3. Solving ODEs with periodic input

1-3

1/1 point (graded)

Consider the system

$$\ddot{x} + 0.1\omega_0 \dot{x} + \omega_0^2 x = \omega_0^2 f(\omega_0 t).$$

with input $f(\omega_0 t)$ where $f(t)$ is the 2π-periodic sawtooth wave $f(t) = t$ for $-\pi < t < \pi$ of period 2π, which has Fourier series

$$f(t) = 2\left(\sin t - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \cdots\right).$$

The Fourier series representation of the steady state response takes the form

$$x(t) = A_1 \cos(\omega_1 t - \phi_1) + A_2 \cos(\omega_2 t - \phi_2) + \cdots.$$

Calculate the largest two amplitudes of the steady state solution exactly, and then approximate them to **exactly one significant digit** . Write the approximate solution as a sum of two sinusoidal waves. (You can check for yourself that the third largest amplitude is less than 1% of the largest amplitude.)

(Enter your answer in terms of t, ω_0 and ϕ_n (for any n) since the phase shifts are not readily determined.)

$x(t) \approx$ 20.0*cos(omega_0*t-phi_1)+0.33259505*cos(2*omega_0 ✔ **Answer:** 20*cos(omega_0*t-phi_1) - (0.3)*cos(2*omega_0*t-phi_2)

FORMULA INPUT HELP

Solution:

First note that

$$f(\omega_0 t) = 2\left(\sin \omega_0 t - \frac{\sin(2\omega_0 t)}{2} + \frac{\sin(3\omega_0 t)}{3} - \cdots\right),$$

thus the $\omega_n = \omega_0 n$ in the Fourier series of the steady state response. Thus to solve this problem, we next determine the steady state $x_{p,n}$ response for

$$\left(D^2 + 0.1\omega_0 D + \omega_0^2\right) x = \omega_0^2 \sin(\omega_0 n t)$$

for each n. Since $\omega_0^2 \sin(\omega_0 n t) = \mathrm{Im}\left(\omega_0^2 e^{i\omega_0 n t}\right)$,

$$x_{p,n} = \mathrm{Im}\left(\frac{\omega_0^2 e^{i\omega_0 n t}}{(i\omega_0 n)^2 + 0.1\omega_0(i\omega_0 n) + \omega_0^2}\right) = \mathrm{Im}\left(\frac{\omega_0^2 e^{i\omega_0 n t}}{\omega_0^2\left(1 - n^2 + 0.1ni\right)}\right) = \frac{1}{\sqrt{(1-n^2)^2 + 0.01n^2}} \cos(\omega_0 n t - \phi_n).$$

Therefore the steady state response is given by

$$x_p = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{1}{\sqrt{(1-n^2)^2 + 0.01n^2}} \cos(\omega_0 n t - \phi_n)\right).$$

The amplitude is proportional to $1/n^3$, therefore we expect the two largest terms to be the $n=1$ and $n=2$ term. Checking this we find

$$\begin{aligned} A_1 &= \frac{2}{\sqrt{0.01}} = 20 \\ A_2 &= \frac{1}{\sqrt{9+0.04}} = 0.3 \\ A_3 &= \frac{2}{3}\frac{1}{\sqrt{64+0.09}} = 0.08. \end{aligned}$$

The sum of the two largest terms is given by

$$20\cos\left(\omega_0 t - \phi_1\right) + 0.3\cos\left(\omega_0 2t - \phi_2\right).$$

Note that slighly more exploration shows that $\phi_1 = \pi$, so the answer

$$-20\cos\left(\omega_0 t\right) + 0.3\cos\left(\omega_0 2t - \phi_2\right)$$

is also accepted.

Furthermore, writing out the first few terms explicitly in terms of cosines and sines is also accepted (with no reference to the phase shifts).

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Answers are displayed within the problem

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Divided by $omega_0$.

discussion posted about a month ago by **supersimonx2000**

In my solution I rewrote the FS as being $2\sum \frac{sin(n\omega_0 t)}{n\omega_0}$. In my final solution I end up dividing my coefficient by ω_0 unnecessarily. Is there an error in my FS?

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3 responses

jfrench (Staff)
about a month ago

You are missing a negative sign on every other term!

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myyukiko
16 days ago

I also got a similar FS, $f\left(\omega_0 t\right) = 2\sum_{n\geq 1} \frac{(-1)^{n+1}}{\omega_0 n} \sin\left(n\omega_0 t\right)$, which lead me to an extra $1/\omega_0$ in the final solution. I read this comment and got the right answer, but still do not understand why my previous solution is not correct.

If we substitute directly $t \to \omega_0 t$ in the given FS, don't we miss some information about the period of the function?

Add a comment

Roger I
15 days ago

I agree. If $f\left(t\right)$ has period 2π then, setting $\omega_0 = 2$ (to make the point concrete) then doesn't $f\left(2t\right)$ have a period of π?

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