Translating $N(\mu, \sigma^2)$ random variables.

Problem: Suppose $X \sim N(\mu, \sigma^2)$ and that $Y = aX + b$ with $a \neq 0$. Does Y have a normal distribution? If so, what is its mean and variance?
Ans. Yes, $Y \sim N(a\mu + b, a^2\sigma^2)$ and here's why: Y has the correct (cumulative) distribution function F.

A computation will show this: suppose that $a > 0$ then

$$
\begin{aligned}
F(y) = P(Y \leq y) = P(aX + b \leq y) &= P\left(X \leq \frac{y-b}{a}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\frac{y-b}{a}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}a\sigma} \int_{-\infty}^{y} e^{-\frac{1}{2}\left(\frac{t-(a\mu+b)}{a\sigma}\right)^2}\, dt
\end{aligned}
$$

This last equality is obtained via the substitution $x = \dfrac{t-b}{a}$. The density function f for Y is F' which, because of the fundamental theorem of calculus, is very easy to compute:

$$
f(y) = F'(y) = \frac{d}{dy}\left(\frac{1}{\sqrt{2\pi}a\sigma} \int_{-\infty}^{y} e^{-\frac{1}{2}\left(\frac{t-(a\mu+b)}{a\sigma}\right)^2}\, dt\right) = \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{1}{2}\left(\frac{y-(a\mu+b)}{a\sigma}\right)^2}\, dt.
$$

We recognize f as the p.d.f. of a $N(\widehat{\mu}, \widehat{\sigma}^2)$ random variable with $\widehat{\mu} = a\mu + b$ and $\widehat{\sigma}^2 = (a\sigma)^2$, i.e., $Y \sim N(a\mu + b, a^2\sigma^2)$.

What if $a < 0$? Let's see what happens in the above computation for F:
(for the integral, use the same substitution $x = \dfrac{t-b}{a}$ as above, being careful to remember that $a < 0$)

$$
\begin{aligned}
F(y) &= P(Y \leq y) = P(aX + b \leq y) \\
&= P\left(X \geq \frac{y-b}{a}\right) \qquad \text{(a is negative, so the inequality reverses)} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{y-b}{a}}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}a\sigma} \int_{y}^{-\infty} e^{-\frac{1}{2}\left(\frac{t-(a\mu+b)}{a\sigma}\right)^2}\, dt \\
&= \frac{1}{\sqrt{2\pi}(-a\sigma)} \int_{-\infty}^{y} e^{-\frac{1}{2}\left(\frac{t-(a\mu+b)}{-a\sigma}\right)^2}\, dt
\end{aligned}
$$

As before we have $f(y) = F'(y) = \dfrac{1}{\sqrt{2\pi}(-a\sigma)} e^{-\frac{1}{2}\left(\frac{y-(a\mu+b)}{-a\sigma}\right)^2}\, dt$ and, since $-a\sigma > 0$, once again f is the p.d.f. of a random variable with distribution $N(a\mu + b, a^2\sigma^2)$.

Application: For us, the main use of this result is that it provides an easy way to convert normally distributed random variables, with arbitray mean μ and variance σ^2, to a *standard normal* random variable. More precisely, if you know $X \sim N(\mu, \sigma)$ and set $Z = \dfrac{X-\mu}{\sigma}$ then $Z \sim N(a\mu + b, a^2\sigma^2) = N(0, 1)$ since $a = 1/\sigma$ and $b = -\mu/\sigma$.

Table V in your text can now be used to "look up" probabilities involving X, no matter its mean and variance, by converting them to probabilites for Z.