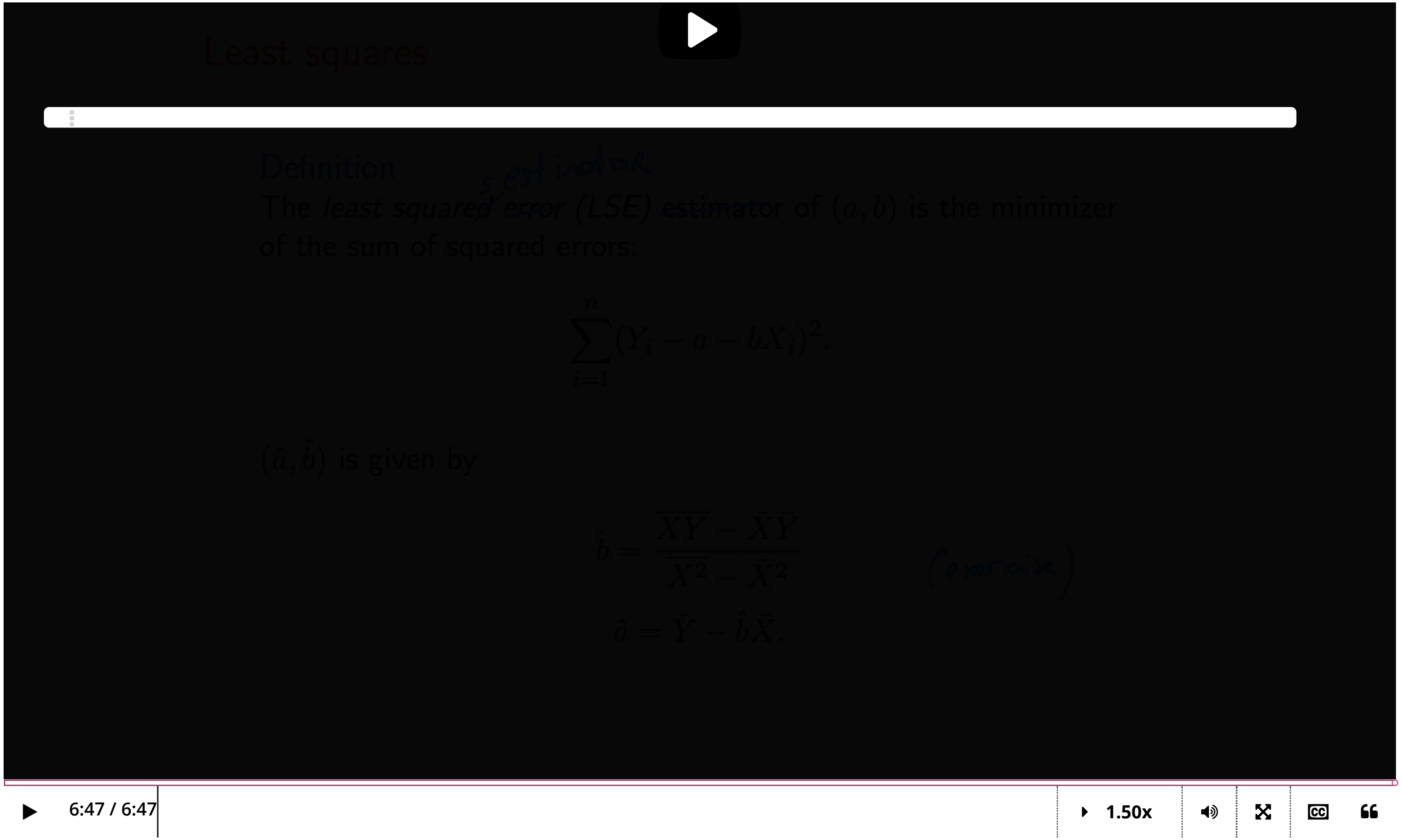
10. Empirical Linear Regression via The Statistical Hammer Least Squares Estimator (LSE)



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In **empirical linear regression** , we are given a collection of points $\{(x_i, y_i)\}_{i=1}^{n}$. The goal is to fit a linear model $Y = a + bX + \varepsilon$ by computing the **Least Squares Estimator** , which minimizes the **loss function**

$$\frac{1}{n}\sum_{i=1}^{n}\left(y_i-(a+bx_i)\right)^2.$$

Using the same technique as in the problems on theoretical linear regression, one obtains the solution

$$\hat{a}=\overline{y}-\frac{\overline{xy}-\overline{x}\cdot\overline{y}}{\overline{x^2}-\overline{x}^2}\overline{x}\qquad \hat{b}=\frac{\overline{xy}-\overline{x}\cdot\overline{y}}{\overline{x^2}-\overline{x}^2}.$$

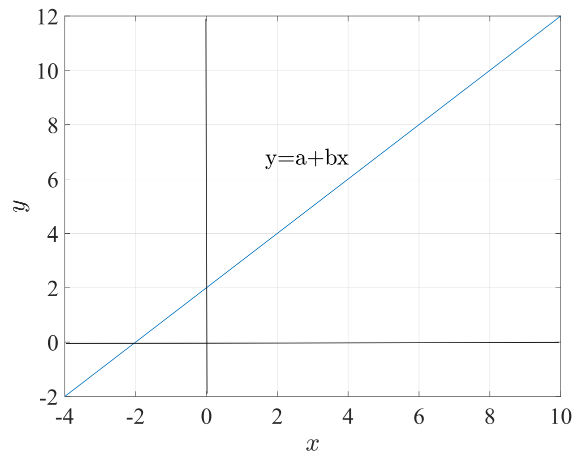
In this particular case, this is precisely what one obtains by taking the least squares solution for the theoretical linear regression problem

$$a=\mathbb{E}\left[Y\right]-\frac{\mathsf{Cov}\left(X,Y\right)}{\mathsf{Var}\left(X\right)}\mathbb{E}\left[X\right]\qquad b=\frac{\mathsf{Cov}\left(X,Y\right)}{\mathsf{Var}\left(X\right)}$$

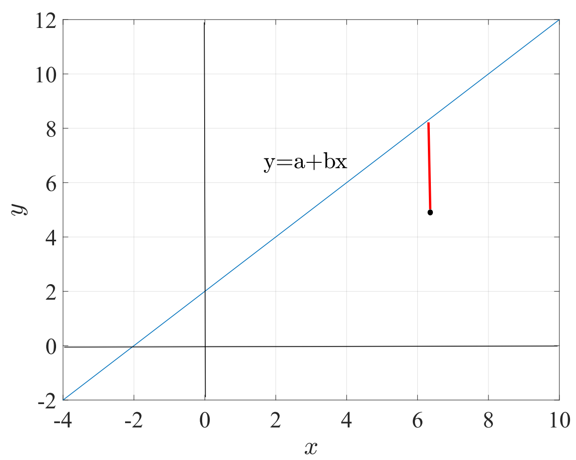
and replacing each term with their empirical counterparts according to the *plug-in principle*, i.e. $\mathbb{E}\left[X\right]$ with $\overline{x}$, $\mathsf{Var}\left(X\right)=\mathbb{E}\left[X^2\right]-\mathbb{E}[X]^2$ with $\overline{x^2}-\overline{x}^2$, etc. It happens to work nicely in this setting, but this trick does not always work out in general! (See if you can reproduce the proof that this formula is the correct one.)

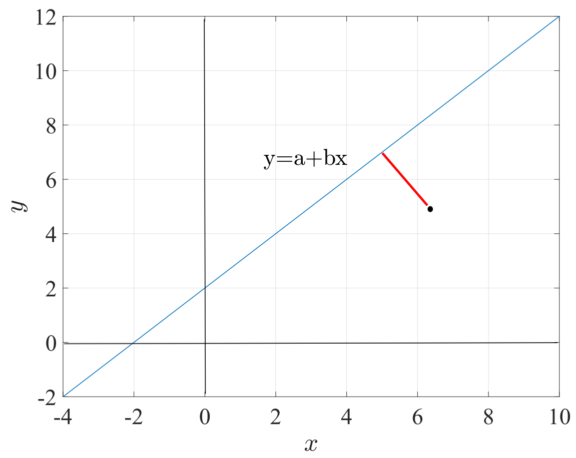
Which Squared Loss?

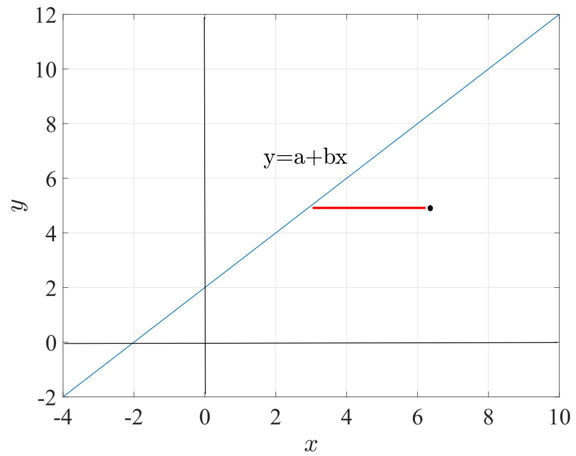
1/1 point (graded)
Consider the line $y=a+bx$, illustrated below.



In each of the following choices, (X, Y) is an arbitrary point, not necessarily on the line. Which choice best illustrates, via a line segment highlighted in red, the distance that is squared by the expression $(Y - (a + bX))^2$?







Solution:

The specified quantity $(Y-(a+bX))^2$ is the squared difference between the Y coordinate of the point (X,Y) and that of the point $(X,a+bX)$ predicted by the line. This represents the **vertical** squared distance between these two points.

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Answers are displayed within the problem

Assumptions of Linear Regression

1/1 point (graded)

Assume that we are given a collection of n points $\{(x_i,y_i)\}_{i=1}^n$. Which one of the following choices, on its own, provides a **sufficient** condition under which a unique least-squares estimator $(\hat{a},\hat{b})$ exists?

- There are at least two total observations; i.e. $(n\geq 2)$.
- There are at least two distinct x's in the collection; i.e. $x_1\neq x_2$.
- There are at least two distinct y's in the collection; i.e. $y_1\neq y_2$.

Solution:

The correct choice is "**There are at least two distinct x's in the collection; i.e. $x_1\neq x_2$**". The other two choices allow collections of points where all of the x's are equal: $x_1=\cdots=x_n$. This makes it so that the empirical variance is equal to zero. Just as in the case of theoretical linear regression, this means that there is an infinite family of lines that minimize the empirical least-squared error. More specifically, any line that crosses $(\bar{x},\bar{y})$, regardless of the slope, is a minimizer.

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Linear Least Squares: A Numerical Example

2/2 points (graded)

Consider the four points $(x_1, y_1) = (-5, -10)$, $(x_2, y_2) = (0, 3)$, $(x_3, y_3) = (2, 11)$ and $(x_4, y_4) = (3, 14)$. The line that minimizes the empirical squared error can be expressed as $y = \hat{a} + \hat{b}x$, where

$\hat{a} =$ 4.5 ✔ **Answer:** 4.5

$\hat{b} =$ 3.0 ✔ **Answer:** 3

Solution:

We may directly apply the formula

$$\hat{b} = \frac{\frac{1}{4}\sum_{i=1}^{4}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right)}{\frac{1}{4}\sum_{i=1}^{4}\left(x_i - \overline{x}\right)^2}$$

$$\hat{a} = \overline{y} - \hat{b}\overline{x}.$$

to obtain the pair $(\hat{a}, \hat{b}) = (4.5, 3)$.

Alternatively, the calculation can be simplified using the observation that $\overline{x} = \frac{1}{4}(-5 + 0 + 2 + 3) = 0$. In fact, recall that Cov $(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$ and the numerator of b is the empirical estimate of Cov (X, Y). Since $\overline{x}$ is zero, the numerator of b reduces to the empirical estimate of $\mathbb{E}[XY]$, or $\frac{1}{4}\sum_{i=1}^{4} x_i y_i$. Therefore, the above calculation can be computed as

$$\hat{b} = \frac{\sum_{i=1}^{4} x_i y_i}{\sum_{i=1}^{4} x_i^2} = \frac{(-5 \cdot -10) + (0 \cdot 3) + (2 \cdot 11) + (3 \cdot 14)}{(-5)^2 + 0^2 + 2^2 + 3^2} = \frac{114}{38} = 3$$

$$\hat{a} = \bar{y} = \frac{-10+3+11+14}{4} = \frac{18}{4} = 4.5.$$

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