

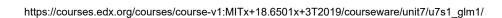


<u>Lecture 21: Introduction to</u> <u>Generalized Linear Models;</u>

<u>Course</u> > <u>Unit 7 Generalized Linear Models</u> > <u>Exponential Families</u>

> 6. The Exponential Family

6. The Exponential Family Exponential Families: Definition





Recall from lecture that a family of distribution $\{\mathbf{P}_{\theta}: \boldsymbol{\theta} \in \Theta\}$, where the parameter space $\Theta \subset \mathbb{R}^k$ is k-dimensional, is called a k-parameter exponential family on \mathbb{R}^q if the pmf or pdf $f_{\theta}: \mathbb{R}^q \to \mathbb{R}$ of \mathbf{P}_{θ} can be written in the form

$$f_{oldsymbol{ heta}}\left(\mathbf{y}
ight) = h\left(\mathbf{y}
ight) \exp\left(oldsymbol{\eta}\left(oldsymbol{ heta}
ight) \cdot \mathbf{T}\left(\mathbf{y}
ight) - B\left(oldsymbol{ heta}
ight)
ight) \qquad ext{where} egin{dcases} oldsymbol{\eta}\left(oldsymbol{ heta}\left(\mathbf{\eta}_{1}\left(oldsymbol{ heta}
ight)
ight) & : \mathbb{R}^{k}
ightarrow \mathbb{R}^{k} \ T_{k}\left(\mathbf{y}
ight) & : \mathbb{R}^{q}
ightarrow \mathbb{R}^{k} \ B\left(oldsymbol{ heta}
ight) & : \mathbb{R}^{k}
ightarrow \mathbb{R}, \end{cases}$$

When k=q=1, this reduces to

$$f_{ heta}\left(y
ight) = h\left(y
ight) \, \exp\left(\eta\left(heta
ight) T\left(y
ight) - B\left(heta
ight)
ight).$$

Note: The following exercises are similar to what will be presented in lecture, but we encourage you to first attempt these yourselves.

Practice: Decomposing the exponent

4/4 points (graded)

For the two following pmfs with one parameter heta that are written in the form

$$f_{ heta}\left(y
ight) \; = \; h\left(y
ight)e^{w\left(heta,y
ight)},$$

first decompose $w\left(heta,y
ight)$ as

$$w(\theta, y) = \eta(\theta) T(y) - B(\theta),$$

then enter the product $\eta\left(heta\right) T\left(y
ight)$ below. Select the distribution that $f_{ heta}$ defines.

1. For
$$f_{ heta}\left(y
ight) = e^{w\left(heta,y
ight)}$$
 where

$$w\left(heta,y
ight) =y\ln \left(heta
ight) +\left(1-y
ight) \ln \left(1- heta
ight)$$

and
$$y=0,1,\, heta\in (0,1)$$
 :

$$\eta\left(heta
ight)T\left(y
ight)=$$
 $y*\ln(heta/(1- heta))$ $y\cdot\ln\left(rac{ heta}{1- heta}
ight)$

What distribution does the pmf $f_{ heta}\left(y
ight)$ define?

- $\bigcirc \mathcal{N}\left(heta,1
 ight)$
- $\bigcirc \mathcal{N}\left(1, heta
 ight)$
- lacksquare Ber (θ)
- \bigcirc Poiss (θ)
- none of the above

~

^{2.} For $f_{ heta}\left(y
ight)=rac{1}{y!}e^{w\left(heta,y
ight)}$ where $w\left(heta,y
ight)=- heta+y\ln\left(heta
ight),$ and $y=0,1,2,\ldots,\, heta\in\left(0,1
ight)$:

$$\eta\left(heta
ight)T\left(y
ight)=$$
 $y*\ln(heta)$ $y\cdot\ln\left(heta
ight)$ Answer: $y*\ln(heta)$

What distribution does the pmf $f_{\theta}\left(y\right)$ define?

- $\bigcirc \mathcal{N}\left(heta,1
 ight)$
- $\bigcirc \mathcal{N}\left(1, heta
 ight)$

 \bigcirc Ber (θ)

lacksquare Poiss (θ)

none of the above



STANDARD NOTATION

Solution:

1. For $f_{ heta}\left(y
ight)=\,e^{w\left(heta,y
ight)}$ where $w\left(heta,y
ight)=y\ln\left(heta
ight)+\left(1-y
ight)\ln\left(1- heta
ight)$ and $y\in\{0,1\},\, heta\in\left(0,1
ight)$:

$$w\left(heta,y
ight) \,=\, y \ln \left(heta
ight) + \left(1-y
ight) \ln \left(1- heta
ight) \,=\, y \left(\ln \left(heta
ight) - \ln \left(1- heta
ight)
ight) + \ln \left(1- heta
ight)$$

Hence, $\eta\left(\theta\right)T\left(y\right)=y\left(\ln\left(\theta\right)-\ln\left(1-\theta\right)\right)$ and $B\left(\theta\right)=-\ln\left(1-\theta\right)$. Rewriting f_{θ} :

$$f_{ heta} \left(y
ight) \; = \; e^{y \ln(heta) + (1-y) \ln(1- heta)} \; = \; heta^y (1- heta)^{(1-y)},$$

we see that f_{θ} is the pmf of a Bernoulli distribution with parameter θ .

2. For $f_{\theta}\left(y\right)=\frac{1}{y!}e^{w\left(\theta,y\right)}$ where $w\left(\theta,y\right)=-\theta+y\ln\left(\theta\right)$, and $y=0,1,2,\ldots,$ $\theta\in\left(0,1\right)$ Hence, $\eta\left(\theta\right)T\left(y\right)=y\ln\left(\theta\right)$ and $B\left(\theta\right)=\theta$. Rewriting f_{θ}

$$f_{ heta}\left(y
ight) \,=\, rac{1}{y!} e^{- heta+y\ln(heta)} \,=\, e^{- heta} rac{ heta^y}{y!},$$

we recognize f_{θ} as the pmf of a Poisson distribution with parameter θ .

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You have used 1 of 3 attempts

• Answers are displayed within the problem

Practice: Normal distribution with known variance

1/1 point (graded)

The normal distribution $\mathcal{N}\left(heta,1
ight)$ with with mean heta and known variance $\sigma^2=1$ has pdf

$$f_{ heta}\left(y
ight) \; = \; rac{1}{\sqrt{2\pi}}e^{-rac{\left(y- heta
ight)^{2}}{2}} \, .$$

Rewrite $f_{ heta}$ in the form

$$f_{ heta}\left(y
ight) \; = \; h\left(y
ight)e^{\eta\left(heta
ight)T\left(y
ight)-B\left(heta
ight)} \qquad ext{where } \eta\left(heta
ight), T\left(y
ight): \mathbb{R}
ightarrow \mathbb{R},$$

and enter the product $\eta\left(\theta\right)T\left(y\right)$ below.

$$\eta\left(heta
ight)T\left(y
ight)=$$
 y*theta $y\cdot heta$ Answer: y*theta

STANDARD NOTATION

Solution:

$$egin{array}{lll} f_{ heta}\left(y
ight) &=& rac{1}{\sqrt{2\pi}}e^{-rac{\left(y- heta
ight)^{2}}{2}} \ &=& rac{1}{\sqrt{2\pi}}e^{-rac{\left(y^{2}-2y heta+ heta^{2}
ight)}{2}} \ &=& rac{1}{\sqrt{2\pi}}e^{-y^{2}/2}e^{rac{2y heta- heta^{2}}{2}} \end{array}$$

$$= \; h\left(y
ight)e^{\eta\left(heta
ight)T\left(y
ight)-B\left(heta
ight)} \qquad ext{where} \; egin{dcases} \eta\left(heta
ight)T\left(y
ight) &=\left(y
ight)\left(heta
ight) \ B\left(heta
ight) &=rac{ heta^{2}}{2} \ h\left(y
ight) &=\left(e^{-rac{y^{2}}{2}}
ight)/\sqrt{2\pi} \end{cases}$$

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You have used 1 of 3 attempts

1 Answers are displayed within the problem

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