3. Gaussian random variables

Moments of Gaussian random variables

5/5 points (graded)

Let X be a Gaussian random variable with mean μ and variance σ^2 . Compute the following moments:

Remember that we use the terms **Gaussian random variable** and **normal random variable** interchangeably.

(Enter your answers in terms of μ and σ.)

$\mathbb{E}\left[X^2\right] =$ mu^2+sigma^2 ✔ **Answer:** sigma^2 + mu^2

$\mu^2 + \sigma^2$

$\mathbb{E}\left[X^3\right] =$ mu^3+3*mu*sigma^2 ✔ **Answer:** 3 * sigma^2 * mu + mu^3

$\mu^3 + 3 \cdot \mu \cdot \sigma^2$

$\mathbb{E}\left[X^4\right] =$ mu^4+6*mu^2*sigma^2 ✔ **Answer:** 3 * sigma^4 + 6 * sigma^2 * mu^2 + mu^4

$\mu^4 + 6 \cdot \mu^2 \cdot \sigma^2 + 3 \cdot \sigma^4$

Var$\left(X^2\right) =$ 4*mu^2*sigma^2+2*sig ✔ **Answer:** 2 * sigma^4 + 4 * sigma^2 * mu^2

$4 \cdot \mu^2 \cdot \sigma^2 + 2 \cdot \sigma^4$

Write $\mathbf{P}\left(X > 0\right)$ in terms of the **cumulative distribution function (cdf)** Φ of the standard Gaussian distribution, that is,

$$\Phi\left(x\right) = \mathbf{P}\left(Z \leq x\right), \quad x \in \mathbb{R},$$

where $Z \sim \mathcal{N}\left(0,1\right)$ is a standard normal variable. (Enter `Phi` for Φ.)

$\mathbf{P}\left(X > 0\right) =$ 1-Phi(-mu/sigma) ✔ **Answer:** 1 - Phi(-mu/sigma)

STANDARD NOTATION

Solution:

We can write a general Gaussian variable $X \sim \mathcal{N}(\mu, \sigma^2)$ as $X = \sigma Z + \mu$, where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. Hence, the calculation can be made by factoring out the corresponding polynomials and calculating (or looking up) the moments of Z:

$$\begin{aligned} \mathbb{E}[Z] &= 0 \\ \mathbb{E}[Z^2] &= 1 \\ \mathbb{E}[Z^3] &= 0 \\ \mathbb{E}[Z^4] &= 3. \end{aligned}$$

As an example, let us compute $\mathbb{E}[X^3]$. Denote the density of a standard normal distribution by $\varphi(z)$, i.e.,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

With this, we calculate

$$\begin{aligned} \mathbb{E}[X^3] &= \int_{-\infty}^{\infty} (\sigma z + \mu)^3 \phi(z)\, dz \\ &= \sigma^3 \mathbb{E}[Z^3] + 3\sigma^2 \mu \mathbb{E}[Z^2] + 3\sigma\mu^2 \mathbb{E}[Z] + \mu^3 \\ &= 3\sigma^2 \mu + \mu^3. \end{aligned}$$

For $\mathsf{Var}\left(X^2\right)$, we can use the formula $\mathsf{Var}\left(X^2\right) = \mathbb{E}\left[X^4\right] - \left(\mathbb{E}\left[X^2\right]\right)^2$.

Similarly, we can express the probability $\mathbf{P}\left(X > 0\right)$ as

$$\begin{aligned}\mathbf{P}\left(X > 0\right) = & \quad \mathbf{P}\left(\sigma Z + \mu > 0\right) = \mathbf{P}\left(\sigma Z > -\mu\right) \\ = & \quad \mathbf{P}\left(Z > -\frac{\mu}{\sigma}\right) = 1 - \Phi\left(-\frac{\mu}{\sigma}\right).\end{aligned}$$

Submit You have used 1 of 4 attempts

Answers are displayed within the problem

Covariance of Gaussians

4/4 points (graded)

Recall that **i.i.d.** stands for **independent and identically distributed** . A collection of random variables $X_1, \ldots, X_n$ are **i.i.d.** if all of them follow the same distribution, and each X_i does not contain information about the other realizations.

Let X, Y be i.i.d. **standard** normal random variables, that is, $X, Y \sim \mathcal{N}\left(0, 1\right)$.

Recall that the **covariance** of two random variables X and Y, denoted by $\mathsf{Cov}\left(X, Y\right)$, is defined as

$$\mathsf{Cov}\left(X, Y\right) \;=\; \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)\left(Y - \mathbb{E}\left[Y\right]\right)\right]. \qquad (1.3)$$

Compute the following variances and covariances.

$\text{Var}(X+Y) =$ 2 ✔ **Answer:** 2

$\text{Var}(XY) =$ 1 ✔ **Answer:** 1

$\text{Cov}(X, X+Y) =$ 1 ✔ **Answer:** 1

$\text{Cov}(X, XY) =$ 0 ✔ **Answer:** 0

STANDARD NOTATION

Solution:

Note that by the definition of a standard Gaussian random variable,

$$\mathbb{E}[X] = \mathbb{E}[Y] = 0 \quad \mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1.$$

With this, compute

$$\begin{aligned} \text{Var}(X+Y) = &\quad \text{Var}(X) + \text{Var}(Y) && \text{(independence)} \\ = &\quad 1 + 1 = 2, \end{aligned}$$

$$\begin{aligned} (XY) = &\quad \mathbb{E}[(XY)^2] - (\mathbb{E}[XY])^2 \\ = &\quad \mathbb{E}[X^2]\,\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y]^2 && \text{(independence)} \end{aligned}$$

$$= \quad 1 \times 1 - 0 = 1,$$

$$\begin{aligned}(X, X+Y) = \quad & \mathbb{E}[X(X+Y)] - \mathbb{E}[X]\,\mathbb{E}[X+Y] \\ = \quad & \mathbb{E}[X^2] + \mathbb{E}[XY] - \mathbb{E}[X](\mathbb{E}[X] + \mathbb{E}[Y]) && \text{(linearity of expectation)} \\ = \quad & \mathbb{E}[X^2] + \mathbb{E}[X]\,\mathbb{E}[Y] - \mathbb{E}[X]^2 - \mathbb{E}[X]\,\mathbb{E}[Y] && \text{(independence)} \\ = \quad & 1,\end{aligned}$$

$$\begin{aligned}(X, XY) = \quad & \mathbb{E}[X(XY)] - \mathbb{E}[X]\,\mathbb{E}[XY] \\ = \quad & \mathbb{E}[X^2]\,\mathbb{E}[Y] - \mathbb{E}[X]^2\mathbb{E}[Y] && \text{(independence)} \\ = \quad & 1 \cdot 0 - 0 \cdot 0 = 0.\end{aligned}$$

: 8. Covariance, 9. Covariance properties, and 10. the variance of a sum in Lecture 12, *Sums of independent random variables; covariance, and correlation*.

Submit

You have used 1 of 3 attempts

Answers are displayed within the problem

True or False: Variance, covariance and independence

2/2 points (graded)

For each of the statements below, determine whether it is true (meaning, always true) or false (meaning, not always true).

- For any two random variables, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.

- True
- False ✔

- If the covariance, $\mathrm{Cov}\,(X,Y)$ between two random variables X,Y is 0, then X and Y are independent.

- True
- False ✔

STANDARD NOTATION

Solution:

- The first item is False. For any two random variables, it is known that,

$$\mathsf{Var}\,(X+Y) = \mathsf{Var}\,(X) + \mathsf{Var}\,(Y) + 2\mathrm{Cov}\,(X,Y)\,.$$

  In particular, if $\mathrm{Cov}\,(X,Y) \neq 0$, this does not hold.

- The second item is also false. As a simple example, let $X \sim \mathrm{Unif}\,[-1,1]$ and let $Y = X^2$. Then,

$$\mathrm{Cov}\,(X,Y) = \mathbb{E}\,[XY] - \mathbb{E}\,[X]\,\mathbb{E}\,[Y] = \mathbb{E}\,[X^3] - \mathbb{E}\,[X]\,\mathbb{E}\,[X^2] = 0,$$

using the fact that X is centered and symmetric around 0, and its odd moments vanish. Even though they are uncorrelated, they are (highly) dependent, Y is obtained from X, intuitively!

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Discussion

Hide Discussion

Topic: Unit 0. Course Overview, Syllabus, Guidelines, and Homework on Prerequisites:Homework 0: Probability and Linear algebra Review / 3. Gaussian random variables

Add a Post

Show all posts ▼ by recent activity ▼

Book on statistics
I have not taken the recommended earlier course mentioned as prerequisite for this course. Can someone recommend any book followed in that ea... 7
Pinned

Covariance of Gaussians 9

For those who are overwhelmed with the first problem (like myself) 30

? Struggling with this section questions
Covariance of Gaussians, a hint will be really helpful
2

? Two dependent random variables covariance=0 example
Are there any examples when Cov(X,Y) between two random variables X,Y is 0 and they are dependent? I can't come up with the one. Any ideas?
4

Last part of first question
I am stuck on the last part of the first question: "Write $\mathbf{P}(X>0)$ in terms of the cumulative distribution function (cdf) Φ of the standard Gaussian distrib...
4

Notation Bug? on last part of Moments of Gaussian random variables question
Is there something wrong with the standard notation Phi(0) to denote the normal cumulative distribution function at zero?
6

Where did you learn this "moments" thing?
I managed to get the correct answers after some googling and tedious calculations but I have never heard of the "moments" until now. It wasn't taug...
14

Exercise 1
Hi guys I am unsure as to why on the last part of the question when it asks you to write P(x>0) as a Cumulative distribution function the answer is no...
3

? [Staff] First part N/A in the mobile app
I almost missed the 5 point worth first part of this question. It doesn't appear in the mobile app. Could you check please? Thank you in advance
1

[Staff] [Typo]
3

Format for question 4
I get: Invalid Input: X, Y not permitted in answer as a variable. Without revealing answers, is there an issue with putting answers as their function for...
2

Learn About Verified Certificates