# 10. Linear dependence and independence

When we describe the set of solutions to $\mathbf{A}\mathbf{x} = \mathbf{0}$ as the span of a set of vectors, we want this set of vectors to be as small as possible. In order to eliminate redundant vectors, we need to generalize the notion of linear dependence to 3 or more vectors.

**Example 10.1** On the previous page, we saw that $\mathrm{Span}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right) = \mathrm{Span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$. This is because the vector $\begin{pmatrix}2\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\1\end{pmatrix}$. In this case, it may seem obvious that this vector is only adding redundant information because it is a scalar multiple of the other vector. We had a term to describe this, we say that these two vectors are linearly dependent.

What does it mean for 3 or more vectors to be linearly dependent?

**Example 10.2** Determine if there are any redundant vectors used to define the subspace $\mathrm{Span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$.

To start, let's see if all vectors are in the span of one vector $\mathrm{Span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)$. If this were the case, we could write $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$ as $c\begin{pmatrix}1\\1\\1\end{pmatrix}$ for some value $c$. This is not possible. So at most one vector is redundant.

Let's now consider the space $\mathrm{Span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$. We ask if $\begin{pmatrix}1\\0\\1\end{pmatrix}$ lies in this subspace. If it does, then it does not add any new information. We see that in fact, it is in this span because

$$\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

This idea of redundence is captured in the definition of what it means for 3 or more vectors to be linearly dependent.

**Geometric definition:** A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ is **linearly dependent** if there is at least one vector $\mathbf{v}_i$ that lies in the subspace spanned by the other vectors.

Vectors that are not linearly dependent are called **linearly independent** .

**Reduction to 2 vectors:**

**Example 10.3** Let's consider this definition of linear dependence in the case of two vectors: $\mathbf{v}_1, \mathbf{v}_2$. These vectors are linearly dependent if one of these vectors, say $\mathbf{v}_j$ is in the subspace $\mathrm{Span}(\mathbf{v}_i)$. Since $\mathrm{Span}(\mathbf{v}_i) = \{c\mathbf{v}_i, c \text{ any number}\}$, for $\mathbf{v}_j$ to be in this subspace implies that $\mathbf{v}_j = c\mathbf{v}_i$ for some number $c$. This is equivalent to saying that both vectors lie on the same line through the origin, that is one is a scalar multiple of the other. (This was the definition we gave for linear dependence of two vectors.)

**More small examples:**

**Example 10.4** The vectors $\begin{pmatrix}3\\5\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$ are linearly dependent because $\begin{pmatrix}3\\5\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$:

$$\begin{pmatrix}3\\5\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\end{pmatrix}.$$

**Example 10.5** Are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ linearly dependent? The vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not in the span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, but we don't know yet if they are linearly independent. They are linearly dependent because $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ is in the span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

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## Testing linear dependence

To have a systematic way of testing for linear dependence, it is helpful to have a definition in terms of an algebraic relationship that we can test.

**Equivalent algebraic definition:** A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly dependent** if there exist scalars $c_1, \ldots, c_n$ **not all zero** such that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0.$$

**Proof that the two definitions are equivalent**

Show

**Remark 10.6** If a collection of $n-1$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ are linearly independent, to test if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are dependent, it is enough to check if $\mathbf{v}_n$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$.

Why? Suppose that the $n$ vectors are linearly dependent. Then there is a nonzero vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix}$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1} + c_n\mathbf{v}_n = 0.$$

If $c_n$ were equal to zero, this would imply that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{n-1}\mathbf{v}_{n-1} = 0.$$

Because we assume these first $n-1$ are linearly independent, this equation is only possible if all the $c_i$ are zero. Therefore it must be the case that $c_n \neq 0$, thus

$$\mathbf{v}_n = \frac{c_1}{c_n}\mathbf{v}_1 + \cdots + \frac{c_{n-1}}{c_n}\mathbf{v}_{n-1},$$

which means that $\mathbf{v}_n$ is in the span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$.

**Example in $\mathbb{R}^3$:**

Suppose we have 2 linearly independent vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. These two vectors span a plane inside of $\mathbb{R}^3$. Now suppose we have a third vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. This vector is **linearly dependent** if and only if lies in the plane spanned by the first two vectors. This is true if and only if we can write it as a linear combination of the first two vectors.

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# Linear independence

Start of transcript. Skip to the end.

(Caption will be displayed when you start playing the video.)

So now I want to say what does it mean for a bunch of vectors

to be independent.

So this is the background that we know.

Now I want to speak about independence.

Let's see.

I can give you the abstract definition, and I will.

0:00 / 6:35 2.0x

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## Algebraic definition of linear independence concept check

1/1 point (graded)

Are the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ linearly independent?

- (•) Yes. They are linearly independent. ✔
- ( ) No. They are linearly dependent.

**Solution:**

Yes. They are linearly independent. Let $c_1$, $c_2$, and $c_3$ be numbers such that

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Simplifying, we see that

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so $c_1$, $c_2$, and $c_3$ must all be zero. Hence the three vectors are linearly independent.

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

## Geometric definition concept check

1/1 point (graded)

Two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^3$ span the plane defined by the equation $x - 2y + z = 0$. Find the value $c$ such that $\mathbf{v}_1, \mathbf{v}_2$ and $\begin{pmatrix} 3 \\ 1 \\ c \end{pmatrix}$ are linearly dependent.

$c =$ -1 **Answer:** -1

**Solution:**

Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are already linearly independent, if follows that for the three vectors to be linearly dependent, $\begin{pmatrix} 3 \\ 1 \\ c \end{pmatrix}$ must lie on the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. This happens when $\begin{pmatrix} 3 \\ 1 \\ c \end{pmatrix}$ satisfies the defining equation of the plane

$$3 - 2(1) + c = 0,$$

that is when $c = -1$.

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## 10. Linear dependence and independence

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### Dependence and Linearly Dependence

question posted 10 days ago by **subsole**

**More small examples:**

**Example 10.4** The vectors $\begin{pmatrix}3\\5\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$ are linearly dependent because $\begin{pmatrix}3\\5\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$:

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**Example 10.5** Are $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}1\\0\\0\end{pmatrix}$, and $\begin{pmatrix}2\\0\\0\end{pmatrix}$ linearly dependent? The vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is not in the span of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}2\\0\\0\end{pmatrix}$, but we don't know yet if they are linearly independent. They are dependent because $\begin{pmatrix}2\\0\\0\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

From example 10.4:

are linearly dependent because $\begin{pmatrix}3\\5\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

From example 10.5:

are dependent because $\begin{pmatrix}2\\0\\0\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$

The reason for being 'linearly dependent' and 'dependent' is the same. So what is 'dependent'? From 10.5, it looks like being 'dependent' doesn't guarantee being 'linearly dependent'... But it looks like... The important concept here is 'linearly dependence' (which is well-defined)... So is 'dependence' something too trivial to notice?

This post is visible to everyone.

**mrBB**

10 days ago - marked as answer 10 days ago by **jfrench** (Staff)

Those two terms mean exactly the same (in this context). "Linear (in)dependence" is probably the most correct term to use, but from the context we can deduce that '(in)dependence' is casual speak/write for 'linear (in)dependence'.

mrBB is exactly correct. If you watch the video, you notice that Prof. Strang points out that he will generally only say dependent or independent but means linearly dependent or linearly independent. Looks like we unintentionally adopted the same convention. I will try to go through and add the word linearly for clarity though!

posted 10 days ago by **jfrench** (Staff)

No problem. Thanks for being kind. I think it's very difficult to explain simple things plainly and simply when people get... advanced. Maybe very few people are willing to do that if they can do so... I think 18.033x is doing that...

posted 10 days ago by **subsole**

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1 other response

**HaugDaniel**

about 3 hours ago

Just a thought: Two non zero vectors in $R^n$ are independent if and only if one is not a *scalar* multiple of the other. And { $w1, w2, \ldots, wk$} is a basis for subspace $W$ of $R^n$ if and only if the column vectors $w1, w2, \ldots, wk$ span $W$ and are independent.

So in general if { $w1, w2, w3...wk$} is a set of of column vectors in $R^n$ a dependence relation in the set is an equation of the form $r1 * w1 + r2 * w2 + \ldots + rk * wk = 0$ with at least one scalar $rj$ being zero. Any two non zero non parallel vectors in $R^2$ form a basis for $R^2$, and the basis for a span of multiple vectors depends on the order in which they are placed as columns in the matrix $A$.

Finding a basis for any subspace $W$ of $R^n$, requires determining whether or not the linear system $Ax = 0$ has a non trivial solution obtained from the augmented matrix $[A|0]$ reduced row echelon reduction *standard basis vector form* $H$ with zeros above and below pivots of numeral *one* in each of the column vectors. And showing dependence would amount to showing a column vector is a linear combination of other column. Thereby deleting columns shown to be dependent and retaining the other columns as a basis for the subspace $W$.

So the vectors are independent if and only if row reduction of the matrix $A$ yields a matrix $H$ with a pivot in every column. I use row interchange and scaling to obtain a non zero entry pivot $p$ in the top row of the first column and then use the fraction { $-r/p$ } as a multiplier for row reduction below, where $r$ represents the non zero entry in the first column, for each row below the row above with a pivot.

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