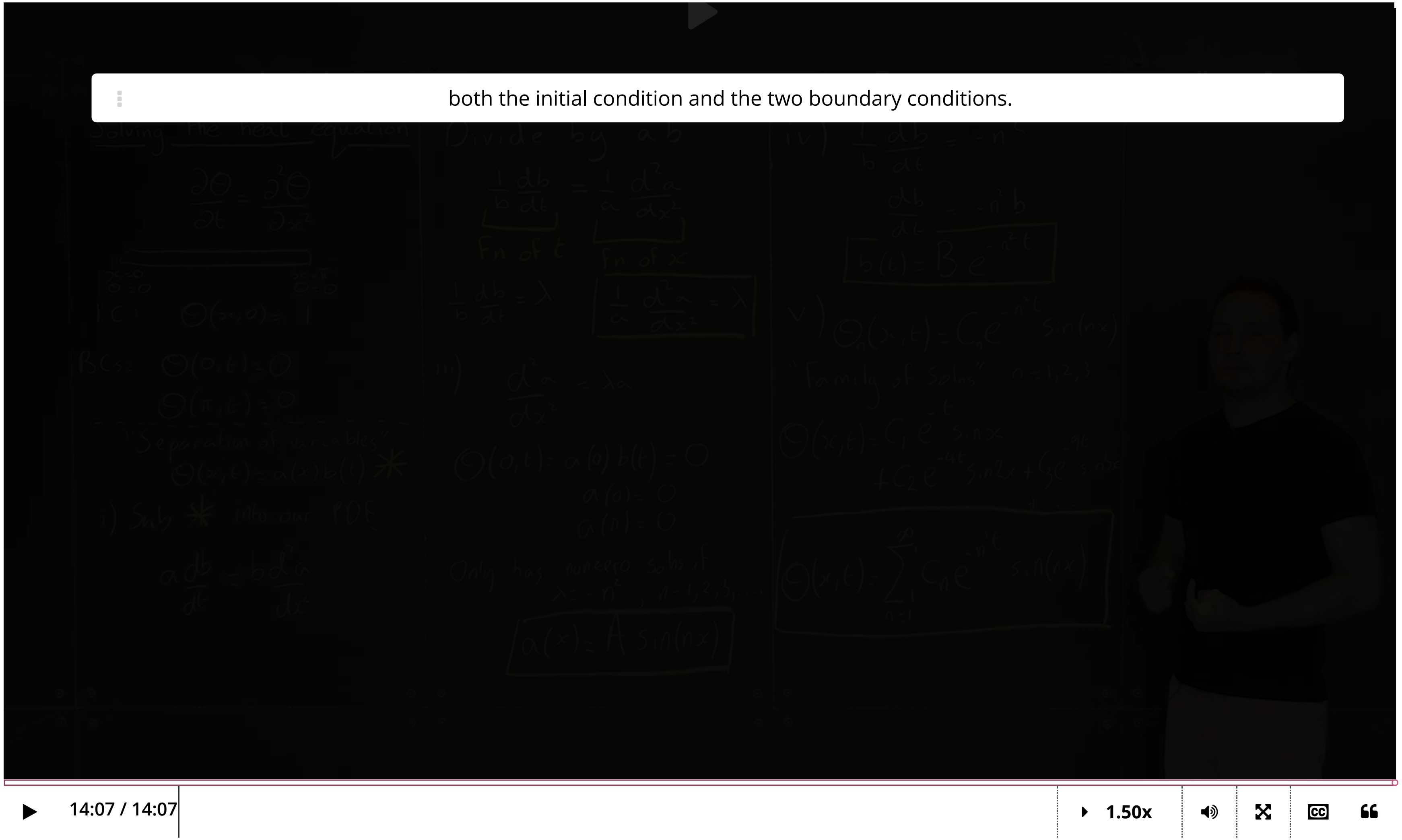
4. Separation of variables: solving a PDE with homogeneous boundary conditions

Worked example: homogeneous boundary conditions



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Let's now try to solve the PDE. For simplicity, suppose that $L = \pi$, $\theta_0 = 1$, and $\nu = 1$. *(The general case is similar. In fact, one could reduce to this special case by changes of variable.)*

So now we are solving

$$\begin{aligned}\frac{\partial \theta}{\partial t} &= \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 < x < \pi,\ t > 0,\\ \theta(0,t) &= 0 \quad \text{for } t \geq 0\\ \theta(\pi,t) &= 0 \quad \text{for } t \geq 0\\ \theta(x,0) &= 1 \quad \text{for } x \in (0,\pi).\end{aligned}$$

Idea (separation of variables): Forget about the initial condition $\theta(x,0) = 1$ for now, but look for nonzero solutions of the form

$$\theta(x,t) = v(x)\ w(t).$$

(Note that we are making a slight change in notation.) Substituting into the PDE gives

$$\begin{aligned}v(x)\,\dot{w}(t) &= w(t)\,v''(x)\\ \frac{\dot{w}(t)}{w(t)} &= \frac{v''(x)}{v(x)}.\end{aligned}$$

(at least where $v(x)$ and $w(t)$ are nonzero).

The only way for a function of x to equal to a function of t is if both functions are the same constant. That is, there is a constant λ such that

$$\frac{v''(x)}{v(x)} = \lambda \quad \text{and} \quad \frac{\dot{w}(t)}{w(t)} = \lambda,$$

or in other words,

$$v''(x) = \lambda\, v(x) \quad \text{and} \quad \dot{w}(t) = \lambda\, w(t)\,.$$

Substituting $\theta(x,t) = v(x)\,w(t)$ into the first boundary condition $\theta(0,t) = 0$ gives $v(0)\,w(t) = 0$ for all t, but $w(t)$ is not the zero function, so this translates into $v(0) = 0$. Similarly, the second boundary condition $\theta(\pi,t) = 0$ translates into $v(\pi) = 0$.

We already solved $v''(x) = \lambda\, v(x)$ subject to the boundary conditions $v(0) = 0$ and $v(\pi) = 0$: nonzero solutions $v(x)$ exist only if $\lambda = -n^2$ for some positive integer n, and in that case $v(x)$ is a scalar times $\sin nx$.

For $\lambda = -n^2$, what is a matching possibility for w? Since $\dot{w} = -n^2 w$, the function w is a scalar times $e^{-n^2 t}$.

This gives rise to one solution

$$\theta_n(x,t) = e^{-n^2 t} \sin nx$$

for each positive integer n, to the PDE with boundary conditions. Each such solution is called a **normal mode** .

Because the boundary conditions are homogeneous, we can get other solutions by taking linear combinations that also satisfy the homogeneous boundary conditions:

$$\theta(x,t) = b_1 e^{-t} \sin x + b_2 e^{-4t} \sin 2x + b_3 e^{-9t} \sin 3x + \cdots. \tag{3.44}$$

This turns out to be the general solution to the PDE $\dfrac{\partial \theta}{\partial t} = \nu \dfrac{\partial^2 \theta}{\partial x^2}$ with the boundary conditions $\theta(0,t) = 0$ and $\theta(\pi,t) = 0$.

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