2

Line Fitting & SVD

Computer Vision Lab

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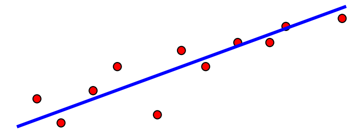
Problem



- Given a set of points P_1, P_2,…, P_N in the plain
- Fit a straight line such that
  - Sum of squared distances of the points from the line is minimized (***geometric error***)
  - Line represented by equations: $c + n_x x + n_y y = 0, \quad n_x^2 + n_y^2 = 1$
    - (n_x, n_y) is the unit normal to the line
    - When P is not on the line, then $|r|$ is its distance from it.
  - ➔ Constrained Least Squares problem: $c + n_x x + n_y y = r$
    - Minimize $\|r\| = \sum_{i=1}^{N} r_i^2$
    - subject to $\underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ 1 & x_N & y_N \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c \\ n_x \\ n_y \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}}_{\mathbf{r}}$ and $n_x^2 + n_y^2 = 1$

Solution

- N may be large ➔ A has many rows
  - Compute **QR** decomposition (Q orthogonal, R upper triangular) of A to reduce the problem to solving a small system
  - Since the norm is invariant to orthogonal transformations (i.e. $\|Q^T r\| = \|r\|$), **we can proceed as follows:**

$$\mathbf{A} = \mathbf{QR} \Rightarrow \quad \mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ n_x \\ n_y \end{bmatrix} = \mathbf{Q}^{\mathrm{T}}\mathbf{r} \quad \text{and} \quad n_x^2 + n_y^2 = 1$$

Solution

- Since the nonlinear constraint only involves 2 unknowns, we have to solve

$$\begin{bmatrix} R_{22} & R_{23} \\ 0 & R_{33} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} \approx 0, \quad \text{subject to} \quad \mathbf{n}_\mathbf{x}^2 + \mathbf{n}_\mathbf{y}^2 = \mathbf{1}$$

  - This is a classical Constrained LSE (CLSE) problem: $\|Bx\| = \min$, subject to $\|x\| = 1$.
    - The min. value is the *smallest singular value* of B
    - The solution is the ***corresponding singular vector***
    - ➔ (n_x, n_y) is obtained by SVD of B
    - ➔ c is then obtained by back substitution

SVD: definition

- Singular Value Decomposition:
  - Any mxn matrix can be written as the product of three matrices

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

U_1 V_1

$$\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} u_{11} \\ u_{21} \\ \\ u_{m1} \end{matrix}} & \begin{matrix} u_{12} \\ u_{22} \\ \\ u_{m2} \end{matrix} & \begin{matrix} u_{1m} \\ u_{2m} \\ \\ u_{mm} \end{matrix} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & \\ 0 & & \sigma_n \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} \boxed{v_{11} \quad v_{21} \quad v_{n1}} \\ v_{12} \quad v_{22} \quad v_{n2} \\ \\ v_{1n} \quad v_{2n} \quad v_{nn} \end{bmatrix}$$

- Singular values σ_i are fully determined by A
  - D is diagonal: $d_{ij} = 0$ if $i \neq j$; $d_{ii} = \sigma_i$ $(i=1,2,\ldots,n)$
  - $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N \geq 0$
- Both U and V are not unique
  - Columns of each are mutual orthogonal vectors

Slides adopted from

CS 395/495-26: Spring 2004

IBMR: Singular Value Decomposition (SVD Review)

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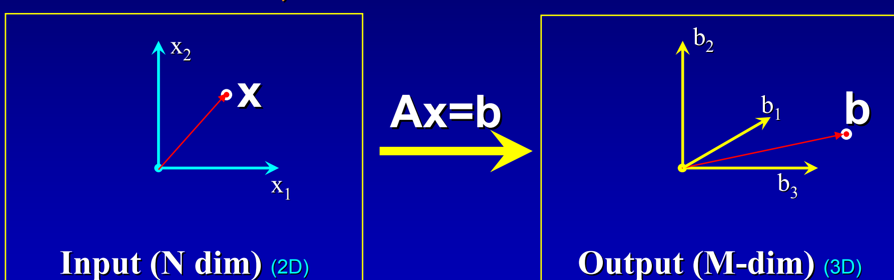
Matrix Multiply: A Change-of-Axes

- Matrix Multiply: $Ax = b$
  - x and b are column vectors
  - A has m rows, n columns

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & & \ldots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \ldots \\ b_m \end{bmatrix}$$



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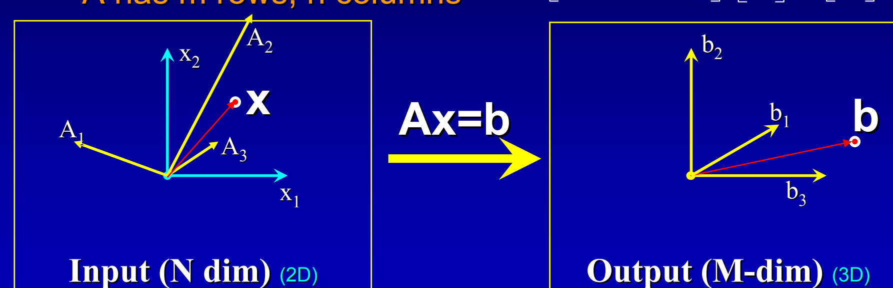
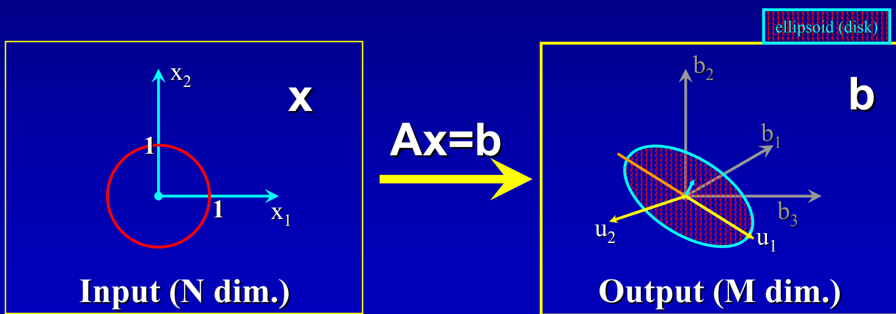


- Matrix multiply is just a set of dot-products; it 'changes coordinates' for vector x, makes b.
  - Rows of A = $A_1, A_2, A_3, \ldots$ = new coordinate axes
  - Ax = a dot product for each new axis

How Does Matrix 'stretch space'?

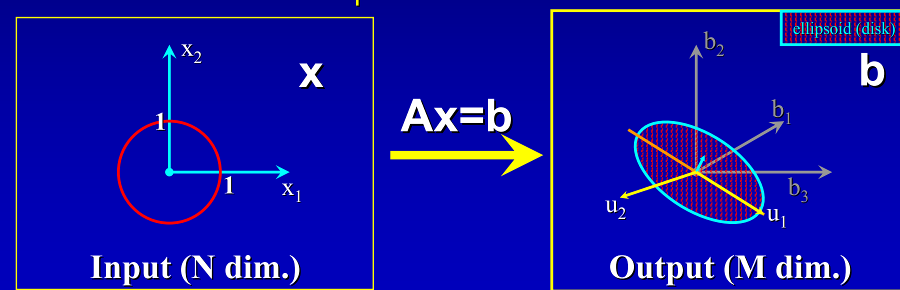
- Sphere of all unit-length **x** → ellipsoid of **b**
- Output ellipsoid's axes are always perpendicular (true for any ellipsoid)



THUS we can make ⊥, unit-length axes…

SVD finds: 'Output' Vectors U, and…

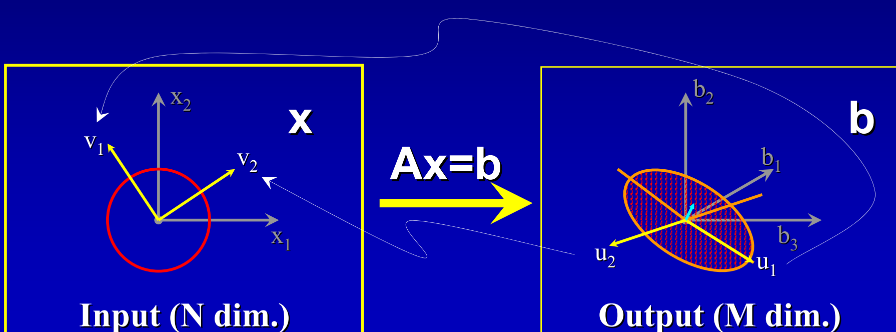
- Sphere of all unit-length **x** → ellipsoid of **b**
- Output ellipsoid's axes form orthonormal basis vectors U_i:
- Basis vectors U_i are columns of U matrix



SVD(A) = **U**SV^T columns of U = **OUTPUT** basis vectors

SVD finds: …'Input' Vectors V, and…

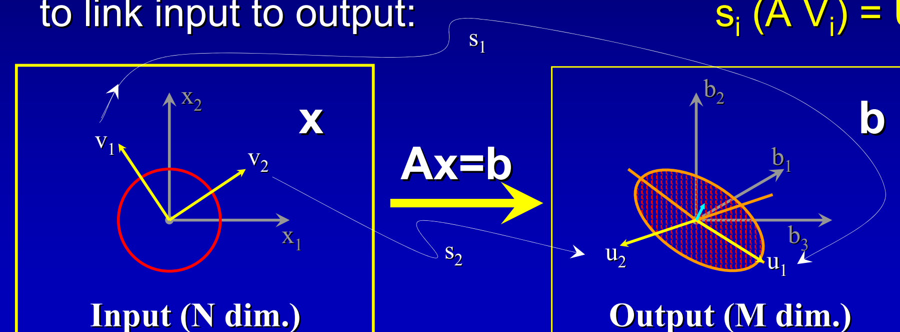
- For each $\mathbf{U_i}$, make a matching $\mathbf{V_i}$ that:
  - Transforms to $\mathbf{U_i}$ (with scaling $\mathbf{s_i}$) : $s_i\,(A\,V_i) = U_i$
  - Forms an orthonormal basis of input space



SVD(A) = US$\mathbf{V}^T$ columns of V = **INPUT** basis vectors

SVD finds: …'Scale' factors S.

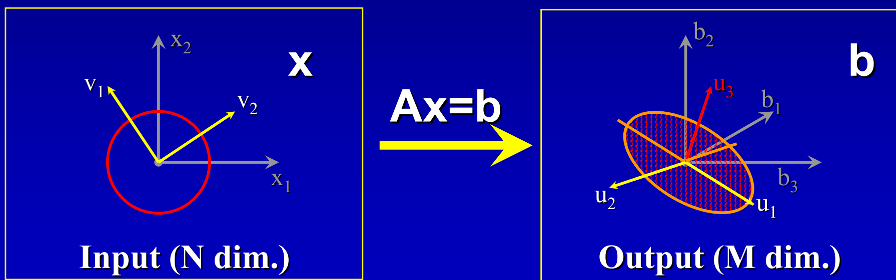
- We have Unit-length Output $\mathbf{U_i}$, vectors,
- Each from a Unit-length Input $\mathbf{V_i}$ vector, so
- So we need 'scale factor' (singular values) $\mathbf{s_i}$ to link input to output: $s_i\,(A\,V_i) = U_i$



SVD(A) = U**S**V^T

SVD Review: What is it?

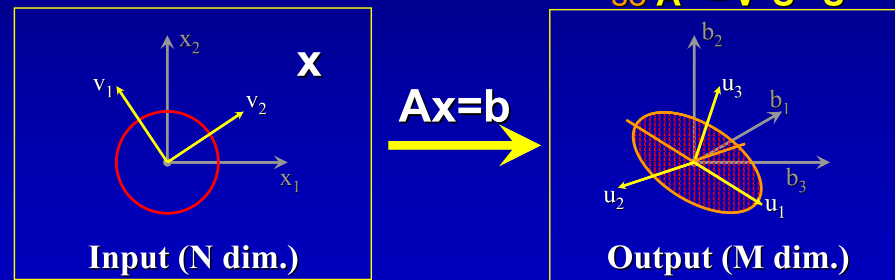
- Finish: SVD(A) = **USV^T**
  - add 'missing' U_i or V_i, define these s_i=0.
  - Singular matrix **S**: diagonals are s_i : 'v-to-u scale factors'
  - Matrix **U**, Matrix **V** have columns U_i and V_i



$A = USV^T$ 'Find Input & Output Axes, Linked by Scale'

'Let SVDs Explain it All for You'

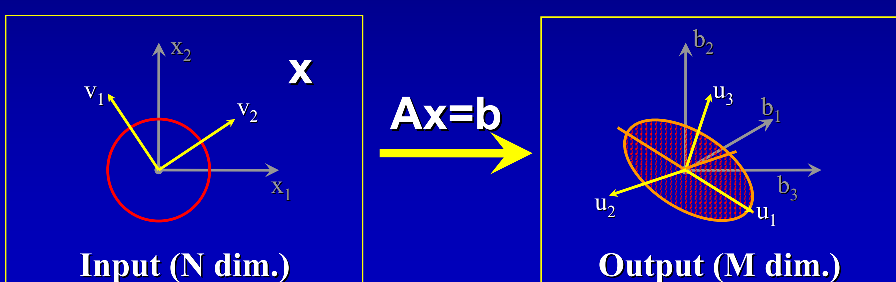
- cool! U and V are Orthonormal! $U^{-1} = U^T$, $V^{-1} = V^T$
- Rank(A)? == # of non-zero singular values s_i
- 'ill conditioned'? Some s_i are nearly zero
- 'Invert' a non-square matrix A ? Amazing!
  pseudo-inverse: $A=USV^T$; $A\,VS^{-1}U^T = I$;
  so $\mathbf{A^{-1} = V\ S^{-1}\ U^T}$



$A = USV^T$ 'Find Input & Output Axes, Linked by Scale'

'Solve for the Null Space' Ax=0

- Easy! x is the V_i axis that doesn't affect the output (s_i = 0)
- are *all* s_i nonzero? then the only answer is x=0.
- Null 'space' because V_i is a DIMENSION--- x = Vi * a
- More than 1 zero-valued s_i? null space >1 dimension
  ... $x = aV_{i1} + bV_{i2} + \ldots$



$A = USV^T$ 'Find Input & Output Axes, Linked by Scale'

SVD: properties

$\mathbf{A} = \mathbf{UDV}^T$

- 1. Singularity and Condition Number
  - nxn A is nonsingular IFF all singular values are nonzero
  - Condition number : degree of singularity of A $C = \sigma_1 / \sigma_n$
    - A is ill-conditioned if 1/C is comparable to the arithmetic precision of your machine; almost singular
- 2. Rank of a square matrix A
  - Rank (A) = number of nonzero singular values
- 3. Inverse of a square Matrix
  - If A is nonsingular $\mathbf{A}^{-1} = \mathbf{VD}^{-1}\mathbf{U}^T$
  - In general, the pseudo-inverse of A $\mathbf{A}^{+} = \mathbf{VD}_0^{-1}\mathbf{U}^T$
- 4. Eigenvalues and Eigenvectors
  - Eigenvalues of both A^TA and AA^T are σ_i^2 ($\sigma_i > 0$)
  - The columns of U are the eigenvectors of AA^T (mxm) $\mathbf{AA}^T\mathbf{u}_i = \sigma_i^2\mathbf{u}_i$
  - The columns of V are the eigenvectors of A^TA (nxn) $\mathbf{A}^T\mathbf{A}\mathbf{v}_i = \sigma_i^2\mathbf{v}_i$

SVD: Application 1

- Least Square $\mathbf{Ax} = \mathbf{b}$
  - Solve a system of m equations for n unknowns **x**(m >= n)
  - A is a mxn matrix of the coefficients
  - b (≠0) is the m-D vector of the data
  - Solution:

$$\underbrace{\mathbf{A}^T\mathbf{A}}_{\text{nxn matrix}}\mathbf{x} = \mathbf{A}^T\mathbf{b} \Rightarrow \mathbf{x} = \underbrace{(\mathbf{A}^T\mathbf{A})^+}_{\text{Pseudo-inverse}}\mathbf{A}^T\mathbf{b}$$

  - How to solve: compute the pseudo-inverse of A^TA by SVD
    - $(A^TA)^+$ is more likely to coincide with $(A^TA)^{-1}$ given m > n
    - Always a good idea to look at the condition number of A^TA

SVD: Application 2

- Homogeneous System $\mathbf{Ax} = \mathbf{0}$
  - m equations for n unknowns **x**(m >= n-1)
  - Rank (A) = n-1 (by looking at the SVD of A)
  - A non-trivial solution (up to a arbitrary scale) by SVD:
  - Simply proportional to the eigenvector corresponding to the only zero eigenvalue of A^TA (nxn matrix)
- Note:
  - All the other eigenvalues are positive because Rank (A)=n-1
  - In practice, the eigenvector (i.e. v_n) corresponding to the minimum eigenvalue of A^TA, i.e. σ_n^2

SVD: Application 3

- Problem Statements
  - Numerical estimate of a matrix A whose entries are not independent
  - Errors introduced by noise alter the estimate to Â
- Enforcing Constraints by SVD
  - Take orthogonal matrix A as an example
  - Find the closest matrix to Â, which satisfies the constraints exactly
    - SVD of Â $\quad \hat{\mathbf{A}} = \mathbf{UDV}^T$
    - Observation: D = I (all the singular values are 1) if A is orthogonal
    - Solution: changing the singular values to those expected $\quad \mathbf{A} = \mathbf{UIV}^T$

Homework

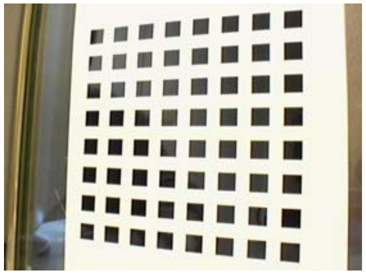


Image taken from http://research.microsoft.com/~zhang/calib/Calibration/CalibIm3.gif

- Detect edges on the calibrating target image (use Canny edge detector from XITE)
- Fit straight lines to the edge points
  - Use your prior knowledge about the calibration pattern