Outer Measure of the complement of a Vitali Set in [0,1] equal to 1

Asked 7 years, 9 months ago Active 1 month ago Viewed 2k times

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I am trying to prove the first part of exercise 33, ch. 1 in Stein and Shakarchi (*Real Analysis*). I am running into some difficulties following the hint though. Here is the problem (note, N is a Vitali set constructed in $[0, 1]$):

Show that the set $[0, 1] - N$ has outer measure $m_*(N^c) = 1$. [Hint: argue by contradiction, and pick a measurable set such that $N^c \subset U \subset [0, 1]$ and $m_*(U) \leq 1 - \epsilon$.

I know that both N and its complement are not measurable, so neither are countable. I know that measurable subsets of non-measurable sets have measure 0. I am not sure how to proceed given the proof though. A point to note: the book does not work with the inner measure at all, and even though I have used the inner measure in a previous course, I do not think I am allowed to for this proof.

real-analysis measure-theory

edited Sep 30 '12 at 23:08
user940

asked Sep 30 '12 at 22:16
joe
71 2

2 You wrote something false. Take a Vitali set in $[0, \frac{1}{2}]$, union the interval $[\frac{1}{2}, 1]$. This set is certainly non-measurable, but it has a measurable subset of measure $\frac{1}{2}$. It is true, however, that the Vitali set has inner measure zero. – Asaf Karagila ♦ Sep 30 '12 at 22:30

One way to see that Vitali set have inner measure zero is Lemma 2 in here – leo Oct 1 '12 at 22:56

2 Answers

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Recall the key property of the Vitali set N is that the sets $N \oplus q_i$, where $\oplus$ is addition mod 1 and $\mathbb{Q} = \{q_1, q_2, \ldots\}$, are disjoint. (Not sure if this is the same notation that S&S use.)

Now, consider the sets $U^c \oplus q_i$. Show that they are disjoint measurable sets and each has the same measure as U^c, which is greater than ϵ. Therefore, what can you say about the measure of $V = \bigcup_i U^c \oplus q_i$? This should give you a contradiction.

answered Jan 8 at 6:18
Nate Eldredge
79.7k 11 100 201

Hint: By Theorem 3.2, $m_*(I) = m_*(U) + m_*(U^c)$.