The Vitali Sets

The most famous examples of non-measurable sets are the Vitali sets, named in honor of the Italian mathematician Giuseppe Vitali, in 1905.

Vitali proved that there is no way of extending λ to apply to a Vitali set while preserving all three of Non-Negativity, Countable Additivity and Uniformity.

I'll start by giving you an intuitive sense of why the Vitali sets are not measurable, and then go through the proof.

The Intuitive Picture

In Lecture 6 we considered a thought experiment: imagine that God has selected a positive integer, but that you have no idea which. What should your credence be that God selected the number 17? More generally: what should your credence be that She selected a given number k?

Our discussion revealed that as long as you think that your credence distribution ought to be *uniform*—as long as you think that the same answer should be given for each value of k — it is a consequence of Countable Additivity that your credences must remain undefined.

For if $p(\text{God selects } k) = 0$ for each k, Countable Additivity entails that $p(\text{God selects a positive integer}) = 0$, which is incorrect. And if $p(\text{God selects } k) = r$ for each k, where r is a positive real number, Countable Additivity entails that $p(\text{God selects a positive integer}) = \infty$, which is also incorrect. (We also saw that there are reasons of principle why appealing to infinitesimals won't save the day, but we won't have to worry about infinitesimals here, since we're presupposing Non-Negativity.)

The moral of our thought experiment is that in the presence of Countable Additivity, there is no such thing as a uniform probability distribution over a countably infinite set of (mutually exclusive and jointly exhaustive) possibilities.

The proof of Vitali's Theorem is a version of this same idea.

We'll partition $[0, 1)$ into a countable infinity of "Vitali Sets", and use Uniformity to show that these sets must all have the same measure, if they have a measure at all. We'll then use Countable Additivity and Non-Negativity to show that the Vitali Sets cannot have a measure.

When I described the thought experiment of Lecture 6, I didn't say anything about the selection procedure God uses to pick a positive integer. In particular, I didn't clarify whether there could be a selection procedure that doesn't have the feature that some numbers are more likely to be selected than others.

You can think of Vitali's Theorem as delivering one such procedure.

Here's how it would work. God starts by partitioning $[0, 1)$ into a countable infinity of Vitali Sets, and assigns each member of the partition a distinct positive integer. (Proving that a partition of the right kind exists requires the Axiom of Choice, so God would have to rely on Her super-human capabilities to identify a suitable partition.) God then selects a real number in $[0, 1)$, using the Standard Coin-Toss Procedure of Lecture 7.1.4.1. Finally, God uses the real number she gets as output from the Coin-Toss Procedure to select the integer that corresponds to the member of the partition to which the real number belongs.

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