9.2.5 Toward a Systematic Approach to Finding All Solutions

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Week 9 due Dec 9, 2023 18:12 IST Completed

9.2.5 Toward a Systematic Approach to Finding All Solutions

Video

Start of transcript. Skip to the end.

Dr. Robert van de Geijn: So in the last units,

we discussed how linear systems can have multiple solutions.

And what we would like to look at now is how

to find such solutions systematically.

Remember from unit 9.2.3, two units

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Reading Assignment

0 points possible (ungraded)

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Calculator

Homework 9.2.5.1

1/1 point (graded)

Which of the below expressions is the general solution (an expression for all solutions) for

$$\begin{pmatrix} 2 & -2 & -4 \\ -2 & 1 & 4 \\ 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.$$

- [ ] $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$
- [ ] $x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$
- [x] $x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

✓

- Set this up as an appended system

$$\left(\begin{array}{ccc|c} 2 & -2 & -4 & 4 \\ -2 & 1 & 4 & -3 \\ 2 & 0 & -4 & 2 \end{array}\right).$$

Now, start applying Gaussian elimination (with row exchanges if necessary).

- Use the first row to eliminate the coefficients in the first column below the diagonal:

$$\left(\begin{array}{ccc|c} \boxed{2} & -2 & -4 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 2 & 0 & -2 \end{array}\right).$$

- Use the second row to eliminate the coefficients in the second column below the second row:

$$\left(\begin{array}{ccc|c} \boxed{2} & -2 & -4 & 4 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This is good news: while we are now left with two equations with three unknowns, the last row translates to $0 = 0$ and hence there is no contradication. There will be an infinite number of solutions.

- Subtract a multiple of the second row from the first row to eliminate the coefficient above the pivot.

$$\left(\begin{array}{ccc|c} \boxed{2} & 0 & -4 & 2 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Calculator

- Divide the first and second row by appropriate values to make the normalize the pivots to one:

$$\left(\begin{array}{rrr|r} \boxed{1} & 0 & -2 & 1 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- The above really represents two appended systems, the second one corresponding to the case where the right-hand side is zero:

$$\left(\begin{array}{rrr|r} \boxed{1} & 0 & -2 & 1 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{rrr|r} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- The free variable is now χ_2 since there is no pivot in the column corresponding to that component of x.

- We translate the appended system on the left back into a linear system but with $\chi_2 = 0$ (since it can be chosen to equal any value and zero is convenient):

$$\begin{array}{rcrcrcr} \chi_0 & & & - & 2\,(0) & = & 1 \\ & & \chi_1 & & & = & -1 \\ & & & & \chi_2 & = & 0 \end{array}$$

  Solving this yields the specific solution $x_s = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

- We translate the appended system on the right back into a linear system but with $\chi_1 = 1$ (we can choose any value except for zero, and one is is convenient):

$$\begin{array}{rcrcrcr} \chi_0 & & & - & 2\,(1) & = & 0 \\ & & \chi_1 & & & = & 0 \\ & & & & \chi_2 & = & 1 \end{array}$$

  Solving this yields the solution $x_n = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

- The general solution now is

$$x = x_s + \beta x_n = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

*
(Recap)

The above seems like a very long-winded way of answering the question. In practice, here is what you will want to do:

1. Set the linear system up as an appended system $$\left(\begin{array}{rrr\|r} 2 & -2 & -4 & 4 \\ -2 & 1 & 4 & -3 \\ 2 & 0 & -4 & 2 \end{array}\right).$$	**2.** Check if you need to pivot (exchange) rows, identify the pivot, and eliminate the elements below the pivot: $$\left(\begin{array}{rrr\|r} \boxed{2} & -2 & -4 & 4 \\ 0 & -1 & 0 & 1 \end{array}\right).$$

	$\begin{array}{ccc\|c} 0 & 2 & 0 & -2 \end{array}$
3. Check if you need to pivot (exchange) rows, identify the pivot, and eliminate the elements below the pivot: $$\left(\begin{array}{ccc\|c} \boxed{2} & -2 & -4 & 4 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$	**4.** Eliminate elements above the pivot: $$\left(\begin{array}{ccc\|c} \boxed{2} & 0 & -4 & 2 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$
5. Divide to normalize the pivots to one: $$\left(\begin{array}{ccc\|c} \boxed{1} & 0 & -2 & 1 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$	**6.** Identify the free variable(s) as corresponding to the column(s) in which there is no pivot. In this case, that is χ_2.
7. Set the free variable(s) to zero and solve. This gives you the specific solution $$x_s = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$	**8.** Set the right-hand side to zero in the transformed system $$\left(\begin{array}{ccc\|c} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$
9. Set the free variables one by one to one (and the others to zero). This gives you vectors that satisfy $Ax = 0$: $$x_n = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$	**10.** The general solution now is $$x = x_s + \beta x_n = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$ (If there is more than one free variable, you will get more terms with vectors that satisfy $Ax = 0$. More on that later.)

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Homework 9.2.5.2

1/1 point (graded)

Find the general solution (an expression for all solutions) for $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$.

- $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$
- $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Calculator

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$x = x_s + \beta x_n = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

- Set this up as an appended system

$$\left(\begin{array}{rrr|r} 2 & -4 & -2 & 4 \\ -2 & 4 & 1 & -3 \\ 2 & -4 & 0 & 2 \end{array}\right).$$

Now, start applying Gaussian elimination (with row exchanges if necessary).

- Use the first row to eliminate the coefficients in the first column below the diagonal:

$$\left(\begin{array}{rrr|r} 2 & -4 & -2 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right).$$

- There is now a zero on the diagonal and no row below it with which to exchange to put a nonzero there. So, we move on and use the second row to eliminate the coefficients in the third column below the second row:

$$\left(\begin{array}{rrr|r} 2 & -4 & -2 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This is good news: while we are now left with two equations with three unknowns, the last row translates to $0 = 0$. Hence, there is no contradication. There will be an infinite number of solutions.

- Subtract a multiple of the second row from the first row to eliminate the coefficient of the last term in the first row:

$$\left(\begin{array}{rrr|r} 2 & -4 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- Divide the first and second row by appropriate values to make the first coefficient in the row (the pivot) equal to one:

$$\left(\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- We argued in this unit that the above really represents two appended systems, the second one corresponding to the case where the right-hand side is zero:

$$\left(\begin{array}{rrr|r} 1 & -2 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{rrr|r} 1 & -2 & 0 & 0 \end{array}\right)$$

Calculator

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- The rule for picking free variables is to look in each row for the first zero to the left of the pivot (as described above) in that row. If that zero is **not** under the pivot for the row above, than that column corresponds to a free variable. This leads us to choose χ_1 as the free variable.

- We translate the appended system on the left back into a linear system but with $\chi_1 = 0$ (since it can be chosen to equal any value and zero is convenient):

$$\begin{array}{rcrcrcr} \chi_0 & - & 2\,(0) & & & = & 1 \\ & & \chi_1 & & & = & 0 \\ & & & & \chi_2 & = & -1 \end{array}$$

  Solving this yields the specific solution $x_s = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

- We translate the appended system on the right back into a linear system but with $\chi_1 = 1$ (we can choose any value except for zero, and one is is convenient):

$$\begin{array}{rcrcrcr} \chi_0 & - & 2\,(1) & & & = & 0 \\ & & \chi_1 & & & = & 1 \\ & & & & \chi_2 & = & 0 \end{array}$$

  Solving this yields the solution $x_n = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

- The general solution now is

$$x = x_s + \beta x_n = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

*
(Recap)

The above seems like a very long-winded way of answering the question. In practice, here is what you will want to do:

1. Set the linear system up as an appended system $\left(\begin{array}{rrr\|r} 2 & -4 & -2 & 4 \\ -2 & 4 & 1 & -3 \\ 2 & -4 & 0 & 2 \end{array}\right).$	**2.** Check if you need to pivot (exchange) rows, identify the pivot, and eliminate the elements below the pivot: $\left(\begin{array}{rrr\|r} \boxed{2} & -4 & -2 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right).$
3. Check if you need to pivot (exchange) rows, identify the pivot, and eliminate the elements below the pivot: $\left(\begin{array}{rrr\|r} \boxed{2} & -4 & -2 & 4 \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$	**4.** Eliminate elements above the pivot in the second row: $\left(\begin{array}{rrr\|r} \boxed{2} & -4 & 0 & 2 \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$

Calculator

5. Divide to normalize the pivots to one: $$\left(\begin{array}{ccc\|c} \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$	**6.** Identify the free variable(s) as corresponding to the column(s) in which there is no pivot. In this case, that is χ_1.
7. Set the free variables to zero and solve. This gives you the specific solution $$x_s = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$	**8.** Set the right-hand side to zero in the transformed system $$\left(\begin{array}{ccc\|c} \boxed{1} & -2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$
9. Set the free variables one by one to one (and the others to zero). This gives you vectors that satisfy $Ax = 0$: $$x_n = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$	**10.** The general solution now is $$x = x_s + \beta x_n = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$ (If there is more than one free variable, you will get more terms with vectors that satisfy $Ax = 0$. More on that later.)

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