## **\*\*** MATHEMATICS

## Differentiating an Inner Product

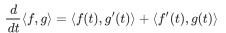
Asked 7 years, 10 months ago Active 7 years, 10 months ago Viewed 24k times



If  $(V, \langle \cdot, \cdot \rangle)$  is a finite-dimensional inner product space and  $f, g : \mathbb{R} \longrightarrow V$  are differentiable functions, a straightforward calculation with components shows that

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This approach is not very satisfying. However, attempting to apply the definition of the derivative directly doesn't seem to work for me. Is there a slick, perhaps intrinsic way, to prove this that doesn't involve working in coordinates?

real-analysis functional-analysis inner-product-space derivatives



asked Jan 4 '12 at 3:21

ItsNotObvious

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**8.341** 6 41

2 Answers



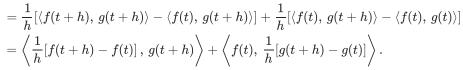
Observe that



$$\frac{1}{h} [\langle f(t+h), g(t+h) \rangle - \langle f(t), g(t) \rangle]$$

$$= \frac{1}{h} [\langle f(t+h), g(t+h) \rangle - \langle f(t), g(t) \rangle]$$







As h o 0 the first expression converges to

$$\frac{d}{dt}\langle f(t), g(t)\rangle$$

and the last expression converges to

$$\langle f'(t), g(t) \rangle + \langle f(t), g'(t) \rangle$$

by definition of the derivative, by continuity of q and by continuity of the scalar product. Hence the desired equality follows.

Note that this doesn't use finite-dimensionality and that the argument is the exact same as the one for the ordinary product rule from calculus.

edited Jan 4 '12 at 18:27

answered Jan 4 '12 at 3:36



66.3k

303



This answer may be needlessly complicated if you don't want such generality, taking the approach of first finding the Fréchet derivative of a bilinear operator.

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If V, W, and Z are normed spaces, and if  $T: V \times W \to Z$  is a continuous (real) <u>bilinear operator</u>, meaning that there exists  $C \ge 0$  such that  $||T(v,w)|| \le C||v|| ||w||$  for all  $v \in V$  and  $w \in W$ , then the <u>derivative</u> of T at  $(v_0,w_0)$  is  $DT|_{(v_0,w_0)}(v,w) = T(v,w_0) + T(v_0,w)$ . (I am assuming that  $V \times W$  is given a norm equivalent with  $||(v,w)|| = \sqrt{||v||^2 + ||w||^2}$ .) This follows from the straightforward computation

$$\frac{\|T(v_0+v,w_0+w)-T(v_0,w_0)-(T(v,w_0)+T(v_0,w))\|}{\|(v,w)\|}=\frac{\|T(v,w)\|}{\|(v,w)\|}\leq C\frac{\|v\|\|w\|}{\|(v,w)\|}\to 0$$

as  $(v, w) \to 0$ .

With  $V=W, Z=\mathbb{R}$  or  $Z=\mathbb{C}$ , and  $T:V\times V\to Z$  the inner product, this gives  $DT_{(v_0,w_0)}(v,w)=\langle v,w_0\rangle+\langle v_0,w\rangle$ . Now if  $f,g:\mathbb{R}\to V$  are differentiable, then  $F:\mathbb{R}\to V\times V$  defined by F(t)=(f(t),g(t)) is differentiable with  $DF|_t(h)=h(f'(t),g'(t))$ . By the chain rule,

$$\left.D(T\circ F)\right|_t(h) = \left.DT\right|_{F(t)}\circ DF\right|_t(h) = h(\left\langle f'(t),g(t)\right\rangle + \left\langle f(t),g'(t)\right\rangle),$$

which means  $\frac{d}{dt}\langle f,g \rangle = \langle f'(t),g(t) \rangle + \langle f(t),g'(t) \rangle$ .

answered Jan 4 '12 at 4:31

Jonas Meyer

43.4k 6 155 268

- I notice that  $\langle v, w_0 \rangle + \langle v_0, w \rangle$  is not linear in (v, w). (I.e. it's conjugate linear in w). So how can this be the derivative? I stumbled upon this page with exactly this question in mind. Eric Auld Jan 16 '15 at 1:02
- 3 \_\_\_ @Eric: It is real linear. That it what is used here, as noted above, "if T... is a continuous (real) bilinear operator...". Jonas Meyer Jan 16 '15 at 2:33 /