The Real Numbers: Preliminaries

We will now verify an astounding claim I made earlier: *some infinities are bigger than others.* We will show, in particular, that there are more real numbers than there are natural numbers.

The set of **real numbers**, $\mathbb{R}$, is the set of numbers that you can write down in decimal notation by starting with a numeral (e.g. "17"), adding a decimal point ("17."), and then an infinite sequence of digits (e.g. "17.8423…"). Any symbol between "0" and "9" is an admissible digit.

Every natural number is a real number, since every natural can all be written out as a numeral followed by an infinite sequence of zeroes (e.g. 17=17.0000…).

The rational numbers are also real numbers. In fact, they are real numbers with the special feature of having *periodic* decimal expansions: after a certain point, the expansions are just repetitions of some finite sequence of digits. For instance, $1318/185 = 7.12\,(432)$, where the brackets surrounding "432" indicate that these three digits are to be repeated indefinitely, so as to get $7.12432432432432\ldots$.

Not every real number is a rational number, however. There are irrational real numbers, such as π and $\sqrt{2}$. Unlike the decimal expansions of rational numbers, the decimal expansions of irrational numbers are never periodic: they never end with an infinitely repeating a finite sequence of digits.

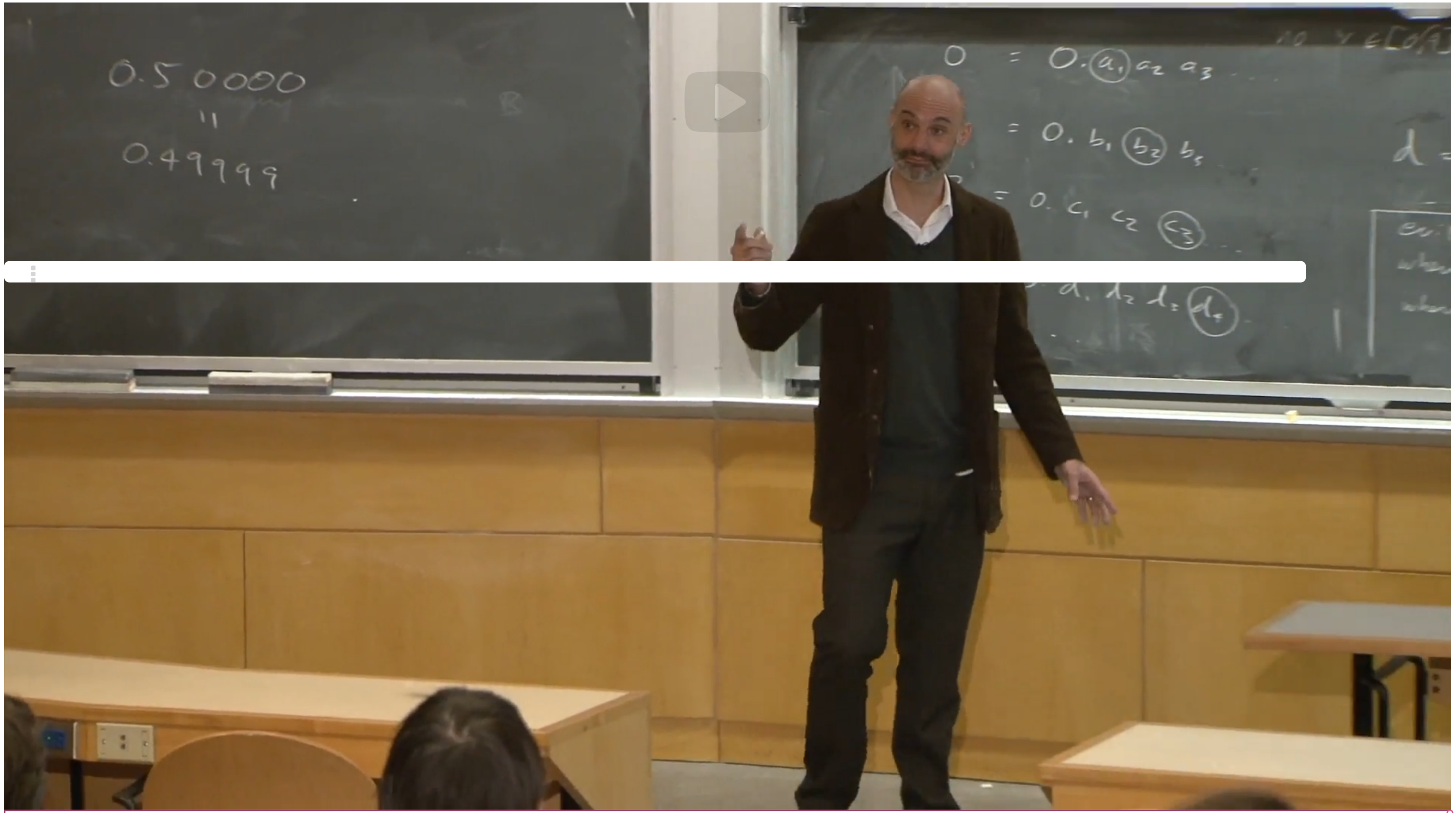
Some real numbers have multiple names in decimal notation. For example, the number $1/2$ is named both by the numeral "$0.5\,(0)$" and by the numeral "$0.4\,(9)$". To see this, note that each of the following inequalities must hold:

$$\begin{aligned}
0.5\,(0) - 0.4\,(9) <& \quad 0.01 \text{ (since } 0.49 < 0.4\,(9))\\
0.5\,(0) - 0.4\,(9) <& \quad 0.001 \text{ (since } 0.499 < 0.4\,(9))\\
0.5\,(0) - 0.4\,(9) <& \quad 0.0001 \text{ (since } 0.4999 < 0.4\,(9))\\
&\vdots
\end{aligned}$$

This means that the difference between 0.5(0) and 0.4(9) must be smaller than each of $0.01, 0.001, 0.0001$, and so forth. Since the difference between real numbers must be a non-negative real number, and since the only non-negative real number smaller than each of $0.01, 0.001, 0.0001, \ldots$ is 0, it follows that the difference between 0.5(0) and 0.4(9) must be 0. So $0.5\,(0) = 0.4\,(9)$.

Having real numbers with multiple names in decimal notation turns out to be a nuisance in the present context. Fortunately, the problem is easily avoided. The crucial observation is that the only real numbers with multiple names are those named by numerals that end in an infinite sequence of 9s. Such numbers always have exactly two names: one ending in an infinite sequence of 9s and one ending in an infinite sequence of 0s. So we can avoid having numbers with multiple names by treating numerals that end in an infinite sequence of 9s as invalid, and naming the relevant numbers using numerals that end in an infinite sequence of 0s.

Video: duplicate decimal notations



1:12 / 1:12 1.50x

Video

[Download video file]

Transcripts

[Download SubRip (.srt) file]

[Download Text (.txt) file]

Problem 1

1/1 point (ungraded)

In the text above I suggested avoiding the problem of multiple numerals by ignoring numerals that end in an infinite sequence of 9s. Could we have also avoided the problem by ignoring numerals that end in an infinite sequence of 0s?

- [ ] Yes. We could have also avoided the problem this way.
- [x] No. We could not have avoided the problem this way.

✔

Explanation
No. The number zero is named by a numeral that ends in an infinite sequence of 0s, but no name that ends in an infinite sequence of 9s.

Submit

Answers are displayed within the problem

Problem 2

1/1 point (ungraded)
Distinct decimal expansions $0.\delta_1^1\delta_2^1\delta_3^1\ldots$ and $0.\delta_1^2\delta_2^2\delta_3^2\ldots$ can only refer to the same number if one of them is of the form $0.s\,(9)$ and the other is of the form $0.s'\,(0)$, where s is a sequence of digits and s' is the result of raising the last digit in s by one.

True or False?

True ✔ **Answer:** True

(If it's true, try proving it. If it's false, find a counterexample.)

Explanation
Since $0.\delta_1^1\delta_2^1\delta_3^1\ldots$ and $0.\delta_1^2\delta_2^2\delta_3^2\ldots$ are distinct, they must differ in at least one digit. Let k be the smallest index such that $\delta_k^1 \neq \delta_k^2$. This means that the truncated expansions $0.\delta_1^1\ldots\delta_k^1$ and $0.\delta_1^2\ldots\delta_k^2$ must differ in value by at least $1/10^k$.
Let us now focus on the remaining digits, $\delta_{k+1}^i\delta_{k+2}^i\delta_{k+3}^i\ldots$ (where i is either 1 or 2). This sequence of digits contributes the following value to $0.\delta_1^i\delta_2^i\delta_3^i\ldots$:

$$\frac{\delta^i_{k+1}}{10^{k+1}} + \frac{\delta^i_{k+2}}{10^{k+2}} + \frac{\delta^i_{k+3}}{10^{k+3}} + \dots$$

This sum must yield a non-negative number smaller or equal to $\frac{1}{10^k}$. The sum will be 0 if every digit in $\delta^i_{k+1}\delta^i_{k+2}\delta^i_{k+3}\dots$ is 0, since

$$\frac{0}{10^{k+1}} + \frac{0}{10^{k+2}} + \frac{0}{10^{k+3}} + \dots = 0$$

And the number will be $\frac{1}{10^k}$, if every digit in $\delta^i_{k+1}\delta^i_{k+2}\delta^i_{k+3}\dots$ is 9, since

$$\frac{9}{10^{k+1}} + \frac{9}{10^{k+2}} + \frac{9}{10^{k+3}} + \dots = \frac{1}{10^k}$$

Notice, moreover, that the *only* way for the sum to be zero is for every digit in $\delta^i_{k+1}\delta^i_{k+2}\delta^i_{k+3}\dots$ to be 0, and that the *only* way for the sum to be $\frac{1}{10^k}$ is for every digit in $\delta^i_{k+1}\delta^i_{k+2}\delta^i_{k+3}\dots$ to be 9.
We are now in a position to verify that $0.\delta^1_1\delta^1_2\delta^1_3\dots$ and $0.\delta^2_1\delta^2_2\delta^2_3\dots$ can only refer to the same number if following two conditions are met:

1. The difference between $0.\delta^1_1\dots\delta^1_k$ and $0.\delta^2_1\dots\delta^2_k$ must be smaller or equal to $1/10^k$. (The reason is that this difference must match the difference between the value contributed by the sequences of digits $\delta^1_{k+1}\delta^1_{k+2}\delta^1_{k+3}\dots$ and the value contributed by sequence of digits $\delta^2_{k+1}\delta^2_{k+2}\delta^2_{k+3}\dots$, which we know to be at most $1/10^k$.) We have also seen, however, that the difference between $0.\delta^1_1\dots\delta^1_k$ and $0.\delta^2_1\dots\delta^2_k$ must be at least $1/10^k$. Putting the two together gives us the result that the difference must be *exactly* $1/10^k$. Assuming, with no loss of generality, that $0.\delta^1_1\dots\delta^1_k < 0.\delta^2_1\dots\delta^2_k$, this means that digit δ^2_k is the result of raising digit δ^1_k by one.

2. The value contributed by $\delta^1_{k+1}\delta^1_{k+2}\delta^1_{k+3}\dots$ must be the value contributed by $\delta^2_{k+1}\delta^2_{k+2}\delta^2_{k+3}\dots$ plus $1/10^k$, in order to compensate for the fact that $0.\delta^2_1\dots\delta^2_k = 0.\delta^1_1\dots\delta^1_k + 1/10^k$. But we know that the value contributed by $\delta^2_{k+1}\delta^2_{k+2}\delta^2_{k+3}\dots$ and the value contributed by $\delta^1_{k+1}\delta^1_{k+2}\delta^1_{k+3}\dots$ must be both be smaller or equal to $1/10^k$. It follows that the value contributed by $\delta^1_{k+1}\delta^1_{k+2}\delta^1_{k+3}\dots$ must be $1/10^k$ and the value contributed by $\delta^2_{k+1}\delta^2_{k+2}\delta^2_{k+3}\dots$ must be 0. And we know that the only way for this to happen is for $\delta^1_{k+1}, \delta^1_{k+2}, \delta^1_{k+3}, \dots$ to all be 9s and $\delta^2_{k+1}, \delta^2_{k+2}, \delta^2_{k+3}, \dots$ to all be 0s.

Submit

Answers are displayed within the problem

Problem 3

1/1 point (ungraded)

Every rational number has a periodic decimal expansion.

True or False?

True

✔ **Answer:** True

(If it's true, try proving it. If it's false, find a counterexample.)

Explanation

It's true. Here's the proof.

Note that $d_0.d_1d_2\ldots$ is a decimal expansion for a/b, where:

- d_0 is the integral part of $\frac{a}{b}$, with remainder r_0.
- d_1 is the integral part of $\frac{10r_0}{b}$, with remainder r_1.
- d_2 is the integral part of $\frac{10r_1}{b}$, with remainder r_2.
- And so forth

Notice, moreover, that there are only finitely many distinct values that the r_k can take, since they must always be smaller than b. Because the values of d_{i+1} and r_{i+1} depend only on the value of r_i, this means that the d_k will eventually start repeating themselves. So our decimal expansion is periodic.

Submit

Answers are displayed within the problem

Problem 4

2/2 points (ungraded)

If p is a period of length n, then $0.(p) = ?$

- $0.p$
- $\frac{p}{10^n-1}$
- $\frac{n}{10^p-1}$
- $p - 0.pppp\ldots$

Is $0.(p)$ a rational number?

Yes

Answer: Yes

Explanation

Note that the first line of the transformation below is a simple algebraic truth, and that successive lines are obtained by substituting the first line's right-hand term for bold-faced occurrences of "p":

$$p = \frac{(10^n - 1)\,p}{10^n} + \frac{\mathbf{p}}{10^n}$$

$$p = \frac{(10^n - 1)\,p}{10^n} + \frac{\frac{(10^n - 1)\,p}{10^n} + \frac{p}{10^n}}{10^n}$$

$$p = \frac{(10^n - 1)\,p}{10^n} + \frac{(10^n - 1)\,p}{10^{2n}} + \frac{\mathbf{p}}{10^{2n}}$$

$$p = \frac{(10^n - 1)\,p}{10^n} + \frac{(10^n - 1)\,p}{10^{2n}} + \frac{\frac{(10^n - 1)\,p}{10^n} + \frac{p}{10^n}}{10^{2n}}$$

$$p = \frac{(10^n - 1)\,p}{10^n} + \frac{(10^n - 1)\,p}{10^{2n}} + \frac{(10^n - 1)\,p}{10^{3n}} + \frac{\mathbf{p}}{10^{3n}}$$

$$\vdots$$

$$p = \frac{(10^n - 1)\,p}{10^n} + \frac{(10^n - 1)\,p}{10^{2n}} + \frac{(10^n - 1)\,p}{10^{3n}} + \frac{(10^n - 1)\,p}{10^{4n}} + \dots$$

$$p = (10^n - 1)\left(\frac{p}{10^n} + \frac{p}{10^{2n}} + \frac{p}{10^{3n}} + \frac{p}{10^{4n}} + \dots\right)$$

From this we may conclude,

$$\frac{p}{10^n - 1} = \frac{p}{10^n} + \frac{p}{10^{2n}} + \frac{p}{10^{3n}} + \dots = 0.(p)$$

Because $\frac{p}{10^n - 1}$ is of the form $\frac{a}{b}$ and therefore a rational number, $0.(p)$ is a rational number.

Submit

Answers are displayed within the problem

Problem 5

1/1 point (ungraded)

Every real number with a periodic decimal notation is a rational number.

True or False?

True

Answer: True

(If it's true, try proving it. If it's false, find a counterexample.)

Explanation

It's true. Here's the proof.

Every real number with a periodic decimal notation is a number of the form, where a is an integer, n is a natural number, and p is a period:

$$\frac{a + 0.(p)}{10^n}$$

But we know that this must be a rational number, because we verified in the previous exercise that $0.(p)$ is a rational number.

Submit

Answers are displayed within the problem

Discussion

Hide Discussion

Topic: Week 1 / The Real Numbers: Preliminaries

Add a Post

Show all posts

by recent activity

Page glitch? 4

evil twin

We say that the infinity of natural Numbers is smaller than the infinity of real numbers between 0 and 1. The reason is, if we allot natural numbers to the real numbers betwe...

7

Tip: Another video about the proof

At least for me, this explanation makes more sense: https://youtu.be/mEEM_dLWY0g

2

Alternate view of problem 4

1