Size Comparisons

Here is a useful piece of notation. If A is a set, we will use $|A|$ to talk about the size, or **cardinality**, of A.

In some of the problems of the previous lecture we verified that bijections are reflexive, symmetrical and transitive. This means that the relation $|A| = |B|$ is itself reflexive, symmetrical and transitive.

In other words, for any sets A, B and C we have:

Reflexivity
$|A| = |A|$

Symmetry
If $|A| = |B|$, then $|B| = |A|$

Transitivity
If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$

A reflexive, symmetric and transitive relation is said to be an **equivalence relation**.

By relying on the fact that $|A| = |B|$ is an equivalence relation, we can provide a beautifully succinct summary of the results so far:

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}^{\geq 0}| = |\mathbb{Q}|$$

These results make it tempting to suppose that any infinite set has the same cardinality as any other. But we'll soon see that this is not so: *there are infinite sets with different cardinalities*.

We'll see, in particular, that the set of real numbers, $\mathbb{R}$ has a greater cardinality than the set of natural numbers:

$$|\mathbb{N}| < |\mathbb{R}|$$

In order to prepare for these results, however, we'll need to get clear about what it means to say that one cardinality is greater than another. Recall that our understanding of *sameness* of cardinality is based on the Bijection Principle, which I restate here using our new notation:

> **The Bijection Principle**
> $|A| = |B|$ if and only if there is a bijection from A to B.

How might we extend this idea to explain what it takes for the cardinality of B is greater than the cardinality of A (in symbols: $|A| < |B|$). We'll start by addressing a closely related question: what does it take for the cardinality of B to be *at least as great* as the cardinality of A (in symbols: $|A| \leq |B|$)?

The answer we will be working with here is based on the following principle:

> **The Injection Principle**
> $|A| \leq |B|$ if and only if there is a bijection from A to a subset of B.

Why call it the Injection Principle? Because an **injection** from A to B is a bijection from A to a subset of B. Notice, moreover, that the Injection Principle makes intuitive sense. Intuitively, the only way for A to have just as many elements as a subset of B is for B to have at least as many elements as A.

(If you'd like some additional help making sense of injections, there's a useful video here. If you watch the video, keep in mind that a bijective function is a function that is both injective and surjective.)

Together, the Bijection and Injection Principles give us everything we need to make cardinality comparisons between infinite sets. This is because $|A| < |B|$ can be defined in terms of $|A| = |B|$ and $|A| \leq |B|$:

$$|A| < |B| \leftrightarrow_{df} |A| \leq |B| \text{ and } |A| \neq |B|$$

(I use "$\leftrightarrow_{df}$" to indicate that the sentence to the left of the biconditional is to be defined in terms of the sentence to its right.)

Note that our definition makes intuitive sense. Intuitively, if the cardinality of B is at least as great as the cardinality of A, and if the cardinalities are not the same, then the cardinality of B must be greater than the cardinality of A.

Notation	How it's defined	Informal notion
$\|A\| = \|B\|$	bijection from A to B	just as many elements in A as in B
$\|A\| \leq \|B\|$	injection from A to B	at most as many elements in A as in B
$\|A\| < \|B\|$	$\|A\| \leq \|B\|$ and $\|A\| \neq \|B\|$	fewer elements in A than in B
$\|A\| \geq \|B\|$	$\|B\| \leq \|A\|$	at least as many elements in A as in B
$\|A\| > \|B\|$	$\|A\| \geq \|B\|$ and $\|A\| \neq \|B\|$	more elements in A than in B

One of the reasons Injection Principle is so attractive is that $\leq$ is a **partial order**. In other words, $\leq$ satisfies the following principles, for any sets A, B and C:

Reflexivity
$|A| \leq |A|$

Anti-symmetry
If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

Transitivity
If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$

We'll have a go at proving the first and third of these principles in the exercises below. The second principle requires a lengthy proof, which we won't reconstruct here. It is sometimes referred to as the **Cantor-Schroeder-Bernstein Theorem**, and it is easy to learn more about it online.

There is a fourth principle which is intuitively correct, and which we'd very much like to be able to prove:

Totality
For any sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$

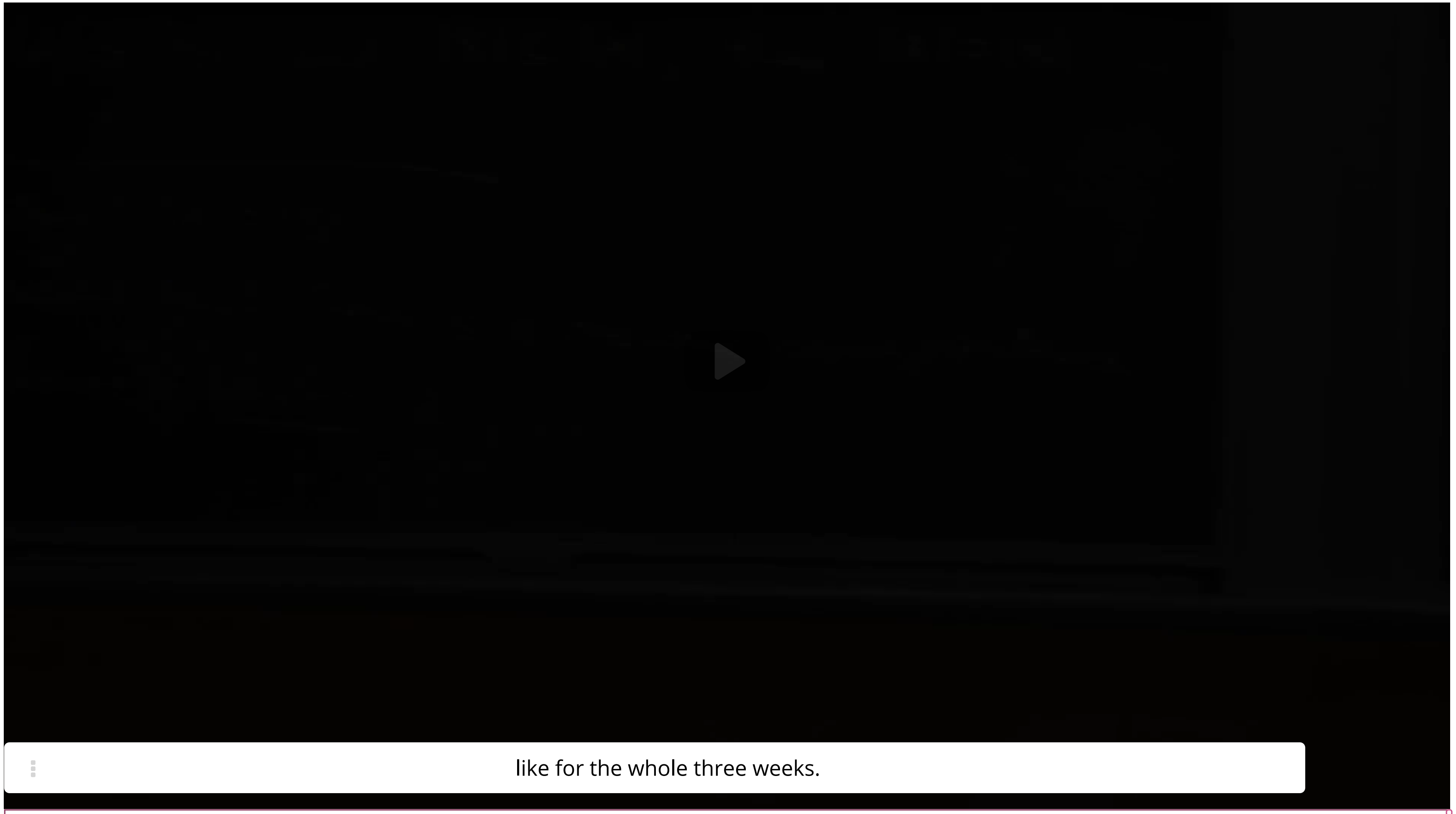
Unfortunately, one can only prove Totality if one assumes a controversial set-theoretic axiom: the Axiom of Choice. I'll have a lot more to say about the Axiom of Choice later on. (We'll come across it again in Lecture 3, and I'll offer a more thorough discussion in Lecture 7.)

Some notation

Earlier I mentioned that bijections go by many names. The same is true of injections. (And I'm afraid I sometimes use other names in the videos associated with this course; the videos were shot several years ago!). Here is a translation-manual:

Notation in text	Notation in videos
f is an injection from A to B	f is a one-one function from A **into** B
f is a bijection from A to B	f is a one-one function from A **onto** B or f is a one-one correspondence between A and B

Video Review: at least as great



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Problem 1

1/1 point (ungraded)

A function f from A to B is an injection (as defined above) if and only if f assigns a differet element of B to each element of A.

True or False?

True

✔ **Answer:** True

Explanation
It's true. Here's why.
Suppose, first, that f is an injection from A to B. In other words: f is a bijection from A to some subset B' of B. By the definition of bijection, this means that f assigns a different element of B' (and therefore B) to each element of A.
Now suppose that f assigns a different element of B to each element of A. If B' is the set of elements of B to which f assigns some element of A, f is a bijection from A to B'.

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Answers are displayed within the problem

Problem 2

1/1 point (ungraded)
For f to be a *surjection* from A to B is for there to be no element of B to which f fails to assign some element of A. Now consider the following proposition:

f is a bijection from A to B if and only if f is both an injection from A to B and a surjection from A to B.

Is it true or false?

True

✔ **Answer:** True

Explanation
A bijection from A to B is a function from A to B such that: (i) each element of A is assigned to a different element of B, and (ii) no element of B is left without an assignment from A. Part (i) of this definition is satisfied if and only if f is an injection, and part (2) is satisfied if and only if f is a surjection.

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Problem 3

1/1 point (ungraded)

Which of the following, if any, is true of $\leq$ (injections)?

- [x] **Reflexivity**
  For any set A, $|A| \leq |A|$.
- [ ] **Symmetry**
  For any sets A and B, if $|A| \leq |B|$, then $|B| \leq |A|$.
- [x] **Transitivity**
  For any sets A, B, and C, if $|A| \leq |B|$, and $|B| \leq |C|$, then $|A| \leq |C|$.

✔

Explanation

The identity function $f(x) = x$ is an injection from A to A, so A is reflexive for any A.

To show that transitivity holds for arbitrary sets A, B and C, assume that f is an injection from A to B and that g is an injection from B to C. Since the composition of injections is injective, $g \circ f$ is an injection from A to C.

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Answers are displayed within the problem

Problem 4

1/1 point (ungraded)

If $|A| < |B|$ and $|B| \leq |C|$, then $|A| < |C|$.

True or False?

True

✔ **Answer:** True

(If it's true, have a go at proving it. If it's false, produce a counterexample.)

Explanation

We will assume that $|A| < |B|$ and $|B| \leq |C|$ and prove $|A| < |C|$. This requires proving two things: $|A| \leq |C|$ and $|A| \neq |C|$.
$|A| \leq |C|$ follows from transitivity, which was verified in the previous exercise.
To prove that $|A| \neq |C|$, we shall assume that $|A| = |C|$ and use this assumption to derive a contradiction.
Since we are assuming that $|A| = |C|$, we are assuming that there is a bijection f from A to C, and therefore a bijection f^{-1} from C to A. Since $|B| \leq |C|$, we know that there is an injection g from B to C. So $h(x) = f^{-1}(g(x))$ is an injection from B to A. But our initial assumptions give us $|A| < |B|$ (and therefore $|A| \leq |B|$). So, by the Cantor-Schroeder-Bernstein Theorem, $|A| = |B|$, which contradicts $|A| < |B|$.

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Answers are displayed within the problem

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Reflexivity

I do not understand how, by the reflexivity rule, the cardinality of A is less than or equal to the cardinality of A. Aren't the two sets the same (A and A) and so would they not h...

2

Size Comparisons, Problem 1 typo?

In the "Explanation" (i.e. solution), it says "By the definition of bijection, this means that *A* assigns a different element of B' (and therefore B) to each element of A." It seems ...

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