Problem 3: A six-sided die

(3/3 points)

A fair, 6-sided die is rolled 6 times independently. Assume that the results of the different rolls are independent. Let $(a_1, \ldots, a_6)$ denote a typical outcome, where each a_i belongs to $\{1, \ldots, 6\}$.

Note: Enter numerical answers; do not enter '!' or combinations. The following table for $\binom{n}{k}$ up to $n = 6$ has been provided for your convenience:

	k						
n	0	1	2	3	4	5	6
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

1. Find the probability that the results of the 6 rolls are all different. (Answer with at least 3 decimal digits.)

0.0154321

Answer: 0.01543

For any outcome $\omega = (a_1, \ldots, a_6)$, let $R(\omega)$ be the **set** $\{a_1, \ldots, a_6\}$; this is the set of numbers that showed up at least once in the different rolls. For example, if $\omega = (2, 2, 5, 2, 3, 5)$, then $R(\omega) = \{2, 3, 5\}$.

2.

Find the probability that $R(\omega)$ has exactly two elements. (Answer with at least 3 decimal digits.)

0.01993313

Answer: 0.01993

3. Find the probability that $R(\omega)$ has exactly three elements.

0.2314815

Answer: 0.23148

Answer:

1. Let A be the event where the results of the 6 rolls are all the same. Since all outcomes are equally likely, $\mathbf{P}(A) = \frac{|A|}{|\Omega|}$.

$$|A| = 6!$$
$$|\Omega| = 6^6$$
$$P(A) = \frac{6!}{6^6} = 5/324 \approx 0.01543$$

2. Let B be the event where $R(\omega)$ has exactly two elements. We can find the number of elements in B by first choosing a pair of distinct numbers for $R(\omega)$ – there are $\binom{6}{2}$ choices. For each pair, we can then count the number of ways they can be assigned to a sequence of length 6 that consists of only those two numbers and has

at least one of each. We see that there are $\binom{6}{k}$ ways of constructing a sequence that consists of k repetitions of the first number, and $6-k$ repetitions of the second number, where $1 \le k \le 5$. Therefore,

$$|B| = \binom{6}{2} \sum_{k=1}^{5} \binom{6}{k}$$
$$|\Omega| = 6^6$$
$$P(B) = \frac{|B|}{|\Omega|} = 155/7776 \approx 0.01993$$

3. Let C be the event where $R(\omega)$ has exactly three elements. We can find the number of elements in C by first choosing a triple of distinct numbers for $R(\omega)$ – there are $\binom{6}{3}$ choices. For each triple, we can then count the number of ways they can be assigned to a sequence of length 6 that consists of only those three numbers and has at least one of each. We see that there are $\binom{6}{k}\binom{6-k}{l}$ ways of constructing a sequence that consists of k repetitions of the first number, l repetitions of the second number, and $6-k-l$ repetitions of the third number, where $1 \le k \le 4$ and $1 \le l \le 5-k$. Therefore,

$$|C| = \binom{6}{3} \sum_{k=1}^{4} \sum_{l=1}^{5-k} \binom{6}{k}\binom{6-k}{l}$$
$$|\Omega| = 6^6$$

By numerically evaluating the various entries in the formula for $|C|$, we find that:

$$P(C) = \frac{|C|}{|\Omega|} = 25/108 \approx 0.23148$$

You have used 1 of 2 submissions