Homework Problem: Alice's Coins

(10 points possible)

Alice has five coins in a bag: two coins are normal, two are double-headed, and the last one is double-tailed. She reaches into the bag and randomly pulls out a coin. Without looking at the coin she drew, she tosses it.

- **(a)** What is the probability that once the coin lands, the side of the coin that is face-down is heads? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  3/5 **Answer:** 3/5

- **(b)** The coin lands and shows heads face-up. What is the probability that the face-down side is heads? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

  2/3 **Answer:** 2/3

Alice discards the first coin **(the one from part (b) that landed and showed heads face-up)**, reaches again into the bag and draws out a random coin. Again, without looking at it, she tosses it.

- **(c)** What is the probability that the coin shows heads face-up? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

13/24

? **Answer:** 13/24

Solution:

- **(a)** What is the probability that once the coin lands, the side of the coin that is face-down is heads?

  Solution: The parts of this problem become much easier to answer when we can condition on which type of coin she drew. Therefore, we will rely on total probability. Using total probability requires us to determine the conditional probability of the event (the face-down side of the coin is heads) and the prior probability of pulling out each kind of coin.

  Let C_F, C_H, and C_T be the events that she pulled a fair coin, double-headed, and double-tailed coin, respectively. These events are collectively exhaustive. Let D_H be the event that the face-down side is heads. We apply total probability:

$$\begin{aligned}\mathbb{P}(D_H) &= \mathbb{P}(D_H|C_F)\mathbb{P}(C_F) + \mathbb{P}(D_H|C_H)\mathbb{P}(C_H) + \mathbb{P}(D_H|C_T)\mathbb{P}(C_T) \\ &= \frac{1}{2}\cdot\frac{2}{5} + 1\cdot\frac{2}{5} + 0\cdot\frac{1}{5} \\ &= \boxed{\frac{3}{5}}.\end{aligned}$$

- **(b)** The coin lands and shows heads face-up. What is the probability that the face-down side is heads?

Towards Infinity in Modeling Uncertainty (Week 4)
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Homework 3 (Week 4)
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Solution: Let U_H be the event that the face-up side is heads. Then we are interested in $\mathbb{P}(D_H|U_H)$. Using the definition of conditional probability, we can write $\mathbb{P}(D_H|U_H) = \mathbb{P}(D_H \cap U_H)/\mathbb{P}(U_H)$. By symmetry of the coins, the marginal (unconditional) probabilities $\mathbb{P}(U_H)$ and $\mathbb{P}(D_H)$ are equal. We can find $\mathbb{P}(D_H \cap U_H)$ using total probability:

$$\begin{aligned}\mathbb{P}(U_H \cap D_H) &= \mathbb{P}(U_H \cap D_H|C_F)\mathbb{P}(C_F) + \mathbb{P}(U_H \cap D_H|C_H)\mathbb{P}(C_H) + \\ &\quad \mathbb{P}(U_H \cap D_H|C_T)\mathbb{P}(C_T) \\ &= 0 \cdot \frac{2}{5} + 1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} \\ &= \frac{2}{5}.\end{aligned}$$

Our final answer is obtained by $\mathbb{P}(D_H \cap U_H)/\mathbb{P}(U_H) = (2/5)/(3/5) = \boxed{\frac{2}{3}}$.

Alice discards the first coin, reaches again into the bag and draws out a random coin. Again, without looking at it, she tosses it.

- **(c)** What is the probability that the coin shows heads face-up?

  Solution: Let $U_{H,2}$ be the event that the face-up side of the second coin is heads. Since reasoning about the second coin would be easier if we knew what the first coin was, we will again use total probability:

$$\begin{aligned}\mathbb{P}(U_{H,2}|U_H) &= \mathbb{P}(U_{H,2}|U_H, C_F)\mathbb{P}(C_F|U_H) + \mathbb{P}(U_{H,2}|U_H, C_H)\mathbb{P}(C_H|U_H) + \\ &\quad \mathbb{P}(U_{H,2}|U_H, C_T)\mathbb{P}(C_T|U_H)\end{aligned}$$

Since $\mathbb{P}(C_T|U_H) = 0$, the last term is 0. Here, U_H and $U_{H,2}$ are *conditionally independent* given C_F: if we know that the first coin was fair, then its outcome won't affect the outcome of the second, and similarly for C_H. Computing $\mathbb{P}(U_{H,2}|C_F)$ proceeds like the calculation we did in part (a), but with one fewer fair coin in the bag. Similarly, $\mathbb{P}(U_{H,2}|C_H)$ is similar to part (a) with one fewer double-headed coin in the bag:

$$\begin{aligned}
\mathbb{P}(U_{H,2}|C_F) &= \frac{1}{2}\cdot\frac{1}{4} + 1\cdot\frac{2}{4} + 0\cdot\frac{1}{4} \\
&= \frac{5}{8} \\
\mathbb{P}(U_{H,2}|C_H) &= \frac{1}{2}\cdot\frac{2}{4} + 1\cdot\frac{1}{4} + 0\cdot\frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

We can compute the probabilities $\mathbb{P}(C_F|U_H)$ and $\mathbb{P}(C_H|U_H)$ using Bayes' rule as $1/3$ and $2/3$ respectively:

$$\begin{aligned}
\mathbb{P}(C_F|U_H) &= \frac{\mathbb{P}(U_H|C_F)\mathbb{P}(C_F)}{\mathbb{P}(U_H)} \\
&= \frac{(1/2)(2/5)}{(1/2)(2/5)+1(2/5)} \\
&= \frac{1}{3}, \\
\mathbb{P}(C_H|U_H) &= \frac{\mathbb{P}(U_H|C_H)\mathbb{P}(C_H)}{\mathbb{P}(U_H)} \\
&= \frac{1(2/5)}{(1/2)(2/5)+1(2/5)} \\
&= \frac{2}{3}.
\end{aligned}$$

Putting the pieces together,

$$\begin{aligned}
\mathbb{P}(U_{H,2}|U_H) &= \mathbb{P}(U_{H,2}|U_H, C_F)\mathbb{P}(C_F|U_H) + \mathbb{P}(U_{H,2}|U_H, C_H)\mathbb{P}(C_H|U_H) \\
&= \frac{5}{8}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{2}{3} \\
&= \boxed{\frac{13}{24}}
\end{aligned}$$

This is lower than our answer from part (a): if we've removed a coin that produces heads, our chance of obtaining heads from the bag decreases.

You have used 0 of 10 submissions