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3. Worked example: solving a homogeneous linear system

Problem 3.1 The reduced row echelon form of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 6 & 2 & -4 & -8 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 2 & -3 & 1 & 4 & -7 & 1 \\ 6 & -9 & 0 & 11 & -19 & 3 \end{pmatrix}$$

is the matrix

$$\mathtt{rref}(\mathbf{A}) = \begin{pmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 0 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Find all solutions to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

Definition 3.2 The set of all solutions to a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ is called the **nullspace of matrix $\mathbf{A}$**, and is denoted $\mathrm{NS}(\mathbf{A})$.

Solution: First note that each variable corresponds to a column in the reduced row echelon matrix

$$\begin{array}{cccccc} x & y & z & u & v & w \end{array}$$
$$\begin{pmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 0 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Any column that contains a pivot is called a **pivot column.** A variable whose corresponding column is a pivot column is called a **dependent variable** or **pivot variable.** The other variables are called **free variables.** In this problem, x, z, w are **dependent variables,** and y, u, v are **free variables.**

Let $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix}$. Then the system $\mathbf{Ax} = \mathbf{0}$ is equivalent to the system of equations

$$x - \frac{3}{2}y \quad + \frac{11}{6}u - \frac{19}{6}v \qquad = \quad 0$$

$$\begin{aligned} z + \frac{1}{3}u - \frac{2}{3}v &= 0 \\ w &= 0 \\ 0 &= 0. \end{aligned}$$

The last equation gives us no information so we discard it.

$$\begin{aligned} x - \frac{3}{2}y \quad + \frac{11}{6}u - \frac{19}{6}v &= 0 \\ z + \frac{1}{3}u - \frac{2}{3}v &= 0 \\ w &= 0 \end{aligned}$$

Note that because we used the reduced row echelon form to solve this system, each equation directly expressed one pivot variable as a linear combination of the free variables. Start by assigning parameters to each of the free variables.

$$\begin{aligned} y &= c_1 \quad \text{for a parameter } c_1, \\ u &= c_2 \quad \text{for a parameter } c_2, \\ v &= c_3 \quad \text{for a parameter } c_3. \end{aligned}$$

Next express each pivot variable in terms of the parameters.

$$\begin{aligned} x &= \frac{3}{2}y - \frac{11}{6}u + \frac{19}{6}v = \frac{3}{2}c_1 - \frac{11}{6}c_2 + \frac{19}{6}c_3 \\ z &= -\frac{1}{3}u + \frac{2}{3}v = -\frac{1}{3}c_2 + \frac{2}{3}c_3 \\ w &= 0 \end{aligned}$$

Therefore the general solution is given by

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{3}{2}c_1 - \frac{11}{6}c_2 + \frac{19}{6}c_3 \\ c_1 \\ -\frac{1}{3}c_2 + \frac{2}{3}c_3 \\ c_2 \\ c_3 \\ 0 \end{pmatrix} = \begin{pmatrix} (3/2)c_1 \\ c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} (-11/6)c_2 \\ 0 \\ (-1/3)c_2 \\ c_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} (19/6)c_3 \\ 0 \\ (2/3)c_3 \\ 0 \\ c_3 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -11/6 \\ 0 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 19/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

In particular, each of the three vectors

$$\begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -11/6 \\ 0 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 19/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

is a solution to the original homogeneous equation. And the set of all solutions is found by taking all linear combinations of these solutions.

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