9. Performing Wald's Test on a Gaussian Data Set

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3/3 points (graded)

Suppose $X_1, \ldots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$. Your goal is to hypothesis test between

$$\begin{aligned} H_0 &: (\mu, \sigma^2) = (0, 1) \\ H_1 &: (\mu, \sigma^2) \neq (0, 1). \end{aligned}$$

Recall Wald's test from a previous problem, which, under the above hypotheses, takes the form

$$\psi_\alpha := \mathbf{1}\left(W_n > q_\alpha\left(\chi_2^2\right)\right) = \mathbf{1}\left(n\left(\hat{\theta}_n^T - \begin{pmatrix} 0 & 1 \end{pmatrix}\right) \mathcal{I}\left((0,1)\right)\left(\hat{\theta}_n - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) > q_\alpha\left(\chi_2^2\right)\right)$$

where $q_\alpha\left(\chi_2^2\right)$ is the α-quantile of χ_2^2. You are given that the technical conditions required for the MLE to be asymptotically normal are satisfied for a Gaussian statistical model with unknown mean and variance.

What is the smallest value of $q_\alpha\left(\chi_2^2\right)$ so that ψ_α is a test with asymptotic level 5%?
(You should use a table (e.g. https://people.richland.edu/james/lecture/m170/tbl-chi.html or software (e.g. R) to answer this question.)

For ψ_α to have level 5%:

$q_\alpha\left(\chi_2^2\right) \geq$ 5.991 ✔ **Answer:** 5.991

Suppose you observe the data set

$$0.2, -0.1, -1.9, -0.4, -1.8$$

What is the value of the test statistic W_5 for this data set?

Hint: Recall that the MLE of a Gaussian $\mathcal{N}\left(\mu, \sigma^2\right)$ is given by

$$\begin{pmatrix} \widehat{\mu}_n^{MLE} \\ \left(\widehat{\sigma^2}\right)_n^{MLE} \end{pmatrix} = \begin{pmatrix} \overline{X}_n \\ \frac{1}{n}\sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2 \end{pmatrix}$$

and the Fisher information is given by

$$\mathcal{I}\left(\mu, \sigma^2\right) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}.$$

$W_5 =$ 3.3299600000000003 ✔ **Answer:** 3.33

Will Wald's test **reject** or **fail to reject** for this data set?

Reject

Fail to reject

Solution:

Since we have assumed that the MLE is asymptotically normal, we have

$$W_n \xrightarrow[n\to\infty]{(d)} \chi_2^2.$$

There are precisely two degrees of freedom since we have two unknowns. The test ψ_α has asymptotic level 5% if $\alpha = 5\%$. Consulting a table, we see that the 0.05-quantile for χ_2^2 is $q_\alpha = 5.991$.

For the given data set, we compute

$$\begin{aligned} \widehat{\mu}_n^{MLE} &= \overline{X}_n \approx -0.8 \\ \widehat{\sigma^2}_n^{MLE} &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 \approx 0.772. \end{aligned}$$

The Fisher information, under the null hypothesis $(\mu, \sigma^2) = (0, 1)$, is

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore,

$$W_5 = 5 \cdot ((-0.8, 0.772) - (0,1)) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} -0.8 \\ 0.772 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^T \approx 3.33.$$

Since $q_{0.05} = 5.991 > 3.33$, we would **fail to reject** the null hypothesis for the given sample.

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Answers are displayed within the problem

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For Wald's test statistic, which Fisher Info to be used- (Theta-MLE) or (Theta-under null hyp.)?

question posted 7 days ago by rmgajera

For Wald's test statistic calculations, which Fisher Info to be used- (Theta-MLE) or (Theta-under null hyp.)? In slide, I-(Theta^MLE) is mentioned, whereby . in the solution of above problem, I-(Theta-null hyp.) is used. I got it wrong as I used the Fisher info with theta^MLE(i.e., by putting values of MLE parameters in fisher matrix).

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1 response

ya_mukhin (Staff)
5 days ago

Hi @rmgajera, this is a very good question! Asymptotically, both estimates of the Fisher information matrix lead to the same statistical test, because MLE is consistent. However, these two choices of the covariance matrix yield different tests in any given finite sample, as you have experienced. For this particular problem, you are given the exact form of the test (including the exact information matrix to use, $\mathcal{I}\,(0,1)$), so it is fair that the grader expects this choice.

Okay! Noted. Thank you for clarifying.

posted 5 days ago by **rmgajera**

And there will be an upcoming recitation to explore this question.

posted 4 days ago by **karenechu** (Staff)

Thank you! Will this recitation explain when and why we should use $I\,(\hat{\theta}^{MLE})$ vs. $I\,(\theta_0)$?

posted about 8 hours ago by **Alexander_Andrianov**

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