4. Characteristic Polynomial

The **characteristic polynomial** of an $n \times n$ matrix $\mathbf{A}$ is defined as

$$P(\lambda) := \det(\lambda \mathbf{I} - \mathbf{A}).$$

(If instead you use $\det(\mathbf{A} - \lambda \mathbf{I})$, you will need to negate the result when n is odd in order to get a polynomial that starts with λ^n instead of $-\lambda^n$.)

The roots of $P(\lambda)$ are the eigenvalues of $\mathbf{A}$.

Reasoning for this

Show

Problem 4.1 What are the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}?$$

Solution: The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{A}) &= \begin{vmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 1 \end{vmatrix} \\ &= (\lambda - 1)\left((\lambda-1)(\lambda-1) - (0)(0)\right) + (-1)\left((0)(0) - (-1)(\lambda-1)\right) \qquad \text{(expansion along the top row)} \\ &= \lambda^3 - 3\lambda^2 + 2\lambda \;=\; \lambda(1-\lambda)(2-\lambda). \end{aligned}$$

The roots of the characteristic polynomial are $0,\ 1,\ 2,$ and therefore these are the eigenvalues of $\mathbf{A}$.

Definition 4.2 The **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

In example above, each of the eigenvalues $0,\ 1,$ and 2 has multiplicity 1.

Example 4.3 For the matrix $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, find the eigenvalues and their multiplicities.

Solution:
The characteristic polynomial is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda + 2 & -1 & -1 \\ -1 & \lambda + 2 & -1 \\ -1 & -1 & \lambda + 2 \end{vmatrix}$$

$$= (\lambda+2)\left((\lambda+2)(\lambda+2)-1\right)-(-1)\left((-1)(\lambda+2)-(-1)(-1)\right)+(-1)\left((-1)(-1)-(-1)(\lambda+2)\right) \qquad \text{(expand along th}$$
$$= \lambda^3+6\lambda^2+9\lambda$$
$$= \lambda(\lambda+3)^2.$$

Therefore, the eigenvalues are $0,$ with multiplicity $1,$ and $-3,$ with multiplicity $2.$

Find the eigenvalues

1/1 point (graded)

Find the eigenvalues of the upper triangular matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 6 \end{pmatrix}.$

(Enter as a list with multiples all listed explicitly separated by commas. For example, type **0,-3,-3** if the eigenvalues are 0 with multiplicity $1,$ and -3 with multiplicity $2,$ as in the above example.)

2,2,6 ✔ **Answer:** 2,2,6

Solution:

The characteristic polynomial is

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \det\begin{pmatrix} \lambda-2 & -3 & -5 \\ 0 & \lambda-2 & 7 \\ 0 & 0 & \lambda-6 \end{pmatrix} = (\lambda-2)(\lambda-2)(\lambda-6).$$

so the eigenvalues are $2,$ with mulitplicity $2,$ and $6,$ with multiplicity 1. Listed with multiplicity, they are $2, 2, 6.$

Remark: In general, for any upper triangular or lower triangular matrix, the eigenvalues are the entries of the matrix on the main diagonal.

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Nullspace

2/2 points (graded)

If a nonzero vector is in the nullspace of a square matrix $\mathbf{A},$ is it an eigenvector of $\mathbf{A}$?

yes ▼ ✔ **Answer:** yes

Which of the following are equivalent to the statement that 0 is an eigenvalue for a given square matrix $\mathbf{A}$? (Choose all that apply.)

- [x] There exists a nonzero solution to $\mathbf{A}\mathbf{v}=\mathbf{0}.$ ✔
- [x] $\det(\mathbf{A})=0$ ✔
- [ ] $\det(\mathbf{A})\neq 0$
- [ ] $\mathrm{NS}(\mathbf{A})=\mathbf{0}$

- [x] $\mathrm{NS}(\mathbf{A}) \neq \mathbf{0}$ ✔

✔

Solution:

- If a vector $\mathbf{v}$ is in the nullspace of $\mathbf{A},$ then $\mathbf{Av} = \mathbf{0} = (0)\mathbf{v}.$ So it is an eigenvector of $\mathbf{A}$ associated to the eigenvalue 0.
- If 0 is an eigenvalue for a matrix $\mathbf{A},$ then by definition, there exists a nonzero solution to $\mathbf{Av} = \mathbf{0};$ that is, $\mathrm{NS}(\mathbf{A}) \neq \mathbf{0},$ and this happens if and only if $\det(\mathbf{A}) = 0$.

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