The Basel Problem (1)

- The proof of the Prime Number Theorem uses the **Riemann zeta function**.

$$\zeta(s) = \sum_{N=1}^{\infty} \frac{1}{N^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

- The special values of ζ(s) were studied by Euler in the 18^{th} century.

The Basel Problem (2)

Basel Problem

Calculate the sum of the inverses of the squares?

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

Pietro Mengoli (1626-1686)

Leonhard Euler (1707-1783)

https://en.wikipedia.org/wiki/Pietro_Mengoli

The Basel Problem (3)

- Basel is Euler's hometown.

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

Leonhard
Euler
(1707-1783)

https://en.wikipedia.org/wiki/Basel

The Basel Problem (4)

Answer (Euler, 1734)

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$
$$= \frac{\pi^2}{6}$$

π **= 3.1415926535...** is the circumference of a circle with diameter 1.

The Basel Problem shows a **mysterious connection** between π and the squares!

The Basel Problem (5)

- Euler used the sine function

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^2}{(3\pi)^2}\right)\left(1-\frac{x^2}{(4\pi)^2}\right)\cdots$$

- He obtained mysterious formulae for ξ (2N). It is a product of π^{2N} and a (mysterious) **rational number**.

$$\zeta(2)=\frac{\pi^2}{6} \qquad \zeta(4)=\frac{\pi^4}{90} \qquad \zeta(6)=\frac{\pi^6}{945}$$

$$\zeta(8)=\frac{\pi^8}{9450} \qquad \zeta(10)=\frac{\pi^{10}}{93555} \qquad \zeta(12)=\frac{691\pi^{12}}{638512875}$$

The Basel Problem (6)

- The rational numbers appearing in Euler's formula of $\xi(2N)$ play an important role in Kummer's theory of cyclotomic fields.
- Kummer proved
  Fermat's Last Thm
  $\mathbf{X^P + Y^P = Z^P}$
  for many P.

Ernst Eduard Kummer (1810-1893)

Kenkichi Iwasawa (1917-1998)

https://en.wikipedia.org/wiki/Ernst_Kummer
https://alchetron.com/Kenkichi-Iwasawa-788267-W