MINIMUM

Problem

Let X_1, X_2, ... , X_{100} be independent random variables, all have the same uniform distribution over the interval $[0, 10]$. Let $W = \min\{X_1, X_2, ..., X_{100}\}$. Find $P(W \le x)$ and $E(W)$.

Solution. Let X be the random variable with uniform distribution over $[0, 10]$. Then

$$f_X(x) = \begin{cases} 1/10 & \text{for } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}, \quad \text{and } F_X(x) = P(X \le x) = \begin{cases} 0 & \text{if } x < 0 \\ x/10 & \text{if } 0 \le x \le 10 \\ 1 & \text{if } x > 1 \end{cases}.$$

Since all the X_n have the same distribution, $P(X_n > x) = P(X > x)$ for each $n = 1, 2, \ldots, 100$. (We will not need $f_X(x)$ below.)

Since

$$\{W \le x\} = \{\min\{X_1, X_2, ..., X_{100}\} > x\}' = \left\{\bigcap_{n=1}^{100} \{X_n > x\}\right\}',$$

we have

$$\begin{aligned} P(W \le x) &= P\left(\left\{\textstyle\bigcap_{n=1}^{100}\{X_n > x\}\right\}'\right) = 1 - P\left(\textstyle\bigcap\{X_n > x\}\right) = \\ &= 1 - P(X_1 > x) \cdots P(X_{100} > x) = 1 - [P(X > x)]^{100} = \\ &= 1 - [1 - P(X \le x)]^{100} = 1 - [1 - F_X(x)]^{100} = \\ &= \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1 - x/10)^{100} & \text{if } 0 \le x \le 10 \\ 1 & \text{if } x > 10 \end{cases} \end{aligned}$$

Since $F_W(x) = P(W \le x)$ and $f_W(x) = F_W'(x)$, we have

$$f_W(x) = \begin{cases} 0 & \text{if } x < 0 \\ 100\,(1 - x/10)^{99}/10 & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x > 10 \end{cases}.$$

So,

$$E(W) = \int_{-\infty}^{\infty} x f_W(x)\, dx = \int_0^{10} 10x\,(1 - x/10)^{99}\, dx = 10^3 \int_0^1 (1-u)u^{99}\, du = 10/101$$

where the substitution $u = 1 - x/10$ is used in evaluating the integral.

Answer

$$P(W \le x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1 - x/10)^{100} & \text{if } 0 \le x \le 10 \\ 1 & \text{if } x > 10 \end{cases}, \qquad E(W) = 10/101.$$

Question. Can you give an intuitive explanation of $E(W)$?

Problem Let X and Y be two independent random variables, X has a uniform distribution over $[5,10]$ and Y has a uniform distribution over $[7,10]$. Let $W = \min(X,Y)$. Find $E(W)$.

Solution 1. We first find $F_W(x) = P(W \le x)$. Since $\{W \le x\} = \{W > x\}' = \{\{X > x\} \cap \{Y > x\}\}'$,

$$P(W \le x) = 1 - P(\{X > x\} \cap \{Y > x\}) = 1 - P(X > x)P(Y > x). \tag{1}$$

We have

$$P(X > x) = \begin{cases} 1 & \text{if } x < 5 \\ (10-x)/5 & \text{if } 5 \le x \le 10 \\ 1 & \text{if } x > 10 \end{cases}, \quad P(Y > x) = \begin{cases} 1 & \text{if } x < 7 \\ (10-x)/3 & \text{if } 7 \le x \le 10 \\ 1 & \text{if } x > 10 \end{cases}.$$

Substituting these into (1), we get

$$F_W(x) = P(W \le x) = \begin{cases} 0 & \text{if } x < 5 \\ 1 - \frac{10-x}{5} & \text{if } 5 \le x \le 7 \\ 1 - \frac{10-x}{5} \cdot \frac{10-x}{3} & \text{if } 7 \le x \le 10 \\ 1 & \text{if } x > 10 \end{cases} \tag{2}$$

Simplifying and differentiating (2), we get

$$f_W(x) = \begin{cases} 0 & \text{if } x < 5 \\ 1/5 & \text{if } 5 \le x \le 7 \\ 2(10-x)/15 & \text{if } 7 \le x \le 10 \\ 0 & \text{if } x > 10 \end{cases} \quad \text{or} \quad f_W(x) = \begin{cases} 1/5 & \text{if } 5 \le x \le 7 \\ 2(10-x)/15 & \text{if } 7 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

So, $E(W) = \displaystyle\int_{-\infty}^{\infty} x f_W(x)\,dx = \int_5^7 x/5\,dx + \frac{2}{15}\int_7^{10} x(10-x)\,dx = \frac{36}{5} = 7.2.$

Solution 2. Since we consider the minimum of only two variables, it can be done using the joint distribution.[1] X and Y are independent, so their joint density is the product of the individual densities:

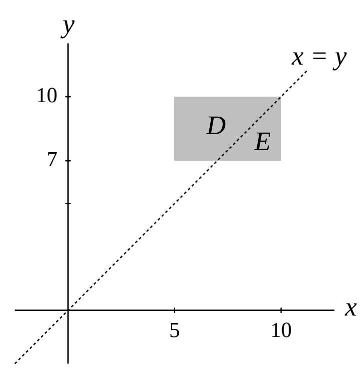
$$f_X(x) = \begin{cases} 1/5 & \text{if } 5 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 1/3 & \text{if } 7 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

imply

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & \text{if } 5 \le x \le 10,\ 7 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}.$$

In the picture to the right, $w = \min(x,y) = x$ in D and $w = \min(x,y) = y$ in E. So,

$$\begin{aligned} E(W) &= \iint w(x,y) f_{X,Y}(x,y)\,dxdy \\ &= \iint_D w/15\,dxdy + \iint_E w/15\,dxdy \\ &= \iint_D x/15\,dxdy + \iint_E y/15\,dxdy \\ &= 24/5 + 12/5 \ = \ 36/5. \end{aligned}$$



[1] If there are more than three variables, the domain of the joint density is in a space of dimensions higher than 3. It is hard to "visualize" the picture.