4. Solving the system with 3 tanks

Solving the system: the first eigenvalue and eigenspace

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Let us review the solution to the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

in the context of the 3 connected storage tanks.

Solution:
Step 1. Eigenvalues

The characteristic equation of $\mathbf{A}$ is

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda+2 & -1 & -1 \\ -1 & \lambda+2 & -1 \\ -1 & -1 & \lambda+2 \end{vmatrix} = 0.$$

Expanding in polynomial form and factoring, we get

$$\begin{aligned} \lambda^3 + 6\lambda^2 + 9\lambda &= 0 \\ \lambda(\lambda^2 + 6\lambda + 9) &= 0 \\ \lambda(\lambda+3)^2 &= 0. \end{aligned}$$

Therefore, the eigenvalues of $\mathbf{A}$ are $0,$ with multiplicity $1,$ and -3 with multiplicity 2.

Step 2. Eigenvectors and the exponential solutions

Let us find the eigenvectors/eigenspace to accompany each eigenvalue.

Eigenspace of $\lambda = 0$:

The eigenspace corresponding to $\lambda_1 = 0$ is the null space of

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

By inspection, we see that the vector of all 1's is an eigenvector. Therefore, the eigenspace is spanned by the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and a normal mode to the system of DE is

$$\mathbf{v}_1 e^{0t} \quad = \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This solution, or a positive scalar multiple of it, corresponds to a constant (or steady state) solution where the tanks all have the same level of fluid and there is no net flow between the tanks.

Solving the system: the repeated eigenvalue

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Let us continue to review the solutions to

$$\dot{\mathbf{x}} \quad = \quad \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

in the context of the three connected fluid tanks.

Eigenspace of $\lambda = -3$:

Next, we find the eigenvectors corresponding to the repeated eigenvalue -3. We need to find the nullspace of the matrix

$$(-3)\mathbf{I} - \mathbf{A} \quad = \quad \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$

We can find 2 eigenvectors by inspection:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Since the multiplicity of the eigenvalue -3 is $2,$ the maximum number of linearly independent eigenvectors is $2,$ and so we have found enough eigenvectors.

Step 3. The general solution

The general solution is a linear combination of the normal modes:

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{v}_1 e^{0t} + c_2\mathbf{v}_2 e^{-3t} + c_3\mathbf{v}_3 e^{-3t} \\ &= c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left[c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right] e^{-3t}. \end{aligned}$$

The three constants $c_1,$ $c_2,$ and c_3 are determined by the initial conditions.

Note that the terms proportional to e^{-3t} will decay to zero as $t \to \infty,$ so every solution, regardless of the initial conditions, will approach a constant solution. This means that the fluid heights in the three tanks will approach the same height as time goes on, as expected from physical experience.

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