The Riemann Zeta Function
Part 1: meromorphic extension onto $\mathbb{C}$

Throughout this note, we use the following definition for the complex log and power functions: For $z = re^{i\theta}$ with $r > 0$ and $0 \le \theta < 2\pi$, define

$$\log z = \ln r + i\theta, \qquad z^s = e^{s \log z} \ (s \in \mathbb{C}).$$

Here, ln is the real natural logarithmic function.

Definition of $\zeta(s)$ for $\operatorname{Re} s > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\operatorname{Re} s > 1).$$

Now we try to extend $\zeta(s)$ onto a meromorphic function on the whole complex plane.

First Observation: Identity $\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \qquad (\operatorname{Re} s > 1)$.

Proof:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt = n^s \int_0^\infty x^{s-1} e^{-nx} dx. \qquad \text{(we have substituted } t = nx\text{)}$$

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\Gamma(s) = \sum_{n=1}^{\infty} \int_0^\infty x^{s-1} e^{-nx} dx = \int_0^\infty \sum_{n=1}^{\infty} x^{s-1} e^{-nx} dx = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx. \quad \blacksquare$$

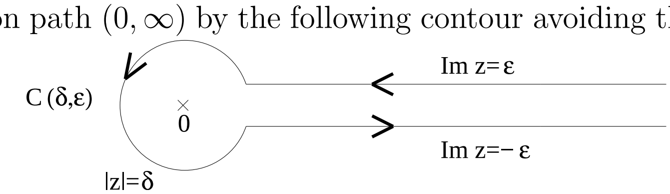
If we can show that the right hand side $\int_0^\infty \frac{x^{s-1}}{e^x-1} dx$ is a meromorphic function of s for all $s \in \mathbb{C}$, then

$$\zeta(s) = \frac{1}{\Gamma(s)} (\text{the right hand side})$$

can be used as a definition of $\zeta(s)$ on the whole plane. Unfortunately, in the present form this improper integral is divergent for $\operatorname{Re} s \le 1$, since the integrand behaves bad at the point $x = 0$:

$$\left| \frac{x^{s-1}}{e^x - 1} \right| \sim x^{\operatorname{Re} s - 2} \qquad (x \sim 0).$$

So this idea does not work directly. Riemann overcame this difficulty by the following trick: Replace the integration path $(0, \infty)$ by the following contour avoiding the singular point $x = 0$:



where $0 < \varepsilon < \delta < 2\pi$ are fixed (small) numbers.

Definition: For $s \in \mathbb{C}$,

$$G(s) = \int_{C(\delta,\varepsilon)} \frac{z^{s-1}}{e^z - 1} dz.$$

This integral behaves well and defines an entire fcuntion.

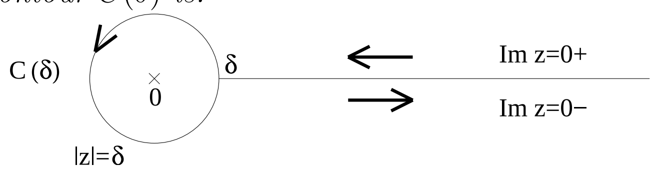
FACT 1: *$G(s)$ is an entire function of $s \in \mathbb{C}$.*

FACT 2: *$G(s)$ is independent of the choices of δ and ε.*

FACT 3: *Fix any $0 < \delta < 2\pi$. For $s \in \mathbb{C}$,*

$$G(s) = \int_{C(\delta)} \frac{z^{s-1}}{e^z - 1} dz = \int_{|z|=\delta} \frac{z^{s-1}}{e^z - 1} dz + (e^{i2\pi s} - 1) \int_{\delta}^{\infty} \frac{x^{s-1}}{e^x - 1} dx,$$

where the integration contour $C(\delta)$ is:



FACT 4: *For* $\operatorname{Re} s > 1$,

$$G(s) = (e^{i2\pi s} - 1) \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = (e^{i2\pi s} - 1)\Gamma(s)\zeta(s).$$

Now we can give a definition of $\zeta(s)$ on the whole plane.

DEFINITION OF $\zeta(s)$ FOR $s \in \mathbb{C}$: $\quad \zeta(s) = \dfrac{G(s)}{(e^{i2\pi s} - 1)\Gamma(s)}.$

Fact 4 shows that this definition is consistent with the original (infinite series) definition of $\zeta(s)$ in the region $\operatorname{Re} s > 1$.

By the new definition, we immediately see that

$$\zeta(s) \textit{ is meromorphic on } \mathbb{C}.$$

In the next note, we'll study some special values of $\zeta(s)$; in particular, we'll be interested in its pole and zeros.

The proofs of the above Facts 1-4 are collected below.

Proof of Fact 1: Cutting off the tail on the right side, we can approximate the infinite path $C(\delta, \epsilon)$ by a family of paths C_n with finite length; the limit of C_n as $n \to \infty$ is $C(\delta, \epsilon)$.

The integrand $g(s, z) = \dfrac{z^{s-1}}{e^z - 1}$ is a continuous function of $(s, z) \in \mathbb{C} \times (\mathbb{C} \setminus [0, \infty))$. For each fixed $z \in \mathbb{C} \setminus [0, \infty)$, $g(s, z)$ is an entire function of s. Hence, for each n, $\displaystyle\int_{C_n} g(s, z) dz$ is an entire function of s.

We have

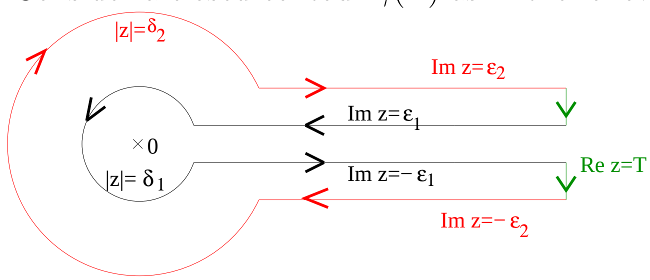
$$\int_{C_n} g(s,z)dz \to \int_{C(\delta,\epsilon)} g(s,z)dz.$$

Since the limit of holomorphic functions is also holomorphic, this shows that $G(s)$ is an entire function of s. ∎

Proof of Fact 2: Let $0 < \varepsilon_1 < \delta_1 < 2\pi$ and $0 < \varepsilon_2 < \delta_2 < 2\pi$ be fixed. Denote by $g(s,z) = z^{s-1}/(e^z - 1)$. We try to show

$$\int_{C(\delta_1,\varepsilon_1)} g(s,z)dz = \int_{C(\delta_2,\varepsilon_2)} g(s,z)dz.$$

Take a large $T > 0$. Consider a closed contour $\gamma(T)$ as in the following figure:



which consists of black, red and green parts.

Since the closed contour $\gamma(T)$ is in the simply connected region $z \in \mathbb{C} \setminus [0,\infty)$ where $g(s,z)$ is holomorphic in z, we have

$$\int_{\gamma(T)} g(s,z)dz = 0.$$

Now pass to the limit as $T \to \infty$:

$$\int_{\text{black}} g(s,z)dz \to \int_{C(\delta_1,\varepsilon_1)} g(s,z)dz, \quad \int_{\text{red}} g(s,z)dz \to -\int_{C(\delta_2,\varepsilon_2)} g(s,z)dz,$$

and

$$\int_{\text{green}} g(s,z)dz \to 0. \qquad ∎$$

Proof of Fact 3: Take the limit as $\varepsilon \downarrow 0$. Notice that for $x > 0$, we have the following limits as $\varepsilon \downarrow 0$:

$$(x+i\varepsilon)^{s-1} \to x^{s-1}, \quad (x-i\varepsilon)^{s-1} \to x^{s-1}e^{i2\pi(s-1)} = x^{s-1}e^{i2\pi s}.$$

Thus, as $\varepsilon \downarrow 0$,

$$\begin{aligned}\int_{C(\delta,\varepsilon)} \frac{z^{s-1}}{e^z-1}dz &\to \int_\infty^\delta \frac{x^{s-1}}{e^x-1}dx + \int_{|z|=\delta} \frac{z^{s-1}}{e^z-1}dz + \int_\delta^\infty \frac{x^{s-1}e^{i2\pi s}}{e^x-1}dx \\ &= \int_{|z|=\delta} \frac{z^{s-1}}{e^z-1}dz + \int_\delta^\infty \frac{x^{s-1}(e^{i2\pi s}-1)}{e^x-1}dx. \qquad ∎\end{aligned}$$

Proof of Fact 4: Use Fact 3 and take the limit as $\delta \downarrow 0$. We only need to show

$$\int_{|z|=\delta} \frac{z^{s-1}}{e^z - 1} dz \to 0.$$

Estimate:

$$\begin{aligned}
\left| \int_{|z|=\delta} \frac{z^{s-1}}{e^z - 1} dz \right| &\leq \int_{|z|=\delta} \frac{|z^{s-1}|}{|\exp(z) - 1|} |dz| \\
&= \int_0^{2\pi} \delta^{\operatorname{Re} s - 1} e^{-\theta \operatorname{Im} s} \left| \exp(\delta e^{i\theta}) - 1 \right|^{-1} \delta d\theta \\
&= \delta^{\operatorname{Re} s - 1} \int_0^{2\pi} e^{-\theta \operatorname{Im} s} \left| \frac{\exp(\delta e^{i\theta}) - 1}{\delta} \right|^{-1} d\theta
\end{aligned}$$

Now, as $\delta \downarrow 0$ we have

$$\frac{\exp(\delta e^{i\theta}) - 1}{\delta} \to e^{i\theta}$$

and hence

$$\int_0^{2\pi} e^{-\theta \operatorname{Im} s} \left| \frac{\exp(\delta e^{i\theta}) - 1}{\delta} \right|^{-1} d\theta \to \int_0^{2\pi} e^{-\theta \operatorname{Im} s} d\theta < \infty.$$

This shows that as $\delta \downarrow 0$, $\int_{|z|=\delta} \frac{z^{s-1}}{e^z - 1} dz$ is of order $O(\delta^{\operatorname{Re} s - 1})$ and thus decays to 0 if $\operatorname{Re} s > 1$. ∎

EXERCISES

1. Complete the proof of Fact 2, by verifying $\displaystyle\int_{\text{green}} g(s,z) dz \to 0$.

2. Riemann's technique summarized above can be used to deal with other functions as well. For instance, use the definition $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$ for $\operatorname{Re} s > 0$. By considering

$$H(s) = \int_{C(\delta,\varepsilon)} z^{s-1} e^{-z} dz,$$

we can extend $\Gamma(s)$ to a meromorphic function on the whole plane.

   (a) The integral defining $H(s)$ converges for any $0 < \varepsilon < \delta < \infty$ and any $s \in \mathbb{C}$.
   (b) $H(s)$ is an entire function of s.
   (c) $H(s)$ is independent of δ and ε.
   (d) Fix $\delta > 0$. For any $s \in \mathbb{C}$,

$$H(s) = \int_{C(\delta)} z^{s-1} e^{-z} dz = \int_{|z|=\delta} z^{s-1} e^{-z} dz + (e^{i2\pi s} - 1) \int_\delta^\infty x^{s-1} e^{-x} dx.$$

   (e) For $\operatorname{Re} s > 0$, $H(s) = (e^{i2\pi s} - 1)\Gamma(s)$.

Remark: In view of (b) and (e), we can use $\Gamma(s) = H(s)/(e^{i2\pi s} - 1)$ as the global definition of $\Gamma(s)$ on $\mathbb{C}$. This provides an alternative treatment of $\Gamma(s)$. Can you find the poles of $\Gamma(s)$ and evaluate residues using this method?