6. Behaviors of Type 1 and Type 2 Errors for One-Sided Tests

How Type 1 Error Changes as Theta decreases

3/3 points (graded)

In the problems on the previous page, as well as in the examples in lecture, the level and power of the one-sided tests are determined by the type 1 and type 2 errors at the **boundary** of Θ_0 and Θ_1. In the following problems, we will explore the qualitative reasons for this.

Setup:

let $X_1, \ldots, X_n \stackrel{iid}{\sim} X \sim \mathbf{P}_{\mu^*}$ where $\mu^* \in \mathbb{R}$ is the true unknown mean of X, and the variance σ^2 of X is fixed. The associated statistical model is $\left(E, \{\mathbf{P}_\mu\}_{\mu\in\mathbb{R}}\right)$ where E is the sample space of X.

We conduct a one-sided hypothesis test with the following hypotheses:

$$\begin{aligned} H_0 : \mu^* \leq \mu_0 \qquad &\Longleftrightarrow \Theta_0 = (-\infty, \mu_0] \\ H_1 : \mu^* > \mu_0 \qquad &\Longleftrightarrow \Theta_1 = (\mu_0, +\infty) \end{aligned}$$

Note the boundary between Θ_0 and Θ_1. You use the statistical test:

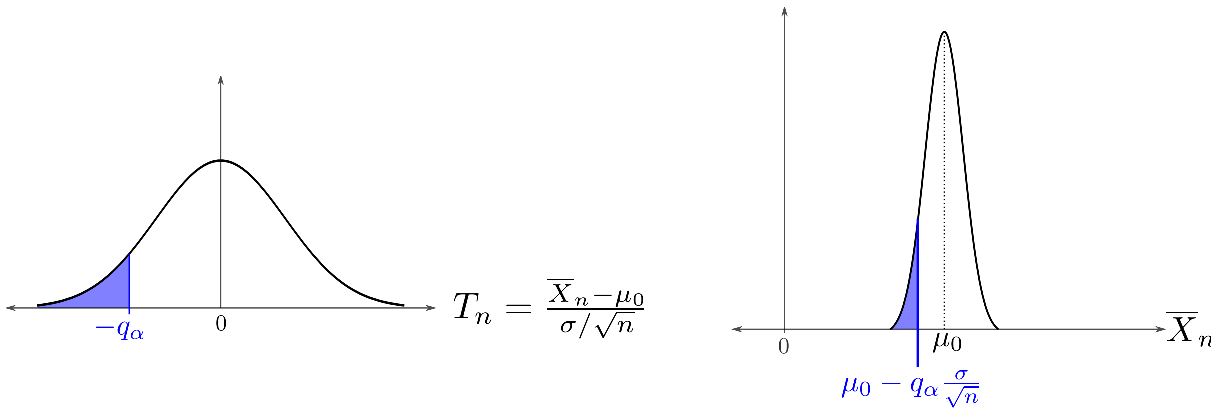
$$\psi_n = \mathbf{1}\left(T_n > q_\alpha\right)$$

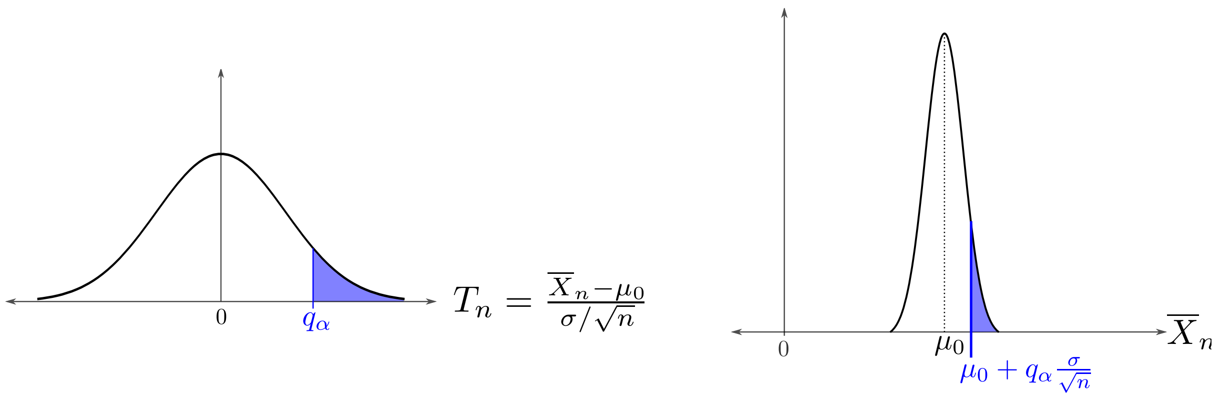
$$\text{where} \quad T_n = \sqrt{n}\frac{\overline{X}_n - \mu_0}{\sigma}.$$

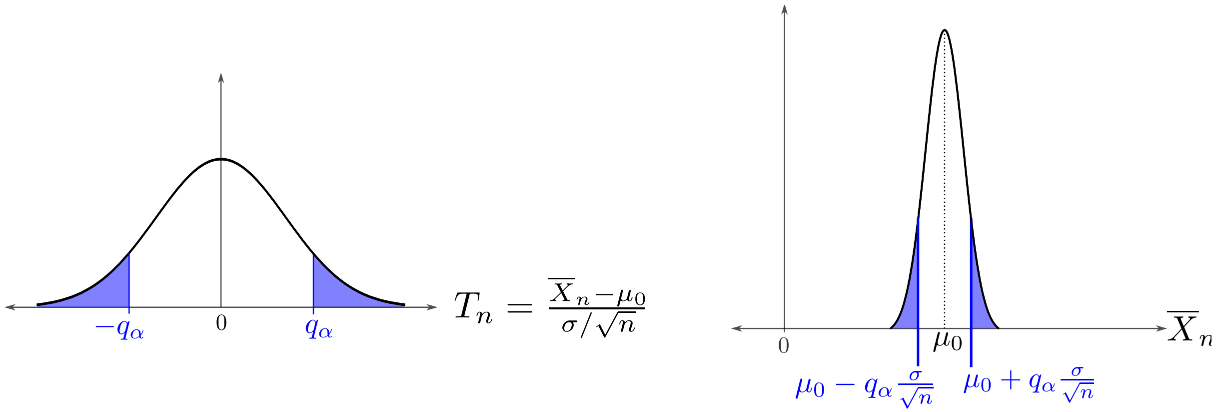
Questions:

Which of following regions correspond the type 1 error $\alpha_{\psi_n}(\mu_0)$ for large n? Note that μ_0 the boundary point of Θ_0 and Θ_1.

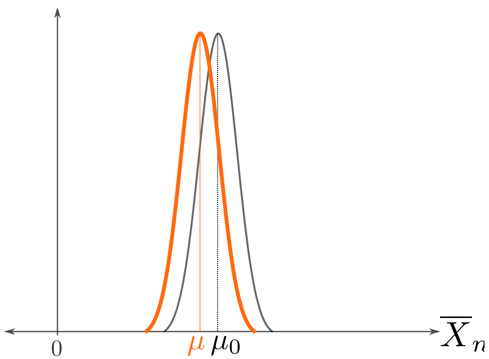
(The figures on left column depicts the distribution of T_n while the ones on the right depict the distribution of $\overline{X}_n$. Figures not drawn to scale.)

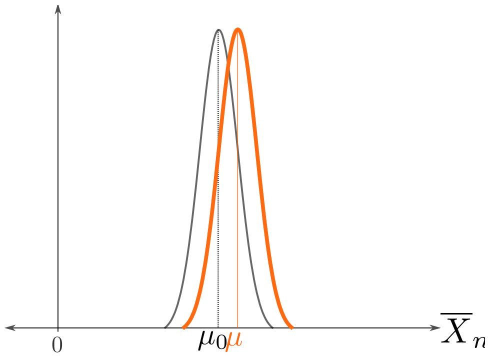




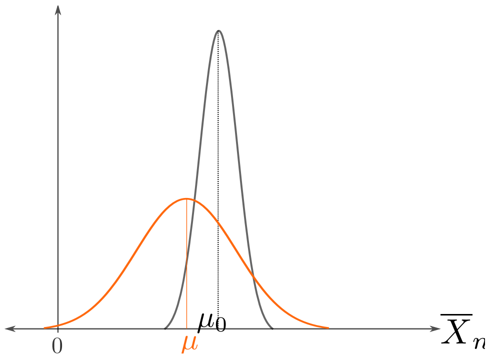


Which orange curve below is the graph of the distribution of $\overline{X}_n$ for $\mu < \mu_0$, (i.e. for μ in the interior of Θ_0)? The grey curve is the graph the distribution of $\overline{X}_n$ for $\mu = \mu_0$.

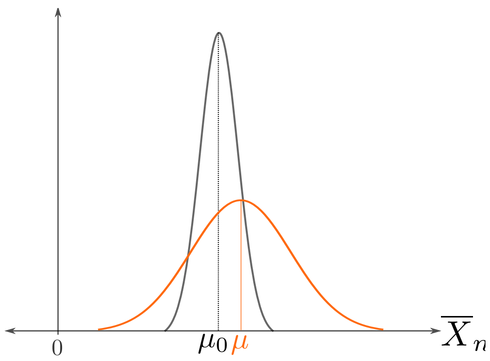




0
μ_0
μ
$\overline{X}_n$



0
μ_0
μ
$\overline{X}_n$

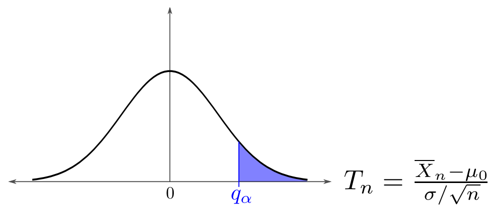


As μ decreases from μ_0 (i.e., moving away from the boundary of Θ_0 and Θ_1), does the type 1 error $\alpha_{\psi_n}(\mu)$ increase, decrease, or not exhibit a simple trend?

- increase
- decrease
- does not exhibit a simple trend

Solution:

At $\mu = \mu_0$ and when n is large, $T_n \sim \mathcal{N}(0,1)$ by the CLT. Therefore, when n is large, the type 1 error $\mathbf{P}_{\mu_0}(T_n > q_\alpha)$ is geometrically approximately the area of the “right tail" of standard normal distribution defined by the line $T_n = q_\alpha$.



The area of the shaded region is the type 1 error of ψ_n at μ_0: $\mathbf{P}_{\mu_0}\left(\overline{T}_n > q_\alpha\right)$.

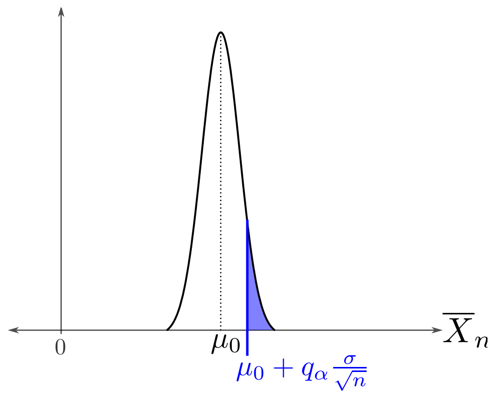
Alternatively, since

$$T_n \;=\; \sqrt{n}\frac{\overline{X}_n - \mu_0}{\sigma} \;>\; q_\alpha \;\iff\; \overline{X}_n >\; \mu_0 + q_\alpha\frac{\sigma}{\sqrt{n}},$$

we have

$$\mathbf{P}_{\mu_0}\left(T_n > q_\alpha\right) \;=\; \mathbf{P}_{\mu_0}\left(\overline{X}_n > \mu_0 + q_\alpha\frac{\sigma}{\sqrt{n}}\right),$$

which is the area of the “right tail" of the distribution of $\overline{X}_n$ to the right of $\overline{X}_n = \mu_0 + q_\alpha\frac{\sigma}{\sqrt{n}}$. By the CLT, for n large, the distribution of $\overline{X}_n$ is approximately Gaussian, with mean $\mathbb{E}\left[X\right]$ and variance $\frac{\sigma}{\sqrt{n}}$.



The area of the shaded region is the type 1 error of ψ_n at μ_0: $\mathbf{P}_{\mu_0}\left(\overline{X}_n > \mu_0 + q_\alpha \dfrac{\sigma}{\sqrt{n}}\right)$.

Since $\mu = \mathbb{E}[X]$, the CLT implies that $\overline{X}_n$ is approximately Gaussian with mean μ for large n. Recall the variance of X is fixed at σ, so the distribution of $\overline{X}_n$ for $\mu < \mu_0$ is a simple shift, without rescaling, to the left of the distribution of $\overline{X}_n$ at μ_0.

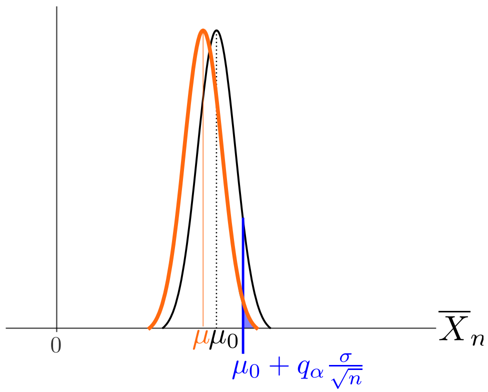
Finally, to look for a trend for the type 1 error $\alpha_{\psi_n(\mu)}$ as μ decreases from μ_0, first observe that the threshold

$$\tau_{n,\alpha} = \mu_0 + q_\alpha \frac{\sigma}{\sqrt{n}}$$

of the test

$$\psi = \mathbf{1}\left(T_n > q_\alpha\right) = \mathbf{1}(\overline{X}_n > \tau_{n,\alpha}$$

does **not** depend on the parameter μ. The only thing that changes as μ changes is the distribution of $\overline{X}_n$, which shifts to the **left** as μ decreases. Since the type 1 error $\alpha_{\psi_n}(\mu) = \mathbf{P}_\mu(\overline{X}_n > \tau)$ is the area of the tail to the **right** of τ, we see that the type 1 error continues to decrease as μ (and the distribution of $\overline{X}_n$) moves to the left.



The distribution of $\overline{X}_n$ at μ_0, the boundary point between Θ_0 and Θ_1; The distribution of $\overline{X}_n$ at $\mu < \mu_0$ (orange curve), a shift to the left from the distribution at μ_0

The type 1 error $\alpha_{\psi_n}(\mu)$ in the interior of Θ_0 is smaller than the type 1 error $\alpha_{\psi_n}(\mu_0)$ at the boundary of Θ_0 and Θ_1.

Remark: The type 2 error $\beta_{\psi_n}(\mu) = 1 - \mathbf{P}_\mu(\overline{X}_n > \tau)$ decreases as μ increases from μ_0: as μ increases, the distribution of $\overline{X}_n$ shifts without rescaling to the right but the threshold τ remains constant. This implies $\mathbf{P}_\mu(\overline{X}_n > \tau)$ continues to increases as μ moves to the right from the boundary of Θ_0 and Θ_1, and hence the Type 2 error continues to decrease.

In conclusion, for any one-sided hypothesis test where the family of distributions is parametrized by the mean of the distribution and the variance is fixed for the entire entire family, the type 1 and type 2 error achieve their suprema (or maxima) at the boundary between Θ_0 and Θ_1. Therefore, the level and power can be read off at the boundary.

Further question: Does the reasoning above works for two-sided tests?

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You have used 2 of 2 attempts

Answers are displayed within the problem

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