5. Application: finding a basis for the span of any collection of vectors

Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$, how can one compute a basis of $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$?

Algorithm for computing a basis for $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$

1. Form the matrix $\mathbf{A}$ whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

2. Find a basis for $\mathrm{CS}(\mathbf{A})$ (by using a row echelon form as discussed earlier).

Example 5.1 Find a basis for the span of the following vectors:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Start by forming the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Find the row echelon form of $\mathbf{A}$ using Gaussian elimination:

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The pivot columns are the first and third columns, therefore a basis is given by

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 3 \\ 1 \end{pmatrix}.$$

Finding a basis for a span

1/1 point (graded)

At least one of the following sets of vectors is a basis for the span of the following three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

Find one basis.

- (selected) $\mathbf{v}_1, \mathbf{v}_2.$ ✔
- $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3.$
- $\mathbf{v}_1, \mathbf{v}_3.$ ✔
- $\mathbf{v}_1.$
- $\mathbf{v}_2.$
- $\mathbf{v}_1, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$ ✔

Solution:

We first put $\mathbf{A} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$ into row echelon form:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the column space.

Observe that

$$\mathbf{v}_2 = \mathbf{v}_1 + \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$
$$\mathbf{v}_3 = \mathbf{v}_1 + 2\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

In particular, $\mathbf{v}_2 = \frac{1}{2}(\mathbf{v}_3 + \mathbf{v}_1)$. Thus since $\mathbf{v}_2$ can be described in terms of $\mathbf{v}_1$ and $\mathbf{v}_3$, an equally valid basis is $\mathbf{v}_1$ and $\mathbf{v}_3$.

These linear relationships say even more. For example all vectors in the space can be described as a span of $\mathbf{v}_1$ and $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. Thus $\mathbf{v}_1$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is also a basis, and in particular, so is $\mathbf{v}_1$ and $\mathbf{v}_1 - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

None of the other options serve as bases because they assume dimensions different from 2.

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