The Viterbi Algorithm

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THE VITERBI ALGORITHM PREFACE

Max-product or min-sum specialized to HMMs results in the Viterbi algorithm, proposed by MIT alum Andrew Viterbi back in 1967, which pre-dates both the forward-backward algorithm (1980) and the sum-product algorithm (1982). The Viterbi algorithm produces an MAP estimate for the sequence of hidden states given the observations of an HMM.

The only exercise in this section is for you to make sure that you too can run the Viterbi algorithm by hand! You will be coding it up in the mini-project on robot localization *and* figuring out how to get the second-best sequence rather than just the best sequence!

The Viterbi Algorithm

6.008.1x - The Viterbi Algorithm

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states we can be in.

So what that looks like is here we're

going to have m 2 to 3 prime of fair

is going to be equal to min over two things, so x 2

being equal to fair and x 2 being equal to biased.

And the observation which is heads, which is going to be,

well, this 1 and 2.

So we're going to have 1, 2.

And then we're going to account for the transition.

So the transition here from fair to fair

is going to be 2 minus gamma.

And the transition here is going to be from biased to fair,

which is going to be 2

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THE VITERBI ALGORITHM (COURSE NOTES)

Let's work through an example. I have two coins, one fair and one biased, and I keep picking a coin and flipping it. You get to observe whether each flip was heads or tails, and from that information, you have to guess the sequence of coins that I used.

At first, I choose either the fair coin or the biased coin uniformly at random. I prefer not to switch between flips: that is, after each flip, my next flip will use the same coin with probability 3/4, and switch coins with probability 1/4. The fair coin is equally likely to be heads or tails, and the biased coin has probability 3/4 of coming up tails and probability 1/4 of coming up heads.

If you observe the sequence $HHTTT$ (where H is heads and T is tails), what's the most likely sequence of coins that I used?

We reuse notation from how we formulated HMM's when we presented the forward-backward algorithm. In particular, for this HMM with 5 hidden states, we define the following potentials:

$$\widetilde{\phi}_i(x_i) = \begin{cases} p_{X_1}(x_1) p_{Y_1|X_1}(y_1|x_1) & \text{for } i = 1, \\ p_{Y_i|X_i}(y_i|x_i) & \text{for } i = 2, \dots, 5, \end{cases}$$

$$\psi(x_{i-1}, x_i) = p_{X_i|X_{i-1}}(x_i \mid x_{i-1}) \quad \text{for } n = 2, \dots, 5.$$

Thus, we have (importantly, observations $HHTTT$ have been folded into the node potentials!):

ψ

		x_i = fair	x_i = biased
x_{i-1}	fair	$3/4$	$1/4$
	biased	$1/4$	$3/4$

$\widetilde{\phi}_1$

x_1	fair	$(1/2) \times (1/2)$
	biased	$(1/2) \times (1/4)$

$\widetilde{\phi}_2$

x_2	fair	$1/2$
	biased	$1/4$

$\widetilde{\phi}_i,\ i = 3, 4, 5$

x_i	fair	$1/2$
	biased	$3/4$

Defining $\gamma \triangleq \log_2(3) \approx 1.6$, we have:

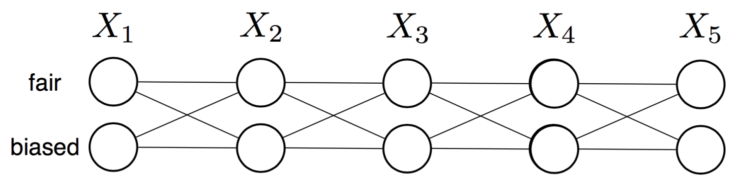
$-\log_2 \psi$		x_i = fair	x_i = biased
x_{i-1}	fair	$2-\gamma$	2
	biased	2	$2-\gamma$

$-\log_2 \widetilde{\phi}_1$		
x_1	fair	2
	biased	3

$-\log_2 \widetilde{\phi}_2$		
x_2	fair	1
	biased	2

$-\log_2 \widetilde{\phi}_i,\ i = 3, 4, 5$		
x_i	fair	1
	biased	$2-\gamma$

Below, we show what is called the *trellis diagram* for this problem: note that while this diagram has nodes and edges, it is *not* a probabilistic graphical model in that the nodes do not correspond to random variables, and the nodes/edges aren't associated with potential tables, etc. Instead, the i-th column consists of all the possible states that X_i can be, and the edges denote all possible transitions, so that a path going from left to right (a path in a trellis diagram can't go rightward and then leftward; it can only go rightward) corresponds to a specific possible sequence of states that $X_1, \ldots, X_n$ can take on.



The trellis diagram for our example

We will see that the Viterbi algorithm just finds a path along this trellis.

For a length-n HMM, the min-sum messages are

$$m'_{1\to 2}(x_2) = \min_{x_1}\left[-\log_2 \widetilde{\phi}_1(x_1) - \log_2 \psi(x_1, x_2)\right],$$

$$t_{2\to 1}(x_2) = \arg\min_{x_1}\left[-\log_2 \widetilde{\phi}_1(x_1) - \log_2 \psi(x_1, x_2)\right],$$

$$m'_{(i-1)\to i}(x_i) = \min_{x_{i-1}}\left[-\log_2 \widetilde{\phi}_{i-1}(x_{i-1}) - \log_2 \psi(x_{i-1}, x_i) + m'_{(i-2)\to(i-1)}(x_{i-1})\right] \quad \text{for } i = 3, \ldots,$$

$$t_{i\to(i-1)}(x_i) = \arg\min_{x_{i-1}}\left[-\log_2 \widetilde{\phi}_{i-1}(x_{i-1}) - \log_2 \psi(x_{i-1}, x_i) + m'_{(i-2)\to(i-1)}(x_{i-1})\right] \quad \text{for } i = 3, \ldots,$$

Let's determine the messages for our length $n = 5$ HMM.

We compute the first message:

$$\begin{aligned}
m_{1\to 2}(\text{fair}) &= \min_{x_1}\left[-\log_2 \widetilde{\phi}_1(x_1) - \log_2(\psi(x_1, \text{fair}))\right] \\
&= \min\Big\{\overbrace{-\log_2 \widetilde{\phi}_1(\text{fair}) - \log_2(\psi(\text{fair}, \text{fair}))}^{x_1=\text{fair}}, \\
&\qquad \underbrace{-\log_2 \widetilde{\phi}_1(\text{biased}) - \log_2(\psi(\text{biased}, \text{fair}))}_{x_1=\text{biased}}\Big\} \\
&= \min\Big\{\underbrace{2 + (2 - \gamma)}_{x_1=\text{fair}}, \ \underbrace{3 + 2}_{x_1=\text{biased}}\Big\} \\
&= \min\Big\{\underbrace{4 - \gamma}_{x_1=\text{fair}}, \ \underbrace{5}_{x_1=\text{biased}}\Big\} \\
&= 4 - \gamma,
\end{aligned}$$

where the $\arg\min$ is $x_1 = \text{fair}$, so $t_{2\to 1}(\text{fair}) = \text{fair}$.

Next:

$$
\begin{aligned}
m_{1\to 2}(\text{biased}) &= \min_{x_1}\Big[-\log_2 \widetilde{\phi}_1(x_1) - \log_2(\psi(x_1, \text{biased}))\Big] \\
&= \min\Big\{\overbrace{-\log_2 \widetilde{\phi}_1(\text{fair}) - \log_2(\psi(\text{fair}, \text{biased}))}^{x_1=\text{fair}}, \\
&\qquad \underbrace{-\log_2 \widetilde{\phi}_1(\text{biased}) - \log_2(\psi(\text{biased}, \text{biased}))}_{x_1=\text{biased}}\Big\} \\
&= \min\{\underbrace{2+2}_{x_1=\text{fair}}, \underbrace{3+(2-\gamma)}_{x_1=\text{biased}}\} \\
&= \min\{\underbrace{4}_{x_1=\text{fair}}, \underbrace{5-\gamma}_{x_1=\text{biased}}\} \\
&= 5-\gamma,
\end{aligned}
$$

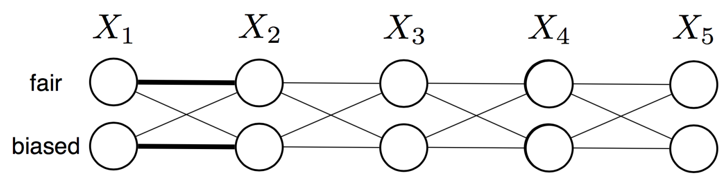
where the $\arg\min$ is $x_1 = \text{biased}$, so $t_{2\to 1}(\text{biased}) = \text{biased}$.

Our message and traceback tables are then:

$$
m'_{1\to 2}(x_2) = \begin{cases} 4-\gamma & \text{if } x_2 = \text{fair} \\ 5-\gamma & \text{if } x_2 = \text{biased} \end{cases}
$$

$$
t_{2\to 1}(x_2) = \begin{cases} \text{fair} & \text{if } x_2 = \text{fair} \\ \text{biased} & \text{if } x_2 = \text{biased} \end{cases}
$$

This traceback table tells us how to select x_1 once we decide on x_2. This is illustrated in the first-step trellis diagram below:



The first traceback

The second message is computed similarly:

$$\begin{aligned}
m'_{2\to 3}(\text{fair}) &= \min_{x_2} \Big[-\log_2 \widetilde{\phi}_2(x_2) - \log_2(\psi(x_2, \text{fair})) + m'_{1\to 2}(x_2) \Big] \\
&= \min \Big\{ \overbrace{-\log_2 \widetilde{\phi}_2(\text{fair}) - \log_2(\psi(\text{fair}, \text{fair})) + m'_{1\to 2}(\text{fair})}^{x_2=\text{fair}}, \\
&\qquad \underbrace{-\log_2 \widetilde{\phi}_2(\text{biased}) - \log_2(\psi(\text{biased}, \text{fair})) + m'_{1\to 2}(\text{biased})}_{x_2=\text{biased}} \Big\} \\
&= \min \Big\{ \underbrace{1 + (2-\gamma) + (4-\gamma)}_{x_2=\text{fair}}, \underbrace{2 + 2 + (5-\gamma)}_{x_2=\text{biased}} \Big\} \\
&= \min \Big\{ \underbrace{7 - 2\gamma}_{x_2=\text{fair}}, \ \underbrace{9 - \gamma}_{x_2=\text{biased}} \ \Big\} \\
&= 7 - 2\gamma,
\end{aligned}$$

where the $\arg\min$ is $x_2 = \text{fair}$, so $t_{3\to 2}(\text{fair}) = \text{fair}$.

Next:

$$
\begin{aligned}
m'_{2\to 3}(\text{biased}) &= \min_{x_2}\Big[-\log_2 \widetilde{\phi}_2(x_2) - \log_2(\psi(x_2,\text{biased})) + m'_{1\to 2}(x_2)\Big] \\
&= \min\Big\{\overbrace{-\log_2 \widetilde{\phi}_2(\text{fair}) - \log_2(\psi(\text{fair},\text{biased})) + m'_{1\to 2}(\text{fair})}^{x_2=\text{fair}}, \\
&\qquad \underbrace{-\log_2 \widetilde{\phi}_2(\text{biased}) - \log_2(\psi(\text{biased},\text{biased})) + m'_{1\to 2}(\text{biased})}_{x_2=\text{biased}}\Big\} \\
&= \min\Big\{\underbrace{1+2+(4-\gamma)}_{x_2=\text{fair}}, \underbrace{2+(2-\gamma)+(5-\gamma)}_{x_2=\text{biased}}\Big\} \\
&= \min\Big\{\underbrace{7-\gamma}_{x_2=\text{fair}}, \underbrace{9-2\gamma}_{x_2=\text{biased}}\Big\} \\
&= 7-\gamma,
\end{aligned}
$$

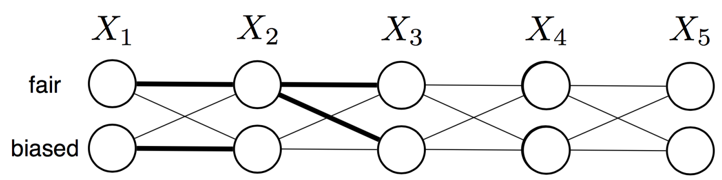
where the $\mathbf{arg\,min}$ is $x_2 = \text{fair}$, so $t_{3\to 2}(\text{biased}) = \text{fair}$.

Our message and traceback tables are then:

$$
m'_{2\to 3}(x_3) = \begin{cases} 7-2\gamma & \text{if } x_3 = \text{fair} \\ 7-\gamma & \text{if } x_3 = \text{biased} \end{cases}
$$

$$
t_{3\to 2}(x_3) = \begin{cases} \text{fair} & \text{if } x_3 = \text{fair} \\ \text{fair} & \text{if } x_3 = \text{biased} \end{cases}
$$

The traceback is illustrated below:



The second traceback

We see that after observing wo heads in a row, we will eventually pick $x_2 =$ fair no matter what x_3 is.

Make sure that you can also compute these messages! The rest of the message and traceback tables are as follows:

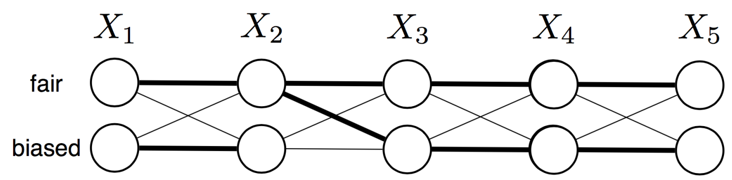
$$m'_{3\to4}(x_4) = \begin{cases} 10-3\gamma & \text{if } x_4 = \text{fair} \\ 11-3\gamma & \text{if } x_4 = \text{biased} \end{cases}$$

$$t_{4\to3}(x_4) = \begin{cases} \text{fair} & \text{if } x_4 = \text{fair} \\ \text{biased} & \text{if } x_4 = \text{biased} \end{cases}$$

$$m'_{4\to5}(x_5) = \begin{cases} 13-4\gamma & \text{if } x_5 = \text{fair} \\ 15-5\gamma & \text{if } x_5 = \text{biased} \end{cases}$$

$$t_{5\to4}(x_5) = \begin{cases} \text{fair} & \text{if } x_5 = \text{fair} \\ \text{biased} & \text{if } x_5 = \text{biased} \end{cases}$$

The traceback is illustrated below:

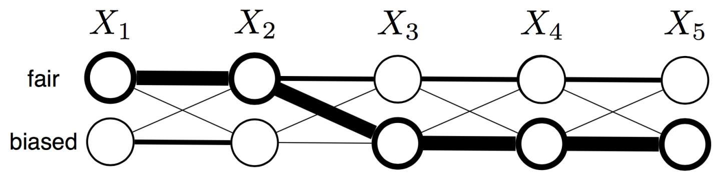


The final traceback

Finally, we can compute the most likely value for X_5:

$$\hat{x}_5 = \arg\min\Big(\underbrace{1+(13-4\gamma)}_{x_5=\text{fair}}, \underbrace{(2-\gamma)+(15-5\gamma)}_{x_5=\text{biased}}\Big) = \text{biased}$$

We can see from the trellis diagram that as soon as we decide on a value for X_5, we just have to follow our “trail of breadcrumbs" from the traceback messages to get the path highlighted below:



The result of the Viterbi algorithm

We can do the same thing with the traceback messages:

$$\begin{aligned}\hat{x}_4 &= t_{5\to4}(\hat{x}_5) = \text{biased}\\ \hat{x}_3 &= t_{4\to3}(\hat{x}_4) = \text{biased}\\ \hat{x}_2 &= t_{3\to2}(\hat{x}_3) = \text{fair}\\ \hat{x}_1 &= t_{2\to1}(\hat{x}_2) = \text{fair}\end{aligned}$$

We conclude that the MAP estimate for the sequence of hidden states given observed sequence $HHTTT$ is “fair, fair, biased, biased, biased".

Discussion

Topic: Special Case - MAP in Hidden Markov Models / The Viterbi Algorithm

Show Discussion