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2. Lecture 2

The following can be done after Lecture 2.

2-1

10/10 points (graded)

Each of the following equations defines an infinite collection S of vectors in $\mathbb{R}^2$, namely the set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ such that x and y satisfy the equation. For which equations is this collection a vector space?

(Check all those that apply.)

- [x] $x = 0$ ✔
- [ ] $x + y = 5$
- [x] $3x + y = 0$ ✔
- [ ] $y = x^2$

✔

Solution:

$x = 0$ and $3x + y = 0$.

$x = 0$: The zero vector $(0,0)$ is in S. Any vector in S has the form $(0,c)$ for some c, and multiplying by any scalar gives another vector of the same form. Adding two such vectors $(0,c)$ and $(0,d)$ gives another vector of the same form. The preceding three sentences together are what it means for S to be a subspace of $\mathbb{R}^2$.

$x + y = 5$: This time, $(0,0)$ is not in S, so the line $x + y = 5$ fails even the first condition to be a subspace.

$3x + y = 0$: The zero vector $(0,0)$ is in S. If (a,b) is a vector in S and c is a scalar, then $3a + b = 0$, so $3(ca) + (cb) = c(3a + b) = c(0) = 0$, which says that the vector $c(a,b) = (ca, cb)$ is in S. If (a,b) and (a',b') are both in S, then $3a + b = 0$ and $3a' + b' = 0$, so $3(a + a') + (b + b') = (3a + b) + (3a' + b') = 0 + 0 = 0$, which says that the vector $(a,b) + (a',b') = (a + a', b + b')$ is in S. The preceding three sentences together are what it means for S to be a subspace of $\mathbb{R}^2$.

$y = x^2$: The zero vector $(0,0)$ is in S. But multiplying the vector $(1,1) \in S$ by 2 gives a vector $(2,2)$ which is not in S. Thus S fails the second condition to be a subspace. (It also fails the third condition.) Since not all three conditions were satisfied, S is not a subspace.

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Answers are displayed within the problem

2-2

10/10 points (graded)

Which of the following lists is a basis for $\mathbb{R}^2$?

(Check all those that apply.)

- [x] $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✔
- [ ] $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- [ ] $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$
- [ ] $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}$
- [x] $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$ ✔

✔

Solution:

The bases for $\mathbb{R}^2$ are $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ and "$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$.

The list $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ is a basis because both of the following hold:

- Every vector $\begin{pmatrix}a\\b\end{pmatrix}$ in $\mathbb{R}^2$ is expressible as a linear combination of the given vectors:

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}.$$

- The vectors in the list are linearly independent (since neither is a linear combination of the other, i.e., neither is a multiple of the other).

$\begin{pmatrix}1\\1\end{pmatrix}$ is not a basis, because its span consists only of the scalar multiples of $\begin{pmatrix}1\\1\end{pmatrix}$ (i.e., the line $y = x$) instead of being all of $\mathbb{R}^2$.

The list $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$ is not a basis, since it is not linearly independent: the third vector is a linear combination of the first two.

The list $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}2\\0\end{pmatrix}$ is not a basis, since it is not linearly independent: the second vector is a linear combination of the first vector (which just means that it is a scalar multiple of the first vector).

The list $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}-1\\1\end{pmatrix}$ is a basis because both of the following hold:

- Every vector in $\mathbb{R}^2$ is expressible as a linear combination of the given vectors: given any vector $\begin{pmatrix}a\\b\end{pmatrix}$, it is possible to find x and y such that

$$\begin{pmatrix}a\\b\end{pmatrix} = x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}-1\\1\end{pmatrix}$$

  (solving the system $x - y = a$, $x + y = b$ shows that $x = (a+b)/2$ and $y = (b-a)/2$ work).

- The vectors in the list are linearly independent (since neither is a multiple of the other).

Another way to rule out $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$ would to be observe that they have the wrong number of vectors (every basis for the 2-dimensional space $\mathbb{R}^2$ must have exactly 2 vectors).

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Answers are displayed within the problem

2-3

5/5 points (graded)

Which of the following are true?
(Check all that apply.)

- [x] Every homogeneous system of linear equations has at least one solution. ✔

- [ ] Every inhomogeneous system of linear equations has at least one solution.

Solution:

Only the first is true. A homogeneous linear system always has $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ as a solution. But some inhomogeneous linear systems have no solution; for example,

$$\begin{aligned} x + y &= 2 \\ x + y &= 3, \end{aligned}$$

has no solution.

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Answers are displayed within the problem

2-4

5/5 points (graded)

For which of the following matrices $\mathbf{A}$ does $\mathbf{Ax} = \mathbf{0}$ have a nonzero solution?

- [x] $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- [x] $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

- [x] $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$

Solution:

The answer is $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

For a square matrix $\mathbf{A}$, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution if and only if the columns of $\mathbf{A}$ are linearly dependent.

For two by two matrices, the two columns are linearly dependent if one column is a scalar multiple of the other.

- The second column of $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is 0 times the first column.
- The second column of $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is -1 times the first column.
- The second column of $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is 2 times the first column.

The matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$ have linearly independent columns, as one is not a scalar multiple of the other.

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Answers are displayed within the problem

2. Lecture 2

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2-4 Solution

I don't recall determinant being explained so far in the course of it's relation to linear independence. Yet ... 8

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