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Problem 4: Trajectory estimation

(5/5 points)

The vertical coordinate (“height") of an object in free fall is described by an equation of the form

$$x(t) = \theta_0 + \theta_1 t + \theta_2 t^2,$$

where θ_0, θ_1, and θ_2 are some parameters and t stands for time. At certain times $t_1, \dots, t_n$, we make noisy observations $Y_1, \dots, Y_n$, respectively, of the height of the object. Based on these observations, we would like to estimate the object's vertical trajectory.

We consider the special case where there is only one unknown parameter. We assume that θ_0 (the height of the object at time zero) is a known constant. We also assume that θ_2 (which is related to the acceleration of the object) is known. We view θ_1 as the realized value of a continuous random variable Θ_1. The observed height at time t_i is $Y_i = \theta_0 + \Theta_1 t_i + \theta_2 t_i^2 + W_i$, $i = 1, \dots, n$, where W_i models the observation noise. We assume that $\Theta_1 \sim N(0,1)$, $W_1, \dots, W_n \sim N(0, \sigma^2)$, and all these random variables are independent.

In this case, finding the MAP estimate of Θ_1 involves the minimization of

$$\theta_1^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \theta_0 - \theta_1 t_i - \theta_2 t_i^2)^2$$

with respect to θ_1.

1. Carry out this minimization and choose the correct formula for the MAP estimate, $\hat{\theta}_1$, from the options below.

- $\hat{\theta}_1 = \frac{\sum_{i=1}^n t_i(y_i-\theta_0-\theta_2 t_i^2)}{\sigma^2}$
- (selected) $\hat{\theta}_1 = \frac{\sum_{i=1}^n t_i(y_i-\theta_0-\theta_2 t_i^2)}{\sigma^2+\sum_{i=1}^n t_i^2}$ ✔
- $\hat{\theta}_1 = \frac{\sum_{i=1}^n t_i(y_i-\theta_0-\theta_2 t_i^2)}{\sigma^2+\sum_{i=1}^n \theta_2 t_i^2}$
- none of the above

2. The formula for $\hat{\theta}_1$ can be used to define the MAP estimator, $\hat{\Theta}_1$ (a random variable), as a function of $t_1, \ldots, t_n$ and the random variables $Y_1, \ldots, Y_n$. Identify whether the following statement is true.

Problem Set 7b due Apr 13, 2016 at 23:59 UTC

Solved problems

Additional theoretical material

Unit summary

- Exam 2
- Unit 8: Limit theorems and classical statistics
- Unit 9: Bernoulli and Poisson processes
- Unit 10: Markov chains
- Exit Survey
- Final Exam

The MAP estimator $\hat{\Theta}_1$ has a normal distribution.

True ▾ ✔ **Answer:** True

3. Let $\sigma = 1$ and consider the special case of only two observations ($n = 2$). Write down a formula for the mean squared error $\mathbf{E}[(\hat{\Theta}_1 - \Theta_1)^2]$, as a function of t_1 and t_2. Enter 't1' for t_1 and 't2' for t_2.

$\mathbf{E}[(\hat{\Theta}_1 - \Theta_1)^2] =$ 1/(1+(t1)^2+(t2)^2) ✔ **Answer:** 1/(1+t1^2+t2^2)

4. Consider the “experimental design" problem of choosing when to make measurements. Under the assumptions of part (3), and under the constraints $0 \leq t_1, t_2 \leq 10$, find the values of t_1 and t_2 that minimize the mean squared error associated with the MAP estimator.

$t_1 =$ 10 ✔ **Answer:** 10

$t_2 =$ 10 ✔ **Answer:** 10

Answer:

1. Setting the partial derivative with respect to θ_1 equal to zero, we obtain

$$\theta_1 - \frac{1}{\sigma^2}\sum_{i=1}^{n} t_i(y_i - \theta_0 - \theta_1 t_i - \theta_2 t_i^2) = 0,$$

yielding the MAP estimate

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n} t_i(y_i - \theta_0 - \theta_2 t_i^2)}{\sigma^2 + \sum_{i=1}^{n} t_i^2}.$$

2. We have

$$\hat{\Theta}_1 = \frac{\sum_{i=1}^{n} t_i(Y_i - \theta_0 - \theta_2 t_i^2)}{\sigma^2 + \sum_{i=1}^{n} t_i^2}.$$

Recall that the observation model is $Y_i = \theta_0 + \Theta_1 t_i + \theta_2 t_i^2 + W_i$, and so we can rewrite the estimator as

$$\begin{aligned}\hat{\Theta}_1 &= \frac{\sum_{i=1}^{n} t_i(\Theta_1 t_i + W_i)}{\sigma^2 + \sum_{i=1}^{n} t_i^2} \\ &= \frac{\Theta_1 \sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} t_i W_i}{\sigma^2 + \sum_{i=1}^{n} t_i^2}.\end{aligned}$$

We see that $\hat{\Theta}_1$ is a linear function of Θ_1 and $W_1, \ldots, W_n$, which are all normal and independent. Since a linear function of independent normal random variables is normal, it follows that $\hat{\Theta}_1$ is normal.

3. Using part (2), we can see, for the special case of $\sigma = 1$ and $n = 2$, that the estimation error is

$$\tilde{\Theta}_1 \triangleq \hat{\Theta}_1 - \Theta_1 = \frac{t_1 W_1 + t_2 W_2 - \Theta_1}{1 + t_1^2 + t_2^2}.$$

Since Θ_1, W_1, W_2 are all zero-mean, independent normal random variables, $\tilde{\Theta}_1$ is also a zero-mean normal random variable. Hence, the mean squared error is

$$\begin{aligned} \mathbf{E}[(\hat{\Theta}_1 - \Theta_1)^2] = \mathbf{E}[\tilde{\Theta}_1^2] = \mathrm{var}(\tilde{\Theta}_1) &= \frac{\mathrm{var}(\Theta_1) + t_1^2 \mathrm{var}(W_1) + t_2^2 \mathrm{var}(W_2)}{\left(1 + t_1^2 + t_2^2\right)^2} \\ &= \frac{1 + t_1^2 + t_2^2}{\left(1 + t_1^2 + t_2^2\right)^2} \\ &= \frac{1}{1 + t_1^2 + t_2^2}. \end{aligned}$$

4. In order to minimize the mean squared error found in part (3), we should choose the observation times to be as large as possible. Under the constraints $0 \le t_1, t_2 \le 10$, we should choose $t_1 = t_2 = 10$.

The intuition is that (since θ_0 and θ_2 are known constants), we are effectively making observations of the form

$$Z_i = \Theta_1 t_i + W_i.$$

Or equivalently, we are making observations of the form

$$Z_i' = \frac{Z_i}{t_i} = \Theta_1 + \frac{W_i}{t_i}.$$

When are these observations most informative? When the noise term is smallest — more precisely, when its variance is smallest. This corresponds to choosing t_i as large as possible.

You have used 1 of 2 submissions

DISCUSSION

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