MITx: 6.041x Introduction to Probability - The Science of Uncertainty

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Exercise: Gambler's ruin

(5/5 points)

Mary loves gambling. She starts out with \$200 and keeps playing rounds of the same game. For each round, she can bet either \$100 or \$200 (assuming she has sufficient funds) and wins with probability p. Assume that whether she wins is independent across rounds and is unaffected by the size of her bet.

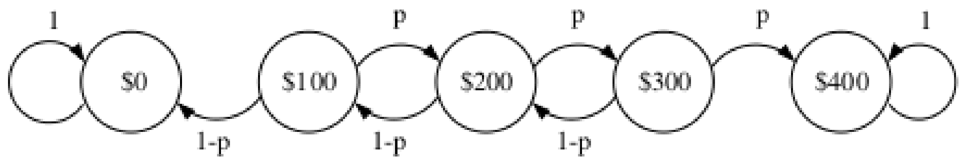
If she wins, she receives back double what she bet, and if she loses, she receives back nothing (i.e., she loses the amount she bet). Assume that Mary stops when she either runs out of money or has reached \$400 or more, whichever comes first. What is her optimal betting strategy? Here, “optimal" means the strategy that gives her the greatest probability of reaching \$400 or more, and “strategy" means a rule saying how much she should bet when she has \$100, \$200, and \$300 (the amount she bets need not be the same in these three cases).

1. What is Mary's optimal strategy when she has \$300?

- (●) Always bet \$100 ✔
- ( ) Always bet \$200

- It depends on p

2. What is Mary's optimal strategy when she has $200?
Hint: For this question, the evaluation of the options can be analyzed using the following transition probability graph:



- Always bet $100
- Always bet $200
- Bet $100 only when $p > 1/2$ ✔
- Bet $100 only when $p > 1/3$
- None of the above

Lec. 26: Absorption probabilities and expected time to absorption
Exercises 26 due May 18, 2016 at 23:59 UTC

Solved problems

Problem Set 10
Problem Set 10 due May 18, 2016 at 23:59 UTC

Exit Survey

Answer:

1. When Mary has \$300, she should bet \$100. Whether she bets \$100 or \$200, she will meet her goal if she wins the next round. If she bets \$100 and loses, she will then have \$200. If she bets \$200 and loses, she will then have \$100. Everything else being equal, it is more advantageous to have \$200 than to have \$100.

2. The more difficult decision is how much to bet when Mary has \$200. We will investigate both possible strategies and decide which is preferable.

   First, Mary can bet \$200. This case is easy to analyze. She will either reach her target \$400 or lose all her money on the next round. Therefore, the probability that she will reach her target if she bets \$200 is p.

   Second, Mary can bet \$100. In this case, Mary will not immediately reach her target or lose her money, so more analysis is required. As an aid, we use the state transition diagram given in the hint. It is clear that when Mary has \$100, she must bet \$100 because she cannot bet \$200. Hence, the possible transitions from state \$200 are to \$0 and \$200. From part (1), we know that her optimal bet when she has \$300 is \$100. Hence, the possible transitions from state \$300 are to \$200 and \$400. Reaching \$400 and \$0 are the two absorbing states.

   We want to find the probability of eventually reaching the \$400 absorbing state, given that she starts with \$200. We will denote the probability that Mary reaches \$400 given that she starts with j hundred dollars by a_j. We have the following system of equations:

$$\begin{aligned} a_1 &= a_2 p \\ a_2 &= a_1(1-p) + a_3 p \end{aligned}$$

$$a_3 \;=\; a_2(1-p)+p.$$

Solving, we obtain $a_2 = \frac{p^2}{1-2p+2p^2}$.

We need to compare p and a_2 to decide whether to bet \$100 or \$200 when she has \$200. Betting \$200 is better when

$$\begin{aligned} p &> a_2 = \frac{p^2}{1-2p+2p^2} \\ 1-2p+2p^2 &> p \\ (1-2p)(1-p) &> 0, \end{aligned}$$

which implies $1-2p>0 \Rightarrow p<1/2$. On the other hand, betting \$100 is better when $p>1/2$. When $p=1/2$, the two bets give the same probability of reaching \$400.

For an intuitive explanation of the result, consider the case where p is small. If Mary bets \$100, she would need to win at least 2 rounds in order to reach \$400, which is relatively unlikely even accounting for the fact that she can move back and forth between \$100, \$200, and \$300. Hence, it is better just to go “all in" by betting \$200 and count on winning a single round.

You have used 1 of 2 submissions