Problem 2: Find the limits

(3/3 points)

Let S_n be the number of successes in n independent Bernoulli trials, where the probability of success for each trial is $1/2$. Provide a numerical value, to a precision of 3 decimal places, for each of the following limits. You may want to refer to the standard normal table .

1.

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{2} - 20 \le S_n \le \frac{n}{2} + 20\right) =$$ 0

Answer: 0

2.

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{2} - \frac{n}{3} \le S_n \le \frac{n}{2} + \frac{n}{3}\right) =$$ 1

Answer: 1

3.

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{2} - \frac{\sqrt{n}}{4} \le S_n \le \frac{n}{2} + \frac{\sqrt{n}}{4}\right) =$$ 0.3829249

Answer: 0.383

Answer:

Note first that $S_n = X_1 + \cdots + X_n$, where the X_i are independent Bernoulli random variables with parameter $1/2$. Hence, $\mathbf{E}[S_n] = n/2$ and $\mathbf{var}(S_n) = n/4$.

1. Let us fix some $\epsilon > 0$. No matter how small ϵ is, as long as n is large enough, we will have $\epsilon\sqrt{n} > 20$. Therefore,

$$\begin{aligned}
\mathbf{P}\left(\frac{n}{2} - 20 \le S_n \le \frac{n}{2} + 20\right) &\le \mathbf{P}\left(\frac{n}{2} - \epsilon\sqrt{n} \le S_n \le \frac{n}{2} + \epsilon\sqrt{n}\right) \\
&= \mathbf{P}\left(-\frac{\epsilon\sqrt{n}}{\sqrt{n/4}} \le \frac{S_n - n/2}{\sqrt{n/4}} \le \frac{\epsilon\sqrt{n}}{\sqrt{n/4}}\right) \\
&= \mathbf{P}\left(-2\epsilon \le \frac{S_n - n/2}{\sqrt{n/4}} \le 2\epsilon\right).
\end{aligned}$$

By the Central Limit Theorem,

$$\lim_{n\to\infty} \mathbf{P}\left(-2\epsilon \le \frac{S_n - n/2}{\sqrt{n/4}} \le 2\epsilon\right) = \Phi(2\epsilon) - \Phi(-2\epsilon).$$

Since this is true for any $\epsilon > 0$, it is also true in the limit as $\epsilon \to 0^+$. The final answer then follows from the fact that

$$\lim_{\epsilon\to 0^+} \Phi(2\epsilon) - \Phi(-2\epsilon) = \Phi(0) - \Phi(0) = 0.$$

Solved problems

Additional theoretical material

Problem Set 8
Problem Set 8 due Apr 27, 2016 at 23:59 UTC

Unit summary

- Unit 9: Bernoulli and Poisson processes
- Unit 10: Markov chains
- Exit Survey
- Final Exam

2. The event $\frac{n}{2} - \frac{n}{3} \le S_n \le \frac{n}{2} + \frac{n}{3}$ is the same as the event $|(S_n/n) - (1/2)| \le 1/3$. By the weak law of large numbers, the probability of this event converges to 1 as $n \to \infty$.

3. By the Central Limit Theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{2} - \frac{\sqrt{n}}{4} \le S_n \le \frac{n}{2} + \frac{\sqrt{n}}{4}\right) &= \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{S_n - n/2}{\sqrt{n/4}}\right| \le \frac{\sqrt{n}/4}{\sqrt{n/4}}\right)\\ &= \Phi(1/2) - \Phi(-1/2)\\ &\approx 0.383.\end{aligned}$$

You have used 1 of 2 submissions

DISCUSSION

Click “Show Discussion" below to see discussions on this problem.