Chapter Five
Combinatorics

5.1 INTRODUCTION

Although its historical roots are found in mathematical recreations and games, the field of combinatorics has recently found a variety of applications both within and outside of mathematics. Its importance in computer science, operations research, and probability is easy to document, and now there are conferences on combinatorial algebra and important results in analysis whose proofs are distinctly combinatorial in nature. Moreover, combinatorics has become these days one of the more active areas of mathematical research, with more than a dozen international journals devoted to publishing new results. The purpose of this chapter is to provide an introduction to some of the basic principles and techniques of this area of mathematics.

Basically, combinatorics is concerned with arranging the elements of sets into definite patterns. Typically, the sets under consideration are discrete, including finite sets such as the set $\{0, 1, \ldots, n-1\}$ of all natural numbers less than the positive integer n, and infinite discrete sets such as the set $\mathbb{Z}$ of integers. Thus, combinatorics is placed under the broad heading of discrete mathematics.

In attempting to arrange the elements of a set into a pattern that satisfies certain specified conditions, a number of issues generally arise (think, for example, of *Sudoku* puzzles):

1. *Existence*. Does such an arrangement exist?

2. *Enumeration or classification*. How many valid arrangements are there? Can they be classified in some way?

3. *Algorithms*. If such arrangements exist, is there a definite method for constructing one, or all, of them?

4. *Generalization*. Does the problem under consideration suggest other, related problems?

We begin by considering a famous combinatorial problem known as the "problem of the 36 officers," first considered in the eighteenth century by the great Swiss mathematician Leonard Euler. The problem is as follows: Given 36 military officers, representing each possible combination of six ranks with six regiments, can they be arranged in a square marching formation of six rows and six columns so that each rank and each regiment is represented in every row and in every column?

To clarify the problem a bit, note that the rank and regiment of an officer can be represented as an ordered pair (r, s), where r denotes the rank and s the regiment of the officer. For simplicity, let us denote the ranks and regiments by the numbers 1, 2, 3, 4, 5, 6. Thus, for instance, the ordered pair $(2, 5)$ denotes the officer with rank 2 and regiment 5. There are 36 officers, one for each of the 36 ordered pairs of the form (r, s), with $r, s \in \{1, 2, 3, 4, 5, 6\}$.

First, consider just the ranks of the officers in any proposed solution to our problem. There are six officers of each rank, so a solution to our problem must have the ranks 1, 2, 3, 4, 5, and 6 arranged in a square array of six rows and six columns so that each number occurs exactly once in any row and in any column.

One possible arrangement for the officer ranks is the array shown in Figure 5.1. For convenience, we index the rows and columns of this array with the numbers 1, 2, 3, 4, 5, and 6 and let r_{ij} denote the rank of the officer in row i and column j. Thus, for example, $r_{14} = 4$, $r_{36} = 2$, and $r_{55} = 3$.

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 4 & 5 & 6 & 1 \\
3 & 4 & 5 & 6 & 1 & 2 \\
4 & 5 & 6 & 1 & 2 & 3 \\
5 & 6 & 1 & 2 & 3 & 4 \\
6 & 1 & 2 & 3 & 4 & 5
\end{array}$$

Figure 5.1 A possible arrangement for the ranks of the 36 officers

Next, let us focus our attention on the regiments. Again, a proposed solution to our problem must have the six regiments 1, 2, 3, 4, 5, 6 arranged in an array of six rows and six columns so that each regiment occurs exactly once in any row and in any column. A possible arrangement for the officer regiments is the array shown in Figure 5.2. Let s_{ij} denote the regiment of the officer in row i and column j of this array, $1 \le i, j \le 6$.

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 4 & 5 & 6 & 1 & 2 \\
5 & 6 & 1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 & 6 & 1 \\
4 & 5 & 6 & 1 & 2 & 3 \\
6 & 1 & 2 & 3 & 4 & 5
\end{array}$$

Figure 5.2 A possible arrangement for the regiments of the 36 officers

We can now consider the array of 36 ordered pairs (r_{ij}, s_{ij}), where $1 \le i, j \le 6$, shown in Figure 5.3. The problem of the 36 officers would be solved in the affirmative if the 36 ordered pairs in this array were distinct, giving us each of the 36 possible rank-regiment pairs. Unfortunately, this is not the case, as can be observed. Notice that the ordered pairs in the first row are repeated in the last row, and that the ordered pairs $(1, 4)$, $(2, 5)$, $(3, 6)$, $(4, 1)$, $(5, 2)$, and $(6, 3)$ do not appear.

$$\begin{array}{cccccc}
(1,1) & (2,2) & (3,3) & (4,4) & (5,5) & (6,6) \\
(2,3) & (3,4) & (4,5) & (5,6) & (6,1) & (1,2) \\
(3,5) & (4,6) & (5,1) & (6,2) & (1,3) & (2,4) \\
(4,2) & (5,3) & (6,4) & (1,5) & (2,6) & (3,1) \\
(5,4) & (6,5) & (1,6) & (2,1) & (3,2) & (4,3) \\
(6,6) & (1,1) & (2,2) & (3,3) & (4,4) & (5,5)
\end{array}$$

Figure 5.3 Putting together the rank and regiment arrays

Consideration of the problem of the 36 officers led Euler to generalize the problem by defining "Latin squares." For a positive integer n, a ***Latin square of order n*** is an arrangement of the integers $1, 2, \ldots, n$ (or any other n distinct symbols) into a square array of n rows and n columns such that each number (symbol) occurs exactly once in any row and in any column.

Given a Latin square R of order n, let r_{ij} denote the entry in row i and column j, where the rows and columns are indexed $1, 2, \ldots, n$. Let S be another Latin square of order n with typical entry s_{ij}. We say that R and S are ***orthogonal Latin squares*** provided the n^2 ordered pairs (r_{ij}, s_{ij}), $1 \leq i, j \leq n$, are distinct, thus giving all n^2 elements in the product set $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. In this case, the n by n array (R, S) of ordered pairs (r_{ij}, s_{ij}) is called a ***Greco-Latin square of order n***. (The names *Latin square* and *Greco-Latin square* come from Euler's practice of using Latin letters a, b, c, ... to denote the elements of one square and the corresponding Greek letters α, β, γ, ... to denote the elements of an orthogonal square.) Then the problem of the 36 officers can be restated as follows:

Do there exist two orthogonal Latin squares of order 6?

or,

Does there exist a Greco-Latin square of order 6?

Euler gave methods for constructing a pair of orthogonal Latin squares of order $n \geq 2$ whenever $n \bmod 4 \in \{0, 1, 3\}$. For instance, if there were 16 officers in Euler's problem instead of 36, with one officer of each possible combination of four ranks with four regiments, then we would be interested in finding two orthogonal Latin squares of order 4. An example of these is shown in Figure 1.4, along with the corresponding Greco-Latin square of order 4.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{array}$$

(a) Two orthogonal Latin squares of order 4

$$\begin{array}{cccc} (1,1) & (2,2) & (3,3) & (4,4) \\ (2,3) & (1,4) & (4,1) & (3,2) \\ (3,4) & (4,3) & (1,2) & (2,1) \\ (4,2) & (3,1) & (2,4) & (1,3) \end{array}$$

(b) The corresponding Greco-Latin square of order 4

Figure 5.4

It can be checked that there do not exist two orthogonal Latin squares of order 2 (see Exercise 1). Euler searched for, but could not find, a pair of orthogonal Latin squares of order 6. Notice that $2 \bmod 4 = 2 = 6 \bmod 4$. This led Euler to conjecture that there do not exist two orthogonal Latin squares of order n when $n \bmod 4 = 2$. Around 1900, using an exhaustive search, G. Tarry [Le probleme des 36 officiers, *Comptes Rendus de L'Association Francaise pour L'Avancement de Science*, 1 (1900): 122—123; 2 (1901): 170—203] showed that Euler's conjecture is true for $n = 6$.

This answers the problem of the 36 officers in the negative. The cases $n = 10, 14, 18, 22, \ldots$ remained unsettled until around 1960, when R.C. Bose, S.S. Shrikhande, and E. T. Parker [Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Canadian Journal of Mathematics* 12 (1960): 189 – 203] showed how to construct a pair of orthogonal Latin squares of order n whenever $n \geq 10$ and $n \bmod 4 = 2$. This amazing discovery showed that Euler's conjecture is false for $n > 6$, so that $n = 2$ and $n = 6$ are the only cases when a pair of orthogonal Latin squares fails to exist.

The story of the Euler conjecture and its solution is a beautiful example of how new mathematics is discovered and communicated. If you are interested in learning more about this problem see, as a start, the November 1959 issue of *Scientific American*. The cover of this issue shows a color-coded version of the Greco-Latin square of order 10 depicted in Figure 5.5. Martin Gardner's "Mathematical Games" column in that issue also discusses the solution of Euler's conjecture. Also recommended is the recent article by Dominic Klyve and Lee Stemkowski, "Greco-Latin Squares and a Mistaken Conjecture of Euler," *The College Mathematics Journal*, 1 (2006), 2 – 15.

$(1,1)$	$(5,8)$	$(2,9)$	$(8,7)$	$(3,10)$	$(10,4)$	$(9,6)$	$(4,5)$	$(7,2)$	$(6,3)$
$(9,7)$	$(2,2)$	$(6,8)$	$(3,9)$	$(8,1)$	$(4,10)$	$(10,5)$	$(5,6)$	$(1,3)$	$(7,4)$
$(10,6)$	$(9,1)$	$(3,3)$	$(7,8)$	$(4,9)$	$(8,2)$	$(5,10)$	$(6,7)$	$(2,4)$	$(1,5)$
$(6,10)$	$(10,7)$	$(9,2)$	$(4,4)$	$(1,8)$	$(5,9)$	$(8,3)$	$(7,1)$	$(3,5)$	$(2,6)$
$(8,4)$	$(7,10)$	$(10,1)$	$(9,3)$	$(5,5)$	$(2,8)$	$(6,9)$	$(1,2)$	$(4,6)$	$(4,1)$
$(7,9)$	$(8,5)$	$(1,10)$	$(10,2)$	$(9,4)$	$(6,6)$	$(3,8)$	$(2,3)$	$(5,7)$	$(4,1)$
$(4,8)$	$(1,9)$	$(8,6)$	$(2,10)$	$(10,3)$	$(9,5)$	$(7,7)$	$(3,4)$	$(6,1)$	$(5,2)$
$(2,5)$	$(3,6)$	$(4,7)$	$(5,1)$	$(6,2)$	$(7,3)$	$(1,4)$	$(8,8)$	$(9,9)$	$(10,10)$
$(3,2)$	$(4,3)$	$(5,4)$	$(6,5)$	$(7,6)$	$(1,7)$	$(2,1)$	$(9,10)$	$(10,8)$	$(8,9)$
$(5,3)$	$(6,4)$	$(7,5)$	$(1,6)$	$(2,7)$	$(3,1)$	$(4,2)$	$(10,9)$	$(8,10)$	$(9,8)$

Figure 5.5 A Greco-Latin square of order 10

Why is there so much interest in what appears to be a mathematical recreation? It turns out that Latin squares and other types of combinatorial designs are important in the design of certain kinds of statistical studies. There is an example of such an application in Exercise 2.

Exercise Set 5.1

1. Find all Latin squares of order 2 and verify that no two of them are orthogonal.

2. A statistical study is to be conducted to test the effect of three different kinds of fertilizer on the yield of three different varieties of corn. A large rectangular field to be used for the study is divided into nine plots arranged into three rows and three columns — one plot for each of the nine combinations of fertilizer type with corn variety. However, growing conditions may not be uniform across the field; for example, one side is bordered by a busy highway, while another side is near a river and may receive additional moisture in the form of dew. To minimize the effect of such factors on the study, it is desired that each kind of fertilizer be applied, and each variety of corn be planted, in exactly one plot in any row and in any column of the field. Explain how to use orthogonal Latin squares to determine how the varieties of corn should be planted and the kinds of fertilizer should be applied.

3. Let n be an odd positive integer. For $0 \leq i, j \leq n-1$, define n by n arrays R and S by

$$r_{ij} = (i + j) \bmod n \qquad \text{and} \qquad s_{ij} = (2i + j) \bmod n$$

(a) Find R and S in the case $n = 5$ and verify that they are orthogonal Latin squares of order 5 (using the symbols 0, 1, 2, 3, and 4).

(b) Verify, in general, that R and S are Latin squares of order n.

(c) Verify, in general, that R and S are orthogonal.

4. Let A be a Latin square of order 4 (using the symbols 0, 1, 2, and 3) and let B be a Latin square of order m (using the symbols 0, 1, . . ., $m - 1$). We can define a Latin square C of order $n = 4m$ as follows. For $0 \leq i, j \leq 4m - 1$, let

$$c_{ij} = a_{uv} + 4b_{xy}$$

where $u = i$ div m, $v = j$ div m, $x = i$ mod m, and $y = j$ mod m. The Latin square C is called the *composition* of A with B; we write $C = A \circ B$.

(a) Consider the case $m = 3$. Let A_1 be a Latin square of order 4 and let B_1 be a Latin square of order 3. Construct the Latin square $C_1 = A_1 \circ B_1$ of order 12.

(b) Let A_2 be a Latin square of order 4 orthogonal to A_1 and let B_2 be a Latin square of order 3 orthogonal to B_1. Construct $C_2 = A_2 \circ B_2$ and verify that C_1 and C_2 are orthogonal Latin squares of order 12.

(c) Is composition of Latin square commutative?

(It might be worthwhile to write a computer program to generate the Latin squares C_1 and C_2 and to check that they are orthogonal.)

5. The following two problems were considered before Euler's work on Latin squares.

(a) Arrange the 16 court cards (aces, kings, queens, jacks) of a standard deck into a square array of four rows and four columns so that each row and column contains exactly one card of every rank and of every suit.

(b) Here's a tougher problem: Arrange these 16 cards so that each row, each column, and each of the two diagonals contains exactly one card of each rank and suit.

6. The idea of composition introduced in Exercise 4 can be generalized. Let A be a Latin square of order k and let B be a Latin square of order m. The ***composition*** of A with B is the Latin square $C = A \circ B$ of order $n = km$ defined as follows. For $0 \leq i, j \leq km - 1$, let

$$c_{ij} = a_{uv} + kb_{xy}$$

where $u = i$ div m, $v = j$ div m, $x = i$ mod m, and $y = j$ mod m.

(a) Show that C is, indeed, a Latin square.

(b) Let A_1 and A_2 be orthogonal Latin squares of order k and let B_1 and B_2 be orthogonal Latin squares of order m. Let $C_1 = A_1 \circ B_1$ and $C_2 = A_2 \circ B_2$. Show that C_1 and C_2 are orthogonal.

(c) It is known that there exist two orthogonal Latin squares of order 2^d for each $d \geq 2$. Use this fact and the result of Exercise 3 to prove Euler's result that there exist two orthogonal Latin squares of order n if n mod $4 = 0$.

7. A ***magic square of order n*** is an n by n array containing the integers 1, 2, . . ., n^2 such that the sum of the numbers in any row, in any column, and in either diagonal is the same value, known as the **magic value** for that order. For example, Figure 5.6 shows a magic square of order 3 and a magic square of order 4. The order-3 magic square is unique and was called the *lo shu* by the ancient Chinese. The order-4 magic square shown is known as *Dürer's magic square.*

(a) Find the magic constant for magic squares of order n.

An n by n array that satisfies the "magic condition" except for one or both of the diagonals is called a ***semi-magic square***.

(b) In his paper, De Quadratis Magicus, Euler gave the following method for producing a semi-magic square of order n from a Greco-Latin square of order n (using the symbols $0, 1, \ldots, n-1$): replace the ordered pair (a, b) by the number $an + b + 1$. Verify that this method works.

(c) Try Euler's method for $n = 5$. Can you find a Greco-Latin square of order 5 that produces a magic square of order 5?

(d) Is Euler's method invertible? That is, given a semi-magic square of order n, can Euler's method be "inverted" to produce a Greco-Latin square of order n?

8	1	6
3	5	7
4	9	2

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 5.6 Magic squares

5.2 ADDITION AND MULTIPLICATION PRINCIPLES

As mentioned in the last section, if S is the set of solutions to some combinatorial problem and S is finite, then we may be interested in enumerating the elements of S or in finding the cardinality of S. It often helps to express S as the union of two nonempty disjoint sets, that is, to express S as $S_1 \cup S_2$, where $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, and $S_1 \cap S_2 = \emptyset$. Or, it may be that S can be expressed as the product of two sets, that is, $S = S_1 \times S_2$. Thus, given that S_1 and S_2 are disjoint finite sets and that $|S_1|$ and $|S_2|$ are known, how do we find $|S_1 \cup S_2|$? Similarly, given that S_1 and S_2 are finite sets, how is $|S_1 \times S_2|$ determined from $|S_1|$ and $|S_2|$? These questions are answered by the addition principle and the multiplication principle, which are introduced in this section.

We begin with the addition principle for two sets. Let S_1 have cardinality m and let S_2 have cardinality n, where m and n are positive integers and $S_1 \cap S_2 = \emptyset$; say, $S_1 = \{x_1, x_2, \ldots, x_m\}$ and $S_2 = \{y_1, y_2, \ldots, y_n\}$. Then $S_1 \cup S_2 = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$, and it is easy to see that $S_1 \cup S_2$ has cardinality $m + n$. This observation yields the following theorem.

Theorem 5.1: If S_1 and S_2 are nonempty, disjoint finite sets, then

$$|S_1 \cup S_2| = |S_1| + |S_2|$$

■

Theorem 5.1 is obvious if $|S_2| = 1$. This case can be used to anchor a rigorous proof by induction on $|S_2|$ (see Exercise 12).

In Theorem 5.1, let $S = S_1 \cup S_2$. Then S_1 and S_2 are two nonempty disjoint subsets of S whose union is S. More generally, we may have n nonempty, pairwise-disjoint subsets $S_1, S_2, \ldots, S_n$ of a set S such that $S = S_1 \cup S_2 \cup \cdots \cup S_n$. We wish to generalize Theorem 5.1 to this case.

Definition 5.1: Let S be a nonempty set and let n be a positive integer. Given that $S_1, S_2, \ldots, S_n$ are n nonempty, pairwise-disjoint subsets of S such that $S = S_1 \cup S_2 \cup \cdots \cup S_n$, then the collection $\{S_1, S_2, \ldots, S_n\}$ is called a ***partition*** of S. The number n of subsets in a partition is called the ***degree*** of the partition, or the ***number of parts*** in the partition.

■

Example 5.1: The collection $\{\{1,2,3\},\{4,5\},\{6,7\}\}$ is a partition of $\{1,2,3,4,5,6,7\}$ into 3 parts. The collection $\{\{1\},\{2,3,5,7\},\{4,6\}\}$ is a different partition of degree 3 of $\{1,2,3,4,5,6,7\}$. Note that, because a partition of a set S is a set (of subsets of S), we know when two partitions of S are, in fact, the same. For instance, $\{\{0,1,2,9\},\{3,4,8\},\{5,6\},\{7\}\}$ and $\{\{7\},\{5,6\},\{3,4,8\},\{0,1,2,9\}\}$ are equal partitions of the set $\{0,1,2,3,4,5,6,7,8,9\}$ into 4 parts. Also note that, given a nonempty set S, there is only one partition of S into 1 part, namely, $\{S\}$.
■

Theorem 5.2 (Addition Principle): Let S be a nonempty finite set, let $n \geq 2$ be an integer, and let $\{S_1, S_2, \ldots, S_n\}$ be a partition of S. Then

$$|S| = |S_1| + |S_2| + \cdots + |S_n|$$

Proof: We proceed by induction on n, letting $P(n)$ denote the following statement (propositional function): For any nonempty finite set S, if $\{S_1, S_2, \ldots, S_n\}$ is a partition of S into n parts, then

$$|S| = |S_1| + |S_2| + \cdots + |S_n|$$

Note that $P(2)$ is true by Theorem 5.1.

Let k be an arbitrary integer, $k \geq 2$, and assume that $P(k)$ holds. More precisely, the induction hypothesis is that for any nonempty finite set S', if $\{S'_1, S'_2, \ldots, S'_k\}$ is a partition of S into k parts, then

$$|S'| = |S'_1| + |S'_2| + \cdots + |S'_k|$$

To complete the proof, it must be shown that $P(k+1)$ holds. To do this, let S be a nonempty finite set and let $\{S_1, S_2, \ldots, S_k, S_{k+1}\}$ be a partition of S into $k+1$ parts; we must show that

$$|S| = |S_1| + |S_2| + \cdots + |S_k| + |S_{k+1}|$$

Note that $\{S_1 \cup \cdots \cup S_k, S_{k+1}\}$ is a partition of S of degree 2. Hence, using Theorem 5.1 and the induction hypothesis (with $S' = S_1 \cup S_2 \cup \cdots \cup S_k$), we have

$$\begin{aligned} |S| &= |S_1 \cup S_2 \cup \cdots \cup S_k| + |S_{k+1}| \\ &= |S_1| + |S_2| + \cdots + |S_k| + |S_{k+1}| \end{aligned}$$

as was to be shown.

■

Example 5.2: The set S of 52 cards in a standard deck may be partitioned as $\{S_1, S_2, S_3, S_4\}$, where S_1 is the set of spades, S_2 is the set of hearts, S_3 is the set of diamonds, and S_4 is the set of clubs. Note that $|S_1| = |S_2| = |S_3| = |S_4| = 13$, and $4(13) = 52$. Another way to partition S is $\{R_1, R_2, R_3\}$, where R_1 is the set of aces, R_2 is the set of face cards (jacks, queens, kings), and

$R_3 = S - (R_1 \cup R_2)$, that is, R_3 is the set of cards that are neither aces nor face cards (twos, threes, ..., nines, tens). Note that $|R_1| = 4$, $|R_2| = 12$, $|R_3| = 36$, and $4 + 12 + 36 = 52 = |S|$.

■

Example 5.3: The addition principle allows us to use a "divide and conquer" approach to solve certain counting problems. For example, consider baseball's World Series, in which the American League champion and the National League champion play a series of games, the winner of the series being the first team to win four games. Let us denote the outcome of a particular series as a k-tuple, $4 \le k \le 7$; for instance, (N, N, A, N, N) (or NNANN) indicates a series that the National League team wins in five games, with the American League team winning the third game, and (A, A, N, N, A, N, A) (or AANNANA) indicates a series that the American League team wins in seven games, with the National League team winning the third, fourth, and sixth games. How many outcomes are there for such a series?

Let S be the set of outcomes. First, we partition S as $\{S_1, S_2\}$, where S_1 is the set of outcomes won by the American League team and S_2 is the set of outcomes won by the National League team. By appealing to symmetry, we can certainly say that $|S_1| = |S_2|$, so that $|S| = 2\,|S_1|$. Now the problem will be solved if we can find $|S_1|$. Let's partition S_1 as $\{A_4, A_5, A_6, A_7\}$, where A_i is the set of outcomes won by the American League team in i games, $4 \le i \le 7$. Then $A_4 = \{\text{AAAA}\}$ and $A_5 = \{\text{NAAAA, ANAAA, AANAA, AAANA}\}$, so that $|A_4| = 1$ and $|A_5| = 4$.

Using techniques to be developed later in this section and in the next section, it can shown that $|A_6| = 10$ and $|A_7| = 20$. Thus, by the addition principle, we have

$$|S| = 2\,|S_1| = 2\,(|A_4|+|A_5|+|A_6|+|A_7|) = 2\,(1 + 4 + 10 + 20) = 70$$

■

An important instance of the addition principle is the following: Let S be a finite set such that $|S|$ is known and let T be a nonempty subset of S such that $|T|$ is to be determined. Often it may be easier to determine $|S - T|$ and then use the addition principle to compute $|T|$. This is done by observing that $\{T, S - T\}$ is a partition of S, so that

$$|T| = |S| - |S - T|$$

Another problem that comes up frequently is to determine $|A \cup B|$ for two finite sets A and B. Often, this problem can be handled by observing that $\{A - B, A \cap B, B - A\}$ is a partition of $A \cup B$ into 3 parts. Thus, to count the elements in A or B, we can count: (1) the number of elements in A but not in B; (2) the number of elements in both A and B; and (3) the number of elements in B but not in A, and then add these three counts to get our answer.

Example 5.4: Refer to Example 5.3 and find the number of outcomes for the World Series in which the American League team wins at least two games.

Solution: Let S be the set of outcomes for the World Series and let T be the set of outcomes in which the American League team wins at least two games. Example 5.3 shows that $|S| = 70$; we are asked to find $|T|$. Note that $S - T$ is the set of outcomes such that the American League team wins at most one game; that is, $S - T$ consists of the one outcome in which the National League team sweeps the series in four games, plus the four outcomes won by the National League team in five games. Thus, $|S - T| = 5$, and so

$$|T| = |S| - |S - T| = 70 - 5 = 65$$

■

Next we turn to the multiplication principle, beginning with the case of two finite sets S_1 and S_2. As an example, let $S_1 = \{0, 1\}$ and $S_2 = \{2, 3, 4\}$. Then

$$S_1 \times S_2 = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$$

Note that

$$|S_1 \times S_2| = 6 = 2 \cdot 3 = |S_1| \cdot |S_2|$$

This example suggests the following result.

Theorem 5.3: For finite sets S_1 and S_2,

$$|S_1 \times S_2| = |S_1| \cdot |S_2|$$

Proof: If S_1 or S_2 is empty, then the result of the theorem is immediate. So we may assume that $|S_1| = m > 0$ and $|S_2| = n > 0$, say $S_1 = \{x_1, x_2, \ldots, x_m\}$ and $S_2 = \{y_1, y_2, \ldots, y_n\}$. We proceed by induction on n, letting $P(n)$ denote the following statement: For any nonempty finite set $S_1 = \{x_1, x_2, \ldots, x_m\}$ of cardinality m, if $S_2 = \{y_1, y_2, \ldots, y_n\}$ is a nonempty finite set of cardinality n, then $|S_1 \times S_2| = |S_1| \cdot |S_2|$.

If $n = 1$, then $S_2 = \{y_1\}$ and $S_1 \times S_2 = \{(x_1, y_1), (x_2, y_1), \ldots, (x_m, y_1)\}$. Thus, $|S_1 \times S_2| = m = m \cdot 1 = |S_1| \cdot |S_2|$, so $P(1)$ holds.

Let k be an arbitrary positive integer and assume that $P(k)$ holds. More precisely, the induction hypothesis is that if S'_1 is any nonempty finite set and S'_2 is any finite set of cardinality k, then

$$|S'_1 \times S'_2| = |S'_1| \cdot |S'_2|$$

To complete the proof, it must be shown that $P(k+1)$ holds. To do this, let S_1 be any nonempty finite set and let S_2 be a finite set of cardinality $k + 1$, say, $S_2 = \{y_1, y_2, \ldots, y_k, y_{k+1}\}$. We must show that

$$|S_1 \times S_2| = |S_1| \cdot |S_2|$$

We partition S_2 as $\{S'_2, \{y_{k+1}\}\}$, where $S'_2 = S_2 - \{y_{k+1}\} = \{y_1, y_2, \ldots, y_k\}$. Note that $|S'_2| = k$ and that

$$S_1 \times S_2 = S_1 \times (S'_2 \cup \{y_{k+1}\}) = (S_1 \times S'_2) \cup (S_1 \times \{y_{k+1}\})$$

Here, we are using the result that, for any sets A, B, and C,

$$A \times (B \cup C) = (A \cup B) \times (A \cup C)$$

In fact, $\{S_1 \times S'_2, S_1 \times \{y_{k+1}\}\}$ is a partition of $S_1 \times S_2$. Thus, using the addition principle and the induction hypothesis, we have

$$\begin{aligned}
|S_1 \times S_2| &= |(S_1 \times S'_2) \cup (S_1 \times \{y_{k+1}\})| \\
&= |S_1 \times S'_2| + |S_1 \times \{y_{k+1}\}| \\
&= |S_1| \cdot |S'_2| + |S_1| \cdot |\{y_{k+1}\}| \\
&= |S_1| \cdot (|S'_2| + |\{y_{k+1}\}|) \\
&= |S_1| \cdot |S'_2 \cup \{y_{k+1}\}| \\
&= |S_1| \cdot |S_2|
\end{aligned}$$

as was to be shown.

■

Theorem 5.3 may be generalized to the product of $n \geq 2$ finite sets, as in Theorem 5.2. We can prove this result by induction on n in a manner similar to the proof of Theorem 5.2; the proof also uses the fact that, for $n \geq 3$,

$$|S_1 \times S_2 \times \cdots \times S_n| = |(S_1 \times S_2 \times \cdots \times S_{n-1}) \times S_n|$$

The complete proof is left for you to develop in Exercise 14.

Theorem 5.4 (Multiplication Principle): For $n \geq 2$ nonempty finite sets $S_1 S_2, \ldots, S_n$, if $S = S_1 \times S_2 \times \cdots \times S_n$, then

$$|S| = |S_1|\,|S_2| \cdots |S_n|$$

■

Corollary 5.5: For a nonempty finite set T, if $S = T \times T \times \cdots \times T$ is the n-fold product of T with itself, then

$$|S| = |T|^n$$

■

The statement of the multiplication principle in the preceding results no doubt seems quite abstract. An alternate formulation applies to a variety of combinatorial problems in which a sequence of elements is chosen.

Multiplication Principle: Suppose that a sequence $(x_1, x_2, \ldots, x_n)$ of $n \geq 2$ elements is to be chosen. If $x_1$can be chosen in k_1 ways and, for each choice of x_1, the element x_2 can be chosen in k_2 ways, and so on, until finally, for each choice of $(x_1, x_2, \ldots, x_{n-1})$, the element x_n can be chosen in k_n ways, then the sequence $(x_1, x_2, \ldots, x_n)$ can be chosen in $k_1 k_2 \cdots k_n$ ways.

■

Example 5.5: To play a certain state lottery, a person must choose a sequence of four digits, for instance, 3738 or 0246. How many ways are there to play the lottery if:

(a) There are no restrictions?
(b) No digit is repeated?
(c) The digit 0 is used at least once?
(d) The last digit cannot be 0 nor can two 0s be consecutive?
(e) Each digit is prime?
(f) At least one of the digits is prime?

Solution: Let $D = \{0, 1, \ldots, 9\}$ be the set of digits. In playing the lottery, a person chooses a sequence (x_1, x_2, x_3, x_4) with each $x_i \in D$. Thus:

(a) The number of ways to play the lottery is

$$|D \times D \times D \times D| = |D|^4 = 10^4 = 10,000$$

by Corollary 5.5.

(b) Let us use the alternate formulation of the multiplication principle to find the number of ways to play the lottery if no digit is repeated. There are 10 choices for x_1. Once x_1 has been chosen, then x_2 must be selected from the set $D - \{x_1\}$. So there are 9 choices for x_2. Then x_3 must be chosen from the set $D - \{x_1, x_2\}$, so there are 8 choices for x_3. Finally, x_4 must be chosen from $D - \{x_1, x_2, x_3\}$, so there are 7 choices for x_4. Thus, altogether, there are $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ ways to play if no digit is repeated.

(c) Let $S = D \times D \times D \times D$ be the set of ways to play the lottery and let T be the set of ways to play using the digit 0 at least once. Here we use the technique of finding $|S - T|$. Note that $S - T$ is the set of ways to play the lottery without using the digit 0, so that

$$S - T = (D - \{0\}) \times (D - \{0\}) \times (D - \{0\}) \times (D - \{0\})$$

Thus,

$$|S - T| = |D - \{0\}|^4 = 9^4$$

so that

$$|T| = |S| - |S - T| = 10^4 - 9^4 = 3439$$

(d) Here the restriction is to disallow sequences such as 9870 that end with a 0, and also to disallow sequences such as 5002 that have two 0s consecutive. We let X be the set of ways to play the lottery under this restriction. Suppose we try to find $|X|$ using the multiplication principle. There are 10 ways to choose x_1. However, having chosen x_1, the number of choices for x_2 depends on whether or not $x_1 = 0$; if $x_1 \neq 0$, then there are 10 choices for x_2, but if $x_1 = 0$, then there are only 9 choices for x_2 since x_1 and x_2 cannot both be 0. When a situation of this kind arises, the solution can often be found by partitioning X and employing the addition principle. Here we partition X as $\{X_1, X_2, X_3\}$, where

$$\begin{aligned}
X_1 &= \{(x_1, x_2, x_3, x_4) \in X \mid x_1 = 0\} \\
X_2 &= \{(x_1, x_2, x_3, x_4) \in X \mid x_1 \neq 0 \text{ and } x_2 = 0\} \\
X_3 &= \{(x_1, x_2, x_3, x_4) \in X \mid x_1 \neq 0 \text{ and } x_2 \neq 0\}
\end{aligned}$$

By using the multiplication principle, it can be checked that

$$|X_1| = 9^2 \cdot 10 \qquad |X_2| = 9^3 \qquad |X_3| = 9^3 \cdot 10$$

Let's check $|X_1|$, for instance. Since $x_1 = 0$, we have $x_2 \in D - \{0\}$, $x_3 \in D$, and $x_4 \in D - \{0\}$. Thus, there are 9 choices for x_2, 10 choices for x_3, and 9 choices for x_4. So $|X_1| = 9^2 \cdot 10$. Now, by the addition principle,

$$|X| = |X_1| + |X_2| + |X_3| = 9^2(10 + 9 + 90) = 8829$$

We'll go over parts (e) and (f) in class.

■

Example 5.6: Find the number of positive factors of $n = 21168 = 2^4 \cdot 3^3 \cdot 7^2$.

Solution: The key observation here is that m is a positive factor of n if and only if

$$m = 2^r \cdot 3^s \cdot 7^t$$

with integers r, s, and t satisfying $0 \leq r \leq 4$, $0 \leq s \leq 3$, and $0 \leq t \leq 2$. Thus, we choose a positive factor of n by choosing values for r, s, and t. For example, if we choose $r = 1$, $s = 0$, and $t = 2$, we obtain $k = 98$, whereas choosing $r = 4$, $s = 3$, and $t = 0$ results in $k = 432$. Hence, the number of positive factors of 21168 is the number of sequences (r, s, t) with $r, s, t \in \mathbb{Z}$ and $0 \leq r \leq 4$, $0 \leq s \leq 3$, and $0 \leq t \leq 2$. This, by the multiplication principle, is $5 \cdot 4 \cdot 3 = 60$.

∎

A convenient way to illustrate the multiplication principle is with the aid of a picture called a ***tree diagram***. Consider the case where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. The tree diagram for $X \times Y$ is shown in Figure 5.7. Notice that there are three arcs leaving the root of the tree; these correspond to the elements of X and are labeled as such. Leaving each node at level 1 are four arcs corresponding to and labeled with the elements of Y. To obtain an element of $X \times Y$, we follow a path from the root, first to a node at level 1, and then to a leaf of the tree; if the first arc of this path is labeled x_i and the second arc is labeled y_j, then the associated element of $X \times Y$ is (x_i, y_j) and the leaf is given this label, as shown in the figure.

More generally, the arcs of a tree diagram may be labeled with sets. (The set labeling an arc might represent, for example, a set of choices that are equivalent, in some sense, at some stage in forming a combinatorial object.) If an arc is labeled with a set X of cardinality k, then we think of this arc as actually representing k multiple arcs, one corresponding to each element of X. In this case, if a path from the root to a leaf of the tree has arcs labeled with the sets $X_1, X_2, \ldots, X_n$, respectively, then the leaf is labeled with $X_1 \times X_2 \times \cdots \times X_n$. Alternately, we may label the leaf with $|X_1 \times X_2 \times \cdots \times X_n|$, which represents the total number of paths from the root to that leaf, in the sense that the arc labeled X_i really represents $|X_i|$ multiple arcs. The following example should help clarify these ideas.

Example 5.7: Consider the state lottery described in Example 5.5. A tree diagram may be used to count the number of ways to play the lottery subject to the restriction of part (d), namely, that the last digit cannot be 0 nor can two 0s be consecutive. This tree diagram is shown in Figure 5.8; the digraph shown is $T = (V, A)$, where $V = \{a, b, c, d, e, f, u, v, w, x, y, z\}$ and

$$A = \{(a, b), (a, c), (b, d), (b, e), (c, f), (d, u), (e, v), (f, w), (u, x), (v, y), (w, z)\}$$

We let $D = \{0, 1, \ldots, 9\}$ and let X be the set of ways to play the lottery subject to the given restriction.

Let $(x_1, x_2, x_3, x_4) \in X$. As noted in Example 5.5, the set of choices for x_2 depends on whether $x_1 \neq 0$ or $x_1 = 0$. Thus, our tree diagram has two arcs leaving the root a; arc (a, b) is labeled with the set $D - \{0\}$ and represents the choice $x_1 \neq 0$, whereas arc (a, c) is labeled with $\{0\}$, representing the choice $x_1 = 0$.

Next, we focus our attention on node b. Since $x_1 \neq 0$ in this case, it is possible for x_2 to be 0, and the set of choices for x_3 depends on whether $x_2 \neq 0$ or $x_2 = 0$. Hence, the tree diagram has two arcs leaving node b; arc (b, d) is labeled $D - \{0\}$ and arc (b, e) is labeled $\{0\}$. Let us focus our

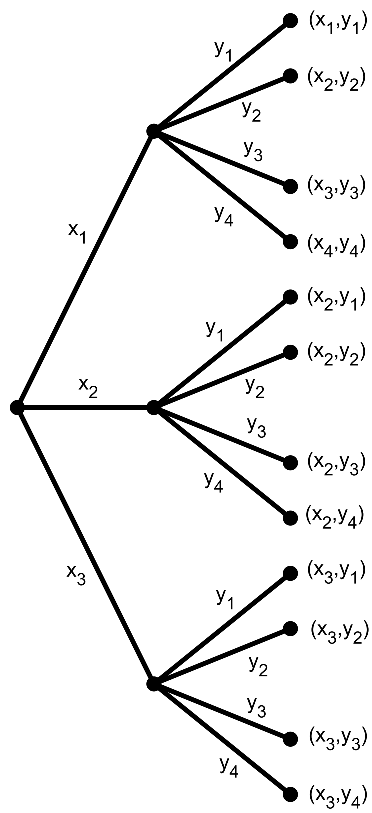

Figure 5.7 A tree diagram

attention next on node c. In this case, $x_1 = 0$, so that x_2 cannot be 0. So there is just one arc (c, f) leaving node c and this arc is labeled $D - \{0\}$.

At node d we have $x_2 \neq 0$, so that x_3 is allowed to be 0 but x_4 is not. Thus, from d there is a path of length two to the leaf x; arc (d, u) is labeled D and arc (u, x) is labeled $D - \{0\}$. The leaf x is labeled 7290; this is the cardinality of the set $(D - \{0\}) \times (D - \{0\}) \times D \times (D - \{0\})$ and is the number of sequences $(x_1, x_2, x_3, x_4) \in X$ with $x_1 \neq 0$ and $x_2 \neq 0$. Notice that $7290 = 9 \cdot 9 \cdot 10 \cdot 9$ is the product of the cardinalities of the sets labeling the arcs along the path from the root a to the leaf x.

At node e we have $x_2 = 0$, so that neither x_3 nor x_4 is allowed to be 0. This is the reason for the path of length two from e to the leaf y, with both arcs (e, v) and (v, y) labeled $D - \{0\}$. The label $729 = 9 \cdot 1 \cdot 9 \cdot 9$ on the leaf y is $|(D - \{0\}) \times \{0\} \times (D - \{0\}) \times (D - \{0\})|$ and is the number of sequences (x_1, x_2, x_3, x_4) in X with $x_1 \neq 0$ and $x_2 = 0$.

The analysis at node f is done in a similar fashion.

The cardinality of X is the total number of paths from the root a to the leaves of the tree. By our analysis, this is just the sum of the labels on the leaves, so that we obtain

$$|X| = 7290 + 729 + 810 = 8829$$

■

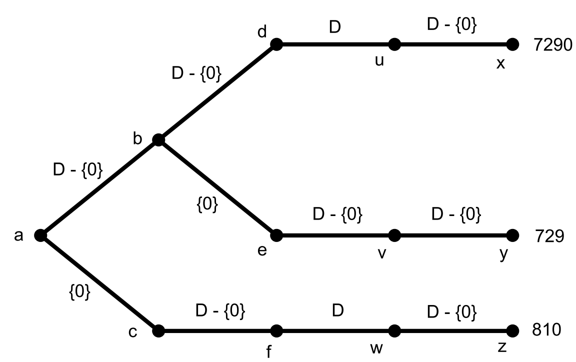

Figure 5.8 Tree diagram for Example 5.7

Exercise Set 5.2

1. Robert, a mathematician, has seven shirts and ten ties. If he is of the opinion that any shirt goes with any tie, how many shirt-tie combinations does he have?

2. A professor has five books on algebra, six books on geometry, and seven books on number theory. How many ways are there for a student to choose two books not both on the same subject?

3. A Social Security number is a sequence of nine digits, for example, 080-55-1617.

   (a) How many possible Social Security numbers are there?

How many possible Social Security numbers are there satisfying the following restrictions?

   (b) The digit 0 is not used.
   (c) The sequence neither begins nor ends with the digit 0.
   (d) No digit is repeated (used more than once).

4. How many different collections of soda pop cans can be formed from three (identical) Pepsi-Cola cans, three (identical) Coca-Cola cans, and six (identical) Seven-Up cans?

5. An identifier in the programming language Ada consists of letters, digits, and underscore characters. (Ada uses the underscore character "_" as a separation character, to make identifiers such as UNIT_COST easier to read.) An identifier must begin with a letter, may not end with a underscore character, and may not contain two consecutive underscore characters. (Also, Ada makes no distinction between uppercase and lowercase letters.)

   (a) How many three-character Ada identifiers are there?
   (b) How many four-character Ada identifiers are there?
   (c) How many five-character Ada identifiers contain at most one underscore character?
   (d) How many five-character Ada identifiers contain exactly two underscore characters?
   (e) How many five-character Ada identifiers are there?

6. How many different five-digit integers can be formed using the five digits $1, 2, 2, 2, 3$?

7. Consider the experiment of choosing, at random, a sequence of 4 cards from a standard deck of 52 playing cards. The cards are chosen one at a time with replacement, meaning that after a card is chosen it is replaced and the deck is reshuffled. We are interested in the sequence of cards that is obtained, for example (two of clubs, ace of hearts, king of spades, two of clubs).

   (a) Determine the number of sequences.

Determine the number of sequences satisfying the following restrictions:

   (b) None of the cards are spades.
   (c) All 4 cards are spades.
   (d) All 4 cards are the same suit.
   (e) The first card is a king and the third card is not an ace.
   (f) At least one of the cards is a spade.

8. How many five-letter strings can be formed using the letters a, b, c, d (with repeated letters allowed)? How many of these do not contain the substring "bad"?

9. A box contains three balls colored red, blue, and green. An experiment consists of choosing at random a sequence of three balls, one at a time, with replacement. An outcome for this experiment is written as an ordered triple, for example, (B, R, G) denotes that the first ball chosen is blue, the second is red, and the third is green.

(a) How many outcomes are there for this experiment?

(b) Draw a tree diagram that helps answer the question, How many outcomes for the experiment have the property that no two balls chosen consecutively have the same color?

(c) Draw a tree diagram that helps answer the question, How many outcomes for the experiment have the property that at least two of the balls chosen are blue?

10. Refer to Example 1.3 and use a tree diagram to help determine the number of outcomes in each of the following cases:

(a) The National League team wins the series in four games.
(b) The American League team wins the series in five games.
(c) The National League team wins the series in six games.
(d) The American League team wins the series in seven games.

11. Three different mathematics final examinations and two different computer science final examinations are to be scheduled during a five-day period. Suppose that each exam is to be scheduled from 1 PM to 4 PM on one of the days.

(a) In how many ways can these examinations be scheduled if there are no restrictions?

In how many ways can these examinations be scheduled under the following restrictions?

(b) No two exams can be scheduled for the same day.

(c) No two of the mathematics exams can be scheduled for the same day, nor can the two computer science exams be scheduled for the same day.

(d) Each mathematics exam must be the only exam scheduled for the day on which it is scheduled.

Hint: Denote the mathematics exams A, B, and C, the computer science exams D and E, and the days 1, 2, 3, 4, 5; a complete schedule for the exams can be expressed as a 5-tuple, for example, $(1, 3, 5, 3, 2)$ indicates that exams A, B, C, D, and E are scheduled for days 1, 3, 5, 3, and 2, respectively.

12. Prove Theorem 1.1 by using induction on $|S_2|$.

13. Consider the set $S = \{1000, 1001, 1002, \ldots, 9999\}$ of 4-digit positive integers.

(a) Find $|S|$.
(b) How many elements of S are even?
(c) How many elements of S contain the digit 8 exactly once?
(d) How many elements of S contain the digit 8 (at least once)?
(e) How many elements of S are palindromes? (A positive integer is a *palindrome* if the integer remains the same when its digits are reversed, as, for example, in 1221 or 37873.)

14. Prove Theorem 1.4 by using induction on n.

15. In the game of Mastermind, one player, the "code maker," selects a sequence of four colors; this is the "code." The colors are chosen from the set {red, blue, green, white, black, yellow}. For example, (green, red, blue, red) and (white, black, yellow, red) are possible codes.

(a) How many codes are there?
(b) How many codes use four colors?

16. A box contains 12 distinct colored balls, numbered 1 through 12. Balls 1, 2, and 3 are red; balls 4, 5, 6, and 7 are blue; and balls 8, 9, 10, 11, and 12 are green. An experiment consists of choosing three balls at random from the box, one at a time, with replacement; the outcome is the sequence of three balls chosen: $(7, 4, 9)$ or $(3, 8, 3)$, for example.

(a) How many outcomes are there?

How many outcomes satisfy the following conditions?

(b) The first ball is red, the second is blue, and the third is green.
(c) The first and second balls are green and the third ball is blue.
(d) Exactly two balls are green.
(e) All three balls are red.
(f) All three balls are the same color.
(g) At least one of the three balls is red.

17. A certain make of lock has the integers 0 through 49 arranged in a circle. A code for opening the lock consists of a sequence of three of these numbers — $(7, 32, 18)$, for instance.

(a) How many codes are there?
(b) How many codes (x_1, x_2, x_3) have the property that $x_1 \neq x_2$ and $x_2 \neq x_3$?

18. Generalizing Example 1.6, let $p_1, p_2, \ldots, p_k$ be distinct primes, let $a_1, a_2, \ldots, a_k$ be positive integers, and let

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot \cdots \cdot p_k^{a_k}$$

Determine the number of positive factors of n.

19. Let $X = \{1, 2, 3, 4, 5\}$ and let $Y = \{1, 2, 3, 4, 5, 6, 7\}$.

(a) Find the number of functions from X to Y. Hint: Think of the function $f\colon X \to Y$ as the sequence $(f(1), f(2), f(3), f(4), f(5))$.
(b) Find the number of one-to-one functions from X to Y.
(c) Find the number of functions from X onto Y.
(d) Find the number of functions f from X to Y with the property that $f(x)$ is odd for at least one $x \in X$.
(e) Find the number of functions f from X to Y with the property that $f(1)$ is odd or $f(5) = 5$. Hint: Use the addition principle.
(f) Find the number of functions f from X to Y with the property that $f^{-1}(\{5\}) = \{5\}$.
(g) Find the number of functions from Y onto X.

5.3 PERMUTATIONS AND COMBINATIONS

For positive integers m and n, let X be a set of cardinality m and let Y be a set of cardinality n. We ask, How many functions are there from X to Y? In particular, we let $X = \{1, 2, \ldots, m\}$, $Y = \{1, 2, \ldots, n\}$, and $f\colon X \to Y$. Then f is determined by its m images $f(1)$, $f(2)$, …, $f(m)$. In fact, f can be specified by giving these images in a sequence: $(f(1), f(2), \ldots, f(m))$. For instance, if $m = 3$, $n = 4$, and we give the sequence $(2, 4, 4)$, then the function $f\colon X \to Y$ is defined by

$f(1) = 2$, $f(2) = 4$, and $f(3) = 4$. Thus, the number of functions from X to Y is the number of sequences $(f(1), f(2), \ldots, f(m))$ with each $f(x) \in Y$. Such a sequence has m elements and there are n choices for each element. Hence, from the multiplication principle, we obtain the following result.

Theorem 5.6: For positive integers m and n, let X be a set of cardinality m and let Y be a set of cardinality n. Then the number of functions from X to Y is n^m.

■

Some books use the notation Y^X to denote the set of functions from the set X to the set Y. Theorem 5.6 can then be stated as follows: For finite, nonempty sets X and Y,

$$| Y^X | = | Y |^{|X|}$$

Next, let us restrict the set of functions from X to Y. Recall that $f\colon X \to Y$ is *one-to-one* provided the condition

$$x_1 \neq x_2 \to f(x_1) \neq f(x_2)$$

holds for all $x_1, x_2 \in X$. For $X = \{1, 2, \ldots, m\}$, this condition is that the m images $f(1)$, $f(2)$, …, $f(m)$ are distinct or, in other words, that

$$|\operatorname{im} f| = |\{f(1), f(2), \ldots, f(m)\}| = m$$

So we ask, For X of cardinality m and Y of cardinality n, how many one-to-one functions are there from X to Y?

Again, we let $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$. As before, a function $f\colon X \to Y$ is determined by its sequence of images $(f(1), f(2), \ldots, f(m))$. If f is one-to-one, then $(f(1), f(2), \ldots, f(m))$ is a sequence of m distinct elements from Y.

Definition 5.2: For a nonempty set Y and a positive integer r, an ***r*-permutation** of Y is an r-tuple $(y_1, y_2, \ldots, y_r)$, where $y_1, y_2, \ldots, y_r$ are distinct elements of Y. If Y is finite and $|Y| = n \geq r$, then the number of r-permutations of Y is denoted by $P(n, r)$ or $(n)_r$; also, an n-permutation of Y is simply called a ***permutation*** of Y.

■

As indicated by the definition, we can form an r-permutation of a nonempty set Y by making an ordered selection of r distinct elements from Y. For instance, suppose $Y = \{1, 2, 3, 4\}$ and $r = 3$, so that we seek the 3-permutations (y_1, y_2, y_3) of Y. If we select, in order, $y_1 = 2$, $y_2 = 1$, and $y_3 = 4$, then we obtain the 3-permutation $(2, 1, 4)$. How many 3-permutations of $\{1, 2, 3, 4\}$ are there? Well, there are four choices for the first element y_1, then three choices for y_2, and then two choices for y_3. By the multiplication principle, there are $4 \cdot 3 \cdot 2 = 24$ choices in all; that is, the number of 3-permutations of $\{1, 2, 3, 4\}$ is $P(4, 3) = 24$. Here is a list of them:

$$\begin{array}{cccccc}
(1,2,3) & (1,3,2) & (2,1,3) & (2,3,1) & (3,1,2) & (3,2,1) \\
(1,2,4) & (1,4,2) & (2,1,4) & (2,4,1) & (4,1,2) & (4,2,1) \\
(1,3,4) & (1,4,3) & (3,1,4) & (3,4,1) & (4,1,3) & (4,3,1) \\
(2,3,4) & (2,4,3) & (3,2,4) & (3,4,2) & (4,2,3) & (4,3,2)
\end{array}$$

Note that each 3-permutations of $\{1, 2, 3, 4\}$ can be viewed as a function $f\colon\ \{1, 2, 3\} \to \{1, 2, 3, 4\}$ such that f is one-to-one. For instance, $(3, 4, 1)$ can be interpreted as the function f defined by $f(1) = 3$, $f(2) = 4$, and $f(3) = 1$. Conversely, any given one-to-one function $f\colon\ \{1, 2, 3\} \to \{1, 2, 3, 4\}$ determines a 3-permutation of $\{1, 2, 3, 4\}$, namely, $(f(1), f(2), f(3))$. Thus, the number of one-to-one functions from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ is $P(4, 3) = 24$.

In general, we let $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$, where $1 \le m \le n$. Given an m-permutation of Y, say, $(y_1, y_2, \ldots, y_m)$, this determines a one-to-one function $f\colon\ X \to Y$, namely, $f(1) = y_1$, $f(2) = y_2$, …, $f(m) = y_m$. Conversely, each one-to-one function $f\colon\ X \to Y$ determines an m-permutation of Y, namely, $(f(1), f(2), \ldots, f(m))$. Thus, we obtain the following result.

Theorem 5.7: For integers m and n with $1 \le m \le n$, let X be a set of cardinality m and let Y be a set of cardinality n. Then the number of one-to-one functions from X to Y is $P(n, m)$.

■

In the context of this result, note that it is impossible to have one-to-one functions from an m-element set to an n-element set if $m > n$. This is one version of the so-called *pigeonhole principle* and is stated as Theorem 3.1, part 1.

In light of the preceding discussion, it would be useful to have a formula for computing the number $P(n, r)$ of r-permutations of an n-element set. Such a formula is easily obtained by applying the multiplication principle.

Theorem 5.8: For integers r and n with $1 \le r \le n$,

$$P(n, r) = \prod_{k=1}^{r} (n - k + 1) = n(n-1)\,\cdots\,(n - r + 1)$$

Proof: Let Y be an n-element set. As noted, an r-permutation of Y is a sequence $(y_1, y_2, \ldots, y_r)$ of distinct elements of Y. There are n choices for y_1; thus, $P(n, 1) = n$ and the result of the theorem holds for the case $r = 1$. So we may assume $2 \le r \le n$. Since y_2 must be selected from the set $Y - \{y_1\}$, there are $n - 1$ choices for y_2. In general, having already selected the first $k - 1$ elements y_1, y_2, …, y_{k-1}, $k \ge 2$, the next element y_k must be selected from the set $Y - \{y_1, y_2, \ldots, y_{k-1}\}$, so there are $n - (k - 1) = n - k + 1$ choices for y_k. The result of the theorem then follows from the multiplication principle.

■

It is easy to remember the formula for $P(n, r)$ given by Theorem 5.8, namely, $P(n, r)$ is a product of r factors, the first factor is n, and each factor thereafter is one less than the preceding factor.

Theorems 5.6 and 5.7 can be restated explicitly in the context of choosing a sequence $(y_1, y_2, \ldots, y_r)$ of r elements from an n-element set Y. If repetition is allowed, then the number of ways to choose such a sequence is, by the multiplication principle, n^r. On the other hand, if repetition is not allowed, then necessarily $r \le n$ and $(y_1, y_2, \ldots, y_r)$ is an r-permutation of Y; thus, the number of ways to choose such a sequence is $P(n, r)$.

For a nonnegative integer n, recall that ***n-factorial*** is written $n!$ and is defined recursively by

$$0! = 1$$
$$n! = n \cdot (n-1)!, \quad n \ge 1$$

Alternately, for $n \geq 1$,

$$n! = \prod_{k=1}^{n} k$$

Thus, we can remark that

$$P(n, r) = \frac{n!}{(n-r)!}$$

We also see that $P(n, n) = n!$, yielding the following corollaries.

Corollary 5.9: For a positive integer n, the number of permutations of an n-element set is n-factorial.

■

Corollary 5.10: For a positive integer n, let X and Y be sets of cardinality n. Then the number of bijections from X to Y is n-factorial.

■

Example 5.8: To play the state lottery of Example 5.5, a person chooses a sequence (x_1, x_2, x_3, x_4) of four digits. As noted, there are 10^4 ways to play this lottery. If someone wishes to play subject to the restriction that no digit is repeated, then (x_1, x_2, x_3, x_4) is a 4-permutation of $D = \{0, 1, \ldots, 9\}$. Hence, there are

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

ways to play subject to this restriction.

■

Example 5.9: Consider forming five-letter code words using the letters a, b, c, d, e, f, g, and h, for example, fabda or aghcd. By the multiplication principle, there are 8^5 such code words.

(a) How many code words have no repeated letter?
(b) How many code words have no repeated letters and include the letter a?
(c) How many code words contain at least one repeated letter?

How many code words contain exactly one a, exactly one b, and have the a before the b?

(d) Answer in the case that repeated letters are allowed.
(e) Answer in the case that repeated letters are not allowed.

Solution: Think of a code word as a sequence $(x_1, x_2, x_3, x_4, x_5)$, where each $x_i \in \{$a, b, c, d, e, f, g, h$\}$.

(a) A code word with no repeated letter is a 5-permutation of $\{$a, b, c, d, e, f, g, h$\}$, so that the number of such words is

$$P(8, 5) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$$

(b) A code word with no repeated letter that includes the letter a can be formed in a unique way by first choosing a 4-permutation of $\{$b, c, d, e, f, g, h$\}$ and then choosing the position of the letter a relative to the other four letters. For instance, if we first choose the 4-permutation $(\mathrm{e}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ and then choose to put the letter a in position four, then we obtain the code word ebcad. Since the number of 4-permutations of {b, c, d, e, f, g, h} is $P(7,4)$, and there are five possible positions for the letter a, the number of code words that have no repeated letter and include the letter a is

$$P(7,4) \cdot 5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 5 = 4200$$

(Exercise 33 suggests an alternate solution using the addition principle.)

(c) By using the addition principle, we can find the answer to this part by subtracting the number of code words with no repeated letters from the total number of code words. Thus, the number of code words containing at least one repeated letter is

$$8^5 - P(8,5) = 32768 - 6720 = 26048$$

We will go over parts (d) and (e) in class.

■

In choosing an r-permutation of a set Y, the order in which the elements are chosen is important. In playing the lottery of Example 5.5, for instance, the sequences 7391 and 1379 are different, even though they both involve the same four elements 1, 3, 7, and 9. However, consider a lottery such as New York's Lotto 54. To play, someone chooses a *subset* of six numbers from the set $S = \{1, 2, \ldots, 54\}$, so that what matters is the set of six numbers selected and not the order in which they are selected. Suppose we wish to determine the number of ways to play Lotto 54, or the number of ways to play subject to the restriction that exactly four of the numbers chosen are prime. Since someone plays Lotto 54 by choosing a 6-element subset of $\{1, 2, \ldots, 54\}$, we wish to determine the number of 6-element subsets of a 54-element set. Similarly, finding the number of ways to play subject to the restriction that four of the numbers chosen are prime involves finding the number of 4-element subsets of a 16-element set, namely, $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53\}$, as well as the number of 2-element subsets of a 38-element set, namely, $S - P$.

Now we introduce some general terminology and notation. We let n be a positive integer, S be an n-element set, and r be an integer with $0 \leq r \leq n$. Then any r-element subset of S is called an ***r-combination*** of S. We denote the number of r-combinations of an n-element set by

$$C(n,r) \quad \text{or} \quad \binom{n}{r}$$

$C(n,r)$ is read "n choose r" and is sometimes referred to as the ***number of combinations of n things taken r at a time***. Note that an r-combination of S is a subset of S and hence is not allowed to contain repeated elements.

To help us determine a formula for $C(n,r)$, we consider the following exercise: List the 3-combinations of $S = \{1, 2, 3, 4\}$, and next to each subset list the 3-permutations of S using the elements from that subset. Here goes:

$$\begin{array}{ll}
\{1,2,3\}: & (1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1) \\
\{1,2,4\}: & (1,2,4),(1,4,2),(2,1,4),(2,4,1),(4,1,2),(4,2,1) \\
\{1,3,4\}: & (1,3,4),(1,4,3),(3,1,4),(3,4,1),(4,1,3),(4,3,1) \\
\{2,3,4\}: & (2,3,4),(2,4,3),(3,2,4),(3,4,2),(4,2,3),(4,3,2)
\end{array}$$

Note that each of the four 3-combinations of S yields six 3-permutations of S. Thus, we see that

$$P(4,3) = 24 = 6 \cdot 4 = P(3,3) \cdot C(4,3) = 3! \cdot C(4,3)$$

This example suggests the following general relationship between $P(n,r)$ and $C(n,r)$.

Theorem 5.11: For integers r and n with $1 \leq r \leq n$, we have

$$P(n,r) = r! \cdot C(n,r)$$

Proof: Consider any r-element subset of an n-element set S. Such a subset gives rise to r-factorial r-permutations of S, because the r elements in the subset can be ordered in $P(r,r) = r!$ ways. Moreover, every r-permutation of S is uniquely determined in this way. Thus, we obtain the relation

$$P(n,r) = r! \cdot C(n,r)$$

■

By Theorem 5.8,

$$P(n,r) = \frac{n!}{(n-r)!}$$

for $1 \leq r \leq n$. Applying Theorem 5.11 then yields

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

for $1 \leq r \leq n$. This formula for $C(n,r)$ in terms of factorials is also seen to hold in the case $r = 0$.

Corollary 5.12: Let n be a positive integer and let r be an integer such that $0 \leq r \leq n$. Then

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

■

The above formula for $C(n,r)$ has theoretical value but should not be used to compute $C(n,r)$. Instead, notice that

$$C(n,0) = C(n,n) = 1$$

since a set $S = \{x_1, x_2, \ldots, x_n\}$ has one 0-element subset (the empty set) and one n-element subset (itself), and

$$C(n,1) = C(n,n-1) = n$$

since S has n one-element subsets, namely, $\{x_1\}$, $\{x_2\}$, …, $\{x_n\}$, and n subsets of cardinality $n-1$, namely, $S - \{x_1\}$, $S - \{x_2\}$, …, $S - \{x_n\}$. For $2 \leq r \leq n-2$, we use the formula

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\,\cdots\,(n-r+1)}{r(r-1)\,\cdots\,1}$$

This formula is easy to remember since both the numerator and denominator are products containing r factors.

Example 5.10: The number of ways to play Lotto 54 is

$$C(54,6) = \frac{P(54,6)}{6!} = \frac{54 \cdot 53 \cdot 52 \cdot 51 \cdot 50 \cdot 49}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 25827165$$

Find the number of ways to play Lotto 54 subject to the following restrictions:

(a) 19 is one of the numbers chosen.
(b) Exactly four of the numbers chosen are prime.
(c) At least one of the numbers chosen is prime.
(d) The difference between the largest number chosen and smallest number chosen is 30.
(e) The number 19 is not chosen.
(f) If 19 is chosen, then 37 is also chosen.

Solution: Let $S = \{1, 2, \ldots, 54\}$ and let

$$P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53\}$$

be the set of primes between 1 and 54.

(a) To play subject to the restriction that 19 is one of the numbers chosen, one must choose five additional numbers from the set $S - \{19\}$. The number of ways to do this is

$$C(53,5) = \frac{P(53,5)}{5!} = \frac{53 \cdot 52 \cdot 51 \cdot 50 \cdot 49}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2869685$$

(b) To play subject to the restriction that four of the numbers chosen are prime, one can first choose four numbers from the set P and then choose two numbers from the set $S - P$. By the multiplication principle, the number of ways to do this is

$$C(16,4) \cdot C(38,2) = \frac{P(16,4)}{4!} \cdot \frac{P(38,2)}{2!} = \frac{16 \cdot 15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{38 \cdot 37}{2 \cdot 1} = 1279460$$

(c) Let X be the set of ways to play Lotto 54 and let Y be the set of ways to play subject to the restriction that at least one of the six numbers chosen is prime. Then $X - Y$ is the set of ways to play without choosing any prime numbers, that is, with all six numbers chosen from $S - P$. Thus, $|X - Y| = C(38,6)$. Then, using part (a) and the addition principle,

$$|Y| = |X| - |X - Y| = C(54,6) - C(38,6) = 23066484$$

(d) To play subject to the restriction that the difference between the largest and smallest numbers chosen is 30, one could first decide on the smallest number to choose; call it k. Then the largest number chosen is $k + 30$, and the other four numbers must be chosen from the 29-element set $\{k+1, k+2, \ldots, k+29\}$. Since $1 \leq k$ and $k + 30 \leq 54$, we have $1 \leq k \leq 24$. Thus, there are 24 choices for k and $C(29,4)$ ways to choose the four numbers other than k and $k + 30$. Hence, by the multiplication principle, the answer to this part is $24 \cdot C(29,4) = 570024$.

We'll go over parts (e) and (f) in class.

■

Example 5.11: Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 3\}$. Find the number of functions from X onto Y.

Solution: Let S be the set of functions from X onto Y and let $f \in S$. Then im $f = Y$; in fact, $\{f^{-1}(\{1\}), f^{-1}(\{2\}), f^{-1}(\{3\})\}$ is a partition of X into 3 parts. Since $|X| = 5$, there are two possibilities for the cardinalities of the sets in this partition: (1) $|f^{-1}(\{y_1\})| = 3$ and $|f^{-1}(\{y_2\})| = 1 = |f^{-1}(\{y_3\})|$, or (2) $|f^{-1}(\{y_1\})| = 2 = |f^{-1}(\{y_2\})|$ and $|f^{-1}(\{y_3\})| = 1$, where $\{y_1, y_2, y_3\} = \{1, 2, 3\}$. In words, either one element of Y has three preimages and the other two elements of Y have one preimage each, or two elements of Y have two preimages each and the other element of Y has one preimage.

This suggests using the addition principle to find $|S|$; let us partition S as $\{S_1, S_2\}$, where

$$S_1 = \{f \in S \mid \text{some } y \text{ in } Y \text{ has three preimages under } f\}$$

and

$$S_2 = \{f \in S \mid \text{some distinct } y_1 \text{ and } y_2 \text{ in } Y \text{ have two preimages each under } f\}$$

To find $|S_1|$, we note that an $f \in S_1$ is uniquely determined by the following sequence of choices: first, choose $y \in Y$; second, choose three preimages in X for y; third, complete f by choosing a bijection from $X - f^{-1}(\{y\})$ to $Y - \{y\}$. For example, choosing $y = 2$, then choosing $f^{-1}(\{2\}) = \{1, 3, 5\}$, and then completing f by choosing $f(2) = 1$ and $f(4) = 3$ yields f from X onto Y defined by $f(1) = 2$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, and $f(5) = 2$. Now, there are $C(3, 1) = 3$ choices for y, then $C(5, 3) = 10$ ways to choose $f^{-1}(\{y\})$ (since $f^{-1}(\{y\})$ is a 3-element subset of X), and, finally, $2! = 2$ bijections from $X - f^{-1}(\{y\})$ to $Y - \{y\}$. Thus, by the multiplication principle,

$$|S_1| = 3 \cdot 10 \cdot 2 = 60$$

To find $|S_2|$, we note that $f \in S_2$ is uniquely determined by the following sequence of choices: first, choose a subset $\{y_1, y_2\}$ of Y; second, choose two preimages in X for y_1; third, choose two preimages in $X - f^{-1}(\{y_1\})$ for y_2. By default, f maps the one remaining element of $X - (f^{-1}(\{y_1\}) \cup f^{-1}(\{y_2\}))$ to the one remaining element y_3 of $Y - \{y_1, y_2\}$. For example, choosing $\{y_1, y_2\} = \{1, 3\}$, then choosing $f^{-1}(\{1\}) = \{1, 5\}$, and then choosing $f^{-1}(\{3\}) = \{2, 4\}$, yields f from X onto Y defined by $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(4) = 3$, and $f(5) = 1$. Now, there are $C(3, 2) = 3$ ways to choose $\{y_1, y_2\}$, then $C(5, 2) = 10$ ways to choose $f^{-1}(\{y_1\})$, and, finally, $C(3, 2) = 3$ ways to choose $f^{-1}(\{y_2\})$. Thus, by the multiplication principle,

$$|S_2| = 3 \cdot 10 \cdot 3 = 90$$

Therefore, using the addition principle,

$$|S| = |S_1| + |S_2| = 60 + 90 = 150$$

Compare this to the number of functions from X to Y, which is $3^5 = 243$.

■

Example 5.12: The purpose of this example is to point out an error that is frequently made in counting problems. Consider a box containing seven distinct colored balls: three are blue, two are red, and two are green. A subset of three of the balls is to be selected at random. How many such subsets contain at least two blue balls?

Incorrect Solution: We begin with an incorrect analysis of this problem. We want a subset that contains at least two blue balls. So, let's choose any two of the three blue balls and, once this has been done, let's choose any one of the remaining five balls to complete our subset.

For our first step, we may choose any two of the three blue balls in $C(3, 2)$ ways. Then, for our second step, any one of the remaining five balls may be chosen as the third ball in the subset. Thus, by the multiplication principle, the number of subsets with at least two blue balls is

$$C(3, 2) \cdot C(5, 1) = 3 \cdot 5 = 15$$

Correct Solution: To see that the above answer is incorrect, we first solve the problem by brute force. The number of balls was purposely kept small in this problem to allow a listing of all the 3-combinations with at least two blue balls.

Denote the set of balls by $\{b_1, b_2, b_3, r_1, r_2, g_1, g_2\}$, where b_1, b_2, and b_3 are the blue balls. Also, let x denote any one of the four non-blue balls. Then the 3-combinations with at least two blue balls are: $\{b_1, b_2, b_3\}$, the four subsets of the form $\{b_1, b_2, x\}$, the four subsets of the form $\{b_1, b_3, x\}$, and the four subsets of the form $\{b_2, b_3, x\}$. Hence, there are 13 such subsets, not 15.

We can, of course, obtain the correct answer by properly applying our counting techniques. If a 3-combination is to contain at least two blue balls, then either it contains exactly two blue balls or it contains exactly three blue balls. The number of 3-combinations with exactly two blue balls is $C(3, 2) \cdot C(4, 1) = 12$. And there is just one 3-combination with exactly three blue balls. Thus, by the addition principle, the number of 3-combinations that contain at least two blue balls is $12 + 1 = 13$.

You are encouraged to find for yourself the flaw in the incorrect analysis.

■

Recall that the *power set* $\mathcal{P}(S)$ of a set S is the set of all subsets of S. For example,

$$\begin{aligned}
\mathcal{P}(\emptyset) &= \{\emptyset\} \\
\mathcal{P}(\{1\}) &= \{\emptyset, \{1\}\} \\
\mathcal{P}(\{1, 2\}) &= \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \\
\mathcal{P}(\{1, 2, 3\}) &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}
\end{aligned}$$

and so on. Note that the result

$$|S| = n \rightarrow |\mathcal{P}(S)| = 2^n$$

holds in each of these cases. We next prove, in general, that a set with n elements has 2^n subsets. We give two proofs of this — first, by induction on n and, second, by a more "combinatorial" justification.

Theorem 5.12: For a nonnegative integer n, if a set S has cardinality n, then its power set $\mathcal{P}(S)$ has cardinality 2^n; that is, any set with n elements has 2^n subsets.

Proof: We first prove the result by induction on n.

As seen above, the only set with cardinality 0 is the empty set, and it has exactly one subset, namely, itself. Thus, the result holds when $n = 0$.

Let k be an arbitrary nonnegative integer and assume the general result holds when $n = k$. More specifically, the induction hypothesis is that if S' is any finite set with cardinality k, then $\mathcal{P}(S')$ has cardinality 2^k.

To complete the proof, it must be shown that any set with $k+1$ elements has 2^{k+1} subsets. To that end, let S be a set such that $|S| = k+1$; we must show that $|\mathcal{P}(S)| = 2^{k+1}$. Choose $s \in S$; note that this is possible since S is nonempty. Then any subset X of S either contains s or does not contain s. Hence, $\mathcal{P}(S)$ may be partitioned as $\{\mathcal{P}_1, \mathcal{P}_2\}$, where

$$\mathcal{P}_1 = \{X \in \mathcal{P}(S) \mid s \in X\} \quad \text{and} \quad \mathcal{P}_2 = \{X \in \mathcal{P}(S) \mid s \notin X\}$$

Now then, note that

$$X \in \mathcal{P}_1 \;\leftrightarrow\; X - \{s\} \in \mathcal{P}(S - \{s\})$$

whereas

$$X \in \mathcal{P}_2 \;\leftrightarrow\; X \in \mathcal{P}(S - \{s\})$$

Thus, by the induction hypothesis,

$$|\mathcal{P}_1| = |\mathcal{P}_2| = |\mathcal{P}(S - \{s\})| = 2^k$$

Hence, by the addition principle,

$$|\mathcal{P}(S)| = |\mathcal{P}_1| + |\mathcal{P}_2| = 2^k + 2^k = 2^{k+1}$$

as was to be shown.

Alternate Proof: The result of the theorem is obvious for $n = 0$, so let S be a nonempty set with $n \geq 1$ elements. In fact, we may assume, without loss of generality, that $S = \{1, 2, \ldots, n\}$. We are to prove that $|\mathcal{P}(S)| = 2^n$. The type of combinatorial proof that we wish to illustrate is what is called a *bijective proof.* The strategy is to construct a bijection (or matching) f from $\mathcal{P}(S)$ to a set $\mathcal{F}$ that is known to have 2^n elements. It then follows that $\mathcal{P}(S)$ has 2^n elements.

Before we can construct f we need a candidate for the set $\mathcal{F}$. What set associated with S is known to have cardinality 2^n? Well, by Theorem 5.6, the number of functions from S to the set {in,out} is 2^n, so we let $\mathcal{F}$ be the set of all functions from S to {in,out}. To complete the proof, we need a bijection f from $\mathcal{P}(S)$ to $\mathcal{F}$. Note that the elements of the domain of f are subsets of S. Given a subset T of S, its image $f(T)$ is to be a function from S to {in,out}; let's denote this function by f_T. Is there a natural choice for f_T?

To get a better feel for what is going on here, let's consider a specific example. Let $S = \{1, 2, 3, 4, 5\}$ and suppose that $T = \{1, 3, 5\}$. Given $x \in S$, note that either $x \notin T$ or $x \in T$. So there is a natural way to decide whether $f_T(x) = \text{in}$ or $f_T(x) = \text{out}$, namely, we let $f_T(x) = \text{in}$ if and only if $x \in T$. Thus, f_T in this specific example is defined by

$$f_T(1) = \text{in} \quad f_T(2) = \text{out} \quad f_T(3) = \text{in} \quad f_T(4) = \text{out} \quad f_T(5) = \text{in}$$

In the general case, we define $f\colon \mathcal{P}(S) \to \mathcal{F}$ as follows: $f(T) = f_T$, where $f_T\colon S \to \{\text{in,out}\}$ is given by

$$f_T(x) = \begin{cases} \text{in} & \text{if } x \in T \\ \text{out} & \text{if } x \notin T \end{cases}$$

The function f_T is called the *characteristic function* of T.

To show that f is one-to-one, we let T_1 and T_2 be different subsets of S; without loss of generality, we assume that $x_0 \in T_1 - T_2$. Then $f_{T_1}(x_0) = $in, whereas $f_{T_2}(x_0) = $out; hence, $f_{T_1} \neq f_{T_2}$. This shows that f is one-to-one.

To show that f is onto, we let g be a function from S to $\{\text{in,out}\}$; we must exhibit a subset T of S such that $f_T = g$. We let T consist of those $x \in S$ such that $g(x) = $in; then

$$g(x) = \text{in} \;\leftrightarrow\; x \in T \;\leftrightarrow\; f_T(x) = \text{in}$$

so that $f_T = g$. This shows that f is onto and thus proves that f is a bijection.

■

Again, we let n be a nonnegative integer and let S be a set with cardinality n. An alternate formula for the cardinality of $\mathcal{P}(S)$ can be obtained by using the addition principle as follows. We partition $\mathcal{P}(S)$ as $\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_n\}$, where $\mathcal{P}_r$ is the set of r-combinations of S. Then $|\mathcal{P}_r| = C(n,r)$, and we obtain, by the addition principle,

$$|\mathcal{P}(S)| = C(n,0) + C(n,1) + \cdots + C(n,n) = \sum_{r=0}^{n} C(n,r)$$

This result, together with Theorem 5.12, gives us the following interesting combinatorial identity.

Corollary 5.13: For any nonnegative integer n,

$$\sum_{r=0}^{n} C(n,r) = 2^n$$

■

We now let n be a positive integer, let S be a set such that $|S| = n$, and let r be an integer with $1 \le r \le n$. A ***circular r-permutation*** of S is a sequence of r distinct elements from S, written

$$\big[x_0, x_1, \ldots, x_{r-1}\big]$$

with the understanding that two circular r-permutations $\big[x_0,x_1,\ldots,x_{r-1}\big]$ and $\big[y_0,y_1,\ldots,y_{r-1}\big]$ are equal provided there is an integer k, $0 \le k \le r-1$, such that for all i, $0 \le i \le r-1$, we have

$$y_i = x_j$$

where $j = (i+k) \bmod r$. A ***circular permutation*** of S is simply a circular n-permutation of S.

For example, let $S = \{1,2,3,4,5\}$. Then the 3-permutations $(1,2,3)$, $(2,3,1)$, and $(3,1,2)$ are different, but they are the same as circular 3-permutations of S; that is,

$$\big[1,2,3\big] = \big[2,3,1\big] = \big[3,1,2\big]$$

(To see that $[1,2,3] = [2,3,1]$, apply the definition of equality with $k = 1$; to see that $[1,2,3] = [3,1,2]$, apply the definition with $k = 2$.) Similarly, $(1,3,2,5)$, $(3,2,5,1)$, $(2,5,1,3)$, and $(5,1,3,2)$ are distinct 4-permutations of S; however,

$$[1,3,2,5] = [3,2,5,1] = [2,5,1,3] = [5,1,3,2]$$

Given a circular r-permutation of an n-element set S, say, $[x_1, x_2, \ldots, x_r]$, note that

$$[x_1, x_2, \ldots, x_r] = [x_2, x_3, \ldots, x_r, x_1] = \cdots = [x_r, x_1, x_2, \ldots, x_{r-1}]$$

whereas, the r r-permutations

$$(x_1, x_2, \ldots, x_r), \quad (x_2, x_3, \ldots, x_r, x_1), \quad \ldots, \quad (x_r, x_1, x_2, \ldots, x_{r-1})$$

are distinct. That is, the number of r-permutations of S corresponding to any fixed circular r-permutation of S is r. This yields the following result.

Theorem 5.14: For integers r and n with $1 \le r \le n$, the number of circular r-permutations of an n-element set is

$$\frac{P(n,r)}{r}$$

■

We remark, as an immediate consequence of Theorem 5.14, that the number of circular permutations of an n-element set is $(n-1)$-factorial.

Example 5.13: Six dignitaries are to be seated at a banquet.

(a) Find the number of ways to seat them in a row of six chairs at the head table.

(b) Find the number of ways to seat them at a circular table with six chairs. Assume that the particular position a dignitary is seated at is unimportant; all that matters to any dignitary is which person is seated to his or her left and which person is seated to his or her right.

Solution: Let S be the set of dignitaries.

(a) Here, each seating arrangement corresponds to a unique permutation of S, so the number of seating arrangements is 6-factorial.

(b) In this part, the number of different seating arrangements is the same as the number of circular permutations of S, so the answer is 5-factorial. To see this, take a given assignment of dignitaries to chairs at the table. If, for some k, $0 \le k \le 5$, each dignitary moves k chairs to the left, then we have the same seating arrangement, since each person still has the same left-neighbor and the same right-neighbor.

■

Exercise Set 5.3

1. Seven runners are entered in the 1600 meter race at a track meet. Different trophies are awarded to the first, second, and third place finishers. In how many ways might the trophies be awarded?

2. If the Yankees have nine players to trade and the Red Sox have seven players to trade, in how many ways might the two teams trade four players for four players?

3. Consider the game of Mastermind (see Exercise 15 in Exercise Set 1.2). How many codes use:

(a) only one color?
(b) exactly two colors?
(c) exactly three colors?
(d) exactly four colors?

4. Ron has n friends who enjoy playing bridge. Every Wednesday evening, Ron invites three of these friends to his home for a bridge game. Ron always sits in the south position at the bridge table; he decides which friends are to sit in the west, north, and east positions. How large (at least) must n be if Ron is able to do this for 210 weeks before repeating a seating arrangement?

5. Ron has n friends who enjoy playing bridge, and he is able to invite a different subset of three of them to his home every Wednesday night for 104 weeks. How large (at least) is n?

6. How many four-digit numbers can be formed from the digits $1, 2, 3, 4, 5, 6$? How many of these have distinct digits?

7. A checkerboard has 64 distinct squares arranged into eight rows and eight columns.

(a) In how many ways can eight identical checkers be placed on the board so that no two checkers are in the same row or in the same column?
(b) In how many ways can two identical red checkers and two identical black checkers be placed on the board so that no two checkers of the opposite color are in the same row or in the same column?

8. For a positive integer n and an integer k, $0 \le k \le n$, consider the identity

$$C(n, k) = C(n, n-k)$$

(a) Verify this identity algebraically, using Corollary 1.12.
(b) Give a "combinatorial" justification for it, arguing that the number of k-combinations of $\{1, 2, \ldots, n\}$ is the same as the number of $(n-k)$-combinations of $\{1, 2, \ldots, n\}$. One way to do this is to give a bijective proof.

9. The Department of Mathematics has four full professors, eight associate professors, and three assistant professors. A four-person curriculum committee is to be selected; this can be done in $C(15, 4)$ ways. How many ways are there to select the committee under the following restrictions?

(a) At least one full professor is chosen.
(b) The committee contains at least one professor of each rank.

10. In how many ways can tickets to eight different football games be distributed among 14 football fans if no fan gets more than one ticket?

11. A university is to interview six candidates for the position of provost; Dr. Deming is a member of the search committee.

(a) After interviewing the candidates, Dr. Deming is to rank the candidates from first to sixth choice. In how many ways can this be done?
(b) Suppose that, rather than ranking all the candidates, Dr. Deming is to give a first choice, a second choice, and a third choice. In how many ways can this be done?
(c) Suppose that, rather than ranking the candidates, Dr. Deming is to indicate a set of three "acceptable" candidates. In how many ways can this be done?
(d) Suppose that, rather than ranking the candidates, Dr. Deming is to indicate a set (possibly empty) of acceptable candidates. (This is called "approval voting.") In how many ways can this be done?

12. How many permutations of $\{1, 2, 3, 4, 5\}$ satisfy the following conditions?

(a) 1 is in the first position
(b) 1 is in the first position and 2 is in the second position
(c) 1 occurs before 2

13. How many binary strings of length 16 contain exactly six 1s?

14. In how many ways can a nonempty subset of people be chosen from six men and six women if the subset has equal numbers of men and women?

15. A hand in the card game of (standard) poker can be regarded as a 5-card subset of the standard deck of 52 cards. How many poker hands contain at least one card from each suit?

16. How many five-digit strings, formed using the digits $1, 2, 3, 4, 5$, satisfy the following conditions?

(a) The string contains the substring 12 or the substring 21.
(b) The string contains one of the substrings 123, 132, 213, 231, 312, or 321.

17. In how many ways can seven distinct keys be put on a key ring? (This question has several possible answers, depending on how one decides whether two ways of placing the keys on the ring are the "same" or "different." For example, is the orientation of the individual keys important? Situations of this sort are not atypical in combinatorics.)

18. Four married couples are to be seated at a circular table with eight chairs. As in Example 1.13, the particular position a person is seated at is unimportant; all that matters to any person is which person is seated to her or his left and which person is seated to her or his right.

(a) How many ways are there to seat these eight people?

How many ways are there to seat them under the following restrictions?

(b) Each man is seated next to two women.
(c) Each husband must be seated next to his wife.
(d) The four men are seated consecutively.

19. How many 5-card poker hands (see Exercise 15) satisfy the following conditions?

(a) The hand contains exactly one pair (not two pairs or three of a kind).
(b) The hand contains exactly three of a rank (not four of a rank or a full house).
(c) The hand is a full house (three of one rank, a pair of another).
(d) The hand is a straight flush (for example, 7, 8, 9, 10, jack, all the same suit).
(e) The hand is a straight (for example, 7, 8, 9, 10, jack, not all the same suit).
(f) The hand is a flush (all the same suit, but not a straight)?

20. If the United States Senate had 52 Democrats and 48 Republicans, how many ways would there be to form a committee of 6 Democrats and 4 Republicans? How many ways would there be to form this committee and then to appoint one of the Democratic members chair and one of the Republican members vice chair of the committee?

21. A pizza automatically comes with cheese and sauce, but any of the following extra items may be ordered: extra cheese, pepperoni, mushrooms, sausage, green peppers, onions, or anchovies.

(a) How many kinds of pizza may be ordered?
(b) How many kinds of pizza with exactly three extra items may be ordered?
(c) If a vegetarian pizza may not contain pepperoni, sausage, or anchovies, how many kinds of vegetarian pizza are there?

22. A member of the Mathematical Association of America (MAA) may belong to any of 11 Special Interest Groups (SIGMAAs). Suppose the MAA wishes to assign each member a code number indicating the SIGMAAs to which that member belongs. How many code numbers are needed?

23. Let m and n be positive integers, let $A = \{1, 2, \ldots, m\}$, and let $B = \{1, 2, \ldots, n\}$.

(a) How many relations are there from A to B?
(b) How many relations are there on A?

24. With A and B as in the previous exercise, find the number of functions from A onto B in each case.

(a) $m = n + 1$
(b) $m = n + 2$
(c) $m = n + 3$

Hint: First work out the special case $n = 3$; see Example 1.11.

25. The vertices of an n-gon are labeled with the numbers 1, 2, ..., n.

(a) How many different labelings are there if two labelings are considered the same provided the sequence of numbers encountered, beginning with the vertex labeled 1 and proceeding clockwise, is the same?
(b) How many different labelings are there if two labelings are considered the same provided they yield the same set of edges.

26. Mary likes to listen to compact discs each evening. How many CDs must she have if she is able to listen each evening for 100 consecutive evenings to:

(a) A different subset of CDs?
(b) A different subset of three CDs?

27. Suppose in Exercise 9 that, in addition to the 4-person curriculum committee, a 3-person administrative committee is also to be selected from the 15 members of the department.

(a) How many ways are there to select both committees (if there are no restrictions)?

How many ways are there to select both committees under the following restrictions?

(b) The committees must be disjoint.
(c) The committees can have at most one member in common.

28. For a positive integer n, we wish to define an ordering of the permutations of $\{1, 2, \ldots, n\}$; this ordering is called ***lexicographic*** (or lex) ***ordering***. Given two permutations $\pi_1 = (x_1, x_2, \ldots, x_n)$ and $\pi_2 = (y_1, y_2, \ldots, y_n)$, we use the notation $\pi_1 < \pi_2$ to indicate that π_1 precedes π_2 in lex order and use the notation $\pi_1 \leq \pi_2$ to indicate that $\pi_1 = \pi_2$ or $\pi_1 < \pi_2$. We then define $\pi_1 < \pi_2$ provided $x_1 < y_1$ or $x_1 = y_1$, $x_2 = y_2$, ..., $x_{i-1} = y_{i-1}$, and $x_i < y_i$ for some i, $1 < i \leq n$.

(a) Show that lex order is a total ordering of the set of permutations of $\{1, 2, \ldots, n\}$.
(b) List the permutations of $\{1, 2, 3, 4\}$ in lex order.
(c) For $n = 7$, what permutation is the immediate successor of $(1, 2, 3, 4, 6, 7, 5)$ (in lex order)?
(d) For $n = 7$, what permutation is the immediate successor of $(1, 6, 2, 7, 5, 4, 3)$?

29. How many partitions $\{A, B, C\}$ of $\{0, 1, 2, \ldots, 9\}$ are there such that $|A| = 2$, $|B| = 3$, and $|C| = 5$?

30. Refer to Exercise 28 and give a recursive definition of $\pi_1 < \pi_2$.

31. In how many ways can 7 cards be chosen from a standard deck of 52 cards so that there are 2 cards from each of any three suits and 1 card from the other suit?

32. Let n be a positive integer and let k be an integer, $0 \leq k \leq n$. We define an ordering, also called ***lexicographic*** (or ***lex***) ***ordering***, on the k-element subsets of $\{1, 2, \ldots, n\}$. Again, given two k-element subsets S_1 and S_2, the notation $S_1 < S_2$ indicates that S_1 precedes S_2 in lex order, and $S_1 \leq S_2$ indicates that $S_1 = S_2$ or $S_1 < S_2$. Now the empty set is the only 0-element subset. For $k \geq 1$, let $S_1 = \{x_1, x_2, \ldots, x_k\}$ and $S_2 = \{y_1, y_2, \ldots, y_k\}$ be two k-element subsets of $\{1, 2, \ldots, n\}$, and assume the elements are listed so that $x_1 < x_2 < \cdots < x_k$ and $y_1 < y_2 < \cdots < y_k$; then $S_1 < S_2$ provided $x_1 < y_1$ or $x_1 = y_1$, $x_2 = y_2$, …, $x_{i-1} = y_{i-1}$, and $x_i < y_i$ for some i, $1 < i \leq k$.

(a) Show that lex order is a total ordering of the k-element subsets of $\{1, 2, \ldots, n\}$.
(b) List the 3-element subsets of $\{1, 2, 3, 4, 5, 6\}$ in lex order.
(c) For $n = 7$, what 4-element subset is the immediate successor of $\{1, 3, 4, 5\}$ (in lex order)?
(d) For $n = 7$, what 4-element subset is the immediate successor of $\{2, 4, 6, 7\}$?

33. Use the addition principle to give an alternate solution to Example 5.9, part (b), resulting in the answer $P(8, 5) - P(7, 5)$.

34. Refer to Exercise 32 and give a recursive definition of $S_1 < S_2$.

35. How many ways are there to distribute m different baseball cards to n children if:

(a) There are no restrictions?
(b) $m < n$ and no child gets more than one card?

36. Let us extend the definition of lex order given in Exercise 32 to define a lexicographic ordering of all the subsets of $\{1, 2, \ldots, n\}$, namely, to $\mathcal{P}(\{1, 2, \ldots, n\})$. First, we agree that the empty subset precedes any nonempty subset. Then, for two nonempty subsets $S_1 = \{x_1, x_2, \ldots, x_k\}$ and $S_2 = \{x_1, x_2, \ldots, x_m\}$ with $x_1 < x_2 < \cdots < x_k$ and $y_1 < y_2 < \cdots < y_m$, we define $S_1 < S_2$ provided $S_1 \subset S_2$, or $x_1 < y_1$, or $x_1 = y_1$, $x_2 = y_2$, …, $x_{i-1} = y_{i-1}$, and $x_i < y_i$ for some i, $1 < i \leq k$.

(a) Show that lex order is a total ordering of $\mathcal{P}(\{1, 2, \ldots, n\})$.
(b) List the subsets of $\{1, 2, 3, 4\}$ in lex order.
(c) For $n = 7$, what subset is the immediate successor of $\{1, 3, 4, 5\}$ (in lex order)?
(d) For $n = 7$, what subset is the immediate successor of $\{1, 2, 3, 4, 5, 6, 7\}$?
(e) For $n = 7$, what subset is the immediate successor of $\{1, 7\}$?

37. Try to answer Exercise 35 if the baseball cards are identical. (One part can be solved using a result discussed in Section 5.4.)

38. Refer to Exercise 36 and give a recursive definition of $S_1 < S_2$.

5.4 ORDERED PARTITIONS AND DISTRIBUTIONS

In how many ways can the letters in GRAPE be permuted? Since we are dealing with five different letters, we see that the answer to this question is $5! = 120$. What about the letters in APPLE? In how many ways can these letters be "permuted"? Suppose, for the moment, that we consider the two P s in APPLE to be different; let us subscript the P s as P_1 and P_2 so we can distinguish them. Then the number of ways to permute these five distinct letters is again 5!. Consider a particular permutation, say, $\mathrm{AP_1LEP_2}$. If we take the natural approach and do not consider the two P s to be different, then this permutation and the permutation $\mathrm{AP_2LEP_1}$ should be considered the same. Thus, the number of ways to permute the letters in APPLE is not 5!, but rather

$$\frac{5!}{2!} = 60$$

That is, we must reduce the number of permutations of five distinct objects by a factor of $2! = 2$, since the two P s are the same.

How many ways are there to permute the letters in BANANA? As an exercise, you are encouraged to show that the answer is

$$\frac{6!}{3!2!} = 60$$

One of the counting techniques developed in this section allows us to answer questions of this sort.

Another counting result allows us to answer questions having the following general form: Given positive integers m and n, in how many ways can m be expressed as a sum of n positive integers:

$$x_1 + x_2 + \cdots + x_n = m$$

where the order of the terms x_1, x_2, …, x_n in the sum is important? For example, if $m = 5$ and $n = 3$, then we have

$$5 = 1 + 1 + 3 = 1 + 2 + 2 = 1 + 3 + 1 = 2 + 1 + 2 = 2 + 2 + 1 = 3 + 1 + 1$$

so the answer is six in this particular case.

We wish to place the preceding questions in the context of finding the number of functions from an m-element set X to an n-element set Y, subject to various restrictions. In particular, for the sake of concreteness, let $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$.

One fact we know already is that the number of functions from X to Y is n^m. This is a direct consequence of the multiplication principle, because a function $f\colon\ X \to Y$ is determined by the m images $f(1)$, $f(2)$, …, $f(m)$, and there are n choices for each of these images.

How many functions from X to Y are one-to-one? We have also answered this question, the answer being $P(n, m)$, if $m \le n$, and 0, otherwise. For if $m \le n$ and $f\colon\ X \to Y$ is one-to-one, then the sequence of images $((f(1), f(2), \ldots, f(m))$ is an m-permutation of the n-element set Y.

A tougher question is the following:

How many functions from X onto Y are there?

Notice that the answer is 0 if $m < n$, so we assume that $m \ge n$. In this case, if $f\colon\ X \to Y$ is onto, then the n preimages $f^{-1}(\{1\})$, $f^{-1}(\{2\})$, …, $f^{-1}(\{n\})$ are nonempty, pairwise-disjoint, and

$$X = \{1, 2, \ldots, m\} = f^{-1}(\{1\}) \cup f^{-1}(\{2\}) \cup \cdots \cup f^{-1}(\{n\})$$

namely, $\{f^{-1}(\{1\}), f^{-1}(\{2\}), \ldots, f^{-1}(\{n\})\}$ is a partition of X into n parts.

Definition 5.3: Let S be a nonempty set and let $\{S_1, S_2, \ldots, S_n\}$ be a partition of S into n parts. Then the n-tuple $(S_1, S_2, \ldots, S_n)$ is called an ***ordered partition of S into n parts***. The number n is called the ***degree*** of the ordered partition. Two ordered partitions $(U_1, U_2, \ldots, U_n)$ and $(V_1, V_2, \ldots, V_k)$ of S are considered to be the same if and only if $n = k$ and $U_i = V_i$ for each i, $1 \le i \le n$.

■

Example 5.14: Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. As pointed out in Example 5.1,

$$\{\{0, 1, 2, 9\}, \{3, 4, 8\}, \{5, 6\}, \{7\}\} \qquad \text{and} \qquad \{\{7\}, \{5, 6\}, \{3, 4, 8\}, \{0, 1, 2, 9\}\}$$

are equal partitions of S into 4 parts. However, the ordered partitions

$$(\{0, 1, 2, 9\}, \{3, 4, 8\}, \{5, 6\}, \{7\}) \qquad \text{and} \qquad (\{7\}, \{5, 6\}, \{3, 4, 8\}, \{0, 1, 2, 9\})$$

are different ordered partitions of S into 4 parts.

■

A function f from $X = \{1, 2, \ldots, m\}$ onto $Y = \{1, 2, \ldots, n\}$ uniquely determines an ordered partition of X into n parts, namely, $(f^{-1}(\{1\}), f^{-1}(\{2\}), \ldots, f^{-1}(\{n\}))$. Conversely, an ordered partition of X into n parts, say, $(X_1, X_2, \ldots, X_n)$, uniquely determines a function f from X onto Y, namely, f is defined by

$$f(x) = y \leftrightarrow x \in X_y$$

Example 5.15: To make the preceding ideas more concrete, let's illustrate the particular case $m = 8$, $n = 4$. Consider the function $f\colon \{1, 2, \ldots, 8\} \to \{1, 2, 3, 4\}$ defined by

$$f(1) = 2 = f(4), \qquad f(2) = 1 = f(3) = f(5), \qquad f(6) = 4 = f(8), \qquad f(7) = 3$$

Then f is onto with $f^{-1}(\{1\}) = \{2, 3, 5\}$, $f^{-1}(\{2\}) = \{1, 4\}$, $f^{-1}(\{3\}) = \{7\}$, and $f^{-1}(\{4\}) = \{6, 8\}$. Notice that $(\{2, 3, 5\}, \{1, 4\}, \{7\}, \{6, 8\})$ is an ordered partition of $\{1, 2, \ldots, 8\}$ into 4 parts. Conversely, suppose we are given some ordered partition of $\{1, 2, \ldots, 8\}$ into 4 parts, say, $(\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\})$, for instance. Then we may define a function f from $\{1, 2, \ldots, 8\}$ onto $\{1, 2, 3, 4\}$ such that $f^{-1}(\{1\}) = \{1, 8\}$, $f^{-1}(\{2\}) = \{2, 7\}$, $f^{-1}(\{3\}) = \{3, 6\}$, and $f^{-1}(\{4\}) = \{4, 5\}$. To do this, we define f as follows:

$$f(1) = 1 = f(8), \qquad f(2) = 2 = f(7), \qquad f(3) = 3 = f(6), \qquad f(4) = 4 = f(5)$$

■

At this point in our discussion, we see that the number of functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, n\}$ is the same as the number of ordered partitions of $\{1, 2, \ldots, m\}$ into n parts, so we would like to know this latter number. Let us attack an easier problem first. Let m_1, m_2, …, m_n be n positive integers whose sum is m. We ask for the number of ordered partitions $(X_1, X_2, \ldots, X_n)$ of $X = \{1, 2, \ldots, m\}$ into n parts with $|X_i| = m_i$, $1 \le i \le n$.

Example 5.16: Determine the number of ordered partitions $(X_1, X_2, \ldots, X_n)$ of $X = \{1, 2, \ldots, 8\}$ with:

(a) $n = 2$, $|X_1| = 3$, $|X_2| = 5$
(b) $n = 4$, $|X_1| = 3$, $|X_2| = |X_3| = 2$, $|X_4| = 1$

Solution:

(a) We may form an ordered partition (X_1, X_2) of X with $|X_1| = 3$ and $|X_2| = 5$ simply by choosing the three elements of X_1, because, once this is done, the remaining five elements of X automatically belong to X_2. So the answer is

$$C(8, 3) = 56$$

(b) We may form an ordered partition (X_1, X_2, X_3, X_4) of X with $|X_1| = 3$, $|X_2| = 2$, $|X_3| = 2$, and $|X_4| = 1$ by first choosing three elements of X for X_1, next choosing two elements of $X - X_1$ for X_2, and then choosing two elements of $X - (X_1 \cup X_2)$ for X_3, with $X_4 = X - (X_1 \cup X_2 \cup X_3)$. The three elements for X_1 can be chosen in $C(8, 3)$ ways. This done, the two elements for X_2 can be chosen in $C(5, 2)$ ways. Finally, the two elements for X_3 can be chosen in $C(3, 2)$ ways. Thus, by the multiplication principle, the answer is

$$C(8, 3)\, C(5, 2)\, C(3, 2) = 1680$$

■

As indicated by part (a) of Example 5.16, we already know a formula for the number of ordered partitions of an m-element set into 2 parts with m_1 elements in the first part and $m_2 = m - m_1$ elements in the second part, namely, $C(m, m_1)$. In the context of ordered partitions, we employ the alternate, yet similar, notation $C(m; m_1, m_2)$.

In general, we let $C(m; m_1, m_2, \ldots, m_n)$ denote the number of ordered partitions of an m-element set into n parts with m_i elements in the ith part, $1 \le i \le n$. Here, of course, $m_1, m_2, \ldots, m_n$ are positive integers satisfying $m_1 + m_2 + \cdots + m_n = m$. A common alternate notation for $C(m; m_1, m_2, \ldots, m_n)$ is

$$\binom{m}{m_1, m_2, \ldots, m_n}$$

Taking our lead from Example 5.16, part (b), we can easily develop a formula for $C(m; m_1, m_2, \ldots, m_n)$.

Theorem 5.15: Given positive integers m and n, and n positive integers $m_1, m_2, \ldots, m_n$ such that $m = m_1 + m_2 + \cdots + m_n$,

$$C(m; m_1, m_2, \ldots, m_n) = \frac{m!}{m_1!\, m_2! \cdots m_n!}$$

Proof: We can uniquely form an ordered partition $(X_1, X_2, \ldots, X_n)$ of an m-element set X into n parts with $|X_i| = m_i$, $1 \le i \le n$, in $n - 1$ steps. The first step is to choose m_1 elements of X for X_1. This can be done in $C(m, m_1)$ ways. For the ith step, $2 \le i \le n - 1$, we choose m_i elements of $X - (X_1 \cup \cdots \cup X_{i-1})$ for X_i. This can be done in

$$C(m - (m_1 + \cdots + m_{i-1}), m_i)$$

ways. Thus, by the multiplication principle, we have that

$$\begin{aligned} C(m; m_1, m_2, \ldots, m_n) &= C(m, m_1)\, C(m - m_1, m_2)\, \cdots\, C(m - (m_1 + \cdots + m_{n-2}), m_{n-1}) \\ &= \frac{m!}{m_1!\, m_2! \cdots m_n!} \end{aligned}$$

Verification of the last equality above is left for you to develop in Exercise 16.

■

The next example shows how Theorem 5.17 can be applied to solve problems of the type mentioned at the beginning of this section.

Example 5.17: Determine the number of different ways to form a string of letters using the letters in the word:

(a) BANANA
(b) MISSISSIPPI

Solution:

(a) A given string $c_1c_2c_3c_4c_5c_6$ of the six letters in BANANA determines an ordered partition (X_1, X_2, X_3) of $\{1, 2, 3, 4, 5, 6\}$ with $|X_1| = 3$, $|X_2| = 1$, and $|X_3| = 2$; namely, let

$$X_1 = \{i \mid c_i = \mathrm{A}\}, \qquad X_2 = \{i \mid c_i = \mathrm{B}\}, \qquad X_3 = \{i \mid c_i = \mathrm{N}\}$$

For instance, the string BANANA itself determines the ordered partition $(\{2, 4, 6\}, \{1\}, \{3, 5\})$ (since the A s occur in positions 2, 4, and 6, the B occurs in position 1, and the N s occur in positions 3 and 5). Conversely, each ordered partition (X_1, X_2, X_3) of $\{1, 2, 3, 4, 5, 6\}$ with $|X_1| = 3$, $|X_2| = 1$, and $|X_3| = 2$ determines a string $c_1c_2c_3c_4c_5c_6$ of the letters in BANANA, namely, c_i =A, B, or N according to whether $i \in X_1$, X_2, or X_3, respectively. For instance, the ordered partition $(\{1, 4, 6\}, \{5\}, \{2, 3\})$ determines the string ANNABA (with the A s in positions 1, 4, and 6, the B in position 5, and the N s in positions 2 and 3).

Thus, the number of different strings using the letters in BANANA is the same as the number of ordered partitions (X_1, X_2, X_3) of $\{1, 2, 3, 4, 5, 6\}$ with $|X_1| = 3$, $|X_2| = 1$, and $|X_3| = 2$. By Theorem 1.17, this number is

$$C(6; 3, 1, 2) = \frac{6!}{3!\, 1!\, 2!} = 60$$

(b) Using the same line of reasoning as in part (a), we see that the number of different strings using the letters in MISSISSIPPI is the same as the number of ordered partitions (X_1, X_2, X_3, X_4) of $\{1, 2, \ldots, 11\}$ with $|X_1| = 4$, $|X_2| = 1$, $|X_3| = 2$, and $|X_4| = 4$. By Theorem 1.17, this number is

$$C(11; 4, 1, 2, 4) = \frac{11!}{4!\, 1!\, 2!\, 4!} = 34650$$

■

Now let's generalize the preceding example. Suppose we have n different types of objects, with m_i identical objects of type i, $1 \le i \le n$, where $m_1 + m_2 + \cdots + m_n = m$. How many different ways are there to list, or permute, these m objects? The number of such "permutations with repetition allowed" is given by the following result, whose formal proof is left for you to work out in Exercise 18.

Theorem 5.16: Given a total of m objects of n different types, with m_i identical objects of type i, $1 \le i \le n$, and $m = m_1 + m_2 + \cdots + m_n$, the number of permutations of these m objects is given by $C(m; m_1, m_2, \ldots, m_n)$.

■

In the context of Theorem 5.16, notice that

$$C(m; 1, 1, \ldots, 1) = m!$$

so the result of Theorem 5.16 agrees with that of Corollary 5.9 in the case when the m objects are all distinguishable.

At this point, we see that the number of functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, n\}$, where $m \ge n$, is given by

$$\sum C(m; m_1, m_2, \ldots, m_n)$$

where the sum is over all n-tuples $(m_1, m_2, \ldots, m_n)$ of positive integers such that $m_1+m_2+\cdots+m_n = m$. How many terms are there in this sum? This brings us to the second question mentioned at the beginning of this section, namely, How many ways are there to express a given positive integer m as a sum of n positive integers $(m \ge n)$, where the order of the terms in the sum is important?

Looking at the problem for $m = 5$ and $n = 3$, let us begin by writing 5 as $1+1+1+1+1$. Now, from among the four plus signs, we choose two of them, and combine the terms between the chosen plus signs. This uniquely yields an expression of 5 as a sum of 3 positive integers. For example, if we choose the first and fourth plus signs, then we obtain

$$5 = 1 + (1 + 1 + 1) + 1 = 1 + 3 + 1$$

Thus, the number of ways to express 5 as a sum of 3 positive integers is the same as the number of ways to choose two of four plus signs, namely, $C(4, 2) = 6$.

Theorem 5.17: Let m and n be positive integers with $m \ge n$. The number of ways to express m as a sum of n positive integers, where the order of the terms in the sum is important, is $C(m - 1, n - 1)$.

Proof: We begin by writing m as $1 + 1 + \cdots + 1$. Then we choose $n - 1$ of the $m - 1$ plus signs, and combine any 1s before the first plus sign chosen, between two consecutive chosen plus signs, and after the last plus sign chosen. Note that this uniquely expresses m as a sum of n positive integers. Therefore, the number of ways to express m as a sum of n positive integers is the number of ways to choose $n - 1$ of $m - 1$ plus signs, that is, $C(m - 1, n - 1)$.

■

Given $m = x_1 + x_2 + \cdots + x_n$, with each x_i a positive integer, we call the n-tuple $(x_1, x_2, \ldots, x_n)$ an ***ordered partition of m into n parts***. So Theorem 5.17 states that the number of ordered partitions of m into n parts is $C(m - 1, n - 1)$. A related result is the following theorem.

Theorem 5.18: The number of ways to express the positive integer m as a sum of n nonnegative integers, where the order of the terms in the sum is important, is $C(m + n - 1, n - 1)$.

Proof: Suppose m is written as

$$m = y_1 + y_2 + \cdots + y_n$$

where each y_i is a nonnegative integer. For $1 \le i \le n$, let $x_i = y_i + 1$. Then we have

$$m + n = x_1 + x_2 + \cdots + x_n$$

Conversely, if $m + n$ is written as above, with each x_i a positive integer, let $y_i = x_i - 1$, for $1 \le i \le n$. Then $m = y_1 + y_2 + \cdots + y_n$. Thus, the number of ways to express m as a sum of n nonnegative integers is the same as the number of ways to express $m + n$ as a sum of n positive integers. This, by Theorem 5.17, is $C(m + n - 1, n - 1)$.

■

Example 5.18: A bakery sells seven kinds of donuts.

(a) How many ways are there to choose 12 donuts?
(b) How many ways are there to choose 12 donuts if there must be at least 1 donut of each kind?

Solution: Assuming that donuts of the same kind are "identical," what matters here is the number of donuts chosen of each kind. Thus, let x_i be the number of donuts of kind i chosen, $1 \le i \le 7$. Then

$$x_1 + x_2 + \cdots + x_7 = 12$$

(a) Here each x_i is a nonnegative integer, so we apply Theorem 5.18. Hence, the answer is $C(12 + 7 - 1, 7 - 1) = C(18, 6)$.

(b) In this part, each x_i is restricted to be positive, so we apply Theorem 5.17. This yields the answer $C(12 - 1, 7 - 1) = C(11, 6)$.

■

The general type of problem suggested by the preceding example occurs frequently in combinatorics. Namely, suppose we have n different types of objects, with a sufficiently large number of identical objects of type i, $1 \le i \le n$. How many different ways are there to choose a collection of m of these objects, with repetition allowed? Such a collection is termed an ***m-combination of n types, with repetition allowed***. (Some books use the term "multiset" to refer to a collection of elements with repeated elements allowed.)

In forming such a collection, what matters is the number m_i of objects of type i selected, $1 \le i \le n$. Notice that $m_1, m_2, \ldots, m_n$ are nonnegative integers such that $m_1 + m_2 + \cdots + m_n = m$. Thus, we obtain the following result as an immediate corollary to Theorem 5.18.

Corollary 5.19: Given objects of n different types, with at least m identical objects of each type, then the number of ways to choose m of these objects is $C(m + n - 1, n - 1)$.

■

Example 5.19: Each of the two squares of a domino contains between zero and six dots. How many different dominoes are there?

Solution: What matters in this problem is the number of dots on each of two squares of a domino. Thus, for $0 \leq i \leq 6$, let x_i be the number of squares with i dots. Then the number of different dominoes is equal to the number of solutions to

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2$$

in nonnegative integers. In other words, we can think of a domino as a 2-combination of seven types, with repetition allowed. Thus, by Corollary 1.19, the number of different dominoes is $C(8, 6) = 28$.

We can also answer the question as follows. Either a domino has the same number of dots on both squares, or not. There are seven different dominoes with equal numbers of dots on their two squares, one for each of the numbers 0, 1, ..., 6. Also, the number of distinct dominoes with different numbers of dots on their two squares is $C(7, 2) = 21$. Thus, the total number of different dominoes is $7 + 21 = 28$.

■

To summarize then, the number of functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, n\}$, where $m \geq n$, is given by

$$\sum C(m; m_1, m_2, \ldots, m_n)$$

where the sum is over all n-tuples $(m_1, m_2, \ldots, m_n)$ of positive integers such that $m_1+m_2+\cdots+m_n = m$. By Theorem 5.17, we know that there are $C(m-1, n-1)$ terms in this sum.

For example, the number of functions from $\{1, 2, 3, 4, 5\}$ onto $\{1, 2, 3\}$ is given by

$$\begin{aligned}
&C(5; 1, 1, 3) + C(5; 1, 2, 2) + C(5; 1, 3, 1) + C(5; 2, 1, 2) + C(5; 2, 2, 1) + C(5; 3, 1, 1)\\
&= \frac{5!}{1!\,1!\,3!} + \frac{5!}{1!\,2!\,2!} + \frac{5!}{1!\,3!\,1!} + \frac{5!}{2!\,1!\,2!} + \frac{5!}{2!\,2!\,1!} + \frac{5!}{3!\,1!\,1!}\\
&= 20 + 30 + 20 + 30 + 30 + 20\\
&= 150
\end{aligned}$$

The problem of determining the number of functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, n\}$ can also be solved using the *principle of inclusion/exclusion*. This principle shows that the number of functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, n\}$ is given by

$$\sum_{k=0}^{n-1} (-1)^k C(m, k)(n-k)^m$$

Another result is that the number of functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, n\}$ is equal to

$$n!S(m, n)$$

where $S(m, n)$ is a special function denoting what are called *Stirling numbers of the second kind*.

Most of the results of this section and Sections 5.2 and 5.3 can be placed in the general context of what are termed *ordered distributions* — distributing a given number of objects to a given number of different (or "ordered") recipients under certain specified conditions. It is convenient here to think of the objects as "balls" and the recipients as "boxes" that are numbered 1, 2, ..., n. Now, we may wish to think of the balls as being different from one another; for example, we may wish to distribute m golf balls, no two of which have exactly the same brand name, color, and number. In this case we can think of the distribution of balls to boxes as a function from the set of balls, say, $\{1, 2, \ldots, m\}$, to the set $\{1, 2, \ldots, n\}$. On the other hand, if the balls are to be considered all the same, then what matters is not which balls are placed in which box, but rather how many balls are placed in each box.

Example 5.20: In each of the following parts, find the number of ways to distribute the m balls to n different boxes.

(a) $m = 8$, $n = 4$, and the balls are different
(b) $m = 4$, $n = 8$, the balls are different, and at most one ball can be put in any box
(c) $m = 5$, $n = 3$, the balls are different, and at least one ball must be placed in each box
(d) $m = 8$, $n = 4$, the balls are different, three balls must be put in box 1, two balls must be put in each of boxes 2 and 3, and one ball must be put in box 4
(e) $m = 8$, $n = 4$, and the balls are identical
(f) $m = 4$, $n = 8$, the balls are identical, and at most one ball can be put in any box
(g) $m = 8$, $n = 4$, the balls are identical, and at least one ball must be placed in each box

Solution:

(a) The number of ways to distribute 8 different balls to 4 different boxes is the same as the number of functions from $\{1, 2, \ldots, 8\}$ to $\{1, 2, 3, 4\}$, which is $4^8 = 65536$ by Theorem 1.6.

(b) The number of ways to distribute 4 different balls to 8 different boxes with at most one ball per box is the same as the number of one-to-one functions from $\{1, 2, 3, 4\}$ to $\{1, 2, \ldots, 8\}$; by Theorem 1.7, this is $P(8, 4) = 1680$.

(c) The number of ways to distribute 5 different balls to 3 different boxes with at least one ball per box is the same as the number of functions from $\{1, 2, 3, 4, 5\}$ onto $\{1, 2, 3\}$, which was found to be 150 prior to this example.

(d) Note that an assignment of 8 different balls to 4 different boxes (with at least one ball per box) is equivalent to an ordered partition (B_1, B_2, B_3, B_4) of the set of balls; namely, B_i is the set of balls assigned to box i. Here we require that $|B_1| = 3$, $|B_2| = |B_3| = 2$, and $|B_4| = 1$. Thus, by Theorem 1.15, the answer is $C(8; 3, 2, 2, 1) = 1680$.

(e) The number of ways to distribute 8 identical balls to 4 different boxes is the same as the number of solutions to $x_1 + x_2 + x_3 + x_4 = 8$ with each x_i a nonnegative integer (x_i is the number of balls placed in box i). By Theorem 1.18, this is $C(8 + 4 - 1, 4 - 1) = C(11, 3) = 165$.

(f) To determine an assignment of 4 identical balls to 8 different boxes with at most one ball in any box, we simply must decide which 4 boxes are to get a ball. Hence the answer is $C(8, 4) = 70$.

(g) The number of ways to distribute 8 identical balls to 4 different boxes with at least one ball per box is the same as the number of solutions to $x_1 + x_2 + x_3 + x_4 = 8$, with each x_i a positive integer; by Theorem 1.17, this is $C(8 - 1, 4 - 1) = C(7, 3) = 35$.

■

Taking our lead from the preceding example, the next theorem summarizes results concerning ordered distributions; the proof is left for you to work out in Exercise 20. A full-semester course on combinatorics might discuss "unordered distributions," namely, what happens in the problem when the boxes are considered to be identical or indistinguishable.

Theorem 5.20: Consider the number of ways to distribute m balls to n different boxes.

1. If the balls are different, then:

(a) The number of ways to distribute the balls is n^m.
(b) If $m \leq n$, then the number of ways to distribute the balls with at most one ball per box is $P(n, m)$.
(c) If $m \geq n$, then the number of ways to distribute the balls with at least one ball per box is the same as the number of functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, n\}$.
(d) If $m = m_1 + m_2 + \cdots + m_n$, with each $m_i \in \mathbb{Z}^+$, then the number of ways to distribute the balls with m_i balls placed in box i is $C(m; m_1, m_2, \ldots, m_n)$.

2. If the balls are identical, then:

(a) The number of ways to distribute the balls is $C(m + n - 1, n - 1)$.
(b) If $m \leq n$, then the number of ways to distribute the balls with at most one ball per box is $C(n, m)$.
(c) If $m \geq n$, then the number of ways to distribute the balls with at least one ball per box is $C(m - 1, n - 1)$.

■

Exercise Set 5.4

1. A store sells six different brands of chewing gum. In how many ways can someone choose to buy 12 packs of gum?

2. Given n men and their wives ($n \geq 3$), how many ways are there to select a subset of cardinality x from these $2n$ people such that the subset does not contain a married couple? Answer for:

(a) $x = 1$ (b) $x = 2$ (c) $x = 3$ (d) a general $x \leq n$

3. The clubhouse at a certain golf course sells five different types of sandwiches. A golfer orders eight sandwiches for herself and her seven friends. From the point of view of the person taking the order, how many orders of eight sandwiches are possible?

4. Consider the set S of positive integers between 100000 and 999999, inclusive.

(a) How many of these integers contain the digit 8 exactly twice?
(b) How many contain three 7s, two 8s, and a 9?

5. A regular six-sided die is rolled seven times and the sequence of results is recorded; for example, $(3, 1, 3, 6, 4, 4, 2)$.

(a) How many such sequences are there?

Determine the number of sequences with the following properties:

(b) Each of the numbers $1, 2, 3, 4, 5, 6$ occurs.
(c) The sequence has three 2s, two 4s, and two 6s.
(d) Only two of the numbers $1, 2, 3, 4, 5, 6$ occur.

6. Every Saturday night for 8 consecutive weeks, Sue invites one of her four boyfriends to a movie. How many ways can she do this subject to the following restrictions?

(a) The order of invitations is not important and each boyfriend receives at least one invitation.
(b) The order of invitations is important; Sue's favorite boyfriend, Andy, receives three invitations, Bob receives one invitation, and Carl and Dave receive two invitations each.
(c) The order of invitations is not important and there are no restrictions.

7. In how many ways can 12 golf balls be distributed to 4 golfers in each of the following cases?

(a) The balls are different.
(b) The balls are identical.
(c) The balls are different and each golfer gets at least one ball.
(d) The balls are identical and each golfer gets at least one ball.
(e) The balls are different and each golfer gets three balls.
(f) The balls are identical and each golfer gets three balls.

8. Ten students enter a local pub one at a time, and each orders a beer. Four of these students always drink Stroh's, three always drink Budweiser, two always drink Coors, and one student always drinks Molson's.

(a) How many sequences of ten orders are possible?
(b) Six of the students decide to have another round. From the bartender's viewpoint, how many orders of six beers are possible?

9. How many ways are there to deal a 5-card poker hand (from a standard deck of 52 cards) to each of four players, Curly, Larry, Moe, and Joe, under the following conditions?

(a) There are no restrictions.
(b) Each player gets an ace.
(c) Curly gets a flush in spades, Larry a flush in hearts, Moe a flush in diamonds, and Joe a flush in clubs?

10. How many nine-letter strings can be formed from two A s, five C s, and two E s under the following conditions?

(a) There are no restrictions.
(b) No two vowels are consecutive.
(c) No two C s are consecutive.

11. In how many ways can a student distribute a good McIntosh apple, a good Golden Delicious apple, a rotten apple, and nine identical oranges to his four professors under the following conditions?

(a) There are no restrictions.
(b) Each professor gets at most one apple and at least one orange.
(c) Each professor gets (exactly) three pieces of fruit.

12. In how many ways can six coins be selected from six pennies, six nickels, six dimes, and six quarters? Assume that coins of the same value are identical.

13. In the game of Yahtzee, five dice are rolled. Assuming the dice are identical, how many outcomes are there?

14. Given n integers $b_1, b_2, \ldots, b_n$, find a formula for the number of ways to express the positive integer m as a sum of n integers:

$$m = m_1 + m_2 + \cdots + m_n$$

where the order of the terms is important and $m_i \geq b_i$, $1 \leq i \leq n$.

15. In how many ways can $2n$ persons be paired to form n bridge teams? (Answer for $n \in \{1, 2, 3, 4\}$ and then for a general n.)

16. Complete the proof of Theorem 5.15 by verifying the identity

$$C(m, m_1)C(m - m_1, m_2) \cdots C(m - (m_1 + \cdots + m_{n-2}), m_{n-1}) = \frac{m!}{m_1!\, m_2! \cdots m_n!}$$

17. In how many ways can 10 (single-scoop) ice cream cones be purchased from an ice cream parlor that offers 26 flavors under the following conditions?

(a) There are no restrictions.
(b) Exactly 3 flavors are used.
(c) At most one cone of any flavor is purchased.

18. Prove Theorem 5.16; in particular:

(a) Give a proof based on the solution of Example 5.17.
(b) Give a proof based on the ideas given at the beginning of this section.

19. How many ways are there to distribute 30 identical slices of pizza among five fraternity brothers

(a) with no restrictions?

How many ways are there to distribute the slices under the following restrictions?

(b) Each brother gets at least 1 slice.
(c) Each brother gets at least 3 slices.

20. Prove the following parts of Theorem 5.20:

(a) part 1(a) (b) part 1(b) (c) part 1(c) (d) part 1(d)
(e) part 2(a) (f) part 2(b) (g) part 2(c)

21. If m_1 identical dice and m_2 identical coins are tossed, how many results are there?

22. Consider the equation

$$x_1 + x_2 + x_3 + x_4 = 8$$

where each x_i is a nonnegative integer. One possible solution to this equation is given by $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 2$. This solution can be conveniently displayed as the string 1&11&111&11 of 1s and &s. The solution $x_1 = 0$, $x_2 = 4$, $x_3 = 0$, and $x_4 = 4$, corresponds to the string &1111&&1111. In fact, any solution to the given equation determines a unique string of eight 1s and three &s. Conversely, any string of eight 1s and three &s determines a unique solution of the equation. Hence, the number of solutions to the equation in nonnegative integers is equal to number of different strings of eight 1s and three &s.

(a) Use this idea to give an alternate proof of Theorem 5.18.
(b) Prove Theorem 5.17 as a corollary to Theorem 5.18.

23. Let's play two modified versions of Yahtzee. In version 1, you roll five dice and win \$1 if you get a pair or better; otherwise, you lose \$1. In version 2, you roll six dice and win \$1 if you get two pairs or better; otherwise, you lose \$1. Which version would you prefer to play?

5.5 BINOMIAL AND MULTINOMIAL THEOREMS

In this section, we prove a very important result called the *binomial theorem*. The binomial theorem involves the numbers $C(n, k)$ and is used to derive several interesting identities relating these numbers. We also consider the *multinomial theorem*, an extension of the binomial theorem involving the numbers $C(m; m_1, m_2, \ldots, m_n)$.

To begin, consider expanding the third power of the binomial $x + y$. We write

$$(x+y)^3 = (x_1+y_1)(x_2+y_2)(x_3+y_3)$$

where $x = x_1 = x_2 = x_3$ and $y = y_1 = y_2 = y_3$. Here the subscripts are used to help us keep track of which factor an x or y comes from. For example, x_2 simply denotes the x from the second factor of $(x + y)^3$. Thus,

$$\begin{aligned}(x+y)^3 &= (x_1+y_1)(x_2+y_2)(x_3+y_3)\\ &= x_1x_2x_3 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + y_1x_2x_3 + y_1x_2y_3 + y_1y_2x_3 + y_1y_2y_3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

We notice that the initial expansion of $(x + y)^3$ has $8 = 2^3$ terms. This is a result of the multiplication principle, because, in forming a term, we must decide, for each of the three factors, whether to choose the x or the y from that factor. Next, combining like terms, we obtain one x^3 term, three x^2y terms, three xy^2 terms, and one y^3 term. Note that each such term has the form

$$c_k x^{3-k} y^k$$

where $0 \le k \le 3$, and that the coefficients c_k satisfy

$$c_0 = 1 = C(3,0) \qquad c_1 = 3 = C(3,1) \qquad c_2 = 3 = C(3,2) \qquad c_3 = 1 = C(3,3)$$

that is, the coefficient c_k of $x^{3-k}y^k$ is $C(3, k)$, for $0 \le k \le 3$.

In general, we desire a formula for expanding $(x + y)^n$, where n is a nonnegative integer. In addition to the expansion of $(x + y)^3$, let us work out expansions for several other small values of n:

$$\begin{aligned}(x+y)^0 &= 1 = C(0,0)\\ (x+y)^1 &= x + y = C(1,0)x + C(1,1)y\\ (x+y)^2 &= x^2 + 2xy + y^2 = C(2,0)x^2 + C(2,1)xy + C(2,2)y^2\\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 = C(3,0)x^3 + C(3,1)x^2y + C(3,2)xy^2 + C(3,3)y^3\\ (x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\\ &= C(4,0)x^4 + C(4,1)x^3y + C(4,2)x^2y^2 + C(4,3)xy^3 + C(4,4)y^4\end{aligned}$$

In general, by expanding the binomial $(x + y)^n$ as a sum of terms of the form $c_k x^{n-k} y^k$, we can conjecture that

$$c_k = C(n, k)$$

for $0 \le k \le n$. This is, in fact, the binomial theorem, and because the numbers $C(n, k)$ occur in this context, they are often referred to as ***binomial coefficients***.

Theorem 5.21 (Binomial Theorem): For every nonnegative integer n and any numbers x and y,

$$(x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k$$

Proof: A combinatorial argument is given. As before, we write $(x+y)^n$ in the form

$$(x+y)^n = (x_1+y_1)(x_2+y_2)\cdots(x_n+y_n)$$

where $x_i = x$ and $y_i = y$ from the i th factor of $x+y$ on the right-hand side. Before suppressing subscripts and combining like terms, this expression expands as a sum of 2^n terms, because an arbitrary term is obtained by choosing, for each $i \in \{1, 2, \ldots, n\}$, either x_i or y_i from the i th factor. If we choose x from $n-k$ of the factors and y from the other k factors, where $0 \le k \le n$, then such a term has the form $x^{n-k}y^k$ when the subscripts are suppressed. The number of ways to obtain such a term is the number of ways to select the k factors, out of n factors, from which a y is chosen. This is precisely $C(n,k)$. Thus, when like terms are combined, the coefficient of $x^{n-k}y^k$ is $C(n,k)$. This proves the result.

■

The binomial theorem can also be proved by induction on n; see Exercise 2.

Example 5.21: Find the coefficient of:

(a) x^7y^3 in the expansion of $(x+y)^{10}$;
(b) x^4y^7 in the expansion of $(2x-y)^{11}$.

Solution:

(a) By the binomial theorem,

$$(x+y)^{10} = \sum_{k=0}^{10} C(10,k)x^{10-k}y^k$$

Thus, the coefficient of x^7y^3 is $C(10,3) = 120$.

(b) In order to apply the binomial theorem, let $x' = 2x$ and $y' = -y$. Then

$$(2x-y)^{11} = (x'+y')^{11} = \sum_{k=0}^{11} C(11,k)(x')^{11-k}(y')^k$$

When $k=7$, this expansion yields the term

$$C(11,7)(x')^4(y')^7 = C(11,7)(2x)^4(-y)^7 = -2^4\,C(11,7)\,x^4\,y^7$$

Thus, the coefficient of x^4y^7 is $-2^4\,C(11,7) = -5280$.

■

The binomial theorem is true for any numbers x and y; in particular, if we replace x by 1 and y by x, then we obtain the following corollary.

Corollary 5.22: For every nonnegative integer n and any number x,

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$$

∎

We can go a step further and replace x by 1 in Corollary 5.22; this yields the identity

$$2^n = \sum_{k=0}^{n} C(n,k)$$

which is Corollary 5.13. If we let $x = -1$ in Corollary 5.22, then we obtain the identity

$$0 = C(n,0) - C(n,1) + \cdots + (-1)^k C(n,k) + \cdots + (-1)^n C(n,n)$$

or

$$0 = \sum_{k=0}^{n} (-1)^k C(n,k)$$

This can be interpreted as saying that, for an n-element set S, the number of subsets of S of even cardinality is the same as the number of subsets of S of odd cardinality; see Exercise 16.

Several other interesting identities can be obtained from Corollary 5.22.

Example 5.22: Show that the identity

$$n \cdot 2^{n-1} = \sum_{k=1}^{n} k\, C(n,k)$$

holds for every positive integer n.

Solution: Our first method is to employ Corollary 5.22 and some calculus. We start with

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$$

This can be interpreted as stating that two (real-valued) functions are identical. Thus, we may differentiate both sides of this identity with respect to x:

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\, C(n,k)\, x^{k-1}$$

(Recall that the derivative of a sum of functions is the sum of the derivatives of those functions.) Setting $x = 1$, we see that

$$n \cdot 2^{n-1} = \sum_{k=1}^{n} k\, C(n,k)$$

Alternate Solution: Our second method is combinatorial; one way to verify an identity is to show that the left-hand side and right-hand side count the same thing.

Consider the left-hand side of this identity:

$$n \cdot 2^{n-1}$$

This has a rather obvious combinatorial interpretation, namely, by the multiplication principle, it is the number of ways to first choose an element x from $S = \{1, 2, \ldots, n\}$ and then choose a subset A of $S - \{x\}$. So, the left-hand side, $n \cdot 2^{n-1}$, counts the number of ordered pairs (x, A) with $x \in S$ and $A \subseteq S - \{x\}$. Now our job is to show that the right-hand side counts the same thing.

Since the right-hand side

$$\sum_{k=1}^{n} k\, C(n, k)$$

is a sum, we should think of applying the addition principle. Note that for a fixed k, $1 \leq k \leq n$, the term $kC(n, k)$ counts the number of ways to choose a k-element subset B of S and then choose $x \in B$. So we have the following recipe for uniquely obtaining an ordered pair (x, A), where $x \in S$ and $A \subseteq S - \{x\}$:

Step 1: Choose k, $1 \leq k \leq n$.
Step 2: Choose a k-element subset B of S.
Step 3: Choose x in B and let $A = B - \{x\}$.

By the addition principle, the number of ways to follow this recipe, and hence the number of ordered pairs (x, A), is

$$\sum_{k=1}^{n} k\, C(n, k)$$

Therefore, it follows that

$$n \cdot 2^{n-1} = \sum_{k=1}^{n} k\, C(n, k)$$

This result can also be shown by induction on n, making use of Theorem 5.23 below.

■

The binomial coefficients $C(n, k)$, $0 \leq k \leq n$, are listed for successive values of n in the following triangular array:

$$\begin{array}{ccccccccc}
 & & & & C(0,0) & & & & \\
 & & & C(1,0) & & C(1,1) & & & \\
 & & C(2,0) & & C(2,1) & & C(2,2) & & \\
 & C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) & \\
C(4,0) & & C(4,1) & & C(4,2) & & C(4,3) & & C(4,4) \\
 & & & & \vdots & & & &
\end{array}$$

or

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

This array is called ***Pascal's triangle***, after the seventeenth-century French mathematician Blaise Pascal. Although he did not discover the triangle, he did derive several of its properties. A careful look at Pascal's triangle reveals some interesting patterns. For example, notice that the array is symmetric with respect to the vertical column containing $C(0,0)$, $C(2,1)$, $C(4,2)$, and so on; that is, for $0 \leq k \leq n$, it appears that

$$C(n,k) = C(n, n-k)$$

This result is addressed in Exercise 8 of Exercise Set 5.3. Also, for $n \geq 2$ and $0 < k < n$, notice that $C(n,k)$ is the sum of the two numbers just above it in the preceding row. In other words, it appears to be the case that

$$C(n,k) = C(n-1, k-1) + C(n-1, k)$$

In fact, this result is known as *Pascal's identity.* It is very useful, deserving special mention and a combinatorial proof.

Theorem 5.23 (Pascal's Identity): For every integer $n \geq 2$ and for each integer k, $0 < k < n$,

$$C(n,k) = C(n-1, k-1) + C(n-1, k)$$

Proof: Let $S = \{1, 2, \ldots, n\}$, where n is an integer, $n \geq 2$. For k an integer, $0 < k < n$, let $\mathcal{P}_k$ denote the set of k-element subsets of S. We partition $\mathcal{P}_k$ as $\{\mathcal{P}_k', \mathcal{P}_k''\}$, where

$$\mathcal{P}_k' = \{T \in \mathcal{P}_k \mid n \in T\}$$
$$\mathcal{P}_k'' = \{T \in \mathcal{P}_k \mid n \notin T\}$$

For $T \in \mathcal{P}_k$, note that

$$T \in \mathcal{P}_k' \leftrightarrow T - \{n\} \subseteq S - \{n\}$$

Thus, there is a one-to-one correspondence between the elements of $\mathcal{P}_k'$ and the $(k-1)$-element subsets of $S - \{n\}$. Hence,

$$|\mathcal{P}_k'| = C(n-1, k-1)$$

In a similar fashion, note that

$$T \in \mathcal{P}_k'' \leftrightarrow T \subseteq S - \{n\}$$

Hence, there is a one-to-one correspondence between the elements of $\mathcal{P}_k''$ and the k-element subsets of $S - \{n\}$. Therefore,

$$|\mathcal{P}_k''| = C(n-1, k)$$

Thus, by the addition principle,

$$C(n,k) = |\mathcal{P}_k| = |\mathcal{P}_k'| + |\mathcal{P}_k''| = C(n-1, k-1) + C(n-1, k)$$

■

Another interesting identity comes from looking at the sum of the elements on a diagonal that slopes upward from left to right (starting in a given row); for example, starting with row 4, observe that

$$\begin{aligned}
C(0,0) + C(1,0) + C(2,0) + C(3,0) + C(4,0) &= 5 = C(5,1)\\
C(1,1) + C(2,1) + C(3,1) + C(4,1) &= 10 = C(5,2)\\
C(2,2) + C(3,2) + C(4,2) &= 10 = C(5,3)\\
C(3,3) + C(4,3) &= 5 = C(5,4)\\
C(4,4) &= 1 = C(5,5)
\end{aligned}$$

Note that each left-hand side gives a sum of binomial coefficients $C(m, r)$, where m ranges from r to n. We can compute such a sum by repeated application of Pascal's identity; for example,

$$\begin{aligned} C(4,4)+C(5,4)+C(6,4)+C(7,4) &= C(5,5)+C(5,4)+C(6,4)+C(7,4) \\ &= C(6,5)+C(6,4)+C(7,4) \\ &= C(7,5)+C(7,4) \\ &= C(8,5) \end{aligned}$$

This suggests a proof by induction of the following result.

Theorem 5.24: For every positive integer n and for each integer r, $0 \le r \le n$,

$$\sum_{m=r}^{n} C(m,r) = C(n+1, r+1)$$

Proof: We proceed by induction on n. To see that the formula in the theorem holds when $n = 1$, we must test it for $r = 0$ and $r = 1$. For $r = 0$, we see that $C(0,0) + C(1,0) = C(2,1)$ holds, whereas for $r = 1$, note that $C(1,1) = C(2,2)$. Therefore, the theorem holds when $n = 1$.

Let k be an arbitrary positive integer and assume the theorem holds when $n = k$. Explicitly, our induction hypothesis is that

$$C(r,r) + C(r+1,r) + \cdots + C(k,r) = C(k+1,r+1) \qquad \bigstar$$

for each r, $0 \le r \le k$. To complete the proof, it must be shown that the theorem holds when $n = k + 1$, namely, that

$$C(r,r) + C(r+1,r) + \cdots + C(k,r) + C(k+1,r) = C(k+2,r+2)$$

for each r, $0 \le r \le k+1$. When $r = k+1$, this statement is that

$$C(k+1,k+1) = C(k+2,k+2)$$

and is clearly true. To see that it holds for $0 \le r \le k$, we add $C(k+1,r)$ to both sides of the identity $\bigstar$, and apply Pascal's identity:

$$\begin{aligned} C(r,r) + C(r+1,r) + \cdots + C(k,r) + C(k+1,r) &= C(k+1,r+1) + C(k+1,r) \\ &= C(k+2,r+1) \end{aligned}$$

as was to be shown.

$\blacksquare$

Example 5.23: Find a (nice, compact) formula for the sum

$$\sum_{k=1}^{n} k(k+1)(k+2) = (1)(2)(3) + (2)(3)(4) + \cdots + n(n+1)(n+2)$$

Solution: We apply Theorem 5.24 to arrive at the value of this sum. The key is to realize that the k th term can be expressed in terms of a binomial coefficient:

$$k(k+1)(k+2) = P(k+2,3) = 3!\,C(k+2,3)$$

Thus,

$$\begin{aligned}\sum_{k=1}^{n} k(k+1)(k+2) &= \sum_{k=1}^{n} 3!\,C(k+2,3)\\ &= 3!\sum_{k=1}^{n} C(k+2,3)\\ &= 3!\sum_{m=3}^{n+2} C(m,3)\\ &= 3!\,C(n+3,4)\end{aligned}$$

■

Further identities and applications of the binomial theorem are explored in the exercises.

We next move on to the multinomial theorem. Just as the binomial theorem deals with expanding the n th power of a binomial, the multinomial theorem deals with expanding the n th power of a general multinomial:

$$(x_1 + x_2 + \cdots + x_k)^n$$

The terms in such an expansion have the general form

$$c\ x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where $n_1, n_2, \ldots, n_k$ are nonnegative integers such that $n_1+n_2+\cdots+n_k = n$. We wish to determine the coefficient c of this general term.

Let's look at the specific case $k = 3$, $n = 4$. It is easy to verify that

$$\begin{aligned}(x_1+x_2+x_3)^4 = x_1^4 &+ 4x_1^3x_2 + 4x_1^3x_3 + 6x_1^2x_2^2 + 12x_1^2x_2x_3 + 6x_1^2x_3^2 + 4x_1x_2^3 + 12x_1x_2^2x_3\\ &+ 12x_1x_2x_3^2 + 4x_1x_3^3 + x_2^4 + 4x_2^3x_3 + 6x_2^2x_3^2 + 4x_2x_3^3 + x_3^4\end{aligned}$$

We note that the expansion of $(x_1+x_2+x_3)^4$ has 15 terms. This is because each term has the form

$$c\ x_1^{m_1} x_2^{m_2} x_3^{m_3}$$

where m_1, m_2, m_3 are nonnegative integers and $m_1 + m_2 + m_3 = 4$. So, by Theorem 1.18, the number of such terms is $C(4+3-1, 3-1) = C(6,2) = 15$. Now, let's focus on the coefficients; for instance, note that 12 is the coefficient of $x_1^2x_2x_3$. Why is this so? Well, $(x_1+x_2+x_3)^4$ is the product of four factors. In the initial expansion of this product, before like terms are combined, we obtain a term of the form $x_1^2x_2x_3$ by choosing x_1 from exactly two of the four factors, then choosing x_2 from one of the two remaining factors, and then choosing x_3 from the remaining factor. So, by the multiplication principle, the number of such terms is

$$C(4,2)C(2,1)C(1,1) = C(4;2,1,1) = 12$$

Let's now consider the initial expansion of

$$(x_1 + x_2 + \cdots + x_k)^n$$

Before like terms are combined, the number of terms of the form $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ is the number of ways to choose x_1 from exactly n_1 of the n factors, then choose x_2 from exactly n_2 of the remaining $n-n_1$ factors, and so on, until finally exactly n_k factors remain and x_k is chosen from each of these. By the multiplication principle, this number is

$$C(n,n_1)C(n-n_1,n_2)\cdots C(n-(n_1+n_2+\cdots+n_{k-2}),n_{k-1})C(n_k,n_k)=\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

When each n_i is positive, this number is written $C(n;n_1,n_2,\ldots,n_k)$ and is the number of ordered partitions $(S_1,S_2,\ldots,S_k)$ of an n-element set into k parts with $|S_i|=n_i$, $1\le i\le k$. However, the above formula holds even when some of the n_is are 0. Thus, it makes sense to call $C(n;n_1,n_2,\ldots,n_k)$ a ***multinomial coefficient*** and to define it, for any nonnegative integers n and $n_1,n_2,\ldots,n_k$ such that $n=n_1+n_2+\cdots+n_k$, by the formula

$$C(n;n_1,n_2,\ldots,n_k)=\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

With this understanding, we can state the multinomial theorem.

Theorem 5.25 (Multinomial Theorem): For every nonnegative integer n, every positive integer k, and any numbers $x_1,x_2,\ldots,x_k$,

$$(x_1+x_2+\cdots+x_k)^n=\sum C(n;n_1,n_2,\ldots,n_k)\,x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$$

where the sum is over all k-tuples $(n_1,n_2,\ldots,n_k)$ of nonnegative integers such that $n_1+n_2+\cdots+n_k=n$.

■

Example 5.24: Find the coefficient of:

(a) $x_1^3x_2^2x_3^1$ in the expansion of $(x_1+x_2+x_3)^6$;
(b) x^4y^3 in the expansion of $(w+x+y+z)^7$;
(c) x^2y^3 in the expansion of $(x+2y+3)^8$;
(d) x^{30} in the expansion of $(x^2+x^3+x^5)^{11}$.

Solution:

(a) We apply the multinomial theorem with $n=6$ and $k=3$ to find that the answer is $C(6;3,2,1)=60$.

(b) We apply the multinomial theorem with $n=7$, $k=4$, $x_1=w$, $x_2=x$, $x_3=y$, and $x_4=z$. Then the coefficient of $x^4y^3=w^0x^4y^3z^0$ is $C(7;0,4,3,0)=35$.

(c) We apply the multinomial theorem with $n=8$, $k=3$, $x_1=x$, $x_2=2y$, and $x_3=3$. Thus,

$$(x+2y+3)^8=\sum C(8;n_1,n_2,n_3)x^{n_1}(2y)^{n_2}3^{n_3}$$

So the coefficient of x^2y^3 is $C(8;2,3,3)2^33^3=120960$.

(d) First, we apply the multinomial theorem with $n = 11$, $k = 3$, $x_1 = x^2$, $x_2 = x^3$, and $x_3 = x^5$. This yields

$$\begin{aligned}(x^2 + x^3 + x^5)^{11} &= \sum C(11; n_1, n_2, n_3)(x^2)^{n_1}(x^3)^{n_2}(x^5)^{n_3} \\ &= \sum C(11; n_1, n_2, n_3)x^{2n_1+3n_2+5n_3}\end{aligned}$$

Thus, the coefficient of x^{30} is

$$\sum C(11; n_1, n_2, n_3)$$

where the sum is over all triples (n_1, n_2, n_3) of nonnegative integers with $n_1 + n_2 + n_3 = 11$ and $2n_1 + 3n_2 + 5n_3 = 30$. Note that this system of two equations in three unknowns is equivalent to $n_1 + n_2 = 11 - n_3$ and $3n_1 + 2n_2 = 25$. This has three solutions: $(7, 2, 2)$, $(5, 5, 1)$, and $(3, 8, 0)$. Hence, the coefficient we are seeking is $C(11; 7, 2, 2) + C(11; 5, 5, 1) + C(11; 3, 8, 0) = 4917$.

■

Exercise Set 5.5

1. Use Pascal's identity (Theorem 5.23) to evaluate the following binomial coefficients:

   (a) $C(4, 2)$ (b) $C(6, 3)$

2. Prove the binomial theorem (Theorem 5.21) by induction on n.
3. Use the binomial theorem to expand each of the following binomials:

   (a) $(x + y)^5$ (b) $(x + y)^6$ (c) $(x + 3y)^7$

4. Prove Pascal's identity (Theorem 5.23) algebraically, using Corollary 5.12.
5. Give a combinatorial proof of Theorem 5.24. Hint: Consider choosing an $(r + 1)$-element subset of $\{1, 2, \ldots, n + 1\}$; partition these subsets into $n - r + 1$ classes according to the largest element chosen, and apply the addition principle.
6. Prove the following property, called the *unimodal property* of the binomial coefficients:

   (a) $C(n, k) \leq C(n, k + 1)$ for $0 \leq 2k < n$
   (b) $C(n, k) > C(n, k + 1)$ for $n \leq 2k \leq 2n - 2$
   (c) $C(n, k) = C(n, k + 1)$ if and only if $2k = n - 1$

7. Find the coefficient of:

   (a) $x^{11}y^4$ in the expansion of $(x + y)^{15}$
   (b) x^6y^4 in the expansion of $(2x + y)^{10}$
   (c) x^3y^5 in the expansion of $(3x - 2y)^8$
   (d) x^5 in the expansion of $(1 + x + x^2)(1 + x)^6$

8. For a positive integer n, prove that

$$\sum_{k=0}^{n} C(n, k)^2 = C(2n, n)$$

   (a) Use the binomial theorem and the fact that $(1 + x)^{2n} = (1 + x)^n(1 + x)^n$.

   (b) Give a combinatorial proof based on the fact that each n-element subset of $\{1, 2, \ldots, 2n\}$ is the union of a k-element subset of $\{1, 2, \ldots, n\}$ and an $(n-k)$-element subset of $\{n+1, n+2, \ldots, 2n\}$ for some k, $0 \leq k \leq n$; partition according to the value of k and apply the addition principle.

9. Use an appropriate identity to simplify each of the following expressions.

(a) $C(12,4)+C(12,5)$
(b) $C(5,5)+C(6,5)+C(7,5)+\cdots+C(11,5)$
(c) $C(10,0)+C(10,1)+C(10,2)+\cdots+C(10,10)$
(d) $1\cdot C(9,1)+2\cdot C(9,2)+3\cdot C(9,3)+\cdots+9\cdot C(9,9)$

10. Consider Pascal's triangle with $n+1$ rows; let $f(n)$ be the sum of the numbers that lie on the median from the lower left-hand corner to the opposite side. For instance, $f(5)=1+4+3=8$ and $f(6)=1+5+6+1=13$.

(a) Express $f(n)$ as a sum of binomial coefficients.
(b) Do you recognize the sequence $(f(0),f(1),f(2),\ldots)$?

11. Let n and k be integers with $0\le k<n$.

(a) Give an algebraic proof (using Corollary 1.12) of the identity

$$(n-k)C(n,k)=nC(n-1,k)$$

(b) Give a combinatorial proof of the same identity. Hint: Let $A=\{1,2,\ldots,n\}$; how many ways are there to select a pair (x,B), where $B\subseteq A$, $|B|=k<n$, and $x\in A-B$?

12. For n an integer, $n>1$, prove the identity

$$\sum_{k=1}^{n}(-1)^{k-1}kC(n,k)=0$$

13. Use the identity of Exercise 11 (and the fact that $C(k,k)=1$) to compute the following binomial coefficients.

(a) $C(4,2)$ (b) $C(6,3)$

14. Let n and k be positive integers.

(a) Find integers a, b, and c such that

$$k^3=aC(k,3)+bC(k,2)+cC(k,1)$$

(b) Use part (a) and appropriate identities to find a formula for

$$1^3+2^3+\cdots+n^3$$

15. For nonnegative integers n and r, consider the identity

$$\sum_{k=0}^{r}C(n+k,k)=C(n+r+1,r)$$

(a) Give a combinatorial proof of this identity. Hint: Consider choosing an r-element subset of $\{1,2,\ldots,n+r+1\}$; partition these subsets according to the largest element not chosen, and apply the addition principle.
(b) Prove the identity by induction on r.

16. Let S be an n-element set, $n > 0$. Show that the number of subsets of S with even cardinality is the same as the number of subsets of S with odd cardinality. What is this number?

17. Let n be a positive integer and let r be a real number.

(a) Use Corollary 1.24 to show that

$$3^n = C(n,0)2^0 + C(n,1)2^1 + \cdots + C(n,n)2^n$$

(b) Generalize part (a) to give a formula for the sum

$$\sum_{k=0}^{n} C(n,k)r^k$$

18. Here is an outline of an alternate proof Corollary 5.22. We know that $f(x) = (1+x)^n$ is a polynomial of degree n, say, $g(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, and we suppose that $f(x) = g(x)$ for all $x \in \mathbb{R}$. We wish to determine the coefficients $c_0, c_1, c_2, \ldots, c_n$. To determine c_0, use the fact that $f(0) = g(0)$; to determine c_1, use the fact that $f'(0) = g'(0)$; to determine c_2, use the fact that $f''(0) = g''(0)$, and so on.

19. Find a formula for

$$\sum_{k=0}^{n} (2k+1)C(n,k)$$

20. For a given nonnegative integer n, the directed graph $G_n = (V_n, A_n)$ is defined as follows:

$$V_n = \{(x,y) \mid x,y \in \mathbb{Z} \text{ and } n \geq x \geq y \geq 0\}$$
$$A_n = \{((x_1,y_1),(x_2,y_2)) \mid x_2 = x_1 + 1 \text{ and } (y_2 = y_1 \text{ or } y_2 = y_1 + 1)\}$$

(a) Draw the directed graph G_4.
(b) In G_n, show that the number of paths from $(0,0)$ to (n,k) is $C(n,k)$.

21. For $n \in \mathbb{Z}^+$, prove that $C(2n+1,n) < 4^n$.

22. Extend the result of Example 5.23 by showing that, for a positive integer m,

$$\sum_{k=1}^{n} P(m+k-1,m) = m!\,C(n+m,m+1)$$

23. Let n and k be integers with $0 \leq k < n$. Give a combinatorial proof of the identity

$$nC(n-1,k) = (k+1)C(n,k+1)$$

24. For a positive integer n, prove that

$$\sum_{k=0}^{n} \frac{C(n,k)}{k+1} = \frac{2^{n+1}-1}{n+1}$$

(a) Use Corollary 5.22 and some integral calculus.
(b) Use the identity of Exercise 23, rewritten as

$$\frac{C(n,k)}{k+1} = \frac{C(n+1,k+1)}{n+1}$$

25. Find the coefficient of:

(a) $x_1^4 x_2^2 x_3^1 x_4^3$ in the expansion of $(x_1 + x_2 + x_3 + x_4)^{10}$
(b) xy^3z^5 in the expansion of $(v + w + x + y + z)^9$
(c) x^2y^3 in the expansion of $(x - 3y + 4)^8$
(d) x^{18} in the expansion of $(1 + x + x^2 + x^3)^7$

CHAPTER PROBLEMS

1. A city police department has 10 detectives — 7 males and 3 females. In how many ways can a team of 3 detectives be chosen to work on a case under the following conditions?

(a) There are no restrictions.
(b) The team must have at least 1 male and at least 1 female.

2. A hand in a game of bridge consists of 13 of the 52 cards in a standard deck.

(a) How many bridge hands are there?

How many bridge hands satisfy the following properties?

(b) The hand has exactly two aces.
(c) The hand has exactly three aces, two kings, one queen, and zero jacks.
(d) The hand has at least one heart.
(e) The hand has four spades, three hearts, three diamonds, and three clubs.
(f) The hand has four cards in one suit and three cards in each of the other three suits.

3. Given 11 different combinatorics books and 7 different number theory books, how many ways are there for Curly to choose a combinatorics book, and then for Larry to choose a combinatorics or a number theory book, and then for Moe to choose both a combinatorics and a number theory book? (Assume that the books are chosen without replacement.)

4. If the numbers from 1 to 100000 are listed, how many times does the digit 5 appear?

5. How many 8-letter sequences, constructed using the 26 (lowercase) letters of the alphabet, satisfy the following conditions?

(a) The sequence contains the letter a exactly three times.
(b) The sequence contains three or four vowels (a, e, i, o, u).
(c) The sequence has no repeated letters.
(d) The sequence contains the letter e an even number of times.

6. A certain make of keyless door lock has five buttons numbered $1, 2, 3, 4, 5$. The lock is set by choosing a code consisting of an ordered pair (A, B) of nonempty subsets of $\{1, 2, 3, 4, 5\}$. To open the door, one first pushes (simultaneously) the buttons corresponding to the elements of A and then pushes the buttons corresponding to the elements of B. How many ways are there to set the lock?

7. The mathematics club at a small college has six senior, five junior, four sophomore, and three freshmen members. Five members are to be chosen to represent the club on the mathematics department curriculum committee. In how many ways can these five students be chosen under the following conditions?

(a) There are no restrictions.
(b) At least two seniors are chosen.
(c) At most one freshman is chosen.
(d) Exactly two sophomores are chosen.

8. Let $P_{m,n}$ denote the set of nonzero polynomials of degree at most n with coefficients chosen from the set $\{0, 1, \ldots, m\}$. Find $|P_{m,n}|$.

9. A jar contains three red balls, four orange balls, five blue balls, and six green balls, numbered 1 through 18, respectively. A subset of four balls is to be selected from the jar at random. How many subsets satisfy the following conditions?

(a) The subset has exactly two orange balls and one green ball.
(b) The subset contains one ball of each color.
(c) It is not the case that all four balls are the same color.
(d) At least two balls in the subset are not blue.

10. How many relations are there on an n-element set? Determine the number of relations on $\{1, 2, \ldots, n\}$ that are:

(a) reflexive
(b) irreflexive
(c) symmetric
(d) antisymmetric

11. Determine the number of ways to arrange the 26 letters A, B, . . . , Z:

(a) in a row.
(b) in a row so that no two vowels are consecutive.
(c) in a circle.
(d) in a circle so that no two vowels are consecutive.

12. Eight students and four faculty members must be seated at two (identical) six-person tables for lunch; what matters is how this set of twelve people is partitioned into two 6-element subsets.

(a) In how many ways can this be done?
(b) In how many ways can it be done so that each table has at least one faculty member?

13. The twelve face cards from a standard deck are arranged in a circle. Four cards are to be chosen.

(a) In how many ways can this be done?
(b) In how many ways can this be done if no two cards that are consecutive (in the circle) are chosen?

14. Let m and n be positive integers with $m < n$. We are interested in the following question: How many permutations of $\{1, 2, \ldots, n\}$ do not contain a permutation of $\{1, 2, \ldots, m\}$ as a "subpermutation"?

(a) First, consider a specific case: How many permutations of $\{1, 2, 3, 4\}$ contain neither the permutation $(1, 2)$ nor the permutation $(2, 1)$?
(b) Try another specific case: How many permutations of $\{1, 2, 3, 4, 5, 6, 7\}$ contain none of the permutations $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, or $(3, 2, 1)$?
(c) Formally define the term subpermutation and answer the general question.

15. Verify that the two arrays shown below are orthogonal Latin squares of order 8.

$A_1 =$

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

$A_2 =$

0	1	2	3	4	5	6	7
2	3	0	1	6	7	4	5
4	5	6	7	0	1	2	3
6	7	4	5	2	3	0	1
3	2	1	0	7	6	5	4
1	0	3	2	5	4	7	6
7	6	5	4	3	2	1	0
5	4	7	6	1	0	3	2

16. Exercises 21 and 22 of Exercise Set 2.5 consider the construction of a field F of order 8. Once constructed, we can rename the elements of this field 0, 1, . . ., 7 so that the array A_1 in the preceding exercise is the addition table for the field and the following is its multiplication table:

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	3	1	7	5
3	0	3	6	5	7	4	1	2
4	0	4	3	7	6	2	5	1
5	0	5	1	4	2	7	3	6
6	0	6	7	1	5	3	2	4
7	0	7	5	2	1	6	4	3

Using F, define the arrays A_1, A_2, . . ., A_7 as follows: (1) The rows of A_i are the same as the rows of A_1, only permuted; (2) the initial column A_i is the same as column i in the multiplication table for F. Show that any two of A_1, A_2, . . ., A_7 are orthogonal Latin squares of order 8.

17. Let n be a positive integer such that $n \bmod 4 \neq 2$. Then either n is odd or n is a multiple of 4. If n is odd, then the result of Exercise 3 of Exercise Set 5.1 can be used to construct a pair of orthogonal Latin squares of order n.

(a) If $n = 2^{k+1}$, with $k \in \mathbb{Z}^+$, show that there exists a pair of orthogonal Latin squares of order n. Hint: Employ the result of problem 15, composition of Latin squares, and induction on k.

(b) Prove: If n is a multiple of 4, then there exists a pair of orthogonal Latin squares of order n. Hint: Write n as

$$n = 2^{k+1} \cdot m$$

with m odd. Apply the result of part (a) and composition of Latin squares.

18. How many ways are there to distribute 20 different homework problems to 10 students under the following conditions?

(a) There are no restrictions.
(b) Each student gets 2 problems.

19. How many different positive integers can be formed using the given nine digits?

(a) $1, 2, 3, 4, 5, 6, 7, 8, 9$ (b) $1, 1, 3, 3, 5, 5, 7, 7, 9$
(c) $3, 3, 3, 6, 6, 6, 9, 9, 9$ (d) $8, 8, 9, 9, 9, 9, 9, 9, 9$

20. A local dairy offers 16 flavors of ice cream.

(a) How many ways are there to choose eight scoops of ice cream?
(b) How many ways are there to choose eight scoops of ice cream of eight different flavors?
(c) How many ways are there to distribute the eight scoops of part (b) to four students so that each student gets two scoops?
(d) How many ways are there to distribute eight (identical) scoops of chocolate ice cream to four students so that each student gets at least one scoop?
(e) If a professor wants to purchase a triple-scoop ice cream cone, how many choices does she have? Does the order of flavors on the cone matter? Answer the question for both cases.

21. Consider choosing a subset of three digits from $\{1, 2, \ldots, 9\}$.

(a) In how many ways can this be done?
(b) In how many ways can this be done if no two of the digits chosen are consecutive?

22. Given five calculus books, three linear algebra books, and two number theory books (all distinct), how many ways are there to perform the following tasks?

(a) Select three books, one in each subject.
(b) Make a row of three books.
(c) Make a row of three books, with one book on each subject.
(d) Make a row of three books with exactly two of the subjects represented.
(e) Make a row of three books all on the same subject.
(f) Place the books on three different shelves, where the order of the books on a shelf matters.

23. A rumor is spread among m college presidents as follows: The person who starts the rumor telephones someone, and then that person telephones someone else, and so on. Assume any person can pass along the rumor to anyone except the person from whom the rumor was heard. How many different paths through the group can the rumor follow in n calls?

24. Given the books in Problem 22, in how many ways can they be arranged on a shelf under the following conditions?

(a) There are no restrictions.
(b) Books on the same subject must be placed together.

25. For $1 \leq k \leq n$, consider the identity

$$kC(n, k) = (n - k + 1)C(n, k - 1)$$

(a) Verify this identity algebraically, using Corollary 5.12.
(b) Give a combinatorial proof of this identity. Hint: Argue that both sides of the identity count the number of ways to choose an ordered pair (B, x), where B is a k-combination of $\{1, 2, \ldots, n\}$ and $x \in B$.

26. Let m, n_1, and n_2 be integers with $0 \leq m \leq n_1 \leq n_2$. Prove *Vandermonde's identity*:

$$C(n_1 + n_2, m) = \sum_{k=0}^{m} C(n_1, k)C(n_2, m - k)$$

(a) Use the binomial theorem and the fact that $(1+x)^{m+n} = (1+x)^m(1+x)^n$.

(b) Give a combinatorial proof based on the fact that each m-combination of $\{1, 2, \ldots, n_1+n_2\}$ is the union of a k-combination of $\{1, 2, \ldots, n_1\}$ and an $(m-k)$-combination of $\{n_1+1, n_1+2, \ldots, n_1+n_2\}$ for some k, $0 \le k \le m$; partition according to the value of k and apply the addition principle.

(c) Prove the result by induction on m.

27. Find the coefficient of x^4 in the expansion of $(10x + \frac{1}{2})^8$.

28. Use the binomial theorem to rewrite $(1+\sqrt{3})^6$ in the form $a+b\sqrt{3}$, where a and b are integers.

29. Determine the coefficient of x^7y^5 in the expansion of $(2x - y/4)^{12}$.

30. Prove the identity

$$1 \cdot C(n,0) + 2 \cdot C(n,1) + \cdots + (n+1) \cdot C(n,n) = (n+2) \cdot 2^{n-1}$$

31. The row of Pascal's triangle corresponding to $n = 6$ is

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Use Pascal's identity to find the rows of Pascal's triangle corresponding to the following values of n:

(a) 7 (b) 8

32. Use binomial coefficients to give a formula for the sum

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n$$

33. Compute $C(6,3)$.

(a) Use Pascal's identity.

(b) Use the identity from Problem 25: $C(n,k) = (n-k+1)C(n,k-1)/k$.

34. Let n be a positive integer. Prove that

$$\left(\sum_{i=0}^{n} C(n,i)\right)^2 = \sum_{j=0}^{2n} C(2n,j)$$

35. Let m and n be integers with $0 \le m \le n$. Prove the identity

$$C(n,0)C(n,m) + C(n,1)C(n,m-1) + \cdots + C(n,m)C(n,0) = C(2n,m)$$

36. Find a formula for the sum

$$\sum_{k=0}^{n} \frac{(-1)^k C(n,k)}{k+1}$$

37. Let m, n_1, and n_2 be positive integers with $m \le n_2$. Prove that

$$C(n_1+n_2, n_1+m) = \sum_{k=0}^{n_2} C(n_1,k)C(n_2,m+k)$$

(Note: By definition, $C(n,i) = 0$ when $n < i$.)

38. Find a formula for

$$\sum_{k=1}^{n} C(n, k-1)C(n, k)$$

39. For a nonnegative integer n, show that

$$\sum_{k=0}^{n} (-1)^k C(n, k)3^{n-k} = 2^n$$

40. For positive integers n and k, show that

$$k^n = \sum C(n; n_1, n_2, \ldots, n_k)$$

where the sum is over all k-tuples $(n_1, n_2, \ldots, n_k)$ of nonnegative integers such that $n_1 + n_2 + \cdots + n_k = n$.

41. Let n be a positive integer. Show that

(a) $\sum_{k=0}^{n} (2k+1)C(2n+1, 2k+1) = (2n+1)2^{2n-1}$
(b) $\sum_{k=1}^{n} 2kC(2n, 2k) = n2^{2n-1}$

42. Let k be a fixed positive integer. Show that any nonnegative integer r is uniquely represented as

$$r = C(m_1, 1) + C(m_2, 2) + \cdots + C(m_k, k)$$

for integers $0 \leq m_1 < m_2 < \cdots < m_k$.

43. Let r and n be positive integers with $2 \leq r \leq n-2$. Give a combinatorial argument to show that

$$C(n, r) = C(n-2, r) + 2C(n-2, r-1) + C(n-2, r-2)$$

44. Given a positive integer n and an integer r, $0 \leq r \leq n! - 1$, show that r is uniquely expressed as

$$\sum_{k=1}^{n-1} d_k k!$$

where each d_k is an integer, $0 \leq d_k \leq k$. This representation is called the *factorial representation* of r. For instance,

$$85 = 3 \cdot 4! + 2 \cdot 3! + 0 \cdot 2! + 1 \cdot 1!$$

so $d_4 = 3$, $d_3 = 2$, $d_2 = 0$, and $d_1 = 1$ in the factorial representation of 85.

45. For positive integers m and n, determine the number of integer solutions to

$$x_1 + x_2 + \cdots + x_n + x_{n+1} + \cdots + x_{2n} = m$$

such that $x_1, \ldots, x_n$ are nonnegative integers and $x_{n+1}, \ldots, x_{2n}$ are positive integers.

46. Let $n_1, n_2, \ldots, n_k$, and n be positive integers with $n = n_1 + n_2 + \cdots + n_k$. Show that the multinomial coefficient $C(n; n_1, n_2, \ldots, n_k)$ is equal to

$$C(n-1; n_1-1, n_2, \ldots, n_k) + C(n-1; n_1, n_2-1, n_3, \ldots, n_k) + \cdots + C(n-1; n_1, n_2, \ldots, n_k-1)$$

47. Determine the constant term of $(x + x^{-1} + 2x^{-4})^{17}$.

48. Let m, k, and n be integers with $0 \leq m \leq k \leq n$.

(a) Show that $C(k,m)C(n,k) = C(n,m)C(n-m,k-m) = C(n,m)C(n-m,n-k)$.
(b) Show that

$$\sum_{k=m}^{n} C(k,m)C(n,k) = C(n,m)2^{n-m}$$

49. Use the multinomial theorem to simplify $(1+\sqrt{2}+\sqrt{5}\,)^4$.

50. Eight scientists work at a secret research building. The building is to have D doors, each with four locks. Each scientist is to be issued a set of K keys. For reasons of security, it is required that at least four scientists be present together in order that, among them, they possess a subset of four keys that fit the four locks on one of the D doors. How many doors must the building have and how many keys must each scientist be issued?

51. Use the binomial theorem to determine whether $(1.01)^{100}$ is greater than 2.

52. If, in Problem 50, there is to be only one door with L locks, what should the values be for L and K if it is required that at least four scientists be present together in order to enter the building?

53. Use the binomial theorem to determine whether $(0.99)^{100}$ is less than 0.4.

54. How many ways are there to distribute 12 different books to three different bookshelves if the order of the books on a shelf is important? (Note: Some shelves may have no books assigned to them.)

55. How many ways are there to place eight rings on four fingers of your left hand? Answer assuming that:

(a) The rings are identical.
(b) The rings are identical and at least one ring must be placed on each finger.
(c) The rings are different and the order of rings on a finger is not important.
(d) The rings are different and the order of rings on a finger is important.
(e) The rings are different, the order of rings on a finger is important, and at least one ring must be placed on each finger.
(f) The rings are different, the order of rings on a finger is important, and exactly two rings must be placed on each finger.

56. Generalize Problems 54 and 55(d) to determine a formula for the number of ways to distribute m different balls to n different boxes if the order in which the balls are distributed to a box is important.

57. How many ways are there to distribute 12 different books to three different bookshelves if the order of the books on a shelf is important and at least 1 book must be placed on each shelf?

58. Generalize Problems 55(e) and 57 to determine a formula for the number of ways to distribute m different balls to n different boxes ($m \geq n$) if the order in which the balls are distributed to a box is important and at least one ball must be placed in each box.

59. Suppose in Problems 54 and 57 that we have a top shelf, a middle shelf, and a bottom shelf. How many ways are there to distribute the 12 books to these three shelves if the order of the books on a shelf is important, 3 books are placed on the top shelf, 4 books are placed on the middle shelf, and 5 books are placed on the bottom shelf.

60. Generalize Problems 55(f) and 59 to determine a formula for the number of ways to distribute m different balls to n different boxes if the order in which the balls are distributed to a box is important and exactly m_i balls are placed in box i, $1 \leq i \leq n$, where each m_i is a nonnegative integer and $m_1 + m_2 + \cdots + m_n = m$.