Fun with Prime Numbers (3)
Invitation to the Mysterious World of Mathematics

Tetsushi Ito

Department of Mathematics,
Kyoto University

What are reciprocity laws?

Theorem

For an odd prime number P,

$P \equiv 1 \pmod 4 \Leftrightarrow P = X^2 + Y^2$ for some X,Y.

A is congruent to B modulo N

$A \equiv B \pmod N \Leftrightarrow A - B$ is divisible by N.

Example

$5 = 1^2 + 2^2$ $13 = 2^2 + 3^2$ $17 = 1^2 + 4^2$ $29 = 2^2 + 5^2$

$37 = 1^2 + 6^2$ $41 = 4^2 + 5^2$ $53 = 2^2 + 7^2$ $61 = 5^2 + 6^2$

$73 = 3^2 + 8^2$ $89 = 5^2 + 8^2$ $97 = 4^2 + 9^2$

What are reciprocity laws? (2)

Theorem

For an odd prime number P,

P≡1 or 3 （mod 8） ⇔ P=X^2+2Y^2 for some X,Y.

Theorem

For a prime number P≠2,5,

P≡1 or 9 （mod 20） ⇔ P=X^2+5Y^2 for some X,Y.

Theorem

For a prime number P≠3,

P≡1 （mod 3） ⇔ P=X^2+3Y^2 for some X,Y.

What are reciprocity laws? (3)

- These results are understood by the Quadratic Reciprocity Law proved by Gauss.

Leonhard Euler (1707-1783)

Adrien-Marie Legendre (1752-1833)

Johann Carl Friedrich Gauss (1777-1855)

Reference   https://en.wikipedia.org/wiki/Leonhard_Euler
https://en.wikipedia.org/wiki/Adrien-Marie_Legendre
https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss

What are reciprocity laws? (4)

- For many quadratic polynomials F(X,Y), the equivalence

$$P \equiv ?? \text{ or } ???.... \pmod{N}$$
$$\Leftrightarrow P=F(X,Y) \text{ for some } X,Y$$

holds for a prime number P not dividing N.

- N is the conductor.

What are reciprocity laws? (5)

- There is a far-reaching generalization of the Quadratic Reciprocity Law, Class Field Theory.

Heinrich Martin Weber (1842-1913)

David Hilbert (1862-1943)

Teiji Takagi (1875-1960)

Emil Artin (1898-1962)

Reference
https://en.wikipedia.org/wiki/Heinrich_Martin_Weber
http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Takagi.html
https://en.wikipedia.org/wiki/David_Hilbert
https://en.wikipedia.org/wiki/Emil_Artin

What are reciprocity laws? (6)

- There are problems on prime numbers, which cannot be answered using Class Field Theory only. We need the Langlands Program (theory of modular forms/theta functions) to fully understand it.
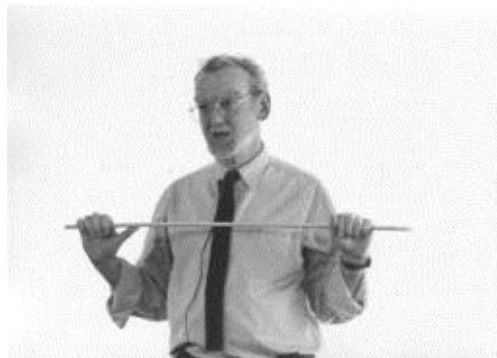
Theorem

There does not exist N with

$$P \equiv ?? \text{ or } ???.... \pmod{N}$$
$$\Leftrightarrow P = 6X^2 + XY + Y^2 \text{ for some } X, Y.$$

What are reciprocity laws? (7)

- Class Field Theory is vastly generalized to the Langlands Program, which is believed to explain 'all Reciprocity Laws' on prime numbers.

Robert Phelan Langlands
(1936-)

Reference   http://publications.ias.edu/rpl/

What are reciprocity laws? (8)

- The Langlands Program is one of the central themes in modern number theory. Even a partial solution leads to striking applications.

Fermat's Last Theorem
For an integer $N \geq 3$, there do not exist $X,Y,Z \geq 1$ satisfying
$$X^N + Y^N = Z^N.$$

Andrew John Wiles
(1953-)