5 Magical Sequences

5-1 Catalan Numbers

组合数学 Combinatorics

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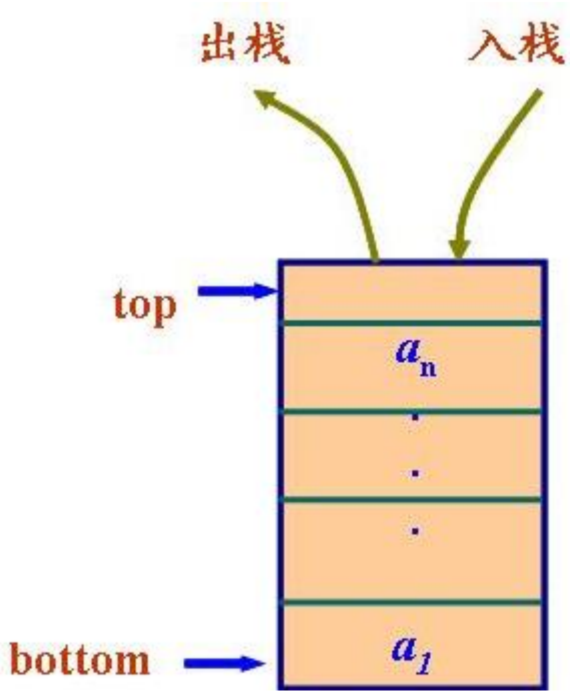
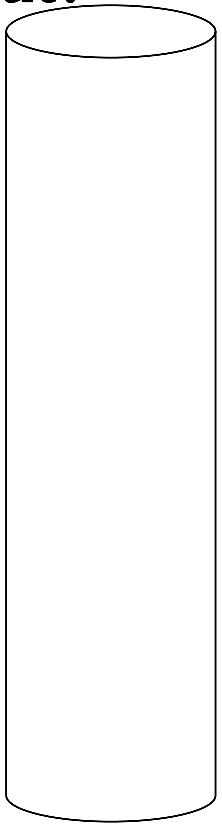
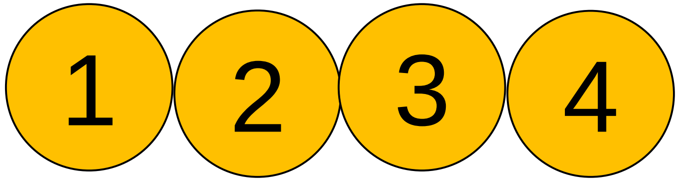
Amazing World of Computers

- **One stack (of infinite size) has the "push" sequence: 1, 2, 3, … n. How many ways can the numbers be popped out?**





1， 2， 3, 4

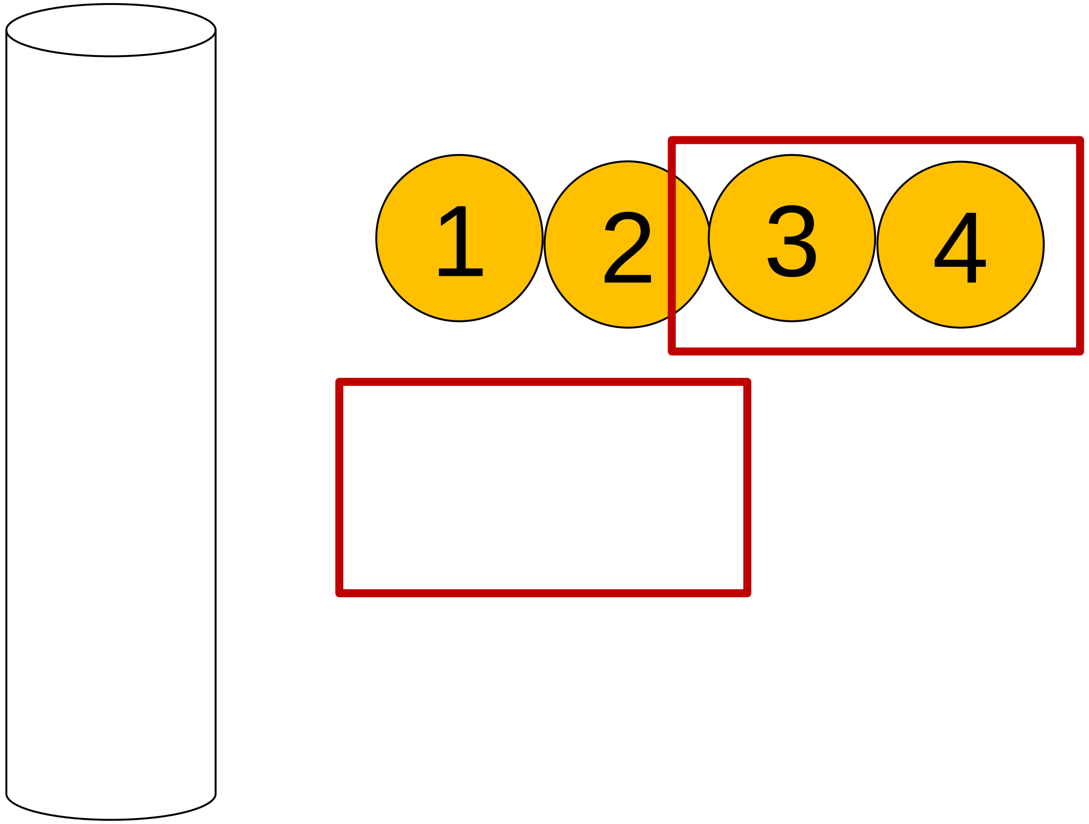


Push 1, Push 2, Pop 2, Pop 1, Push 3, Pop 3, Push 4, Pop 4

Amazing World of Computers

- **One stack (of infinite size) has the "push" sequence: 1, 2, 3, … n. How many ways can the numbers be popped out?**

Partition into steps when the stack was first empty.



The first time (Seq. 1) when the stack is empty, k elements are popped out.

Partition the n sequences into two sub-sequences, where one is Seq. 1~k-1 where there are k-1 elements, and the other is Seq. k+1~n, where there are n-k elements.

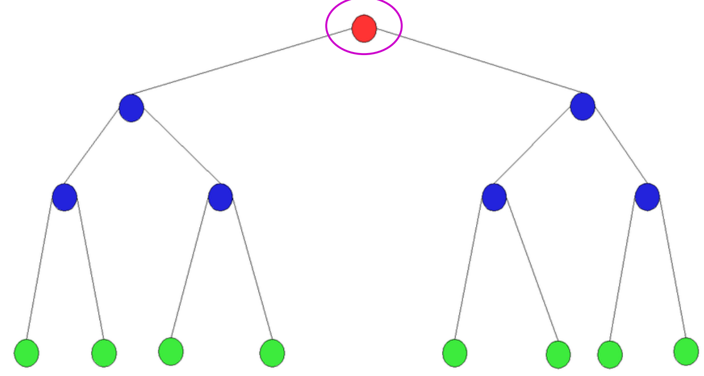
If f(n) is the sequence with n elements, then:

$f(n)= f(k-1)* f(n-k)$

$k=1\sim n$

$$f(n) = f(0)*f(n-1) + f(1)*f(n-2) + \ldots\ldots + f(n-2)*f(1) + f(n-1)*f(0)$$

Binary Tree



- ***How many different shapes can a binary tree with n nodes have?***
- The root will obviously contain one node. Assume T(i, j) stands the left subtree of the root containing i nodes, and right subtree with j nodes.
- Not counting the root, there are n-1 nodes that could be structured as: $T(0, n\text{-}1), T(1, n\text{-}2), \ldots T(n\text{-}1, 0)$.
- Suppose the solution is f(n):

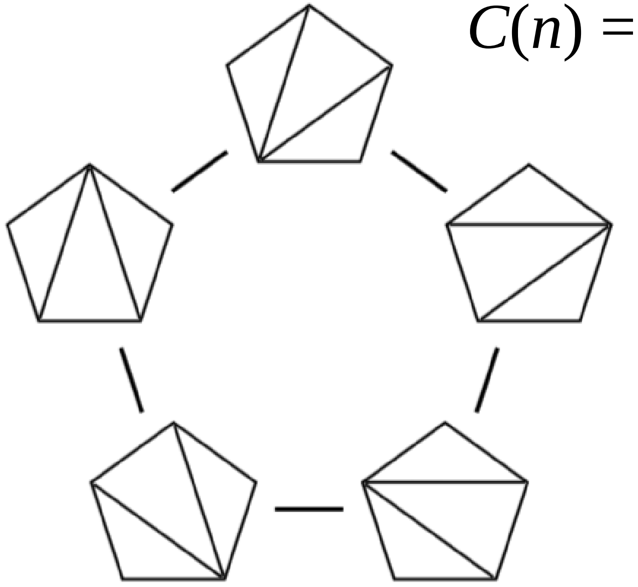
$$f(n) = f(0)*f(n-1) + f(1)*f(n-2) + \ldots\ldots + f(n-2)*f(1) + f(n-1)*f(0)$$

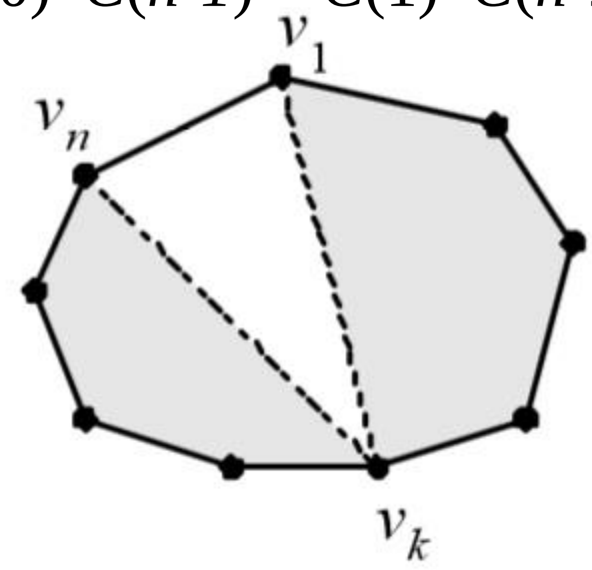
- If $f(0) = 1$， then $f(1) = 1$, $f(2) = 2$, $f(3) = 5$。

Catalan Numbers

- In 1751, in a letter to Christian Goldbach, Euler discussed about the following problem:
  - How many triangulations of a n-gon are there?

$$C(n) = C(0)*C(n\text{-}1) + C(1)*C(n\text{-}2) + \text{.......}+ C(n\text{-}2)*C(1) + C(0)*C(n\text{-}2)$$





$$\frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (n-1)}$$

Christian Goldbach (1690 – 1764)

History

- In1758, Johann Segner gave a recurrence formula for this question

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

- In 1838， this became a hot research topic
  - Gabriel Lame gave a complete proof and a simpler expression.
  - Eugène Charles Catalan discovered the connection to parenthesized expressions during his exploration of the Towers of Hanoi puzzle.

…

  - In 1900 Eugen Netto in his book named these numbers *Catalan* numbers.

History

- According to research papers in 1988 and 1999, the first person to discover Catalan numbers wasn’t Euler.
  - In 1753, Euler described the Catalan sequence while dividing a polygon into triangles.
  - In 1730, however, Minggantu, a Chinese mathematician (Mongolian) during Qing dynasty, had used the Catalan sequence. See “Ge Yuan Mi Lu Jie Fa”. This research was completed and published by his student in 1774.

Catalan Numbers

- Named after the Belgian mathematician Eugene Charles Catalan (1814–1894)
- OEIS A000108
- 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, ...

$$C(n) = C(0)*C(n\text{-}1) + C(1)*C(n\text{-}2) + \ldots\ldots. + C(n\text{-}2)*C(1) + C(n\text{-}1)*C(0)$$

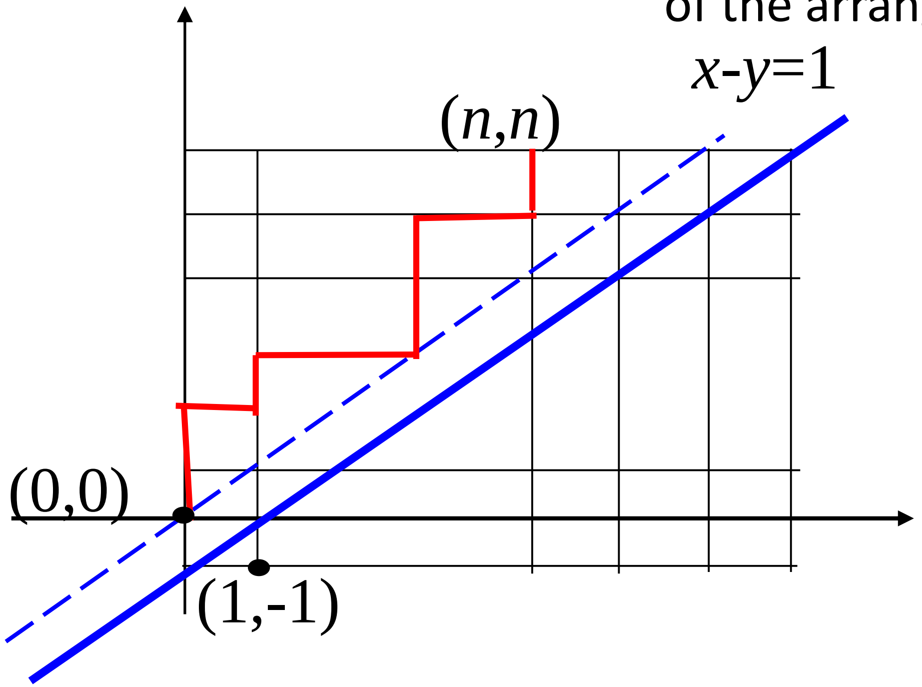
$$
\begin{array}{rcl}
C_1 & = & C_0C_0 \\
C_2 & = & C_1C_0 + C_0C_1 \\
C_3 & = & C_2C_0 + C_1C_1 + C_0C_2 \\
C_4 & = & C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3 \\
\cdots & & \cdots
\end{array}
$$

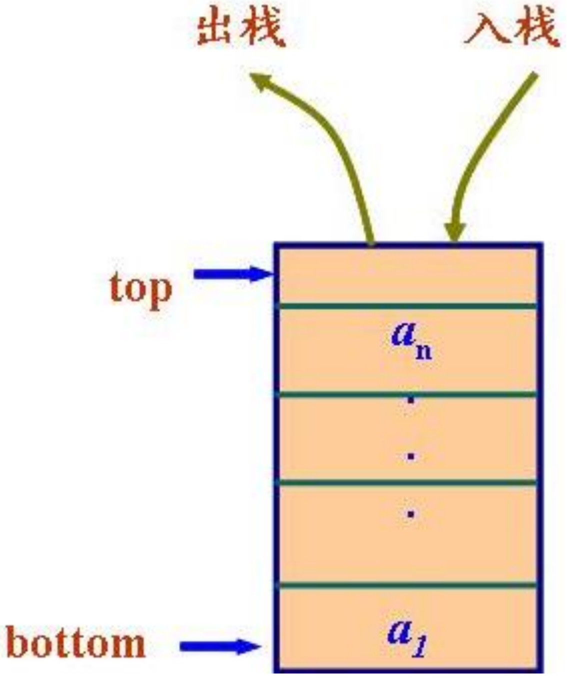
$C(n) = C(0)*C(n\text{-}1) + C(1)*C(n\text{-}2) + \text{.......}+ C(n\text{-}2)*C(1) + C(n\text{-}1)*C(0)$

Stack and Lattice Paths

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

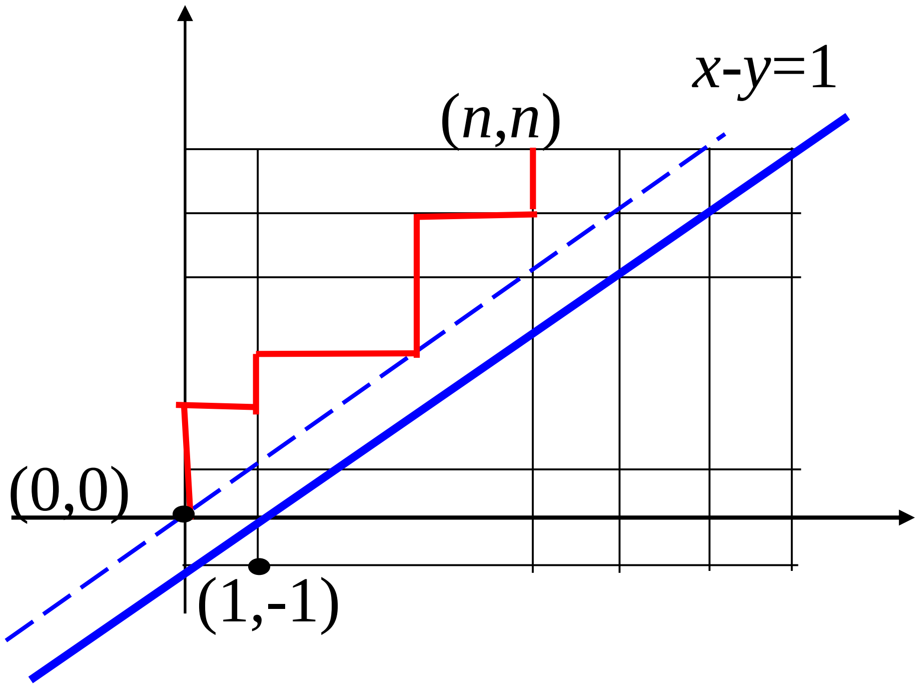
- HW: W2-G1: 8 people is lining up to purchase tickets, 4 people are holding 10 Yuan, 4 people are holding 20 Yuan, the ticket price is 10 Yuan. The ticket booth does not have money at the first place, find out how many different possible ways of the arrangement of 8 people that they can successful purchase the tickets.





Catalan Numbers?

Dyck Path



Limit the line to either move one grid down or one grid to the right. $x-y=1$;

The problem turns into a lattice path going from (0,0) to (n,n) without touching the x-y=1 line.

The symmetry point of (0,0) about $x-y=1$ is (1,-1).

Using the method used in the last question, the number of lattice paths equal to:

$$C_n = C(2n,n) - C(2n,n-1)$$

$$= \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k}$$

Proof using generating functions

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

- Generating function $$c(x) = \sum_{n=0}^{\infty} C_n x^n.$$

$$c(x)^2 = C_0C_0 + (C_1C_0 + C_0C_1)x + (C_2C_0 + C_1C_1 + C_0C_2)x^2 + \ldots\ldots$$

$$\underbrace{C_0C_0}_{C_1} \quad \underbrace{(C_1C_0 + C_0C_1)}_{C_2} \quad \underbrace{(C_2C_0 + C_1C_1 + C_0C_2)}_{C_3}$$

$$c(x)^2 = C_1 + C_2x + C_3x^2 + \ldots\ldots$$

$$c(x) = C_0 + C_1x + C_2x^2 + \ldots\ldots$$

$$\begin{aligned} C_1 &= C_0C_0 \\ C_2 &= C_1C_0 + C_0C_1 \\ C_3 &= C_2C_0 + C_1C_1 + C_0C_2 \\ C_4 &= C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3 \\ \ldots & \quad \ldots \end{aligned}$$

$C_0=1$

$$c(x) = 1 + xc(x)^2;$$

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!} x^n \qquad \alpha \in R$$

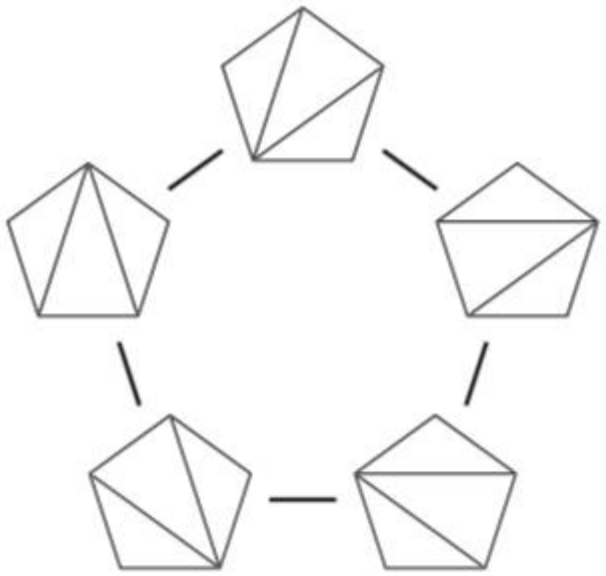
$$c(x) = 1 + xc(x)^2;$$

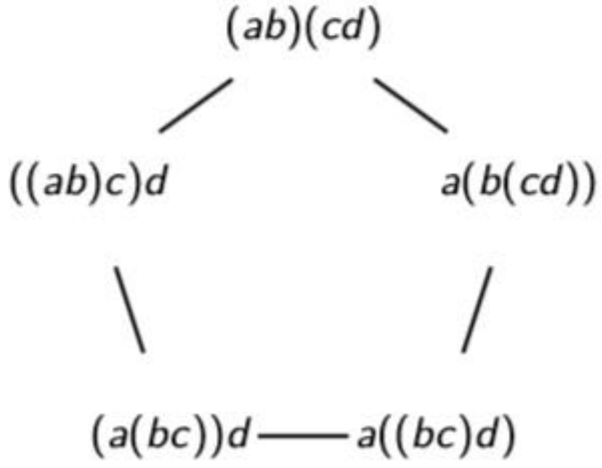
$$\lim_{x \to 0+} c(x) = C_0 = 1 \qquad c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$c(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}.$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Same problem





If we remove the parenthesis, what will the calculation be like?

$$((2-1)\cdot(-(1+2))\cdot 4)\div((1+2)\cdot 2)$$

$$2-1\cdot -1+2\cdot 4\div 1+2\cdot 2$$

$$(()(()))(())$$

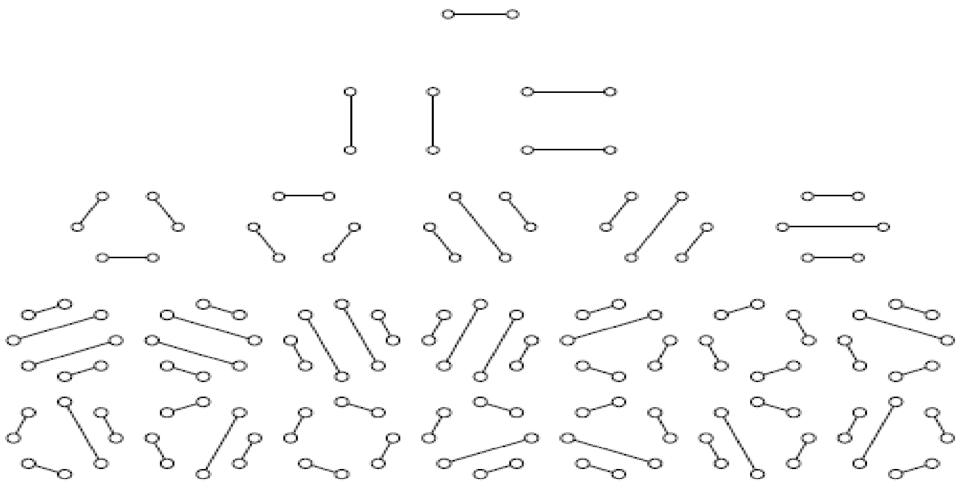
- How many pairs of parenthesis can be correctly matched?
- C_n is the number of Dyck words of length 2n.
- Here, a Dyck word is a string consisting of n X’s and n Y’s such that no initial segment of the string has more Y’s than X’s. For example, the following are Dyck words of length 6.
  XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY

Dyck Language

- Dyck Language is a very interesting language in the field of Computer Science. In the theory of formal languages of CS, Mathematics, and Linguistics, Dyck language is the language consisting of balanced strings of parentheses [and].
- It is important in the parsing of expressions that must have a correctly nested sequence of parentheses, such as arithmetic or algebraic expressions.
- It is named after the mathematician Walther von Dyck.

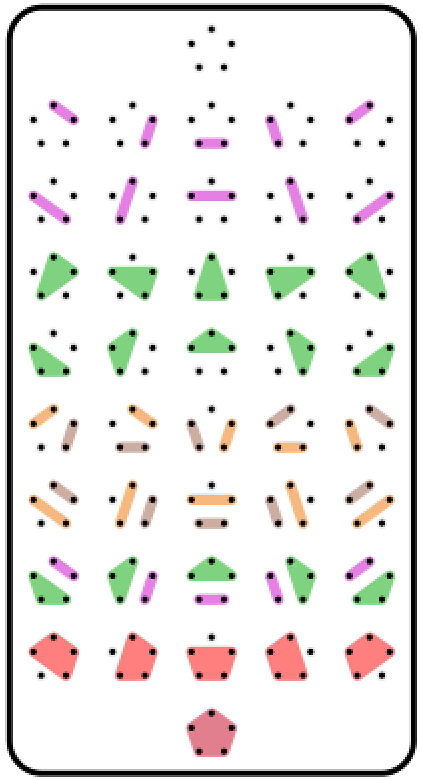
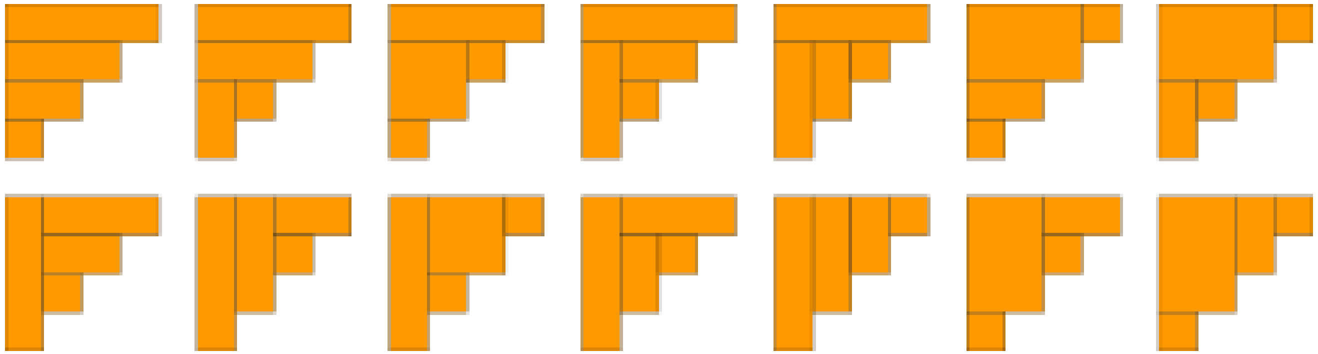
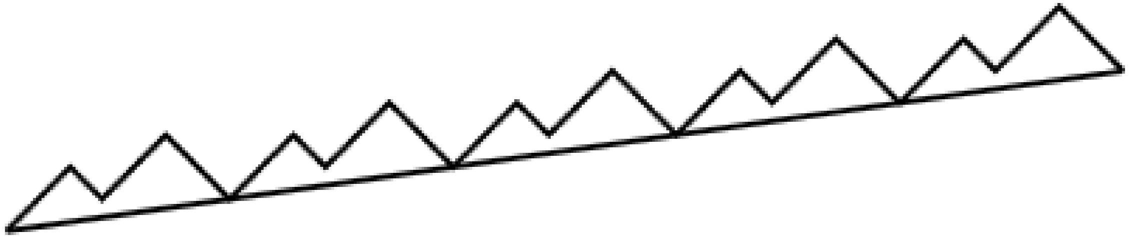
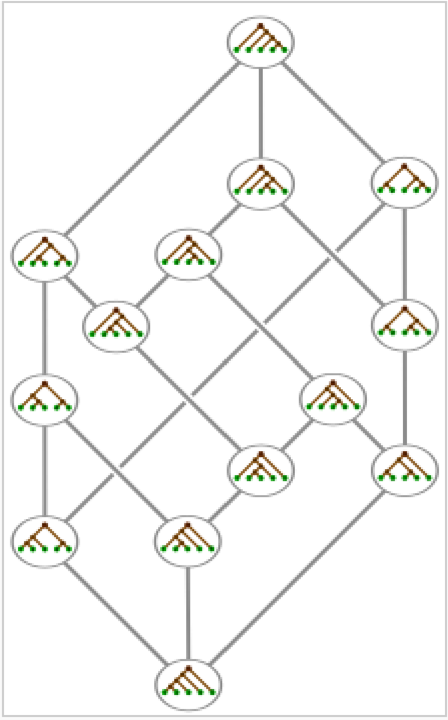
Hands across the table

- If n people are seated around a circular table, in how many ways can all of them be simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other?
- The result is Catalan sequence. This is called the Hands across the table" problem. There's also a romantic movie with the same name!



Catalan Numbers

- Hot research topic in the 20th century
  - M.Kuchinski found 31 structures that could be enumerated by Catalan numbers.
  - As of 08/21/2010, R. Stanley counts 190 structures counted by Catalan numbers.

5 Magical Sequences

5-2 Exponential Generating Functions

组合数学 Combinatorics

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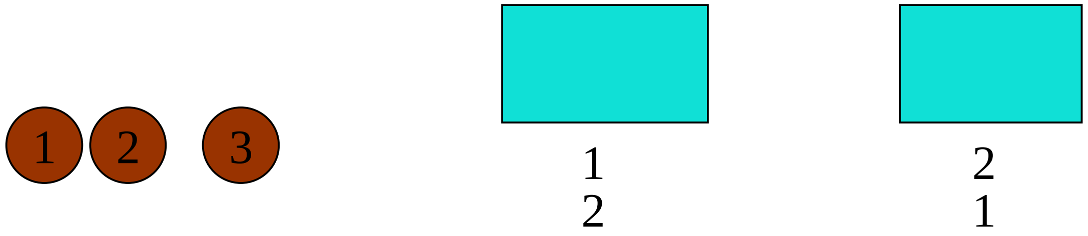
Exponential Generating Functions

- Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots$$

- Generating function of C(n,k)
- The coefficient is the combination of selecting k terms from n



$$C(n,k)k! = P(n,k)$$

$$(1+x)^n = (1+x)(1+x)(1+x)(1+x)......(1+x)(1+x)$$

$$x_0 \quad 1 \quad x_1 \quad x_2 \quad \ldots \quad x_{k-1} \quad 1$$

$$(1+x)^n = \sum_{k=0}^{\infty} C(n,k)x^k = \sum_{k=0}^{\infty} \frac{C(n,k)k!}{k!}x^k = \sum_{k=0}^{\infty} P(n,k)\frac{x^k}{k!}$$

To compute permutations, and not combinations, use these terms: $\{1, x, \frac{x^2}{2!}, \ldots, \frac{x^n}{n!}\}$

Full r-permutations

- How many different permutations do the 8 letters in "pingpang" have?
- Adding indices to differentiate: $$\begin{pmatrix} & & 8 & & \\ 2 & 2 & 2 & 1 & 1 \end{pmatrix} = \frac{8!}{2!2!2!}$$

  $p_1p_2n_1n_2g_1g_2ia$

Full r-permutations

- $3a_1$, $2a_2$, $3a_3$
  - How many combinations of length k?

$$G(x) = (1 + x + x^2 + x^3)(1 + x + x^2)(1 + x + x^2 + x^3)$$

$$= (1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5) \cdot (1 + x + x^2 + x^3)$$

$$= 1 + 3x + 6x^2 + 9x^3 + 10x^4 + 9x^5 + 6x^6 + 3x^7 + x^8$$

$$G(x) = 1 + 3x + 6x^2 + 9x^3 + 10x^4 + 9x^5 + 6x^6 + 3x^7 + x^8$$

$3a_1, 2a_2, 3a_3$; How many combinations of length 4?

- From the coefficient of x^4, we get 10 combinations of length 4 from these 8 elements. Expanding the expression yields:

$$(1 + x_1 + x_1^2 + x_1^3)(1 + x_2 + x_2^2)(1 + x_3 + x_3^2 + x_3^3)$$

$$= [1 + (x_1 + x_2) + (x_1^2 + x_1 x_2 + x_2^2) + (x_1^3 + x_1^2 x_2 + x_1 x_2^2)$$

$$+ (x_1^3 x_2 + x_1^2 x_2^2) + x_1^3 x_2^2] \cdot (1 + x_3 + x_3^2 + x_3^3)$$

$$= 1 + (1 + x_1 + x_2 + x_3)$$

$$+ (x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^3)$$

$$+ (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 + x_3^3)$$

$$+ (\underline{x_1 x_3^3 + x_2 x_3^3 + x_1^2 x_3^2 + x_1 x_2 x_3^2 + x_2^2 x_3^2 + x_1^3 x_3 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3}$$

$$\underline{+ x_1^3 x_3 + x_1^2 x_2^2}) + \cdots$$

Full r-permutations

$3a_1, 2a_2, 3a_3$; How many permutations of length 4?

$x_1^2 x_3^2$ using 2 of each as an example, the different permutations equal to:

$$\frac{4!}{2!2!} = 6$$

$a_1a_1a_3a_3,\ a_1a_3a_1a_3,\ a_3a_1a_3a_1,\ a_1a_3a_3a_1,$
$a_3a_3a_1a_1,\ a_3a_1a_1a_3,$ 6 ways. Similarly for one a_1 and 3 a_3, the different permutations equal: $\frac{4!}{3!} = 4$

$a_1a_3a_3a_3,\ a_3a_1a_3a_3,\ a_3a_3a_1a_3,\ a_3a_3a_3a_1,$

can be solved for, in a similar manner.

$$x_1x_3^3+x_2x_3^3+x_1^2x_3^2+x_1x_2x_3^2+x_2^2x_3^2+x_1^3x_3$$

$$+x_1^2x_2x_3+x_1x_2^2x_3+x_1^3x_3+x_1^2x_2^2$$

Therefore, the total no. of different permutations:

$$4!\left(\frac{1}{1!3!}+\frac{1}{1!3!}+\frac{1}{2!2!}+\frac{1}{1!1!2!}+\frac{1}{2!2!}+\frac{1}{3!1!}+\frac{1}{2!1!1!}+\frac{1}{1!2!1!}+\frac{1}{3!1!}+\frac{1}{2!2!}\right)$$

$$=4!\left(\frac{4}{3!}+\frac{3}{2!2!}+\frac{3}{2!}\right)=4!\frac{4\cdot 2!\cdot 2!+3\cdot 3!+3\cdot 2!\cdot 3!}{2!2!3!}$$

$$=16+18+36=70$$

$$G(x) = (1 + x + x^2 + x^3)(1 + x + x^2)(1 + x + x^2 + x^3)$$

For easier computation, using the aforementioned property:

$$G_e(x) = (1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!})(1 + \frac{x}{1!} + \frac{x^2}{2!})(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!})$$

$$= (1 + 2x + 2x^2 + \frac{7}{6}x^3 + \frac{5}{12}x^4 + \frac{1}{12}x^5) \cdot (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3)$$

$$= 1 + 3x + \frac{9}{2}x^2 + \frac{14}{3}x^3 + \frac{35}{12}x^4 + \frac{17}{12}x^5 + \frac{35}{72}x^6 + \frac{8}{72}x^7 + \frac{1}{72}x^8$$

$$G_e(x) = 1! + \frac{3}{1!}x + \frac{9}{2!}x^2 + \frac{28}{3!}x^3 + \frac{70}{4!}x^4 + \frac{170}{5!}x^5 + \frac{350}{6!}x^6 + \frac{560}{7!}x^7 + \frac{560}{8!}x^8$$

$$= 4!(\frac{4}{3!} + \frac{3}{2!2!} + \frac{3}{2!}) = 16 + 18 + 36 = 70$$

- Generating Function **Combinations**

For a seq. $a_0, a_1, a_2, \cdots$, construct:

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots,$$

G(x) is known as the generating function of $a_0, a_1, a_2, \cdots$

- Exponential Generating Function **Permutations**

For a seq. $a_0, a_1, a_2, \cdots$, construct:

$$G_e(x) = a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \cdots + \frac{a_k}{k!}x^k + \cdots$$

G(x) is known as the exponential generating function of $a_0, a_1, a_2, \cdots$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \ldots}$$

Is the exponential generating function of {1,1,…,1}

Exponential Generating Function

Full r-permutations: If there are n_1 a_1's, n_2 a_2's, …, n_k a_k's, then: Constructing a permutation of length n, with different permutations equals:

$$\frac{n!}{n_1!n_2!\cdots n_k!} \qquad n = n_1 + n_2 + \cdots + n_k$$

r-permutations: If there are n_1 a_1's, n_2 a_2's, …, n_k a_k's, then: From n elements choosing r permutations, where all permutations are different equal to p_r. Therefore, for the sequence, p_0, p_1, … , p_n, the exponential generating function is:

$$G_e(x) = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n_1}}{n_1!}\right)$$

$$\cdot \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n_2}}{n_2!}\right) \cdots \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n_k}}{n_k!}\right)$$

Exponential Generating Function is perfect for counting permutations of multisets.

Example： Using the digits 1, 2, 3 and 4, construct a five digit number where 1 doesn’t appear more than 2 times, but has to appear at least once; 2 can’t appear more than once; 3 can appear up to 3 times but can also not appear in the number; 4 can only appear an even number of times. How many numbers can satisfy these conditions?

1 doesn't appear more than twice, but has to appear at least once; 2 can't appear more than twice; 3 can appear up to 3 times but can also not appear in the number; 4 can only appear an even number of times.

An r digit number that satisfies the condition is a_r; the exponential generating function for the seq. $a_0, a_1, \ldots, a_{10}$ is:

$$G_e(x) = \left(\frac{x}{1!}+\frac{x^2}{2!}\right)(1+x)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}\right)$$

$$= \left(x+\frac{3}{2}x^2+\frac{1}{2}x^3\right)\left(1+x+x^2+\frac{2}{3}x^3+\frac{7}{24}x^4+\frac{1}{8}x^5+\frac{x^6}{48}+\frac{x^7}{144}\right)$$

$$= x+\frac{5}{2}x^2+3x^3+\frac{8}{3}x^4+\frac{43}{24}x^5+\frac{43}{48}x^6+\frac{17}{48}x^7+\frac{1}{288}x^8+\frac{1}{48}x^9+\frac{1}{288}x^{10}$$

$$= \frac{x}{1!}+5\frac{x^2}{2!}+18\frac{x^3}{3!}+64\frac{x^4}{4!}+215\frac{x^5}{5!}+645\frac{x^6}{6!}+1785\frac{x^7}{7!}$$
$$+140\frac{x^8}{8!}+7650\frac{x^9}{9!}+12600\frac{x^{10}}{10!}$$

There are 215 5-digit numbers that satisfy the conditions.

Example

With digits 1, 3, 5, 7, 9, how many n-digit numbers are there, where 3 and 7 appear an even number of times, and 1, 5, 9 don't have any conditions.

Assume an r-digit number that satisfies the conditions is a_r. Then, the exponential generating function of the seq. of a_1, a_2, …, a_3 is:

$$G_e(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3$$

$$G_e(x) = \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3$$

Since $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$,

$$\therefore \quad 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots = \frac{1}{2}(e^x+e^{-x}).$$

$$G_e(x) = \frac{1}{4}(e^x+e^{-x})^2 e^{3x} = \frac{1}{4}(e^{2x}+2+e^{-2x})e^{3x}$$

$$= \frac{1}{4}(e^{5x}+2e^{3x}+e^x) = \frac{1}{4}\left(\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n + 2\sum_{n=0}^{\infty}\frac{3^n x^n}{n!} + \sum_{n=0}^{\infty}\frac{x^n}{n!}\right)$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(5^n+2\cdot 3^n+1)\frac{x^n}{n!}.$$

$$\therefore \quad a_n = \frac{1}{4}(5^n+2\cdot 3^n+1).$$

5 Magical Sequence

5-3 Derangements

组合数学 Combinatorics

Changing Partners

- We were waltzing together to a dreaming melody
  When they called out 'Change Partners.'
  And you waltzed away from me.
  Now my arms feel so empty,
  As I gaze around the floor.
  And I'll keep on changing partners,
- There are n couples of male and female dancing, and after each song, the partners must change. In this scenario, how many ways can the partners change?

Derangement

- Derangement problems were first researched by Nicholas Bernoulli, since then it has also been called the Bernoulli-Euler Problem of the Misaddressed Letters.
- An ordered sequence of n elements has n! different permutations. If in a permutation, none of the objects in their original place, it is known as derangement.

◦

A **derangement** is a permutation of the elements of a set such that none of the elements appear in their original position

Derangement

1 2 has only one derangement: 2 1

1 2 3 has two derangements: 3 1 2，2 3 1这

This can be seen as a derangement of 1 2, where 3 replaces 1 and 2.

2 1 3

3 1 2

2 1 3

2 3 1

Derangement

Derangement of 1 2 3 4

4 3 2 1， 4 1 2 3， 4 3 1 2，
3 4 1 2， 3 4 2 1， 2 4 1 3，
2 1 4 3， 3 1 4 2， 2 3 4 1。

4 switches positions with 123, and then a derangement of the leftover two elements is done.

1 2 3 4→4 3 2 1，
1 2 3 4→3 4 1 2，
1 2 3 4→ 2 1 4 3

4 switches positions with 312
(Derangement of 123)

3 1 2 4→4 1 2 3，
3 1 2 4→3 4 2 1，
3 1 2 4→ 3 1 4 2

4 switches positions with 231

2 3 1 4→4 3 1 2，
2 3 1 4→2 4 1 3，
2 3 1 4→ 2 3 4 1

Derangement

From the above analysis, we now have intuition to construct derangements.

Suppose the derangement of n elements in a sequence 1, 2, … n is D_n, for every i^{th} number, it has to switch with the other n-1 numbers, followed by a derangement of the other n-2 elements. We will have a total of $(n-1)D_{n-2}$ number of derangements.

For the other part, not considering the number i, there is a derangement of the other n-1 elements to be done, followed by switching i with the other numbers yielding a total of $(n-1)D_{n-1}$ number of derangements.

$$D_n = (n-1)(D_{n-1} + D_{n-2}),$$

$$D_1 = 0, D_2 = 1,$$

$$\therefore \quad D_0 = 1.$$

$$D_n = (n-1)(D_{n-2} + D_{n-1}), \qquad n \geq 3.$$

- Proof: Consider the D_n derangements of {1,2,…,n}.
- Partition according to which of the integers 2,3,4…,n is in the first position.
  - n-1 partitions:

    $2i_2i_3\ldots i_n \quad i_2\neq 2, i_3\neq 3, \ldots, i_n\neq n,$
  - *d_n is the number of derangements in each partition*

    $Dn = (n\text{-}1)\, d_n$
- For each Partition such as $2i_2i_3\ldots i_n$, we can also partition into 2 subparts according to whether i_2=1 or not.
  - *2* partitions:

    $21i_3\ldots i_n \quad i_3\neq 3, \ldots, i_n\neq n,$ *d'_n is the number of derangements in this partition*

    $d'_n = D_{n-2}$

    $2i_2i_3\ldots i_n \quad i_2\neq 1, i_3\neq 3, \ldots, i_n\neq n,$ *d''_n is the number of derangements in this partition*

    $d''_n = D_{n-1}$
  - $d_n = d'_n + d''_n$

$$D_n = (n-1)(D_{n-2} + D_{n-1}), \qquad n \geq 3.$$

$$\boldsymbol{D_n = (n-1)(D_{n-1} + D_{n-2}),} \qquad \boldsymbol{D_1 = 0, D_2 = 1}$$

The non-constant coefficient recurrence relation.

$$\therefore \quad D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}]$$

$$= (-1)^2[D_{n-2} - (n-2)D_{n-3}]$$

$$= \cdots$$

$$= (-1)^{n-1}(D_1 - D_0)$$

Since $D_1 = 0$, $D_0 = 1$, we can get a recurrence relation of D_n

$$D_n - nD_{n-1} = (-1)^n.$$

$$D_n - nD_{n-1} = (-1)^n.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

Suppose $G_e(x) = D_0 + D_1 x + \frac{D_2}{2!}x^2 + \frac{D_3}{3!}x^3 + \cdots$

$$\frac{x}{1!} \quad D_1 = 1D_0 + (-1)^1$$

$$\frac{x^2}{2!} \quad D_2 = 2D_1 + (-1)^2$$

$$\frac{x^3}{3!} \quad D_3 = 3D_2 + (-1)^3$$

$$\vdots$$

Then, $G_e(x) = xG_e(x) + e^{-x}$

$$G_e(x) = \frac{e^{-x}}{1-x} = (1 - x + \frac{x^{2`}}{2!} - \cdots)/(1-x).$$

$$\therefore \quad D_n = (1 - 1 + \frac{1}{2!} - \cdots \pm \frac{1}{n!}) \cdot n!$$

$$D_n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots \pm \frac{n!}{n!} = \sum_{k=0}^{n} (-1)^k C(n,k)(n-k)!$$

5 Magical Sequence

5-4 Stirling Numbers

组合数学 Combinatorics

清华大学 马昱春

James Stirling(1692-1770)

- Stirling’s approximation
- Stirling Permutation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

  - Permutation of multisets (1, 1, 2, 2,…,k, k)
  - Require that all numbers between the two repeated numbers are larger than these two numbers:
  - (1, 1, 2, 2)
  - 1221, 1122, 2211

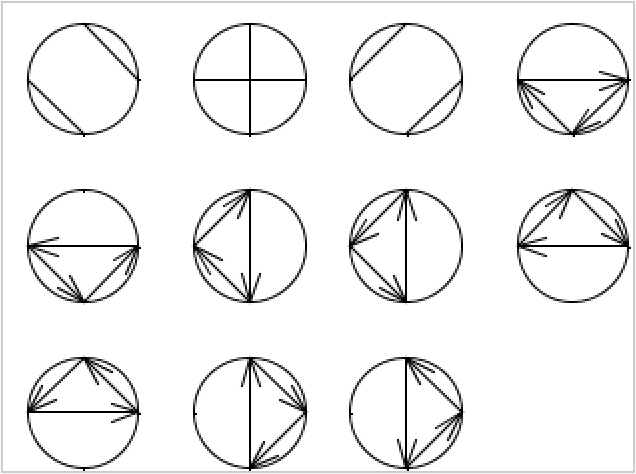
Stirling numbers of the first kind

Stirling Numbers

Stirling numbers of the second kind

Stirling numbers of the first kind

- *n people group dancing, divided into m disjoint circular groups.*
  - *A, B, C, D: 4 people dancing in two groups:*
    - {A,B},{C,D} {A,C},{B,D} {A,D},{B,C}
    - {A},{B,C,D} {A},{B,D,C} {B},{A,C,D} {B},{A,D,C} {C},{A,B,D} {C},{A,D,B} {D},{A,B,C} {D},{A,C,B}
  - Stirling numbers of the first kind s(n,m)
  - $s(4,2)=11$



s(4,2)=11

Stirling numbers of the first kind

- Stirling numbers of the first kind $s(n,m)$
  - $s(n,0)=0$, $s(1,1)=1$
  - $\mathbf{s(n+1,m)=s(n, m\text{-}1)+n\ s(n, m)}$
    - The n+1st person can dance alone, and the others form m-1 groups.
    - The n+1st person joins a group, and will have n choices for groups, while the other n people have s(n, m) ways of forming groups.

Stirling numbers of the first kind

- $s(n+1,m)=s(n, m-1)+n\, s(n, m)$

  Unsigned Stirling numbers of the first kind

$$s(n+1,k)=s(n,k-1)-ns(n,k).$$

Signed Stirling numbers of the first kind

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1)$$

Rising factorial

$$=s(n,0)+s(n,1)x+s(n,2)x^2+\ldots\ldots+s(n,n)x^n$$

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1)$$

Falling factorial

$$=s(n,0)+s(n,1)x+s(n,2)x^2+\ldots\ldots+s(n,n)x^n$$

Stirling numbers of the first kind

$$s(n+1,k)=s(n,k-1)-ns(n,k).$$

Signed Stirling numbers of the first kind

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1)$$

$$=s(n,0)+s(n,1)x+s(n,2)x^2+\ldots\ldots+s(n,n)x^n$$

$$[x]_{n+1}=[s(n,0)+s(n,1)x+\cdots+s(n,n)x^n](x-n)$$

$$=s(n+1,0)+s(n+1,1)x+\cdots s(n+1,n+1)x^{n+1},$$

The coefficient of x^k :

$$s(n,k-1)-ns(n,k)=s(n+1,k)$$

Stirling numbers of the second kind

- Partition n balls into k groups, there is no ordering within the groups.
  - If you have four balls of colors red, blue, yellow and white, and are asked to partition them into one group, there is only one possibility.
  - If you have to partition the same into four groups, there is also only one possibility.
  - How about partitioning into two groups though?
  - **Please complete the following quiz.**

Stirling numbers of the second kind

- Partition n balls into k groups, there is no ordering within the groups.
  – Placing balls model
  – *Placing n balls into m identical boxes, where boxes are non-empty:*

	1	2	3	4	5	6	7
First box	r	y	b	w	ry	rb	Rw
Second box	ybw	rbw	ryw	ryb	bw	yw	yb

$$\therefore \quad S(4,2) = 7.$$

Stirling numbers

Definition: *The number of ways n non-identical balls in m identical boxes where none of the boxes are empty is known as the Stirling number of the second kind.*

Theorem: Stirling number of the second kind S(n, m) has the following properties:

$$(a)\ \ S(n,0)=0, \qquad (b)\ \ S(n,1)=1,$$

$$(c)\ \ S(n,2)=2^{n-1}-1, \qquad (d)\ \ S(n,n)=1.$$

n \ k	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

Proof:

(a), (b), (d) are obvious.

Proof of (c):

$$(c)\ S(n,2) = 2^{n-1} - 1$$

(c) Assume there are n non-identical balls $b_1, b_2, \dots, b_n$, from which we pick b_1. Now for the other n-1 balls, there are two possibilities, either they can be in the same box as b1 or a box that doesn't contain b1. But it can only satisfy one of these conditions, therefore there are 2^{n-1} -1 ways of placing the balls.

$$S(n,2) = 2^{n-1} - 1$$

First kind: $s(n+1,k)=s(n,k-1)-ns(n,k).$ $s(n+1,k)=s(n,k-1)+ns(n,k).$

Stirling Numbers

Theorem: Stirling number of the second kind satisfies the following recurrence relation.

$$S(n,m)=mS(n-1,m)+S(n-1,m-1) \quad (n>1, m\geq 1).$$

Proof: Suppose there are n non-identical balls b_1, b_2, …, b_n, from which we pick a ball b_1. Placing n balls into m boxes where no box is empty can be classified into two possibilities.

1. b_1 is placed on its own in a box, therefore, the count is S(n-1, m-1)
2. b_1 is not the only ball in its box. This is equivalent to placing n-1 balls in m boxes, where no box is empty. This yields the count S(n-1, m). Out of these boxes, b_1 can be placed in any of these m boxes, so the total number of ways in this possibility equals to m*S(n-1, m)

Using the sum rule: $S(n,m)=S(n-1,m-1)+mS(n-1,m).$

Stirling Numbers

Place 5 balls (Red, Yellow, Blue, White and Green) into two boxes:

$$S(5,2) = 2S(4,2) + S(4,1) = 2\times 7 + 1 = 15,$$

There are 15 different ways of doing so.

First take away the green ball, and place the other four balls into two boxes. This is the same problem as discussed in a previous example. Like in the previous example, r, y, b, w stand for red, yellow, blue and white balls, while g stands for the green ball.

Stirling Numbers

g is not the only ball in the box				g is the only ball in the box	
Box 1	Box 2	Box 1	Box 2	Box 1	Box 2
rg	ybw	r	ybwg	g	rybw
yg	rbw	y	rbwg		
bg	ryw	b	rywg		
wg	ryb	w	rybg		
ryg	bw	ry	bwg		
rbg	yw	rb	ywg		
rwg	yb	rw	ybg		

Stirling Numbers

$$S(n,m)=\frac{1}{m!}\sum_{h=0}^{m}C(m,h)(-1)^h(m-h)^n$$

- *The ways to place n non-identical balls in m identical boxes where no box is empty equal S(n,m)*
- *The ways to place n* non-identical balls in m **non-identical** boxes, where no box is empty equal m!S(n, m)
  - Equivalent to permutation of picking n from m non-identical elements with repetition.

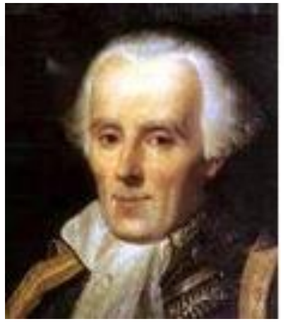
$$G_e(x)=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)^m=(e^x-1)^m=\sum_{h=0}^{m}C(m,h)(-1)^h e^{(m-h)x}$$

$$e^{(m-h)x}=1+\frac{m-h}{1!}x+\frac{(m-h)^2}{2!}x^2+\frac{(m-h)^3}{3!}x^3+\dots=\sum_{n=0}^{\infty}\frac{(m-h)^n}{n!}x^n$$

$$G_e(x)=\sum_{h=0}^{m}C(m,h)(-1)^h\sum_{n=0}^{\infty}\frac{(m-h)^n}{n!}x^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{h=0}^{m}C(m,h)(-1)^h(m-h)^n$$

$$m!S(n,m)=\sum_{h=0}^{m}C(m,h)(-1)^h(m-h)^n$$

Generating Function



Pierre-Simon Laplace
1812

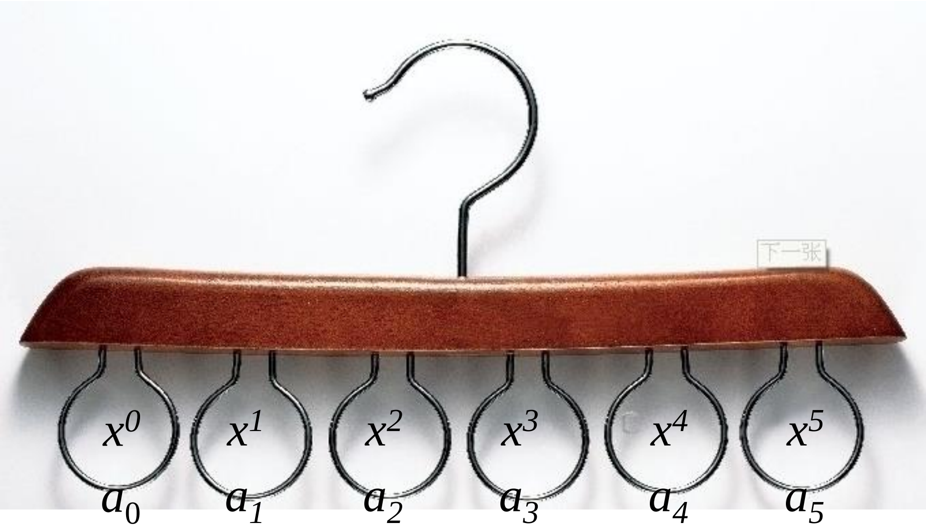
Definition: For a sequence a_0, a_1, a_2…, construct a function

$$G(x)= a_0+a_1x+a_2x^2+\ldots,$$

G(x) is the generating function of a_0, a_1, a_2…sequence

A generating function is a clothesline on which we hang up a sequence of numbers.

— Herbert Wilf



Linear recurrence relation with constant coefficients

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0,$$

$$\boxed{C(x) = x^k + C_1 x^{k-1} + \cdots + C_{k-1} x + C_k = 0}$$

Distinct non-zero real roots

$$a_n = l_1 a_1^n + l_2 a_2^n + \cdots + l_k a_k^n$$

Conjugate complex roots

$$a_n = A\rho^n \cos n\theta + B\rho^n \sin n\theta$$

Real multiple roots

$$(A_0 + A_1 n + \cdots + A_{k-1} n^{k-1})\alpha_1^n$$ **or**

$$(A_0 + A_1\binom{n}{1} + A_2\binom{n}{2} + A_3\binom{n}{3} + \ldots + A_{k-1}\binom{n}{k-1})\alpha_1^n$$

- Generating Function

Combinations

For a seq. $a_0, a_1, a_2, \cdots$, construct:

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots,$$

G(x) is known as the generating function of $a_0, a_1, a_2, \cdots$

- Exponential Generating Function

Permutations

For a seq. $a_0, a_1, a_2, \cdots$, construct:

$$G_e(x) = a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \cdots + \frac{a_k}{k!}x^k + \cdots$$

G(x) is known as the exponential generating function of $a_0, a_1, a_2, \cdots$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \dots}$$

Is the exponential generating function of {1,1,…,1}

Summary of placing balls into boxes problem

With n balls and m boxes, 8 different problems could be constructed by changing constraints on the identity of boxes, balls and whether a box is allowed to be empty ($2^3=8$).

- n non-identical balls， m non-identical boxes， empty box allowed
- n non-identical balls， m non-identical boxes， empty box not allowed
- n non-identical balls， m identical boxes， empty box allowed
- n non-identical balls， m identical boxes， empty box not allowed
- n identical balls， m non-identical boxes， empty box allowed
- n identical balls， m non-identical boxes， empty box not allowed
- n identical balls， m identical boxes， empty box allowed
- n identical balls， m identical boxes， empty box not allowed

?