MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Problem 3: Hypothesis test with a continuous observation

(5/5 points)

Let Θ be a Bernoulli random variable that indicates which one of two hypotheses is true, and let $\mathbf{P}(\Theta = 1) = p$. Under the hypothesis $\Theta = 0$, the random variable X is uniformly distributed over the interval $[0, 1]$. Under the alternative hypothesis $\Theta = 1$, the PDF of X is given by

$$f_{X|\Theta}(x \mid 1) = \begin{cases} 2x, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the MAP rule for deciding between the two hypotheses, given that $X = x$.

1. Suppose for this part of the problem that $p = 3/5$. The MAP rule can choose in favor of the hypothesis $\Theta = 1$ if and only if $x \ge c_1$. Find the value of c_1.

   $c_1 =$ 1/3 ✔ **Answer:** 0.33333

2. Assume now that p is general such that $0 \le p \le 1$. It turns out that there exists a constant c such that the MAP rule always decides in favor of the hypothesis $\Theta = 0$ if and only if $p < c$. Find c.

$c =$ 1/3 ✔ **Answer:** 0.33333

3. For this part of the problem, assume again that $p = 3/5$. Find the conditional probability of error for the MAP decision rule given that the hypothesis $\Theta = 0$ is true.

$\mathbf{P}(\text{error} \mid \Theta = 0) =$ 2/3 ✔ **Answer:** 0.66667

4. Find the probability of error associated with the MAP rule as a function of p. Express your answer in terms of p using standard notation .

When $p \leq 1/3$, $\mathbf{P}(\text{error}) =$ p ✔ **Answer:** p

When $p \geq 1/3$, $\mathbf{P}(\text{error}) =$ 3/2-1/(4*p)-(5*p)/4 ✔ **Answer:** 3/2-5*p/4-0.25/p

Answer:

1. If $0 < p < 1$, we can choose in favor of the hypothesis $\Theta = 1$ if and only if

$$\begin{aligned} f_{X|\Theta}(x \mid 1) p_\Theta(1) &\geq f_{X|\Theta}(x \mid 0) p_\Theta(0), \\ 2x \cdot p &\geq 1 \cdot (1 - p), \\ x &\geq \frac{1-p}{2p}. \end{aligned}$$

If $p = 3/5$, the rule above corresponds to $x \geq 1/3$.

Additional theoretical material

Unit summary

- Exam 2
- Unit 8: Limit theorems and classical statistics
- Unit 9: Bernoulli and Poisson processes
- Unit 10: Markov chains
- Exit Survey
- Final Exam

2. We will be forced to choose in favor of the hypothesis $\Theta = 0$ if the condition $x \geq (1-p)/(2p)$ can never hold. Since $x \in [0,1]$, this will be the case if and only if the threshold $(1-p)/(2p)$ exceeds 1, that is, $1 - p > 2p$, or $p < 1/3$.

3. If the hypothesis $\Theta = 0$ is true, an error occurs when we decide in favor of the hypothesis $\Theta = 1$. For $p = 3/5$, this corresponds to the event $\{X \geq 1/3\}$. Therefore,

$$\mathbf{P}(\text{error} \mid \Theta = 0) = \mathbf{P}(X \geq 1/3 \mid \Theta = 0) = \int_{1/3}^{1} f_{X|\Theta}(x \mid 0)\, dx = \int_{1/3}^{1} 1\, dx = \frac{2}{3}.$$

4. Similar to the computation above, we find that for $p \geq 1/3$,

$$\begin{aligned}
\mathbf{P}(\text{error} \mid \Theta = 0) &= \mathbf{P}\left(X \geq \frac{1-p}{2p} \mid \Theta = 0\right) \\
&= \int_{\frac{1-p}{2p}}^{1} f_{X|\Theta}(x \mid 0)\, dx \\
&= \int_{\frac{1-p}{2p}}^{1} 1\, dx \\
&= 1 - \frac{1-p}{2p} \\
&= \frac{3p-1}{2p},
\end{aligned}$$

$$\begin{aligned}\mathbf{P}(\text{error} \mid \Theta = 1) &= \mathbf{P}\left(X < \frac{1-p}{2p} \mid \Theta = 1\right) \\ &= \int_0^{\frac{1-p}{2p}} f_{X|\Theta}(x \mid 1)\, dx \\ &= \int_0^{\frac{1-p}{2p}} 2x\, dx \\ &= \left(\frac{1-p}{2p}\right)^2.\end{aligned}$$

Using the total probability theorem, we find

$$\begin{aligned}\mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \mid \Theta = 0)p_\Theta(0) + \mathbf{P}(\text{error} \mid \Theta = 1)p_\Theta(1) \\ &= \frac{(3p-1)(1-p)}{2p} + \frac{(1-p)^2}{4p} = \frac{(1-p)(5p-1)}{4p}, \text{ for } p \geq 1/3.\end{aligned}$$

For $p < 1/3$, we will always decide $\Theta = 0$, and the resulting probability of error is

$$\mathbf{P}(\text{error}) = \mathbf{P}(\text{error} \mid \Theta = 0)p_\Theta(0) + \mathbf{P}(\text{error} \mid \Theta = 1)p_\Theta(1) = 0 \cdot (1-p) + 1 \cdot p = p.$$

For the boundary case of $p = 1/3$, both formulas yield $\mathbf{P}(\text{error}) = 1/3$.

You have used 1 of 2 submissions

DISCUSSION

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