4. Rank and the Rank-Nullity Theorem

Problem 4.1 Let $\mathbf{A}$ be the 3×5 matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & -2 & 9 & 10 & 11 \\ 1 & 2 & 9 & 11 & 13 \end{pmatrix}.$$

1. Find a basis for $\mathrm{CS}(\mathbf{A})$.
2. What are $\dim \mathrm{NS}(\mathbf{A})$ and $\dim \mathrm{CS}(\mathbf{A})$?

Solution:

1. First we find a row echelon form. Add the first row to the second, and add -1 times the first row to the third:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 12 & 14 & 16 \\ 0 & 0 & 6 & 7 & 8 \end{pmatrix}.$$

Add $-1/2$ times the second row to the third:

$$\mathbf{B} := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 12 & 14 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is in row echelon form.

The pivots of $\mathbf{B}$ are in the first and third columns.

Basis for $\mathrm{CS}(\mathbf{A})$: first and third columns of $\mathbf{A}$, i.e., $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 9 \end{pmatrix}$.

2.
$$\dim \mathrm{NS}(\mathbf{A}) = \# \textbf{ non-pivot columns of } \mathbf{B} = \mathbf{3}.$$

$$\dim \mathrm{CS}(\mathbf{A}) = \# \textbf{ pivot columns of } \mathbf{B} = \mathbf{2}.$$

Definition 4.2 The **nullity** of $\mathbf{A}$ is defined as the number

$$\text{nullity}(\mathbf{A}) = \dim \mathrm{NS}(\mathbf{A}).$$

The **rank** of $\mathbf{A}$ is defined as the number

$$\text{rank}(\mathbf{A}) = \dim \mathrm{CS}(\mathbf{A}).$$

Rank concept check I

1/1 point (graded)

What can the rank of a 3 by 5 matrix be?

- [x] 0 ✔
- [x] 1 ✔
- [x] 2 ✔
- [x] 3 ✔
- [ ] 4

☐ 5

✔

Solution:

The rank is equal to the dimension of the column space of the matrix. The rank must be less than or equal to 3 because the column space is a subspace of $\mathbb{R}^3$. All of the ranks $0,\ 1,\ 2$ and 3 can be realized as shown by the following matrices:

Rank	Example
0	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
1	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

The rank of any matrix $\mathbf{A}$ is always between 0 and $\min(m, n)$. To see why, note that the column space is a subspace of $\mathbb{R}^m$, so the rank is less than or equal to m. However $\mathrm{CS}(\mathbf{A})$ is spanned by n columns, hence its dimension is less than or equal to n and so is rank of $\mathbf{A}$. Putting this together, this means that the rank is less than or equal to the minimum of m and n. To see that every rank between 0 and $\min(m, n)$ is possible, we can create matrices with these ranks in the same way as for the specific 3×5 example above.

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Answers are displayed within the problem

Theorem 4.3 For any $m \times n$ matrix $\mathbf{A}$,

$$\boxed{\dim \mathrm{NS}(\mathbf{A}) + \mathrm{rank}(\mathbf{A}) = n.}$$

This theorem is called the **rank-nullity theorem** .

Proof

Proof. Let $\mathbf{B}$ be a row echelon form of $\mathbf{A}$.

$$\begin{aligned} \dim \mathrm{NS}(\mathbf{A}) + \mathrm{rank}(\mathbf{A}) &= \dim \mathrm{NS}(\mathbf{A}) + \dim \mathrm{CS}(\mathbf{A}) \\ &= (\text{\# non-pivot columns of } \mathbf{B}) + (\text{\# pivot columns of } \mathbf{B}) \\ &= \text{\# columns of } \mathbf{B} \\ &= n. \end{aligned}$$

□

Rank-nullity concept check I

1/1 point (graded)

A 2×3 matrix $\mathbf{A}$ has $\dim \mathrm{NS}(\mathbf{A}) = 0.$ Which of the following are true?

- [ ] There exists a 2×3 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathrm{rank}(\mathbf{A}) = 0$.
- [ ] There exists a 2×3 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathrm{rank}(\mathbf{A}) = 1$.
- [ ] There exists a 2×3 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathrm{rank}(\mathbf{A}) = 2$.
- [ ] There exists a 2×3 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathrm{rank}(\mathbf{A}) = 3$.
- [x] Such a matrix cannot exist. ✔

✔

Solution:

The matrix has 3 columns but only 2 rows, so the rank cannot be grater than $2.$ Then $\dim \mathrm{NS}(\mathbf{A}) = n - \mathbf{rank}(\mathbf{A}) \geq 1,$ which shows that it is impossible to have a matrix of this size with a 0-dimensional nullspace.

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Rank-nullity concept check II

1/1 point (graded)

A 3×2 matrix $\mathbf{A}$ has $\dim \mathrm{NS}(\mathbf{A}) = 0.$ Which of the following are true?

- [ ] There exists a 3×2 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathbf{rank}(\mathbf{A}) = 0$.
- [ ] There exists a 3×2 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathbf{rank}(\mathbf{A}) = 1$.
- [x] There exists a 3×2 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathbf{rank}(\mathbf{A}) = 2$. ✔
- [ ] There exists a 3×2 matrix $\mathbf{A}$ with $\dim \mathrm{NS}(\mathbf{A}) = 0$ and $\mathbf{rank}(\mathbf{A}) = 3$.
- [ ] Such a matrix cannot exist.

✔

Solution:

The matrix has 3 rows but only 2 columns, so the rank cannot be grater than $2.$ Since $\dim \mathrm{NS}(\mathbf{A}) + \mathbf{rank}(\mathbf{A}) = n$ and $\dim \mathrm{NS}(\mathbf{A}) = 0,$ we get $\mathbf{rank}(\mathbf{A}) = n = 2,$ so all columns of $\mathbf{A}$ are pivot columns.

An example of such a matrix is the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

When the rank of a matrix equals the number of its columns we say that this matrix has a **full column rank.** Whenever this happens, the matrix has a 0−dimensional nullspace.

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Rank-nullity concept check III

1/1 point (graded)

The $m \times n$ matrix $\mathbf{A}$ has $\textbf{rank}(\mathbf{A}) = 2$ and $\dim \text{NS}(\mathbf{A}) = 4.$ What are all the possibilities for m and n?

- $m = 2$ and $n = 4$
- $m = 4$ and $n = 6$
- $m = 6$ and $n = 4$
- $m \geq 2$ and $n = 6$ ✔
- $m \leq 2$ and $n \geq 4$

Solution:

By the rank nullity theorem, $\dim \text{NS}(\mathbf{A}) + \dim \text{CS}(\mathbf{A}) = n$, so the total number of columns is $n = 6.$ The rank of $\mathbf{A}$ is dimension of $\text{CS}(\mathbf{A})$, which is a subspace of $\mathbb{R}^m$. So m is at least 2. The following examples that show that each of the values $m \geq 2$ are possible:

- $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
- (Keep adding rows of zeros to get all m!)

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4. Rank and the Rank-Nullity Theorem

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