STAT 141 **10/21/04**

Reading for next time: Chapter 6-7.

Thur. 11 Nov. Midterm 6.30-8.30.

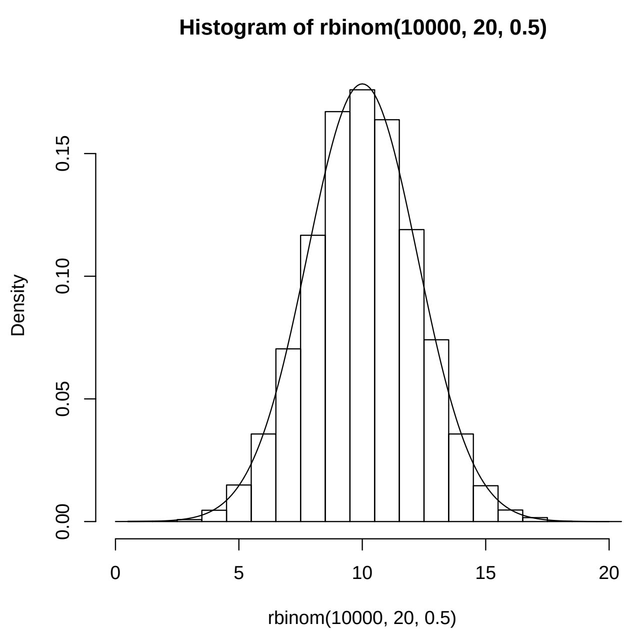
- Normal Approximation to the Binomial.
- Confidence Intervals: intuition and graphics.
- Confidence Intervals: formulas.

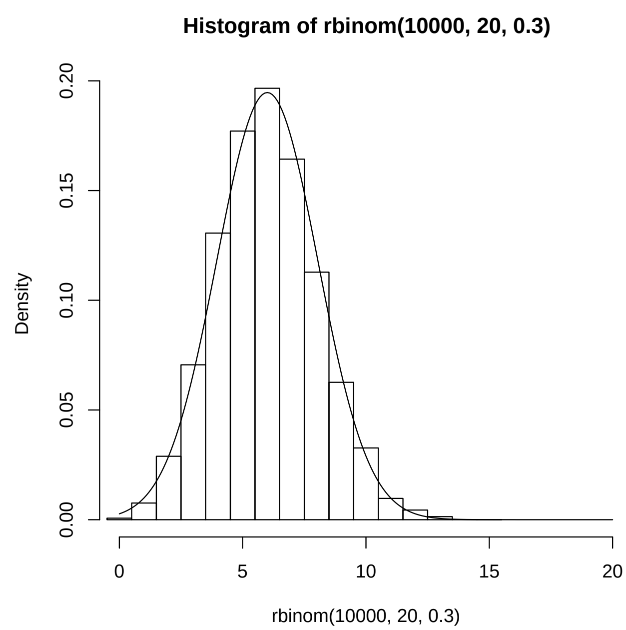
Normal Approximation to the Binomial

1. Sum of many independent 0/1 components with probabilities equal p (with n large enough such that $npq \geq 3$), then the binomial number of success in n trials can be approximated by the Normal distribution with mean $\mu = np$ and standard deviation $\sqrt{np(1-p)}$.
2. For n large, the sampling distristribution of $\hat{p}$ can be approximated by a normal distribution with mean=p and standard deviation $\sqrt{\frac{p(1-p)}{n}}$.

```
hist(rbinom(10000,20,0.5),xlim=c(0,20),
   probability=T,breaks=seq(0.5,20.5,1))
lines(seq(0,20,0.1),dnorm(seq(0,20,0.1),
         10,sqrt(5)))
```

```
#Non symmetric binomial
hist(rbinom(10000,20,0.3),xlim=c(0,20),
    probability=T,breaks=seq(-0.5,15.5,1))
 lines(seq(0,20,0.1),dnorm(seq(0,20,0.1),
          6,sqrt(4.2)))
```





Continuity Correction:

$$P(a \leq X \leq b) \simeq P(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}} \leq Z \leq \frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}})$$

"Statisticians are the only people who insist on being wrong 5% of the time"

CONFIDENCE INTERVALS (S& W Chap 6)
Confidence interval for unknown μ (with known σ)
Interpretation of C.I.- repeated sampling and the confidence stack
What a confidence interval depends on: C, n and σ
Choice of sample size

Two Remarks to complement the last lecture on normal approximation and CLT:

1. Example: Consider incomes in town, where $\mu = 39.97$ and $\sigma = 13.75$: X_1 NOT normal.
Sample, n=50 ,$P(\bar{X}_{50} \geq 44)$?
$\bar{X}_{50} \sim \mathcal{N}(39.97, \frac{13.75}{\sqrt{50}})$

$\bar{X}_{50}$ is approximately normally distributed with mean around 40 and sd 1.94,

$$P = P(\bar{X}_{50} \geq 44) = P(\frac{\bar{X}_{50} - 40}{1.94} > \frac{44 - 40}{1.94}) \simeq P(Z > 2.06) = 2\%$$

2. Remark. Adding independent variables brings the sum closer to being normal.

Hence, if you start at the normal, you should stay there!

If $X \sim N(\mu, \sigma^2)$ then $\bar{X} = \frac{X_1+X_2 \cdots X_n}{n} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$ exactly.

More generally, if X and Y are normal, independent, then aX+bY Normal

for any constants a, b (— a *linear combination*). What are the mean & variance of *aX+bY* ?

Typical poll says “support for Bush is 52% with margin of error of 4%” This is an example of a confidence interval.

C.I.’s are one of the strangest animals in the statistical zoo, and one has to be careful with their interpretation. There has been quite a lot of philosophical debate about them, but neverthess they remain a very useful tool for assessing the accuracy of estimates.

CONFIDENCE INTERVAL Estimate +/- Margin of Error: E +/- M

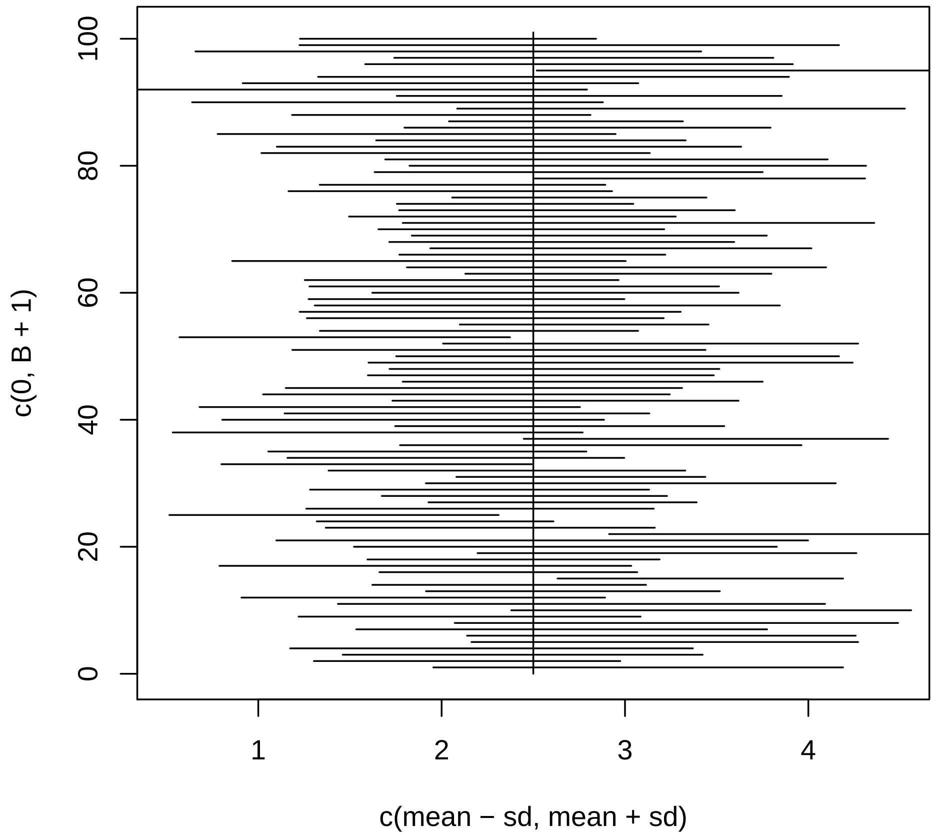
2 key components:

1) interval (E-M, E+M) (with estimate E at center)

2) confidence **level** C 95%, 99% or other

C = Probability that *the method* yields an interval containing the true value (of the unknown parameter).

The confidence stack: Imagine drawing lots of samples – each generating a 95% C.I.




```
lower=rep(0,25)
upper=rep(0,25)
meanx=rep(0,25)
stdex=rep(0,25)
plot(c(0,5),c(0,26),type='n')
for ( i in (1:25)){
samplex=rnorm(15,2.5,2)
meanx[i]=mean(samplex)
stdex[i]=sqrt(var(samplex)/15)
lower[i]=meanx[i]-1.96*stdex[i]
upper[i]=meanx[i]+1.96*stdex[i]
lines(c(lower[i],upper[i]),c(i,i))  }
lines(c(2,2),c(0,26))
```

```
cis=function(n=15,mean=2.5,sd=2,B=25){
lower=rep(0,B)
upper=rep(0,B)
meanx=rep(0,B)
stdex=rep(0,B)
plot(c(mean-sd,mean+sd),c(0,B+1),type='n')
for ( i in (1:B)){
samplex=rnorm(n,mean,sd)
meanx[i]=mean(samplex)
stdex[i]=sqrt(var(samplex)/n)
lower[i]=meanx[i]-1.96*stdex[i]
upper[i]=meanx[i]+1.96*stdex[i]
lines(c(lower[i],upper[i]),c(i,i))}
lines(c(mean,mean),c(0,B+1))  }
cis(B=100)
```

Some intervals do not overlap with the true value μ, the randomness comes from the sample chosen NOT the mean which has a fixed unknown value.

Examples:

a) C.I. for population mean μ , with **known** popn SD σ

b) C.I. for pop mean μ, unknown σ.

c) C.I. for difference in two means, unknown σ.

Preparation: Book's notation: z_α = location on standard normal curve with area $1 - 2\alpha$ under $(-z_\alpha, z_\alpha)$: quantiles

Conf. Interval for mean μ **, with** known σ

Suppose a random variable X has mean μ (unknown) and SD σ (known), and that we have n independent observations $x_1, x_2, \ldots, x_n$ of this r.v.

$$\text{A level C, or } 100(1-2\alpha)\% \text{ confidence interval for } \mu \text{ is } [\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}, \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}]$$

The interval is *"exact"* if X itself has a normal distribution *approximately correct* (by the CLT) for any X if n is *large* , usually we suppose $n > 20$.

Standard error of the sample mean (and other sample statistics)

If σ known, then SD of sample mean, $\sigma(\bar{x}) = \frac{\sigma}{\sqrt{n}}$, when σ is unkown, we use the estimated standard error of the mean:

$$s_{\bar{x}} = SE_{\bar{x}} = \frac{s}{\sqrt{n}}$$

The sample mean is an example of a *statistic T*, (a quantity derived from a sample of data, such as $\bar{x}$). Other examples of statistics include the sample standard deviation *s*, sample coefficient of variation *CV* sample skewness and kurtosis.

Warning about names for variability of random variables and statistics: Important to distinguish between the *population value* of the variability of a statistic, (which is generally unknown, since it depends on the whole population), and a *sample estimate* which is based on observed data from a probability sample.The latter is a random quantity (if we drew another sample, we would get a different estimate).

The term "*standard error*" is usually reserved for the SD of the sample mean The term "*standard error of T*" refers to the SD of a sample statistic *T.*

Example Confidence interval for the mean of IQs, for a population whose known variance is $\sigma^2 = 225 = 15^2$, Sample size n=50. $\bar{x} = 113.9$ observed mean. Special feature of IQs: normally distributed, and $\sigma = 15$ is known, so C=95%, $z_{\frac{\alpha}{2}} = 1.96$ margin of error $M = 1.96 \times 15/\sqrt{50} = 1.96 \times 2.12 = 4.2$

$$\text{95\% CI is } [113.9 - 4.2, 113.9 + 4.2] = [109.7, 118.1]$$

A level C, or $100(1-\alpha)$ % confidence interval for μ is

$$[\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}]$$

But to return to reality, we don't know σ. Thus we must estimate the standard deviation of $\bar{X}$ with:

$$SE_{\bar{X}} = \frac{s}{\sqrt{n}}$$

But s is just a function of our X_i's and thus is a random variable too – it has a sampling distribution too.

Before we could say if we knew σ

$$P(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}) = 1 - \alpha$$

which after algebra gave the confidence interval.

[Remember for any s, z_s is **defined** as where $1 - 2s$ of the area falls in $(-z_s, z_s)$. So $z_s = \texttt{qnorm}(1-s) = -\texttt{qnorm}(s) = 1 - s$ quantile. i.e. z_s is the positive side.]

Now we want a similar setup, so that:

$$P(?? < \frac{\bar{X} - \mu}{SE_{\bar{X}}} < ??) = \alpha$$

We need know the probability distribution of $T = \frac{\bar{X} - \mu}{SE_{\bar{X}}}$. T has the Student's t-distribution with $n - 1$ degrees of freedom. We write this as $T \sim t_{n-1}$. The degrees of freedom=ν is the only parameter of this distribution.