MATHEMATICS

Calculate the closed form of the following series

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8

$$\sum_{m=r}^{\infty}\binom{m-1}{r-1}\frac{1}{4^m}$$

The answer given is

4

$$\frac{1}{3^r}$$

I tried expanding the expression so it becomes

$$\sum_{m=r}^{\infty}\frac{(m-1)!}{(r-1)!(m-r)!}\frac{1}{4^m}$$

but I do not know how to follow.

Any help will be appreciated, thanks.

calculus summation binomial-coefficients closed-form

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edited Sep 2 at 2:43
RobPratt
26.4k 3 16 38

asked Sep 1 at 9:41
q19a
73 6

are you sure that m starts at r and gets larger and larger? What are combinatorial numbers then? – user376343 Sep 1 at 9:46

1 The answer cannot be $1/3^m$ since we sum over m. $1/3^r$ would be correct – Claude Leibovici Sep 1 at 9:52

You are right, it was a typo. – q19a Sep 1 at 9:58

And yes, I am sure it starts at r and gets larger and larger, it is a solution of an exam given by the professor, I can give you the whole problem if you want. – q19a Sep 1 at 9:59

6 Answers

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You can use recurrence relation, thanks to the Pascal's identity:

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$$\binom{m-1}{k-1} = \binom{m}{k} - \binom{m-1}{k}.$$

As suggested by Claude Leibovici in the comment we can have a more general result. Let

$$S(k) = \sum_{m=k}^{\infty} \binom{m-1}{k-1} \frac{1}{x^m}.$$

It seems like $S(k)$ converges as long as $|x| > 1$.

Using Pascal's identity we have

$$\sum_{m=k}^{\infty} \binom{m-1}{k-1} \frac{1}{x^m} = x \sum_{m=k}^{\infty} \binom{m}{k} \frac{1}{x^{m+1}} - \sum_{m=k}^{\infty} \binom{m-1}{k} \frac{1}{x^m}$$

Which give us $S(k) = xS(k) - S(k-1)$ or

$$S(k-1) = (x-1)S(k) \tag{1}$$

Now $S(1)$ is just the geometric series

$$\sum_{m=1}^{\infty} \frac{1}{x^m} = \frac{1}{x-1}$$

Which gives us the solution for (1) is

$$S(k) = \frac{1}{(x-1)^k} = \sum_{m=k}^{\infty} \binom{m-1}{k-1} \frac{1}{x^m} \tag{2}$$

Setting $x = 4$ gives us the desired result.

Note the similarity with negative binomial theorem:

$$\frac{1}{(x+a)^k} = \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} x^j a^{k-j}$$

when $a = -1$ which converges for $|x| < 1$.

We can then combine $S(k)$ with this to get a (Laurent) series expansion of $\frac{1}{(x-1)^k}$ as a corollary.

Share Cite Edit Follow Flag edited Sep 1 at 11:50

answered Sep 1 at 10:22
Azlif
1,986 4 19

Now I get it, thanks! – q19a Sep 1 at 10:36

1 May I suggest you make it more general. Replace $\frac{1}{4}$ by x and get $S(k) = \left(\frac{x}{1-x}\right)^k$. It would be interesting for other users. Thanks and cheers. – Claude Leibovici Sep 1 at 10:50

@ClaudeLeibovici Sure! Thank you for the suggestion. – Azlif Sep 1 at 11:23

3

Say we have an unfair coin that yields head with probability $\frac{3}{4}$ and we toss it repeatedly until we get r heads. The probability that the r-th head is from the m-th toss is given by the following expression:

$$\binom{m-1}{r-1}\left(\frac{1}{4}\right)^{m-r}\left(\frac{3}{4}\right)^{r}$$

If we sum the probability for all possible m then we'll get one

$$1=\sum_{m=r}^{\infty}\binom{m-1}{r-1}\left(\frac{1}{4}\right)^{m-r}\left(\frac{3}{4}\right)^{r}$$

$$=3^r\sum_{m=r}^{\infty}\binom{m-1}{r-1}\left(\frac{1}{4}\right)^{m}$$

or equivalently

$$\left(\frac{1}{3}\right)^{r}=\sum_{m=r}^{\infty}\binom{m-1}{r-1}\left(\frac{1}{4}\right)^{m}$$

For a general case in which probability of getting head is $1-x$ where $0\le x<1$:

$$1=\sum_{m=r}^{\infty}\binom{m-1}{r-1}x^{m-r}(1-x)^{r}$$

$$=\left(\frac{1-x}{x}\right)^{r}\sum_{m=r}^{\infty}\binom{m-1}{r-1}x^{m}$$

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edited Sep 1 at 11:27

answered Sep 1 at 11:15

R Rezha Adrian Tanuharja
7,386 5 21

Lovely solution. – NoName Sep 1 at 11:38

1

We obtain

$$\begin{aligned}\sum_{m=r}^{\infty}\binom{m-1}{r-1}\frac{1}{4^m}&=\sum_{m=0}^{\infty}\binom{m+r-1}{m}\frac{1}{4^{m+r}} &(1)\\ &=\frac{1}{4^r}\sum_{m=0}^{\infty}\binom{-r}{m}\left(-\frac{1}{4}\right)^m &(2)\\ &=\frac{1}{4^r}\frac{1}{\left(1-\frac{1}{4}\right)^r} &(3)\\ &=\frac{1}{3^r}\end{aligned}$$

and the claim follows.

Comment:

- In (1) we shift the index to start with $m=0$ and we also use the identity $\binom{p}{q}=\binom{p}{p-q}$.
- In (2) we factor out $\frac{1}{4^r}$ and apply the binomial identity $\binom{-p}{q}=\binom{p+q-1}{q}(-1)^q$.
- In (3) we use the *binomial series expansion*.

Share Cite Edit Follow Flag edited Sep 3 at 14:14 answered Sep 2 at 19:30

epi163sqrt
89.8k 6 85 208

1

Consider the series

$$S:=\sum_{m=r}^{\infty}\binom{m-1}{r-1}x^m$$

We have the following factorial relation:

$$\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$$

So that $\binom{m-1}{r-1}=\frac{r}{m}\binom{m}{r}$. Hence we have

$$S:=\sum_{m\geq r}\binom{m-1}{r-1}x^m=\sum_{m\geq r}\frac{r}{m}\binom{m}{r}x^m=\sum_{k\geq 0}\frac{r}{k+r}\binom{k+r}{r}x^{k+r}$$

For $\beta\in\mathbb{C}$ we have the binomial series $\frac{1}{(1-z)^{\beta+1}}=\sum_{k\geq 0}\binom{k+\beta}{k}z^k$.

Writing $\frac{1}{k+r}=\int_0^1 y^{k+r-1}\,dy$ we have:

$$\begin{aligned}S &= rx^r \sum_{k\geq 0} \binom{k+r}{r} x^k \int_0^1 y^{r+k-1}\,\mathrm{d}y \\ &= rx^r \int_0^1 \sum_{k\geq 0} \binom{k+r}{r} x^k y^{r+k-1}\,\mathrm{d}y \\ &= rx^r \int_0^1 y^{r-1} \sum_{k\geq 0} \binom{k+r}{r} (xy)^k\,\mathrm{d}y \\ &= rx^r \int_0^1 y^{r-1} \frac{1}{(1-xy)^{r+1}}\,\mathrm{d}y \\ &= \frac{rx^r}{r(1-x)^r} \\ &= \frac{x^r}{(1-x)^r}.\end{aligned}$$

The case where $x = \frac{1}{4}$ gives $S = \frac{1}{3^r}$.

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edited Sep 2 at 2:42
RobPratt
26.4k 3 16 38

answered Sep 1 at 11:28
NoName
2,185 1 7 19

0

Change the index of summation to $n := m - 1$. Your sum is then

$$\sum_{n=r-1}^{\infty} \binom{n}{r-1} \frac{1}{4^{n+1}} = \frac{1}{4} \sum_{n=r-1}^{\infty} \binom{n}{r-1} \frac{1}{4^n}.$$

Now apply the identity

$$\sum_{n=k}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}$$

with $k := r - 1$ and $x := \frac{1}{4}$ to obtain the result.

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answered Sep 1 at 18:46
grand_chat
30.3k 1 24 50

0

We can use the following identity (used to compute the power of an infinite Taylor series) directly to compute the sum:

$$\left(\sum_{k=1}^{\infty} x^k\right)^r = \sum_{m=r}^{\infty} \binom{m-1}{r-1} x^m$$

(refer to https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics) (example 4)

- We can understand the above equality with combinatorics, by using stars and bars method.

- Consider the ways to form m^{th} power of x on the RHS.
- The above is equivalent to placing m identical balls (RHS) into r boxes (LHS), so that each box contains at least one ball
- It can be done in $\binom{m-r+r-1}{r-1} = \binom{m-1}{r-1}$ ways.

Now, choosing $x = \frac{1}{4}$, from the RHS of the above identity, we have,

$$\sum_{m=r}^{\infty} \binom{m-1}{r-1} \left(\frac{1}{4}\right)^m = \left(\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k\right)^r = \left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right)^r = \frac{1}{3^r}$$

by using the formula for a sum of an infinite GP series.