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Proving the Theorem

Proof of the Vitali Theorem

to have the same measure.

So by uniformity they cannot have a measure at all.

So they are non-measurable.

There is no such thing as the Lebesgue measure of a Vitali

Set.

Amazing!

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Here is a sketch the proof of the non-measurability of Vitali Sets. (Some of the technical details are assigned as exercises below.)

We'll start by **partioning** $[0, 1)$.

In other words, we'll divide $[0, 1)$ into a family of non-overlapping "cells", whose union is $[0, 1)$.

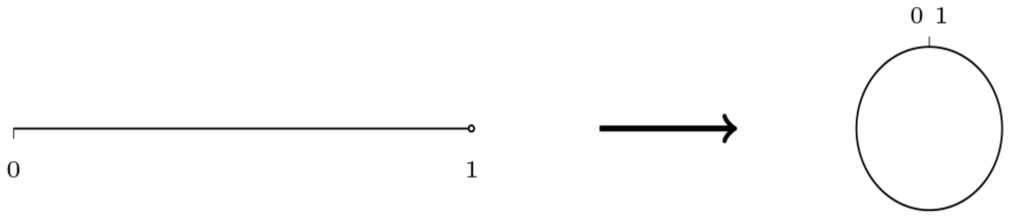
The cells are characterized as follows: for $a, b \in [0, 1)$, a and b are in the same cell if and only if $a - b$ is a rational number. For instance, $\frac{1}{2}$ and $\frac{1}{6}$ are in the same cell because $\frac{1}{2} - \frac{1}{6} = \frac{2}{3}$, which is a rational number. Similarly, $\pi - 3$ and $\pi - \frac{25}{8}$ are in the same cell because $(\pi - 3) - (\pi - \frac{25}{8}) = \frac{1}{8}$, which is a rational number. But $\pi - 3$ and $\frac{1}{2}$ are not in the same cell because $\pi - \frac{7}{2}$ is not a rational number.

We'll call this partition of $[0, 1)$ $\mathcal{U}$, because it has uncountably many cells. The next step of our proof will be to use $\mathcal{U}$ to characterize a partition of $[0, 1)$ with countably many cells, which we'll call $\mathcal{C}$. Each cell of $\mathcal{C}$ will be a set V_q for $q \in \mathbb{Q}^{[0,1)}$. ($\mathbb{Q}^{[0,1)}$ is the set a rational numbers in $[0, 1)$.)

I will now explain which elements of $[0, 1)$ to include in a given cell V_q of $\mathcal{C}$.

The first part of the process is to pick a representative from each cell in $\mathcal{U}$. (In other words: we need a *choice set* for $\mathcal{U}$. It is a consequence of Solovey's result, mentioned above, that it is impossible to *define* a choice set for $\mathcal{U}$. In other words: it is impossible to specify a criterion that that could be used to single out exactly one element from each cell in $\mathcal{U}$. But it follows from the Axiom of Choice that a choice set for $\mathcal{U}$ must nonetheless exist. And its existence is all we need here, because all we're aiming to show is that non-measurable sets exist.)

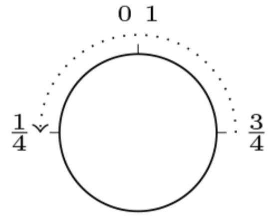
We will now use our representatives from each cell in $\mathcal{U}$ to populate the cells of $\mathcal{C}$ with elements of $[0, 1)$. The first step is to think of $[0, 1)$ as a line segment of length 1 (which is missing one of its endpoints), and bend it into a circle:



Recall that the difference between any two elements in a cell $\mathcal{U}$ is always a rational number.

From this it follows that each member of $[0,1)$ can be reached by starting at the representative of its cell in $\mathcal{U}$, and traveling some rational distance around the circle, going counter-clockwise.

Suppose, for example, that $\frac{3}{4}$ is in our choice set for $\mathcal{U}$, and has therefore been selected as the representative of its cell in $\mathcal{U}$. (Call this cell $C_{\frac{3}{4}}$.) Now consider a second point in $C_{\frac{3}{4}}$: as it might be, $\frac{1}{4}$. Since $\frac{3}{4}$ and $\frac{1}{4}$ are in the same cell of $\mathcal{U}$, one can reach $\frac{1}{4}$ by starting at $\frac{3}{4}$, and traveling a rational distance around the circle, going counter-clockwise—in this case a distance of $\frac{1}{2}$:



If a is point in $[0,1)$, let us say that $\delta(a)$ is the distance one would have to travel on the circle, going counter-clockwise, to get to a from the representative for a's cell in $\mathcal{U}$. In our example, $\delta\left(\frac{1}{4}\right)=\frac{1}{2}$.

It is now straightforward to explain how to populate the cells of our countable partition $\mathcal{C}$ with elements of $[0,1)$: each cell V_q $(q \in \mathbb{Q}^{[0,1)})$ of $\mathcal{C}$ is populated with those $a \in [0,1)$ such that $\delta(a)=q$. As you'll be asked to verify below, this definition guarantees that the V_q $(q \in \mathbb{Q}^{[0,1)})$ form a countable partition of $[0,1)$.

Let a **Vitali Set** be a cell V_q $(q \in \mathbb{Q}^{[0,1)})$ of $\mathcal{C}$.

All that remains to complete our proof is to verify that the Vitali Sets must all have the same measure, if they have a measure at all.

The basic idea is straightforward. Recall that V_q is the set of points at a distance of q from their cell's representative, going counter-clockwise. From this it follows that V_q can be obtained by *rotating* V_0 on the circle counter-clockwise, by a distance of q. So one can use Uniformity to show that V_0 and V_q have the same measure, if they have a measure at all (and therefore that all Vitali Sets have the same measure, if they have a measure at all).

We are now in a position to wrap up our proof. We have seen that $[0, 1)$ can be partitioned into countably many Vitali Sets, and that these sets must all have the same measure, if they have a measure at all. But, for reasons rehearsed in Lecture 7.2.2.1, we know that in the presence of Non-Negativity and Countable Additivity there can be no such thing as uniform measure over a countable family of (mutually exclusive and jointly exhaustive) subsets of a set of measure 1. So there can be no way of expanding the notion of Lebesgue measure to Vitali sets, without giving up on Non-Negativity, Countable Additivity or Uniformity.

Problem 1

1/1 point (ungraded)

The relation R, which holds between a and b if and only if $a - b$ is a rational number, satisfies which of the following three properties?

☑ *Reflexivity:* For every x in $[0, 1)$, xRx

☑ *Symmetry:* For every x and y in $[0, 1)$, if xRy then yRx.

☑ *Transitivity:* For every x, y and z in $[0, 1)$, if xRy and yRz then xRz.

✓

(If R satisfies all three, then $\mathcal{U}$ is a partition of $[0, 1)$.)

Explanation

First, reflexivity. We want to show that every number in $[0, 1]$ differs from itself by a rational number. Easy: every real number differs from itself by 0, and 0 is a rational number.
Next, symmetry. For $x, y \in [0, 1]$ we want to show that if $x - y \in \mathbb{Q}$, then $y - x \in \mathbb{Q}$. If $x - y = r$, then $y - x = -r$. But $r \in \mathbb{Q}$, then $-r \in \mathbb{Q}$.
Finally, transitivity. For $x, y, z \in [0, 1]$ we want to show that if $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$, then $x - z \in \mathbb{Q}$. Suppose $x - y = r$ and $y - z = s$. Then $y = x - r$ and $y = s + z$. So $x - r = s + z$, and therefore $x - z = s + r$. But if $r, s \in \mathbb{Q}$, then $s + r \in \mathbb{Q}$.

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ℹ Answers are displayed within the problem

Problem 2

1/1 point (ungraded)

Each cell of $\mathcal{U}$ has how many members?

● Countably many

○ Uncountably many

○ None of the above

✓

Explanation
Let C be a cell in $\mathcal{U}$, and let $a \in C$. Every number in C differs from a by some rational number. Since there are only countably many rational numbers, this means that there must be at most countably many numbers in C.

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Problem 3

1 point possible (ungraded)
Show that $\mathcal{U}$ has uncountably many cells.

☐ done

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Problem 4

2/2 points (ungraded)

Every real number in $[0, 1)$ belongs to some V_q ($q \in \mathbb{Q}^{[0,1)}$).

True or false?

● True

○ False

✓

Explanation

We verify that every real number in $[0,1)$ belongs to some V_q $(q \in \mathbb{Q}^{[0,1)})$.
Let $a \in [0,1)$. Since $\mathcal{U}$ is a partition, a must be in some cell C of $\mathcal{U}$. Let r be C's representative. By the definition of $\mathcal{U}$, $a - r$ is a rational number. So $\delta(a)$, which is the distance one would have to travel on the circle to get from r to a going counter-clockwise, must be a rational number in $[0,1)$. So by the definition of $\mathcal{C}$, a must be in $V_{\delta(a)}$.

No real number in $[0,1)$ belongs to more than one V_q $(q \in \mathbb{Q}^{[0,1)})$.

True or false?

● True

○ False

✓

Explanation

We verify that no real number in $[0,1)$ belongs to more than one V_q $(q \in \mathbb{Q}^{[0,1)})$. Suppose otherwise. Then some $a \in [0,1)$ belongs to both V_q and V_p $(p \neq q,\ q, p \in \mathbb{Q}^{[0,1)})$. By the definition of $\mathcal{C}$ this means that $\delta(a)$ must be equal to both p and q, which is impossible.

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Problem 5

1/1 point (ungraded)

In showing that the Vitali Sets all have the same measure if they have a measure at all, we proceeded somewhat informally, by thinking of $[0,1)$ as a circle. When $[0,1)$ is instead thought of as a line-segment, one can get from V_0 to V_q by translating V_0 by q, and then subtracting 1 from any points that end up outside $[0,1)$. More precisely, V_0 can be transformed into V_q in three steps. One first divides the points in V_0 into two subsets, depending on whether they are smaller than $1 - q$:

- $V_{\overleftarrow{0}} = V_0 \cap [0, 1 - q)$
- $V_{\vec{0}} = V_0 \cap [1 - q, 1)$

Next, one translates $V_{\overleftarrow{0}}$ by q and $V_{\overleftarrow{0}}$ by $q - 1$, yielding $(V_{\overleftarrow{0}})^q$ and $(V_{\vec{0}})^{q-1}$, respectively. Finally, one takes the union of the translated sets: $(V_{\overleftarrow{0}})^q \cup (V_{\vec{0}})^{q-1}$.

Consider the following claim:

$$V_q = (V_{\overleftarrow{0}})^q \cup (V_{\vec{0}})^{q-1} \; (q \in \mathbb{Q}^{[0,1)}).$$

True or false?

● True

○ False

✓

Explanation

We verify that $V_q = (V_{\overleftarrow{0}})^q \cup (V_{\vec{0}})^{q-1} \; (q \in \mathbb{Q}^{[0,1)})$.

Let $a \in V_q$, and let r be the representative of a's cell in $\mathcal{U}$. V_q, recall, is the set of $x \in [0, 1)$ such that $\delta(x) = q$. There are two cases, $r + q < 1$ and $r + q \geq 1$. Let us consider each of them in turn, and show that $a \in V_q$ if and only if $a \in (V_{\overleftarrow{0}})^q \cup (V_{\vec{0}})^{q-1}$.

- Assume $r + q < 1$. Since $\delta(a) = q$, one would have to travel a distance of q on the circle, going counter-clockwise, to get to a from r. Since we have $r + q < 1$, this means that $a = r + q$.

  Now suppose that $r \in V_0$. We have $r \in V_{\overleftarrow{0}}$, since $V_{\overleftarrow{0}} = V_0 \cap [0, 1 - q)$ and $r + q < 1$. So $a = r + q$ entails $a \in (V_{\overleftarrow{0}})^q$. Suppose, conversely, that $a \in (V_{\overleftarrow{0}})^q$. Then $a = r + q$ entails that $r \in V_{\overleftarrow{0}}$, and therefore that $r \in V_0$.

- Assume $r + q \geq 1$. Since $\delta(a) = q$, one would have to travel a distance of q on the circle, going counter-clockwise, to get to a from r. But $r + q \geq 1$, so one must cross the 0 point on the circle when traveling from r to a. Since since $q < 1$ (and since the circle has circumference 1), this means that $a = r + q - 1$.

Now suppose that $r \in V_0$. We have $r \in V_{\vec{0}}$, since $V_{\vec{0}} = V_0 \cap [1-q, 1)$ and $r + q \geq 1$. So $a = r + q - 1$ entails $a \in (V_{\vec{0}})^{q-1}$. Suppose, conversely, that $a \in (V_{\vec{0}})^{q-1}$. Then $a = r + q - 1$ entails that $r \in V_{\overleftarrow{0}}$, and therefore that $r \in V_0$.

(Note that from this it follows that V_0 and V_q have the same measure, if they have a measure at all, and therefore that all Vitali Sets have the same measure, if they have a measure at all. For it follows from Uniformity that $V_{\overleftarrow{0}}$ and $V_{\vec{0}}$ must have the same measures as their translations, if they have measures at all. And it follows from Countable Additivity that the measure of V_0 must be the sum of the measures of $V_{\overleftarrow{0}}$ and $V_{\vec{0}}$ (if the latter have measures), and that the measure of V_q must be the sum of the measures of $(V_{\overleftarrow{0}})^q$ and $(V_{\vec{0}})^{1-q}$, if the latter have measures.)

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