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Lebesgue Measurable B

August 9, 2015 • Analysis

The Basic Idea

Our goal for today is to construct a Lebesgue measurable se set exists because the Lebesgue measure is the *completion* o collection $\mathscr{B}$ of Borel sets is generated by the open sets, whe measurable sets $\mathscr{L}$ is generated by both the open sets *and* z the containment is a proper one.

To produce a set in $\mathscr{L} \setminus \mathscr{B}$, we'll assume two facts:

1. Every set in $\mathscr{L}$ with positive measure contains a non (L
2. 97.3% of all counterexamples in real analysis involve th

Okay okay, the last one isn't *really* a fact, but it may not surp central to today's discussion. In summary, we will define a h function with a continuous inverse) from $[0,1]$ to $[0,2]$ which set) of measure 0 to a set of measure 1. By fact #1, this set of measurable subset, say N. And the preimage of N will be Le a Borel set. We'll fill in the details below, and while we do, ke a homeomorphism - a merely continuous function just won' Cantor set, we'll see that *homeomorphisms* (much less contin *preserve measure*. It's because of this that we can produce a not Borel.

From English to Math

Begin by defining a function $f : [0,1] \to [0,2]$ by

$$f(x) = c(x) + x$$

where $c : [0,1] \to [0,1]$ is the Cantor function. The graph of the horizontal lines are now all tilted with a slope of 1. I've dr iterations. This function has the following properties:

f is strictly increasing

- since $f' = 1$ almost everywhere (recall $c' = 0$ almost everywhere)

f is continuous

- since both c and x are continuous

f^{-1} exists

- f is 1-1 since it's strictly increasing; it's onto by the Intermediate Value Theorem: since $f(0) = 0$, $f(1) = 2$ f is continuous, it assumes all values in between 0 and

f^{-1} is continuous (hence f is a homeomorphism

- see footnote *

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We should also observe that f maps the intervals of $[0,1]$ wh construction of the Cantor set $\mathscr{C}$ to intervals of $[0,2]$ of the s

$$\mu(f([0,1] \setminus \mathscr{C})) = \mu([0,1] \setminus \mathscr{C}$$

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But since $[0,2] = f(\mathscr{C}) \sqcup f([0,1] \setminus \mathscr{C})$, we see that
$2 = \mu([0,2]) = \mu(f(\mathscr{C})) + \mu(f([0,1] \setminus \mathscr{C})) = \mu(f(\mathscr{C})) + 1$

$$\mu(f(\mathscr{C})) = 1.$$

From this we deduce that $f(\mathscr{C}) \subset [0,2]$ contains a non-mea
in the introduction). And here is where we make our

Claim: $f^{-1}(N)$ is Lebesgue measurable but *not* I

This is easy to prove, but its substance lies in the following

Lemma: A strictly increasing function defined or sets to Borel sets.

Proof of Lemma

We follow exercises #45-47 of ch. 2 in Royden's *Real Analysis* increasing function defined on some interval. By our analysis function is a homeomorphism. This fact enables us to show t sets. To do so, it suffices to show that for any continuous fur

$$\mathscr{A} = \{E : g^{-1}(E) \text{ is Borel}$$

is a σ-algebra containing the open sets. Once we show this, the Borel sets and therefore, taking g to be f^{-1} (which we kr $(f^{-1})^{-1}(E) = f(E)$ is Borel for any Borel set E, which is wh

Showing $\mathscr{A}$ is a σ-algebra (the first two bullets) which conta is simple enough (recall that $\mathscr{B}$ denotes the Borel sets):

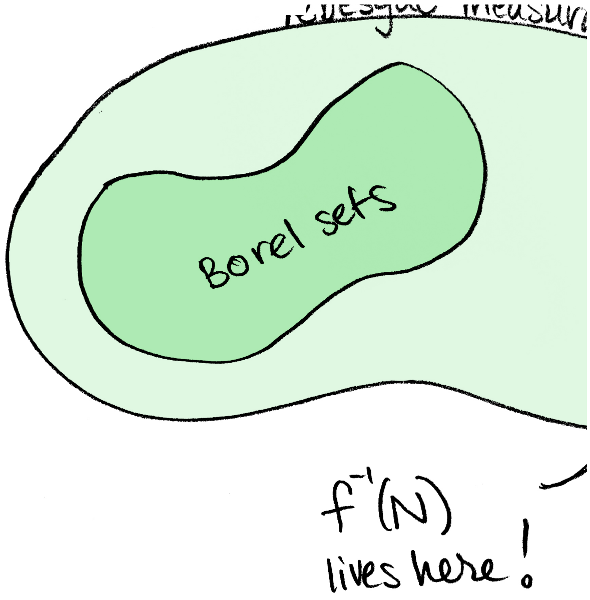
- If $\{E_i\} \subset \mathscr{A}$ then $f^{-1}(\cup E_i) = \cup f^{-1}(E_i) \in \mathscr{B}$ since $\mathscr{B}$
- If $E \subset \mathscr{A}$ then $f^{-1}(E^c) = (f^{-1}(E))^c \in \mathscr{B}$ since $\mathscr{B}$ is a
- If U is open, then $f^{-1}(U)$ is open and thus an element

We are now ready for the

Proof of Claim

Since $N \subset f(\mathscr{C})$, we know that $f^{-1}(N) \subset \mathscr{C}$ is measurable (subset of a zero set and the Lebesgue measure is complete. I were, then since f maps Borel sets to Borel sets by our Lemr is Borel. But that's impossible since N isn't even measurable!

Lebesgue meas...



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Footnotes

**Proof*: Let $h = f^{-1} : [0,2] \to [0,1]$ and suppose $U \subset [0,1]$
and hence closed (and bounded). Since f is continuous, $f([0$
rewrite this as

$$\begin{aligned} f([0,1] \smallsetminus U) &= f([0,1]) \smallsetminus f \\ &= [0,2] \smallsetminus f(U) \\ &= [0,2] \smallsetminus h^{-1}(\end{aligned}$$

which allows us to conclude $h^{-1}(U)$ is open.

***Proof*: This follows simply because c is constant on any int
interval $(a,b) \subset [0,1] \smallsetminus \mathscr{C}$, we have $c(a) = c(b)$ and so

$$\begin{aligned} \mu((f(a), f(b))) &= f(b) - f(a) \\ &= c(b) + b - c(\\ &= b - a. \end{aligned}$$

References

- Much of today's discussion is taken from here.

- see also *Real Analysis* (4ed) by Royden, section 2.7, Proposit

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la flaca • 3 years ago

Hello, I am frequent reader of your blog, which
student helps me a lot. I just have a question:
When you prove (in the footnote section) that t
of f is continuous you take the set [0,1]\U whicl
and conclude that its image under f is also clos
this? I don't see how the continuity of f implies

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Tai-Danae Bradley ➦ la flaca • 3 years ago

Hi la flaca, I'm so glad you enjoy the blo
glad you asked the question. The 2nd/3
sentences of the footnote aren't worded
Here's the idea:

[0,1] \ U is both closed and bounded. Th
(by Heine Borel) it's compact. Because
continuous, the image f([0,1] \ U) is also
and therefore (again, by Heine Borel) it
closed.

^ | ˅ • Reply • Share ›

la flaca ➦ Tai-Danae Bradley • 3 years

Of course! I forgot about Heine-
it's clear to me. Thank you very
your quick response and thanks
your great articles, they do what
books should do: explain things
and comprehensive way.

^ | ˅ • Reply • Share ›

Jorge Garcia • 2 years ago

Thank you so much for the article! It really help
But I have a question: when you say "The colle
Borel sets is generated by the open sets, when
of Lebesgue measurable sets is generated by
open sets and zero sets." What do you mean b
"zero sets"? I had never Heard of this generati
Lebesgue measurable sets before and I am ve
interested in understanding it!

^ | ˅ • Reply • Share ›

Tai-Danae Mod ➦ Jorge Garcia • 2 years ago

Hi Jorge, I'm glad you found the math in
A "zero set" can be thought of as a set
very small. It's a bit like: if you tried to m
with a ruler, then it would be *so* tiny, y
do it! Or if you tried to put it on a scale,
weigh nothing! But the ruler/scale is rea
something called the Lebesgue measu
more, you might enjoy the book *Real
Mathematical Analysis* by Charles Pug
discussion of zero sets is on p. 365). Yo
find more resources under the 'real ana
section of this link:
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Jorge Garcia ➦ Tai-Danae • 2 years a

Hi Tai-Danae! Thanks for your a