Exercise: The Expectation of a Geometric Distribution

(4/4 points)

In this exercise, we use the law of total expectation to find the expected value of a geometric random variable. The law of total expectation says that for a random variable X (with alphabet $\mathcal{X}$) and a partition $\mathcal{B}_1, \ldots, \mathcal{B}_n$ of the sample space,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid \mathcal{B}_i] \mathbb{P}(\mathcal{B}_i),$$

where

$$\mathbb{E}[X \mid \mathcal{B}_i] = \sum_{x \in \mathcal{X}} x p_{X|\mathcal{B}_i}(x) = \sum_{x \in \mathcal{X}} x \frac{\mathbb{P}(X = x, \mathcal{B}_i)}{\mathbb{P}(\mathcal{B}_i)}.$$

Let $X \sim \text{Geo}(p)$ be the number of tosses until we get heads for the first time, where the probability of heads is p. Let $\mathcal{B}$ be the event that we get heads in 1 try. Let $\mathcal{B}^c$ be the event that we get heads in more than 1 try. Note that $\mathcal{B}$ and $\mathcal{B}^c$ form a partition of the sample space.

- What is $\mathbb{P}(\mathcal{B})$?

p

Answer: p

- What is $\mathbb{E}[X \mid \mathcal{B}]$?

1

Answer: 1

- What is $\mathbb{E}[X \mid \mathcal{B}^c]$? Write your answer in terms of $m \triangleq \mathbb{E}[X]$. Note that we do not know $\mathbb{E}[X]$ for right now, but we can still relate $\mathbb{E}[X \mid \mathcal{B}^c]$ to $\mathbb{E}[X]$.

  Hint: If you do not get heads the first time, then starting from the second toss, the distribution for the number of tosses remaining is still geometric!

  In this part, please provide your answer as a mathematical formula (and not as Python code). Use ^ for exponentiation, e.g., x^2 denotes x^2. Explicitly include multiplication using *, e.g. x*y is xy.

1+m

Answer: 1 + m

$1 + m$

- Using the law of total expectation,

$$\mathbb{E}[X] = \mathbb{E}[X \mid \mathcal{B}]\mathbb{P}(\mathcal{B}) + \mathbb{E}[X \mid \mathcal{B}^c](1 - \mathbb{P}(\mathcal{B})).$$

Using your answers to the previous part, you should now have a recursive equation, meaning that the unknown quantity $\mathbb{E}[X]$ appears on both sides of the equation, and so you can solve for it.

What is $\mathbb{E}[X]$?

In this part, please provide your answer as a mathematical formula (and not as Python code). Use ^ for exponentiation, e.g., x^2 denotes x^2. Explicitly include multiplication using *, e.g. x*y is xy.

1/p

Answer: 1/p

$\frac{1}{p}$

Solution:

- What is $\mathbb{P}(\mathcal{B})$?

  Solution: This is just the probability of getting heads the first time, which is $\boxed{p}$.

- What is $\mathbb{E}[X \mid \mathcal{B}]$?

  Solution: If we get heads in 1 try, then the number of times until tossing heads is $\boxed{1}$.

- What is $\mathbb{E}[X \mid \mathcal{B}^c]$? Write your answer in terms of $m \triangleq \mathbb{E}[X]$. Note that we do not know $\mathbb{E}[X]$ for right now, but we can still relate $\mathbb{E}[X \mid \mathcal{B}^c]$ to $\mathbb{E}[X]$.

  Hint: If you do not get heads the first time, then starting from the second toss, the distribution for the number of tosses remaining is still geometric!

Solution: So we tossed the coin and it was tails, so this took up 1 toss. The number of tosses that remains is just another $\text{Geo}(p)$ random variable (remember: the tosses are all independent so that initial toss doesn't affect any of the future tosses)!

Thus, the expectation of X given that the first toss was tails (i.e., it takes more than 1 try) is

$$\mathbb{E}[X \mid \mathcal{B}^c] = 1 + \mathbb{E}[X].$$

The 1 appears because that's the first toss where we got tails.

So what goes in the answer box is $\boxed{1+m}$, where $m \triangleq \mathbb{E}[X]$.

- Using the law of total expectation,

$$\mathbb{E}[X] = \mathbb{E}[X \mid \mathcal{B}]\mathbb{P}(\mathcal{B}) + \mathbb{E}[X \mid \mathcal{B}^c](1 - \mathbb{P}(\mathcal{B})).$$

Using your answers to the previous part, you should now have a recursive equation, meaning that the unknown quantity $\mathbb{E}[X]$ appears on both sides of the equation, and so you can solve for it.

What is $\mathbb{E}[X]$?

Solution: Putting together the pieces from the previous parts,

$$\mathbb{E}[X] \;=\; 1 \cdot p + (1 + \mathbb{E}[X]) \cdot (1 - p)$$

$$
\begin{aligned}
&= p + 1 - p + \mathbb{E}[X] - \mathbb{E}[X]p \\
&= 1 + \mathbb{E}[X] - \mathbb{E}[X]p.
\end{aligned}
$$

Rearranging terms yields

$$\mathbb{E}[X] = \boxed{\frac{1}{p}}.$$

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