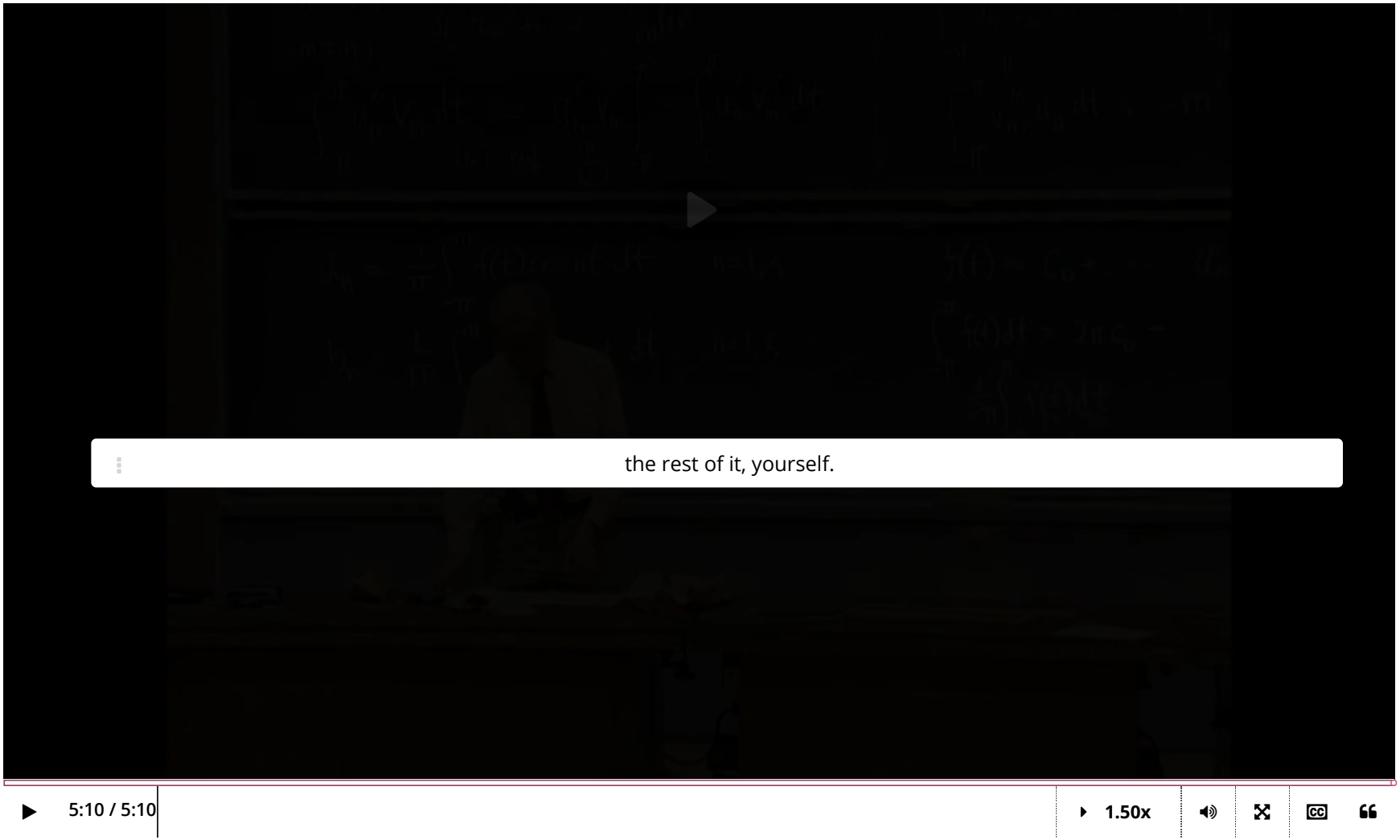
10. The Fourier series of the square wave

Video note: In the example video below, there is an error. Find the error. **Warning,** this is the most common error you will make in computing Fourier coefficients!

Square wave example



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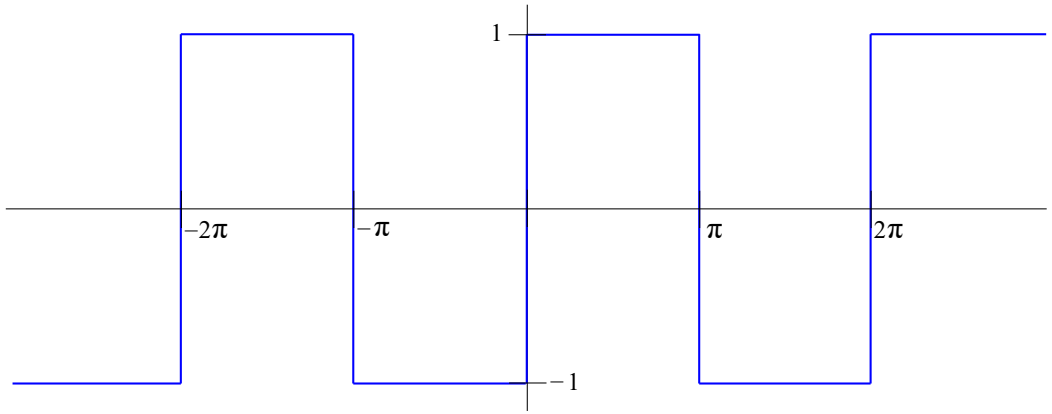
Here we provide the full computation of the Fourier series for the square wave of period 2π

$$\mathrm{Sq}\,(t) := \begin{cases} 1, & \text{if } 0 < t < \pi, \\ -1 & \text{if } -\pi < t < 0. \end{cases}$$



First we compute the constant term,

$$\frac{a_0}{2} = \frac{\text{Signed area of one period}}{2\pi} = \frac{\pi - \pi}{2\pi} = 0.$$

Next we compute the rest of the Fourier coefficients a_n, $n > 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \mathrm{Sq}\,(t)\cos(nt)\ dt\right) \\ &= \frac{1}{\pi}\left(-\int_{-\pi}^{0} \cos(nt)\ dt + \int_{0}^{\pi} \cos(nt)\ dt\right) \\ &= \frac{1}{\pi}\left(-\frac{\sin(nt)}{n}\Big|_{-\pi}^{0} + \frac{\sin(nt)}{n}\Big|_{0}^{\pi}\right) = 0 \end{aligned}$$

Finally, compute the coefficients b_n.

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \mathrm{Sq}\,(t)\sin(nt)\ dt\right) \\ &= \frac{1}{\pi}\left(-\int_{-\pi}^{0}\sin(nt)\ dt + \int_{0}^{\pi}\sin(nt)\ dt\right) \\ &= \frac{1}{\pi}\left(\frac{\cos(nt)}{n}\Big|_{-\pi}^{0} - \frac{\cos(nt)}{n}\Big|_{0}^{\pi}\right) \end{aligned}$$

Note that $\cos(n\pi) = \cos(-n\pi) = (-1)^n$, therefore

$$\begin{aligned} \frac{1}{\pi}\left(\frac{\cos(nt)}{n}\Big|_{-\pi}^{0} - \frac{\cos(nt)}{n}\Big|_{0}^{\pi}\right) &= \frac{1}{\pi}\left[\left(\frac{1-(-1)^n}{n}\right) - \left(\frac{(-1)^n - 1}{n}\right)\right] \\ &= \frac{2}{\pi}\left(\frac{1-(-1)^n}{n}\right) \\ &= \begin{cases} \frac{4}{n\pi} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases} \end{aligned}$$

Therefore the Fourier series for the square wave is

$$\mathrm{Sq}\,(t) \text{ ”=” } \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right) = \frac{4}{\pi}\sum_{n\ \mathrm{odd}}^{\infty}\frac{\sin nt}{n}.$$

Video error

In the video above, the error is that the factor of $1/\pi$ is missing.

Hide

Remark 10.1 Note that while the square wave was undefined at the jump discontinuities at every integer multiple of π, the Fourier series is well defined and is 0 at each integer multiple of π (because $\sin(n\pi) = 0$ for all n).

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