Proof of Central Limit Theorem

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Preliminary Inequalities: In order to utilize the result (sometimes called the continuity theorem) that convergence in distribution is equivalent to pointwise convergence of the corresponding characteristic functions, we need the following estimates about Taylor expansions of exponential functions.

1. If $u \geq 0$, then
$$0 \leq e^{-u} - 1 + u \leq u^2/2.$$
2. If t is real, then
$$|e^{it} - 1 - it| \leq |t|^2/2 \quad \text{and} \quad |e^{it} - 1 - it - (it)^2/2| \leq |t|^3/6.$$

Central Limit Theorem: Let $\{X_n\}$ be a sequence of i.i.d. (independent identically distributed) random variables with common mean 0 and common variance 1. Then, if $Z \sim N(0,1)$ and $S_n = X_1 + X_2 + \cdots + X_n$, we have $S_n/\sqrt{n} \to Z$ in distribution as $n \to \infty$. In other words, for every $x \in \mathbf{R}$,

$$\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

Proof: Let $\hat{F}$ be the characteristic function of the common distribution of the $\{X_n\}$. Then for every $t \in \mathbf{R}$, the characteristic function of $S_n/\sqrt{n}$ is given by

$$E\left(e^{itS_n/\sqrt{n}}\right) = E\left(e^{it\sum_{k=1}^{n} X_k/\sqrt{n}}\right) = \left[\hat{F}\left(t/\sqrt{n}\right)\right]^n.$$

Consequently, our task is to prove that, for each $t \in \mathbf{R}$,

$$\lim_{n\to\infty} \left[\hat{F}\left(t/\sqrt{n}\right)\right]^n = e^{-t^2/2}.$$

Note that there is nothing to prove if $t = 0$.

We begin our estimation by noting that

$$\begin{aligned} \left|\left[\hat{F}\left(t/\sqrt{n}\right)\right]^n - e^{-t^2/2}\right| &= \left|\left[\hat{F}\left(t/\sqrt{n}\right)\right]^n - \left[e^{-t^2/2n}\right]^n\right| \\ &\leq n\left|\hat{F}\left(t/\sqrt{n}\right) - e^{-t^2/2n}\right| \end{aligned}$$

since $|\hat{F}\left(t/\sqrt{n}\right)| \leq 1$ and $0 \leq e^{-t^2/2n} \leq 1$.

For the next step, we use the triangle inequality to see that

$$n\left|\hat{F}\left(t/\sqrt{n}\right)-e^{-t^2/2n}\right|\leq n\left|\hat{F}\left(t/\sqrt{n}\right)-(1-t^2/2n)\right|+n\left|(1-t^2/2n)-e^{-t^2/2n}\right|.$$

By our first estimate, letting $u=t^2/2n\geq 0$, we see that

$$n\left|(1-t^2/2n)-e^{-t^2/2n}\right|\leq n(t^2/2n)^2/2=t^4/8n$$

and this approaches 0 as $n\to\infty$.

In the first term we note that

$$\begin{aligned}n\left|\hat{F}\left(t/\sqrt{n}\right)-(1-t^2/2n)\right| &= n\left|E\left[e^{itX/\sqrt{n}}-(1+itX/\sqrt{n}+i^2t^2X^2/2n)\right]\right| \\ &\leq nE\left[\left|\ e^{itX/\sqrt{n}}-(1+itX/\sqrt{n}+i^2t^2X^2/2n)\right|\ \right],\end{aligned}$$

where X is a random variable with characteristic function $\hat{F}$, since $E(X)=0$ and $Var(X)=E(X^2)=1$. Now, on the one hand,

$$\begin{aligned}\left|\ e^{itX/\sqrt{n}}-(1+itX/\sqrt{n}+i^2t^2X^2/2n)\right| &\leq \left|\ e^{itX/\sqrt{n}}-(1+itX/\sqrt{n})\right|+t^2X^2/2n \\ &\leq t^2X^2/2n+t^2X^2/2n \\ &= t^2X^2/n,\end{aligned}$$

using the triangle inequality and the first of the complex exponential estimates. On the other hand,

$$\left|\ e^{itX/\sqrt{n}}-(1+itX/\sqrt{n}+i^2t^2X^2/2n)\right|\leq |t|^3|X|^3/6n^{3/2},$$

using the second of the complex exponential estimates.

For any $\delta>0$ and positive integer n, let $A=A(\delta,n)=\{|X|>\delta\sqrt{n}\}$. Then

$$\left|\ e^{itX/\sqrt{n}}-(1+itX/\sqrt{n}+i^2t^2X^2/2n)\right|\leq (t^2X^2/n)I_A+(|t|^3|X|^3/6n^{3/2})I_{A^c}.$$

Consequently,

$$\begin{aligned}&nE\left[\left|\ e^{itX/\sqrt{n}}-(1+itX/\sqrt{n}+i^2t^2X^2/2n)\right|\ \right] \\ &\leq nE\left[(t^2X^2/n)I_A\right]+nE\left[(|t|^3|X|^3/6n^{3/2})I_{A^c}\right] \\ &\leq t^2E(X^2I_{\{|X|>\delta\sqrt{n}\}})+|t|^3\delta/6\end{aligned}$$

for every positive integer n, since $E(X^2)=1$. Therefore, given $\varepsilon>0$, we first choose $\delta>0$ so that $|t|^3\delta/6\leq\varepsilon/2$ and then for this δ we choose the positive integer N so that if $n\geq N$ we have

$$t^2E(X^2I_{\{|X|>\delta\sqrt{n}\}})=t^2[1-E(X^2I_{\{|X|\leq\delta\sqrt{n}\}})]\leq\varepsilon/2.$$

The last inequality is a consequence of either the monotone or dominated convergence theorems.