3. Column space

Definition 3.1 The **column space** of a matrix $\mathbf{A}$ is the span of its columns. The notation for it is $\mathrm{CS}(\mathbf{A})$. (It is also called the **image** of $\mathbf{A}$, and written $\mathrm{Im}\,(\mathbf{A})$.)

Since $\mathrm{CS}(\mathbf{A})$ is a span, it is a vector space.

Example 3.2 Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$. Its column space is given by

$$\mathrm{CS}(\mathbf{A}) = \text{the span of } \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

Using that the column vectors are all constant multiples of the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we find a basis for the column space consisting of the single vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and find that $\mathrm{CS}(\mathbf{A})$ is the line $y = 2x$ in $\mathbb{R}^2$.

Example 3.3 Find a basis for the column space of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix}.$$

The column space is defined as $\mathrm{Span}\left(\begin{pmatrix}1\\-1\\1\\0\end{pmatrix},\begin{pmatrix}2\\-2\\2\\0\end{pmatrix},\begin{pmatrix}3\\-3\\3\\9\end{pmatrix}\right)$. But the second vector is a multiple of the first vector, so it is redundant. Therefore, the column space can be described more simply as the span of the first and third columns:

$$\mathrm{CS}(\mathbf{A}) = \mathrm{Span}\left(\begin{pmatrix}1\\-1\\1\\0\end{pmatrix},\begin{pmatrix}3\\-3\\3\\9\end{pmatrix}\right).$$

These two vectors are linearly independent, so we do need both vectors in the basis for this column space.

Column space concept check I

1/1 point (graded)

What is the column space of the matrix $\mathbf{A} = \begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$? Check all that apply.

☐ $\mathrm{CS}\left(\mathbf{A}\right) = \mathbf{0}.$

☑ $\mathrm{CS}\left(\mathbf{A}\right) = \mathrm{Span}\begin{pmatrix}1\\1\end{pmatrix}.$ ✓

☐ $\mathrm{CS}\left(\mathbf{A}\right) = \mathrm{Span}\begin{pmatrix}1\\0\end{pmatrix}.$

☐ $\mathrm{CS}\left(\mathbf{A}\right) = \mathrm{Span}\begin{pmatrix}3\\0\end{pmatrix}.$

☐ $\mathrm{CS}\left(\mathbf{A}\right) = \mathbb{R}^2.$

☐ $\mathrm{CS}\left(\mathbf{A}\right)=\mathbb{R}^3$.

☐ $\mathrm{CS}\left(\mathbf{A}\right)=\mathrm{Span}\begin{pmatrix}1\\2\\3\end{pmatrix}$.

☑ $\mathrm{CS}\left(\mathbf{A}\right)=\mathrm{Span}\begin{pmatrix}\sqrt{2}/3\\\sqrt{2}/3\end{pmatrix}$. ✓

✓

Solution:

Because all of the columns of multiples of a single vector $\begin{pmatrix}1\\1\end{pmatrix}$, we know that a basis for the column space is given by the span of this one vector:

$$\mathrm{CS}\left(\mathbf{A}\right)=\mathrm{Span}\begin{pmatrix}1\\1\end{pmatrix}.$$

However, this basis is not unique. Any scalar multiple of this vector is also a basis for the column space:

$$\mathrm{CS}\left(\mathbf{A}\right)=\mathrm{Span}\begin{pmatrix}1\\1\end{pmatrix}=\mathrm{Span}\begin{pmatrix}\sqrt{2}/3\\\sqrt{2}/3\end{pmatrix}.$$

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Answers are displayed within the problem

Column space concept check II

1/1 point (graded)

If you switch two columns of a matrix $\mathbf{A}$, will the new matrix have the same column space?

● Yes. ✓

○ No.

○ It depends on the matrix.

Solution:

Yes, the column spaces are the same. Since the column space of a matrix is the span of its columns, as long as we don't change the columns themselves, we will have a span of the same set of vectors. This span will be the same regardless of the order of the columns.

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Steps to compute a basis for $\mathrm{CS}(\mathbf{A})$:

1. Perform Gaussian elimination to convert $\mathbf{A}$ to a matrix $\mathbf{B}$ in row echelon form.

2. Identify the pivot columns of $\mathbf{B}$.

3. The corresponding columns of $\mathbf{A}$ are a basis for $\mathrm{CS}(\mathbf{A})$.

Proof that the algorithm finds a basis for column space. (*)

Recall that $\mathbf{B}$ is a row echelon form of $\mathbf{A}$. Let $\mathbf{C}$ be the *reduced* row echelon form of $\mathbf{A}$. If

$$\text{fifth column} = 3(\text{first column}) + 7(\text{second column})$$

is true for a matrix, it will remain true after any row operation.

Similarly, any linear relation between columns is preserved by row operations. So the linear relations between columns of $\mathbf{A}$ are the same as the linear relations between columns of $\mathbf{C}$. The condition that certain numbered columns (say the first, second, and fourth) of a matrix form a basis is expressible in terms of which linear relations hold. So if certain columns form a basis for $\mathrm{CS}(\mathbf{C})$, the same numbered columns will form a basis for $\mathrm{CS}(\mathbf{A})$.

Recall that $\mathbf{B}$ is a row echelon form of $\mathbf{A}$. We can obtain the reduced row echelon form $\mathbf{C}$ by performing Gauss-Jordan elimination on $\mathbf{B}$. This process does not change the pivot locations. Thus it will be enough to show that the pivot columns of $\mathbf{C}$ form a basis of $\mathrm{CS}(\mathbf{C})$. Since $\mathbf{C}$ is in reduced row echelon form, the pivot columns of $\mathbf{C}$ are the first r of the m standard basis vectors for $\mathbb{R}^m$, where r is the number of nonzero rows of $\mathbf{C}$. These columns are linearly **independent,** and every other column is a linear combination of them, since the entries of $\mathbf{C}$ below the first r rows are all zeros. Thus the pivot columns of $\mathbf{C}$ form a basis of $\mathrm{CS}(\mathbf{C})$.

(The symbol (*) after the title means that you are not required to know this proof for any exam in this class.)

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In particular,

$$\boxed{\mathbf{\dim CS(A) = \# \text{ pivot columns of } B.}}$$

Warning: Usually $\mathrm{CS}(\mathbf{A}) \neq \mathrm{CS}(\mathbf{B})$.

Practice with algorithm

1/1 point (graded)

Which of the following sets of vectors is a basis for the column space of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 1 & -2 \\ 2 & 3 & 4 \end{pmatrix}?$$

Hint: a row echelon form for $\mathbf{A}$ is the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

○ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$

● $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ ✓

○ $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

○ $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

Solution:

The first and the second columns are the pivot columns in the row echelon form of $\mathbf{A}$. Therefore, the first and the second columns $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ of $\mathbf{A}$ form a basis for its column space.

The algorithm gives one basis of CS(A), but of course it has many other different bases. Let's check then that none other option gives such a basis. The first option is wrong since the vectors spanned by that basis have third coorinate 0, so we can't get any of the columns of A in their span. Note here that pivot columns of B aren't related to CS(A) themselves and only point to columns of A that span CS(A)

The algorithm gives one basis of $\mathrm{CS}(\mathbf{A})$, but of course it has many other bases. Let's check that no other option gives such a basis. The first option is wrong since the vectors spanned by that basis have third coordinate $\mathbf{0}$, so we can't get any of the columns of $\mathbf{A}$ in their span. Note here that pivot columns of $\mathbf{B}$ aren't related to $\mathrm{CS}(\mathbf{A})$ themselves, and only point to columns of $\mathbf{A}$ that span $\mathrm{CS}(\mathbf{A})$.

All bases of $\mathrm{CS}(\mathbf{A})$ should have the same number of vectors. The basis we found in our case has 2 vectors, so should have any other correct answer as well. Therefore both choices three and four are not correct because they have the wrong dimension.

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You have used 2 of 3 attempts

Answers are displayed within the problem

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[Typo] Repeated paragraph

In the solution of the last exercise on this page, the paragraph starting with "The algorithm gives..." is pri... 1

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