Famous Theorems of Mathematics/Algebra/Linear Transformations

Lemma for the eigenspace

All eigenvectors of the linear transformation A that correspond to the eigenvalue λ form a subspace $L^{(\lambda)}$ in L.

Proof by Shilov (1969)

In fact, if $Ax_1 = \lambda x_1$, and $Ax_2 = \lambda x_2$, then

$$A(\alpha x_1 + \beta x_2) = \alpha Ax_1 + \beta Ax_2 = \alpha\lambda x_1 + \beta\lambda x_2 = \lambda\ (\alpha x_1 + \beta x_2)$$

with which the statement in the lemma is proven.

Lemma for linear independence of eigenvectors

Eigenvectors x_1, x_2, ... , x_n of the (linear) transformation A with respective pairwise distinct eigenvalues λ_1, λ_2, ... , λ_n, are linearly independent.

Proof by Shilov (1969)

This statement is proved by induction to number n. It is obvious that for $n = 1$ the lemma is true. Suppose that the lemma is true for all $n - 1$ eigenvalues of the transformation A; it remains to show that it is true for all n eigenvectors of the transformation A. Suppose a linear combination of n eigenvectors of the transformation A is 0:

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0.$$

Applying transformation A to this identity, one has

$$\alpha_1\lambda_1 x_1 + \alpha_2\lambda_2 x_2 + \ldots + \alpha_n\lambda_n x_n = 0.$$

Multiply the first equation by λ_n and subtract from the second one; one obtains

$$\alpha_1(\lambda_1 - \lambda_n)x_1 + \alpha_2(\lambda_2 - \lambda_n)x_2 + \ldots + \alpha_{n-1}(\lambda_{n-1} - \lambda_n)x_{n-1} = 0,$$

from where by induction all coefficients must be zero. Distinct eigenvalues have nonzero difference, so each $\alpha_i = 0$ for $i < n$; the first equation reduces to

$$\alpha_n x_n = 0$$

which means $\alpha_n = 0$, too. Consequently, all coefficients α_i are 0. Therefore, the vectors x_1, x_2, ..., x_n are linearly independent.