9. Inhomogeneous linear systems: structure of the solution set

Recall that for an inhomogeneous linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, there are two possibilities:

1. The vector $\mathbf{b}$ is not in $\mathrm{CS}(\mathbf{A})$; there are no solutions.
2. The vector $\mathbf{b}$ is in $\mathrm{CS}(\mathbf{A})$; there exists a solution.

In case 2 we will describe the solution in terms of the nullspace of $\mathbf{A}$ and any one solution.

If an inhomogeneous equation has solutions, then we can express the general solution in terms of any one particular solution plus the general solution to the associated homogeneous equation: If $\mathbf{x}_p$ is a particular solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_h$ is the general solution to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$, then $\mathbf{x} := \mathbf{x}_p + \mathbf{x}_h$ is the general solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Here is why it works: Suppose that a solution exists; let $\mathbf{x}_p$ be one, so $\mathbf{A}\mathbf{x}_p = \mathbf{b}$. If $\mathbf{x}_h$ satisfies $\mathbf{A}\mathbf{x}_h = \mathbf{0}$, adding the two equations gives

$$\mathbf{A}(\mathbf{x}_p + \mathbf{x}_h) = \mathbf{A}(\mathbf{x}_p) + \mathbf{A}(\mathbf{x}_h) = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

so adding $\mathbf{x}_p$ to $\mathbf{x}_h$ produces a solution $\mathbf{x}$ to the inhomogeneous equation. Every solution $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{b}$ arises this way.

This is analogous to the shape of solutions to inhomogeneous differential equations.

Problem 9.1 The inhomogeneous linear system

$$\begin{aligned} x+2y \quad\quad +2v+3w &= 4 \\ -y+2z+3v+\ w &= 5 \\ 2w &= 6 \end{aligned}$$

has augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 4 \\ 0 & -1 & 2 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{array}\right)$$

in row echelon form.

Write the general solution to the system as one particular solution plus the general solution to the corresponding homogeneous system.

Solution: (We've solved this example before in the lecture on elimination. Let's review the process here quickly.) This system is already in row echelon form. Solve the equations in reverse order, and substitute into previous equations:

$$w \quad = 3$$

There is no equation for v in terms of the later variable w, so v can be any number; set

$$v \quad = c_1 \quad \text{for a parameter } c_1.$$

There is no equation for z in terms of v, w, so set

$$\begin{aligned} z &= c_2 \quad \text{for a parameter } c_2. \\ -y+2c_2+3c_1+3 &= 5 \\ y &= 3c_1+2c_2-2 \end{aligned}$$

$$\begin{aligned} x + 2(3c_1 + 2c_2 - 2) + 2c_1 + 3(3) &= 4 \\ x &= -8c_1 - 4c_2 - 1. \end{aligned}$$

Conclusion: The general solution is

$$\begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} = \begin{pmatrix} -8c_1 - 4c_2 - 1 \\ 3c_1 + 2c_2 - 2 \\ c_2 \\ c_1 \\ 3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 3 \end{pmatrix}}_{\text{particular solution}} + \underbrace{c_1 \begin{pmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{general homogeneous solution}},$$

where c_1, c_2 are parameters.

Analogy with Solutions to Differential Equations : If $p(D)$ is a linear differential operator, then the general solution to $p(D)\,x = f$ is $x = x_p + x_h$, where x_p is any particular solution, and x_h is the general solution to the homogeneous equation $p(D)\,x = 0$. The general solution to a linear system $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_h$ is the general solution to the homogeneous system. The differential operator $p(D)$ is the analogue of the matrix $\mathbf{A}$.

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