MITx 18.02.1x

Multivariable Calculus 1: Vectors and Derivatives

Help sandipan_dey

Course Progress Dates Calendar Discussion Notes

Course / Unit 5: Curves and Surfaces / Problem Set 5B

Previous ✓ ✓ Next

4. Ray tracing

Bookmark this page

Calculator Hide Notes

Part B due Oct 5, 2021 20:30 IST

Apply

Suppose we wish to render a 3D scene to a computer screen. This is the main problem of Computer Graphics. This problem arises in numerous contexts: creating 3D simulations, making special effects for a film, developing 3D video games, and several other applications.

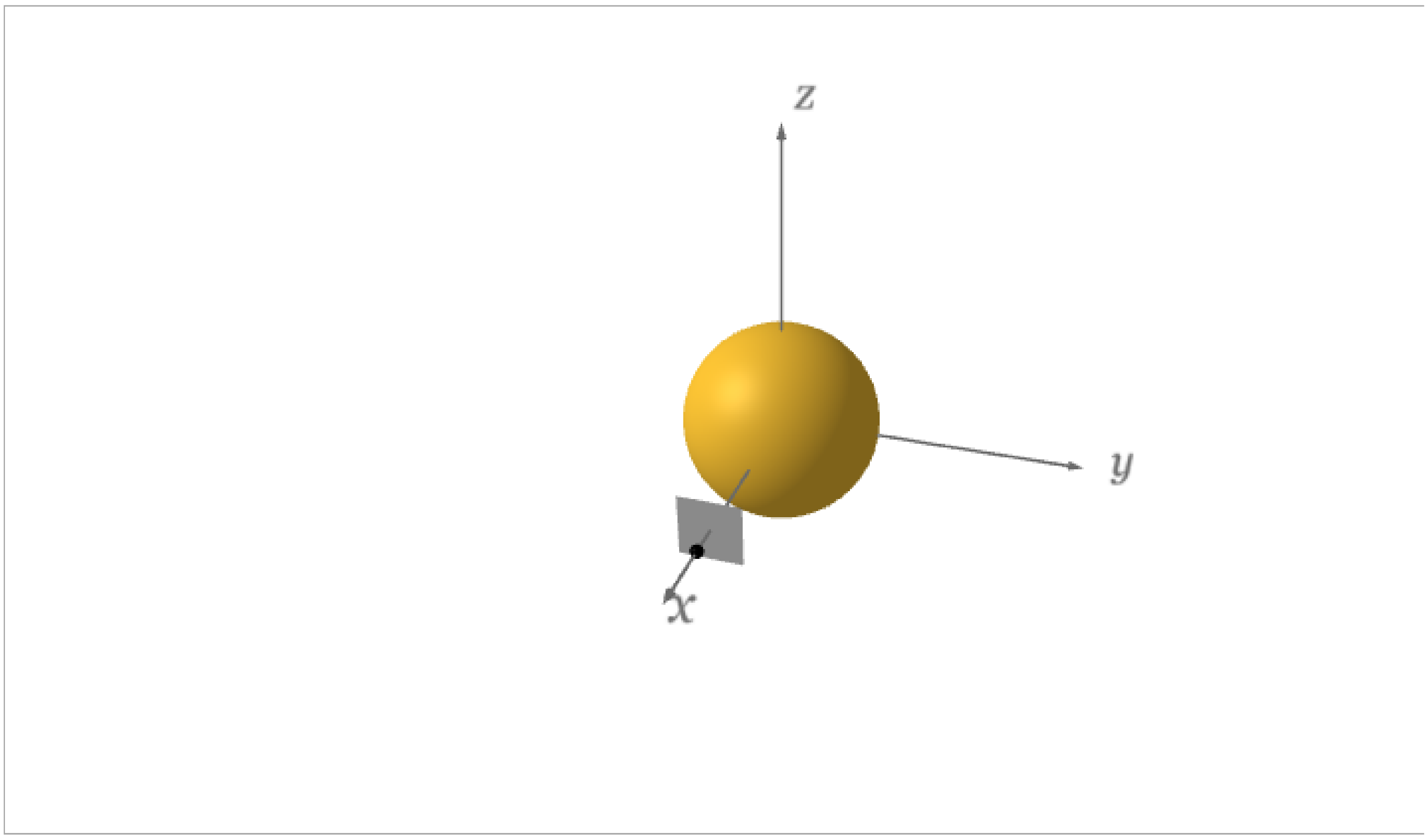
There are various methods used in the industry for solving this problem. One of the most common is "ray tracing," which you will learn about in this problem.

In this example, suppose our scene consists of an orange sphere of radius 13 at the origin.

Suppose that our computer screen is a 2D blank screen of width 200 pixels by 200 pixels. For each one of the 40,000 pixels, we need to decide if it should have the color of the sphere (orange) or not (black).

The idea behind ray tracing is to simulate the way rays of light would bounce into a camera. Let's suppose the camera is placed at position $(20, 0, 0)$ and is facing towards the origin. We imagine an "image plane" that is a small plane placed in front of the camera that represents our 2D blank screen. Let's imagine the plane's vertices are $(19, \pm 1, \pm 1)$.

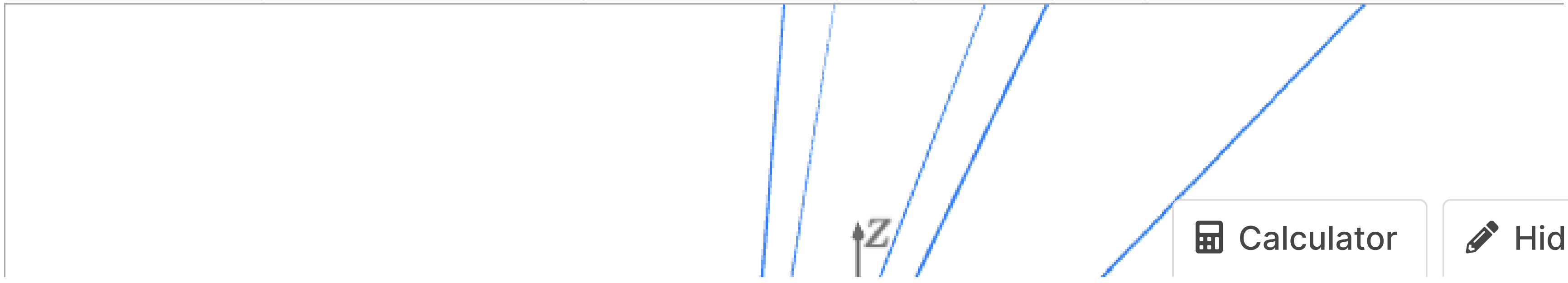
▶ **Sphere, camera, and screen setup**



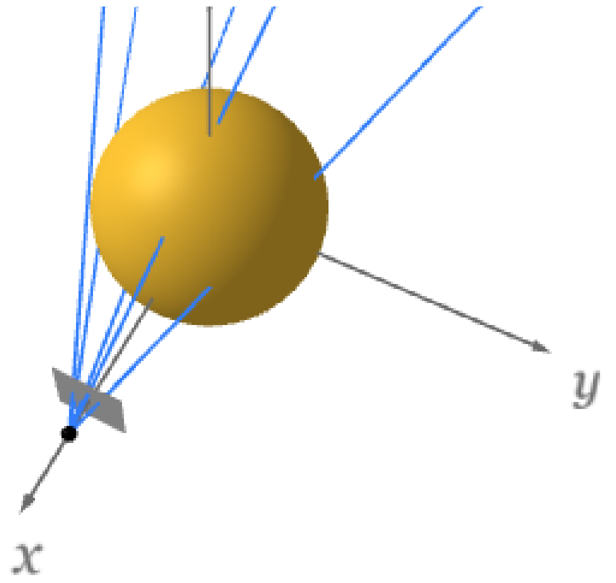
Now for each pixel in the 2D image, we construct a corresponding point in the image plane, and use that to form a ray that goes from the camera's position through that point. With the ray in hand, we can use parametric equations to figure out if/where the ray intersects the sphere. That tells us what color to make the pixel.

▼ **Ray tracing**

Interactive 3D Image Below: Click and drag to rotate the image. Right clicking changes the focus of the rotation.



Calculator Hide Notes



Ray direction

1/1 point (graded)

Suppose the pixel of interest is in the direction of the **unit** vector $\hat{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. What is the equation for the ray $\vec{q}\,(t)$ such that $\vec{q}\,(0) = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix}$, whose trajectory moves in the positive direction of $\hat{u}$ with unit speed?

Your answer will contain u_1, u_2, u_3 and t.

$\vec{q}\,(t) =$ [20+t*u_1, t*u_2, t*u_3] ✓ **Answer:** [20+t*u_1,t*u_2,t*u_3]

Solution:

We may write $\vec{q}\,(t)$ as

$$\vec{q}\,(t) = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 20 + u_1 t \\ u_2 t \\ u_3 t \end{pmatrix} \tag{6.299}$$

Submit You have used 1 of 9 attempts

Answers are displayed within the problem

Ray intersection

1/1 point (graded)
Continuing with the ray $\vec{q}\,(t)$ from the previous question, which of the following gives the t value(s) when the ray $\vec{q}\,(t)$ strikes the sphere of radius 13 centered at the origin?

● $-20u_1 \pm \sqrt{400u_1^2 - 231}$

○ $169u_1^2 \pm \sqrt{400 + 169u_1}$

○ $400u_1^2 \pm \sqrt{231 + 100u_1}$

○ $-40u_1 \pm \sqrt{1600u_1^2 - 924}$

Calculator Hide Notes

✓

Solution:

The ray strikes the sphere at the t-values when $\vec{q}\,(t)\cdot\vec{q}\,(t) = 13^2 = 169$.

We have

$$\begin{aligned}\vec{q}\,(t)\cdot\vec{q}\,(t) &= (20+u_1t)^2+(u_2t)^2+(u_3t)^2 && (6.300)\\ &= 400+40u_1t+u_1^2t^2+u_2^2t^2+u_3^2t^2 && (6.301)\end{aligned}$$

Therefore, the desired values of t solve a quadratic equation at^2+bt+c with

$$\begin{aligned}a &= u_1^2+u_2^2+u_3^2=1 && (6.302)\\ b &= 40u_1 && (6.303)\\ c &= 231 && (6.304)\end{aligned}$$

By the quadratic formula, the values of t are given by

$$\begin{aligned}t &= \frac{-40u_1 \pm \sqrt{(-40u_1)^2-4\,(1)\,(231)}}{2} && (6.305)\\ &= -20u_1 \pm \sqrt{400u_1^2-231} && (6.306)\end{aligned}$$

Submit You have used 1 of 9 attempts

ℹ Answers are displayed within the problem

Image

2/2 points (graded)

Suppose the pixel of interest corresponds to the point $(19,a,b)$ for values $-1 \le a,b \le 1$. Find a mathematical condition on a,b that describes all a,b values such that the corresponding ray strikes the sphere.

The “corresponding ray" is the ray starting from $(20,0,0)$ and going through $(19,a,b)$.

(The answer is a number. No variables should appear in your answer box).

Formula	**Relationship**	**Number**	
a^2+b^2	○ $=$ ○ $\neq$ ● $\leq$ ○ $\geq$	169/231	✓ **Answer:** 169/231

✓

Solution:

The ray strikes the sphere if the discriminant appearing in the previous problem is nonnegative. Thus, the condition on u_1 is:

Calculator Hide Notes

$$400u_1^2 - 231 \geq 0 \tag{6.307}$$

Equivalently,

$$u_1^2 \geq 231/400 \tag{6.308}$$

For the given point $(19, a, b)$, the value of u_1 is $\dfrac{1}{\sqrt{1^2 + a^2 + b^2}}$.

Therefore the condition becomes

$$\frac{1}{1 + a^2 + b^2} \geq 231/400 \tag{6.309}$$

This can be written as

$$\begin{aligned} 1 + a^2 + b^2 &\leq 400/231 && (6.310) \\ a^2 + b^2 &\leq 169/231 && (6.311) \end{aligned}$$

Submit You have used 2 of 9 attempts

Answers are displayed within the problem

Pixel

2/2 points (graded)

In our 200×200 image, the pixel in row 121 and column 172 can be represented by the point $(19, 0.21, 0.72)$ in the image plane.

The corresponding ray $\vec{q}\,(t)$ as defined above strikes the sphere at two t-values, t_0 and t_1 with $t_0 < t_1$.

Find t_0 and t_1.

Note: it may be helpful to know that $\sqrt{1 + (0.21)^2 + (0.72)^2} = 5/4$.

$t_0 =$ 11 ✓ **Answer:** 11

$t_1 =$ 21 ✓ **Answer:** 21

Solution:

Let $\vec{u} = \begin{pmatrix} -1 \\ 0.21 \\ 0.72 \end{pmatrix}$. Note that $|\vec{u}| = 5/4$.

Letting $\hat{u} = \dfrac{\vec{u}}{|\vec{u}|} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, we have $u_1 = -4/5$.

We form the ray $\vec{q}\,(t) = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + t\dfrac{\vec{u}}{|\vec{u}|}$.

Calculator Hide Notes

By a previous problem, $\vec{q}\,(t)$ strikes the sphere at the t-values

$$t = -20u_1 \pm \sqrt{400u_1^2 - 231} \qquad (6.312)$$

In this case, $u_1 = -4/5$. Note that $400(-4/5)^2 - 231 = 25$. Therefore,

$$t_0 \;= 16 - \sqrt{25} = 11 \qquad (6.313)$$
$$t_1 \;= 16 + \sqrt{25} = 21 \qquad (6.314)$$

Submit

You have used 1 of 9 attempts

Answers are displayed within the problem

Strike the Sphere

3/3 points (graded)

At what point (x_S, y_S, z_S) does the ray from $\begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix}$ through $\begin{pmatrix} 19 \\ 0.21 \\ 0.72 \end{pmatrix}$ strike the sphere of radius 13 centered at the origin for the first time?

Note: you should use either t_0 or t_1, but not both. Choose carefully!

Do not round your answers. Three decimal points should be sufficient.

$x_S =$ 11.2 ✓ **Answer:** 11.2

$y_S =$ 1.848 ✓ **Answer:** 1.848

$z_S =$ 6.336 ✓ **Answer:** 6.336

Solution:

We form the ray

$$\vec{q}\,(t) \;= \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + t\frac{\vec{u}}{|\vec{u}|} \qquad (6.315)$$

$$= \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + \frac{4}{5}t \begin{pmatrix} -1 \\ 0.21 \\ 0.72 \end{pmatrix} \qquad (6.316)$$

In order to get the point where the ray first strikes the sphere, we use the value t_0 computed in a previous problem:

$$t_0 = 11 \qquad (6.317)$$

Therefore, the answer is

$$\begin{pmatrix} x_S \\ y_S \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \end{pmatrix} + \frac{4}{5}(11) \begin{pmatrix} -1 \\ 0.21 \end{pmatrix} = \begin{pmatrix} 11.2 \\ 1.848 \end{pmatrix}$$

Calculator Hide Notes

$$z_S \qquad 0 \qquad 0.72 \qquad 6.336$$

Submit You have used 1 of 9 attempts

Answers are displayed within the problem

Shading

1/1 point (graded)

Suppose the brightness of the sphere at the point (x, y, z) is given by $\dfrac{13-x}{26}$.

What should be the brightness of the pixel in row 121 and column 172 (using the point $(19, 0.21, 0.72)$ in the image plane)?

Round your answer to three decimal places.

(This type of shading is known as “directional lighting" and is used to model light from a sun-like source).

0.06923077 ✓ **Answer:** 0.069

Solution:

From the previous problem, we know that the corresponding ray strikes the sphere at a point with x-coordinate equal to 11.2. Using the formula for brightness, this corresponds to a brightness of

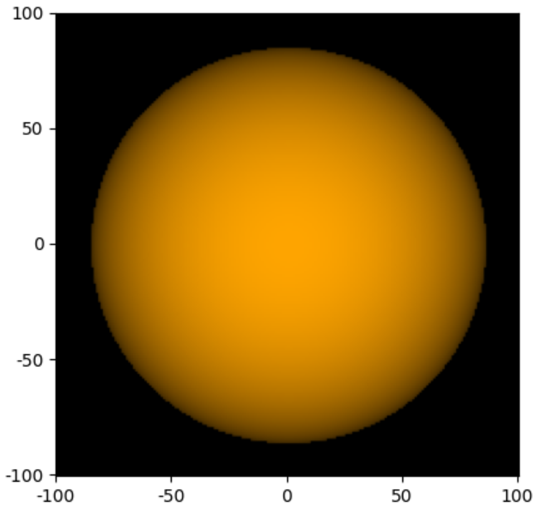
$$\frac{13-11.2}{26} = 0.069 \tag{6.319}$$

Submit You have used 1 of 9 attempts

Answers are displayed within the problem

Result

Carrying out the above procedure for all pixels in the 200×200 image, we can produce the following image:



Footnote: to produce this image, we multiplied the orange color by the eighth power of the “brightness" multiplier to exaggerate the shading effect.

Calculator Hide Notes

4. Ray tracing

Hide Discussion

Topic: Unit 5: Curves and Surfaces / 4. Ray tracing

Add a Post

Show all posts ⌄ by recent activity ⌄

There are no posts in this topic yet.

✖

< Previous Next >

edX

About
Affiliates
edX for Business
Open edX
Careers
News

Legal

Terms of Service & Honor Code
Privacy Policy
Accessibility Policy
Trademark Policy
Sitemap

Connect

Blog
Contact Us
Help Center
Media Kit
Donate

Calculator Hide Notes