3. Partial autocorrelation

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Exercises due Nov 10, 2021 17:29 IST Completed

Partial Autocorrelation

Let $X_0, \ldots, X_n$ be a stationary time series. Recall that the autocorrelation function of the series at lag h is defined

$$\rho_X(h) = \text{Corr}(X_h, X_0) = \mathbf{E}\left[(X_h - \mathbf{E}[X_0])(X_0 - \mathbf{E}[X_0])\right] / \text{Var}(X_0).$$

The partial autocorrelation between X_h and X_0 is the correlation between X_h and X_0 with the correlation due to the intermediate terms of the series $X_1, \ldots, X_{h-1}$ removed. That is, the correlation between X_h and X_0 after the intermediate terms of the series $X_1, \ldots, X_{h-1}$ have been partialled-out. Formally, the partial autocorrelation of time series X_t at lag h is

$$\alpha_X(h) := \text{Corr}\left(X_h - \hat{X}_h^{\text{lin}_{h-1}},\ X_0 - \hat{X}_0^{\text{lin}_{h-1}}\right),$$

where $\hat{X}_h^{\text{lin}_{h-1}}$ is the linear regression (projection) of X_h on $X_1, \ldots, X_{h-1}$, and $\hat{X}_0^{\text{lin}_{h-1}}$ is the linear regression (projection) of X_0 on $X_1, \ldots, X_{h-1}$. That is, $\alpha_X(h)$ is the correlation between X_h and X_0 with the best linear predictions $\hat{X}_h^{\text{lin}_{h-1}}$ of X_h and $\hat{X}_0^{\text{lin}_{h-1}}$ of X_0 based on the intermediate terms of the series $X_1, \ldots, X_{h-1}$ removed from X_h and X_0 respectively.

A convenient method to compute the partial autocorrelation $\alpha_X(h)$ is to use the Frisch-Waugh-Lovell theorem. Specifically, FWL theorem says that $\alpha_X(h)$ is the regression coefficient on regressor X_{t-h} in the regression of X_t on the set of regressors $X_{t-1}, \ldots, X_{t-h}$: If

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_h X_{t-h} + \tilde{X}_t, \quad \text{where} \quad \mathbf{E}[X_{t-j}\tilde{X}_t] = 0, \quad j = 1, \ldots, h,$$

is the best linear prediction of X_t in the linear span of $X_{t-1}, \ldots, X_{t-h}$ (i.e., the linear regression), then $\alpha_X(h) = \phi_h$. The regression coefficients $\phi_1, \ldots, \phi_h$ can be estimated by e.g., method of moments with Yule-Walker equations.

The partial autocorrelation function $\alpha_X(h)$ is used to determine the order of the $\text{AR}(p_0)$ process:

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_{p_0} X_{t-p_0} + W_t, \qquad W_t \sim \text{WN}(\sigma_W^2), \quad \mathbf{E}[X_t] = 0.$$

We have seen that the autocorrelation function $\gamma_X(h) \neq 0$ for all time lags $h > p_0$. This happens because X_{t-j} for $j = 1, \ldots, p_0$ are linearly related to X_{t-h} through the recursive definition of the process. This means that we cannot use the autocovariance function to determine the order of the process.

Recap of AR model

2/2 points (graded)

Consider $\text{AR}(1)$ process $X_t = \phi X_{t-1} + W_t$, $|\phi| < 1$.

Is X_t stationary?

- [x] True
- [ ] False

✔

Which one of the following is the correct formula of $\gamma_X(h)$?

- ○ $\gamma_X(h) = \phi^{h-1}\sigma_W^2/(1-\phi^2)$
- ○ $\gamma_X(h) = \phi^{h-1}\sigma_W^2/\phi^2$
- ○ $\gamma_X(h) = \phi^{h}\sigma_W^2/\phi^2$
- ◉ $\gamma_X(h) = \phi^{h}\sigma_W^2/(1-\phi^2)$

✔

Solution:

Write

$$\begin{aligned} X_t &= \phi X_{t-1} + W_t \\ &= \phi\left[\phi X_{t-2} + W_{t-1}\right] + W_t \\ &\vdots \\ &= \sum_{j=0}^{1} \phi^j W_{t-j} + \phi^2 X_{t-2} \\ &\vdots \\ &= \sum_{j=0}^{h-1} \phi^j W_{t-j} + \phi^h X_{t-h} \quad \vdots \\ &= \sum_{j=0}^{\infty} \phi^j W_{t-j} \end{aligned}$$

From the last line, we find that $\mathsf{Var}(X_t) = \sigma_W^2 \frac{1}{1-\phi^2}$ by taking the variance on the right hand side, noting that white noise terms are uncorrelated and summing the geometric series with the common ration ϕ^2 (by the property of squaring a constant taken outside of the variance).

Furthermore, from the second to last line, we find that

$$\mathsf{Cov}(X_t, X_{t-h}) = \mathsf{Cov}\left(\sum_{j=0}^{h-1} \phi^j W_{t-j} + \phi^h X_{t-h}, X_{t-h}\right) = \phi^h \mathsf{Cov}(X_{t-h}, X_{t-h}) = \phi^h \frac{\sigma_W^2}{1-\phi^2}.$$

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PACF of AR(1)

4/4 points (graded)

Consider $\mathsf{AR}(1)$ process $X_t = 0.5X_{t-1} + W_t$. Compute the PACF $\alpha_X(h)$ for lag h= 0,1,2 and 3.

Compute $\alpha_X(0)$.

1 ✔ **Answer:** 1

Compute $\alpha_X(1)$.

0.5 ✔ **Answer:** 0.5

Compute $\alpha_X(2)$.

0 ✔ **Answer:** 0

Compute $\alpha_X(3)$.

0 ✔ **Answer:** 0

Solution:

Consider any AR (1) process $X_t = \phi X_{t-1} + W_t$, $|\phi| < 1$.

The coefficient of regression of X_t on itself is always 1, so $\alpha_X(0) = 1$.

Next, from the definition of the autoregressive process, we see that ϕ is the regression coefficient of X_t on X_{t-1} because the contemporary white noise term W_t is orthogonal to the past observation X_{t-1}. That is, $\alpha_X(1) = \phi = 0.5$ in this case.

Furthermore, we can write the AR (1) process as an AR (h) process:

$$X_t = \phi X_{t-1} + 0 \cdot X_{t-2} + \cdot + 0 \cdot X_{t-h} + W_t$$

and note that the contemporaneous white noise term W_t is orthogonal to all previous realizations of the process $(X_{t-1}, \ldots, X_{t-h})$. Hence, the coefficients in the expression above $(\phi, 0, \ldots, 0)$ are in fact least squares coefficients in the regression of X_t on $X_{t-1}, \ldots, X_{t-h}$. Therefore, $\alpha_X(h) = 0$ for all $h > 1$.

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[STAFF] Partial autocorrelation definition?

discussion posted 2 months ago by **trungaero**

I am quite still confused about PCAF. In the same language to define ACF, is it correct to define PCAF as the conditional correlation between Xh and X0 conditioning on any Xt in between, or:

$$\alpha(h) := corr(X_h, X_0 | X_1 .. X_{h-1}) = \frac{E[X_h - E[X_h|X_1..X_{h-1}]|X_1..X_{h-1}] . E[X_0 - E[X_0|X_1..X_{h-1}]|X_1..X}{var(X_0|X_1..X_{h-1})}$$

In the probability language, I understand that we try to figure out the conditional independence between X0 and Xh in the conditional world. And the use of linear regression is just for estimating the conditional expectation $E[X_h|X_1..X_{h-1}]$. Please correct my understanding if it's wrong? Thanks.

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1 response

silunz (Staff)
about a month ago

I think your understanding is correct. Just the right-hand side should be

$$\frac{E\left[\left(X_h - E\left[X_h|X_1..X_{h-1}\right]\right)\left(X_0 - E\left[X_0|X_1..X_{h-1}\right]\right)|X_1..X_{h-1}\right]}{\sqrt{var\left(X_0|X_1..X_{h-1}\right)var\left(X_h|X_1..X_{h-1}\right)}}.$$

Thank you for the answer and for your correction!

posted about a month ago by **trungaero**

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