Exercise: Multiple observations and unknowns

(4/4 points)

Let Θ_1, Θ_2, W_1, and W_2 be independent standard normal random variables. We obtain two observations,

$$X_1 = \Theta_1 + W_1, \qquad X_2 = \Theta_1 + \Theta_2 + W_2.$$

Find the MAP estimate $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ of (Θ_1, Θ_2) if we observe that $X_1 = 1$, $X_2 = 3$. (You will have to solve a system of two linear equations.)

$\hat{\theta}_1 =$ 1 **Answer:** 1

$\hat{\theta}_2 =$ 1 **Answer:** 1

Answer:
As usual, we focus on the exponential term in the numerator of the expression given by Bayes' rule. The prior contributes a term of the form

$$e^{-\frac{1}{2}(\theta_1^2+\theta_2^2)}.$$

Conditioned on $(\Theta_1, \Theta_2) = (\theta_1, \theta_2)$, the measurements are independent. In the conditional universe, X_1 is normal with mean θ_1, X_2 is normal with mean $\theta_1 + \theta_2$, and both variances are 1. Thus, the term $f_{X_1,X_2|\Theta_1,\Theta_2}$ makes a contribution of the form

$$e^{-\frac{1}{2}(x_1-\theta_1)^2} \cdot e^{-\frac{1}{2}(x_2-\theta_1-\theta_2)^2}.$$

We substitute $x_1 = 1$ and $x_2 = 3$, and in order to find the MAP estimate, we minimize the expression

$$\frac{1}{2}\left(\theta_1^2 + \theta_2^2 + (\theta_1 - 1)^2 + (\theta_1 + \theta_2 - 3)^2\right).$$

Setting the derivatives (with respect to θ_1 and θ_2) to zero, we obtain:

$$\hat{\theta}_1 + (\hat{\theta}_1 - 1) + (\hat{\theta}_1 + \hat{\theta}_2 - 3) = 0, \qquad \hat{\theta}_2 + (\hat{\theta}_1 + \hat{\theta}_2 - 3) = 0,$$

or

$$3\hat{\theta}_1 + \hat{\theta}_2 = 4, \qquad \hat{\theta}_1 + 2\hat{\theta}_2 = 3.$$

Either by inspection, or by substitution, we obtain the solution $\hat{\theta}_1 = 1$, $\hat{\theta}_2 = 1$.

You have used 1 of 3 submissions