8. Jacobian Shortcut

Explore

Shortcut

The previous step-by-step algorithm for linearization is correct, but there is a shortcut to find the linearization systematically. Suppose we want to linearize a transformation $x, y \implies A, B$ of the form

$$A = A(x, y) \tag{5.129}$$

$$B = B(x, y) \tag{5.130}$$

at the point (x_0, y_0).

Theorem

Theorem The linearization of $x, y \implies A, B$ at (x_0, y_0) is given by the matrix:

$$\left.\begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix}\right|_{(x,y)=(x_0,y_0)} \tag{5.131}$$

For example, before we had

$$A = \quad xy \tag{5.132}$$

$$B = \quad x^2 - y^2 \tag{5.133}$$

The matrix of partial derivatives is given by

$$\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} \tag{5.134}$$

Evaluating at the desired point $(1, 2)$ we get the same answer as before:

$$\begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix}. \tag{5.135}$$

It can be faster to work with the matrix of partial derivatives in this way.

Why does the shortcut work?

The shortcut works because the general formula for linear approximation says that

$$A(x_0 + \Delta x, y_0 + \Delta y) - A(x_0, y_0) \quad \approx A_x(x_0, y_0)\,\Delta x + A_y(x_0, y_0)\,\Delta y \tag{5.136}$$

$$B(x_0 + \Delta x, y_0 + \Delta y) - B(x_0, y_0) \quad \approx B_x(x_0, y_0)\,\Delta x + B_y(x_0, y_0)\,\Delta y \tag{5.137}$$

Looking carefully, we can extract the hidden matrix:

$$\begin{pmatrix} A(x_0+\Delta x, y_0+\Delta y) - A(x_0,y_0) \\ B(x_0+\Delta x, y_0+\Delta y) - B(x_0,y_0) \end{pmatrix} \approx \underbrace{\begin{pmatrix} A_x(x_0,y_0) & A_y(x_0,y_0) \\ B_x(x_0,y_0) & B_y(x_0,y_0) \end{pmatrix}}_{\text{Linearization}} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \tag{5.138}$$

The Jacobian Matrix

The matrix appearing in the above Theorem is called the “Jacobian matrix".

Definition 8.2 The **Jacobian matrix** of the transformation $x, y \implies A, B$ at the point (x_0, y_0) is the matrix

$$\begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix}\Bigg|_{(x,y)=(x_0,y_0)} \tag{5.139}$$

In a phrase, **the Jacobian is the hidden matrix that drives linear approximation.** When you hear “Jacobian" think of “linear approximation matrix."

Some care must be taken to keep the terminology straight. The "Jacobian matrix at (x_0, y_0)" is a matrix of numbers, and therefore it represents a linear function, J with two inputs and two outputs. The input to this J should be a small change in the input, $\Delta x, \Delta y$, and the output is the approximate change in the output, $\Delta A, \Delta B$. In symbols:

$$J\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \approx \begin{pmatrix} \Delta A \\ \Delta B \end{pmatrix} \tag{5.140}$$

However, "the Jacobian matrix" (before evaluating at (x_0, y_0)) is a matrix of partial derivatives, so it will typically contain messy formulas with x's and y's. Only after plugging in $(x, y) = (x_0, y_0)$ do we obtain the linearization at (x_0, y_0). Summarizing with the example above:

$$\underbrace{\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}}_{\text{Jacobian matrix before evaluating at base point}} \qquad \underbrace{\begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix}}_{\text{Jacobian matrix after evaluating at base point}} \tag{5.141}$$

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