1. Mixed boundary conditions

Finding eigenvalues and eigenfunctions

2/2 points (graded)

Consider a thin insulated metal rod of length 1, which satisfies the differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 < x < 1,\ t > 0.$$

Initially at $t = 0$, the temperature of the rod is given by $\theta(x, 0) = f(x)$. Then the left end is placed in an ice bath and held at $0^\circ \mathrm{C}$, and the right end is insulated.

Use separation of variables $\theta(x, t) = v(x)\, w(t)$ to reduce this PDE to the system

$$\begin{aligned} \frac{d^2}{dx^2} v(x) &= \lambda v(x) \\ \frac{d}{dt} w(t) &= \lambda w(t). \end{aligned}$$

Find all eigenvalues λ_k and eigenfunctions $v_k(x)$ that satisfy the boundary conditions specified in this problem for $k = 0, 1, 2, \ldots$.

For $k = 0, 1, 2, 3, \ldots,\ \lambda_k =$ -(2*k+1)^2*pi^2/4 ✔ **Answer:** -((2*k+1)*pi/2)^2

$$-\frac{(2\cdot k+1)^2 \cdot \pi^2}{4}$$

For $k = 0, 1, 2, 3, \ldots,$ $v_k(x) =$ sin((2*k+1)*pi/2*x) ✔ **Answer:** sin((2*k+1)*pi*x/2)

$\sin\left(\frac{(2\cdot k+1)\cdot\pi}{2}\cdot x\right)$

FORMULA INPUT HELP

Solution:

Note that the boundary condition are

$$\begin{aligned} \theta(0,t) &= 0, \qquad t > 0 \\ \frac{\partial\theta}{\partial x}(1,t) &= 0, \qquad t > 0 \end{aligned}$$

Setting $\theta(x,t) = v(x)\,w(t)$ gives boundary conditions

$$\begin{aligned} v(0) &= 0 \\ \frac{dv}{dx}(1) &= 0. \end{aligned}$$

Start by solving $\frac{d^2}{dx^2}v(x) = \lambda v(x)$.

Case 1: If $\lambda > 0$, then the general solution is

$$v(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

The boundary conditions give

$$\begin{aligned} v(0) &= c_1 + c_2 = 0 \qquad \longrightarrow c_2 = -c_1 \\ v'(1) &= c_1\sqrt{\lambda}e^{\sqrt{\lambda}} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}} \end{aligned}$$

$$= c_1\sqrt{\lambda}\left(e^{\sqrt{\lambda}}+e^{-\sqrt{\lambda}}\right)=0$$

Because $e^y > 0$ for any value of y, it follows that the only way for these boundary condition to hold are if $c_1 = c_2 = 0$, the zero solution.

Case 2: If $\lambda = 0$, then the general solution is

$$v\left(x\right)=c_1 x+c_2$$

The boundary conditions give

$$\begin{aligned} v\left(0\right) &= c_2=0 \\ v'\left(1\right) &= c_1=0 \end{aligned}$$

Again, only the zero solution.

As usual, the only case in which there are multiple solutions is when $\lambda < 0$. In this case, the general solution for $v\left(x\right)$ is

$$v\left(x\right)=A\cos\left(\sqrt{-\lambda}x\right)+B\sin\left(\sqrt{-\lambda}x\right).$$

Solving for the boundary conditions we find

$$\begin{aligned} v\left(0\right) &= A=0 \\ v'\left(1\right) &= B\sqrt{-\lambda}\cos\left(\sqrt{-\lambda}\right)=0 \end{aligned}$$

Note that this equality holds if $B = 0$ (the zero solution) or if $\cos\left(\sqrt{-\lambda}\right)=0$. This holds whenever

$$\sqrt{-\lambda}=\frac{2k+1}{2}\pi,\qquad k=0,1,2,\ldots.$$

In other words, the eigenvalues are $\lambda = -\left(\frac{(2k+1)\pi}{2}\right)^2$, for $k = 0, 1, 2, \ldots$, and the corresponding eigenfunctions of amplitude 1 (i.e. setting $B = 1$) are

$$v_k(x) = \sin\left(\frac{(2k+1)\pi}{2}x\right).$$

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You have used 3 of 7 attempts

Answers are displayed within the problem

Sketch the eigenfunctions

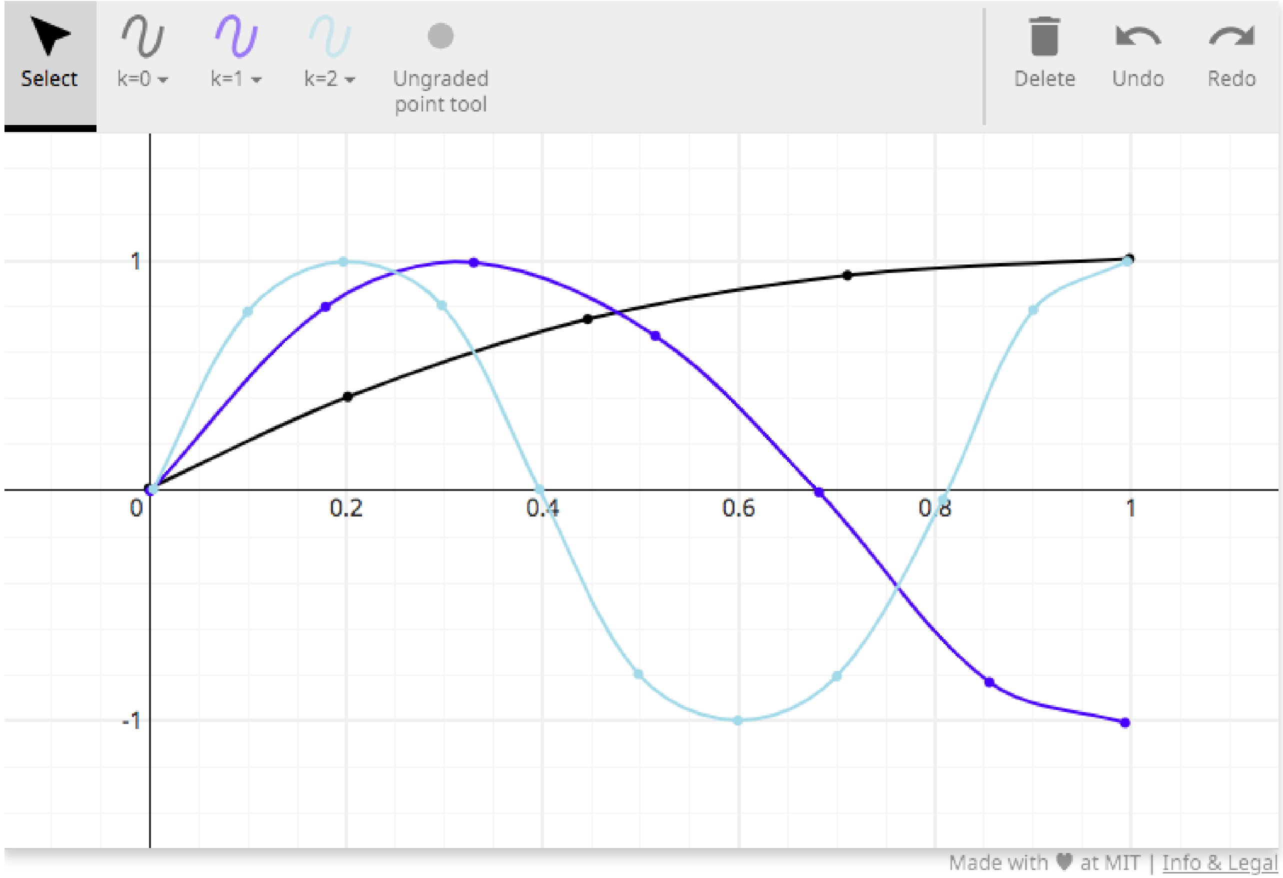
1.0/1 point (graded)

Sketch the first three eigenfunctions (of amplitude 1) for $k = 0$, $k = 1$, and $k = 2$ in the graph below.

(Note that you must use the specified drawing tool for each eigenfunction. Do not use the same tool to draw all three functions.)

Note that you can use the freeform tool, or a spline tool to draw these functions. Click the arrow in the the tool select menu to select a different tool.

- The freeform tools draws what you draw (slightly smoothed for readability). This is the tool we have mainly used in this course.
- The spline tool lets you click on a series of points, and will automatically generate a curve connecting these points. You can edit the curve by clicking select and clicking and draging the clicked points after you finish the curve.



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Find the normal modes

1/1 point (graded)

Find the normal modes $\theta_k(x,t) = v_k(x)\,w_k(t)$ for $k = 0, 1, 2, \ldots$.

(Enter with amplitude 1.)

sin((2*k+1)*pi/2*x)*e^(-(2*k+1)^2*pi^2/4*t)

✔ **Answer:** e^(-((2*k+1)*pi/2)^2*t)*sin((2*k+1)*pi*x/2)

$\sin\left(\frac{(2\cdot k+1)\cdot\pi}{2}\cdot x\right)\cdot e^{-\frac{(2\cdot k+1)^2\cdot\pi^2}{4}\cdot t}$

FORMULA INPUT HELP

Solution:

The corresponding normal modes are found by multiplying the eigenfunctions $v_k(x)$ by the corresponding $w_k(t)$.

For $\lambda = -\left(\frac{(2k+1)\pi}{2}\right)^2$, the functions $w_k(t)$ are

$$w_k(t) = e^{-\left(\frac{(2k+1)\pi}{2}\right)^2 t}.$$

Hence the normal modes (of amplitude 1) are

$$\theta_k(x,t) = v_k(x)\,w_k(t) = e^{-\left(\frac{(2k+1)\pi}{2}\right)^2 t}\sin\left(\frac{(2k+1)\pi}{2}x\right).$$

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You have used 1 of 7 attempts

Answers are displayed within the problem

Identify the appropriate initial condition

0.88/1 point (graded)

In order to solve

$$f\left(x\right)=\sum_{k=0}^{\infty}b_{k}v_{k}\left(x\right),$$

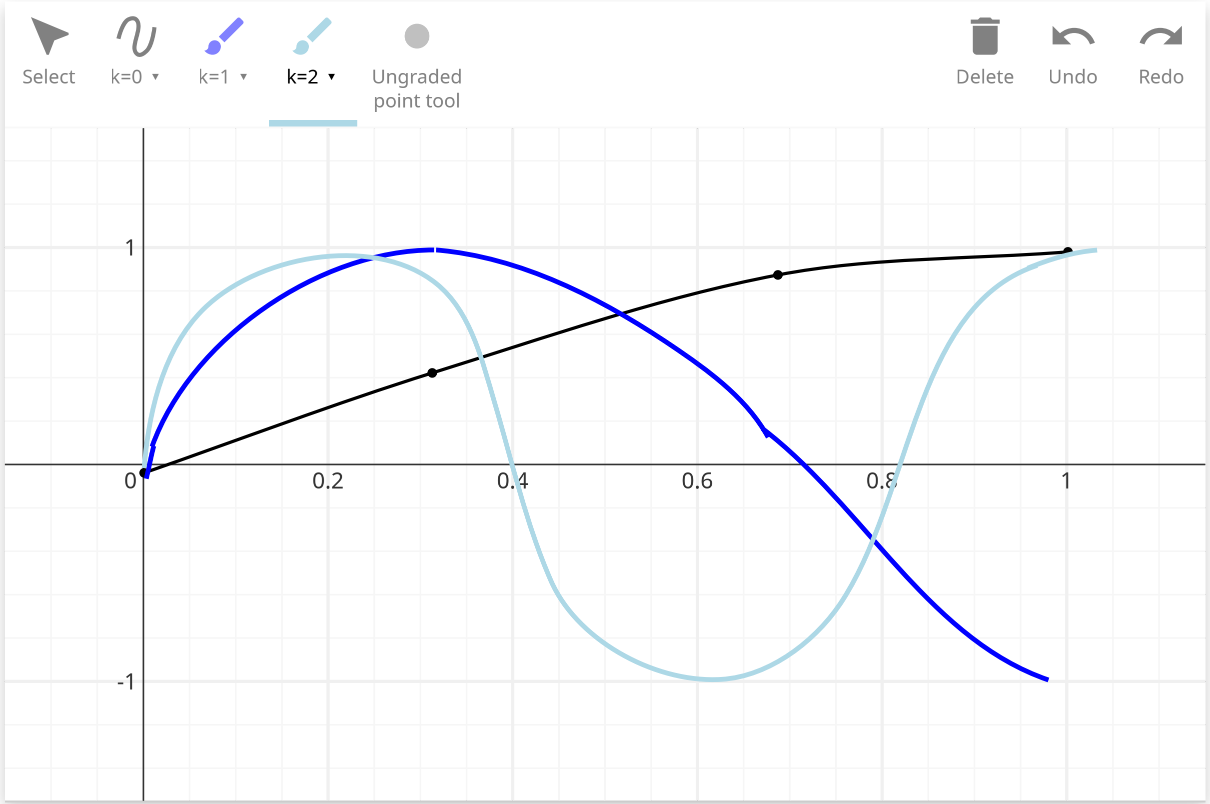
what must be true of the type of periodic extension of $f\left(x\right)$?

- [ ] the periodic extension of $f\left(x\right)$ must be even
- [x] the periodic extension of $f\left(x\right)$ must be odd ✱
- [ ] the periodic extension of $f\left(x\right)$ must have period 1
- [ ] the periodic extension of $f\left(x\right)$ must have period 2
- [ ] the periodic extension of $f\left(x\right)$ must have period 3
- [x] the periodic extension of $f\left(x\right)$ must have period 4 ✱
- [ ] the periodic extension of $f\left(x\right)$ must be symmetric about the line $x=1$ ✓
- [ ] the periodic extension of $f\left(x\right)$ must be antisymmetric about the line $x=1$

✱

Solution:

When $t=0$, we must solve



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Answer: .

Good job!

Solution:

- The $k=0$ eigenfunction of amplitude 1 is $\sin\left(\pi x/2\right)$.
- The $k=1$ eigenfunction of amplitude 1 is $\sin\left(3\pi x/2\right)$.
- The $k=2$ eigenfunction of amplitude 1 is $\sin\left(5\pi x/2\right)$.

$$f(x) = \sum_{k=0}^{\infty} b_k \sin\left(\frac{2k+1}{2}\pi x\right)$$

Because the Fourier series involves only sine terms of base period 4 (the period of $\sin(\pi x/2)$), we must extend $f(x)$ to be an odd periodic function of period 4 in order to find coefficients b_k that solve this initial condition.

However, note that such a Fourier series has the form

$$\sum_{k=0}^{\infty} b_k \sin\left(\frac{k}{2}\pi x\right).$$

In order for the sine series to involve only the odd periodic terms, there must be a further symmetry. This further symmetry requires that $f(x)$ is symmetric about the line $x = 1$. Note that in this case we can define $f(x)$ on the interval $0 < x < 2$ by

$$F(x) = \begin{cases} f(x) & 0 < x < 1 \\ f(2-x) & 1 < x < 2 \end{cases}$$

Computing the coefficients for such a function using the fact that $f(x)$ is odd and of period $2L = 4$ we find

$$\begin{aligned} b_k &= \frac{2}{2}\int_0^2 F(x)\sin\left(\frac{k}{2}\pi x\right)\,dx \\ &= \int_0^1 f(x)\sin\left(\frac{k}{2}\pi x\right)\,dx + \int_1^2 f(2-x)\sin\left(\frac{k}{2}\pi x\right)\,dx \\ &= \int_0^1 f(x)\sin\left(\frac{k}{2}\pi x\right)\,dx - \int_1^0 f(u)\sin\left(\frac{k}{2}\pi(2-u)\right)\,du \\ &= \int_0^1 f(x)\sin\left(\frac{k}{2}\pi x\right)\,dx + \int_0^1 f(u)\sin\left(\frac{k}{2}\pi(2-u)\right)\,du, \end{aligned}$$

where the second to last line is obtained by making the substitution $u = 2 - x$ in the second integral. The sine addition forumla tells us that

$$\begin{aligned}
\sin\left(\frac{k}{2}\pi(2-u)\right) &= \sin(k\pi)\cos\left(\frac{k}{2}\pi u\right) - \sin\left(\frac{k}{2}\pi u\right)\cos(k\pi) \\
&= -\sin\left(\frac{k}{2}\pi u\right)(-1)^k \\
&= (-1)^{k+1}\sin\left(\frac{k}{2}\pi u\right).
\end{aligned}$$

Plugging back into the the last line we get

$$\begin{aligned}
b_k &= \int_0^1 f(x)\sin\left(\frac{k}{2}\pi x\right)\,dx + (-1)^{k+1}\int_0^1 f(u)\sin\left(\frac{k}{2}\pi u\right)\,du \\
&= \begin{cases} 2\int_0^1 f(x)\sin\left(\frac{k}{2}\pi x\right)\,dx & k \text{ odd} \\ 0 & k \text{ even.} \end{cases}
\end{aligned}$$

Thus the necessary symmetry requires that $f(x)$ be odd, period 4, and symmetric about $x = 1$.

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You have used 3 of 3 attempts

Answers are displayed within the problem

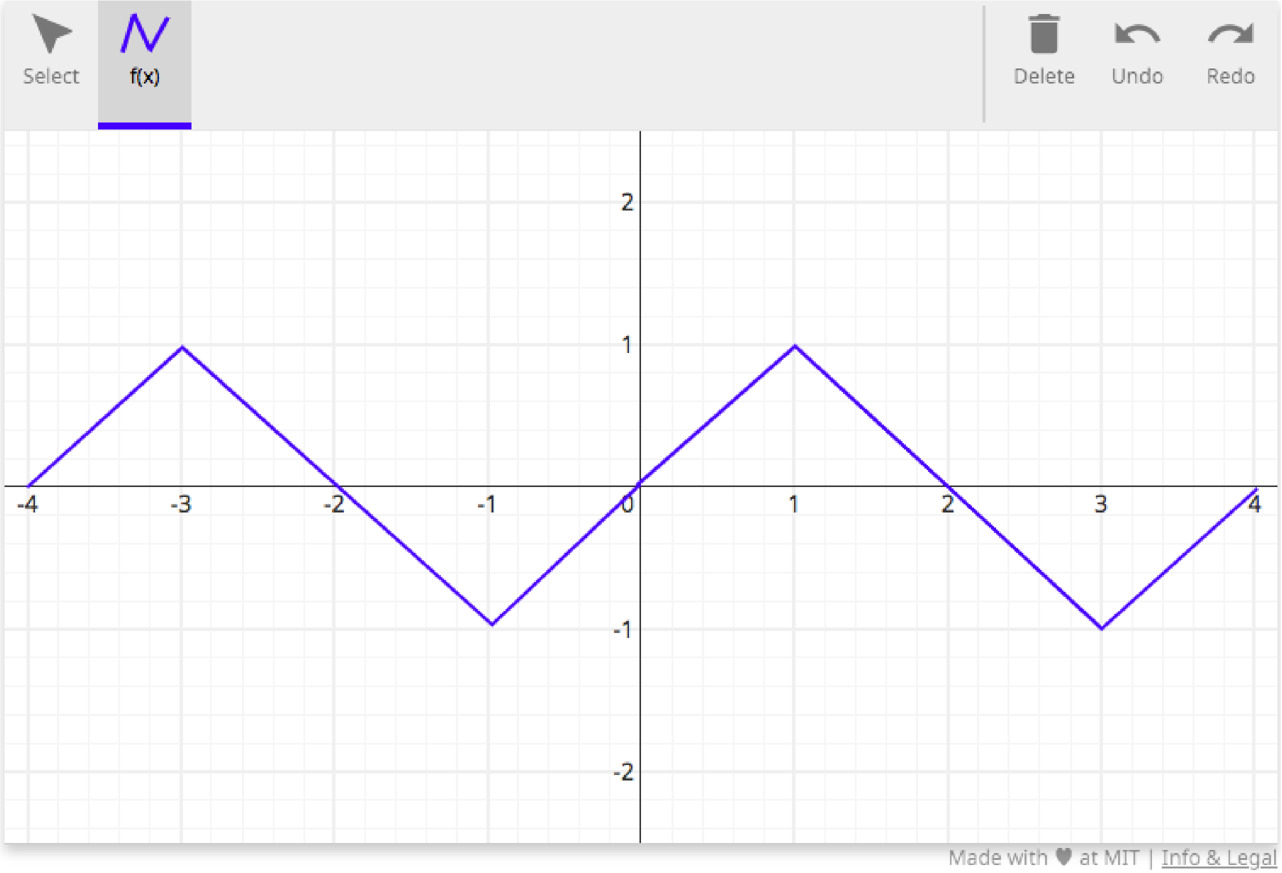
Find the initial conditions part 2

1.0/1 point (graded)

(We are in the same setup as the previous problems.) We have a thin insulated metal rod of length 1, which satisfies the differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 < x < 1, \; t > 0.$$

The sketch of the 4-periodic, odd function that is symmetric about $x = 1$ and is equal to x on $0 < x < 1$ is shown below.



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Solve

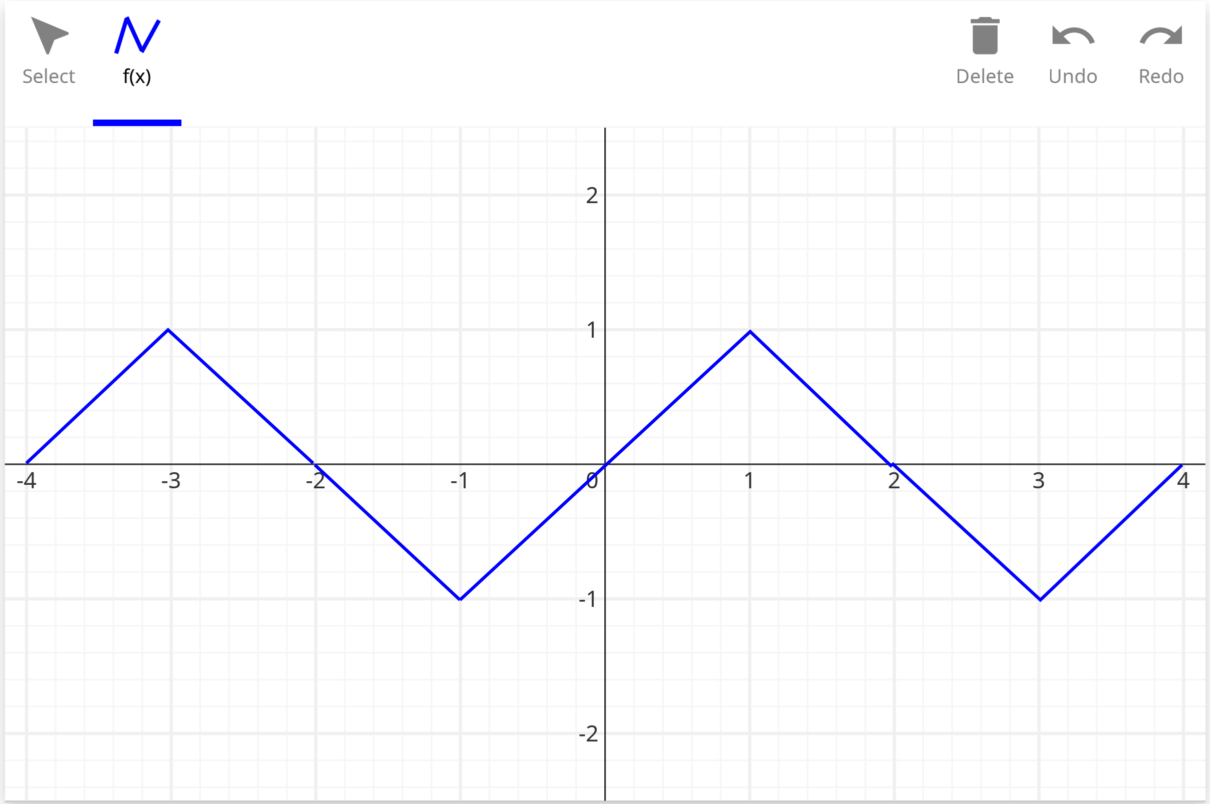
1.0/1 point (graded)

(We are in the same setup as the previous problems.) We have a thin insulated metal rod of length 1, which satisfies the differential equation

The left end is placed in an ice bath and held at $0^\circ\mathrm{C}$, and the right end is insulated.

Initially at $t = 0$, the temperature of the rod is given by $\theta(x, 0) = x$. Sketch the extended initial condition $f(x)$ with the properties determined by the previous problem.

(Sketch the function $f(x)$ on the interval $-4 \le x \le 4$.)



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Answer: .

Good job!

Solution:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 < x < 1,\ t > 0.$$

The left end is placed in an ice bath and held at $0°\mathrm{C}$, and the right end is insulated.

Initially at $t = 0$, the temperature of the rod is given by $\theta(x, 0) = x$. Find the coefficients in the general solution

$$\theta(x, t) = \sum_{k=0}^{\infty} c_k w_k(t) v_k(x).$$

$c_k =$ 8*sin((2*k+1)*pi/2)/((2*k **Answer:** 8*(-1)^k/(pi*(2*k+1))^2

Solution:

You can carry out the computation in numerous ways. The simplest in this case may be to compute directly using the formula derived two problems prior.

$$c_k = 2\int_0^1 x \sin\left(\frac{2k+1}{2}\pi x\right)\, dx = \frac{8(-1)^k}{(\pi(2k+1))^2}.$$

Submit You have used 5 of 10 attempts

Answers are displayed within the problem

1. Mixed boundary conditions

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Requesting the STAFF to check my c_k

discussion posted about 11 hours ago by **sandipan_dey**

I got the graph correct, but stuck on c_k , @STAFF, could you please check whether I am on the right track for c_k , I double-checked my values, but could not find what is wrong, thank you very much in advance.

Also, should not the sum be from $k = 1$ instead of $k = 0$, since we are going to have the $sin(.)$ term for $k = 0$ vanished anyway.

The symmetry argument for the odd periodic extension is not something that I could get intuitively, as in why should we have it on the first place, requesting a more intuitive explanation (as in, in which class of problems we should consider it etc.).

Finally, is not it a little counter-intuitive to have the period of the odd extension of $f(x)$ different from the length of the rod? does it have any physical meaning? is it just an accident that this happens here for the mixed-BC, or there is any known pattern?

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2 responses

jfrench (Staff)
about 3 hours ago

(1) Your c_k is almost correct. Note that sometimes, your coefficient is zero for terms that should not be zero! :) Try expressing it without the sin term, and I think you'll have it correct.

(2) It is typically when using series that only involve sin to start the sum from $k = 1$ since the constant term only exists for even series.

(3) Technically, you could do this another way, but the easiest extension that is zero at the left end point is the odd extension.

(4) I actually don't think there is a physical reason for this length/period discrepancy per se. It is really just a mathematical trick for finding the solution, more than a real physical procedure.

Add a comment

sandipan dey
18 minutes ago

Thank you very much @jfrench for your reply.

(1) & (2): I was equating the coefficients of two slightly different basis functions, that's why the answer was wrong, now I got it correct. Although my final answer involves $sin\left(.\right)$ basis, but the indices are different (so that we must start from $k=0$) and the basis can be easily converted to a $cos\left(.\right)$ basis.

(3): I was interested to know when we need to use the symmetry argument and how to think about it on the first place, for example without the hint given in the question, it seems that i could not be able to think about using the symmetry argument by myself (so that i can use the argument at similar problems later).

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