Area of the triangle formed by circumcenter, incenter and orthocenter

Asked 5 years, 2 months ago Active 1 year, 6 months ago Viewed 3k times

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Lets say we have $\triangle ABC$ having O, I, H as its circumcenter, incenter and orthocenter. How can I go on finding the area of the $\triangle HOI$.

I thought of doing the question using the distance (length) between HO,HI and OI and then using the Heron's formula, but that has made the calculation very much complicated. Is there any simple way to crack the problem?

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geometry trigonometry

Share Cite Edit Follow Flag edited Jun 16 '16 at 13:22 asked Jun 16 '16 at 9:53

Harsh Sharma
2,240 16 32

5 Answers

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Given any triangle $\triangle ABC$, we will abuse notation and use the same letter to represent both a vertex and the angle at that vertex. Let

- a, b, c be the side lengths $|BC|$, $|CA|$ and $|AB|$.
- $L = a + b + c$ be the perimeter.
- c_X, s_X, t_X be $\cos X, \sin X, \tan X$ for any angle $X \in \{A, B, C\}$.
- R be the circumradius.

Let I, O, H be the incenter, circumcenter and orthocenter.
Their barycentric coordinates are given by

$$\begin{array}{rcccc}\alpha_I : \beta_I : \gamma_I & = & \sin A : \sin B : \sin C & = & t_A c_A : t_B c_B : t_C c_C\\ \alpha_O : \beta_O : \gamma_O & = & \sin 2A : \sin 2B : \sin 2C & = & 2t_A c_A^2 : 2t_B c_B^2 : 2t_C c_C^2\\ \alpha_H : \beta_H : \gamma_H & = & \tan A : \tan B : \tan C & = & t_A : t_B : t_C\end{array}$$

Let $\mathcal{A}_0$ and $\mathcal{A}$ be the area of $\triangle ABC$ and $\triangle IOH$, their ratio is given by

$$\frac{\mathcal{A}}{\mathcal{A}_0} = \left|\det\begin{bmatrix}\alpha_I & \beta_I & \gamma_I\\ \alpha_O & \beta_O & \gamma_O\\ \alpha_H & \beta_H & \gamma_H\end{bmatrix}\right| = \frac{\mathcal{N}}{\delta_I \delta_O \delta_H}$$

where $\begin{cases} \delta_I &= \sin A + \sin B + \sin C \\ \delta_O &= \sin 2A + \sin 2B + \sin 2C \\ \delta_H &= \tan A + \tan B + \tan C \end{cases}$ and

$$\begin{aligned}\mathcal{N} &= \left|\det\begin{bmatrix} t_A c_A & t_B c_B & t_C c_C \\ 2t_A c_A^2 & 2t_B c_B^2 & 2t_C c_C^2 \\ t_A & t_B & t_C\end{bmatrix}\right| = 2t_At_Bt_C\left|\det\begin{bmatrix} c_A & c_B & c_C \\ c_A^2 & c_B^2 & c_C^2 \\ 1 & 1 & 1\end{bmatrix}\right|\\ &= 2t_At_Bt_C|(c_A-c_B)(c_B-c_C)(c_C-c_A)|\end{aligned}$$

Since $A+B+C=\pi$, $\delta_H = t_A+t_B+t_C = t_At_Bt_C$. Together with the relations,

$$\begin{cases} \mathcal{A}_0 &= \frac12 R^2(\sin 2A + \sin 2B + \sin 2C) \\ L &= 2R(\sin A + \sin B + \sin C)\end{cases} \implies \begin{cases} \delta_O = \frac{2\mathcal{A}_0}{R^2} \\ \delta_I = \frac{L}{2R}\end{cases}$$

We find

$$\frac{\mathcal{A}}{\mathcal{A}_0} = \frac{2}{\delta_I\delta_O}|(c_A-c_B)(c_B-c_C)(c_C-c_A)| = \frac{2R^3}{\mathcal{A}_0 L}|(c_A-c_B)(c_B-c_C)(c_C-c_A)|$$

Notice

$$c_A - c_B = \frac{-a^2+b^2+c^2}{2bc} - \frac{a^2-b^2+c^2}{2ac} = \frac{(b-a)L(L-2a)}{2abc}$$

and similar expressions for $c_B - c_C, c_C - c_A$, we find

$$\frac{\mathcal{A}}{\mathcal{A}_0} = \left(\frac{2R^3}{\mathcal{A}_0 L}\cdot\frac{L^3(L-2a)(L-2b)(L-2c)}{8a^3b^3c^3}\right)|(a-b)(b-c)(c-a)|$$

Recall the Heron's formula and a beautiful relation between $\mathcal{A}_0$ and R:

$$16\mathcal{A}_0^2 = L(L-2a)(L-2b)(L-2c) \quad\text{and}\quad 4\mathcal{A}_0 R = abc$$

What's inside the parenthesis above can be simplified as

$$\frac{2R^3}{\mathcal{A}_0 L}\cdot\frac{L^2\cdot 16{\mathcal{A}_0}^2}{8(4\mathcal{A}_0R)^3} = \frac{L}{16\mathcal{A}_0^2} = \frac{1}{(L-2a)(L-2b)(L-2c)}$$

This leads to a reasonably simple ratio one can use to compute the area $\mathcal{A}$.

$$\frac{\mathcal{A}}{\mathcal{A}_0} = \frac{|(a-b)(b-c)(c-a)|}{(-a+b+c)(a-b+c)(a+b-c)}$$

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answered Jun 16 '16 at 14:28
achille hui
113k 6 157 307

@JackD'Aurizio thanks for the compliment. – achille hui Jun 17 '16 at 0:52

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Hint:

$AH = 2R\cos A, AI = 4R\sin\frac{B}{2}\sin\frac{C}{2}, OA = R, \angle OAH = 2\angle OAI = B - C$

$\triangle HOI = \triangle AHO - \triangle AHI - \triangle AIO$

Final answer

$$2R^2 \sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$$

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edited Jun 12 '20 at 10:38
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answered Aug 13 '18 at 14:22
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From the information of incentre and Euler line, we have:

$$\begin{aligned} r &= \frac{\Delta}{s} \\ R &= \frac{abc}{4\Delta} \\ IH &= \sqrt{4R^2 + 2r^2 - \frac{1}{2}(a^2+b^2+c^2)} \\ IO &= \sqrt{R(R-2r)} \\ OH &= 9R^2 - (a^2+b^2+c^2) \end{aligned}$$

It's do-able by using simple program in a computer. You can also find the area bound ($\Delta HOI = \frac{d}{2} \times OH$) by knowing the distance bound between the incentre and the Euler line.

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edited Jun 16 '16 at 11:05

answered Jun 16 '16 at 10:59
Ng Chung Tak
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2

If you compute the exact barycentric coordinates of O, H, I, the ratio $\frac{[OHI]}{[ABC]}$ is given by a simple determinant. So, just exploit the first table on this page and perform some computation.

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answered Jun 16 '16 at 12:48
Jack D'Aurizio
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A área do triângulo HOI do triângulo ABC de lados a=|BC|, b=|AC| e c=|AB| é dada por:

$$AreaHOI = |\frac{1}{4}.\frac{\begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}}|$$

Dedução: Seja o triângulo ABC, com o vértice A na origem do Sistema Cartesiano e C no semi eixo positivo x. Assim:

$$A = (0, 0)$$

$$B = (\frac{-a^2+b^2+c^2}{2b}, \frac{2S}{b})$$

$$C = (b, 0)$$

$$S = \sqrt{p(p-a)(p-b)(p-c)}$$

$$p = \frac{a+b+c}{2}$$

$$AreaHOI = 3.AreaGIO$$

$$AreaGIO = \frac{1}{2}\begin{vmatrix} xG & yG & 1 \\ xI & yI & 1 \\ xO & yO & 1 \end{vmatrix}$$

"Obtendo as coordenadas de G, I e O através de suas respectivas fórmulas e fazendo substituições obtemos:"

$$AreaHOI = 3.\frac{1}{2}\begin{vmatrix} \frac{-a^2+3b^2+c^2}{6b} & \frac{2S}{3b} & 1 \\ \frac{-a+b+c}{2} & \frac{2S}{a+b+c} & 1 \\ \frac{b}{2} & \frac{b(a^2-b^2+c^2)}{8S} & 1 \end{vmatrix}$$

"Desenvolvendo, substituindo S, p e fatorando obtemos:"

$$AreaHOI = \left|\frac{(a-b)(a-c)(b-c)(a+b+c)}{16S}\right|$$

"Substituindo as expressões por determinantes (S no formato de área de Cayley-Menger), obtemos:"

$$AreaHOI = |\frac{1}{4}.\frac{\begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}}|$$

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