# 10. Concavity in higher dimensions and Eigenvalues

## Concavity in 2 dimensions: Compute the Hessian

4/4 points (graded)

What is the Hessian $\mathbf{H}f$ of the function $f(x,y) = -2x^2 + \sqrt{2}xy - \frac{5}{2}y^2$? Fill in the values of the entries of $\mathbf{H}f$.

$(\mathbf{H}f)_{11} =$ -4 ✔ **Answer:** -4 $(\mathbf{H}f)_{12} =$ sqrt(2) ✔ **Answer:** sqrt(2)

$(\mathbf{H}f)_{21} =$ sqrt(2) ✔ **Answer:** sqrt(2) $(\mathbf{H}f)_{22} =$ -5 ✔ **Answer:** -5

**Solution:**

We compute that

$$(\mathbf{H}f)_{11} = \frac{\partial^2 f}{\partial \lambda^2} = -4, \qquad (\mathbf{H}f)_{12} = \frac{\partial^2 f}{\partial \lambda \partial y} = \sqrt{2}$$
$$(\mathbf{H}f)_{21} = \frac{\partial^2 f}{\partial \lambda \partial y} = \sqrt{2}, \qquad (\mathbf{H}f)_{22} = \frac{\partial^2 f}{\partial y^2} = -5.$$

So this implies that

$$\mathbf{H}f = \begin{pmatrix} -4 & \sqrt{2} \\ \sqrt{2} & -5 \end{pmatrix}.$$

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Answers are displayed within the problem

## (Optional) Concavity in 2 dimensions: Positive Definiteness and Eigenvalues

0 points possible (ungraded)

A symmetric (real-valued) $d \times d$ matrix $\mathbf{A}$ is **positive semi-definite** (*resp.* **positive definite**) if and only if all of its eigenvalues are **non-negative** (*resp.* **positive**).

Analogously, it is **negative semi-definite** (*resp.* **negative definite**) if and only if all of its eigenvalues are **non-positive** (*resp.* **negative**).

As above, consider $f(x,y) = -2x^2 + \sqrt{2}xy - \frac{5}{2}y^2$.

What are the eigenvalues $\lambda_1, \lambda_2$ of $\mathbf{H}f$? Assume that $\lambda_1 < \lambda_2$.

$\lambda_1 =$ -6 ✔ **Answer:** -6 $\lambda_2 =$ -3 ✔ **Answer:** -3

Based on your answer to the last question, $f$ is ...

- ○ Convex
- ◉ Concave

None of the Above

**Solution:**

Recall from the previous problem that the Hessian of $f$ is

$$\mathbf{H}f = \begin{pmatrix} -4 & \sqrt{2} \\ \sqrt{2} & -5 \end{pmatrix}.$$

To find the eigenvalues, we need to solve for $\lambda$ such that

$$\det\left(\mathbf{H}f - \lambda I\right) = \det\left(\begin{pmatrix} -4-\lambda & \sqrt{2} \\ \sqrt{2} & -5-\lambda \end{pmatrix}\right) = \lambda^2 + 9\lambda + 18 = 0.$$

Factoring the quadratic: $\lambda^2 + 9\lambda + 18 = (\lambda + 6)(\lambda + 3)$ shows that $\lambda_1 = -6$ and $\lambda_2 = -3$.

The function $f$ is twice-differentiable, so it is concave if $x^T \mathbf{H}f x \leq 0$ for all $x \in \mathbb{R}^2$. By the remark in the problem statement, this is equivalent to all of the eigenvalues of $\mathbf{H}f$ being negative. Hence, $f$ is concave (in fact it is *strictly* concave).

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Typo in solution to the first problem 1

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