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Pairwise vs. Three-way Independence

This is a very classic example, reported in any book on Probability:

Example 1. We throw two dice. Let A be the event "the sum of the points is 7", B the event "die #1 came up 3", and C the event "die #2 came up 4". Now, $P[A] = P[B] = P[C] = \frac{1}{6}$. Also,

$$P[A \cap B] = P[A \cap C] = P[B \cap C] = \frac{1}{36}$$

so that all events are pairwise independent. However,

$$P[A \cap B \cap C] = P[B \cap C] = \frac{1}{36}$$

while

$$P[A]P[B]P[C] = \frac{1}{216}$$

so they are not independent as a triplet.

First, note that, indeed, $P[A \cap B] = P[B \cap C] = \frac{1}{36}$, since the fact that A *and* B occurred is the same as the fact that B and C occurred.

Example 2. Another example is the case of Ω consisting of four equally likely points, a_1, a_2, a_3, a_4. Let $A = \{a_1, a_2\}, B = \{a_2, a_3\}, C = \{a_3 a_1\}$. The three are not independent, but they are pairwise.

However, it is also true that, as long as we consider only specific events (that is, we don't take into consideration their complements, or, more generally, other members of their algebra), that mutual (3-way) independence *does not imply pairwise independence*!

Here is a somewhat trivial example:

Example 3. Let $P[A] = p, P[B] = q, P[A \cap B] \neq pq, P[C] = 0$ then, trivially,

$$P[A \cap B \cap C] \leqslant P[C] = 0, \text{and}\, P[A]P[B]P[C] = 0$$

but A and B are not pairwise independent.

A less trivial example is the following:

Example 4. Consider the toss of two distinct dice. The sample space is partitioned into equally likely events of the form $(i,\ j)$, where i and j are the points on the first, respectively second die. Obviously, $P[(i, j)] = \frac{1}{36}$. Now, consider the three events

$$A_1 = '' i = 1, 2, \text{or}\, 3'' \;\; A_2 = '' i = 3, 4, \text{or}\, 5'' \;\; A_3 = i + j = 9$$

We have

$$
\begin{aligned}
A_1 \cap A_2 &= \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\} \\
A_1 \cap A_3 &= \{(3,6)\} \\
A_2 \cap A_3 &= \{(3,6),(4,5),(5,4)\} \\
A_1 \cap A_2 \cap A_3 &= \{(3,6)\}
\end{aligned}
$$

We have the following probabilities:

$$
\begin{aligned}
P[A_1] = P[A_2] = \frac{1}{2}, P[A_3] &= \frac{1}{9} \\
P[A_1 \cap A_2 \cap A_3] = \frac{1}{36} &= P[A_1]P[A_2]P[A_3]
\end{aligned}
$$

but

$$
\begin{aligned}
P[A_1 \cap A_2] &= \frac{1}{6} \neq \frac{1}{4} \\
P[A_1 \cap A_3] &= \frac{1}{36} \neq \frac{1}{18} \\
P[A_2 \cap A_3] &= \frac{1}{12} \neq \frac{1}{18}
\end{aligned}
$$

Note, referring to Example 2, that $P[C^c] = 1$, so that $P[C^c \cap A \cap B] = P[A \cap B] \neq 1 \cdot p \cdot q$, so that considering the complement of one of the sets makes the new triplet dependent. Similarly, referring to Example 3, $P[A_3^c] = \frac{8}{9}$, and

$$A_1 \cap A_2 \cap A_3^c = \{(3,1),(3,2),(3,3),(3,4),(3,5)\}$$

which has probability $\frac{5}{36} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{8}{9} = \frac{2}{9}$. Note that this fact does not apply to pairs of events:

Fact: If A is independent of B, then so are, pairwise, A^c and B, A and B^c, and A^c and B^c. That's because, for example, $P[A \cap B^c] = P[A \cap (\Omega \backslash B)]$ and $P[A \cap (\Omega \backslash B)] + P[A \cap B] = P[A]$, so $P[A \cap B^c] + P[A]P[B] = P[A]$, hence $P[A \cap B^c] = P[A] - P[A]P[B] = P[A](1 - P[B]) = P[A]P[B^c]$. Similarly for the other cases.

This points to a better definition of independence of multiple events:

Theorem: *Suppose events A, B, C satisfy the conditions*

$$P[X \cap Y \cap Z] = P[X]P[Y]P[Z]$$

where X, Y, Z are, respectively, A, or A^c, B, or B^c, and C, or C^c. Then they are also pairwise independent. The result extends to any finite collection of events, in an obvious way.

Proof: We can write $P[A \cap B] = P[(A \cap B \cap C) \cup (A \cap B \cap C^c)] = P[A]P[B]P[C] + P[A]P[B]P[C^c]$, because the two parts are disjoint. This is equal to $P[A]P[B](P[C] + P[C^c]) = P[A]P[B]$. All other cases are treated in the same way.

Remark: Checking all intersections of the sets and their complement can be seen as checking independence of all couples built from the *minimal algebra generated by each of the events*, which, for an event A, is the collection $\{A, A^c, \Omega, \emptyset\}$. Of course, trivially, Ω, and $\emptyset$ are independent of any event.

While some scholars have looked at the taxonomy of events that are $k-$independent, but not $h-$independent for $h < k$, this is not a very exciting subject, since, in considering independence and, more generally, conditional probabilities, it is much more significant to look at all events in the algebras the events belong to naturally - at the very least the ones generated by each event and its complement.

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Example 1. We throw two dice. Let A be the event "the sum of the points is 7", B the event "die #1 came up 3", and C the event "die #2 came up 4". Now, $P[A] = P[B] = P[C] = \frac{1}{6}$. Also,

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A less trivial example is the following:

Example 3. Consider the toss of two distinct dice. The sample space is partitioned into equally likely events of the form $(i,\ j)$, where i and j are the points on the first, respectively second die. Obviously, $P[(i, j)] = \frac{1}{36}$. Now, consider the three events

$$A_1 = '' i = 1, 2, \text{or } 3'' \quad A_2 = '' i = 3, 4, \text{or } 5'' \quad A_3 = i + j = 9$$

We have

$$\begin{aligned} A_1 \cap A_2 &= \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\} \\ A_1 \cap A_3 &= \{(3,6)\} \\ A_2 \cap A_3 &= \{(3,6), (4,5), (5,4)\} \\ A_1 \cap A_2 \cap A_3 &= \{(3,6)\} \end{aligned}$$

We have the following probabilities:

$$P[A_1] = P[A_2] = \frac{1}{2}, P[A_3] = \frac{1}{9}$$
$$P[A_1 \cap A_2 \cap A_3] = \frac{1}{36} = P[A_1]P[A_2]P[A_3]$$

but

$$P[A_1 \cap A_2] = \frac{1}{6} \neq \frac{1}{4}$$
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Note, referring to Example 2, that $P[C^c] = 1$, so that $P[C^c \cap A \cap B] = P[A \cap B] \neq 1 \cdot p \cdot q$, so that considering the complement of one of the sets makes the new triplet dependent. Similarly, referring to Example 3, $P[A_3^c] = \frac{8}{9}$, and

$$A_1 \cap A_2 \cap A_3^c = \{(3,1), (3,2), (3,3), (3,4), (3,5)\}$$

which has probability $\frac{5}{36} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{8}{9} = \frac{2}{9}$.

While some scholars have looked at the taxonomy of events that are $k-$ independent, but not $h-$ independent for $h < k$, this is not a very exciting subject, since, in considering independence and, more generally, conditional probabilities, it is much more significant to look at all events in the algebras the events belong to naturally - at the very least the ones generated by each event and its complement.

2 Three-Card Games

We have discussed three variations of the same game in class (yet another variation is the "two-drawer cabinets" problem in the assignment for Week 2).

1. We have three cards: one is white on both sides, one is black on both sides, and one is black on one side, and white on the other. We draw one at random and put it face up, hiding the back. What are the odds that the hidden side is of the same color?

2. Three prisoners have been condemned to death, but one has been pardoned. However, the ruler has ordered the jailer not to reveal who the pardoned prisoner is. One of the prisoners walks to the jailer and asks him to tell him, at least, the name of another prisoner who has *not* been pardoned. After a moment's reflection, the jailer agrees that he would not betray the king's trust if he did, and tells him one name. Is the prisoner right in thinking that his probability of being pardoned have gone from $\frac{1}{3}$ before, to $\frac{1}{2}$ now?

3. The "Monty Hall" TV game. A contestant is presented with three closed doors, one of which hides a prize, and has to pick one. After she has made her choice, the host of the show opens one of the remaining two doors and shows that no prize is behind *that* door. Now, the contestant has the option of switching her original choice. Is there any advantage for her to do so?

A closer look shows that puzzles 2 and 3 are identical. The only difference is in the story line, in that in case #2 the prisoner does not get the option of switching his fate with the remaining convict. Puzzle 1 is actually the same as well, except for an interesting twist in the problem of modeling the problems that is present in puzzles 2 and 3.

There are several ways of solving these puzzles, and we'll present three here. The first is notable because, even though it may be the least "explanatory" (if you guessed the wrong answer, this solution might not convince you in your guts), but is the "surest": you don't have to "understand" the mechanism of the puzzle to find the correct solution - just crank the math, and it comes out by itself.

2.1 Three Cards

2.1.1 Bayes' Rule

Let WW, BB, WB denote the event that the 2-white-faces card, 2-black-faces card and black-and-white-faces card, respectively, has been drawn. Suppose, for definiteness, that the open face turns out to be white. Let A be the event "the open face is white". Obviously, assuming a "random drawing",

$$P[WW] = P[BB] = P[WB] = \frac{1}{3}$$

Also, it is intuitively clear that $P[A] = \frac{1}{3}$ (there are as many white as black faces), but to hammer it in, we can apply the "Total probability formula": since

$$P[A\,|WW] = 1$$

$$P[A\,|BB] = 0$$

$$P[A\,|WB] = \frac{1}{2}$$

we have

$$P[A] = P[A\,|WW]\,P[WW] + P[A\,|BB]\,P[BB] + P[A\,|WB]\,P[WB] =$$

$$= 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}$$

If the open face is white, the hidden face is also white in case the chosen card is WW. Thus, the event "the hidden face is of the same color as the open face" is, in our case, WW. We know that A happened, hence we are looking for $P[WW\,|A]$. Applying Bayes' Rule, we have

$$P[WW\,|A] = \frac{P[A\,|WW]\,P[WW]}{P[A]} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Now, necessarily, $P[WB\,|A] = \frac{1}{3}$, as can also be computed directly from Bayes' Rule:

$$P[WB\,|A] = \frac{P[A\,|WB]\,P[WB]}{P[A]} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

2.1.2 Direct Computation of Conditional Probabilities

Bayes' Rule is nothing more than a rewriting of the definition of conditional probability, so it's no surprise that going straight to the definition will produce very much the same calculations. Precisely,

- $P[A] = \frac{1}{2}$ because there are as many white as black faces
- The event "the open face is white, and so is the hidden face" is $A \cap WW$. Of course, $P[A \cap WW] = P[A\,|WW]\,P[WW] = 1 \cdot \frac{1}{3} = \frac{1}{3}$.

Hence,

$$P[WW\,|A] = \frac{P[A \cap WW]}{P[A]} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

2.1.3 Counting Favorable Cases

Though trickier, this method also works - if applied correctly. Once a white face has come up, we are restricted to consider only two of the three cards. But the possible cases are

1. The hidden face is black (the card is the black-an-white card)
2. The card we picked is the white-white one, and the face showing is face #1
3. The card we picked is the white-white card, and the face showing is face #2

Two out of three of these cases are in favor of "same color", and thus the probability is $\frac{2}{3}$. Note that we have to distinguish between *two distinct cases* in describing "a white face is showing" and "the card has two white faces", much as we distinguish between points on die #1, and die #2 when throwing two dice and registering a $1, 2$: there are two possibilities - $\{1, 2\}$, and $\{2, 1\}$.

There is an even faster method for counting. Remember that we are considering the event "the hidden face is of the same color as the open face" - we were talking about the open face being white only to be definite, but the argument only refers to the hidden face being of the same color. Now, having three cards, for the hidden face to be different from the open one, it is *necessary* that the card we picked was the black-and-white, and this only happens one time out of three!

2.2 Monty Hall or The Three Prisoners

2.2.1 Bayes' Rule

As noted, these two puzzles are identical. We choose the less gruesome as the working example. There are three doors, and, conventionally, we will call "first" the door the contestant chooses. Let A be the event "first door wins", B "second door wins", and C "third door wins". Clearly, $P[A] = P[B] = P[C] = \frac{1}{3}$. Let a, b, c be the events "host opens first, second, third door", respectively.

Now, $P[a] = 0$, because the contestant chose the first door. Also, $P[b|B] = P[c|C] = 0$, because the point of the game is that the host opens a non-winning door. Consequently, $P[c|B] = P[b|C] = 1$. We are left to determine what $P[b|A]$ and $P[c|A]$ are. Let us stay vague, and assume $P[b|A] = p, P[c|A] = 1 - p$ (that is, if the contestant picked the winning door, the host will open door #2 with probability p). We can now check what the conditional probabilities are for the event "first door - i.e., originally chosen door - is a winner", given the action of the host (note that, since $P[a] = 0$, , $P[A|a]$ is undefined):

$$P[A|b] = \frac{P[b|A]P[A]}{P[b]} = \frac{p \cdot \frac{1}{3}}{P[b|A]P[A] + P[b|B]P[B] + P[b|C]P[C]} =$$

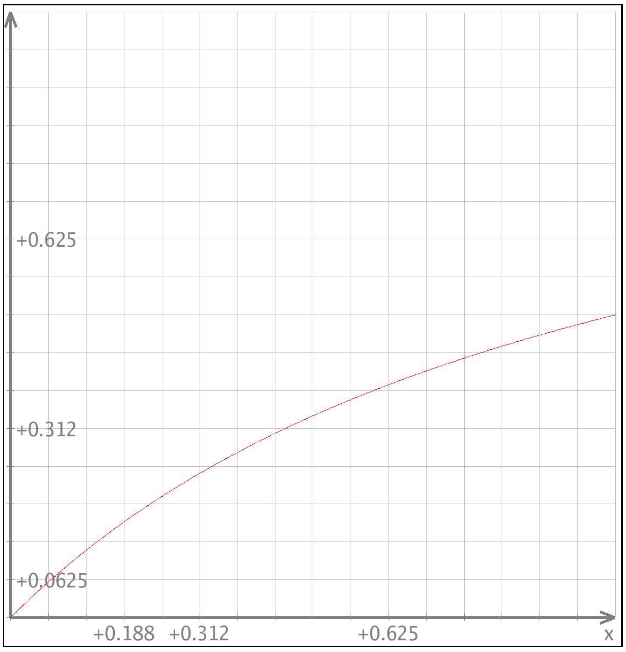
$$= \frac{\frac{p}{3}}{\frac{p}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{p}{p+1}$$

The calculation for $P[A|c]$ is similar, after interchanging $1-p$ for p: we get $\frac{1-p}{2-p}$. Now, p ranges from 0 to 1, even though we would probably instinctively assume $p = \frac{1}{2}$. Still, this parameter affects the strategy of the game significantly! If $p = \frac{1}{2}$, we have that

$$\frac{p}{p+1} = \frac{1-p}{2-p} = \frac{1}{3}$$

i.e., *if the conductor picks which door to open completely at random* (when he has a choice), then switching doors is a much better strategy, since it wins two times out of three. In the case of the prisoners, this means that if, in case the inquisitive prisoner was the one who had been pardoned, the jailer gives up the name of one of the other two at random, the prisoner's chance of being the pardoned one is still stuck at $\frac{1}{3}$!

What if the conductor or the jailer have a bias? Like, having a choice, they'd much rather open door #2, say, than door #3 (or reveal the fate of one rather than the other of the other two prisoners). In this case, we have that our conditional probability, as a function of p, is strictly monotone, from 0 to $\frac{1}{2}$, as p goes from 0 to 1:



For instance, if $p = 1$ (given a choice, conductor always chooses door #2 - or, better, since the naming of doors 2 and 3 is arbitrary, we are calling his "favorite" door #2),

$$\frac{p}{p+1} = \frac{1}{1+1} = \frac{1}{2}$$

i.e., it makes no difference if we switch or not (or, for the prisoner, his favorable odds have indeed increased). Since this function of p is increasing between $\frac{1}{2}$ and 1, we see that we have a whole range of nuances, from one extreme (indifference between doors) to the other (strong preference for the door that is opened).

What happens if $p < \frac{1}{2}$? That means, that the host opens his "least favorite" door (which he could, as long as $p < 1$). Then, of course,

$$\frac{p}{p+1} < \frac{1}{3}$$

and the positive effect of switching is even stronger!

Of course, if you *do* end up playing the Monty Hall game, you'd have presumably no information on the right value of p, so you might as well switch doors: after all, the worst case would be that your chances would not improve,but they'd not worsen either.

2.2.2 Straight Calculation

Here is a funny argument, which seems to contradict what we just proved. Let us consider the events A, "the contestant chooses the winning door", and S, "switching doors is the losing strategy". Obviously, $A = E,$. $P[A] = P[S] = \frac{1}{3}$. I.e., the result we got when $p = \frac{1}{2}$ seems to be true, regardless of the value of p, contrary to our previous argument!

Fact is, this is the ***absolute*** (***a priori***) probability of these events, ***before*** the host operated. Or, put differently, the strategy suggested at the end of the previous subsection, will win 2/3 of the times (essentially, the suggestion was to ignore the host!). This is perfectly consistent with our previous calculation, if we apply the total probability formula, with the values we already found:

$$P[A] = P[A\,|a]\,P[a] + P[A\,|b]\,P[b] + P[A\,|c]\,P[c] =$$
$$= 0 + \frac{p}{p+1} \cdot \left(\frac{p}{3} + \frac{1}{3}\right) + \frac{1-p}{2-p} \cdot \left(\frac{1-p}{3} + \frac{1}{3}\right) = \frac{1}{3}$$

How to read these two results? The point is that the conditional probability calculation is useful (aside of its value as a fun exercise on Bayes' Rule) *when we know the "strategy" used by the host* (or the jailer), when making his choice (i.e., the value of p). If we don't know it, we'll have varying probabilities, depending on how "preferred" the door chosen by the host was, but, over all cases of choices, over the long run, they will end adding up to $\frac{1}{3}$ for our original choice to be a winner, no matter the value of p. Of course, you might want to watch a long run of Monty Hall games and evaluate the empirical frequency of the occurrence of A, to infer a statistical estimate of the "true" value of p.

Remark: You might have noticed that, when $p = \frac{1}{2}$, $P[A] = P[A\,|b] = P[A\,|c] = \frac{1}{3}$. In other words, if $p = \frac{1}{2}$, *looking at the host's (or jailer's) choice does not change our probabilities.* This is an example of *independent events*: our observation does not add or subtract anything to our prior knowledge (much to the chagrin of the inquisitive prisoner).

2.2.3 Verbal Arguments

The argument in the preceding section can be easily summed up in words: one third of the time the contestant will chose the winning door. In this third of all possible cases, the conductor will have a choice of which door to open, and the contestant should stick to her choice (of course, she won't know whether hers is one of these "good" cases). Two thirds of the time, the chosen door is a loser, and the host is obligated to open the other losing door - in effect pointing the contestant to the winning door.

The "naive" solution of probability $\frac{1}{2}$ for the strategy of switching choice (after the conductor's opening of a door, "we are left with two doors, one of which is a winner, the other a loser, so it doesn't matter") is wrong because it does not take into account the information gained from the conductor, which, in two out of three cases, does not arbitrarily pick a door, but is actually telling us which is the winning one. The game would be indeed an even game if the conductor opened a door *before* the contestant made her choice: this would, indeed, reduce her choice to two equally likely cases.

Note, finally, that the discussion in the previous subsection can be also treated in a less formal way. Suppose the conductor is "biased", and, given a choice (i.e., if our choice, door #1 is the winning door), will always open, say, door #3 - *and we know it.* Then, if we see him opening door #2, *we know for sure that door #3 is the winner.* On the other hand, if we see him opening door #3, we can only deduce that , either #2 is the winner, or #1 is the winner, and he exercised his preference, leaving us, indeed, with a choice with probability $\frac{1}{2}$. So, depending on which door is actually the winner, we'll go from certain victory to an even game. As the formal calculations show, over the long run, the odds even out at a $\frac{2}{3}$ probability for winning when switching!

Basic Algebra of Sets

	Algebra of sets	*Algebra of numbers*
	Union $\cup$	sum "+"
	Intersection $\cap$	product "$\cdot$"
1	$A \cup B = B \cup A$	$a + b = b + a$
2	$A \cap B = B \cap A$	$a \cdot b = b \cdot a$
3	$A \cup (B \cup C) = A \cup B \cup C$	$a + (b + c) = a + b + c$
4	$A \cap (B \cap C) = A \cap B \cap C$	$a \cdot (b \cdot c) = a \cdot b \cdot c$
5	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$a \cdot (b + c) = a \cdot b + a \cdot c$
6	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	see below

Since $A \cap A = A$, $(A \cap B) \subset A$, $(A \cap C) \subset A$, and

$$(a + b)(a + c) = a \cdot a + a \cdot b + a \cdot c + b \cdot c$$

Line 6 in the table above follows from

$$(A \cup B) \cap (A \cup C) = \underbrace{(A \cap A) \cup (A \cap B) \cup (A \cap C)}_{=A} \cup (B \cap C) = A \cup (B \cap C)$$

This illustrates that set algebra has its own rules.

De Morgan's laws:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \tag{2.1}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{2.2}$$

Similarly,

$$\overline{A \cup B \cup C} = \overline{A \cup D} = \overline{A} \cap \overline{D} = \overline{A} \cap \overline{B \cup C} = \overline{A} \cap (\overline{B} \cap \overline{C}) = \overline{A} \cap B \cap \overline{C}$$

$$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}$$

$$\overline{A_1 \cap \cdots \cap A_n} = \overline{A_1} \cup \cdots \cup \overline{A_n}$$

$$\overline{(A_1 \cup A_2) \cap (A_3 \cup A_4)} = (\overline{A_1} \cap \overline{A_2}) \cup (\overline{A_3} \cap \overline{A_4})$$

Rules: (1) interchange $\cup$ and $\cap$; (2) interchange $(*)$ and $\overline{(*)}$. However, care should be taken when dealing with multiple nests, as demonstrated below.

Example

$$\overline{\underbrace{(\overline{A} \cap B)}_{D} \cup \overline{C}} = \overline{\overline{D} \cup \overline{C}} = D \cap C = (\overline{A} \cap B) \cap C = \overline{A \cup \overline{B}} \cap C \tag{2.3}$$