7. Fundamental matrix

Introducing the fundamental matrix for 2 by 2 systems

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7:51 / 7:51 2.0x HD

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Fundamental matrix for n by n systems

We build fundamental matrices for $n \times n$ systems in the same way that we build fundamental matrices for 2×2 systems.

We will now write the solutions to an $n \times n$ system in a more compact form using what is known as the fundamental matrix.

Consider an $n \times n$ homogeneous linear constant coefficient system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

The set of solutions is an n-dimensional vector space. Let $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ be a basis of solutions, that is, $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent. Write $\mathbf{x}_1, \ldots, \mathbf{x}_n$ as column vectors side-by-side to form an $n \times n$ matrix

$$\mathbf{X}(t) := \begin{pmatrix} | & & | \\ \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \\ | & & | \end{pmatrix}.$$

Any such $\mathbf{X}(t)$ is called a **fundamental matrix** for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. (This is a matrix-valued **function** , since each $\mathbf{x}_i$ is a vector-valued function of t.)

Example 7.1 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ has eigenvalues and eigenvectors

$$\lambda_1 = 2 \qquad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3 \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence, the functions

$$e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix} \quad \textbf{and} \quad e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

are a basis of solutions. Therefore, one fundamental matrix of the system is

$$\mathbf{X}(t) = \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}.$$

Criteria of a fundamental matrix

Theorem 7.2 A matrix-valued function $\mathbf{X}(t)$ is a fundamental matrix for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ if and only if

- $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ and
- the matrix $\mathbf{X}(0)$ is nonsingular, namely, $\det(\mathbf{X}(0)) \neq 0$.

The first property above is equivalent to saying that each column of $\mathbf{X}(t)$ is a solution. This is because differentiation on the left-hand side of $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ and the matrix multiplication on the right hand side can be done column-by-column.

The second property $\det\mathbf{X}(0) \neq 0$ says that the column vectors $\mathbf{x}_1(0), \ldots, \mathbf{x}_n(0)$ are linearly independent. Remarkably, once $\det\mathbf{X}(0) \neq 0$ we also have $\det\mathbf{X}(t) \neq 0$ for all t. In the example, $\det(\mathbf{X}(t)) = e^{5t} \neq 0$ for all t.

Proof using the Wronskian

Let $\mathbf{x}_1(t), \ldots, , \mathbf{x}_n(t)$ be solutions to the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$ and define the matrix

$$\mathbf{X}(t) \quad = \quad \begin{pmatrix} | & & | \\ \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \\ | & & | \end{pmatrix}.$$

The following theorem implies that if $\det \mathbf{X}(0) \neq 0,$ then the columns of $\mathbf{X}(t)$ are linearly independent, implying $\mathbf{X}(t)$ is a fundamental matrix.

Theorem 7.3 The set of vector functions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ solving the system above are linearly independent if and only if $\det(\mathbf{X}(t)) \neq 0$ for all t. The determinant $\det(\mathbf{X}(t))$ is called the **Wronskian** .

Proof:
($\Leftarrow$) First, we prove the backwards direction by proving the contrapositive. If $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent, then there are constants $c_1, c_2, \ldots, c_n,$ which are not all zeroes, such that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t) = 0$ for all t. Hence, $\det(\mathbf{X}(t)) = 0$ for all t.

($\Rightarrow$) Now we prove the forward direction. Suppose that $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent, but suppose that there is a point t_0 where $\det(\mathbf{X}(t_0)) = 0.$ This means there are constants $c_1, c_2, \ldots, c_n,$ not all zeroes, such that $c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) + \ldots + c_n\mathbf{x}_n(t_0) = 0.$ But this implies, for example, that both $c_1\mathbf{x}_1(t)$ and $-(c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t))$ satisfy the same initial conditions. By the existence and uniqueness theorem, these must be equal, implying $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t) = 0.$

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