Finite Sequences of Natural Numbers

In this section we will verify that the set of finite sequences of natural numbers ($\mathfrak{F}$) has the same cardinality as the set of natural numbers. In other words: $|\mathbb{N}| = |\mathfrak{F}|$.

We could, if we wanted, try to prove this result directly, by defining a bijection from $\mathbb{N}$ to $\mathfrak{F}$. But I'd like to tell you about a trick, which makes the proof a lot easier. Rather than trying to define a bijection from $\mathbb{N}$ to $\mathfrak{F}$, we'll verify each of the following:

1. $|\mathbb{N}| \leq |\mathfrak{F}|$

2. $|\mathfrak{F}| \leq |\mathbb{N}|$

Then we'll use the Cantor-Schroeder-Bernstein Theorem to conclude $|\mathbb{N}| = |\mathfrak{F}|$.

To verify $|\mathbb{N}| \leq |\mathfrak{F}|$, we need an injective function from $\mathbb{N}$ to $\mathfrak{F}$. But this is totally straightforward, since one such function is the function that maps each natural number n to the one-membered sequence $\langle n \rangle$. To verify $|\mathfrak{F}| \leq |\mathbb{N}|$, we need an injective function f from $\mathfrak{F}$ to $\mathbb{N}$. Here is one way of doing so:

$$f(\langle n_1, n_2, \ldots, n_k \rangle) = p_1^{n_1+1} \cdot p_2^{n_2+1} \cdot \ldots \cdot p_k^{n_k+1}$$

where p_i is the ith prime number. For example:

$$
\begin{aligned}
f(\langle 4,0,1\rangle) &= 2^{4+1} \cdot 3^{0+1} \cdot 5^{1+1} \\
&= 32 \cdot 3 \cdot 25 = 2400
\end{aligned}
$$

Our function f function certainly succeeds in assigning a natural number to each finite sequence of natural numbers. And it is guaranteed to be injective, because of the following important result:

> **Fundamental Theorem of Arithmetic**
> Every positive integer greater than 1 has a unique decomposition into primes.

We have now verified $|\mathbb{N}| \leq |\mathfrak{F}|$ and $|\mathfrak{F}| \leq |\mathbb{N}|$. So the Cantor-Schroeder-Bernstein Theorem gives us $|\mathbb{N}| = |\mathfrak{F}|$. This concludes our proof!

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I understand why the cardinality of the set of natural numbers is less than or equal to the cardinality of the set of finite sequences of natural numbers. However, I don't get th... 3

[verify |N|≤|F| with an injective function or with an bijection?](#)
The Injection Principle was: "|A|≤|B| if and only if there is a bijection from A to a subset of B." Above is stated "To verify |N|≤|F|, we need an injective function from N to F. " ... 5