



MITx: 6.041x Introduction to Probability - The Science of Uncertainty



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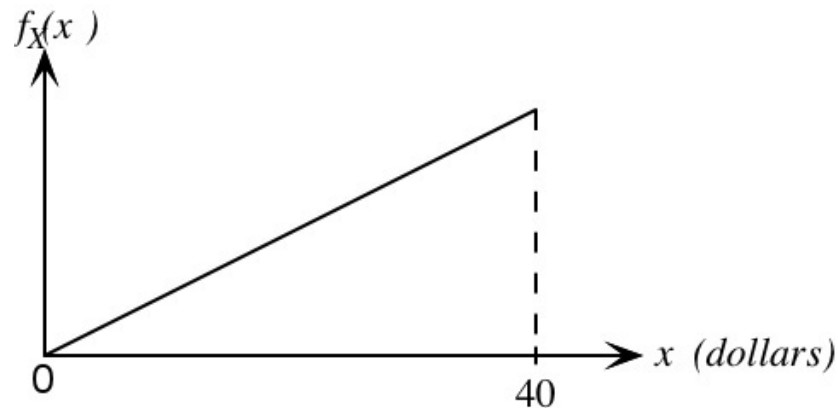
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Bookmark

Problem 4: Paul goes to the casino

(7/7 points)

Paul is vacationing in Monte Carlo. On any given night, he takes \mathbf{X} dollars to the casino and returns with \mathbf{Y} dollars. The random variable \mathbf{X} has the PDF shown in the figure. Conditional on $\mathbf{X} = \mathbf{x}$, the continuous random variable \mathbf{Y} is uniformly distributed between zero and $\mathbf{2x}$.




1. Determine the joint PDF $f_{X,Y}(x, y)$.


If $0 < x < 40$ and $0 < y < 2x$,

Unit overview


Lec. 8: Probability density functions

Exercises 8 due Mar 18, 2016 at 23:59 UTC 

Lec. 9: Conditioning on an event; Multiple r.v.'s

Exercises 9 due Mar 18, 2016 at 23:59 UTC 


Lec. 10: Conditioning on a random variable; Independence; Bayes' rule

Exercises 10 due Mar 18, 2016 at 23:59 UTC 

Standard normal table

Solved problems

Problem Set 5

Problem Set 5 due Mar 18, 2016 at 23:59 UTC 

Unit summary

- ▶ Unit 6: Further topics on random variables
- ▶ Unit 7: Bayesian inference

$$f_{X,Y}(x, y) = \boxed{1/1600}$$

✓ Answer: 0.000625

If $y < 0$ or $y > 2x$,

$$f_{X,Y}(x, y) = \boxed{0}$$

✓ Answer: 0

2. On any particular night, Paul makes a profit of $Z = Y - X$ dollars. Find the probability that Paul makes a positive profit (i.e., $\mathbf{P}(Z > 0)$):

$$\boxed{1/2}$$

✓ Answer: 0.5

3. Find the PDF of Z . Express your answers in terms of z using standard notation. *Hint: Start by finding $f_{Z|X}(z|x)$.*

$$\text{If } 0 < z < 40, f_Z(z) = \boxed{1/40 - z/1600}$$

✓ Answer: (40-z)/1600

$$\text{If } -40 < z < 0, f_Z(z) = \boxed{1/40 + z/1600}$$

✓ Answer: (40+z)/1600

$$\text{If } z < -40 \text{ or } z > 40, f_Z(z) = \boxed{0}$$

✓ Answer: 0

4. What is $\mathbf{E}[Z]$?

- ▶ Exam 2
- ▶ Unit 8: Limit theorems and classical statistics
- ▶ Unit 9: Bernoulli and Poisson processes
- ▶ Unit 10: Markov chains
- ▶ Exit Survey
- ▶ Final Exam

 $\mathbf{E}[Z] =$

✓ Answer: 0

Answer:

1. By the multiplication rule, $f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y | x)$. We have $f_X(x) = ax$, as shown in the figure. Furthermore,

$$1 = \int_0^{40} ax \, dx = 800a.$$

Hence, $f_X(x) = x/800$. From the problem statement, $f_{Y|X}(y | x) = 1/(2x)$ if $0 \leq y \leq 2x$. Therefore,

$$f_{X,Y}(x, y) = \begin{cases} 1/1600, & \text{if } 0 \leq x \leq 40 \text{ and } 0 \leq y \leq 2x, \\ 0, & \text{otherwise.} \end{cases}$$

2. Paul makes a positive profit if and only if $Y > X$. This occurs with probability

$$\mathbf{P}(Y > X) = \int_{-\infty}^{\infty} \int_x^{\infty} f_{X,Y}(x, y) \, dy \, dx = \int_0^{40} \int_x^{2x} \frac{1}{1600} \, dy \, dx = \frac{1}{2}.$$

We could have also arrived at this answer by realizing that for each possible value of X , there is a $1/2$ probability that $Y > X$, and therefore by the total probability theorem,

$$\begin{aligned}
 \mathbf{P}(Y > X) &= \int_0^{40} \mathbf{P}(Y > X | X = x) f_X(x) dx \\
 &= \int_0^{40} \frac{1}{2} f_X(x) dx \\
 &= \frac{1}{2}.
 \end{aligned}$$

3. The joint PDF of X and Z satisfies $f_{X,Z}(x, z) = f_X(x) f_{Z|X}(z|x)$. Given $X = x$, Y is uniformly distributed on $[0, 2x]$, which implies that $Y - x$ is uniformly distributed on $[-x, x]$. Thus, given $X = x$, $Z = Y - X$ is uniformly distributed on $[-x, x]$. Hence, $f_{Z|X}(z | x) = \frac{1}{2x}$ for $-x \leq z \leq x$.

Therefore, $f_{X,Z}(x, z) = (x/800) \cdot (1/2x) = 1/1600$ for $0 \leq x \leq 40$ and $-x \leq z \leq x$ and is 0 elsewhere. Graphically, on the x - z plane, the joint PDF is a constant $1/1600$ on the triangle with vertices at $(0, 0)$, $(40, -40)$, and $(40, 40)$.

To calculate the marginal PDF $f_Z(z)$, we integrate the joint PDF over x . The joint PDF as found above is nonzero only for z between -40 and 40 . For z in this range, the joint PDF is nonzero when x is between $|z|$ and 40 . Therefore,

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Z}(x, z) dx = \int_{|z|}^{40} \frac{1}{1600} dx = \frac{40 - |z|}{1600}, \text{ if } |z| < 40,$$

and $f_Z(z) = 0$ if $|z| > 40$.

4.

We observe that $\mathbf{E}[Y|X = x] = x$ for any $x \in [0, 40]$. Thus, using the total expectation theorem,

$$\begin{aligned}\mathbf{E}[Y] &= \int_0^{40} \mathbf{E}[Y|X = x] f_X(x) dx \\ &= \int_0^{40} x f_X(x) dx \\ &= \mathbf{E}[X].\end{aligned}$$

We conclude that $\mathbf{E}[Z] = \mathbf{E}[Y] - \mathbf{E}[X] = 0$.

You have used 3 of 3 submissions

DISCUSSION

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