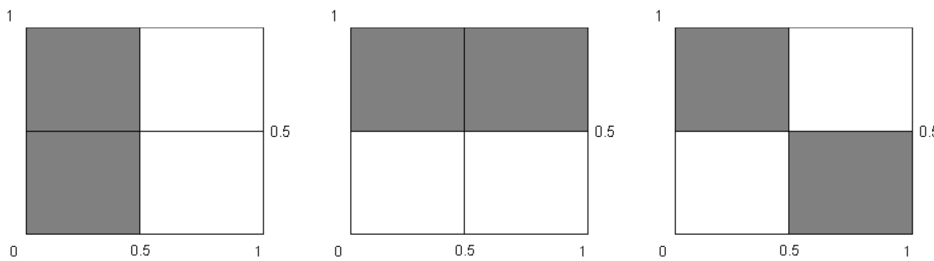
Does pairwise independence imply independence?

In the book on page 28 it has been shown that mutual independence of A and B together with mutual independence of B and C does not imply mutual independence of A and C. More generally, consider a sequence A_j of events such that for any pair $i \neq j$, A_i and A_j are mutually independent. This is called pairwise independence. The question is now: Does pairwise independence imply independence?

The answer is No! As a counter example, consider the following three subsets of the unit square $[0,1] \times [0,1]$, indicated by the shaded area.



A B C

The probabilities involved are the shaded areas themselves. Thus

$$P(A) = P(B) = P(C) = 1/2.$$

Moreover, note that

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = 1/4.$$

Therefore,

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \end{aligned}$$

However, $P(A \cap B \cap C) = P(A \cap B) = 1/4$ whereas $P(A)P(B)P(C) = 1/8$, hence, A, B and C are not independent.