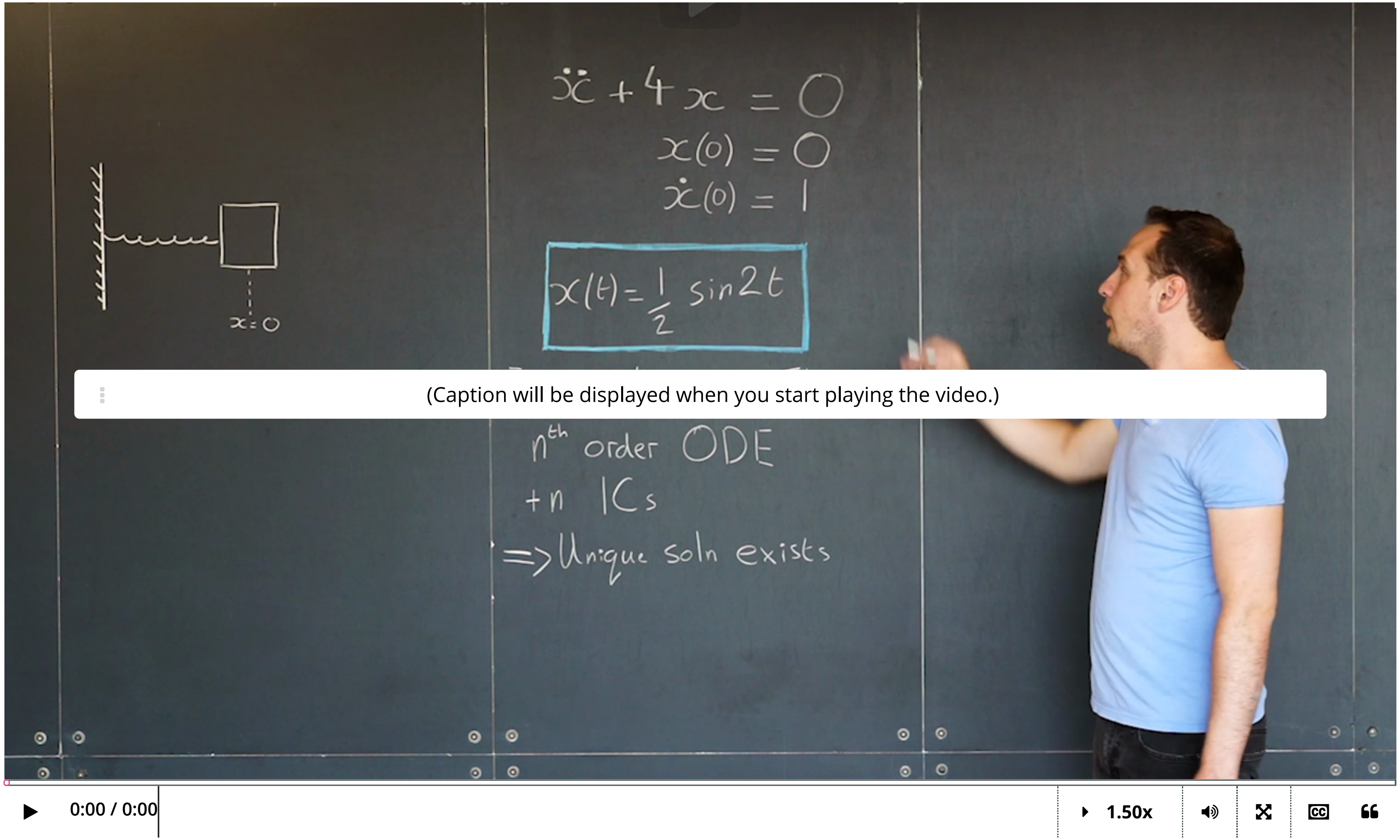
3. Boundary value problems (BVP)

A comparison of IVP and BVP



Video

[Download video file]

Transcripts

[Download SubRip (.srt) file]

[Download Text (.txt) file]

Suppose we have the same spring-mass system as the initial value problem you just solved. However, instead of specifying the initial position and velocity, we want to specify different conditions at different points in time.

The mass is initially at the origin at time $t = 0$, but now we are interested in finding a solution that ensures the mass passes through the point $x = 1$ at some later time $t = t_1$. The governing differential equation is the same

$$\frac{d^2}{dt^2}x + 4x = 0.$$

But now we are interested in finding a solution on a time interval $0 \leq t \leq t_1$, where we impose a condition at each end of the interval:

$$\begin{aligned} x(0) &= 0 \\ x(t_1) &= 1. \end{aligned}$$

We call these conditions **boundary conditions.**

This type of problem is known as a **boundary value problem,** or **BVP** for short.

Solution: The general solution is the same as for the initial value problem:

$$x(t) = A\cos(2t) + B\sin(2t).$$

We find A and B by substituting in $t = 0$ and $t = t_1$, and solving the equations.

$$\begin{aligned} x(0) = 0 &= A\cos(0) + B\sin(0), \qquad \longrightarrow A = 0 \\ x(t_1) = 1 &= B\sin(2t_1), \qquad \longrightarrow B = \frac{1}{\sin(2t_1)} \end{aligned}$$

The solution to this boundary value problem is

$$x(t) = \frac{\sin(2t)}{\sin(2t_1)}.$$

Observe that this solution only makes sense for values of t_1 such that $\sin(2t_1) \neq 0$. For example, when $t_1 = \pi/2$, this boundary value problem does not have a solution at all!

A new feature of boundary value problems is that they need not obey existence and uniqueness. There are three possibilities:

1. The boundary value problem has a unique solution.

2. The boundary value problem has no solution.

3. The boundary value problem has many solutions.

In this example, you see a failure of existence for $t_1 = n\pi/2$, where $n = 1, 2, 3, \ldots$. However, the solution exists and is unique for values of t_1 such that $\sin(2t_1) \neq 0$.

Boundary value problem

1/1 point (graded)

Consider the same spring-mass system modeled by the differential equation

$$\ddot{x} + 4x = 0.$$

How many solutions does this differential equation have on the interval $0 \leq t \leq t_1$, which satisfy the **boundary conditions**

$$\begin{aligned} x(0) &= 0 \\ x(t_1) &= 0. \end{aligned}$$

How many solutions are there for any one value of t_1?

either 0 or 1 solutions.

either 1 or 2 solutions.

either 0 or infinitely many solutions.

either 1 or infinitely many solutions.

Solution:

The general solution takes the form

$$x\left(t\right)=A\cos\left(2t\right)+B\sin\left(2t\right).$$

To satisfy the boundary conditions, we must have

$$x\left(0\right)\ =\ A\cos\left(0\right)+B\sin\left(0\right)=A=0.$$

Therefore, plugging in $A=0$ and using our second boundary condition we get

$$x\left(t_1\right)\ =\ B\sin\left(2t_1\right)=0.$$

This boundary condition is satisfied if either

1. $B=0$
2. $\sin\left(2t_1\right)=0.$

If $B=0$, this gives us the trivial solution $x\left(t\right)=0$, which does satisfy the differential equation and boundary conditions. Thus there is at least one solution for every value of t_1.

If $\sin(2t_1) = 0$, this means that $2t_1 = n\pi$ for n a positive integer, $n = 1, 2, 3 \ldots$. In particular whenever $t_1 = \frac{n\pi}{2}$, then

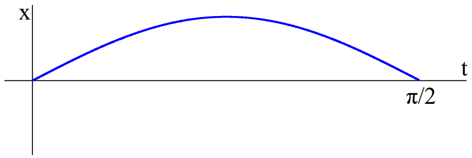
$$x(t) = B\sin(2t)$$

satisfies the boundary conditions and differential equation for all values of B, therefore there are infinitely many solutions.

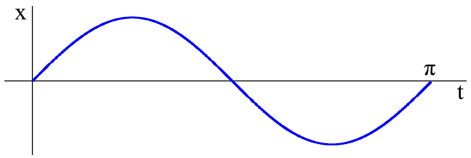
Let's look at what these solutions look like for the various values of t_1 for which this boundary value problem has infinitely many solutions.

n t_1 **Graph of $B\sin(2t)$ for $B = 1$**

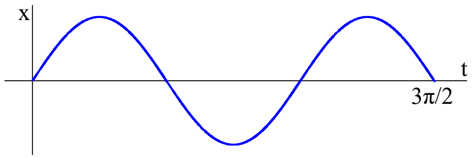
1 $\pi/2$



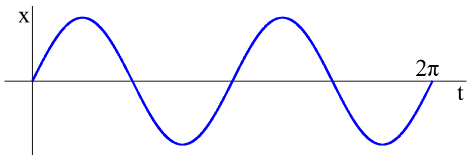
2 π



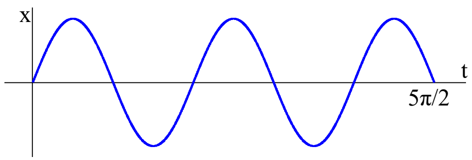
3 $3\pi/2$



4 2π



5 $5\pi/2$



Submit

You have used 1 of 3 attempts

Answers are displayed within the problem

3. Boundary value problems (BVP)

Hide Discussion

Topic: Unit 2: Boundary value problems and PDEs / 3. Boundary value problems (BVP)

Add a Post

All Posts

Boundary Value Problem Solutions

discussion posted 3 days ago by Ruhagura

Could someone please help me understand how any other solutions could possibly come about save for the trivial one in this question?

This post is visible to everyone.

Add a Response

1 response

showrov

2 days ago

The condition to be satisfied is $B\sin\left(2t_1\right) = 0$.
If the end points of the interval, where we are searching for our solution, is such that $t_1 = n\frac{\pi}{2}$,
the criterion to be followed by the solution $x = B\sin\left(2t\right)$ in obeying $x\left(t_1\right) = 0$ ends up as $B.0 = 0$...
This is satisfied by any B, and thus the solution isn't unique.

Add a comment

Showing all responses

Add a response:

Preview

Submit