2. Lecture 5

The following can be done after Lecture 5.

Please enter solutions in terms of π rather than numerical approximations to guarantee a correct grading. Simply type **pi** into the answer box and treat as any other variable, using * to denote multiplication, / to denote division, and $\wedge$ to denote exponents.

5-1

5.0/5.0 points (graded)

Which of the following are true of the heat equation $\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}$, where α is a positive constant ?

Check all that apply.

- [x] The set of solutions is a vector space.
- [ ] Every solution has the form $\theta(x,t) = v(x)\, w(t)$ for some functions $v(x)$ and $w(t)$.

✔

Solution:

Only the first statement is true.

The zero function is a solution, multiplying any solution by a scalar gives another solution, and adding any two solutions gives another solution, so the set of solutions is a vector space.

Although there is a basis of solutions consisting of functions of the form $v(x)\, w(t)$, there are other solutions that are linear combinations of these, and most of these are not products $v(x)\, w(t)$; for example, $e^{-t} \sin x + e^{-9t} \sin 3x$ is not such a product.

How can you tell that there is not some sneaky way to write

$$e^{-t} \sin x + e^{-9t} \sin 3x = v(x)\; w(t)?$$

If such an identity existed, setting $x = \pi/3$ would show that $w(t)$ is a scalar multiple of e^{-t}, but then

$$\frac{e^{-t} \sin x + e^{-9t} \sin 3x}{w(t)} = v(x)$$

is proportional to $\sin x + e^{-8t} \sin 3x$, which is a contradiction since $v(x)$ does not depend on t.

Submit

You have used 1 of 2 attempts

Answers are displayed within the problem

5-2

15.0/15.0 points (graded)

The function $\theta(x,t)$ for $x \in [0,1]$ and $t \geq 0$ is a solution to the heat equation $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$ with conditions $\theta(x,0) = 2 \sin \pi x + 32 \sin 2\pi x$, $\theta(0,t) = 0$, and $\theta(1,t) = 0$. What is $\theta(x, \pi^{-2} \ln 2)$?

Note: The answer box will recognize sin, cos, tan, sinh, cosh, etc.; simply put the argument in round parentheses; e.g. sin(pi*x/L).

$\theta(x, \pi^{-2}\ln 2) =$ sin(pi*x)+2*sin(2*pi*x) **Answer:** sin(pi*x) + 2*sin(2*pi*x)

$\sin(\pi \cdot x) + 2 \cdot \sin(2 \cdot \pi \cdot x)$

Solution:

For each positive integer n, the function $e^{-n^2\pi^2 t}\sin n\pi x$ is the solution to the heat equation satisfying $\theta(0,t) = 0$ and $\theta(1,t) = 0$. So the linear combination

$$\theta(x,t) = 2e^{-\pi^2 t}\sin \pi x + 32e^{-4\pi^2 t}\sin 2\pi x$$

is a solution that satisfies all the conditions. Then

$$\theta(x, \pi^{-2}\ln 2) = 2e^{-\ln 2}\sin \pi x + 32e^{-4\ln 2}\sin 2\pi x = \sin \pi x + 2\sin 2\pi x.$$

Submit You have used 1 of 15 attempts

Answers are displayed within the problem

5-3

10/10 points (graded)

Consider an insulated uniform metal rod of length π with exposed ends and with thermal diffusivity 1. Suppose that at $t = 0$, the temperature profile is

$$\theta(x,0) = 10 + \sin 3x + 20\sin 5x + 2\sin 7x,$$

but then the ends are held in ice at $0^\circ\mathrm{C}$. When t is large, the temperature profile is closely approximated by a sinusoidal function of x whose amplitude is decaying to 0. What is the angular frequency of that sinusoidal function?

(Hint: Start with the general solution to the heat equation with boundary conditions, and then match it to the given initial condition.)

1

Answer: 1

Solution:

The answer is 1.

The general solution to the heat equation with boundary conditions $\theta(0,t)=0$ and $\theta(\pi,t)=0$ is

$$\theta(x,t)=b_1e^{-t}\sin x+b_2e^{-4t}\sin 2x+b_3e^{-9t}\sin 3x+\cdots.$$

As $t\to\infty$, the sinusoidal functions of x have amplitude decaying at different rates, and it is the first nonzero term in the series that decays the slowest and that hence will eventually become most prominent.

Substituting $t=0$ and plugging the initial condition into the left hand side gives

$$10+\sin 3x+20\sin 5x+2\sin 7x=b_1\sin x+b_2\sin 2x+b_3\sin 3x+\cdots$$

for all x in $(0,\pi)$.

To find all the numbers b_n, we need to figure out how to express the constant function 10 as a sum of sines on the interval $(0,\pi)$. To do so, extend it to an odd function on $(-\pi,\pi)$ and then extend this to a periodic function of period 2π:

$$10\mathrm{Sq}(t)=\frac{40}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right).$$

Thus

$$\frac{40}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right)+\sin 3x+20\sin 5x+2\sin 7x=b_1\sin x+b_2\sin 2x+b_3\sin 3x+\cdots,$$

so

$$\begin{aligned}
b_1 &= \frac{40}{\pi}\\
b_2 &= 0\\
b_3 &= \frac{40}{3\pi}+1\\
b_4 &= 0\\
b_5 &= \frac{40}{5\pi}+20\\
&\vdots
\end{aligned}$$

In particular, the first nonzero b_n is b_1, so the dominant term in $\theta(x,t)$ for large t is the first term,

$$\frac{40}{\pi}e^{-t}\sin x,$$

a sinusoidal function of x of angular frequency 1 whose amplitude is decaying to 0.

Submit

You have used 2 of 3 attempts

Answers are displayed within the problem

5-4

10.0/10.0 points (graded)

Let $\theta(x,t)$ be the steady-state solution to the heat equation $\dfrac{\partial\theta}{\partial t}=\dfrac{\partial^2\theta}{\partial x^2}$ for an insulated metal rod of length 10 meters with left end held at $10^\circ\mathrm{C}$ and right end held at $30^\circ\mathrm{C}$. What is the value of $\theta(x,t)$ at a point 4 meters from the left end, in degrees Celsius?

18

✔ **Answer:** 18

18

Solution:

The answer is 18.

In the steady-state solution, $\frac{\partial\theta}{\partial t}=0$, $\theta=\theta(x)$ is a function of x alone. The heat equation then forces $\frac{\partial^2\theta}{\partial x^2}=0$, which says that $\theta(x)=ax+b$ for some constants a and b. Since $\theta(0)=10$ and $\theta(10)=30$ (if x measures distance along the rod measured from the left end), we get $\theta(x)=2x+10$, and $\theta(4)=18$.

Submit

You have used 1 of 15 attempts

Answers are displayed within the problem

5-5

10.0/10.0 points (graded)

Consider an insulated uniform metal rod of length π with insulated left end (at $x=0$) and with exposed right end (at $x=\pi$) held at $0^\circ\mathrm{C}$. Let $\theta(x,t)$ be its temperature in degrees Celsius at a position x units from the left end after t seconds. If $\theta(x,t)$ has the form $v(x)\,w(t)$ for some not-identically-zero functions $v(x)$ and $w(t)$, which of the following must be true of $v(x)$?

(Check all that apply.)

- [ ] $v(0) = 0.$
- [x] $v'(0) = 0.$
- [x] $v(\pi) = 0.$
- [ ] $v'(\pi) = 0.$

Solution:

The answer is that $v'(0) = 0$ and $v(\pi) = 0$ must hold, but not the others.

To say that the left end is insulated means that the heat flux across $x = 0$ is 0, and heat flux is proportional to $-\frac{\partial\theta}{\partial x}$, so $\frac{\partial\theta}{\partial x} = 0$ whenever $x = 0$. In other words $v'(x)\,w(t) = 0$ whenever $x = 0$, so $-v'(0)\,w(t) = 0$. But $w(t)$ is not identically zero, so $v'(0) = 0$.

To say that the right end is exposed and held at $0^\circ\mathrm{C}$ means that $\theta(\pi, t) = 0$ for all $t > 0$, so $v(\pi)\,w(t) = 0$ for all $t > 0$, but $w(t)$ is not identically zero, so $v(\pi) = 0$.

The heat equation with these boundary conditions is going to have a solution of the form $\theta(x,t) = e^{-ct}\cos(x/2)$ for some constant $c > 0$. Thus it is possible that $v(x) = \cos(x/2)$, in which case $v'(x) = -\frac{1}{2}\sin(x/2)$; this shows that $v(0)$ and $v'(\pi)$ can be nonzero.

Submit

You have used 1 of 3 attempts

Answers are displayed within the problem

2. Lecture 5

Hide Discussion

Topic: Unit 2: Boundary value problems and PDEs / 2. Lecture 5

Add a Post

All Posts

5-3

discussion posted 16 days ago by agiri123

`When` t `is large, the temperature profile is closely approximated by a sinusoidal function of` x `whose amplitude is decaying to 0 . What is the angular frequency of that sinusoidal function?`

Once you match the general solution $\theta(x,t)$ with $\theta(x,0)$ you end up with a function that decays exponentially but it is a linear combination of multiple sinusoids each with a different angular frequency decaying at a different rate, with larger angular frequencies decaying faster. So what is being meant by `angular frequency of that sinusoidal function` ?

This post is visible to everyone.

Add a Response

2 responses

horsleyt

15 days ago

The angular frequency is the coefficient of the sinusoid. Eg. angular frequency of cos(12*x) is 12, etc. The question asks what is the angular frequency of the sinusoid that makes the best approximation in the limit.

For anyone else just as confused as i was, here is how i finally solved it.

Check out https://courses.edx.org/courses/course-v1:MITx+18.03Fx+2T2020/courseware/unit2/28_heatequation/ The summary section, especially the initial conditions section lays out clearly how to apply initial conditions and match it to general solution $\theta(x,t)$. Be careful with the constant term in $\theta(x,0)$

posted 14 days ago by agiri123

could you give an input on how you handled the constant term? getting blocked on this

posted 9 days ago by alexba91

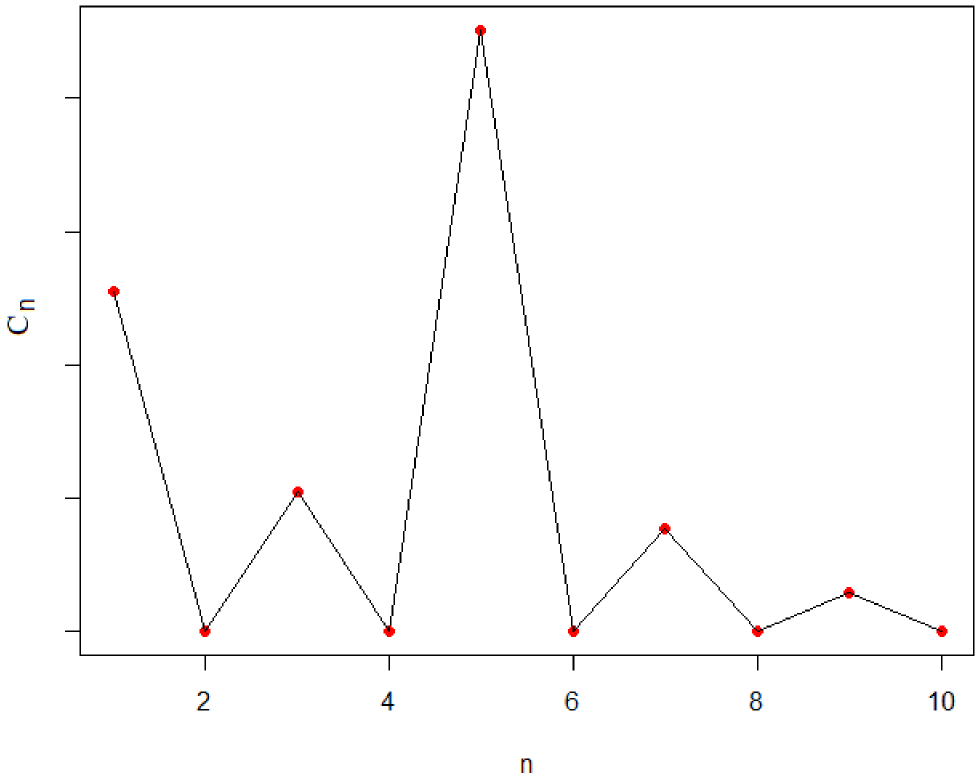
If you're still talking about question 5-3, I found the Fourier series of the appropriate function that had the value of 10 in $(0, \pi)$, or at least gave some thought as to what it would look like.

posted 8 days ago by **Steve_Nicodemus** (Community TA)

Add a comment

sandipan_dey
about 12 hours ago

The below figure shows the magnitudes of the coefficients (C_n) for $\theta(x, 0)$ I obtained, by replacing the constant term with an appropriate Fourier series infinite-sum and adding the additional sinusoidal terms. Clearly, n=5 has the highest amplitude and the dominating sinusoidal, but grader rejected. I have no clue what I am doing wrong, any help will be appreciated, thanks in advance.



Update

My bad, i was not considering the exponentially decaying amplitude, now got the correct output.

posted less than a minute ago by **sandipan_dey**

Add a comment

Showing all responses

Add a response:

Preview

Submit