5. Statistics of random walk

Exercises due Nov 10, 2021 17:29 IST Completed

To compute the basic statistics (mean, variance, covariance) of the random walk, it is useful to write X_t as a sum of the perturbations $\{W_h\}_{h=1}^t$ that accumulate over time:

$$\begin{aligned} X_t &= X_{t-1} + W_t \\ &= \left[X_{t-2} + W_{t-1}\right] + W_t \\ &\vdots \\ &= X_0 + \sum_{h=1}^{t} W_h. \end{aligned}$$

Similarly, for the random walk with drift Y_t we have:

$$\begin{aligned} Y_t &= \delta + Y_{t-1} + W_t \\ &= \delta + \left[\delta + Y_{t-2} + W_{t-1}\right] + W_t \\ &\vdots \\ &= \delta \cdot t + Y_0 + \sum_{h=1}^{t} W_h. \end{aligned}$$

Using these representations we can find the marginal mean function, the covariance function and the autocorrelation function:

$$\begin{aligned} \mu_X(t) &= \mathbf{E}[X_t] \\ &= \mathbf{E}[X_0 + \sum_{h=1}^{t} W_h] \\ &= \mathbf{E}[X_0] \end{aligned}$$

since W_t is white noise and has mean zero.

$$\begin{aligned} \sigma_X^2(t) &= \mathsf{Var}(X_t) \\ &= \mathsf{Var}\Big(X_0 + \sum_{h=1}^{t} W_h\Big) \\ &= \mathsf{Var}(X_0) + \sum_{h=1}^{t}\left[2\mathsf{Cov}(X_0, W_h) + \mathsf{Var}(W_h)\right] + 2\sum_{1\le h<j\le t} \mathsf{Cov}(W_h, W_j) \\ &= \mathsf{Var}(X_0) + t\cdot\sigma_W^2 \end{aligned}$$

since W_h is uncorrelated with X_0 and with W_j for $j \neq h$.

$$\begin{aligned} \gamma_X(s,t) &= \mathsf{Cov}(X_s, X_t) \\ &= \mathsf{Cov}\Big(X_0 + \sum_{h=1}^{s} W_h,\ X_0 + \sum_{h=1}^{t} W_h\Big) \\ &= \mathsf{Var}(X_0) + \sum_{h=1}^{\min(s,t)} \mathsf{Var}(W_h) \\ &= \mathsf{Var}(X_0) + \min(s,t)\cdot\sigma_W^2 \end{aligned}$$

since W_h is uncorrelated with X_0 and with W_j for $j \neq h$.

Note that typically X_0 is assumed to be deterministic, thus $\text{Var}(X_0) = 0$.

Statistics of random walk

3/3 points (graded)

Let X be a random walk with random perturbations that have $\sigma_W^2 = 1$.

Compute $\gamma_X(5, 10)$.

5 ✔ **Answer:** 5

Compute $\gamma_X(10, 15)$.

10 ✔ **Answer:** 10

Does the autocovariance function $\gamma_X(s, t)$ only depend on time gap $|s - t|$?

- ○ True
- ◉ False

✔

Solution:

From the above calculations, $\gamma_X(5, 10) = \min(5, 10) \cdot 1 = 5$ and $\gamma_X(10, 15) = 10$. Therefore, the autocovariance of a random walk is not stationary since the gap between time stamps in these two computations is the same but the covariances are different.

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Answers are displayed within the problem

Statistics of random walk with drift

As an exercise, repeat the calculation for the marginal mean function $\mu_Y(t) = \mathbf{E}[Y_t]$, the marginal variance function $\sigma_Y^2(t) = \text{Var}(Y_t)$ and the autocovariance function $\gamma_Y(s, t) = \text{Cov}(Y_s, Y_t)$ of the random walk with drift Y_t.

Computation of marginal mean, marginal variance, and autocovariance function

From the expression above we have

$$\begin{aligned} \mu_Y(t) &= \mathbf{E}[\delta t + Y_0 + \sum_{h=1}^{t} W_h] \\ &= \delta t + \mathbf{E}[Y_0] \end{aligned}$$

by linearity of expectation and zero mean property of white noise.

$$\text{Var}(Y_t) = \text{Var}\Big(Y_0 + \delta t + \sum_{h=1}^{t} W_t\Big)$$

$$= \text{Var}\left(Y_0 + \sum_{h=1}^{t} W_t\right)$$

$$= \text{Var}\,(Y_0) + t\sigma_W^2$$

by property of the variance (that a shift of a distribution does not change the spread of the distribution) and assumptions about white noise.

Similarly, $\text{Cor}\,(Y_t, Y_s) = \text{Var}\,(Y_0) + \min\,(t, s)\,\sigma_W^2$.

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Random walk and stationarity

2/2 points (graded)
Is the random walk with drift process stationary?

- ○ True
- ◉ False ✔

Is the random walk process stationary?

- ○ True
- ◉ False ✔

Solution:

False. The random walk is not stationary because the variance is growing with time and the autocovariance depends on the smallest of the two time stamps rather than on the difference.

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Differencing random walk

2/2 points (graded)
Consider the first difference ∇Y_t of a random walk Y_t with drift.

Calculate the marginal mean function $\mu_{\nabla Y}\,(t)$, the marginal variance function $\sigma^2_{\nabla Y}\,(t)$ and the autocovariance function $\gamma_{\nabla Y}\,(s, t)$ of ∇Y_t.

Select all correct statements.

- [ ] ∇Y_t is random walk
- [x] ∇Y_t is white noise plus constant $(\delta + W_t)$
- [x] $\mu_{\nabla Y}\,(t)$ is constant

- [x] $\sigma^2_{\nabla Y}(t)$ is constant
- [x] $\gamma_{\nabla Y}(s,t) = 0$
- [ ] $\gamma_{\nabla Y}(s,t)$ is constant, but not necessarily 0

Is ∇Y_t a stationary time series?

- (●) True
- ( ) False

Solution:

The first difference of the random walk is $\nabla Y_t = Y_t - Y_{t-1} = \delta + W_t$ white noise. Therefore $\mu_{\nabla Y}(t) = \delta$ is constant, $\sigma^2_{\nabla Y}(t) = \sigma^2_W$ is constant, and $\gamma_{\nabla Y}(s,t) = 0$ for $s \neq t$.
Yes, a shifted white noise $\{\delta + W_t\}$ is a stationary time series.

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(Optional) Estimations of random walk model

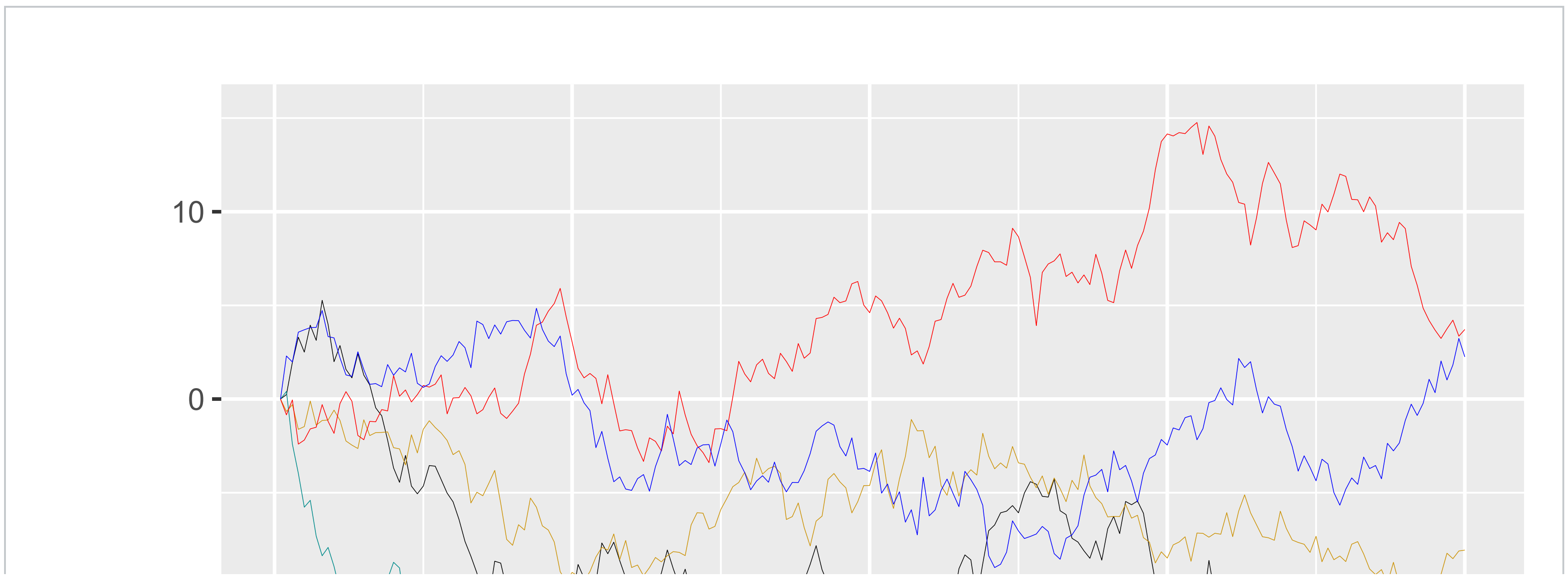
0 points possible (ungraded)
If X_t is a random walk, would our estimators $\hat{\sigma}^2_X(1)$ and $\hat{\gamma}_X(h)$ be consistent for σ^2_X and $\gamma_X(1,h)$?

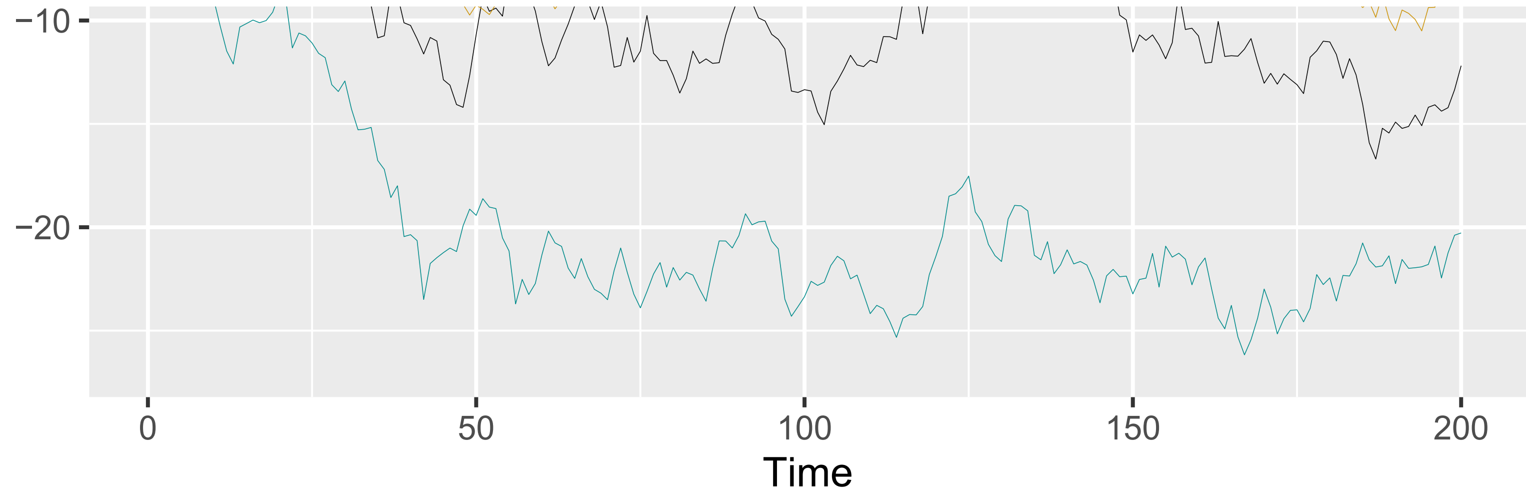
- ( ) True
- (●) False

Solution:

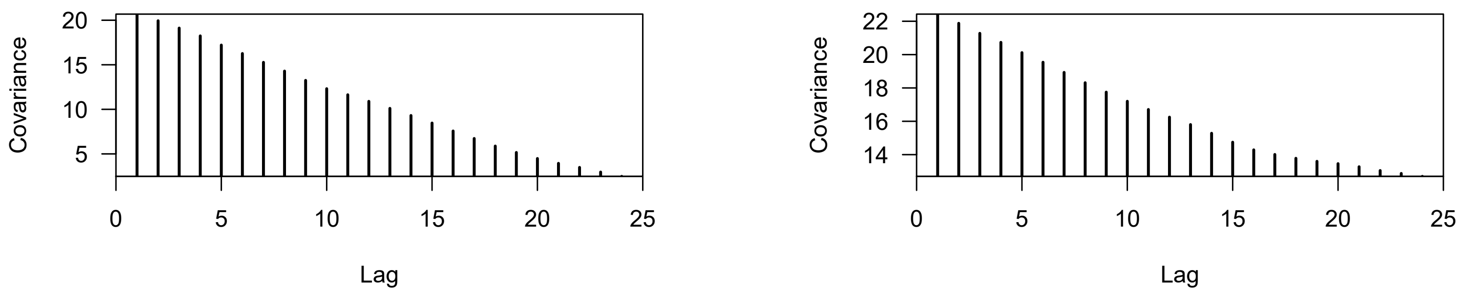
No, consistency of the estimated acf function requires **stationarity** which random walk does not have.

We can illustrate this with a plot of an estimated acf:





Random Walk



Estimated acf for black and red paths

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