Total Orderings

Even though Anti-Symmetry and Transitivity are non-trivial, they are relatively weak ordering conditions.

For example, they are both satisfied by an "empty" precedence relation $<$, according to which $a < b$ is never the case.

They are also compatible with orderings that have tree-like structures. Consider, for example, the set of Queen Victoria's descendants, ordered as follows: a precedes b if and only if a is b's (direct-line) ancestor. This ordering can be represented as a tree, with Victoria at the base, each of her children branching off from the base, each of her grandchildren branching off from their parents nodes, and so forth.

A feature of tree-like orderings is that they allow for individuals who don't bear the precedence relation to one another. This is true, for example, of any two of Victoria's children. Since neither is an ancestor of the other, neither of them precedes the other in our ancestry-based ordering.

Here we will restrict our attention to orderings on which any two objects are such that one precedes the other. What this means, formally speaking, is that we will restrict our attention to total orderings.

A **total ordering** $<$ on A is an ordering that satisfies the following condition whenever a and b are distinct elements of A:

Totality
$a < b$ or $b < a$.

Here is an example of a total ordering on the set of Victoria's descendants: a precedes b if and only if a was born before b. Since no two of Victoria's descendants were born at exactly the same time, any two of them will be such that one of them precedes the other.

Another example of a total ordering is the standard ordering of the integers, $<_{\mathbb{Z}}$:

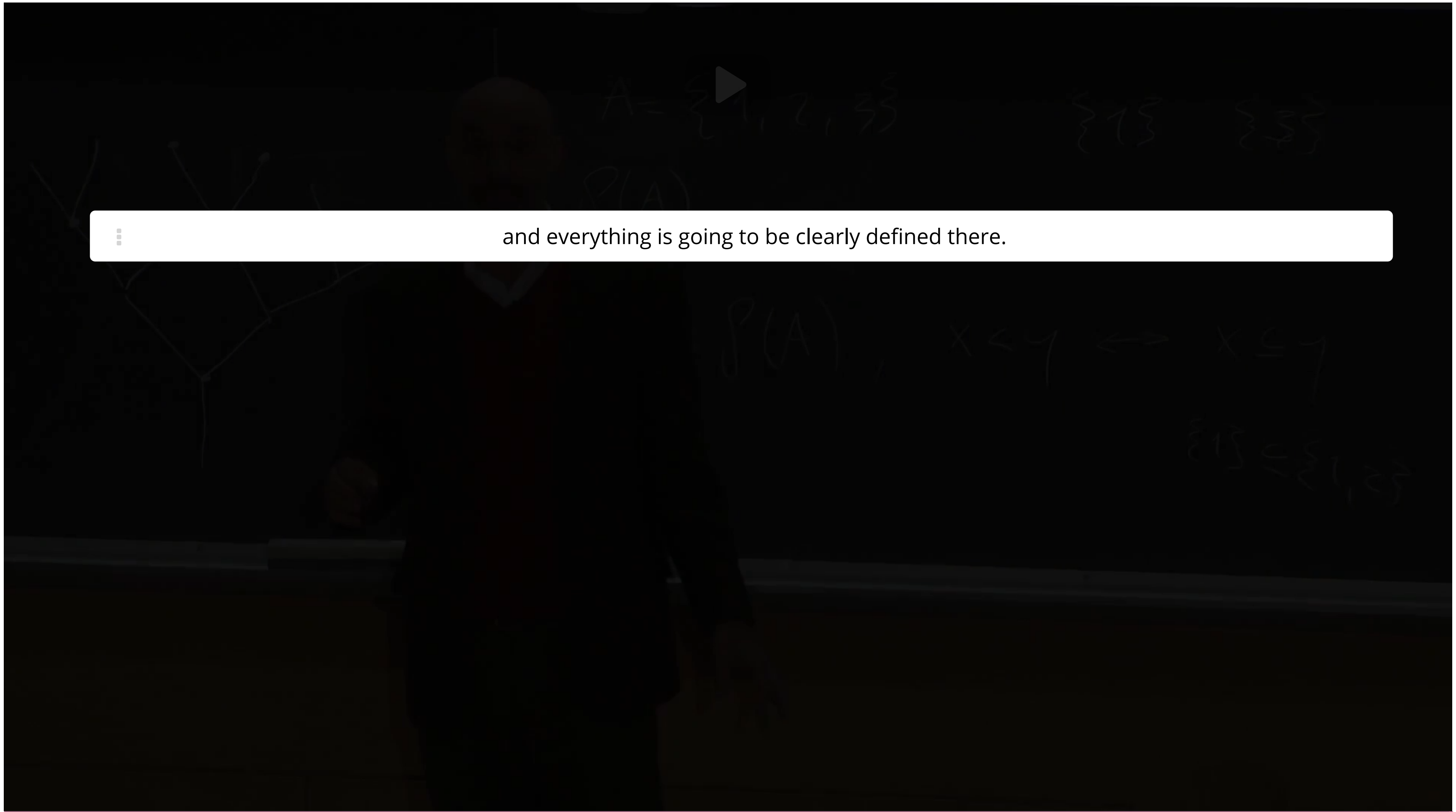
$$\ldots \; <_{\mathbb{Z}} -2 <_{\mathbb{Z}} -1 <_{\mathbb{Z}} 0 <_{\mathbb{Z}} 1 <_{\mathbb{Z}} 2 <_{\mathbb{Z}} 3 <_{\mathbb{Z}} \ldots$$

Integer a precedes integer b on this ordering if and only if $b = a + n$, for n a positive integer. Since any two integers are such that there is some positive difference between them, any two of them are such that one of them $<_{\mathbb{Z}}$-precedes the other.

Notation

Total orderings go by different names. In the videos I call them *linear orderings*. Sorry about that!

Video Review: Total Orderings (i.e. Linear Orderings)



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Problem 1

1/1 point (ungraded)

Is the following ordering a total ordering?

The integers $\ldots, -2, -1, 0, 1, 2, \ldots$, under an "inverse" ordering $<^{-1}$ such that $a <^{-1} b$ if and only if $b <_{\mathbb{Z}} a$, where $<_{\mathbb{Z}}$ is the standard ordering of the integers, as characterized in the main text.

- [x] Yes. It is a total ordering.
- [ ] No. It is not a total ordering.

✔

Explanation

Yes, $<^{-1}$ is a total ordering of $\mathbb{Z}$:

$$\ldots \;<^{-1} 2 <^{-1} 1 <^{-1} 0 <^{-1} -1 <^{-1} -2 <^{-1} -3 <^{-1} \ldots$$

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Problem 2

1/1 point (ungraded)

Is the following ordering a total ordering?

The power set of $\{0, 1\}$, under an ordering $\subset$ such that $a \subset b$ if and only if a is a proper subset of b. (a is a proper subset of b if and only if $a \neq b$ and every element of a is an element of b.)

- [ ] Yes. It is a total ordering.

No. It is not a total ordering.

Explanation

No, $\subset$ is not a total ordering of $\mathscr{P}(\{0,1\})$. Totality fails because none of the following holds: $\{0\} = \{1\}$, $\{0\} \subset \{1\}$, $\{1\} \subset \{0\}$.

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Problem 3

1/1 point (ungraded)

Is the following ordering a total ordering?

> The real numbers in the interval $[0,1]$, under the standard ordering $<_{\mathbb{R}}$, which is such that $a <_{\mathbb{R}} b$ if and only if $b = r + a$ for r a positive real number.

Yes. It is a total ordering.

No. It is not a total ordering.

Explanation

Yes, $<_{\mathbb{R}}$ is a total ordering of $[0,1]$. One indication of this is that the elements of $[0,1]$ can be represented as points in a line segment.

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Problem 4

1/1 point (ungraded)

Is the following ordering a total ordering?

The set of countably infinite sequences $\langle d_0, d_1, \ldots \rangle$ (where each d_i is a digit $0, 1, \ldots, 9$), under an ordering $<$ such that $\langle d_0^1, d_1^1, \ldots \rangle < \langle d_0^2, d_1^2, \ldots \rangle$ if and only if $0.d_0^1 d_1^1, \ldots <_{\mathbb{R}} 0.d_0^2 d_1^2, \ldots$ (where $<_{\mathbb{R}}$ is defined as in the previous question).

- ( ) Yes. It is a total ordering.
- (•) No. It is not a total ordering.

✔

Explanation

No, $<$ is not a total ordering of the relevant set of sequences. Totality fails because $0.5\,(0) = 0.4\,(9)$, so none of the following holds $\langle 5, 0, 0, 0, \ldots \rangle = \langle 4, 9, 9, 9, \ldots \rangle$, $\langle 5, 0, 0, 0, \ldots \rangle < \langle 4, 9, 9, 9, \ldots \rangle$, $\langle 4, 9, 9, 9, \ldots \rangle < \langle 5, 0, 0, 0, \ldots \rangle$.

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Problem 5

1/1 point (ungraded)

Is the following ordering a total ordering?

The set of soldiers, under an ordering $<$ such that $x < y$ if and only if x outranks y.

Yes. It is a total ordering.

No. It is not a total ordering.

Explanation
A typical set of soldiers is not totally ordered by $<$, since there will be soldiers a and b such that neither outranks the other. But one could, of course, imagine an army in which every rank is occupied by exactly one soldier. Such an army would be totally ordered by $<$.

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? Question about problem 1 on previous page: anti-symmetry implies anti-reflexivity
Hello! I have a quick question on the Problem on the Orderings page, where we have to indicate whether anti-reflexivity follows from the anti-symmetry condition of the prec... 1