The Power Set of Natural Numbers

We have now proved two important results about the relative sizes of infinite sets.

First, we saw that there are more real numbers than natural numbers: $|\mathbb{N}| < |\mathbb{R}|$.

Then we proved Cantor's Theorem, from which it follows that there are more subsets of natural numbers than natural numbers: $|\mathbb{N}| < |\mathscr{P}(\mathbb{N})|$.

These results entail that $\mathscr{P}(\mathbb{N})$ and $\mathbb{R}$ both have cardinalities greater than $\mathbb{N}$.

But they do not settle the relative sizes of $\mathscr{P}(\mathbb{N})$ and $\mathbb{R}$. As it turns out, the two sets have exactly the same size:

$$|\mathscr{P}(\mathbb{N})| = |\mathbb{R}|$$

I find this is a deeply satisfying result, so I'd like to end this chapter by showing you how to prove it.