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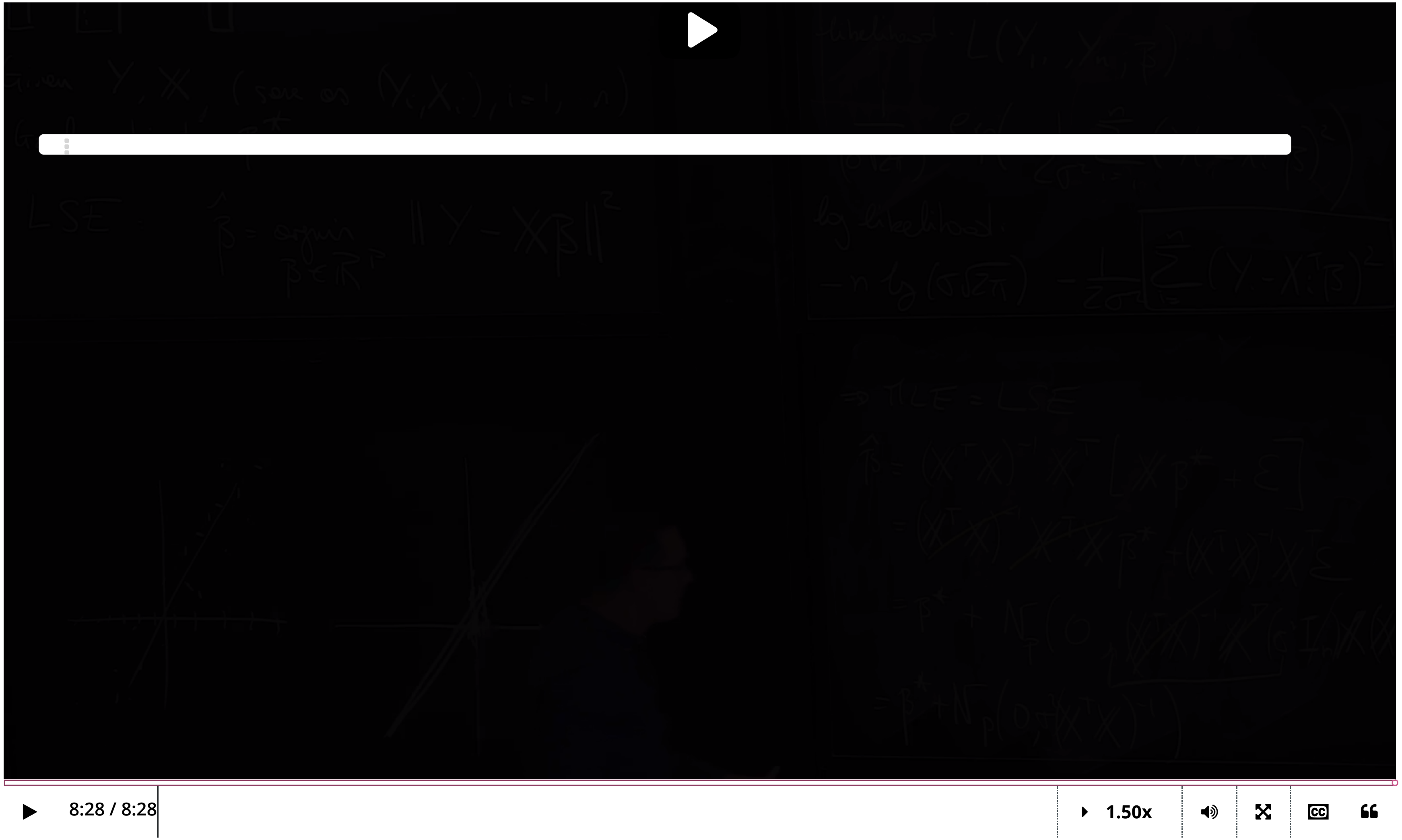
# 7. Distribution of the Least Square Estimator

## Distribution of the Least Square Estimator



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## Gaussian Noise

3/3 points (graded)

Recall that the Least-Squares Estimator $\hat{\beta}$ has the formula

$$\hat{\boldsymbol{\beta}} = (\mathbb{X}^T \mathbb{X})^{-1} \mathbb{X}^T \mathbf{Y}.$$

If we assume that the vector $\boldsymbol{\varepsilon}$ is an $n$-dimensional Gaussian with mean $0$ and covariance $\sigma^2 I_n$ for some known $\sigma^2 > 0$, then:

“The model is **homoscedastic** ; i.e. $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d."

● True

○ False

✓

“In the deterministic design setting, the LSE $\hat{\boldsymbol{\beta}}$ is a Gaussian random variable."

● True

○ False

✓

“If $\mathbb{X}$ is a random variable, then the LSE $\hat{\boldsymbol{\beta}}$ is still a Gaussian random variable."

○ True

● False

✓

**Solution:**

- “**The model is homoscedastic ; i.e. $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d.**" is true. The covariance matrix is a diagonal matrix. Recall the useful fact that the $i$th coordinate of a multi-dimensional gaussian is also a gaussian. In the case where the covariance matrix is diagonal, the coordinates also happen to be independent. Therefore, the first statement is **true**.

- “**In the deterministic design setting, the LSE $\hat{\beta}$ is a Gaussian random variable**" is true. We have

$$\hat{\beta} = \left(\mathbb{X}^T \mathbb{X}\right)^{-1} \mathbb{X}^T \mathbf{Y} = \beta + \left(\mathbb{X}^T \mathbb{X}\right)^{-1} \mathbb{X}^T \epsilon.$$

By using the result of the first exercise, we arrive at the conclusion that $\hat{\beta}$ is also Gaussian: $\hat{\beta} \sim \mathcal{N}\left(\beta, \sigma^2 \left(\mathbb{X}^T \mathbb{X}\right)^{-1}\right)$.

- "**If $\mathbb{X}$ is a random variable, then the LSE $\hat{\beta}$ is still a Gaussian random variable**" is false. *The assumption that $\mathbb{X}$ is deterministic/constant is crucial.* If $\mathbb{X}$ were a generic random variable, then $\hat{\beta}$ might no longer be Gaussian.

  Perhaps the simplest example is the case where $\mathbb{X}$ is determined by an unbiased coin flip. Specifically, consider what happens if $\mathbb{X}$ takes value $\mathbb{X}_1$ if the coin comes up heads, otherwise it takes value $\mathbb{X}_2$. Then by the law of total probability, $\hat{\beta}$ has density $\frac{f_1}{2} + \frac{f_2}{2}$ where $f_1, f_2$ are densities of Gaussians $\mathcal{N}\left(\beta, \sigma^2 \left(\mathbb{X}_1^T \mathbb{X}_1\right)^{-1}\right), \mathcal{N}\left(\beta, \sigma^2 \left(\mathbb{X}_2^T \mathbb{X}_2\right)^{-1}\right)$ respectively. If $\mathbb{X}_1 \neq \mathbb{X}_2$, this is not a Gaussian distribution, but a "mixture" of two Gaussians. (Note: this is not to be confused with the density of the **sum of two Gaussian random variables**, which IS a Gaussian random variable. Summing the densities is different from summing the random variables!) In general, it can be very difficult to write down the distribution of $\hat{\beta}$ in terms of the distribution of $\mathbb{X}$.

Submit You have used 1 of 1 attempt

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ℹ Answers are displayed within the problem

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