MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Bookmarks

Bookmark

Problem 4: Ships

(7/7 points)

All ships travel at the same speed through a wide canal. Each ship takes t days to traverse the length of the canal. Eastbound ships (i.e., ships traveling east) arrive as a Poisson process with an arrival rate of λ_E ships per day. Westbound ships (i.e., ships traveling west) arrive as an independent Poisson process with an arrival rate of λ_W ships per day. A pointer at some location in the canal is always pointing in the direction of travel of the most recent ship to pass it.

In each part below, your answers will be algebraic expressions in terms of $\lambda_E, \lambda_W, x, t, v$ and/or k. Enter 'LE' for λ_E and 'LW' for λ_W, and use 'exp()' for exponentials. Do **not** use 'fac()' or '!' for factorials; instead, calculate out the numerical value of any factorials. Follow standard notation .

For parts (1) and (2), suppose that the pointer is currently pointing west.

1) What is the probability that the next ship to pass will be westbound?

(LW) / (LE+LW) ✔ **Answer:** LW/(LW+LE)

2) Determine the PDF, $f_X(x)$, of the remaining time, X, until the pointer changes direction.

For $x \geq 0$, $f_X(x) =$ (LE)*exp(-(LE)*x) ✔ **Answer:** LE*exp(-LE*x)

For the remaining parts of this problem, we make no assumptions about the current direction of the pointer.

3) What is the probability that an eastbound ship does not pass by any westbound ships during its eastward journey through the canal?

exp(-(LW)*(2*t)) ✔ **Answer:** exp(-2*LW*t)

4) Starting at an arbitrary time, we monitor the cross-section of the canal at some fixed location along its length. Let V be the amount of time that will have elapsed (since we began monitoring) by the time we observe our seventh eastbound ship. Determine the PDF of V.

For $v \geq 0$, $f_V(v) =$ (LE)^7*v^6*exp(-(LE)*v ✔ **Answer:** LE^7*v^6*exp(-LE*v)/720

5) What is the probability that the next ship to arrive causes a change in the direction of the pointer?

2*(LE*LW)/(LE+LW)^2 ✔ **Answer:** 2*LW*LE/((LW+LE)^2)

6) If we begin monitoring a fixed cross-section of the canal at an arbitrary time, determine the probability mass function $p_K(k)$ for K, the total number of ships we observe up to and including the seventh eastbound ship we see. The answer will be of the form $p_K(k) = \binom{a}{6} \cdot b$, for suitable algebraic expressions in place of a and b.

Solved problems

Additional theoretical material

Problem Set 9
Problem Set 9 due May 11, 2016 at 23:59 UTC

Unit summary

Unit 10: Markov chains

Exit Survey

$a =$ k-1 ✔ **Answer:** k-1

$b =$ ((LE)^7*(LW)^(k-7))/(LE+LW)^k ✔ **Answer:** LE^7*LW^(k-7)/(LE+LW)^k

Answer:
In parts (1) and (2), we are given that the last ship to pass the pointer was westbound.

1. The direction of the next ship is independent of previous ships. Therefore, we are simply looking for the probability that the next arrival is westbound. Considered the Poisson process resulting from merging the eastbound and westbound Poisson processes, which are independent. This merged process has rate $\lambda_E + \lambda_W$, and we are looking for the probability that any given arrival came from the westbound process, which is

$$\mathbf{P}(\text{next ship is westbound}) = \frac{\lambda_W}{\lambda_E + \lambda_W}.$$

2. The pointer will change direction on the next arrival of an eastbound ship. Because eastbound arrivals occur according to a Poisson process with rate λ_E, the remaining time until such an arrival is exponential with parameter λ_E, and so

$$f_X(x) = \lambda_E e^{-\lambda_E x}\ , \quad x \geq 0.$$

3. Suppose that an eastbound ship enters the canal at time t_0. During its traversal, this ship will meet any westbound ship that entered the canal between times $t_0 - t$ and $t_0 + t$. Thus, the desired probability is the probability that there are no westbound ship arrivals during an interval of length $2t$, and using the Poisson PMF, it is equal to $e^{-\lambda_W \cdot (2t)}$.

4. The time until we see the seventh eastbound ship is an Erlang random variable of order 7, with parameter λ_E. Thus the PDF of V is

$$f_V(v) = \frac{\lambda_E^7 v^6 e^{-\lambda_E v}}{6!}, \quad v \geq 0.$$

5. This is the probability that a westbound ship passed last (making the pointer point west) and an eastbound ship will pass next, or an eastbound ship passed last (making the pointer point east) and a westbound ship will pass next. As in part (1), consider the Poisson process obtained by merging the two independent eastbound and westbound processes. Any given arrival in the merged process is eastbound with probability $\frac{\lambda_E}{\lambda_E+\lambda_W}$ and is westbound with probability $\frac{\lambda_W}{\lambda_E+\lambda_W}$. Thus, the desired probability is

$$\left(\frac{\lambda_E}{\lambda_E + \lambda_W}\right)\left(\frac{\lambda_W}{\lambda_E + \lambda_W}\right) + \left(\frac{\lambda_W}{\lambda_E + \lambda_W}\right)\left(\frac{\lambda_E}{\lambda_E + \lambda_W}\right) = \frac{2\lambda_E \lambda_W}{(\lambda_E + \lambda_W)^2}.$$

6.

Consider again the merged process. We view each ship arrival as an independent Bernoulli trial, and each eastbound ship as a “success". Each trial is a success with probability $p = \lambda_E/(\lambda_E + \lambda_W)$. We are interested in the PMF of the number of trials until the seventh success. This is a Pascal PMF of order seven, with parameter p:

$$\binom{k-1}{6}\left(\frac{\lambda_E}{\lambda_E+\lambda_W}\right)^7\left(\frac{\lambda_W}{\lambda_E+\lambda_W}\right)^{k-7}, \qquad k = 7, 8, \ldots.$$

You have used 3 of 3 submissions

DISCUSSION

Click “Show Discussion" below to see discussions on this problem.