15. Examples of diagonalization

Steps to diagonalize an $n \times n$ matrix $\mathbf{A}$:

Step 1. Find the eigenvalues and eigenvectors of $\mathbf{A}$.

Step 2. Check that there are enough linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to form a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$).

If yes, set

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{pmatrix} \qquad \text{where } \mathbf{v}_i \text{ corresponds to } \lambda_i.$$

Important: The eigenvectors must be listed in the same order as their eigenvalues.

If any of the eigenspaces is deficient, there will not be enough linearly independent eigenvectors, and $\mathbf{A}$ is **not** diagonalizable.

Step 3. . Write $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.

Example 15.1 Let us diagonalize the matrix $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

Step 1. The eigenvalues and eigenvectors of $\mathbf{A}$:

$$\begin{array}{ccc} \text{Eigenvalue} & & \text{Eigenvectors (basis of eigenspace)} \\ \lambda = 0 & ; & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}$$

$$\lambda = -2 \quad ; \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Step 2. The eigenvalues are of multiplicity 1, so the matrix is complete. Indeed, the two eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^2$. Set

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

where the first column of $\mathbf{S}$ is the eigenvector of the eigenvalue $0,$ listed in the first diagonal entry of $\mathbf{D},$ and the second column of $\mathbf{S}$ is the eigenvector of $-2,$ listed in the second diagonal entry in $\mathbf{D}.$

Step 3. Write $\mathbf{A} = \mathbf{SDS}^{-1}$. If desired, compute $\mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ to get the explicit formula:

$$\mathbf{A} = \mathbf{SDS}^{-1}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Problem 15.2 Diagonalize $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$

Solution

Show

Concept Check 1

1/1 point (graded)

Suppose that $\mathbf{S}$ is a matrix each one of whose columns is an eigenvector of a square matrix $\mathbf{A}$. Assume also that $\mathbf{S}$ is square. Now, if the columns of $\mathbf{S}$ are linearly independent, then:

- [ ] $\mathbf{A}$ is invertible
- [x] $\mathbf{S}$ is invertible ✔
- [x] $\mathbf{A}$ is diagonalizable ✔
- [ ] $\mathbf{S}$ is diagonalizatble

✔

Solution:

If the columns of $\mathbf{S}$ are independent, then it has full rank, hence it is invertible. Also, if $\mathbf{S}$ is invertible, then the matrix $\mathbf{A}$ has a diagonalization.

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You have used 1 of 3 attempts

Answers are displayed within the problem

Concept Check 2

0/1 point (graded)

If the eigenvalues of $\mathbf{A}$ are $2,\ 2,$ and $5,$ then $\mathbf{A}$ is:

- [ ] invertible ✔
- [ ] diagonalizable
- [x] not diagonalizable

✖

Solution:

The determinant of $\mathbf{A}$ is the product of the eigenvalues, which is not equal to zero. Thus $\mathbf{A}$ is invertible. However, we do not know if $\mathbf{A}$ is diagonalizable or not because there is a repeated eigenvalue. If the eigenspace is 2 dimensional, then $\mathbf{A}$ is diagonalizable; otherwise $\mathbf{A}$ is not.

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Concept Check 3

1/1 point (graded)

If the only eigenvector of $\mathbf{A}$ is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, then $\mathbf{A}$ has:

- [ ] no inverse
- [x] a repeated eigenvalue ✔
- [x] no diagonalization ✔

✔

Solution:

The matrix $\mathbf{A}$ must have a repeated eigenvalue, and a deficient eigenspace. This implies that there is no diagonalization.

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