Problem 1: True or false

(4/4 points)
We are told that events A and B are conditionally independent, given a third event C, and that $\mathbf{P}(B \mid C) > 0$. For each one of the following statements, decide whether the statement is "Always true", or "Not always true."

1. A and B are conditionally independent, given the event C^c.

   Not always true ▾ ✔ **Answer:** Not always true

2. A and B^c are conditionally independent, given the event C.

   Always true ▾ ✔ **Answer:** Always true

3. $\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid B)$

   Not always true ▾ ✔ **Answer:** Not always true

4. $\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid C)$

Always true ▾ ✔ **Answer:** Always true

Answer:

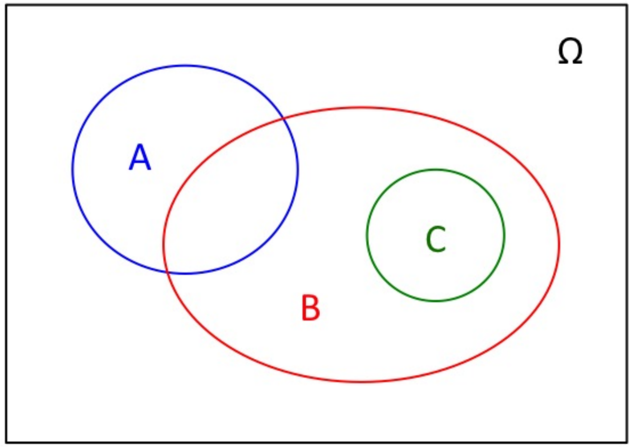
1. Not always true. Counterexample: Let X, Y be binary random variables. Consider a model with the following properties:
   Conditioned on C, X and Y are independent.
   Conditioned on C^c, X and Y are dependent.

   Let $A = \{X = 1\}$ and $B = \{Y = 1\}$. Then, A and B are conditionally independent given C, but they will be generically dependent conditioned on C^c.

2. Always true.

$$\begin{aligned} \mathbf{P}(A \mid C) &= \mathbf{P}(A \cap B \mid C) + \mathbf{P}(A \cap B^c \mid C) \\ \mathbf{P}(A \mid C) &= \mathbf{P}(A \mid C)\mathbf{P}(B \mid C) + \mathbf{P}(A \cap B^c \mid C) \\ \mathbf{P}(A \mid C)(1 - \mathbf{P}(B \mid C)) &= \mathbf{P}(A \cap B^c \mid C) \\ \mathbf{P}(A \mid C)\mathbf{P}(B^c \mid C) &= \mathbf{P}(A \cap B^c \mid C). \end{aligned}$$

3. Not always true. Counterexample: Let $\mathbf{P}(A) > 0$, $\mathbf{P}(B) > 0$, $\mathbf{P}(C > 0)$, $\mathbf{P}(A \cap B) > 0$ and $\mathbf{P}(A \cap C) = 0$. Furthermore, let $C \subset B$.



Show that A and B are conditionally independent given C:
$\mathbf{P}(A \cap B \mid C) = 0 = (0)(1) = \mathbf{P}(A \mid C)\mathbf{P}(B \mid C)$

Show that $\mathbf{P}(A \mid B \cap C) \neq \mathbf{P}(A \mid B)$:
$\mathbf{P}(A \mid B \cap C) = 0 \neq \mathbf{P}(A \mid B) > 0$

4. Always true. This is equivalent to the definition of independence of A and B in the conditional universe where C has occurred.

You have used 1 of 1 submissions