4. Interlude: Square Roots of Matrices

Interlude: Square root of a positive semi-definite matrix

Recall that a matrix $\mathbf{A}$ of size $d \times d$ is **positive semi-definite** if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$. Two example classes of positive semi-definite matrices are:

- Diagonal matrices with non-negative entries: $\mathbf{D} = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & c_d \end{pmatrix}$ where $c_i \geq 0$ for all i. (You have shown in exercise in a previous lecture that indeed $\mathbf{x}^T\mathbf{D}\mathbf{x} \geq 0$ for all $\mathbf{x}$.
- Matrix products $\mathbf{P}^T\mathbf{D}\mathbf{P}$ where $\mathbf{P}$ is an invertible (square) matrix and $\mathbf{D}$ is a diagonal matrix with non-negative entries (as above). **Proof:** $\mathbf{x}^T\left(\mathbf{P}^T\mathbf{D}\mathbf{P}\right)\mathbf{x} = \left(\mathbf{P}\mathbf{x}\right)^T\mathbf{D}\left(\mathbf{P}\mathbf{x}\right) = \mathbf{y}^T\mathbf{D}\mathbf{y} \geq 0$ for all vectors $\mathbf{x}$.

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The **positive semi-definite square root** (or simply the square root) of a positive semi-definite matrix $\mathbf{A}$ is another positive semi-definite matrix, denoted by $\mathbf{A}^{1/2}$, satisfying $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$. It is the case that for any positive semi-definite matrix (positive definite matrix, respectively), the positive semi-definite square root (positive definite square root, respectively) is unique.

Square Root of a Matrix

1/1 point (graded)

Using the definition above of the square root of a matrix, find the square root $\mathbf{D}^{1/2}$ of $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

(Enter your answer as a matrix, e.g. by typing **[[1,2],[5,1]]** for the matrix $\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$. Note the square brackets, and the commas as separators.)

$\mathbf{D}^{1/2} =$ [[sqrt(2),0],[0,0]] ✔ **Answer:** [[sqrt(2),0],[0,0]]

STANDARD NOTATION

Solution:

Since

$$\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

We have $\mathbf{D}^{1/2} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$.

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(Optional): Square Root of a Matrix

0 points possible (ungraded)
Let

$$\mathbf{A} = \mathbf{P}^T\mathbf{D}\mathbf{P} \qquad \text{where } \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Note that $\mathbf{P}^T = \mathbf{P}^{-1}$.

Find the square root $\mathbf{A}^{1/2}$ of the matrix $\mathbf{A}$.
Hint: $\mathbf{P}^T\mathbf{B}^2\mathbf{P} = \mathbf{P}^T\mathbf{B}\left(\mathbf{P}\mathbf{P}^T\right)\mathbf{B}\mathbf{P}$.

(Enter your answer as a matrix, e.g. by typing **[[1,2],[5,-1]]** for the matrix $\begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix}$. Note the square brackets, and commas as separators.)

$\mathbf{A}^{1/2} =$ [[sqrt(3)/2,-sqrt(3)/2],[-sqrt(3)/2,sqrt(3)/2]]

Answer: sqrt(3)/2*[[1,-1],[-1,1]]

STANDARD NOTATION

Solution:

$$\begin{aligned}
\left(\mathbf{P}^T\mathbf{D}^{1/2}\mathbf{P}\right)\left(\mathbf{P}^T\mathbf{D}^{1/2}\mathbf{P}\right) &= \mathbf{P}^T\mathbf{D}^{1/2}\left(\mathbf{P}\mathbf{P}^T\right)\mathbf{D}^{1/2}\mathbf{P} \\
&= \mathbf{P}^T\mathbf{D}^{1/2}\left(\mathbf{P}\mathbf{P}^{-1}\right)\mathbf{D}^{1/2}\mathbf{P} \qquad \text{since } \mathbf{P}^T = \mathbf{P}^{-1} \\
&= \mathbf{P}^T\mathbf{D}^{1/2}\mathbf{D}^{1/2}\mathbf{P} \\
&= \mathbf{P}^T\mathbf{D}\mathbf{P}
\end{aligned}$$

Hence $\mathbf{A}^{1/2} = \mathbf{P}^T\mathbf{D}^{1/2}\mathbf{P}$. Plugging in the values of $\mathbf{D}$ and $\mathbf{P}$, we get

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$$\begin{aligned}\mathbf{A}^{1/2} = \mathbf{P}^T\mathbf{D}^{1/2}\mathbf{P} &= \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\right)\begin{pmatrix}\sqrt{3} & 0\\ 0 & 0\end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\right)\\ &= \frac{1}{2}\begin{pmatrix}\sqrt{3} & 0\\ -\sqrt{3} & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\ &= \frac{\sqrt{3}}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\end{aligned}$$

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