12. Examples of Maximum Likelihood Estimators

Note: The following problem will be presented in lecture (at the bottom of this page), but we encourage you to attempt it first.

Maximum Likelihood Estimator of a Bernoulli Statistical Model I

3/3 points (graded)

In the next two problems, you will compute the MLE (maximum likelihood estimator) associated to a Bernoulli statistical model.

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Ber}(p^*)$ for some unknown $p^* \in (0,1)$. You construct the associated statistical model $(\{0,1\}, \{\mathrm{Ber}(p)\}_{p\in(0,1)})$. Let L_n denote the likelihood of this statistical model. Recall that in the fourth problem "Likelihood of a Bernoulli Statistical Model" from two slides ago that you derived the formula

$$L_n(x_1, \ldots, x_n, p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}.$$

Oftentimes for computating the MLE it is more convenient to work with and optimize the **log-likelihood** $\ell(p) := \ln L_n(x_1, \ldots, x_n, p)$.

The derivative of the log-likelihood can be written

$$\frac{\partial}{\partial p}\ln L_n\left(x_1,\ldots,x_n,p\right) = A/p - \left(n - A\right)/B$$

where A can be expressed in terms of $\sum_{i=1}^{n} x_i$ and B can be expressed in terms of p. Fill in the blanks with the appropriate values for A and B

(Enter **Sigma_n** for entire sum $\sum_{i=1}^{n} x_i$).

$A =$ Sigma_n ✔ **Answer:** Sigma_n

Σ_n

$B =$ 1-p ✔ **Answer:** 1-p

$1-p$

For which p does $\frac{\partial}{\partial p}\ln L_n\left(x_1,\ldots,x_n,p\right) = 0$? Denote this critical point by $\hat{p}$.

- ○ $\hat{p} = 0$
- ○ $\hat{p} = 1$
- ○ $\hat{p} = \sum_{i=1}^{n} x_i$
- ● $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i$

✔

STANDARD NOTATION

Solution:

Observe that

$$\begin{aligned}\ln L_n(x_1,\ldots,x_n,p) &= \ln\left(p^{\sum_{i=1} x_i}(1-p)^{n-\sum_{i=1} x_i}\right)\\ &= \left(\sum_{i=1}^n x_i\right)\ln p + \left(n-\sum_{i=1}^n x_i\right)\ln(1-p).\end{aligned}$$

Taking the derivative with respect to p,

$$\frac{\partial}{\partial p}\ln L_n(x_1,\ldots,x_n,p) = \frac{\sum_{i=1}^n x_i}{p} - \frac{n-\sum_{i=1}^n x_i}{1-p}.$$

We set this to be 0 and solve for p:

$$\begin{aligned}\frac{\sum_{i=1}^n x_i}{p} - \frac{n-\sum_{i=1}^n x_i}{1-p} &= 0 \Leftrightarrow\\ \frac{(1-p)\sum_{i=1}^n x_i - p\left(n-\sum_{i=1}^n x_i\right)}{p(1-p)} &= 0 \Leftrightarrow\\ \frac{\sum_{i=1}^n x_i - np}{p(1-p)} &= 0.\end{aligned}$$

Since the derivative blows up at $p=0,1$, we can assume $0<p<1$ and ignore the denominator for the purpose of solving for p. Hence $\hat{p} = \frac{1}{n}\sum_{i=1}^n x_i$ is the unique critical point of the log-likelihood.

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

Maximum Likelihood Estimator of a Bernoulli Statistical Model: Second Derivative Test

5/5 points (graded)

Setup:

As above, let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Ber}\,(p^*)$ for some unknown $p^* \in (0,1)$. You construct the associated statistical model $(\{0,1\}, \{\mathrm{Ber}\,(p)\}_{p\in(0,1)})$. Let L_n denote the likelihood of this statistical model. Recall from a previous problem that

$$L_n\,(x_1, \ldots, x_n, p) = \prod_{i=1}^{n} p^{x_i}\,(1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i}\,(1-p)^{n-\sum_{i=1}^n x_i}.$$

As stated, it will be more convenient to work with the **log-likelihood** $\ell\,(p) = \ln L_n\,(x_1, \ldots, x_n, p)$.

Question:

Next we will do the second derivative test to see if the critical point $\hat{p}$ obtained from the previous question is a local maximum. The second derivative of the log-likelihood can be written

$$\frac{\partial^2}{\partial p^2} \ln L_n\,(x_1, \ldots, x_n, p) = -\frac{C}{p^2} - \frac{n-C}{D}$$

where C depends on $\sum_{i=1}^n x_i$ and D depends on p. Fill in the blanks with the correct values of C and D.

(Type **Sigma_n** for the entire sum $\sum_{i=1}^n x_i$)

$C =$ Sigma_n ✔ **Answer:** Sigma_n

Σ_n

$D =$ (1-p)^2 ✔ **Answer:** (1-p)^2

$(1-p)^2$

Next we will test the endpoints of our optimization problem. Fill in the blanks with the correct values:
(Note that here we are working with the **likelihood**, *not* the **log-likelihood**)

$L_n(x_1, \ldots, x_n, 0) =$ 0 ✔ **Answer:** 0.0

$L_n(x_1, \ldots, x_n, 1) =$ 0 ✔ **Answer:** 0.0

What is the maximum likelihood estimator (MLE) $\hat{p}_n^{MLE}$ for the true parameter p^*?

- ○ 0
- ○ 1
- ○ $\sum_{i=1}^n X_i$
- ● $\frac{1}{n}\sum_{i=1}^n X_i$

✔

Solution:

The second derivative is

$$\frac{\partial}{\partial\theta}\left(\frac{\sum_{i=1}^n X_i}{p} - \frac{n - \sum_{i=1}^n X_i}{1-p}\right) = -\frac{\sum_{i=1}^n x_i}{p^2} - \frac{n - \sum_{i=1}^n x_i}{(1-p)^2}.$$

Since this expression is always negative, this implies that the critical point $\hat{p}$ is a **local maximum**.

Testing the endpoints we see

$$\begin{aligned} L_n(x_1, \ldots, x_n, 0) &= 0^{\sum_{i=1}^n x_i} (1)^{n-\sum_{i=1}^n x_i} = 0 \\ L_n(x_1, \ldots, x_n, 1) &= 1^{\sum_{i=1}^n x_i} (0)^{n-\sum_{i=1}^n x_i} = 0 \end{aligned}$$

Since the likelihood is non-negative, the endpoints are actually **global minima**.

Hence, the global maximum is achieved at $\hat{p} = \dfrac{1}{n}\displaystyle\sum_{i=1}^n x_i$. Plugging in the random variables $X_1, \ldots, X_n$, we derive the MLE

$$\hat{p}_n^{MLE} = \frac{1}{n}\sum_{i=1}^n X_i$$

which is precisely the **sample mean**.

Remark 1: This problem illustrates the conceptually nice fact that the **maximum likelihood estimator** for a Bernoulli statistical model is the **sample mean**.

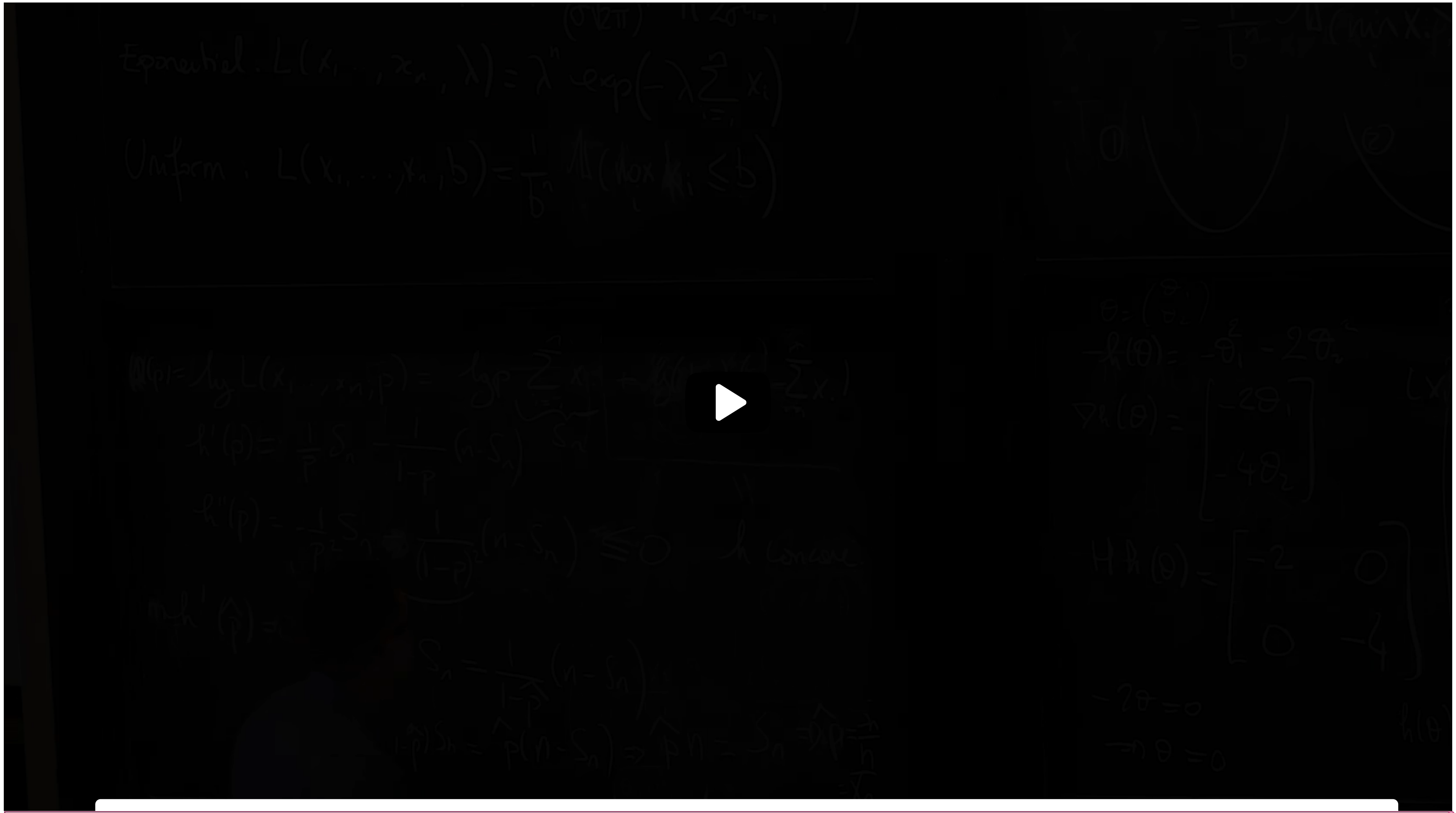
Remark 2: Note that for this problem, we derived the maximum likelihood estimator by optimizing $\ln L_n$ treating $x_1, \ldots, x_n$ as abstract variables. At the end, we plugged in our random samples $X_1, \ldots, X_n$. In practice, we would have access to observations $X_1 = x_1, \ldots, X_n = x_n$, and we can simply plug in $x_1, \ldots, x_n$ for the values of $X_1, \ldots, X_n$ in the expression for the MLE to get our estimate of the true parameter.

Remark 3: Alternatively, to get the estimate for p^*, we can first plug in the observations $X_1 = x_1, \ldots, X_n = x_n$ and then optimize the log-likelihood $\ln L_n(x_1, \ldots, x_n, p)$ as a function of p. You will get the same answer either way.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

Maximum Likelihood Estimator of Bernoulli Trials



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? [STAFF] Clarification required on diagonal of hessian
This statement from the professor " More generally, if you want to disprove the fact that it's the hessian of a concave function, you have to just exhibit an x such that the x tra... 3

The result of h''(p) when S_n = n 2

Don't think solution to second problem is technically correct even if it's easy to infer the intent.
The answer has a calculation with powers that doesn't seem to be correct in a certain trivial case and it seems the likelihood would be different then wouldn't it? 2

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