MATHEMATICS

Stochastic Processess question clarification

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A grocery store is stocked every morning with a pack of cookies. The grocer has noticed that the daily demand is a random variable X with distribution:

$$P[X=0]=\frac{1}{4},$$

$$P[X=1]=\frac{1}{2},$$

$$P[X=2]=\frac{1}{6},$$

$$P[X=3]=\frac{1}{12}$$

.

Describe the quantity of cookies that each night the grocery store has as a Markov Chain and find its stationary distribution.

If last night there was no pack of cookies at the grocery store, what is the expected number of days until the next time the grocery store runs out of cookies?

My first calculations:

For $k=0$

$$k=0\left\{\begin{matrix} p(0,1)=\frac{1}{4} \\ p(0,0)=\frac{3}{4} \end{matrix}\right\}$$

For $k=1$

$$k=1\left\{\begin{matrix} p(1,2)=\frac{1}{4} \\ p(1,1)=\frac{1}{2} \\ p(1,0)=\frac{1}{4} \end{matrix}\right\}$$

For $k=2$

$$k\geq 2\left\{\begin{matrix} p(k,k+1)=\frac{1}{4} \\ p(k,k)=\frac{1}{2} \\ p(k,k-1)=\frac{1}{6} \end{matrix}\right\}$$

$$\left. p(k,k-2) = \tfrac{1}{12} \right\}$$

$$\sum_x \pi(x) = 1$$

$$\pi_x = \sum_y \pi(y)p(y,x)$$

$$\begin{aligned}\pi_0 &= \frac{3}{4}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{12}\pi_2 \\ \pi_0 &= \pi_1 + \frac{1}{3}\pi_2\end{aligned}$$

For $k \geq 2$ we have :

$$\begin{aligned}\pi_{k-1} - \pi_k + \pi_{k+1} + \pi_{k+2} &= 0 \\ \frac{1}{4}\pi_{k-1} - \frac{1}{2}\pi_k + \frac{1}{6}\pi_{k+1} + \frac{1}{12}\pi_{k+2} &= \pi_k \\ \frac{1}{4}\pi_{k-1} + \frac{1}{6}\pi_{k+1} + \frac{1}{12}\pi_{k+2} &= \frac{1}{2}\pi_k \\ \frac{24}{4}\pi_{k-1} + \frac{24}{6}\pi_{k+1} + \frac{24}{12}\pi_{k+2} &= 12\pi_k \\ 6\pi_{k-1} + 4\pi_{k+1} + 2\pi_{k+2} &= 12\pi_k \\ 3\pi_{k-1} + 2\pi_{k+1} + \pi_{k+2} &= 6\pi_k \\ -6\pi_k + 3\pi_{k-1} + 2\pi_{k+1} + \pi_{k+2} &= 0\end{aligned}$$

Substituting we have:

$$\lambda^3 + 2\lambda^2 - 6\lambda + 3 = 0$$

Simplifying:

$$(\lambda - 1)(\lambda^2 + 3\lambda - 3)$$

So $\lambda_1 = 1$

$$\lambda_{2,3} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \left\{ \begin{array}{l} \lambda_2 = \frac{-3+\sqrt{21}}{2} \\ \lambda_3 = \frac{-3-\sqrt{21}}{2} \end{array} \right\}$$

λ_3 is excluded because is negative and we are left with λ_2 .

Someone else step from here:

A friend of mine suggested that because $\alpha_0 = 0$ as $k \to \infty$:

$$\begin{aligned}\pi_k &= \alpha_0 1^k + \alpha_1\lambda_2^k \\ \pi_k &= \alpha_1\lambda_2^k\end{aligned}$$

Therefore:

$$\sum_k \pi_k = \alpha_1$$
$$\sum_{k=0}^{\infty} \lambda_2^k = \alpha_1 \frac{1}{1-\lambda_2}$$
$$\alpha_1 = (1-\lambda_2)$$

Thus:

$$\pi_k = \frac{2}{2} - \frac{-3+\sqrt{21}}{2} = \frac{5-\sqrt{21}}{2} = 0.2087122$$

and the expected time is :

$$\mathbb{E}_0[T_0^+] = \frac{1}{\pi_0} = \frac{2}{5-\sqrt{21}} = 4.791288$$

Question:

I found that $\pi_0 = \pi_1 + \frac{1}{3}\pi_2 \quad (1).$

Now probably he did :

$$\pi_k = \alpha_0 \lambda_1^k + \alpha_1 \lambda_1^k + \alpha_2 \lambda_2^k$$

and he applied it to (1) as :

$$\alpha_0 + \alpha_1 \lambda_2^0 = \alpha_0 + \alpha_1 \lambda_2^1 + \frac{1}{3}\alpha_0 + \frac{1}{3}\alpha_1 \lambda_2^2$$

doing the calculation I ended to :

$$\frac{1}{3}\alpha_0 = \alpha_1(1-\lambda_2 - \frac{1}{3}\lambda_2^2)$$

1)

and from here why he said $\frac{1}{3}\alpha_0 = 0$,so $\alpha_0 = 0$? and

2) how he concluded that $\sum_{k=0}^{\infty} \lambda_2^k = \alpha_1 \frac{1}{1-\lambda_2}$?$

probability-theory stochastic-processes markov-chains

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asked 4 hours ago

Hungry Homer
11 3

1 Answer

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I think

(1) is because π_k represents a (steady state) probability, so $0 \le \pi_k \le 1$ and hence it can't have the term 1^k as part of general solution of the recurrence relation, since $\sum\limits_k \pi_k = 1$ but $\sum\limits_k 1^k$ is divergent as an infinite series sum, so the only thing possible here is that the coefficient of 1^k, i.e., α_0 is 0.

(2) Actually, there is a line-break in between, it should be

$$\sum_k \pi_k = \alpha_1 \sum_{k=0}^{\infty} \lambda_2^k = \alpha_1 \frac{1}{1-\lambda_2} = 1$$
$$\implies \alpha_1 = (1-\lambda_2)$$

The sum being a sum of an infinite GP series with $|\lambda_2| < 1$.

Also, another way, with the following row-stochastic Markov Transition probability matrix, we can compute the stationary distribution with power-iteration as follows, for example with `python` :

Markov Transition Matrix

	# Cookies in (i+1)-th night													
#Cookies i-th night	0	1	2	3	4	5	6	…	k-2	k-1	k	k+1	…
0	3/4	1/4	0	0	0	0	0	…					…
1	1/4	1/2	1/4	0	0	0	0						
2	1/12	1/6	1/2	1/4	0	0	0						
3	0	1/12	1/6	1/2	1/4	0	0						
…	…							…					…
k	0	0	0	0	0	0	0	…	1/12	1/6	1/2	1/4	…
…	…							…					…

```
n = 100
A = np.zeros((n+1, n+1))
A[0,1], A[0,0] = 1/4, 3/4
A[1,2], A[1,1], A[1,0] = 1/4, 1/2, 1/4
for k in range(2, n):
    A[k,k+1], A[k,k], A[k,k-1], A[k,k-2] = 1/4, 1/2, 1/6, 1/12
A[n,n], A[n,n-1], A[n,n-2] = 3/4, 1/6, 1/12

# power iteration
pA = A
for i in range(10000):
   pA = pA @ A

# stationary distribution
pa
#array([[2.08712153e-01, 1.65151390e-01, 1.30682288e-01, ...,
#     2.29223403e-11, 1.71917552e-11, 1.71917552e-11],
#    [2.08712153e-01, 1.65151390e-01, 1.30682288e-01, ...,
#     2.29223403e-11, 1.71917552e-11, 1.71917552e-11],
#    [2.08712153e-01, 1.65151390e-01, 1.30682288e-01, ...,
#     2.29223403e-11, 1.71917552e-11, 1.71917552e-11],
#
```

```
#    ...,
#    [2.08712153e-01, 1.65151390e-01, 1.30682288e-01, ...,
#     2.29223403e-11, 1.71917552e-11, 1.71917552e-11],
#    [2.08712153e-01, 1.65151390e-01, 1.30682288e-01, ...,
#     2.29223403e-11, 1.71917552e-11, 1.71917552e-11],
#    [2.08712153e-01, 1.65151390e-01, 1.30682288e-01, ...,
#     2.29223403e-11, 1.71917552e-11, 1.71917552e-11]])
```

with probability that grocery has no cookies pack (next night) as 0.208712153, irrespective of the number of cookies pack in the last night.

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answered 4 mins ago

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