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Problem 3: Hypothesis test with a continuous observation

(5/5 points)

Let Θ be a Bernoulli random variable that indicates which one of two hypotheses is true, and let $\mathbf{P}(\Theta = 1) = p$. Under the hypothesis $\Theta = 0$, the random variable \mathbf{X} is uniformly distributed over the interval $[0, 1]$. Under the alternative hypothesis $\Theta = 1$, the PDF of \mathbf{X} is given by

$$f_{\mathbf{X}|\Theta}(x | 1) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the MAP rule for deciding between the two hypotheses, given that $\mathbf{X} = x$.

1. Suppose for this part of the problem that $p = 3/5$. The MAP rule can choose in favor of the hypothesis $\Theta = 1$ if and only if $x \geq c_1$. Find the value of c_1 .

 $c_1 =$

1/3



Answer: 0.33333


2. Assume now that p is general such that $0 \leq p \leq 1$. It turns out that there exists a constant c such that the MAP rule always decides in favor of the hypothesis $\Theta = 0$ if and only if $p < c$. Find c .

on random variables


▼ Unit 7: Bayesian inference

Unit overview


Lec. 14: Introduction to Bayesian inference

Exercises 14 due Apr 06, 2016 at 23:59 UTC 


Lec. 15: Linear models with normal noise

Exercises 15 due Apr 06, 2016 at 23:59 UTC 


Problem Set 7a

Problem Set 7a due Apr 06, 2016 at 23:59 UTC 


Lec. 16: Least mean squares (LMS) estimation

Exercises 16 due Apr 13, 2016 at 23:59 UTC 

Lec. 17: Linear least mean squares (LLMS) estimation

Exercises 17 due Apr 13, 2016 at 23:59 UTC 

Problem Set 7b

Problem Set 7b due Apr 13, 2016 at 23:59 UTC 

Solved problems

$$c = \boxed{1/3}$$

✓ Answer: 0.33333

3. For this part of the problem, assume again that $p = 3/5$. Find the conditional probability of error for the MAP decision rule given that the hypothesis $\Theta = 0$ is true.

$$\mathbf{P}(\text{error} \mid \Theta = 0) = \boxed{2/3}$$

✓ Answer: 0.66667

4. Find the probability of error associated with the MAP rule as a function of p . Express your answer in terms of p using standard notation .

$$\text{When } p \leq 1/3, \mathbf{P}(\text{error}) =$$

$$\boxed{p}$$

✓ Answer: p

$$\text{When } p \geq 1/3, \mathbf{P}(\text{error}) =$$

$$\boxed{3/2 - 1/(4 \cdot p) - (5 \cdot p)/4}$$

✓ Answer: 3/2-5*p/4-0.25/p

Answer:

1. If $0 < p < 1$, we can choose in favor of the hypothesis $\Theta = 1$ if and only if

$$f_{X|\Theta}(x \mid 1)p_{\Theta}(1) \geq f_{X|\Theta}(x \mid 0)p_{\Theta}(0),$$

$$2x \cdot p \geq 1 \cdot (1 - p),$$

$$x \geq \frac{1 - p}{2p}.$$

If $p = 3/5$, the rule above corresponds to $x \geq 1/3$.

Additional theoretical material

Unit summary

- ▶ Exam 2
- ▶ Unit 8: Limit theorems and classical statistics
- ▶ Unit 9: Bernoulli and Poisson processes
- ▶ Unit 10: Markov chains
- ▶ Exit Survey
- ▶ Final Exam

2. We will be forced to choose in favor of the hypothesis $\Theta = 0$ if the condition $x \geq (1 - p)/(2p)$ can never hold. Since $x \in [0, 1]$, this will be the case if and only if the threshold $(1 - p)/(2p)$ exceeds 1, that is, $1 - p > 2p$, or $p < 1/3$.

3. If the hypothesis $\Theta = 0$ is true, an error occurs when we decide in favor of the hypothesis $\Theta = 1$. For $p = 3/5$, this corresponds to the event $\{X \geq 1/3\}$. Therefore,

$$\mathbf{P}(\text{error} \mid \Theta = 0) = \mathbf{P}(X \geq 1/3 \mid \Theta = 0) = \int_{1/3}^1 f_{X|\Theta}(x \mid 0) dx = \int_{1/3}^1 1 dx = \frac{2}{3}.$$

4. Similar to the computation above, we find that for $p \geq 1/3$,

$$\begin{aligned} \mathbf{P}(\text{error} \mid \Theta = 0) &= \mathbf{P}\left(X \geq \frac{1-p}{2p} \mid \Theta = 0\right) \\ &= \int_{\frac{1-p}{2p}}^1 f_{X|\Theta}(x \mid 0) dx \\ &= \int_{\frac{1-p}{2p}}^1 1 dx \\ &= 1 - \frac{1-p}{2p} \\ &= \frac{3p-1}{2p}, \end{aligned}$$

$$\begin{aligned}
 \mathbf{P}(\text{error} \mid \Theta = 1) &= \mathbf{P}\left(X < \frac{1-p}{2p} \mid \Theta = 1\right) \\
 &= \int_0^{\frac{1-p}{2p}} f_{X|\Theta}(x \mid 1) dx \\
 &= \int_0^{\frac{1-p}{2p}} 2x dx \\
 &= \left(\frac{1-p}{2p}\right)^2.
 \end{aligned}$$

Using the total probability theorem, we find

$$\begin{aligned}
 \mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \mid \Theta = 0)p_{\Theta}(0) + \mathbf{P}(\text{error} \mid \Theta = 1)p_{\Theta}(1) \\
 &= \frac{(3p-1)(1-p)}{2p} + \frac{(1-p)^2}{4p} = \frac{(1-p)(5p-1)}{4p}, \text{ for } p \geq 1/3.
 \end{aligned}$$

For $p < 1/3$, we will always decide $\Theta = 0$, and the resulting probability of error is

$$\mathbf{P}(\text{error}) = \mathbf{P}(\text{error} \mid \Theta = 0)p_{\Theta}(0) + \mathbf{P}(\text{error} \mid \Theta = 1)p_{\Theta}(1) = 0 \cdot (1-p) + 1 \cdot p = p.$$

For the boundary case of $p = 1/3$, both formulas yield $\mathbf{P}(\text{error}) = 1/3$.

You have used 1 of 2 submissions

DISCUSSION

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