# Fun with Prime Numbers (3)

*Invitation to the Mysterious World of Mathematics*

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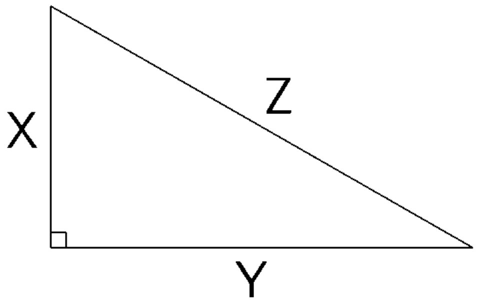
# Pythagoras' Theorem

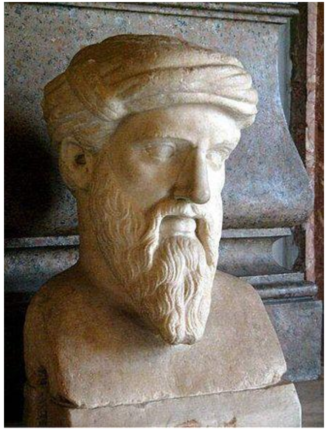
<u>Theorem</u>

For a right triangle with side length X,Y,Z, where X,Y are the legs and Z is the hypotenuse, we have:

$$X^2 + Y^2 = Z^2.$$




$$X^2 + Y^2 = Z^2$$



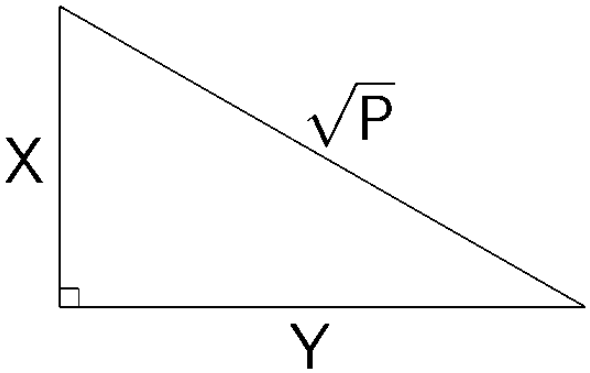
Pythagoras of Samos (570-495 BC)

Reference
https://en.wikipedia.org/wiki/Pythagoras

# Pythagoras' Theorem (2)

- Let us interpret Fermat's theorem on sums of two squares

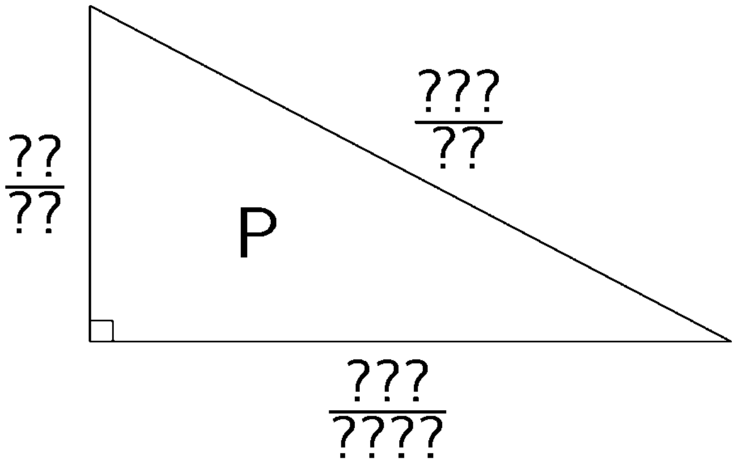
$P \equiv 1 \pmod 4 \iff$ There exists a right triangle with hypotenuse $\sqrt{P}$ whose legs are integers.




$$X^2 + Y^2 = P$$

# Mystery of Triangles

<u>Problem</u> (Congruent Number Problem)
For a prime number P, does there exist a right triangle with area P whose sides are rational numbers? (P is called a congruent number.)



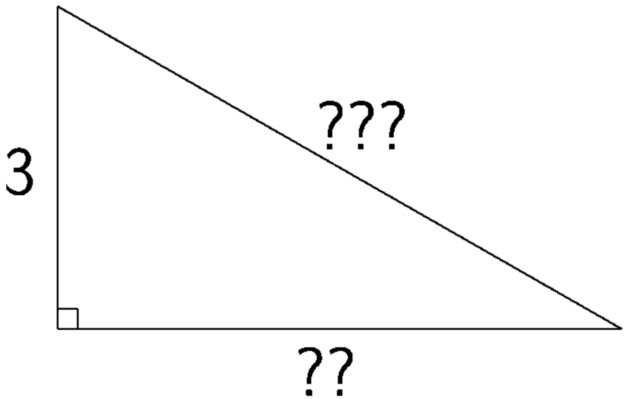

Sides = Rational Numbers

Area = Prime Number

P-triangle

# Mystery of Triangles (2)

(P=5) Does there exist a 5-triangle?
Can we take the length of a leg = 3?




Area $= 5$

5-triangle

# Mystery of Triangles (3)

Does there exist a 5-triangle?
Can we take the length of a leg = 3?

- The answer is 'No'.
- Assume X=3, Y,Z be the sides of a 5-triangle.

$XY \div 2 = 5 \quad 3^2+Y^2 = Z^2.$

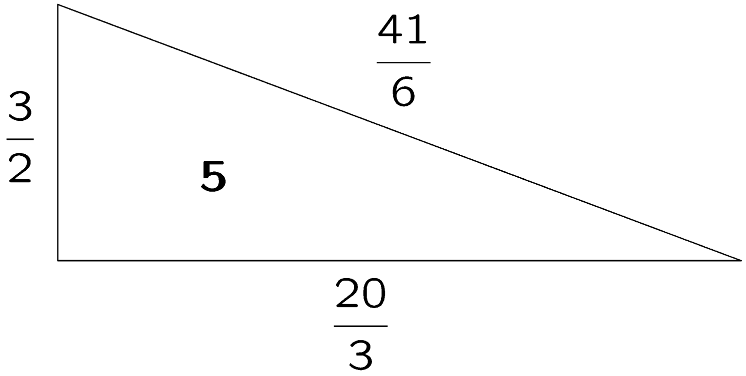
⇨ $Y = 10/3 \quad Z^2 = 9 + 100/9 = 181/9$

⇨ $Z = \sqrt{181}/3$ : not rational number!

# Mystery of Triangles (4)

Does there exist a 5-triangle?

- The answer is 'Yes'.
- It is not so easy to find 5-triangles.
- There are infinitely many 5-triangles.



# Mystery of Triangles (5)

Problem (Congruent Number Problem)
For a prime number P, does there exist a P-triangle? (A right triangle with area P whose sides are rational numbers.)

- Very difficult problem.
- There is a 'conjectural answer' related to certain Reciprocity Laws.