Classical Optimization: Unconstrained Optimization

Three Motivating Problems

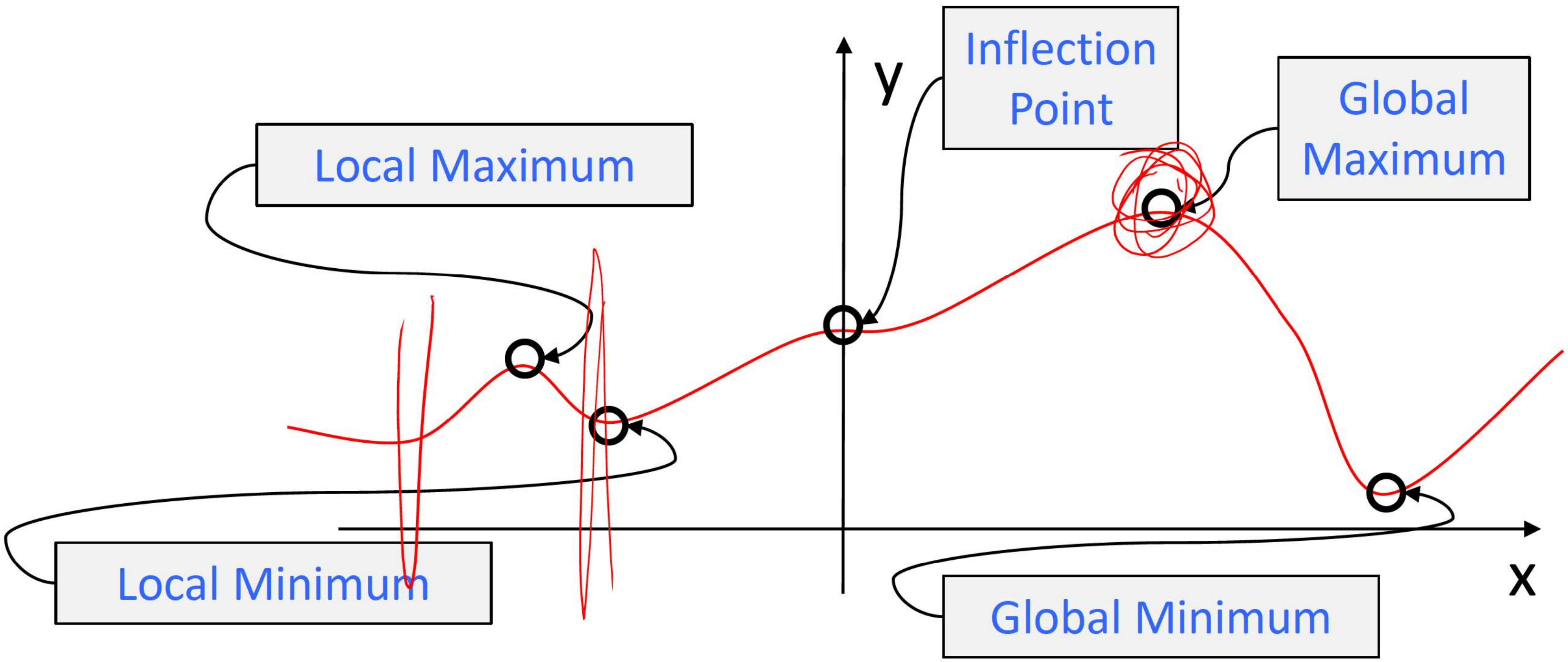
- Profit Maximization - iWidget
  - Given cost and demand functions, find the price for the iWidget that produces maximum profit.
- Inventory Replenishment Policy – Gears Unlimited
  - Given annual demand and costs for ordering and holding, calculate the re-order quantity that minimizes total cost.
- Package Optimization – boxy.com
  - Calculate the package dimensions that maximize total usable volume given a specific cardboard sheet.

Common Features

- Each of these problems . . .
  - Requires the use of a Prescriptive Model,
  - Utilizes a math function to make the decision,
  - Looks for an "extreme point" solution, and
  - Are unconstrained in that there is not a resource limit.
- What is an extreme point of a function?
  - The point, or points, where the function takes on an extreme value, typically either a minimum or a maximum.
  - The point(s) where the slope or "rate of change" of the function is equal to zero.

Extreme Points

- Types of Extreme points
  - Minimum, Maximum, or Inflection Points
  - The minimum and maximum points are either global or local



How do we find these "extreme point" solutions?

We'll use differential calculus to find where the slope is equal to zero!

(relax – it really is easier than you might think!)

Classical Optimization

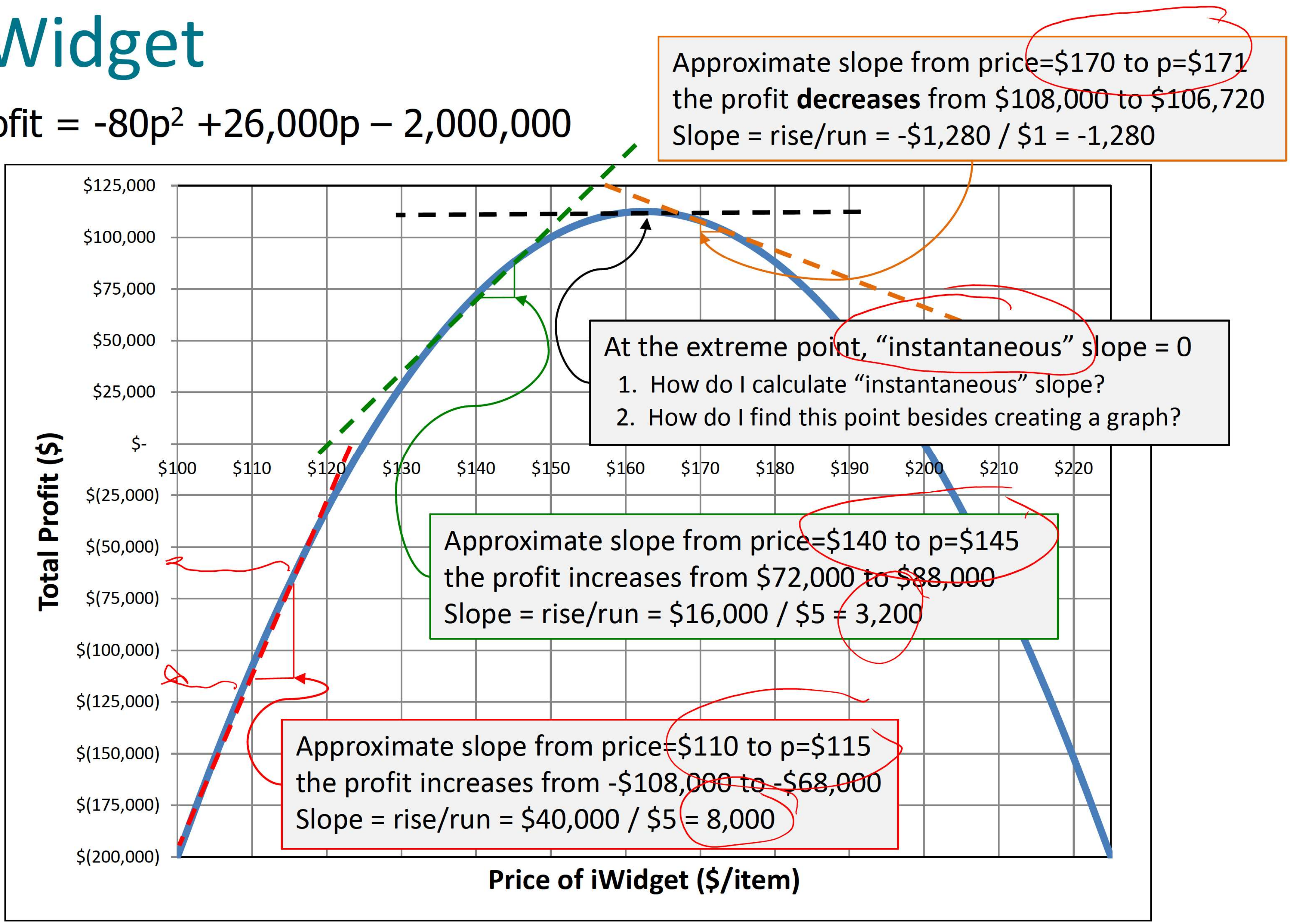
Classical Optimization

- Use differential calculus to find extreme solutions
  - Look for where the rate of change, the slope, goes to zero
  - Check for sufficiency conditions
  - Continuity and convexity come into play

- Example: iWidget
  - We are manufacturing a product where we know:
    - The cost function = f(# made) = 500,000 + 75x
    - The demand function = f(price) = 20,000 – 80p
    - And therefore the profit function = $-80p^2 + 26{,}000p - 2{,}000{,}000$
  - We want to find the price, p, that maximizes profits.

iWidget

Profit = $-80p^2 + 26{,}000p - 2{,}000{,}000$



Finding the Instantaneous Slope:
The First Derivative

Calculating Instantaneous Slope

- Given any function, y=f(x), we define y'=f'(x) as the instantaneous rate of change at point x.

$$f'(x) = \lim_{\delta \to 0}\left(\frac{f(x+\delta) - f(x)}{\delta}\right)$$

Different nomenclature is often used, but they all mean the same thing – the instantaneous rate of change of function f with respect to x.

$$y' = f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$$

- Example: Suppose we have f(x) = -4x² + 5x.

$$f'(x) = \left(\frac{\left[-4(x+\delta)^2 + 5(x+\delta)\right] - \left[-4x^2 + 5x\right]}{\delta}\right) = \left(\frac{-4\left(x^2 + 2\delta x + \delta^2\right) + 5x + 5\delta + 4x^2 - 5x}{\delta}\right)$$

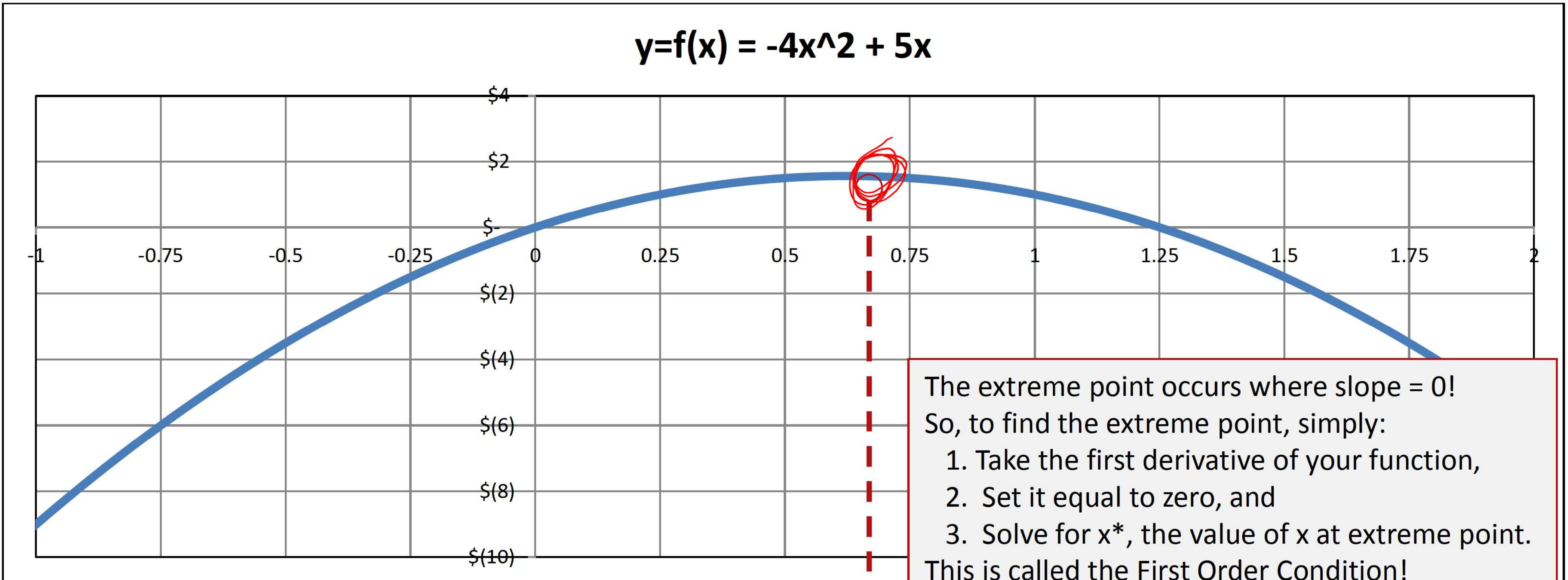
$$= \left(\frac{-4x^2 - 8\delta x - 4\delta^2 + 5x + 5\delta + 4x^2 - 5x}{\delta}\right) = \left(\frac{\cancel{-4x^2} + \cancel{4x^2} + \cancel{5x} - \cancel{5x} - 8\delta x + 5\delta - 4\delta^2}{\delta}\right)$$

$$= \left(\frac{-8\cancel{\delta}x + 5\cancel{\delta} - 4\delta^{\cancel{2}}}{\cancel{\delta}}\right) = -8x + 5 - \cancel{4\delta}$$

But, as δ approaches zero, this becomes:

$$y' = f'(x) = -8x + 5$$

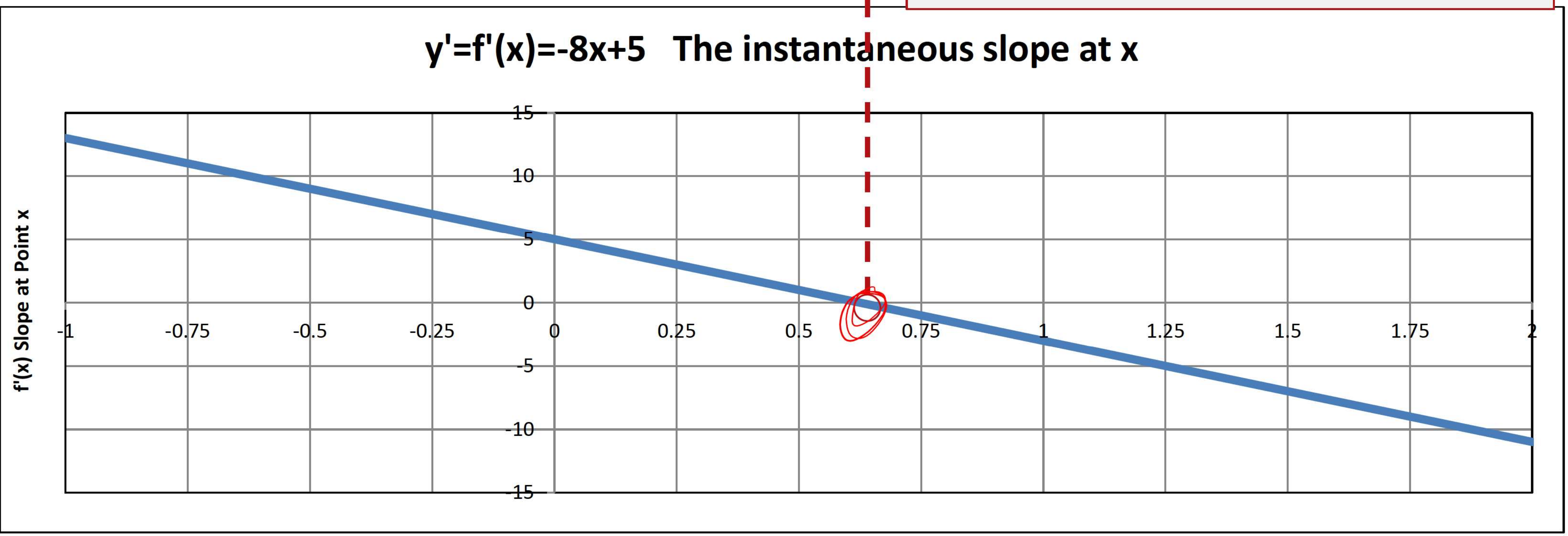
Which is the function that gives me the instantaneous slope at any point!



The extreme point occurs where slope = 0!
So, to find the extreme point, simply:

1. Take the first derivative of your function,
2. Set it equal to zero, and
3. Solve for x*, the value of x at extreme point.

This is called the First Order Condition!



Rule for Differentiation

- Luckily, we have some simple rules for finding derivatives!

$$y = f(x) = a \quad \rightarrow \quad y' = f'(x) = 0$$

$$y = f(x) = ax^n \quad \rightarrow \quad y' = f'(x) = anx^{n-1}$$

$y=3x + 12 = 3x^1 + 12$ $\quad$ $y'=3(1)x^{(1-1)} + 0 = 3$

It is a linear function! The slope of a linear function does not change!

$y=-4x^2 +5x$ $\quad$ $y'=-4(2)x^{(2-1)} +5(1)x^{(1-1)} = -8x + 5$

Finding x^* => $-8x+5 = 0$ or $-8x=-5$
so $x^*=5/8 = 0.625$

$y=4/x + 3x$
$= 4x^{-1}+3x^1$ $\quad$ $y'=4(-1)x^{(-1-1)} +3(1)x^{(1-1)} = -4x^{-2} + 3$

Finding x^* => $-4x^{-2}+3 = 0$
or $-4x^{-2}= -3$ or $4 = 3x^2$
so $x^*=\sqrt{(4/3)} = 1.15$

There are many different shortcuts and rules for finding derivatives of more complex functions – but we will use the Power Rule almost all of the time.

Solving the iWidget Problem

iWidget Solution

- Find the price, p, that maximizes the profit function:

$$y = -80p^2 + 26{,}000p - 2{,}000{,}000$$

- Solution
  1. Take the first derivative:

$$y' = dy/dp = -80(2)p^{(2-1)} + 26{,}000(1)p^{(1-1)} = -160p + 26{,}000$$

  2. Set the first derivative equal to zero:

$$-160p + 26{,}000 = 0$$

  3. Solve for p*:

$$-160p = -26{,}000 \quad \text{so that } p^* = 26{,}000 / 160$$

$$p^* = \$162.50$$

Set price at \$162.50 in order to maximize profit.

Expected profit will be \$112,500.

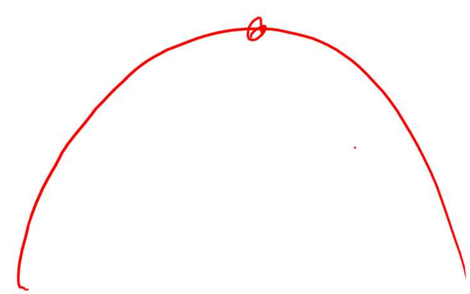
$$\textit{profit} = (-80)(162.5)^2 + (26{,}000)(162.5) - 2{,}000{,}000$$

Two Questions . . .

1. How do I know this is a maximum and not a minimum?
2. How do I know whether this is global or local?

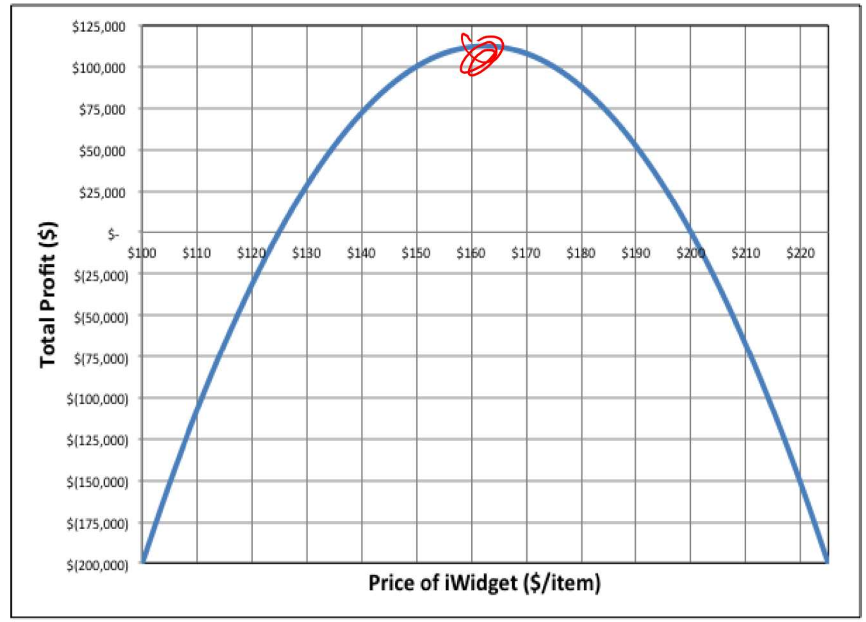
Necessary and Sufficient Conditions

- In order to determine x* at the max/min of an unconstrained function:
- Necessary Condition – the slope has to be zero, that is, $f'(x^*)=0$
- Sufficient Conditions – determines whether extreme point is min or max by taking the Second Derivative, $f''(x)$.
  - If $f''(x) > 0$ then the extreme point is a local minimum
  - If $f''(x) < 0$ then the extreme point is a local maximum
  - If $f''(x) = 0$ then it is inconclusive
- Special Cases (yea!)
  - If f(x) is **convex** then f(x*) is a **global minimum**
  - If f(x) is **concave** then f(x*) is **global maximum**



iWidget Solved

- By observation, we know that p*=162.50 is a global optimal, but lets do it formally.



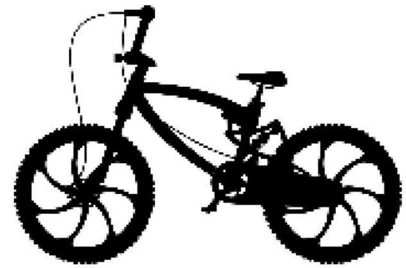
- Checking Second Order Conditions
  - $y = f(p) = -80p^2 + 26{,}000p - 2{,}000{,}000$
  - $y' = f'(p) = -160p + 26{,}000$
  - $y'' = f''(p) = (-160)(1)p^{(1-1)} + 0 = -160$

- Because -160 < 0 , this is local maximum

- But, since f(p) is a concave function, we know that it is a global maximum!

Example: Gears Unlimited

Gears Unlimited



Inventory Replenishment Policy

Gears Unlimited distributes specialty gears, derailleurs, and brakes for high-end mountain and BMX bikes. One of their most steady selling items is the PK35 derailleur. They sell about 1500 of the PK35's a year. They cost $75 each to procure from a supplier and Gears Unlimited assumes that the cost of capital is 20% a year. It costs about $350 to place and receive an order of the PK35s, regardless of the quantity of the order.

How many PK35s should Gears Unlimited order at a time to minimize the average annual cost in terms of purchase cost, ordering costs, and holding costs?

1. What do we know?

D = Demand = 1,500 items/year

c = Unit cost = 75 $/item

A = Ordering cost = 350 $/order

r = Cost of capital = 0.20 $/$/year

2. What do we want to find?

Q = Order Quantity (items/order)

Find Q* that minimizes Total Cost

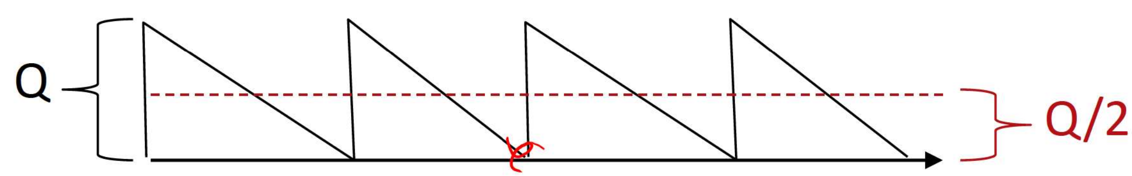
3. What is my objective function?

TotalCost = PurchaseCost + OrderCost + HoldingCost

Purchase Cost = cD = (75)(1500) = 112,500 $/yr

OrderCost = A(D/Q) = (350)(1500)/Q = 525,000/Q $/yr

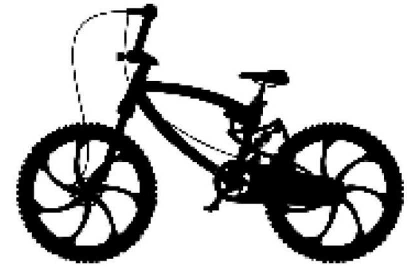
HoldingCost = rc(Q/2) = (.20)(75)(Q/2) = 7.5Q $/yr



$$TC(Q) = cD + A(D/Q) + rc(Q/2)$$

$$TC(Q) = 112{,}500 + 525{,}000/Q + 7.5Q$$

Gears Unlimited Solution



1. Determine the Objective Function

$$TC = f(Q) = cD + A(D/Q) + rc(Q/2)$$
$$= 112{,}500 + 525{,}000/Q + 7.5Q$$

2. Take first derivative

$$f'(Q) = 0 + (525{,}000)(-1)Q^{(-1-1)} + (7.5)(1)Q^{(1-1)}$$
$$= -525{,}000/Q^2 + 7.5$$

3. Set 1st derivative equal to zero and solve for Q*

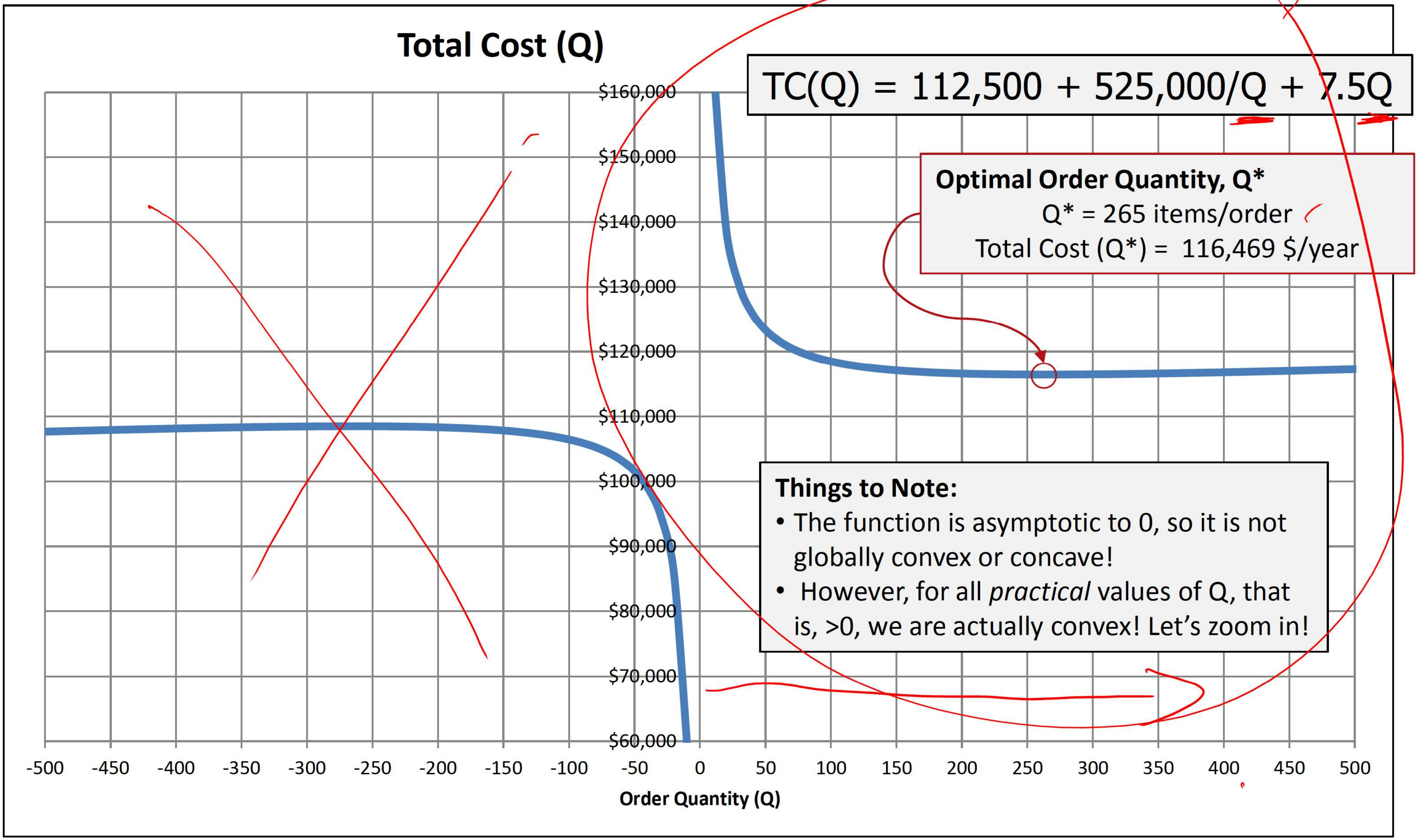
$$f'(Q^*) = -525{,}000/Q^{*2} + 7.5 = 0$$
$$-525{,}000/Q^{*2} = -7.5$$
$$Q^{*2} = 525{,}000/7.5 = 70{,}000$$
$$Q^* = \sqrt{(70{,}000)} = 264.6 \cong 265 \text{ items/order}$$

4. Check 2nd order conditions

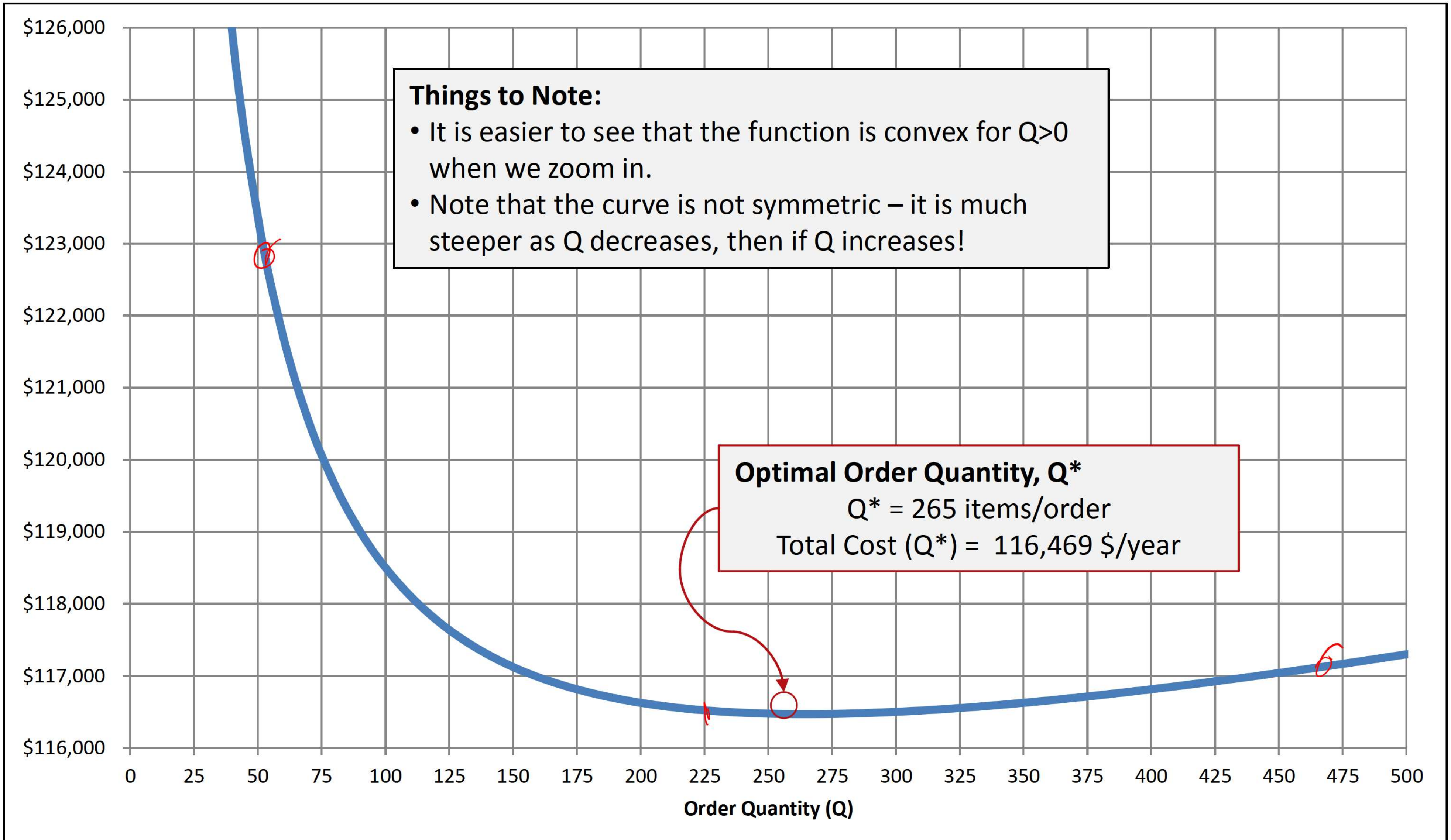
$$f''(Q^*) = -525{,}000(-2)/Q^{*3} = (1{,}050{,}000)/Q^{*3} > 0$$

Because Q* will always be greater than zero, we know that this Q* is a local minimum.

Gears Unlimited – Total Cost Function



Gears Unlimited – Total Cost Function



Things to Note:
- It is easier to see that the function is convex for Q>0 when we zoom in.
- Note that the curve is not symmetric – it is much steeper as Q decreases, then if Q increases!

Optimal Order Quantity, Q*
Q* = 265 items/order
Total Cost (Q*) = 116,469 $/year

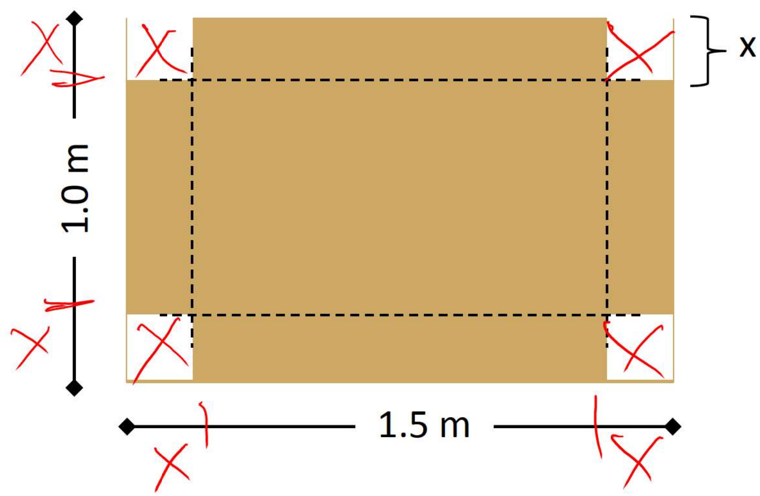
Example: boxy.com

boxy.com

Optimal Design

You are consulting with boxy.com, the premier online corrugated packaging company. They just received a large quantity of heavy duty cardboard from a third party at an extremely low cost. All of the sheets are 1 meter by 1.5 meters in dimension. You have been asked to come up with the design that maximizes the total volume of a box made from this sheet. The only cutting that can be made, however, are equal-sized squares from each of the four corners. The edges then fold up to form the box.

How big should the square cut-outs be to maximize the box's volume?



1. What do we know?

W= Width = 1 m
L = Length = 1.5 m
x = Height of box (also the amount cut)

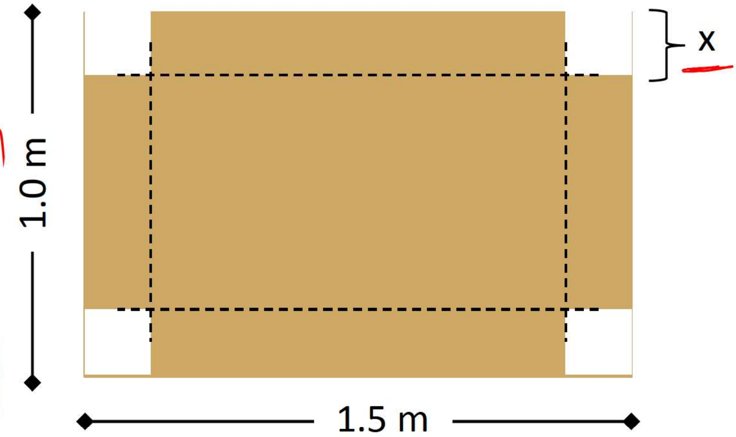
2. What do we want to find?

Find x* that maximizes Volume
V = Volume = (Width)(Length)(Height)
= (W-2x)(L-2x)(x)

3. What is my objective function?

$$\max V = (W-2x)(L-2x)(x)$$
$$= (WL-2xL-2Wx+4x^2)x = 4x^3-2Wx^2-2Lx^2+WLx$$
$$= 4x^3 - 2x^2 - 3x^2+1.5x = \underline{4x^3-5x^2+1.5x}$$

boxy.com Solution



1. Determine the Objective Function

$$V = f(x) = = 4x^3-2Wx^2-2Lx^2+WLx$$

$$= 4x^3-5x^2+1.5x$$

2. Take first derivative

$$f'(x) = (4)(3)x^{(3-1)} - (5)(2)x^{(2-1)} + 1.5(1)x^{(1-1)}$$

$$= 12x^2 - 10x + 1.5$$

3. Set 1st derivative equal to zero and solve for x*

Recall . . .

$$y = ax^2 + bx + c$$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$f'(x^*) = 12x^2 - 10x + 1.5 = 0$$

$$r_1, r_2 = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(12)(1.5)}}{2(12)} = \frac{10 \pm \sqrt{100-72}}{24} = \frac{10 \pm \sqrt{28}}{24}$$

$$r_1 = \frac{10+\sqrt{28}}{24} = \cancel{0.637m}$$

$$r_2 = \frac{10-\sqrt{28}}{24} = 0.196m$$

4. Check 2nd order conditions

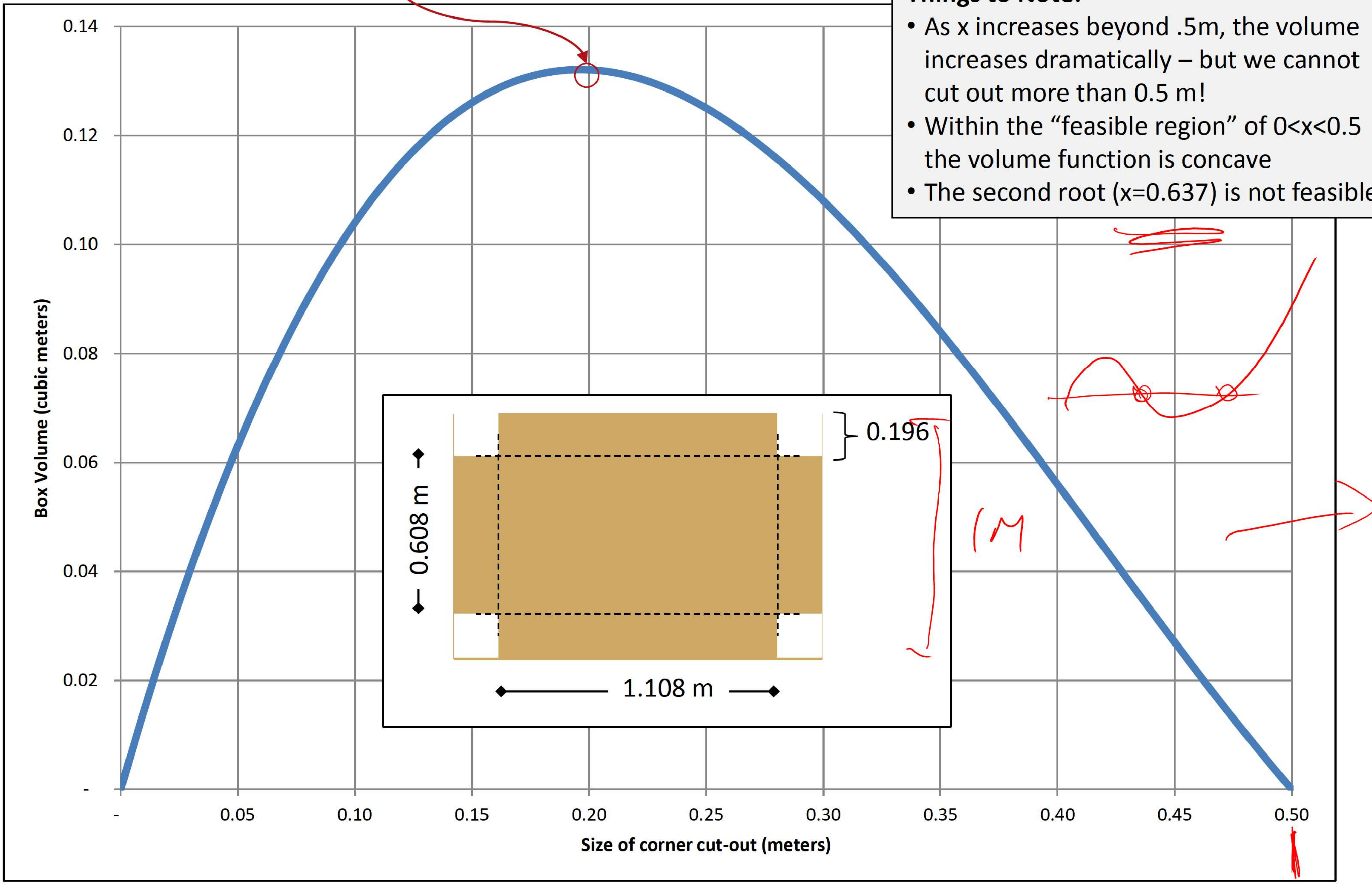
$$f''(x^*) = 12(2)x^* - 10(1) = 24x^* - 10 < 0$$

The function at x*=0.196 is a local maximum.

Maximum volume = 0.132 m^3

boxy.com

Optimal Cut-Out, x^* = 0.196 m
Max Volume (x^*) = 0.132 m^3



Things to Note:

- As x increases beyond .5m, the volume increases dramatically – but we cannot cut out more than 0.5 m!
- Within the "feasible region" of 0<x<0.5 the volume function is concave
- The second root (x=0.637) is not feasible

Key Points from Lesson

Key Points from Lesson

Power Rule

$$y = f(x) = ax^n \quad \square \quad y' = f'(x) = anx^{n-1}$$

- Unconstrained Optimization
  - Extreme points of a function
  - Typically a minimum or a maximum
  - Extreme points occur where slope = 0
- To Find Extreme Point Solutions, $f(x^*)$
  - First Order (Necessary) Condition
    - $f'(x^*) = 0$
    - First derivative used to find instantaneous rate of change
  - Second Order (Sufficiency) Condition
    - If $f''(x) > 0$ then the extreme point is a local minimum
    - If $f''(x) < 0$ then the extreme point is a local maximum
    - If $f''(x) = 0$ then it is inconclusive
    - If $f(x)$ is **convex** then $f(x^*)$ is a **global minimum**
    - If $f(x)$ is **concave** then $f(x^*)$ is **global maximum**

Questions, Comments, Suggestions?
Use the Discussion Forum!

"Griffin and Cody wish they were unconstrained"
Yankee Golden Retriever Rescued Dogs
(www.ygrr.org)

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