4. Maximum Likelihood Estimation, Tests, and Confidence Intervals

Setup :

Let $X_1, \ldots, X_n \overset{iid}{\sim} X$ be distributed i.i.d. with probability density function

$$f_\theta(x) = (x/\theta^2)\exp(-x^2/2\theta^2)\,\mathbf{1}(x \geq 0), \theta > 0.$$

(a)

3/3 points (graded)
Let $l(\theta) = \ln L(X_1, \ldots, X_n, \theta)$ denote the log likelihood. Find the critical point of $l(\theta)$. (The critical point is unique because KL divergence is definite.)

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(If applicable, enter **barX_n** for $\overline{X}_n$ and **bar(X_n^2)** for $\overline{X_n^2}$.)

Critical point of $l(\theta)$ is at $\theta =$ sqrt(bar(X_n^2) /2) ✔

Find the second derivative $l'' = \frac{d^2 l}{d\theta^2}$ of $l(\theta)$. Your answer should be a function of θ and the data $X_1, \ldots, X_n$.

(Do **not** evaluate l'' at the critical point at this stage.)

(If applicable, enter **Sigma_i(X_i)** for $\sum_{i=1}^{n} X_i$ and and **Sigma_i(X_i^2)** for $\sum_{i=1}^{n} X_i^2$.)

$l'' = \frac{d^2 l}{d\theta^2} =$ 2*n/theta^2-3* Sigma_i(X_i^2)/theta^4 ✔

Using the second derivative test, is the critcal point you obtain above a global maximum, a global minimum, or neither of $l(\theta)$ in the domain $\theta > 0$?

- (●) global maximum
- ( ) global minimum
- ( ) neither

✔

What can you conclude about the maximum likelihood estimator $\hat{\theta}$ for θ?
(There is no answer box for this question.)

STANDARD NOTATION

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✔ Correct (3/3 points)

(b)

1/1 point (graded)
What is the Fisher information $I(\theta)$ of the random variables X_i?

$I(\theta) =$ 4/theta^2 ✔

$\frac{4}{\theta^2}$

STANDARD NOTATION

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✔ Correct (1/1 point)

(c)

2/2 points (graded)
Use the theorem for the MLE to write down the asymptotic distribution of the MLE $\hat{\theta}$.

Give an asymptotic 95% confidence interval $\mathcal{I}_{\text{plug-in}}$ for θ using the plug-in method. (You may use I in the answer box below to denote $I(\hat{\theta})$, the Fisher Information, which you found in the previous part, evaluated at $\hat{\theta}$.)

(If applicable, enter **I** for $I(\hat{\theta})$, **hattheta** for $\hat{\theta}$, and **q(alpha)** for q_α for any numerical value α. Recall q_α denotes the value such that $\mathbf{P}(Z \geq q_\alpha) = \alpha$ for $Z \sim \mathcal{N}(0,1)$.)

(Do not worry if the parser does not render properly; the graders will work independently. To render properly, add parentheses around **q(alpha)**, i.e. enter **(q(alpha))**.)

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$\mathcal{I}_{\text{plug-in}} = [A, B]$ where

$A =$ hattheta-(q(0.025))/sqrt(n*I)

$B =$ hattheta+(q(0.025))/sqrt(n*I)

STANDARD NOTATION

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Correct (2/2 points)

(d)

1/1 point (graded)

Use the results from the previous parts to give a test with asymptotic level α for testing

$$H_0 : \theta = 1 \quad \text{v.s.} \quad H_1 : \theta \neq 1.$$

Suppose $n = 100$ and the data gives $\overline{X}_n = 1.5$ and $\overline{X_n^2} = 4.0$. Find the p-value associated to this data for this hypothesis test.

(If applicable, enter **Phi(z)** for the cdf $\Phi(z)$ of a normal variable Z, **q(alpha)** for q_α for any numerical value α.)

p-value: 2-2*Phi(8.28)

Correction Note: An earlier version gave the data $\bar{X}_n = 2.5$ instead, which led to the variance being negative, i.e. impossible data! The grader has no issue.

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STANDARD NOTATION

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✔ Correct (1/1 point)

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