8. Covariance Matrices

Note: Now is a good time to review the matrix exercises in Homework 0.

Note on Notation: In this course, we assume all vectors to be column vectors. Therefore, while

$$\mathbf{X} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(d)} \end{bmatrix},$$

we sometimes write it as $\mathbf{X} = \left(X^{(1)}, \ldots, X^{(d)}\right)^T$ to be more compact in representation.

Example of Covariance II

4/4 points (graded)

Let X, Y be random variables such that

- X takes the values ± 1 each with probability 0.5
- (Conditioned on X) Y is chosen uniformly from the set $\{-3X-1, -3X, -3X+1\}$.

(Round all answers to 2 decimal places.)

What is Cov (X, X) (equivalent to Var (X))?

Cov $(X, X) =$ 1 ✔ **Answer:** 1.0

What is Cov (Y, Y) (equivalent to Var (Y))?

Cov $(Y, Y) =$ 29/3 ✔ **Answer:** 9.67

What is Cov (X, Y)?

Cov $(X, Y) =$ -3 ✔ **Answer:** -3.00

What is Cov (Y, X)?

Cov $(Y, X) =$ -3 ✔ **Answer:** -3.00

Solution:

Observe that $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ are both zero, since X is uniformly distributed over $\{\pm 1\}$ and Y is uniformly distributed over the set $\{-4, -3, -2, 2, 3, 4\}$.

- Cov (X, X) is the variance of X, which equals $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = p + (1-p) = 1$.
- Cov (Y, Y) is the variance of Y, which equals $\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{16+9+4+4+9+16}{6} = \frac{29}{3} \approx 9.67$.

- Cov (X, Y) and Cov (Y, X) are always equal, by the symmetry of the definition. Observe that the joint density of (X, Y) is uniform over the pairs $(1, -4), (1, -3), (1, -2), (-1, 2), (-1, 3), (-1, 4)$. Thus, either covariance can be computed as $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{-4-3-2-2-3-4}{6} = -3.$

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Answers are displayed within the problem

Covariance Matrix

4/4 points (graded)

Given random variables $X^{(1)}, X^{(2)}, \ldots, X^{(d)}$, one can write down the **covariance matrix** Σ, where $\Sigma_{i,j} = \text{Cov}(X^{(i)}, X^{(j)})$.

Let $X^{(1)}$, $X^{(2)}$ be random variables such that

- $X^{(1)}$ takes the values ± 1 each with probability 0.5
- (Conditioned on $X^{(1)}$) $X^{(2)}$ is chosen uniformly from the set $\{-3X^{(1)} - 1, -3X^{(1)}, -3X^{(1)} + 1\}$.

What is the covariance matrix Σ?

$\Sigma_{1,1} =$ 1 ✔ **Answer:** 1.0 $\Sigma_{1,2} =$ -3 ✔ **Answer:** -3.00

$\Sigma_{2,1} =$ -3 ✔ **Answer:** -3.00 $\Sigma_{2,2} =$ 29/3 ✔ **Answer:** 9.67

Solution:

Using the answer to the previous question, the 2×2 covariance matrix Σ evaluates to

$$\Sigma = \begin{pmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -3 & \frac{29}{3} \end{pmatrix}$$

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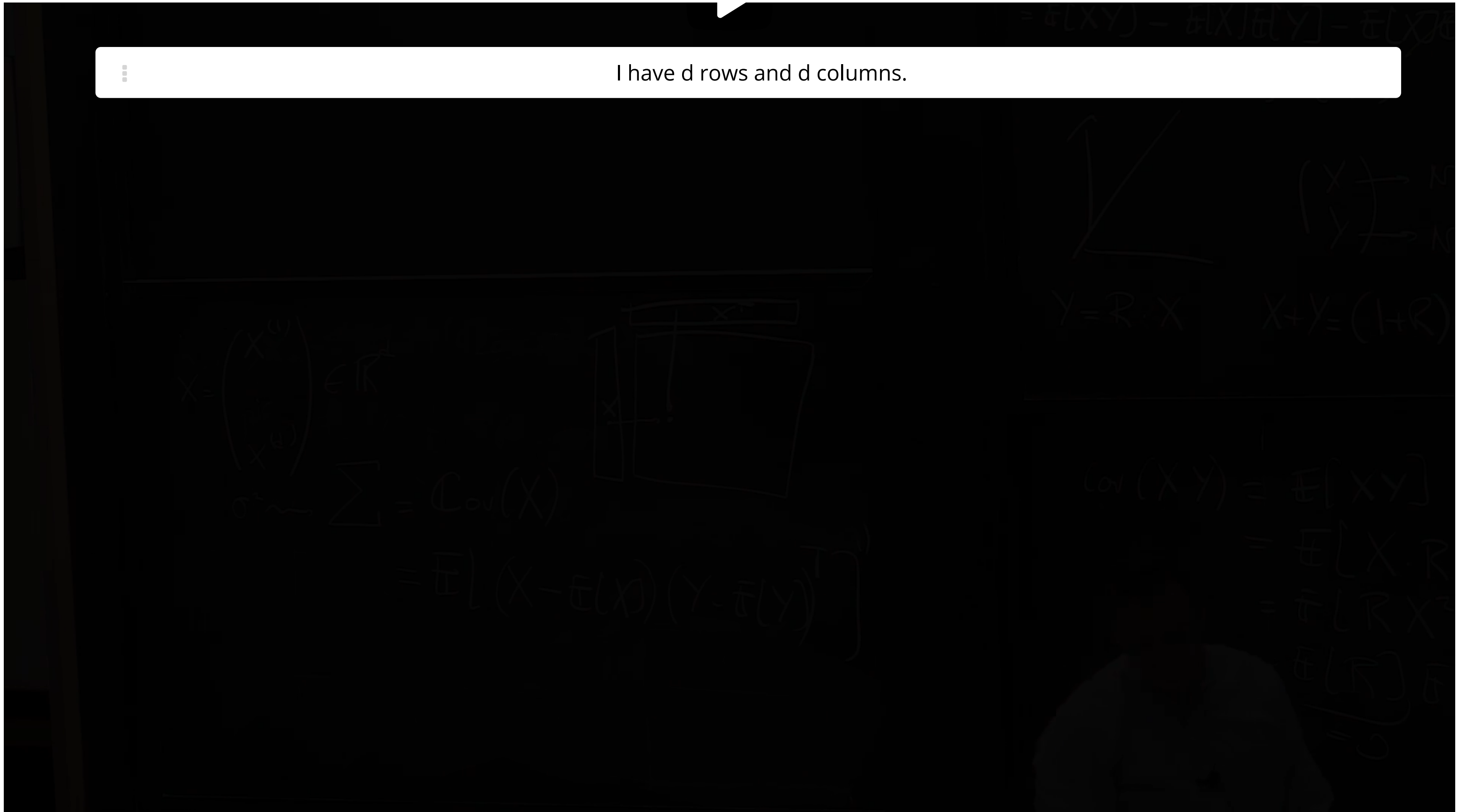
Covariance Matrix: Definitions



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Here is a compact formula for the covariance matrix using vector notation.

Let $\mathbf{X} = \begin{pmatrix} X^{(1)} \\ \vdots \\ X^{(d)} \end{pmatrix}$ be a random vector of size $d \times 1$.

Let $\mu \triangleq \mathbb{E}[\mathbf{X}]$ denote the **entry-wise** mean, i.e.

$$\mathbb{E}[\mathbf{X}] \quad = \quad \begin{pmatrix} \mathbb{E}[X^{(1)}] \\ \vdots \\ \mathbb{E}[X^{(d)}] \end{pmatrix}.$$

Consider the vector outer product (refer to Homework 0) $(\mathbf{X}-\mu)(\mathbf{X}-\mu)^T$, which is a random $d \times d$ matrix. Then the **covariance matrix** Σ can be written as

$$\Sigma = \mathbb{E}\left[(\mathbf{X}-\mu)(\mathbf{X}-\mu)^T\right].$$

Note: The following exercises will be discussed as properties of covariance in the video that follows, but we encourage you attempt these exercises before watching the video.

Covariance Matrix: Properties I

1/1 point (graded)

Let $\mathbf{X}$ be a random vector and let $\mathbf{Y} = \mathbf{X} + B$, where B is a constant vector. Let $\mu_{\mathbf{X}}$ be the mean vector of $\mathbf{X}$ and let $\Sigma_{\mathbf{X}}$ be the covariance matrix of $\mathbf{X}$. Select from the following all statements that are correct.

- [ ] The covariance matrix of $\mathbf{Y}$ could potentially be equal to $\Sigma_{\mathbf{X}}$ only under some conditions imposed on B
- [x] The covariance matrix of $\mathbf{Y}$ is the same as $\Sigma_{\mathbf{X}}$ for all vectors B

- [x] The covariance matrix of $\mathbf{Y}$ has the same size as the matrix $\Sigma_{\mathbf{X}}$

- [ ] The covariance matrix of $\mathbf{Y}$ is the same as $\Sigma_{\mathbf{X}}$ if and only if vector B is equal to 0

✔

Solution:

Choices 2 and 3 are correct. Let the covariance matrix of $\mathbf{Y}$ be denoted $\Sigma_{\mathbf{Y}}$. Note that $\mathbb{E}[\mathbf{X}+B] = \mu_{\mathbf{X}} + B$ for any vector B.

$$\Sigma_{\mathbf{Y}} = \mathbb{E}\left[(\mathbf{X}+B-\mu_{\mathbf{X}}-B)(\mathbf{X}+B-\mu_{\mathbf{X}}-B)^T\right] = \Sigma_{\mathbf{X}}$$

Since choice 2 is correct, choices 1 and 4 that impose certain conditions on B are technically incorrect as we do not require that B satisfiy some conditions for $\Sigma_{\mathbf{Y}}$ to be the same as $\Sigma_{\mathbf{X}}$.

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Covariance Matrix: Properties II

1/1 point (graded)

Let $\mathbf{X}$ be a random vector of size $d \times 1$ and let $\mathbf{Y} = A\mathbf{X} + B$, where A is a constant matrix of size $n \times d$ and B is a constant vector of size $n \times 1$. Let $\mu_{\mathbf{X}}$ be the mean vector of $\mathbf{X}$ and let $\Sigma_{\mathbf{X}}$ be the covariance matrix of $\mathbf{X}$. Let $\mu_{\mathbf{Y}}$ be the mean vector of $\mathbf{Y}$ and let $\Sigma_{\mathbf{Y}}$ be the covariance matrix of $\mathbf{Y}$.

Select from the following all statements that are correct.

- [x] $\Sigma_{\mathbf{Y}}$ is the same as covariance matrix of $A\mathbf{X}$

- [x] $\Sigma_{\mathbf{Y}}$ is of size $n \times n$
- [ ] $\Sigma_{\mathbf{Y}} = A^2 \Sigma_{\mathbf{X}}$
- [x] $\Sigma_{\mathbf{Y}} = A \Sigma_{\mathbf{X}} A^T$
- [ ] $\Sigma_{\mathbf{Y}} = A^T \Sigma_{\mathbf{X}} A$

✔

Solution:

As $\mathbf{Y}$ is an $n \times 1$ random vector, $\Sigma_{\mathbf{Y}}$ is of size $n \times n$.

From the previous problem we know that $\Sigma_{\mathbf{Y}}$ is the same as the covariance matrix of $A\mathbf{X}$. Therefore, it suffices to find this matrix, which we denote $\Sigma_{A\mathbf{X}}$.

$$\begin{aligned}
\Sigma_{A\mathbf{X}} &= \mathbb{E}\left[\left(A\mathbf{X} - A\mu_X\right)\left(A\mathbf{X} - A\mu_X\right)^T\right] \\
&= \mathbb{E}\left[A\left(\mathbf{X} - \mu_X\right)\left(\mathbf{X}^T A^T - \mu_X^T A^T\right)\right] \\
&= \mathbb{E}\left[A\left(\mathbf{X} - \mu_X\right)\left(\mathbf{X} - \mu_X\right)^T A^T\right] \\
&= A\mathbb{E}\left[\left(\mathbf{X} - \mu_X\right)\left(\mathbf{X} - \mu_X\right)^T\right] A^T \\
&= A\Sigma_{\mathbf{X}} A^T.
\end{aligned}$$

Therefore, choices 1, 2, and 4 are correct.

Choices 3 and 5 are not correct in general (even if A is a square matrix) because matrix multiplication is not commutative.

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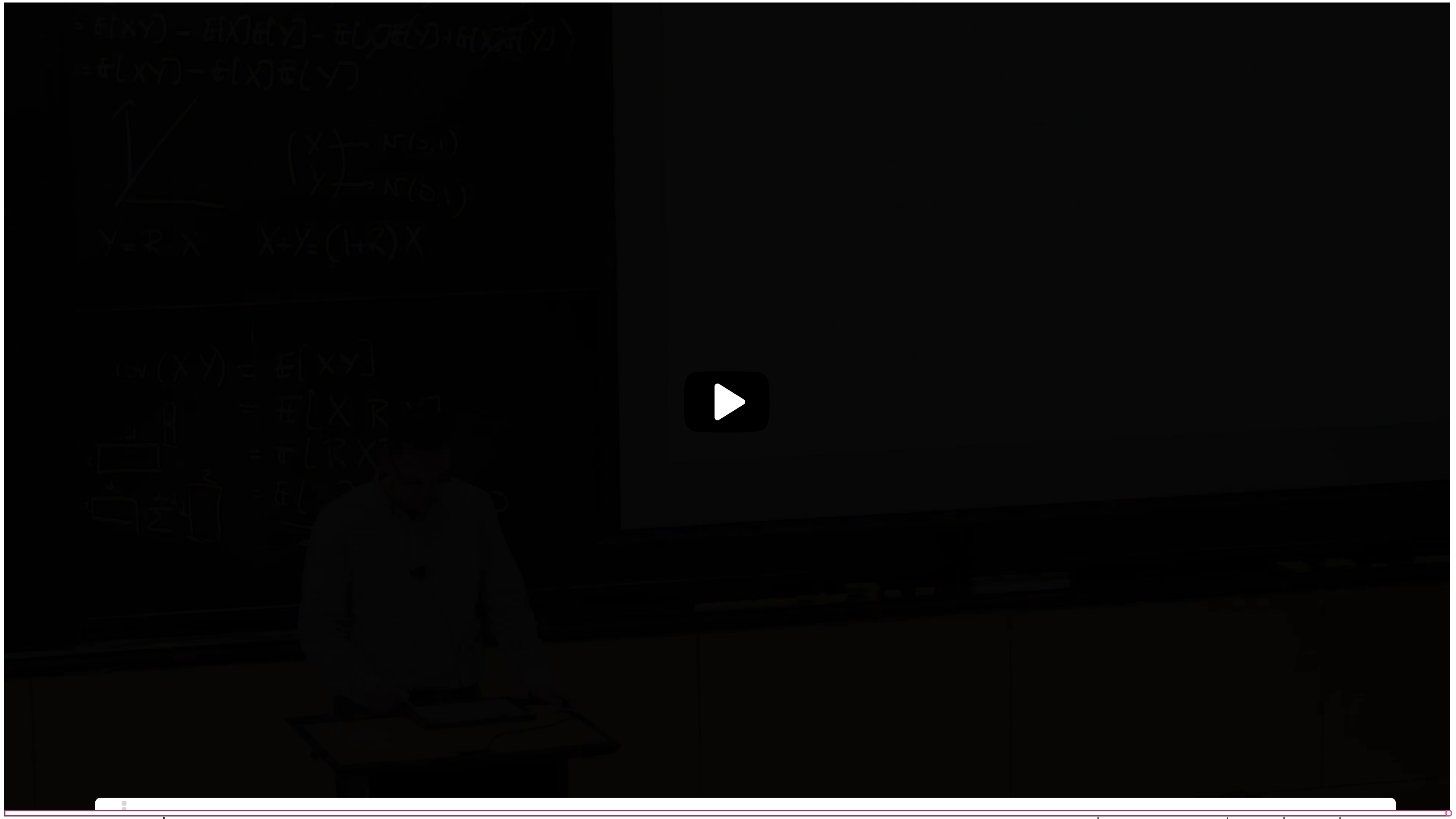
Covariance Matrix: Affine Transformation



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Effect of Linear Transformations of Covariance Matrix

4/4 points (graded)

Let $\mathbf{X} = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}$ be a random vector with covariance Matrix $\Sigma_{\mathbf{X}} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$.

Let $\mathbf{Y} = M\mathbf{X}$, where $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Observe that $Y^{(1)} = X^{(1)} - X^{(2)}$ and $Y^{(2)} = X^{(1)} + X^{(2)}$. What is the new covariance matrix Σ_Y?

$(\Sigma_{\mathbf{Y}})_{1,1} =$ 1 ✔ **Answer:** 1 $(\Sigma_{\mathbf{Y}})_{1,2} =$ 0 ✔ **Answer:** 0.0

$(\Sigma_{\mathbf{Y}})_{2,1} =$ 0 ✔ **Answer:** 0.0 $(\Sigma_{\mathbf{Y}})_{2,2} =$ 3 ✔ **Answer:** 3

Solution:

Recall from an earlier problem that for any pair of random variables A, B with the same variance $\text{Var}(A) = \text{Var}(B) = \sigma^2$, $\text{Cov}(A - B, A + B) = \text{Var}(A) - \text{Var}(B) = 0$.

Therefore, given the matrix M, $\Sigma_{\mathbf{Y}}$ must be a diagonal matrix.

We have
$\text{Cov}(Y^{(1)}, Y^{(1)}) = \text{Cov}(X^{(1)} - X^{(2)}, X^{(1)} - X^{(2)}) = \text{Cov}(X^{(1)}, X^{(1)}) - 2\text{Cov}(X^{(1)}, X^{(2)}) + \text{Cov}(X^{(2)}, X^{(2)}) = 1 - 1 + 1 = 1.$

Similarly,
$\text{Cov}(Y^{(2)}, Y^{(2)}) = \text{Cov}(X^{(1)} + X^{(2)}, X^{(1)} + X^{(2)}) = \text{Cov}(X^{(1)}, X^{(1)}) + 2\text{Cov}(X^{(1)}, X^{(2)}) + \text{Cov}(X^{(2)}, X^{(2)}) = 1 + 1 + 1 = 3.$

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Answers are displayed within the problem

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- ? Variance tells us variation of a r.v., Covariance tells us variation b/w 2 r.v. is there [Something] that tells us about variation of m r.v — 13
- The last problem seems simpler to just solve... — 3
- One of the Cov(A, A) questions seems to be done in terms of X and then cancelling out X — 2
  But keep getting this marked as incorrect. Is there a way to compute this, or should I also iterate through the values of X. This for Question... **Example of Covariance II**
- Example of Covariance II — 5
  In the first exercise I can't get cov(x,y) right although I did get cov(x,x) and cov(y, y) correct so not sure what may be wrong? Once I displayed all possible x,y pairs in an excel sh...

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