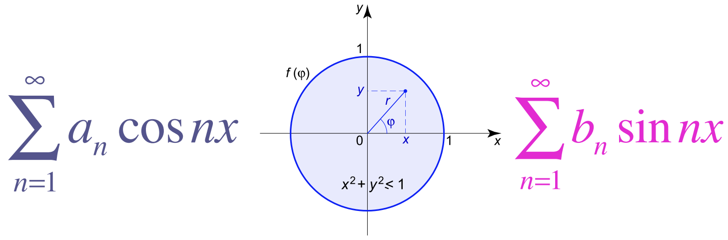
Calculus

Fourier Series



Complex Form of Fourier Series

Let the function $f(x)$ be defined on the interval $[-\pi, \pi]$. Using the well-known Euler's formulas

Recommended Pages

Definition of Fourier Series and Typical Examples

Fourier Series of Functions with an Arbitrary Period

Even and Odd Extensions

Complex Form of Fourier Series

Applications of Fourier Series to Differential Equations

we can write the Fourier series of the function in complex form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{inx} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Here we have used the following notations:

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}.$$

The coefficients c_n are called complex Fourier coefficients. They are defined by the formulas

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx} dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

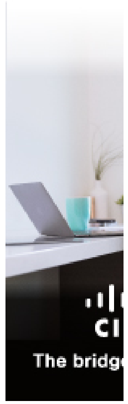
If necessary to expand a function $f(x)$ of period $2L$, we can use the following expressions:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-\frac{in\pi x}{L}} dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

The complex form of Fourier series is algebraically simpler and more symmetric. Therefore, it is



Solved Problems

Click or tap a problem to see the solution.

Example 1

Using complex form, find the Fourier series of the function

$$f\left(x\right) = \text{sign}\, x = \begin{cases} -1, & -\pi \le x \le 0 \\ 1, & 0 < x \le \pi \end{cases}.$$

Example 2

Using complex form find the Fourier series of the function $f\left(x\right) = x^2$, defined on the interval $\left[-1, 1\right]$.

Example 3

Using complex form find the Fourier series of the function

$$f\left(x\right) = \frac{a \sin x}{1 - 2a \cos x + a^2}, \quad \left|a\right| < 1.$$

Using complex form, find the Fourier series of the function

$$f(x) = \operatorname{sign} x = \begin{cases} -1, & -\pi \le x \le 0 \\ 1, & 0 < x \le \pi \end{cases}.$$

Solution.

We calculate the coefficients c_0 and c_n for $n \neq 0$:

$$c_0 = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \left[\int\limits_{-\pi}^{0} (-1)\, dx + \int\limits_{0}^{\pi} dx \right] = \frac{1}{2\pi} \left[(-x)\big|_{-\pi}^{0} + x\big|_{0}^{\pi} \right]$$

$$= \frac{1}{2\pi} (-\cancel{\pi} + \cancel{\pi}) = 0,$$

$$c_n = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(x)\, e^{-inx} dx = \frac{1}{2\pi} \left[\int\limits_{-\pi}^{0} (-1)\, e^{-inx} dx + \int\limits_{0}^{\pi} e^{-inx} dx \right]$$

$$= \frac{1}{2\pi} \left[-\frac{\left(e^{-inx}\right)\big|_{-\pi}^{0}}{-in} + \frac{\left(e^{-inx}\right)\big|_{0}^{\pi}}{-in} \right] = \frac{i}{2\pi n} \left[-\left(1 - e^{in\pi}\right) + e^{-in\pi} - 1 \right]$$

$$= \frac{i}{2\pi n} \left[e^{in\pi} + e^{-in\pi} - 2 \right] = \frac{i}{\pi n} \left[\frac{e^{in\pi} + e^{-in\pi}}{2} - 1 \right] = \frac{i}{\pi n} [\cos n\pi - 1]$$

$$= \frac{i}{\pi n} \left[(-1)^n - 1 \right].$$

If $n = 2k$, then $c_{2k} = 0$. If $n = 2k - 1$, then $c_{2k-1} = -\frac{2i}{(2k-1)\pi}$.

Hence, the Fourier series of the function in complex form is

$$f(x) = \operatorname{sign} x = -\frac{2i}{\ldots} \sum^{\infty} \frac{1}{\ldots} e^{i(2k-1)x}.$$

We can transform the series and write it in the real form. Rename: $n = 2k - 1$, $n = \pm 1, \pm 2, \pm 3, \ldots$ Then

$$f(x) = \operatorname{sign} x = -\frac{2i}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{i(2k-1)x} = -\frac{2i}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n}$$
$$= -\frac{2i}{\pi} \sum_{n=1}^{\infty} \left(\frac{e^{-inx}}{-n} + \frac{e^{inx}}{n} \right) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{e^{inx} - e^{-inx}}{2in} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$
$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

Graph of the function and its Fourier approximation for $n = 5$ and $n = 50$ are shown in Figure 1.

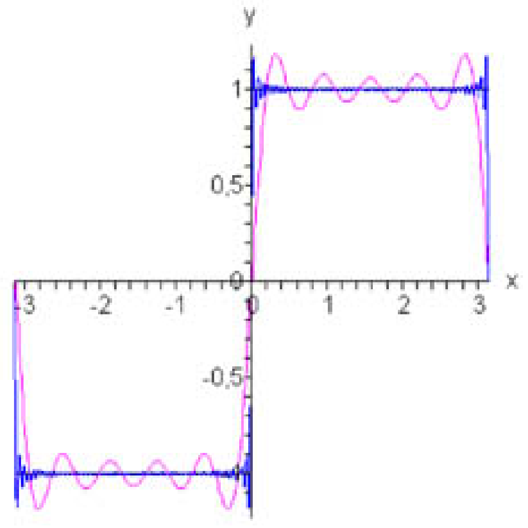


Figure 1, n = 5, n = 50