Integration by Parts

Dr. Philippe B. Laval
Kennesaw State University

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Abstract

This handout describes an integration method called Integration by Parts. Integration by parts is to integrals what the product rule is to derivatives.

1 Integration by Parts (5.6)

1.1 General Concept

If f and g are two differentiable functions, then the product rule gives us:

$$(f(x)\,g(x))' = f(x)\,g'(x) + f'(x)\,g(x)$$

If we integrate both sides, we get

$$\int (f(x)\,g(x))'\,dx = \int f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx$$

From the fundamental theorem of calculus, we know that the integral of the derivative of a function is the function itself, therefore we have:

$$f(x)\,g(x) = \int f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx$$

Or, solving for the first integral

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

This is the integration by parts formula. However, we usually do not remember it this way. If we let $u = f(x)$, then $du = f'(x)\,dx$. Similarly, if $v = g(x)$, then $dv = g'(x)\,dx$. Doing the two substitutions gives us:

$$\int u\,dv = uv - \int v\,du \tag{1}$$

The goal when using this formula is to pretend that the integral we are given is of the form $\int udv$. We find u and dv that accomplish this. Once we have u and dv, we find du and v. Then, we can rewrite the given integral as $uv - \int vdu$. This will work if the new integral we obtained can be evaluated or is easier to evaluate than the one we started with. Also, once we have selected dv, v is an antiderivative of dv. therefore, we must be able to find an antiderivative for dv.

For definite integrals, the integration by parts formula becomes

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x)|_a^b - \int_a^b f'(x)\, g(x)\, dx$$

We illustrate this technique with several examples. For clarity, the quantities we select from the integral (u and dv) will be in bold characters. The quantities we deduce from our selection (du and v) will be in normal characters.

Example 1 *Find* $\int x \sin x dx$
If we select

$$\begin{array}{cc} \mathbf{u = x} & du = dx \\ v = -\cos x & \mathbf{dv} = \sin \mathbf{xdx} \end{array}$$

Then, applying formula 1 gives us:

$$\begin{aligned} \int x \sin x dx &= -x \cos x - \int -\cos x dx \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

Remark 2 *There is usually more than one way to select u and dv. However, we should do it so that when we apply the integration by parts formula, we obtain an integral we can solve, or at least one that does not seem more complicated. In the above example, if we had selected*

$$\begin{array}{cc} \mathbf{u} = \sin \mathbf{x} & du = \cos x dx \\ v = \dfrac{x^2}{2} & \mathbf{dv = xdx} \end{array}$$

then the integration by parts formula would have given us

$$\int x \sin x dx = \frac{x^2}{2} \sin x - \frac{1}{2} \int x^2 \cos x dx$$

which is more complicated than the integral we started with.

Example 3 *Find* $\int xe^x dx$
Following the previous example, we select

$$\begin{array}{cc} \mathbf{u = x} & du = dx \\ v = e^x & \mathbf{dv = e^x dx} \end{array}$$

Then, applying formula 1 gives

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

Example 4 *Find* $\int_1^e \ln x dx$
If we select

$$\begin{array}{ll}\mathbf{u} = \ln \mathbf{x} & du = \frac{1}{x} dx \\ v = x & \mathbf{dv} = \mathbf{dx}\end{array}$$

Then, applying formula 1 gives

$$\begin{aligned}\int_1^e \ln x dx &= x \ln x|_1^e - \int_1^e dx \\ &= x \ln x|_1^e - x|_1^e \\ &= (e \ln e - \ln 1) - (e - 1) \\ &= 1\end{aligned}$$

Remark 5 *You will note that in the problem we just did, we found that*

$$\int \ln x dx = x \ln x - x + C$$

Example 6 *Find* $\int \tan^{-1} x dx$
If we select

$$\begin{array}{ll}\mathbf{u} = \tan^{-\mathbf{1}} x & du = \frac{1}{1+x^2} dx \\ v = x & \mathbf{dv} = \mathbf{dx}\end{array}$$

Then, applying formula 1 gives

$$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx \tag{2}$$

We will do $\int \frac{x}{1+x^2} dx$ *separately, using the substitution* $u = 1 + x^2$. *Then,* $du = 2xdx$ *therefore*

$$\begin{aligned}\int \frac{x}{1+x^2} dx &= \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln \left|1 + x^2\right| + C \\ &= \frac{1}{2} \ln \left(1 + x^2\right) + C\end{aligned}$$

Using what we just found in equation 2, gives us

$$\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln \left(1 + x^2\right) + C$$

1.2 Repeated Integration by Parts

In some cases, applying the integration by parts formula one time will not be enough. You may need to apply it twice, or more. We look at some example to illustrate the various cases which can occur.

Example 7 *Find* $\int x^2 e^x dx$
If we select

$$\begin{array}{rr} \mathbf{u = x^2} & du = 2xdx \\ v = e^x & \mathbf{dv = e^x dx} \end{array}$$

Then, applying formula 1 gives

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

At this point, we are left with an integral that we still cannot do. However, we see that it looks simpler than the one we started with. So, we try the integration by parts formula again. In fact, we already did this integral in an example above, so we will simply use the result.

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2\left(xe^x - e^x + C\right) \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

Note that we replaced $-2C$ *by* C*. We only wish to denote that there is some constant.*

Example 8 *Find* $\int x^2 \sin x dx$
If we select

$$\begin{array}{rr} \mathbf{u = x^2} & du = 2xdx \\ v = -\cos x & \mathbf{dv} = \sin \mathbf{xdx} \end{array}$$

Then, applying formula 1 gives

$$\int x^2 \sin x dx = -x^2 \cos x + 2\int x \cos x dx \tag{3}$$

We apply the integration by parts formula to $\int x \cos x dx$*. If we select*

$$\begin{array}{rr} \mathbf{u = x} & du = dx \\ v = \sin x & \mathbf{dv} = \cos \mathbf{xdx} \end{array}$$

Then, applying formula 1 gives

$$\begin{aligned} \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C \end{aligned}$$

If we use what we just found in equation 3, we obtain

$$\begin{aligned}\int x^2 \sin x dx &= -x^2 \cos x + 2\left(x \sin x + \cos x + C\right) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C\end{aligned}$$

Here again, we replaced $2C$ *by* C.

In these two examples, we applied the integration by parts formula several time. We noticed that every time, the integral was getting easier. So, our hope was that we would eventually be able to find it, which we did. In some cases, a different situation will present itself, as illustrated in the next example.

Example 9 *Find* $\int e^x \sin x dx$
If we select

$$\begin{array}{cc} \mathbf{u = e^x} & du = e^x dx \\ v = -\cos x & \mathbf{dv} = \sin \mathbf{xdx} \end{array}$$

Then, applying formula 1 gives

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx \tag{4}$$

The integral we obtain is not simpler, but it is not more difficult either. Furthermore, it looks like the original, $\sin x$ *having been replaced by* $\cos x$. *We apply the integration by parts formula to it. If we select*

$$\begin{array}{cc} \mathbf{u = e^x} & du = e^x dx \\ v = \sin x & \mathbf{dv} = \cos \mathbf{xdx} \end{array}$$

Then, applying formula 1 gives

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

If we replace what we just found in equation 4, we obtain

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

Therefore, we can solve for $\int e^x \sin x dx$. *We obtain*

$$\begin{aligned}\int e^x \sin x dx + \int e^x \sin x dx &= -e^x \cos x + e^x \sin x \\ 2\int e^x \sin x dx &= e^x \left(\sin x - \cos x\right) \\ \int e^x \sin x dx &= \frac{e^x}{2}\left(\sin x - \cos x\right) + C\end{aligned}$$

Note, we inserted the constant C *in the last step.*

Using integration by parts, we can prove important formulae, called reduction formulae. The most important ones are listed as a theorem.

Theorem 10 *Let n be an integer such that $n \geq 2$.*

1. $\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$
2. $\int \cos^n x dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx$
3. $\int (\ln x)^n dx = x (\ln x)^n - n \int (\ln x)^{n-1} dx$
4. $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

Proof. *We only prove part 1. The other parts are left as an exercise.*

1. *Prove that* $\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$. *If we select*

$$\begin{array}{ll} \mathbf{u} = \sin^{\mathbf{n-1}} x & du = (n-1) \sin^{n-2} x \cos x dx \\ v = -\cos x & \mathbf{dv} = \sin \mathbf{xdx} \end{array}$$

Then, applying formula 1 gives

$$\begin{aligned} \int \sin^n x dx &= -\sin^{n-1} x \cos x + (n-1) \int \cos^2 x \sin^{n-2} x dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \left(1 - \sin^2 x\right) \sin^{n-2} x dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx \\ \int \sin^n x dx + (n-1) \int \sin^n x dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx \\ n \int \sin^n x dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx \\ \int \sin^n x dx &= \frac{-1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx \end{aligned}$$

2. *see exercise 34*
3. *see exercise 37*
4. *do as an exercise*

∎

We illustrate with an example how these formulas might be used.

Example 11 *Find* $\int (\ln x)^3 \, dx = 6x \left(\frac{1}{6} \ln^3 x - \frac{1}{2} \ln^2 x + \ln x - 1 \right)$
Using part 3 of theorem 10, we obtain

$$\int (\ln x)^3 \, dx = x (\ln x)^3 - 3 \int (\ln x)^2 \, dx$$

To find $\int (\ln x)^2 \, dx$, *we apply the same formula again.*

$$\int (\ln x)^2 \, dx = x (\ln x)^2 - 2 \int \ln x dx$$

The last integral was found in an example above (otherwise, we would find it using integration by parts). It was found to be

$$\int \ln x dx = x \ln x - x$$

Combining the three gives us

$$\begin{aligned} \int (\ln x)^3 \, dx &= x (\ln x)^3 - 3 \int (\ln x)^2 \, dx \\ &= x (\ln x)^3 - 3 \left[x (\ln x)^2 - 2 \int \ln x dx \right] \\ &= x (\ln x)^3 - 3x (\ln x)^2 + 6 \int \ln x dx \\ &= x (\ln x)^3 - 3x (\ln x)^2 + 6 [x \ln x - x] \\ &= x (\ln x)^3 - 3x (\ln x)^2 + 6x \ln x - 6x \end{aligned}$$

Remark 12 *As this example shows, the reduction formula may have to be applied several times before we find the answer.*

1.3 Things to know

- Know and be able to apply the integration by parts formula. Remember that this formula may have to be applied more than once.
- Using integration by parts, be able to prove and use the following reduction formulas

$$\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int (\ln x)^n \, dx = x (\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

- Related homework assigned:
  - # 1-19 odd #, 25, 33, 34, 35a, 35b, 37, 39, 45 on pages 398, 399.
  - Find $\int \sin^{-1} x dx$
  - Finish proving theorem 10