3. Baking a cake

Baking a cake is actually a complicated process. Here, we vastly simplify the process by modeling the baking of a 1-dimensional cake of length L that starts at room temperature (20 $^\circ\mathrm{C}$). One end is exposed directly to the oven temperature (200 $^\circ\mathrm{C}$). The thin 1-D cake is insulated and the other end is insulated by the pan.

The main difference in this model is that we assume that both the thermal conductivity $k\left(\theta\right)$ and the heat capacity $c\left(\theta\right)$ vary with respect to temperature. We ignore all other baking effects.

In this case, the partial differential equation is

$$c\left(\theta\right)\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left(k\left(\theta\right)\frac{\partial\theta}{\partial x}\right)\qquad 0<x<L,\ t>0.$$

The initial condition is

$$\theta\left(x,0\right)=20,\qquad 0<x<L.$$

The boundary conditions are

$$\theta\left(0,t\right)=200,\quad \frac{\partial}{\partial x}\theta\left(L,t\right)=0,\qquad t>0.$$

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This partial differential equation cannot be solved by separation of variables. However, it can be solved numerically. In the following problems, we will guide you through developing a function to model the thermal conductivity $k(\theta)$ and heat capacity $c(\theta)$, and then discretize and solve this PDE for cakes of different sizes to compare the baking time.

Model the conductivity and capacity (External resource)

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$$c(\theta)\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}k(\theta)\frac{\partial\theta}{\partial x}, \qquad 0 < x < L,\ t > 0$$

with the boundary conditions $\theta(0,t) = 200$ and $\dfrac{\partial}{\partial x}\theta(L,t) = 0$ and initial condition $\theta(x,0) = 20$.

We will use a **forward in time, centered in space** numerical scheme. Let θ_j^i denote the solution at time $i\Delta t$ and position $j\Delta x$.

Then a discrete forward time time derivative is

$$\frac{\partial\theta}{\partial t} \approx \frac{\theta_j^{i+1} - \theta_j^i}{\Delta t}.$$

To create the centered space derivative, we will take a backwards in space derivative the first time, and then a forwards in space derivative the second time.

What do we do about the values of $c(\theta)$ and $k(\theta)$?

- Evaluate $c(\theta)$ at the initial point in time.
- Take an average of the values of $k(\theta)$ between the two points in space.

Substituting the discrete time and space derivatives into the heat equation above gives

$$\begin{aligned} c(\theta_j^i)\frac{\theta_j^{i+1} - \theta_j^i}{\Delta t} &= \frac{\partial}{\partial x}\left(\frac{k(\theta_j^i) + k(\theta_{j-1}^i)}{2}\right)\left(\frac{\theta_j^i - \theta_{j-1}^i}{\Delta x}\right) \\ &= \frac{\left(\frac{k(\theta_{j+1}^i)+k(\theta_j^i)}{2}\right)\left(\frac{\theta_{j+1}^i-\theta_j^i}{\Delta x}\right) - \left(\frac{k(\theta_j^i)+k(\theta_{j-1}^i)}{2}\right)\left(\frac{\theta_j^i-\theta_{j-1}^i}{\Delta x}\right)}{\Delta x} \end{aligned}$$

$$\theta_j^{i+1} = \theta_j^i + \frac{\Delta t}{2c\left(\theta_j^i\right)(\Delta x)^2}\left[\left(k\left(\theta_{j+1}^i\right)+k\left(\theta_j^i\right)\right)\left(\theta_{j+1}^i-\theta_j^i\right)-\left(k\left(\theta_j^i\right)+k\left(\theta_{j-1}^i\right)\right)\left(\theta_j^i-\theta_{j-1}^i\right)\right]$$

where at each time step i we impose the boundary conditions $\theta_1^i = 200$ and $\theta_N^i = \theta_{N-1}^i$.

Find the times to bake different sized cakes

2/2 points (graded)

A cake is baked once it has reached $100^\circ\mathrm{C}$. Use the code we provide for you and modify it to find the length of time it takes to bake a small cake ($L = 0.1$m) and a large cake ($L = 0.2$m).

(*Hin: Use an if then statement to break the animation once the cake is baked.*)

Copy and paste the following into MATLAB Online to run the script. (Use short cut commands for copy and paste: ctrl-c and ctrl-v on windows, and cmd-c, cmd-v on a MAC.)

```
url = 'https://courses.edx.org/asset-v1:MITx+18.03Fx+3T2018+type@asset+block@heatEqn_cake_bake.m';
websave('heatEqn_cake_bake.m',url)
```

Small cake bake time:

38.46 ✔ **Answer:** 38.46

Large cake bake time:

153.96 ✔ **Answer:** 153.96

Solution:

The easiest way to determine the time is to set the max time to some large number (tmax = 200 for example) and add a bit of code that stops the run time once the center (which is the last to cook) reaches $100^\circ\mathrm{C}$.

That is, before the end of the for loop that runs through the time steps, you can add a bit of code that says

```
if u(i+1,end)>100 break end
```

If you do this, you find that a cake of length 0.1 meters is fully cooked when $t = 38.46$, and a cake of length 0.2 meters is cooked when $t = 153.96$.

Even this simple calculation shows why professional cake bakers bake small cakes and assemble them into larger structures. The much longer baking time for larger cakes leads to over cooking or burning of the edges before the middle is cooked.

Submit You have used 2 of 5 attempts

Answers are displayed within the problem

3. Baking a cake

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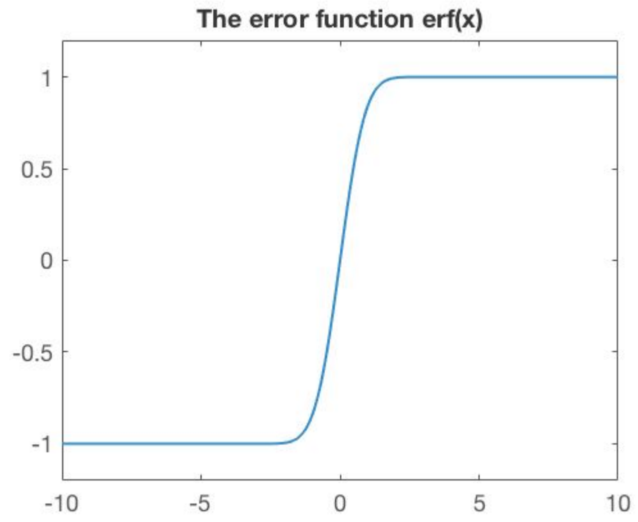
Raw cake batter has more thermal conductivity and a larger heat capacity as it contains a large amount of water. As the water bakes off, the cake batter begins to resemble a foam, which has a lower thermal conductivity, and lower heat capacity. In this model, we assume that the cake is baked once it reaches $100°\mathrm{C}$, and it is at this point that the thermal properties change as well.

Based on a paper that both experimentally and numerically modeled the baking of a cake, and measured these quantities, we will model raw cake batter with a thermal conductivity 0.31 $(\mathrm{Watts}/(\mathrm{meter}\cdot°\mathrm{Celsius}))$ at room temperature and heat capacity of 2800 $(\mathrm{Joules}/\mathrm{kg}°\mathrm{Celsius})$, and cooked cake batter with a thermal conductivity 0.19 $(\mathrm{W}/(\mathrm{m}°\mathrm{C}))$ and a heat capacity 2200 $(\mathrm{J}/\mathrm{kg}°\mathrm{C})$. We suppose that it varies continuously between these two values by using an error function. (We will ignore the steepness of this function to simplify our model.)

The error function

```
erf(x)
```

varies smoothly, but sharply, between -1 and 1 at $x = 0$.



Your job is to scale and translate this error function to create a function modeling the thermal conductivity $k(\theta)$ as a function of temperature that is 0.31 for temperatures less than $100°\mathrm{C}$ and 0.19 for temperatures greater $100°\mathrm{C}$. We let the transition happen at $100°\mathrm{C}$.

Similarly, create a function $c(\theta)$ modeling the heat capacity that is 2800 for temperatures less than $100°\mathrm{C}$, and is 2200 for temperatures greater than $100°\mathrm{C}$.

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Script

Save Reset MATLAB Documentation (https://www.mathworks.com/help/)

```
theta = linspace(20,200,1000);

%Create a vector k that is the correct scaled/translated erf function
% which is evaluated at the vector theta and is 0.31 for theta < 100,
% and is 0.19 for theta > 100
k = 0.25 + 0.06*erf(100-theta); %(1+erf(theta-100))/2;
%k = (k < 0) * (0.31) + (k > 0) * (0.19)
plot(theta, k);

%Create a vector c that is the correct scaled/translated erf function
% which is evaluated at the vector theta and is 2800 for theta < 100,
% and is 2200 for theta > 100
c = 2500 + 300*erf(100-theta);
%plot(theta, c);

```

Run Script

Assessment: All Tests Passed

Submit

Check k

Check c

Output

Discretize the PDE

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Let's numerically solve the heat equation