HW7 Solutions

1. (15 pts.) James Bond

James Bond, my favorite hero, has again jumped off a plane. The plane is traveling from from base A to base B, distance 100 km apart. Now suppose the plane takes off from the ground at base A, climbs at an angle of 45 degrees to an altitude of 10 km, flies at that altitude for a while, and then descends at an angle of 45 degrees to land at base B. All along the plane is assumed to fly at the same *speed*. James Bond jump off at a time uniformly distributed over the duration of the journey. You can assume that James Bond, being who he is, violates the laws of physics and descends vertically after he jumps.

Compute the expectation and variance of this position. How do they compare to the case when the plane flies at constant velocity (with no ascending and descending)?

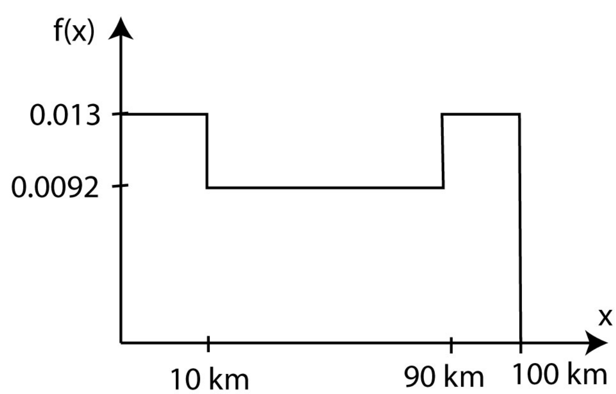
Answer:

The density function of the position James Bond lands is given by

$$f(x) = \begin{cases} \frac{1}{10} \times \frac{10\sqrt{2}}{20\sqrt{2}+80} & \text{if } x \in [0,10) \\ \frac{1}{80} \times \frac{80}{20\sqrt{2}+80} & \text{if } x \in [10,90) \\ \frac{1}{10} \times \frac{10\sqrt{2}}{20\sqrt{2}+80} & \text{if } x \in [90,100] \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{\sqrt{2}}{20\sqrt{2}+80} & \text{if } x \in [0,10) \\ \frac{1}{20\sqrt{2}+80} & \text{if } x \in [10,90) \\ \frac{\sqrt{2}}{20\sqrt{2}+80} & \text{if } x \in [90,100] \\ 0 & \text{otherwise.} \end{cases}$$

The density function looks like this:



The expectation will be

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^{100} x f(x)\, dx \\
&= \int_0^{10} x \frac{\sqrt{2}}{20\sqrt{2}+80} dx + \int_{10}^{90} x \frac{1}{20\sqrt{2}+80} dx + \int_{90}^{100} x \frac{\sqrt{2}}{20\sqrt{2}+80} dx \\
&= \frac{1}{20\sqrt{2}+80} \left(\left[\sqrt{2}\frac{x^2}{2} \right]_{x=0}^{10} + \left[\frac{x^2}{2} \right]_{x=10}^{90} + \left[\sqrt{2}\frac{x^2}{2} \right]_{x=90}^{100} \right) \\
&= \frac{1}{20\sqrt{2}+80} \left(\frac{1}{2}\left(1-\sqrt{2}\right)(90^2-10^2) + \frac{1}{2}\left(\sqrt{2}\right)100^2 \right) \\
&= \frac{1}{20\sqrt{2}+80} \left(\frac{1}{2}\left(2000\sqrt{2}+8000\right) \right) \\
&= 50
\end{aligned}
$$

(A sanity check: The distribution is symmetric about the midpoint of the journey (50 km), so it makes sense that the expectation is 50.)

The expectation is the same as when the plane travels at the same altitude for the entire trip.

The variance will be

$$
\begin{aligned}
\mathbf{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \int_0^{100} x^2 f(x)\, dx - 50^2 \\
&= \int_0^{10} x^2 \frac{\sqrt{2}}{20\sqrt{2}+80} dx + \int_{10}^{90} x^2 \frac{1}{20\sqrt{2}+80} dx + \int_{90}^{100} x^2 \frac{\sqrt{2}}{20\sqrt{2}+80} dx - 50^2 \\
&= \frac{1}{20\sqrt{2}+80} \left(\left[\sqrt{2}\frac{x^3}{3} \right]_{x=0}^{10} + \left[\frac{x^3}{3} \right]_{x=10}^{90} + \left[\sqrt{2}\frac{x^3}{3} \right]_{x=90}^{100} \right) - 50^2 \\
&= \frac{1}{20\sqrt{2}+80} \left(\frac{1}{3}\left(1-\sqrt{2}\right)(90^3-10^3) + \frac{1}{2}\left(\sqrt{2}\right)100^3 \right) - 50^2 \\
&= \frac{1}{20\sqrt{2}+80} \left(\frac{1}{3}\left(272000\sqrt{2}+728000\right) \right) - 50^2 \\
&= \frac{1}{3} \times \frac{34\sqrt{2}+91}{2\sqrt{2}+8} \times 800 - 2500 \\
&\approx 925.
\end{aligned}
$$

In the uniform case where the plane flies at a constant speed at the same altitude, the variance is $\frac{100^2}{12} \approx 833$. The variance is greater here because the ascent/descent increases the probability that Bond jumps close to base A or B, i.e., the probability that he jumps far away from the expectation (the middle).

2. (20 pts.) Laplace Distribution

A continuous random variable X is said to have a Laplace distribution with parameter λ if its pdf is given by:

$$
f(x) = A \exp\left(-\lambda |x|\right), \qquad -\infty < x < \infty
$$

for some constant A.

(a) (2 pts.) Can the parameter λ be negative? Can λ be zero? Explain.

Answer: No. No. We know that for a valid pdf,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \ .$$

If we plug in for $f(x)$, we get

$$\int_{-\infty}^{\infty} A \exp(-\lambda|x|)\, dx = 2\int_{0}^{\infty} A \exp(-\lambda x)\, dx = 2\left[\frac{-A}{\lambda}\exp(-\lambda x)\right]_0^{\infty} .$$

If $\lambda \leq 0$, this evaluates to infinity, and then there is no way to choose A so that the area under the pdf is 1. Therefore, it must be that $\lambda > 0$.

(b) (3 pts.) Compute the constant A in terms of λ. Sketch the pdf.

Answer: Following part (a), we know

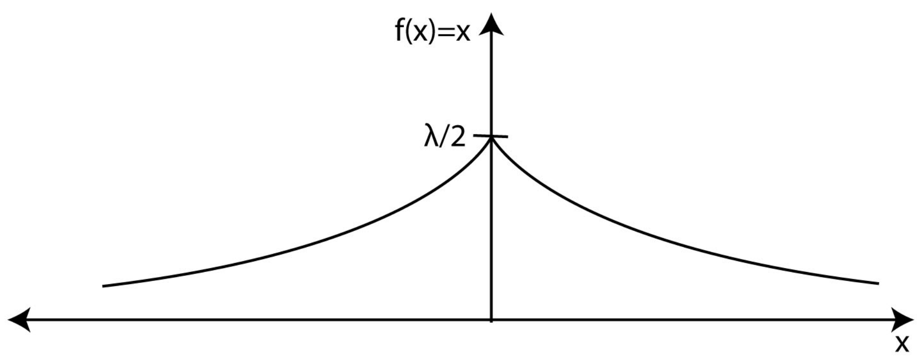
$$\int_{-\infty}^{\infty} A \exp(-\lambda|x|)\, dx = 2\int_{0}^{\infty} A \exp(-\lambda x)\, dx = 1 \ .$$

Also,

$$\begin{aligned} 2\int_{0}^{\infty} A \exp(-\lambda x)\, dx &= 2\left[\frac{-A}{\lambda}\exp(-\lambda x)\right]_{x=0}^{\infty} \\ &= 0 - 2 \times \left(-\frac{A}{\lambda}\right) \\ &= 2\frac{A}{\lambda}. \end{aligned}$$

Therefore, we must have $2A/\lambda = 1$, or equivalently, $A = \lambda/2$.

The pdf looks like this:



(c) (4 pts.) Compute the mean and variance of X in terms of λ.

Answer: Using the definition of the expectation:

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x\frac{\lambda}{2}\exp(-\lambda|x|)\, dx \\ &= \int_{-\infty}^{0} x\frac{\lambda}{2}\exp(-\lambda|x|)\, dx + \int_{0}^{\infty} x\frac{\lambda}{2}\exp(-\lambda|x|)\, dx \\ &= \int_{0}^{\infty} -x\frac{\lambda}{2}\exp(-\lambda|x|)\, dx + \int_{0}^{\infty} x\frac{\lambda}{2}\exp(-\lambda|x|)\, dx \\ &= 0 \end{aligned}$$

(Alternatively, we could note that the distribution is symmetric around $x = 0$, from which it follows that the mean is zero.)

For the variance, we have

$$\begin{aligned}
\mathbf{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] \\
&= \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} \exp\left(-\lambda|x|\right) dx \\
&= 2\int_{0}^{\infty} x^2 \frac{\lambda}{2} \exp\left(-\lambda x\right) dx \\
&= \left[-\frac{\lambda x(\lambda x + 2) - 2}{\lambda^2} \exp\left(-\lambda x\right) \right]_{x=0}^{\infty} \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

(The integral in the 3rd line is done using integration by parts.)

(d) (4 pts.) Compute the cumulative distribution function (cdf) of X.

Answer:

$$\begin{aligned}
\mathbf{P}(X \leq x) &= \int_{-\infty}^{x} f(u) du \\
&= \int_{-\infty}^{x} \frac{\lambda}{2} \exp\left(-\lambda|u|\right) du \quad .
\end{aligned}$$

When $x \leq 0$,

$$\begin{aligned}
\mathbf{P}(X \leq x) &= \int_{-\infty}^{x} \frac{\lambda}{2} \exp\left(-\lambda(-u)\right) du \\
&= \frac{1}{2}[\exp(\lambda u)]_{-\infty}^{x} \\
&= \frac{1}{2}\exp(\lambda x).
\end{aligned}$$

When $x > 0$,

$$\begin{aligned}
\mathbf{P}(X \leq x) &= \int_{0}^{x} \frac{\lambda}{2} \exp\left(-\lambda u\right) du + \frac{1}{2} \\
&= \frac{1}{2}[-\exp(-\lambda u)]_{0}^{x} + \frac{1}{2} \\
&= \frac{1}{2}(-\exp(-\lambda x) + 1) + \frac{1}{2} \\
&= 1 - \frac{1}{2}\exp(-\lambda x).
\end{aligned}$$

(e) (3 pts.) For $s, t > 0$, compute $\mathbf{P}[X \geq s + t | X \geq s]$.

Answer: We begin with the definition of conditional probability:

$$\mathbf{P}(X \geq s + t | X \geq s) = \frac{\mathbf{P}(X \geq s + t \cap X \geq s)}{\mathbf{P}(X \geq s)} \quad .$$

Since $t > 0$, if $X \geq s + t$ then $X \geq t$, so we have

$$\mathbf{P}(X \geq s + t \cap X \geq s) = \mathbf{P}(X \geq s + t).$$

Also, since s and t are both positive, $\mathbf{P}(X \geq s+t) = \exp(-\lambda(s+t))$ and $\mathbf{P}(X \geq s) = \exp(-\lambda s)$. Therefore,

$$\mathbf{P}(X \geq s+t | X \geq s) = \frac{\mathbf{P}(X \geq s+t)}{\mathbf{P}(X \geq s)} = \frac{\frac{\exp(-\lambda(s+t))}{2}}{\frac{\exp(-\lambda s)}{2}} = \exp(-\lambda t).$$

(f) (4 pts.) Let $Y = |X|$. Compute the pdf of Y. (Hint: you may consider starting from the cdf of X that you have computed in part (d).)

Answer: Note that for $y \geq 0$,

$$\mathbf{P}(Y \leq y) = 1 - \mathbf{P}(Y > y) = 1 - (\mathbf{P}(X > y) + \mathbf{P}(X < -y)) \quad .$$

Thus,

$$\mathbf{P}(Y \leq y) = 1 - \left(\frac{1}{2}\exp(-\lambda y) + \frac{1}{2}\exp(-\lambda y)\right) = 1 - \exp(-\lambda y) \quad .$$

This is the cumulative distribution function for Y. Finally, we differentiate $\mathbf{P}(Y \leq y)$ to get the pdf of Y, noting that

$$\frac{d}{dy}(1 - \exp(-\lambda y)) = \lambda \exp(-\lambda y).$$

Consequently, the pdf for Y is given by

$$f(y) = \begin{cases} \lambda \exp(-\lambda y) & \text{if } y \geq 0, \\ 0 & \text{if } y < 0. \end{cases}$$

3. (15 pts.) Bus waiting times

You show up at a bus stop at a random time. There are three buses running on a periodic schedule. They all go to the same destination (which is where you want to go). Each of the three buses arrives at your bus stop once every 10 minutes, but at different offsets from each other. The offsets of the three buses are uniformly random and independent. You will get on the first bus to arrive at your bus stop. Let the random variable T denote the number of minutes you have to wait until the first bus arrives.

(a) **(6 pts.)** Compute an expression for the probability density function (pdf) and the cumulative distribution function (cdf) for T.

Answer: Let X_1, X_2, X_3 be random variables denoting the number of minutes you have to wait for bus 1, 2, or 3 respectively. They have a uniform distribution: $X_1, X_2, X_3 \sim \text{Uniform}(0 \ldots 10)$. Also $T = \min(X_1, X_2, X_3)$. Let f_1, f_2, f_3 be the probability density functions for X_1, X_2, X_3, and let F_1, F_2, F_3 be the cumulative distribution functions. Let g be the probability density function for T, and G be the cumulative distribution function for T. Then,

$$\begin{aligned} f_i(x) &= \frac{1}{10} \text{ for } x \in [0, 10] \\ F_i(x) &= \frac{x}{10} \text{ for } x \in [0, 10] \end{aligned}$$

To obtain the probability density function $g(t)$, we first compute the cumulative density function $G(t) = \mathbf{P}(T \leq t)$, and then we calculate its derivative. Suppose $0 \leq t \leq 10$. To compute $\mathbf{P}(T \leq t)$, we compute the probability of the complement event:

$$\begin{aligned} \mathbf{P}(T > t) &= \mathbf{P}(\min(X_1, X_2, X_3) > t) \\ &= \mathbf{P}(X_1 > t \text{ and } X_2 > t \text{ and } X_3 > t) \\ &= \mathbf{P}(X_1 > t) \times \mathbf{P}(X_2 > t) \times \mathbf{P}(X_3 > t) \\ &= \left(1 - \frac{t}{10}\right)^3. \end{aligned}$$

Therefore,

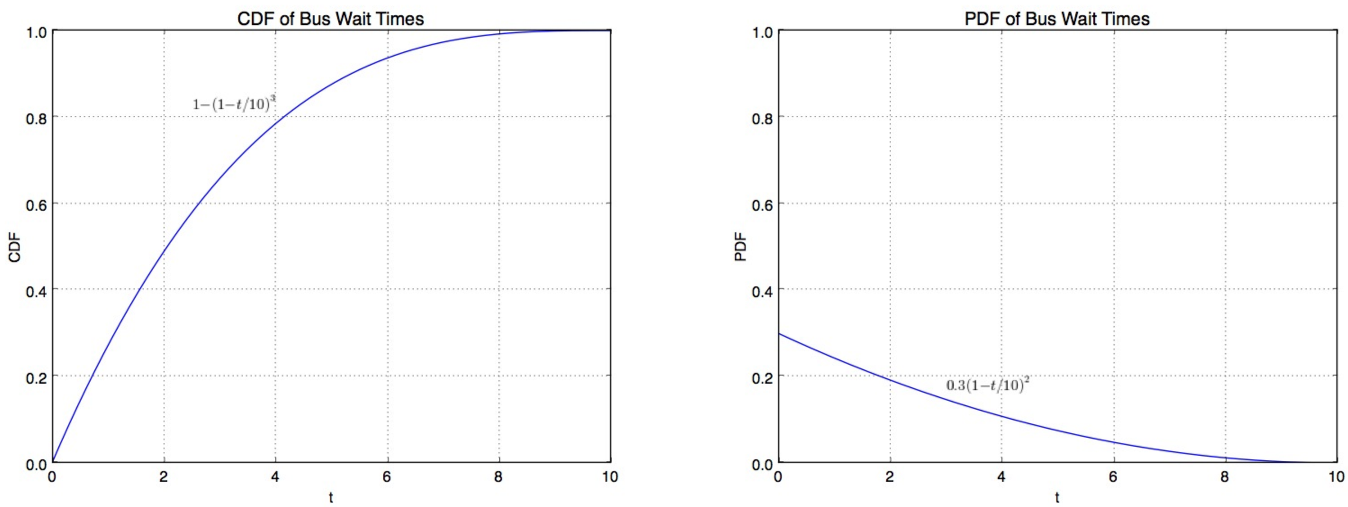
$$G(t) = \mathbf{P}(T \leq t) = 1 - \mathbf{P}(T > t) = 1 - \left(1 - \frac{t}{10}\right)^3 \quad \text{for } t \in [0, 10].$$

Now the pdf is the derivative of the cdf:

$$g(t) = \frac{d}{dt}G(t) = 3 \times \left(1 - \frac{t}{10}\right)^2 \times \frac{1}{10} = \frac{3}{10}\left(1 - \frac{t}{10}\right)^2 \quad \text{for } t \in [0, 10].$$

(b) **(3 pts.)** Plot the pdf and cdf.

Answer:



(c) **(6 pts.)** Compute $\mathbb{E}[T]$ and $\text{Var}(T)$.
Answer:

$$\begin{aligned}
\mathbb{E}[T] &= \int_0^{10} t \times g(t)\, dt \\
&= \int_0^{10} t \times \frac{3}{10} \times \left(1 - \frac{t}{10}\right)^2 dt \\
&= \frac{3}{10} \int_0^{10} \left(t - \frac{t^2}{5} + \frac{t^3}{100}\right) dt \\
&= \frac{3}{10} \left(\frac{t^2}{2} - \frac{t^3}{15} + \frac{t^4}{400}\right)\Big|_0^{10} \\
&= \frac{3}{10} \left(\frac{100}{2} - \frac{1000}{15} + \frac{10000}{400}\right) \\
&= \frac{5}{2}.
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[T^2] &= \int_0^{10} t^2 \times g(t)\, dt \\
&= \int_0^{10} t^2 \times \frac{3}{10} \times \left(1 - \frac{t}{10}\right)^2 dt \\
&= \frac{3}{10} \int_0^{10} \left(t^2 - \frac{t^3}{5} + \frac{t^4}{100}\right) dt \\
&= \frac{3}{10} \left(\frac{t^3}{3} - \frac{t^4}{20} + \frac{t^5}{500}\right)\Big|_0^{10} \\
&= 10.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(T) &= \mathbb{E}[T^2] - \mathbb{E}[T]^2 \\
&= 10 - (5/2)^2 \\
&= \frac{15}{4}.
\end{aligned}
$$

4. (20 pts.) Exponential Distribution

We begin by proving two very useful properties of the exponential distribution. We then use them to solve a problem in photography

(a) (4 pts) Let r.v. X have geometric distribution with parameter p. Show that, for any positive m, n, we have

$$P(X > m + n | X > m) = P(X > n).$$

This is the memoryless property of the geometric distribution. Why do you think this property is called memoryless?

Answer: We first calculate the probability $\mathbf{P}(X > m)$ for a geometric random variable X with parameter p; we know that $X > m$ when the first m trials were failures; therefore, we get:

$$\mathbf{P}(X > m) = (1 - p)^m$$

Using this, we can now compute the conditional probability:

$$
\begin{aligned}
\mathbf{P}(X > m + n | X > m) &= \frac{\mathbf{P}((X > m + n) \cap (X > m))}{\mathbf{P}(X > m)} \\
&= \frac{\mathbf{P}(X > m + n)}{\mathbf{P}(X > m)} \qquad \text{[Since } (X > m + n) \cap (X > m) \equiv (X > m + n)\text{]} \\
&= \frac{(1 - p)^{m+n}}{(1 - p)^m} \\
&= (1 - p)^n \\
&= \mathbf{P}(X > n)
\end{aligned}
$$

This is exactly what we wanted to show. This property is called memoryless because even knowing that we have waited m trials and have not yet seen a success, the probability of seeing a success in the next n trials is exactly the same as if we had not seen any trials at all.

(b) (4 pts) Let r.v. X have exponential distribution with parameter λ. Show that, for any positive s,t, we have

$$P(X > s + t | X > t) = P(X > s).$$

[This is the memoryless property of the exponential distribution.]

Answer: We first calculate the probability $\mathbf{P}(X > t)$ for an (2 points) exponential random variable X with parameter λ.

$$\mathbf{P}(X > t) \; = \; \int_t^\infty \lambda e^{-\lambda x} dx \; = \; e^{-\lambda t}$$

Using this, it is now easy to compute the given conditional probability

$$\begin{aligned}
\mathbf{P}(X > s + t | X > t) &= \frac{\mathbf{P}((X > s + t) \cap (X > t))}{\mathbf{P}(X > t)} \\
&= \frac{\mathbf{P}(X > s + t)}{\mathbf{P}(X > t)} \qquad \text{[Since } (X > s + t) \cap (X > t) \equiv (X > s + t)\text{]} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\
&= e^{-\lambda s} \; = \; \mathbf{P}(X > s)
\end{aligned}$$

(c) (4 pts) Let r.v.s X_1, X_2 be independent and exponentially distributed with parameters λ_1, λ_2. Show that the r.v. $Y = \min\{X_1, X_2\}$ is exponentially distributed with parameter $\lambda_1 + \lambda_2$. [Hint: work with cdfs.]

Answer: Using the hint given, we will work with the CDF; from part b, remember that if X is exponential with parameter λ, then $\mathbf{P}(X \leq t) = 1 - \mathbf{P}(X > t) = 1 - e^{-\lambda t}$. So, if we can show that the CDF of Y is also of this form, then we can conclude that Y is also an exponential random variable. So we get:

$$\begin{aligned}
\mathbf{P}(Y \leq t) &= 1 - \mathbf{P}(Y > t) \\
&= 1 - \mathbf{P}(\min\{X_1, X_2\} > t) \\
&= 1 - \mathbf{P}((X_1 > t) \cap (X_2 > t)) \\
&= 1 - \mathbf{P}(X_1 > t) \cdot \mathbf{P}(X_2 > t) \qquad \text{[Since } X_1, X_2 \text{ are independent]} \\
&= 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} \\
&= 1 - e^{-(\lambda_1 + \lambda_2) t}
\end{aligned}$$

This is exactly the CDF for an exponential random variable with parameter $\lambda_1 + \lambda_2$.

(d) (4 pts) You have a digital camera that requires two batteries to operate. You purchase n batteries, labeled $1, 2, ..., n$, each of which has a lifetime that is exponentially distributed with parameter λ and is independent of all the other batteries. Initially you install batteries 1 and 2. Each time a battery fails, you replace it with the lowest-numbered unused battery. At the end of this process you will be left with just one working battery. What is the expected total time until the end of the process? Justify your answer

Answer: Let T be the time when we are left with only one battery. We want to compute $\mathbb{E}[T]$.To facilitate the calculation, we can break up T into the intervals

$$T \; = \; T_1 + T_2 + \ldots + T_{n-1}$$

where T_i is the time between the $(i-1)^{th}$ failure and i^{th} failure. After $n-1$ failures, we are left with only one battery and the process ends. Note that the i^{th} failure *need not* correspond to the battery labeled i.

We can calculate $\mathbb{E}[T]$ as

$$\mathbb{E}[T] = \mathbb{E}[T_1] + \mathbb{E}[T_2] + \ldots + \mathbb{E}[T_{n-1}]$$

It now remains to find $\mathbb{E}[T_i]$ for each i. To calculate $\mathbb{E}[T_i]$, note that after $i-1$ failures, we have the battery labeled i and another battery labeled j for $j < i$ installed in the camera.

The lifetime of the battery labeled i is distributed as an exponential random variable with parameter λ. Also, because of the memoryless property proved in part (a), the *remaining* lifetime of the battery labeled j, conditioned on the battery being functional till the time of the $(i-1)^{th}$ failure, is *also* distributed as an exponential random variable with parameter λ. Hence, T_i is the minimum of two exponential random variables, each with parameter λ. By part (b), T_i is exponentially distributed with parameter 2λ. This gives

$$\mathbb{E}[T_i] = \frac{1}{2\lambda}$$

and hence,

$$\mathbb{E}[T] = \mathbb{E}[T_1] + \mathbb{E}[T_2] + \ldots + \mathbb{E}[T_{n-1}] = \frac{n-1}{2\lambda}$$

(e) (4 pts) In the scenario of part (d), what is the probability that battery i is the last remaining working battery, as a function of i?

Answer: We first calculate that if at any point, the battery labeled i and one labeled j, (for some $j \neq i$) are installed, then what is the probability that the one labelled j fails first. If neither of the batteries have failed, then conditioned on this fact, the distributions of their remaining lifetimes are identical (exponential with parameter λ). Hence, by symmetry, the probability that the one labeled j fails first is $1/2$.

For batteries $2, 3, \ldots, n$, the battery labelled i is installed after $i-2$ batteries have failed. The probability that it is the last one left, is the probability that in each of the remaining $n-i+1$ failures that remain, it is the other installed battery in the camera which fails first and not the battery labeled i. From the previous calculation, each of these events happen with probability $\frac{1}{2}$, conditioned on the history. Hence, the probability that the battery labeled i is the last one left is $1/2^{n-i+1}$. However, this formula is slightly off for the battery labeled 1; like battery 2, it also needs to survive $n-1$ failures, not n failures, and so we get:

$$\mathbf{P}(\text{last battery is battery } i) = \begin{cases} \frac{1}{2^{n-1}} & i = 1 \\ \frac{1}{2^{n-i+1}} & 2 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$$

5. (15 pts.) Misprints

A textbook has on average one misprint per page.

(a) (7 pts) What is the chance that you see exactly 4 misprints on page 1?

Answer: We assume that misprints are "rare events" that follows the Poisson distribution. Let X_1 be a Poisson random variable representing the number of misprints on page 1. Since a Poisson random variable with parameter λ has expectation λ, we know $X_1 \sim \text{Poiss}(1)$. Hence

$$\mathbf{P}(X_1 = 4) = \frac{e^{-1}}{4!} = \frac{e^{-1}}{24} \approx 0.0153$$

(b) (8 pts) What is the chance that you see exactly 4 misprints on some page in the textbook, if the textbook is 250 pages long?

Answer: Let X_i be a Poisson random variable representing the number of misprints on page i. By the independence of $X_1, \ldots, X_{250}$,

$$\begin{aligned}\mathbf{P}(X_i = 4 \text{ for some } i) &= 1 - \mathbf{P}(X_i \neq 4 \text{ for all } i) \\ &= 1 - \left(1 - \frac{e^{-1}}{24}\right)^{250} \\ &\approx 0.979\end{aligned}$$

6. (15 pts.) Functions of a r.v.

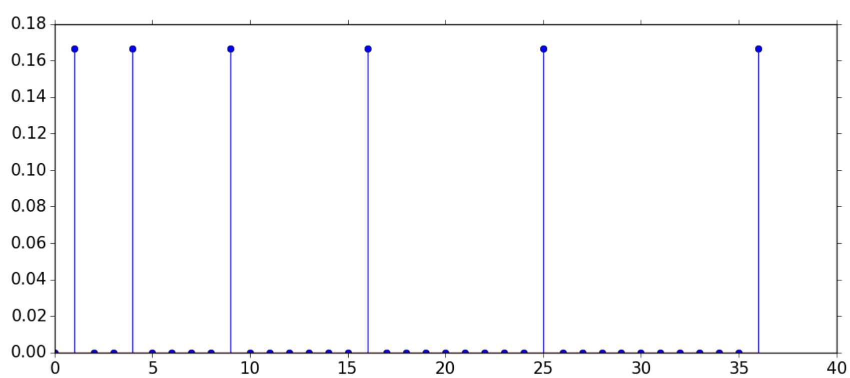
(a) (5 pts.) Suppose X is a discrete random variable uniform in $\{1, 2, \ldots, n\}$. What is the pmf of the random variable $Y = X^2$? Is it uniform on its range?

Answer: The range of Y is $\{1, 4, 9, \ldots, n^2\}$.

$$\begin{aligned}\mathbf{P}(Y = m) &= \mathbf{P}(X^2 = m) \\ &= \mathbf{P}(X = \sqrt{m}) \\ &= \frac{1}{n}\end{aligned}$$

when there exists $i \in \{1, 2, \ldots, n\}$ such that $m = i^2$. $\mathbf{P}(Y = m) = 0$ for all other values of m. Hence Y is also uniform on its range.

Plot of pmf for $n = 6$:



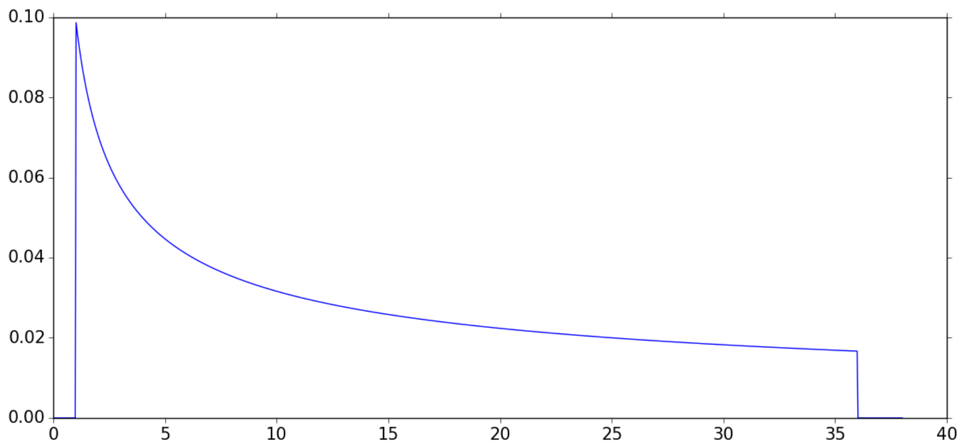
(b) (5 pts.) Suppose X is a continuous random variable uniform in the interval $[1, n]$. What is the pdf of the random variable $Y = X^2$? Is it uniform on its range? Sketch the pdf, and give an intuitive explanation of the shape of the curve.

Answer: The range of Y is the interval $[1, n^2]$. We start by computing the cdf of Y. If $y < 1$, $\mathbf{P}(Y \leq y) = 0$. If $y > n^2$, $\mathbf{P}(Y \leq y) = 1$. For $y \in [1, n^2]$ we have

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) \\ &= \mathbf{P}(X^2 \leq y) \\ &= \mathbf{P}(X \leq \sqrt{y}) \\ &= \int_1^{\sqrt{y}} \frac{1}{n-1} dx \\ &= \frac{1}{n-1}(\sqrt{y} - 1) \end{aligned}$$

Therefore, $f_Y(y) = \frac{1}{2(n-1)\sqrt{y}}$. The pdf of Y is decreasing, meaning that the probabilities of having larger values is smaller. Intuitively, this is because intervals of X get mapped to larger and larger intervals for Y.

Plot of pdf for $n = 6$:



(c) (5 pts.) Suppose X is a continuous random variable uniform in the interval $[-n, n]$. What is the pdf of $Y = X^2$? Sketch the pdf.

Answer: The range of Y is the interval $[0, n^2]$. If $y < 0$, $\mathbf{P}(Y \leq y) = 0$. If $y > n^2$, $\mathbf{P}(Y \leq y) = 1$. For $y \in [0, n^2]$ we have

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) \\ &= \mathbf{P}(X^2 \leq y) \\ &= \mathbf{P}(|X| \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2n} dx \\ &= \frac{1}{n}(\sqrt{y}) \end{aligned}$$

Therefore, $f_Y(y) = \frac{1}{2n\sqrt{y}}$.

Plot of pdf for $n = 6$: