11. Complex eigenvalues and eigenvectors

Recall that complex non-real eigenvalues come in pairs of complex conjugates. It turns out the eigenvectors of a complex eigenvalue are the complex conjugates of the eigenvectors of the conjugate eigenvalue.

Example 11.1 Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have found previously that the eigenvalues are $1, \pm i$. Let us now find the eigenvectors. We will start with those of the complex eigenvalue i.

Eigenspace of i **:** $\mathrm{NS}(i\mathbf{I} - \mathbf{A})$.

We reduce $i\mathbf{I} - \mathbf{A}$ to row-echelon form;

$$i\mathbf{I} - \mathbf{A} = \begin{pmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & i-1 \end{pmatrix} \to \begin{pmatrix} i & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By back substitution (the first equation is $ix + y = 0$), the solutions are $c\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$ for any scalar c. That is,

$$\text{Eigenspace of } i = \mathrm{NS}\left(i\mathbf{I} - \mathbf{A}\right) = \mathrm{Span}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}.$$

Eigenspace of $-i$: $\mathrm{NS}(-i\mathbf{I}-\mathbf{A})$.

Since $\mathbf{A}$ is a real matrix, $-i\mathbf{I}-\mathbf{A}$ is the complex conjugate of $i\mathbf{I}-\mathbf{A},$ and every step in the Gaussian elimination is the same as before except with i replaced by $-i$:

$$-i\mathbf{I}-\mathbf{A} = \begin{pmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -i-1 \end{pmatrix} \to \begin{pmatrix} i & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the row-echelon form of $-i\mathbf{I}-\mathbf{A}$ is the complex conjugate of the row echelon form of $i\mathbf{I}-\mathbf{A}$ and the vectors in the nullspace of $-i\mathbf{I}-\mathbf{A}$ are the complex conjugates of the vectors in the nullspace of $i\mathbf{I}-\mathbf{A}$. Hence,

$$\text{Eigenspace of } -i = \mathrm{NS}\left(-i\mathbf{I}-\mathbf{A}\right) = \text{Span}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

Eigenspace of 1: $\mathrm{NS}(\mathbf{I}-\mathbf{A})$.

We start by reducing $\mathbf{I}-\mathbf{A}$:

$$\mathbf{I}-\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives the eigenspace

$$\text{Eigenspace of } 1 = \mathrm{NS}\left(\mathbf{I}-\mathbf{A}\right) \quad = \quad \text{Span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(You could also have guessed the solution $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and plugged it into $\mathbf{A}\mathbf{v}$ to check that indeed it is an eigenvector of eigenvalue 1.)

Conclusion : The eigenvalues and corresponding eigenspaces of the matrix $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are:

$$\begin{array}{rcl} \textbf{Eigenvalue} & & \textbf{Corresponding eigenspace} \\ \lambda = i & ; & \text{Span} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \\ \lambda = -i & ; & \text{Span} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \\ \lambda = 1 & ; & \text{Span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

(The eigenvectors for each eigenvalue are all vectors in the corresponding eigenspace.)

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