3. Bayesian Estimation and Linear Regression

We will now explore what linear regression looks like from a particular Bayesian Framework. The answers that you find here may be surprising to you, hopefully in a pleasant way.

Suppose that:

- $Y_1, \ldots, Y_n$ are independent given the pair (β_0, β_1)
- $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$, where each ϵ_i are i.i.d. $\mathcal{N}(0, 1/\tau)$ (which has variance $1/\tau$)
- the $X_i \in \mathbb{R}$ are deterministic.

We will think of β_0, β_1 and τ as being random variables.

Suppose that we place an improper prior on (β_0, β_1, τ):

$$\pi(\beta_0, \beta_1, \tau) = 1/\tau.$$

Since this expression on the right hand side does not depend on the β's, we may take the conditional distribution $\pi(\beta_0, \beta_1 | \tau)$ to be the "uniform" improper prior: $\pi(\beta_0, \beta_1 | \tau) = 1$.

Answer the following problems given these assumptions. As a reminder, we let $\mathbb{X}$ be the design matrix, where the ith row is the row vector $(1, X_i)$, and let $\mathbf{Y}$ be the column vector $\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$.

(a) The Bayesian setup: The posterior distribution

2/2 points (graded)

Observe that if β_0, β_1 and τ are given, then each Y_i is a gaussian: $Y_i | (\beta_0, \beta_1, \tau) \sim \mathcal{N}(\beta_0 + \beta_1 X_i, 1/\tau)$.

Therefore, the likelihood function of the vector $(Y_1, \ldots, Y_n)$ given (β_0, β_1, τ) is of the form

$$\left(\frac{1}{\sqrt{2\pi/\tau}}\right)^n \exp\left(-\frac{\tau}{2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 X_i)^2\right)$$

It turns out that the distribution of (β_0, β_1) given τ and $Y_1, \ldots, Y_n$ is a 2-dimensional Gaussian. In terms of $\mathbb{X}$, $\mathbf{Y}$ and τ, what is its mean and covariance matrix?

Hint: look ahead and see what part (b) is asking. What answer do you hope would come out, at least for one of these two things?

(Type **X** for $\mathbb{X}$, **trans(X)** for the transpose $\mathbb{X}^T$, and **X^(-1)** for the inverse $\mathbb{X}^{-1}$ of a matrix $\mathbb{X}$.)

Mean: (trans(X)*X)^(-1)*(trans(X)*Y) ✔

Covariance: (trans(X)*X)^(-1)/tau ✔

STANDARD NOTATION

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✔ Correct (2/2 points)

(b)

1/1 point (graded)

What is the Bayes estimator $\widehat{(\beta_0, \beta_1)}^{\text{Bayes}}$ for (β_0, β_1)?

Hint: Use your answer from part (a). However, as hinted: the answer here is guessable, even if you didn't solve the previous part.

(Answer in terms of $\mathbb{X}$, $\mathbf{Y}$ and τ.)

(Type **X** for $\mathbb{X}$, **trans(X)** for the transpose $\mathbb{X}^T$ of a matrix $\mathbb{X}$, and **X^(-1)** for the inverse $\mathbb{X}^{-1}$ of a matrix $\mathbb{X}$.)

$\widehat{(\beta_0, \beta_1)}^{\text{Bayes}} =$ (trans(X)*X)^(-1)*(trans(X)*Y) ✔

STANDARD NOTATION

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✔ Correct (1/1 point)

(c)

1/1 point (graded)

Given our improper prior, we ought to take the posterior distribution of $\tau | (\beta_0, \beta_1)$ to also be $\pi(\tau | \beta_0, \beta_1) = \frac{1}{\tau}$, for each realization of τ.

What type of distribution is the posterior distribution of τ given the **triple** $(\beta_0, \beta_1, \mathbf{Y})$?

- Gaussian
- Chi-Squared

- Gamma
- Uniform
- Exponential
- Other (not listed above)

✔

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