

The Riemann Hypothesis

Summary:

When studying the distribution of prime numbers Riemann extended Euler's zeta function (defined just for s with real part greater than one)

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

to the entire complex plane (*sans* simple pole at $s = 1$). Riemann noted that his zeta function had trivial zeros at $-2, -4, -6, \dots$; that all nontrivial zeros were symmetric about the line $\text{Re}(s) = 1/2$; and that the few he calculated were on that line. **The Riemann hypothesis is that all nontrivial zeros are on this line.** Proving the Riemann Hypothesis would allow us to greatly sharpen many number theoretical results. For example, in 1901 von Koch showed that the Riemann hypothesis is equivalent to:

$$\pi(x) = \text{Li}(x) + O(x^{1/2} \ln x)$$

But it would not make factoring any easier! There are a couple standard ways to generalize the Riemann hypothesis.

1. The Riemann Hypothesis:

Euler studied the sum

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for integers $s > 1$ (clearly $\zeta(1)$ is infinite). Euler discovered a formula relating $\zeta(2k)$ to the Bernoulli numbers yielding results such as $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. But what has this got to do with the primes? The answer is in the following product taken over the primes p (also discovered by Euler):

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

Euler wrote this as

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \frac{2^s}{2^s - 1} \frac{3^s}{3^s - 1} \frac{5^s}{5^s - 1} \frac{7^s}{7^s - 1} \dots$$

Riemann later extended the definition of $\zeta(s)$ to all complex numbers s (except the simple pole at $s=1$ with residue one). Euler's product still holds if the real part of s is greater than one. Riemann derived the functional equation of the Riemann zeta function:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

where the gamma function $\Gamma(s)$ is the well-known extension of the factorial function ($\Gamma(n+1) = n!$ for non-negative integers n):

$$\Gamma(s) = \int_0^{\infty} e^{-u} u^{s-1} du = \frac{1}{s e^{\gamma s}} \prod_{n=1}^{\infty} \frac{e^{s/n}}{1 + s/n}$$

(Here the integral form holds if the real part of s is greater than one, and the product form holds for all complex numbers s .)

The Riemann zeta function has the **trivial zeros** at $-2, -4, -6, \dots$ (the poles of $\Gamma(s/2)$). Using the Euler product (with the functional equation) it is easy to show that all the other zeros are in the **critical strip** of non-real complex numbers with $0 \leq \text{Re}(s) \leq 1$, and that they are symmetric about the **critical line** $\text{Re}(s)=1/2$. The unproved **Riemann hypothesis** is that all of the nontrivial zeros are actually on the critical line.

In 1986 it was shown that the first 1,500,000,001 nontrivial zeros of the Riemann zeta function do indeed have real part one-half [VTW86]. Hardy proved in 1915 that an infinite number of the zeros do occur on the critical line and in 1989 Conrey showed that over 40% of the zeros in the critical strip are on the critical line [Conrey89]. However, there is still a chance that the Riemann hypothesis is false. From August of 2001 through 2005, Sebastian Wedeniwski ran ZetaGrid which verified that the first 100 billion zeros were on the critical line.

2. Who cares?

In 1900 Hilbert listed proving or disproving this hypothesis as one of the most important unsolved problems confronting modern mathematics and it is central to understanding the overall distribution of the primes. When Hadamard and de la Vallee Poussin proved the [prime number theorem](#), they actually showed

$$\pi(x) = \text{Li}(x) + O(xe^{-a\sqrt{\ln x}})$$

for some positive constant a , and they did this by bounding the real part of the zeros in the critical strip away from 0 and 1. The error term is directly dependent on what was known about the zero-free region within the critical strip. As our knowledge of the size of this region increases, the error term decreases. In fact, in 1901 von Koch showed that the Riemann hypothesis is equivalent to

$$\pi(x) = \text{Li}(x) + O(x^{1/2} \ln x)$$

There are many results like this, see, for example [BS96].

Generalizations of RH

Recall again our starting point from Euler:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Why should the numerators all be one? One important way to alter the series is to replace the numerators with functions $\chi(n)$ called Dirichlet characters (these can be viewed as functions for which there exists a positive integer k with $\chi(n+k) = \chi(n)$ for all n , and with $\chi(n) = 0$ whenever $\gcd(n, k) > 1$). The resulting infinite sum $L(\chi, s)$ is a Dirichlet L-function. Once again we analytically continue the function to one that is meromorphic on the entire complex plane. The **extended Riemann Hypothesis** is that for every Dirichlet character χ and the zeros $L(\chi, s) = 0$ with $0 \leq \text{Re}(s) \leq 1$, have real part $1/2$. The distributions of the zeros of these L-functions are closely related to the number of primes in arithmetic progressions with a fixed difference k . Should the extended Riemann Hypothesis be proven, then [Miller's test](#) would provide an efficient primality proof for general numbers. See, for example, [BS96 8.5-6].

Another way to generalize Euler's sum is to leave the field of rational numbers, and replace the denominators with the norms of the non-zero ideals (special sets of elements) in a finite field extension of the rationals K (called a number field). The resulting sum is the Dedekind zeta-function of K and can again be analytically continued. These zeta functions also have a simple pole at zero and infinitely many zero in the critical region. The **generalized Riemann Hypothesis** is again that the zeros in the critical region all have real part $1/2$. See, for example, [BS96 8.7].

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