Problem 3. Let m be a positive integer and consider the number $N = m(m+2013)$.

a. Prove that if N is a perfect square, the m cannot be prime.

b. Find a positive integer m such that N is a perfect square.

Solution.

a. If N is a perfect square, then N has prime factorization

$$N = p_1^{2r_1} p_2^{2r_2} \cdots p_k^{2r_k},$$

where each of $r_1, r_2, \ldots, r_k$ is a positive integer. In particular ever prime factor of N would occur an even number of times. This if m is a prime, then m must also be a factor of $m+2013$. It follows that m is also a factor of $2013 = 3 \cdot 11 \cdot 61$. It follows that $m = 3$ or $m = 11$ or $m = 61$, that is, m must be one of the prime factors of 2013. However, it is easy to check that none of

$$3(3+2013), \qquad 11(11+2013) \qquad \text{and} \qquad 61(61+2013)$$

is prime. This proves that if $N = m(m+2013)$ is a perfect square, then m cannot be prime.

b. There are many values of m for which $m(m+2013)$ is a square. One easy way to find such am m is to recall that

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

Thus if we take

$$m = 1+3+5+\cdots+2011 = 1006^2, \qquad \text{then} \qquad m+2103 = 1+3+5+\cdots+2011+2013 = 1007^2,$$

and

$$N = 1006^2 \cdot 1007^2$$

is a perfect square.

Similarly, noting that $2013 = 669 + 671 + 673$ leads to $m = 334^2$, and $2013 = 173 + 175 + \cdots + 193$ leads to $m = 86^2$. In addition, one can also write 2013 as a sum of 31 consecutive odd numbers,

$$2013 = 29 + 31 + \cdots + 93,$$

which leads to the solution $m = 14^2$.

It is not hard to show that if m is *not* a perfect square, then m has the form $m = 3^a 11^b 61^c k^2$ where each of a, b, c is either 0 or 1. There are seven ways to chose $(a,\ b,\ c)$ with these restrictions, and in each case the equation $m(m+2013) = n^2$ reduces to an equation of the form $k^2(k^2 + d) = \ell^2$ where d is a factor of 2013. This last equation can be solved as above and leads to the rest of the solutions: 671, 976, 1875, 4575, 9251, 15616, 29700, 91091, and 336675.