Integer and Mixed Integer Linear Programs

MIT Center for Transportation & Logistics

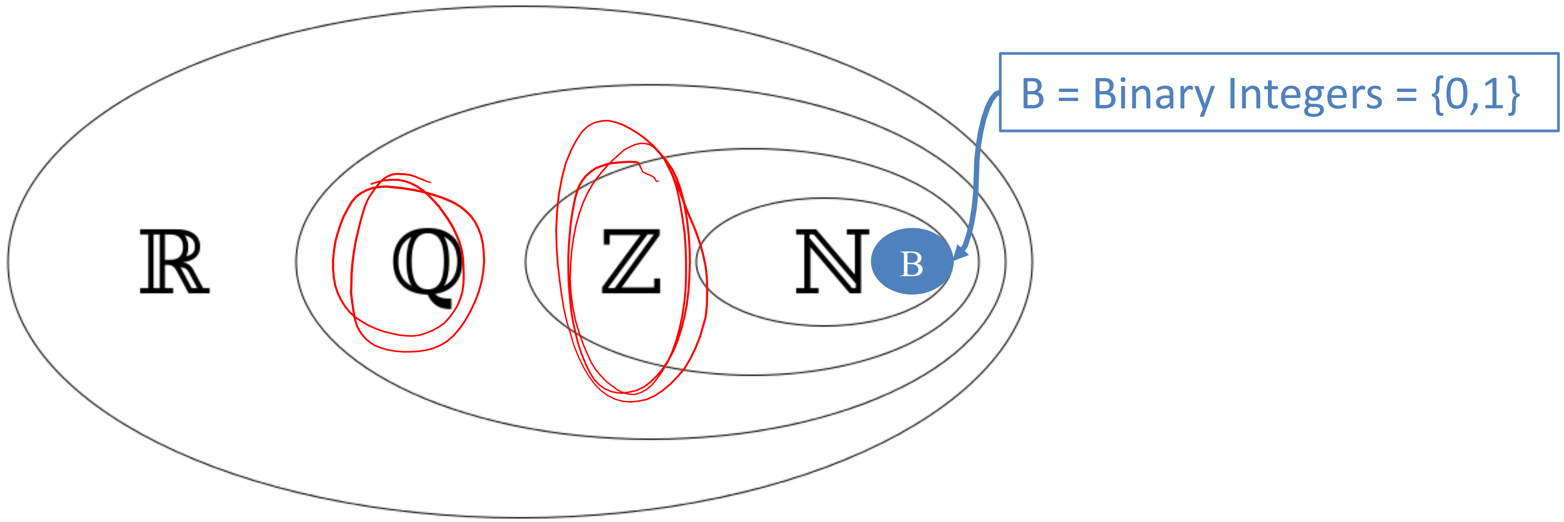
Numbers, Numbers, Everywhere!

N = Natural, *Whole*, or Counting Numbers = 0, 1, 2, 3, . . .

Z = Integers = . . . -3, -2, -1, 0, 1, 2, 3, . . .

Q = Rational Numbers = any fraction of Integers, 1/2 , -5/9, 0/22, . . . etc.

R = Real Numbers = all Rational and Irrational numbers, i.e, π, √2, e, . . . etc.



Why the heck do we care?

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Integer Variables

- Why use them?
  - When its physically impossible to have fractional solutions
    - For example; number of people to hire, number of ships to make
    - However, if dealing with large numbers, continuous is fine
  - Allows for modeling logical conditions (Binary)
    - If Then:

      If we have product leaving plant A **then** we must open it
    - Either Or:

      We can **either** produce ≥1000 units **or** none at all.
    - Select From:

      We must **select** ≥4 DCs to open **from** the 10 possible

      We must **select** ≤5 products to make **from** the 15 available
- Why do we have to treat them differently?

Banner Chemicals II: IP Example

Motivating Problem – Banner Chemicals II

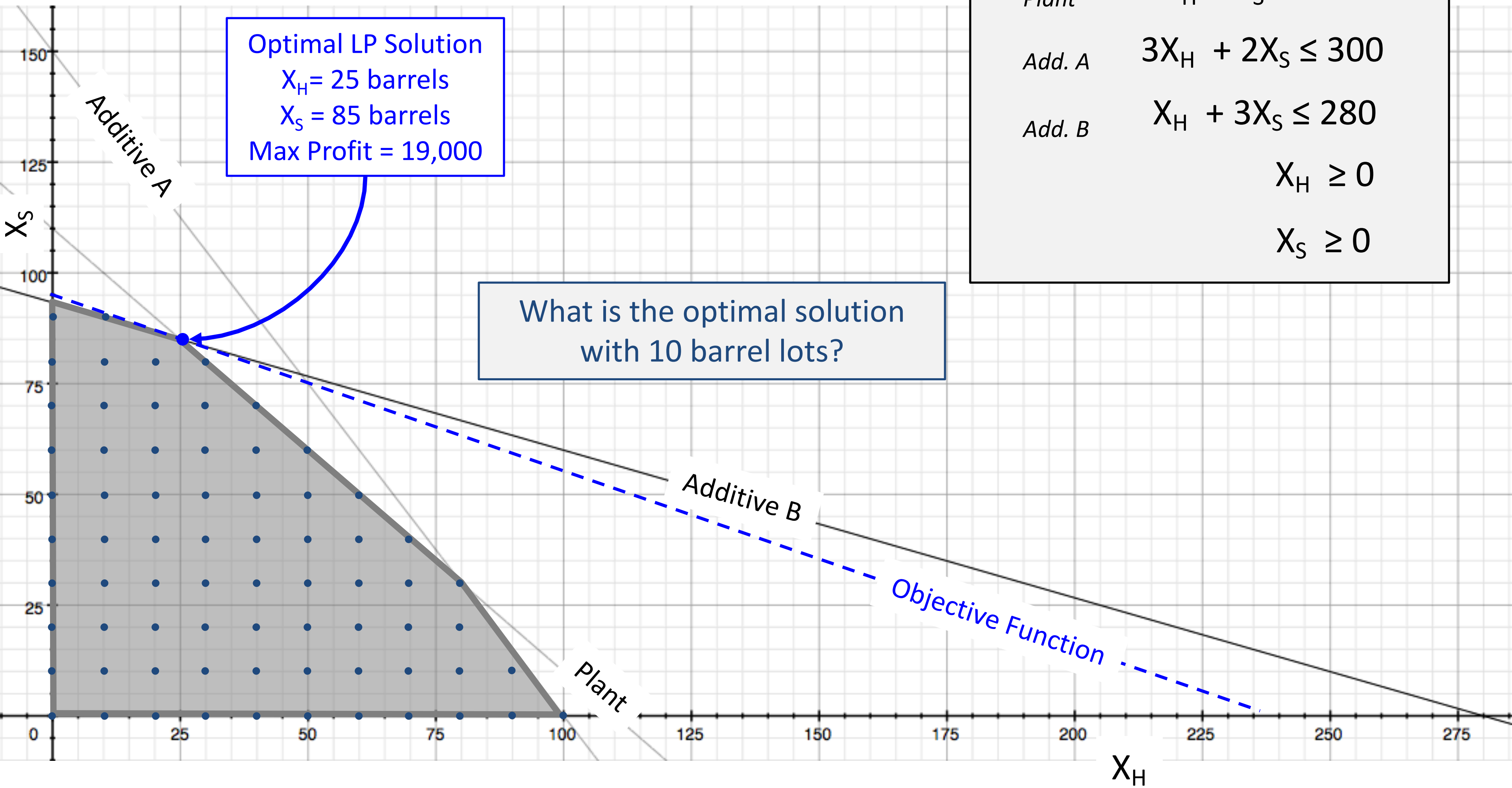
- Situation
  - Banner Chemicals manufactures specialty chemicals. One of their products comes in two grades, high and supreme. The capacity at the plant is 110 barrels per week.
  - The high and supreme grade products use the same basic raw materials but require different ratios of additives. The high grade requires 3 gallons of additive A and 1 gallon of additive B per barrel while the supreme grade requires 2 gallons of additive A and 3 gallons of additive B per barrel.
  - The supply of both of these additives is quite limited. Each week, this product line is allocated only 300 gallons of additive A per week and 280 gallons of additive B.
  - A barrel of the high grade has a profit margin of $80 per barrel while the supreme grade has a profit margin of $200 per barrel.
- Question
  - How many barrels of High and Supreme grade should Banner Chemicals produce each week assuming you can only produce in 10 barrel lots?

Banner Chemicals

$$\text{Max } z(X_H, X_S) = 80X_H + 200X_S$$

s.t.

Plant	$X_H + X_S \leq 110$
Add. A	$3X_H + 2X_S \leq 300$
Add. B	$X_H + 3X_S \leq 280$
	$X_H \geq 0$
	$X_S \geq 0$



What is the optimal solution with 10 barrel lots?

Notes:

- Feasible region becomes a collection of points, no longer a convex hull
- We cannot rely on “corner” solutions anymore – solution space is much bigger!

How to find solution to IP?

$$\text{Max } z(X_H, X_S) = 80X_H + 200X_S$$

s.t.

Plant	$X_H + X_S \leq 110$
Add. A	$3X_H + 2X_S \leq 300$
Add. B	$X_H + 3X_S \leq 280$
	$X_H \geq 0$
	$X_S \geq 0$

- Let's try "rounding" the solution to the closest acceptable integer values?
  - LP Solution:
    - X_H=25 barrels X_S= 85 barrels
  - Rounding to closest "10 barrel" solution for (X_H, X_S):
    1. $z_{LOT}(30, 90)$ = \$20,400 but it is infeasible (Plant constraint)
    2. $z_{LOT}(30, 80)$ = \$18,400 feasible
    3. $z_{LOT}(20, 90)$ = \$19,600 but it is infeasible (Additive B constraint)
  - So, using this approach z^*_{LOT}= \$18,400 with X_H=30, X_S=80
  - But, is it the best?
- Let's solve all of the points to make sure!

This approach is called Mass Enumeration.

Mass Enumeration of Banner Chemical

Max $z(X_H, X_S) = 80X_H + 200X_S$

s.t.

Plant	$X_H + X_S \le 110$
Add. A	$3X_H + 2X_S \le 300$
Add. B	$X_H + 3X_S \le 280$
	$X_H \ge 0$
	$X_S \ge 0$

Optimal IP Solution
X_H= 10 barrels
X_S = 90 barrels
Max Profit = 18,800

Optimal LP Solution
X_H= 25 barrels
X_S = 85 barrels
Max Profit = 19,000

Closest "rounded" LP Solution
X_H= 30 barrels
X_S = 80 barrels
Max Profit = 18,400

Each cell shows $z = 80X_H + 200X_S$
x indicates infeasible solution

Barrels of Xs	0	10	20	30	40	50	60	70	80	90	100
100	x	x	x	x	x	x	x	x	x	x	x
90	\$ 18,000	\$ 18,800	x	x	x	x					
80	\$ 16,000	\$ 16,800	\$ 17,600	\$ 18,400	x	x					
70	\$ 14,000	\$ 14,800	\$ 15,600	\$ 16,400	\$ 17,200	x					
60	\$ 12,000	\$ 12,800	\$ 13,600	\$ 14,400	\$ 15,200	\$ 16,000	x	x	x	x	x
50	\$ 10,000	\$ 10,800	\$ 11,600	\$ 12,400	\$ 13,200	\$ 14,000	\$ 14,800	x	x	x	x
40	\$ 8,000	\$ 8,800	\$ 9,600	\$ 10,400	\$ 11,200	\$ 12,000	\$ 12,800	\$ 13,600	x	x	x
30	\$ 6,000	\$ 6,800	\$ 7,600	\$ 8,400	\$ 9,200	\$ 10,000	\$ 10,800	\$ 11,600	\$ 12,400	x	x
20	\$ 4,000	\$ 4,800	\$ 5,600	\$ 6,400	\$ 7,200	\$ 8,000	\$ 8,800	\$ 9,600	\$ 10,400	x	x
10	\$ 2,000	\$ 2,800	\$ 3,600	\$ 4,400	\$ 5,200	\$ 6,000	\$ 6,800	\$ 7,600	\$ 8,400	\$ 9,200	x
0	\$ -	\$ 800	\$ 1,600	\$ 2,400	\$ 3,200	\$ 4,000	\$ 4,800	\$ 5,600	\$ 6,400	\$ 7,200	\$ 8,000
	Barrels of Xh										

Notes:
- Rounding the optimal LP solution will not always lead to an optimal IP solution
- Mass enumeration is very time consuming – not always possible for real problems!
- IP solution can never be better than the LP solution!
- IPs are much, much, much harder to solve than LPs!

Formulation Changes . . . not much!

Max $z(X_H, X_S) = 80X_H + 200X_S$

s.t.

Plant $X_H + X_S \leq 110$

Add. A $3X_H + 2X_S \leq 300$

Add. B $X_H + 3X_S \leq 280$

$X_H \geq 0$

$X_S \geq 0$

- In order to solve in integer values of "lots of ten", we need to:
  - Convert Decision Variables
    - $X_{HL} = X_H / 10$ $X_{SL} = X_S / 10$
  - Scale the coefficients and constraint RHS
    - e.g. 110 barrels becomes 11 lots of ten
  - Indicate that the new DVs are Integers

Max $z(X_{HL}, X_{SL}) = 800X_{HL} + 2000X_{SL}$

s.t.

Plant $X_{HL} + X_{SL} \leq 11$

Add. A $3X_{HL} + 2X_{SL} \leq 30$

Add. B $X_{HL} + 3X_{SL} \leq 28$

$X_{HL}, X_{SL} \geq 0$ Integers

GoNuts Juice Company: Model 1

GoNuts Juice Company

GoNuts manufactures different juices made entirely of various exotic nuts. Their primary market is China and they operate three plants located in Ethiopia, Tanzania, and Nigeria. You have been asked to help them determine where to manufacture the two newest juices they offer, Gingko Nut and Kola Nut. Each plant has a different variable cost structure and capacity for manufacturing the different juices. Also, each juice has an expected demand.

Cost/Unit	Ginko	Kola
Ethiopia	¥21.00	¥22.50
Tanzania	¥22.50	¥24.50
Nigeria	¥23.00	¥25.50

Capacity	Units/Month
Ethiopia	425
Tanzania	400
Nigeria	750

Demand	Units/Month
Ginko	550
Kola	450

How much of each juice should be made at each plant in order to minimize total cost while meeting demand and adhering to plant capacity?

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Formulating GoNuts

Step 1. Determine Decision Variables

$x_{G,E}$ = Number of Ginko Juice units made in Ethiopia plant
$x_{K,E}$ = Number of Kola Juice units made in Ethiopia plant
$x_{G,T}$ = Number of Ginko Juice units made in Tanzania plant
$x_{K,T}$ = Number of Kola Juice units made in Tanzania plant
$x_{G,N}$ = Number of Ginko Juice units made in Nigeria plant
$x_{K,N}$ = Number of Kola Juice units made in Nigeria plant

x_{ij} = Number of units of product i made in plant j
$x_{ij} \geq 0$ for all i,j

Step 2. Formulate Objective Function

Minimize z = Cost = $21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N}$

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij}$$

where:

x_{ij} = Number of units of product i made in plant j
c_{ij} = Cost per unit of product i made at plant j

Formulating GoNuts

Step 3. Formulate Constraints

Plant Capacity

$$x_{G,E} + x_{K,E} \leq 425$$
$$x_{G,T} + x_{K,T} \leq 400$$
$$x_{G,N} + x_{K,N} \leq 750$$

$$\sum_i x_{ij} \leq C_j \qquad \forall j$$

where:

x_{ij} = Number of units of product i made in plant j

C_j = Capacity in units at plant j

Product Demand

$$x_{G,E} + x_{G,T} + x_{G,N} \geq 550$$
$$x_{K,E} + x_{K,T} + x_{K,N} \geq 450$$

$$\sum_j x_{ij} \geq D_i \qquad \forall i$$

where:

x_{ij} = Number of units of product i made in plant j

D_i = Demand for product i in units

Formulating GoNuts

Minimize $z = \text{Cost} = 21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N}$

subject to

$$x_{G,E} + x_{K,E} \leq 425$$
$$x_{G,T} + x_{K,T} \leq 400$$
$$x_{G,N} + x_{K,N} \leq 750$$
$$x_{G,E} + x_{G,T} + x_{G,N} \geq 550$$
$$x_{K,E} + x_{K,T} + x_{K,N} \geq 450$$
$$x_{G,E}, x_{K,E}, x_{G,T}, x_{K,T}, x_{G,N}, x_{K,N} \geq 0$$

c_{ij}	**i=1** G	**i=2** K
j=1	¥21.00	¥22.50
j=2	¥22.50	¥24.50
j=3	¥23.00	¥25.50

Capacity	C_j
j=1	425
j=2	400
j=3	750

Demand	D_i
i=1	550
i=2	450

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij}$$

$$s.t.$$

$$\sum_i x_{ij} \leq C_j \quad \forall j \qquad 3$$

$$\sum_j x_{ij} \geq D_i \quad \forall i \qquad 2$$

$$x_{ij} \geq 0 \quad \forall ij$$

where:

x_{ij} = Number of units of product i made in plant j

c_{ij} = Cost per unit of product i made at plant j

C_j = Capacity in units at plant j

D_i = Demand for product i in units

Optimal Solution

	Ginko	**Kola**
Ethiopia	0	425
Tanzania	375	25
Nigeria	175	0

Total min cost = ¥ 22,637.50

GoNuts Juice Company: Model 2

GoNuts Juice Company: Model 2

GoNuts manufactures different juices made entirely of various exotic nuts. Their primary market is China and they operate three plants located in Ethiopia, Tanzania, and Nigeria. You have been asked to help them determine where to manufacture the two newest juices they offer, Gingko Nut and Kola Nut. Each plant has a different fixed and variable cost structure and capacity for manufacturing the different juices. The fixed cost only applies if the plant produces any juice. Also, each juice has an expected demand.

Cost/Unit	Ginko	Kola
Ethiopia	¥21.00	¥22.50
Tanzania	¥22.50	¥24.50
Nigeria	¥23.00	¥25.50

Capacity	Units/Month	Fixed (¥/Month)
Ethiopia	425	¥1,500
Tanzania	400	¥2,000
Nigeria	750	¥3,000

Demand	Units/Month
Ginko	550
Kola	450

How much of each juice should be made at each plant in order to minimize total cost while meeting demand and adhering to plant capacity?

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Formulating GoNuts 2

Step 1. Determine Decision Variables

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 otherwise

Step 2. Formulate Objective Function

Min $z = 21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N} + 1500y_E + 2000y_T + 3000y_N$

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij} + \sum_j f_j y_j$$

where:

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 o.w.
c_{ij} = Cost per unit of product i made at plant j
f_j = Fixed cost per month if plant j is used

Formulating GoNuts 2

Step 3. Formulate Constraints

Capacity

$x_{G,E} + x_{K,E} \leq 425$
$x_{G,T} + x_{K,T} \leq 400$
$x_{G,N} + x_{K,N} \leq 750$

Demand

$x_{G,E} + x_{G,T} + x_{G,N} \geq 550$
$x_{K,E} + x_{K,T} + x_{K,N} \geq 450$

$$\sum_i x_{ij} \leq C_j \qquad \forall j$$
$$\sum_j x_{ij} \geq D_i \qquad \forall i$$

where:

x_{ij} = Number of units of product i made in plant j
C_j = Capacity in units at plant j
D_i = Demand for product i in units

- Is this enough? Try solving it!
  - You need to ensure that **if** a plant produces product, **then** it is actually opened!
  - If Then conditions require both a
    - Binary Variable
    - Linking Constraint

If Then Conditions

If-Then Condition

Looking at the Nigeria Plant . . .

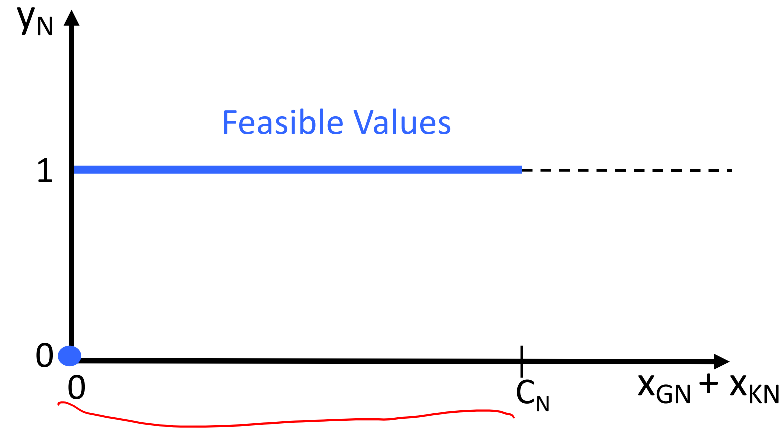
- How do y_N, x_{GN} and x_{KN} interact?

IF . . .	THEN	
$x_{GN}+x_{KN}$	$y_N = 0$	$y_N = 1$
$= 0$	YES	YES
>0 and $\leq C_N$	NO	YES

$$\sum_i x_{ij} \leq M y_j \quad \forall j$$

where:

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 o.w.
M = a big number (such as C_j in this case)



$$x_{GN} + x_{KN} \leq 750\, y_N$$

IF $x_{GN}+x_{KN}$ =	THEN y_N =	
0	0 or 1	$0 \leq 750\, y_N$
99	1	$99 \leq 750\, y_N$
1	1	$1 \leq 750\, y_N$

If the X values >0, then Y MUST be equal to 1! Otherwise, it would violate the constraint.

Formulating GoNuts 2

Step 3. Formulate Constraints

Capacity

$$x_{G,E} + x_{K,E} \leq 425$$
$$x_{G,T} + x_{K,T} \leq 400$$
$$x_{G,N} + x_{K,N} \leq 750$$

Demand

$$x_{G,E} + x_{G,T} + x_{G,N} \geq 550$$
$$x_{K,E} + x_{K,T} + x_{K,N} \geq 450$$

Linking

$$x_{G,E} + x_{K,E} - 425y_E \leq 0$$
$$x_{G,T} + x_{K,T} - 400y_T \leq 0$$
$$x_{G,N} + x_{K,N} - 750y_N \leq 0$$

$$\sum_i x_{ij} \leq C_j \qquad \forall j$$
$$\sum_j x_{ij} \geq D_i \qquad \forall i$$
$$\sum_i x_{ij} - My_j \leq 0 \qquad \forall j$$

where:

- x_{ij} = Number of units of product i made in plant j
- y_j = 1 if plant j is opened; = 0 o.w.
- M = a big number (such as C_j in this case)
- C_j = Capacity in units at plant j
- D_i = Demand for product i in units

GoNuts Juice Company Model 2: With If Then Conditions

Formulation of GoNuts Model 2

Min z = $21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N} + 1500y_E + 2000y_T + 3000y_N$

subject to

$$x_{G,E} + x_{K,E} \leq 425$$
$$x_{G,T} + x_{K,T} \leq 400$$
$$x_{G,N} + x_{K,N} \leq 750$$
$$x_{G,E} + x_{G,T} + x_{G,N} \geq 550$$
$$x_{K,E} + x_{K,T} + x_{K,N} \geq 450$$
$$x_{G,E} + x_{K,E} - 425y_E \leq 0$$
$$x_{G,T} + x_{K,T} - 400y_T \leq 0$$
$$x_{G,N} + x_{K,N} - 750y_N \leq 0$$
$$x_{G,E}, x_{K,E}, x_{G,T}, x_{K,T}, x_{G,N}, x_{K,N} \geq 0$$
$$y_E, y_T, y_N = \{0, 1\}$$

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij} + \sum_j f_j y_j$$

s.t.

$$\sum_i x_{ij} \leq C_j \quad \forall j$$
$$\sum_j x_{ij} \geq D_i \quad \forall i$$
$$\sum_i x_{ij} - My_j \leq 0 \quad \forall j$$
$$x_{ij} \geq 0 \quad \forall ij$$
$$y_j = \{0,1\}$$

where:

- x_{ij} = Number of units of product i made in plant j
- y_j = 1 if plant j is opened; = 0 o.w.
- c_{ij} = Cost per unit of product i made at plant j
- C_j = Capacity in units at plant j
- D_i = Demand for product i in units
- M = a big number (such as C_j in this case)

Demand	D_i
i=1	550
i=2	450

Capacity	C_j	f_j
j=1	425	1500
j=2	400	2000
j=3	750	3000

c_{ij}	i=1	i=2
j=1	¥21.00	¥22.50
j=2	¥22.50	¥24.50
j=3	¥23.00	¥25.50

Solution: GoNuts Models 1 & 2

Model 1 – only variable costs

z*= ¥ 22,637.50

	Ginko	Kola
Ethiopia	0	425
Tanzania	375	25
Nigeria	175	0

Model 2 – with fixed plant costs

z*= ¥ 27,350.00

	Ginko	Kola
Ethiopia	0	425
Tanzania	0	0
Nigeria	550	25

GoNuts Juice Company: Model 3
Adding Either Or Conditions

GoNuts Juice Company: Model 3

GoNuts manufactures different juices made entirely of various exotic nuts. Their primary market is China and they operate three plants located in Ethiopia, Tanzania, and Nigeria. You have been asked to help them determine where to manufacture the two newest juices they offer, Gingko Nut and Kola Nut. Each plant has a different fixed and variable cost structure and both minimum and maximum capacities for manufacturing the different juices if the plant opens. The fixed cost only applies if the plant produces any juice. Also, each juice has an expected demand.

Cost/Unit	Ginko	Kola
Ethiopia	¥21.00	¥22.50
Tanzania	¥22.50	¥24.50
Nigeria	¥23.00	¥25.50

Capacity (units/Month)	Max Capacity	Min Capacity	Fixed (¥/Month)
Ethiopia	425	100	¥1,500
Tanzania	400	250	¥2,000
Nigeria	750	600	¥3,000

Demand	Units/Month
Ginko	550
Kola	450

If the Nigeria plant opens, it must produce at least 600 units

How much of each juice should be made at each plant in order to minimize total cost while meeting demand and adhering to plant capacity?

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Formulating GoNuts 3

Step 1. Determine Decision Variables No Change!

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 otherwise

Step 2. Formulate Objective Function No Change!

Min z = $21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N} + 1500y_E + 2000y_T + 3000y_N$

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij} + \sum_j f_j y_j$$

where:

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 o.w.
c_{ij} = Cost per unit of product i made at plant j
f_j = Fixed cost per month if plant j is used

Formulating GoNuts 3

Step 3. Formulate Constraints

Capacity

$$x_{G,E} + x_{K,E} \le 425$$
$$x_{G,T} + x_{K,T} \le 400$$
$$x_{G,N} + x_{K,N} \le 750$$

Demand

$$x_{G,E} + x_{G,T} + x_{G,N} \ge 550$$
$$x_{K,E} + x_{K,T} + x_{K,N} \ge 450$$

Linking

$$x_{G,E} + x_{K,E} - 425y_E \le 0$$
$$x_{G,T} + x_{K,T} - 400y_T \le 0$$
$$x_{G,N} + x_{K,N} - 750y_N \le 0$$

$$\sum_i x_{ij} \le C_j \qquad \forall j$$
$$\sum_j x_{ij} \ge D_i \qquad \forall i$$
$$\sum_i x_{ij} - My_j \le 0 \quad \forall j$$

where:

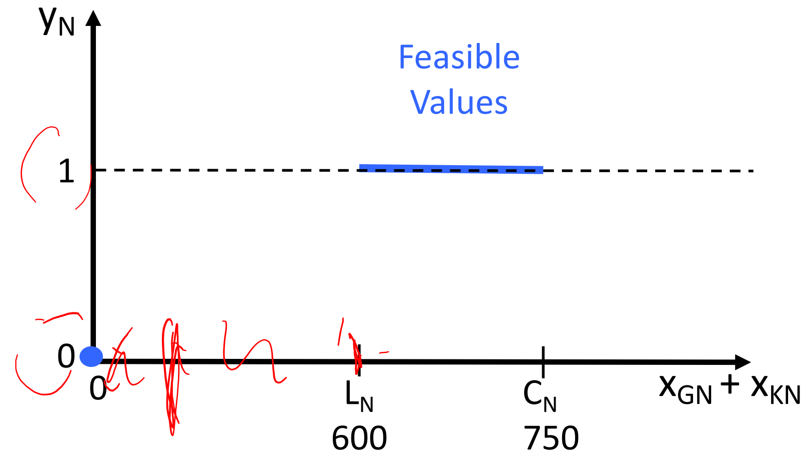
- x_{ij} = Number of units of product i made in plant j
- y_j = 1 if plant j is opened; = 0 o.w.
- M = a big number (such as C_j in this case)
- C_j = Maximum capacity in units at plant j
- L_j = Minimum level of production at plant j
- D_i = Demand for product i in units

We need to add a constraint that ensures that if we DO use plant j, that the volume is between the minimum allowable level, L_j, and the maximum capacity, C_j. This is sometimes called an Either-Or condition.

Either Or Condition

Looking at the Nigeria Plant . . .

- How do y_N, x_{GN} and x_{KN} interact?



IF . . .	THEN	
$x_{GN}+x_{KN}$	$y_N = 0$	$y_N = 1$
$= 0$	YES	NO
>0 and $< L_N$	NO	NO
$\geq L_N$ and $\leq C_N$	NO	YES

IF $x_{GN}+x_{KN}$	THEN y_N $\leq 750y_N$	THEN y_N $\geq 600y_N$
0	0 or 1	0
200	1	0
600	1	0 or 1

$$\sum_i x_{ij} \leq M y_j \quad \forall j$$
$$\sum_i x_{ij} \geq L_j y_j \quad \forall j$$

where:

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 o.w.
M = a big number (such as C_j in this case)
L_j = Minimum level of production at plant j

If the X values >0, then they must be ≥L, the lower limit, and ≤C, the maximum capacity!

Formulating GoNuts 3

Min z = $21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N} + 1500y_E + 2000y_T + 3000y_N$

subject to

$$x_{G,E} + x_{K,E} \le 425$$
$$x_{G,T} + x_{K,T} \le 400$$
$$x_{G,N} + x_{K,N} \le 750$$
$$x_{G,E} + x_{G,T} + x_{G,N} \ge 550$$
$$x_{K,E} + x_{K,T} + x_{K,N} \ge 450$$
$$x_{G,E} + x_{K,E} - 425y_E \le 0$$
$$x_{G,T} + x_{K,T} - 400y_T \le 0$$
$$x_{G,N} + x_{K,N} - 750y_N \le 0$$
$$x_{G,E} + x_{K,E} - 100y_E \ge 0$$
$$x_{G,T} + x_{K,T} - 250y_T \ge 0$$
$$x_{G,N} + x_{K,N} - 600y_N \ge 0$$
$$x_{G,E}, x_{K,E}, x_{G,T}, x_{K,T}, x_{G,N}, x_{K,N} \ge 0$$
$$y_E, y_T, y_N = \{0, 1\}$$

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij} + \sum_j f_j y_j$$

s.t.

$$\sum_i x_{ij} \le C_j \quad \forall j$$
$$\sum_j x_{ij} \ge D_i \quad \forall i$$
$$\sum_i x_{ij} - M y_j \le 0 \quad \forall j$$
$$\sum_i x_{ij} - L_j y_j \ge 0 \quad \forall j$$
$$x_{ij} \ge 0 \quad \forall ij$$
$$y_j = \{0,1\}$$

where:

- x_{ij} = Number of units of product i made in plant j
- y_j = 1 if plant j is opened; = 0 o.w.
- c_{ij} = Cost per unit of product i made at plant j
- C_j = Maximum capacity in units at plant j
- L_j = Minimum level of production at plant j
- D_i = Demand for product i in units
- M = a big number (such as C_j in this case)

Products	D_i
i=1	550
i=2	450

c_{ij}	i=1	i=2
j=1	¥21.00	¥22.50
j=2	¥22.50	¥24.50
j=3	¥23.00	¥25.50

Plants	C_j	L_j	f_j
j=1	425	100	1500
j=2	400	250	2000
j=3	750	600	3000

Solution: GoNuts Models 1, 2, & 3

Model 1 – only variable costs

z*= ¥ 22,637.50

	Ginko	Kola
Ethiopia	0	425
Tanzania	375	25
Nigeria	175	0

Model 2 – with fixed plant costs

z*= ¥ 27,350.00

	Ginko	Kola
Ethiopia	0	425
Tanzania	0	0
Nigeria	550	25

Model 3 – with fixed plant costs and minimum production levels

z*= ¥ 27,425.00

	Ginko	Kola
Ethiopia	0	400
Tanzania	0	0
Nigeria	550	50

GoNuts Juice Company: Model 4
Adding Select From Conditions

GoNuts Juice Company: Model 4

GoNuts manufactures different juices made entirely of various exotic nuts. Their primary market is China and they operate three plants located in Ethiopia, Tanzania, and Nigeria. You have been asked to help them determine where to manufacture the two newest juices they offer, Gingko Nut and Kola Nut. Each plant has a different variable cost structure and a maximum capacity. GoNuts can only operate 2 plants at a maximum. Also, each juice has an expected demand.

Cost/Unit	Ginko	Kola
Ethiopia	¥21.00	¥22.50
Tanzania	¥22.50	¥24.50
Nigeria	¥23.00	¥25.50

Capacity (units/Month)	Max Capacity
Ethiopia	425
Tanzania	400
Nigeria	750

Demand	Units/Month
Ginko	550
Kola	450

How much of each juice should be made at each plant in order to minimize total cost while meeting demand and adhering to plant capacity?

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Formulating GoNuts 4

Step 1. Determine Decision Variables No Change!

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 otherwise

Step 2. Formulate Objective Function Slight Change!

Min z = $21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N}$

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij}$$

where:

x_{ij} = Number of units of product i made in plant j
y_j = 1 if plant j is opened; = 0 o.w.
c_{ij} = Cost per unit of product i made at plant j

Formulating GoNuts 4

Step 3. Formulate Constraints

Capacity

$x_{G,E} + x_{K,E} \leq 425$
$x_{G,T} + x_{K,T} \leq 400$
$x_{G,N} + x_{K,N} \leq 750$

Demand

$x_{G,E} + x_{G,T} + x_{G,N} \geq 550$
$x_{K,E} + x_{K,T} + x_{K,N} \geq 450$

Linking

$x_{G,E} + x_{K,E} - 425y_E \leq 0$
$x_{G,T} + x_{K,T} - 400y_T \leq 0$
$x_{G,N} + x_{K,N} - 750y_N \leq 0$

Max Plants

$y_E + y_T + y_N \leq 2$

$$\sum_i x_{ij} \leq C_j \qquad \forall j$$
$$\sum_j x_{ij} \geq D_i \qquad \forall i$$
$$\sum_i x_{ij} - My_j \leq 0 \quad \forall j$$
$$\sum_j y_j \leq N$$

where:

- x_{ij} = Number of units of product i made in plant j
- y_j = 1 if plant j is opened; = 0 o.w.
- M = a big number (such as C_j in this case)
- C_j = Maximum capacity in units at plant j
- N = Number of plants allowed to be opened
- D_i = Demand for product i in units

We need to add a constraint that ensures that only N plants are used! We will use the Binary Variables, y_j, the Linking Constraints, and a new constraint that says the sum of the Binary Variables must not exceed N. This is sometimes called an Select-From condition.

Formulating GoNuts 4

$$\text{Min } z = 21x_{G,E} + 22.5x_{K,E} + 22.5x_{G,T} + 24.5x_{K,T} + 23x_{G,N} + 25.5x_{K,N}$$

subject to

$$x_{G,E} + x_{K,E} \leq 425$$
$$x_{G,T} + x_{K,T} \leq 400$$
$$x_{G,N} + x_{K,N} \leq 750$$
$$x_{G,E} + x_{G,T} + x_{G,N} \geq 550$$
$$x_{K,E} + x_{K,T} + x_{K,N} \geq 450$$
$$x_{G,E} + x_{K,E} - 425y_E \leq 0$$
$$x_{G,T} + x_{K,T} - 400y_T \leq 0$$
$$x_{G,N} + x_{K,N} - 750y_N \leq 0$$
$$y_E + y_T + y_N \leq 2$$
$$x_{G,E}, x_{K,E}, x_{G,T}, x_{K,T}, x_{G,N}, x_{K,N} \geq 0$$
$$y_E, y_T, y_N = \{0, 1\}$$

Products	D_i
i=1	550
i=2	450

N = 2

c_{ij}	i=1	i=2
j=1	¥21.00	¥22.50
j=2	¥22.50	¥24.50
j=3	¥23.00	¥25.50

Plants	C_j
j=1	425
j=2	400
j=3	750

$$Min \quad z = \sum_i \sum_j c_{ij} x_{ij}$$

$s.t.$

$$\sum_i x_{ij} \leq C_j \quad \forall j$$
$$\sum_j x_{ij} \geq D_i \quad \forall i$$
$$\sum_i x_{ij} - My_j \leq 0 \quad \forall j$$
$$\sum_j y_j \leq N$$
$$x_{ij} \geq 0 \quad \forall ij$$
$$y_j = \{0,1\}$$

where:

- x_{ij} = Number of units of product i made in plant j
- y_j = 1 if plant j is opened; = 0 o.w.
- c_{ij} = Cost per unit of product i made at plant j
- C_j = Maximum capacity in units at plant j
- D_i = Demand for product i in units
- M = a big number (such as C_j in this case)
- N = Number of plants allowed to be opened

Solution: GoNuts All Models

Model 1 – only variable costs

z*= ¥ 22,637.50

	Ginko	Kola
Ethiopia	0	425
Tanzania	375	25
Nigeria	175	0

Model 2 – with fixed plant costs

z*= ¥ 27,350.00

	Ginko	Kola
Ethiopia	0	425
Tanzania	0	0
Nigeria	550	25

Model 3 – with fixed plant costs and minimum production levels

z*= ¥ 27,425.00

	Ginko	Kola
Ethiopia	0	400
Tanzania	0	0
Nigeria	550	50

Model 4 – only variable costs but with maximum number of plants allowed

z*= ¥ 22,850.50

	Ginko	Kola
Ethiopia	0	425
Tanzania	0	0
Nigeria	550	25

Key Points from Lesson

Key Points from Lesson (1/2)

- IPs and MILPs are different from LPs
  - Much harder to solve since solution space expands!
  - Formulations
    - LPs a correct formulation is generally a good formulation
    - For IPs a correct formulation is necessary but not sufficient to guarantee solvability
  - IPs require solving multiple LPs to establish bounds – relaxing the Integer constraints
  - Can't just "round" the LP solution – might not be feasible
- When using integer (not binary) variables, solve the LP first to see if it is sufficient.

Key Points from Lesson (2/2)

- Binary variables are very powerful and can be used for modeling logical conditions
  - If Then – links continuous to binary variables

$$\boxed{\sum_i x_{ij} - My_j \le 0 \quad \forall j}$$

  - Either Or – ensures a minimum level if used at all

$$\boxed{\sum_i x_{ij} - My_j \le 0 \quad \forall j \qquad \sum_i x_{ij} - L_j y_j \ge 0 \quad \forall j}$$

  - Select From – picks the best X of Y choices (min or max)

$$\boxed{\sum_i x_{ij} - My_j \le 0 \quad \forall j \qquad \sum_j y_j \le N}$$

Questions, Comments, Suggestions?
Use the Discussion Forum!

“Athena – before and after completing the MITx MicroMasters Credential.
(photos courtesy of Lana Scott)

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