STAT 414 / 415 Probability Theory and Mathematical Statistics

The Standard Normal and The Chi-Square

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We have one more theoretical topic to address before getting back to some practical applications on the next page, and that is the relationship between the normal distribution and the chi-square distribution. The following theorem clarifies the relationship.

Theorem. If X is normally distributed with mean μ and variance $\sigma^2 > 0$, then:

$$V = \left(\frac{X-\mu}{\sigma}\right)^2 = Z^2$$

is distributed as a chi-square random variable with 1 degree of freedom.

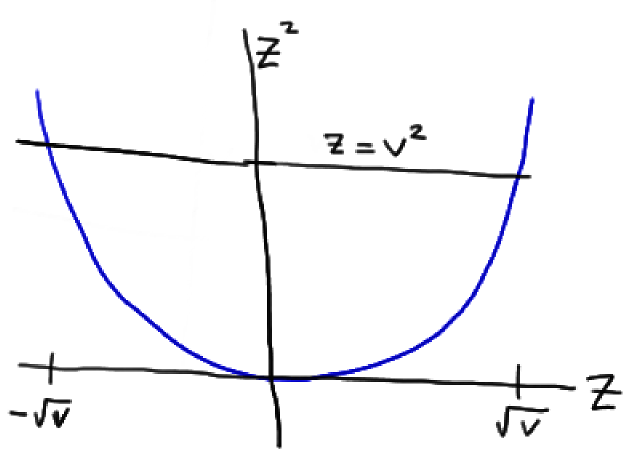
Proof. To prove this theorem, we need to show that the p.d.f. of the random variable V is the same as the p.d.f. of a chi-square random variable with 1 degree of freedom. That is, we need to show that:

$$g(v) = \frac{1}{\Gamma(1/2)2^{1/2}} v^{\frac{1}{2}-1} e^{-v/2}$$

The strategy we'll take is to find $G(v)$, the cumulative distribution function of V, and then differentiate it to get $g(v)$, the probability density function of V. That said, we start with the definition of the cumulative distribution function of V:

$$G(v) = P(V \leq v) = P(Z^2 \leq v)$$

That second equality comes, of course, from the fact that $V = Z^2$. Now, taking note of the behavior of a parabolic function:



we can simplify $G(v)$ to get:

$$G(v) = P(-\sqrt{v} < Z < \sqrt{v})$$

Now, to find the desired probability we need to integrate, over the given interval, the probability density function of a standard normal random variable Z. That is:

$$G(v) = \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

By the symmetry of the normal distribution, we can integrate over just the positive portion of the integral, and then multiply by two:

$$G(v) = 2\int_{0}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

Okay, now let's do the following change of variables:

Let $z = \sqrt{y} = y^{1/2}$

So $dz = \frac{1}{2} y^{-1/2} dy = \frac{1}{2\sqrt{y}} dy$

And $z^2 = y$ and $z = 0 \Rightarrow y = 0$

$z = \sqrt{v} \Rightarrow y = v$

Doing so, we get:

$$G(v) = 2\int_0^v \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right)\left(\frac{1}{2\sqrt{y}}\right) dy$$

$$G(v) = \int_0^v \frac{1}{\sqrt{\pi}\sqrt{2}} y^{\frac{1}{2}-1} \exp\left(-\frac{y}{2}\right) dy$$

for $v > 0$. Now, by one form of the Fundamental Theorem of Calculus:

$$\text{Given } \int_a^x f(t)\,dt = F(x) - F(a)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = F'(x) = f(x)$$

we can take the derivative of $G(v)$ to get the probability density function $g(v)$:

$$g(v) = G'(v) = \frac{1}{\sqrt{\pi}\sqrt{2}} v^{\frac{1}{2}-1} e^{-v/2}$$

for $0 < v < \infty$. If you compare this $g(v)$ to the first g(v) that we said we needed to find way back at the beginning of this proof, you should see that we are done if the following is true:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

It is indeed true, as the following argument illustrates. Because g(v) is a p.d.f., the integral of the p.d.f. over the support must equal 1:

$$\int_0^\infty \frac{1}{\sqrt{\pi}\sqrt{2}} v^{\frac{1}{2}-1} e^{-v/2} dv = 1$$

Now, change the variables by letting $v = 2x$, so that $dv = 2\,dx$. Making the change, we get:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{2}} (2x)^{\frac{1}{2}-1} e^{-x} 2dx = 1$$

Rewriting things just a bit, we get:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} x^{\frac{1}{2}-1} e^{-x} 2dx = 1$$

And simplifying, we get:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx = 1$$

Now, it's just a matter of recognizing that the integral is the gamma function of 1/2:

$$\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1$$

Our proof is complete.

So, now that we've taken care of the theoretical argument. Let's take a look at an example to see that the theorem is, in fact, believable in a practical sense.

Example

Find the probability that the standard normal random variable Z falls between -1.96 and 1.96 in two ways:

1. using the standard normal distribution
2. using the chi-square distribution

Solution. The standard normal table (Table V in the textbook) yields:

$$P(-1.96 < Z < 1.96) = P(Z < 1.96) - P(Z > 1.96) = 0.975 - 0.025 = 0.95$$

The chi-square table (Table IV in the textbook) yields the same answer:

$$P(-1.96 < Z < 1.96) = P(|Z| < 1.96) = P(Z^2 < 1.96^2) = P(\chi^2_{(1)}) < 3.8416) = 0.95$$

‹ Normal Properties (../153/) up (../91/) Some Applications › (../155/)