Homework Problem: Alice Hunts Dragons

(10 points possible)

When she is not calculating marginal distributions, Alice spends her time hunting dragons. For every dragon she encounters, Alice measures its fire power X (measured on a scale from 1 to 4) and its roar volume Y (measured on a scale from 1 to 3). She notices that the proportion of dragons with certain fire power and roar volume in the population behaves as the following function:

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the joint probability table $p_{X,Y}$ is of the form

$$p_{X,Y}(x,y) = c f(x,y) \qquad \text{for } x \in \{1,2,3,4\}, y \in \{1,2,3\},$$

for some constant $c > 0$ that you will determine.

- **(a)** Determine the constant c, which ensures that $p_{X,Y}$ is a valid probability distribution. (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

1/72

? **Answer:** 1/72

- **(b)** Determine $\mathbb{P}(Y < X)$. (Note that $\{Y < X\}$ is an event. Think about what outcomes are in it.)

  (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

47/72

? **Answer:** 47/72

- **(c)** Determine $\mathbb{P}(X < Y)$. (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

23/72

? **Answer:** 23/72

- **(d)** Determine $\mathbb{P}(Y = X)$. (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

1/36

? **Answer:** 2/72

- **(e)** Determine $\mathbb{P}(Y = 3)$. (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

2/3

? **Answer:** 48/72

Towards Infinity in Modeling Uncertainty (Week 4)
Exercises due Oct 13, 2016 at 02:30 IST

Homework 3 (Week 4)
Homework due Oct 13, 2016 at 02:30 IST

- **(f)** Find the probability tables for p_X and p_Y. Express your answers as Python dictionaries. (Your answer should be the Python dictionary itself, and *not* the dictionary assigned to a variable, so please do not include, for instance, "`prob_table =`" before specifying your answer. You can use fractions. If you use decimals instead, please be accurate and use at least 5 decimal places.)

  p_X probability table (the dictionary keys should be the Python integers 1, 2, 3, 4):

  {1:1/6,2:1/4,3:0,4:7/12} ? **Answer:** {1: 12/72, 2: 18/72, 3: 0, 4: 42/72}

  p_Y probability table (the dictionary keys should be the Python integers 1, 2, 3):

  {1:1/3,2:0,3:2/3} ? **Answer:** {1: 24/72, 2: 0, 3: 48/72}

Solution:

(a) Determine the constant c, which ensures that $p_{X,Y}$ is a valid probability distribution.

Solution: From the definition of f it follows that there are six coordinate pairs (x, y) with nonzero probabilities of occurring. These are $(1, 1), (1, 3), (2, 1), (2, 3), (4, 1)$ and $(4, 3)$. The probability of a pair is proportional to the sum of the squares of the coordinates of the pair, $x^2 + y^2$. Therefore there is a constant c such that the PMF $p_{X,Y}(x, y)$:

$$p_{X,Y}(x,y) = \begin{cases} c(x^2+y^2) & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\} \\ 0 & \text{otherwise.} \end{cases}$$

Because the probability of the entire sample space must equal 1, we have $(1+1)c + (1+9)c + (4+1)c + (4+9)c + (16+1)c + (16+9)c = 1$ which implies that $c = \frac{1}{72}$ and therefore:

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{72}(x^2+y^2) & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Determine $\mathbb{P}(Y < X)$.

Solution: There are three sample points for which $y < x$:
$P(Y < X) = P((2,1)) + P((4,1)) + P((4,3)) = \frac{5}{72} + \frac{17}{72} + \frac{25}{72} = \frac{47}{72}$

(c) Determine $\mathbb{P}(X < Y)$.

Solution: There are two sample points for which $y > x$:
$P(X > Y) = P((1,3)) + P((2,3)) = \frac{10}{72} + \frac{13}{72} = \frac{23}{72}$

(d) Determine $\mathbb{P}(Y = X)$.

Solution: There is only one sample point for which $y = x$: $P(Y = X) = P((1,1)) = \frac{2}{72}$

(e) Determine $\mathbb{P}(Y = 3)$.

Solution: There are three sample points for which $y = 3$.
$P(Y = 3) = P((1,3)) + P((2,3)) + P((4,3)) = \frac{10}{72} + \frac{13}{72} + \frac{25}{72} = \frac{48}{72}$

(f) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.

Solution: In general for two discrete random variables X and Y for which a joint PMF is defined, we have:

$$p_X(x) = \sum_y p_{X,Y}(x,y) \quad p_Y(y) = \sum_x p_{X,Y}(x,y)$$

In this problem, the ranges of X and Y are quite restricted so we can determine the marginal PMF by enumeration: $p_X(2) = P((2,1)) + P((2,3)) = \frac{18}{72}$

Performing the required computations:

$$p_X(x) = \begin{cases} 12/72 & \text{if } x = 1 \\ 18/72 & \text{if } x = 2 \\ 42/72 & \text{if } x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

And

$$p_Y(y) = \begin{cases} 24/72 & \text{if } y = 1 \\ 48/72 & \text{if } y = 3 \\ 0 & \text{otherwise.} \end{cases}$$

You have used 0 of 5 submissions