# Problem 4: Convolution calculations

(6/6 points)

1. Let the discrete random variable $X$ be uniform on $\{0,1,2\}$ and let the discrete random variable $Y$ be uniform on $\{3,4\}$. Assume that $X$ and $Y$ are independent. Find the PMF of $X+Y$ using convolution. Determine the values of the constants $a$, $b$, $c$, and $d$ that appear in the following specification of the PMF.

$$p_{X+Y}(z) = \begin{cases} a, & z=3, \\ b, & z=4, \\ c, & z=5, \\ d, & z=6, \\ 0, & \text{otherwise.} \end{cases}$$

$a =$ 1/6 **Answer:** 0.16667

$b =$ 1/3 **Answer:** 0.33333

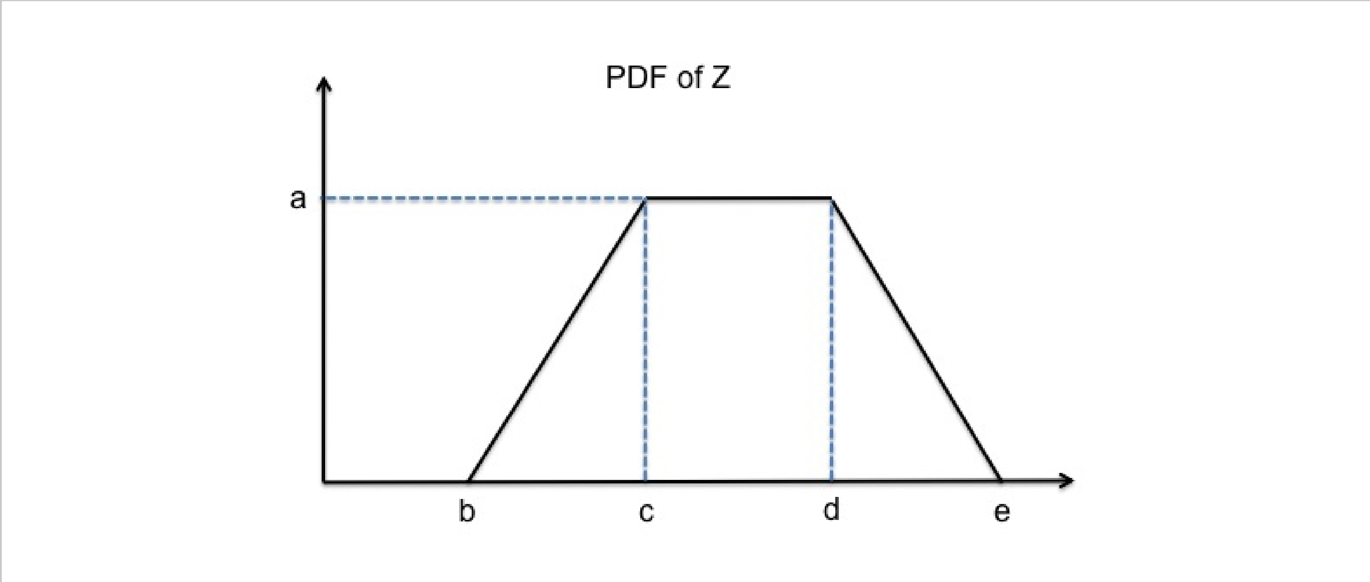
$c =$ 1/3 **Answer:** 0.33333

- Unit 6: Further topics on random variables



$d =$ 1/6 ✔ **Answer:** 0.16667

2. Let the random variable $X$ be uniform on $[0, 2]$ and the random variable $Y$ be uniform on $[3, 4]$. (Note that in this case, $X$ and $Y$ are continuous random variables.) Assume that $X$ and $Y$ are independent. Let $Z = X + Y$. Find the PDF of $Z$ using convolution. The following figure shows a plot of this PDF. Determine the values of $a$, $b$, $c$, $d$, and $e$.




$a =$ 1/2 ✔ **Answer:** 0.5

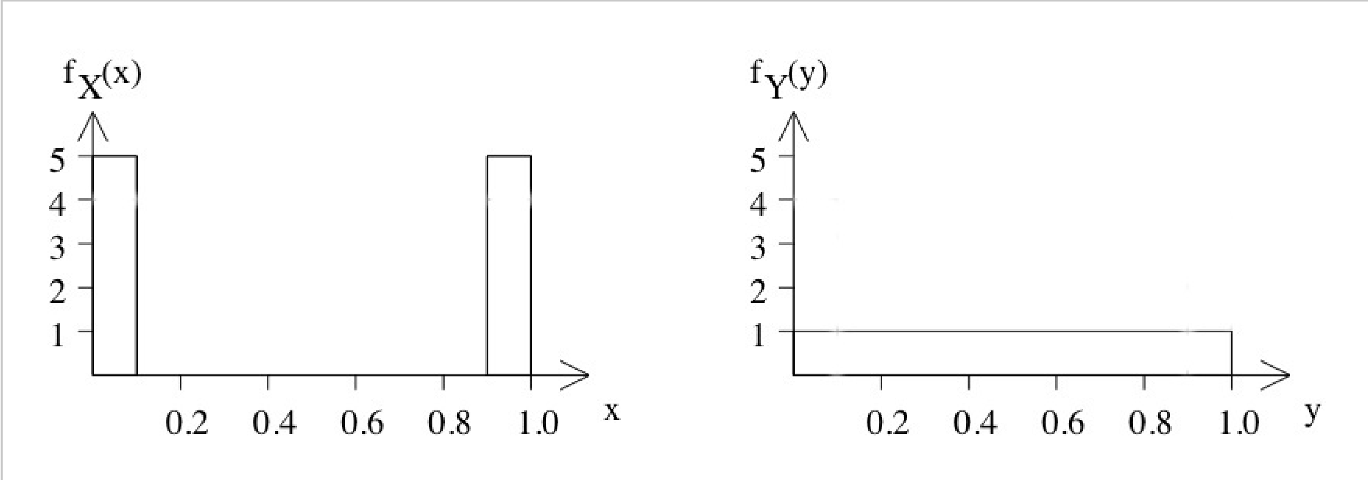
$b =$ 3 ✔ **Answer:** 3

$c =$ 4 ✔ **Answer:** 4

$d =$ 5 **Answer:** 5

$e =$ 6 **Answer:** 6

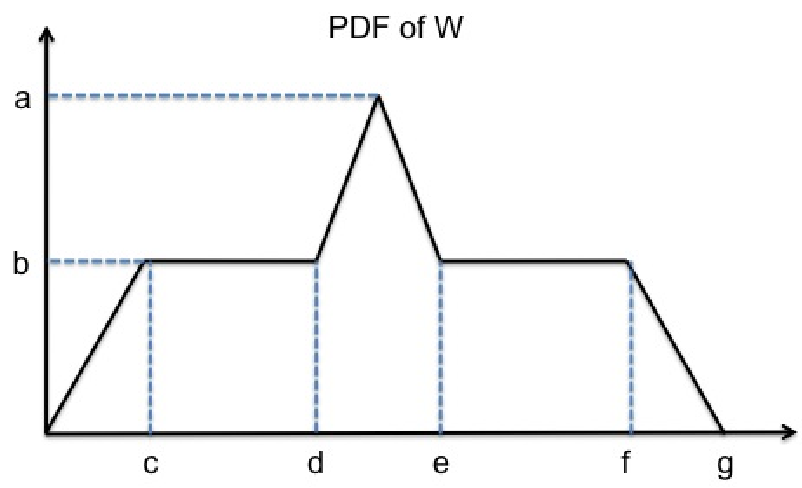
3. Let $X$and $Y$ be two independent random variables with the PDFs shown below. below.




$$f_X(x) = \begin{cases} 5, & \text{if } 0 \le x \le 0.1 \text{ or } 0.9 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & \text{if } 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = X + Y$. The following figure shows a plot of the PDF of $W$. Determine the values of $a$, $b$, $c$, $d$, $e$, $f$, and $g$.




$a =$ 1.0 ✔ **Answer:** 1

$b =$ 0.5 ✔ **Answer:** 0.5

$c =$ 0.1 ✔ **Answer:** 0.1

$d =$ 0.9 ✔ **Answer:** 0.9

$e =$ 1.1 ✔ **Answer:** 1.1

$f =$ 1.9 ✔ **Answer:** 1.9

$g =$ 2 ✔ **Answer:** 2

Answer:

1.
$$p_{X+Y}(z) = \begin{cases} 1/6, & z \in \{3, 6\} \\ 1/3, & z \in \{4, 5\} \\ 0, & \text{otherwise.} \end{cases}$$

2. If $3 \leq z \leq 6$, we have

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx \\ &= \int_{\max(0,z-4)}^{\min(2,z-3)} \frac{1}{2}\, dx \\ &= (\min(2, z-3) - \max(0, z-4))/2. \end{aligned}$$

The PDF of $X+Y$ is then

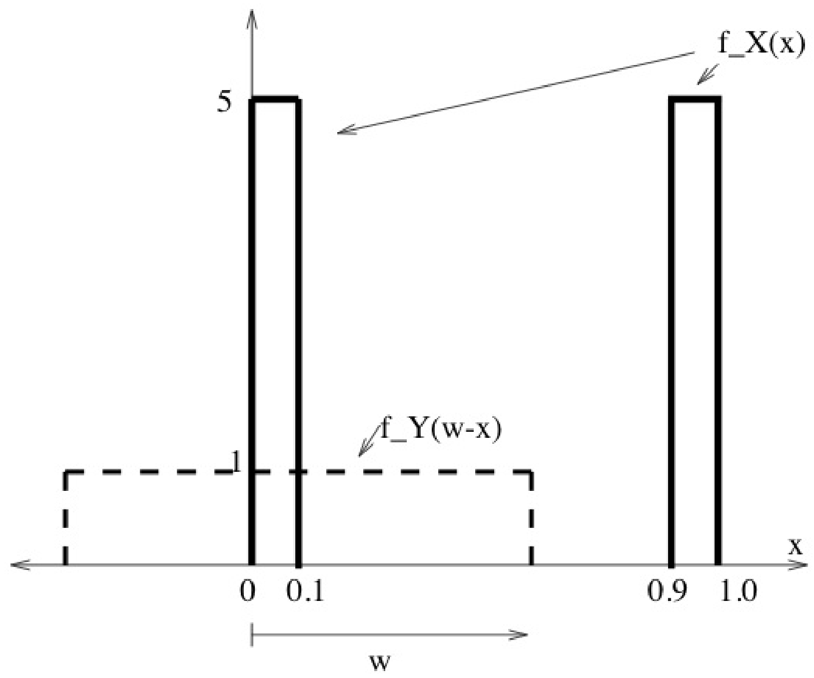
$$f_{X+Y}(z) = \begin{cases} \frac{z-3}{2}, & 3 \leq z < 4, \\ \frac{1}{2}, & 4 \leq z < 5, \\ \frac{6-z}{2}, & 5 \leq z \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$

This answer can also be found by calculating the convolution graphically.

3. We have

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx.$$

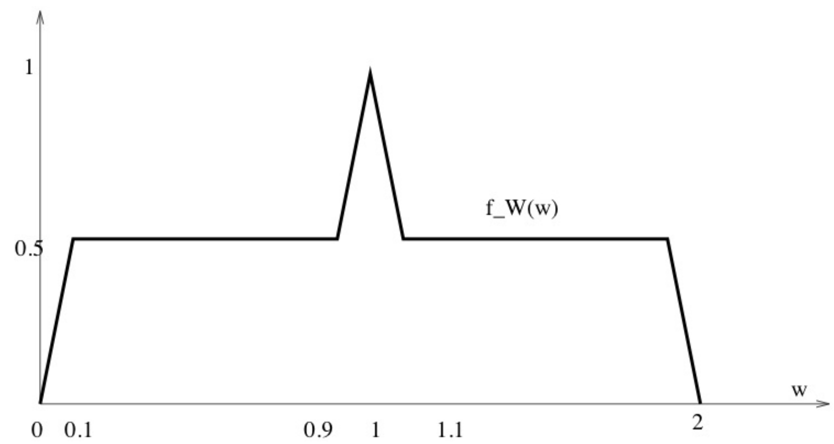
Graphically, $f_Y(w-x)$, as a function of $x$, is obtained by “flipping" $f_Y(x)$ about the $x = 0$ axis, then shifting the plot to the right by $w$. The PDF $f_X(x)$ is then sketched on the same plot.



From this graph we compute the integral of the product of the curves for any given $w$. By visualizing the graph as $w$ is varied, we obtain

$$f_W(w) = \begin{cases} 5w, & 0 \le w \le 0.1, \\ 0.5, & 0.1 \le w \le 0.9, \\ 5(0.1 + (w - 0.9)), & 0.9 \le w \le 1.0, \\ 5(0.1 + (1.1 - w)), & 1.0 \le w \le 1.1, \\ 0.5, & 1.1 \le w \le 1.9, \\ 5(2.0 - w), & 1.9 \le w \le 2.0, \\ 0, & \text{otherwise.} \end{cases}$$

Pictorially,



*You have used 1 of 2 submissions*

## DISCUSSION

Click “Show Discussion" below to see discussions on this problem.