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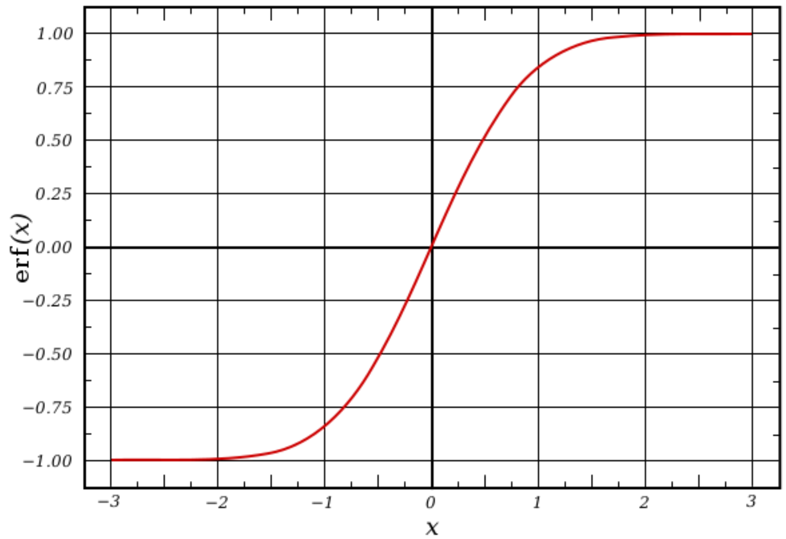
Error function

In mathematics, the **error function** (also called the **Gauss error function**) is a special function (non-elementary) of sigmoid shape that occurs in probability, statistics, and partial differential equations describing diffusion. It is defined as:[1][2]

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2}\, dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt.$$



Plot of the error function

In statistics, for nonnegative values of x, the error function has the following interpretation: for a random variable Y that is normally distributed with mean 0 and variance 0.5, $\operatorname{erf}(x)$ describes the probability of Y falling in the range $[-x, x]$.

There are several closely related functions, such as the complementary error function, the imaginary error function, and others.

Contents

Name

The name "error function" and its abbreviation erf were proposed by J. W. L. Glaisher in 1871 on account of its connection with "the theory of Probability, and notably the theory of Errors."[3] The error function complement was also discussed by Glaisher in a separate publication in the same year.[4] For the "law of facility" of errors whose density is given by

$$f(x)=\left(\frac{c}{\pi}\right)^{\frac{1}{2}}e^{-cx^2}$$

(the normal distribution), Glaisher calculates the chance of an error lying between p and q as:

$$\left(\frac{c}{\pi}\right)^{\frac{1}{2}}\int_p^q e^{-cx^2}\,dx=\tfrac{1}{2}\left(\operatorname{erf}(q\sqrt{c})-\operatorname{erf}(p\sqrt{c})\right).$$

Applications

When the results of a series of measurements are described by a normal distribution with standard deviation σ and expected value 0, then $\operatorname{erf}\left(\frac{a}{\sigma\sqrt{2}}\right)$ is the probability that the error of a single measurement lies between $-a$ and $+a$, for positive a. This is useful, for example, in determining the bit error rate of a digital communication system.

The error and complementary error functions occur, for example, in solutions of the heat equation when boundary conditions are given by the Heaviside step function.

The error function and its approximations can be used to estimate results that hold with high probability. Given random variable $X \sim \mathrm{Norm}[\mu, \sigma]$ and constant $L < \mu$:

$$\Pr[X \leq L] = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{L-\mu}{\sqrt{2}\sigma}\right) \approx A\exp\left(-B\left(\frac{L-\mu}{\sigma}\right)^2\right)$$

where A and B are certain numeric constants. If L is sufficiently far from the mean, i.e. $\mu - L \geq \sigma\sqrt{\ln k}$, then:

$$\Pr[X \leq L] \leq A\exp(-B\ln k) = \frac{A}{k^B}$$

so the probability goes to 0 as $k \to \infty$.

Properties

The property $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ means that the error function is an odd function. This directly results from the fact that the integrand e^{-t^2} is an even function.

For any complex number z:

$$\operatorname{erf}(\overline{z}) = \overline{\operatorname{erf}(z)}$$

where $\overline{z}$ is the complex conjugate of z.

The integrand $f = \exp(-z^2)$ and $f = \operatorname{erf}(z)$ are shown in the complex z-plane in figures 2 and 3. Level of $\operatorname{Im}(f) = 0$ is shown with a thick green line. Negative integer values of $\operatorname{Im}(f)$ are shown with thick red lines. Positive integer values of $\operatorname{Im}(f)$ are shown with thick blue lines. Intermediate levels of $\operatorname{Im}(f)$ = constant are shown with thin green lines. Intermediate levels of $\operatorname{Re}(f)$ = constant are shown with thin red lines for negative values and with thin blue lines for positive values.

The error function at $+\infty$ is exactly 1 (see Gaussian integral). At the real axis, $\operatorname{erf}(z)$ approaches unity at $z \to +\infty$ and -1 at $z \to -\infty$. At the imaginary axis, it tends to $\pm i\infty$.

Taylor series

The error function is an entire function; it has no singularities (except that at infinity) and its Taylor expansion always converges.

The defining integral cannot be evaluated in closed form in terms of elementary functions, but by expanding the integrand e^{-z^2} into its Maclaurin series and integrating term by term, one obtains the error function's Maclaurin series as:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} - \cdots \right)$$

which holds for every complex number z. The denominator terms are sequence A007680 in the OEIS.

For iterative calculation of the above series, the following alternative formulation may be useful:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(z \prod_{k=1}^{n} \frac{-(2k-1)z^2}{k(2k+1)} \right) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{z}{2n+1} \prod_{k=1}^{n} \frac{-z^2}{k}$$

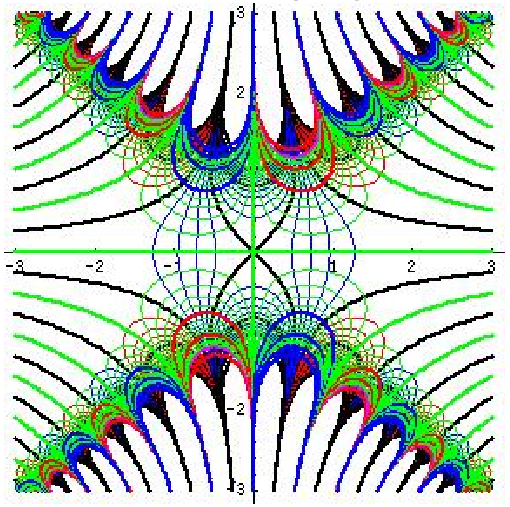
because $\frac{-(2k-1)z^2}{k(2k+1)}$ expresses the multiplier to turn the k^{th} term into the $(k+1)^{\text{th}}$ term (considering z as the first term).

The imaginary error function has a very similar Maclaurin series, which is:

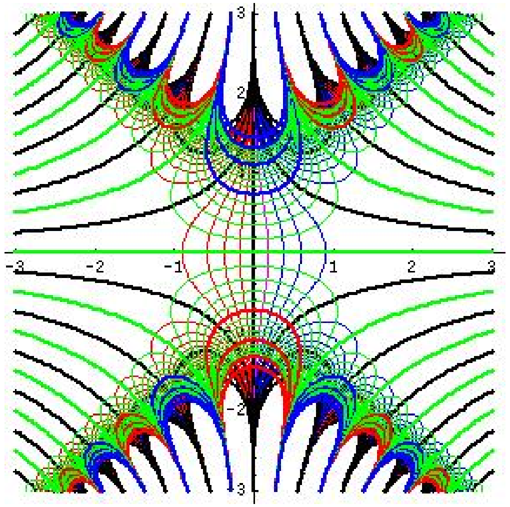
$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left(z + \frac{z^3}{3} + \frac{z^5}{10} + \frac{z^7}{42} + \frac{z^9}{216} + \cdots \right)$$

which holds for every complex number z.

Plots in the complex plane



Integrand $\exp(-z^2)$



$\operatorname{erf}(z)$

Derivative and integral

The derivative of the error function follows immediately from its definition:

$$\frac{d}{dz} \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}.$$

From this, the derivative of the imaginary error function is also immediate:

$$\frac{d}{dz}\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} e^{z^2}.$$

An antiderivative of the error function, obtainable by integration by parts, is

$$z \operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}.$$

An antiderivative of the imaginary error function, also obtainable by integration by parts, is

$$z \operatorname{erfi}(z) - \frac{e^{z^2}}{\sqrt{\pi}}.$$

Higher order derivatives are given by

$$\operatorname{erf}^{(k)}(z) = \frac{2(-1)^{k-1}}{\sqrt{\pi}} H_{k-1}(z) e^{-z^2} = \frac{2}{\sqrt{\pi}} \frac{d^{k-1}}{dz^{k-1}}\left(e^{-z^2}\right), \qquad k = 1, 2, \dots$$

where H are the physicists' Hermite polynomials.[5]

Bürmann series

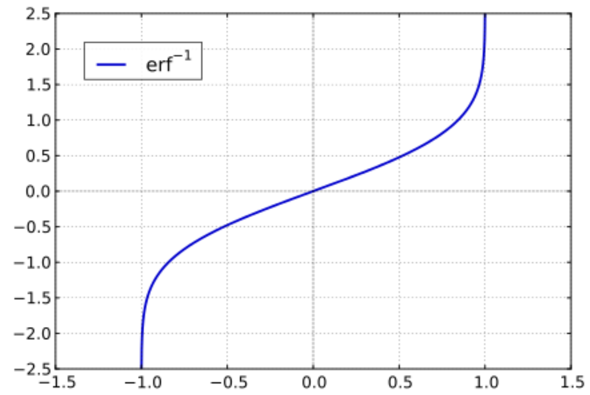
An expansion,[6] which converges more rapidly for all real values of x than a Taylor expansion, is obtained by using Hans Heinrich Bürmann's theorem:[7]

$$\begin{aligned}\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \operatorname{sgn}(x) \sqrt{1 - e^{-x^2}} \left(1 - \frac{1}{12}\left(1 - e^{-x^2}\right) - \frac{7}{480}\left(1 - e^{-x^2}\right)^2 - \frac{5}{896}\left(1 - e^{-x^2}\right)^3 - \frac{787}{276480}\left(1 - e^{-x^2}\right)^4 - \cdots\right) \\ &= \frac{2}{\sqrt{\pi}} \operatorname{sgn}(x) \sqrt{1 - e^{-x^2}} \left(\frac{\sqrt{\pi}}{2} + \sum_{k=1}^{\infty} c_k e^{-kx^2}\right).\end{aligned}$$

By keeping only the first two coefficients and choosing $c_1 = \frac{31}{200}$ and $c_2 = -\frac{341}{8000}$, the resulting approximation shows its largest relative error at $x = \pm 1.3796$, where it is less than $3.6127 \cdot 10^{-3}$:

$$\operatorname{erf}(x) \approx \frac{2}{\sqrt{\pi}} \operatorname{sgn}(x) \sqrt{1-e^{-x^2}} \left(\frac{\sqrt{\pi}}{2} + \frac{31}{200} e^{-x^2} - \frac{341}{8000} e^{-2x^2} \right).$$

Inverse functions



Inverse error function

Given complex number z, there is not a *unique* complex number w satisfying $\operatorname{erf}(w) = z$, so a true inverse function would be multivalued. However, for $-1 < x < 1$, there is a unique *real* number denoted $\operatorname{erf}^{-1}(x)$ satisfying

$$\operatorname{erf}\left(\operatorname{erf}^{-1}(x)\right) = x.$$

The **inverse error function** is usually defined with domain (−1,1), and it is restricted to this domain in many computer algebra systems. However, it can be extended to the disk $|z| < 1$ of the complex plane, using the Maclaurin series

$$\operatorname{erf}^{-1}(z) = \sum_{k=0}^{\infty} \frac{c_k}{2k+1} \left(\frac{\sqrt{\pi}}{2} z \right)^{2k+1},$$

where $c_0 = 1$ and

$$c_k = \sum_{m=0}^{k-1} \frac{c_m c_{k-1-m}}{(m+1)(2m+1)} = \left\{ 1, 1, \frac{7}{6}, \frac{127}{90}, \frac{4369}{2520}, \frac{34807}{16200}, \ldots \right\}.$$

So we have the series expansion (common factors have been canceled from numerators and denominators):

$$\operatorname{erf}^{-1}(z) = \tfrac{1}{2}\sqrt{\pi} \left(z + \frac{\pi}{12} z^3 + \frac{7\pi^2}{480} z^5 + \frac{127\pi^3}{40320} z^7 + \frac{4369\pi^4}{5806080} z^9 + \frac{34807\pi^5}{182476800} z^{11} + \cdots \right).$$

(After cancellation the numerator/denominator fractions are entries OEIS: A092676/OEIS: A092677 in the OEIS; without cancellation the numerator terms are given in entry OEIS: A002067.) The error function's value at ±∞ is equal to ±1.

For $|z| < 1$, we have $\operatorname{erf}\left(\operatorname{erf}^{-1}(z)\right) = z$.

The **inverse complementary error function** is defined as

$$\operatorname{erfc}^{-1}(1-z) = \operatorname{erf}^{-1}(z).$$

For *real* *x*, there is a unique *real* number $\operatorname{erfi}^{-1}(x)$ satisfying $\operatorname{erfi}\left(\operatorname{erfi}^{-1}(x)\right) = x$. The **inverse imaginary error function** is defined as $\operatorname{erfi}^{-1}(x)$.[8]

For any real *x*, Newton's method can be used to compute $\operatorname{erfi}^{-1}(x)$, and for $-1 \leq x \leq 1$, the following Maclaurin series converges:

$$\operatorname{erfi}^{-1}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c_k}{2k+1} \left(\frac{\sqrt{\pi}}{2} z\right)^{2k+1},$$

where c_k is defined as above.

Asymptotic expansion

A useful asymptotic expansion of the complementary error function (and therefore also of the error function) for large real *x* is

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n}\right] = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n},$$

where $(2n - 1)!!$ is the double factorial of $(2n - 1)$, which is the product of all odd numbers up to $(2n - 1)$. This series diverges for every finite *x*, and its meaning as asymptotic expansion is that, for any $N \in \mathbb{N}$ one has

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{N-1} (-1)^n \frac{(2n-1)!!}{(2x^2)^n} + R_N(x)$$

where the remainder, in Landau notation, is

$$R_N(x) = O\left(x^{1-2N} e^{-x^2}\right)$$

as $x \to \infty$.

Indeed, the exact value of the remainder is

$$R_N(x) := \frac{(-1)^N}{\sqrt{\pi}} 2^{1-2N} \frac{(2N)!}{N!} \int_x^\infty t^{-2N} e^{-t^2}\, dt,$$

which follows easily by induction, writing

$$e^{-t^2} = -(2t)^{-1} \left(e^{-t^2}\right)'$$

and integrating by parts.

For large enough values of x, only the first few terms of this asymptotic expansion are needed to obtain a good approximation of erfc(x) (while for not too large values of x, the above Taylor expansion at 0 provides a very fast convergence).

Continued fraction expansion

A continued fraction expansion of the complementary error function is:[9]

$$\operatorname{erfc}(z) = \frac{z}{\sqrt{\pi}} e^{-z^2} \cfrac{1}{z^2 + \cfrac{a_1}{1 + \cfrac{a_2}{z^2 + \cfrac{a_3}{1 + \cdots}}}} \qquad a_m = \frac{m}{2}.$$

Integral of error function with Gaussian density function

$$\int_{-\infty}^{\infty} \operatorname{erf}(ax+b) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \operatorname{erf}\left[\frac{a\mu+b}{\sqrt{1+2a^2\sigma^2}}\right], \qquad a, b, \mu, \sigma \in \mathbb{R}$$

Factorial series

The inverse factorial series

$$\operatorname{erfc} z = \frac{e^{-z^2}}{\sqrt{\pi}\,z} \sum_{n=0}^{\infty} \frac{(-1)^n Q_n}{(z^2+1)^{\bar{n}}} = \frac{e^{-z^2}}{\sqrt{\pi}\,z}\left(1 - \frac{1}{2}\frac{1}{(z^2+1)} + \frac{1}{4}\frac{1}{(z^2+1)(z^2+2)} - \cdots\right)$$

converges for $\operatorname{Re}(z^2) > 0$. Here

$$Q_n \stackrel{\text{def}}{=} \frac{1}{\Gamma(1/2)} \int_0^\infty \tau(\tau-1)\cdots(\tau-n+1)\tau^{-1/2} e^{-\tau}\, d\tau = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{\bar{k}} s(n,k),$$

$z^{\bar{n}}$ denotes the rising factorial, and $s(n,k)$ denotes a signed Stirling number of the first kind.[10][11]

Numerical approximations

Approximation with elementary functions

- Abramowitz and Stegun give several approximations of varying accuracy (equations 7.1.25–28). This allows one to choose the fastest approximation suitable for a given application. In order of increasing accuracy, they are:

$$\operatorname{erf}(x) \approx 1 - \frac{1}{(1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)^4}, \qquad x \geq 0$$

(maximum error: 5×10^{-4})

where a_1 = 0.278393, a_2 = 0.230389, a_3 = 0.000972, a_4 = 0.078108

$$\operatorname{erf}(x) \approx 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2}, \quad t = \frac{1}{1+px}, \qquad x \geq 0$$ (maximum error: 2.5×10^{-5})

where p = 0.47047, a_1 = 0.3480242, a_2 = −0.0958798, a_3 = 0.7478556

$$\operatorname{erf}(x) \approx 1 - \frac{1}{(1 + a_1 x + a_2 x^2 + \cdots + a_6 x^6)^{16}}, \qquad x \geq 0$$ (maximum error: 3×10^{-7})

where a_1 = 0.0705230784, a_2 = 0.0422820123, a_3 = 0.0092705272, a_4 = 0.0001520143, a_5 = 0.0002765672, a_6 = 0.0000430638

$$\operatorname{erf}(x) \approx 1-(a_1t+a_2t^2+\cdots+a_5t^5)e^{-x^2}, \quad t=\frac{1}{1+px}$$ (maximum error: 1.5×10^{-7})

where p = 0.3275911, a_1 = 0.254829592, a_2 = −0.284496736, a_3 = 1.421413741, a_4 = −1.453152027, a_5 = 1.061405429

All of these approximations are valid for $x \geq 0$. To use these approximations for negative x, use the fact that erf(x) is an odd function, so erf(x) = −erf(−x).

- Exponential bounds and a pure exponential approximation for the complementary error function are given by [12]

$$\begin{aligned}\operatorname{erfc}(x) &\leq \frac{1}{2}e^{-2x^2}+\frac{1}{2}e^{-x^2} \leq e^{-x^2}, \qquad x>0\\ \operatorname{erfc}(x) &\approx \frac{1}{6}e^{-x^2}+\frac{1}{2}e^{-\frac{4}{3}x^2}, \qquad x>0.\end{aligned}$$

- A tight approximation of the complementary error function for $x \in [0,\infty)$ is given by Karagiannidis & Lioumpas (2007)[13] who showed for the appropriate choice of parameters $\{A,B\}$ that

$$\operatorname{erfc}(x) \approx \frac{\left(1-e^{-Ax}\right)e^{-x^2}}{B\sqrt{\pi}x}.$$

They determined $\{A,B\}=\{1.98,1.135\}$, which gave a good approximation for all $x \geq 0$.

- A single-term lower bound is[14]

$$\operatorname{erfc}(x) \geq \sqrt{\frac{2e}{\pi}}\frac{\sqrt{\beta-1}}{\beta}e^{-\beta x^2}, \qquad x\geq 0, \beta>1,$$

where the parameter β can be picked to minimize error on the desired interval of approximation.

- Another approximation is given by Sergei Winitzki using his "global Padé approximations":[15][16]:2–3

$$\operatorname{erf}(x) \approx \operatorname{sgn}(x)\sqrt{1-\exp\left(-x^2\frac{\frac{4}{\pi}+ax^2}{1+ax^2}\right)}$$

where

$$a=\frac{8(\pi-3)}{3\pi(4-\pi)} \approx 0.140012.$$

This is designed to be very accurate in a neighborhood of 0 and a neighborhood of infinity, and the *relative* error is less than 0.00035 for all real *x*. Using the alternate value *a* ≈ 0.147 reduces the maximum relative error to about 0.00013.[17]

This approximation can be inverted to obtain an approximation for the inverse error function:

$$\operatorname{erf}^{-1}(x) \approx \operatorname{sgn}(x)\sqrt{\sqrt{\left(\frac{2}{\pi a}+\frac{\ln(1-x^2)}{2}\right)^2-\frac{\ln(1-x^2)}{a}}-\left(\frac{2}{\pi a}+\frac{\ln(1-x^2)}{2}\right)}.$$

Polynomial

An approximation with a maximal error of 1.2×10^{-7} for any real argument is:[18]

$$\operatorname{erf}(x)=\begin{cases}1-\tau & x\geq 0\\ \tau-1 & x<0\end{cases}$$

with

$$\begin{aligned}\tau=t\cdot\exp\bigl(&-x^2-1.26551223+1.00002368t+0.37409196t^2+0.09678418t^3-0.18628806t^4\\&+0.27886807t^5-1.13520398t^6+1.48851587t^7-0.82215223t^8+0.17087277t^9\bigr)\end{aligned}$$

and

$$t=\frac{1}{1+0.5|x|}.$$

Table of values

x	erf(x)	1-erf(x)
0	0	1
0.02	0.022 564 575	0.977 435 425
0.04	0.045 111 106	0.954 888 894
0.06	0.067 621 594	0.932 378 406
0.08	0.090 078 126	0.909 921 874
0.1	0.112 462 916	0.887 537 084
0.2	0.222 702 589	0.777 297 411
0.3	0.328 626 759	0.671 373 241
0.4	0.428 392 355	0.571 607 645
0.5	0.520 499 878	0.479 500 122
0.6	0.603 856 091	0.396 143 909
0.7	0.677 801 194	0.322 198 806
0.8	0.742 100 965	0.257 899 035
0.9	0.796 908 212	0.203 091 788
1	0.842 700 793	0.157 299 207
1.1	0.880 205 07	0.119 794 93
1.2	0.910 313 978	0.089 686 022
1.3	0.934 007 945	0.065 992 055
1.4	0.952 285 12	0.047 714 88
1.5	0.966 105 146	0.033 894 854
1.6	0.976 348 383	0.023 651 617
1.7	0.983 790 459	0.016 209 541
1.8	0.989 090 502	0.010 909 498
1.9	0.992 790 429	0.007 209 571
2	0.995 322 265	0.004 677 735
2.1	0.997 020 533	0.002 979 467

2.2	0.998 137 154	0.001 862 846
2.3	0.998 856 823	0.001 143 177
2.4	0.999 311 486	0.000 688 514
2.5	0.999 593 048	0.000 406 952
3	0.999 977 91	0.000 022 09
3.5	0.999 999 257	0.000 000 743

Related functions

Complementary error function

The **complementary error function**, denoted erfc, is defined as

$$\begin{aligned}\operatorname{erfc}(x) &= 1-\operatorname{erf}(x)\\ &= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt\\ &= e^{-x^2}\operatorname{erfcx}(x),\end{aligned}$$

which also defines erfcx, the **scaled complementary error function**[19] (which can be used instead of erfc to avoid arithmetic underflow[19][20]). Another form of $\operatorname{erfc}(x)$ for non-negative x is known as Craig's formula, after its discoverer:[21]

$$\operatorname{erfc}(x\mid x\geq 0)=\frac{2}{\pi}\int_0^{\pi/2}\exp\left(-\frac{x^2}{\sin^2\theta}\right)d\theta.$$

This expression is valid only for positive values of x, but it can be used in conjunction with $\operatorname{erfc}(x) = 2 - \operatorname{erfc}(-x)$ to obtain $\operatorname{erfc}(x)$ for negative values. This form is advantageous in that the range of integration is fixed and finite.

Imaginary error function

The **imaginary error function**, denoted *erfi*, is defined as

$$\begin{aligned}\operatorname{erfi}(x) &= -i\operatorname{erf}(ix)\\ &= \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt\\ &= \frac{2}{\sqrt{\pi}}e^{x^2}D(x),\end{aligned}$$

where $D(x)$ is the Dawson function (which can be used instead of erfi to avoid arithmetic overflow[19]).

Despite the name "imaginary error function", $\operatorname{erfi}(x)$ is real when x is real.

When the error function is evaluated for arbitrary complex arguments z, the resulting **complex error function** is usually discussed in scaled form as the Faddeeva function:

$$w(z) = e^{-z^2}\operatorname{erfc}(-iz) = \operatorname{erfcx}(-iz).$$

Cumulative distribution function

The error function is essentially identical to the standard normal cumulative distribution function, denoted Φ, also named norm(x) by software languages, as they differ only by scaling and translation. Indeed,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{\frac{-t^2}{2}}\,dt = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = \frac{1}{2}\operatorname{erfc}\left(-\frac{x}{\sqrt{2}}\right)$$

or rearranged for erf and erfc:

$$\begin{aligned}\operatorname{erf}(x) &= 2\Phi\left(x\sqrt{2}\right)-1\\ \operatorname{erfc}(x) &= 2\Phi\left(-x\sqrt{2}\right) = 2\left(1-\Phi\left(x\sqrt{2}\right)\right).\end{aligned}$$

Consequently, the error function is also closely related to the Q-function, which is the tail probability of the standard normal distribution. The Q-function can be expressed in terms of the error function as

$$Q(x) = \frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right).$$

The inverse of Φ is known as the normal quantile function, or probit function and may be expressed in terms of the inverse error function as

$$\operatorname{probit}(p) = \Phi^{-1}(p) = \sqrt{2}\operatorname{erf}^{-1}(2p-1) = -\sqrt{2}\operatorname{erfc}^{-1}(2p).$$

The standard normal cdf is used more often in probability and statistics, and the error function is used more often in other branches of mathematics.

The error function is a special case of the Mittag-Leffler function, and can also be expressed as a confluent hypergeometric function (Kummer's function):

$$\operatorname{erf}(x) = \frac{2x}{\sqrt{\pi}} M\left(\frac{1}{2}, \frac{3}{2}, -x^2\right).$$

It has a simple expression in terms of the Fresnel integral.

In terms of the regularized gamma function P and the incomplete gamma function,

$$\operatorname{erf}(x) = \operatorname{sgn}(x) P\left(\frac{1}{2}, x^2\right) = \frac{\operatorname{sgn}(x)}{\sqrt{\pi}} \gamma\left(\frac{1}{2}, x^2\right).$$

$\operatorname{sgn}(x)$ is the sign function.

Generalized error functions

Some authors discuss the more general functions:

$$E_n(x) = \frac{n!}{\sqrt{\pi}} \int_0^x e^{-t^n}\,dt = \frac{n!}{\sqrt{\pi}} \sum_{p=0}^{\infty} (-1)^p \frac{x^{np+1}}{(np+1)p!}.$$

Notable cases are:

- $E_0(x)$ is a straight line through the origin: $E_0(x) = \dfrac{x}{e\sqrt{\pi}}$
- $E_2(x)$ is the error function, erf(x).

After division by $n!$, all the E_n for odd n look similar (but not identical) to each other. Similarly, the E_n for even n look similar (but not identical) to each other after a simple division by $n!$. All generalised error functions for $n > 0$ look similar on the positive x side of the graph.

These generalised functions can equivalently be expressed for $x > 0$ using the gamma function and incomplete gamma function:

$$E_n(x) = \frac{1}{\sqrt{\pi}}\Gamma(n)\left(\Gamma\left(\frac{1}{n}\right) - \Gamma\left(\frac{1}{n}, x^n\right)\right), \qquad x > 0.$$

Therefore, we can define the error function in terms of the incomplete Gamma function:

$$\operatorname{erf}(x) = 1 - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, x^2\right).$$

Iterated integrals of the complementary error function

The iterated integrals of the complementary error function are defined by[22]

$$\begin{aligned}
\mathrm{i}^n\operatorname{erfc}(z) &= \int_z^\infty \mathrm{i}^{n-1}\operatorname{erfc}(\zeta)\,d\zeta \\
\mathrm{i}^0\operatorname{erfc}(z) &= \operatorname{erfc}(z) \\
\mathrm{i}^1\operatorname{erfc}(z) &= \operatorname{ierfc}(z) = \frac{1}{\sqrt{\pi}}e^{-z^2} - z\operatorname{erfc}(z) \\
\mathrm{i}^2\operatorname{erfc}(z) &= \frac{1}{4}\left[\operatorname{erfc}(z) - 2z\operatorname{ierfc}(z)\right]
\end{aligned}$$

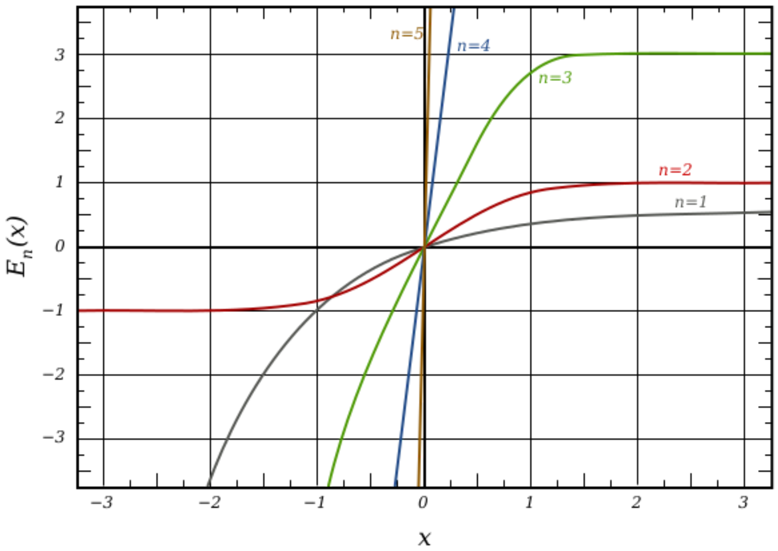
The general recurrence formula is

$$2n\,\mathrm{i}^n\operatorname{erfc}(z) = \mathrm{i}^{n-2}\operatorname{erfc}(z) - 2z\,\mathrm{i}^{n-1}\operatorname{erfc}(z)$$

They have the power series

$$i^n\operatorname{erfc}(z) = \sum_{j=0}^{\infty}\frac{(-z)^j}{2^{n-j}j!\,\Gamma\left(1+\frac{n-j}{2}\right)},$$

from which follow the symmetry properties



Graph of generalised error functions $E_n(x)$:
grey curve: $E_1(x) = (1 - e^{-x})/\sqrt{\pi}$
red curve: $E_2(x) = \operatorname{erf}(x)$
green curve: $E_3(x)$
blue curve: $E_4(x)$
gold curve: $E_5(x)$.

$$i^{2m}\operatorname{erfc}(-z) = -i^{2m}\operatorname{erfc}(z) + \sum_{q=0}^{m} \frac{z^{2q}}{2^{2(m-q)-1}(2q)!(m-q)!}$$

and

$$i^{2m+1}\operatorname{erfc}(-z) = i^{2m+1}\operatorname{erfc}(z) + \sum_{q=0}^{m} \frac{z^{2q+1}}{2^{2(m-q)-1}(2q+1)!(m-q)!}.$$

See also

Related functions

- Gaussian integral, over the whole real line
- Gaussian function, derivative
- Dawson function, renormalized imaginary error function
- Goodwin–Staton integral

In probability

- Normal distribution
- Normal cumulative distribution function, a scaled and shifted form of error function
- Probit, the inverse or quantile function of the normal CDF
- Q-function, the tail probability of the normal distribution

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External links

- MathWorld – Erf (http://mathworld.wolfram.com/Erf.html)
- A Table of Integrals of the Error Functions (http://nvlpubs.nist.gov/nistpubs/jres/73B/jresv73Bn1p1_A1b.pdf)