# What to do about non-measurability?

In proving Vitali's non-measurability theorem, we have learned a horrifying truth. We have learned that there is a precise sense in which it is impossible to assign a respectable measure to certain subsets of $[0, 1]$. How should one make sense of such a result? In this lecture we will explore some possible answers.

**An Experiment to the Rescue?**

The intimate connection between measurability and probability raises an intriguing possibility. What if there was an "experiment" that allowed us to assign a measure to $V$? Imagine a situation in which we are able to apply the Coin Toss Procedure again and again, to select points from $[0, 1]$. Couldn't we count the number of times we get a point in $V$, and use this information to assign a measure to $V$?

It goes without saying that it is impossible to carry out such an experiment in practice. Each application of the Coin Toss Procedure requires an infinite sequence of coin tosses. But let us idealize, and suppose that we are able to apply the Coin Toss Procedure (and indeed apply it infinitely many times). Notice, moreover, that in proving the theorem we didn't actually lay down a definite criterion for membership in $V$. All we did was show that some suitable set $V$ must exist. But let us idealize once more, and imagine that we are able to fix upon a specific Vitali set $V$, by bypassing the Axiom of Choice and selecting a particular member from each family of "siblings" in $[0, 1]$.

Would such an idealized experiment help us assign a measure to $V$? It seems to me that it would not. The first thing to note is that it is not enough to count the number of times the Coin Toss Procedure yields points $V$ and compare it with the number of times it yields points outside $V$. To see this, imagine applying the Coin Toss Procedure $\aleph_0$-many times, and getting the following results:

- Number of times procedure yielded a number in $V$: $\aleph_0$.
- Number of times procedure yielded a number outside $V$: $\aleph_0$.

Should you conclude that $P(V) = 50\%$? No! Notice, for example, that in such an experiment one would expect to get $\aleph_0$-many numbers in $[0, \frac{1}{4}]$ and $\aleph_0$-many numbers outside $[0, \frac{1}{4}]$. But that is no reason to think that $p([0, \frac{1}{4}]) = p\left(\overline{[0, \frac{1}{4}]}\right) = 50\%$. If the experiment is to be helpful, we need more than just information about the *cardinality* of outputs within $V$ and outputs outside $V$. We need a method for determining the *proportion* of total outputs that fall within $V$. And, as we have seen, cardinalities need not determine proportions.

One might wonder whether there is a strategy that one could use to try to identify the needed proportions "experimentally". Suppose one carries out the Coin Toss Procedure infinitely many times, once for each natural number. One then determines, for each finite $n$, how many of the first $n$ tosses fell within a given subset $A$ of $[0, 1]$. (Call this number $|A_n|$.) Finally, one lets the measure of $A$ be calculated experimentally, as follows:

$$\mu(A) = \lim_{n\to\infty} \frac{|A_n|}{n}$$

The hope is then that if one actually ran the experiment one would get the result that $\mu([0, \frac{1}{4}]) = \frac{1}{4}$. More generally, one might hope to get the result that $\mu(A)$ turns out to be the Lebesgue Measure of $A$ whenever $A$ is a Borel Set (or, indeed, a Lebesgue Measurable set).

Let us suppose that we run such an experiment in an effort to identify a measure for our Vitali set $V$. What sort of results one might expect? One possibility is that $\mu(V)$ fails to converge. In that case the experiment would give us no reason to think that $V$ has a measure. But suppose we run the experiment again and again, and that $\mu(V)$ always converges to the same number $m \in [0, 1]$. Would this give us any grounds for thinking that $m$ is the measure of $V$? It is not clear that it would. For consider the following question: what values might our experiment assign to $\mu(V_q)$ $(q \in \mathbb{Q}^{[0,1)})$?

There is no comfortable answer to this question. If we get the result that $\mu(V_q) = m$ for each $q \in \mathbb{Q}^{[0,1)]}$, it will thereby be the case that $\mu$ fails to be countably additive, since there is no real number $m$ such that $m + m + \cdots = 1$. So we will have grounds for thinking that something has gone wrong with our experiment. If, on the other hand, we get the result that $\mu(V_q)$ is not uniformly $m$ for each $q \in \mathbb{Q}^{[0,1)]}$, it will thereby be the case that $\mu$ fails to be uniform, since the $V_q$ $(q \in \mathbb{Q}^{[0,1)})$ are essentially copies of $V$, just positioned elsewhere in $[0, 1)$. So, again, we will have grounds for thinking that something has gone wrong with our experiment.

The moral is that it is not clear that we could use an experiment to help find a measure for $V$, even when we take certain idealizations for granted. For should we find that $\mu(V)$ converges to some value, it is not clear that we would be left with grounds for assigning a measure to $V$, rather than grounds for thinking that something went wrong with the experiment. In retrospect, this should come as no surprise. The non-measurability of $V$ follows from three very basic features of Lebesgue Measure: Non-Negativity, Countable Additivity and Uniformity. So we know that no version of our experiment could deliver a value for $V$ without violating at least one of these assumptions. And the price of violating the assumptions is that it's no longer clear that the experiment delivers something worth calling a Lebesgue Measure.

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