1. Implicit hypothesis testing

Given n i.i.d. samples $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, we want to find a test with asymptotic level 5% for the hypotheses

$$H_0 : \mu \geq \sigma \quad \text{vs} \quad H_1 : \mu < \sigma. \tag{7.1}$$

(a)

1/1 point (graded)

As a first step, define the maximum likelihood estimators

$$\hat{\mu} = \bar{X}_n, \quad \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

Give a function $g(x, y)$ such that

$$g\left(\hat{\mu}, \widehat{\sigma^2}\right) \xrightarrow[n\to\infty]{\mathbf{P}} \mu - \sigma.$$

$g(x,y) =$ x-sqrt(y)

$x - \sqrt{y}$

Submit You have used 1 of 3 attempts

(b)

1/1 point (graded)
Note: To avoid too much double jeopardy, you will be able to see the solution to this part once you answered it correctly, and used all your attempts.

What is the asymptotic variance of $g\left(\hat{\mu}, \widehat{\sigma^2}\right)$ that you found in part (a)?

$V\left(g\left(\hat{\mu}, \widehat{\sigma^2}\right)\right) =$ 3*sigma^2/2 **Answer:** 3/2*sigma^2

STANDARD NOTATION

Solution:

First, by the Theorem giving asymptotic normality for maximum likelihood estimators, we have

$$\sqrt{n}\begin{pmatrix} \hat{\mu} \\ \widehat{\sigma^2} \end{pmatrix} - \sqrt{n}\begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \xrightarrow[n\to\infty]{(\mathrm{D})} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, I(\mu, \sigma^2)^{-1}\right),$$

where $I\left(\mu, \sigma^2\right)$ denotes the Fisher information that we computed earlier to be

$$I\left(\mu,\sigma^2\right)=\begin{pmatrix}\frac{1}{\sigma^2} & 0\\ 0 & \frac{1}{2\sigma^4}\end{pmatrix}.$$

Hence,

$$\sqrt{n}\begin{pmatrix}\hat{\mu}\\ \widehat{\sigma^2}\end{pmatrix}-\sqrt{n}\begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\xrightarrow[n\to\infty]{\text{(D)}}\mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\frac{1}{\sigma^2} & 0\\ 0 & \frac{1}{2\sigma^4}\end{pmatrix}^{-1}\right).$$

Now, defining

$$g:\mathbb{R}\times(0,\infty)\to\mathbb{R},\quad (x,y)\mapsto x-\sqrt{y},$$

we can compute

$$\nabla g\left(x,y\right)=\begin{pmatrix}1\\ -\frac{1}{2\sqrt{y}}.\end{pmatrix}$$

Then, apply the multivariate Delta method to obtain

$$\sqrt{n}\left(\hat{\mu}-\sqrt{\widehat{\sigma^2}}-(\mu-\sigma)\right)\xrightarrow[n\to\infty]{\text{(D)}}\mathcal{N}\left(0,\nabla g(\mu,\sigma^2)^T I(\mu,\sigma^2)^{-1}\nabla g\left(\mu,\sigma^2\right)\right)=\mathcal{N}\left(0,\frac{3}{2}\sigma^2\right).$$

That means

$$V\left(g\left(\hat{\mu}, \widehat{\sigma^2}\right)\right) = \frac{3}{2}\sigma^2.$$

Submit

You have used 2 of 3 attempts

Answers are displayed within the problem

(c)

0/1 point (graded)

Using your result from part (b) together with a plug-in estimator for the asymptotic variance, give a test for

$$H_0 : \mu \geq \sigma \quad \text{vs} \quad H_1 : \mu < \sigma. \tag{7.2}$$

that is with asymptotic level 5% and of the form

$$\psi = \mathbf{1}\{f(\hat{\mu}, \widehat{\sigma^2}) > 0\},$$

where

$$f(\hat{\mu}, \widehat{\sigma^2}) = -T(\hat{\mu}, \widehat{\sigma^2}) - q$$

for some function T and $q > 0$.

(Enter **hatmu**, **hat(sigma^2)** for $\hat{\mu}$, $\widehat{\sigma^2}$, respectively. Use the quantile function q for best results. E.g: enter $q(0.01)$ for the 0.99-quantile.)

$f(\hat{\mu}, \widehat{\sigma^2}) =$ -(hatmu-sqrt(hat(sigma^2)))/(sqrt((3/2)*hat(sigma^2/n)))-q(0.05) ✖

STANDARD NOTATION

Submit You have used 4 of 4 attempts

✖ Incorrect (0/1 point)

(d)

0/1 point (graded)

Using the same test as in part (c), give the (asymptotic) p-value of the test given observations $\hat{\mu}$ and $\widehat{\sigma^2}$.

(Enter **Phi(x)** for the cdf $\Phi\left(x\right)$ of a Normal distribution. Enter **hatmu**, **hat(sigma^2)** for $\hat{\mu}$, $\widehat{\sigma^2}$, respectively.)

p-value = Phi((hatmu-sqrt(hat(sigma^2)))/(sqrt((3/2)*hat(sigma^2/n)))) ✖

STANDARD NOTATION

Submit You have used 1 of 4 attempts

✖ Incorrect (0/1 point)

(e)

4/4 points (graded)

What is the (asymptotic) p-value if the sample size is $n = 100$, $\hat{\mu} = 2.41$, and $\widehat{\sigma^2} = 5.20$?

p-value = 0.6787543683455306 ✔

What if $n = 100$, $\hat{\mu} = 3.28$, and $\widehat{\sigma^2} = 15.95$?

p-value = 0.07225394750622968

In the second case, at level 10%, do you reject H_0 ?

- (•) Yes
- ( ) No

At 5%, do you reject H_0 ?

- ( ) Yes
- (•) No

Submit

You have used 1 of 3 attempts

Correct (4/4 points)

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Part C - Asymptotic distribution

question posted 8 days ago by JoseHernandezSC

Are we supposed to take the test based on an asymptotic Chi-squared distribution or a Normal Distibution? This would change the Test-Statistic.

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3 responses

Erocha (Community TA)
8 days ago

I suspect there would be a hint if $q(\alpha)$ were also representing the χ^2 quantile, so your safe bet is that it only represents our good and old normal quantile.

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ya_mukhin (Staff)
7 days ago

Would you be able to construct a 1-sided test with the chi-square asymptotic distribution?

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sandipan_dey
7 days ago

I got my answer in the form of [...], for some α, but got rejected by the grader, any clue where I am doing wrong? thanks in advance.

I guess your numerator must also depend on 'n'.

posted 7 days ago by **markarjaybajo**

Make sure you use the answer of previous question and use the plug-in estimator as the question asks.

posted 7 days ago by **Erocha** (Community TA)

Thank you @Erocha, I was not using the plug-in estimator at one place.

Also in this case we have $H_0 : \mu > \sigma$ and not $\mu = \sigma$, so can we still replace $\mu - \sigma$ by 0 under H_0?

In other words, is the test $H_0 : \mu \geq \sigma$ and $H_1 : \mu < \sigma$ same as the given test $H_0 : \mu > \sigma$ and $H_1 : \mu \leq \sigma$?

posted 7 days ago by **sandipan_dey**

The hypothesis in (c) is $H_0 : \mu > \sigma \quad \text{vs} \quad H_1 : \mu \leq \sigma.$, but I think it should better be $H_0 : \mu \geq \sigma \quad \text{vs} \quad H_1 : \mu < \sigma.$

posted 6 days ago by **Erocha** (Community TA)

Hi @sandipan_dey, great point, thank you! It is not possible to test the null $H_0 : \mu > \sigma$ without also testing the limit points of $\{\mu = \sigma\}$ of that subset of the parameter space! In other words, if you reject the null as it is written, you are also rejecting the combinations of mean and variance with equality. This is because it is impossible to distinguish, even in infinite samples / asymptotics, a value $\mu = \sigma$ and *all* the parameter values in the null that are arbitrarily close to equality. I have fixed our code, thanks!

posted 6 days ago by **ya_mukhin** (Staff)

Thank you for the confirmation @ya_mukhin.

posted 6 days ago by **sandipan_dey**

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