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Area of the triangle formed by circumcenter, incenter and orthocenter

Asked 5 years, 2 months ago Active 1 year, 6 months ago Viewed 3k times



Lets say we have $\triangle ABC$ having O, I, H as its circumcenter, incenter and orthocenter. How can I go on finding the area of the $\triangle HOI$.





I thought of doing the question using the distance (length) between HO,HI and OI and then using the Heron's formula, but that has made the calculation very much complicated. Is there any simple way to crack the problem?





geometry trigonometry

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edited Jun 16 '16 at 13:22



5 Answers

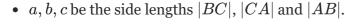




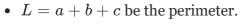
Given any triangle $\triangle ABC$, we will abuse notation and use the same letter to represent both a vertex and the angle at that vertex. Let













• c_X, s_X, t_X be $\cos X, \sin X, \tan X$ for any angle $X \in \{A, B, C\}$.



• R be the circumradius.

Let I, O, H be the incenter, circumcenter and orthocenter.

Their barycentric coordinates are given by

$$egin{array}{lcl} lpha_I:eta_I:\gamma_I&=&\sin A:\sin B:\sin C&=&t_Ac_A:t_Bc_B:t_Cc_C\ lpha_O:eta_O:\gamma_O&=&\sin 2A:\sin 2B:\sin 2C&=&2t_Ac_A^2:2t_Bc_B^2:2t_Cc_C^2\ lpha_H:eta_H:\gamma_H&=& an A: an B: an C&=&t_A:t_B:t_C \end{array}$$

Let A_0 and A be the area of $\triangle ABC$ and $\triangle IOH$, their ratio is given by

$$egin{aligned} rac{\mathcal{A}}{\mathcal{A}_0} = \left| \det egin{bmatrix} lpha_I & eta_I & \gamma_I \ lpha_O & eta_O & \gamma_O \ lpha_H & eta_H & \gamma_H \ \end{bmatrix}
ight| = rac{\mathcal{N}}{\delta_I \delta_O \delta_H} \end{aligned}$$

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$$egin{array}{lll} ext{where} & \left\{ egin{array}{lll} \delta_I & = \sin A + \sin B + \sin C \ \delta_O & = \sin 2A + \sin 2B + \sin 2C \ \delta_H & = an A + an B + an C \end{array}
ight. \end{array}$$

$$\mathcal{N} = \left| \det egin{bmatrix} t_A c_A & t_B c_B & t_C c_C \ 2t_A c_A^2 & 2t_B c_B^2 & 2t_C c_C^2 \ t_A & t_B & t_C \end{bmatrix}
ight| = 2t_A t_B t_C \left| \det egin{bmatrix} c_A & c_B & c_C \ c_A^2 & c_B^2 & c_C^2 \ 1 & 1 & 1 \end{bmatrix}
ight| = 2t_A t_B t_C |(c_A - c_B)(c_B - c_C)(c_C - c_A)|$$

Since $A + B + C = \pi$, $\delta_H = t_A + t_B + t_C = t_A t_B t_C$. Together with the relations,

$$\left\{egin{array}{ll} \mathcal{A}_0 &= rac{1}{2}R^2(\sin 2A + \sin 2B + \sin 2C) \ L &= 2R(\sin A + \sin B + \sin C) \end{array}
ight. \Longrightarrow \left\{egin{array}{ll} \delta_O &= rac{2\mathcal{A}_0}{R^2} \ \delta_I &= rac{L}{2R} \end{array}
ight.$$

We find

$$rac{\mathcal{A}}{\mathcal{A}_0} = rac{2}{\delta_I \delta_O} |(c_A - c_B)(c_B - c_C)(c_C - c_A)| = rac{2R^3}{\mathcal{A}_0 L} |(c_A - c_B)(c_B - c_C)(c_C - c_A)|$$

Notice

$$c_A - c_B = rac{-a^2 + b^2 + c^2}{2bc} - rac{a^2 - b^2 + c^2}{2ac} = rac{(b-a)L(L-2a)}{2abc}$$

and similar expressions for $c_B - c_C, c_C - c_A$, we find

$$rac{\mathcal{A}}{\mathcal{A}_0} = \left(rac{2R^3}{\mathcal{A}_0 L} \cdot rac{L^3 (L-2a)(L-2b)(L-2c)}{8a^3b^3c^3}
ight) |(a-b)(b-c)(c-a)|$$

Recall the <u>Heron's formula</u> and a beautiful relation between A_0 and R:

$$16A_0^2 = L(L-2a)(L-2b)(L-2c)$$
 and $4A_0R = abc$

What's inside the parenthesis above can be simplified as

$$rac{2R^3}{\mathcal{A}_0 L} \cdot rac{L^2 \cdot 16 {\mathcal{A}_0}^2}{8(4 \mathcal{A}_0 R)^3} = rac{L}{16 A_0^2} = rac{1}{(L-2a)(L-2b)(L-2c)}$$

This leads to a reasonably simple ratio one can use to compute the area A.

$$rac{\mathcal{A}}{\mathcal{A}_0} = rac{|(a-b)(b-c)(c-a)|}{(-a+b+c)(a-b+c)(a+b-c)}$$

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@JackD'Aurizio thanks for the compliment. - achille hui Jun 17 '16 at 0:52

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Hint:



$$AH = 2R\cos A, AI = 4R\sin\frac{B}{2}\sin\frac{C}{2}, OA = R, \angle OAH = 2\angle OAI = B - C$$

(1)

$$\triangle HOI = \triangle AHO - \triangle AHI - \triangle AIO$$

Final answer

$$2R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}$$

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edited Jun 12 '20 at 10:38









From the information of <u>incentre</u> and <u>Euler line</u>, we have:

3





$$egin{aligned} r &= rac{\Delta}{s} \ R &= rac{abc}{4\Delta} \ IH &= \sqrt{4R^2 + 2r^2 - rac{1}{2}(a^2 + b^2 + c^2)} \ IO &= \sqrt{R(R-2r)} \ OH &= 9R^2 - (a^2 + b^2 + c^2) \end{aligned}$$

It's do-able by using simple program in a computer. You can also find the area bound $(\Delta HOI = \frac{d}{2} \times OH)$ by knowing the <u>distance bound</u> between the incentre and the Euler line.

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edited Jun 16 '16 at 11:05

answered Jun 16 '16 at 10:59



Ng Chung Tak

7.3k

10 4

If you compute the exact barycentric coordinates of O, H, I, the ratio $\frac{[OHI]}{[ABC]}$ is given by a simple determinant. So, just exploit the first table on this page and perform some

computation.



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answered Jun 16 '16 at 12:48



Jack D'Aurizio

3331 38

43 //1



A área do triângulo HOI do triângulo ABC de lados $a=|BC|,\,b=|AC|$ e c=|AB| é dada por:

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$$AreaHOI = |rac{1}{4}. rac{egin{array}{c|cccc} a & a^3 & 1 \ b & b^3 & 1 \ c & c^3 & 1 \ \end{array}}{egin{array}{c|ccccc} 0 & a^2 & b^2 & 1 \ a^2 & 0 & c^2 & 1 \ b^2 & c^2 & 0 & 1 \ 1 & 1 & 1 & 0 \ \end{array}}|_{C}$$

Dedução: Seja o triângulo ABC, com o vértice A na origem do Sistema Cartesiano e C no semi eixo positivo x. Assim:

$$egin{aligned} A &= (0,0) \ B &= (rac{-a^2+b^2+c^2}{2b},rac{2S}{b}) \ C &= (b,0) \ S &= \sqrt{p(p-a)(p-b)(p-c)} \ p &= rac{a+b+c}{2} \end{aligned}$$

$$AreaHOI = 3.AreaGIO$$

$$AreaGIO = rac{1}{2}egin{bmatrix} xG & yG & 1 \ xI & yI & 1 \ xO & yO & 1 \ \end{bmatrix}$$

"Obtendo as coordenadas de G, I e O através de suas respectivas fórmulas e fazendo substituições obtemos:"

$$AreaHOI = 3.rac{1}{2}igg| egin{array}{cccc} rac{-a^2+3b^2+c^2}{6b} & rac{2S}{3b} & 1 \ rac{-a+b+c}{2} & rac{2S}{a+b+c} & 1 \ rac{b}{2} & rac{b(a^2-b^2+c^2)}{8S} & 1 \ \end{array}$$

"Desenvolvendo, substituindo S, p e fatorando obtemos:"

$$AreaHOI = \left| \frac{(a-b)(a-c)(b-c)(a+b+c)}{16S} \right|$$

"Substituindo as expressões por determinantes (S no formato de área de Cayley-Menger), obtemos:"

$$AreaHOI = |rac{1}{4}. rac{egin{array}{c|c} a & a^3 & 1 \ b & b^3 & 1 \ c & c^3 & 1 \ \hline 0 & a^2 & b^2 & 1 \ a^2 & 0 & c^2 & 1 \ b^2 & c^2 & 0 & 1 \ 1 & 1 & 1 & 0 \ \hline \end{pmatrix}$$

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