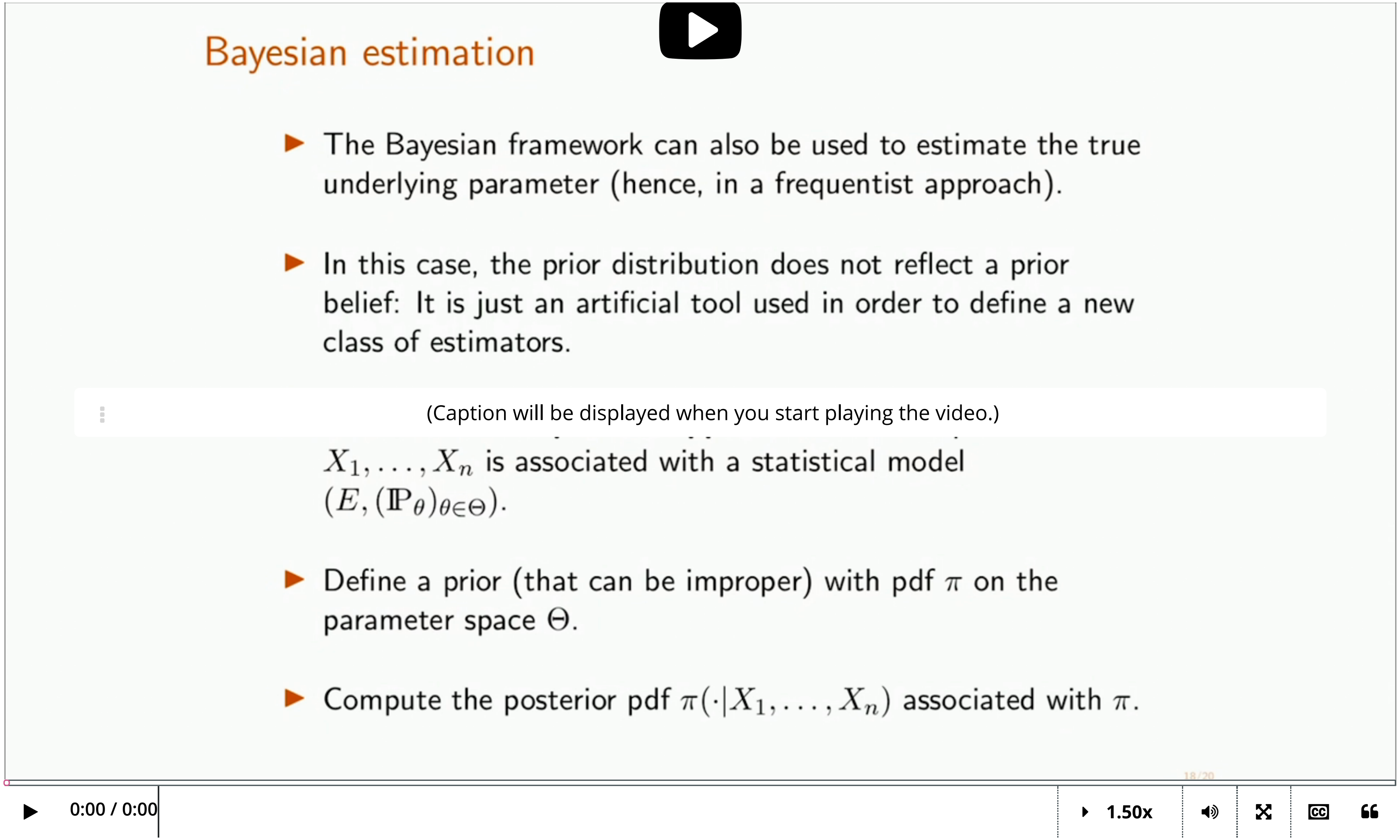
10. Bayesian Statistics for Estimation

Bayesian Estimation

Generating Speech Output



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(a)

4/4 points (graded)

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Consider the posterior distribution derived in the worked example from the previous lecture (here and here).

To recap, our parameter of interest is λ, prior distribution $\textsf{Exp}\,(a)$, and likelihood $\textsf{Poiss}\,(\lambda)$ for n observations $X_1, \ldots, X_n$. This is a Gamma distribution with parameters q_0 and λ_0 that you must get from the last two answerboxes in Worked Example Part II.

As before, recall the **Gamma distribution** , which is a probability distribution with parameters $q > 0$ and $\lambda > 0$, has support on $(0, \infty)$, and whose density is given by $f(x) = \frac{\lambda^q x^{q-1} e^{-\lambda x}}{\Gamma(q)}$. Here, Γ is the Euler Gamma function.

Of the four sample statistics (mean, median, mode, variance), the Gamma distribution has a simple closed form for three of them. Look up statistics for the Gamma distribution, then for the three that have a simple closed form, calculate them and express your answer in terms of a, n, and $\sum_{i=1}^{n} X_i$ (use **SumXi**), otherwise enter -1.

Note that depending on your source, the format of the Gamma distribution may be different, so you must make sure that you have the correct corresponding parameters.

mean:

(SumXi+1)/(a+n) ✔ **Answer:** (SumXi+1)/(a+n)

$\frac{SumXi+1}{a+n}$

median:

-1 ✔ **Answer:** -1

-1

mode:

(SumXi)/(a+n) ✔ **Answer:** SumXi/(a+n)

$\frac{SumXi}{a+n}$

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(SumXi+1)/(a+n)^2

Answer: (SumXi+1)/(a+n)^2

$\frac{SumXi+1}{(a+n)^2}$

STANDARD NOTATION

Solution:

Checking our answer from Worked Example Part II indicates that our parameters for the Gamma distribution are

$$q_0 = (\sum_{i=1}^{n} X_i) + 1$$

and

$$\lambda_0 = a + n.$$

Looking up the statistics for the Gamma distribution, we get that

- Mean is $\frac{q_0}{\lambda_0} = \boxed{\frac{(\sum_{i=1}^{n} X_i) + 1}{a+n}}$.
- Median: no closed form, so we enter $\boxed{-1}$.
- Mode is $\frac{q_0 - 1}{\lambda_0} = \boxed{\frac{\sum_{i=1}^{n} X_i}{a+n}}$.
- Variance is $\frac{q_0}{\beta^2} = \boxed{\frac{(\sum_{i=1}^{n} X_i) + 1}{(a+n)^2}}$.

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Answers are displayed within the problem

(b)

4/4 points (graded)

Suppose we have the improper prior $\pi(\lambda) \propto e^{-a\lambda}$, $\lambda \in \mathbb{R}$ (and $a \geq 0$). Conditional on λ, we have observations $X_1, X_2, \cdots, X_n \overset{\text{i.i.d}}{\sim} \mathsf{N}(\lambda, 1)$. Compute the posterior distribution $\pi(\lambda | X_1, X_2, \ldots, X_n)$, then provide the following statistics on the posterior distribution.

Use **SumXi** for $\sum_{i=1}^{n} X_i$.

mean:

(SumXi -a)/n

✔ **Answer:** (SumXi-a)/n

$\frac{SumXi-a}{n}$

variance:

1/n

✔ **Answer:** 1/n+a*0+SumXi*0

$\frac{1}{n}$

$q_{0.025}$ (cutoff for highest 2.5%):

(SumXi -a)/n+1.96/sqrt(n

✔ **Answer:** (SumXi-a)/n+1.96/sqrt(n)

$\frac{SumXi-a}{n} + \frac{1.96}{\sqrt{n}}$

STANDARD NOTATION

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True or False: The variance of this distribution models our uncertainty about the value of the parameter λ.

- [x] True
- [ ] False

✔

Solution:

In order to calculate statistics about the posterior distribution, we need to compute it first. To do this, we use Bayes' theorem, combining the prior $\pi(\lambda)$ and the likelihood function $L_n(X_1,\ldots,X_n|\lambda)$.

We can easily get that

$$\pi(\lambda) = \exp(-a\lambda)$$

and

$$L_n(X_1,\ldots,X_n|\lambda) \propto \exp\left(\sum_{i=1}^{n} -\frac{(X_i-\lambda)^2}{2}\right).$$

Using Bayes' formula, un-normalized, then gives

$$\begin{aligned}
\pi(\lambda|X_1,\ldots,X_n) &\propto \pi(\lambda) L_n(X_1,\ldots,X_n|\lambda) \\
&\propto \exp(-a\lambda)\exp\left(\sum_{i=1}^{n} -\frac{(X_i-\lambda)^2}{2}\right) \\
&= \exp\left(-a\lambda + \sum_{i=1}^{n} -\frac{(X_i-\lambda)^2}{2}\right) \\
&= \exp\left(-\frac{n}{2}\lambda^2 + \left(\left(\sum_{i=1}^{n} X_i\right) - a\right)\lambda - \frac{1}{2}\sum_{i=1}^{n} X_i^2\right)
\end{aligned}$$

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This is an exponential of a quadratic polynomial in λ, that is, it has the form $\alpha\lambda^2 + \beta\lambda + \gamma$. We get the equivalence $\alpha = -\frac{n}{2}$, $\beta = (\sum_{i=1}^{n} X_i) - a$, and $\gamma = -\frac{1}{2}\sum_{i=1}^{n} X_i^2$.

We have derived in a previous exercise that this corresponds to a Gaussian distribution. In this part, we allowed for an improper prior that's supported on the whole real line, so there's no truncation involved and the posterior distribution is indeed a Gaussian. Now, we calculate its parameters.

- Its mean μ is $\frac{-\beta}{2\alpha} = \boxed{\frac{(\sum_{i=1}^{n} X_i) - a}{n}}$.
- Its variance σ^2 is $\frac{-1}{2\alpha} = \boxed{\frac{1}{n}}$.

From this, we calculuate

$$q_{0.025} = \mu + 1.95\sqrt{\sigma^2} = \boxed{\frac{(\sum_{i=1}^{n} X_i) - a}{n} + \frac{1.96}{\sqrt{n}}}.$$

The variance of our posterior distribution reflects the spread of possible values of λ once both our prior and the observations are taken into account, so it indeed models our uncertainty about the value of λ.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

(c)

1/2 points (graded)

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Now, suppose that we instead have the proper prior $\pi(\lambda) \sim$ Exp (a) $(a > 0)$. Again, just as in part (b): conditional on λ, we have observations $X_1, X_2, \cdots, X_n \overset{\text{i.i.d}}{\sim}$ N $(\lambda, 1)$. You may assume that $a < \sum_{i=1}^{n} X_i$. Compute the posterior distribution $\pi(\lambda | X_1, X_2, \ldots, X_n)$, then provide the following statistics on the posterior distribution. Write `Phi` for the CDF function $\Phi()$ and `PhiInv` for its inverse.

Use **SumXi** for $\sum_{i=1}^{n} X_i$.

median:

(SumXi-a)/n+(PhiInv(0.75

✖ **Answer:** (SumXi-a)/n+1/sqrt(n)*PhiInv(1-0.5*Phi(1/sqrt(n)*(SumXi-a)))

mode:

(SumXi-a)/n

✔ **Answer:** (SumXi-a)/n

$\frac{SumXi-a}{n}$

STANDARD NOTATION

Solution:

The calculations done in the previous part may be repeated, up to the point where we derive that the distribution is proportional to a Gaussian distribution. (Notice that Exp (a) exactly corresponds to $\pi(\lambda) \propto e^{-a\lambda}$, just with a support of $[0, \infty]$ instead of $\mathbb{R}$.

In this part, however, because our prior is only defined over $[0, \infty]$, we have to truncate our distribution. Hence, our posterior is the Gaussian distribution with parameters $\mu = \frac{(\sum_{i=1}^{n} X_i) - a}{n}$ and $\sigma^2 = \frac{1}{n}$, truncated such that only $\lambda > 0$ is considered.

The mode is still easy to calculate due to the given assumption $a < \sum_{i=1}^{n} X_i$, which implies that the tip of the Gaussian distribution is positive, so this is definitely the mode. The tip of the Gaussian distribution is the same as the mean in the non-truncated distribution, so we get that the mode is $\boxed{\frac{\sum_{i=1}^{n} X_i - a}{n}}$.

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The median is a bit more complex, but is easily resolved by considering quantiles of the truncated and the full Gaussian distribution.

We first calculate the proportion of the distribution that's left. The area below 0 in a Gaussian disteribution with given μ and σ^2 will be $\Phi\left(-\frac{\mu}{\sigma}\right)$, so the amount remaining is $1-\Phi\left(-\frac{\mu}{\sigma}\right)=\Phi\left(\frac{\mu}{\sigma}\right)$. Hence, the area above the median of the truncated distribution is half the amount remaining, so it is $\frac{1}{2}\Phi\left(\frac{\mu}{\sigma}\right)$. From this, we could calculate the area up to the median of the truncated distribution to be $1-\frac{1}{2}\Phi\left(\frac{\mu}{\sigma}\right)$.

Finally, we get the z-score of the median of the truncated distribution in terms of the whole distribution to be $\Phi^{-1}\left(1-\frac{1}{2}\Phi\left(\frac{\mu}{\sigma}\right)\right)$. Substituting back $\mu=\frac{(\sum_{i=1}^{n}X_i)-a}{n}$ and $\sigma^2=\frac{1}{n}$, and then using these to convert the z-score to the actual value, gives the answer

$$\frac{\sum_{i=1}^{n}X_i-a}{n}+\frac{1}{\sqrt{n}}\Phi^{-1}\left(1-\frac{1}{2}\Phi\left(\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}X_i-a\right)\right)\right).$$

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Answers are displayed within the problem

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Topic: Unit 5 Bayesian statistics:Lecture 18: Jeffreys Prior and Bayesian Confidence Interval / 10. Bayesian Statistics for Estimation

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Disagreement with Wikipedia page?

question posted 2 days ago by kirill94

I believe the last part (c) disagrees with wikipedia article on trunkated normal: https://en.wikipedia.org/wiki/Truncated_normal_distribution

Substituting α and β I've got another answer to (c)...

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3 responses

aajagnow
2 days ago

I double checked that page (thanks for the link, by the way). The stats there worked for me, but I had to be very, very careful.

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markweitzman (Community TA)
2 days ago

I did the calculation and then compared with the wikipedia page, and they agree and get green check marks.

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sandipan dey
15 minutes ago

I thought the median will be $\mu + \Phi^{-1}\left(\frac{\Phi(0)+\Phi(\infty)}{2}\right)\sigma = \mu + \Phi^{-1}\left(\frac{0.5+1}{2}\right)\sigma = \mu + \Phi^{-1}(0.75)\,\sigma$, where $\mu = \frac{\sum_i X_i - a}{n}$ and $\sigma^2 = \frac{1}{n}$, since posterior is $\propto \mathcal{N}(\mu, \sigma^2)$ with the left part with the corresponding standardized value ≤ 0 being truncated. But I got it marked wrong.

With the following code in python, I tried for a few different μ, σ and the median returned by python's *truncnorm.median()* function and my computed value were exactly same, as shown below (also wikipedia's formula $\mu + \Phi^{-1}\left(\frac{\Phi(-\mu/\sigma)+\Phi(\infty)}{2}\right)\sigma$ with the standardized values do not seem to work, as shown below):

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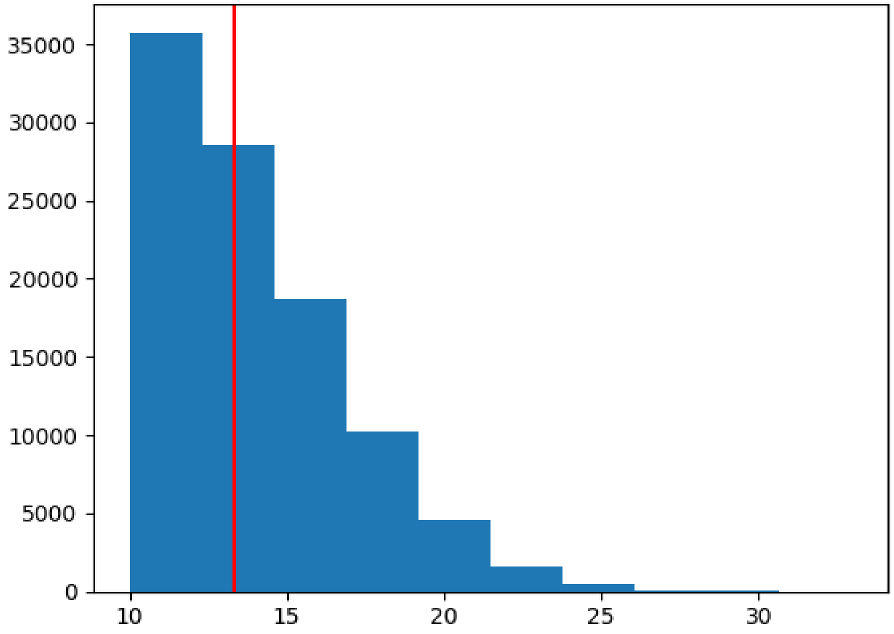
```
from scipy.stats import truncnorm
mu = 10
sigma = 5
print(truncnorm.median(a=0, b=np.Inf, loc=mu, scale=sigma))   # scipy's implementation
# 13.372448750980409
print(mu+norm.ppf(0.75)*(sigma))                                                    # my formula
# 13.372448750980409
print(mu+norm.ppf((1+norm.cdf(-mu/sigma))/2)*(sigma))          # wikipedia's formula
# 10.142584632954586
```



Did not get what's going wrong here, can anyone help? thanks in advance.

Also, I was thinking of writing some MCMC / variational inference code to approximate the posterior distribution, but never done it before (may be some examples in **computational Bayesian** will be useful, since I am a beginner in probabilistic programming).

[UPDATE] I think I got the issue, a parameter value passed to the scipy function was wrong (it needs to be standardized), wiki's formula is correct.

The correct code should be:

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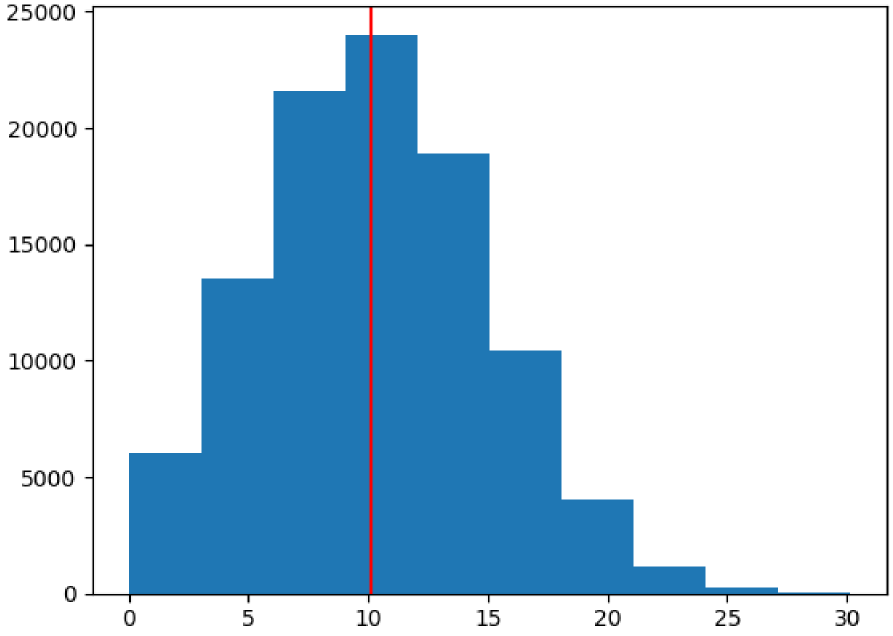
```
print(truncnorm.median(a=-mu/sigma, b=np.Inf, loc=mu, scale=sigma))   # scipy's implementation
# 10.142584632954586
```



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