7.2 Classification of states

Unit 7: Markov Chains

Adapted from Blitzstein-Hwang Chapter 11.

In this section we introduce terminology for describing the various characteristics of a Markov chain. The states of a Markov chain can be classified as *recurrent* or *transient*, depending on whether they are visited over and over again in the long run or are eventually abandoned. States can also be classified according to their *period*, which is a positive integer summarizing the amount of time that can elapse between successive visits to a state. These characteristics are important because they determine the long-run behavior of the Markov chain, which we will study in Section 7.3.1.

The concepts of recurrence and transience are best illustrated with a concrete example. In the Markov chain shown on the left of Figure 7.2.1 (previously featured in Example 7.1.10), a particle moving around between states will continue to spend time in all 4 states in the long run, since it is possible to get from any state to any other state. In contrast, consider the chain on the right of Figure 7.2.1, and let the particle start at state 1. For a while, the chain may linger in the triangle formed by states 1, 2, and 3, but eventually it will reach state 4, and from there it can never return to states 1, 2, or 3. It will then wander around between states 4, 5, and 6 forever. States 1, 2, and 3 are *transient* and states 4, 5, and 6 are *recurrent*.

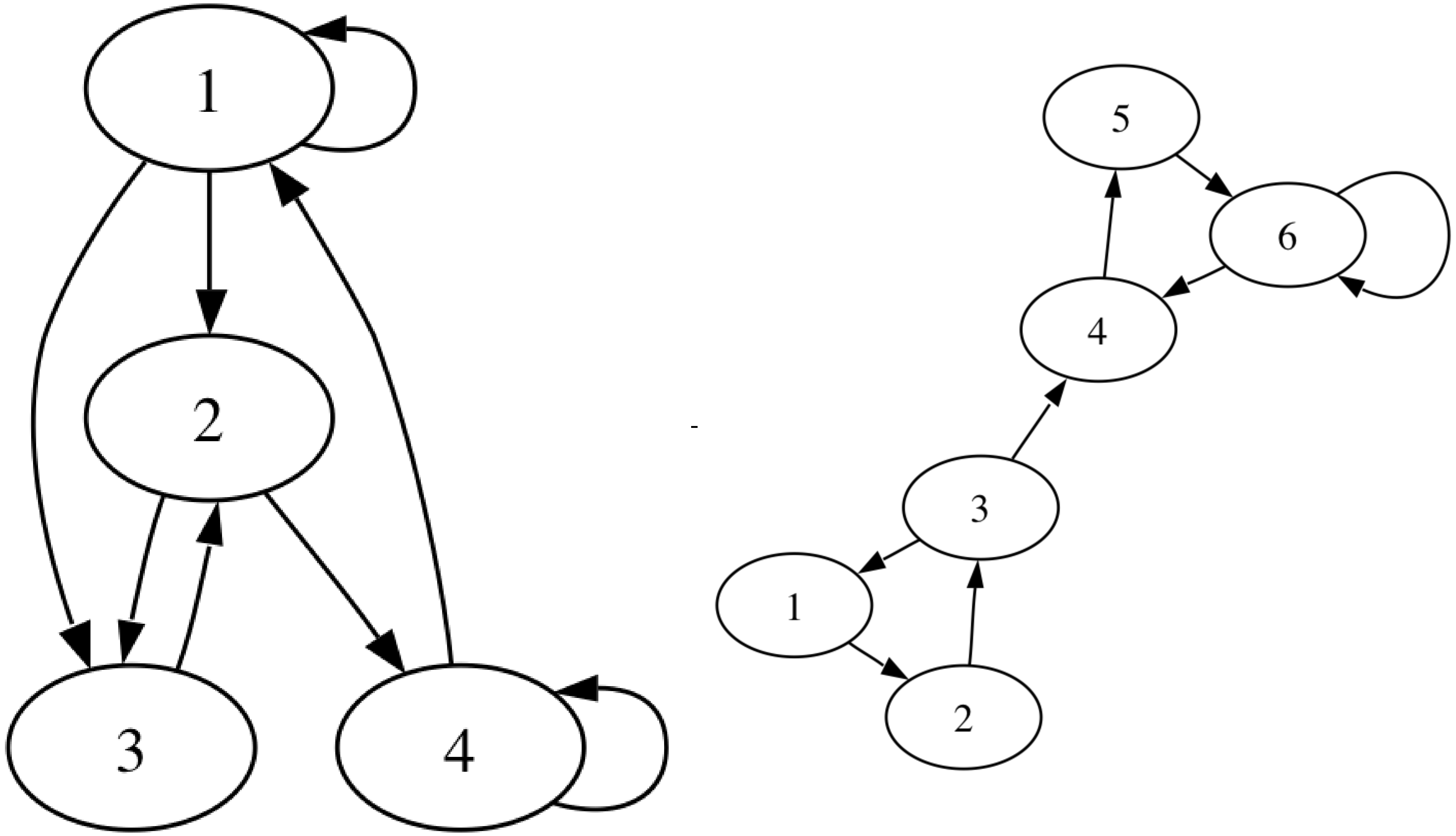

Figure 7.2.1:
Left: 4-state Markov chain with all states recurrent. View Larger Image. Image Description.
Right: 6-state Markov chain with states 1, 2, and 3 transient. View Larger Image. Image Description.

In general, these concepts are defined as follows.

> **Definition 7.2.2 (Recurrent and transient states).**
>
> State i of a Markov chain is *recurrent* if starting from i, the probability is 1 that the chain will eventually return to i. Otherwise, the state is *transient*, which means that if the chain starts from i, there is a positive probability of never returning to i.

In fact, although the definition of a transient state only requires that there be a positive probability of never returning to the state, we can say something stronger: as long as there is a positive probability of leaving i forever, the chain eventually *will* leave i forever. In the long run, anything that can happen, will happen (with a finite state space). We can make this statement precise with the following proposition.

Proposition 7.2.3 (Number of returns to transient state is Geometric).

Let i be a transient state of a Markov chain. Suppose the probability of never returning to i, starting from i, is a positive number $p > 0$. Then, starting from i, the number of times that the chain returns to i before leaving forever is distributed $\mathrm{Geom}(p)$.

Proof

We use the story of the Geometric distribution: each time that the chain is at i, we have a Bernoulli trial which results in "failure" if the chain eventually returns to i and "success" if the chain leaves i forever; these trials are independent by the Markov property. The number of returns to state i is simply the number of failures before the first success, which is the story of the Geometric distribution. In particular, since a Geometric random variable always takes finite values, this proposition tells us that after a finite number of visits, the chain will leave state i forever.

If the number of states is not too large, one way to classify states as recurrent or transient is to draw a diagram of the Markov chain and use the same kind of reasoning that we used when analyzing the chains in Figure 7.2.1. A special case where we can immediately conclude all states are recurrent is when the chain is *irreducible*, meaning that it is possible to get from any state to any other state.

> **Definition 7.2.4 (Irreducible and reducible chain).**
>
> A Markov chain with transition matrix Q is *irreducible* if for any two states i and j, it is possible to go from i to j in a finite number of steps (with positive probability). That is, for any states i, j there is some positive integer n such that the (i, j) entry of Q^n is positive. A Markov chain that is not irreducible is called *reducible*.

Proposition 7.2.5 (Irreducible implies all states recurrent).

In an irreducible Markov chain with a finite state space, all states are recurrent.

Proof

It is clear that at least one state must be recurrent; if all states were transient, the chain would eventually leave all states forever and have nowhere to go! So assume without loss of generality that state 1 is recurrent, and consider any other state i. We know that $q_{1i}^{(n)}$ is positive for some n, by the definition of irreducibility. Thus, every time the chain is at state 1, it has a positive probability of going to state i in n steps. Since the chain visits state 1 infinitely often, we know the chain *will* eventually reach state i from state 1; think of each visit to state 1 as starting a trial, where "success" is defined as reaching state i in at most n steps. From state i, the chain will return to state 1 because state 1 is recurrent, and by the same logic, it will eventually reach state i again. By induction, the chain will visit state i infinitely often. Since i was arbitrary, we conclude that all states are recurrent.

The converse of the proposition is false; it is possible to have a reducible Markov chain whose states are all recurrent. An example is given by the Markov chain below, which consists of two "islands" of states.

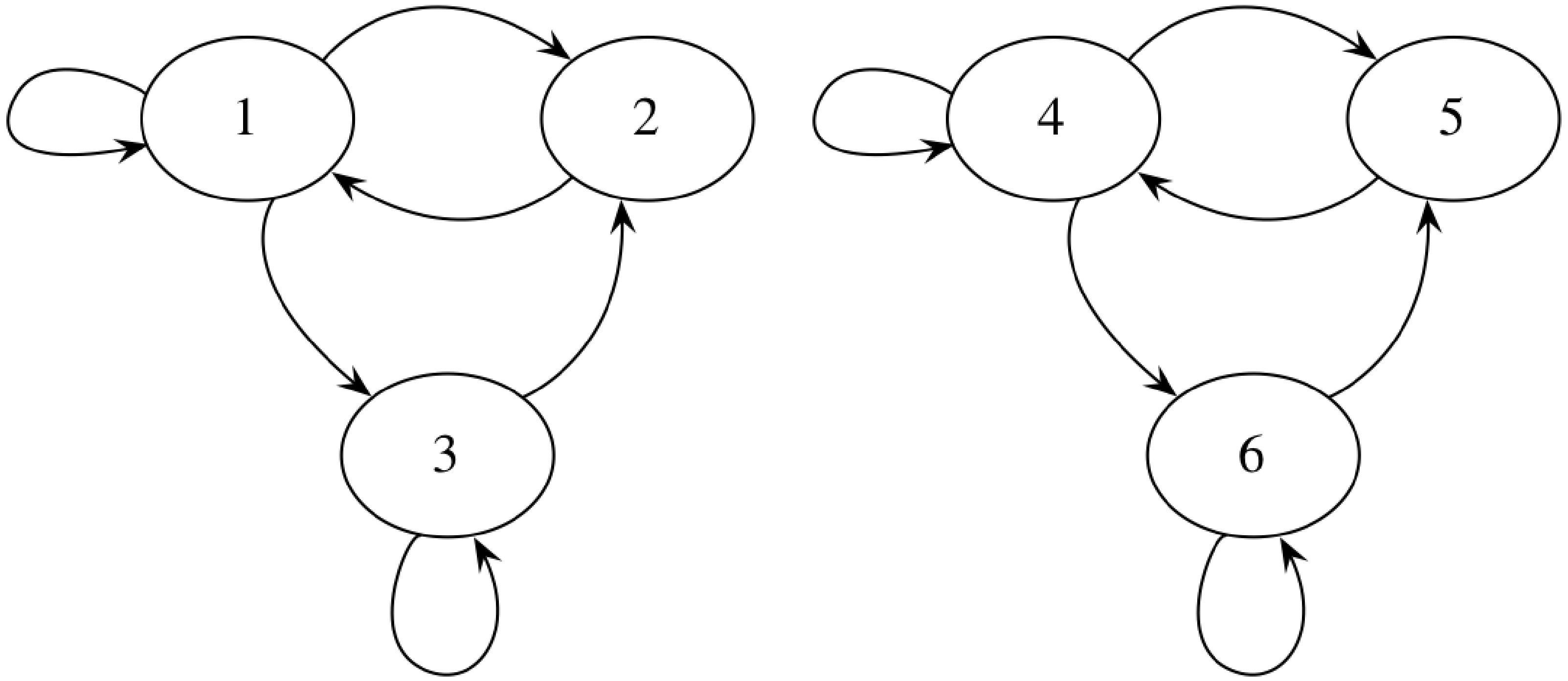

Figure 7.2.6

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Image Description

> **Definition 7.2.7 (Period of a state, periodic and aperiodic chain).**
>
> The *period* of a state i in a Markov chain is the greatest common divisor (gcd) of the possible numbers of steps it can take to return to i when starting at i. That is, the period of i is the greatest common divisor of numbers n such that the (i, i) entry of Q^n is positive. (The period of i is undefined if it's impossible ever to return to i after starting at i.) A state is called *aperiodic* if its period equals 1, and *periodic* otherwise. The chain itself is called *aperiodic* if all its states are aperiodic, and *periodic* otherwise.

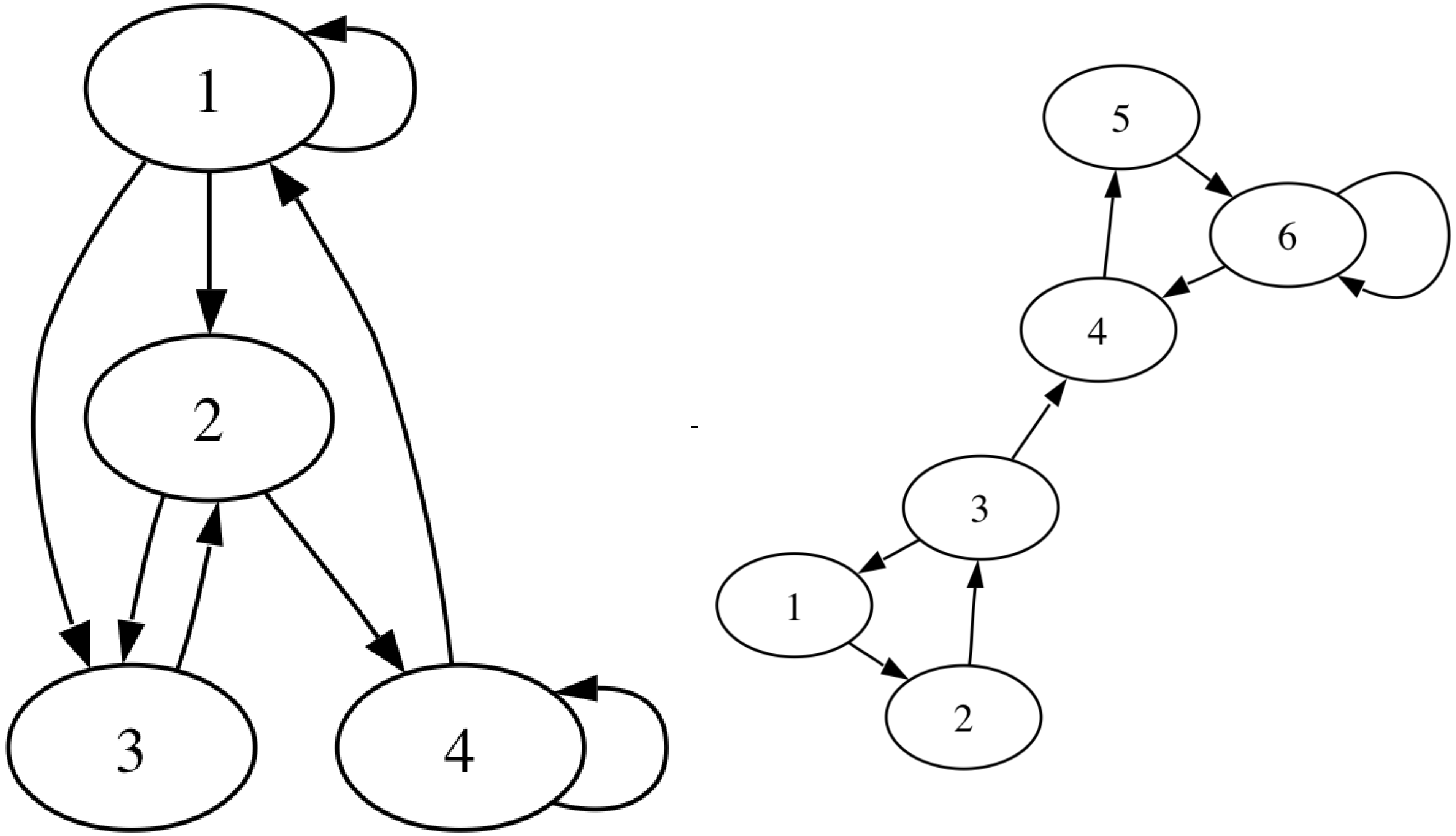

Figure 7.2.8:
Left: an aperiodic Markov chain. View Larger Image. Image Description.
Right: a periodic Markov chain in which states 1, 2, and 3 have period 3. View Larger Image. Image Description.

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