MITx: 15.053x Optimization Methods in Business Analytics

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Problem 5

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PART A

1/1 point (graded)

This problem will be based on the game of Fiver. It is also sometimes called called Lights Out and you can try your hand at the game here and for more details check out the Wikipedia page here.

Assume we have a 5x5 grid that initially consists of all white blocks. When we click on a block, it flips its color (from white to black or from black to white) and that of adjacent (vertical and horizontal) blocks. Can you make all of the blocks black? The game is illustrated in this figure. For a documented illustration, click here.

We wish to write an optimization problem whose solution solves the problem in the fewest moves.

We will model it is an integer program. Let $x(i,j) \geq 0$ denote the number of times we click on the block in row i and column j.

Consider the element in row 3, column 2. Write a constraint that guarantees it turns black.

Hint: Consider the outcome when a square is selected $1, 2 \text{ or } 3$ *times.*

- (selected) $x(2,2) + x(3,1) + x(3,2) + x(3,3) + x(4,2)$ is odd ✔
- $x(2,2) + x(3,1) + x(3,3) + x(4,2)$ is odd

- Exit Survey

- $x(2,2)+x(3,1)+x(3,2)+x(3,3)+x(4,2)$ is even
- $x(2,2)+x(3,1)+x(3,3)+x(4,2)$ is even

EXPLANATION

Solution

The correct answer is:
$x(2,2)+x(3,1)+x(3,2)+x(3,3)+x(4,2)$ is odd

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PART B

1/1 point (graded)

Transform the following constraint into integer linear constraints.

x is odd $, x \geq 0, x$ integer

- $x-y=1, x, y \geq 0; x, y$ integer
- $x-2y=1, x, y \geq 0; x, y$ integer ✔

- $2x - y = 1, x, y \geq 0; x, y$ integer
- $x - 2y = 1, y \geq x \geq 0; x, y$ integer

EXPLANATION

Solution

The correct answer is:

$x - 2y = 1, x, y \geq 0,$ integer

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PART C

1/1 point (graded)

Is it always true that in an optimal answer each of the variables $x(i, j)$ will be binary?

- False
- True ✔

EXPLANATION

Solution

The correct answer is:

True

Clicking on a block twice is equivalent to doing nothing at all. The second click undoes the changes made in the first click.

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PART D

1/1 point (graded)

Consider the element in row 3, column 2 again. Using the results from PARTS A, B, and C, write a constraint that guarantees it turns black. For the answers, assume that there is a constraint

$$x(i,j) + x(i,j-1) + x(i,j+1) + x(i-1,j) + x(i+1,j) = w(i,j)$$

Assume also that $x(i',j')$ is binary for all i',j' and $y(i,j) \in \{0,1,2\}$.

- $w(i,j) + 2y(i,j) = 1, \text{ for } (i,j) = (3,2)$
- $w(i,j) + 2y(i,j) = -1, \text{ for } (i,j) = (3,2).$
- $w(i,j) - 2y(i,j) = 1, \text{ for } (i,j) = (3,2).$ ✔
- $w(i,j) - 2y(i,j) = -1, \text{ for } (i,j) = (3,2).$

EXPLANATION

Solution

The correct answer is:

$w(i,j) - 2y(i,j) = 1, \text{ for } (i,j) = (3,2)$

A correct statement (but not something allowed in an integer program) would be "$w(3,2)$ is odd." We could also say "$w(3,2) = 1 \text{ or } 3 \text{ or } 5$" because $w(3,2) \leq 5$. This is equivalent to " $w(3,2) - 2y(3,2) = 1$, and $y(3,2) \in \{0,1,2\}$."

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PART E

1/1 point (graded)

Using the formulation suggested by PART A, one can, in principle, write an integer program to optimize the game of fiver. If one did it in the most straightforward manner, there would be 25 different constraints, one for each block. That would take a very long time (and a lot of space) to have to write out in detail.

Perhaps one could express it more succinctly in a manner similar to the way that the constraint in PART D is given. Ideally, one would only need to write a constraint for Block(i,j) and let i vary from 1 to 5 and let j vary from 1 to 5. One runs into a difficulty though. The constraint for Blocks (1, 1) includes just three of the x decision variables. It would be

$$x(1,1) + x(1,2) + x(2,1) - 2y(1,1) = 1,$$

where all four of these variables are binary. (Do you see why $y(1,1)$ can be restricted to be binary?) Similarly for Blocks (1, 5), (5, 1) and (5, 5). Other Blocks in which i or j = 1 or 5 include 4 decision variables. It would seem difficult to be able to express all 25 constraints very succinctly. However, there is a way to do so. It involves creating "dummy decision variables." How many additional dummy variables do you think are needed? HINT: it is more than 23 and less than 25. (Yes, the previous question was intended as a joke. But now that you know the answer, see if you can figure out a clever way of writing the constraints succinctly if you are allowed the 24 extra dummy variables. This number arises because $7^2 - 5^2 = 24$. The explanation for this exercise will be given when the solutions for the problem set are released.)

24

Answer: 24

24

EXPLANATION

We first create binary decision variables $x(i,j)$ for whether a block is clicked on. One would normally let i and j vary from 1 to 5. This would lead to 25 binary variables. However, we permit i and j to each vary from 0 to 6. The variable $x(i,j)$ is a dummy variable if i or j is equal to 0 or 6. We require each of these dummy variables to be 0. In other words, when these variables occur in a constraint, they do nothing. (They are just placeholders.) The total number of x variables is 49. The number of original x variables is 25. Thus there are 24 dummy variables.

$$\left.\begin{array}{lr}
\textbf{min} & \sum_{i,j=1,...,5} x(i,j) \\
\textbf{s.t.:} & \\
 & x(i,j) + x(i,j-1) + x(i,j+1) + x(i-1,j) + x(i+1,j) - 2y(i,j) = 1 \\
 & i = 1,\ldots,5, j = 1,\ldots,5 \\
 & x(i,j) = 0 \text{ if either } i \text{ or } j = 0 \text{ or } 6 \\
 & x(i,j) \in \{0,1\} \text{ and } y(i,j) \in \{0,1,2\} \\
 & \text{for } i = 0,\ldots,6, j = 0,\ldots,6
\end{array}\right\}$$

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PS3. Problem 5

Topic: Week 3 / PS3. Problem 5

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