



<u>Course</u> > <u>Unit 4 Hypothesis testing</u> > <u>Homework 7</u> > 1. Implicit hypothesis testing

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1. Implicit hypothesis testing

Given n i.i.d. samples $X_1,\ldots,X_n\sim\mathcal{N}\left(\mu,\sigma^2\right)$ with $\mu\in\mathbb{R}$ and $\sigma^2>0$, we want to find a test with asymptotic level 5% for the hypotheses

$$H_0: \mu > \sigma \quad \text{vs} \quad H_1: \mu < \sigma.$$
 (7.1)

(a)

1/1 point (graded)

As a first step, define the maximum likelihood estimators

$$\hat{\mu} = ar{X}_n, \quad \widehat{\sigma^2} = rac{1}{n} \sum_{i=1}^n \left(X_i - ar{X}_n
ight)^2.$$

Give a function g(x,y) such that

$$g(\hat{\mu},\widehat{\sigma^2}) \overset{\mathbf{P}}{\underset{n o \infty}{\longrightarrow}} \mu - \sigma.$$

Submit

You have used 1 of 3 attempts

(b)

1/1 point (graded)

Note: To avoid too much double jeopardy, you will be able to see the solution to this part once you answered it correctly, and used all your attempts.

What is the asymptotic variance of $\,g\,(\hat{\mu},\widehat{\sigma^2})\,$ that you found in part (a)?

$$V\left(g\left(\hat{\mu},\widehat{\sigma^2}
ight)
ight)= \boxed{$$
 3*sigma^2/2

STANDARD NOTATION

Solution:

First, by the Theorem giving asymptotic normality for maximum likelihood estimators, we have

$$\sqrt{n}\left(rac{\hat{\mu}}{\widehat{\sigma^2}}
ight) - \sqrt{n}\left(rac{\mu}{\sigma^2}
ight) \stackrel{ ext{(D)}}{\longrightarrow} \mathcal{N}\left(\left(rac{0}{0}
ight), I(\mu, \sigma^2)^{-1}
ight),$$

where $I\left(\mu,\sigma^2\right)$ denotes the Fisher information that we computed earlier to be

$$I\left(\mu,\sigma^2
ight) = \left(egin{array}{cc} rac{1}{\sigma^2} & 0 \ 0 & rac{1}{2\sigma^4} \end{array}
ight).$$

Hence,

$$\sqrt{n} \left(rac{\hat{\mu}}{\widehat{\sigma^2}}
ight) - \sqrt{n} \left(rac{\mu}{\sigma^2}
ight) \stackrel{ ext{(D)}}{\longrightarrow} \mathcal{N} \left(\left(egin{matrix} 0 \ 0 \end{matrix}
ight), \left(egin{matrix} rac{1}{\sigma^2} & 0 \ 0 & rac{1}{2\sigma^4} \end{matrix}
ight)^{-1}
ight).$$

Now, defining

$$g: \mathbb{R} imes (0,\infty) o \mathbb{R}, \quad (x,y) \mapsto x - \sqrt{y},$$

we can compute

$$abla g\left(x,y
ight) = \left(egin{array}{c} 1 \ -rac{1}{2\sqrt{y}}. \end{array}
ight)$$

Then, apply the multivariate Delta method to obtain

$$\sqrt{n}\left(\hat{\mu}-\sqrt{\widehat{\sigma^2}}-\left(\mu-\sigma
ight)
ight) \xrightarrow[n o \infty]{(\mathrm{D})} \mathcal{N}\left(0,
abla g(\mu,\sigma^2)^T I(\mu,\sigma^2)^{-1}
abla g\left(\mu,\sigma^2
ight)
ight) = \mathcal{N}\left(0,rac{3}{2}\sigma^2
ight).$$

That means

$$V\left(g\left(\hat{\mu},\widehat{\sigma^2}
ight)
ight)=rac{3}{2}\sigma^2.$$

Submit

You have used 2 of 3 attempts

• Answers are displayed within the problem

(c)

0/1 point (graded)

Using your result from part (b) together with a plug-in estimator for the asymptotic variance, give a test for

$$H_0: \mu \geq \sigma \quad \text{vs} \quad H_1: \mu < \sigma.$$
 (7.2)

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that is with asymptotic level 5% and of the form

$$\psi = \mathbf{1}\{f(\hat{\mu},\widehat{\sigma^2})>0\},$$

where

$$f(\hat{\mu},\widehat{\sigma^2}) = -T(\hat{\mu},\widehat{\sigma^2}) - q$$

for some function $\,T\,$ and $\,q>0$.

(Enter **hatmu**, **hat(sigma^2**) for $\hat{\mu}$, $\widehat{\sigma^2}$, respectively. Use the quantile function q for best results. E.g. enter q(0.01) for the 0.99-quantile.)

$$f(\hat{\mu},\widehat{\sigma^2})=$$
 -(hatmu-sqrt(hat(sigma^2)))/(sqrt((3/2)*hat(sigma^2/n)))-q(0.05)

STANDARD NOTATION

Submit

You have used 4 of 4 attempts

★ Incorrect (0/1 point)

(d)

0/1 point (graded)

Using the same test as in part (c), give the (asymptotic) p-value of the test given observations $\hat{\mu}$ and $\widehat{\sigma^2}$.

(Enter **Phi(x)** for the cdf $\Phi(x)$ of a Normal distribution. Enter **hatmu**, **hat(sigma^2)** for $\hat{\mu}$, $\widehat{\sigma^2}$, respectively.)

p-value = Phi((hatmu-sqrt(hat(sigma^2)))/(sqrt((3/2)*hat(sigma^2/n))))

STANDARD NOTATION

Submit

You have used 1 of 4 attempts

★ Incorrect (0/1 point)

(e)

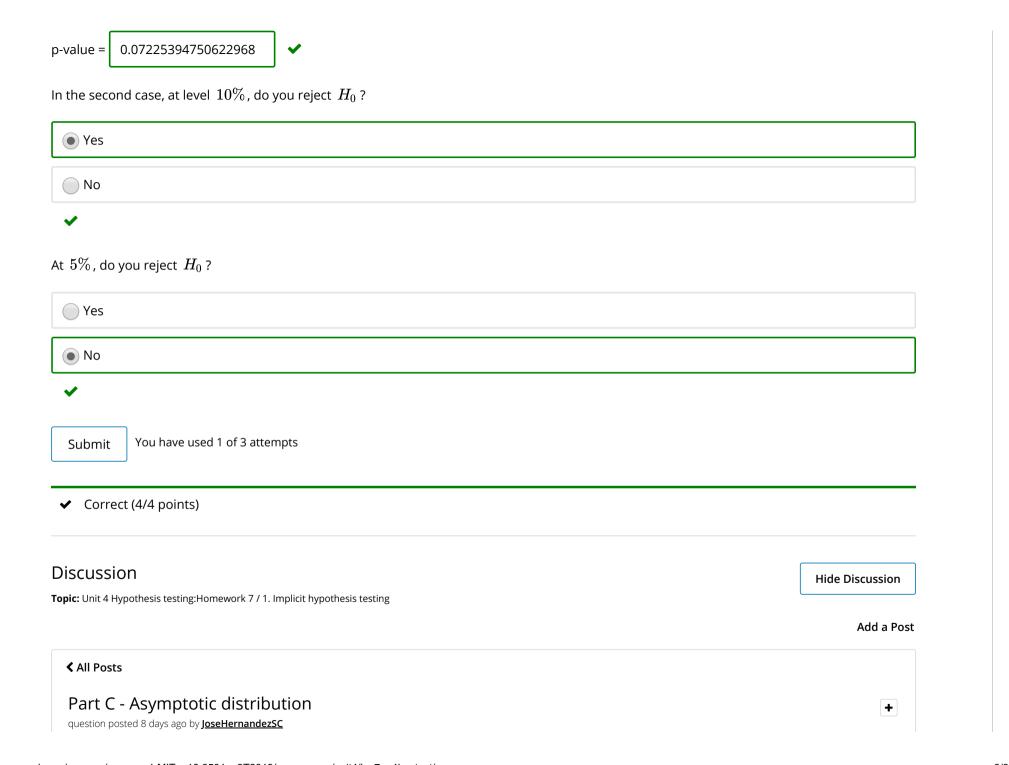
4/4 points (graded)

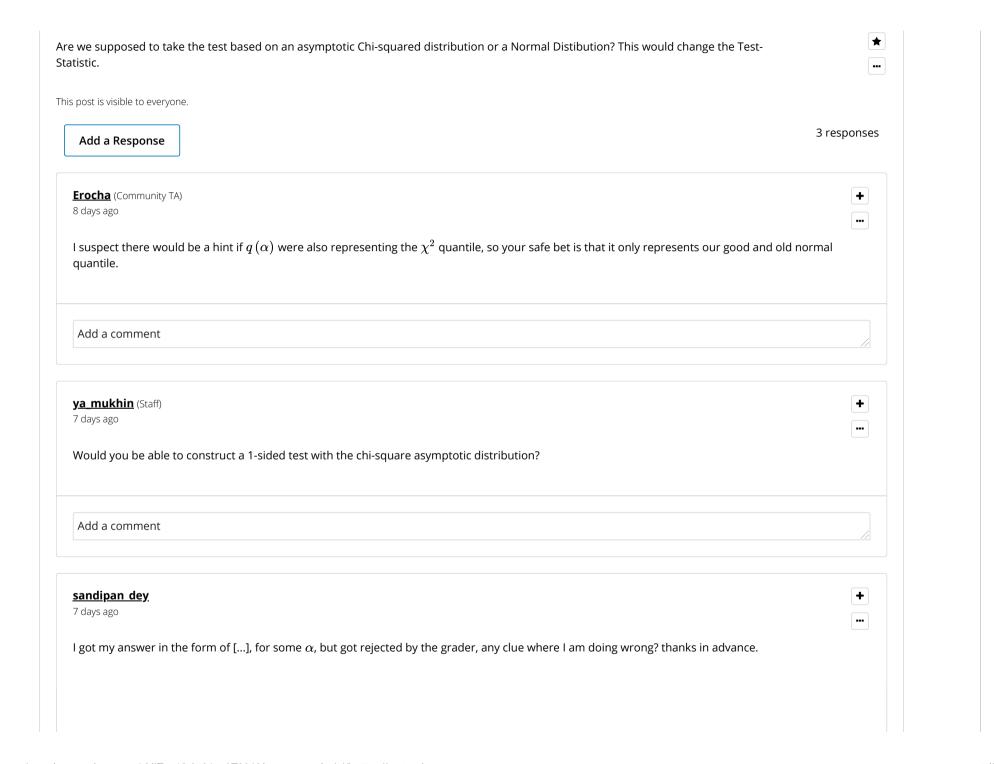
What is the (asymptotic) p-value if the sample size is $\,n=100$, $\,\hat{\mu}=2.41$, and $\,\widehat{\sigma^2}=5.20$?

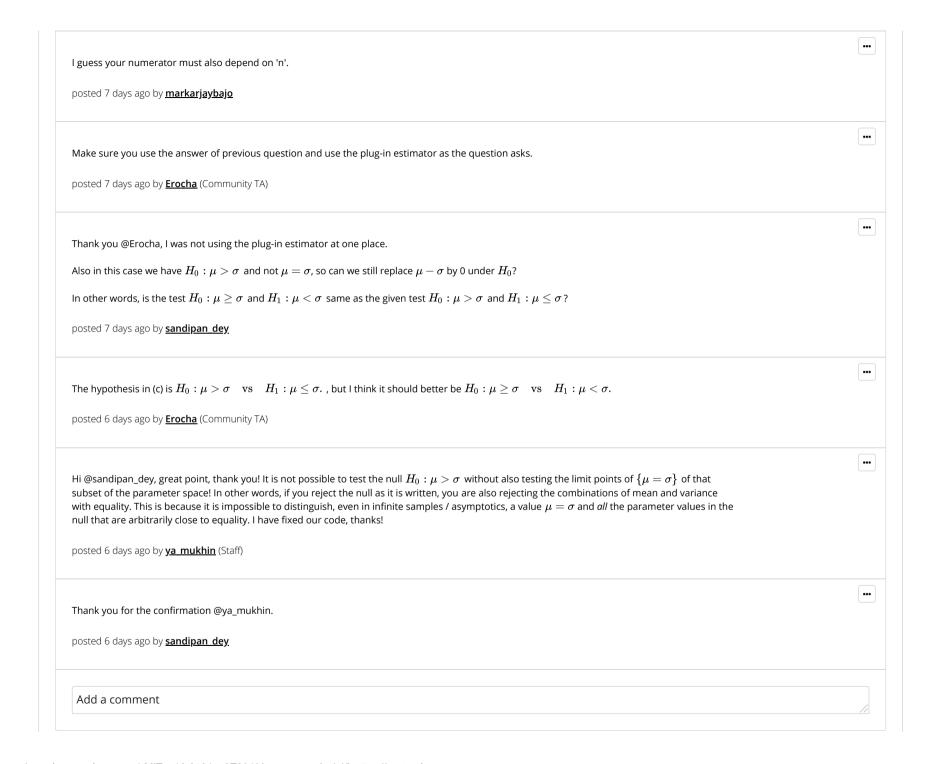
p-value = 0.6787543683455306

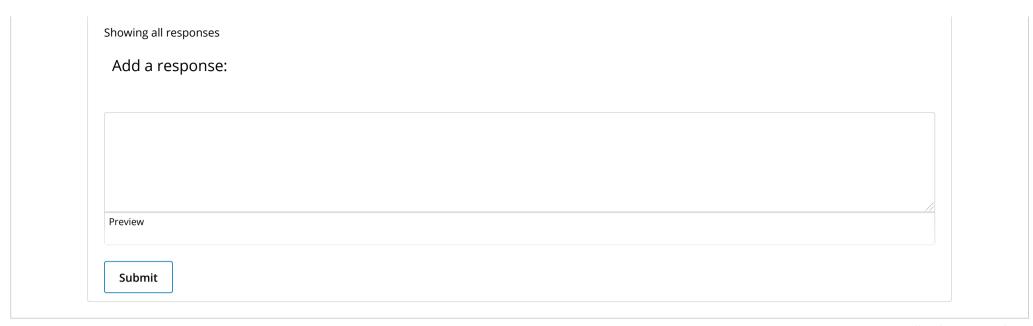
What if n=100 , $\hat{\mu}=3.28$, and $\widehat{\sigma^2}=15.95$?

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