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# Homework

Homework due Aug 5, 2020 21:30 IST

The exercises below will count towards your grade. **You have only one chance to answer these questions.** Take your time, and think carefully before answering.

## Problem 1

28.5/30 points (graded)

Or each of the follow sets of real numbers, determine whether it is a Borel Set.

If it *is* a Borel Set, specify its Lebesgue Measure.

And if it is *not* a Borel Set, enter '$0$' into the numeric entry field.

(You may avail yourself of the fact that every Borel Set has a Lebesgue Measure.)

As usual, $[a, b] = \{x : a \leq x \leq b\}$ and $[a, b) = \{x : a \leq x < b\}$.

$[\frac{1}{4}, \frac{5}{6}]$

● Borel Set

○ Not a Borel Set

✓

7/12 ✓ **Answer:** 7/12

$\frac{7}{12}$

**Explanation**

Borel set. $\lambda\left[\frac{1}{4}, \frac{5}{6}\right] = \frac{5}{6} - \frac{1}{4} = \frac{7}{12}$.

$\{\frac{5}{6}\}$

● Borel Set

○ Not a Borel Set

✓

0 ✓ **Answer:** 0

$0$

**Explanation**

Borel set. $\lambda\{\frac{5}{6}\} = 0$.

$[\frac{1}{4}, \frac{5}{6})$

● Borel Set

○ Not a Borel Set

✓

7/12 ✓ **Answer:** 7/12

$\frac{7}{12}$

**Explanation**

Borel set. Since $\lambda\left[\frac{1}{4}, \frac{5}{6}\right] = \lambda\left[\frac{1}{4}, \frac{5}{6}\right) + \lambda\{\frac{5}{6}\}$, it follows from the previous exercises that $[\frac{1}{4}, \frac{5}{6}) = \frac{7}{12}$.

$[\frac{1}{4}, \frac{5}{6}] - [\frac{1}{3}, \frac{1}{2}]$

● Borel Set

○ Not a Borel Set

✓

5/12

✓ **Answer:** 5/12

$\frac{5}{12}$

**Explanation**

Borel set. Since $\lambda\left[\frac{1}{4},\frac{5}{6}\right] = \lambda\left(\left[\frac{1}{4},\frac{5}{6}\right] - \left[\frac{1}{3},\frac{1}{2}\right]\right) + \lambda\left[\frac{1}{3},\frac{1}{2}\right]$, and since $\lambda\left[\frac{1}{3},\frac{1}{2}\right] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, $\lambda\left(\left[\frac{1}{4},\frac{5}{6}\right] - \left[\frac{1}{3},\frac{1}{2}\right]\right) = \frac{7}{12} - \frac{1}{6} = \frac{5}{12}$.

$\left[\frac{1}{4},\frac{5}{6}\right] - \left\{\frac{1}{3},\frac{1}{2}\right\}$

● Borel Set

○ Not a Borel Set

✓

7/12

✓ **Answer:** 7/12

$\frac{7}{12}$

**Explanation**

Borel set. Since $\lambda\left[\frac{1}{4},\frac{5}{6}\right] = \lambda\left(\left[\frac{1}{4},\frac{5}{6}\right] - \left\{\frac{1}{3},\frac{1}{2}\right\}\right) + \lambda\left\{\frac{1}{3},\frac{1}{2}\right\}$, and since $\lambda\left\{\frac{1}{3},\frac{1}{2}\right\}$, we have $\lambda\left(\left[\frac{1}{4},\frac{5}{6}\right] - \left\{\frac{1}{3},\frac{1}{2}\right\}\right) = \frac{7}{12} - 0 = \frac{7}{12}$.

$\left\{\frac{1}{2^1},\frac{1}{2^2},\frac{1}{2^3},\ldots\right\}$

● Borel Set

○ Not a Borel Set

✓

0

✓ **Answer:** 0

$0$

**Explanation**

Borel set. Since $\left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$ is countable, $\lambda\left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\} = 0$.

$[\frac{4}{5}, \frac{5}{6}] \cup \{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$

● Borel Set

○ Not a Borel Set

✓

7/12

✗ **Answer:** 1/30

$\frac{7}{12}$

**Explanation**

Borel set.

$$\lambda\left(\left[\frac{4}{5}, \frac{5}{6}\right] \cup \left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}\right) = \lambda\left[\frac{4}{5}, \frac{5}{6}\right] + \lambda\left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\} = \left(\frac{5}{6} - \frac{4}{5}\right) + 0 = \frac{1}{30}$$

.

$[0, \frac{1}{2}] - \{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$

● Borel Set

○ Not a Borel Set

✓

1/2 ✓ **Answer:** 1/2

$\frac{1}{2}$

**Explanation**

BBorel set. Since $\lambda\left([0,\frac{1}{2}]\right) = \lambda\left([0,\frac{1}{2}] - \left\{\frac{1}{2^1},\frac{1}{2^2},\frac{1}{2^3},\ldots\right\}\right) + \lambda\left\{\frac{1}{2^1},\frac{1}{2^2},\frac{1}{2^3},\ldots\right\}$, we have $\lambda\left([0,\frac{1}{2}] - \left\{\frac{1}{2^1},\frac{1}{2^2},\frac{1}{2^3},\ldots\right\}\right) = \frac{1}{2} - 0 = \frac{1}{2}$.

a Vitali set

○ Borel Set

● Not a Borel Set

✓

0 ✓ **Answer:** 0

$0$

**Explanation**

Not a Borel set. This can be verified by noting that all Borel sets have Lebesgue measure, but Vitali sets do not.

the complement of a Vitali set

○ Borel Set

● Not a Borel Set

✓

0 ✓ **Answer:** 0

$0$

**Explanation**

Not a Borel set. If it were a Borel set, its complement would be a Borel set too, and we know that it isn't.

Submit

You have used 1 of 1 attempt

ℹ Answers are displayed within the problem

## Problem 2

30/30 points (graded)

Consider the following procedure for picking points at random from $[0, 1]$. We use *two* coins, coin $O$ and coin $E$. Coin $O$ has a $\frac{1}{5}$ chance of landing heads. Coin $E$ has a $\frac{1}{8}$ chance of landing heads. You then do as follows:

> You toss a coin once for each natural number. For the first toss, you use coin $O$. For the second toss, you use coin $E$. For the third toss, you use coin $O$. In general, you use coin $O$ on the $n$-th toss if $n$ is odd, and coin $E$ on the $n$-th toss if $n$ is even. Each time the coin lands Heads you write down a $0$, and each time it lands Tails you write down a $1$. Then you pick whichever number in $[0, 1]$ has as its binary expansion '$0$' followed by a decimal point followed by the infinite sequence of zeroes and ones you got from your coin tosses.

Answer the following questions, inputting your answer as a whole number, or as a fraction of whole numbers.

Using this procedure, what is the probability of selecting a number in $[0, \frac{1}{2}]$?

1/5

✓ **Answer:** 1/5

$\frac{1}{5}$

**Explanation**

There are two ways to select a number in $[0, \frac{1}{2}]$. Either you get Heads on the first toss, or you get Tails on the first toss, and Heads on all subsequent tosses. By Additivity, it follows that the probability of getting a number in $[0, \frac{1}{2}]$ is the probability of the getting Heads on the first toss, plus the probability of getting Tails on the first toss and Heads thereafter. Since the latter event has probability $0$, this means that we can ignore it in our calculations: the probability of getting a number in $[0, \frac{1}{2}]$ is simply the probability of the getting Heads on the first toss. Since you use coin $O$ on the first toss, the probability of getting Heads on the first toss is $\frac{1}{5}$. So the probability of selecting a number in $[0, \frac{1}{2}]$ is $\frac{1}{5}$.

Using this procedure, what is probability of selecting a number in the interval $(0, 1]$? Note that $(0, 1] = [0, 1] - \{0\}$.

1

✓ **Answer:** 1

$1$

**Explanation**
We know that the probability that you will select a number in $[0, 1]$ is $1$. And, by Additivity, we know that the probability of selecting a number in $1$ is equal to the probability of selecting $0$, plus the probability of selecting a number in $(0, 1]$. Of course, the probability that $0$ is selected is $0$. So it must be that the probability of selecting a number in $(0, 1]$ is $1$.

Using this procedure, what is probability of selecting a number greater than or equal to $\frac{15}{16}$ ?

49/100

✓ **Answer:** 784/1600

$\frac{49}{100}$

**Explanation**
In order for a number greater than or equal to $\frac{15}{16}$ to be selected the coin must land Tails on each of your first four coin tosses. Here's why: the first toss of Tails puts you in the interval $[\frac{1}{2}, 1]$, the second in $[\frac{3}{4}, 1]$, the third in $[\frac{7}{8}, 1]$, and the fourth in $[\frac{15}{16}, 1]$. (Actually, there's another way of getting a number greater than or equal to $\frac{15}{16}$: if the coins land Tails, Tails, Tails Heads, and Tails tereafter, you'll end up at the $\frac{15}{16}$ point. But since this is an event of probability zero, we may ignore it.) With the first toss, you toss coin $O$, which gives you a $\frac{4}{6}$ chance at Tails. With the second, you toss with $E$, which gives you a $\frac{7}{8}$ chance at Tails. And so forth. So when we calculate the chance that you land Tails four times in a row this is what we get: $\frac{4}{5} \times \frac{7}{8} \times \frac{4}{5} \times \frac{7}{8} = \frac{784}{1600}$.

Submit

You have used 1 of 1 attempt

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ℹ Answers are displayed within the problem

## Problem 3

10/10 points (graded)

Is the following claim true or false?

> There is a definite probability that the **Standard Coin-Toss Procedure** from the "Uniform and Non-Uniform Measures" section will output a number in $[\frac{1}{2^{n+1}}, \frac{1}{2^n})$, for $1 \leq n$.

● True

○ False

✓

**Explanation**

The Standard Coin Toss Procedure will output a number in $[\frac{1}{2^{n+1}}, \frac{1}{2^n})$ just in case the coin lands Heads the first $n$ tosses and tails on the $n+1$th toss, which is an event of probability $\frac{1}{2^{n+1}}$

Now suppose the set $A$ is the union of finitely many intervals of the form $[\frac{1}{2^{n+1}}, \frac{1}{2^n})$, for $1 \leq n$.

Is the following claim true or false?

> There is a definite probability that the **Standard Coin-Toss Procedure** from the "Uniform and Non-Uniform Measures" section will output a number in $A$. (You may assume that the Standard Coin Toss Procedure delivers a probability function.)

● True

○ False

✓

**Explanation**

Since distinct intervals of the form $[\frac{1}{2^{n+1}}, \frac{1}{2^n})$ do not overlap, the result follows immediately from Additivity.

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You have used 1 of 1 attempt

Answers are displayed within the problem

## Problem 4

10/10 points (graded)

Let the $A_1, A_2, A_3 \ldots$ be subsets of $[0,1]$ such that:

$$A_1 = \{\frac{1}{2}\}$$

$$A_2 = \{\frac{1}{4}, \frac{1}{8}\}$$

$$A_3 = \{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\}$$

$$\ldots \text{etc.}$$

Suppose we are picking numbers from $[0,1]$ using the **Standard Coin-Toss Procedure** from the "Uniform and Non-Uniform Measures". Consider the probability of picking a number in the the union of the $A_i$s.:

$$\bigcup_{i \in \mathbb{N}} (A_i)$$

Assuming *Countable Additivity*, what is the probability of picking a point in that union? (Enter your answer as a fraction)

0

✓ **Answer:** 0

$0$

**Explanation**

The answer is 0. For every $i \in \mathbb{N}$, $A_i$ has finitely many members. So, given that we are using the coin-toss method, for every $i \in \mathbb{N}$ $p(A_i) = 0$. $p\left(\bigcup_{i \in \mathbb{N}} (A_i)\right)$ is then just summing 0 with itself (a countably infinite number of times), which of course results in 0!

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You have used 1 of 1 attempt

ℹ Answers are displayed within the problem

## Problem 5

20/20 points (graded)

Recall that a *choice set* for set $A$ is a set containing exactly one element from each member of $A$. And recall the *Axiom of Choice*:

> **Axiom of Choice**: Any set of non-empty, non-overlapping sets has a choice set.

On a first reading, the Axiom of Choice is likely to sound trivial. This exercise is aimed at helping you understand why it is not. (It is a variant of an explanation given long ago by British philosopher Bertrand Russell.)

Here are two standard set-theoretic axioms:

> **Union**: If a set $A$ exists, then so does its union, $\bigcup A$. (Recall that $\bigcup A$ is the set $\{x :$ $x$ is a member of some element of $A\}$ (i.e. the set of members of members of $A$.)
>
> **Separation**: Let $\phi(x)$ be any formula of the form "$x$ is such and such" (for instance, "$x$ is a natural number"). Then if set $A$ exists, the following set also exists: $\{x : x \in A$ and $\phi(x)\}$ (i.e. the set of objects that are members of $A$ and satisfy condition $\phi(x)$).

Here is an example of how Separation might be used. Let $\phi(x)$ be the formula "$x$ is an octopus". Separation entails that if the set $A = \{x : x \text{ is an animal}\}$ exists, so does $\{x : x \in A \text{ and } x \text{ is an octopus}\}$, which is the set of octopuses.

Now, let $S$ be an infinite set, each member of which is a set of two shoes: a right shoe and a left shoe. Assume that no two elements of $S$ have any shoes in common. Now suppose you'd like to have a choice set for $S$. The Axiom of Choice guarantees that a choice set exists, but it doesn't give you much information about what it looks like. Let's see if we can do better than that. It follows from Union that $\bigcup S$ exists.

Question: Is there an application of Separation to $\bigcup S$ that delivers a choice set for $S$?

● Yes

○ No

✓

### Explanation

Yes. Separation entails the existence of $\{x : x \in \bigcup S$ and $x$ is a right shoe $\}$ which is a choice set for $S$.

Next, let $S$ be an infinite set, each member of which is a set of two socks. (Assume that all socks are alike, and, in particular, that there is no such thing as a "right" sock or a "left" sock.) Assume that no two elements of $S$ have any socks in common. Now suppose you'd like to have a choice set for $S$. The Axiom of Choice guarantees that a choice set exists, but it doesn't give you much information about what it looks like. Let's see if we can do better than that. It follows from Union that $\bigcup S$ exists.

Question: Is there an application of Separation to $\bigcup S$ that delivers a choice set for $S$?

○ Yes

● No

✓

**Explanation**

No. We can't use Separation to get a choice set because we are not able to articulate a feature of socks that applies to exactly one element from each pair, so we are not able to come up with a suitable choice for $\phi(x)$.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem