4. Critical points definition

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Critical points of two variable functions

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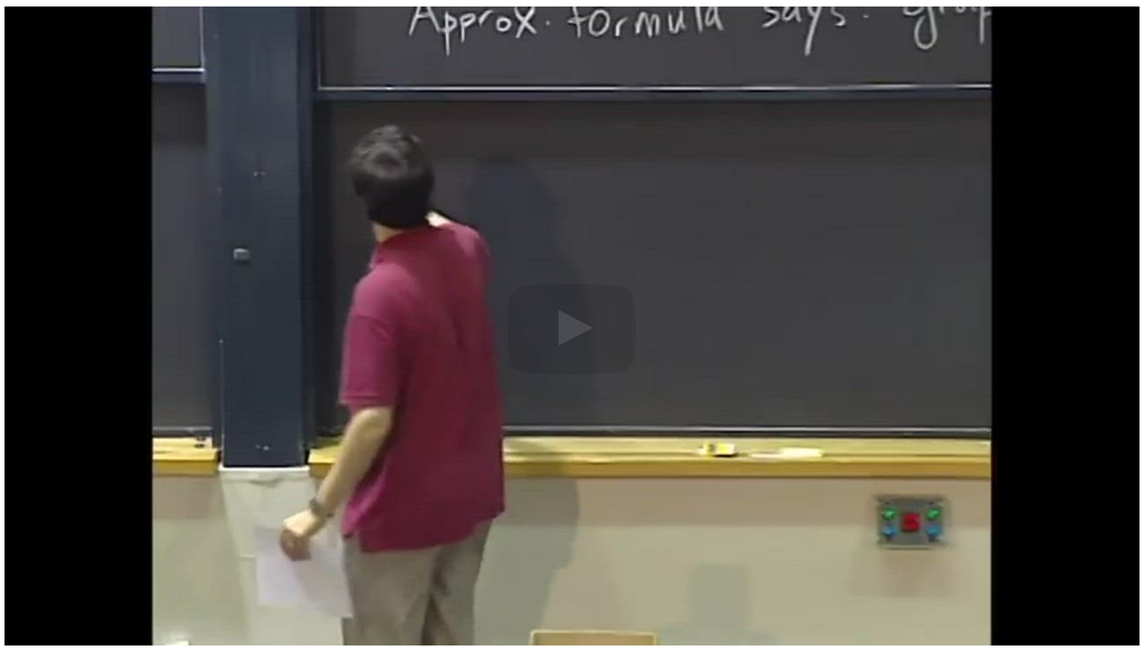

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PROFESSOR: So that's going to be an application

of partial derivatives, to look at optimization problems.

OK.

So maybe 10 years from now when you have a real job,

your job might be to actually minimize the cost of something,

or maximize the profit of something,

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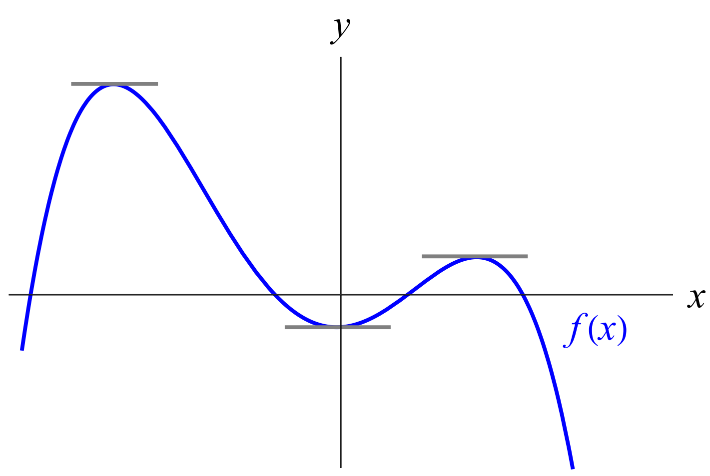
Goal: We are interested in optimization problems involving functions of several variables. That is, we want to find maxima and minima of a function of two variables. To get started, we need to figure out the mathematical characteristics of local maxima and minima in terms of partial derivatives.

Recall from single variable calculus that we locate the extrema (maxima and minima) of a function of one variable $f(x)$ by finding critical points, and then testing other properties of the first or second derivatives. Critical points are points where $f'(x) = 0$ (or is undefined). Geometrically, this means that the graph of $f(x)$ has a horizontal tangent line at its critical point (if $f'(x)$ exists there). An illustration of this is shown below.



The notion of critical points can be generalized to multivariable functions as follows.

> **Definition 4.1** Let $f(x,y)$ be a function of two variables. A **critical point** of $f(x,y)$ is a point (a,b) at which $\nabla f(a,b) = \vec{0}$. In other words, when $f_x(a,b) = 0$ and $f_y(a,b) = 0$ simultaneously.

The partial derivatives being both zero is a necessary but not sufficient condition for a critical point to be a maximum or a minimum. Below we will explore why it is necessary. In two pages we w

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sufficient.

Suppose that a function $f(x,y)$ has a local maximum (or minimum) at a point (a,b). Then we know that if we vary x or y independently, we should not be able to increase (or decrease) the value of f. That is,

$$\Delta z \approx 0$$

near the point (a,b).

From our approximation formula, we know that

$$\Delta z \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y.$$

Therefore the only way to have a maximum or minimum point is for $\nabla f(a,b) = \vec{0}$.

Geometrically, if $\nabla f(a,b) = \vec{0}$, then the equation for the tangent plane for $f(x,y)$ at a point (a,b) is given by

$$\begin{aligned} f(x,y) &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) && (4.9)\\ &\approx f(a,b), && (4.10)\end{aligned}$$

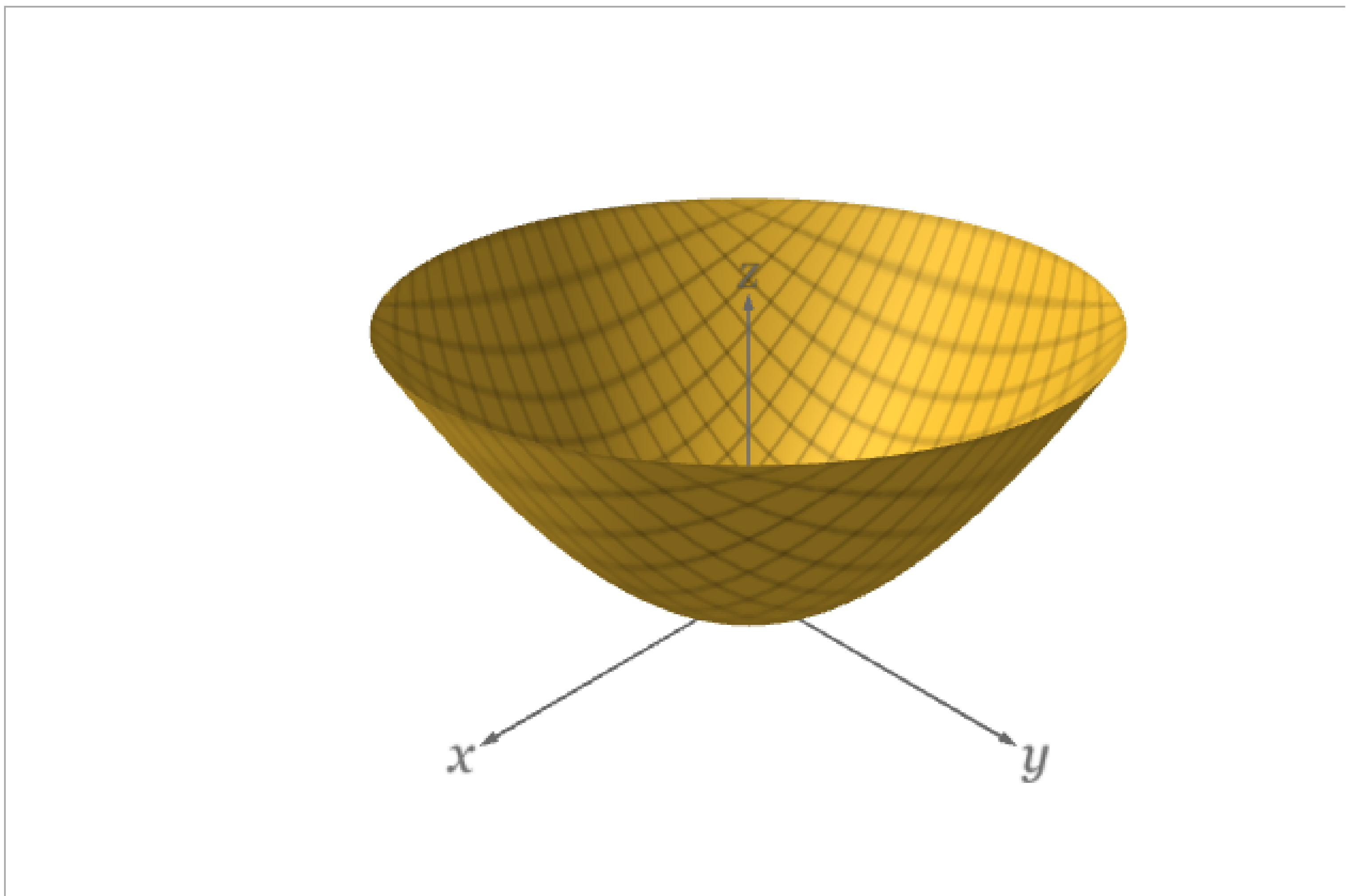
which is a constant. This means that the tangent plane is horizontal at the point (a,b).

Example 4.2 Consider the function

$$f(x,y) = x^2 + y^2 \qquad (4.11)$$

shown below.

▶ PARABOLOID



First, compute $f_x(x,y) = 2x$, which is equal to 0 when $x = 0$. Setting $x = 0$ in the original function gives

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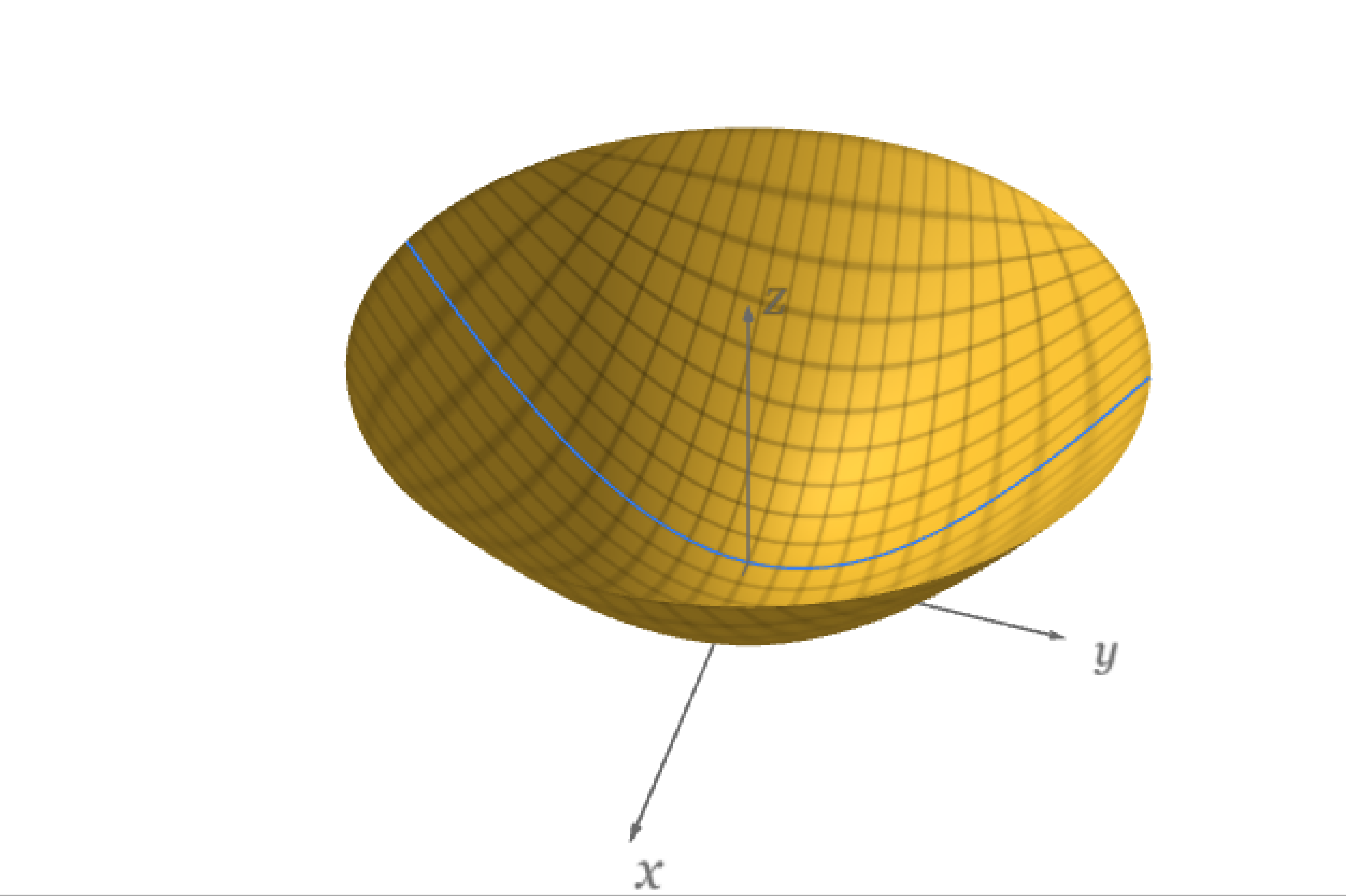
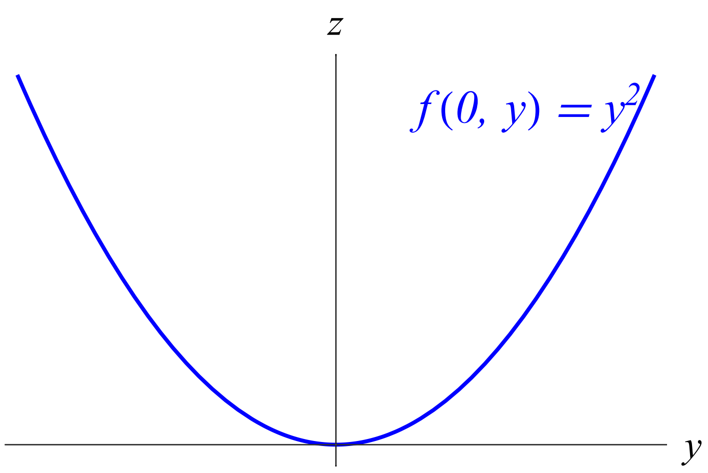
$$f(0,y) = y^2. \quad (4.12)$$

This describes a slice of the graph given by the parabola shown as thick blue curve in the figure below.

▶ PARABOLOID AND CURVE OF INTERSECTION WITH THE PLANE $x = 0$

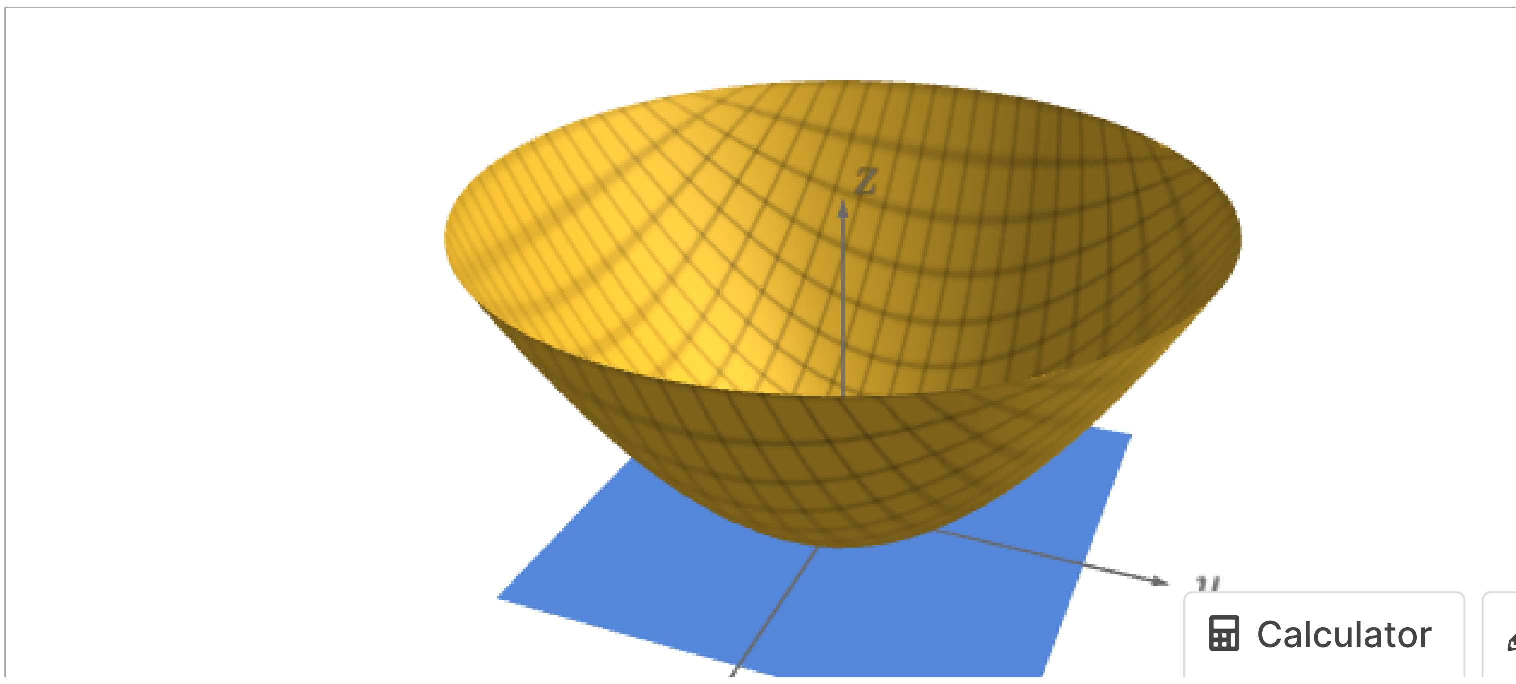


We can also view this slice in the yz-plane as shown below.



If we move along the the curve $f(0,y) = y^2$, the value of f can increase or decrease. In order to find a critical point, we must locate the point where $f_y = 0$ as well. Doing so gives $f_y(x,y) = 2y$ which means a critical point of $f(x,y)$ occurs at $(x,y) = (0,0)$. Notice that this is exactly where $\nabla f(x,y) = \vec{0}$. The image below shows the function $f(x,y) = x^2 + y^2$ along with the horizontal tangent plane at the critical point $(x,y) = (0,0)$.

▶ PARABOLOID WITH HORIZONTAL TANGENT PLANE AT CRITICAL POINT



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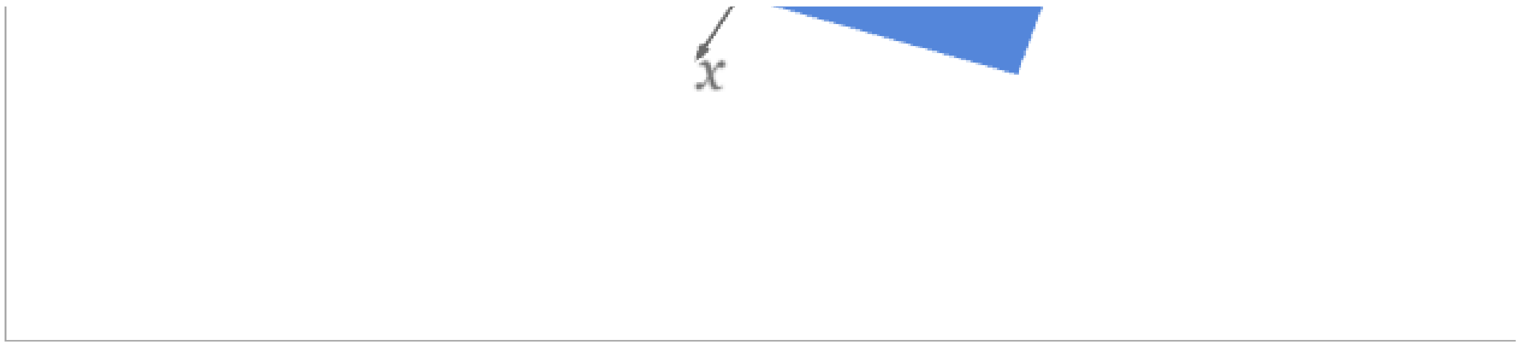

Spoiler: Extension of critical points to higher dimensions

Let $f(x_1, x_2, \ldots, x_n)$ be a function of n variables. A **critical point** of $f(x_1, x_2, \ldots, x_n)$ is a point $(x_1^*, x_2^*, \ldots, x_n^*)$ at which $\nabla f(x_1^*, x_2^*, \ldots, x_n^*) = \vec{0}$. In other words, when $f_{x_1}(x_1^*, x_2^*, \ldots, x_n^*) = 0$, $f_{x_2}(x_1^*, x_2^*, \ldots, x_n^*) = 0$, ..., and $f_{x_n}(x_1^*, x_2^*, \ldots, x_n^*) = 0$ simultaneously.

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Remark 4.3 You can also think about points where ∇f is undefined, but this can get complicated in higher dimensions. For now, we will focus our attention on the given definition of critical points where we assume ∇f exists and is equal to the zero vector.

Which are critical points?

1/1 point (graded)

Which of the following points are critical points of the function $g(x, y) = x^2/2 - xy + y$ whose partial derivatives are given by

$$g_x(x,y) = x - y \quad (4.13)$$
$$g_y(x,y) = -x + 1 \quad (4.14)$$

- [ ] $(0, 0)$
- [ ] $(1, -1)$
- [ ] $(-1, 1)$
- [ ] $(-1, -1)$
- [x] None of the above

Solution:

Both partial derivatives must be equal to zero. Thus we solve the system of equations

$$g_x(x,y) = x - y = 0 \quad (4.15)$$
$$g_y(x,y) = -x + 1 = 0 \quad (4.16)$$

The second equation tells us that $x = 1$. Plugging $x = 1$ into the first equation, we get that $y = 1$ as well. This option is not present in the choices, so the answer is “None of the above".

Submit You have used 1 of 5 attempts

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Answers are displayed within the problem

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[STAFF] Strange selection for Which are critical points? 3
So I selected the correct answer and it said I had 0.6 points, finally when I looked at the correct answer it also includes none of the a...

Requirement to have delta_z = 0 3

[STAFF] Type in "Goal" 3
In the second line of "Goal" under the video, there is a missing comma after "That is."

[staff] typo in the formula 2

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