The Proof

Now for our proof of $|\mathscr{P}(\mathbb{N})| = |\mathbb{R}|$.

What we will actually prove is $|\mathscr{P}(\mathbb{N})| = |[0,1]|$, but this suffices to give us what we want because we know that $|\mathbb{R}| = |[0,1]|$.

Notice, first, that if A is a subset of $\mathbb{N}$, one can represent A as an infinite sequence of 1s and 0s; namely: the sequence that contains a 1 in the nth position if n is a member of A, and a 0 in the nth position otherwise. So, for instance, the set of odd numbers is represented by the sequence $\langle 0, 1, 0, 1, 0, 1, \ldots \rangle$.

Notice, moreover, that each such sequence can be used to characterize a different binary expansion in $[0,1]$; namely: the expansion that starts with "0.", and is followed by digits corresponding to the members of that sequence. So, for instance, the sequence $\langle 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$ yields the binary expansion "$0.0101010\ldots$", which names the number $1/3$.

This gives us a bijection from $\mathscr{P}(\mathbb{N})$ to the set B_0 of binary expansions of the form "$0.b_1 b_2 \ldots$". So all we need to complete the proof is a bijection from $[0,1]$ to B_0. This final step would be trivial if every number in $[0,1]$ was named by exactly one binary expansion in B_0. But we have seen that some numbers in $[0,1]$ are named by two binary expansions in B_0.

Fortunately, there is a nice way of getting around the problem.

Recall that there are only countably many members of $[0,1]$ with more than one name in B_0, since only rational numbers have multiple names. And we know from the No Countable Difference Principle that adding countably many members to an infinite set doesn't change the set's cardinality. So there must be a bijection from $[0,1]$ to B_0.

We have established that there is a bijection from $\mathscr{P}(\mathbb{N})$ to B_0, and that there is a bijection from $[0,1]$ to B_0. So it follows from the symmetry and transitivity of bijections that there must be a bijection from $\mathscr{P}(\mathbb{N})$ to $[0,1]$.

This completes the proof.

Problem 1

0.0/1.0 point (ungraded)

In the text above I gave a relatively informal proof for the claim that there is a bijection from $[0,1]$ to B_0. Give a more rigorous version of the proof.

☐ Done

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Problem 2

1.0/1.0 point (ungraded)

Let F be the set of functions from natural numbers to natural numbers.

Is it the case that $|\mathbb{N}| < |F|$?

Yes ✔

(If it's true, show why. If it's false, explain why not.)

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Binary set B 5

A Set of Size in between |N| and |R|. 7
Are there any set S such that |N| < |S| < |R|?