Some Additional Results

We have shown that $|\mathbb{N}| < |\,[0,1)\,|$, and therefore that $|\mathbb{N}| < |\mathbb{R}|$. Notice, however, that this does not immediately settle the question of whether $[0,1)$ and $\mathbb{R}$ have the same cardinality, since nothing we've proved so far rules out $|\mathbb{N}| < |\,[0,1)\,| < |\mathbb{R}|$. As you'll be asked to prove below, they do have the same cardinality:

$$|\,[0,1)\,| = |\mathbb{R}|$$

This is a non-trivial result. It means that just like there are different infinite sets that have the same size as the natural numbers, so there are different infinite sets that have the same size as the real numbers.

The exercises below will ask you to show that the following sets have the same cardinality as the real numbers:

Set	Also known as	Members
$(0,1)$	unit interval (open)	real numbers larger than 0 but smaller than 1
$[0,1]$	unit interval (closed)	real numbers larger or equal to 0 but smaller or equal to 1
$[0,a]$	arbitrarily sized interval	real numbers larger or equal to 0 but smaller or equal to a $(a>0)$
$[0,1]\times[0,1]$	unit square	pairs of real numbers larger or equal to 0 but smaller or equal to 1
$[0,1]\times[0,1]\times[0,1]$	unit cube	triples of real numbers larger or equal to 0 but smaller or equal to 1
$\underbrace{[0,1]\times\cdots\times[0,1]}_{n \text{ times}}$	n-dimensional hypercube	n-tuples of real numbers larger or equal to 0 but smaller or equal to 1

Some examples of sets with the same cardinality as $\mathbb{R}$.

As you work through the exercises, you'll find it useful to avail yourself of the following result: *adding natural-number-many members to an infinite set doesn't change the set's cardinality,.*

More precisely:

No Countable Difference Principle
$|S| = |S \cup A|$, whenever S is infinite and A is countable.

(For a set S to be **infinite** is for it to be the case that $|\mathbb{N}| \leq |S|$; for a set A to be **countable** is for it to be the case that $|A| \leq |\mathbb{N}|$.)

Problem 1

0.0/1.0 point (ungraded)

Have a go at proving the No Countable Difference Principle. (After you've thought about it for a bit, feel free to look at the video below.)

No Countable Difference Principle
$|S| = |S \cup A|$, whenever S is infinite and A is countable.

If you'd like some help, click *Hint* below.

☐ Done

? **Hint (1 of 1):** Try proceeding in the following two steps:

Next Hint

- **Step 1.** Since S is infinite (and therefore $|\mathbb{N}| \leq |S|$) we know that there is an injective function from the natural numbers to S. Let the range of that function be the set $S^{\mathbb{N}} = s_0, s_1, s_2, \ldots$, and show that $|S^{\mathbb{N}}| = |S^{\mathbb{N}} \cup A|$.
- **Step 2.** Use the fact that that $|S^{\mathbb{N}}| = |S^{\mathbb{N}} \cup A|$ to show that $|S| = |S \cup A|$.

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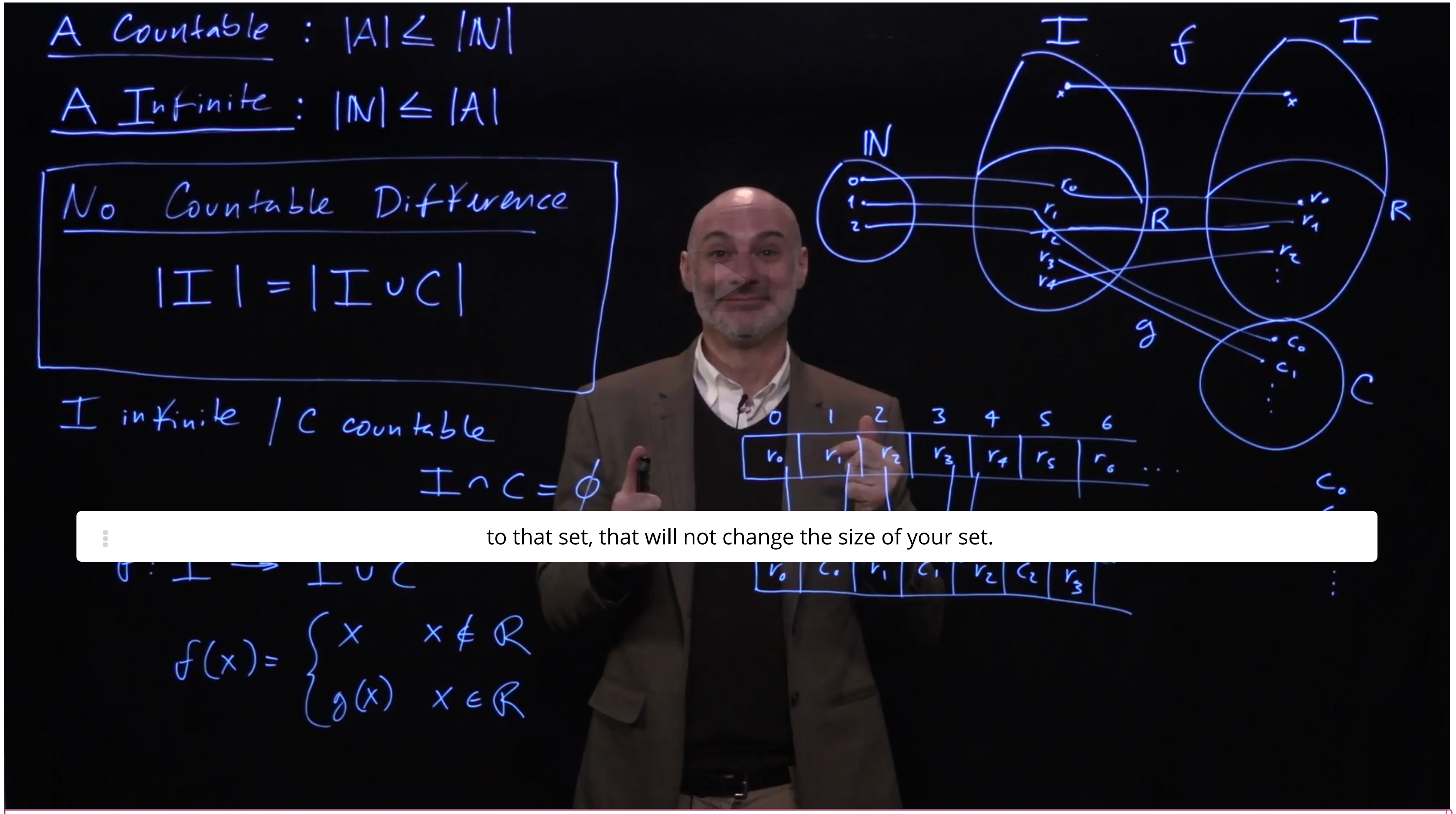
Video: the No Countable Difference Principle



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Video

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Problem 2

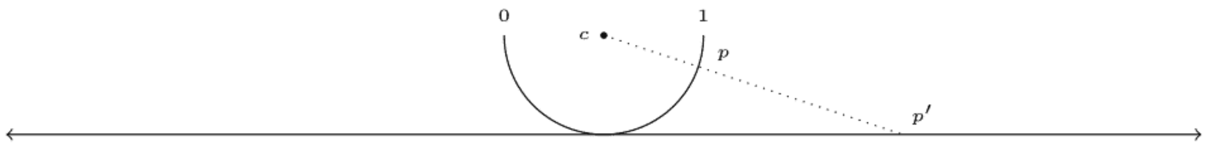
1/1 point (ungraded)

Recall that the open interval $(0, 1)$ is the set of of the real numbers x such that $0 < x < 1$.

Is it true that $|\,(0, 1)\,| = |\mathbb{R}|$?

Yes, it's true. ✔ **Answer:** Yes, it's true.

Explanation
Here's one way of defining a bijection from $(0, 1)$ to $\mathbb{R}$.
Start by bending the open line segment $(0, 1)$ into the bottom half of a semicircle, as shown in the figure below. Next, rest the semicircle on the real line. Let c be the center of the semicircle, and consider the family of "rays" emanating from c that cross the semicircle and reach the real line. The dotted line in the figure below is an example of one such ray: it crosses the semicircle at p and reaches the real line at p'. Note that each ray in our family crosses exactly one point on the semicircle and one point on the real line. Notice, moreover, that each point on the semicircle, and each point on the real line, is crossed by exactly one ray. This means that one can define a bijection from $(0, 1)$ to $\mathbb{R}$ by assigning each point on the semicircle to the point on the real line with whom it shares ray. (Note that it is essential to this proof that we work with the open interval $(0, 1)$ rather than the closed interval $[0, 1]$, since a ray crossing the semicircle at point 0 or 1 would fail to reach the real line.)



A bijection from $(0, 1)$ to $\mathbb{R}$.

A version of the same idea can be used to define a bijection from $(0, 1)$ to $\mathbb{R}$ analytically rather than geometrically. First note that the tangent function $tan\,(x)$ is a bijection from $(-\pi/2, \pi/2)$ to the reals. Next, note that the function $f\,(x) = \pi\,(x - \frac{1}{2})$ is a bijection from $(0, 1)$ to $(-\pi/2, \pi/2)$. Putting the two together, we get the result that $tan \circ f$ is a bijection from $(0, 1)$ to the reals.

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Answers are displayed within the problem

Problem 3

1/1 point (ungraded)

Recall that the closed interval $[0,1]$ is the set of the real numbers x such that $0 \leq x \leq 1$, and that $[0,1)$ is the set of the real numbers x such that $0 \leq x < 1$.

Is it true that $|\,[0,1]\,| = |\mathbb{R}|$?

Yes, it's true. ✔ **Answer:** Yes, it's true.

Is it true that $|\,[0,1)\,| = |\mathbb{R}|$?

Yes, it's true. ✔ **Answer:** Yes, it's true.

(If it is true, try proving it. If it isn't true, try to explain why.)

Explanation

It is easy to verify that $(0,1)$ is infinite. (For instance, $f(n) = \frac{1}{n+2}$ is an injective function from $\mathbb{N}$ to $(0,1)$.)
Since $|\,(0,1)\,| = |\,(0,1) \cup \{0,1\}|$, it follows from the No Countable Difference Principle that $|\,(0,1)\,| = |\,[0,1]\,|$.
Since $|\,(0,1)\,| = |\,(0,1) \cup \{0\}|$, it follows from the No Countable Difference Principle that $|\,(0,1)\,| = |\,[0,1)\,|$.
But in an earlier exercise (Problem 2, above) we showed that $|\,(0,1)\,| = |\mathbb{R}|$. So it follows from the symmetry and transitivity of bijections that $|\,[0,1]\,| = |\mathbb{R}|$ and $|\,[0,1)\,| = |\mathbb{R}|$.

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Problem 4

1/1 point (ungraded)

For any $a \in \mathbb{R}$ such that $a > 0$, $|\,[0,a]\,| = |\mathbb{R}|$.

True or False?

True ⌄ ✔ **Answer:** True

(If it's true, try to prove it. If it's false, try to explain why.)

Explanation
It's true. Here's why:
$f(x) = ax$ is a bijection from $[0,1]$ to $[0,a]$. Since we know that $|\,[0,1]\,| = |\mathbb{R}|$, it follows from the symmetry and transitivity of bijections that $|\,[0,a]\,| = |\mathbb{R}|$

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Answers are displayed within the problem

Problem 5

1/1 point (ungraded)
Just like one can think of the interval $[0,1]$ as representing a line of length 1, so one can think of the points in the cartesian product $[0,1] \times [0,1]$ as representing a square of side-length 1. (In general, the **Cartesian product** $A \times B$ is the set of pairs $\langle a, b \rangle$ such that $a \in A$ and $b \in B$.)

Is it true that there are just as many points on the unit line as there are points on the unit square?:

$$|\,[0,1]\,| = |\,[0,1] \times [0,1]\,|$$

If you'd like some help, click *Hint* below.

Yes, it's true. ⌄ ✔ **Answer:** Yes, it's true.

Explanation
Step 1. The following f is a bijection from D to $D \times D$:

$$f(\langle d_0, d_1, d_2, \ldots \rangle) = \langle \langle d_0, d_2, d_4, \ldots \rangle, \langle d_1, d_3, d_5, \ldots \rangle \rangle$$

Step 2. Think of each sequence $\langle d_0, d_1, d_2, \ldots \rangle$ in D as the decimal expansion

$$0.d_0, d_1, d_2, \ldots$$

Then every real number in $[0, 1]$ is named by some sequence in D, and every sequence in D names some real number in $[0, 1]$. Unfortunately, this does not yet give us a bijection between D and $[0, 1]$ because D includes decimal expansions that end in an infinite sequence of 9s. So there are numbers in $[0, 1]$ that are named by two different sequences in D.

Let D^- be the result of removing from D every sequence ending in an infinite sequence of 9s. Since D^- consists of exactly one name for each number in $[0, 1]$, we have $|\,[0, 1]\,| = |D^-|$. By the symmetry and transitivity of bijections, this means that in order to prove $|\,[0, 1]\,| = |D|$, it suffices to show that $|D| = |D^-|$.

Notice, however, that there are only countably sequences in D that are not in D^-. (There are only countably many rational numbers, and every decimal expansion ending in an infinite sequence of 9s is a rational number.) So it follows form the No Countable Difference Principle that $|D| = |D^-|$.

Step 3. The upshot of Step 2 is that there is a bijection f from $[0, 1]$ to D. One can use f to define a bijection g from $[0, 1] \times [0, 1]$ to $D \times D$, as follows:

$$\text{For each } \langle x, x' \rangle \text{ in } [0, 1] \times [0, 1], \text{ let } g\left(\langle x, x' \rangle\right) = \langle f\left(x\right), f\left(x'\right) \rangle.$$

Step 4. The result follows immediately from the symmetry and transitivity of bijections.

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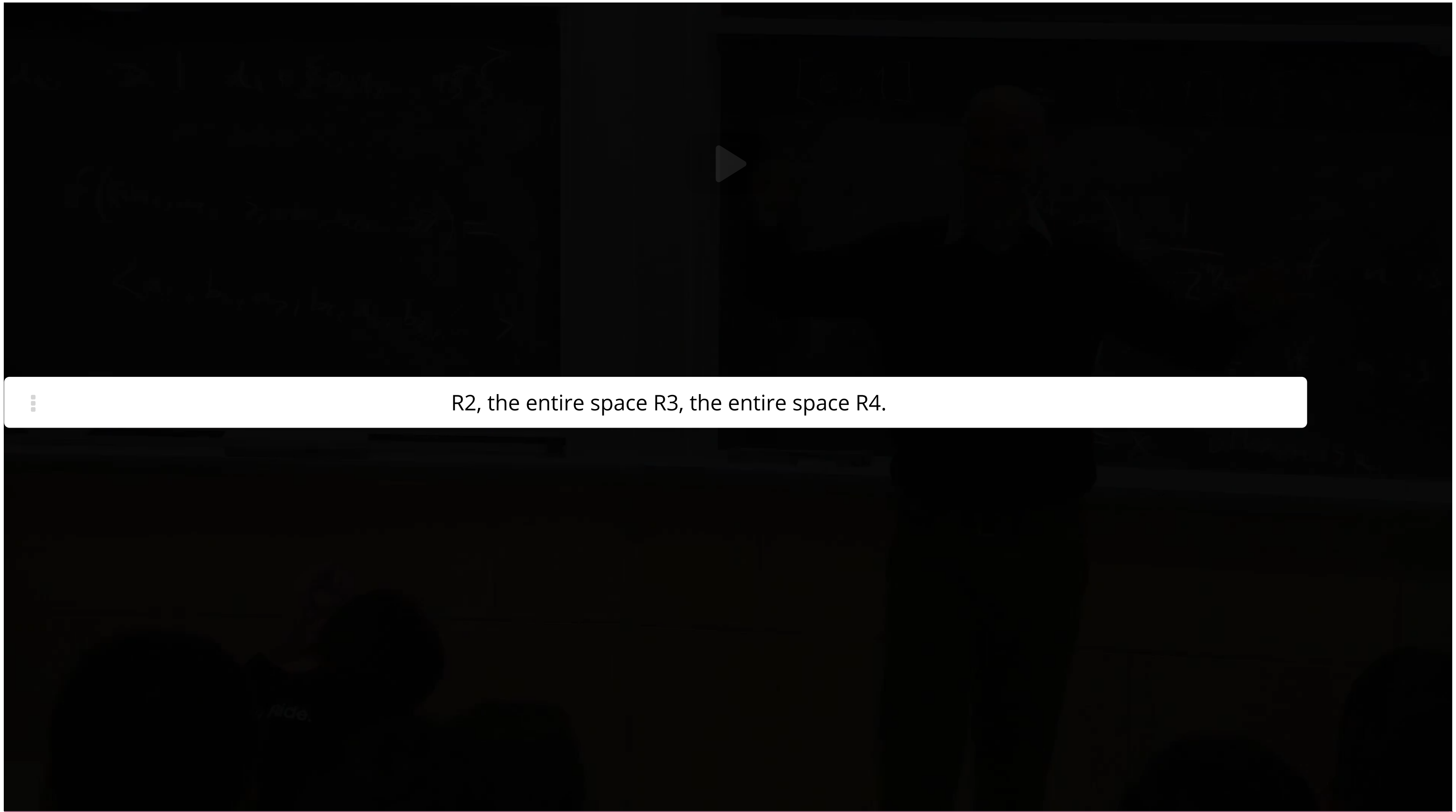
Answers are displayed within the problem

Video: Proving that |[0,1]| = |[0,1] x [0,1]|



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Video
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Problem 6

1/1 point (ungraded)

Just like one can think of the points in the cartesian product $[0,1] \times [0,1]$ as representing a square of side length 1, so one can think of the points in $\underbrace{[0,1] \times \ldots \times [0,1]}_{n \text{ times}}$ as an n-dimensional *hypercube*.

Is it true that $|\,[0,1]\,| = |\underbrace{[0,1] \times \ldots \times [0,1]}_{n \text{ times}}|$?

Yes, it's true. ✔ **Answer:** Yes, it's true.

(If it's true, try to prove it. If it's false, try to explain why.)

Explanation

We know from the previous result that there is a bijection f from $[0,1]$ to $[0,1] \times [0,1]$. Notice that f can be used to define a bijection f' from $[0,1]$ to $[0,1] \times [0,1] \times [0,1]$, as follows:

$$\text{if } f(x) = \langle x_1, x_2 \rangle, \text{ then } f'(x) = \langle x_1, f_2(x_2) \rangle$$

This technique can be iterated to generate a bijection from $[0,1]$ to $\underbrace{[0,1] \times \ldots \times [0,1]}_{n \text{ times}}$ for any finite n.

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Typo in Solution of Problem 6? 3

A more general relation 6

I have heard a more general theorem if $|A|$ is infinite and $|B|$ is smaller than or equal to $|A|$ then $|A \cup B| = |A|$, what is the proof?