The Borel Sets

A **Borel Set** is a set that you can get to by starting with the family of line segments and carrying out finitely many applications of the operations of *complementation* and *countable union*:

- The **complementation operation** takes each set A to $\mathbb{R} - A$, where $\mathbb{R}$ is the set of real numbers. (I will sometimes refer to the complement of A as $\overline{A}$.) So, for instance, the result of applying the complement operation to $[0,1]$ is the set $\overline{[0,1]} = \mathbb{R} - [0,1]$, which consists of every real number except for the elements of $[0,1]$.

- The **countable union operation** takes each countable (i.e. finite or countably infinite) family of sets $A_1, A_2, A_3, \ldots$ to their union: $\bigcup\{A_1, A_2, A_3 \ldots\}$. So, for instance, the result of applying the countable union operation to the sets $[0, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, \frac{7}{8}], \ldots$ is the set $[0,1)$, which consists of the real numbers x such that $0 \leq x < 1$. (Note that the round bracket on the right-hand side of "$[0,1)$" is used to indicate that the end-point 1 is not included in the set.)

Formally, a Borel Set is any member of the smallest class $\mathscr{B}$ such that: (i) every line-segment is in $\mathscr{B}$, (ii) if a set is in $\mathscr{B}$, then so is its complement, and (iii) if a countable family of sets is in $\mathscr{B}$, then so is its union.

To get some practice working with Borel Sets, let us verify that the set of *irrational* real numbers $\overline{\mathbb{Q}}$ is a Borel Set. ($\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers.) What we need to show is that the set $\overline{\mathbb{Q}}$ can be generated by starting with a family of line segments $[a,b]$ and applying complementation and countable union as many times as needed. This can be done in three steps:

- *Step 1*: For every rational number $q \in \mathbb{Q}$, the singleton set $\{q\}$ is identical to the point-sized line-segment $[q,q]$. Since every line-segment is a Borel Set, it follows that every singleton set $\{q\}$ $(q \in \mathbb{Q})$ is a Borel Set.

- *Step* 2: Since the set of rational numbers, $\mathbb{Q}$, is countable, it is the countable union of the family of Borel Sets $\{q\}$, for $q \in \mathbb{Q}$. And since the countable union of Borel Sets is a Borel Set, $\mathbb{Q}$ must be a Borel Set too.

- *Step* 3: Since $\mathbb{Q}$ is a Borel Set, $\overline{\mathbb{Q}}$ is the complement of a Borel Set. So $\overline{\mathbb{Q}}$ must be a Borel Set too.

(Note that a procedure of this kind can be used to show that any countable set is a Borel Set, as is its complement.)

Problem 1

1/1 point (ungraded)
If $A_1, A_2, A_3, \ldots$ is a countable family of Borel Sets, then $\bigcap\{A_1, A_2, A_3, \ldots\}$ is a Borel Set.

(In general, the **intersection** of a set $\{A_1, A_2, A_3, \ldots\}$ (in symbols: $\bigcap\{A_1, A_2, A_3, \ldots\}$) is the set of individuals x such that x is a member of each set $A_1, A_2, A_3, \ldots$.)

True or false?

- (selected) True
- False

✔

Explanation
The key observation is that

$$\bigcap\{A_1, A_2, A_3, \ldots\} = \overline{\bigcup\{\overline{A_1}, \overline{A_2}, \overline{A_3}, \ldots\}}$$

where $\overline{A} = \mathbb{R} - A$. One can therefore verify that $\bigcap\{A_1, A_2, A_3, \ldots\}$ is a Borel Set as follows. Since each of $A_1, A_2, A_3, \ldots$ is a Borel Set, we can use complementation to show that each of $\overline{A_1}, \overline{A_2}, \overline{A_3}, \ldots$ is a Borel Set. But by countable union, this means that $\bigcup\{\overline{A_1}, \overline{A_2}, \overline{A_3}, \ldots\}$ must also be a Borel Set. So, by complementation, $\bigcap\{A_1, A_2, A_3, \ldots\}$ must also be a Borel Set.

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Problem 2

1/1 point (ungraded)

Assume that A and B are Borel Sets. Is it true that $A - B$ (i.e. the set of elements in A which are not in B) is a Borel Set?

- (●) Yes
- ( ) No

Explanation

The key observation is that $A - B = A \cap \overline{B}$. Since the complement of a Borel Set is a Borel Set, and since (as verified in the previous exercise), the intersection of Borel Sets is a Borel Set, this means that $A - B$ is a Borel Set.

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"line segment"
What is the purpose of talking of a "line segment" in this context? Is "line segment" to be understood... 5

What is the point of Borel sets?
Maybe we are getting there, but what is the intuitive point of defining something as a Borel set? I get... 4