6. Analogy with eigenvalue-eigenvector problems

To describe a function $v(x)$, one needs to give infinitely many numbers, namely its values at all the different input x-values. Thus $v(x)$ is like a vector of infinite length.

The linear differential operator $\frac{d^2}{dx^2}$ maps each function to a function, just as a 2×2 matrix defines a linear transformation mapping each vector in $\mathbb{R}^2$ to another vector in $\mathbb{R}^2$. Thus $\frac{d^2}{dx^2}$ is like an $\infty \times \infty$ matrix.

The ODE $\frac{d^2}{dx^2} v = \lambda v$ (with boundary conditions) amounts to an infinite system of equations: the ODE consists of one equality of numbers at each x in the interval $(0, \pi)$, and boundary conditions are equalities at the endpoints. Thus the ODE with boundary conditions is like a system of equations $A\mathbf{v} = \lambda\mathbf{v}$. Nonzero solutions $v(x)$ to $\frac{d^2}{dx^2} v = \lambda v$ exist only for special values of λ, namely

$$\lambda = -1, \ -4, \ -9, \ \ldots,$$

just as $A\mathbf{v} = \lambda\mathbf{v}$ has a nonzero solution $\mathbf{v}$ only for special values of λ, namely the eigenvalues of A. But the differential operator $\frac{d^2}{dx^2}$ has infinitely many eigenvalues, as one would expect for an $\infty \times \infty$ matrix.

The nonzero solutions $v(x)$ to $\frac{d^2}{dx^2} v = \lambda v$ satisfying the boundary conditions are called **eigenfunctions** , since they act like eigenvectors.

Summary of the analogies:

Eigenvector Problem	Eigenfunction Problem

vector $\mathbf{v}$	function $v(x)$
$n \times n$ matrix A	the linear operator $\frac{d^2}{dx^2}$
eigenvalue-eigenvector problem	boundary value problem
$A\mathbf{v} = \lambda\mathbf{v}$	$\frac{d^2}{dx^2}v = \lambda v$ for $0 < x < \pi$, $v(0) = 0$, $v(\pi) = 0$
no more than n eigenvalues λ	eigenvalues $\lambda = -1, -4, -9, \ldots$
no more than n eigenvectors $\mathbf{v}$	eigenfunctions $v(x) = \sin(\sqrt{-\lambda}x), \; \lambda = -1, -4, -9, \ldots$

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