Fun with Prime Numbers (4)

Invitation to the Mysterious World of Mathematics

Tetsushi Ito

Department of Mathematics,
Kyoto University

Polynomial analogue

- Let us consider a polynomial analogue of the ABC conjecture.
- A polynomial is an expression consisting of variables and coefficients (complex numbers).
- deg f(x) = degree of the polynomial f(x)

Example

$2x-1 \qquad x^2+x-1 \qquad (3x+1)^2+(x-1)^3$

Polynomial analogue (2)

Analogy between integers and polynomials

- Operations

  $A+B \quad A-B \quad A \times B$

- Division

  $$A = SB+R$$
  $$|R| < |B|$$

- Prime factorization

  $$N = P_1 \times \cdots \times P_M$$

- Operations

  $f(x)+g(x) \quad f(x)-g(x) \quad f(x)g(x)$

- Division

  $$f(x)=p(x)g(x)+r(x)$$
  $$\deg r(x) < \deg g(x)$$

- Irreducible decomposition

  $$f(x) = p_1(x)\cdots p_M(x)$$

Polynomial analogue (3)

Definition

- A triple of non-constant polynomials $(f(x), g(x), h(x))$ is a polynomial ABC triple if

$$f(x) + g(x) = h(x)$$

$$\deg f(x),\ \deg g(x) \leq \deg h(x)$$

and no irreducible polynomial $p(x)$ (of degree ≥ 1) divides both of $f(x)$ and $g(x)$.

Polynomial analogue (4)

<u>Definition</u>

- For a polynomial ABC triple $(f(x),g(x),h(x))$, let $N(x)$ be the product of all distinct irreducible factors of $f(x)g(x)h(x)$.

- $N(x)$ is the conductor polynomial.
- $\deg N(x)$ is an analogue of the conductor of an ABC triple.

Polynomial analogue (5)

<u>Theorem</u> (Polynomial ABC Conjecture)
For every polynomial ABC triple
(f(x),g(x),h(x)), we have

$$\deg h(x) < \deg N(x).$$

<u>Differences</u>

- We do not need $(1+\varepsilon)$.
- There are no exceptions.

Polynomial analogue (6)

Example

- $f(x) = x^3 \quad g(x) = -3x^2 + 3x - 1 \quad h(x) = (x-1)^3$
- $f(x) + g(x) = h(x)$

  $\Rightarrow$ $(f(x),g(x),h(x))$: polynomial ABC triple
- $N(x)$ = product of all distinct irreducible factors of $f(x)g(x)h(x)$

  $= x\,(x-1)\,(-3x^2+3x-1)$
- $\deg h(x) = 3 < \deg N(x) = 4$

Proof of the Polynomial ABC Conjecture

- We shall use the derivative f' (x) of a polynomial f(x).

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$$f'(x) = n\, a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

Proof of the Polynomial ABC Conjecture (2)

- $(f(x),g(x),h(x))$: polynomial ABC triple
- (Claim 1) $f(x)$ and $g(x)$ divides

$$P(x) = \big(f(x)\, g'(x) - f'(x)\, g(x)\big)\, N(x).$$

- Assume α is a complex root of $f(x)=0$.

  If $(x-\alpha)^e$ divides $f(x)$, we see that $(x-\alpha)^{e-1}$ divides $f'(x)$, and $(x-\alpha)$ divides $N(x)$.

  Hence $(x-\alpha)^e$ divides $P(x)$.
- Similarly, $g(x)$ divides $P(x)$.

Proof of the Polynomial ABC Conjecture (3)

- (Claim 2) $h(x)$ divides $P(x)$.

  Since $f(x) + g(x) = h(x)$, we have

  $$f(x)\, g'(x) - f'(x)\, g(x)$$

  $$= \bigl(h(x) - g(x)\bigr)\, g'(x) - \bigl(h'(x) - g'(x)\bigr)\, g(x)$$

  $$= h(x)\, g'(x) - h'(x)\, g(x).$$

  Hence $P(x) = \bigl(h(x)\, g'(x) - h'(x)\, g(x)\bigr)\, N(x)$.

  The rest of the proof is the same as before.

Proof of the Polynomial ABC Conjecture (4)

- By (Claim 1)+(Claim 2), $f(x)g(x)h(x)$ divides $P(x)$. Hence we have

$$\deg f(x) + \deg g(x) + \deg h(x)$$

$$\leq \deg P(x)$$

$$< \deg f(x) + \deg g(x) + \deg N(x)$$

$$\deg h(x) < \deg N(x)$$

Q.E.D.

Conclusion

- Since the time of Euclid, many results on prime numbers are known. Mathematicians are still working on them.
- There are many challenging open problems.
- I hope you will solve open problems in the future, and discover new phenomena on prime numbers. Good luck!