7. The Chi-Squared Distribution and its Properties

The Chi-Squared Distribution and its Expectation

3/3 points (graded)

Note: This problem introduces the chi-squared distribution and is intended as an exercise in probability that you are encouraged to attempt before watching the following video.

The χ^2_d **distribution with d degrees of freedom** is given by the distribution of

$$Z_1^2 + Z_2^2 + \cdots + Z_d^2,$$

where $Z_1, \ldots, Z_d \overset{iid}{\sim} \mathcal{N}(0,1)$.

What is the smallest possible sample space of χ^2_d?

- $\mathbb{Z}_{\geq 0}$
- $\mathbb{Z}$
- $\mathbb{R}_{\geq 0}$
- $\mathbb{R}$

✔

If $X \sim \chi^2_d$, what is $\mathbb{E}[X]$? Give your answer in terms of d.

d

✔ **Answer:** d

d

STANDARD NOTATION

Let $\mathbf{Z} \sim \mathcal{N}(0, I_{d\times d})$ denote a random vector whose components are standard Gaussians: $Z^{(1)}, \ldots, Z^{(d)} \sim \mathcal{N}(0,1)$. Which one of the following random variables has a chi-squared distribution with d degrees of freedom?

- $\max\left(Z^{(1)}, \ldots, Z^{(d)}\right)$
- $|Z^{(1)}| + |Z^{(2)}| + \cdots |Z^{(d)}|$
- $\|\mathbf{Z}\|_2$
- $\|\mathbf{Z}\|_2^2$

✔

Solution:

The smallest sample space of a Gaussian random variable Z is $\mathbb{R}$. Hence, the smallest possible sample space of Z^2 is $\mathbb{R}_{\geq 0}$. And the same holds for the sum

$$Z_1^2 + Z_2^2 + \cdots + Z_d^2,$$

so the smallest possible sample space for χ_d^2 is $\mathbb{R}_{\geq 0}$.

Next, by linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}[Z_1^2 + Z_2^2 + \cdots + Z_d^2] = d \cdot 1 = d,$$

because $Z_1, \ldots, Z_d \overset{iid}{\sim} \mathcal{N}(0,1)$.

The ℓ_2 norm $\|\cdot\|_2$ measures the Euclidean distance from the origin. Hence, if $\mathbf{Z} \sim \mathcal{N}(0, I_{d\times d})$, then

$$\|\mathbf{Z}\|_2^2 = \left(Z^{(1)}\right)^2 + \left(Z^{(2)}\right)^2 + \cdots + \left(Z^{(d)}\right)^2 \sim \chi_d^2.$$

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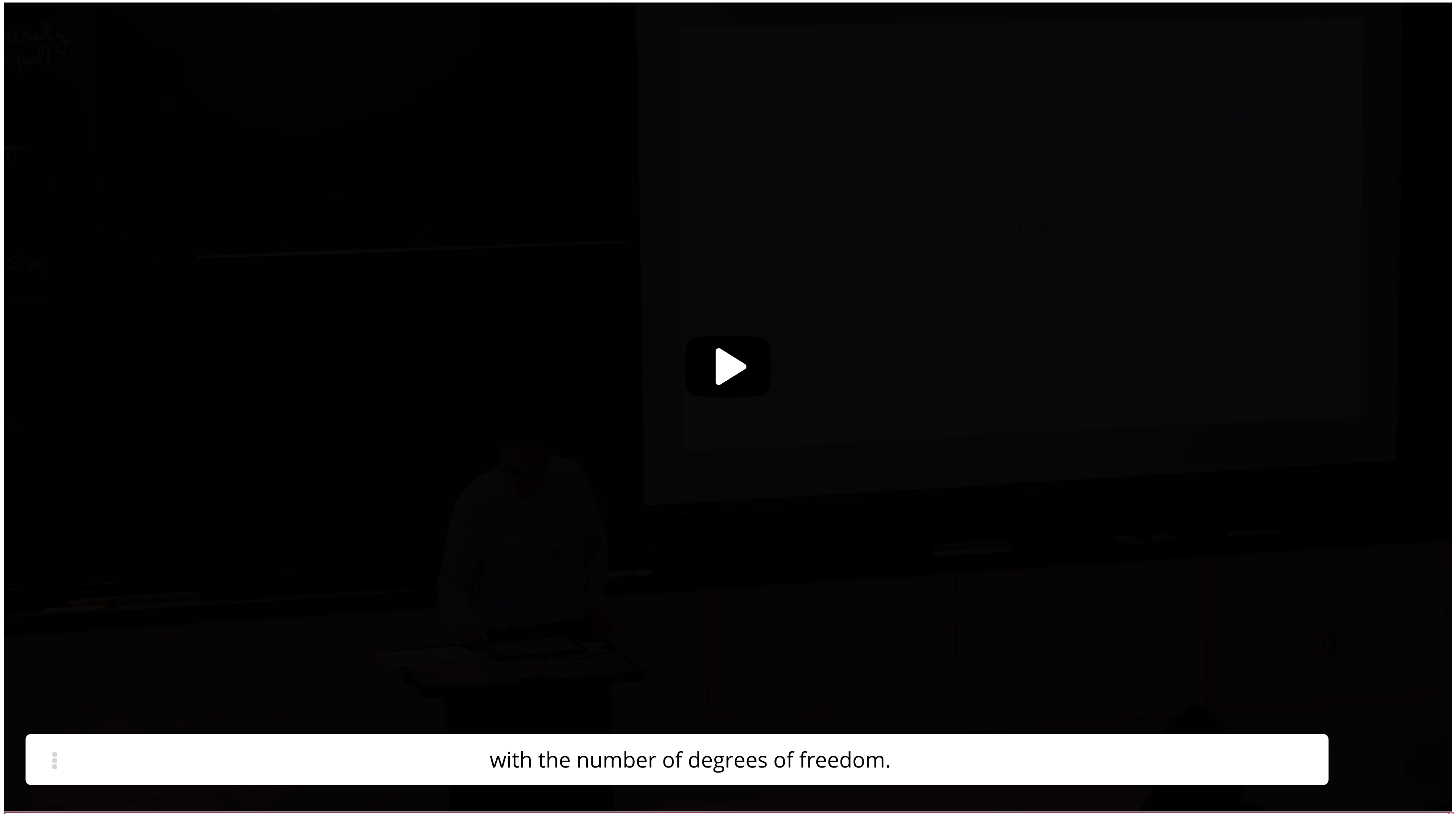
Answers are displayed within the problem

Distribution of Sample Variance of Gaussian: The Chi-Squared Distribution



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Throwing Darts

1/1 point (graded)

You are playing darts on a dart-board that is represented by the entire plane, $\mathbb{R}^2$. You get a 'bullseye' if the dart lands inside of the unit disc $D^1 := \{(x,y) : x^2 + y^2 \leq 1\}$. You dart throws are modeled by a Gaussian random vector $\mathbf{Z}$, where $Z^{(1)}, Z^{(2)} \overset{iid}{\sim} \mathcal{N}(0,1)$.

Let f_d represent the density of the χ^2_d distribution.

Which of the following equals the probability of getting a bullseye?

- $\int_0^1 f_1(x)\ dx$
- $\int_0^1 f_2(x)\ dx$
- $\int_1^\infty f_2(x)\ dx$
- $\int\int_{D^1} f_2(x)\ dxdy$

Solution:

A bullseye is given by the event $\left(Z^{(1)}\right)^2 + \left(Z^{(2)}\right)^2 \leq 1$. Since $\left(Z^{(1)}\right)^2 + \left(Z^{(2)}\right)^2 \sim \chi^2_2$, it follows that

$$P(\text{bullseye}) = \int_0^1 f_2(x)\ dx.$$

Remark: The $d = 2$ case is special, because it turns out that $\chi^2_2 = \text{Exp}(1/2)$. This can be seen using the explicit formula for the density of a χ^2_2, but it is not necessary for this course to know the density of a chi-squared random variable with d degrees of freedom by heart.

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Properties of the Chi-Squared Distribution

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The Chi-Squared Distribution and the Sample Second Moment

2/2 points (graded)

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2)$ and let

$$V_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

denote the sample second moment. For an appropriate expression A given in terms of n and σ^2, we have that $AV_n \sim \chi^2$.

What is A?

$A =$ n/sigma^2 ✔ **Answer:** n/sigma^2

$\frac{n}{\sigma^2}$

How many degrees of freedom does the above χ-squared random variable have? (Give your answer in terms of n.)

n ✔ **Answer:** n

n

STANDARD NOTATION

Solution:

Observe that

$$\frac{n}{\sigma^2}V_n = \sum_{i=1}^{n} \frac{X_i^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i}{\sigma}\right)^2,$$

and $X_i/\sigma \sim \mathcal{N}(0,1)$ because $X_i \sim \mathcal{N}(0,\sigma^2)$. Hence, $\frac{n}{\sigma^2}V_n$ is a χ^2_n random variable.

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Answers are displayed within the problem

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[Staff] Throwing darts problem
can you please add a reasoning in the solution, why the incorrect answer choices are incorrect?
5

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