MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Bookmarks

- Unit 0: Overview
- Entrance Survey
- Unit 1: Probability models and axioms
- Unit 2: Conditioning and independence
- Unit 3: Counting
- **Unit 4: Discrete random variables**

Unit overview

Lec. 5: Probability mass functions and expectations
Exercises 5 due Mar 02, 2016 at 23:59 UTC

Unit 4: Discrete random variables > Problem Set 4 > Problem 1 Vertical: Tosses of a biased coin

Bookmark

Problem 1: Tosses of a biased coin

(9/9 points)

Consider 10 independent tosses of a biased coin with the probability of Heads at each toss equal to p, where $0 < p < 1$.

1. Let A be the event that there are 6 Heads in the first 8 tosses. Let B be the event that the 9th toss results in Heads.

   Find $\mathbf{P}(B \mid A)$ and express it in terms of p using standard notation .

   p

   Answer: p

2. Find the probability that there are 3 Heads in the first 4 tosses and 2 Heads in the last 3 tosses. Express your answer in terms of p using standard notation . Remember not to use factorials or combinations in your answer.

   12*p^5*(1-p)^2

   Answer: 12*p^5*(1-p)^2

3. Given that there were 4 Heads in the first 7 tosses, find the probability that the 2nd Heads occurred at the 4th toss. Give a numerical answer.

9/35 **Answer:** 0.25714

4. We are interested in calculating the probability that there are 5 Heads in the first 8 tosses and 3 Heads in the last 5 tosses. Give the numerical values of a, b, c, d, e, and f that would match the answer $ap^7(1-p)^3 + bp^c(1-p)^d + ep^f(1-p)^f$.

$a =$ 15 **Answer:** 15

$b =$ 60 **Answer:** 60

$c =$ 6 **Answer:** 6

$d =$ 4 **Answer:** 4

$e =$ 10 **Answer:** 10

$f =$ 5 **Answer:** 5

Answer:

1. Event A refers to the first 8 tosses and event B refers to the 9th toss. Since tosses are independent, the 9th toss is independent of the first 8 tosses, and so events A and B are independent. Thus, $\mathbf{P}(B \mid A) = \mathbf{P}(B) = p$.

2. Let C be the event "3 Heads in the first 4 tosses" and let D be the event "2 Heads in the last 3 tosses". Since there is no overlap in the tosses involved in events C and D, these two events are independent. Therefore,

$$\begin{aligned}\mathbf{P}(C \cap D) &= \mathbf{P}(C)\mathbf{P}(D) \\ &= \binom{4}{3}p^3(1-p) \cdot \binom{3}{2}p^2(1-p) \\ &= 12p^5(1-p)^2.\end{aligned}$$

3. Let E be the event “4 Heads in the first 7 tosses" and let F be the event “2nd Heads occurred on the 4th toss". We are asked to find $\mathbf{P}(F \mid E) = \mathbf{P}(F \cap E)/\mathbf{P}(E)$.

   The event $F \cap E$ occurs if there is 1 Heads in the first 3 tosses, Heads on the 4th toss, and 2 Heads in the next 3 tosses. Thus, we have

$$\begin{aligned}\mathbf{P}(F \mid E) &= \frac{\mathbf{P}(F \cap E)}{\mathbf{P}(E)} \\ &= \frac{\binom{3}{1}p(1-p)^2 \cdot p \cdot \binom{3}{2}p^2(1-p)}{\binom{7}{4}p^4(1-p)^3} \\ &= \frac{\binom{3}{1} \cdot 1 \cdot \binom{3}{2}}{\binom{7}{4}} \\ &= \frac{9}{35}.\end{aligned}$$

   Alternatively, we can solve this problem by counting. We are given that 4 Heads occurred in the first 7 tosses. Each sequence of 7 tosses with 4 Heads is equally probable, and so the discrete uniform probability law can be used here. There are $\binom{7}{4}$ elements in E. For the event $E \cap F$, there are $\binom{3}{1}$ ways to arrange 1 Heads in the first 3 tosses, 1 way to arrange the 2nd Heads in the 4th toss, and $\binom{3}{2}$ ways to arrange 2 Heads in the next 3 tosses. Therefore,

$$\mathbf{P}(F \mid E) = \frac{\binom{3}{1} \cdot 1 \cdot \binom{3}{2}}{\binom{7}{4}} = \frac{9}{35}.$$

4. Let G be the event “5 Heads in the first 8 tosses" and let H be the event “3 Heads in the last 5 tosses". These two events are not independent as there is some overlap in the tosses (the 6th, 7th, and 8th tosses). To compute the probability of interest, we partition the set $G \cap H$ into three (disjoint) subsets by considering separately the possible numbers of Heads in tosses 6 through 8:

$$\begin{aligned} G \cap H = \; & \{\text{2 Heads in tosses 1-5, 3 Heads in tosses 6-8, 0 Heads in tosses 9-10}\} \\ \cup \; & \{\text{3 Heads in tosses 1-5, 2 Heads in tosses 6-8, 1 Heads in tosses 9-10}\} \\ \cup \; & \{\text{4 Heads in tosses 1-5, 1 Heads in tosses 6-8, 2 Heads in tosses 9-10}\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{P}(G \cap H) \; &= \binom{5}{2} p^2 (1-p)^3 \cdot \binom{3}{3} p^3 \cdot \binom{2}{0} (1-p)^2 \\ &\quad + \binom{5}{3} p^3 (1-p)^2 \cdot \binom{3}{2} p^2 (1-p) \cdot \binom{2}{1} p(1-p) \end{aligned}$$

$$+ \binom{5}{4} p^4 (1-p) \cdot \binom{3}{1} p (1-p)^2 \cdot \binom{2}{2} p^2$$
$$= 15p^7(1-p)^3 + 60p^6(1-p)^4 + 10p^5(1-p)^5.$$

You have used 2 of 3 submissions

Printable problem set available here .

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