Problem 1: Convergence in probability

(6/6 points)

For each of the following sequences, determine the value to which it converges in probability.

(a) Let $X_1, X_2, \ldots$ be independent continuous random variables, each uniformly distributed between -1 and 1.

1. Let $U_i = \dfrac{X_1 + X_2 + \cdots + X_i}{i}, \quad i = 1, 2, \ldots.$

   What value does the sequence U_i converge to in probability?

   0

   ✔ **Answer:** 0

2. Let $W_i = \max(X_1, X_2, \ldots, X_i), \quad i = 1, 2, \ldots.$

   What value does the sequence W_i converge to in probability?

   1

   ✔ **Answer:** 1

3.

Let $V_i = X_1 \cdot X_2 \cdot \cdots \cdot X_i, \quad i = 1, 2, \ldots$.
What value does the sequence V_i converge to in probability?

0 ✔ **Answer:** 0

(b) Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbf{E}[X_i] = 2$ and $\mathbf{var}(X_i) = 9$, and let $Y_i = X_i / 2^i$.

1. What value does the sequence Y_i converge to in probability?

   0 ✔ **Answer:** 0

2. Let $A_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$. What value does the sequence A_n converge to in probability?

   0 ✔ **Answer:** 0

3. Let $Z_i = \frac{1}{3} X_i + \frac{2}{3} X_{i+1}$ for $i = 1, 2, \ldots$, and let $M_n = \frac{1}{n} \sum_{i=1}^{n} Z_i$ for $n = 1, 2, \ldots$.
   What value does the sequence M_n converge to in probability?

   2 ✔ **Answer:** 2

Solved problems

Additional theoretical material

Problem Set 8
Problem Set 8 due Apr 27, 2016 at 23:59 UTC

Unit summary

- Unit 9: Bernoulli and Poisson processes
- Unit 10: Markov chains
- Exit Survey
- Final Exam

Answer:
(a)

1. The sequence converges to 0. From the weak law of large numbers, we have convergence in probability to $\mathbf{E}[X_i]$, which is zero in this case.

2. The sequence converges to 1. Since $-1 \leq W_i \leq 1$, we have $|W_i - 1| \leq 2$ and so for $\epsilon > 2$, we trivially have $\lim_{i\to\infty} \mathbf{P}(|W_i - 1| \geq \epsilon) = \lim_{i\to\infty} 0 = 0.$

   Assuming $\epsilon \in (0, 2]$, we have

$$\begin{aligned}\lim_{i\to\infty} \mathbf{P}(|W_i - 1| \geq \epsilon) &= \lim_{i\to\infty} \mathbf{P}(1 - W_i \geq \epsilon)\\ &= \lim_{i\to\infty} \mathbf{P}(W_i \leq 1 - \epsilon)\\ &= \lim_{i\to\infty} \mathbf{P}(\max\{X_1, \ldots, X_i\} \leq 1 - \epsilon)\\ &= \lim_{i\to\infty} \mathbf{P}\left(X_1 \leq 1 - \epsilon\right) \cdots \mathbf{P}\left(X_i \leq 1 - \epsilon\right)\\ &= \lim_{i\to\infty} \left(1 - \frac{\epsilon}{2}\right)^i\\ &= 0.\end{aligned}$$

3. The sequence converges to 0. Note that $|X_k| \leq 1$ for all k, and so $|V_i| = |X_1||X_2|\cdots|X_i| \leq \min\{|X_1|, |X_2|, \ldots, |X_i|\} \leq 1$.

   Hence, for any $\epsilon > 1$, we trivially have $\lim_{i\to\infty} \mathbf{P}(|V_i - 0| \geq \epsilon) = \lim_{i\to\infty} 0 = 0$.

For $\epsilon \in (0,1]$, we have

$$\begin{aligned}
\lim_{i\to\infty} \mathbf{P}(|V_i - 0| \geq \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|X_1 X_2 \cdots X_i| \geq \epsilon) \\
&= \lim_{i\to\infty} \mathbf{P}(|X_1||X_2|\cdots|X_i| \geq \epsilon) \\
&\leq \lim_{i\to\infty} \mathbf{P}(\min\{|X_1|, |X_2|, \ldots, |X_i|\} \geq \epsilon) \\
&= \lim_{i\to\infty} \mathbf{P}(|X_1| \geq \epsilon)\mathbf{P}(|X_2| \geq \epsilon)\cdots\mathbf{P}(|X_i| \geq \epsilon) \\
&= \lim_{i\to\infty} (1-\epsilon)^i \\
&= 0.
\end{aligned}$$

(b)

1. The sequence converges to 0. We have $\mathbf{E}[Y_i] = \mathbf{E}[X_i]/2^i = 2/2^i = 1/2^{i-1}$ and $\mathbf{var}(Y_i) = \mathbf{var}(X_i)/(2^i)^2 = 9/2^{2i}$. By the Chebyshev inequality, for any $\epsilon > 0$,

$$\mathbf{P}\left(\left|Y_i - \frac{1}{2^{i-1}}\right| \geq \epsilon\right) \leq \frac{9}{2^{2i} \cdot \epsilon^2}.$$

Taking the limit as $i \to \infty$, we have

$$\lim_{i\to\infty} \mathbf{P}(|Y_i - 0| \geq \epsilon) = 0.$$

2. The sequence converges to 0. We have

$$\begin{aligned}
\mathbf{E}[A_n] &= \mathbf{E}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] \\
&= \frac{1}{n}\mathbf{E}\left[\sum_{i=1}^{n}\frac{X_i}{2^i}\right] \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}\frac{2}{2^i}\right) \\
&= \frac{1}{n}\left(2-\frac{2}{2^n}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{var}(A_n) &= \mathbf{var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) \\
&= \frac{1}{n^2}\mathbf{var}\left(\sum_{i=1}^{n}\frac{X_i}{2^i}\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n}\frac{9}{2^{2i}}\right)
\end{aligned}$$

$$= \frac{1}{n^2}\left(3 - \frac{3}{2^{2n}}\right).$$

Note that $\lim_{n\to\infty} \mathbf{E}[A_n] = 0$ and $\lim_{n\to\infty} \mathbf{var}(A_n) = 0$.

By the Chebyshev inequality, for any $\epsilon > 0$,

$$\mathbf{P}\left(\left|A_n - \frac{1}{n}\left(2 - \frac{2}{2^n}\right)\right| \geq \epsilon\right) \leq \frac{1}{n^2\epsilon^2}\left(3 - \frac{3}{2^{2n}}\right).$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbf{P}(|A_n - 0| \geq \epsilon) = 0.$$

3. The sequence converges to 2. Note that

$$M_n = \frac{1}{3} \cdot \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{2}{3} \cdot \frac{1}{n}\sum_{i=1}^{n} X_{i+1}.$$

By the weak law of large numbers, the first term converges in probability to $(1/3)\cdot\mathbf{E}[X_i]$ and the second term converges in probability to $(2/3)\cdot\mathbf{E}[X_i]$. As discussed in lecture, if two sequences of random variables each converge in probability, then their sum also converges in probability to the sum of the two limits. Therefore, M_n converges in probability to $(1/3)\cdot\mathbf{E}[X_i]+(2/3)\cdot\mathbf{E}[X_i]=2$.

You have used 1 of 2 submissions

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