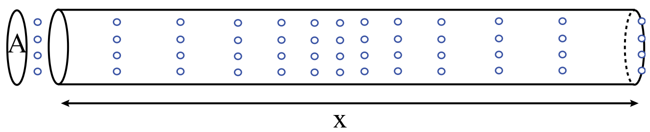
1. Woodwinds and pressure waves

When we looked at waves propogating in a string, we were interested in how vertical displacements of the string changed over time. That is, all displacements were transverse, and we called these **transverse waves.** The equation modeling this motion is the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

The same wave equation models **longitudinal waves,** which occur when all of the motion occurs along the major axis. An example of a longitudinal wave is a **sound wave.** (For more about how sound waves works, you may want to check out the following introductory videos from Khan Academy.)

In this recitation, we explore how the wave equation applies in the context of sound waves moving through a flute (modeled as a thin cylinder with open ends in air).



Let us define the variables and assumptions.

Variables and functions: Define

L length of the cylinder

A	cross-sectional area of cylinder
t	time
x	position along the cylinder (from 0 to L)
$u(x,t)$	*horizontal* displacement of the air molecules
p_0	ambient air pressure
$p(x,t)$	the difference in pressure away from the ambient air pressure

Here

- L, A, p_0 are constants;
- t, x are independent variables; and
- $u = u(x,t)$ and $p = p(x,t)$ are functions defined for $x \in [0, L]$ and $t \geq 0$. The horizontal displacement is measured relative to the equilibrium position in which (statistically) air molecules are evenly spaced and at ambient pressure.

Assumptions: All statistically significant air molecule motion occurs in the x-direction, and is small. Changes in pressure are small compared to the ambient air pressure.

Both the pressure and the horizontal displacement satisfy the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,$$

where c is the speed of sound.

To understand boundary conditions, we need to understand how the pressure and horizontal displacement are related. The physics of sound involves three main ideas.

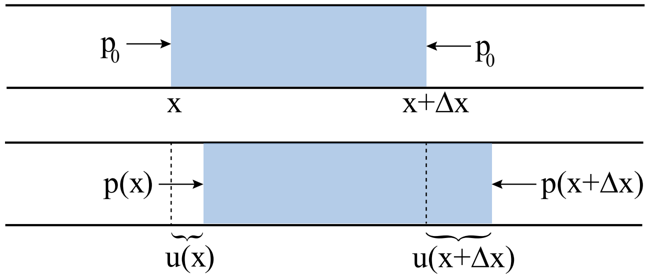
1. Moving air molecules cause changes in air density.

2. Changes in density correspond to changes in pressure.

3. Pressure differences generate motion of air molecules.

Let's look at what happens when we look at a small volume of air in our cylinder at equilibrium pressure p_0. The volume V at equilibrium pressure is given by

$$V = A\Delta x.$$



If the pressure is changed from equilibrium to $p\left(x\right)$, then the change in volume from equilibrium is

$$\Delta V = A\left(u\left(x+\Delta x\right) - u\left(x\right)\right) = A\Delta u.$$

All that is left is to understand how changes in pressure affect changes in volume. What we know is that if we increase the pressure on a given volume, the volume decreases. The decrease is proportional to the original volume V and the variation in pressure p. This is captured in the equation

$$\Delta V = -\kappa V p,$$

where κ is a physical constant.

Plugging in the expressions we found from the image for V and ΔV involving Δx and Δu into our last equation, we find a relationship between pressure p and displacement u:

$$\begin{aligned} \Delta V &= -\kappa V p \\ A\Delta u &= -\kappa\left(A\Delta x\right)p \\ \frac{\Delta u}{\Delta x} &= -\kappa p. \\ \frac{\partial u}{\partial x} &= -\kappa p. \end{aligned}$$

Understanding boundary conditions

1/1 point (graded)
If we have a cylinder of length L with two open ends, we know that the end of the cylinder cannot hold any pressure beyond the ambient pressure p_0. Therefore $p\left(0,t\right) = p\left(L,t\right) = 0$ for all $t > 0$.

What are the corresponding boundary conditions on u? (Choose all that apply.)

- [ ] $u\left(0,t\right) = 0$
- [ ] $u\left(L,t\right) = 0$
- [x] $\frac{\partial u}{\partial x}(0,t) = 0$
- [x] $\frac{\partial u}{\partial x}(L,t) = 0$
- [ ] $\frac{\partial^2 u}{\partial x^2}(0,t) = 0$

$\frac{\partial^2 u}{\partial x^2}(L,t) = 0$

Solution:

We are given that

$$\frac{\partial u}{\partial x} = -\kappa p.$$

Therefore if $p\,(0,t) = p\,(L,t) = 0$, the boundary conditions for the corresponding wave equation for displacement are

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.$$

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