5.3 Unit 5 Practice Problems

Unit 5: Averages

Adapted from Blitzstein-Hwang Chapters 4 and 5.

Problem 1

1/1 point (graded)

Randomly, k distinguishable balls are placed into n distinguishable boxes, with all possibilities equally likely. Find the expected number of empty boxes as a function of k and n.

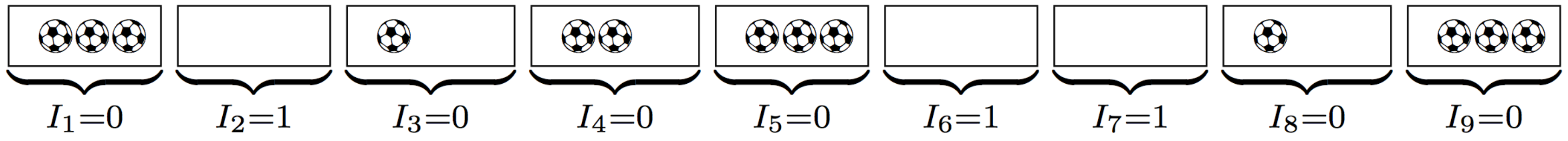
● $n(1-1/n)^k$ ✓

○ $n(1-1/n^k)$

○ $n(1-k/n)$

○ $n(1-k/n)^k$

Solution



Let I_j be the indicator random variable for the j-th box being empty, so $I_1+\cdots+I_n$ is the number of empty boxes (the above picture illustrates a possible outcome with 3 empty boxes, for $n=9, k=13$). Then $E(I_j)=P(I_j=1)=(1-1/n)^k$. By linearity,

$$E\left(\sum_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} E(I_j) = n(1-1/n)^k.$$

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ℹ Answers are displayed within the problem

For Problem 2:

Alice and Bob have just met, and wonder whether they have a mutual friend. Each has 50 friends, out of 1000 other people who live in their town. They think that it's unlikely that they have a friend in common, saying "each of us is only friends with 5% of the people here, so it would be very unlikely that our two 5%'s overlap."

Assume that Alice's 50 friends are a random sample of the 1000 people (equally likely to be any 50 of the 1000), and similarly for Bob. Also assume that knowing who Alice's friends are gives no information about who Bob's friends are.

Problem 2a

1/1 point (graded)

(a) Compute the expected number of mutual friends Alice and Bob have.

2.5 ✓ **Answer:** 2.5

2.5

Solution:

Let I_j be the indicator r.v. for the jth person being a mutual friend. Then

$$E\left(\sum_{j=1}^{1000} I_j\right) = 1000E(I_1) = 1000P(I_1 = 1) = 1000 \cdot \left(\frac{5}{100}\right)^2 = 2.5.$$

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Problem 2b

1/1 point (graded)

(b) Let X be the number of mutual friends they have. Find $P(X = 2)$.

0.266 ✓ **Answer:** 0.266

0.266

Solution

Condition on who Alice's friends are, and then count the number of ways that Bob can be friends with exactly k of them. This gives

$$P(X = k) = \frac{\binom{50}{k}\binom{950}{50-k}}{\binom{1000}{50}}$$

for $0 \leq k \leq 50$ (and 0 otherwise). So:

$$P(X = 2) = \frac{\binom{50}{2}\binom{950}{48}}{\binom{1000}{50}} \approx 0.266$$

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Problem 2c

1/1 point (graded)

(c) Which of the following distributions does X have?

○ Binomial

○ Poisson

○ Geometric

● Hypergeometric ✓

Solution

It is Hypergeometric, as shown by the PMF from the solutions to (b) or by thinking of "tagging" Alice's friends and then seeing how many tagged people there are among Bob's friends.

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For Problem 3:

A group of n people play "Secret Santa" as follows. Each person puts their name on a slip of paper in a hat. Then each person picks a name randomly from the hat (without replacement), and buys a gift for that person. Unfortunately, they overlook the possibility of drawing one's own name, so some may have to buy gifts for themselves (on the bright side, some may like self-selected gifts better). Assume $n \geq 2$.

Problem 3a

1/1 point (graded)

(a) Find the expected value of the number X of people who pick their own names.

1 ✓ **Answer:** 1

1

Solution

Let \(I_j\) be the indicator r.v. for the \(j\)-th person picking their own name. Then \(E(I_j) = P(I_j = 1) = \frac{1}{n}\). By linearity, the expected number is \(n\cdot E(I_j) = 1\).

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Problem 3b

1/1 point (graded)

(b) Let \(X\) be the number of people who pick their own names. What is the *approximate* distribution of \(X\) if \(n\) is large (specify the parameter value or values)?

● \(\textrm{Pois}(1)\) ✓

○ \(\textrm{Pois}(2)\)

○ \(\textrm{Geom}(1/2)\)

○ \(\mathcal{N}(1,1)\)

Solution

By Poisson approximation, \(X\) is approximately Pois(1) for large \(n\).

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FOR PROBLEM 4:

Let $Z \sim \mathcal{N}(0,1)$. A measuring device is used to observe Z, but the device can only handle positive values, and gives a reading of 0 if $Z \leq 0$; this is an example of *censored data*. So assume that $X = ZI_{Z>0}$ is observed rather than Z, where $I_{Z>0}$ is the indicator of $Z > 0$.

Problem 4a

1/1 point (graded)
(a) Find $E(X)$.

0.3989423 ✓ **Answer:** 0.399

0.3989423

Solution
By LOTUS,

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I_{z>0} z e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} z e^{-z^2/2} dz.$$

Letting $u = z^2/2$, we have

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u} du = \frac{1}{\sqrt{2\pi}} \approx 0.399.$$

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Problem 4b

1/1 point (graded)
(b) Find $\mathrm{Var}(\mathrm{X})$.

0.340845 ✓ **Answer:** 0.341

0.340845

Solution
To obtain the variance, note that

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-z^2/2} dz = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz = \frac{1}{2},$$

since a $\mathcal{N}(0,1)$ r.v. has variance 1. Thus,

$$\mathrm{Var}(X) = E(X^2) - (EX)^2 = \frac{1}{2} - \frac{1}{2\pi} \approx 0.341.$$

Note that X is neither purely discrete nor purely continuous, since $X = 0$ with probability $1/2$ and $P(X = x) = 0$ for $x \neq 0$. So X has neither a PDF nor a PMF; but LOTUS still works, allowing us to work with the PDF of Z to study expected values of functions of Z.

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