3. How does the Fourier series really work?

The square wave of height $\pi/4$, which we denote by $S(t) = \frac{\pi}{4}\mathrm{Sq}(t)$ has Fourier series

$$\begin{aligned} S(t) &= \frac{\pi}{4}\mathrm{Sq}(t) = \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots \\ &= \sum_{n \text{ odd}}^{\infty} \frac{1}{n}\sin nt. \end{aligned}$$

Note that the discontinuities are at the integer multiples of π.

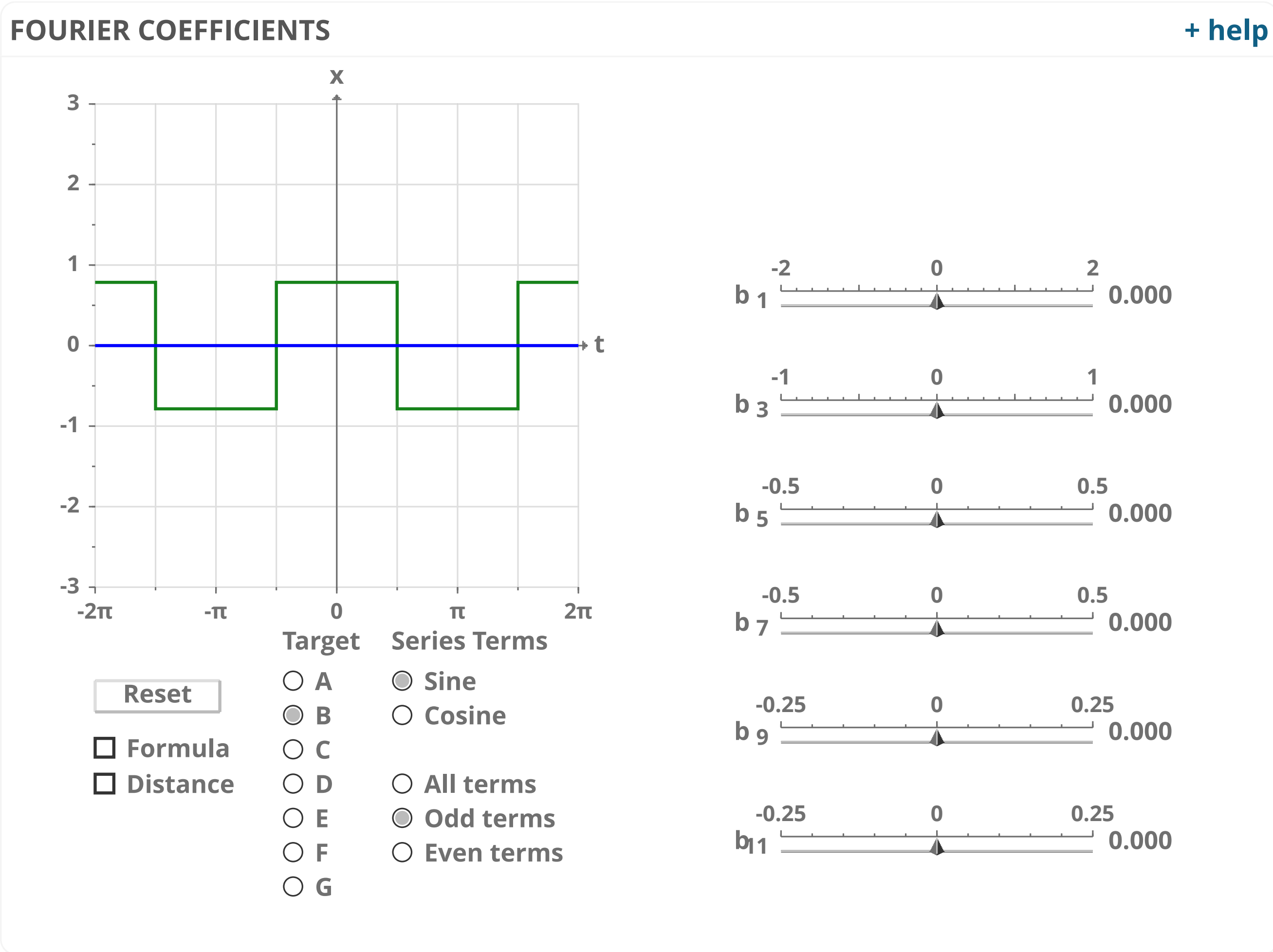
You can visualize the sum of the first few terms of this Fourier series using the mathlet below. The partial sum

$$b_1 \sin t + \cdots + b_{11} \sin 11t$$

is displayed in blue. (Initially these coefficients are all zero, so the display shows a horizontal blue line.)

- Choose the radio button for **Target A** to see the graph of $S(t)$.
- We know that the function is odd, so all the $a_n = 0$. Thus we represent this as a **Sine series** .
- We know that the coefficients b_n involve only odd terms, so we can also select the radio button **Odd terms** .

Use the sliders to select the coefficients b_n for $S(t)$. Observe how each successive b_n creates a better and better approximation to $S(t)$.



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How to understand partial sums of the Fourier series

Very roughly, the idea of the Fourier series of a function, such as $S(t)$, is that it is the closest possible sum of sines that approximate the function on the interval, and at each degree, it is the best.

In what sense is it the closest?

We might want to say that it is the closest possible at every point. This is not possible since at points of discontinuity this automatically fails. The next thought is that we might want to say that it is closest in that the area between the function and the sine term of frequency n is as small as possible. This is more forgiving at discontinuities. Remarkably, there is a correct notion of closest, and it is the integral of the difference of the functions quantity squared that is minimized.

$$\int (f(x) - g(x))^2 \, dx$$

(This is called the **root-mean-squared distance** between two functions.)

Remark 3.1 Extending our analogy with vectors used in the intoduction of scalar product, this new distance is (almost) the same as the usual distance for vectors: $|a - b| = \sqrt{\langle a - b, a - b \rangle}$.

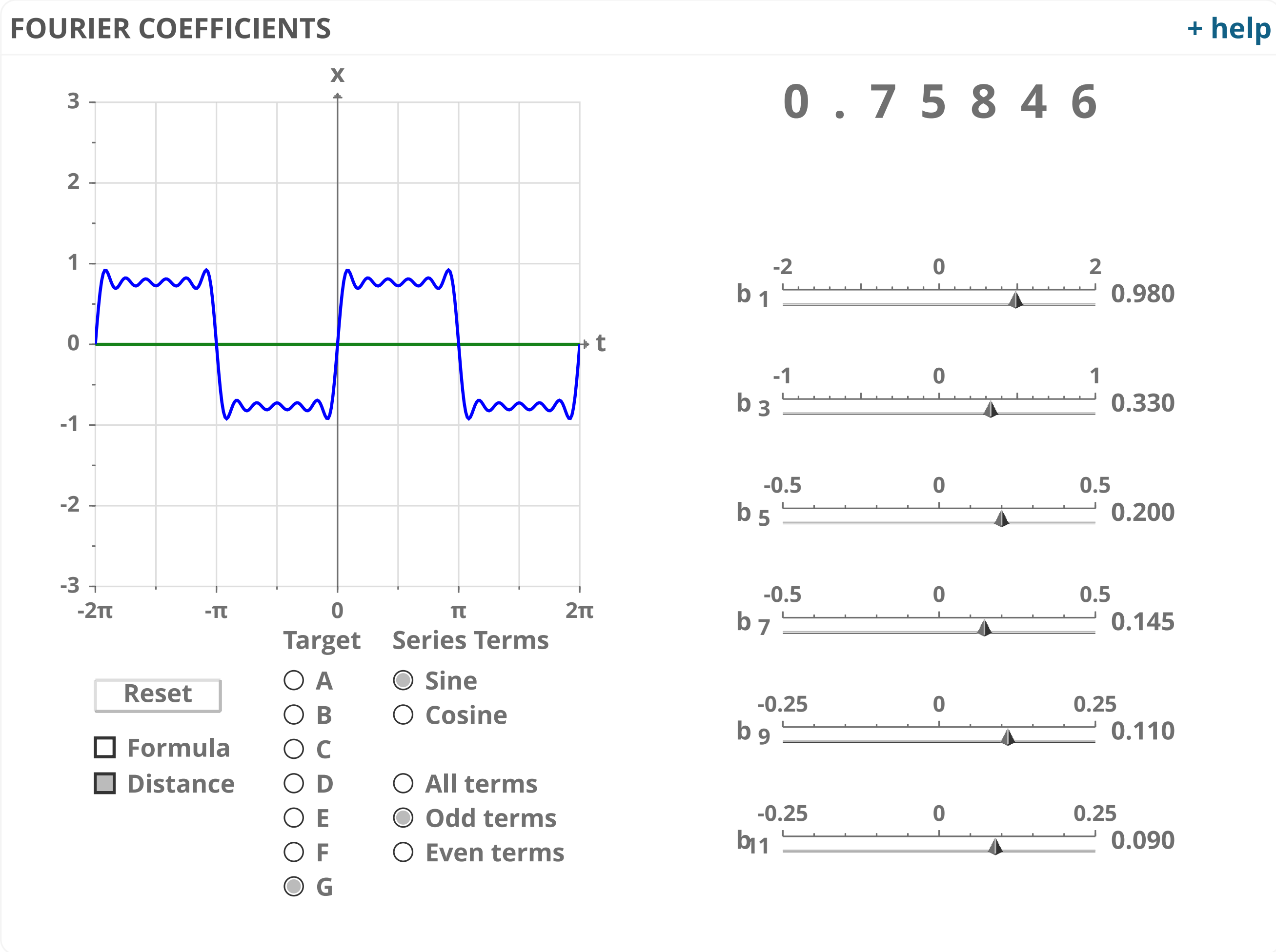
Minimize the distance

6/6 points (graded)

Again choose **Target A**, and choose the **Odd terms** radio button for the sine series. Click the **Distance** check button. This button displays the root-mean-squared distance between the function $S(t)$ and the partial sum $b_1 \sin(t) + b_3 \sin(3t) + \cdots + b_{11} \sin(11t)$ in the sense described above.

In this problem, find the coefficients b_1, b_3, ... , b_{11} by moving the sliders until the **Distance** displayed at the top is a small as possible for each slider. Record the values you find in the answer boxes below.

Hint: Hit tab to enter keyboard accessibility mode and you can tab through the sliders and use the arrow keys to move the sliders with more accuracy.



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$b_1 =$ 1 ✔ **Answer:** 1

$b_3 =$ 1/3 ✔ **Answer:** 0.330

$b_5 =$ 1/5 ✔ **Answer:** 0.200

$b_7 =$ 1/7 ✔ **Answer:** 0.145

$b_9 =$ 1/9 ✔ **Answer:** 0.111

$b_{11} =$ 1/11 ✔ **Answer:** 0.091

Solution:

The values that you find for the coefficients by minimizing the distance is exactly what we would expect (up to some rounding error). In this way, the finite sum of terms is finding the best approximation to the function $S(t)$ with each term. Observe that you can find these coefficients in any order. It turns out that the optimality of each coefficient is independent of all the others.

Note that as good as each approximation is, the approximation does not look very good at the point of discontinuity. What you see is an overshoot referred to as Gibb's phenomenon, which is discussed on the next page.

Submit You have used 1 of 150 attempts

Answers are displayed within the problem

3. How does the Fourier series really work?

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Unclear meaning
Hi! What is the meaning of the Fourier series being the best approximation at each degree? Thanks! 3

Found some typos 2

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Could the due date be extended please? Still finishing the N xN course. 5

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