Problem 6: Correlation coefficients

(6/6 points)

Consider the random variables X, Y and Z, which are given to be pairwise uncorrelated (i.e., X and Y are uncorrelated, X and Z are uncorrelated, and Y and Z are uncorrelated). Suppose that

- $\mathbf{E}[X] = \mathbf{E}[Y] = \mathbf{E}[Z] = 0$,
- $\mathbf{E}[X^2] = \mathbf{E}[Y^2] = \mathbf{E}[Z^2] = 1$,
- $\mathbf{E}[X^3] = \mathbf{E}[Y^3] = \mathbf{E}[Z^3] = 0$,
- $\mathbf{E}[X^4] = \mathbf{E}[Y^4] = \mathbf{E}[Z^4] = 3$.

Let $W = a + bX + cX^2$ and $V = dX$, where a, b, c, and d are constants, all greater than 0.

Find the correlation coefficients $\rho(X-Y, X+Y)$, $\rho(X+Y, Y+Z)$, $\rho(X, Y+Z)$ and $\rho(W, V)$.

1.

$\rho(X-Y, X+Y) =$ 0 ✔ **Answer:** 0

2.

$\rho(X+Y, Y+Z) =$ 1/2 ✔ **Answer:** 0.5

3.

$\rho(X, Y+Z) =$ 0 ✔ **Answer:** 0

4. $\rho(W, V) =$

- ○ $\frac{b}{\sqrt{b^2+c^2}}$
- ○ $\frac{b^2}{\sqrt{b^2+2c^2}}$
- ○ $\frac{bd}{\sqrt{b^2+2c^2}}$
- ◉ $\frac{b}{\sqrt{b^2+2c^2}}$ ✔

Answer:

1. We have

$$\begin{aligned}\mathrm{cov}(X-Y, X+Y) &= \mathbf{E}\big[(X-Y)(X+Y)\big] - \mathbf{E}[X-Y]\mathbf{E}[X+Y] \\ &= \mathbf{E}[X^2 - Y^2] - 0 \\ &= \mathbf{E}[X^2] - \mathbf{E}[Y^2] \\ &= 0.\end{aligned}$$

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Hence, $\rho(X-Y, X+Y) = 0$.

2. Since X and Y are uncorrelated, with zero means, we have $\mathbf{E}[XY] = \mathbf{cov}(X,Y) = 0$. Similarly, we have $\mathbf{E}[XZ] = 0$ and $\mathbf{E}[YZ] = 0$.
Hence,

$$\begin{aligned} \mathbf{cov}(X+Y, Y+Z) &= \mathbf{E}[(X+Y)(Y+Z)] - \mathbf{E}[X+Y]\mathbf{E}[Y+Z] \\ &= \mathbf{E}[XY + XZ + Y^2 + YZ] \\ &= \mathbf{E}[Y^2] \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} \mathbf{var}(X+Y) &= \mathbf{E}[(X+Y)^2] - (\mathbf{E}[X+Y])^2 \\ &= \mathbf{E}[X^2 + 2XY + Y^2] - 0 \\ &= 2. \end{aligned}$$

Similarly, $\mathbf{var}(Y+Z) = 2$.

Therefore, $\rho(X+Y, Y+Z) = \frac{\mathbf{cov}(X+Y,Y+Z)}{\sqrt{\mathbf{var}(X+Y)\mathbf{var}(Y+Z)}} = \frac{1}{2}$.

3.

$$\begin{aligned} \mathbf{cov}(X, Y+Z) &= \mathbf{E}[X(Y+Z)] - \mathbf{E}[X]\mathbf{E}[Y+Z] \\ &= \mathbf{E}[XY + YZ] - 0 \\ &= 0. \end{aligned}$$

Hence, $\rho(X, Y+Z) = 0$.

4.

$$\begin{aligned}
\mathrm{cov}(W,V) &= \mathbf{E}[WV] - \mathbf{E}[W]\mathbf{E}[V] \\
&= \mathbf{E}[adX + bdX^2 + cdX^3] - 0 \\
&= bd, \\
\mathrm{var}(W) &= \mathbf{E}[W^2] - (\mathbf{E}[W])^2 \\
&= \mathbf{E}[a^2 + 2abX + (2ac+b^2)X^2 + 2bcX^3 + c^2X^4] - (a + b\mathbf{E}[X] + c\mathbf{E}[X^2])^2 \\
&= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + 2ac + c^2) \\
&= b^2 + 2c^2, \\
\mathrm{var}(V) &= \mathbf{E}[d^2X^2] - (\mathbf{E}[dX])^2 = d^2.
\end{aligned}$$

Hence, $\rho(W,V) = \frac{bd}{\sqrt{d^2(b^2+2c^2)}} = \frac{b}{\sqrt{b^2+2c^2}}$.

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