

Math 230, Fall 2012: HW 6 Solutions

Problem 1 (p.202 #4). SOLUTION. We know that

$$V(X_1 X_2) = E[X_1^2 X_2^2] - (E[X_1 X_2])^2$$

By independence,

$$\begin{aligned} E[X_1 X_2] &= \mu_1 \mu_2 \\ E[X_1^2 X_2^2] &= E[X_1^2] E[X_2^2] \end{aligned}$$

and

$$\begin{aligned} E[X_1^2] &= V(X_1) + \mu_1^2 \\ &= \sigma_1^2 + \mu_1^2 \end{aligned}$$

and similarly $E[X_2^2] = \sigma_2^2 + \mu_2^2$.

Problem 2 (p. 202 # 8). SOLUTION. Let $A = A_1$, $B = A_2$ and $C = A_3$, for simplicity. Then:

(a) $N = 1_A + 1_B + 1_C$.

(b) Hence, $E[N] = P(A) + P(B) + P(C) = .2 + .25 + 1/3 \simeq .783$.

Before we do the rest, note that $\text{Var}[N] = E[N^2] - E[N]^2$. Thus, all we need to do is compute $E[N^2]$. Using properties of indicators,

$$\begin{aligned} E[N^2] &= E[(1_A + 1_B + 1_C)^2] \\ &= E[1_A + 1_B + 1_C + 21_{AC} + 21_{AB} + 21_{BC}] \\ &= P(A) + P(B) + P(C) + 2P(AC) + 2P(AB) + 2P(BC). \end{aligned}$$

Hence,

(c) $\text{Var}[N] \simeq .783 - (.783)^2 \simeq .17$.

(d) $\text{Var}[N] \simeq .783 - (.783)^2 + 2[(.2)(.25) + .2(1/3) + .25(1/3)] \simeq .57$.

(e) This is done similarly. Just use the fact that $A \subset B \subset C$ to conclude that $AB = A$, $AC = A$, $BC = B$.

Problem 3 (p. 203 # 13). SOLUTION. (a) Note that by Chebychev's inequality

$$P[X \geq 130] = P[X - 100 \geq 3 * 10] \leq P[|X - 100| \geq 3 * 10] \leq 1/3^2 = 1/9.$$

Therefore there can be at most 111112 individuals with scores exceeding 130.

(b) Note that, by symmetry about 100,

$$P[X \geq 130] = P[X \leq 70] = P[X - 100 \leq -3 * 10].$$

In particular,

$$P[X \geq 130] = .5P[|X - 100| \geq 3 * 10] \leq 1/18.$$

Hence, there can be at most 55556 individuals.

(c) Now supposing that the distribution $X - 100/10 \sim \mathcal{N}(0, 1)$ approximately,

$$P[X \geq 130] = P[(X - 100)/10 \geq 3] \simeq 1 - \Phi(3) \simeq 1 - .9987 = .0013.$$

Hence, there can be at most 1300 individuals.

Problem 4 (p.204#15). SOLUTION. Let $X = D_1 - D_2$. Since D_1 and D_2 are two draws at random with replacement from the same population, they are independent and identically distributed. Therefore $E(X) = 0$ and $V(X) = V(D_1) + V(D_2) = 4 + 4 = 8$. Chebyshev's inequality implies

$$P(|X - E(X)| > kSD(X)) \leq \frac{1}{k^2}$$

$$\Rightarrow P(|D_1 - D_2 - 0| > k\sqrt{8}) \leq \frac{1}{k^2}$$

so we choose $k = 10$ to satisfy the specified bound, and $c = 10\sqrt{8}$.

Problem 5 (p.204 #16). SOLUTION. For the distribution of X , note that

$$P(X = -2) = P(X = -1) = P(X = 0) = P(X = 3) = \frac{1}{4}$$

The mean and variance of X can be computed from this and we find

$$E(X) = \frac{-2}{4} + \frac{-1}{4} + 0 + \frac{3}{4} = 0$$

$$V(X) = E(X^2) = 4\frac{1}{4} + 1\frac{1}{4} + 0\frac{1}{4} + 9\frac{1}{4} = \frac{7}{2}$$

For part (b), we can use the normal approximation to the sample total $S_{100} = \sum_{i=1}^{100} X_i$, where X_i is the net gain in game i , so $E(X_i) = 0, V(X_i) = \frac{7}{2}$. The iterations of the game are independent.

Problem 6 (p. 205 #26). (a) Note that $\mu = E[X] = (1/6)[1 + 2 + 3 + 4 + 5 + 6] = 7/2$. Then

$$E[|X - \mu|] = \frac{1}{6}[2.5 + 1.5 + .5 + .5 + 1.5 + 2.5] = 9/6 = 3/2.$$

(b) Note that

$$0 \leq \text{Var}[|X - \mu|] = E[|X - \mu|^2] - (E[|X - \mu|])^2,$$

so

$$E[|X - \mu|] \leq \sqrt{E[|X - \mu|^2]} = \text{SD}[X].$$

Note also that $\text{SD}[X] = E[|X - \mu|]$ if and only if $\text{Var}[X] = (E[|X - \mu|])^2$ if and only if $E[|X - \mu|^2 - (E[|X - \mu|])^2] = 0$ if and only if $|X - \mu|$ is constant.

Problem 7 (p.217 #1). SOLUTION. Part (a) is a binomial (n,p) probability, where $n = 9$. Let $q = 1 - p$. We get

$$P(\text{exactly 5 heads in first 9 tosses}) = \binom{9}{5} p^5 q^4.$$

Part (b) is a geometric distribution; in order for the first head to appear at the 7th toss, the previous 6 tosses must be tails and the 7th toss must be heads. By independence, this is $q^6 p$.

Part (c) is called a negative binomial distribution. In order for the fifth head to appear on the 12th toss, the 12th toss must be a heads, and there must be exactly 4 heads in the previous 11 tosses. Thus

$$P(5\text{th head appears on 12th toss}) = \binom{11}{4} p^4 q^7 p = \binom{11}{4} p^5 q^7.$$

For part (d), observe that by independence of trials, the random variables

$$\begin{aligned} X_1 &= [\text{number of heads in the first 8 tosses}] \\ X_2 &= [\text{number of heads in the next 5 tosses}] \end{aligned}$$

are independent. Also $X_1 \sim B(8, p)$, and $X_2 \sim B(5, p)$. Hence

$$\begin{aligned} P(\text{same \# of H in first 8 as in next 5 tosses}) &= \sum_{k=0}^5 P(X_1 = k, X_2 = k) \\ &= \sum_{k=0}^5 P(X_1 = k)P(X_2 = k) \\ &= \sum_{k=0}^5 \binom{8}{k} p^k q^{8-k} \binom{5}{k} p^k q^{5-k} \end{aligned}$$

Problem 8 (p.218 #4). **SOLUTION.** The probability of one of the coins landing differently from the other two is the complement of the probability that all three land heads or all three land tails. So

$$P(\text{game ends after one round}) = 1 - \left[\frac{1}{8} + \frac{1}{8} \right] = \frac{3}{4}.$$

The probability the game ends after one round is $\frac{3}{4}$.

If game ends after r rounds, the first $r - 1$ rounds must have ended with no coin landing differently from the others, and one coin landing differently from the other two on the r th round. By independence, this is

$$P(\text{game lasts } r \text{ rounds}) = \left[\frac{1}{4} \right]^{r-1} \left[\frac{3}{4} \right].$$

Since the number of rounds of play is a geometric random variable with success probability $\frac{3}{4}$, the expected duration of play is $\frac{4}{3}$ rounds.

Problem 9 (p.218 #6). **SOLUTION.** If there are k failures before the first success, then the first k trials must be failures, and the $k + 1$ st trial must be a success. By independence of trials, the probability of k failures before the first success is $q^k p$, which is exactly the distribution of W . Next,

$$\begin{aligned} P(W > k) &= \sum_{i=k+1}^{\infty} q^i p \\ &= \frac{q^{k+1} p}{1 - q} = q^{k+1} \end{aligned}$$

where the last equality is the sum of a geometric series with common ratio q and first term pq^{k+1} . Now, if \tilde{W} represents the number of trials until the first success, then $\tilde{W} = W + 1$. We already know how to calculate the mean and variance for \tilde{W} . Hence we have

$$\begin{aligned} E[W] &= E[\tilde{W}] - 1 = \frac{1}{p} - 1 \\ V[W] &= V[\tilde{W}] = \frac{q}{p^2} \end{aligned}$$

Problem 10 (p.218 #7). SOLUTION. Suppose that A tosses the first head and that this occurs on trial k , $k = 2n + 1$ for some $n > 0$, because k must be odd if A is to toss the first head. Then the previous $k - 1 = 2n$ trials must have been tails. So the probability that A tosses the first head and this occurs on trial $2n + 1$ is $(1 - p)^{2n}p$. The probability that A tosses the first head is the sum of these probabilities over all nonnegative integers n :

$$\begin{aligned} P(A \text{ tosses first head}) &= \sum_{n=0}^{\infty} (1 - p)^{2n} p \\ &= \frac{p}{1 - (1 - p)^2} = \frac{p}{2p - p^2} = \frac{p}{p(2 - p)} = \frac{1}{2 - p} \end{aligned}$$

Suppose that B tosses the first head on trial $2n$, where $n \geq 1$. This probability is $(1 - p)^{2n-1}p$. So the probability that B tosses the first head is

$$\begin{aligned} P(B \text{ tosses first head}) &= \sum_{n=1}^{\infty} (1 - p)^{2n-1} p \\ &= \frac{p(1 - p)}{1 - (1 - p)^2} = \frac{p(1 - p)}{p(2 - p)} = \frac{1 - p}{2 - p} \end{aligned}$$

The sketches of these probabilities as a function of p is left to you.

Under the scheme where A tosses first, then B tosses twice, etc., the probability that A tosses the first heads, denoted $P(A)$, is

$$\begin{aligned} P(A \text{ tosses first head}) &= \sum_{n=0}^{\infty} (1 - p)^{3n} p \\ &= \frac{p}{1 - (1 - p)^3} = \frac{1}{3 - 3p + p^2} \end{aligned}$$

Under the scheme where A tosses first, then B tosses twice, etc, the probability B wins is $1 - P(A)$. These are equal when $P(A) = 1/2$, i.e. for p such that

$$\frac{1}{2} = \frac{1}{3 - 3p + p^2}$$

which yields $p = \frac{3 - \sqrt{5}}{2}$. To compute B 's chances of tossing the first head, note that $1 - P(A)$ is

$$P(B \text{ tosses first heads}) = \frac{2 - 3p + p^2}{3 - 3p + p^2}$$

and in the limit as $p \rightarrow 0$, this approaches $2/3$. Note also that as $p \rightarrow 1$, this approaches 0, since for very heads-biased coins, A is likely to toss the first head.

Problem 11 (p. 219 #12). SOLUTION. For part (a) we compute directly using independence in

the second line

$$\begin{aligned}
P(W_1 = W_2) &= \sum_{k=1}^n P(W_1 = k, W_2 = k) \\
&= \sum_{k=1}^n P(W_1 = k)P(W_2 = k) \\
&= \sum_{k=1}^n [(1 - p_1)(1 - p_2)]^{k-1} p_1 p_2 \\
&= \frac{p_1 p_2}{1 - (1 - p_1)(1 - p_2)}
\end{aligned}$$

Next, we consider $P(W_1 < W_2)$; the probability of $W_1 > W_2$ can be computed from this and from part (a).

$$\begin{aligned}
P(W_1 < W_2) &= \sum_{k=1}^{\infty} P(W_1 = k, W_2 > k) \\
&= \sum_{k=1}^{\infty} P(W_1 = k)P(W_2 > k) \\
&= \sum_{k=1}^{\infty} [(1 - p_1)^{k-1} p_1] (1 - p_2)^k \\
&= \frac{p_1(1 - p_2)}{1 - (1 - p_1)(1 - p_2)}
\end{aligned}$$

To compute the distribution of the maximum, note that

$$\begin{aligned}
P(\max(W_1, W_2) \leq k) &= P(W_1 \leq k, W_2 \leq k) = P(W_1 \leq k)P(W_2 \leq k) \\
&= [1 - q_1^k] [1 - q_2^k]
\end{aligned}$$

To compute the distribution of the minimum, note that

$$\begin{aligned}
P(\min(W_1, W_2) > k) &= P(W_1 > k, W_2 > k) = P(W_1 > k)P(W_2 > k) \\
&= q_1^k q_2^k
\end{aligned}$$

So $P((\min(W_1, W_2) \leq k) = 1 - q_1^k q_2^k$.

Problem 12 (p.219 #14). SOLUTION. We need to compute the probability $P(V_n = k)$, where $n \leq k \leq 2n - 1$. (Why are these the appropriate lower and upper bounds for k ?). If $V_n = k$, then we note that the k th trial is either a success S or failure F . If the k th trial is a success, then there must have been exactly $n - 1$ successes in the previous $k - 1$ trials; if the k th trial is a failure, then there must have been exactly $n - 1$ failures in the previous $k - 1$ trials. So we get

$$\begin{aligned}
P(V_n = k | S \text{ on trial } k) &= \binom{k-1}{n-1} p^{n-1} (1-p)^{k-n} \\
P(V_n = k | F \text{ on trial } k) &= \binom{k-1}{n-1} (1-p)^{n-1} p^{k-n}
\end{aligned}$$

So we obtain

$$\begin{aligned} P(V_n = k) &= P(V_n = k | S \text{ on trial } k)P(S) + P(V_n = k | F \text{ on trial } k)P(F) \\ &= \binom{k-1}{n-1} p^{n-1} (1-p)^{k-n} p + \binom{k-1}{n-1} (1-p)^{n-1} p^{k-n} (1-p) \end{aligned}$$

for $n \leq k \leq 2n-1$ and 0 otherwise.

Problem 13 (p. 234 #2). How many raisins must cookies contain on average for the chance of a cookie containing at least one raisin to be at least 0.99? Assume a Poisson random scatter of raisins.

SOLUTION. Let N = the number of raisins per cookie. If N is a Poisson random variable with parameter λ , then

$$P(N \geq 1) = 1 - P(N = 0) = 1 - \exp(-\lambda)$$

and for this to be at least 0.99, we need

$$\lambda \geq -\ln(0.01)$$

Recall that $\lambda = E[N]$ is the average number of raisins per cookie.

Problem 14 (p. 234 #5). Microbes are smeared over a plate at an average density of 5000 per square inch. The view field of a microscope is 10^{-4} square inches. What is the chance that at least one microbe is in the viewing field? What assumptions are you making?

SOLUTION. If we assume that no two microbes lie on top of one another and they are evenly smeared across the entire plate, then we may assume that the microbes are distributed across the plate according to a Poisson random scatter. The number N of microbes in the viewing field is a Poisson random variable with mean $\mu = 5000 \times 10^{-4} = 0.5$. So the probability of at least one microbe is

$$P(N \geq 1) = 1 - P(N = 0) = 1 - e^{-0.5}$$

Problem 15 (p. 234 #11). Let X , Y , and Z be independent Poisson random variables with mean 1. Find the following:

- a) $P(X + Y = 4)$;
- b) $E[(X + Y)^2]$;
- c) $P(X + Y + Z = 4)$

SOLUTION. Since X, Y, Z are independent Poisson random variables, we know that $X + Y$ is a Poisson random variable with mean $E[X] + E[Y] = 2$, and $X + Y + Z$ is a Poisson random variable with mean 3. This gives

- a) $P(X + Y = 4) = \frac{e^{-2} 2^4}{4!}$;
- b) $E[(X + Y)^2] = V[(X + Y)] + (E[X + Y])^2 = 2 + 4 = 6$
- c) $P(X + Y + Z = 4) = \frac{e^{-3} 3^4}{4!}$

Problem 16 (p. 235 #16). On average, one cubic inch of Grandma's cookie dough contains 2 chocolate chips and 1 marshmallow.

- a) Suppose Grandma makes a cookie using three cubic inches of dough. Find the chance that the cookie contains at most four chocolate chips, and state your assumptions.
- b) Assume the number of marshmallows in Grandma's dough is independent of the number of chocolate chips. Suppose three cookies are taken, one of which is made with two cubic inches of dough, and the other two with three cubic inches each. What is the chance that at most one of these cookies contains neither chocolate chips nor marshmallows?

SOLUTION. For part (a), if we assume the number of chips is distribution according to a Poisson scatter through a given volume of cookie dough, and if we define N as the number of chips in 3 cubic inches of dough, then N is a Poisson random variable with mean $\mu = 6$ chips. So

$$P(N \leq 4) = \sum_{k=0}^4 \frac{e^{-6} 6^k}{k!}$$

For part (b), let M_1, M_2, M_3 be the number of marshmallows in cookies 1, 2 and 3; and let C_1, C_2 , and C_3 be the number of chips in cookies 1, 2, and 3. By assumption M_i and C_i , where $i = 1, 2$ or 3, are independent Poisson random variables, and the parameter for M_1 is 2; the parameter for M_2 and M_3 is 3; the parameter for C_1 is 4; the parameter for C_2 and C_3 is 6. Now, since the sums of independent Poisson random variables is also Poisson, we know that $N_i = M_i + C_i$ are independent Poisson random variables with parameters that are simply the sums of the parameters for M_i and C_i .

$$\begin{aligned} &P(\text{at most one cookie has no marshmallows and no chips}) \\ &= P(N_i > 0 \text{ for all } i) + P(N_1 = 0, N_2 > 0, N_3 > 0) + P(N_1 > 0, N_2 = 0, N_3 > 0) \\ &\quad + P(N_1 > 0, N_2 > 0, N_3 = 0) \\ &= (1 - e^{-6})(1 - e^{-9})^2 + e^{-6}(1 - e^{-9})^2 + (1 - e^{-6})e^{-9}(1 - e^{-9}) + (1 - e^{-6})(1 - e^{-9})e^{-9} \\ &= (1 - e^{-9})^2 + 2e^{-9}(1 - e^{-6})(1 - e^{-9}) \end{aligned}$$