to my local version of the file, in which "Grid ..." is replaced by a constant '======' marker. Your solution using re.split is a good one and more general than my solution, but I wanted to keep it simple.

6 ^ | v • Reply • Share ›

Martin d'Anjou • 5 years ago

Something is strange with this one, it cannot be solved, yet it has 17 values in it:

Something is strange with this one, it cannot be solved, yet it has 17 values in it:

```
>>> qs='...8.1..........435............7.8........1...2..3....6.......75..34........2..6..'
>>> solve_all([qs])
>>> display(parse_grid(qs))
23479 3679 4679 | 8 249 1 | 5 269 269
1289 1689 1689 | 7 29 5 | 29 4 3
5 189 1489 | 3 249 6 | 7 1289 1289
------------------+------------------+------------------
1349 1369 14569 | 156 7 24 | 8 23569 269
3479 3679 45679 | 56 8 24 | 1 23569 2679
178 2 15678 | 156 3 9 | 4 56 67
------------------+------------------+------------------
6 4 2 | 9 1 8 | 3 7 5
189 5 3 | 4 6 7 | 29 1289 1289
1789 1789 1789 | 2 5 3 | 6 189 4
```

We need to give it a hand:

see more

2 ∧ | ∨ • Reply • Share ›

Mark Lawrence ➦ Martin d'Anjou • 5 years ago

You are mistaken. I've tried this with both Peter's solution and with the excellent Sudoku Explainer and they tally exactly, there is no need to give this solver a helping hand.

2 ∧ | ∨ • Reply • Share ›

Martin d'Anjou ➦ Mark Lawrence • 5 years ago

Please show me the commands. I'd like to know what I am missing.

1 ∧ | ∨ • Reply • Share ›

jssk • 7 months ago

Hi Peter,

I'm an amateur programmer reading around the web looking for ideas to improve on my sudoku solving algorithm or grids that would help expose its bugs, when I came across your page. It is very interesting and your program was so nicely written. Mine was about 1000 lines!

I understand the last grid was impossible, but it probably shouldn't have taken 1400 seconds to figure out it was actually an unsolvable grid. In fact, right off the bat, it becomes apparent that, an instance of number 1 , 5 , and 6 must be placed in either H5 or I5 (under your indexing method) locations, i.e. we have to fit 3 numbers in 2 spots, which immediately makes it an invalid grid.

It took my program 4 iterations to reach that conclusion, which took about 5 seconds. It was probably because of my old slow computer or something was wrong with my implementation in Python, because in my original Matlab code anything under 30 iterations took a fraction of a second.

Anyway, thanks for sharing your thoughts and code. Cheers!

Ning

1 ∧ | ∨ • Reply • Share ›

beka • 2 years ago

nice article. Peter, do you have more puzzles like the impossible one? I generated about 4 million random puzzles (without "at least different 8" rule), but they are all solved in less than 1s (including yours).

1 ∧ | ∨ • Reply • Share ›

Ed • 3 years ago

I am a novice python programmer that happened to stumble on this page by trying to solve the Euler sudoku puzzle. I have to say I have learned so much from the beauty and simplicity of your code. My code for equivalent puzzle. I have to say I have learned so much from the beauty and simplicity of your code. My code for equivalent tasks is so bloated!!! Going through your code and trying to rewrite parts of my own algorithm has really been a great lesson in python for me. Thanks so much.

I am wondering... will I ever be able to write such concise code? Its a little discouraging to think about it, actually.

1 • Reply • Share ›

Udi Meiri • 5 years ago

So how about an Iterative Deepening DFS? It could limit your exposure to bad heuristic choices, decreasing run time on the harder puzzles.

1 • Reply • Share ›

Peter Norvig Mod → Udi Meiri • 5 years ago

Normally you wouldn't use Iterative Deepening on a problem where every terminal state is the same distance from the start. But I suppose that with the automatic inferences., you have terminals at different distances. However, I'm not sure that the solution would tend to me nearer the start.

• Reply • Share ›

yahoo-LYMRKB5SBFCKXFCWPMBX7OOTNQ • 5 years ago

Is it possible to use this algorithm to help solving a scheduling problem?

My rationale is as follow:
1) Having a list of trips, calculate the incompatible set for each one (the same structure as the one used here to explicit that no number can be in the same row,column or region)
2) from the above dictionary calculate the chromatic number CN (will be the maximum set length of any given dictionary entry)
3) initialize the peers dictionary corresponding set to be empty for each trip (at this stage trips are not allocated)
4) create CN number of duties (trips will be allocated to duties)
5) add the duties to the incompatible dictionary (mentioned in 1) with empty set
6) allocate the CN th trips (ordered by maximum set length) to the duties (one in each)
5) update the peers,incompatible and possible values dictionaries for the allocated trips and propagate to the peers

If my rationale is correct then we would have limited significantly the search space. My idea is to then apply a genetic algorithm, but with special rules for mutation and breading, for example:
- encoding only the non allocated values
- allowing mutation to pick from the available values for a trip
- etc...

I would appreciate any comments/ suggestions. Is it feasible? Would it be efficient?

Thanks in advance for your help and most of all for the inspiration.

1 • Reply • Share ›

Guest • 4 years ago

Just discovered this excellent page. I'm surprised that nobody pointed out that each square has 24 peers, not 20. When a number is put in location i,j in a 3x3 unit, then it cannot appear in the i,j position in any other 3x3 unit. Each number appears in a different position in each of the 3x3 units. This is the first pattern I saw when I learned the rules to Sudoku. So I added a line to the peer def:

```
unitlist = ([cross(rows, c) for c in cols] +
[cross(r, cols) for r in rows] +
[cross(rs, cs) for rs in ('ABC','DEF','GHI') for cs in ('123','456','789')] +
[cross(rs, cs) for rs in ('ADG','BEH','CFI') for cs in ('147','258','369')])
```

and this reduced the running time from:
Solved 50 of 50 easy puzzles (avg 0.01 secs (116 Hz), max 0.01 secs).
Solved 95 of 95 hard puzzles (avg 0.03 secs (32 Hz), max 0.15 secs).
Solved 11 of 11 hardest puzzles (avg 0.01 secs (89 Hz), max 0.02 secs).
Solved 99 of 99 random puzzles (avg 0.01 secs (107 Hz), max 0.01 secs).

to:
Solved 50 of 50 easy puzzles (avg 0.01 secs (174 Hz), max 0.01 secs).
Solved 95 of 95 hard puzzles (avg 0.01 secs (176 Hz), max 0.01 secs).
Solved 11 of 11 hardest puzzles (avg 0.01 secs (150 Hz), max 0.01 secs).
Solved 98 of 99 random puzzles (avg 0.01 secs (89 Hz), max 0.02 secs).
1 • Reply • Share ›

Guest → Guest • 4 years ago
No this is wrong.
7 • Reply • Share ›

ferenc lazar → Guest • 3 years ago
each square has 24 peers if you count 3x8, but there are 4 overlaps (2 block-row, 2 block-column)
inspirational python coding...
2 • Reply • Share ›

R U sure → Guest • 3 years ago
Is what you say about "a number being in a location I,j in a 3x3 unit means that it cannot appear in the i,j position in any other 3x3 unit" really correct?
In the solved examples above, the number 4 appears in the same place in two units (first solved example - top left and center unit) and 2 appears in the same place in two units (top left and middle right units).
There may be others, but two examples is probably sufficient.
1 • Reply • Share ›

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

miss a → miss a • 3 years ago
:P
2 • Reply • Share ›

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

mr a → miss a • 3 years ago
no
2 • Reply • Share ›

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

This comment is awaiting moderation. Show comment.

simons ➦ richard • 3 years ago

like like like

1 ^ | v • Reply • Share ›

This comment is awaiting moderation. Show comment.

mr a ➦ simons • 3 years ago

hi

1 ^ | v • Reply • Share ›

Erik Colban • 25 days ago

I really enjoyed this essay. Thank you. The code contains many gems. One that caught my attention was the function `some()`. It's easy to miss that a generator is passed to this function, so only the first solution is computed. At first I thought all solutions were computed but only the first was kept. I noticed the subtlety when rewriting the function as:

```
def some(seq):
"Return some element of seq that is true."
return next((e for e in seq if e), False)
```

To ensure that a puzzle has exactly one solution, I replaced `some()` by `unique()`:

```
def unique(seq):
"Return the unique element of seq that is true."
g = (e for e in seq if e)
first = next(g, False)
second = next(g, False)
return first if first and not second else False
```

^ | v • Reply • Share ›

Luke Schoen • a month ago

In Depth-First Search, do we need to consider ALL possibilities p under values[s] = d such that for every different for s would result in a successful outcome if used, and then benchmark all p, and choose the fastest p? Or do we only have to consider possibilities p until we find the first one that would result in a successful outcome if used only have to consider possibilities p until we find the first one that would result in a successful outcome if used, and then choose that p (even if it using it may be slower than if we used other possibilities that we could have found if we continued searching)?

Based on the above, with regard to the statement: "...call it a depth-first search because we (recursively) consider all possibilities under values[s] = d before we consider a different value for s", shouldn't it instead state "...call it a depth-first search because we (recursively) consider all possibilities under values[s] = d before we consider a different value for s, however if when considering the possible values we find one that would result in a successful outcome if used, then we stop considering the result of any remaining possibilities under values[s] = d, and we no longer need to consider any more values for s"?

Reply • Share ›

Boris • 2 months ago

If anybody is interested - hard1 puzzle has 148357268 solutions. I double checked it.

Reply • Share ›

Lud a. r. • 2 months ago

Thanks for sharing the approach and, especially, the info on the heavy-tailed distribution of run-times! The latter bit made the problem intriguing to me again.

In a naive test I simply ran the problem in parallel - not sub-dividing the work, but simply running the full problem in a "race to finish" from different "entry points" - and found that this made a dramatic improvement:
- A small suite of 20 'hard' valid problems decreased 40ms -> 20ms
- Your hardest invalid problem (the 1439s one) decreased from 6 s -> 8 ms !

By 'entry point' I just mean that whenever I search for the square with the least candidates remaining, I start from a different index. By iterating all 81 of them, I found in the hard-problem that there was a range of 17 indexes where it would run this quickly (~8ms), and for many other entry points it would run in ~200-600ms. (In some layman speculation, I imagine that the hard-problem takes a long time because it requires recursing & back-tracking over a long distance, but when 'guesses' are made by fixing squares within this range, it breaks down the longest possible recursion ... but that's just a hypothesis at this stage.)

In fact I wonder to what extent this may have solved the heavy tail problem - how much the worst case is reduced or if it has just been pushed further out. Unfortunately I think that if it does work it would only be because Sudoku has a fixed-size, so it may not scale to other problems (like those I encounter in my day job) but I think it's an interesting observation nonetheless.

see more

Reply • Share ›

Load more comments