Problem 1: Independent uniform random variables

(3/3 points)

Let X and Y be independent random variables, each uniformly distributed on the interval $[0, 2]$.

1. Find the mean and variance of XY.

$\mathbf{E}[XY] =$ 1 ✔ **Answer:** 1

$\mathbf{var}[XY] =$ 7/9 ✔ **Answer:** 0.77778

2. Find the probability that $XY \geq 1$. Enter a numerical answer.

$\mathbf{P}(XY \geq 1) =$ 0.4034264 ✔ **Answer:** 0.40343

Answer:

1. X and Y are independent, so

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y] = 1.$$

The variance is found using

$$\begin{aligned}
\mathbf{var}(XY) &= \mathbf{E}[(XY)^2] - (\mathbf{E}[XY])^2 \\
&= \mathbf{E}[X^2]\mathbf{E}[Y^2] - (\mathbf{E}[X]\mathbf{E}[Y])^2 \\
&= (\mathbf{var}(X) + \mathbf{E}[X]^2)(\mathbf{var}(Y) + \mathbf{E}[Y]^2) - 1 \\
&= (2^2/12 + 1)(2^2/12 + 1) - 1 \\
&= 7/9.
\end{aligned}$$

2. The joint PDF of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 1/4, & \text{if } 0 \le x, y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

To calculate the probability of interest, we find the area over which we should integrate this joint PDF.

Consider the curve $XY = 1$. Since $0 \le X, Y \le 2$, this curve is defined only for $1/2 \le X \le 2$ and $Y = 1/X$. Since the event of interest is $XY \ge 1$, we are looking for the area above this curve. Hence, we should integrate the joint PDF of X and Y over the range $x \in [1/2, 2]$ and $y \in [1/x, 2]$:

$$\begin{aligned}
\mathbf{P}(XY \geq 1) &= \int_{1/2}^{2} \int_{1/x}^{2} f_{X,Y}(x,y)\, dy\, dx \\
&= \int_{1/2}^{2} \int_{1/x}^{2} \frac{1}{4}\, dy\, dx \\
&= \frac{1}{4} \int_{1/2}^{2} (2 - 1/x)\, dx \\
&= \frac{1}{4}(2x - \ln(x))\Big|_{1/2}^{2} \\
&= \frac{1}{4}(3 - (\ln(2) - \ln(1/2))) \\
&= \frac{1}{4}(3 - \ln(4)) \\
&\approx 0.40343.
\end{aligned}$$

You have used 1 of 2 submissions