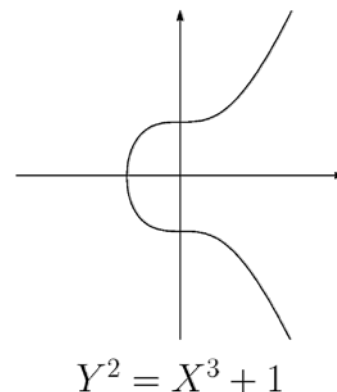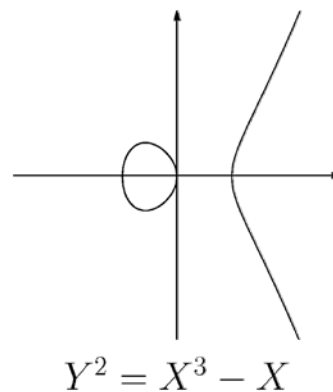# Problem 6

The **elliptic curve**
$$Y^2 = X^3 + 2X + 1$$
has only **three integral points** (S,T) with T>0. Two of them are (S,T)=(0,1), (1,2). Find the third point.

➤ It is difficult to find all the integral points of an elliptic curve.

$$Y^2 = X^3 - X \qquad Y^2 = X^3 + 1$$

Problem 6

$Y^2 = X^3 + 2X + 1$

➢ If $S \leq -1$,

   $S^3 + 2S + 1 = S^3 + 2(S+1) - 1 < 0.$

➢ It is enough to check

   $S = 0, 1, 2, 3, 4, \cdots$

➢ If $S = 8$,

   $S^3 + 2S + 1 = 512 + 16 + 1 = 529$

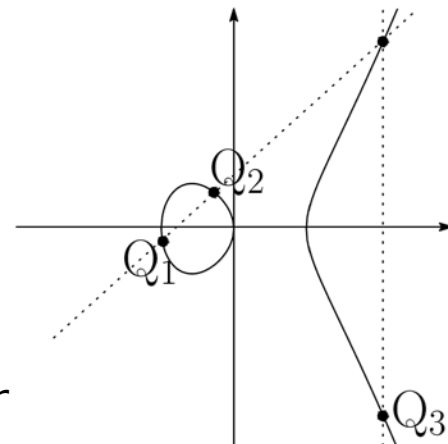   **$= 23×23$**

Problem 6

Answer   (S,T) = (8, 23)

➢ Finding rational points is more difficult. This elliptic curve has **infinitely many rational points**!

Bryan John Birch
(1931-)

Peter Swinnerton-Dyer
(1927-)

https://en.wikipedia.org/wiki/Bryan_John_Birch
https://en.wikipedia.org/wiki/Peter_Swinnerton-Dyer