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4. The geometry of solutions to homogeneous linear systems

Consider the vector space $\mathbb{R}^3$. This is a 3 dimensional vector space. If we have one equation, $ax + by + cz = 0$, where a, b, and c are not all zeros, then the solutions to this equation describe a plane through the origin. Therefore the dimension of the solution space is 2, which is one less than all possible vectors in $\mathbb{R}^3$. If we add a second equation what happens?

When we add a second equation, the dimension of the solution space can at most decrease by 1 again. However, if the second equation describes the same plane as the first equation, the solution set remains unchanged!

Principle: Each additional equation reduces the dimension of the solution space by at most 1. Therefore m equations reduces the dimension of the solution space by at most m.

If our m equations have n variables, then we started with the n dimensional space $\mathbb{R}^n$. The smallest possible dimension of the solution space is $n - m$. This suggests that for a general system with m equations in n variables, the dimension of the solution space is at most n and at least $n - m$ since each equation cuts the dimension of the solution space by at most 1.

Reformulated in terms of matrices, this says that for an m by n matrix $\mathbf{A}$, the solution space to the homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ can be at most be n dimensional, and at least $n - m$ dimensional. By the end of this lecture, we'll make what we mean by n dimensions more precise, and we will have an algorithm for determining the dimension of the solutions to any homogeneous system.

How many solutions?

1/1 point (graded)

Let $\mathbf{A}$ be an m by n matrix. Which of the following *must* be true about solutions to the system $\mathbf{A}\mathbf{x} = \mathbf{0}$?

(Hint: think about what $\mathbf{A}$ does to the zero vector.)

- $\mathbf{Ax} = \mathbf{0}$ has exactly one solution.
- $\mathbf{Ax} = \mathbf{0}$ has at least one solution. ✔
- $\mathbf{Ax} = \mathbf{0}$ has no solutions.

Solution:

Since $\mathbf{A0} = \mathbf{0}$, the vector $\mathbf{x} = \mathbf{0}$ is always a solution. Therefore homogeneous linear systems always have at least one solution. It is not always true that there is exactly one solution. There can be more as we saw in the balancing a chemical equation example.

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Answers are displayed within the problem

Solution vectors

1/1 point (graded)

Suppose that

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 6 & 2 & -4 & -8 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 2 & -3 & 1 & 4 & -7 & 1 \\ 6 & -9 & 0 & 11 & -19 & 3 \end{pmatrix}.$$

Which of the following is true for any solution to $\mathbf{Ax} = \mathbf{0}$?

- $\mathbf{x}$ is a row vector with 4 entries

- $\mathbf{x}$ is a column vector with 4 entries
- $\mathbf{x}$ is a row vector with 6 entries
- $\mathbf{x}$ is a column vector with 6 entries ✔

Solution:

The matrix $\mathbf{A}$ has 4 rows and 6 columns. Therefore the vector $\mathbf{x}$ must be a column vector with $\mathbf{6}$ entries. Note that this makes sense when we think of $\mathbf{A}$ as a function from $\mathbb{R}^{\mathbf{6}}$ to $\mathbb{R}^{\mathbf{4}}$, so $\mathbf{x}$ must be a column vector in $\mathbb{R}^{\mathbf{6}}$.

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