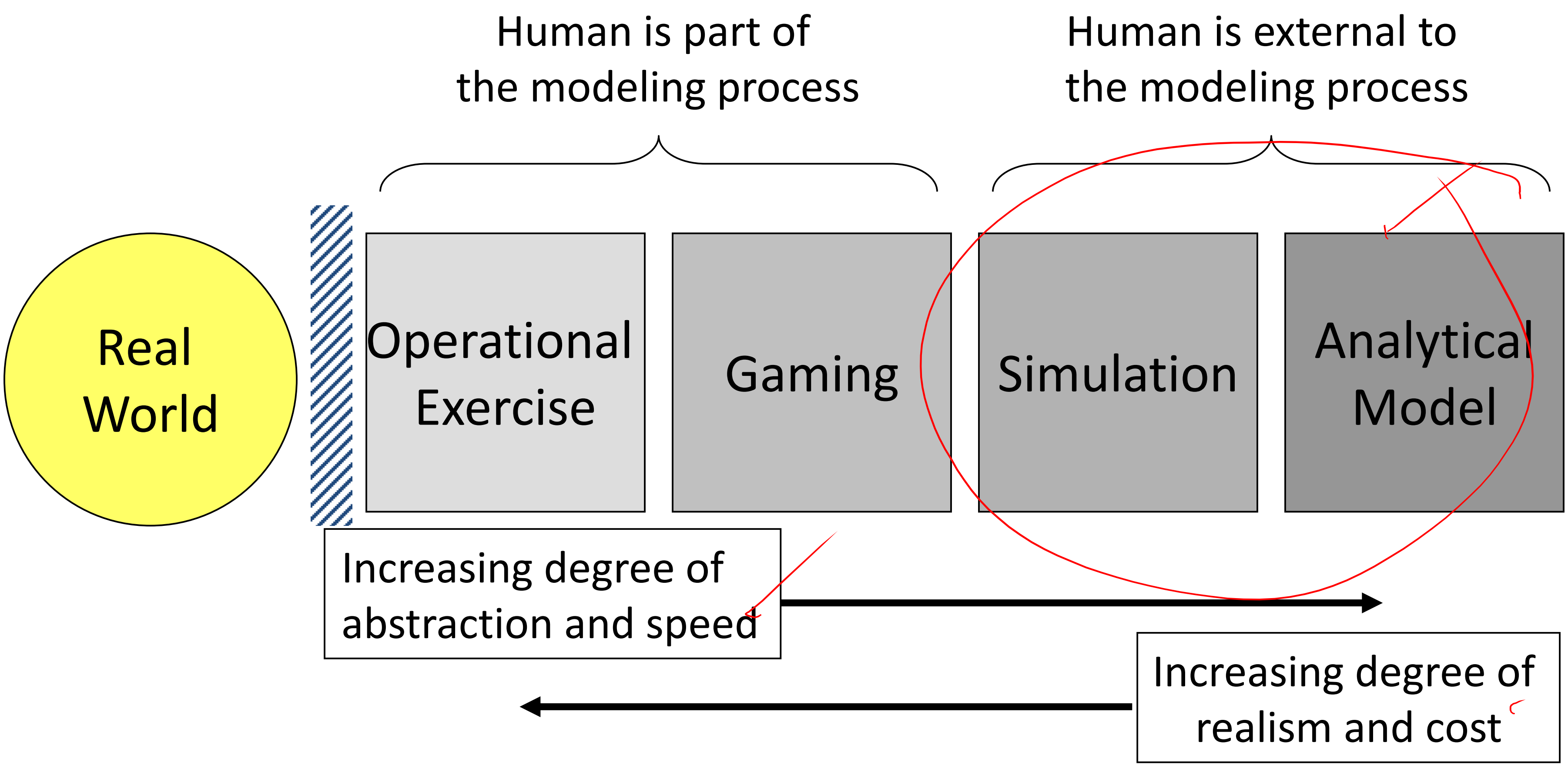
Analytics Basics: Models, Algebra, & Functions

Decision making is at the core of supply chain management.

- How many facilities should I open and where?
- What transportation option should I use?
- How should I trade-off service and cost?
- Where should I source my raw material from?
- How should I share risk with my customers/suppliers?
- How much inventory should I have?
- What is my demand for next year?
- How can I make my supply chain more resilient?

Analytical models are used to make supply chain decisions

Model Classification



Source: Bradley, Hax, Magnanti 1977

Classification of Models

	Strategy Evaluation	Strategy Generation
Certainty	Deterministic Simulation Econometric Models Systems of Simultaneous Equations Input-Output Models	Linear Programming Network Models Integer and MILP Non-Linear Programming Control Theory
Uncertainty	Monte-Carlo Simulation Econometric Models Stochastic Processes Queuing Theory Reliability Theory	Decision Theory Dynamic Programming Inventory Theory Stochastic Programming Stochastic Control Theory

Source: Bradley, Hax, Magnanti 1977

Categories of Mathematical Models

Model Category	Functional Form $f(\cdot)$	Independent Variables	OR/MS Techniques
Descriptive What has happened?	known, well-defined	unknown or uncertain	Simulation, PERT, Queueing Theory, Inventory Models
Predictive What could happen?	unknown, ill-defined	known or under decision maker's control	Regression Analysis, Time Series Analysis, Discriminant Analysis
Prescriptive What should we do?	known, well-defined	known or under decision maker's control	Classic Opt., LP, MILP, CPM, EOQ, NLP,

Source: Ragsdale, 2004

Roadmap for the Course

- Deterministic – Prescriptive Modeling
  - Basic functions & algebra
  - Classical optimization (calculus)
  - Math programming (LPs, IPs, MILPs, & Non-Linear)
- Stochastic/Uncertainty – Predictive & Descriptive
  - Basic probability and distributions
  - Statistical analysis (hypothesis testing)
  - Econometric modeling (regression)
  - Simulation

SCx Approach to Modeling
- Educating Drivers not Mechanics!

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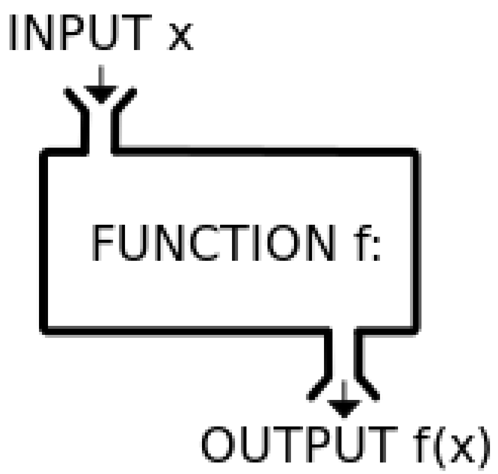
Mathematical Functions

Mathematical Functions



" . . . a relation between a set of inputs and a set of permissible outputs with the property that each input is related to <u>exactly one output</u>."

source: Wikipedia

$$y = f(x)$$

we say:
"f of x" or that "y is a function of x"

If given a value for x, then I can compute the value for y.

Example: $f(x) = x^2$

$x = 2$	then $y = f(2) = 2^2 = 4$
$x = 3.4$	then $y = f(3.4) = 3.4^2 = 11.56$
$x = -2$	then $y = f(-2) = (-2)^2 = 4$

Image by By Wvbailey Public Domain, https://commons.wikimedia.org/w/index.php?curid=10562739

Linear Functions

Typically, constants are denoted by letters from the start of the alphabet (a, b, c, . . .) while variables are letters from the end of the alphabet (x, y, z).

"y changes linearly with x"

dependent variable

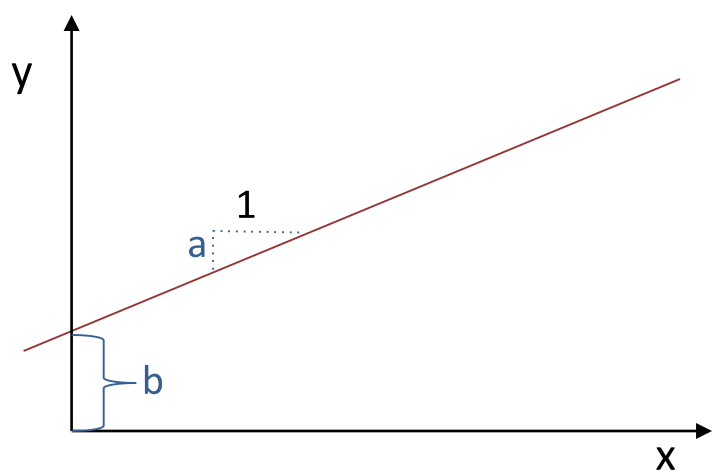
This is the function, f(x).

$$y = \overbrace{ax + b}$$

slope or variable value

independent variable

constant, fixed value, or y-intercept



Examples: Linear Functions

- Truckload Transportation Costs:

  cost = f(distance) = \$200 + 1.35 \$/km * (distance)

- Warehousing Costs

  cost = f(# cases) = €2,500 + 2.5 €/case * (# cases)

- Profit Equation

  profit = f(volume) = (r-c) * v + d

  where:

  r = revenue per item(¥/item)
  c= cost per item (¥/item)
  v = volume sold (items)
  d = fixed cost (¥)

Quadratic Functions

Quadratic Functions

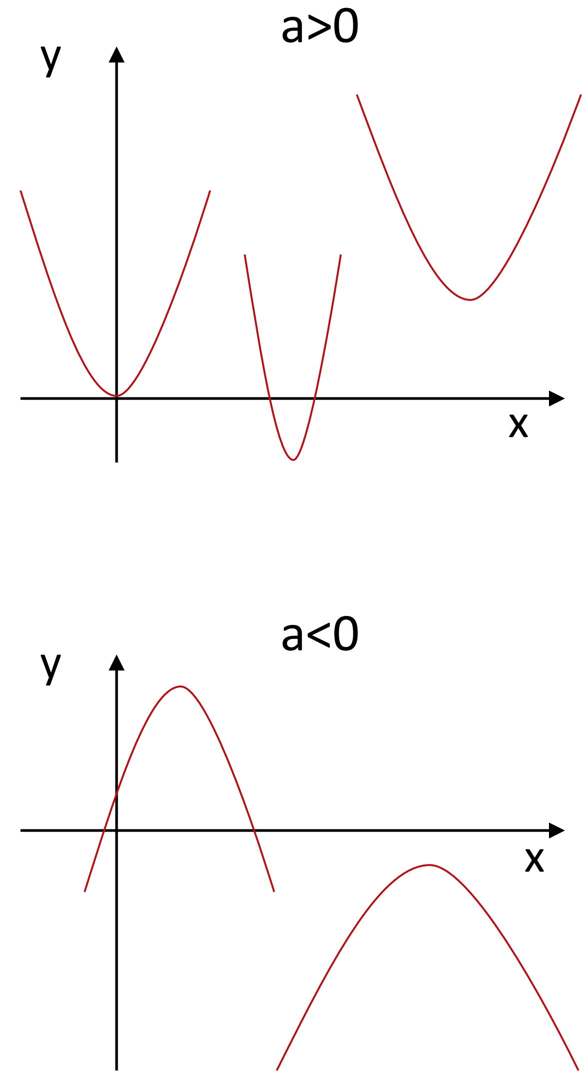
Parabola - Polynomial function of degree 2
where a, b, and c are numbers and a≠0

dependent variable

$$y = \overbrace{ax^2 + bx + c}^{\text{This is the function, } f(x).}$$

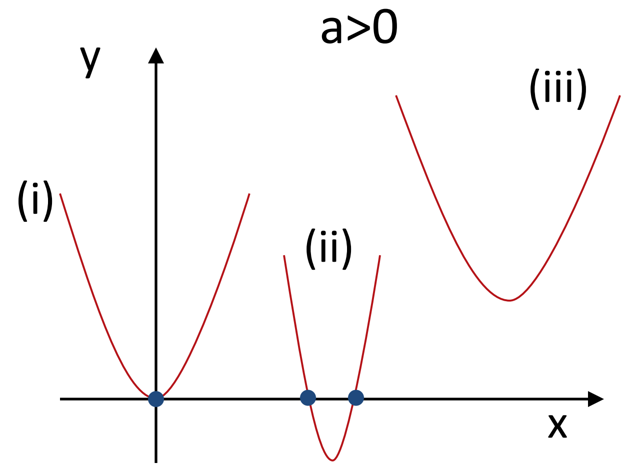
independent variable

constants



- When a>0, the function is convex (or concave up)
- When a<0, the function is concave down

Finding Roots of Quadratic



- The root(s) of a quadratic
  - Values of x for when y=0
  - There can be 2, 1, or 0 roots
- Two methods for finding roots
  - Factoring:
    - Find r_1 and r_2 such that $ax^2+bx+c = a(x-r_1)(x-r_2)$
  - Quadratic equation

(i) $y = 2x^2$
(ii) $y = 2x^2 - 6x +4$
(iii) $y = 3x^2 - 4x + 2$

$$r_1,r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Finding Roots

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i) $y = 2x^2$ so that a=2, b=c=0

$$r_1, r_2 = \frac{-0 \pm \sqrt{0^2 - 4(2)(0)}}{2(2)} = \frac{0}{4} = 0$$

(ii) $y = 2x^2 - 6x + 4$ so that a=2, b=-6, c=4

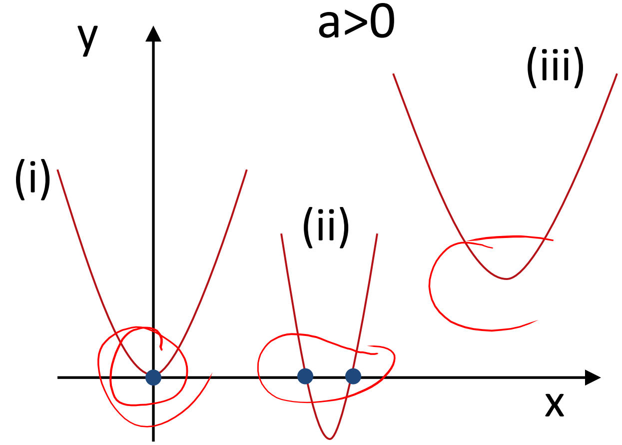
$$r_1, r_2 = \frac{6 \pm \sqrt{(-6)^2 - 4(2)(4)}}{2(2)} = \frac{6 \pm \sqrt{36 - 32}}{4} = \frac{6 \pm 2}{4}$$

$$r_1 = \frac{8}{4} = 2 \qquad r_2 = \frac{4}{4} = 1$$

(iii) $y = 3x^2 - 4x + 2$ so that a=3, b=-4, c=2

$$r_1, r_2 = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(2)}}{2(3)} = \frac{4 \pm \sqrt{16 - 24}}{6} = \frac{4 \pm \sqrt{-8}}{6}$$

r_1, r_2 are complex numbers

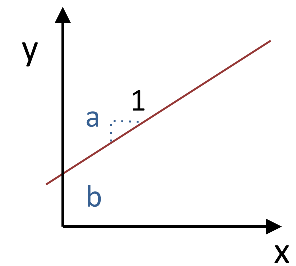


(i) $y = 2x^2$
(ii) $y = 2x^2 - 6x + 4$
(iii) $y = 3x^2 - 4x + 2$

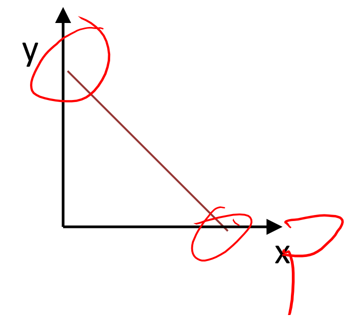
Quadratic Functions in Practice

Quadratic Functions in Practice

- Example: Manufacturing iWidgets – what price to set?
  - Cost of producing iWidgets is a linear function of the number produced, x:
    - cost =f(# made) = 500,000 + 75x



  - Demand for iWidgets is also a linear function of the price, p:
    - unit sales = f(price) = 20,000 – 80p



  - So then:
    - Revenue = (20,000-80p)p = 20,000p – 80p^2
    - Costs = 500,000+75(20,000-80p) = 2,000,000 – 6000p
    - Profit = Revenue – Costs =

      = 20,000p – 80p^2 – (2,000,000 – 6,000p)

      = -80p^2 + 26,000p – 2,000,000

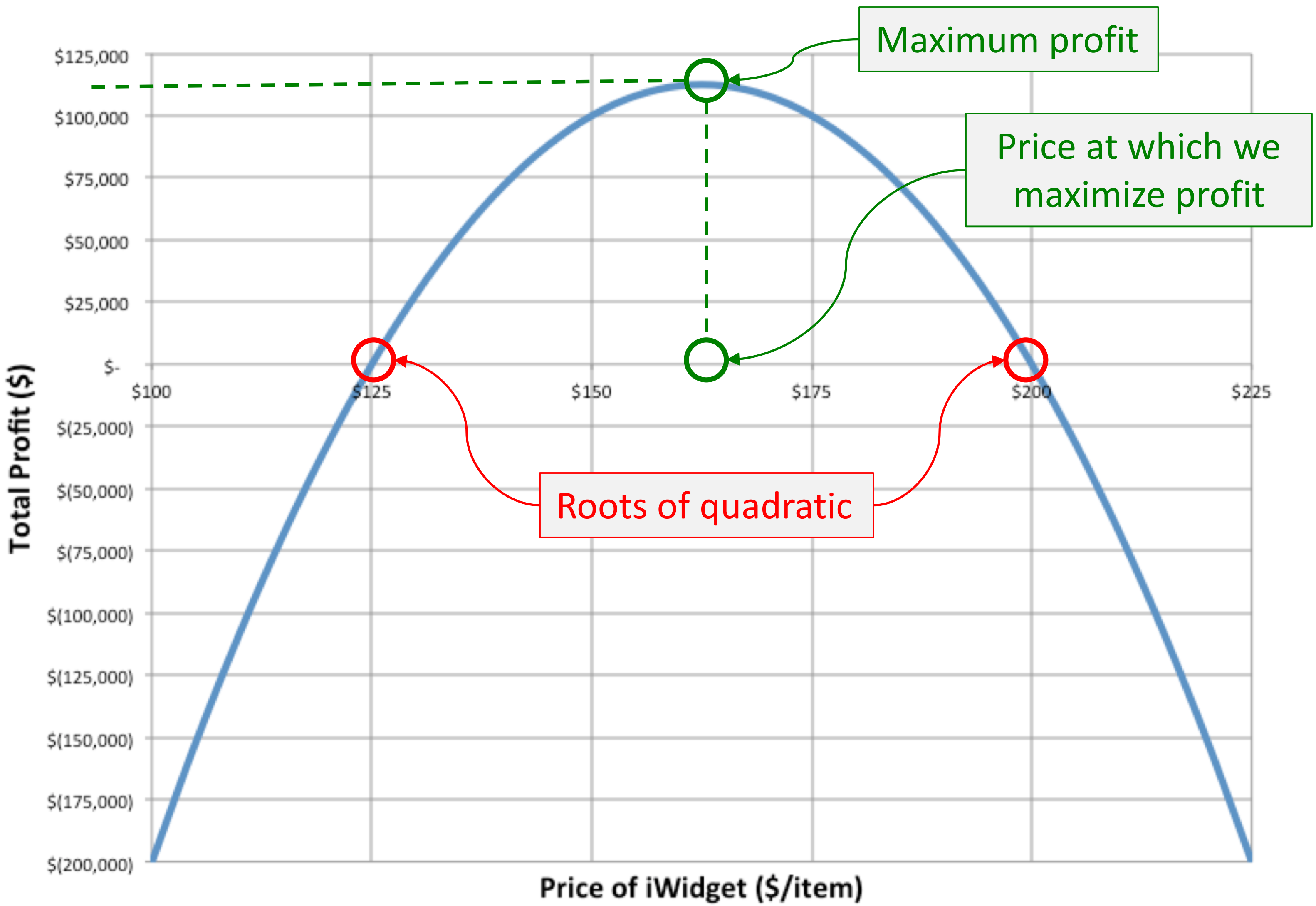
$$\boxed{r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$r_1 = 125$
$r_2 = 200$

What are the root(s) of this equation?

iWidget Example

Profit = $-80p^2 + 26{,}000p - 2{,}000{,}000$

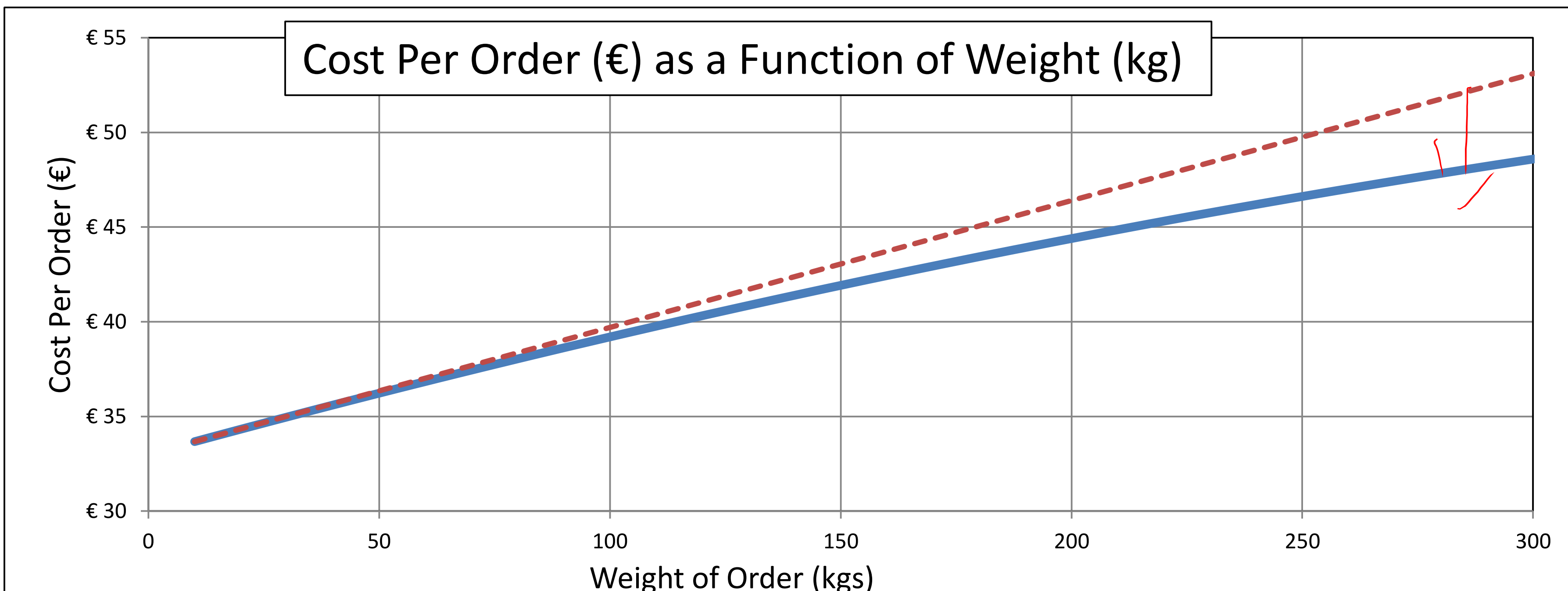


Quadratic Functions in Practice

- Example: Parcel Trucking – impact of weight

  Parcel carriers combine many orders into a single shipment.
  The cost of an individual order is a function of its weight, w.
  However, it is not linear – it is tapering.

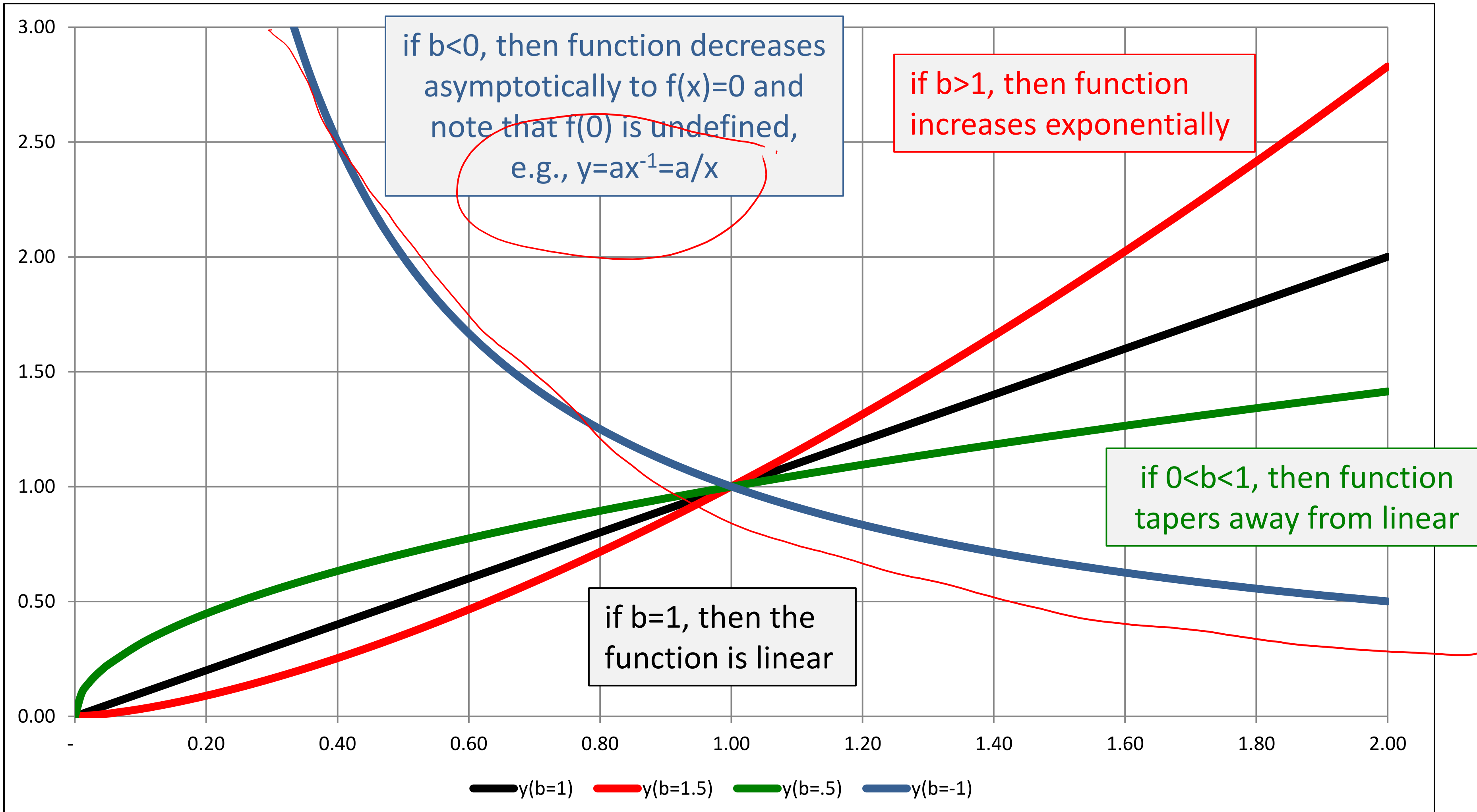
  cost =f(weight) = 33 + 0.067w – 0.00005w^2



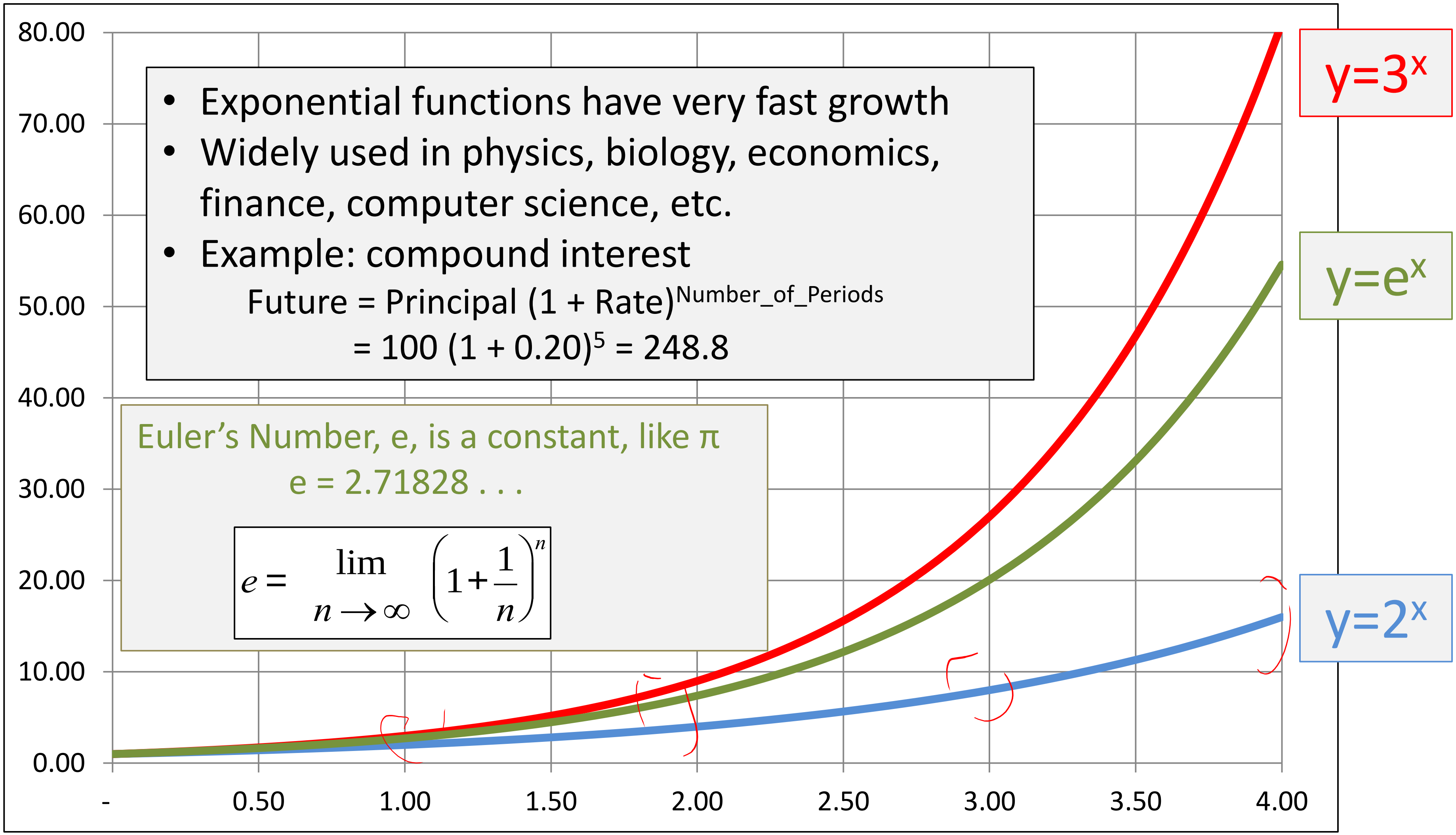
Other Common Functional Forms

Power Function $y=f(x) = ax^b$

The shape of the curve is dictated by the value of b



Exponential Functions $y=ab^x$



- Exponential functions have very fast growth
- Widely used in physics, biology, economics, finance, computer science, etc.
- Example: compound interest

  Future = Principal (1 + Rate)$^{\text{Number\_of\_Periods}}$

  = 100 (1 + 0.20)^5 = 248.8

Euler's Number, e, is a constant, like π

e = 2.71828 . . .

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Logarithms

Other Inverse Relationships

y=ax	←→	x=y÷a
y=a+x	←→	x=y-a

Logarithms

$$y=b^x \leftrightarrow \log_b(y)=x$$

y is the value of b raised to the x^{th} power.

x is the power that I need to raise the base, b, to equal y.

$100 = 10^x$	$\log_{10}(100) = x$	$x=2$
$5 = 10^x$	$\log(5) = x$	$x\cong 0.7$
$1 = e^x$	$\log_e(1) = \ln(1)=x$	$x=0$
$e = e^x$	$\ln(e) = x$	$x=1$

Properties of Logarithms

- $\log(xy) = \log(x) + \log(y)$
- $\log(x/y) = \log(x) - \log(y)$
- $\log(x^a) = a \log(x)$

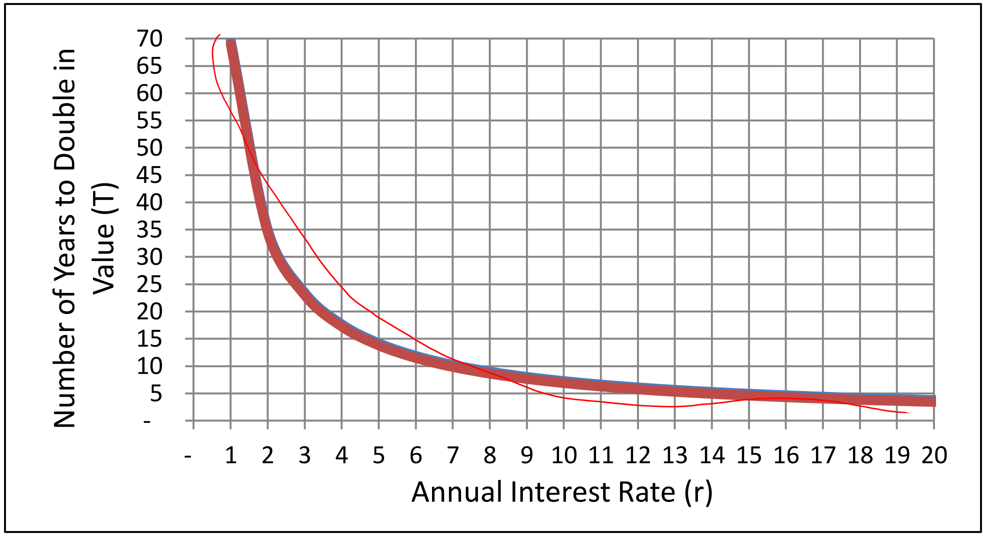
Examples:

- $\ln(3*5) = \ln(3)+\ln(5) = 2.71$
- $\ln(12/7) = \ln(12) - \ln(7) = 0.54$
- $\ln(3^6) = 6 \ln(3) = 6.59$
- $\log(3*5^2) = \log(3) + 2 \log(5) = 1.88$

Practical Example: Doubling Time

- You have invested a sum of money that has an interest rate of 7% annually. How many years, T, will it take to double in value?
  - We know that $F=P(1+r)^n$ and we want to find the n where $F=2P$
  - $F=2P=P(1+r)^n$ which reduces to $2=(1+r)^n=(1.07)^n$
  - We can transform this by taking the ln or log of both sides:
    - $\ln(2) = n \ln(1.07)$
    - Rearranging gives us: $n = \ln(2) / \ln(1.07) = 0.693 / 0.182 = 10.24 = T$
    - We could also use $\log_{10}$ where $T = \log(2)/\log(1.07) = 10.24$
  - The investment will double in value in 10.24 years.

- Can we come up with a general equation or approximation?
  - We know that $T = \ln(2) / \ln(1 + r)$
  - Plotting this for $T=f(r)$ . . . looks like $T=ar^{-1}=a/r$
  - Turns out $T \approx 70/r$

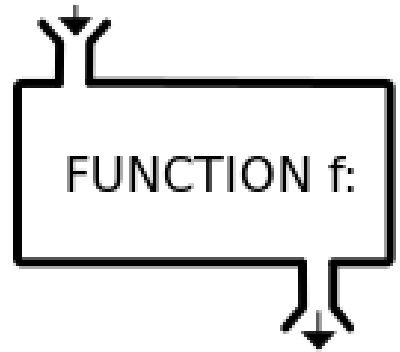


Multivariate Functions

Multivariate Functions

These are just functions with more than one independent variable.

INPUT $(x_1, x_2, \ldots x_n)$



OUTPUT $y = f(x_1, x_2, \ldots x_n)$

still just a single output

we still just say: "y is a function of $x_1, x_2, \ldots x_n$)"

$$y = f(x_1, x_2, \ldots x_n)$$

Example: $f(x_1, x_2) = x_1 + 2x_2 + 5x_1x_2$

$x_1 = 2, x_2 = 4$ then $y = f(2,4) = 2 + 2(4) + 5(2)(4) = 50$

$x_1 = -1, x_2 = 0$ then $y = f(-1,0) = -1 + 2(0) + 5(-1)(0) = -1$

$x_1 = 0, x_2 = -½$ then $y = f(0, -½) = 0 + 2(-½) + 5(0)(-½) = -1$

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Examples: Multivariate Functions

- Parcel Trucking – impact of weight & distance

  Parcel carriers combine many orders into a single shipment. The cost of an individual order is a function of its weight, w, and the distance.

  $$\text{cost} = f(\text{weight, distance}) = c_1 + c_2 w + c_3 w^2 + c_4 d + c_5 d^2 + c_6 dw$$

- Total Logistics Cost Equation

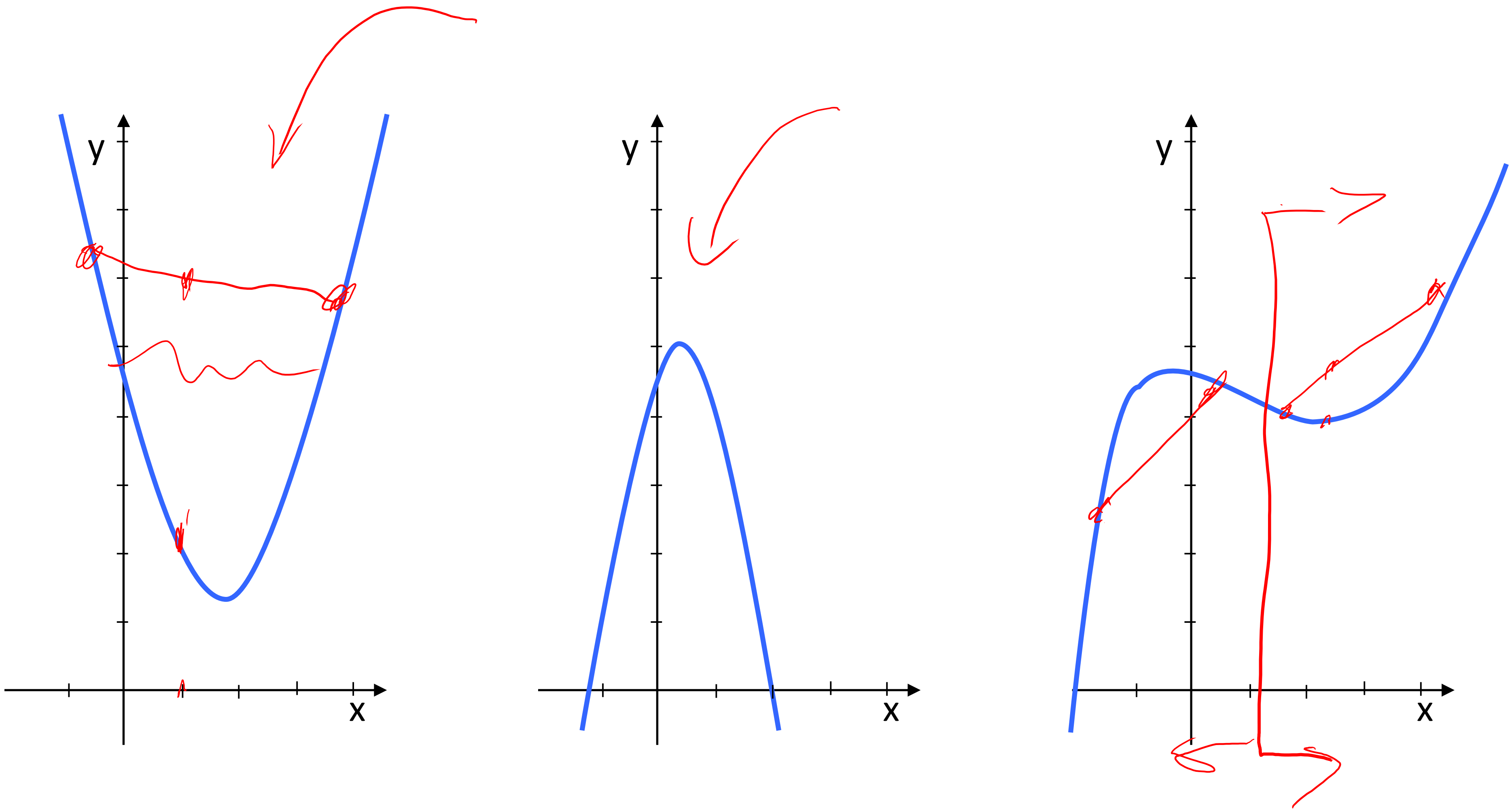
  $\text{cost} = f(\text{Demand, Order Cost, Order Size, }) = cD + AD/Q$ where:

  D = annual demand (items)
  c = cost per item (¥/item)
  A = cost per order (¥/order)
  Q = order size (items/order)

Properties of Functions

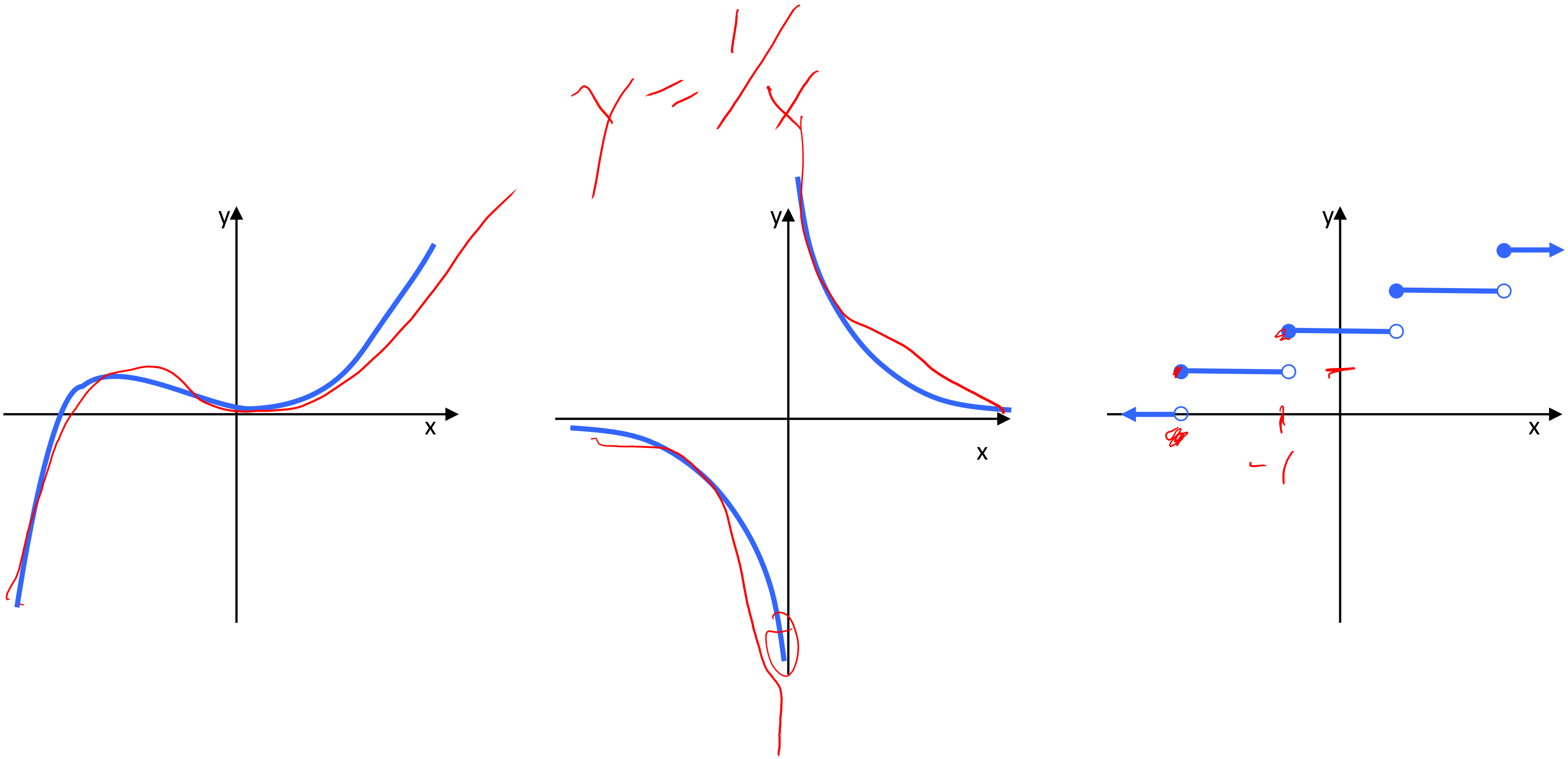
Properties of a Function: Convexity

A function is convex if it "holds water"



Properties of a Function: Continuity

A function is continuous if you can draw it without lifting pen from paper!



Key Points from Lesson

Key Points from Lesson (1/2)

- Different models used for different purposes
  - Descriptive – what has happened?
  - Predictive – what could happen?
  - Prescriptive – what should we do?
- Functions $y=f(x)$
  - Linear functions where $y= ax + b$
  - Quadratic functions where $y= ax^2 + bx + c$
  - Power functions where $y= ax^b$
  - Exponential functions where $y= ab^x$

Key Points from Lesson (2/2)

- Logarithms
  - $y=b^x$ is equivalent to $\log_b(y) = x$
  - Natural log $\ln(y) = \log_e(y)$
- Multivariate functions $y=f(x_1, x_2, \ldots x_n)$
  - Multiple inputs still lead to single output value
- Properties of functions
  - Convexity – does the function "hold water"?
  - Continuity – can I draw the function without lifting my pencil

Questions, Comments, Suggestions?
Use the Discussion Forum!

"Wilson – realizing he is asymptotic to the door"
Yankee Golden Retriever Rescued Dog
(www.ygrr.org)