6. The Exponential Family

Exponential Families: Definition

(Caption will be displayed when you start playing the video.)

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Recall from lecture that a family of distribution $\{\mathbf{P}_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Theta\},$ where the parameter space $\Theta \subset \mathbb{R}^k$ is k-dimensional, is called a **k-parameter exponential family** on $\mathbb{R}^q$ if the pmf or pdf $f_{\boldsymbol{\theta}} : \mathbb{R}^q \to \mathbb{R}$ of $\mathbf{P}_{\boldsymbol{\theta}}$ can be written in the form

$$f_{\boldsymbol{\theta}}(\mathbf{y}) = h(\mathbf{y})\,\exp\left(\boldsymbol{\eta}(\boldsymbol{\theta})\cdot\mathbf{T}(\mathbf{y}) - B(\boldsymbol{\theta})\right) \qquad \text{where} \begin{cases} \boldsymbol{\eta}(\boldsymbol{\theta}) = \begin{pmatrix}\eta_1(\boldsymbol{\theta})\\ \vdots \\ \eta_k(\boldsymbol{\theta})\end{pmatrix} & :\mathbb{R}^k \to \mathbb{R}^k \\ \mathbf{T}(\mathbf{y}) = \begin{pmatrix}T_1(\mathbf{y})\\ \vdots \\ T_k(\mathbf{y})\end{pmatrix} & :\mathbb{R}^q \to \mathbb{R}^k \\ B(\boldsymbol{\theta}) & :\mathbb{R}^k \to \mathbb{R} \\ h(\mathbf{y}) & :\mathbb{R}^q \to \mathbb{R}. \end{cases}$$

When $k = q = 1$, this reduces to

$$f_\theta(y) = h(y)\,\exp\left(\eta(\theta)\,T(y) - B(\theta)\right).$$

Note: The following exercises are similar to what will be presented in lecture, but we encourage you to first attempt these yourselves.

Practice: Decomposing the exponent

4/4 points (graded)

For the two following pmfs with one parameter θ that are written in the form

$$f_\theta(y) \;=\; h(y)\,e^{w(\theta,y)},$$

first decompose $w(\theta, y)$ as

$$w(\theta,y) \;=\; \eta(\theta)\,T(y) - B(\theta),$$

then enter the product $\eta(\theta)\,T(y)$ below. Select the distribution that f_θ defines.

1. For $f_\theta(y) \;=\; e^{w(\theta,y)}$ where

$$w(\theta,y) = y\ln(\theta) + (1-y)\ln(1-\theta)$$

and $y = 0, 1,\ \theta \in (0,1)$:

$\eta(\theta)\, T(y) =$ y*ln(theta/(1-theta)) ✔ **Answer:** y*(ln(theta)-ln(1-theta))

$y{\cdot}\ln\left(\frac{\theta}{1-\theta}\right)$

What distribution does the pmf $f_\theta(y)$ define?

- ◯ $\mathcal{N}(\theta, 1)$
- ◯ $\mathcal{N}(1, \theta)$
- ◉ $\mathsf{Ber}(\theta)$
- ◯ $\mathsf{Poiss}(\theta)$
- ◯ none of the above

✔

2. For $f_\theta(y) = \frac{1}{y!} e^{w(\theta, y)}$ where $w(\theta, y) = -\theta + y \ln(\theta)$, and $y = 0, 1, 2, \ldots, \theta \in (0, 1)$:

$\eta(\theta)\, T(y) =$ y*ln(theta) ✔ **Answer:** y*ln(theta)

$y{\cdot}\ln(\theta)$

What distribution does the pmf $f_\theta(y)$ define?

- ◯ $\mathcal{N}(\theta, 1)$
- ◯ $\mathcal{N}(1, \theta)$

- Ber (θ)
- Poiss (θ)
- none of the above

STANDARD NOTATION

Solution:

1. For $f_\theta(y) = e^{w(\theta,y)}$ where $w(\theta,y) = y\ln(\theta) + (1-y)\ln(1-\theta)$ and $y \in \{0,1\}$, $\theta \in (0,1)$:

$$w(\theta,y) = y\ln(\theta) + (1-y)\ln(1-\theta) = y(\ln(\theta) - \ln(1-\theta)) + \ln(1-\theta)$$

Hence, $\eta(\theta)T(y) = y(\ln(\theta) - \ln(1-\theta))$ and $B(\theta) = -\ln(1-\theta)$. Rewriting f_θ:

$$f_\theta(y) = e^{y\ln(\theta)+(1-y)\ln(1-\theta)} = \theta^y(1-\theta)^{(1-y)},$$

we see that f_θ is the pmf of a Bernoulli distribution with parameter θ.

2. For $f_\theta(y) = \frac{1}{y!}e^{w(\theta,y)}$ where $w(\theta,y) = -\theta + y\ln(\theta)$, and $y = 0,1,2,\ldots$, $\theta \in (0,1)$ Hence, $\eta(\theta)T(y) = y\ln(\theta)$ and $B(\theta) = \theta$. Rewriting f_θ

$$f_\theta(y) = \frac{1}{y!}e^{-\theta+y\ln(\theta)} = e^{-\theta}\frac{\theta^y}{y!},$$

we recognize f_θ as the pmf of a Poisson distribution with parameter θ.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Practice: Normal distribution with known variance

1/1 point (graded)

The normal distribution $\mathcal{N}(\theta, 1)$ with with mean θ and known variance $\sigma^2 = 1$ has pdf

$$f_\theta(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}}.$$

Rewrite f_θ in the form

$$f_\theta(y) = h(y)\, e^{\eta(\theta)T(y) - B(\theta)} \qquad \text{where } \eta(\theta), T(y) : \mathbb{R} \to \mathbb{R},$$

and enter the product $\eta(\theta)\, T(y)$ below.

$\eta(\theta)\, T(y) =$ y*theta ✔ **Answer:** y*theta

$y \cdot \theta$

STANDARD NOTATION

Solution:

$$\begin{aligned} f_\theta(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(y^2-2y\theta+\theta^2)}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{\frac{2y\theta-\theta^2}{2}} \end{aligned}$$

$$= \quad h(y)\, e^{\eta(\theta)T(y) - B(\theta)} \qquad \text{where} \begin{cases} \eta(\theta)\, T(y) & = (y)\,(\theta) \\ B(\theta) & = \frac{\theta^2}{2} \\ h(y) & = (e^{-\frac{y^2}{2}})\, / \sqrt{2\pi} \end{cases}$$

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

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