10. Worked example: resonance with damping

In real life, there is always damping, and this prevents the runaway growth in the pure resonance scenario of the previous section.

Problem 10.1 Describe the steady state solution to

$$\ddot{x} + 0.1\dot{x} + 49x = \frac{\pi}{4}\mathrm{Sq}\,(t)\,.$$

Remark 10.2 The term $0.1\dot{x}$ is the damping term.

Recall: The steady state solution is the periodic solution. (Other solutions will be a sum of the steady state solution with a transient solution solving the homogeneous ODE

$$\ddot{x} + 0.1\dot{x} + 49x = 0;$$

these transient solutions tend to 0 as $t \to \infty$, because the coefficients of the characteristic polynomial are positive (in fact, this is an underdamped system).

Solution: First let's solve

$$\ddot{x} + 0.1\dot{x} + 49x = \sin nt.$$

Before doing that, solve the complex replacement ODE

$$\ddot{z} + 0.1\dot{z} + 49z = e^{int}.$$

The characteristic polynomial is $P(r) = r^2 + 0.1r + 49$, so ERF gives

$$z = \frac{1}{P(in)} e^{int} = \frac{1}{(49 - n^2) + (0.1n)\, i} e^{int},$$

with complex gain $\dfrac{1}{(49 - n^2) + (0.1n)\, i}$ and gain

$$g_n := \frac{1}{|(49 - n^2) + (0.1n)\, i|}.$$

Thus

$$x = \operatorname{Im}\left(\frac{1}{(49 - n^2) + (0.1n)\, i} e^{int}\right);$$

this is a sinusoid of amplitude g_n, so $x = g_n \cos(nt - \phi_n)$ for some ϕ_n.

The input signal

$$\frac{\pi}{4}\operatorname{Sq}(t) = \sum_{n \geq 1,\ \text{odd}} \frac{\sin nt}{n},$$

elicits the system response

$$
\begin{aligned}
x(t) &= \sum_{n\geq 1,\,\text{odd}} g_n \frac{\cos(nt-\phi_n)}{n} \\
&\approx 0.021\cos(t-\phi_1) + 0.008\cos(3t-\phi_3) + 0.008\cos(5t-\phi_5) \\
&\quad +0.204\cos(7t-\phi_7) + 0.003\cos(9t-\phi_9) + (\text{even smaller terms}).
\end{aligned}
$$

Conclusion: The system response is almost indistinguishable from a pure sinusoid of angular frequency 7.

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[Staff] Don't understand a couple of lines in the solution 6