edX   **MITx: 6.041x Introduction to Probability - The Science of Uncertainty**

Bookmarks

🔖 **Bookmark**

▸ Unit 0: Overview

▸ Entrance Survey

▸ Unit 1: Probability models and axioms

▸ Unit 2: Conditioning and independence

▸ Unit 3: Counting

▸ Unit 4: Discrete random variables

▾ Exam 1

Exam 1
Exam 1 due Mar 09, 2016 at 23:59 UTC ✎

▸ Unit 5: Continuous

Problem 2: A binary communication system - Part 2

(1/2 points)
Note: The problem statement from part 1 has been repeated here for your convenience.

A binary communication system is used to send one of two messages:

(i) message A is sent with probability 2/3, and consists of an infinite sequence of zeroes,

(ii) message B is sent with probability 1/3, and consists of an infinite sequence of ones.

The ith received bit is "correct" (i.e., the same as the transmitted bit) with probability 3/4, and is "incorrect" (i.e., a transmitted 0 is received as a 1, and vice versa), with probability 1/4. We assume that **conditioned on any specific message sent,** the received bits, denoted by $Y_1, Y_2, \ldots$ are independent.

1. Is $Y_2 + Y_3$ independent of Y_1?

[Yes ▾]   ✖   **Answer:** No

2. Is $Y_2 - Y_3$ independent of Y_1?

[Yes ▾]   ✔   **Answer:** Yes

Answer:

1. No, they are not independent. Let $X = Y_2 + Y_3$. Using the total probability theorem we find:

$$
\begin{aligned}
\mathbf{P}(X = 0) &= \mathbf{P}(A)\mathbf{P}(X = 0 \mid A) + \mathbf{P}(B)\mathbf{P}(X = 0 \mid B) \\
&= (2/3)(3/4)^2 + (1/3)(1/4)^2 \\
&= 19/48 \approx 0.396.
\end{aligned}
$$

Conditioning on the event $Y_1 = 0$, we now find that:

$$
\begin{aligned}
\mathbf{P}(X = 0 \mid Y_1 = 0) &= \frac{\mathbf{P}(X = 0 \cap Y_1 = 0)}{\mathbf{P}(Y_1 = 0)} \\
&= \frac{\mathbf{P}(A)\mathbf{P}(X = 0 \cap Y_1 = 0 \mid A) + \mathbf{P}(B)\mathbf{P}(X = 0 \cap Y_1 = 0 \mid B)}{\mathbf{P}(Y_1 = 0)} \\
&= \frac{(2/3)(3/4)^3 + (1/3)(1/4)^3}{7/12} \\
&= 55/112 \approx 0.491.
\end{aligned}
$$

Since we have shown $\mathbf{P}(X = 0 \mid Y_1 = 0) \neq \mathbf{P}(X = 0)$ with $\mathbf{P}(Y_1 = 0) > 0$, we conclude that X and Y_1 are not independent. Intuitively, knowing $Y_1 = 0$ increases the likelihood that message A was transmitted, which increases the likelihood that $Y_2 + Y_3 = 0$.

2. Yes, they are independent. Let $Z = Y_2 - Y_3$. We want to show that $p_{Y_1}(y_1)p_Z(z) = p_{Y_1,Z}(y_1, z)$, for all (y_1, z). We have already found the PMF of Y_1 in part (1):

$$p_{Y_1}(y_1) = \begin{cases} 7/12 & \text{if } y_1 = 0 \\ 5/12 & \text{if } y_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Using the total probability theorem, similar to how we calculated $\mathbf{P}(X = 0)$ in part (4), we find the PMF of Z:

$$p_Z(z) = \begin{cases} 3/16 & \text{if } z = -1 \text{ or } z = 1 \\ 5/8 & \text{if } z = 0 \\ 0 & \text{otherwise} \end{cases}$$

The joint PMF can also be found using the total probability theorem:

$$p_{Y_1,Z}(y_1, z) = \begin{cases} 7/64 & \text{if } (y_1, z) = \{(0, -1), (0, 1)\} \\ 35/96 & \text{if } (y_1, z) = (0, 0) \\ 5/64 & \text{if } (y_1, z) = \{(1, -1), (1, 1)\} \\ 25/96 & \text{if } (y_1, z) = (1, 0) \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $p_{Y_1}(y_1)p_Z(z) = p_{Y_1,Z}(y_1, z)$, for all (y_1, z). Intuitively, Z is independent of what message was transmitted, and so, even though Y_1 provides information on which message was transmitted, it ultimately does not provide information on Z.

You have used 1 of 1 submissions