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15. Summary

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Summarize

Big Picture

Vectors are objects with magnitude and direction that have an algebra that allows us to perform mathematical operations such as addition, scaling, and more.

Mechanics

Definition of vectors

> **Definition 15.1** A **vector** is a quantity that has both magnitude and direction.

A vector in two dimensions has two components and can be written as

$$\vec{v} = \langle v_1, v_2 \rangle \tag{3.51}$$

or

$$\vec{v} = v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}} \tag{3.52}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors in the x and y directions, respectively. In other words,

$$\begin{aligned} \hat{\mathbf{i}} &= \langle 1, 0 \rangle \\ \hat{\mathbf{j}} &= \langle 0, 1 \rangle. \end{aligned}$$

A vector in three dimensions has three components and can be written as

$$\vec{v} = \langle v_1, v_2, v_3 \rangle \tag{3.53}$$

or

$$\vec{v} = v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}} + v_3 \hat{\mathbf{k}} \tag{3.54}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors in the x, y, and z directions, respectively. In other words,

$$\begin{aligned} \hat{\mathbf{i}} &= \langle 1, 0, 0 \rangle \\ \hat{\mathbf{j}} &= \langle 0, 1, 0 \rangle \\ \hat{\mathbf{k}} &= \langle 0, 0, 1 \rangle. \end{aligned}$$

Magnitude

Definition 15.2 The **magnitude** of a vector $\vec{v}$ is equal to its length and is denoted by $|\vec{v}|$.

By the Pythagorean theorem, the magnitude of a 2-dimensional vector $\vec{v} = \langle v_1, v_2 \rangle$ is given by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}. \quad (3.55)$$

Extension to higher dimension: Magnitude

Consider a vector with n components given by $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$. The magnitude of $\vec{v}$ is given by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}. \quad (3.56)$$

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Scalar multiplication

We can multiply vectors by real numbers (called **scalars**). Multiplication of a vector $\vec{v}$ by a scalar c will scale v by c. If $c > 0$, the vector $c\vec{v}$ will be in the same direction as $\vec{v}$ and have length $c|\vec{v}|$. If $c < 0$, the vector $c\vec{v}$ will be in the opposite direction of $\vec{v}$ and have length $|c||\vec{v}|$.

To multiply a vector $\vec{v} = \langle v_1, v_2 \rangle$ by a scalar c, we multiply each component by c as follows:

$$c\vec{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle. \quad (3.57)$$

Extension to higher dimension: Scalar Multiplication

Consider a scalar c and a vector with n components given by $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$. Then $c\vec{v}$ is given by

$$c\vec{v} = c\langle v_1, v_2, \cdots, v_n \rangle = \langle cv_1, cv_2, \cdots, cv_n \rangle. \quad (3.58)$$

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Vector addition

We can add two vectors of the same length by adding each of their components. For example, the vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ add up to

$$\vec{v} + \vec{w} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle. \quad (3.59)$$

Extension to higher dimension: Vector addition

Consider two vectors with n components each given by $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\vec{w} = \langle w_1, w_2, \ldots, w_n \rangle$. Then the sum $\vec{v} + \vec{w}$ is given by

$$\vec{v} + \vec{w} = \langle v_1, v_2, \cdots, v_n \rangle + \langle w_1, w_2, \cdots, w_n \rangle = \langle v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n \rangle. \quad (3.60)$$

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Dot products

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Definition 15.3 The **dot product** between vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ is defined as

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2. \qquad (3.61)$$

Extension to higher dimension: Dot product

Consider two vectors of length n given by $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\vec{w} = \langle w_1, w_2, \ldots, w_n \rangle$. The dot product of these vectors is given by

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2, \ldots, v_n \rangle \cdot \langle w_1, w_2, \ldots, w_n \rangle = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n. \qquad (3.62)$$

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The dot product between two vectors $\vec{v}$ and $\vec{w}$ can also be computed as

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta) \qquad (3.63)$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

Ask Yourself

Given a vector, how do you find a unit vector that points in the same direction?

Recall that $\hat{v}$ is defined to be a vector with the same direction as $\vec{v}$ and with a length of 1. Since $|c\vec{v}| = |c||\vec{v}|$, we can choose $c = \frac{1}{|\vec{v}|}$ to obtain the desired $\hat{v}$. In summary, to get $\hat{v}$, we divide $\vec{v}$ by its length $|v|$.

(Remember that the length of a vector $\vec{v} = \langle v_1, v_2 \rangle$ is $\sqrt{v_1^2 + v_2^2}$.)

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Is the dot product useful?

Yes! For one thing, computing the dot product can tell you if two vectors are perpendicular or not. Without the dot product, it would be quite hard to know that without drawing a picture.

More generally, by dividing the dot product by the lengths of the two vectors, you can obtain the cosine of the angle between them. This comes in useful for solving many different problems.

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15. Summary

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[Staff] Small typo 2
In the text below formula 3.52 should be "in the x and y directions" instead of "in the xand y directions". Space needed between wor...

[Staff]Typo in Ask Yourself

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In the first box of Ask Youself, |cv| should be c|**v**|. Thank you.

def 15.2
small point, |v| should have an arrow on top to signify a vector. 2

? [Staff] Length = magnitude and number of components of a vector? 3

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