16. Level of a Statistical test

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Testing the Support of a Uniform Variable: Level and Threshold

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2/2 points (graded)

As in the problems on the previous page, let $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Unif}[0, \theta]$ for an unknown parameter θ and we designed the statistical test

$$\psi_n = \mathbf{1}(\max_{1 \le i \le n} X_i > 1/2)$$

to decide between the null and alternative hypotheses

$$H_0 : \theta \ \le 1/2$$
$$H_1 : \theta \ > 1/2.$$

Let $\alpha_{\psi_n}(\theta)$ and $\beta_{\psi_n}(\theta)$ be the type 1 and type 2 errors respectively.

Recall from lecture that a test ψ has **level** α if

$$\alpha \ \ge \ \alpha_\psi(\theta) \qquad \text{for all } \theta \in \Theta_0,$$

where $\alpha_\psi = \mathbf{P}_\theta(\psi = 1)$ is the type 1 error. We will often use the word "level" to mean the "smallest" such level, i.e. the least upper bound of the type 1 error, defined as follows:

$$\alpha \ = \ \sup_{\theta \in \Theta_0} \alpha_\psi(\theta)$$

Here, $\sup_{\theta \in \Theta_0}$ stands for the supremum over all values of θ within Θ_0. If Θ_0 is a closed (*resp.* closed half-interval), and if $\alpha_\psi(\theta)$ is continuous (*resp.* continuous and decreasing as it approaches infinity), then its supremum equals the maximum.

Using the graph of the errors on the previous page, what is the smallest level α of the test ψ_n?

$\alpha =$ 0 ✔ **Answer:** 0

How should the threshold of the test be changed to increase the smallest level α? In other words, consider tests of the form

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$$\psi_{n,C} = \mathbf{1}(\max_{1 \le i \le n} X_i > C)$$

where C is the threshold. In the original test above, $C = 1/2$. What should the value of C be so that the level of $\psi_{n,C}$ is greater than the level of the $\psi_{n,1/2}$?
(Think of how the graph of $\mathbf{P}_\theta(\psi_C)$ changes with the threshold C.)

- $C > 1/2$
- $C < 1/2$

Solution:

Since the type 1 error $\alpha_{\psi_n}(\theta)$ is constantly zero over Θ_0, the smallest level of this test ψ is $\alpha = 0$.

To increase the smallest level α from 0, note that $\mathbf{P}_\theta\left(\max_{1 \le i \le n} X_i > C\right) = 0$ if and only if $\theta \le C$. This means the constant zero region of graph of $\mathbf{P}_\theta(\psi_C) = 0$ shifts to the right as C increases from $1/2$, and to the left as C decreases from $1/2$. Since the maximum of type 1 error occurs at the boundary $\theta = 1/2$, this means $C < 1/2$ is required for the level to be positive.
Remark: The reason behind increasing the level in this example is to increase the power of the test from 0. In general, one of the first requirements of a test is to have a small-enough level so that the probability of concluding a false positive, (i.e. rejecting the null while the null is true) is controlled.

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Answers are displayed within the problem

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Testing the Support of a Uniform Variable: Determine the Threshold

1/1 point (graded)

As above, let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Unif}\,[0, \theta]$ for an unknown parameter θ and consider tests of the form

$$\psi_{n,C} = \mathbf{1}(\max_{1 \le i \le n} X_i > C)$$

to decide between the null and alternative hypotheses

$$\begin{aligned} H_0 &: \theta \ \le 1/2 \\ H_1 &: \theta \ > 1/2. \end{aligned}$$

Let $\alpha_{\psi_{n,C}}(\theta)$ and $\beta_{\psi_{n,C}}(\theta)$ be the type 1 and type 2 errors respectively.

Determine the smallest threshold C such that the test $\psi_{n,C}$ has level α.

(Enter the roots of x as a power of x, e.g. enter **x^(1/3)** for $\sqrt[3]{x} = x^{1/3}$.)

$C =$ (1-alpha)^(1/n)/2 ✔ **Answer:** 1/2*(1-alpha)^(1/n)

$\frac{(1-\alpha)^{\frac{1}{n}}}{2}$

STANDARD NOTATION

Solution:

Following similar computation as in a previous problem where $C = 1/2$, we have $\mathbf{P}_\theta\left(\psi_{n,C} = 1\right) \ = 1 - \left(\frac{C}{\theta}\right)^n$. Since the smallest level is

$$\begin{aligned} \alpha \ &= \ \max_{\theta \in \Theta_0} p_\theta\left(\psi_{n,C} = 1\right) \\ &= \ p_{1/2}\left(\psi_{n,C} = 1\right) \ = \ 1 - \left(\frac{C}{1/2}\right)^n, \end{aligned}$$

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a test with threshold $C = \frac{1}{2}\sqrt[n]{1-\alpha}$ or smaller will have level α.

Remark: Notice the threshold C depends on n, α, as well as the value of θ at the boundary of Θ_0 and Θ_1.

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Answers are displayed within the problem

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Confused about the shape of the curve

question posted 3 days ago by XuYang2015

I am a little confused here. According to the curve, it seems that level alpha is equal to power pi. It is not true, there must be something wrong with the curve. Actually, I don't understand why the curve can be continuous without interruption, a "S" shape. The x-axis of the curve is true mu? If it is, in the interval of theta 0, the curve should be a constant, because sqrt(n)*(X_bar_n-mu)/sigma is asymptotically normal, meanwhile X_bar_n is a r.v and C is a constant here, P(I(sqrt(n)*(X_bar_n-mu)/sigma > C)) is a constant. Am I right?

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2 responses

Erocha (Community TA)
3 days ago

Do you mean this graph:

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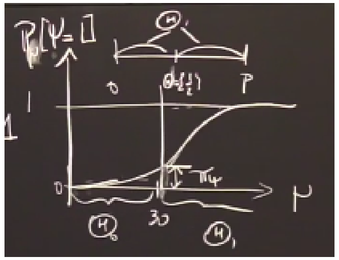

note this is the value of $P_\mu[\psi = 1]$ (a.k.a. power function), and it is continuous indeed. If you have access to the book "All of Statistics" see example 10.2 in Chapter 10.

See this:

Example 11.2 *Let* $X_1, \ldots, X_n \sim N(\mu, \sigma)$ *where* σ *is known. We want to test* $H_0 : \mu \leq 0$ *versus* $H_1 : \mu > 0$. *Hence,* $\Theta_0 = (-\infty, 0]$ *and* $\Theta_1 = (0, \infty)$. *Consider the test:*

$$\text{reject } H_0 \text{ if } T > c$$

where $T = \overline{X}$. *The rejection region is* $R = \left\{x^n : \; T(x^n) > c\right\}$. *Let* Z *denote a standard normal random variable. The power function is*

$$\begin{aligned}
\beta(\mu) &= P_\mu\left(\overline{X} > c\right) \\
&= P_\mu\left(\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} > \frac{\sqrt{n}(c - \mu)}{\sigma}\right) \\
&= P\left(Z > \frac{\sqrt{n}(c - \mu)}{\sigma}\right) \\
&= 1 - \Phi\left(\frac{\sqrt{n}(c - \mu)}{\sigma}\right).
\end{aligned}$$

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This function is increasing in μ. *Hence*

$$\text{size} = \sup_{\mu<0} \beta(\mu) = \beta(0) = 1 - \Phi\left(\frac{\sqrt{n}c}{\sigma}\right).$$

To get a size α *test, set this equal to* α *and solve for* c *to get*

$$c = \frac{\sigma\Phi^{-1}(1-\alpha)}{\sqrt{n}}.$$

So we reject when $\overline{X} > \sigma\Phi^{-1}(1-\alpha)/\sqrt{n}$. *Equivalently, we reject when*

$$\frac{\sqrt{n}\,(\overline{X} - 0)}{\sigma} > z_\alpha. \quad \blacksquare$$

posted 2 days ago by **Cool7**

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XuYang2015

2 days ago

Thank you for your detailed explanations. Just a small question to check if I correctly understand the example you gave (example 11.2). In the example, because the power function is continuously increasing, thus the sup of the power function in the interval Theta0 (-inf, 0] is equal to the inf of the power function in the interval Theta1 (0, inf). It means that if we control the level of the test statistics at alpha, the power of the test statistics is also alpha? Am I right?

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Yes, with my understanding in this example.

$\alpha = \Phi\left(\frac{\sqrt{n}(c-0)}{\sigma}\right)$, $\text{power} = \lim_{\mu \to 0^+} \Phi\left(\frac{\sqrt{n}(c-\mu)}{\sigma}\right)$, giving this function is increasing in μ.

and we know $\Phi\left(\frac{\sqrt{n}(c-\mu)}{\sigma}\right)$ is continuous at 0 when $\sigma \neq 0$, thus $\alpha = power$

posted 2 days ago by Cool7

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