5.5 Law of the unconscious statistician (LOTUS)

Unit 5: Averages

Adapted from Blitzstein-Hwang Chapters 4, 5, and 10.

As we saw in the St. Petersburg paradox, $E(g(X))$ does *not* equal $g(E(X))$ in general if g is not linear. So how do we correctly calculate $E(g(X))$? Since $g(X)$ is an r.v., one way is to first find the distribution of $g(X)$ and then use the definition of expectation. Perhaps surprisingly, it turns out that it is possible to find $E(g(X))$ directly using the distribution of X, without first having to find the distribution of $g(X)$. This is done using the *law of the unconscious statistician* (LOTUS).

> **Theorem 5.5.1 (LOTUS).**
>
> If X is a discrete r.v. and g is a function from $\mathbb{R}$ to $\mathbb{R}$, then
>
> $$E(g(X)) = \sum_x g(x)P(X = x),$$
>
> where the sum is taken over all possible values of X.

This means that we can get the expected value of $g(X)$ knowing only $P(X = x)$, the PMF of X; we don't need to know the PMF of $g(X)$. The name comes from the fact that in going from $E(X)$ to $E(g(X))$ it is tempting just to change x to $g(x)$ in the definition, which can be done very easily and mechanically, perhaps in a state of unconsciousness. On second thought, it may sound too good to be true that finding the distribution of $g(X)$ is not needed for this calculation, but LOTUS says it *is* true. We will omit a general proof of LOTUS, but let's see why it is true in some special cases. Let X have support $0, 1, 2, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$, so the PMF is $P(X = n) = p_n$. Then X^3 has support $0^3, 1^3, 2^3, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$, so

$$E(X) = \sum_{n=0}^{\infty} n p_n,$$

$$E(X^3) = \sum_{n=0}^{\infty} n^3 p_n.$$

As claimed by LOTUS, to edit the expression for $E(X)$ into an expression for $E(X^3)$, we can just change the n in front of the p_n to an n^3; the p_n is unchanged, and we can still use the PMF of X.

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