# 10. Companion Systems

Recall from the course *Differential equations: 2 by 2 systems* that any second order linear ODE can be converted to a first order linear system called its **companion system**.

**Problem 10.1** Convert $\ddot{x}+5\dot{x}+6x=0$ to a first-order system of ODEs.

**Solution**

Define $y := \dot{x}$. Then

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= \ddot{x} = -5\dot{x}-6x = -6x-5y.\end{aligned}$$

In matrix form, this is $\dot{\mathbf{x}} = A\mathbf{x}$ with $A=\begin{pmatrix}0 & 1\\ -6 & -5\end{pmatrix}$. (The matrix $\begin{pmatrix}0 & 1\\ -6 & -5\end{pmatrix}$ arising this way is called the **companion matrix** of the polynomial $x^2+5x+6$.)

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Conversely, given a first-order system, one can eliminate function variables to find a higher-order ODE satisfied by one of the functions. But usually we just leave it as a system.

**Problem 10.2** Given that

$$\begin{aligned}\dot{x} &= 2x-y\\ \dot{y} &= 5x+7y,\end{aligned}$$

eliminate $y$ to find a 2nd order ODE involving only $x$.

**Solution**

Solve for $y$ in the first equation ($y = 2x - \dot{x}$) and substitute into the second:

$$2\dot{x} - \ddot{x} = 5x + 7(2x - \dot{x}).$$

This simplifies to

$$\ddot{x} - 9\dot{x} + 19x = 0.$$

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Similarly, we can make companion systems for higher order linear differential equations by introducing new function variables for the derivatives.

Given an $n^{\text{th}}$ order homogeneous linear ODE

$$D^n x + a_{n-1}\left(D^{n-1}x\right) + a_{n-2}\left(D^{n-2}x\right) + \cdots + a_1 (Dx) + a_0 x \quad = \quad 0, \qquad \text{where } D = \frac{d}{dt},$$

define

$$\begin{aligned} x_0 &= x \\ x_1 &= Dx \\ x_2 &= Dx_1 = D^2 x \\ &\vdots \\ x_{n-1} &= Dx_{n-2} = D^{n-1}x. \end{aligned}$$

Then the original order $n$ DE can be rewritten as

$$Dx_{n-1} = -a_{n-1}x_{n-1} - a_{n-2}x_{n-2} - \cdots - a_1 x_1 - a_0 x_0.$$

Therefore, we have a system of $n$ first order equations: the definitions of $x_1, \cdots, x_{n-1}$, and the original DE rewritten as above. In matrix form, the first order $n \times n$ system is

$$\frac{d}{dt}\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix}.$$

The $n \times n$ matrix on the right hand side is called the **companion matrix,** and this $n \times n$ system is the **companion system** of the degree $n$ ODE.

Note that the dimension of the companion system corresponds to the order of the single ODE we started with.

**Remark 10.3** The same procedure of converting to a first order system works even if the starting ODE is inhomogeneous:

$$D^n x + a_{n-1}\left(D^{n-1}x\right) + a_{n-2}\left(D^{n-2}x\right) + \cdots + a_1\left(Dx\right) + a_0 x \quad = \quad b.$$

The resulting companion system will have an extra vector on the right hand side:

$$\frac{d}{dt}\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{pmatrix}.$$

**Remark 10.4** For constant coefficient ODEs, the characteristic polynomial of the higher order ODE (scaled, if necessary, to have leading coefficient $1$) equals the characteristic polynomial of the matrix of the first-order system.

More generally, we can also convert higher-order **systems** of ODEs to first order systems. For example, a system of $4$ fifth order ODEs can be converted to a $20 \times 20$ first order system ODEs. This is why it is enough to study first order systems.

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## Companion system of third order ODE

1/1 point (graded)

Find the companion matrix $\mathbf{A}$ of

$$D^3 x + 4D^2 x + Dx - x = 0, \qquad (D = \frac{d}{dt})$$

(Enter **[a,b;c,d]** for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. That is, use commas to separate entries within the same row and **semicolon** to separate rows, and **square bracket** the entire matrix. Similarly for larger matrices.)

$\mathbf{A} =$ [0,1,0;0,0,1;1,-1,-4]

**Answer:** [0,1,0;0,0,1;1,-1,-4]

**Solution:**

Given a third order linear ODE

$$D^3 x + 4D^2 x + Dx - x = 0, \qquad (D = \frac{d}{dt})$$

Define

$$\begin{aligned} y &= Dx, \\ z &= Dy = D^2 x. \end{aligned}$$

Then the DE can be written as

$$Dz = -4z - 1y + x.$$

Therefore, we have a system of three equations, which in vector notation is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The $3 \times 3$ matrix on the right hand side is the companion matrix.

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Answers are displayed within the problem

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