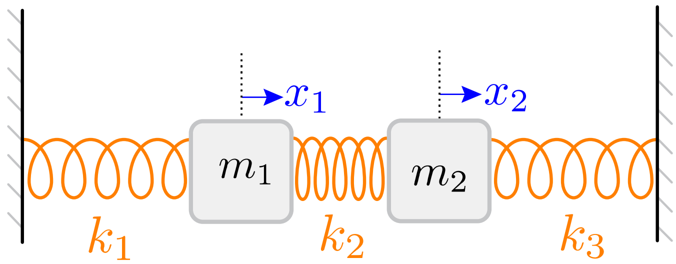
12. Modeling the unforced coupled oscillator

Let us now look at a different system of coupled oscillator. This system is simpler than the one modelling the swaying building with the tuned mass damper inside in the sense that there is no damping, and no external force. But this system does have a third spring.



This system is modeled by two coupled second order constant coefficient ODEs. We will solve this 2×2 system of second order ODEs by first converting it to its companion system. We will find that the eigenvalues are complex and will use the usual procedure of taking the real and imaginary parts of exponential solutions to get a basis of the real solutions.

Modeling :

Let us model the unforced coupled oscillator.

- The extension or compression of spring 1 is given by x_1.
- The extension or compression of spring 2 is given by $x_2 - x_1$.
- And the extension or compression of spring 3 is given by $-x_2$.

Simplifying assumptions

- We are assuming ideal springs, that is, there is no damping.
- The displacements x_1 and x_2 are small compared to the relaxed length of the middle spring. If this assumption is not satisfied, two things may happen: the masses may collide, and the spring forces may no longer be linear in the displacements.

Force on mass 1:
There are two spring forces acting on mass 1: the force F_1 due to spring 1 and the force F_2 due to spring 2. These are given by

$$F_1 = -k_1 x_1; \qquad F_2 = k_2 (x_2 - x_1).$$

Combining these using Newton's second law, we have

$$m_1 \ddot{x}_1 \quad = \quad F_1 + F_2 \quad = \quad -k_1 x_1 + k_2 (x_2 - x_1) \quad = \quad -(k_1 + k_2) x_1 + k_2 x_2.$$

Force on mass 2:
Similarly, there are two forces acting on mass 2: the force F_2 from spring 2, which acts on m_2 with the same magnitude but opposite direction as it acts on m_1, and the force F_3 from spring 3, given by

$$F_3 = -k_3 x_2.$$

Again using Newton's second law, we have

$$m_2\ddot{x}_2 \quad = \quad -F_2 + F_3 \quad = \quad -k_2(x_2 - x_1) - k_3 x_2 \quad = \quad k_2 x_1 - (k_3 + k_2)x_2.$$

The differential equations:
The two equations above together form a second order system:

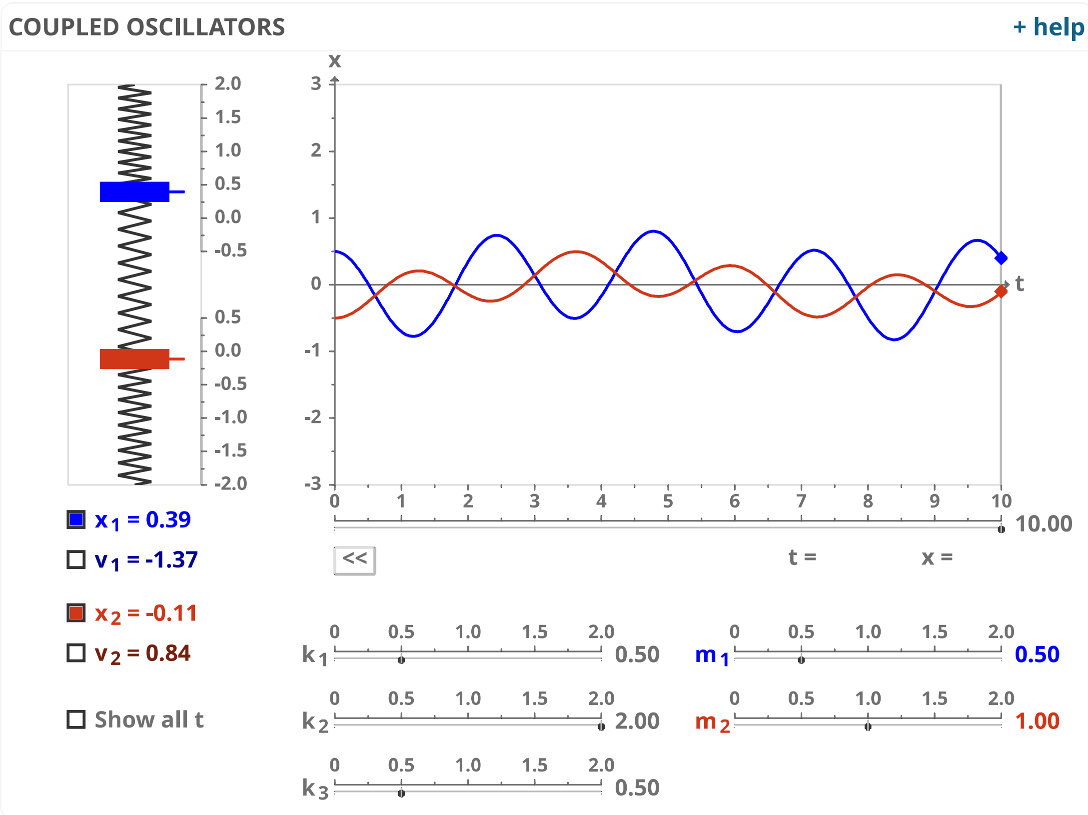
$$\begin{aligned} m_1\ddot{x}_1 &= -(k_2 + k_1)x_1 + k_2 x_2, \\ m_2\ddot{x}_2 &= k_2 x_1 - (k_2 + k_3)x_2, \end{aligned}$$

or equivalently in matrix form:

$$\begin{pmatrix} m_1\ddot{x}_1 \\ m_2\ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(k_2 + k_1) & k_2 \\ k_2 & -(k_2 + k_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Mathlet

The coupled oscillator is simulated in the mathlet below. Adjust the values of the spring constants k_1, k_2, k_3 and the masses m_1, and m_2 and hit the “play" button to see the action!



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