A Proof of Cantor's Theorem

Let us refer to the set of A's subsets as A's **powerset**; in symbols: $\mathscr{P}(A)$. Cantor's Theorem can then be restated as follows:

> **Cantor's Theorem**
> For any set A, $|A| < |\mathscr{P}(A)|$

In order to prove this result, we'll need to verify each of the following two statements:

1. $|A| \leq |\mathscr{P}(A)|$

2. $|A| \neq |\mathscr{P}(A)|$

The first of these statements is straightforward, since the function $f(x) = \{x\}$ is an injection from A to $\mathscr{P}(A)$.

So what we really need to verify is the second statement: $|A| \neq |\mathscr{P}(A)|$.

Our proof of $|A| \neq |\mathscr{P}(A)|$ will proceed by *reductio*. We will assume the negation of what we wish to prove, and use this assumption to prove a contradiction. Since we want to prove $|A| \neq |\mathscr{P}(A)|$, this means that we will assume $|A| = |\mathscr{P}(A)|$. We will assume, in other words, that there is a bijection f from A to $\mathscr{P}(A)$.

Note that for each a in A, $f(a)$ is a member of $\mathscr{P}(A)$, and therefore a subset of A. So we can consider the question of whether a is a member of $f(a)$. In symbols:

$$a \in f(a)?$$

Note that this question will be answered negatively for whichever $a \in A$ is mapped by f to the empty set. Notice, moreover, that (if A is non-empty) the question will be answered positively for whichever member of A is mapped by f to A itself.

Let D be the set of members of A for which the question is answered negatively. In other words:

$$D = \{x \in A : x \notin f(x)\}$$

Since D is a subset of A, f must map some d in A to D. Now consider the following question:

$$d \in D?$$

The definition of D tells us that d is a member of D if and only if d is not a member of $f(d)$. In symbols:

$$d \in D \leftrightarrow d \notin f(d)$$

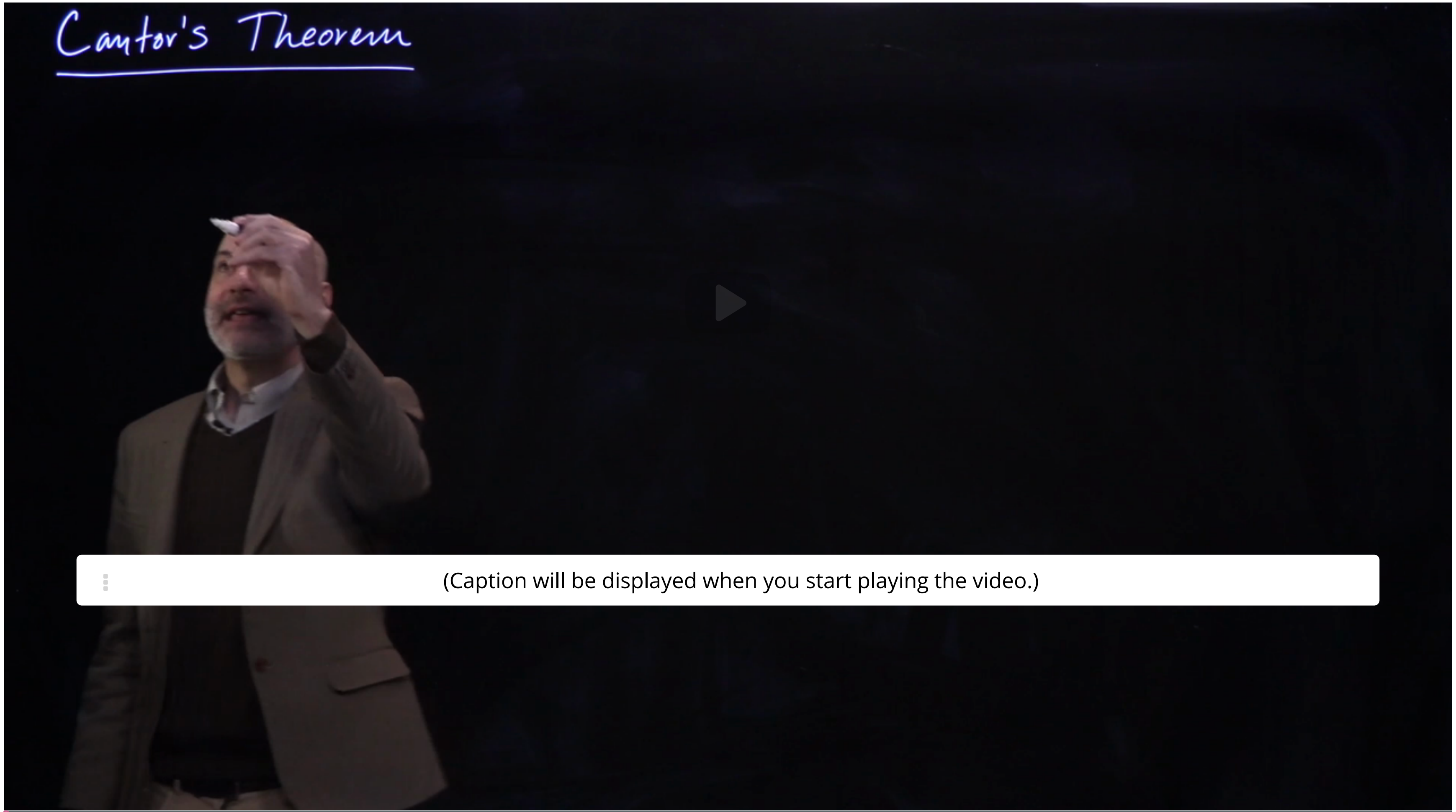
But the definition of d tells us that $f(d) = D$. So we have:

$$d \in D \leftrightarrow d \notin D$$

That statement is a contradiction: it tells us that something is the case if and only if it is not the case. So we have concluded our proof. We have assumed $|A| = |\mathscr{P}(A)|$, and used it to prove a contradiction. From this it follows that $|A| = |\mathscr{P}(A)|$ is false, and therefore that $|A| \neq |\mathscr{P}(A)|$ is true.

Cantor's Theorem is an amazing result. It shows that regardless of how many individuals there are in a set A, and regardless of whether A is finite or infinite A, there must be even more individuals in A's power set. So there are infinitely many sizes of infinity!

Video Review: Proving Cantor's Theorem



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W ant more infinities
how to make even larger infinities because we can only apply the powerset a natural amount of times. what about whenwe can not do new method anymore

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