

# Inverse Gaussian distribution

In probability theory, the **inverse Gaussian distribution** (also known as the **Wald distribution**) is a two-parameter family of continuous probability distributions with support on  $(0,\infty)$ .

Its probability density function is given by

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2x}\right]$$

for  $x > 0$ , where  $\mu > 0$  is the mean and  $\lambda > 0$  is the shape parameter.<sup>[1]</sup>

As  $\lambda$  tends to infinity, the inverse Gaussian distribution becomes more like a normal (Gaussian) distribution. The inverse Gaussian distribution has several properties analogous to a Gaussian distribution. The name can be misleading: it is an "inverse" only in that, while the Gaussian describes a Brownian motion's level at a fixed time, the inverse Gaussian describes the distribution of the time a Brownian motion with positive drift takes to reach a fixed positive level.

Its cumulant generating function (logarithm of the characteristic function) is the inverse of the cumulant generating function of a Gaussian random variable.

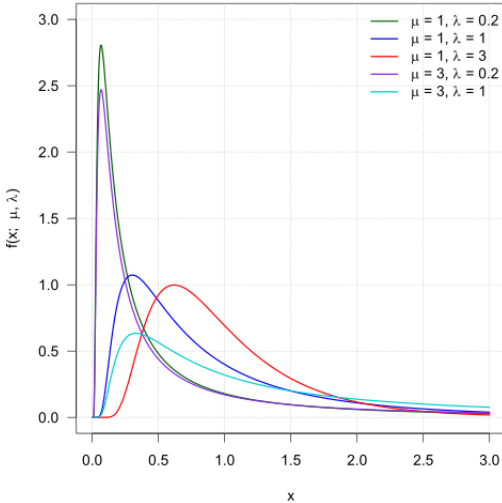
To indicate that a random variable  $X$  is inverse Gaussian-distributed with mean  $\mu$  and shape parameter  $\lambda$  we write  $X \sim \text{IG}(\mu, \lambda)$ .

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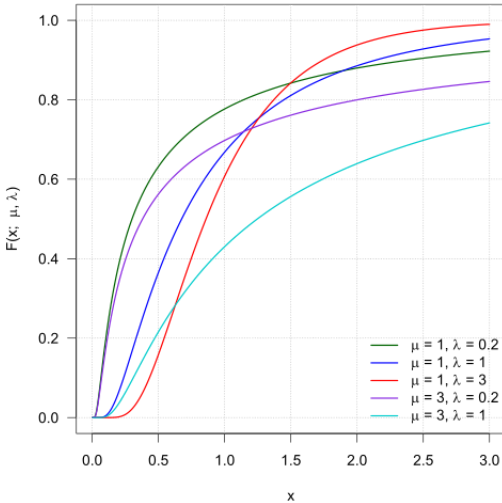
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### Inverse Gaussian

Probability density function



Cumulative distribution function



Notation	$\text{IG}(\mu, \lambda)$
Parameters	$\mu > 0$

# Properties

## Single parameter form

The probability density function (pdf) of inverse Gaussian distribution has a single parameter form given by

$$f(x;\mu,\mu^2)=\frac{\mu}{\sqrt{2\pi x^3}}\exp\left[-\frac{(x-\mu)^2}{2x}\right].$$

In this form, the mean and variance of the distribution are equal,  $\mu=\sigma^2$ .

Also, the cumulative distribution function (cdf) of the single parameter inverse Gaussian distribution is related to the standard normal distribution by

$$\begin{aligned}\Pr(X < x) &= \Phi(z_1) + e^\mu \Phi(z_2), & \text{for } 0 < x \leq \mu, \\ \Pr(X > x) &= \Phi(-z_1) - e^\mu \Phi(z_2), & \text{for } x \geq \mu.\end{aligned}$$

where  $z_1 = \frac{\mu}{x^{1/2}} - x^{1/2}$  and  $z_2 = \frac{\mu}{x^{1/2}} + x^{1/2}$ , where the  $\Phi$  is the cdf of standard normal distribution. The variables  $z_1$  and  $z_2$  are related to each other by the identity  $z_2^2 = z_1^2 + 4\mu$ .

In the single parameter form, the MGF simplifies to

$$M(t) = \exp[\mu(1 - \sqrt{1 - 2t})].$$

An inverse Gaussian distribution in double parameter form  $f(x;\mu,\lambda)$  can be transformed into a single parameter form  $f(y;\mu_0,\mu_0^2)$  by appropriate scaling  $y = \frac{\mu^2 x}{\lambda}$ , where  $\mu_0 = \mu^3/\lambda$ .

The standard form of inverse Gaussian distribution is

$$f(x;1,1)=\frac{1}{\sqrt{2\pi x^3}}\exp\left[-\frac{(x-1)^2}{2x}\right].$$

## Summation

If  $X_i$  has an  $\text{IG}(\mu_0 w_i, \lambda_0 w_i^2)$  distribution for  $i = 1, 2, \dots, n$  and all  $X_i$  are independent, then

$$S = \sum_{i=1}^n X_i \sim \text{IG}\left(\mu_0 \sum w_i, \lambda_0 \left(\sum w_i\right)^2\right).$$

Note that

	$\lambda > 0$
Support	$x \in (0, \infty)$
PDF	$\sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]$
CDF	$\Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\right)$  where $\Phi$ is the standard normal (standard Gaussian) distribution c.d.f.
Mean	$\mathbb{E}[X] = \mu$  $\mathbb{E}[\frac{1}{X}] = \frac{1}{\mu} + \frac{1}{\lambda}$
Mode	$\mu\left[\left(1+\frac{9\mu^2}{4\lambda^2}\right)^{\frac{1}{2}}-\frac{3\mu}{2\lambda}\right]$
Variance	$\text{Var}[X] = \frac{\mu^3}{\lambda}$  $\text{Var}[\frac{1}{X}] = \frac{1}{\mu\lambda} + \frac{2}{\lambda^2}$
Skewness	$3\left(\frac{\mu}{\lambda}\right)^{1/2}$
Ex. kurtosis	$\frac{15\mu}{\lambda}$
MGF	$\exp\left[\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2 t}{\lambda}}\right)\right]$
CF	$\exp\left[\frac{\lambda}{\mu}\left(1-\sqrt{1-\frac{2\mu^2 it}{\lambda}}\right)\right]$

$$\frac{\mathbf{Var}(X_i)}{\mathbf{E}(X_i)} = \frac{\mu_0^2 w_i^2}{\lambda_0 w_i^2} = \frac{\mu_0^2}{\lambda_0}$$

is constant for all *i*. This is a necessary condition for the summation. Otherwise *S* would not be Inverse Gaussian distributed.

Scaling

For any *t* > 0 it holds that

$$X \sim \text{IG}(\mu, \lambda) \quad \Rightarrow \quad tX \sim \text{IG}(t\mu, t\lambda).$$

Exponential family

The inverse Gaussian distribution is a two-parameter exponential family with natural parameters  $-\lambda/(2\mu^2)$  and  $-\lambda/2$ , and natural statistics *X* and 1/*X*.

Relationship with Brownian motion

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The stochastic process *X<sub>t</sub>* given by

$$\begin{aligned} X_0 &= 0 \\ X_t &= \nu t + \sigma W_t \end{aligned}$$

(where *W<sub>t</sub>* is a standard Brownian motion with drift **ν** > 0).

Then the first passage time for a fixed level **α** > 0 by *X<sub>t</sub>* is distributed according to an inverse-Gaussian:

$$T_\alpha = \inf\{t > 0 \mid X_t = \alpha\} \sim \text{IG}\left(\frac{\alpha}{\nu}, \left(\frac{\alpha}{\sigma}\right)^2\right) = \frac{\alpha}{\sigma\sqrt{2\pi x^3}} \exp\left[-\frac{(\alpha - \nu x)^2}{2\sigma^2 x}\right]$$

(cf. Schrödinger<sup>[2]</sup> equation 19, Smoluchowski<sup>[3]</sup>, equation 8, and Folks<sup>[4]</sup>, equation 1).

When drift is zero

A common special case of the above arises when the Brownian motion has no drift. In that case, parameter *μ* tends to infinity, and the first passage time for fixed level *α* has probability density function

$$f\left(x; 0, \left(\frac{\alpha}{\sigma}\right)^2\right) = \frac{\alpha}{\sigma\sqrt{2\pi x^3}} \exp\left(-\frac{\alpha^2}{2\sigma^2 x}\right)$$

(see also Bachelier<sup>[5]:74[6]:39</sup>). This is a Lévy distribution with parameters **c** =  $\left(\frac{\alpha}{\sigma}\right)^2$  and **μ** = 0.

Maximum likelihood

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The model where

$$X_i \sim \text{IG}(\mu, \lambda w_i), \quad i = 1, 2, \dots, n$$

with all  $w_i$  known,  $(\mu, \lambda)$  unknown and all  $X_i$  independent has the following likelihood function

$$L(\mu, \lambda) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \left( \prod_{i=1}^n \frac{w_i}{X_i^3} \right)^{\frac{1}{2}} \exp \left( \frac{\lambda}{\mu} \sum_{i=1}^n w_i - \frac{\lambda}{2\mu^2} \sum_{i=1}^n w_i X_i - \frac{\lambda}{2} \sum_{i=1}^n w_i \frac{1}{X_i} \right).$$

Solving the likelihood equation yields the following maximum likelihood estimates

$$\hat{\mu} = \frac{\sum_{i=1}^n w_i X_i}{\sum_{i=1}^n w_i}, \quad \frac{1}{\hat{\lambda}} = \frac{1}{n} \sum_{i=1}^n w_i \left( \frac{1}{X_i} - \frac{1}{\hat{\mu}} \right).$$

$\hat{\mu}$  and  $\hat{\lambda}$  are independent and

$$\hat{\mu} \sim \text{IG} \left( \mu, \lambda \sum_{i=1}^n w_i \right), \quad \frac{n}{\hat{\lambda}} \sim \frac{1}{\lambda} \chi_{n-1}^2.$$

## Generating random variates from an inverse-Gaussian distribution

The following algorithm may be used.<sup>[7]</sup>

Generate a random variate from a normal distribution with a mean of 0 and 1 standard deviation

$$\nu = N(0, 1).$$

Square the value

$$y = \nu^2$$

and use this relation

$$x = \mu + \frac{\mu^2 y}{2\lambda} - \frac{\mu}{2\lambda} \sqrt{4\mu\lambda y + \mu^2 y^2}.$$

Generate another random variate, this time sampled from a uniform distribution between 0 and 1

$$z = U(0, 1).$$

If  $z \leq \frac{\mu}{\mu + x}$  then return  $x$  else return  $\frac{\mu^2}{x}$ .

Sample code in Java:

```

public double inverseGaussian(double mu, double lambda) {
    Random rand = new Random();
    double v = rand.nextGaussian(); // sample from a normal distribution with a mean of 0 and 1 standard deviation
    double y = v*v;
    double x = mu + (mu*mu*y)/(2*lambda) - (mu/(2*lambda)) * Math.sqrt(4*mu*lambda*y + mu*mu*y*y);
    double test = rand.nextDouble(); // sample from a uniform distribution between 0 and 1
    if (test <= (mu)/(mu + x))
        return x;
    else
        return (mu*mu)/x;
}

```

And to plot Wald distribution in Python using matplotlib and NumPy:

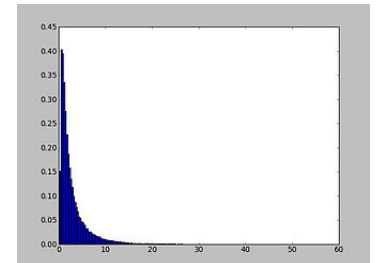
```

import matplotlib.pyplot as plt
import numpy as np

h = plt.hist(np.random.wald(3, 2, 100000), bins=200, normed=True)

plt.show()

```



Wald distribution using Python with aid of matplotlib and NumPy

## Related distributions

- If  $X \sim \text{IG}(\mu, \lambda)$ , then  $kX \sim \text{IG}(k\mu, k\lambda)$  for any number  $k > 0$ .<sup>[1]</sup>
- If  $X_i \sim \text{IG}(\mu, \lambda)$  then  $\sum_{i=1}^n X_i \sim \text{IG}(n\mu, n^2\lambda)$
- If  $X_i \sim \text{IG}(\mu, \lambda)$  for  $i = 1, \dots, n$  then  $\bar{X} \sim \text{IG}(\mu, n\lambda)$
- If  $X_i \sim \text{IG}(\mu_i, 2\mu_i^2)$  then  $\sum_{i=1}^n X_i \sim \text{IG}\left(\sum_{i=1}^n \mu_i, 2\left(\sum_{i=1}^n \mu_i\right)^2\right)$

The convolution of an inverse Gaussian distribution (a Wald distribution) and an exponential (an ex-Wald distribution) is used as a model for response times in psychology,<sup>[8]</sup> with visual search as one example.<sup>[9]</sup>

## History

This distribution appears to have been first derived in 1900 by Louis Bachelier<sup>[5][6]</sup> as the time a stock reaches a certain price for the first time. In 1915 it was used independently by Erwin Schrödinger<sup>[2]</sup> and Marian v. Smoluchowski<sup>[3]</sup> as the time to first passage of a Brownian motion. In the field of reproduction modeling it is known as the Hadwiger function, after Hugo Hadwiger who described it in 1940.<sup>[10]</sup> Abraham Wald re-derived this distribution in 1944<sup>[11]</sup> as the limiting form of a sample in a sequential probability ratio test. The name inverse Gaussian was proposed by Maurice Tweedie in 1945.<sup>[12]</sup> Tweedie investigated this distribution in 1956<sup>[13]</sup> and 1957<sup>[14][15]</sup> and established some of its statistical properties. The distribution was extensively reviewed by Folks and Chhikara in 1978.<sup>[4]</sup>

## Numeric computation and software

Despite the simple formula for the probability density function, numerical probability calculations for the inverse Gaussian distribution nevertheless require special care to achieve full machine accuracy in floating point arithmetic for all parameter values.<sup>[16]</sup> Functions for the inverse Gaussian distribution are provided for the R programming language by several packages including rmutil,<sup>[17][18]</sup> SuppDists,<sup>[19]</sup> STAR,<sup>[20]</sup> invGauss,<sup>[21]</sup> LaplacesDemon,<sup>[22]</sup> and statmod.<sup>[23]</sup>

## See also

- Generalized inverse Gaussian distribution
- Tweedie distributions—The inverse Gaussian distribution is a member of the family of Tweedie exponential dispersion models
- Stopping time

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## Further reading

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## External links

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- Inverse Gaussian Distribution (<http://mathworld.wolfram.com/InverseGaussianDistribution.html>) in Wolfram website.
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