Paradox?

Hilbert's Hotel might seem to give rise to paradox.

Say that the "old guests" are the guests that were occupying the hotel prior to any new arrivals, and that the "new guests" are the old guests plus Oscar. We seem to be trapped between conflicting considerations:

- On the one hand, we want to say that there there are *more* new guests than old guests. (After all: the new guests include every old guest, plus Oscar.)
- On the other hand, we want to say that there are *just as many* new guests as there are old guests. (After all: the new guests and the old guests can be accommodated using the exact same rooms, without any multiple occupancies or empty rooms.)

You'll notice that this is the same problem I introduced in my preface to the class, when I talked about Galileo's treatment of infinity. Consider the squares: $1^2, 2^2, 3^2, 4^2, \ldots$, and their roots: $1, 2, 3, 4, \ldots$.

- On the one hand, we want to say that there are *more* roots than squares. (After all: the roots include every square, and more.)
- On the other hand, we want to say that there are *just as many* roots as there are squares. (As Galileo put it: every square has its own root and every root has its own square, while no square has more than one root and no root has more than one square.")

Here is a more abstract way of thinking about the matter. There are two principles each of which seems eminently sensible, but they turn out to be incompatible with one another in the presence of infinite sets. The first principle is this:

The Proper Subset Principle
Suppose everything in A is also in B, but not vice-versa. Then A and B are *not* of the same size: B has more elements than A.

For example, suppose that A is the set of kangaroos and B is the set of mammals. Since every kangaroo is a mammal but not vice-versa, the Proper Subset Principle tells us that that the set of kangaroos and the set of mammals are not of the same size: there are more mammals than kangaroos.

The second principle is this:

The Bijection Principle
Set A has the same size as set B if and only if there is a *bijection* between A and B.

What is a bijection? Suppose you have some beetles and some boxes, and that you *pair* the beetles and the boxes by placing each beetle in a different box and leaving no empty boxes. As it might be:

$$\begin{array}{ccccc} \text{Beetle} & \text{Beetle} & \text{Beetle} & \text{Beetle} & \text{Beetle} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{Box} & \text{Box} & \text{Box} & \text{Box} & \text{Box} \end{array}$$

A bijection is a pairing of this kind. Accordingly, Hilbert's Hotel gives us a way of defining a bijection between the set of old guests and the set of new guests:

$$\begin{array}{ccccc} \text{Guest } 0 & \text{Guest } 1 & \text{Guest } 2 & \text{Guest } 3 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{Oscar} & \text{Guest } 0 & \text{Guest } 1 & \text{Guest } 2 & \dots \end{array}$$

(In general, a **bijection** from set A to set B is a function from A to B such that (i) each element of A is assigned to a different element of B, and (ii) no element of B is left without an assignment from A.)

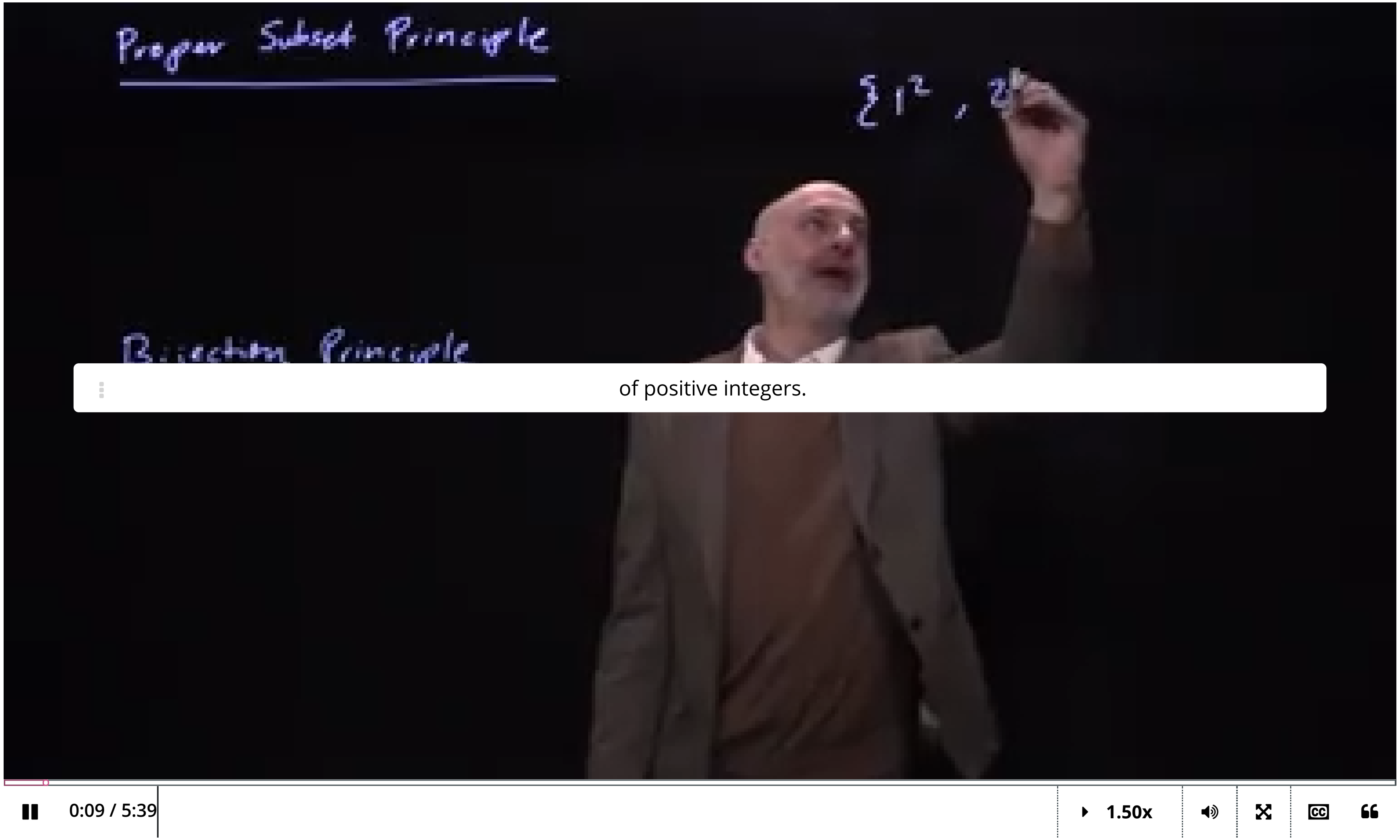
The reason the Proper Subset Principle and the Bijection Principle both seem plausible is that our intuitions about size are almost exclusively developed in the context of finite sets, and both principles are true when attention is restricted to finite sets. The lesson of Hilbert's Hotel, and of Galileo's example of the perfect squares and their roots, is that the principles cannot both be true when it comes to *infinite sets*. For whereas the Bijection Principle entails that the set of old guests and the set of new guests have the same size, the Proper Subset Principle entails that they do not.

As it turns out, the most fruitful way of developing a theory of infinite size is to give up on the Proper Subset Principle, and keep the Bijection Principle. The crucial insight came in 1873, when the great German mathematician Georg Cantor discovered that there are infinite sets between which there can be no bijections.

Cantor went on to develop a rigorous theory of infinite size based on the Bijection Principle. This work made it possible, for the first time in more than two centuries, to find our way out of the paradox that Galileo had articulated so powerfully in 1638. (David Hilbert described Cantor's work on the infinite as "the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity.")

Throughout this book I will follow Cantor's strategy: I will keep the Bijection Principle and discard the Proper Subset Principle. Accordingly, when I say that set A and set B "have the same size", or when I say that A has "just as many" members as B, what I will mean is that there is a bijection from A to B.

Video Review: The Paradox



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Alternative Notation

Bijections go by many different names, and I'm afraid you'll sometimes hear me use other names in the videos associated with this course.)

Here are some alternative ways of speaking about a bijection from A to B:

- a one-one correspondence between A and B
- a one-one correspondence from A to B
- a one-one mapping from A onto B
- a one-one function from A onto B

Discussion

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?	Rejection of the first principle How was it decided that the first principle will not be considered? The first principle can also be used to find a way out of paradox.	1
☑	not defined vs does not exist Is there anyone who believes that there is a difference between something not existing and something not being defined? (It's a very open question, I just need to understand...	5
☑	basis of rejecting a theorem If there are 2 contradictory theorems, we can not reject any of them without any solid reason. Just because one of them leads to results that look good to us is not a valid reas...	4
☑	The Proper Subset Principle as a basis to analyze infinities? Based on the description thus far, it seems as if the decision to reject the Proper Subset Principle and accept the Bijection Principle was—at least mathematically—mostly arbi...	8
	Proper Subset Principle	11
	Analogy to Newton / Einstein Does this analogy work/help? Stated near the end of this video : The bijection principle simplifies to the proper subset principle for finite sets. In a similar way, general relativit...	6
	Philosophy <--> Mathematics Is mathematics a branch of philosophy, or philosophy a branch of mathematics, or neither, or both?	11
	Bijection principle Square of -1 , -2 , -3.. are respectively, 1, 4,9, ... Therefore, could we have bijection principle apply to multiple sets? A - B and say, C = B where A is set of positive squares, C a se...	5

[Wait, there is something fishy here!] 6

[general] 4

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