5. Vector spaces

The geometry of vectors

Our ability to visualize vectors in dimensions larger than 2 or 3 limits our ability to draw images. But we continue to use our geometric intuition in lower dimensions to inform our algebraic treatment of higher dimensional vectors.

In $\mathbb{R}^2$ a vector has 2 entries.

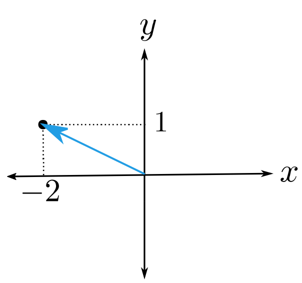
Example: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

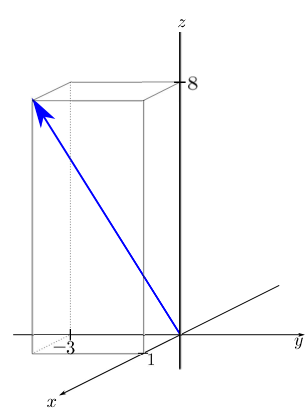
In $\mathbb{R}^3$ a vector has 3 entries.

Example: $\begin{pmatrix} 1 \\ -3 \\ 8 \end{pmatrix}$.

In $\mathbb{R}^N$ a vector has N entries.

Example: $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$.





A video refresher if needed: Vectors, what are they?

Vector spaces

We know that vectors can be added together,

$$\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\\1\\-1\end{pmatrix}=\begin{pmatrix}1-1\\2+1\\3-1\\4+1\\5-1\end{pmatrix}=\begin{pmatrix}0\\3\\2\\5\\4\end{pmatrix}$$

or they can be multiplied by a real (or complex) scalar

$$c\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}=\begin{pmatrix}c\\2c\\3c\\4c\\5c\end{pmatrix}$$

to obtain new vectors. A set of vectors is called **closed under addition** if the sum of any vectors is in the set. Similarly a set of vectors is **closed under scalar multiplication** if for every scalar c and vector $\mathbf{x}$ the vector $c\mathbf{x}$ is in the set.

Example 5.1 Solutions to homogeneous linear systems $\mathbf{A}\mathbf{x}=\mathbf{0}$ are closed under scalar multiplication and (vector) addition:

1. (Closed under scalar multiplication.) If $\mathbf{x}_1$ is a solution, then $\mathbf{A}c\mathbf{x}_1=c\mathbf{A}\mathbf{x}_1=c\mathbf{0}=0$ implies that $c\mathbf{x}_1$ is also a solution.

2. (Closed under addition.) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions then

$$\mathbf{A}(\mathbf{x}_1+\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1+\mathbf{A}\mathbf{x}_2=\mathbf{0}+\mathbf{0}$$

implies that $\mathbf{x}_1+\mathbf{x}_2$ is also a solution.

The notion of a **vector space** captures these properties.

Definition 5.2 A **vector space** (of vectors) over the real (or complex) numbers is a set V such that

0. The zero vector $\mathbf{0}$ is in V.

1. Multiplying any one vector $\mathbf{v}$ in V by a real (or complex) scalar c gives another vector $c\mathbf{v}$ which is also in V.

2. Adding any two vectors $\mathbf{v}$ and $\mathbf{w}$ in V gives another vector $\mathbf{u} = \mathbf{v} + \mathbf{w}$ also in V.

More examples

- The set of real numbers $\mathbb{R}$ can be thought of as a vector space.

- The plane $\mathbb{R}^2$ is the vector space given by the set

$$\mathbb{R}^2 = \{\text{all} \quad \mathbf{v} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{with} \quad c_1, c_2 \text{ real numbers}\}.$$

- A line L through the origin in the plane is a vector space. This equals the set of all scalar multiples of one nonzero vector $\begin{pmatrix} a \\ b \end{pmatrix}$:

$$L = \{\text{all } c \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{with} \quad c \text{ any real number }, a, b \text{ fixed}\}.$$

- The vector space $\mathbb{R}^n$ is the set of all vectors $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ where $c_1, \ldots, c_n$ are real numbers.

- The complex vector space $\mathbb{C}^2$ is the set of all vectors with complex entries:

$$\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \text{ such that } z_1, z_2 \text{ are complex numbers}\}.$$

Nonexample:

The set in $\mathbb{R}^2$ consisting of the single point $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ containing a single point is not a vector space. It fails all three conditions.

0. The point $\begin{pmatrix}0\\0\end{pmatrix}$ is not in the set.

1. The scalar multiple $-1\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$ is not in the set.

2. The sum $\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$ is not in the set.

Vector spaces of functions

1/1 point (graded)

One can also define vector spaces of functions. Here functions play the role of vectors. For example, the constant function that is zero everywhere plays the role of the zero vector.

Which of the following sets of functions are vector spaces? (Choose all that apply.)

(Note that some of the options make use of operator notation for differential equations.)

- [x] The set of polynomials of degree less than or equal to n with real coefficients $\{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \textbf{ where } a_0, \ldots, a_n \textbf{ are real numbers}\}$. ✔
- [ ] The set of polynomials of degree n with real coefficients and constant term equal to 1 $\{1 + a_1x + a_2x^2 + \cdots + a_nx^n \textbf{ where } a_1, \ldots, a_n \textbf{ are real numbers}\}$.
- [x] The set of solutions to a homogeneous differential equation $P(D)x = 0$. ✔
- [ ] The set of solutions to an inhomogeneous differential equation $P(D)x = e^{rt}$.
- [x] The set consisting only of the zero function. ✔

✔

Solution:

Let's check the properties of being closed under scalar multiplication and addition of functions for each option.

- Yes, $\{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \text{ where } a_0, \ldots, a_n \text{ are real numbers}\}$ is a vector space. Scalar multiples of such polynomials and the sum of two such polynomials are still polynomials of degree not higher than n.

- No, $\{1 + a_1 x + a_2 x^2 + \cdots + a_n x^n \text{ where } a_1, \ldots, a_n \text{ are real numbers}\}$ is not a vector space. Multiplying any of these polynomials by a scalar $a \neq 1$ gives a polynomial that is no longer in this set. Another reason is that the constant function 0 is not in this set.

- Yes, $\{\text{functions } x \text{ such that } P(D)x = 0\}$ is a vector space. If x_1 and x_2 are two solutions, then $P(D)(x_1 + x_2) = P(D)x_1 + P(D)x_2 = 0$, so $x_1 + x_2$ is also a solution. Similarly for cx_1 where c is a real number.

- No, $\{\text{functions } x \text{ such that } P(D)x = e^{rt}\}$ is not a vector space. Solutions to inhomogeneous differential equations do not form a vector space because the zero function cannot be a solution to $P(D)0 = f$ if f is nonzero.

- Yes, $\{\text{the zero function } 0\}$ is a vector space. The zero function times any constant is zero. And zero plus zero is zero.

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You have used 1 of 5 attempts

Answers are displayed within the problem

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is it a vector space?
If line doesn't go through the origin in the plane - is this line a vector space? My guess is not, since (0,0) v... 2

A small note/question on the vector space definition 5.2
Point 0. seems to be an immediate consequence of point 1., so I was wondering why 0. was included in t... 5

Vector spaces of functions
I am unable to understand the reason as to why the set of polynomials of degree n with 1 as constant co... 11

Vector spaces of functions
Shouldn't the question be phrased as, "Which of the following sets of functions are **real** vector space... 2

Vector space: single points in 1D and one single point in 2D 4

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