Probability mass function of product of two binomial variables

Asked 2 months ago Active 2 months ago Viewed 114 times

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I have two i.i.d. binomial variables X and Y with given n and p. What is probability mass function of $Z = X \times Y$? I need pmf as function $f(Z, n, p)$.

binomial pdf

edited Sep 19 at 0:56
Michael Hardy
5,749 16 31

asked Sep 18 at 19:48
user2579566
154 6

1 It's going to be messy to express, because the values with positive probability are all the possible products of two integers in the set $\{0, 1, \ldots, n\}$, which contains a complex pattern of gaps. Could you therefore specify the form in which you need this pmf or what you hope to use it for? – whuber ♦ Sep 18 at 21:26

@whuber I've fixed my question. – user2579566 Sep 18 at 21:34

1 Answer

5

There are various ways you could write the mass function of this distribution. All of them will be messy, since they involve checking the possible products that give a stipulated value for the product variable. Here is the most obvious way to write the distribution.

Let $X, Y \sim \text{IID Bin}(n, p)$ and let $Z = XY$ be their product. For any integer $0 \leqslant z \leqslant n^2$ we define the set of pairs of values:

$$\mathcal{S}(z) \equiv \{(x, y) \in \mathbb{N}_{0+}^2 \mid \max(x, y) \leqslant n, xy = z\}.$$

This is the set of all pairs of values within the support of the binomial that multiply to the value z. (Note that it will be an empty set for some values of z.) We then have:

$$= \sum_{(x,y) \in \mathcal{S}(z)} \binom{n}{x} \binom{n}{y} \cdot p^{x+y} (1-p)^{2n-x-y}.$$

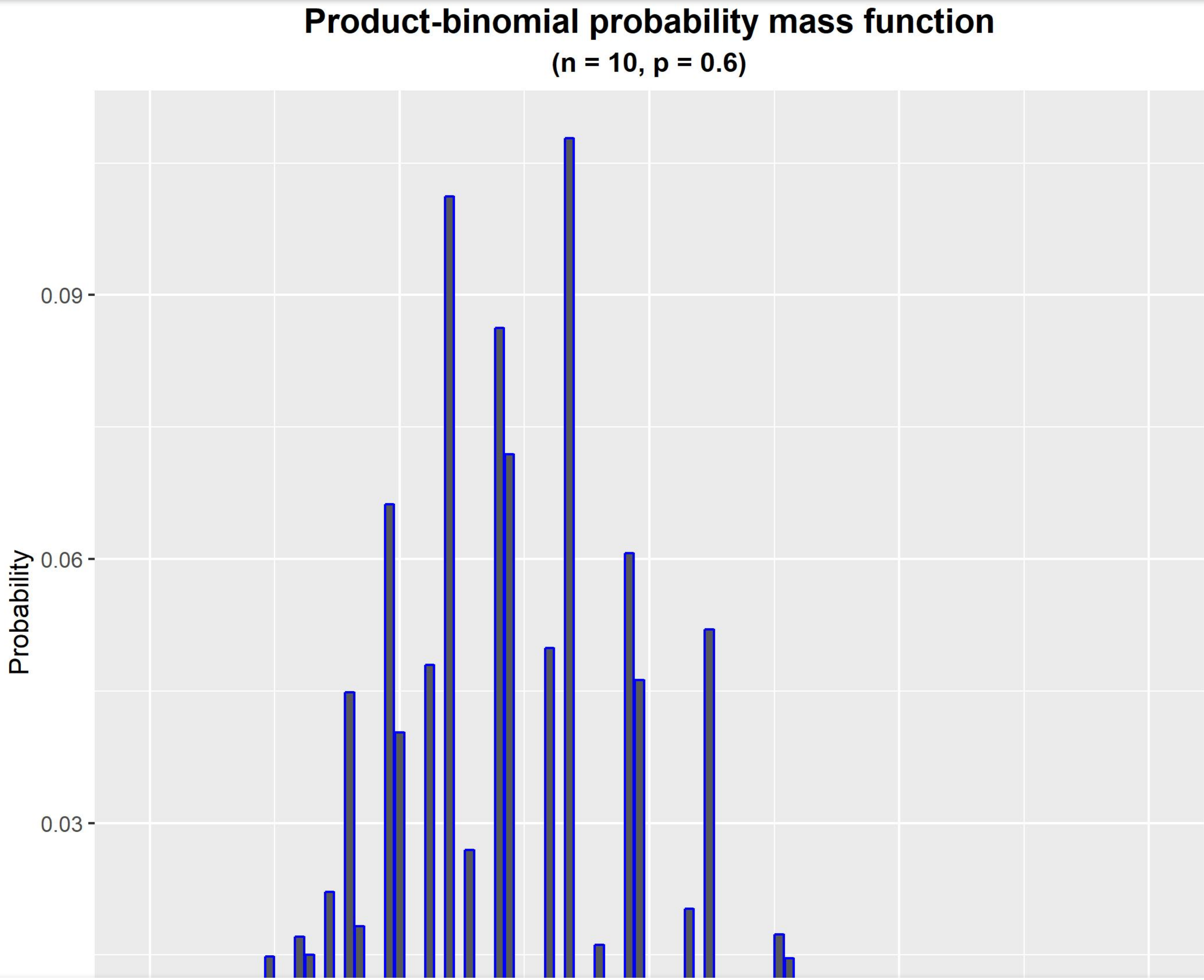
Computing this probability mass function requires you to find the set $\mathcal{S}(z)$ for each z in your support. The distribution has mean and variance:

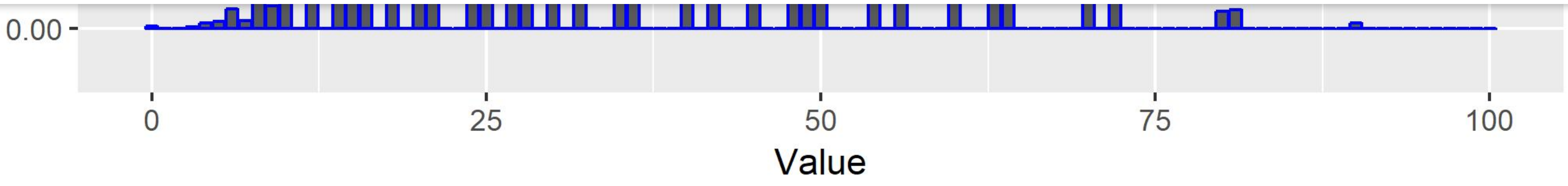
$$\mathbb{E}(Z) = (np)^2 \qquad\qquad \mathbb{V}(Z) = (np)^2 [(1-p+np)^2 - (np)^2].$$

The distribution will be quite jagged, owing to the fact that it is the distribution of a product of discrete random variables. Notwithstanding its jagged distribution, as $n \to \infty$ you will have convergence in probability to $Z/n^2 \to p^2$.

Implementation in `R`: The easiest way to code this mass function is to first create a matrix of joint probabilities for the underlying random variables X and Y, and then allocate each of these probabilities to the appropriate resulting product value. In the code below I will create a function `dprodbinom` which is a vectorised function for the probability mass function of this "product-binomial" distribution.

```
#Create function for PMF of the product-binomial distribution
dprodbinom <- function(Z, size, prob, log = FALSE) {

  #Check input vector is numeric
  if (!is.numeric(Z))     { stop('Error: Input values are not numeric'); }

  #Set parameters
  n <- size;
  p <- prob;

  #Generate matrix of joint probabilities
  SS <- matrix(-Inf, nrow = n+1, ncol = n+1);
  XX <- dbinom(0:n, size = n, prob = p, log = TRUE);
  for (x in 0:n) {
  for (y in 0:n) {
    SS[x+1, y+1] <- XX[x+1] + XX[y+1]; } }

  #Compute the log-mass function of the product random variable
  LOGPMF <- rep(-Inf, n^2+1);
  for (x in 0:n) {
  for (y in 0:n) {
    LOGPMF[x*y+1] <- matrixStats::logSumExp(c(LOGPMF[x*y+1], SS[x+1, y+1])); } }

  #Generate the output vector
  OUT <- rep(-Inf, length(Z));
  for (i in 1:length(Z)) {
    if (Z[i] %in% 0:(n^2)) {
      OUT[i] <- LOGPMF[Z[i]+1]; } }

  #Give the output of the function
  if (log) { OUT } else { exp(OUT) } }
```


Product-binomial probability mass function
(n = 10, p = 0.6)
Probability
0.09
0.06
0.03




```
#Load required libraries
library(matrixStats);
library(ggplot2);

#Generate the mass function
n <- 10;
p <- 0.6;
PMF <- dprodbinom(0:100, size = n, prob = p, log = FALSE);

#Plot the mass function
THEME  <- theme(plot.title = element_text(hjust = 0.5, size = 14, face = 'bold'),
                plot.subtitle = element_text(hjust = 0.5, face = 'bold'));
DATA   <- data.frame(Value = 0:100, Probability = PMF);
FIGURE <- ggplot(aes(x = Value, y = Probability), data = DATA) +
            geom_bar(stat = 'identity', colour = 'blue') +
            THEME +
            ggtitle('Product-binomial probability mass function') +
            labs(subtitle = paste0('(n = ', n, ', p = ', p, ')'));
FIGURE;
```

edited Sep 19 at 10:58

answered Sep 18 at 21:38

Reinstate Monica
43.2k 2 53 180

I think you might find it easier and more insightful to express the result in terms of the prime factorization of z. BTW, the $\max(x, y) \leq n$ condition is superfluous. – whuber ♦ Sep 18 at 21:55

What is "$S(z)$"? – whuber ♦ Sep 18 at 22:21