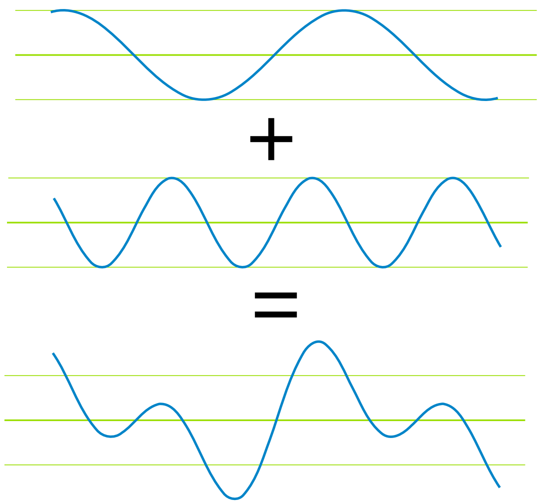
Fourier Series

Sine and cosine waves can make other functions!

Here two different sine waves add together to make a new wave:



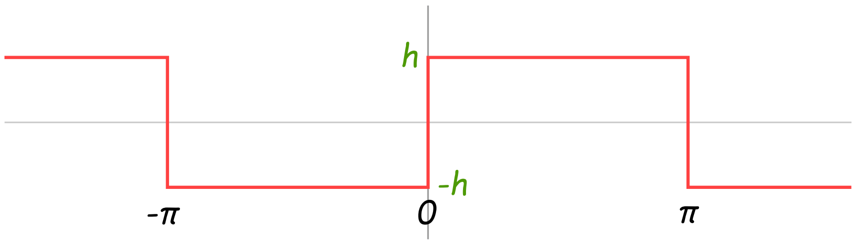
Try "sin(x)+sin(2x)" at the function grapher.

(You can also **hear it** at Sound Beats.)

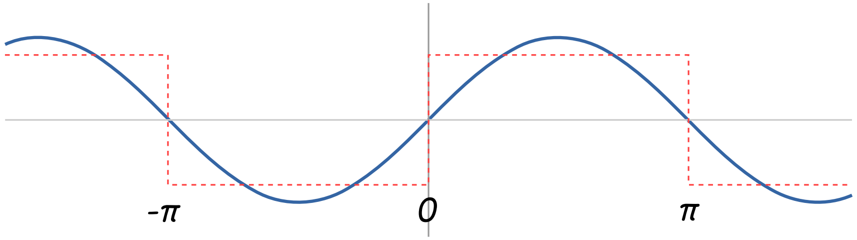
Square Wave

Can we use sine waves to make a **square wave**?

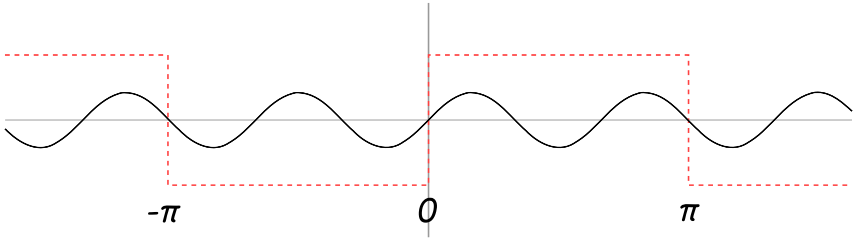
Our target is this square wave:



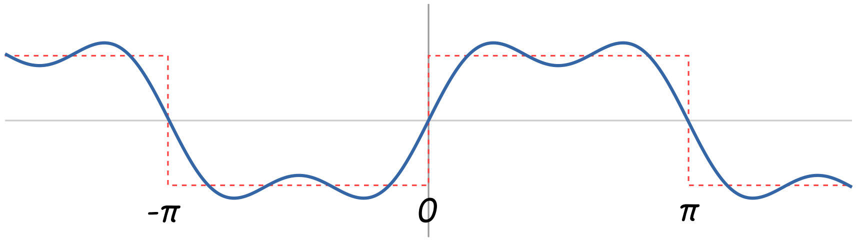
Start with **sin(x)**:



Then take **sin(3x)/3**:

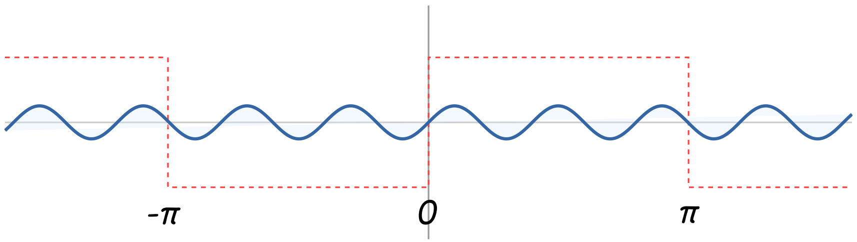


And add it to make **sin(x)+sin(3x)/3**:

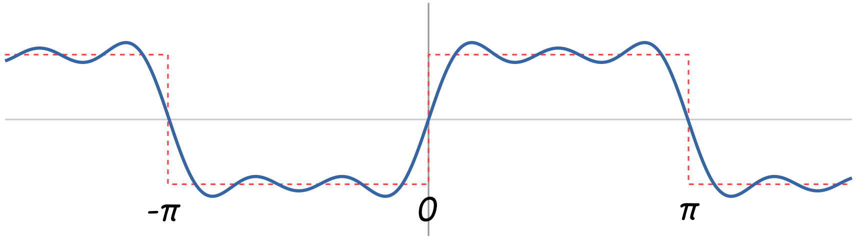


Can you see how it starts to look a little like a square wave?

Now take **sin(5x)/5**:

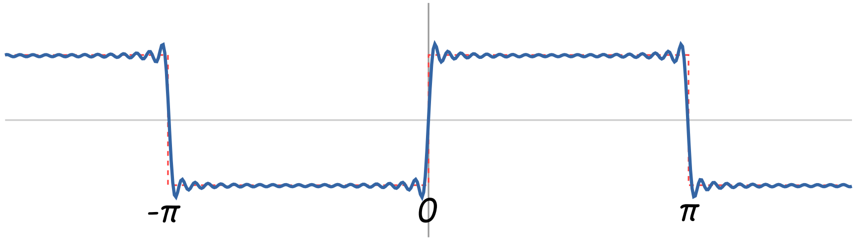


Add it also, to make **sin(x)+sin(3x)/3+sin(5x)/5**:

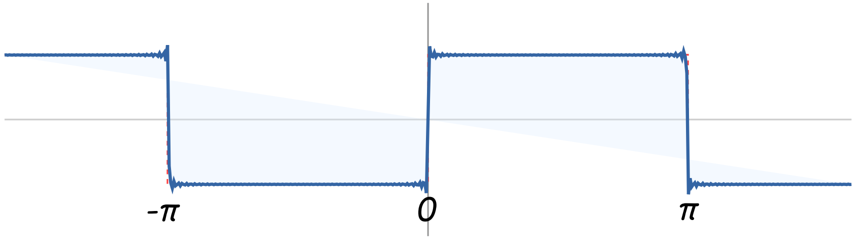


Getting better! Let's add a lot more sine waves.

Using 20 sine waves we get **sin(x)+sin(3x)/3+sin(5x)/5 + ... + sin(39x)/39**:



Using 100 sine waves we get **sin(x)+sin(3x)/3+sin(5x)/5 + ... + sin(199x)/199**:



And if we could add infinite sine waves in that pattern we would **have** a square wave!

So we can say that:

a square wave = sin(x) + sin(3x)/3 + sin(5x)/5 + ... (infinitely)

That is the idea of a Fourier series.

By adding infinite sine (and or cosine) waves we can make other functions, even if they are a bit weird.

You might like to have a little play with:

The Fourier Series Grapher

And it is also fun to use Spiral Artist and see how circles make waves.

They are designed to be experimented with, so play around and get a feel for the subject.

Finding the Coefficients

How did we know to use sin(3x)/3, sin(5x)/5, etc?

There are formulas!

First let us write down a full series of sines and cosines, with a name for all coefficients:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx\,\pi/L) + \sum_{n=1}^{\infty} b_n \sin(nx\,\pi/L)$$

Where:

- f(x) is the function we want (such as a square wave)
- L is **half of the period** of the function
- a_0, a_n and b_n are **coefficients** that we need to calculate!

What does $\sum_{n=1}^{\infty} a_n \cos(nx\,\pi/L)$ mean?

It uses Sigma Notation to mean **sum** up the series of values starting at n=1:

- $a_1 \cos(1x\,\pi/L)$
- $a_2 \cos(2x\,\pi/L)$

- etc

We do not (yet) know the values of a_1, a_2 etc.

All we need are the coefficients a_0, a_n and b_n. And these are the formulas:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(nx\, \pi/L)\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(nx\, \pi/L)\, dx$$

What does $\int_{-L}^{L}$ **f(x) sin(nx π/L) dx** mean?

It is an integral, but in practice it just means to find the **net area** of

f(x) sin(nx π/L)

between −L and L

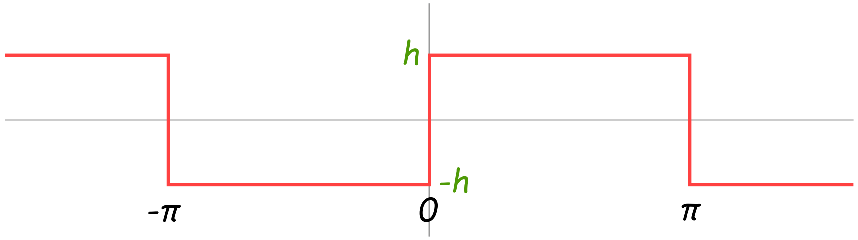
We can often find that area just by sketching and using basic calculations, but other times we may need to use the Integration Rules.

So this is what we do:

- Take our **target function, multiply it by sine** (or cosine) and **integrate** (find the area)
- Do that for n=0, n=1, etc to calculate each coefficient
- And after we calculate all coefficients, we put them into the series formula above.

Each step is not that hard, but it does take a long time to do! But once you know how, it becomes fairly routine.

Example: This Square Wave:

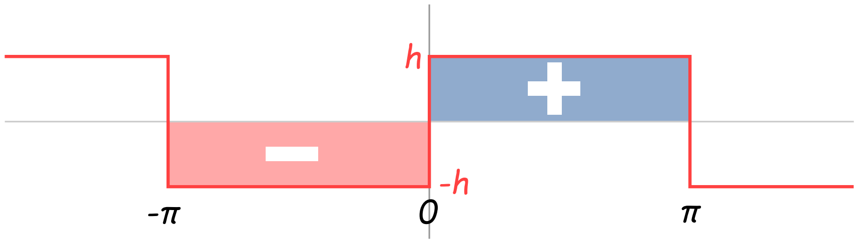


- **L =** π *(the Period is 2π)*
- The square wave is from **−h** to **+h**

Now our job is to calculate $\mathbf{a_0}$, $\mathbf{a_n}$ and $\mathbf{b_n}$

$\mathbf{a_0}$ is the net area between −L and L, then divided by 2L. It is basically an **average** of f(x) in that range.

Looking at this sketch:



The net area of the square wave from −L to L is **zero**.

So we know that:

$$a_0 = 0$$

For $\mathbf{a_1}$ we know that n=1 and L=π, so:

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(1x\ \pi/\pi)\ dx$$

Which simplifies to:

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(x)\ dx$$

Now, because the square wave changes abruptly at x=0 we need to break the calculation into **−π to 0** and **0 to** π,

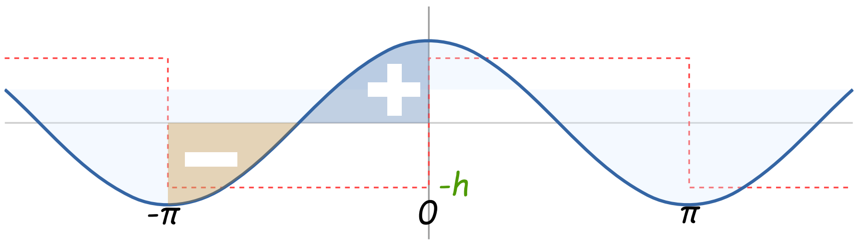
From −π to 0 we know f(x) is simply equal to **−h**:

$$\frac{1}{\pi} \int_{-\pi}^{0} -h \cos(x)\ dx$$

We can move the constant **−h** outside the integral:

$$\frac{-h}{\pi} \int_{-\pi}^{0} \cos(x)\ dx$$

Let's sketch **cos(x)**:

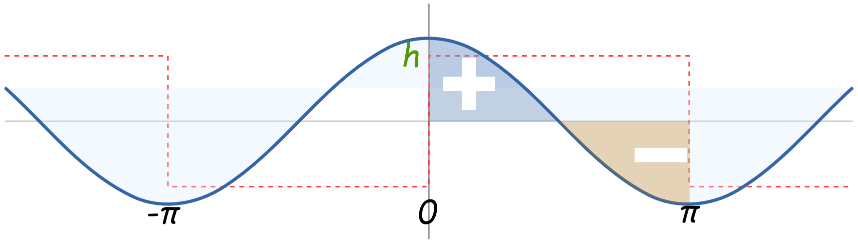


The net area of cos(x) from -π to 0 is **zero**.

So the net area must be 0:

$$\frac{-h}{\pi} \int_{-\pi}^{0} \cos(x)\, dx = \mathbf{0}$$

The same idea applies from **0 to** π,



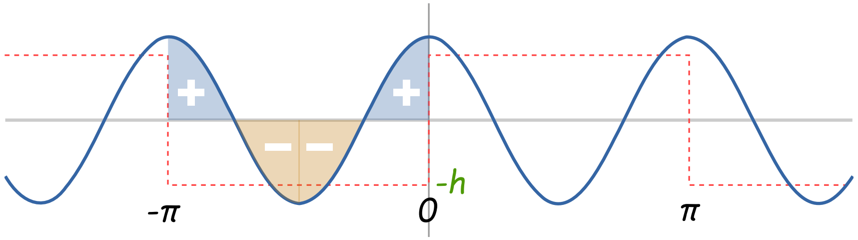
The net area of cos(x) from 0 to π is **zero**.

and so we can conclude that:

$$a_1 = 0$$

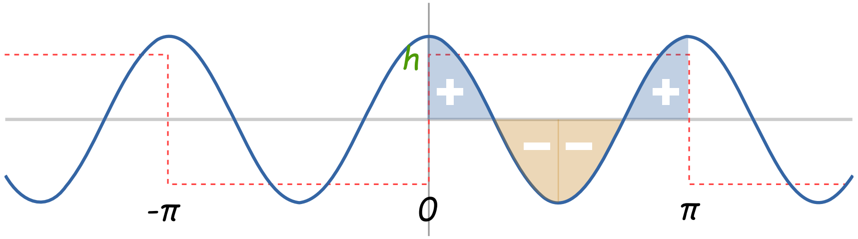
Now let us look at a_2

Aaaand ... the same thing happens!



The net area of cos(2x) from **-π to 0** is **zero**.

And:



The net area of cos(2x) from **0 to** π is also **zero**.

So we know that:

$$a_2 = 0$$

In fact we can extend this idea to every value of **a** and conclude that:

$$a_n = 0$$

So far there has been no need for any major calculations! A few sketches and a little thought have been enough.

*But now on to the **sine** function!*

For $\mathbf{b_1}$ we know that n=1 and L=π, so:

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(1x\ \pi/\pi)\, dx$$

Which simplifies to:

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(x)\, dx$$

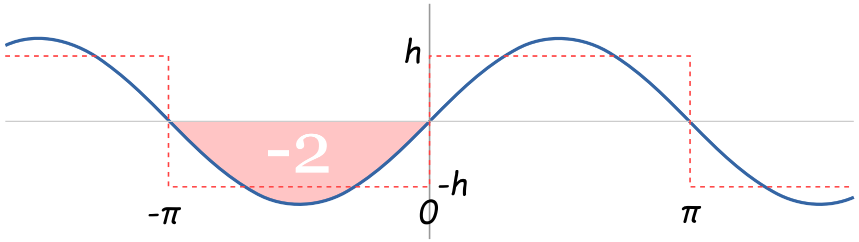
and as before, because of the abrupt change at x=0 we need to break the calculation into **−π to 0** and **0 to** π, So, just looking at the integral **from −π to 0**, we know f(x) = −h:

$$\frac{1}{\pi}\int_{-\pi}^{0} -h \sin(x)\, dx$$

We can move the constant **−h** outside the integral:

$$\frac{-h}{\pi}\int_{-\pi}^{0} \sin(x)\, dx$$

And **sin(x)** looks like this:



How do we know the area is −2?

First we use [Integration Rules] to find the integral of **sin(x)** is **−cos(x)**:

Then we calculate the [definite integral] between −π and 0 by calculating the value of −cos(x) for **0** and for **−π** and subtracting, like this:

$$[-\cos(0)] - [-\cos(-\pi)] = -1 - 1 = -2$$

So, between −π and 0 we get

$$\frac{-h}{\pi}(-2)$$

Next we look at the integral from **0 to** π:

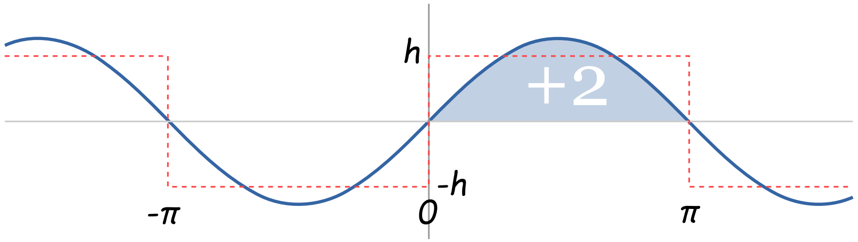
$$\frac{1}{\pi}\int_{0}^{\pi} h \sin(x)\, dx$$

Putting **h** outside the integral:

$$\frac{h}{\pi}\int_{0}^{\pi} \sin(x)\, dx$$

And its integral is:

$$[-\cos(\pi)] - [-\cos(0)] = 1 - [-1] = 2$$



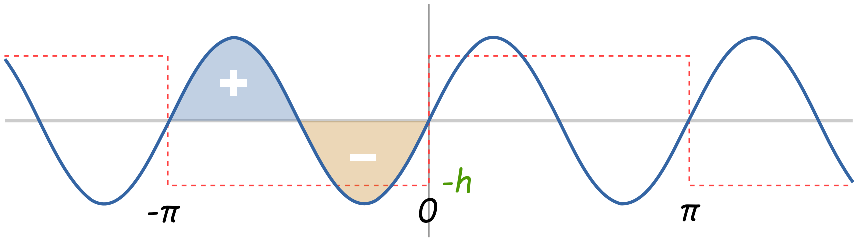
Now, combining both sides we get:

$$b_1 = \frac{1}{\pi}[\ (-h) \times (-2) + (h) \times (2)\ ] = \frac{4h}{\pi}$$

For $\mathbf{b_2}$ we have this integral:

$$\frac{-h}{\pi}\int_{-\pi}^{0} \sin(2x)\, dx$$

Which looks like:



The net area of sin(2x) from **−π to 0** is **zero**.

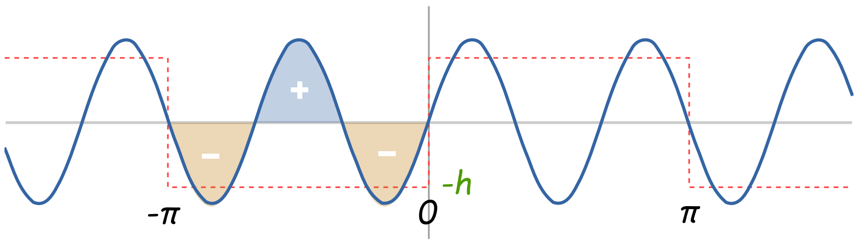
And we have seen this kind of thing before, so we conclude that:

$$b_2 = 0$$

For $\mathbf{b_3}$ we have this integral:

$$\frac{-h}{\pi} \int_{-\pi}^{0} \sin(3x)\,dx$$

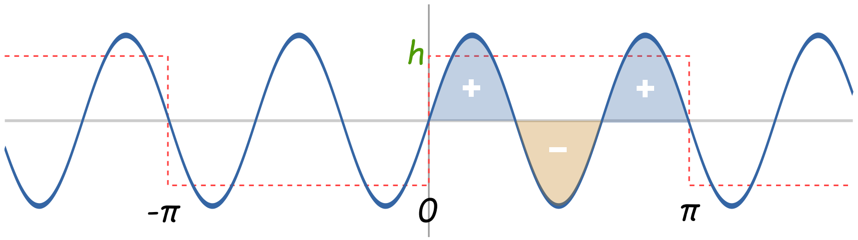
And we get this interesting situation:



Two areas cancel, but the third one is important!

So it is like the b_1 integral, but with only one-third of the area.

For **0 to** π we have:

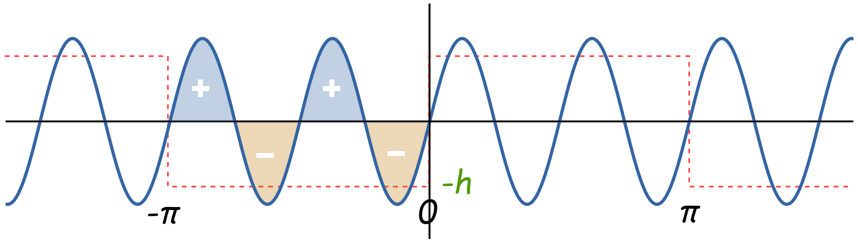


Again two areas cancel, but not the third

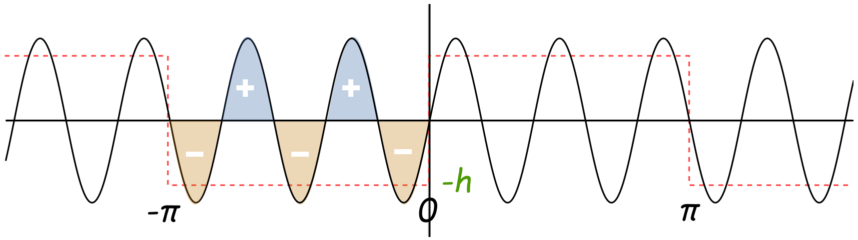
And we can conclude:

$$b_3 = \frac{b_1}{3} = \frac{4h}{3\pi}$$

The pattern continues:



When n is even the areas cancel for a result of zero.



When n is odd, all except one area cancel for a result of 1/n.

So we can say

$b_n = \frac{4h}{n\pi}$ when n is odd, but **0** otherwise

And we arrive at our last step: putting the coefficients into the master formula:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx\,\pi/L) + \sum_{n=1}^{\infty} b_n \sin(nx\,\pi/L)$$

- $a_0 = 0$
- all $a_n = 0$,
- $b_n = 0$ when n is even
- $b_n = \frac{4h}{n\pi}$ when n is odd, so:

$$f(x) = \frac{4h}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \ldots \right)$$

In conclusion:

- Think about each coefficient, sketch the functions and see if you can find a pattern,
- put it all together into the series formula at the end

And when you are done go over to:

The Fourier Series Grapher

and see if you got it right!

Why not try it with "sin((2n-1)*x)/(2n-1)", the 2n-1 neatly gives odd values, and see if you get a square wave.

Other Functions

Of course we can use this for many other functions!

But we must be able to work out all the coefficients, which in practice means that we work out the **area** of:

- the function
- the function times sine
- the function times cosine

But as we saw above we can use tricks like breaking the function into pieces, using common sense, geometry and calculus to help us.

Here are a few well known ones:

Wave	Series	Fourier Series Grapher
Square Wave	sin(x) + sin(3x)/3 + sin(5x)/5 + ...	sin((2n-1)*x)/(2n-1)
Sawtooth	sin(x) + sin(2x)/2 + sin(3x)/3 + ...	sin(n*x)/n
Pulse	sin(x) + sin(2x) + sin(3x) + ...	sin(n*x)*0.1
Triangle	sin(x) − sin(3x)/9 + sin(5x)/25 − ...	sin((2n-1)*x)*(-1)^n/(2n-1)^2

Footnote. Different versions of the formula!

On this page we used the general formula:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx\,\pi/L) + \sum_{n=1}^{\infty} b_n \sin(nx\,\pi/L)$$

But when the function f(x) has a period from -π to π we can use a simplified version:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Or even this one, where a_0 is rolled into the first sum (now n=**0** to ∞):

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

But I prefer the one we use here, as it is more practical allowing for different periods.