SOLUTION 10

Problem 1. Let A be an $n \times n$ matrix, then the distinct eigenvalues of A are linearly independent.

a. Let $\{v_1, \ldots, v_k\}$ be a set of eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of A.

Suppose that $k = 1$, then we have one eigenvector v_1. Since eigenvectors are nonzero, the equation $c_1 v_1 = 0$ has only the trivial solution $c_1 = 0$. Thus, the set $\{v_1\}$ is linearly independent.

b. Because of part a, we can assume that the vectors $\{v_1, \ldots, v_p\}$ are linearly independent, where $1 \leq p < k$.

- If v_{p+1} is linearly dependent on the vectors $\{v_1, \ldots, v_p\}$, then v_{p+1} is a linear combination of the vectors $\{v_1, \ldots, v_p\}$. Therefore,

$$c_1 v_1 + \cdots + c_p v_p = v_{p+1}. \tag{0.1}$$

- We multiply both sides of (0.1) by A, since Ax is a linear transformation we have

$$c_1 A v_1 v_1 + \cdots + c_p A v_p = A v_{p+1}. \tag{0.2}$$

- Since $v_1, \ldots, v_p$ are eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A, (0.2) can be written as

$$c_1 \lambda_1 v_1 + \cdots + c_p \lambda_p v_p = \lambda_{p+1} v_{p+1}. \tag{0.3}$$

- Subbing in (0.1) into (0.3) and subtract to the other side gives

$$c_1(\lambda_1 - \lambda_{p+1}) v_1 + \cdots + c_p(\lambda_p - \lambda_{p+1}) v_{p+1} = 0. \tag{0.4}$$

c. Since the vectors $\{v_1, \ldots, v_p\}$ are linearly independent (0.4) must have only the trivial solution. Moreover, the eigenvalues are distinct, thus $\lambda_1 - \lambda_{p+1} \neq 0, \ldots, \lambda_p - \lambda_{p+1} \neq 0$. It follows that $c_1 = \cdots = c_p = 0$. However, (0.1) then implies that $v_{p+1} = 0$, which contradicts v_{p+1} being an eigenvector. Therefore, our original assumption that v_{p+1} is linearly dependent is false, and it follows that the vectors $\{v_1, \ldots, v_p, v_{p+1}\}$ are linearly independent.

Remark: This concludes the proof by induction. We established our base case in part a. In part b-c we have shown that if the result holds for the vectors $v_1, \ldots, v_p$ then it must also hold for the vectors $v_1, \ldots, v_p, v_{p+1}$. Therefore, all eigenvectors $v_1, \ldots, v_k$ corresponding to distinct eigenvalues must be linearly independent.

Problem 2. Let A be an $n \times n$ matrix with n distinct eigenvalues. Then the eigenvectors of A form a basis for $\mathbb{R}^n$.

Proof. From problem 1 we know that the eigenvectors $v_1, \ldots, v_n$ corresponding to the n distinct eigenvalues will be linearly independent. Therefore, we have n linearly independent vectors in R^n, which therefore form a basis for R^n. $\square$

> **Remark:** I am assuming that A is a real matrix and all the eigenvalues are real. This is not always the case, but can be guaranteed when the matrix A is symmetric. If the eigenvalues are not real, then the corresponding eigenvectors will be complex and ultimately form a basis for $\mathbb{C}^n$.

Problem 3. Let $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$.

a. The eigenvalues of A are the roots of the characteristic polynomial

$$\begin{aligned} \det(A - \lambda I) &= (4-\lambda)(1-\lambda) - 10 \\ &= \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1). \end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 6$.

b.

- $A - \lambda_1 I = \begin{bmatrix} 5 & 5 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, an eigenvector of A corresponding to λ_1 is $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- $A - \lambda_2 I = \begin{bmatrix} -2 & 5 \\ 2 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 5 \\ 0 & 0 \end{bmatrix}$, an eigenvector of A corresponding to λ_2 is $v_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

c. Since the eigenvalues are distinct, the eigenvectors of A are linearly independent. Therefore, we have two linearly independent vectors in $\mathbb{R}^2$, it follows that they must form a basis for $\mathbb{R}^2$.

Problem 4.

a. Let $S = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$.

b. Note that $S^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -5 \\ 1 & 1 \end{bmatrix}$. Moreover,

$$\begin{aligned} S^{-1}AS &= \frac{1}{7}\begin{bmatrix} 2 & -5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 30 \\ 1 & 12 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} -7 & 0 \\ 0 & 42 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}. \end{aligned}$$

c. Let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we want to find scalars c_1 and c_2 such that

$$x = c_1 v_1 + c_2 v_2.$$

To this end, we compute

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{2}{7} \end{bmatrix}.$$

Therefore, $[x]_\beta = \begin{bmatrix} -\frac{3}{7} \\ \frac{2}{7} \end{bmatrix}$. Moreover, we find the image of x under T by noting

$$\begin{aligned} T(x) = T\left(c_1 v_1 + c_2 v_2\right) &= c_1 T(v_1) + c_2 T(v_2) \\ &= c_1\left(\lambda_1 v_1\right) + c_2\left(\lambda_2 v_2\right) \\ &= -\frac{3}{7}\left(-\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \frac{2}{7}\left(6\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right). \end{aligned}$$

Problem 5.

a. We will show that S is invertible by finding the inverse of S:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{array}\right].$$

We see that S has 3 pivots and is therefore invertible, and

$$S^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

b. Compute $B = S^{-1} A S$

$$\begin{aligned} &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 4 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & \frac{17}{4} & \frac{3}{4} \\ 0 & \frac{31}{4} & \frac{5}{4} \\ 0 & \frac{7}{4} & -\frac{3}{4} \end{bmatrix}. \end{aligned}$$

The eigenvalues of B must be the same as the eigenvalues of A, since similarity transformations preserve eigenvalues.

c. One eigenvalue of A is $\lambda_1 = 2$, the other two eigenvalues of A are stored in the smaller submatrix

$$\hat{A} = \begin{bmatrix} \frac{31}{4} & \frac{5}{4} \\ \frac{7}{4} & -\frac{3}{4} \end{bmatrix}.$$

We can find the eigenvalues of $\hat{A}$ by computing the roots of the characteristic polynomial

$$\begin{aligned} \det\left(\hat{A} - \lambda I\right) &= \left(\frac{31}{4} - \lambda\right)\left(\frac{-3}{4} - \lambda\right) - \frac{35}{16} \\ &= \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1). \end{aligned}$$

Therefore, the other two eigenvalues of A are $\lambda_2 = 8$ and $\lambda_3 = -1$.