1.4.3 Qualitative Analysis of a System of Differential Equations via Phase Planes

We want to understand how fishing affected the proportions of predators and prey caught during World War I. We'll start by looking at the behavior of fish populations without fishing. This is a common strategy of mathematicians – look at a simpler situation first.

Here's a specific example of the Lotka-Volterra model with predator and prey fish. Here $S(t)$ is the population size of sardines (in hundreds of thousands) and $M(t)$ is the population size of marlin (in hundreds), and t represents time in years. We'll model these using the following differential equations:

$$\frac{dS}{dt} = 0.5S - 0.4SM$$

$$\frac{dM}{dt} = -0.2M + 0.03SM$$

- Why are the units different? We measure the marlin M population in hundreds and the sardine S population in hundreds of thousands because there are many, many more sardines than marlin. One marlin must eat many, many sardines to survive – in a population with similar numbers of marlin and sardines, most of the marlin would starve.

- To get a feel for the problem, look at pictures or video clips of marlin and sardines together, such as those in the article *Speared! Fearless striped marlin means business as it spots its lunch and goes straight for the kill* from The Daily Mail or in the video *HD footage of Sailfish attacking a bait ball at Isla Mujeres Mexico* from SZTV (on YouTube)

- Why is the synthesis constant $\mathbf{d} = 0.03$ so much smaller than the predation constant $\mathbf{b} = 0.4$? When a marlin eats a sardine, the effect on the sardine population is negative and dramatic: one of the members is removed. The effect on the marlin population is positive (the marlin is now less hungry), but much less dramatic. A marlin must eat many sardines in order to sustain itself to reproduce with another marlin; the fact that d is about one-tenth of b is discussed at the end of Population Dynamics Part II.

To get a sense of what happens to the populations over time, we use a qualitative analysis similar to what we did for the logistic differential equation. For a system of two autonomous first-order differential equations, this qualitative analysis is called a **phase plane analysis.** A **phase plane** is a way to visualize the relationship between S and M, by thinking about how S is changing with time and how M is changing with time. This will help us see the general direction of a **solution trajectory**, a curve representing the path traced out by the points $(S(t), M(t))$ for all time t.

If you're not familiar with phase plane analysis, check out the next video. Peter gives a step-by-step explanation of how to do phase plane analysis for a slightly different predator-prey system, using the variables x and y instead of S and M, to be more mathematically familiar.

If you are familiar with phase plane analysis, move onto the next quiz which will guide you through the phase plane analysis of the sardine-marlin system.

English ▼