Problem 4: Manhole covers

(2/3 points)

Manhole explosions (usually caused by gas leaks and sparks) are on the rise in your city. On any given day, the manhole cover near your house explodes with some unknown probability, which is the same across all days. We model this unknown probability of explosion as a random variable Q, which is uniformly distributed between 0 and 0.1. Let X_i be a Bernoulli random variable that indicates whether the manhole cover near your house explodes on day i (where today is day 1).

Give numerical answers for parts (1) and (2).

1. $\mathbf{E}[X_i] =$ 0.05 ✔ **Answer:** 0.05

2. $\textbf{var}(X_i) =$ 0.0475 ✔ **Answer:** 0.0475

3. Let A be the event that the manhole cover did not explode yesterday (i.e., $X_0 = 0$). Find the conditional PDF of Q given A. Express your answer in terms of q using standard notation .

For $0 \leq q \leq 0.1$, $f_{Q|A}(q) =$ 10/(1-5*q^2) ✖ **Answer:** 200*(1-q)/19

Answer:

1. Using the law of iterated expectations, we have

$$\mathbf{E}[X_i] = \mathbf{E}\left[\mathbf{E}[X_i \mid Q]\right] = \mathbf{E}[Q] = 0.05.$$

2. Using the law of total variance, we have

$$\begin{aligned}
\mathrm{var}(X_i) &= \mathbf{E}[\mathrm{var}(X_i \mid Q)] + \mathrm{var}\left(\mathbf{E}[X_i \mid Q]\right) \\
&= \mathbf{E}[Q(1-Q)] + \mathrm{var}(Q) \\
&= \mathbf{E}[Q] - \mathbf{E}[Q^2] + \mathrm{var}(Q) \\
&= \mathbf{E}[Q] - \left(\mathrm{var}(Q) + (\mathbf{E}[Q])^2\right) + \mathrm{var}(Q) \\
&= \mathbf{E}[Q] - (\mathbf{E}[Q])^2 \\
&= 0.05 - 0.05^2 \\
&= 0.0475.
\end{aligned}$$

3. Using Bayes' rule, we have for $0 \leq q \leq 0.1$,

$$f_{Q|A}(q) = \frac{f_Q(q)\mathbf{P}(A \mid Q = q)}{\mathbf{P}(A)}$$

$$
\begin{aligned}
&= \frac{f_Q(q)\mathbf{P}(A \mid Q=q)}{\int_0^{0.1} f_Q(q)\mathbf{P}(A \mid Q=q)\,dq} \\
&= \frac{10(1-q)}{\int_0^{0.1} 10(1-q)\,dq} \\
&= \frac{10(1-q)}{1-0.05} \\
&= \frac{200(1-q)}{19}.
\end{aligned}
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You have used 4 of 4 submissions