Unit 12: Quiz

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The problems in the quiz will be automatically graded.

You are given only **ONE** attempt for each problem. You will be able to see the solution after you submit your answers.

If the answer is numeric and has more than 4 decimal places, just give your answers to **4 decimal places (0.0001 accuracy)** unless otherwise stated.

e.g. for 0.123456... you just need to round it up to 0.1235

Problem 1

3/3 points (graded)

1. Consider three independent continuous Uniform random variables, each of which has a constant density on $[0, 10]$.

a. Find the density $f_{X_{(1)}}(x_1)$ of the 1st order statistic (i.e., find the density of the min). Then compute $f_{X_{(1)}}(5)$.

$f_{X_{(1)}}(5) =$ 3/40 ✔ **Answer:** 0.075

b. Find the density $f_{X_{(2)}}(x_2)$ of the 2nd order statistic. Then compute $f_{X_{(2)}}(5)$.

$f_{X_{(2)}}(5) =$ 3/20 ✔ **Answer:** 0.15

c. Find the density $f_{X_{(3)}}(x_3)$ of the 3rd order statistic (i.e., find the density of the max). Then compute $f_{X_{(3)}}(5)$.

$f_{X_{(3)}}(5) =$ 3/40 ✔ **Answer:** 0.075

Explanation

1a. We have $f_{X_{(1)}}(x_1) = \left(\frac{3}{0!1!2!}\right)\left(\frac{1}{10}\right)\left(\frac{x_1}{10}\right)^0\left(1-\frac{x_1}{10}\right)^2 = \left(\frac{3}{10}\right)\left(1-\frac{x_1}{10}\right)^2$ for $0 < x_1 < 10$, and $f_{X_{(1)}}(x_1) = 0$ otherwise.

1b. We have $f_{X_{(2)}}(x_2) = \left(\frac{3}{1!1!1!}\right)\left(\frac{1}{10}\right)\left(\frac{x_2}{10}\right)^1\left(1-\frac{x_2}{10}\right)^1 = \left(\frac{3}{50}\right)(x_2)\left(1-\frac{x_2}{10}\right)$ for $0 < x_2 < 10$, and $f_{X_{(2)}}(x_2) = 0$ otherwise.

1c. We have $f_{X_{(3)}}(x_3) = \left(\frac{3}{2!1!0!}\right)\left(\frac{1}{10}\right)\left(\frac{x_3}{10}\right)^2\left(1-\frac{x_3}{10}\right)^0 = \frac{3x_3^2}{1000}$ for $0 < x_3 < 10$, and $f_{X_{(3)}}(x_3) = 0$ otherwise.

Submit You have used 1 of 1 attempt

Problem 2

3/3 points (graded)

2. Same setup as question #1.

a. Find $\mathbb{E}(X_{(1)})$.

2.5

✔ **Answer:** 2.5

b. Find $\mathbb{E}(X_{(2)})$.

5

✔ **Answer:** 5

c. Find $\mathbb{E}(X_{(3)})$.

7.5

✔ **Answer:** 7.5

d. Since the sum of the three random variables and the sum of the three order statistics must be the same (always), then their expected values are the same, i.e.,
$X_1 + X_2 + X_3 = X_{(1)} + X_{(2)} + X_{(3)}$.
So $\mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_{(1)} + X_{(2)} + X_{(3)})$.
We also know that $\mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 5 + 5 + 5 = 15$. Use this to double check your answers to parts a, b, c. Do the answers sum up to 15?

Explanation

2a. We have $\mathbb{E}(X_{(1)}) = \int_0^{10} (x_1)(\frac{3}{10})(1 - \frac{x_1}{10})^2\,dx_1 = 5/2$.
2b. We have $\mathbb{E}(X_{(2)}) = \int_0^{10} (x_2)(\frac{3}{50})(x_2)(1 - \frac{x_2}{10})\,dx_2 = 5$.

2c. We have $\mathbb{E}(X_{(3)}) = \int_0^{10} (x_3)(\frac{3x_3^2}{1000})\,dx_3 = 15/2$.
2d. Indeed, we get $5/2 + 5 + 15/2 = 15$, as we knew we must.

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Problem 3

2/2 points (graded)

3. Suppose X_1, X_2 are independent continuous random variables with $f_{X_1,X_2}(x_1, x_2) = (1/8)^2(4 - x_1)(4 - x_2)$ on the square $0 < x_1 < 4$ and $0 < x_2 < 4$, and $f_{X_1,X_2}(x_1, x_2) = 0$ otherwise.

a. Find the density $f_{X_{(1)}}(x_1)$ of the 1st order statistic (i.e., find the density of the min). Then compute $f_{X_{(1)}}(2)$.

$f_{X_{(1)}}(2) =$ 0.125 **Answer:** 0.125

b. Find the density $f_{X_{(2)}}(x_2)$ of the 2nd order statistic (i.e., find the density of the max). Then compute $f_{X_{(2)}}(2)$.

$f_{X_{(2)}}(2) =$ 0.375 **Answer:** 0.375

Explanation

3a. We see that X_1 and X_2 each have density $(1/8)(4-x)$ for $0 < x < 4$, and therefore each have CDF $\int_0^a (1/8)(4-x)\,dx = (a/16)(8-a)$ for $0 < a < 4$.

Therefore, we have

$$
\begin{aligned}
f_{X_1}(x_1) \\
&= \binom{2}{0,1,1}(1/8)(4-x_1)((x_1/16)(8-x_1))^0(1-(x_1/16)(8-x_1))^1 \\
&= \left(\frac{1}{64}\right)(4-x_1)^3 = 1-(3/4)x_1+(3/16)x_1^2-(1/64)x_1^3
\end{aligned}
$$

for $0 < x_1 < 4$.

3b. We have

$$
\begin{aligned}
f_{X_2}(x_2) \\
&= \binom{2}{1,1,0}(1/8)(4-x_2)((x_2/16)(8-x_2))^1(1-(x_2/16)(8-x_2))^0 \\
&= \left(\frac{x_2}{64}\right)(4-x_2)(8-x_2) = (1/64)x_2^3-(3/16)x_2^2+(1/2)x_2
\end{aligned}
$$

for $0 < x_2 < 4$.

Submit You have used 1 of 1 attempt

Problem 4

2/2 points (graded)

4. Same setup as question #3.

a. Find $\mathbb{E}(X_{(1)})$.

0.8 ✔ **Answer:** 0.8

b. Find $\mathbb{E}(X_{(2)})$.

1.866667

✔ **Answer:** 1.8667

c. Since the sum of the two random variables and the sum of the two order statistics must be the same (always), then their expected values are the same, i.e., $X_1 + X_2 = X_{(1)} + X_{(2)}$. So $\mathbb{E}(X_1 + X_2) = \mathbb{E}(X_{(1)} + X_{(2)})$. We also know that $\mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 4/3 + 4/3 = 8/3$. Use this to double check your answers to parts a, b. Do the answers sum up to $8/3$?

Explanation

4a. We have $\mathbb{E}(X_{(1)}) = \int_0^4 (x_1)(1 - (3/4)x_1 + (3/16)x_1^2 - (1/64)x_1^3)\,dx_1 = 4/5$.

4b. We have $\mathbb{E}(X_{(2)}) = \int_0^4 (x_2)((1/64)x_2^3 - (3/16)x_2^2 + (1/2)x_2)\,dx_2 = 28/15$.

4c. Indeed, we get $4/5 + 28/15 = 8/3$, as we knew we must.

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Problem 5

2/2 points (graded)

5. Suppose that the number of errors a student makes on his exam has a Poisson distribution, with an average of 3. Let X denote the number of errors.

a. Find the moment generating function $M_X(t)$ of X.

- (selected) $e^{3(e^t-1)}$ ✔
- $e^{1-3(e^t)}$
- $e^{3(t-1)}$
- $e^{3(1-t)}$

b. Compute $M'_X(0)$. Hint: You should get 3 for your answer, since $M'_X(0) = \mathbb{E}(X)$.

3 ✔ **Answer:** 3

Explanation

5a. We have

$$M_X(t) = \mathbb{E}(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-3}3^x}{x!}$$
$$= e^{-3} \sum_{x=0}^{\infty} \frac{(e^t 3)^x}{x!} = e^{-3} e^{(3e^t)} = e^{3(e^t-1)}$$

5b. We have $M'_X(t) = \frac{d}{dt} e^{3(e^t-1)} = (e^{3(e^t-1)})(3e^t)$, so $M'_X(0) = (e^{3(e^0-1)})(3e^0) = 3$.

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Problem 6

2/2 points (graded)

6. Use X to denote the time (in seconds) that Mary waits for her next text to arrive. Suppose that X has an Exponential distribution, and $\mathbb{E}(X) = 15$.

a. Find the moment generating function $M_X(t)$ of X.

- ( ) $\frac{1/15}{t-(1/15)}$
- (•) $\frac{1/15}{(1/15)-t}$ ✔
- ( ) $\frac{1/15}{e^t-(1/15)}$
- ( ) $\frac{1/15}{(1/15)-e^t}$

b. Compute $M_X'(0)$. Hint: You should get 15 for your answer, since $M_X'(0) = \mathbb{E}(X)$.

15 ✔ **Answer:** 15

Explanation

6a. We compute $M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty (e^{tx})(\frac{1}{15}e^{-x/15})\,dx = \frac{1/15}{(1/15)-t}$.

6b. We have $M_X'(t) = \frac{d}{dt}\frac{1/15}{(1/15)-t} = \frac{1/15}{((1/15)-t)^2}$. So $\mathbb{E}(X) = M_X'(0) = \frac{1/15}{((1/15)-0)^2} = 15$.

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Problem 7

2/2 points (graded)

7. Same setup as #6.

a. Compute $M_X''(0)$. This is equal to $\mathbb{E}(X^2)$.

450 ✔ **Answer:** 450

b. Use your solutions to **6b** and **7a** to compute $\mathrm{Var}(X)$. Does this agree with the formula that you know, for the variance of an Exponential random variable?

225 ✔ **Answer:** 225

Explanation

7a. From **6b**, we have $M_X'(t) = \frac{d}{dt}\frac{1/15}{(1/15)-t} = \frac{1/15}{((1/15)-t)^2}$. Taking another derivative with respect to t, we get $M_X''(t) = \frac{d}{dt}\frac{1/15}{((1/15)-t)^2} = (2)\frac{1/15}{((1/15)-t)^3}$. So $\mathbb{E}(X^2) = M_X''(0) = (2)\frac{1/15}{((1/15)-0)^3} = 2(15^2)$.

7b. We have $\mathbf{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2(15^2) - (15)^2 = 15^2$. This is what we knew we would get for the answer, since for an Exponential random variable, the variance is equal to the square of the mean.

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Problem 8

3/3 points (graded)

8. Suppose that random variable X has probability mass function $P(X = x) = (27/40)(1/3)^x$, for integers $0 \leq x \leq 3$.

a. Verify that this is a valid probability mass function.

b. Manually compute the expected value of X.

0.45 ✔ **Answer:** 0.45

c. Find the moment generating function $M_X(t)$ of X. (If you think for a moment, it is possible to write $M_X(t)$ without using any summation signs or addition symbols.)

- ( ) $\left(\frac{40}{27}\right)\frac{1-(e^t/3)^3}{1-e^t/3}$

- $\left(\frac{27}{40}\right)\frac{1-(e^t/3)^3}{1-e^t/3}$
- $\left(\frac{40}{27}\right)\frac{1-(e^t/3)^4}{1-e^t/3}$
- $\left(\frac{27}{40}\right)\frac{1-(e^t/3)^4}{1-e^t/3}$ ✔

d. Compute $M'_X(0)$. Hint: Your answer should agree with your answer for **8b**.

0.45

✔ **Answer:** 0.45

Explanation

8a. All of the values $p_X(x) = P(X = x)$ are nonnegative, and we have $\sum_{x=0}^{3}(27/40)(1/3)^x = 0.675 + 0.225 + 0.075 + 0.025 = 1$. So $p_X(x)$ is a valid probability mass function.

8b. We compute $(0)(0.675) + (1)(0.225) + (2)(0.075) + (3)(0.025) = 0.45$.

8c. We have

$$M_X(t) = \mathbb{E}(e^{tX}) = \sum_{x=0}^{3} e^{tx}\left(\tfrac{27}{40}\right)(1/3)^x$$
$$= \left(\tfrac{27}{40}\right)\sum_{x=0}^{3}(e^t/3)^x = \left(\tfrac{27}{40}\right)\frac{1-(e^t/3)^4}{1-e^t/3}.$$

8d. We have

$$M'_X(t) = \frac{d}{dt}\left(\tfrac{27}{40}\right)\frac{1-(e^t/3)^4}{1-e^t/3} = \left(\tfrac{27}{40}\right)\frac{(1-e^t/3)(-4(e^t/3)^3(1/3))-(1-(e^t/3)^4)(-e^t/3)}{(1-e^t/3)^2}.$$

So

$$\mathbb{E}(X) = M'_X(0) = \left(\frac{27}{40}\right)\frac{(1-e^0/3)(-4(e^0/3)^3(1/3))-(1-(e^0/3)^4)(-e^0/3)}{(1-e^0/3)^2}$$
$$= \left(\frac{27}{40}\right)\frac{(2/3)(-4(1/3)^4)-(1-(1/3)^4)(-1/3)}{(2/3)^2} = 0.45.$$

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Problem 9

4/4 points (graded)

9. Children are decorating rocks with paint and sparkly material, to give as gifts. The weights of the rocks are assumed to be uniformly distributed between 0.5 and 2.2 pounds. Let X denote the weight of such a rock. Suppose that the cost of the materials to be used on such a rock is $Y = (2/5)X + 0.1$.

a. Find the probability density function $f_Y(y)$ of Y. Be sure to specify where $f_Y(y)$ is nonzero.

$f_Y(y) =$ 25/17 ✔ **Answer:** 1.4706 for 0.3 ✔ **Answer:** $0.3 \leq y \leq$ 0.98 ✔ **Answer:** 0.98

and $f_Y(y) = 0$ otherwise.

b. Use $f_Y(y)$ to find the probability that Y is less than 0.60.

15/34 ✔ **Answer:** 0.44

c. Check your answer by using $f_X(x)$ to find the probability that $(2/5)X + 0.1$ is less than 0.60.

Explanation

9a. We have $0.5 \le X \le 2.2$, so $0.3 \le Y \le 0.98$. Since X is uniformly distributed on $[0.5, 2.2]$, and $(2/5)X + 0.1$ is a linear function of X (i.e., just a scaling and shifting), then Y must be uniformly distributed on $[0.3, 0.98]$, so $f_Y(y) = 1/(0.98 - 0.3) = 1/0.68 = 1.47$ for $0.3 \le y \le 0.98$, and $f_Y(y) = 0$ otherwise. If you prefer, we can calculate the CDF of Y. For $0.3 \le y \le 0.98$, we have $P(Y \le y) = \frac{y-0.5}{0.98-0.3}$, and differentiating with respect to y yields $f_Y(y) = 1/(0.98 - 0.3) = 1.47$.

9b. We have $P(Y \le 0.60) = \int_{0.3}^{0.6} 1.47\,dy = (0.3)(1.47) = 0.44$.

9c. We have $P(Y \le 0.60) = P((2/5)X + 0.1 \le 0.60)$

$$= P(X \le 1.25) = \frac{1.25-0.5}{2.2-0.5} = 0.44.$$

Submit You have used 1 of 1 attempt

Problem 10

4/4 points (graded)

10. Same setup as #9.

a. What are the mean and standard deviation of the cost Y of the materials used on such a rock?

$\mathbb{E}(Y) =$ 0.64 ✔ **Answer:** 0.64

$\sigma_Y =$ 0.1962991 ✔ **Answer:** 0.2

b. Now suppose that 100 such rocks are to be decorated, and their weights are independent. Use X_j to denote the weight of the jth rock. Thus, the cost of materials used to decorate the jth rock is $Y_j = (2/5)X_j + 0.1$. Find a good approximation for the distribution of the total cost, namely, $Y_1 + \cdots + Y_{100}$.

The distribution of Y is approximately Normal with mean 64 ✔ **Answer:** 64

and variance 3.853334 ✔ **Answer:** 4.0

Explanation

10a. Since Y is uniformly distributed on $[0.3, 0.98]$, then from our formulas for the mean and variance of a Continuous Uniform random variable, we know $\mathbb{E}(Y) = (0.3 + 0.98)/2 = 0.64$ and $\text{Var}(Y) = (0.98 - 0.3)^2/12 = 0.039$.

We can also calculate: $\mathbb{E}(Y) = \int_{0.3}^{0.98} (y)(1.47)\,dy = 0.64$ and $\mathbb{E}(Y^2) = \int_{0.3}^{0.98} (y^2)(1.47)\,dy = 0.45$ so $\text{Var}(Y) = 0.45 - (0.64)^2 = 0.04$, and the standard deviation is $\sigma_Y = \sqrt{0.04} = 0.2$.

10b. Since Y is a sum of 100 independent random variables, each with mean 0.64 and variance 0.039, then the distribution of Y is approximately Normal with mean $(100)(0.64) = 64$ and variance $(100)(0.039) = 3.9$.

Submit You have used 1 of 1 attempt

✔ Correct (4/4 points)

Problem 11

4/4 points (graded)

11. Suppose that X is a continuous random variable that is uniformly distributed on the interval $(0, 3)$. Suppose that we define $Y = (X + 3)(X - 3)$.

a. What is the probability density function $f_Y(y)$ of Y? Compute $f_Y(-5)$. For which values of y is the density nonzero?

$f_Y(-5) =$ 1/12 ✔ **Answer:** 0.0833

The density nonzero when -9 ✔ **Answer:** -9 $< y \leq$

0 ✔ **Answer:** 0

b. Use $f_Y(y)$ to get the mean of Y, as $\mathbb{E}(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$.

-6 ✔ **Answer:** -6

c. Use $f_X(x)$ to get the mean of Y indirectly, as $\mathbb{E}(Y) = \int_{-\infty}^{\infty} (x+3)(x-3)\, f_X(x)\, dx$. Your solution should agree with **11b**.

Explanation

11a. Since $Y = (X + 3)(X - 3) = X^2 - 9$, and $0 \leq X \leq 3$, then $-9 \leq Y \leq 0$. For $-9 \leq y \leq 0$, we have

$$F_Y(y) = P(Y \leq y) = P(X^2 - 9 \leq y)$$

$$= P(X \leq \sqrt{y+9}) = \frac{\sqrt{y+9} - 0}{3 - 0} = \frac{1}{3}\sqrt{y+9}.$$

Differentiating with respect to y, we get $f_Y(y) = \frac{1}{6}(y + 9)^{-1/2}$.

11b. We have $\mathbb{E}(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_{-9}^{0} (y)(1/6)(y+9)^{-1/2}\, dy$.
Using $u = y + 9$, this gives

$$\begin{aligned}\mathbb{E}(Y) &= \int_0^9 (u-9)(1/6)(u)^{-1/2}\, du \\ &= (1/6)\int_0^9 (u^{1/2} - 9u^{-1/2})\, du \\ &= (1/6)((2/3)u^{3/2} - 18u^{1/2})\Big|_{u=0}^{9} \\ &= (1/6)((2/3)9^{3/2} - (18)9^{1/2}) \\ &= (1/6)((2/3)(27) - (18)(3)) = (1/6)(18-54) \\ &= (1/6)(-36) = -6.\end{aligned}$$

11c. We compute

$$\begin{aligned}\mathbb{E}(Y) &= \mathbb{E}((X+3)(X-3)) = \mathbb{E}(X^2 - 9) \\ &= \int_0^3 (x^2-9)(1/3)\, dx = (1/3)(x^3/3 - 9x)\Big|_{x=0}^{3} \\ &= (1/3)(3^3/3 - (9)(3)) = (1/3)(9-27) \\ &= (1/3)(-18) = -6.\end{aligned}$$

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✔ Correct (4/4 points)

Problem 12

2/4 points (graded)

12. Suppose that the joint distribution of X and Y is uniform in the triangular region of the (x, y)-plane with corners at the origin and $(5, 0)$ and $(5, 2)$.

a. Find $\mathbb{E}(X)$.

10/3

✔ **Answer:** 3.3333

b. Find $\mathbb{E}(Y)$.

2/3

✔ **Answer:** 0.6667

c. Find $\mathbb{E}(XY)$.

5

✖ **Answer:** 2.5

d. Use your solutions to parts a, b, c to find the covariance of X and Y.

$\mathrm{Cov}(X,Y) =$ 25/9

✖ **Answer:** 0.2778

Explanation

12a. We have $\mathbb{E}(X) = \int_0^5 \int_0^{(2/5)x} (x)(1/5)\,dy\,dx = 10/3$.

12b. We have $\mathbb{E}(Y) = \int_0^5 \int_0^{(2/5)x} (y)(1/5)\,dy\,dx = 2/3$.

12c. We have $\mathbb{E}(XY) = \int_0^5 \int_0^{(2/5)x} (xy)(1/5)\,dy\,dx = 5/2$.

12d. We conclude that

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)\\ &= 5/2 - (10/3)(2/3) = 5/18 = 0.2778.\end{aligned}$$

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