11. Chi-Squared Test for a Family of Discrete Distributions

In the problems on this page, you will apply the χ^2 goodness of fit test to determine whether or not a sample has a binomial distribution.

So far, we have used the χ^2 test to determine if our data had a categorical distribution with specific parameters (e.g. uniform on an N element set).

For the problems on this page, we extend the discussion on χ^2 tests **beyond** what was discussed in lecture to the following more general statistical set-up.

Let $X_1, \ldots, X_n \overset{iid}{\sim} X \sim \mathbf{P}$ denote iid discrete random variables supported on $\{0, \ldots, K\}$. We will decide between the following null and alternative hypotheses:

$$\begin{aligned} H_0 : &\quad \mathbf{P} \in \{\text{Bin}(K, \theta)\}_{\theta \in (0,1)} \\ H_1 : &\quad \mathbf{P} \notin \{\text{Bin}(K, \theta)\}_{\theta \in (0,1)}, \end{aligned}$$

where the null hypothesis can be rephrased as:

$$H_0 : \quad \text{there exists } \theta \in (0,1) \text{ such that for all } j = 0, \ldots, K, \text{ we have } P(X = j) = \binom{K}{j} \theta^j (1-\theta)^{K-j}.$$

Review: Log-likelihood for a Binomial Distribution

2/2 points (graded)

Let $\left(\{0,\ldots,K\},\{\mathrm{Bin}\,(K,\theta)\}_{\theta\in(0,1)}\right)$ denote a binomial statistical model. Let $X_1,\ldots,X_n \overset{iid}{\sim} \mathrm{Bin}\,(K,\theta^*)$ for some unknown parameter $\theta^* \in (0,1)$.

The log-likelihood of this statistical model can be written

$$C + A\log B + (nK - A)\log\left(1 - B\right)$$

where C is independent of θ, A depends on $\sum_{i=1}^{n} X_i$, and B depends on θ.

What is A?

Use **Sigma** to stand for $\sum_{i=1}^{n} X_i$.

Sigma

✔ **Answer:** Sigma

Σ

What is B?

theta

✔ **Answer:** theta

θ

STANDARD NOTATION

Solution:

The pmf of $\mathrm{Bin}\,(K,\theta)$ is

$$j \mapsto \binom{K}{j} \theta^j (1-\theta)^{K-j}$$

for $j \in \{1, \ldots, K\}$.

Therefore, the likelihood is given by

$$\begin{aligned} L_n(X_1, \ldots, X_n, \theta) &= \prod_{i=1}^{n} \left(\binom{K}{X_i} \theta^{X_i} (1-\theta)^{K-X_i} \right) \\ &= \left(\prod_{i=1}^{n} \binom{K}{X_i} \right) \theta^{\sum_{i=1}^{n} X_i} (1-\theta)^{nK - \sum_{i=1}^{n} X_i}. \end{aligned}$$

Taking the logarithm, we have

$$\log L_n(X_1, \ldots, X_n, \theta) = \log \left(\prod_{i=1}^{n} \binom{K}{X_i} \right) + \left(\sum_{i=1}^{n} X_i \right) \log \theta + \left(nK - \sum_{i=1}^{n} X_i \right) \log(1-\theta).$$

Therefore, $A = \sum_{i=1}^{n} X_i$ and $B = \theta$.

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You have used 1 of 4 attempts

Answers are displayed within the problem

Review: MLE for a Binomial Distribution

1/1 point (graded)

As above, let $(\{0, \ldots, K\}, \{\mathrm{Bin}(K, \theta)\}_{\theta \in (0,1)})$ denote a binomial statistical model. Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Bin}(K, \theta^*)$ for some unknown parameter $\theta^* \in (0,1)$.

Which of the following denotes the MLE for θ^*?

○ $\sum_{i=1}^n X_i$

○ $\frac{1}{n}\sum_{i=1}^n X_i$

○ $\frac{1}{K}\sum_{i=1}^n X_i$

● $\frac{1}{nK}\sum_{i=1}^n X_i$

✓

Solution:

Recall from the previous problem that

$$\log L_n\left(X_1,\ldots,X_n,\theta\right) = C + \left(\sum_{i=1}^{n} X_i\right)\log\theta + \left(nK - \sum_{i=1}^{n} X_i\right)\log\left(1-\theta\right)$$

where C does not depend on θ.

To compute the MLE, we need to maximize the above with respect to the parameter θ. We set the derivative to be 0:

$$0 = \frac{\sum_{i=1}^n X_i}{\theta} - \frac{nK - \sum_{i=1}^n X_i}{1-\theta}.$$

The above holds when

$$p = \frac{1}{nK}\sum_{i=1}^{n} X_i.$$

Therefore, the right-hand side is the MLE for this statistical model.

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You have used 1 of 2 attempts

Answers are displayed within the problem

χ^2-Test for a Family of Distributions :

Now, we return to the following more general statistical set-up.

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathbf{P}$ denote iid discrete random variables supported on $\{0, \ldots, K\}$. We will decide between the following null and alternative hypotheses.

$$H_0 : \quad \mathbf{P} \in \{\mathrm{Bin}\,(K, \theta)\}_{\theta \in (0,1)}.$$

$$H_1 : \quad \mathbf{P} \notin \{\mathrm{Bin}\,(K, \theta)\}_{\theta \in (0,1)}.$$

Let f_θ denote the pmf of the distribution $\mathrm{Bin}\,(K, \theta)$, and let $\hat{\theta}$ denote the MLE of the parameter θ from the previous problem.

Further, let N_j denote the number of times that j ($j \in \{0, 1, \ldots, K\}$) appears in the data set $X_1, \ldots, X_n$ (so that $\sum_{j=0}^{K} N_j = n.$) The χ^2 **test statistic for this hypothesis test** is defined to be

$$T_n := n \sum_{j=0}^{K} \frac{\left(\frac{N_j}{n} - f_{\hat{\theta}}\,(j)\right)^2}{f_{\hat{\theta}}\,(j)}.$$

This statistic is different from before. Previously, under the null hypothesis, $\mathbf{P}\left(X=j\right)=p_j$ for some fixed p_j. Here, instead, we use $f_{\hat{\theta}}\left(j\right)$ to estimate $\mathbf{P}\left(X=j\right)$. This statistic still converges in distribution to a χ^2 distribution, but the number of degrees of freedom is smaller.

Degrees of Freedom for χ^2 Test for a Family of Distribution

More generally, to test if a distribution $\mathbf{P}$ is described by some member of a family of discrete distributions $\{\mathbf{P}_\theta\}_{\theta\in\Theta\subset\mathbb{R}^d}$ where $\Theta\subset\mathbb{R}^d$ is d-dimensional, with support $\{0,1,2,\ldots,K\}$ and pmf f_θ, i.e. to test the hypotheses:

$$\begin{aligned} H_0: &\quad \mathbf{P}\in\{\mathbf{P}_\theta\}_{\theta\in\Theta}\\ H_1: &\quad \mathbf{P}\notin\{\mathbf{P}_\theta\}_{\theta\in\Theta}, \end{aligned}$$

then if indeed $\mathbf{P}\in\{\mathbf{P}_\theta\}_{\theta\in\Theta\subset\mathbb{R}^d}$ (*i.e.*, the null hypothesis H_0 holds), and if in addition some technical assumptions hold, then we have that

$$T_n := n\sum_{j=0}^{K}\frac{\left(\frac{N_j}{n}-f_{\hat{\theta}}\left(j\right)\right)^2}{f_{\hat{\theta}}\left(j\right)} \xrightarrow[n\to\infty]{(d)} \chi^2_{(K+1)-d-1}.$$

Note that $K+1$ is the support size of $\mathbf{P}_\theta$ (for all θ.)

In our example testing for a binomial distribution, the parameter θ is one-dimensional, i.e. $d=1$. Therefore, under the null hypothesis H_0, it holds that

$$T_n \xrightarrow[n\to\infty]{(d)} \chi^2_{(K+1)-1-1}=\chi^2_{K-1}.$$

Chi-squared Test for a Binomial Distribution on a Sample Data Set I

1/1 point (graded)

Consider the same statistical set-up as above. In particular, we have the test statistic

$$T_n := n \sum_{j=0}^{K} \frac{\left(\frac{N_j}{n} - f_{\hat{\theta}}\left(j\right)\right)^2}{f_{\hat{\theta}}\left(j\right)}.$$

where $\hat{\theta}$ is the MLE for the binomial statistical model $\left(\{0,1,\ldots,K\}, \{\mathrm{Bin}\left(K,\theta\right)\}_{\theta\in(0,1)}\right)$.

We define our test to be

$$\psi_n = \mathbf{1}\left(T_n > \tau\right),$$

where τ is a threshold that you will specify. For the remainder of this page, we will assume that $K=3$ (the sample space is $\{0,1,2,3\}$).

What value of τ should be chosen so that ψ_n is a test of asymptotic level 5%? Give a numerical value with at least 3 decimals.

(Use this table or software to find the quantiles of a chi-squared distribution.)

$\tau =$ 5.991464547107979 ✓ **Answer:** 5.991

Solution:

Since $K=3$ and $d=1$, we know that the limiting distribution of T_n is χ_2^2. Thus, the asymptotic level is the value τ such that

$$\lim_{n\to\infty} P\left(T_n > \tau\right) = P\left(Z > \tau\right) = 0.05$$

where $Z \sim \chi_2^2$. Hence, τ should be chosen to be 5.991 (from the given table).

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You have used 1 of 2 attempts

🛈 Answers are displayed within the problem

Chi-squared Test for a Binomial Distribution on a Sample Data Set II

3/3 points (graded)

Consider the same statistical set-up as above. Suppose we observe a data set consisting of 1000 observations as described in the following (format: i, number of observations of i):

i	N_i
0	339
1	455
2	180
3	26

What is the value of the test statistic T_n for this data set? Give a numerical value with at least 4 decimals. (You are encouraged to use computational software.)

$T_n =$ 0.8828551921498 ✓ **Answer:** 0.8829

What is the p-value of this data set with respect to the test ψ_{1000}? Give a numerical value with at least 4 decimals.

Use this tool to find the tail probabilities of a χ^2 distribution (you may also use any other software). If you are using this tool, note that you need to set "Choose Type of Control" to "Adjust X-axis quantile (Chi square) value" to find the tail probability associated with an x-axis value for a chi-squared distribution with degrees of freedom set in the "Degrees of Freedom" box.

p-value: 0.6431176531870 ✓ **Answer:** 0.6431

If ψ_n is designed to have level 5%, would you **reject** or **fail to reject** on the given data set?

○ Reject

● Fail to reject

✓

Solution:

Observe that the MLE is given by

$$\hat{p} = \frac{1}{3 \cdot 1000}(455 + 2 \cdot 180 + 3 \cdot 26) \approx 0.29767.$$

Thus for this data set,

$$\begin{aligned} T_n = 1000 \quad & \cdot \Bigg(\frac{\left(\frac{339}{1000} - \binom{3}{0}\left(0.2977^0\right)(0.7023)^{3-0}\right)^2}{\binom{3}{0}\left(0.2977^0\right)(0.7023)^{3-0}} + \frac{\left(\frac{455}{1000} - \binom{3}{1}\left(0.2977^1\right)(0.7023)^{3-1}\right)^2}{\binom{3}{1}\left(0.2977^1\right)(0.7023)^{3-1}} + \\ & \frac{\left(\frac{180}{1000} - \binom{3}{2}\left(0.2977^2\right)(0.7023)^{3-2}\right)^2}{\binom{3}{2}\left(0.2977^2\right)(0.7023)^{3-2}} + \frac{\left(\frac{26}{1000} - \binom{3}{3}\left(0.2977^3\right)(0.7023)^{3-3}\right)^2}{\binom{3}{3}\left(0.2977^3\right)(0.7023)^{3-3}} \Bigg) \\ & \approx 0.8829 \end{aligned}$$

The asymptotic p-value for this data set is given by

$$\lim_{n \to \infty} P\left(T_n > 0.8829\right) = P\left(Z > 0.8829\right).$$

where $Z \sim \chi_2^2$. Consulting the suggested link, we see that $P\left(Z > 0.8829\right) \approx 0.6431$.

According to the golden rule of p-values, since $0.6431 > 0.05$, we should **fail to reject** the null hypothesis that $X_1, \ldots, X_{1000}$ are distributed as $\mathrm{Bin}(3, \mathrm{p})$ for some value of the parameter p.

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You have used 2 of 3 attempts

Answers are displayed within the problem

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Help with PMF - Last Problem

question posted 3 days ago by **corderfj**

I'm a bit confused about how to get $f_{\hat{\theta}}\left(j\right)$ I got the question for the MLE right, but following that formula I get a different estimated $\hat{\theta}$ for each j. Thinking about it, I assume that we should get a SINGLE $\hat{\theta}$ based on all the observations reported, but not sure how to get there from the MLE formula in the section above. What am I doing wrong? Any tips?

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1 response

Gaylyn

3 days ago

Using what you found for the MLE for a Binomial Distribution above, think about how you calculate the sum of the X_i's.

Thank you. I finally figured it out in my last available try! It is not very intuitive what this sum is - At least not for me :)

posted 2 days ago by **corderfj**

Glad you got it! I had to stop and think about it too.

posted 2 days ago by **Gaylyn**

Are we looking for a single theta MLE or one for each K? I keep getting the same thing for $\frac{Nj}{n}$ and $f_{\theta}(j)$ leading to zero. My guess is I'm using the MLE incorrectly. Any futher tips on Xi. Is it just the value of the observation * the number of times it appears all summed?

posted about 3 hours ago by **jtourkis**

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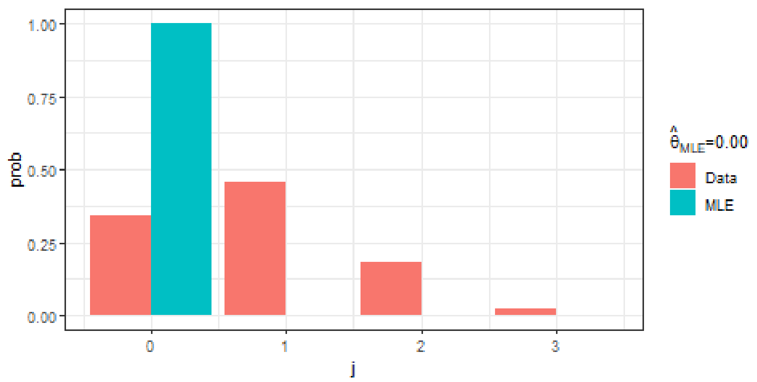
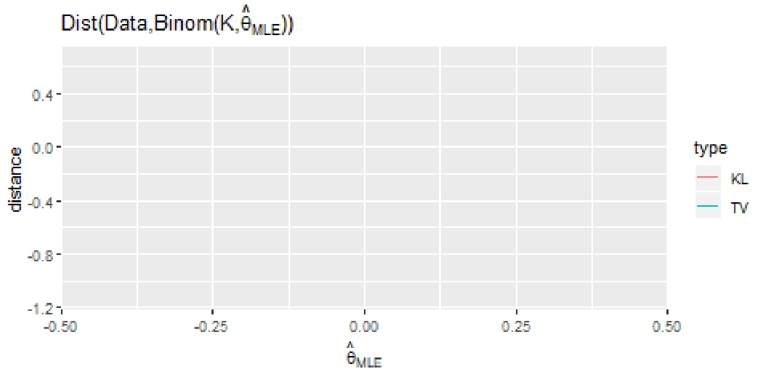
sandipan_dey
less than a minute ago

Just plotted the data and the $Binom\left(K, \hat{\theta}_{MLE}\right)$ distribution for different values of $\hat{\theta}_{MLE}$ (to compute with simulation).

A few points worth noticing and some questions:

1. Both KL and TV distance in between the data and $Binom\left(K, \theta\right)$ get minimized at $\theta = \hat{\theta}_{MLE}$.
2. With $D\left(p||q\right) = KL\left(p\left(.\right), q\left(.\right)\right) = \sum\limits_{x} p\left(x\right)\left(log\left(p\left(x\right)\right) - log\left(q\left(x\right)\right)\right)$ and $log\left(0\right) := 0$, KL distance seems to be have values > 1 at extremes (is it the property of KL, what can be the maximum value achievable of KL)?
3. Will TV and KL between data and the distribution with the estimated parameter always get minimized at $\hat{\theta}_{MLE}$? Can we prove it? How about other estimators (unbiased , MAP) of θ? will the distances get minimized at other estimator values too?





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