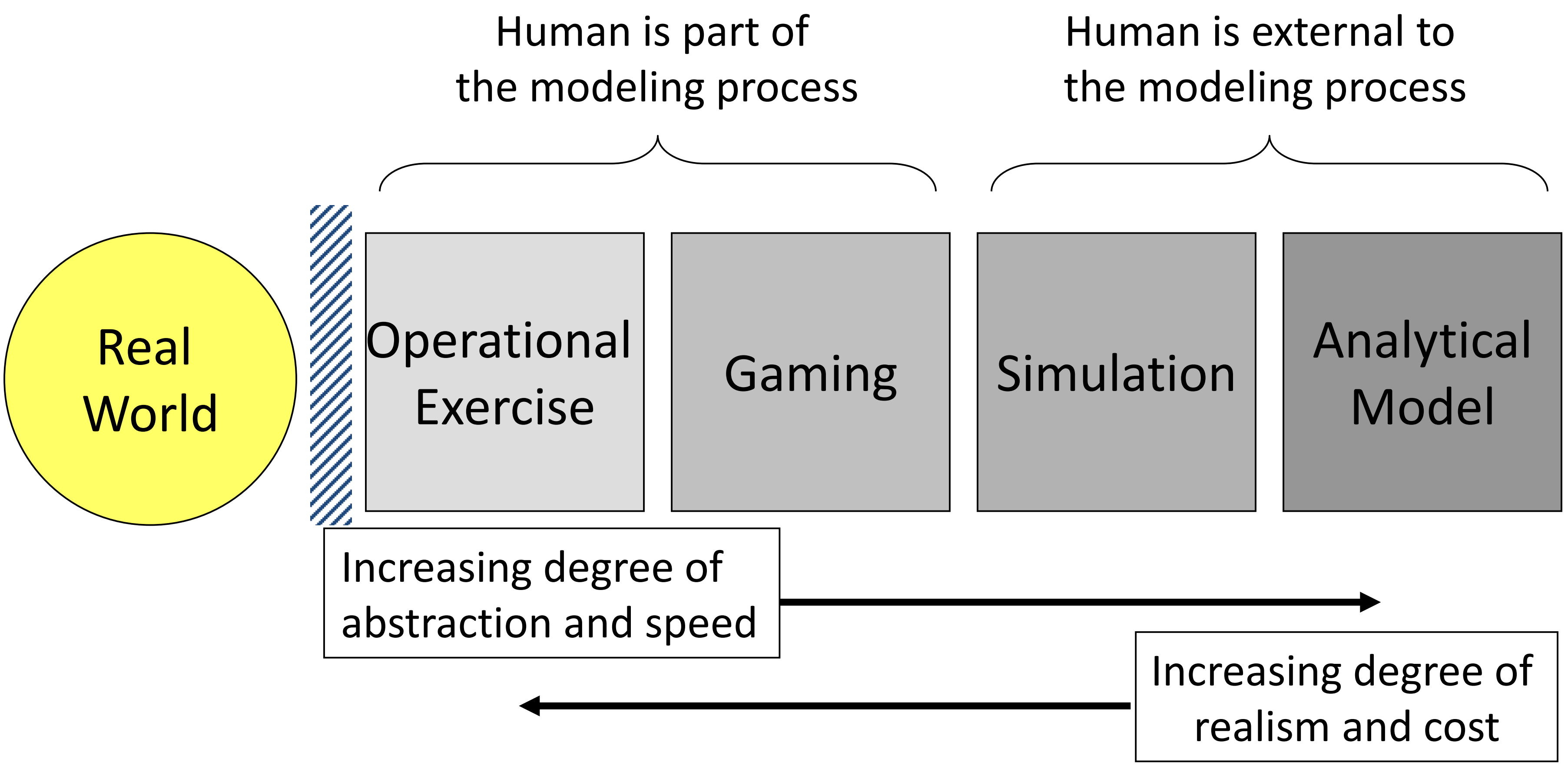
Analytics Basics: Models, Algebra, & Functions

Decision making is at the core of supply chain management.

- How many facilities should I open and where?
- What transportation option should I use?
- How should I trade-off service and cost?
- Where should I source my raw material from?
- How should I share risk with my customers/suppliers?
- How much inventory should I have?
- What is my demand for next year?
- How can I make my supply chain more resilient?

Analytical models are used to make supply chain decisions

Model Classification



Classification of Models

	Strategy Evaluation	Strategy Generation
Certainty	Deterministic Simulation Econometric Models Systems of Simultaneous Equations Input-Output Models	Linear Programming Network Models Integer and MILP Non-Linear Programming Control Theory
Uncertainty	Monte-Carlo Simulation Econometric Models Stochastic Processes Queuing Theory Reliability Theory	Decision Theory Dynamic Programming Inventory Theory Stochastic Programming Stochastic Control Theory

Source: Bradley, Hax, Magnanti 1977

Categories of Mathematical Models

Model Category	Functional Form $f(\cdot)$	Independent Variables	OR/MS Techniques
Descriptive What has happened?	known, well-defined	unknown or uncertain	Simulation, PERT, Queueing Theory, Inventory Models
Predictive What could happen?	unknown, ill-defined	known or under decision maker's control	Regression Analysis, Time Series Analysis, Discriminant Analysis
Prescriptive What should we do?	known, well-defined	known or under decision maker's control	Classic Opt., LP, MILP, CPM, EOQ, NLP,

Source: Ragsdale, 2004

Roadmap for the Course

- Deterministic – Prescriptive Modeling
  - Basic functions & algebra
  - Classical optimization (calculus)
  - Math programming (LPs, IPs, MILPs, & Non-Linear)
- Stochastic/Uncertainty – Predictive & Descriptive
  - Basic probability and distributions
  - Statistical analysis (hypothesis testing)
  - Econometric modeling (regression)
  - Simulation

SCx Approach to Modeling
- Educating Drivers not Mechanics!

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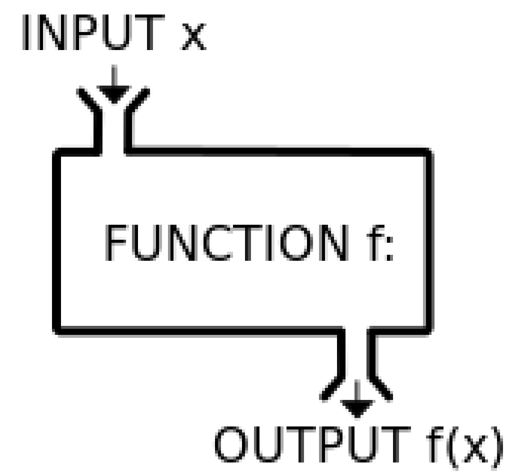
Mathematical Functions

Mathematical Functions

" . . . a relation between a set of inputs and a set of permissible outputs with the property that each input is related to <u>exactly one output</u>."

source: Wikipedia



$$y = f(x)$$

we say:
"f of x" or that "y is a function of x"

If given a value for x, then I can compute the value for y.

Example: $f(x) = x^2$

$x = 2$	then $y = f(2) = 2^2 = 4$
$x = 3.4$	then $y = f(3.4) = 3.4^2 = 11.56$
$x = -2$	then $y = f(-2) = (-2)^2 = 4$

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Linear Functions

Typically, constants are denoted by letters from the start of the alphabet (a, b, c, . . .) while variables are letters from the end of the alphabet (x, y, z).

"y changes linearly with x"

dependent variable

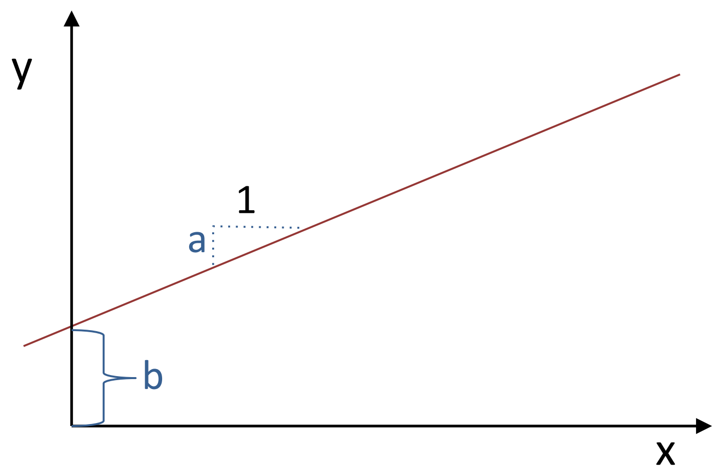
This is the function, f(x).

$$y = \overbrace{ax + b}$$

constant, fixed value, or y-intercept

slope or variable value

independent variable



Examples: Linear Functions

- Truckload Transportation Costs:

  cost = f(distance) = \$200 + 1.35 \$/km * (distance)

- Warehousing Costs

  cost = f(# cases) = €2,500 + 2.5 €/case * (# cases)

- Profit Equation

  profit = f(volume) = (r-c) * v + d

  where:

  r = revenue per item(¥/item)
  c= cost per item (¥/item)
  v = volume sold (items)
  d = fixed cost (¥)

Quadratic Functions

Quadratic Functions

Parabola - Polynomial function of degree 2
where a, b, and c are numbers and a≠0

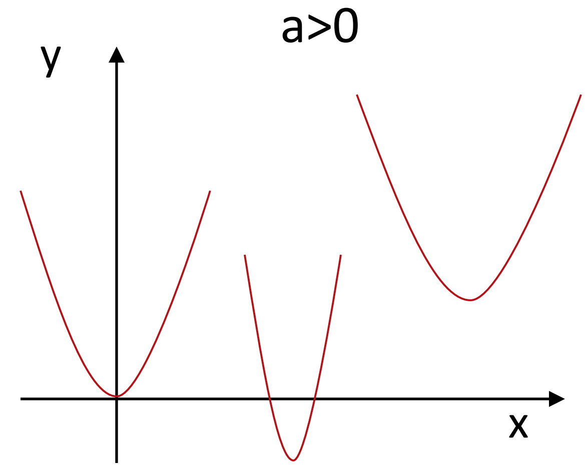
dependent variable

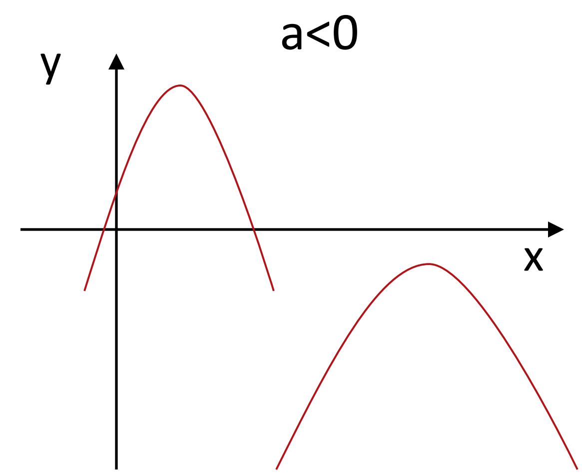
This is the function, f(x).

$$y = \underbrace{ax^2 + bx + c}_{f(x)}$$

independent variable

constants

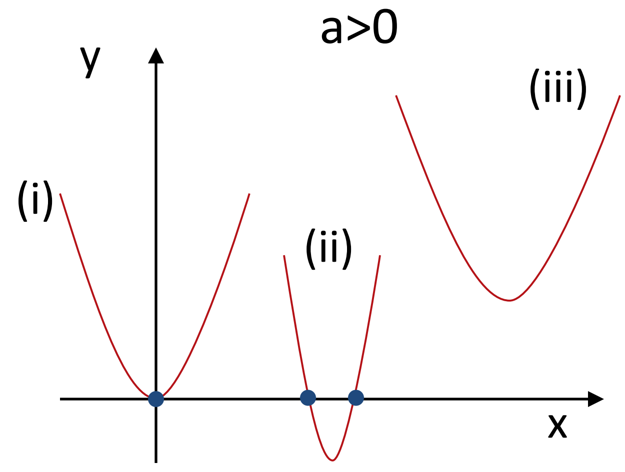




- When a>0, the function is convex (or concave up)
- When a<0, the function is concave down

Finding Roots of Quadratic

- The root(s) of a quadratic
  - Values of x for when y=0
  - There can be 2, 1, or 0 roots
- Two methods for finding roots
  - Factoring:
    - Find r_1 and r_2 such that $ax^2+bx+c = a(x-r_1)(x-r_2)$
  - Quadratic equation



(i) $y = 2x^2$
(ii) $y = 2x^2 - 6x + 4$
(iii) $y = 3x^2 - 4x + 2$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Finding Roots

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i) $y = 2x^2$ so that a=2, b=c=0

$$r_1, r_2 = \frac{-0 \pm \sqrt{0^2 - 4(2)(0)}}{2(2)} = \frac{0}{4} = 0$$

(ii) $y = 2x^2 - 6x + 4$ so that a=2, b=-6, c=4

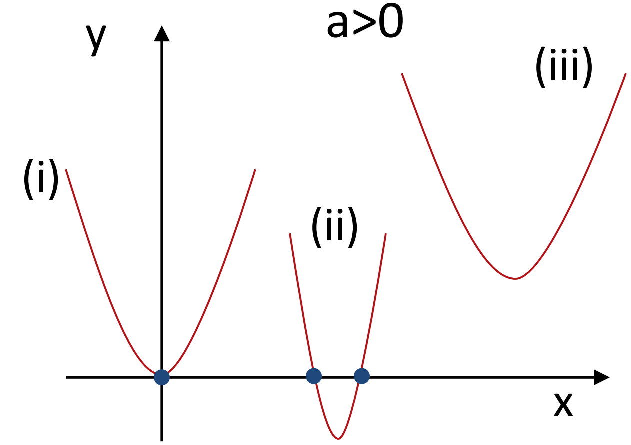
$$r_1, r_2 = \frac{6 \pm \sqrt{(-6)^2 - 4(2)(4)}}{2(2)} = \frac{6 \pm \sqrt{36 - 32}}{4} = \frac{6 \pm 2}{4}$$

$$r_1 = \frac{8}{4} = 2 \qquad r_2 = \frac{4}{4} = 1$$

(iii) $y = 3x^2 - 4x + 2$ so that a=3, b=-4, c=2

$$r_1, r_2 = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(2)}}{2(3)} = \frac{4 \pm \sqrt{16 - 24}}{6} = \frac{4 \pm \sqrt{-8}}{6}$$

r_1, r_2 are complex numbers

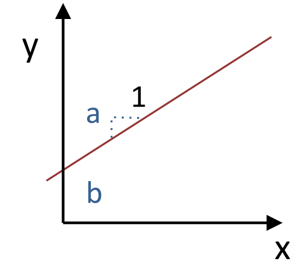


(i) $y = 2x^2$
(ii) $y = 2x^2 - 6x + 4$
(iii) $y = 3x^2 - 4x + 2$

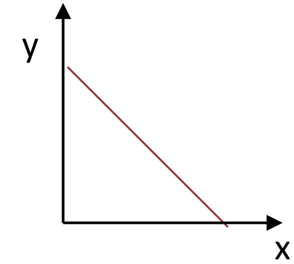
Quadratic Functions in Practice

Quadratic Functions in Practice

- Example: Manufacturing iWidgets – what price to set?
  - Cost of producing iWidgets is a linear function of the number produced, x:
    - cost =f(# made) = 500,000 + 75x



  - Demand for iWidgets is also a linear function of the price, p:
    - unit sales = f(price) = 20,000 – 80p



  - So then:
    - Revenue = (20,000-80p)p = 20,000p – 80p^2
    - Costs = 500,000+75(20,000-80p) = 2,000,000 – 6000p
    - Profit = Revenue – Costs =
      = 20,000p – 80p^2 – (2,000,000 – 6,000p)
      = -80p^2 + 26,000p – 2,000,000

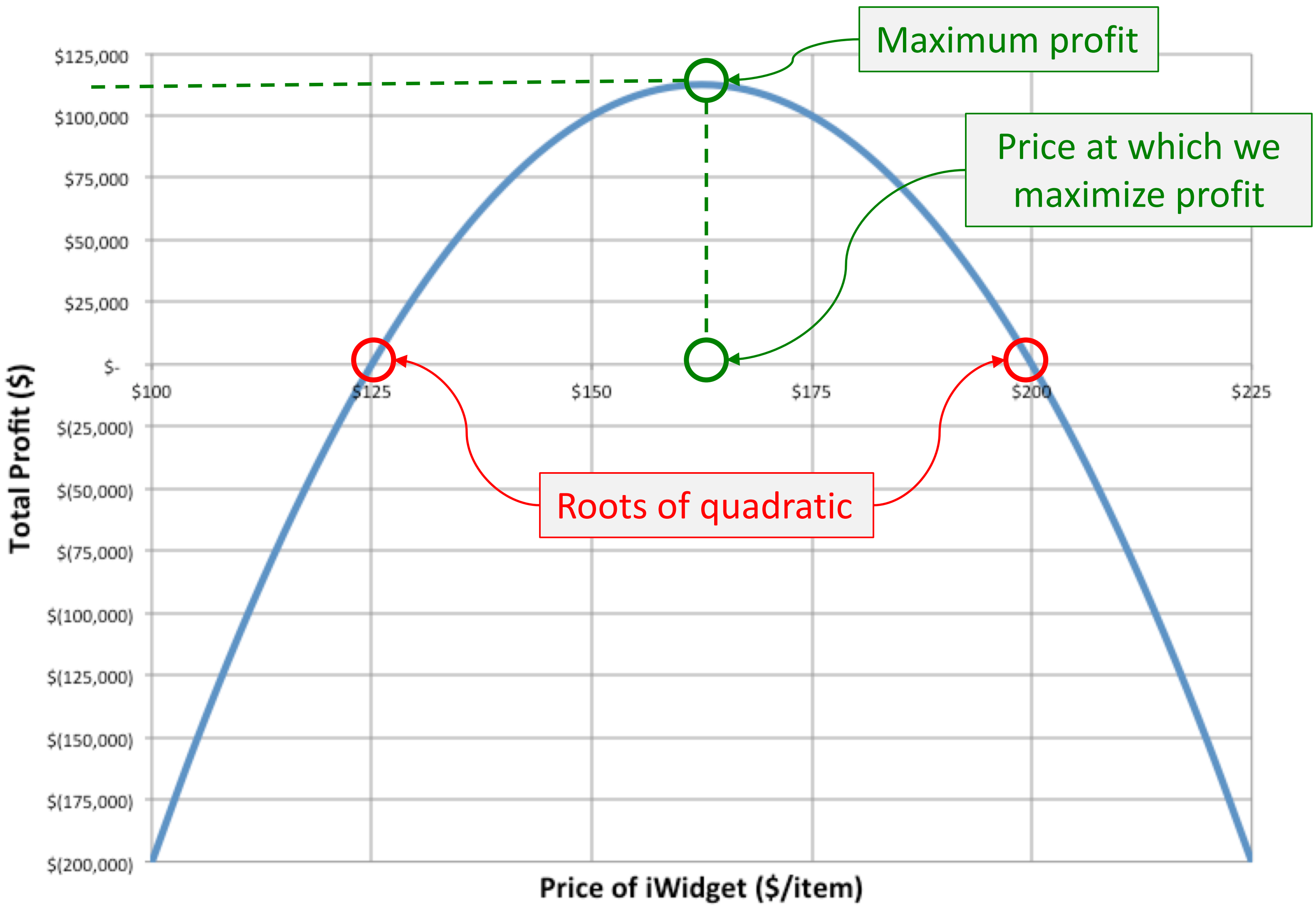
$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

r_1 = 125
r_2= 200

What are the root(s) of this equation?

iWidget Example

$\text{Profit} = -80p^2 + 26{,}000p - 2{,}000{,}000$

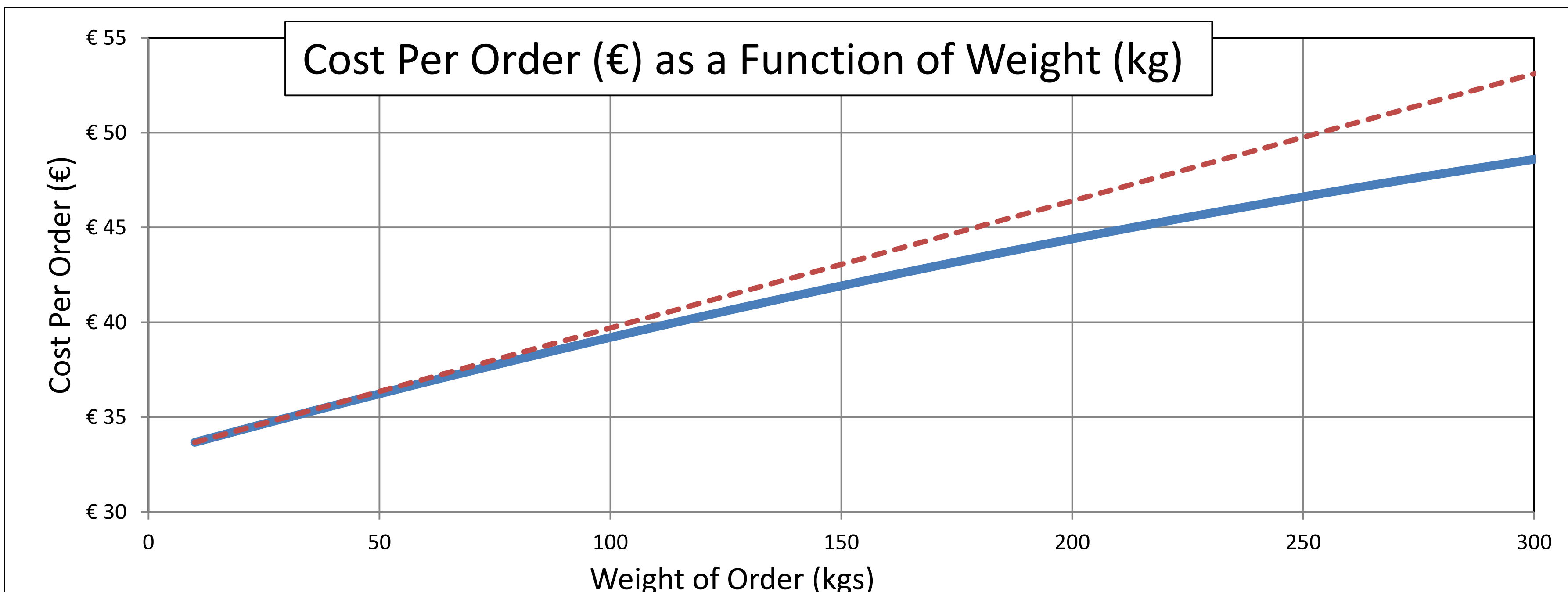


Quadratic Functions in Practice

- Example: Parcel Trucking – impact of weight

  Parcel carriers combine many orders into a single shipment. The cost of an individual order is a function of its weight, w. However, it is not linear – it is tapering.

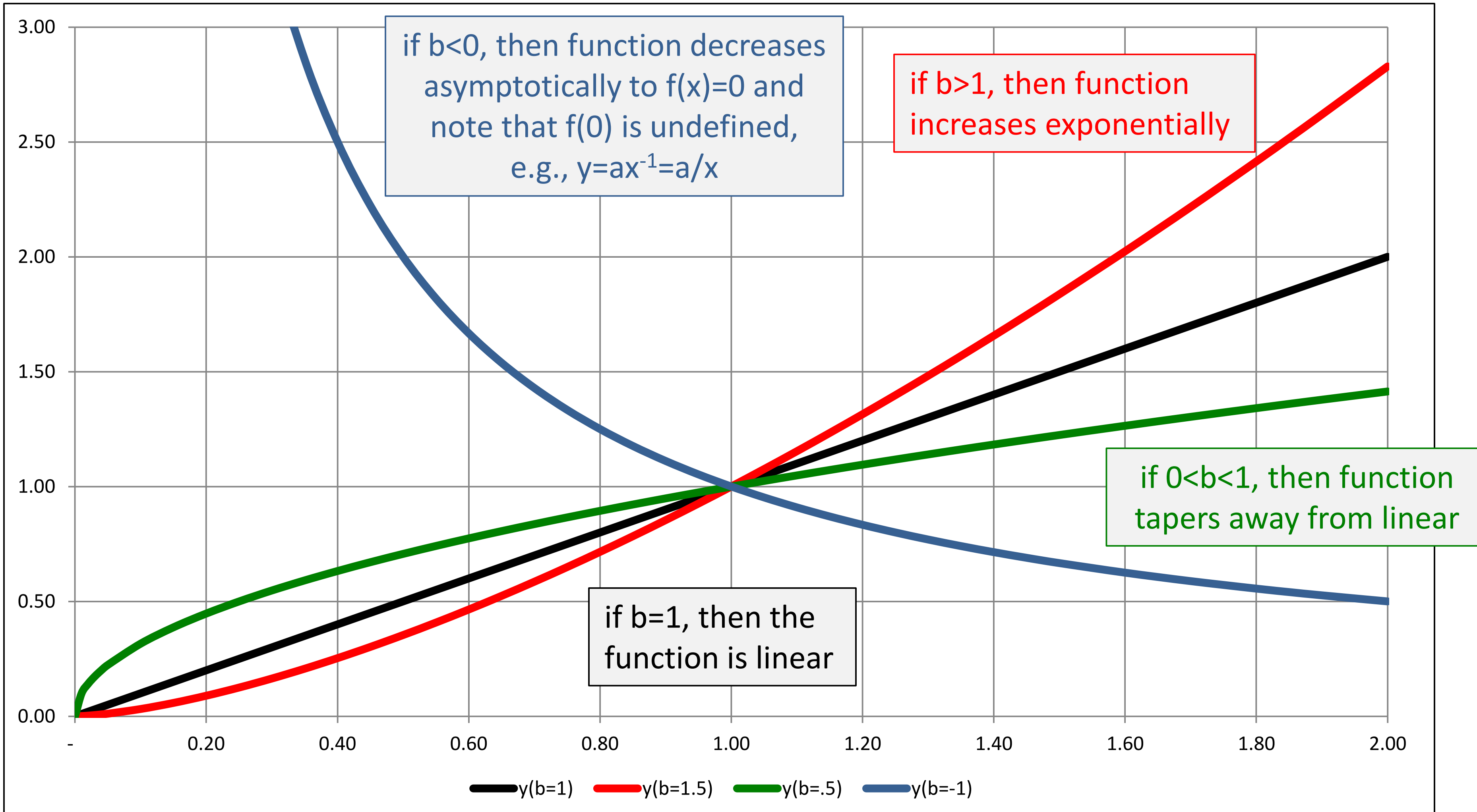
  $$\text{cost} = f(\text{weight}) = 33 + 0.067w - 0.00005w^2$$



Other Common Functional Forms

Power Function $y=f(x) = ax^b$

The shape of the curve is dictated by the value of b



Exponential Functions $y=ab^x$

- Exponential functions have very fast growth
- Widely used in physics, biology, economics, finance, computer science, etc.
- Example: compound interest

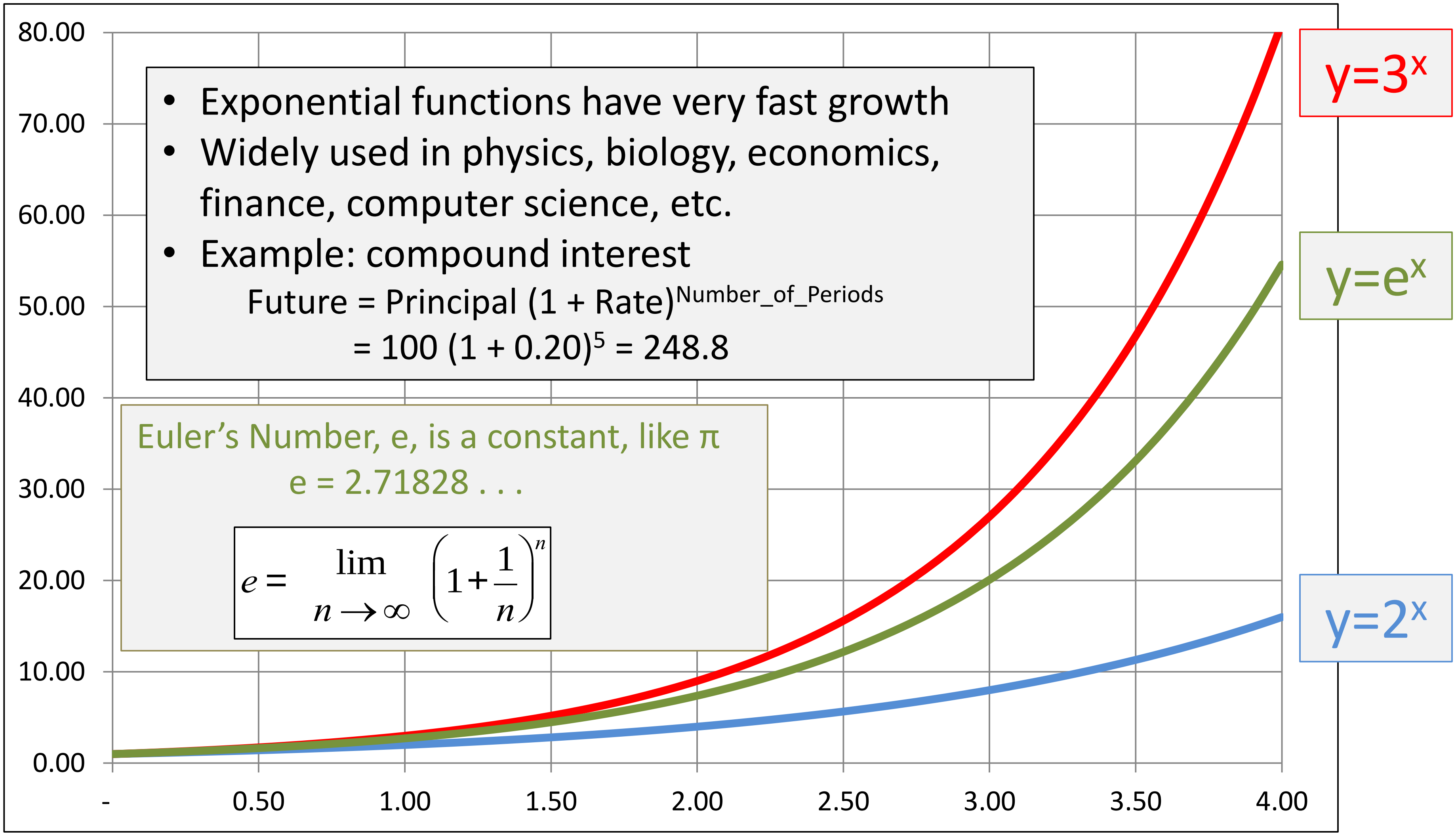
  Future = Principal (1 + Rate)$^{\text{Number\_of\_Periods}}$

  = 100 (1 + 0.20)^5 = 248.8

Euler's Number, e, is a constant, like π

e = 2.71828 . . .

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$



Logarithms

Other Inverse Relationships

$y=ax$	$\leftrightarrow$	$x=y\div a$
$y=a+x$	$\leftrightarrow$	$x=y-a$

Logarithms

$$y=b^x \quad \leftrightarrow \quad \log_b(y)=x$$

y is the value of b raised to the x^{th} power.	x is the power that I need to raise the base, b, to equal y.

$100 = 10^x$	$\log_{10}(100) = x$	$x=2$
$5 = 10^x$	$\log(5) = x$	$x\cong 0.7$
$1 = e^x$	$\log_e(1) = \ln(1)=x$	$x=0$
$e = e^x$	$\ln(e) = x$	$x=1$

Properties of Logarithms

- $\log(xy) = \log(x) + \log(y)$
- $\log(x/y) = \log(x) - \log(y)$
- $\log(x^a) = a \log(x)$

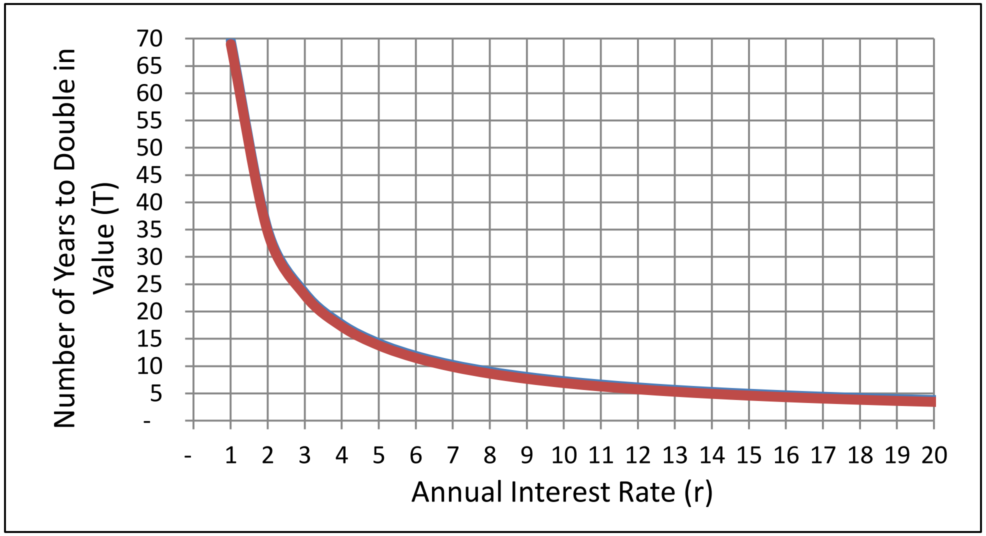
Examples:

- $\ln(3*5) = \ln(3)+\ln(5) = 2.71$
- $\ln(12/7) = \ln(12) - \ln(7) = 0.54$
- $\ln(3^6) = 6\ln(3) = 6.59$
- $\log(3*5^2) = \log(3) + 2\log(5) = 1.88$

Practical Example: Doubling Time

- You have invested a sum of money that has an interest rate of 7% annually. How many years, T, will it take to double in value?
  - We know that $F=P(1+r)^n$ and we want to find the n where $F=2P$
  - $F=2P=P(1+r)^n$ which reduces to $2=(1+r)^n=(1.07)^n$
  - We can transform this by taking the ln or log of both sides:
    - $\ln(2) = n \ln(1.07)$
    - Rearranging gives us: $n = \ln(2) / \ln(1.07) = 0.693 / 0.182 = 10.24 = T$
    - We could also use $\log_{10}$ where $T = \log(2)/\log(1.07) = 10.24$
  - The investment will double in value in 10.24 years.

- Can we come up with a general equation or approximation?
  - We know that $T = \ln(2) / \ln(1 + r)$
  - Plotting this for $T=f(r)$ . . . looks like $T=ar^{-1}=a/r$
  - Turns out $T \approx 70/r$

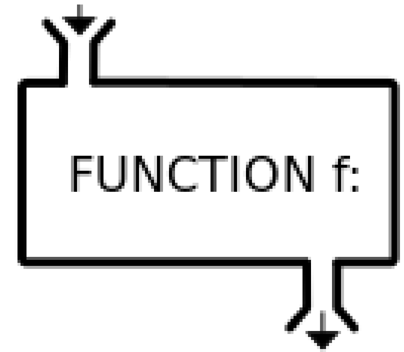


Multivariate Functions

Multivariate Functions

These are just functions with more than one independent variable.

INPUT $(x_1, x_2, \ldots x_n)$



OUTPUT $y = f(x_1, x_2, \ldots x_n)$

still just a single output

we still just say:
"y is a function of $x_1, x_2, \ldots x_n$)"

$$y = f(x_1, x_2, \ldots x_n)$$

Example: $f(x_1, x_2) = x_1 + 2x_2 + 5x_1x_2$

$x_1 = 2,\ x_2 = 4$ then $y = f(2,4) = 2 + 2(4) + 5(2)(4) = 50$

$x_1 = -1,\ x_2 = 0$ then $y = f(-1,0) = -1 + 2(0) + 5(-1)(0) = -1$

$x_1 = 0,\ x_2 = -½$ then $y = f(0, -½) = 0 + 2(-½) + 5(0)(-½) = -1$

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Examples: Multivariate Functions

- Parcel Trucking – impact of weight & distance

  Parcel carriers combine many orders into a single shipment. The cost of an individual order is a function of its weight, w, and the distance.

  $$\text{cost} = f(\text{weight, distance}) = c_1 + c_2 w + c_3 w^2 + c_4 d + c_5 d^2 + c_6 dw$$

- Total Logistics Cost Equation

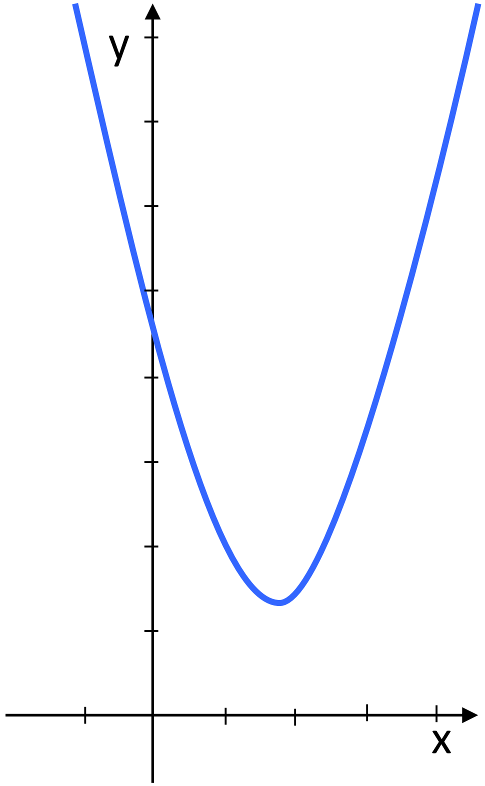
  cost = f(Demand, Order Cost, Order Size,) = $cD + AD/Q$ where:

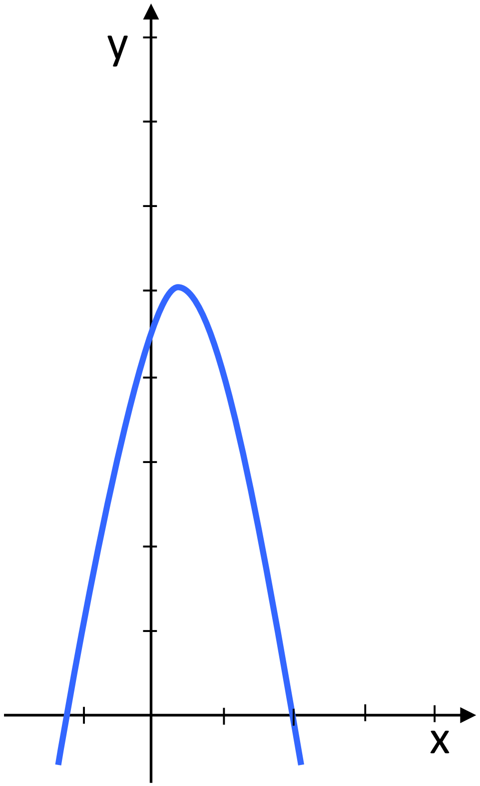
  D = annual demand (items)
  c = cost per item (¥/item)
  A = cost per order (¥/order)
  Q = order size (items/order)

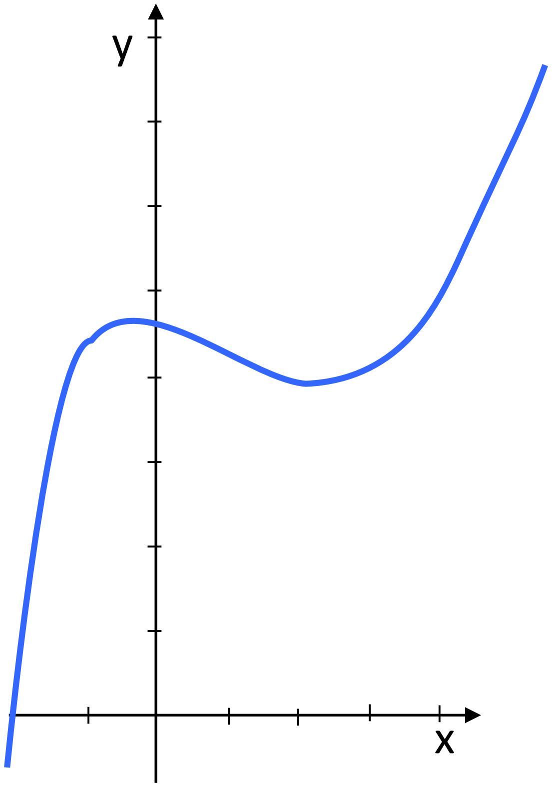
Properties of Functions

Properties of a Function: Convexity

A function is convex if it “holds water”

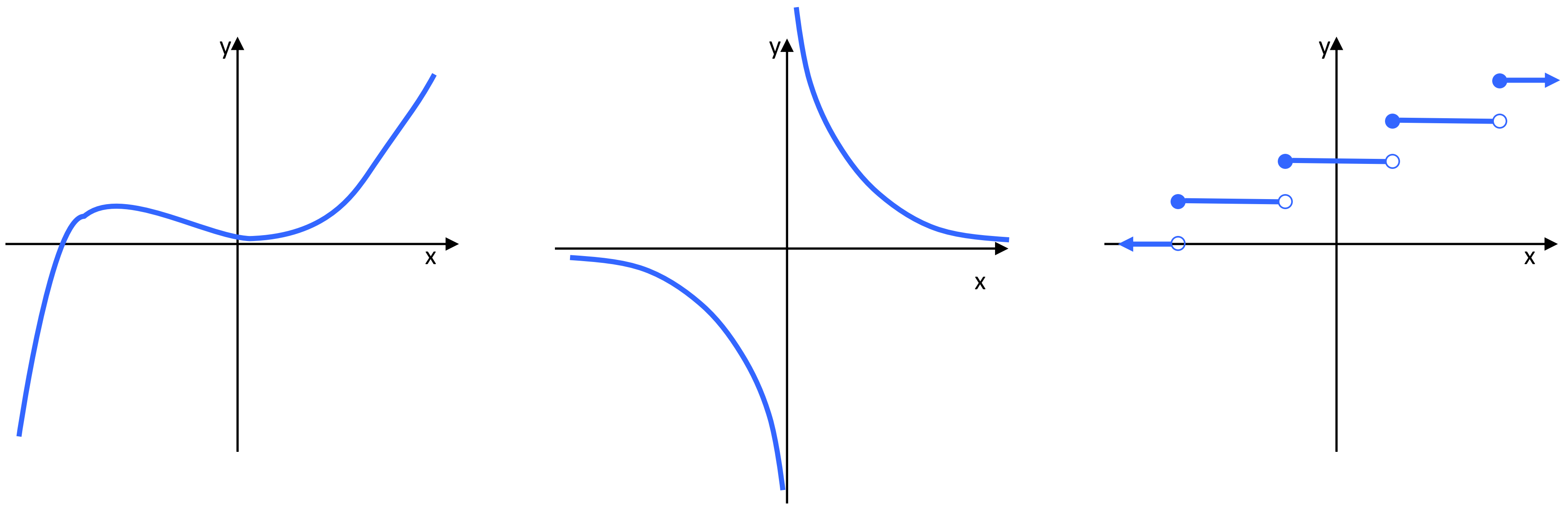






Properties of a Function: Continuity

A function is continuous if you can draw it without lifting pen from paper!



Key Points from Lesson

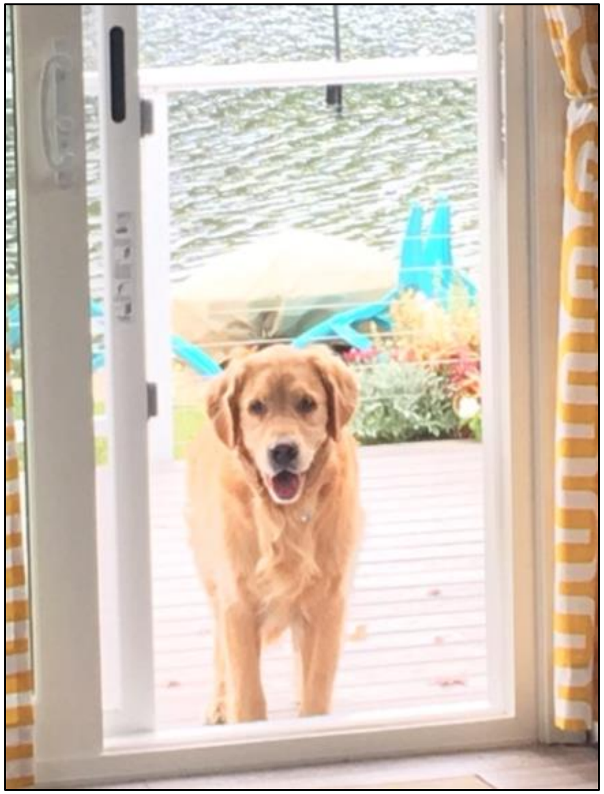
Key Points from Lesson (1/2)

- Different models used for different purposes
  - Descriptive – what has happened?
  - Predictive – what could happen?
  - Prescriptive – what should we do?
- Functions $y=f(x)$
  - Linear functions where $y= ax + b$
  - Quadratic functions where $y= ax^2 + bx + c$
  - Power functions where $y= ax^b$
  - Exponential functions where $y= ab^x$

Key Points from Lesson (2/2)

- Logarithms
  - $y=b^x$ is equivalent to $\log_b(y) = x$
  - Natural log $\ln(y) = \log_e(y)$
- Multivariate functions $y=f(x_1, x_2, \ldots x_n)$
  - Multiple inputs still lead to single output value
- Properties of functions
  - Convexity – does the function "hold water"?
  - Continuity – can I draw the function without lifting my pencil

Questions, Comments, Suggestions?
Use the Discussion Forum!



"Wilson – realizing he is asymptotic to the door"
Yankee Golden Retriever Rescued Dog
(www.ygrr.org)

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