Unit 8: Quiz

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Unit 8: Quiz

The problems in the quiz will be automatically graded.

You are given only **ONE** attempt for each problem. You will be able to see the solution after you submit your answers.

If the answer is numeric and has more than 4 decimal places, just give your answers to **4 decimal places (0.0001 accuracy)** unless otherwise stated.

e.g. for 0.123456... you just need to round it up to 0.1235

Problem 1

2/3 points (graded)

1. Suppose X and Y have joint probability density function

$$f_{X,Y}(x,y) = 70e^{-3x-7y}$$

for $0 < x < y$; and $f_{X,Y}(x,y) = 0$ otherwise.

1a. For $x > 0$, find the density $f_X(x)$ of X. Compute the value of $f_X(0.1)$.

$f_X(0.1) =$ 9.048374 ✖ **Answer:** 3.678794

1b. For $x > 0$, use your answer to **a** to find the conditional density $f_{Y|X}(y \mid x)$ of Y given $X = x$, then compute the value of $f_{Y|X}(0.2 \mid 0.1)$

$f_{Y|X}(0.2 \mid 0.1) =$ 3.476097 ✔ **Answer:** 3.476097

1c. When $x = 1/10$, verify that the conditional probability density function $f_{Y|X}(y \mid \frac{1}{10})$ is a valid density, i.e., that (1) it is nonnegative and (2) we get 1 when integrating over the relevant y's.

1d. Find the conditional probability that $Y > 1/4$, given $X = 1/10$, i.e., $P(Y > 1/4 \mid X = 1/10)$.

0.3499377 ✔ **Answer:** 0.3499

Explanation

1a. For $x > 0$, we have $f_X(x) = \int_x^\infty 70e^{-3x-7y}\,dy = 10e^{-10x}$; for $x \le 0$, we have $f_X(x) = 0$.

1b. For $x > 0$, the conditional density is $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{70e^{-3x-7y}}{10e^{-10x}} = 7e^{7x-7y}$.

1c. We have $f_{Y|X}(y \mid \frac{1}{10}) = 7e^{7(1/10)-7y}$, which is nonnegative, and $\int_{1/10}^\infty 7e^{7(1/10)-7y}\,dy = 1$.

1d. We have $P(Y > 1/4 \mid X = 1/10) = \int_{1/4}^\infty f_{Y|X}(y \mid \frac{1}{10}) = e^{-21/20} = 0.3499$.

Submit You have used 1 of 1 attempt

Problem 2

1/2 points (graded)

2a. How do you setup a calculation to compute $P(Y > 1/4 \mid X > 1/10)$? Do you need the conditional probability density function $f_{Y|X}(y \mid x)$ for this calculation? (Notice that we are now conditioning on $X > 1/10$ instead of $X = 1/10$.) Go ahead and calculate $P(Y > 1/4 \mid X > 1/10)$. It might help to draw separate pictures for the numerator and denominator, so that you get the regions of integration right.

$P(Y > 1/4 \mid X > 1/10) =$ 0.6458221 ✔ **Answer:** 0.6458

2b. Find the conditional probability that $Y < 1/3$, given $X > 1/10$, i.e., $P(Y < 1/3 \mid X > 1/10)$.

$P(Y < 1/3 \mid X > 1/10) =$ 0.3490748 ✘ **Answer:** 0.575

Explanation

2a. We don't need $f_{Y|X}(y \mid x)$ for this part at all. Instead, we use the basic definition of conditional probability from Problem Set 4 (second week of class).

We need to compute $P(Y > 1/4 \mid X > 1/10) = \frac{P(Y>1/4 \,\&\, X>1/10)}{P(X>1/10)}$.

For the numerator,

$P(Y > 1/4 \,\&\, X > 1/10) = \int_{1/4}^{\infty} \int_{1/10}^{y} 70e^{-3x-7y}\, dx\, dy$

$= \int_{1/4}^{\infty} \left(\frac{70}{3} e^{-(3/10)-7y} - \frac{70}{3} e^{-10y}\right) dy$

$= \frac{10}{3}e^{-41/20} - \frac{7}{3}e^{-5/2} = 0.2376.$

For the denominator,

$P(X > 1/10) = \int_{1/10}^{\infty} \int_{x}^{\infty} 70e^{-3x-7y}\,dy\,dx$

$= \int_{1/10}^{\infty} 10e^{-10x}\,dx = e^{-1} = 0.3679.$

So we get

$P(Y > 1/4 \mid X > 1/10) = \frac{P(Y>1/4\ \&\ X>1/10)}{P(X>1/10)}$

$= 0.2376/0.3679 = 0.6458.$

2b. We need to compute $P(Y < 1/3 \mid X > 1/10) = \frac{P(Y<1/3\ \&\ X>1/10)}{P(X>1/10)}$.

For the numerator,

$P(Y < 1/3\ \&\ X > 1/10) = \int_{1/10}^{1/3} \int_{1/10}^{y} 70e^{-3x-7y}\,dx\,dy$

$= \int_{1/10}^{1/3} \left(\frac{70}{3}e^{-(3/10)-7y} - \frac{70}{3}e^{-10y}\right) dy$

$= e^{-1} - \frac{10}{3}e^{-79/30} + \frac{7}{3}e^{-10/3} = 0.2117.$

The denominator is

$P(X > 1/10) = e^{-1} = 0.3679$, just as in part **2a**.

So we get

$P(Y < 1/3 \mid X > 1/10) = \frac{P(Y<1/3\ \&\ X>1/10)}{P(X>1/10)} = 0.2117/0.3679 = 0.575.$

Submit You have used 1 of 1 attempt

✱ Partially correct (1/2 points)

Problem 3

2/3 points (graded)

3. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,8)$.

3a. For $0 \le x \le 2$, find the conditional density $f_{Y|X}(y \mid x)$ of Y, given $X = x$.

- (selected, correct) $\frac{1}{8-4x}$
- $\frac{8-4x}{8}$
- $\frac{1}{8}$
- $\frac{8}{8-4x}$

3b. Find the conditional probability that $Y \le 4$, given $X = 1/2$. I.e., find $P(Y \le 4 \mid X = 1/2)$.

2/3

Answer: 0.6667

3c. Find the conditional probability that $Y \le 4$, given $X \le 1/2$. I.e., find $P(Y \le 4 \mid X \le 1/2)$.

32/7

Answer: 0.5714286

Explanation

3a. For $0 \le x \le 2$, $f_X(x) = \int_0^{8-4x} 1/8\,dy = (8-4x)/8$.
We have
$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/8}{(8-4x)/8} = \frac{1}{8-4x}$.
3b. $f_{Y|X}(y \mid 1/2) = \frac{1}{8-4(1/2)} = 1/6$.
Thus $P(Y \le 4 \mid X = 1/2) = \int_0^4 1/6\,dy = 4/6 = 2/3$.
3c. We have $P(Y \le 4 \mid X \le 1/2) = \frac{P(Y \le 4 \text{ \& } X \le 1/2)}{P(X \le 1/2)}$. Both the numerator and denominator can be calculated by ratios of areas, since the joint density is constant. So we calculate
$P(Y \le 4 \mid X \le 1/2) = \frac{P(Y \le 4 \text{ \& } X \le 1/2)}{P(X \le 1/2)} = \frac{2/8}{(7/2)/8} = \frac{2}{7/2} = 4/7$.

Submit You have used 1 of 1 attempt

✱ Partially correct (2/3 points)

Problem 4

2/2 points (graded)
4a. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0,0)$, $(5,0)$, and $(0,5)$. For a (fixed) value of x with $0 \le x \le 5$, find the conditional density $f_{Y|X}(y \mid x)$ of Y, given $X = x$.

- $\frac{2}{25}$

- $\frac{2}{25(5-x)}$
- $\frac{2}{5-x}$
- $\frac{1}{5-x}$ ✔

4b. Can you generalize this? Suppose that $c > 0$ is a fixed constant. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0, 0)$, $(c, 0)$, and $(0, c)$. For a (fixed) value of x with $0 \leq x \leq c$, find the conditional density $f_{Y|X}(y \mid x)$ of Y, given $X = x$.

- $\frac{2}{c^2}$
- $\frac{1}{c-x}$ ✔
- $\frac{2}{c^2(c-x)}$
- $\frac{2}{c-x}$

Explanation

4a. For $0 \leq x \leq 5$, we have $f_X(x) = \int_0^{5-x} 2/25\,dy = (2/25)(5-x)$. So we get $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2/25}{(2/25)(5-x)} = \frac{1}{5-x}$.

4b. For $0 \leq x \leq c$, we have $f_X(x) = \int_0^{c-x} 2/c^2\,dy = (2/c^2)(c-x)$. So we get $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2/c^2}{(2/c^2)(c-x)} = \frac{1}{c-x}$.

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✔ Correct (2/2 points)

Problem 5

1/1 point (graded)

5. Suppose X and Y have joint probability density function

$$f_{X,Y}(x,y) = 70e^{-3x-7y}$$

for $0 < x < y$; and $f_{X,Y}(x,y) = 0$ otherwise. Find $\mathbb{E}(X)$. (You may either use the joint density given here, or the density $f_X(x)$ that was found in **1a**.)

1/10 ✔ **Answer:** 0.1

Explanation

5. One method is that we can compute
$\mathbb{E}(X) = \int_0^\infty \int_x^\infty (x)(70e^{-3x-7y})\,dy\,dx$
$= \int_0^\infty (x)(70e^{-3x}) \int_x^\infty e^{-7y}\,dy\,dx = \int_0^\infty (x)(70e^{-3x})(1/7)e^{-7x}\,dx,$
which simplifies to

$$\mathbb{E}(X) = \int_0^\infty (x)(10e^{-10x})\,dx = 1/10.$$

FYI, if you decided (instead) to just directly use the density of X, namely, $f_X(x) = 10e^{-10x}$ for $x > 0$, we get exactly the line above, $\mathbb{E}(X) = \int_0^\infty (x)(10e^{-10x})\,dx = 1/10$.

Submit You have used 1 of 1 attempt

✔ Correct (1/1 point)

Problem 6

1/1 point (graded)

6. For the setup in question **5**, find $\mathbb{E}(Y)$. (In this example, there are tradeoffs to the order of integration that you choose to use, i.e., to whether you integrate with respect to x or y first. You might find it instructive to try it both ways and compare the difficulties; this would also enable you to double-check your answer.)

0.2428571

Answer: 0.2428571

Explanation

6. One method is that we can compute

$\mathbb{E}(Y) = \int_0^\infty \int_x^\infty (y)(70e^{-3x-7y})\,dy\,dx$

$= \int_0^\infty (70e^{-3x}) \int_x^\infty ye^{-7y}\,dy\,dx = \int_0^\infty (70e^{-3x}) \frac{7x+1}{49} e^{-7x}\,dx,$

which simplifies to

$$\mathbb{E}(Y) = \int_0^\infty (70/49)(7x+1)e^{-10x}\,dx = 17/70.$$

A second method is that we can compute

$\mathbb{E}(Y) = \int_0^\infty \int_0^y (y)(70e^{-3x-7y})\,dx\,dy$

$= \int_0^\infty (y)(70e^{-7y}) \int_0^y e^{-3x}\,dx\,dy = \int_0^\infty (y)(70e^{-7y})(1/3)(1-e^{-3y})\,dy,$

which simplifies to

$$\mathbb{E}(Y) = \int_0^\infty (y)(70/3)(e^{-7y} - e^{-10y})\,dy = 10/21 - 7/30 = 17/70.$$

Submit You have used 1 of 1 attempt

✔ Correct (1/1 point)

Problem 7

1/1 point (graded)

7. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,8)$. Find $\mathbb{E}(X)$.

2/3

✔ **Answer:** 0.6666667

Explanation

7. One method is that we can compute

$\mathbb{E}(X) = \int_0^2\int_0^{8-4x}(x)(1/8)\,dy\,dx$

$= \int_0^2(x)(1/8)\int_0^{8-4x}1\,dy\,dx = \int_0^2(x)(1/8)(8-4x)\,dx$, which simplifies to

$$\mathbb{E}(X) = \int_0^2(x)\Big(\frac{8-4x}{8}\Big)\,dx = 2/3.$$

FYI, if you decided (instead) to just directly use the density of X, namely, $f_X(x) = \frac{8-4x}{8}$ for $0 \le x \le 2$, we get exactly the line above, $\mathbb{E}(X) = \int_0^2(x)(\frac{8-4x}{8})\,dx = 2/3$.

Submit You have used 1 of 1 attempt

✔ Correct (1/1 point)

Problem 8

2/2 points (graded)

8a. Suppose that Y is an exponential random variable with probability density function $f_Y(y) = 5e^{-5y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise. Compute $\mathbb{E}(Y)$.

1/5

✔ **Answer:** 0.2

8b. Generalize the result in **8a**. In other words, suppose that $\lambda > 0$ is a fixed constant, and suppose that Y is an exponential random variable with probability density function $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise. Compute $\mathbb{E}(Y)$.

- ○ λ
- ◉ $1/\lambda$ ✔
- ○ $-1/\lambda$
- ○ $-\lambda$

Explanation

8a. We have

$$\begin{aligned}\mathbb{E}(Y) &= \int_0^\infty (y)(5e^{-5y})\,dy \\ &= (y)(-e^{-5y})\Big|_{y=0}^\infty - \int_0^\infty -e^{-5y}\,dy \\ &= -(1/5)e^{-5y}\Big|_{y=0}^\infty = 1/5.\end{aligned}$$

8b. We have

$$\begin{aligned}\mathbb{E}(Y) &= \int_0^\infty (y)(\lambda e^{-\lambda y})\,dy \\ &= (y)(-e^{-\lambda y})\Big|_{y=0}^\infty - \int_0^\infty -e^{-\lambda y}\,dy \\ &= -(1/\lambda)e^{-\lambda y}\Big|_{y=0}^\infty = 1/\lambda.\end{aligned}$$

Submit You have used 1 of 1 attempt

✔ Correct (2/2 points)

Problem 9

2/2 points (graded)

9. Suppose X and Y have joint probability density function

$$f_{X,Y}(x,y) = 70e^{-3x-7y}$$

for $0 < x < y$; and $f_{X,Y}(x,y) = 0$ otherwise.

9a. Find $\mathbb{E}(X^2)$.

0.02

Answer: 0.02

9b. Find $\mathbf{Var}(X)$.

0.01

Answer: 0.01

Explanation

9a. One method is that we can compute

$$\mathbb{E}(X^2) = \int_0^\infty \int_x^\infty (x^2)(70e^{-3x-7y})\,dy\,dx$$
$$= \int_0^\infty (x^2)(70e^{-3x}) \int_x^\infty e^{-7y}\,dy\,dx = \int_0^\infty (x^2)(70e^{-3x})(1/7)e^{-7x}\,dx,$$

which simplifies to

$$\mathbb{E}(X^2) = \int_0^\infty (x^2)(10e^{-10x})\,dx$$
$$= (x^2)(-e^{-10x})\Big|_{x=0}^\infty - \int_0^\infty (-e^{-10x})(2x)\,dx$$
$$= 0 + 2\int_0^\infty xe^{-10x}\,dx$$

We already computed in Problem 5: $10\int_0^\infty xe^{-10x}\,dx = 1/10$, and thus

$$\mathbb{E}(X^2) = 2\int_0^\infty xe^{-10x}\,dx = (2/10)(1/10) = 2/100.$$

9b. We have $\mathbf{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2/100 - (1/10)^2 = 1/100.$

Submit You have used 1 of 1 attempt

Correct (2/2 points)

Problem 10

4/4 points (graded)

10. Consider a pair of random variables X, Y with constant joint density on the rectangle with vertices $(0,0)$, $(5,0)$, $(5,8)$, $(0,8)$.

10a. Find $\mathbb{E}(XY)$.

10 ✔ **Answer:** 10

10b. Are X and Y independent?

- (selected) Yes ✔
- No

Now use **10c** and **10d** to double-check your solution to **10a**:

10c. Find $\mathbb{E}(X)$.

5/2 ✔ **Answer:** 2.5

10d. Find $\mathbb{E}(Y)$.

4 ✔ **Answer:** 4

Explanation

10a. We have $\mathbb{E}(XY) = \int_0^5 \int_0^8 (xy)(1/40)\,dy\,dx = \int_0^5 (x)(4/5)\,dx = 10$.

10b. Yes, X and Y are independent. Their joint density $1/40$ can be factored into $1/5$ and $1/8$, and the joint density is defined on a rectangle.

10c. We have $\mathbb{E}(X) = \int_0^5 (x)(1/5)\,dx = 5/2$.

10d. We have $\mathbb{E}(Y) = \int_0^8 (y)(1/8)\,dy = 4$.

Thus, we can use parts **10b**, **10c**, **10d** to double check that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = (5/2)(4) = 10$. (We emphasize that we can only multiply the expected values this way because the X and Y are independent.)

Submit You have used 1 of 1 attempt

✔ Correct (4/4 points)

Problem 11

2/2 points (graded)

11. Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,8)$.

11a. Find $\mathbb{E}(X^2)$.

2/3 ✔ **Answer:** 0.667

11b. Find $\mathbb{E}(XY)$.

4/3

Answer: 1.333

Explanation

11a. One method is that we can compute

$\mathbb{E}(X^2) = \int_0^2 \int_0^{8-4x} (x^2)(1/8)\,dy\,dx$

$= \int_0^2 (x^2)(1/8) \int_0^{8-4x} 1\,dy\,dx = \int_0^2 (x^2)(1/8)(8-4x)\,dx,$

which simplifies to

$$\mathbb{E}(X^2) = \int_0^2 (x^2)\Big(\frac{8-4x}{8}\Big)\,dx = 2/3.$$

FYI, if you decided (instead) to just directly use the density of X, namely, $f_X(x) = \frac{8-4x}{8}$ for $0 \le x \le 2$, we get exactly the line above, $\mathbb{E}(X^2) = \int_0^2 (x^2)(\frac{8-4x}{8})\,dx = 2/3.$

11b. One method is that we can compute

$\mathbb{E}(XY) = \int_0^2 \int_0^{8-4x} (xy)(1/8)\,dy\,dx$

$= \int_0^2 (x)(1/8) \int_0^{8-4x} y\,dy\,dx$

$= \int_0^2 (x)(1/8)(8x^2 - 32x + 32)\,dx = 4/3.$

You could also have changed the order of integration and the bounds, as another possible method of solution.

Submit

You have used 1 of 1 attempt

✔ Correct (2/2 points)

Problem 12

2/2 points (graded)

12a. Suppose that Y is an exponential random variable with probability density function $f_Y(y) = 5e^{-5y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise.

12a. Compute $\mathbb{E}(Y^2)$.

2/25 ✔ **Answer:** 0.08

12b. Compute $\mathrm{Var}(Y)$. (You already have $\mathbb{E}(Y)$ from Problem 8.)

1/25 ✔ **Answer:** 0.04

Explanation

12a. We have

$$\begin{aligned}\mathbb{E}(Y^2) &= \int_0^\infty (y^2)(5e^{-5y})\,dy \\ &= (y^2)(-e^{-5y})\Big|_{y=0}^{\infty} - \int_0^\infty (2y)(-e^{-5y})\,dy\end{aligned}$$

which simplifies to $\mathbb{E}(Y^2) = (2)\int_0^\infty (y)(-e^{-5y})\,dy$.

We saw in **8a** that $5\int_0^\infty (y)(-e^{-5y})\,dy = 1/5$,

so it follows that $\mathbb{E}(Y^2) = (2/5)(1/5) = 2/25$.

12b. We have $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/25 - (1/5)^2 = 1/25$.

Submit You have used 1 of 1 attempt

✔ Correct (2/2 points)