Interactive Real Analysis

Next | Previous | Glossary | Map

7.3. Measures

Theorem 7.3.9: Properties of Lebesgue measure

1. All intervals are measurable and the measure of an interval is its length
2. All open and closed sets are measurable
3. The union and intersection of a finite or countable number of measurable sets is again measurable
4. If A is measurable and A is the union of countable number of measurable sets A_n, then $m(A) \leq \sum m(A_n)$
5. If A is measurable and A is the union of countable number of disjoint measurable sets A_n, then $m(A) = \sum m(A_n)$

Context

Proof:

First let's review a few facts we have shown before:

1. The outer measure of an interval is its length
2. Intervals of the form (a, ∞) are measurable
3. Complements of measurable sets are measurable
4. Union and intersection of two measurable sets are measurable

Also, we need two lemmas that will be useful to show statements 3 and 5.

Lemma 1: Suppose O is a collection of sets such that the union of two elements and the complement of every element from O is again part of O (such a collection is called an *Algebra of Sets*). If $\{ E_n \}$ is any countable collection of elements from O then there is another countable collection $\{ F_n \}$ of *disjoint* elements from O such that $\cup E_n = \cup F_n$.

Lemma 2: If $E_1, E_2, ..., E_n$ are finitely many disjoint measurable sets, then

$$m(A \cap (E_1 \cup \ldots \cup E_n)) = m(A \cap E_1) + \ldots + m(A \cap E_n)$$

We will first prove the proposition and give the proofs of these lemmas at the end.

Proof of statement 3: We have already shown that the union and intersection of two measurable sets is again measurable, so we need to prove the statement for countable unions and intersections. Because of facts (3) and (4) measurable sets form an algebra of sets so that it is sufficient to show statement 3 for countable unions of *disjoint* measurable sets.

Let $\{ E_n \}$ be a countable collection of disjoint measurable sets and E be their union. Define

$$F_n = E_1 \cup E_2 \cup \ldots \cup E_n$$

Because of lemma 2 we have that

$$m^*(A \cap F_n) = m(A \cap E_1) + \ldots + m(A \cap E_n)$$

Because $comp(E) \subset comp(F_n)$ we know

$$m^*(comp(E)) \leq m^*(comp(F_n))$$

Thus, for any set A we have:

$$\begin{aligned} m(A \cap E_1) + \ldots + m(A \cap E_n) + m^*(A \cap comp(E)) &= \\ = m^*(A \cap F_n) + m^*(A \cap comp(E)) & \\ \leq m^*(A \cap F_n) + m^*(A \cap comp(F_n)) &= \\ = m^*(A) & \end{aligned}$$

because the set F_n is measurable. But this is true for any integer n so that

$$\begin{aligned} m^*(A) &\geq \sum m(A \cap E_n) + m^*(A \cap comp(E)) \\ &\geq m(A \cap E) + m^*(A \cap comp(E)) \end{aligned}$$

because of subadditivity. But that proves that the countable union E of the E_n is measurable. That countable intersections of measurable sets are measurable follows from de Morgan laws and because complements of measurable sets are measurable.

Proof of statement 1: Let's focus on an open interval (a, b). We know that (a, ∞) and $(-\infty, b] = comp(b, \infty))$ are both measurable. But

$$(-\infty, b) = \cup (-\infty, b-1/n]$$

Since each set on the right is measurable and countable unions of measurable sets are measurable, intervals of the form $(-\infty, b)$ are also measurable. But

$$(a, b) = (a, \infty) \cap (-\infty, b)$$

so that (a, b) is measurable. Similarly, $[a, b]$ is measurable. Since the outer measure of an interval is its length, and intervals are now measurable, their (Lebesgue) measure must also be their length.

Proof of statement 2: We have shown before that an open set $\boldsymbol{U} \subset \boldsymbol{R}$ can be written as a countable union of open intervals. By (1) intervals are measurable and by (3) countable unions of measurable sets are measurable. Therefore open sets are measurable. But closed sets are the complements of open sets, and complements of measurable sets are measurable. Therefore closed sets are measurable.

Proof of statement 4: This follows immediately from the subadditivity property of outer measure so there is nothing to prove.

Proof of statement 5: In lemma 2 we can set $\boldsymbol{A} = \boldsymbol{R}$ to get that if $\boldsymbol{E}_1, \boldsymbol{E}_2, ..., \boldsymbol{E}_n$ are finitely many disjoint measurable sets, then

$$m(\boldsymbol{E}_1 \cup ... \cup \boldsymbol{E}_n) = m(\boldsymbol{E}_1) + ... + m(\boldsymbol{E}_n)$$

We now need to make the step to countably many sets. If $\{ \boldsymbol{E}_j \}$ is a countable collection of disjoint measurable sets, then

$$\bigcup_{j=1}^{n} E_j \subset \bigcup_{j=1}^{\infty} E_j \text{ so that } m\left(\bigcup_{j=1}^{n} E_j\right) \le m\left(\bigcup_{j=1}^{\infty} E_j\right)$$

But for finitely many sets we know that

$$m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j) \text{ so that } m\left(\bigcup_{j=1}^{\infty} E_j\right) \ge \sum_{j=1}^{n} m(E_j) \text{ for all } n$$

Because that statement holds for all n we conclude that

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \ge \sum_{j=1}^{\infty} m(E_j)$$

The reverse inequality follows immediately from subadditivity (statement 4), so that we have proved equality, and hence statement 5.

It remains to prove the lemmas we have used.

Proof of Lemma 1: Because $\boldsymbol{A} \cup \boldsymbol{B} = comp(comp(\boldsymbol{A}) \cap comp(\boldsymbol{B}))$ we know that intersections of two sets from $\boldsymbol{O}$ must also be part of $\boldsymbol{O}$. The same is true (by induction) for finite unions, intersections, or complements of sets in $\boldsymbol{O}$.

Now let $\{ \boldsymbol{E}_n \}$ be a countable collection of sets in $\boldsymbol{O}$ and recursively define sets $\boldsymbol{F}_n$ as follows:

$$\boldsymbol{F}_1 = \boldsymbol{E}_1$$
$$\boldsymbol{F}_n = \boldsymbol{E}_n - (\boldsymbol{E}_1 \cup ... \cup \boldsymbol{E}_{n-1}$$

for $n > 1$. Because $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} \cap comp(\boldsymbol{B})$ all $\boldsymbol{F}_n$ are part of $\boldsymbol{O}$. It is left as an exercise to show that (i) the $\boldsymbol{F}_n$ are disjoint and (ii) the union of the $\boldsymbol{F}_n$ is the same as the union of the $\boldsymbol{E}_n$.

Proof of Lemma 2: First let's show that for two disjoint measurable sets $\boldsymbol{E}$ and $\boldsymbol{F}$ we have

$$m(\boldsymbol{A} \cap (\boldsymbol{E} \cup \boldsymbol{F})) = m(\boldsymbol{A} \cap \boldsymbol{E}) + m(\boldsymbol{A} \cap \boldsymbol{F})$$

We know that $\boldsymbol{F}$ is measurable, hence

$$m^*(\boldsymbol{A} \cap (\boldsymbol{E} \cup \boldsymbol{F})) = m^*(\boldsymbol{A} \cap (\boldsymbol{E} \cup \boldsymbol{F}) \cap \boldsymbol{F}) + m^*(\boldsymbol{A} \cap (\boldsymbol{E} \cup \boldsymbol{F}) \cap comp(\boldsymbol{F}))$$

But $\boldsymbol{E}$ and $\boldsymbol{F}$ are disjoint so that

$$\boldsymbol{A} \cap (\boldsymbol{E} \cup \boldsymbol{F}) \cap \boldsymbol{F} = \boldsymbol{A} \cap \boldsymbol{F}$$
$$\boldsymbol{A} \cap (\boldsymbol{E} \cup \boldsymbol{F}) \cap comp(\boldsymbol{F}) = \boldsymbol{A} \cap \boldsymbol{E}$$

Therefore

$$m^*(\boldsymbol{A} \cap (\boldsymbol{E} \cup \boldsymbol{F})) = m^*(\boldsymbol{A} \cap \boldsymbol{F}) + m^*(\boldsymbol{A} \cap \boldsymbol{E})$$

It is now easy (and left as an exercise) to use induction to finish the proof.

Next | Previous | Glossary | Map