4. Homogeneous and inhomogeneous boundary conditions

Problem 4.1 How many solutions are there to the initial value problem

$$\frac{d^2}{dx^2}v\left(x\right)=\lambda v\left(x\right),\qquad v\left(0\right)=3,\ \frac{dv}{dx}\left(0\right)=5,$$

where λ is a parameter that can take on any real number value?

Answer

There is 1 solution, by the existence and uniqueness theorem.

Recall the existence and uniqueness theorem for linear ODEs. Let $p_{n-1}\left(t\right),\ldots,p_0\left(t\right),q\left(t\right)$ be continuous functions on an open interval I. Let a be in the interval $I,$ and let $b_0,\ldots,b_{n-1}$ be given numbers. Then there exists a unique solution to the n^{th} order linear ODE

$$y^{(n)}+p_{n-1}\left(t\right)y^{(n-1)}+\cdots+p_1\left(t\right)\dot{y}+p_0\left(t\right)y=q\left(t\right)$$

satisfying the n initial conditions

$$y(a) = b_0, \quad \dot{y}(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

Existence means that there is at least one solution; **uniqueness** means that there is only one solution.

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This is an initial value problem since the conditions are the value and first derivative **at the same point.** In contrast, a **boundary value problem** has conditions at different points.

Consider the following family of examples, one for each λ:

Problem 4.2 Find all functions $v(x)$ on $[0, \pi]$ satisfying $\frac{d^2}{dx^2} v(x) = \lambda v(x)$ for a constant λ and satisfying the **boundary conditions** $v(0) = 0$ and $v(\pi) = 0$.

Even though there are the right number of conditions, there is no longer any guarantee that they specify a unique solution — there is no existence and uniqueness theorem for boundary value problems. For most values of λ, it will turn out that this problem has a unique solution (namely, 0), but for special values of λ, we will see that it has infinitely many solutions!

- The example in problem 4.2 is called a **homogeneous linear boundary value problem,** and can have one solution or infinitely many solutions.
- An **inhomogeneous linear boundary value problem** could have zero solutions, one solution, or infinitely many solutions.
- The situation for **nonlinear boundary value problems** is even more complicated.

Our goal is to use boundary conditions for PDEs, but ODEs can have boundary conditions as well. In general, ODEs are much easier to solve than PDEs, so we now move our discussion to ODEs with boundary conditions to get a sense of how to solve these **boundary value problems** in a more comfortable setting.

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