# 1.5 Story Proofs

## Unit 1: Probability and Counting

**Adapted from Blitzstein-Hwang Chapter 1.**

A *story proof* is a proof by interpretation. For counting problems, this often means counting the same thing in two different ways, rather than doing tedious algebra. A story proof often avoids messy calculations and goes further than an algebraic proof toward *explaining* why the result is true. The word ''story" has several meanings, some more mathematical than others, but a story proof (in the sense in which we're using the term) is a fully valid mathematical proof. Here are some examples of story proofs, which also serve as further examples of counting.

**Example 1.5.1 (The team captain).**

For any positive integers $n$ and $k$ with $k \leq n$,

$$n\binom{n-1}{k-1} = k\binom{n}{k}.$$

This is again easy to check algebraically, using the fact that $m! = m(m-1)!$ for any positive integer $m$, but a story proof is more insightful.

**Story proof**

Consider a group of $n$ people, from which a team of $k$ will be chosen, one of whom will be the team captain. To specify a possibility, we could first choose the team captain and then choose the remaining $k-1$ team members; this gives the left-hand side. Equivalently, we could first choose the $k$ team members and then choose one of them to be captain; this gives the right-hand side.

**Example 1.5.2 (Vandermonde's identity).**

A famous relationship between binomial coefficients, called *Vandermonde's identity*, says that

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

This identity will come up several times in this course. Trying to prove it with a brute force expansion of all the binomial coefficients would be a nightmare. But a story proves the result elegantly and makes it clear *why* the identity holds.

**Story proof**

Consider a group of $m$ peacocks and $n$ toucans, from which a set of size $k$ birds will be chosen. There are $\binom{m+n}{k}$ possibilities for this set of birds. If there are $j$ peacocks in the set, then there must be $k-j$ toucans in the set. The right-hand side of Vandermonde's identity sums up the cases for $j$.