101 help

Determine density of $\min(X,Y)$ and $\max(X,Y)$ for independently uniform distributed variables

Two independent random variables, X and Y, are uniformly distributed on the unit interval $(-1,1)$.

Determine the density for $U = \min(X,Y)$ and for $W = \max(X,Y)$

self-study random-variable pdf uniform extreme-value

edited May 8 '15 at 11:51
RattusRattus
70 7

asked Feb 20 '13 at 4:08
Michael
13 1 3

1 This looks like a standard problem set for students. Is this homework, or otherwise related to coursework or a question from a text or test? Do you know the general approach for dealing with such questions? What region do you need to integrate? – Glen_b ♦ Feb 20 '13 at 4:11

This is for a review question for an upcoming exam. The hint that was given is to use (X'=(x+1)/2 , Y'=(y+1)/2) since these variables follow uniform(0,1). – Michael Feb 20 '13 at 4:24

I have added the homework tag (it's self study of a standard problem for coursework and falls under the scope of the tag). The tag also means that 'helpful hints' is what you should expect to get here. – Glen_b ♦ Feb 20 '13 at 4:27

If you come back with more information about what you have tried, I may expand on my hints, or respond to your attempts. – Glen_b ♦ Feb 20 '13 at 7:18

Thanks for your help. I think I solved it via your second suggestion. Ultimately I got F(w)= [(w+1)/2]^2 and F(u)=1-[1-(u+1)/2]^2 as the cdf's. – Michael Feb 20 '13 at 7:45

2 Answers

You will need to either

1) look at the bivariate distribution of X and Y in order to figure out what region of the pdf for (X, Y) corresponds to U and W, or

2) alternatively, make an algebraic argument in terms of the cdf - e.g. $P(W \leq w) = P(X \leq w, Y \leq w)$ ...

That hint you mentioned doesn't help you any if you don't know what you're supposed to do with the standard uniforms.

answered Feb 20 '13 at 4:31

Glen_b ♦
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I think the hint given for this problem is not very helpful. *Even* if the joint distribution of the minimum and maximum of two independent $U(0,1)$ random variables has been solved as an example in class or in the textbook, teaching a student to rely on plugging-and-chugging from formulas instead of *thinking* about the problem is very bad pedagogical practice, and even more so in this particular case because the general result is not too difficult to derive.

If $Z = \min(X, Y)$ and $W = \max(X, Y)$, then for $w > z$,

$$\begin{aligned}
F_{Z,W}(z,w) &= P\{Z \leq z, W \leq w\} \\
&= P\left[\{X \leq z, Y \leq w\} \cup \{X \leq w, Y \leq z\}\right] \\
&= P\{X \leq z, Y \leq w\} + P\{X \leq w, Y \leq z\} - P\{X \leq z, Y \leq z\} \\
&= F_{X,Y}(z,w) + F_{X,Y}(w,z) - F_{X,Y}(z,z)
\end{aligned}$$

while for $w < z$,

$$\begin{aligned}
F_{Z,W}(z,w) &= P\{Z \leq z, W \leq w\} = P\{Z \leq w, W \leq w\} \\
&= P\{X \leq w, Y \leq w\} \\
&= F_{X,Y}(w,w).
\end{aligned}$$

Consequently, if X and Y are jointly continuous random variables, then

$$f_{Z,W}(z,w) = \frac{\partial^2}{\partial z \partial w} F_{Z,W}(z,w) = \begin{cases} f_{X,Y}(z,w) + f_{X,Y}(w,z), & \text{if } w > z, \\ 0, & \text{if } w < z. \end{cases}$$

One can even think of this end result geometrically. Consider the joint density $f_{X,Y}(x,y)$ as a solid (of volume 1) sitting on the x-y plane. Slice it with a vertical cut along the line $x = y$ and *flip over* the part below the line $x = y$ so that it sits on top of the part above the line $x = y$. The resulting solid is the joint density of the minimum and the maximum.

For example, if the solid is a rectangular parallelepiped whose base is the square with vertices $(1,1),(-1,1),(-1,-1),(1,-1)$, the slicing and flipping over gives a right triangular prism of twice the height as the parallelepiped whose base has vertices $(1,1),(-1,1),(-1,-1)$.

If only the marginal densities are desired and not the joint density, the solution is even easier for the case of iid $U(-1,1)$ random variables. For $-1 \le z \le 1$,

$$\begin{aligned} 1 - F_Z(z) &= P\{Z > z\} = P\{\min(X,Y) > z\} \\ &= P\{X > z, Y > z\} = P\{X > z\}P\{Y > z\} = \left(\frac{1}{2}(1-z)\right)^2 \end{aligned}$$

giving, upon taking the derivative with respect to z that

$$f_Z(z) = \begin{cases} \frac{1-z}{2}, & -1 \le z \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, for $-1 \le z \le 1$,

$$\begin{aligned} F_W(z) &= P\{W \le w\} = P\{\max(X,Y) \le w\} \\ &= P\{X \le w, Y \le w\} = P\{X \le w\}P\{Y \le w\} = \left(\frac{1}{2}(w-(-1))\right)^2 \end{aligned}$$

giving, upon taking the derivative with respect to w that

$$f_W(w) = \begin{cases} \frac{1+w}{2}, & -1 \le w \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

edited Feb 20 '13 at 19:17 answered Feb 20 '13 at 13:25

Dilip Sarwate
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2 Although we have had this conversation before, and I appreciate your focus on the didactic and pedagogical elements, I still can't help wondering whether this answer is unnecessarily labored. The crux is a one-liner: independence of X and Y asserts that $\Pr(\max(X,Y)\le t)$ = $\Pr(X\le t)\Pr(Y\le t)$ = $((t+1)/2)^2$ and differentiation wrt t yields $(t+1)/2$ for the PDF of the max; the PDF of the min is obtained from $\max(X,Y)=-\min(-X,-Y)$. The simplicity of this approach makes it more likely its answer is correct-- even though it differs from yours (which integrates to 2). – whuber ♦ Feb 20 '13 at 18:12

2 @whuber I got the same result $((t+1)/2)^2$ for the CDF of $\max(X,Y)$ as you did, essentially via a one-liner and by the same argument that you used, but then I messed up in the differentiation of the CDFs, in that I forgot the factor of $\frac{1}{2}$ which occurs in the application of the chain rule when one differentiates $(1+t)/2$ w.r.t t; writing the derivative as 1 instead of $\frac{1}{2}$. Thanks for pointing out the mistake. I have corrected my answer. My point really was that mapping the RVs to $U(0,1)$ as suggested, using a plug-and-chug formula, and then mapping back is overkill – Dilip Sarwate Feb 20 '13 at 19:16