6. Existence and uniqueness

The following theorems are straight-forward generalizations of the ones for 2×2 systems.

Dimension theorem for a homogeneous linear system of ODEs. For any first-order homogeneous linear system of n ODEs in n unknown functions

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x},$$

the set of solutions is an n-dimensional vector space.

The dimension theorem is a consequence of the existence and uniqueness theorem:

Existence and uniqueness theorem for a linear system of ODEs. Let $\mathbf{A}(t)$ be a square matrix-valued function and let $\mathbf{r}(t)$ be a vector-valued function, both continuous on an open time interval I. Let $a \in I$, and let $\mathbf{b}$ be a vector. Then there exists a unique solution $\mathbf{x}(t)$ to the system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} + \mathbf{r}(t)$$

satisfying the initial condition

$$\mathbf{x}(a) = \mathbf{b}.$$

Note that for $n \times n$ systems, the initial condition really consists of n conditions, one for each component of $\begin{pmatrix} x_1(a) \\ \vdots \\ x_n(a) \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

In other words, to pin down one solution to a first order linear system, we need to specify the initial condition $\mathbf{x}(a) = \mathbf{b},$ which consists of n equations, one for each component, for $n \times n$ systems.

How does the existence and uniqueness theorem lead to the dimension theorem?

First, set $\mathbf{r}(t) = \mathbf{0}$ for a homogenous system. Secondly, by the existence and uniqueness theorem, once the starting time a is fixed, the solutions to the system are in 1-to-1 correspondence to the possibilities for the initial condition vector $\mathbf{b}.$ Therefore, the set of solutions to the homogeneous system is n-dimensional.

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In most cases, we will consider the differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{r}(t).$$

In other words, we assume $\mathbf{A}$ is **constant, independent of** t, i.e. $\mathbf{A}(t) = \mathbf{A}.$ We will also assume in most cases that $\mathbf{A}$ is **complete**, i.e. the dimension of the eigenspace for each eigenvalue is equal to the multiplicity of the eigenvalue. When $\mathbf{r}(t) = \mathbf{0},$ the general solution to the homogeneous equation, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, takes the form

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + \cdots c_n \mathbf{v}_n e^{\lambda_n t},$$

where the eigenvalues λ_i may not be distinct, but the exponential solutions $\mathbf{v}_i e^{\lambda_i t}$ are linearly independent. This is consistent with the dimension theorem because this is an n-dimensional family of solutions.

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