Stability of Implicit Methods
sandipan_dey 3d

I was wondering (for a given IV problem) whether there exists a generalized way of obtaining a threshold for the time step Δt below which an explicit method such as forward Euler always has a stable solution (can we derive one theoretically, if so how, if not why it is not possible)?

Also, assuming the following: $\exists$ is a solution to a linear system of equations by an implicit method $\Leftrightarrow$ $\exists$ a solution with an explicit method too for some value of time step Δt, such a threshold (however small) always exists for which the solution converges with an explicit method, is that a valid assumption ? if not is there a counter example?

Some insights along these lines will be really helpful. Thanks in advance.

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Hello @darmofal thank you very much for the reply.

A little confusion here: by convergence, don't we mean $\lim_{n\to\infty}|u^n - u_{true}| = 0$, for example for the scaler case, the true value of the solution u_{true} can be different from 0. But if we let $|g(\Delta t)| \leq 1$, does not it mean $\lim_{n\to\infty} u^n = \lim_{n\to\infty}|g(\Delta t)|^n u^0 = 0$, when $|g(\Delta t)| < 1$, u^n always converges to zero? Do we want the amplification factor on the error instead $|u^n - u_{true}| = (g(\Delta t)^n|u^0 - u_{true}|)$ with $|g(\Delta t)| < 1$ instead?

(Just a digression from the main qurstion: When we can't analytically compute or guess / have no idea about the true solution, how to evaluate the numerical methods? Convergence to a value in terms of $|u^{n+1} - u^n| \to 0$ may be a necessary but not sufficient for correctness, e.g. the value may be wrong. How do we evaluate the correctness of the solution obtained using a numerical method then?)

Also, for IVP system, how can we get to $Av = \lambda v$ to have constraints depending on the eigenvalues starting from the amplification factor on the growth and the r.h.s. $Au^n + b$?

Is not the Dahlquist Equivalence Theorem (here) about the explicit methods only, or it enables us to connect implicit and explicit methods with some statement like the following: if there exists a solution for an IVP with an implicit method then there exists a stable / consistent explicit method too for the solution of the IVP?

This brings out more questions, e.g., single-step (e.g., RK) vs. multistep (e.g., Adams–Bashforth) methods for IVP, when are the multistep methods preferred over the single step ones? are they used as ode solvers in any realworld applications (e.g., by python, R or matlab libraries)?

Thank you very much @STAFF for your patience for answering the questions, appreciate it really, despite my questions may be naive, out of curosity.

darmofal Staff 1d

Thank you again for the insightful follow-up. I was a too short with my description of the stability analysis for a linear IVP. The way the analysis works is a bit more conceptually nuanced. It starts with a discrete solution u^n and determines if a perturbed discrete solution exists which is unstable. So, the first step in the analysis is to define the perturbed discrete solution v^n and the perturbation w^n

$$v^n = u^n + w^n$$

Substituting this into the numerical method, you will be able to cancel the u^n and b terms. For example, consider Forward Euler,

$$u^{n+1} = u^n + \Delta t \left[A u^n + b(t^n) \right]$$

Then, we are looking for the existence of a perturbed solution to the Forward Euler method:

$$v^{n+1} = v^n + \Delta t \left[A v^n + b(t^n) \right]$$

Subtracting these two equations gives

$$w^{n+1} = w^n + \Delta t A w^n$$

Now, finally make the substitution for the amplification factor, $w^n = [g(\Delta t)]^n w_0$ which gives:

$$g^{n+1} w^0 = g^n w^0 + \Delta t A g^n w^0$$

Important: g is being raised to a power in the above equation (i.e. it is not an index). Now, you can factor and re-arrange to give,

$$[g - I - \Delta t A] g^n w^0 = 0$$

We are looking for the values g for which this equation is satisfied. There are n roots of $g = 0$ from the g^n factor. And then the expression in the square brackets gives,

$$g = I + \Delta t A$$

Then, the last step of the analysis is to express g in terms of the eigenvalues of A and require that $|g| \leq 1$. This turns out to be equivalent to requiring,

$$|1 + \Delta t \lambda| \leq 1$$

where λ are the eigenvalues of A.

Dahlquist applies to explicit and implicit methods.

NOTE: all of the above is well beyond what we are teaching in this course, and would be covered in a more advanced course on numerical analysis. But, we offer it to give you a little more insight.

darmofal Staff 2d

Hi sandipan_dey. Great questions.

For a general IVP, there is no guaranteed way to determine the Δt below which a stable solution can be obtained. The issue is with nonlinear IVP. For linear IVP, there is a guaranteed approach to do such an analysis. The idea is to assume that the numerical method has a solution of the following form: $u^n = [g(\Delta t)]^n u^0$ where $g(\Delta t)$ is known as the growth or amplification factor. If the IVP is a scalar, than g is a scalar. If the IVP is a system, than g is a matrix. Then, after finding $g(\Delta t)$ for the given numerical method, you determine the timestep for which the magnitude $|g(\Delta t)| \leq 1$. For an IVP system, this will involve the eigenvalues of the A matrix.

Now, while finding the Δ limit is in general not possible for a nonlinear IVP and a numerical method, it is possible to prove convergence using what is known as the Dahlquist Equivalence Theorem. This requires determining if the method if consistent and zero stable, and these are possible for arbitrary IVP. What this means is we can prove that a method will converge in the limit that $\Delta t \to 0$, which implies that there is some (possibly very small timestep) which will produce a stable solution. This answers your last question: specifically, if an explicit method is convergent, there will be Δt which will produce a stable answer, though that could be a very small Δt.

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sandipan_dey right now

Thank you very much for the details @darmofal.

Requesting you to offer a more advanced numerical analysis course on edX as a follow-up course, it will be very helpful.

Add a response