9. Quadratic approximation

Lecture due Sep 13, 2021 20:30 IST Completed

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Quadratic approximation

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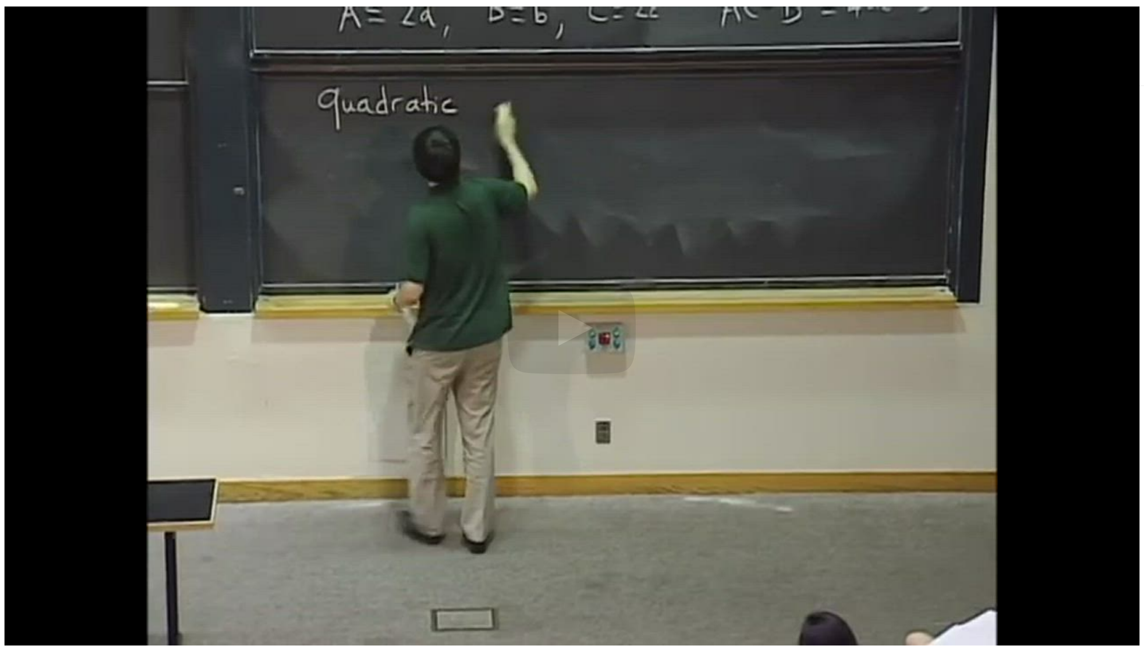

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PROFESSOR: Let me just do, here, quadratic approximation.

So quadratic approximation tells me the following thing.

It tells me if I have a function f of (x,y),

and I want to understand the change in f when I change x

and y a little bit

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In order to formulate a second derivative test for a general function $f(x,y)$, we need a way to connect the work we did for a quadratic equation $w(x,y)$. This connection will come in the form of the quadratic approximation.

Recall that the linear approximation of a function of two variables $f(x,y)$ at a point (x_0,y_0) gives us a way to approximate how f changes near (x_0,y_0).

$$\Delta f = f(x,y) - f(x_0,y_0) \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0). \tag{4.74}$$

The terms involving the first derivative tell us how the function deviates from the value $f(x_0,y_0)$. When (x_0,y_0) happens to be a critical point, the linear approximation becomes

$$\Delta f(x,y) \approx 0 \tag{4.75}$$

because both partial derivatives are zero. We saw that this led to f having a horizontal tangent plane at (x_0,y_0). This means that the tangent plane approximation is not telling us much about the shape of the function $f(x,y)$ near (x_0,y_0). To retrieve this information, we need to think about the quadratic approximation.

Definition 9.1

For a function of two variables $f(x,y)$, the quadratic approximation near the point (x_0,y_0) is given by

$$f(x,y) \quad \approx \quad \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{\text{linear part}}$$

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$$\underbrace{+\frac{1}{2}f_{xx}\left(x_0,y_0\right)\left(x-x_0\right)^2+f_{xy}\left(x_0,y_0\right)\left(x-x_0\right)\left(y-y_0\right)+\frac{1}{2}f_{yy}\left(x_0,y_0\right)\left(y-y_0\right)^2.}_{\text{quadratic part}}$$

The quadratic approximation is a quadratic polynomial that has all the same first and second derivatives as $f\left(x,y\right)$ at the point $\left(x_0,y_0\right)$, and therefore it makes sense that it is a good approximation.

If $\left(x_0,y_0\right)$ is a critical point, then the above equation reduces to

$$\Delta f=f\left(x,y\right)-f\left(x_0,y_0\right)\approx\frac{1}{2}f_{xx}\left(x_0,y_0\right)\left(x-x_0\right)^2+f_{xy}\left(x_0,y_0\right)\left(x-x_0\right)\left(y-y_0\right)+\frac{1}{2}f_{yy}\left(x_0,y_0\right)\left(y-y_0\right)^2. \quad (4.78)$$

The term $1/2f_{xx}$ is the term $a=A/2$ we saw before. The term $f_{xy}=b=B$, and $1/2f_{yy}=c=C/2$.

Thus what we have done is analyze a general function's shape at a critical point by reducing it to its quadratic approximation near that point. In the degenerate case, the behavior of the function depends on derivatives of higher order than 2, so the second derivative test is inconclusive.

Understanding quadratic approximations

6/6 points (graded)

Let's explore the quadratic approximation of a function $g\left(x,y\right)$ at a point that is not a critical point.

Let $g\left(x,y\right)=\sin\left(x\right)\sin\left(y\right)$. Let's try to understand the quadratic approximation at the point $\left(\pi/3,\pi/6\right)$.

We compute all of the partial derivatives below.

$$\begin{aligned} g_x &= \cos\left(x\right)\sin\left(y\right) & g_{xx} &= -\sin\left(x\right)\sin\left(y\right) & (4.79)\\ g_y &= \sin\left(x\right)\cos\left(y\right) & g_{yy} &= -\sin\left(x\right)\sin\left(y\right) & (4.80)\\ & & g_{xy} &= \cos\left(x\right)\cos\left(y\right) & (4.81)\end{aligned}$$

The quadratic approximation at $\left(\pi/3,\pi/6\right)$ takes the form:

$$g\left(x,y\right)\approx A+B\left(x-\pi/3\right)+C\left(y-\pi/6\right)+D(x-\pi/3)^2+E\left(x-\pi/3\right)\left(y-\pi/6\right)+F(y-\pi/6)^2$$

Find the constants in the expression above. (Enter as exact mathematical expressions or decimals correct to two decimal places.)

$A=$ sqrt(3)/4 ✔ **Answer:** sqrt(3)/4

$B=$ 1/4 ✔ **Answer:** 1/4

$C=$ 3/4 ✔ **Answer:** 3/4

$D=$ -sqrt(3)/8 ✔ **Answer:** -sqrt(3)/8

$E=$ sqrt(3)/4 ✔ **Answer:** sqrt(3)/4

$F=$ -sqrt(3)/8 ✔ **Answer:** -sqrt(3)/8

? INPUT HELP

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Solution:

First we evaluate the function and its partial derivatives at the point of interest $(\pi/3, \pi/6)$:

$$\begin{aligned} g(\pi/3,\pi/6) &= \sqrt{3}/4 & g_{xx}(\pi/3,\pi/6) &= -\sqrt{3}/4 & (4.82)\\ g_x(\pi/3,\pi/6) &= 1/4 & g_{yy}(\pi/3,\pi/6) &= -\sqrt{3}/4 & (4.83)\\ g_y(\pi/3,\pi/6) &= 3/4 & g_{xy}(\pi/3,\pi/6) &= \sqrt{3}/4 & (4.84)\end{aligned}$$

The quadratic approximation is given by

$$\begin{aligned} g(x,y) \approx\ & \sqrt{3}/4 + 1/4\,(x-\pi/3) + 3/4\,(y-\pi/6) & (4.85)\\ & -\sqrt{3}/8(x-\pi/3)^2 + \sqrt{3}/4\,(x-\pi/3)\,(y-\pi/6) - \sqrt{3}/8(y-\pi/6)^2. & (4.86)\end{aligned}$$

Submit You have used 1 of 4 attempts

Answers are displayed within the problem

Comparing functions at a critical point

7/7 points (graded)

Let $h(x,y)$ be a function with a critical point at (x_0, y_0).

Let $p(x,y)$ be the function

$$p(x,y) = \frac{1}{2}h_{xx}(x_0,y_0)(x-x_0)^2 + h_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2}h_{yy}(x_0,y_0)(y-y_0)^2.$$

Let $q(x,y)$ be the quadratic approximation of $h(x,y)$ at the point (x_0,y_0).

Let's compare these three functions. For each function p and q determine if it must have the same value, or same partial derivatives as the function h at the point (x_0,y_0) as specified in the table. (Select all that are correct.)

The value $h(x_0,y_0)$ is equal to:

- [ ] $p(x_0,y_0)$
- [x] $q(x_0,y_0)$
- [ ] Neither

The value $h_x(x_0,y_0)$ is equal to:

- [x] $p_x(x_0,y_0)$
- [x] $q_x(x_0,y_0)$
- [ ] Neither

The value $h_y(x_0,y_0)$ is equal to:

- [x] $p_y(x_0,y_0)$
- [x] $q_y(x_0,y_0)$
- [ ] Neither

The value $h_{xx}(x_0,y_0)$ is equal to:

- [x] $p_{xx}(x_0,y_0)$
- [x] $q_{xx}(x_0,y_0)$
- [ ] Neither

The value $h_{xy}(x_0,y_0)$ is equal to:

- [x] $p_{xy}(x_0,y_0)$
- [x] $q_{xy}(x_0,y_0)$
- [ ] Neither

The value $h_{yy}(x_0,y_0)$ is equal to:

- [x] $p_{yy}(x_0,y_0)$
- [x] $q_{yy}(x_0,y_0)$
- [ ] Neither

The value of any third partial derivative of $h(x_0,y_0)$ is equal to:

- [ ] Any third partial derivative of $p(x_0, y_0)$

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- [ ] Any third partial derivative of $p(x_0, y_0)$
- [ ] Any third partial derivative of $q(x_0, y_0)$
- [x] Neither

✔

Solution:

At a critical point, the first x and y partial derivatives are zero. Thus all of the second partial derivatives match, however the value at the point in question only agrees for the quadratic approximation.

Submit You have used 3 of 4 attempts

Answers are displayed within the problem

Comparing functions at a generic point

7/7 points (graded)

The same question as above, but this time, the function $h(x, y)$ does NOT have a critical point at (x_0, y_0).

Let $p(x, y)$ be the function

$$p(x, y) = \frac{1}{2} h_{xx}(x_0, y_0)(x - x_0)^2 + h_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2} h_{yy}(x_0, y_0)(y - y_0)^2.$$

Let $q(x, y)$ be the quadratic approximation of $h(x, y)$ at the point (x_0, y_0).

Let's compare these three functions. For each function p and q determine if it must have the same value, or same partial derivatives as the function h at the point (x_0, y_0) as specified in the table. (Select all that are correct.)

The value $h(x_0, y_0)$ is equal to:

- [ ] $p(x_0, y_0)$
- [x] $q(x_0, y_0)$
- [ ] Neither

✔

The value $h_x(x_0, y_0)$ is equal to:

- [ ] $p_x(x_0, y_0)$
- [x] $q_x(x_0, y_0)$
- [ ] Neither

✔

The value $h_y(x_0, y_0)$ is equal to:

- [ ] $p_y(x_0, y_0)$
- [x] $q_y(x_0, y_0)$
- [ ] Neither

✔

The value $h_{xx}(x_0, y_0)$ is equal to:

- [x] $p_{xx}(x_0, y_0)$
- [x] $q_{xx}(x_0, y_0)$
- [ ] Neither

✔

The value $h_{xy}(x_0, y_0)$ is equal to:

- [x] $p_{xy}(x_0, y_0)$
- [x] $q_{xy}(x_0, y_0)$
- [ ] Neither

✔

The value $h_{yy}(x_0, y_0)$ is equal to:

- [x] $p_{yy}(x_0, y_0)$
- [x] $q_{yy}(x_0, y_0)$
- [ ] Neither

✔

The value of any third partial derivative of $h(x_0, y_0)$ is equal to:

- [ ] Any third partial derivative of $p(x_0, y_0)$
- [ ] Any third partial derivative of $q(x_0, y_0)$

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Any third partial derivative of $q(x_0, y_0)$

- [x] Neither

Solution:

At a point that is not critical point, the first x and y partial derivatives are most likely not zero. Thus all only the full quadratic approximation that includes the linear terms has first partial derivatives that agree with the original function. However, all of the second partial derivatives of all three functions match. As before, the value at the point in question only agrees for the quadratic approximation.

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Answers are displayed within the problem

9. Quadratic approximation

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- The Term f_xy — How can I understand why there should be the term f_xy in quadratic approximation? — 6
- Question about "any third partial derviative" — 7
- Clarification for the solutions? — 4
- 1st question — Why are D and F [...edited...] rather than [...edited...]? — 4
- Quadratic approximations... — Once again, thank you for covering quadratic approximations in 18.01, yeah I know I sound like a broken record but this is such a ... — 2
- [Staff] Need some help here. — Hi Staff, I am having trouble in building an intuition for the comparing functions (both C.Point and generic) question. Can you ple... — 3
- Not discussed in this class — the professor mentioned that the degenerative case of the second derivative test will not be discussed further in this class. may ... — 2
- Comparing functions at a critical/generic point — Is this at all similar to the question from unit 1 about comparing the tangent plane approximation to the function that is being app... — 2
- [staff] typo — 3
- [staff] duplicated paragraph — Community TA — 4
- [Staff] Small Correction — 2
- How is any arbitrary function guaranteed to be equal to its approximation? — ie. h(x,y)=q(q,y) I must be missing something. — 4
- unfortunate choice of words — I wish this question was worded properly. There is an obvious repetition which was not corrected. I don't get the idea--why thro... — 2
- [STAFF] Small technical error in Definition 9.1 — The entire expression, Eq (4.76) + Eq (4.77) is the quadratic approximation. Eq (4.77) should be labeled the quadratic term as on — 4

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