MTH 515a: Inference-II
Assignment No. 4: Bayes and Minimax Estimation

1. Let $X_1, \ldots, X_n$ be a random sample from $\text{Bin}(1, \theta)$ distribution, where $\theta \in \Theta = (0, 1)$. Consider estimation of θ under the squared error loss function $L(\theta, a) = (a - \theta)^2,\ a, \theta \in \Theta = \mathcal{A}$. Consider the randomized decision rule δ_0 defined by:

$$\delta_0(a|\underline{x}) = \begin{cases} \frac{n}{n+1}, & \text{if } a = \overline{x} \\ \frac{1}{n+1}, & \text{if } a = \frac{1}{2} \end{cases}.$$

   Compare the supremum of risks of $\overline{X}$ and δ_0 and hence conclude that $\overline{X}$ is not minimax.

2. What are the conjugate priors for:

   (a) $\text{N}_k(\underline{\theta}, I_k),\ \underline{\theta} \in \mathbb{R}^k$;

   (b) $\text{Bin}(n, \theta),\ \theta \in (0, 1)$; (Binomial distribution with known number of trials $n \in \mathbb{N}$)

   (c) $\text{U}(0, \theta),\ \theta > 0$; (Uniform distribution)

   (d) $\text{E}(0, \theta),\ \theta > 0$; (Exponential Distribution)

   (e) $\text{Bin}(n, \theta),\ n \in \{1, 2, \ldots\}$; (Binomial distribution with known success probability $\theta \in (0, 1)$)

3. Let $X_1, \ldots, X_n$ be i.i.d. Poisson(θ) random variables, where $\theta \in \Theta = (0, \infty)$. For a positive real number r, consider estimation of $g(\theta) = \theta^r$ under the SEL function and Gamma(α_0, μ_0) prior ($\alpha_0, \mu_0 > 0$). Find the Bayes estimator. Also show that $\delta_0(\underline{X}) = \overline{X}$ can not be Bayes estimator of θ with respect to any proper prior distribution.

4. Let X be a random variable having a p.d.f. $f_\theta(x),\ \theta \in \Theta,\ x \in \chi$, and let π be a prior distribution on $\Theta \subseteq \mathbb{R}$. For a real-valued function $g(\theta)$ and a non-negative function $w(\theta)$, such that $\int_\Theta w(\theta) g(\theta) d\pi(\theta) < \infty$, consider estimation of $g(\theta)$ under the loss function $L(\theta, a) = w(\theta)(a - g(\theta))^2$. Show that the Bayes action is

$$\delta_\pi(x) = \frac{\int_\Theta w(\theta) g(\theta) f_\theta(x) d\pi(\theta)}{\int_\Theta w(\theta) f_\theta(x) d\pi(\theta)},\ x \in \chi.$$

5. For $i = 1, \ldots, p$, let δ_i be a Bayes estimator of θ_i under the SEL function. For real constants $c_1, \ldots, c_p$, show that $\sum_{j=1}^p c_j \delta_j$ is a Bayes estimator of $\sum_{j=1}^p c_j \theta_j$ under the squared error loss.

6. Let $X_1, \ldots, X_n$ be a random sample from $\text{N}(\theta, 1)$ distribution, where $\theta \in \mathbb{R}$, and let $\pi \sim \text{DE}(0, 1)$ (Double Exponential distribution). Obtain the Bayes action under the squared error loss.

7. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{N}(\mu, \sigma^2)$ random variables, where $\underline{\theta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_{++}$ is unknown. Consider estimation of $g_1(\underline{\theta}) = \mu$ and $g_2(\underline{\theta}) = \sigma^2$ under SEL functions and the prior for $(\mu, \tau) = (\mu, \frac{1}{2\sigma^2})$ such that conditional prior distribution of μ given τ is $\mathrm{N}(\mu_0, \frac{\sigma_0^2}{\tau})$ and the marginal prior distribution of τ is $\mathrm{Gamma}(\alpha_0, v_0)$ ($\mu_0 \in \mathbb{R}, \sigma_0, \alpha_0, v_0 > 0$). Find Bayes estimators of $g_1(\underline{\theta})$ and $g_2(\underline{\theta})$.

8. Let $X \sim \mathrm{N}(\mu, \sigma^2)$, with a known $\sigma > 0$ and unknown $\mu > 0$. Consider estimating μ under the squared error loss and non-informative prior $\pi =$ the Lebesgue measure on $(0, \infty)$. Show that the generalized Bayes action is

$$\delta(x) = x + \sigma \cdot \frac{\phi(\frac{x}{\sigma})}{\Phi(\frac{x}{\sigma})}, \ x \in \mathbb{R}.$$

9. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{N}(\theta, \sigma_0^2)$ random variables, where $\theta \in \Theta = \mathbb{R}$ is unknown and σ_0 (> 0) is known. Consider estimation of $g(\theta) = \theta$ under the SEL function and $\mathrm{N}(\mu_0, \tau_0^2)$ prior for θ ($\mu_0, \tau_0 > 0$). Find the Bayes estimator. Also show that $\delta_0(\underline{X}) = \overline{X}$ can not be Bayes with respect to any proper prior distribution but it is an admissible and minimax estimator. Further show that $\delta_0(\underline{X}) = \overline{X}$ is the generalized Bayes estimator under the SEL function and non-informative prior $\pi =$ the Lebesgue measure on $\mathbb{R}$.

10. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{N}(\theta, \sigma_0^2)$ random variables, where $\theta \in \Theta = \mathbb{R}$ is unknown and σ_0 (> 0) is known. Consider estimation of $g(\theta) = \theta$ under the SEL function. Let $\delta_{a,b}(\underline{X}) = a\overline{X} + b, a, b \in \mathbb{R}$. Show that $\delta_{a,b}$ is admissible whenever $0 < a < 1$, or, $a = 1$ but $b = 0$, and it is inadmissible whenever $a > 1$, or, $a < 0$, or, $a = 1$ but $b \neq 0$.

11. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{Exp}(\theta)$ ($E_\theta(X_1) = \theta$) random variables, where $\theta \in \Theta = (0, \infty)$. Consider estimation of $g(\theta) = \theta$ under the SEL function and $\mathrm{Gamma}(\alpha_0, \mu_0)$ prior for $\frac{1}{\theta}$ ($\alpha_0, \mu_0 > 0$). Find the Bayes estimator. Show that $\delta_0(\underline{X}) = \overline{X}$ can not be Bayes with respect to any proper prior distribution. Among estimators of the type $\delta_{a,b}(\underline{X}) = a\overline{X} + b, a, b \in \mathbb{R}$, find admissible and inadmissible estimators. Can a minimax estimator be found?

12. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{Gamma}(\alpha, \theta)$, where $\theta \in (0, \infty) = \Theta$ is unknown and $\alpha > 0$ is known. Consider estimation of $g(\theta) = \theta$ under the loss function $L(\theta, a) = (\frac{a}{\theta} - 1)^2, a, \theta \in \Theta = \mathcal{A}$. Show that $\delta_b(\underline{X}) = \frac{n}{n\alpha+1}\overline{X} + b$ is an admissible estimator of θ for any $b \geq 0$. Also show that $\delta_0(\underline{X}) = \frac{n}{n\alpha+1}\overline{X}$ is an admissible and minimax estimator.

13. Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{Bin}(m, \theta)$ distribution, where $\theta \in \Theta = (0, 1)$ is unknown and m is a known positive integer.

    (a) Find the minimax estimator of θ under the SEL function and show that $\delta_0(\underline{X}) = \frac{\overline{X}}{m}$ is not a minimax estimator;

(b) Show that $\delta_0(\underline{X}) = \frac{\overline{X}}{m}$ is an admissible and minimax estimator of θ under the loss function $L(\theta, a) = \frac{(a-\theta)^2}{\theta(1-\theta)},\ a, \theta \in (0, 1)$;

(c) Show that $\overline{X}$ is an admissible estimator of θ under the squared error loss function.

14. Let $X_1, \ldots, X_n$ ne i.i.d. $\mathrm{N}(\mu, \sigma^2)$ random variables, where $\underline{\theta} = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_{++}$ is unknown. Consider estimation of $g(\underline{\theta}) = \mu$ under SEL function. For any estimator δ show that $\sup_{\underline{\theta} \in \Theta} R_\delta(\underline{\theta}) = \infty$. If $\Theta = \mathbb{R} \times (0, c]$, for some positive constant c, show that $\delta_0(\underline{X}) = \overline{X}$ is minimax.

15. Let $X \sim \mathrm{N}(\theta, 1),\ \mu \in \mathbb{R}$. For estimating θ under the absolute error loss function $L(\theta, a) = |a - \theta|,\ a, \theta \in \mathbb{R}$ show that X is an admissible and minimax estimator.

16. Let $X \sim \mathrm{Poisson}(\theta)$, where $\theta \in (0, \infty) = \Theta$ is unknown. Consider estimation of $g(\theta) = \theta$ under the SEL function. Show that, for any estimator δ, $\sup_{\theta \in \Theta} R_\delta(\theta) = \infty$.

17. Let $X \sim \mathrm{G}(\theta)$ (Geometric distribution with support $\{1, 2, \ldots\}$), where $\theta \in (0, 1)$ is unknown. Show that $I_{\{1\}}(X)$ is a minimax estimator of θ under the loss function $L(\theta, a) = \frac{(a-\theta)^2}{\theta(1-\theta)},\ a, \theta \in (0, 1)$.

18. Let $X \sim \mathrm{NB}(r, \theta)$, where $\theta \in \Theta = (0, 1)$ is unknown and r is a fixed positive integer. Consider estimation of $g(\theta) = \frac{1}{\theta}$ under the loss function $L(\theta, a) = \theta^2(a - \frac{1}{\theta})^2, a, \theta \in \Theta = \mathcal{A}$, and $\mathrm{Beta}(\alpha, \beta)$ $(\alpha, \beta > 0)$ prior. Find the Bayes estimator. Show that $\delta_0(X) = \frac{X+1}{r+1}$ is admissible under the SEL function.

19. Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$ distribution, where $\underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty)$ is unknown. Show that $\overline{X}$ is minimax estimator of μ under the loss function $L(\underline{\theta}, a) = \frac{(a-\mu)^2}{\sigma^2},\ a \in \mathbb{R},\ \underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty)$.

20. In a two-sample normal problem (independent random samples) with unknown means find a minimax estimator of $\Delta = \mu_2 - \mu_1$ under the squared error loss function when:

    (a) the variances are known (possibly unequal);

    (b) the variances σ_1^2 and σ_2^2 are unknown but $\sigma_i^2 \in (0, c_i],\ i = 1, 2$, for some known positive constants c_1 and c_2.

21. Let $X \sim \mathrm{N}(\theta, 1),\ \theta \in \mathbb{R}$ and $d\pi(\theta) = e^\theta d\theta,\ \theta \in \mathbb{R}$. For estimating θ under the squared error loss function show that $\delta_{GB}(X) = X + 1$ is a generalized-Bayes estimator but neither minimax nor admissible.

22. Let $X_1, \ldots, X_n$ be a random sample from a distribution F_θ, where $\theta \in \Theta$ is unknown. Find a minimax estimator of θ under the squared error loss function when $F_\theta \sim$:

    (a) $\mathrm{E}(\theta, \sigma_0), \theta \in \Theta = \mathbb{R},\ \sigma_0 > 0$ is known;

(b) $\mathrm{U}(\theta - \frac{1}{2}, \theta + \frac{1}{2}),\ \theta \in \Theta = \mathbb{R}$;

23. Using heuristic (geometric) arguments show that:

    (a) an admissible rule may not be Bayes;

    (b) a Bayes rule may not be admissible;

    (c) Bayes rule may not be unique;

    (d) minimax rule may not be unique;

    (e) minimax rule may not be admissible;

    (f) Bayes rule may not be minimax;

    (g) minimax rule may not be Bayes.