4.5 Exponential

Unit 4: Continuous Random Variables

Adapted from Blitzstein-Hwang Chapter 5.

The Exponential distribution is a continuous distribution that is widely used as a simple model for the waiting time for a certain kind of event to occur, e.g., the time until the next email arrives.

Definition 4.5.1 (Exponential distribution).

A continuous r.v. X is said to have the *Exponential distribution* with parameter λ, where $\lambda > 0$, if its PDF is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

We denote this by $X \sim \mathrm{Expo}(\lambda)$.

The corresponding CDF is

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0.$$

The $\mathrm{Expo}(1)$ PDF and CDF are plotted in Figure 4.5.2.

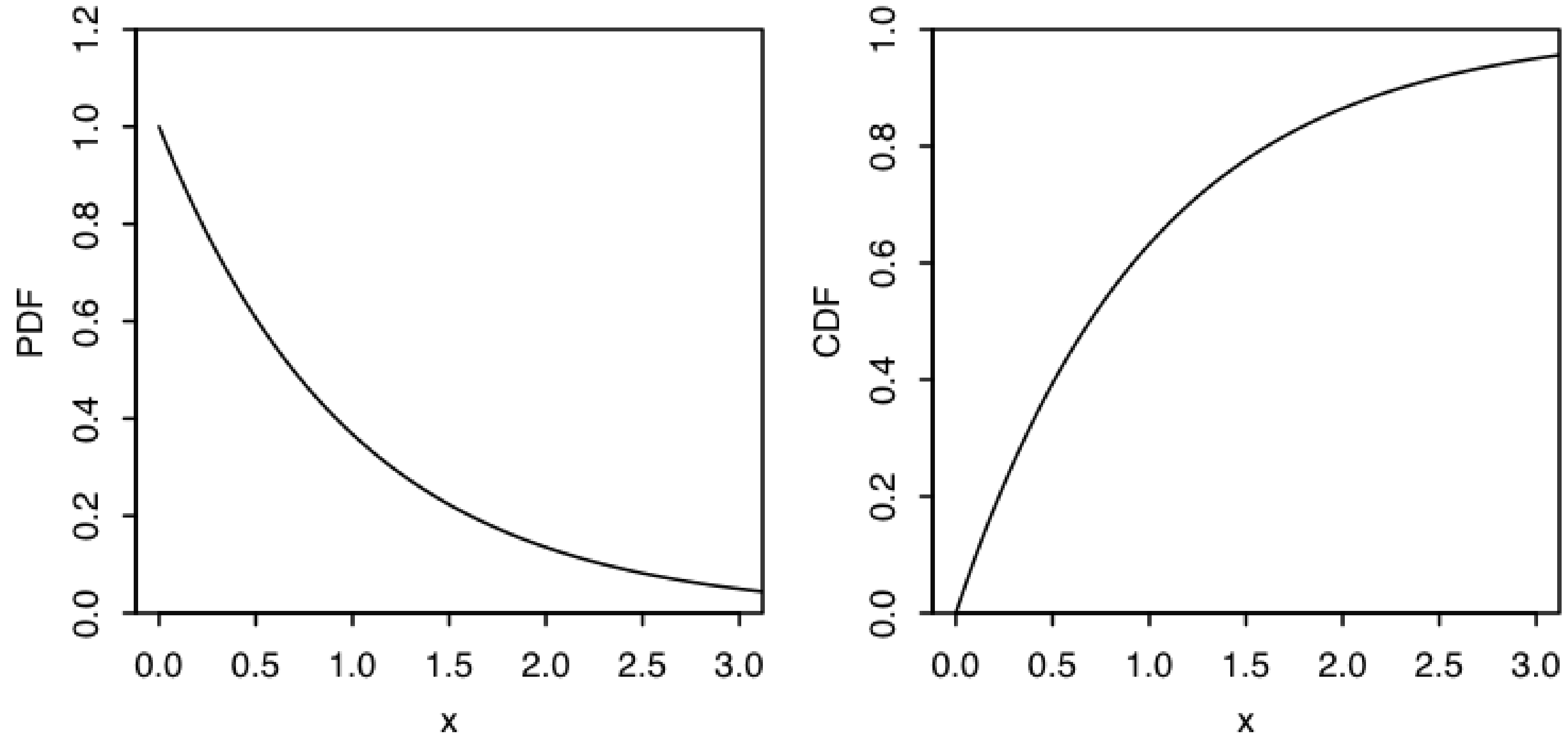

Figure 4.5.2: $\mathrm{Expo}(1)$ PDF and CDF.

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We've seen how all Uniform and Normal distributions are related to one another via location-scale transformations, and we might wonder whether the Exponential distribution allows this too. Exponential r.v.s are defined to have support $(0, \infty)$, and shifting would change the left endpoint. But scale transformations work nicely, and we can use scaling to get from the simple $\mathrm{Expo}(1)$ to the general $\mathrm{Expo}(\lambda)$: if $X \sim \mathrm{Expo}(1)$, then

$$Y = \frac{X}{\lambda} \sim \mathrm{Expo}(\lambda),$$

since

$$P(Y \leq y) = P\left(\frac{X}{\lambda} \leq y\right) = P(X \leq \lambda y) = 1 - e^{-\lambda y}, \quad y > 0.$$

Conversely, if $Y \sim \mathrm{Expo}(\lambda)$, then $\lambda Y \sim \mathrm{Expo}(1)$. The Exponential distribution has a very special property called the *memoryless property*. If the waiting time for a certain event to occur is Exponential, then the memoryless property says that no matter how long you have waited so far, your additional waiting time is still Exponential (with the same parameter).

Definition 4.5.3 (Memoryless property).

A distribution is said to have the *memoryless property* if a random variable X from that distribution satisfies

$$P(X \geq s+t | X \geq s) = P(X \geq t)$$

for all $s, t > 0$.

Here s represents the time you've already spent waiting; the definition says that after you've waited s minutes, the probability you'll have to wait another t minutes is exactly the same as the probability of having to wait t minutes with no previous waiting time under your belt. Another way to state the memoryless property is that conditional on $X \geq s$, the additional waiting time $X - s$ is still distributed $\mathrm{Expo}(\lambda)$.

Using the definition of conditional probability, we can directly verify that the Exponential distribution has the memoryless property. Let $X \sim \mathrm{Expo}(\lambda)$. Then

$$P(X \geq s+t | X \geq s) = \frac{P(X \geq s+t)}{P(X \geq s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \geq t).$$

What are the implications of the memoryless property? If you're waiting at a bus stop and the time until the bus arrives has an Exponential distribution, then conditional on your having waited 30 minutes, the bus isn't due to arrive soon. The distribution simply *forgets* that you've been waiting for half an hour, and your remaining wait time is the same as if you had just shown up to the bus stop. If the lifetime of a machine has an Exponential distribution, then no matter how long the machine has been functional, conditional on having lived that long, the machine is as good as new: there is no wear-and-tear effect that makes the machine more likely to break down soon. If human lifetimes were Exponential, then conditional on having survived to the age of 80, your remaining lifetime would have the same distribution as that of a newborn baby!

Clearly, the memoryless property is not an appropriate description for human or machine lifetimes. Why then do we care about the Exponential distribution?

1. Some physical phenomena, such as radioactive decay, truly do exhibit the memoryless property, so the Exponential is an important model in its own right.
2. The Exponential distribution is well-connected to other named distributions. In the next section, we'll see how the Exponential and Poisson distributions can be united by a shared story.
3. The Exponential serves as a building block for more flexible distributions, such as a distribution known as the *Weibull*, that allow for a wear-and-tear effect (where older units are due to break down) or a survival-of-the-fittest effect (where the longer you've lived, the stronger you get). To understand these distributions, we first have to understand the Exponential.