# 4. Review of linearization techniques

Recall that the best linear approximation of a function of 1 variable $f(x)$ near a point $x = a$ is

$$f(x) \quad \approx \quad f(a) + f'(a)(x-a).$$

Recall that for a function of 2 variables $f(x,y),$ the best linear approximation near a point $(x,y) = (a,b)$ is

$$f(x,y) \quad \approx \quad f(a,b) + \frac{\partial f}{\partial x}(a,b)\ (x-a) + \frac{\partial f}{\partial y}(a,b)\ (y-b)$$

In the course *Differential equations: 2x2 systems*, we saw that to linearize a $2\times 2$ system of autonomous equations near a critical point $(x,y) = (a,b),$

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y), \end{aligned}$$

we need to replace **both** $f(x,y)$ and $g(x,y)$ by the best linear approximations near $(a,b),$ which are given by

$$\begin{aligned} f(x,y) &\approx f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)} (y-b) \;=\; f(a,b) + f_x(a,b)\ (x-a) + f_y(a,b)\ (y-b) \\ g(x,y) &\approx g(a,b) + \left.\frac{\partial g}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial g}{\partial y}\right|_{(a,b)} (y-b) \;=\; g(a,b) + g_x(a,b)\ (x-a) + g_y(a,b)\ (y-b). \end{aligned}$$

Using the definition of matrix multiplication, the best linear approximation of the vector-valued function $\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ near the point $(x,y) = (a,b)$ is

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \approx \underbrace{\begin{pmatrix} f(a,b) \\ g(a,b) \end{pmatrix}}_{\text{value at } (a,b)} + \underbrace{\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\Bigg|_{\mathbf{a}}}_{\text{derivative at } (a,b)} (\mathbf{x} - \mathbf{a}), \qquad \text{where } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

We can abbreviate this notation as

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{a}) + \mathbf{J}(\mathbf{a})(\mathbf{x} - \mathbf{a}),$$

where

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \\ \mathbf{f}(\mathbf{a}) &= \begin{pmatrix} f(a,b) \\ g(a,b) \end{pmatrix}, \\ \mathbf{J} &= \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}, \end{aligned}$$

and the matrix $\mathbf{J}(\mathbf{a})$ is the matrix of partial derivatives evaluated at $\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix}$.

The matrix $\mathbf{J}$ is called the **Jacobian matrix** .

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## Stability of fourth critical point

2 points possible (graded, results hidden)

We are modeling two populations via the model used on the previous page

$$\begin{aligned} \dot{x} &= x - x^2 - axy \\ \dot{y} &= 3y - 2y^2 - bxy. \end{aligned}$$

Find conditions on the parameters $a$ and $b$ so that the critical point you found on the previous page is a stable critical point.

$0 < a <$ 2/3

$0 < b <$ 3

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## Identify the long term behavior, case 1

2 points possible (graded, results hidden)

Identify the long term population ($x_\infty$, $y_\infty$) of the system below, assuming that the starting populations are both positive.

$$\begin{aligned}\dot{x} &= x - x^2 - 0.5xy \\ \dot{y} &= 3y - 2y^2 - 2xy.\end{aligned}$$

(If the long term behavior is oscillatory, or there are two possible outcomes, enter none in both answer boxes.)

$x_\infty =$ 0.5

$0.5$

$y_\infty =$ 1

$1$

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## Identify the long term behavior, case 2

2 points possible (graded, results hidden)

Identify the long term population ($x_\infty$, $y_\infty$) of the system below, assuming that the starting populations are both positive.

$$\begin{aligned}\dot{x} &= x - x^2 - xy \\ \dot{y} &= 3y - 2y^2 - xy.\end{aligned}$$

(If the long term behavior is oscillatory, or there are two possible outcomes, enter none in both answer boxes.)

$x_\infty =$ 0

$0$

$y_\infty =$ 3/2

$\frac{3}{2}$

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## 4. Review of linearization techniques

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