4.6 Poisson processes

Unit 4: Continuous Random Variables

Adapted from Blitzstein-Hwang Chapter 5.

The Exponential distribution is closely connected to the Poisson distribution, as suggested by our use of λ for the parameters of both distributions. In this section we will see that the Exponential and Poisson are linked by a common story, the *Poisson process*.

> **Definition 4.6.1 (Poisson process).**
>
> A process of arrivals in continuous time is called a *Poisson process* with rate λ if the following two conditions hold.
>
> 1. The number of arrivals that occur in an interval of length t is a $\text{Pois}(\lambda t)$ random variable.
> 2. The numbers of arrivals that occur in disjoint intervals are independent of each other. For example, the numbers of arrivals in the intervals $(0, 10)$, $[10, 12)$, and $[15, \infty)$ are independent.

A sketch of a Poisson process is pictured in Figure 4.6.2. Each X marks the spot of an arrival.

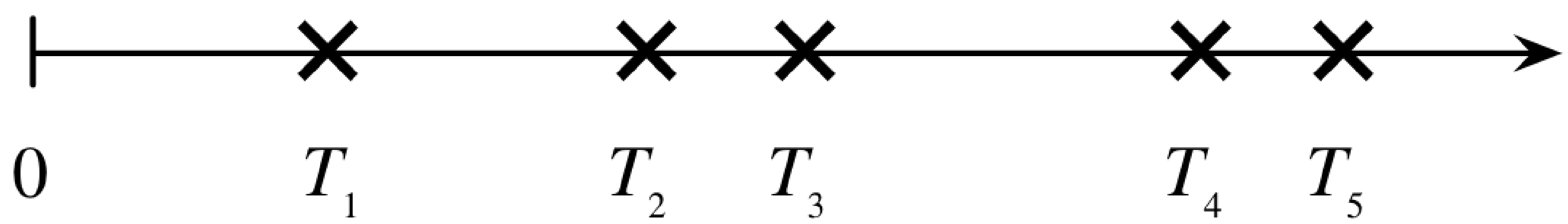

Figure 4.6.2: Poisson process.

View Larger Image
Image Description

For concreteness, suppose the arrivals are emails landing in an inbox according to a Poisson process with rate λ. There are several things we might want to know about this process. One question we could ask is: in one hour, *how many* emails will arrive? The answer comes directly from the definition, which tells us that the number of emails in an hour follows a $\text{Pois}(\lambda)$ distribution. Notice that the number of emails is a nonnegative integer, so a discrete distribution is appropriate.

But we could also flip the question around and ask: *how long* does it take until the first email arrives (measured relative to some fixed starting point)? The waiting time for the first email is a positive real number, so a continuous distribution on $(0, \infty)$ is appropriate. Let T_1 be the time until the first email arrives. To find the distribution of T_1, we just need to understand one crucial fact: saying that the waiting time for the first email is greater than t is the same as saying that *no emails* have arrived between 0 and t. In other words, if N_t is the number of emails that arrive at or before time t, then

$$T_1 > t \textbf{ is the same event as } N_t = 0.$$

We call this the *count-time duality* because it connects a discrete r.v., N_t, which *counts* the number of arrivals, with a continuous r.v., T_1, which marks the *time* of the first arrival.

If two events are the same, they have the same probability. Since $N_t \sim \textrm{Pois}(\lambda t)$ by the definition of Poisson process,

$$P(T_1 > t) = P(N_t = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}.$$

Therefore $P(T_1 \le t) = 1 - e^{-\lambda t}$, so $T_1 \sim \textrm{Expo}(\lambda)$! The time until the first arrival in a Poisson process of rate λ has an Exponential distribution with parameter λ.

What about $T_2 - T_1$, the time between the first and second arrivals? Since disjoint intervals in a Poisson process are independent by definition, the past is irrelevant once the first arrival occurs. Thus $T_2 - T_1$ is independent of the time until the first arrival, and by the same argument as before, $T_2 - T_1$ also has an Exponential distribution with rate λ. Similarly, $T_3 - T_2 \sim \textrm{Expo}(\lambda)$ independently of T_1 and $T_2 - T_1$. Continuing in this way, we deduce that all the interarrival times are i.i.d. $\textrm{Expo}(\lambda)$ random variables. To summarize what we've learned: in a Poisson process of rate λ,

- the number of arrivals in an interval of length 1 is $\textrm{Pois}(\lambda)$, and
- the times between arrivals are i.i.d. $\textrm{Expo}(\lambda)$.

Thus, Poisson processes tie together two important distributions, one discrete and one continuous, and the use of a common symbol λ for both the Poisson and Exponential parameters is felicitous notation, for λ is the arrival rate in the process that unites the two distributions.

The story of the Poisson process provides intuition for the fact that the minimum of independent Exponential r.v.s is another Exponential r.v.

Example 4.6.3 (Minimum of independent Expos).

Let $X_1, \dots, X_n$ be independent, with $X_j \sim \textrm{Expo}(\lambda_j)$. Let $L = \min(X_1, \dots, X_n)$. Show that $L \sim \textrm{Expo}(\lambda_1 + \cdots + \lambda_n)$, and interpret this intuitively.

Solution

We can find the distribution of L by considering its *survival function* $P(L > t)$, since the survival function is 1 minus the CDF.

$$\begin{aligned} P(L > t) &= P(\min(X_1, \ldots, X_n) > t) = P(X_1 > t, \ldots, X_n > t) \\ &= P(X_1 > t) \cdots P(X_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} = e^{-(\lambda_1 + \cdots + \lambda_n)t}. \end{aligned}$$

The second equality holds since saying that the minimum of the X_j is greater than t is the same as saying that all of the X_j are greater than t. The third equality holds by independence of the X_j. Thus, L has the survival function (and the CDF) of an Exponential distribution with parameter $\lambda_1 + \cdots + \lambda_n$.

Intuitively, we can interpret the λ_j as the rates of n independent Poisson processes. We can imagine, for example, X_1 as the waiting time for a green car to pass by, X_2 as the waiting time for a blue car to pass by, and so on, assigning a color to each X_j. Then L is the waiting time for a car with any of these colors to pass by, so it makes sense that L has a combined rate of $\lambda_1 + \cdots + \lambda_n$.