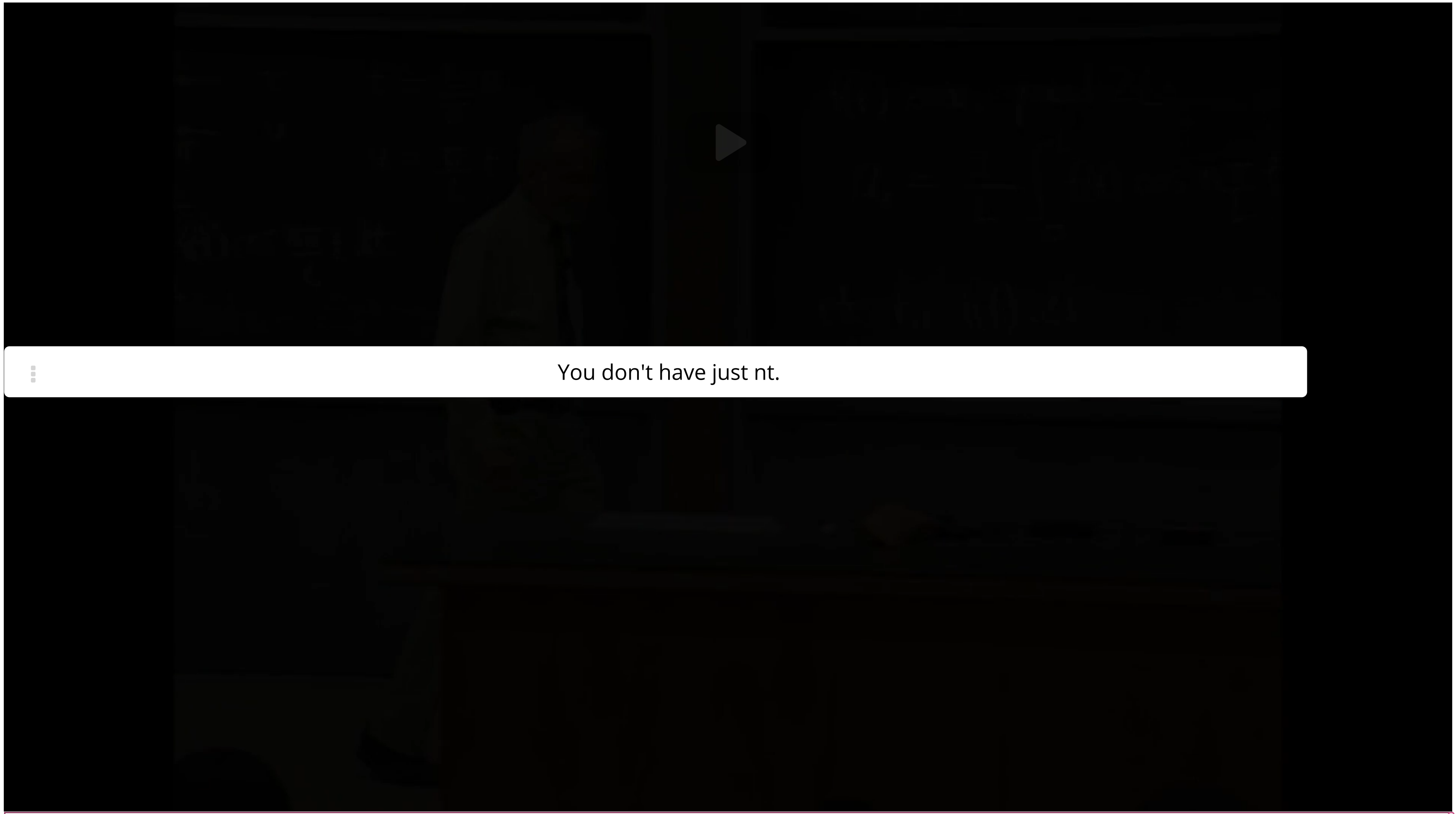
8. Summary: Fourier coefficient formulas

Period 2L formulas



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- Fourier's theorem: “Every" periodic function f of period $2L$ is a Fourier series

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos \frac{n\pi t}{L} + \sum_{n \geq 1} b_n \sin \frac{n\pi t}{L}}.$$

- Given f, the Fourier coefficients a_n and b_n can be computed using:

$$\boxed{\frac{a_0}{2} = \frac{1}{2L} \int_{-L}^{L} f(t)\ dt = \frac{\langle f(t), 1 \rangle}{\langle 1, 1 \rangle}}$$

$$\boxed{a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L}\ dt = \frac{\langle f(t), \cos\left(\frac{n\pi}{L} t\right) \rangle}{\langle \cos\left(\frac{n\pi}{L} t\right), \cos\left(\frac{n\pi}{L} t\right) \rangle}} \text{ for all } n \geq 1,$$

$$\boxed{b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}\ dt = \frac{\langle f(t), \sin\left(\frac{n\pi}{L} t\right) \rangle}{\langle \sin\left(\frac{n\pi}{L} t\right), \sin\left(\frac{n\pi}{L} t\right) \rangle}} \text{ for all } n \geq 1.$$

- If f is even, then only the cosine terms (including the $a_0/2$ term) appear.

- If f is odd, then only the sine terms appear.

Problem 8.1 Define

$$s(t) := \begin{cases} 8, & \text{if } \ 0 < t < 5, \\ 2, & \text{if } \ -5 < t < 0. \end{cases}$$

and extend it to a periodic function of period 10. Find the Fourier series for $s(t)$.

Solution: One way would be to use the Fourier coefficient formulas directly. But we will instead obtain the Fourier series for $s(t)$ from the Fourier series for $\mathrm{Sq}(t)$, by stretching and shifting.

First, stretch horizontally by a factor of $5/\pi$ to get

$$\mathrm{Sq}\left(\frac{\pi t}{5}\right)=\begin{cases}1, & \text{if } 0<t<5,\\ -1, & \text{if } -5<t<0.\end{cases}$$

Here the difference between the upper and lower values is 2, but for $s\left(t\right)$ we want a difference of 6, so multiply by 3:

$$3\mathrm{Sq}\left(\frac{\pi t}{5}\right)=\begin{cases}3, & \text{if } 0<t<5,\\ -3, & \text{if } -5<t<0.\end{cases}$$

Finally add 5:

$$5+3\mathrm{Sq}\left(\frac{\pi t}{5}\right)=\begin{cases}8, & \text{if } 0<t<5,\\ 2, & \text{if } -5<t<0.\end{cases}$$

Since

$$\mathrm{Sq}\left(t\right)=\frac{4}{\pi}\sum_{n\geq 1,\text{ odd}}\frac{1}{n}\sin nt,$$

we get

$$\begin{aligned} s\left(t\right) &= 5+3\mathrm{Sq}\left(\frac{\pi t}{5}\right)\\ &= 5+3\left(\frac{4}{\pi}\right)\sum_{n\geq 1,\text{ odd}}\frac{1}{n}\sin\frac{n\pi t}{5}\\ &= 5+\sum_{n\geq 1,\text{ odd}}\frac{12}{n\pi}\sin\frac{n\pi t}{5}.\end{aligned}$$

Find the Fourier coefficients

2/2 points (graded)

Use the fact that the sawtooth wave of period 2π

$$f(u) = u, \quad -\pi < u < \pi$$

has Fourier series

$$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nu)$$

to find the Fourier series of the odd periodic function (of period 4)

$$g(t) = t/2, \quad -2 < t < 2.$$

Check your answer using the formulas to find the coefficient directly.

FORMULA INPUT HELP

$b_n =$ (-1)^(n+1)*2/(n*pi) ✔ **Answer:** 2*(-1)^(n+1)/(pi*n)

nth term of the Fourier series: (-1)^(n+1)*2/(n*pi)*sin(n*pi*t/2) ✔ **Answer:** 2*(-1)^(n+1)/(pi*n)*sin(n*pi*t/2)

Solution:

Stretch the horizontal axis we find

$$\begin{aligned} f(u) &= \sum \frac{2(-1)^{n+1}}{n} \sin(nu) \\ u &= \frac{\pi t}{2} \\ g(t) &= \frac{1}{\pi} f(u) = \frac{1}{\pi} f\left(\frac{\pi t}{2}\right) \\ &= \sum \frac{2(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi t}{2}\right). \end{aligned}$$

Check the formula directly:

$$\begin{aligned} b_n &= \frac{2}{2} \int_0^2 \frac{t}{2} \sin\left(\frac{n\pi t}{2}\right) dt \\ &= \frac{1}{2} \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt \\ &= \frac{1}{2} \left[\frac{-2t}{n\pi} \cos\left(\frac{n\pi t}{2}\right) \Big|_0^2 + \int_0^2 \frac{2}{n\pi} \cos\left(\frac{n\pi t}{2}\right) dt \right. \\ &= \frac{-2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}. \end{aligned}$$

Therefore the Fourier series is

$$\sum_n \frac{2(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi t}{2}\right),$$

which is what we found above.

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Answers are displayed within the problem

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