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Expected value of conditional Poisson process

Asked 2 days ago Active today Viewed 63 times

1

I have been trying to solve this issue for quite a while. So, lets say that we have a Poisson process, $N = (N_t, t \geq 0)$ and the $\lambda = 3$. Lets say that $Y = (N_2|N_6 = 3)$. Find Ee^Y. However, I am stuck on understanding the notation of Y and how should I process it to continue the calculation of expected value. Should I find joint density function?

self-study poisson-distribution poisson-process

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asked 2 days ago
Gvidas Pranauskas
13 2

New contributor

Distribution of Y is the conditional distribution of N_2 given $N_6 = 3$. Can you figure out this conditional distribution? – StubbornAtom 2 days ago

3 Answers

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1

Let us assume we have a Poisson process with an arrival rate of λ. After some time t, N_t unobserved arrivals have occurred. After some more time, say τ, we observe that $N_{t+\tau}$ arrivals have occurred. What is the distribution of N_t given the observed value of $N_{t+\tau}$?

As it happens, the memoryless property of the Poisson process implies that the time of any given arrival is uniformly distributed over, in this case, $[0, t+\tau]$. This implies that the probability p that any given arrival in $[0, t+\tau]$ actually shows up in the interval $[0, t]$ is just $p = t/(t+\tau)$, the fraction of the total time that occurred before t, and is independent of the time of any other arrival. If we have $N_{t+\tau}$ arrivals overall, the number that arrive in $[0, t]$ is therefore distributed Binomial$(N_{t+\tau}, \frac{t}{(t+\tau)})$

In this case, we have $t = 2$, $t+\tau = 6$, and $N_{t+\tau} = 3$. Substituting gives us the probability distribution of $Y = (N_2|N_6 = 3)$, which is a Binomial$(3, 1/3)$ distribution. Getting from this to $\mathbb{E}e^Y$ is a straightforward calculation.

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answered yesterday
jbowman
30.3k 7 53 105

I understand that it seem straightforward calculation for most of you, but it is quite opposite for me, unfortunately. I suppose that this calculation should be done like this:

$$Ee^Y = \sum_{y=0}^{3} e^y \binom{3}{y} \frac{1}{3}^y \frac{2}{3}^{3-y}$$

? Am I right on this? – Gvidas Pranauskas yesterday

Yes, you're right. You can save yourself some effort if you have some familiarity with moment generating functions (as per @StubbornAtom 's excellent answer.) – jbowman yesterday

Thank you for thoroughly explained solution to this problem. – Gvidas Pranauskas 14 hours ago

3

The notation '$Y = (N_2 \mid N_6 = 3)$' implies that the distribution of the random variable Y is the conditional distribution of N_2 given $N_6 = 3$.

Now if $(N_t)_{t\geq 0}$ is a Poisson process with intensity parameter $\lambda(> 0)$, then the following holds:

- $N_t \sim \text{Poisson}(\lambda t)$.
- $N_{t+s} - N_s \sim \text{Poisson}(\lambda t)$ is independent of N_s.

Using this information, one can find the conditional distribution of N_s given N_t for $0 < s < t$. This turns out to be a standard distribution, and you are required to find/recall the moment generating function of this distribution.

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answered yesterday
StubbornAtom
7,862 1 19 59

Thanks for reminding the moment generating function :) – Gvidas Pranauskas 14 hours ago

-1

Due to memoryless / independent properties of Poisson process, we have

$$P(Y = y) = P(N_2 = y|N_3 = 6) = P(N_2 = y) = \frac{(2\lambda)^y e^{-2\lambda}}{y!}$$

$$\therefore E[e^Y] = \sum_{y=0}^{\infty} P(Y = y).\, e^y = \sum_{y=0}^{\infty} \frac{(2\lambda)^y e^{-2\lambda}}{y!}.\, e^y = e^{-2\lambda} \sum_{y=0}^{\infty} \frac{(2\lambda e)^y}{y!} = e^{-2\lambda}.\, e^{2\lambda e}.$$
$$= e^{2\lambda(e-1)}$$

Now, if we want to compute expectation the conditional Poisson Process for the same interval, let's say, if in a given interval of length 6λ we have 3 arrivals, what's the expected number of arrivals in a subinterval of length 2λ of that interval, then we can proceed as follows.

$N_6 = 3$ restricts $N_2 \leq 3$ (since there is 3 arrivals in interval of length 6λ, then in length 2λ sub-interval of it, we must have less or equal arrivals).

Hence, $P(Y=y)=P(N_2=y)=\frac{e^{-2\lambda y}(2\lambda)^y}{y!}$, where $y \in \{0,1,2,3\}$ (considering memoryless and independence properties of the Poisson process).

$$\therefore E[e^Y]=\sum_{y=0}^{3}P(Y=y).e^y=\sum_{y=0}^{3}\frac{e^{-2\lambda y}(2\lambda)^y}{y!}.e^y=\sum_{y=0}^{3}\frac{e^{(1-2\lambda)y}.(2\lambda)^y}{y!}$$, where we have $\lambda=3$

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answered 2 days ago
Sandipan Dey
258 1 8

1 Note that your solution doesn't depend on the fact that it's N_6 that equals 3. Consider what the probability distributions of N_2 would be if we knew $N_3=3$, or alternatively that $N_{10000}=3$. Would you expect them to be the same? – jbowman 2 days ago