# 5. Failure of existence and uniqueness

Let's continue exploring the family of homogeneous boundary value problems, one for each value of $\lambda$ as in the previous page. But here we restrict our interest to the case where there are nonzero solutions.

**Problem 5.1** Find all **nonzero** functions $v\,(x)$ on $[0, \pi]$ satisfying $\frac{d^2}{dx^2}v\,(x) = \lambda\, v\,(x)$ for a constant $\lambda$ and satisfying the **boundary conditions** $v\,(0) = 0$ and $v\,(\pi) = 0$.

**Solution to the problem:** The equation $v''\,(x) = \lambda\, v\,(x)$ is a homogeneous linear ODE with characteristic polynomial $r^2 - \lambda$.

**Case 1:** $\lambda > 0$**.** Then the general solution is $ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$, and the boundary conditions say

$$a + b = 0 \qquad (3.1)$$

$$ae^{\sqrt{\lambda}\pi} + be^{-\sqrt{\lambda}\pi} = 0. \qquad (3.2)$$

Since

$$\det \begin{pmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\pi} & e^{-\sqrt{\lambda}\pi} \end{pmatrix} \neq 0, \qquad (3.3)$$

the only solution to this linear system is $(a, b) = (0, 0)$. Thus there are no nonzero solutions $v$.

**Case 2:** $\lambda = 0$**.** Then the general solution is $a + bx$, and the boundary conditions say

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$$
\begin{aligned}
a &= 0 && (3.4)\\
a + b\pi &= 0. && (3.5)
\end{aligned}
$$

Again the only solution to this linear system is $(a,b) = (0,0)$. Thus there are no nonzero solutions $v$.

**Case 3:** $\lambda < 0$. We can write $\lambda = -\omega^2$ for some $\omega > 0$. Then the roots of the characteristic polynomial are $\pm i\omega$, and the general solution is $a\cos\omega x + b\sin\omega x$. The first boundary condition says $a = 0$, so $v = b\sin\omega x$. The second boundary condition then says $b\sin\omega\pi = 0$. We are looking for nonzero solutions $v$, so we can assume that $b \neq 0$. Then $\sin\omega\pi = 0$, so $\omega$ is an integer $n$. It is enough to consider $n > 0$ since $\sin(-\omega x) = -\sin(\omega x)$.

**Conclusion:** There exist nonzero solutions if and only if $\lambda = -n^2$ for some positive integer $n$; in that case, all solutions are of the form $b\sin nx$.

**We will use this conclusion as one step in the solution of the Heat Equation in the next lecture.**

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? Case 3: a=0
"The first boundary condition says a=0." I don't understand how this is determined. And if a=0, how is it that b does not equal zero? The first boundary condition is a+b=0, isn'... 3

? Case 2
since it is a second order equation the solution must be of the form: y = a*e^x + b*t*e^x, how do we derive the solution of the form: a+b*x. Thank you in advance for your help. 3

Explain the conclusion
If n is integer, v=0 how explain if you say nonzero solution for some positive integer n ? 3



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