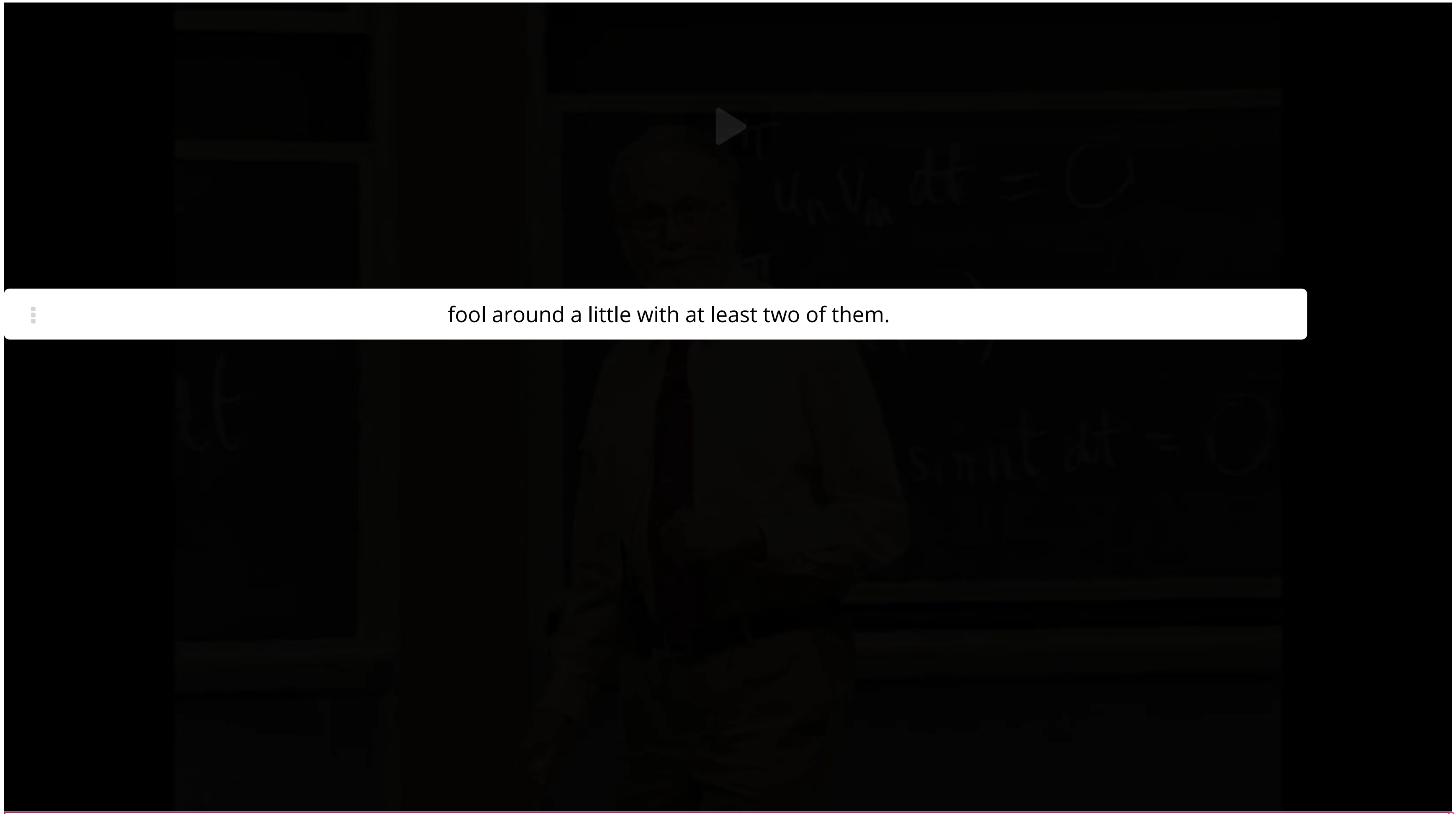
7. (Optional) proof of Orthogonality

Proof of orthogonality



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Proof of orthogonality

Here we prove that if $m \neq n$, then $\sin(nt)$ and $\cos(nt)$ are orthogonal to $\sin(mt)$ and $\cos(mt)$. To do this we use that fact that both $\sin(nt)$ and $\cos(nt)$ solve the differential equation $\ddot{x} + n^2 x = 0$, and both $\sin(mt)$ and $\cos(mt)$ solve the differential equation $\ddot{x} + m^2 x = 0$.

Supose that $u(t)$ solves the differential equation $\ddot{x} + n^2 x = 0$. And suppose $v(t)$ solves the differential equation $\ddot{x} + m^2 x = 0$ and $n \neq m$.

Then in particular

$$\int_{-\pi}^{\pi} \ddot{u}(t) v(t)\ dt \quad = \quad \dot{u}(t) v(t)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \dot{u}(t) \dot{v}(t)\ dt.$$

Note that $\dot{u}(t) v(t)|_{-\pi}^{\pi} = 0$. Both $u(t)$ and $v(t)$ are both periodic on $(-\pi, \pi]$, therefore so is $\dot{u}(t) v(t)$. Therefore the difference $\dot{u}(\pi) v(\pi) - \dot{u}(-\pi) v(-\pi) = 0$.

Thus this reduces to the expression

$$\int_{-\pi}^{\pi} \ddot{u}(t) v(t)\ dt = -\int_{-\pi}^{\pi} \dot{u}(t) \dot{v}(t)\ dt.$$

A similar argument shows that

$$\int_{-\pi}^{\pi} u(t) \ddot{v}(t)\ dt = -\int_{-\pi}^{\pi} \dot{u}(t) \dot{v}(t)\ dt.$$

However, by using the substitution from the differential equation, we have that $\ddot{u}(t) = -n^2 u(t)$ and $\ddot{v}(t) = -m^2 v(t)$. Therefore it follows that

$$n^2 \int_{-\pi}^{\pi} u(t)\, v(t)\ dt = m^2 \int_{-\pi}^{\pi} u(t)\, v(t)\ dt.$$

Note that since we have assumed that $m \neq n$, the only way for this to hold is for the integral $\int_{-\pi}^{\pi} u(t)\, v(t)\ dt = 0$.

Remark 7.1 In disguise, this is the same as showing that two eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

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- Generalization? 6
  Can this proof be generalized with another set of basis functions that satisfy a different ODE? In other words if {u,v} satisfy some linear ODE and form a basis set for some vec...
- Do we have resources for the other two ways to prove it? 3
  Can you please give insights or provide external resources on the proof of orthogonality using trig identities and complex exponential? I think it could be interesting to compa...
- eigenvectors corresponding to distinct eigenvalues of a symmetric matrix 2
- What's the intuition behind considering the given integral in the 'Using ODE' method? 6
  In the video, Prof. Mattuck says that this is the general method for discovering orthogonal bases, but he doesn't explain the reason for selecting the integral as the starting po...