MITx: 6.041x Introduction to Probability - The Science of Uncertainty

Exercise: Normal unknown and additive noise

(3/4 points)

As in the last video, let $X = \Theta + W$, where Θ and W are independent normal random variables and W has mean zero.

a) Assume that W has positive variance. Are X and W independent?

No ✔ **Answer:** No

b) Find the MAP estimator of Θ based on X if $\Theta \sim N(1,1)$ and $W \sim N(0,1)$, and evaluate the corresponding estimate if $X = 2$.

$\hat{\theta} =$ 3/2 ✔ **Answer:** 1.5

c) Find the MAP estimator of Θ based on X if $\Theta \sim N(0,1)$ and $W \sim N(0,4)$, and evaluate the corresponding estimate if $X = 2$.

$\hat{\theta} =$ 2/5 ✔ **Answer:** 0.4

d) For this part of the problem, suppose instead that $X = 2\Theta + 3W$, where Θ and W are standard normal random variables. Find the MAP estimator of Θ based on X under this model and evaluate the corresponding estimate if $X = 2$.

$\hat{\theta} =$ 16/25 ✘ **Answer:** 0.30769

Answer:

a) They are not independent. This is intuitively clear because W has an effect on X. Another way to see it is that we have (by independence of Θ and W) that $\mathbf{E}[\Theta W] = \mathbf{E}[\Theta]\,\mathbf{E}[W] = 0$, which leads to

$$\mathbf{E}[XW] = \mathbf{E}[(\Theta + W)W] = \mathbf{E}[W^2] \neq 0 = \mathbf{E}[X]\,\mathbf{E}[W],$$

which in turn implies that X and W are not independent.

b) If we focus on the terms that involve θ, the posterior is of the form

$$c(x)e^{-(\theta-1)^2/2}e^{-(x-\theta)^2/2}.$$

To find the MAP estimate, we set the derivative with respect to θ of the exponent to zero, so that $(\hat{\theta}-1)+(\hat{\theta}-x)=0$, or $\hat{\theta}=(1+x)/2$, which, when $x=2$, evaluates to $3/2$.

c) If we focus on the terms that involve θ, the posterior is of the form

$$c(x)e^{-\theta^2/2}e^{-(x-\theta)^2/(2\cdot 4)}.$$

To find the MAP estimate, we set the derivative with respect to θ of the exponent to zero, so that $\hat{\theta}+(\hat{\theta}-x)/4=0$, or $\hat{\theta}=x/5$, which, when $x=2$, evaluates to $2/5$.

d) Note that conditional on $\Theta=\theta$, the random variable X is normal with mean 2θ and variance 9. If we focus on the terms that involve θ, the posterior is of the form

$$c(x)e^{-\theta^2/2}e^{-(x-2\theta)^2/(2\cdot 9)}.$$

To find the MAP estimate, we set the derivative with respect to θ of the exponent to zero, so that $\hat{\theta}+2(2\hat{\theta}-x)/9=0$, or $\hat{\theta}=2x/13$, which, when $x=2$, evaluates to $4/13$.

You have used 3 of 3 submissions