13. Complex Fourier series

Here's an idea: Euler's formula

$$e^{it} = \cos t + i \sin t$$

tells us that complex exponentials can be written as a sum of a sine and a cosine function. This suggests that we might be able to write a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

with real coefficients a_n and b_n as a series of complex exponentials $\sum_{n=-\infty}^{\infty} c_n e^{int}$, for some **complex** coefficients c_n. As it turns out, this is true, that is, we can always write a Fourier series in terms of complex exponentials. Since the two series turn out to be equal, we'll also call the series in terms of complex exponentials a Fourier series.

So let's walk through the process of converting a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

to a series of the form $\sum_{n=-\infty}^{\infty} c_n e^{int}$.

Step 1: Rewriting the sum.

Using Euler's formula and the fact that $\sin t$ is an odd function and $\cos t$ is an even function, we notice that

$$\sin t = \frac{i}{2} \left(e^{-it} - e^{it} \right)$$

and

$$\cos t = \frac{1}{2} \left(e^{it} + e^{-it} \right).$$

Then we see that given any Fourier series f, we can write

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} \left(e^{int} + e^{-int} \right) + \frac{ib_n}{2} \left(e^{-int} - e^{int} \right) \right) \\ &= \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left((a_n - ib_n) e^{int} + (a_n + ib_n) e^{-int} \right). \end{aligned}$$

Step 2: Defining coefficients.

We can see that f can be written as a sum of complex exponentials. Let's write the coefficients of these exponentials nicely so that we can easily convert back and forth between the two forms. Define $c_0 := a_0/2$. For $n > 0$, define

$$c_n := \frac{a_n - ib_n}{2},$$

and

$$c_{-n} := \bar{c}_n = \frac{a_n + ib_n}{2}.$$

Then we can write f compactly as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

Step 3: Relating coefficients to inner products.

Remark about inner products:

Note that for real valued 2π-periodic functions f and g we define an inner product as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f \cdot g \, dt.$$

If f and g are complex valued functions, we must define the inner product as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f \cdot \overline{g} \, dt,$$

where $\overline{g}$ is the complex conjugate of g.

Example 13.1

$$\left\langle e^{int}, e^{int}\right\rangle = \int_{-\pi}^{\pi} e^{int}\cdot e^{-int}\,dt = \int_{-\pi}^{\pi} dt = 2\pi.$$

If $m \neq n$,

$$\begin{aligned}\left\langle e^{int}, e^{imt}\right\rangle &= \int_{-\pi}^{\pi} e^{int}\cdot e^{-imt}\,dt \\ &= \int_{-\pi}^{\pi} e^{i(n-m)t}\,dt \\ &= \left.\frac{e^{i(n-m)t}}{i\,(n-m)}\right|_{-\pi}^{\pi} \\ &= \frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{i\,(n-m)} = \frac{2}{n-m}\sin\left((n-m)\,\pi\right) = 0.\end{aligned}$$

Therefore the e^{int} are orthogonal.

In particular, notice that for $n > 0$, we can compute c_n by the formula

$$c_n = \frac{\left\langle f, e^{int}\right\rangle}{\left\langle e^{int}, e^{int}\right\rangle}.$$

This definition is in agreement with the definition of $c_n = \frac{a_n - ib_n}{2}$. (You can check this!)

Check worked out

Show

Key properties of complex Fourier series

- Often integrals involving complex exponentials are actually much easier to compute than integrals involving sines and cosines.

- We can also make sense of Fourier series of complex-valued functions more easily in this setting.

Complex Fourier series concept check

2/2 points (graded)

Let $\mathrm{Sq}\,(t)$ be the 2π-periodic square wave that is equal to 1 for $0 \le t < \pi$ and equal to -1 for $-\pi \le t < 0$. For $n \neq 0$, write a general formula for the nth coefficient of the complex Fourier coefficient c_n of $\mathrm{Sq}\,(t)$.

Tip : Type **a^b** for a^b, type **pi** for π, and **i** for $i = \sqrt{-1}$.

n even, $c_n =$ 0 ✔ **Answer:** 0

0

n odd, $c_n =$ -2*i/(pi*n) ✔ **Answer:** 2/(i*n*pi)

$-\frac{2\cdot i}{\pi\cdot n}$

Solution:

There are multiple ways to approach this problem, one is to use the coefficients a_n and b_n (which we already have calculated) to write c_n, another (better) approach is to calculate c_n directly; this is the approach we'll take. We see that for $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Sq}\,(t)\, e^{-int}\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-int}\, dt + \frac{1}{2\pi}\int_{0}^{\pi} e^{-int}\, dt \\ &= -\frac{1}{2\pi}\left(-\frac{1}{in}\right) e^{-int}\Big|_{-\pi}^{0} + \frac{1}{2\pi}\left(-\frac{1}{in}\right) e^{-int}\Big|_{0}^{\pi} \\ &= \frac{1}{2\pi in}\left(e^0 - e^{in\pi} - \left(e^{-in\pi} - e^0\right)\right) \\ &= \frac{1}{2\pi in}\left(2 - \left(e^{in\pi} + e^{-in\pi}\right)\right). \end{aligned}$$

Examining the cases where n is even and where n is odd seperately, we notice that

$$\begin{aligned} c_n &= \begin{cases} 0, & n \text{ even} \\ \frac{2}{in\pi}, & n \text{ odd} \end{cases} \\ &= \frac{1-(-1)^n}{in\pi}. \end{aligned}$$

In the case that $n = 0$, we see that

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Sq}\,(t)\,dt = 0.$$

Hence the Fourier series for $\mathrm{Sq}\,(t)$ is

$$\mathrm{Sq}\,(t) = \sum_{n=-\infty}^{\infty} \frac{1-(-1)^n}{in\pi} e^{int}.$$

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Answers are displayed within the problem

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? Confused about inner products with complex functions 5

? i m lost at step 2....
i don't seem to understand why the coefficients could be compacted? 9

Step 3 4

Different approach - same result 2

I did something different and I don't know why it happens to be correct. I just took F.S of the Sq(t) and transfer it into complex F.S where I transfered: 4/(n*pi) *sin n*t ---> 2/(n...

Inner product when n != m 3

Confusion regarding the question 3

The question asks for the nth complex coefficients, where n is either odd or even. My confusion is that since the coefficient depends on whether n is positive or negative, sho...

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