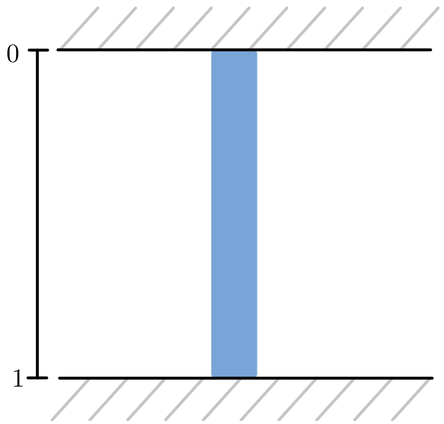
3. Solving boundary value problems numerically

Let's explore how to solve a differential equation with boundary conditions numerically.

We won't be able to use `ODE45`, which requires *initial conditions*. Here we will detail how one can discretize the problem and reduce it to a problem in linear algebra instead!

Our example problem will be the vertical elastic beam, fixed at both ends, with force acting along its vertical axis.



This system satisfies the differential equation

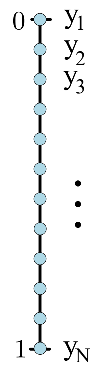
$$\frac{d^2u}{dy^2} = \frac{f(y)}{E},$$

with boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0,$$

where u is the vertical displacement, $f(y)$ is the stress acting per unit length on the beam, and E is the Young's Modulus (a constant determined by material properties of the beam).

The first step is to discretize the beam. The height y is a continuous variable from 0 to 1. Create a series of evenly spaced points $y_1, \ldots y_N$ with $y_1 = 0$ and $y_N = 1$. Then $\Delta y = y_{i+1} - y_i$.



Define $u_i = u(y_i)$.

What is the discrete version of $\frac{du}{dy}$ at the point y_i? This is a subtle question because there isn't just one answer. The approach is to use a secant line approximation. The question is which points to use in the approximation. There are two equally good answers that seem the most natural.

Answer 1	Answer 2
$\frac{u_{i+1} - u_i}{\Delta y}$	$\frac{u_i - u_{i-1}}{\Delta y}$

Answer 1 might be called a forwards derivative; Answer 2, a backwards derivative.

What is the second derivative at y_i? Now there are even more options.

- $\dfrac{u_{i+2} - 2u_{i+1} + u_i}{(\Delta y)^2}$ Forwards
- $\dfrac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2}$ Centered
- $\dfrac{u_i - 2u_{i-1} + u_{i-2}}{(\Delta y)^2}$ Backwards

We will use the centered second derivative here. Let's see how to write out our differential equation as a system of linear equations.

For each i, we get

$$\begin{aligned} \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} &= \frac{f(y_i)}{E} \\ u_{i+1} - 2u_i + u_{i-1} &= \frac{f(y_i)\,\Delta y^2}{E} \end{aligned}$$

We can write this as a matrix equation

$$\begin{pmatrix} ? & ? & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & ? & ? \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \frac{(\Delta y)^2}{E} \begin{pmatrix} f(y_1) \\ f(y_2) \\ \vdots \\ f(y_N) \end{pmatrix}$$

The question of course is what to do at the end points.

The first thing to note is that we must enforce the boundary conditions $u(0) = u_1 = 0$ and $u(1) = u_N = 0$. Therefore the first equation in the matrix must be an equation setting this boundary condtion, and the last equation must set the other boundary condition. We can write this as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \frac{(\Delta y)^2}{E} \begin{pmatrix} 0 \\ f(y_2) \\ \vdots \\ f(y_{N-1}) \\ 0 \end{pmatrix}.$$

This matrix equation is just a system of linear equations and can be solved using MATLAB or other linear algebra techniques.

MATLAB problem (External resource) (1.0 points possible)

- 10 points => 9 gaps — 5
- ? Still lack intuitive understanding of what "u" means. — 4

Differential equation with boundary conditions

Solve the differential equation

$$\frac{d^2u}{dy^2} = -0.01/E, \qquad 0 < y < 1$$

subject to the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0.$$

Follow the steps outlined below:

1. Create a 10x10 matrix $\mathbf{A}$ corresponding to the discrete second derivative and boundary condtions as seen above.
2. Create a 10 element column vector $\mathbf{b}$ that is 0 at the ends and -0.01 everywhere else.
3. Solve $\mathbf{Au} = \frac{(\Delta y)^2}{E}\mathbf{b}$ for $E = 3.2$ to find the solution $\mathbf{u}$.
4. Create a column vector $\mathbf{y}$ of containing 10 evenly spaced points from 0 to 1.
5. Plot $\mathbf{u}$ on the vertical axis, and $\mathbf{y}$ on the horizontal axis.

Script

Save Reset MATLAB Documentation (https://www.mathworks.com/help/)

```
%Create a matrix corresponding to the discrete second derivative and boundary condtions
A = full(gallery('tridiag', 10, 1, -2, 1));
A(1,1) = 1;
A(1,2) = 0;
A(10,10) = 1;
A(10,9) = 0;

% Create a column vector b with 10 entries that is zero at the ends (boundary conditions) and -0.01 in every other entry.
b = ones(10,1)*(-0.01);
b(1) = 0;
b(10) = 0;

% Create a vector u that solves Au = (Delta y)^2/E b.
u = A\(((0.1)^2/3.2)*b);

%Create a column vector of 10 evenly spaced points between 0 and 1 for plotting
y = linspace(0,1,10)';

```

```
%Plot  on the vertical axis, and  on the horizontal axis.
plot(y, u);
```

Run Script

Assessment: All Tests Passed (100%)

Submit

Check A 50% (50%)

Check b 10% (10%)

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- Boundary conditions, Matrix adjustment. — 3
  I am not so comfortable with the adjust done on matrix to enforce boundary conditions. Why the first line is 1 0 0 0 ...? The matrix is the coefficient of the differential equation...
- Writing A and b at once, or more shortly? — 6
  Hello, I wrote A and b in more than one step or command. Is there any way to do it with one command so that the initial conditions are directly included without using A(1,1),...
- Direction of displacement? — 2
  In the formulation above, displacement u(y) is positive. But heuristically, buckling can happen in either direction, correct? Then how would one go about describing such nuan...
- MATLAB possible grader bug — 7
  if one inserts a "clear all" directive on top of script, the grader does not work.
- Not sure why my code is not accepted as I checked it a lot and it ran just fine. — 5
  %Create a matrix corresponding to the discrete second derivative and boundary condtions A = full(gallery('tridiag',10,1,-2,1)); A(1,1) = 1; A(1,2) = 0; A(10,9) = 0; A(10,10) = 1; % C...