2. Review of Fundamentals

Setup:

For all problems on this page, let $X_1, \ldots, X_n \sim X$ be i.i.d. **standard normal** variables.

Square of a standard normal: Warmup

1.0/1.0 point (graded)

What is the mean $\mathbb{E}\left[X^2\right]$ and variance $\mathsf{Var}\left[X^2\right]$ of the random variable X^2?

$\mathbb{E}\left[X^2\right] =$ 1 **Answer:** 1

$\mathsf{Var}\left[X^2\right]$ 2 **Answer:** 2

STANDARD NOTATION

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Sum of squares of standard normal variables

2.0/2.0 points (graded)

Recall that $X_1, \ldots, X_n$ are i.i.d. standard normal variables. Denote by A_n the sample mean of the **squares** of these variables:

$$A_n := \overline{X_n^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{1}{n}\left(X_1^2 + X_2^2 + \ldots + X_n^2\right).$$

What kind of distribution does $nA_n = \left(X_1^2 + X_2^2 + \ldots + X_n^2\right)$ follow?

- ○ normal
- ○ nonparametric
- ○ Cauchy
- ○ Student t
- ○ Chi squared with 1 degrees of freedom
- ● Chi squared with n degrees of freedom
- ○ Gamma
- ○ Beta
- ○ Binomial
- ○ unknown

✔

What is the mean and variance of A_n ($A_n\overline{X_n^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$)? (Note that you are not asked about $n\overline{X_n^2}$; there is no factor n in front.)

$\mathbb{E}\left[\overline{X_n^2}\right] =$ 1 **Answer:** 1

1

$\text{Var}\left[\overline{X_n^2}\right] =$ 2/n **Answer:** 2/n

$\frac{2}{n}$

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Law of Large Numbers

1/1 point (graded)

Does $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ converge in probability to a constant a? If yes, enter the value of a below; if no, enter “DNE".

$A_n \xrightarrow{P} a$ for

$a =$ 1 **Answer:** 1

1

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Central Limit Theorem

1/1 point (graded)

Recall $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. For very large n, the distribution of $\sqrt{n}\,(A_n - a)$ is approximated best by ...

(In the choices below, the parameter for the normal distributions $\mathcal{N}\,(\mu, \sigma^2)$ are the mean μ and the variance σ^2.)

- ( ) $\mathcal{N}\,(0, 1)$
- (•) $\mathcal{N}\,(0, 2)$
- ( ) $\mathcal{N}\,(0, n)$
- ( ) $\mathcal{N}\,(0, 2n)$
- ( ) χ_n^2

✔

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Approximation via Central Limit Theorem

1.0/1.0 point (graded)

Recall $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Use the CLT and the fact that $q_{0.05}(A_1) = 3.84$ to approximate the 0.95-quantile $q_{0.05}(A_n)$ of the random variable A_n for sample sizes $n = 100$ and $n = 10^6$.

(Recall the $1-\alpha$ quantile $q_\alpha(Y)$ of a variable Y is defined by $P(Y > q_\alpha(Y)) = \alpha$.)

(Enter an answer accurate to at least 3 decimal places.)

$q_{0.05}(A_{100}) =$ 1.232617 ✔ **Answer:** 1.284

$q_{0.05}(A_{10^6}) =$ 1.002326 ✔ **Answer:** 1.00284

Solution:

By the CLT, when n is large, $A_n - 1$ is approximated by $\mathcal{N}(0, 2/n)$. Using only $q_{0.05}(A_1) = 3.84$ where $A_1 = X^2$ is a χ^2-variable and the CLT, we get approximations of $q_{0.05}(A_100)$ and $q_{0.05}(A_1000000)$ to be

$$\begin{aligned} q_{0.05}(A_100) &= 1 + \frac{q_{0.05}(A_1) - 1}{\sqrt{100}} = 1.284 \\ q_{0.05}(A_100) &= 1 + \frac{q_{0.05}(A_1) - 1}{\sqrt{1000000}} = 1.00284 \end{aligned}$$

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Continuous Mapping Theorem

1/1 point (graded)

Recall $A_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Define a sequence of random variables $B_n = e^{A_n}$.

Does the sequence of random variables $B_n = e^{A_n}$ converge in probability to a constant b? If yes, enter the value of b below; if no, enter "DNE".

(Enter **e** for the constant e.)

$B_n \xrightarrow{P} b =$ e

e

✔ **Answer:** e

STANDARD NOTATION

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Delta method

2.0/2.0 points (graded)

As above, let a be the limit in probability of A_n, i.e. $A_n \xrightarrow{P} a$, and b be the limit in probabiity of $B_n = e^{A_n}$, i.e. $B_n \xrightarrow{P} b$, if these limits exist.

Does the sequence of random variables $\sqrt{n}\,(B_n - b)$ converge in distribution? Choose the correct characterization of the limit distribution:

- ○ $\mathcal{N}\,(0, 1)$
- ○ $\mathcal{N}\,(0, e^b \mathsf{Var}\,(X^2))$
- ○ $\mathcal{N}\,(0, e^a \mathsf{Var}\,(X^2))$
- ● $\mathcal{N}\,(0, e^{2a} \mathsf{Var}\,(X^2))$
- ○ $\mathcal{N}\,(0, e^2 \mathsf{Var}\,(A_n))$

Does not converge in distribution

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Answers are displayed within the problem

Cochran's Theorem

1/1 point (graded)

Let

$$S_n := A_n - \left(\overline{X}_n\right)^2 = \overline{X_n^2} - \left(\overline{X}_n\right)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2.$$

Using the fact that

$$P\left(-0.3 < \overline{X}_9 < 0.3\right) = 0.63 \quad \text{and} \quad P\left(0.9 < S_9 < 1.1\right) = 0.15,$$

Can $P\left(-0.3 < \overline{X}_9 < 0.3,\ 0.9 < S_9 < 1.1\right)$ be determined? If yes, enter the value below; if no, enter **DNE**.

$P\left(-0.3 < \overline{X}_9 < 0.3,\ 0.9 < S_9 < 1.1\right) =$ 0.0945 **Answer:** 0.0945

0.0945

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