Is dot product a kind of linear transformation

Asked 1 year, 4 months ago Active 1 year, 4 months ago Viewed 2k times

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I wonder in the field of Linear Algebra, if the dot product also referred to as an inner product:

$$\langle u, v\rangle = u \cdot v = u^T v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n, \quad \text{for} \quad u, v \in \mathbb{R}^n$$

can be categorized in a type of linear transformation.

I'm quite confused here, that the definition of $\langle\cdot,\cdot\rangle : V \times V \to F$, is a map of all vectors in a vector space, which is similar to **Matrix Multiplication** (a way to represent linear transformation), but the properties in the **inner product** is a bit different from the former.

It's kind of transforming from one vector space to another.

So the image or codomain of this linear transformation is the inner product space, while the domain of this linear transformation is the original vector space.

linear-algebra linear-transformations inner-product-space

edited Jul 19 '18 at 3:57
zipirovich
11.9k 1 19 32

asked Jul 19 '18 at 2:39
Witnes Chan
71 4

2 It's a function of 2 vector variables, which doesn't fit the definition of a linear transformation, so you need to adapt it somehow. For example, if you hold the first variable constant and look at it as a function of the second variable, than it will be a linear transformation. (And knowing that $< u, v >= u^T v$ actually explains this, because you can look at it as multiplying u^T, which is a 1xn matrix, by a vector v) – GuySa Jul 19 '18 at 2:54

@MrFatzo, I understand, thank you. – Witnes Chan Jul 19 '18 at 3:25

No problem. If you haven't learned about it, you can read about bilinear forms on Wikipedia. Those are functions of 2 vector variables that can be represented with matrix multiplication (multiplying one variable from the left and one variable from the right) – GuySa Jul 19 '18 at 3:29

I will, I use the book <linear algebra and differential equations> which I don't see this concept : **bilinear** – Witnes Chan Jul 19 '18 at 3:39

2 @GuySa Please note there are special symbols `\langle` & `\rangle` in LaTeX/MathJax to make angle brackets: $\langle\ \rangle$, different from 'less-than' and 'greater-than' symbols. – CiaPan Jul 19 '18 at 8:48

3 Answers

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There are a couple ways to view a dot product as a linear map by changing your view slightly.

The map $\langle\cdot,\cdot\rangle : V \times V \to F$ is not linear, it is what we call *bilinear*, which means that it is linear in each variable. I.e. a map $B : V \times V' \to W$ is *bilinear* if for all fixed $v \in V$, and all fixed $v' \in V'$ the maps $u' \mapsto B(v, u')$ and $u \mapsto B(u, v')$ are linear. (Though these maps are equal in the case of a dot product, since it is symmetric, so you just need to check that one is linear. Symmetric meaning that $\langle v, w\rangle = \langle w, v\rangle$)

The other view is perhaps a little more faithful to the idea of viewing the dot product as a linear map, but essentially equivalent. Though perhaps a little more abstract (though such judgments are inherently subjective).

The idea is that we can take a bilinear map $B : V \times V' \to W$ and turn it into a linear map $\tilde{B} : V \to \operatorname{Hom}_F(V', W)$. Where $\operatorname{Hom}_F(V', W)$ denotes the vector space of F-linear maps from V' to W. We define $\tilde{B}(v) = v' \mapsto B(v, v')$. Then one can use the defining property of bilinear maps given above to show that $\tilde{B}$ is linear, and for any $v \in V$, $\tilde{B}(v)$ is a linear map from V' to W. This process is called currying. Then $\tilde{B}$ is basically the same as B, since we can recover B from $\tilde{B}$ from the fact that $B(v, v') = (\tilde{B}v)v'$ (sorry for changing notation to parenthesis-less function application, I just think it's much more readable here).

Thus one can curry the dot product to get a linear map, call it D from V to $\operatorname{Hom}_F(V, F)$. In general, $\operatorname{Hom}_F(V, F)$ is a vector space called V-*dual*, often written V^*, so we can say D is a linear map from V to V^*. I.e., we can view the dot product as being equivalent to a particular nice linear map from V to V^*.

edited Jul 19 '18 at 11:36
Henning Makholm
260k 18 348 591

answered Jul 19 '18 at 3:05
jgon
20.3k 3 25 50

2 Just to point something out even more: fixing one of the vectors makes it a linear map. In short, the fixed vector gives the matrix of the transformation (with respect to some coordinates). – Kyle Miller Jul 19 '18 at 3:12

1 @KyleMiller, I think I basically said that in my definition of bilinear map, though I could have been a little more explicit I suppose (and didn't mention matrices). But thanks for making that more explicit :) – jgon Jul 19 '18 at 3:14

I am upvoting your answer, but would like to say something different. I will post it as a separate answer. – P Vanchinathan Jul 19 '18 at 3:35

@PVanchinathan Go for it! It's always good to have different perspectives on a question :) (Thanks for the upvote btw) – jgon Jul 19 '18 at 3:36

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The dot product isn't a linear transformation, but it gives you a lot of linear transformations: if you think of $\langle v, w\rangle$ as a function of v, with w fixed, then it is a linear transformation $\mathbb{R}^n \to \mathbb{R}$, sending an n-dimensional vector v to the one dimensional vector $\langle v, w\rangle$.

You can also fix v, and think of $\langle v, w\rangle$ as a function of w. Then this also defines a linear transformation $\mathbb{R}^n \to \mathbb{R}$.

In other words, the dot product is linear in each variable.

Actually, there is a way in which you can think of the dot product as a linear transformation. Consider the *function* (not a linear transformation, just a function) $f : \mathbb{R}^n \times \mathbb{R}^n \to \operatorname{Mat}_n(\mathbb{R})$ given by the formula

$$f(a_1, \ldots, a_n, b_1, \ldots, b_n) = C$$

where C is the n by n matrix whose ijth entry is $a_i b_j$. Then there exists a *unique* linear transformation $T : \text{Mat}_n(\mathbb{R}) \to \mathbb{R}$ such that

$$T(f(v,w))$$

is equal to the dot product of v and w for all $v, w \in \mathbb{R}^n$ (can you see what T is, and why it is unique?). This is a special case of the more general principle that multilinear maps identify with linear transformations on the tensor product.

answered Jul 19 '18 at 3:51
D_S
21.1k 6 17 58

can you explain more specific on that matrix -> R example, and I have no idea why it is unique. thankyou. – Witnes Chan Jul 19 '18 at 5:26

Define T to be the linear transformation sending $C = (c_{ij})$ to $c_{11} + \cdots + c_{nn}$. It's easy to check that $T(f(v,w)) = \langle v, w \rangle$. – D_S Jul 19 '18 at 13:40

For uniqueness, suppose S is a linear transformation satisfying $S(f(v,w)) = \langle v, w \rangle$. Look at what happens when v and w are all possible combinations of standard basis elements of $\mathbb{R}^n$. – D_S Jul 19 '18 at 13:41

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This answer considers not just the dot product but in general bilinear mappings to a different vector space: $U \times V \to W$. Formally their definition goes along the lines given in the answer of jgon. Only the domain and codomain are allowed to be more general. There is a construction called tensor product of two vector spaces U and V. This tensor product, denoted by $U \otimes V$ is a device to "convert" bilinear mappings to linear mappings. So every bilinear mapping above will be equivalently represented by a linear transformation $U \otimes V \to W$.

This is akin to quotient construction, say in groups, any homomorphism $f\colon G \to H$ is equivalent to an injective homomorphism $\tilde{f} : G/\ker f \to H$. Please wikipedia pages for construction of tensor products.

answered Jul 19 '18 at 3:47
P Vanchinathan
16.4k 1 22 37

Oh yes, good point! I wasn't thinking of introducing tensor product, but it certainly belongs as an answer here. Perhaps the best possible answer honestly. – jgon Jul 19 '18 at 3:50

1 How about the example of U being $\ell \times m$ matrices, V being $m \times n$ matrices, and W being $\ell \times n$ matrices, with the bilinear map being matrix multiplication? In the case that $n = 1$, we have matrix-vector multiplication, and by fixing the matrix we get a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^\ell$. Similarly, scalar multiplication for a vector space V is a map $\mathbb{R} \otimes V \to V$ (and is in fact an isomorphism). – Kyle Miller Jul 19 '18 at 5:13