Beta function

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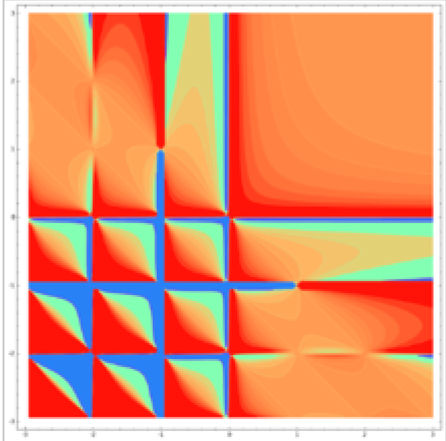
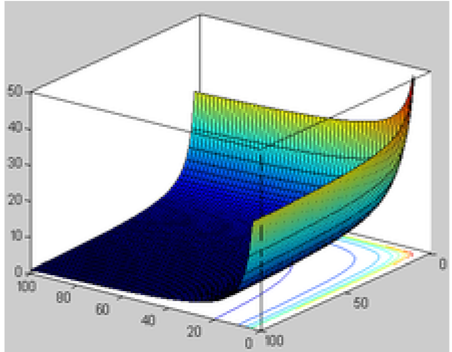
In mathematics, the **beta function**, also called the Euler integral of the first kind, is a special function defined by

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

for $\mathrm{Re}(x), \mathrm{Re}(y) > 0.$



Contour plot of the beta function



A plot of the beta function for positive x and y values

The beta function was studied by Euler and Legendre and was given its name by Jacques Binet; its symbol Β is a Greek capital β rather than the similar Latin capital B.

Contents

Properties

The beta function is symmetric, meaning that

$$\mathrm{B}(x,y)=\mathrm{B}(y,x).$$ [1]

A key property of the Beta function is its relationship to the Gamma function; proof is given below in the section on relationship between gamma function and beta function

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$[1]

When x and y are positive integers, it follows from the definition of the gamma function Γ that:

$$\mathrm{B}(x,y)=\frac{(x-1)!\,(y-1)!}{(x+y-1)!}$$

$$\mathrm{B}(x,y)=2\int_0^{\pi/2}(\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,\mathrm{d}\theta,\qquad \mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0$$[2]

$$\mathrm{B}(x,y)=\int_0^\infty\frac{t^{x-1}}{(1+t)^{x+y}}\,\mathrm{d}t,\qquad \mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0$$[2]

$$\mathrm{B}(x,y)=\sum_{n=0}^{\infty}\frac{\binom{n-y}{n}}{x+n},$$

$$\mathrm{B}(x,y)=\frac{x+y}{xy}\prod_{n=1}^{\infty}\left(1+\frac{xy}{n(x+y+n)}\right)^{-1},$$

The Beta function satisfies several interesting identities, including

$$\mathrm{B}(x,y)=\mathrm{B}(x,y+1)+\mathrm{B}(x+1,y)$$

$$\mathrm{B}(x+1,y)=\mathrm{B}(x,y)\cdot\frac{x}{x+y}$$

$$\mathrm{B}(x,y+1)=\mathrm{B}(x,y)\cdot\frac{y}{x+y}$$

$$\mathrm{B}(x,y)\cdot(t\mapsto t_+^{x+y-1})=(t\to t_+^{x-1})*(t\to t_+^{y-1})\qquad x\geq 1,y\geq 1,$$

$$\mathrm{B}(x,y)\cdot\mathrm{B}(x+y,1-y)=\frac{\pi}{x\sin(\pi y)},$$

where $t\mapsto t_+^x$ is a truncated power function and the star denotes convolution. The lowermost identity above shows in particular $\Gamma(\frac{1}{2})=\sqrt{\pi}$. Some of these identities, e.g. the trigonometric formula, can be applied to deriving the volume of an n-ball in Cartesian coordinates.

Euler's integral for the beta function may be converted into an integral over the Pochhammer contour C as

$$(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})\mathrm{B}(\alpha,\beta)=\int_C t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t.$$

This Pochhammer contour integral converges for all values of α and β and so gives the analytic continuation of the beta function.

Just as the gamma function for integers describes factorials, the beta function can define a binomial coefficient after adjusting indices:

$$\binom{n}{k}=\frac{1}{(n+1)\mathrm{B}(n-k+1,k+1)}.$$

Moreover, for integer n, B can be factored to give a closed form, an interpolation function for continuous values of k:

$$\binom{n}{k}=(-1)^n n!\frac{\sin(\pi k)}{\pi\prod_{i=0}^n(k-i)}.$$

The beta function was the first known scattering amplitude in string theory, first conjectured by Gabriele Veneziano. It also occurs in the theory of the preferential attachment process, a type of stochastic urn process.

Relationship between gamma function and beta function

A simple derivation of the relation can be found in Emil Artin's book **The Gamma Function**, page 18-19.[3]

To derive the integral representation of the beta function, write the product of two factorials as

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty e^{-u}u^{x-1}\,\mathrm{d}u\int_0^\infty e^{-v}v^{y-1}\,\mathrm{d}v\\ &= \int_0^\infty\int_0^\infty e^{-u-v}u^{x-1}v^{y-1}\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Changing variables by $u = f(z,t) = zt$ and $v = g(z,t) = z(1-t)$ shows that this is

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_{z=0}^\infty\int_{t=0}^1 e^{-z}(zt)^{x-1}(z(1-t))^{y-1}|J(z,t)|\,\mathrm{d}t\,\mathrm{d}z\\ &= \int_{z=0}^\infty\int_{t=0}^1 e^{-z}(zt)^{x-1}(z(1-t))^{y-1}z\,\mathrm{d}t\,\mathrm{d}z\\ &= \int_{z=0}^\infty e^{-z}z^{x+y-1}\,\mathrm{d}z\int_{t=0}^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t\\ &= \Gamma(x+y)\mathrm{B}(x,y),\end{aligned}$$

where $|J(z,t)|$ is the absolute value of the Jacobian determinant of $u = f(z,t)$ and $v = g(z,t)$.

The stated identity may be seen as a particular case of the identity for the integral of a convolution. Taking

$f(u) := e^{-u}u^{x-1}1_{\mathbb{R}_+}$ and $g(u) := e^{-u}u^{y-1}1_{\mathbb{R}_+}$, one has:

$$\Gamma(x)\Gamma(y) = \left(\int_{\mathbb{R}} f(u)\mathrm{d}u\right)\left(\int_{\mathbb{R}} g(u)\mathrm{d}u\right) = \int_{\mathbb{R}}(f*g)(u)\mathrm{d}u = \mathrm{B}(x,y)\,\Gamma(x+y).$$

Derivatives

We have

$$\frac{\partial}{\partial x}\mathrm{B}(x,y) = \mathrm{B}(x,y)\left(\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(x+y)}{\Gamma(x+y)}\right) = \mathrm{B}(x,y)(\psi(x) - \psi(x+y)),$$

where $\psi(x)$ is the digamma function.

Integrals

The Nörlund–Rice integral is a contour integral involving the beta function.

Approximation

Stirling's approximation gives the asymptotic formula

$$\mathrm{B}(x,y) \sim \sqrt{2\pi}\frac{x^{x-\frac{1}{2}}\,y^{y-\frac{1}{2}}}{(x+y)^{x+y-\frac{1}{2}}}$$

for large *x* and large *y*. If on the other hand *x* is large and *y* is fixed, then

$$\mathrm{B}(x,y) \sim \Gamma(y)\,x^{-y}.$$

Incomplete beta function

The **incomplete beta function**, a generalization of the beta function, is defined as

$$\mathrm{B}(x;\,a,b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,\mathrm{d}t.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function. The relationship between the two functions is like that between the gamma function and its generalization the incomplete gamma function.

The **regularized incomplete beta function** (or **regularized beta function** for short) is defined in terms of the incomplete beta function and the complete beta function:

$$I_x(a,b) = \frac{\mathrm{B}(x;\,a,b)}{\mathrm{B}(a,b)}.$$

The regularized incomplete beta function is the cumulative distribution function of the Beta distribution, and is related to the cumulative distribution function of a random variable *X* from a binomial distribution, where the "probability of success" is *p* and the sample size is *n*:

$$F(k;n,p) = \Pr(X \le k) = I_{1-p}(n-k,k+1) = 1 - I_p(k+1,n-k).$$

Properties

$$I_0(a,b) = 0$$
$$I_1(a,b) = 1$$
$$I_x(a,1) = x^a$$
$$I_x(1,b) = 1 - (1-x)^b$$
$$I_x(a,b) = 1 - I_{1-x}(b,a)$$
$$I_x(a+1,b) = I_x(a,b) - \frac{x^a(1-x)^b}{aB(a,b)}$$
$$I_x(a,b+1) = I_x(a,b) + \frac{x^a(1-x)^b}{bB(a,b)}$$

Multivariate beta function

The beta function can be extended to a function with more than two arguments:

$$\mathrm{B}(\alpha_1,\ldots,\alpha_K) = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_K)}{\Gamma(\alpha_1+\ldots+\alpha_K)} = \frac{\prod_{i=1}^K \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^K \alpha_i\right)}.$$

This multivariate beta function is used in the definition of the Dirichlet distribution.

Software implementation

Even if unavailable directly, the complete and incomplete beta function values can be calculated using functions commonly included in spreadsheet or computer algebra systems. In Excel, for example, the complete beta value can be calculated from the GammaLn function:

Value = Exp(GammaLn(a) + GammaLn(b) − GammaLn(a + b))

An incomplete beta value can be calculated as:

*Value = BetaDist(x, a, b) * Exp(GammaLn(a) + GammaLn(b) − GammaLn(a + b))*.

These result follow from the properties listed above.

Similarly, *betainc* (incomplete beta function) in MATLAB and GNU Octave, *pbeta* (probability of beta distribution) in R, or *special.betainc* in Python's SciPy package compute the regularized incomplete beta function—which is, in fact, the cumulative beta distribution—and so, to get the actual incomplete beta function, one must multiply the result of *betainc* by the result returned by the corresponding *beta* function.

See also

- Beta distribution
- Binomial distribution
- Jacobi sum, the analogue of the beta function over finite fields.
- Negative binomial distribution
- Yule–Simon distribution
- Uniform distribution (continuous)
- Gamma function
- Dirichlet distribution

References

1. Davis (1972) 6.2.2 p.258
2. Davis (1972) 6.2.1 p.258
3. Artin, Emil. *The Gamma Function* (PDF). pp. 18–19.

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External links

- Hazewinkel, Michiel, ed. (2001), "Beta-function", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- "Evaluation of beta function using Laplace transform". *PlanetMath*.
- Arbitrarily accurate values can be obtained from:

- The Wolfram Functions Site (http://functions.wolfram.com): Evaluate Beta Regularized Incomplete beta (http://functions.wolfram.com/webMathematica/FunctionEvaluation.jsp?name=BetaRegularized)
- danielsoper.com: Incomplete Beta Function Calculator (https://web.archive.org/web/20070120151547/http://www.danielsoper.com:80/statcalc/calc36.aspx), Regularized Incomplete Beta Function Calculator (https://web.archive.org/web/20070120151557/http://www.danielsoper.com:80/statcalc/calc37.aspx)

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