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11. Worked example: decoupling

Change of coordinates

Video

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Note on video: The matrix called $\mathbf{D}$ in the video is not the diagonal matrix of eigenvalues, which is what we call $\mathbf{D}$ in the text.

Recall that a matrix $\mathbf{A}$ is **diagonalizable** if it can be written as

$$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} \quad \text{where} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}.$$

where λ_i and $\mathbf{v}_i$ are corresponding eigenvalue-eigenvector pairs, and the set of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Given a diagonalizable matrix $\mathbf{A}$ the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ can be decoupled as follows. First, define a new vector variable $\mathbf{y}$ by the equation

$$\mathbf{x} = \mathbf{S}\mathbf{y}, \quad \text{where } \mathbf{S} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}.$$

Then substitute this in and rewrite the system in terms of $\mathbf{y}$:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ \mathbf{S}\dot{\mathbf{y}} &= \mathbf{A}\mathbf{S}\mathbf{y} \qquad (\text{since } \mathbf{S} \text{ is a constant matrix}) \\ \mathbf{S}\dot{\mathbf{y}} &= \mathbf{S}\mathbf{D}\mathbf{y} \qquad (\text{since } \mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{D}) \\ \dot{\mathbf{y}} &= \mathbf{D}\mathbf{y} \qquad (\text{ multiply by } \mathbf{S}^{-1} \text{on the left}). \end{aligned}$$

The system in terms of $\mathbf{y}$ is decoupled, because $\mathbf{D}$ is a diagonal matrix.

We can then solve for each coordinate function of $\mathbf{y}$, and then compute $\mathbf{x} = \mathbf{S}\mathbf{y}$.

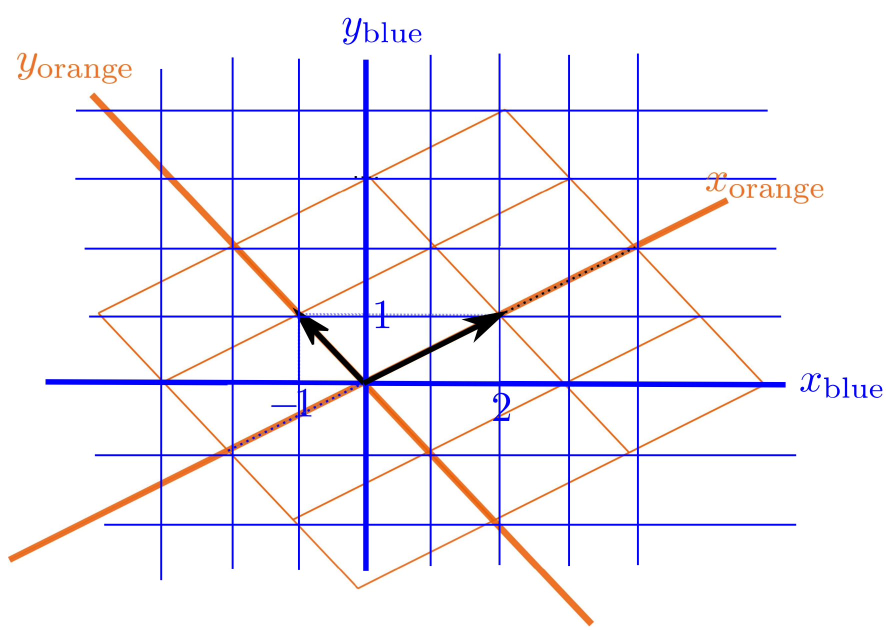
We can view decoupling as rewriting the system in a new coordinate system in which the basis vectors are the eigenvectors of $\mathbf{A}$:

$$\mathbf{x} \quad = \quad \mathbf{S}\mathbf{y}, \qquad \text{where } \mathbf{S} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}.$$

If the eigenvalues are complex, the corresponding eigenvectors will also be complex, i.e., have complex components. All of the above remains formally true, provided we allow all the matrices to have complex entries. This means the new variables $\mathbf{u}$ and $\mathbf{v}$ will be expressed in terms of $\mathbf{x}$ and $\mathbf{y}$ using complex coefficients, and the decoupled system will have complex coefficients. In most branches of science and engineering, this is perfectly acceptable, and one gets in this way a complex decoupling, and the differential equations can still be solved individually. If one insists on using real variables only, decoupling is not possible with complex eigenvalues. This is the assumption that Professor Mattuck used in the video above.

Change of coordinates

3/3 points (graded)



The orange and blue grids form two different coordinate systems of $\mathbb{R}^2$.
The distances between two adjacent grid lines is 1 in both coordinate system.

Consider the vector written as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the orange coordinate system. This is the vector pointing roughly northeast in the figure above. What is this vector written in the blue coordinate system?

(Enter **[a;c]** for the matrix $\begin{pmatrix} a \\ c \end{pmatrix}$. That is, use **semicolon** to separate rows, and **square bracket** the entire vector.)

$\begin{pmatrix} x_{\text{blue}} \\ y_{\text{blue}} \end{pmatrix} =$ [2;1] ✔ **Answer:** [2;1]

Similarly, consider the vector written as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the orange coordinate system. This is the vector pointing roughing northwest in the figure above. What is this vector written in the blue coordinate system?

(Enter **[a;c]** for the matrix $\begin{pmatrix} a \\ c \end{pmatrix}$. That is, use **semicolon** to separate rows, and **square bracket** the entire vector.)

$\begin{pmatrix} x_{\text{blue}} \\ y_{\text{blue}} \end{pmatrix} =$ [-1;1] ✔ **Answer:** [-1;1]

In general, how do we change from the orange coordinate system to the blue coordinate system? In other words, find the matrix $\mathbf{B}$ in the following equation:

$$\begin{pmatrix} x_{\text{blue}} \\ y_{\text{blue}} \end{pmatrix} = \mathbf{B} \begin{pmatrix} x_{\text{orange}} \\ y_{\text{orange}} \end{pmatrix}$$

where the x_{orange} and y_{orange} are the coordinates of the vector in the orange coordinate system, and $x_{\text{blue}}, y_{\text{blue}}$ are the coordinates of the same vector in the blue coordinate system.

(Enter **[a,b;c,d]** for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. That is, use commas to separate entries within the same row and **semicolon** to separate rows, and **square bracket** the entire matrix.)

$\mathbf{B} =$ [2,-1;1,1] ✔ **Answer:** [2,-1;1,1]

Solution:

We need to read the coordinates in the blue system for the required vectors:

- The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the orange system reads $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in the blue system.
- The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the orange system reads $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ in the blue system.

In general, to change coordinates, verify using matrix multiplication that we can use the following equation:

$$\begin{pmatrix} x_{\text{blue}} \\ y_{\text{blue}} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{\text{orange}} \\ y_{\text{orange}} \end{pmatrix}$$

Notice the columns of the matrix on the right hand side are the unit basis vectors of the old orange system written in the new blue system.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

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? change of coordinates
I got it correct on the first attempt only by fumbling around. I guess that I don't understand how the solu... 1

[Staff] Typo in the solution to "Change of coordinates"
The last sentence in the solution to the "Change of coordinates" problem reads: > Notice the columns of ... 2

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