Math 135, Spring 2012: HW 7

Problem 1 (p. 234 #2)**.** SOLUTION. Let $N =$ the number of raisins per cookie. If N is a Poisson random variable with parameter λ, then

$$P(N \geq 1) = 1 - P(N = 0) = 1 - \exp(-\lambda)$$

and for this to be at least 0.99, we need

$$\lambda \geq -\ln(0.01)$$

Recall that $\lambda = E[N]$ is the average number of raisins per cookie.

Problem 2 (p. 234 #5)**.** SOLUTION. If we assume that no two microbes lie on top of one another and they are evenly smeared across the entire plate, then we may assume that the microbes are distributed across the plate according to a Poisson random scatter. The number N of microbes in the viewing field is a Poisson random variable with mean $\mu = 5000 \times 10^{-4} = 0.5$. So the probability of at least one microbe is

$$P(N \geq 1) = 1 - P(N = 0) = 1 - e^{-0.5}$$

Problem 3 (p. 234 #11)**.** SOLUTION. Since X, Y, Z are independent Poisson random variables, we know that $X + Y$ is a Poisson random variable with mean $E[X] + E[Y] = 2$, and $X + Y + Z$ is a Poisson random variable with mean 3. This gives

a) $P(X + Y = 4) = \frac{e^{-2}2^4}{4!}$;

b) $E\left[(X+Y)^2\right] = V\left[(X+Y)\right] + \left(E\left[X+Y\right]\right)^2 = 2 + 4 = 6$

c) $P(X + Y + Z = 4) = \frac{e^{-3}3^4}{4!}$

Problem 4 (p. 235 #16)**.** SOLUTION. For part (a), if we assume the number of chips is distribution according to a Poisson scatter through a given volume of cookie dough, and if we define N as the number of chips in 3 cubic inches of dough, then N is a Poisson random variable with mean $\mu = 6$ chips. So

$$P(N \leq 4) = \sum_{k=0}^{4} \frac{e^{-6}6^k}{k!}$$

For part (b), let M_1, M_2, M_3 be the number of marshmallows in cookies 1, 2 and 3; and let C_1, C_2, and C_3 be the number of chips in cookies 1, 2, and 3. By assumption M_i and C_i, where $i = 1, 2$ or 3, are independent Poisson random variables, and the parameter for M_1 is 2; the parameter for M_2 and M_3 is 3; the parameter for C_1 is 4; the parameter for C_2 and C_3 is 6. Now, since the sums of independent Poisson random variables is also Poisson, we know that $N_i = M_i + C_i$ are independent Poisson random variables with parameters that are simply the sums of the parameters for M_i and C_i.

$$\begin{aligned}
&P(\text{at most one cookie has no marshmallows and no chips}) \\
&= P(N_i > 0 \text{ for all i}) + P(N_1 = 0, N_2 > 0, N_3 > 0) + P(N_1 > 0, N_2 = 0, N_3 > 0) \\
&\quad + P(N_1 > 0, N_2 > 0, N_3 = 0) \\
&= (1 - e^{-6})(1 - e^{-9})^2 + e^{-6}(1 - e^{-9})^2 + (1 - e^{-6})e^{-9}(1 - e^{-9}) + (1 - e^{-6})(1 - e^{-9})e^{-9} \\
&= (1 - e^{-9})^2 + 2e^{-9}(1 - e^{-6})(1 - e^{-9})
\end{aligned}$$

Problem 5 (p.276 #4)**.** SOLUTION. Note that c must be the value that guarantees that the integral of the density over the whole real line is 1:

$$\int_0^1 cx^2(1-x)^2dx = 1$$
$$\Rightarrow c = \frac{1}{\int_0^1 x^2(1-x)^2dx}$$

to evaluate this integral, expand the integrand:

$$x^2(1-x)^2 = x^2(1-2x+x^2) = x^2 - 2x^3 + x^4$$

the final evaluation is left to you; you will find that $c = 30$.
Now compute $E(X)$ and $E(X^2)$

$$E(X) = \int_0^1 cx^3(1-x)^2dx = 30\left[\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right] = \frac{1}{2}$$
$$E(X^2) = \int_0^1 cx^4(1-x)^2dx = 30\left[\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right] = \frac{30}{105} = \frac{2}{7}$$

so the variance is $V(X) = E(X^2) - [E(X)]^2 = \frac{1}{28}$.

Problem 6 (p.276 #5)**.** SOLUTION. The graph of f is left to you. To find $P(-1 \le X \le 2)$ we write

$$\int_{-1}^0 \frac{1}{2(1-x)^2}dx + \int_0^2 \frac{1}{2(1+x)^2}dx$$

each of these integrals can be evaluated by substitution: put $u = x - 1$, for instance, and then the first integral reduces to

$$\int_{-2}^{-1} \frac{1}{2}u^{-2}du = 0.5[1 - 0.5] = 0.25.$$

So $P(-1 \le X \le 2) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$. To find $P(|X| > 1)$, note that

$$\begin{aligned} P(|X| > 1) &= 1 - [P(-1 \le X \le 0) + P(0 < X \le 1)] \\ &= 1 - \left[\int_{-1}^0 \frac{1}{2(1-x)^2}dx + \int_0^1 \frac{1}{2(1+x)^2}dx\right] \\ &= 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

Finally, $E(X)$ is not defined because the expectation of $|X|$ is given by an integral which does not converge:

$$\begin{aligned} E(|X|) &= \int_{-\infty}^0 \frac{-x}{2(1-x)^2}dx + \int_0^\infty \frac{x}{2(1+x)^2}dx \\ &= 2\int_0^\infty \frac{x}{2(1+x)^2}dx \end{aligned}$$

To see that the above integral diverges either use integration by parts or substitute $u = 1 + x$.

Problem 7 (p.276 #8)**.** SOLUTION. For part (a), let X_i be the weight of a single lump of metal

$$P\left(11.8 \leq X_i \leq 12.2\right) = P\left(\frac{11.8-12}{1.1} \leq X_i \leq \frac{12.2-12}{1.1}\right)$$
$$= \Phi\left(\frac{12.2-12}{1.1}\right) - \Phi\left(\frac{11.8-12}{1.1}\right)$$

For part (b), we apply the Central Limit Theorem to the sample average $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{100}$. Here, we do *not* have to assume that the individual measurements are normally distributed, because the CLT guarantees that the sample mean will be asymptotically normal for large n regardless of the distribution of the individual random variables X_i, provided the variables X_i are independent, identically distributed, and have finite variance.

$$P(11.8 \leq \bar{X} \leq 12.2) = P\left(\frac{11.8-12}{1.1/\sqrt{100}} \leq \frac{\bar{X}-\mu}{1.1/\sqrt{100}} \leq \frac{12.2-12}{1.1/\sqrt{100}}\right)$$
$$\approx \Phi\left(\frac{12.2-12}{1.1/\sqrt{100}}\right) - \Phi\left(\frac{11.8-12}{1.1/\sqrt{100}}\right)$$

Problem 8 (p. 276 #12)**.** SOLUTION. Let X be the x coordinate, randomly selected from the region in parts (a), (b), (c), and (d). Let also A denote the area of the region.

(a) First note that if $-2 \leq a < b \leq 0$, then

$$\begin{aligned} P[a \leq X \leq b] &= \frac{1}{A}\int_a^b (x+2) - (-x-2)\,dx \\ &= \frac{1}{A}\int_a^b (2x+4)\,dx. \end{aligned}$$

If, on the other hand, $0 \leq a < b \leq 2$

$$\begin{aligned} P[a \leq X \leq b] &= \frac{1}{A}\int_a^b (-x+2) - (x-2)\,dx \\ &= \frac{1}{A}\int_a^b (-2x+4)\,dx. \end{aligned}$$

Therefore, X has density $f_X(x)$ given by

$$\begin{aligned} f_X(x) &= \frac{1}{A}\begin{cases} 2x+4 & \text{if } -2 \leq x \leq 0 \\ -2x+4 & \text{if } 0 < x \leq 2. \end{cases} \\ &= \begin{cases} \frac{x}{4}+\frac{1}{2} & \text{if } -2 \leq x \leq 0 \\ -\frac{x}{4}+\frac{1}{2} & \text{if } 0 < x \leq 2. \end{cases} \end{aligned}$$

(b) In a similar fashion, for $-2 \leq a < b \leq 0$ we see that

$$P[a \leq X \leq b] = \frac{1}{A}\int_a^b x+2\,dx$$

and if $0 \leq a < b \leq 1$

$$P[a \leq X \leq b] = \frac{1}{A} \int_a^b -2x + 2\, dx$$

Therefore the density $f_X(x)$ is precisely

$$\begin{aligned} f_X(x) &= \frac{1}{A} \begin{cases} x + 2 & \text{if } -2 \leq x \leq 0 \\ -2x + 2 & \text{if } 0 < x \leq 1 \end{cases} \\ &= \begin{cases} \frac{x}{3} + \frac{2}{3} & \text{if } -2 \leq x \leq 0 \\ -\frac{2}{3}x + \frac{2}{3} & \text{if } 0 < x \leq 1 \end{cases} \end{aligned}$$

(c) For $-1 \leq a < b \leq 0$

$$\begin{aligned} P[a \leq X \leq b] &= \frac{1}{A} \int_a^b (2x + 2) - (-\frac{1}{2}x - \frac{1}{2})\, dx \\ &= \frac{1}{A} \int_a^b \frac{5}{2}x + \frac{5}{2}\, dx, \end{aligned}$$

for $0 \leq a < b \leq 1$,

$$\begin{aligned} P[a \leq X \leq b] &= \frac{1}{A} \int_a^b (-\frac{x}{2} + 2) - (-\frac{1}{2}x - \frac{1}{2})\, dx \\ &= \frac{1}{A} \int_a^b \frac{5}{2}\, dx, \end{aligned}$$

and for $1 \leq a < b \leq 2$

$$\begin{aligned} P[a \leq X \leq b] &= \frac{1}{A} \int_a^b (-\frac{x}{2} + 2) - (2x - 3)\, dx \\ &= \frac{1}{A} \int_a^b -\frac{5}{2}x + 5\, dx, \end{aligned}$$

Hence

$$\begin{aligned} f_X(x) &= \frac{1}{A} \begin{cases} \frac{5}{2}x + \frac{5}{2} & \text{if } -1 \leq x < 0 \\ \frac{5}{2} & \text{if } 0 \leq x < 1 \\ -\frac{5}{2}x + 5 & \text{if } 1 \leq x \leq 2 \end{cases} \\ &= \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{if } -1 \leq x < 0 \\ \frac{1}{2} & \text{if } 0 \leq x < 1 \\ -\frac{1}{2}x + 1 & \text{if } 1 \leq x \leq 2 \end{cases} \end{aligned}$$

Problem 9 (p.293 #1)**.** SOLUTION: Let $I_i(t)$ be the indicator of the event that the i^{th} atom is still present at time t. So for part (a) we have $P(I_1(t) = 1) = e^{-\lambda t}$, where λ is the rate at which an atom dies. Since the half-life is one year, $e^{-1\lambda} = \frac{1}{2}$ and $\lambda = \log 2$. Therefore $P(I_1(5) = 1) = e^{-(\log 2)5} = (1/2)^5$.

For part (b), suppose we start with n atoms. We want to find t so that:

$$\sum_{i=1}^{n} EI_i(t) = 0.1n$$

Since $EI_i(t) = e^{-(\log 2)t}$ for every i, we wish to solve

$$\begin{aligned} ne^{-(\log 2)t} &= 0.1n \\ t &= \frac{\log 10}{\log 2} \approx 3.32 \text{ years.} \end{aligned}$$

For part (c) we solve for t in:

$$\begin{aligned} \sum_{i=1}^{1024} EI_i(t) &= 1 \\ 1024\, e^{-(\log 2)t} &= 1 \\ t &= 10. \end{aligned}$$

For part (d) we assume that the atoms survive independently, so:

$$\begin{aligned} P(I_1(10) = I_2(10) = \cdots = I_{1024}(10) = 0) &= \prod_{i=1}^{1024} P(I_i(10) = 0) \\ &= [1 - e^{-(\log 2)10}]^{1024} \\ &= [1 - 1/1024]^{1024} \\ &\approx e^{-1} \approx 0.3679 \end{aligned}$$

Problem 10 (p.293 #5)**.** SOLUTION. Let T_4 and T_3 be the times of the fourth and third calls, respectively. We know that the interarrival times in a Poisson arrival process have an exponential distribution, so $T_4 - T_3$ is an exponential random variables with parameter $\lambda = 1$. Thus

$$P(T_4 - T_3 < 2) = 1 - e^{-2}$$

Next, let N_t the number of calls that have occurred by time t. This is a Poisson random variable with parameter $\mu = t$, so

$$P(N_5 \geq 4) = 1 - P(N_5 = 0) - P(N_5 = 1) - P(N_5 = 2) - P(N_5 = 3)$$

and the probability $P(N_5 = k) = \frac{e^{-5}5^k}{k!}$. The time of the fourth call is a sum of the interarrival call times between the first, second, third, and fourth calls, so this has a gamma distribution with shape parameter $r = 4$ and $\lambda = 1$, so $E[T_4] = \frac{4}{1} = 4$.

Problem 11 (p. 293 #6)**.** SOLUTION. The hits recorded by a Geiger counter can be viewed as a Poisson arrival process with rate $\lambda = 1$. In this case, the times between hits are also exponential random variables with parameter $\lambda = 1$, so T_3 has a gamma distribution with $r = 3, \lambda = 1$. You can integrate by parts twice in the following integral to get a numerical answer:

$$P(2 \leq T_3 \leq 4) = \int_2^4 t^2 e^{-t} dt$$

Alternatively (and perhaps more easily), you can write the probability in terms of $N_t \sim$ Poisson(t), the number of arrivals by time t

$$\begin{aligned} P(2 \le T_3 \le 4) &= P(T_3 \le 4) - P(T_3 < 2) \\ &= P(N_4 \ge 3) - P(N_2 \ge 3) \\ &= 1 - e^{-4}[1 + 4 + 4^2/2] - (1 - e^{-2}[1 + 2 + 2^2/2]) \\ &= 5e^{-2} - 13e^{-4} \end{aligned}$$

Problem 12 (p. 294 #9)**.** Consider the gamma function $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$, where $r > 0$.

a) Use integration by parts to show that $\Gamma(r+1) = r\Gamma(r)$.

SOLUTION. $\Gamma(r+1) = \int_0^\infty x^r e^{-x}dx$. Let $U(x) = x^r, dU/dx = rx^{r-1}$. Let $V(x) = -e^{-x}$. Then integration-by-parts yields

$$\begin{aligned} \Gamma(r+1) &= \int_0^\infty x^r e^{-x}dx \\ &= \lim_{b\to\infty} \left[-b^r e^{-b} + 0\right] + \int_0^\infty rx^{r-1}e^{-x}dx \\ &= 0 + r\int_0^\infty x^{r-1}e^{-x}dx \\ &= r\Gamma(r) \end{aligned}$$

b) Deduce from (a) that $\Gamma(r) = (r-1)!$

SOLUTION. This follows immediately from (a) by induction and from the fact that $\Gamma(1) = 1$. Note that $\Gamma(3) = 2\Gamma(2) = 2[1\Gamma(1)]$. So $\Gamma(4) = 3\Gamma(3) = 3 \times 2$, etc.

c) If T has exponential distribution with rate 1, then show that $E[T^n] = n!$ and $SD(T) = 1$.

SOLUTION. Observe that if $f(t)$ is the exponential density with rate 1, then $f(t) = e^{-t}$, and so

$$E[T^n] = \int_o^\infty t^n f(t)dt = \int_0^\infty t^n e^{-t}dt = \Gamma(n+1) = n!$$

by using the results from parts (a) and (b). The variance calculation is straightforward

$$\begin{aligned} V(T) &= E[T^2] - (E[T])^2 \\ &= 2 - 1 = 1 \end{aligned}$$

d) If T has exponential distribution with rate λ, then show λT has exponential distribution with rate 1, hence $E(T^n) = \frac{n!}{\lambda^n}$, and $SD(T) = \frac{1}{\lambda}$.
SOLUTION. Observe that if $\tilde{T} = \lambda T$, then

$$\begin{aligned} P(\tilde{T} > t) &= P(\lambda T > t) \\ &= P(T > t/\lambda) = e^{\left(-\lambda\frac{t}{\lambda}\right)} \\ &= e^{-t} \end{aligned}$$

Therefore, $\lambda^n E[T^n] = E[(\lambda T)^n] = E[\tilde{T}^n] = n!$ by the previous part, because $\tilde{T}$ is exponential with rate 1. The standard deviation again follows easily.

Problem 13 (p. 294, #10)**.** SOLUTION. Let $Z = \text{int}(T)$. Then

$$P(Z \geq k) = P(\text{int}(T) \geq k) = P(T \geq k) = e^{-\lambda k} = \left[e^{-\lambda}\right]^k = q^k$$

where $q = e^{-\lambda}$ is the probability of failure on each trial; therefore $p = 1 - e^{-\lambda}$ is the probability of success on each trial.

Next, suppose T has exponential distribution with parameter λ. Let $T_m = \text{int}(mT)/m$ so

$$\begin{aligned} P(mT_m \geq k) &= P(\text{int}(mT) \geq k) \\ &= P(mT \geq k) = P(T \geq \frac{k}{m}) \\ &= \left[e^{-\lambda/m}\right]^k \\ &= q_m^k \end{aligned}$$

where $q_m = e^{-\lambda/m}$, so $p_m = 1 - e^{-\lambda/m}$. Conversely, suppose mT_m has geometric p_m distribution for all nonnegative integers m. Let $q_m = 1 - p_m$ be the failure probability on any given trial. The key point is to notice the following chain:

$$\begin{aligned} q_m^m &= P(mT_m \geq m) \\ &= P(\text{int}(mT) \geq m) \\ &= P(\text{int}(T) \geq 1) \\ &= P(T_1 \geq 1) = q_1 \end{aligned}$$

and as a result, $\log(q_1) = m \log(q_m)$ and therefore for all m, $m \log(q_m)$ is *independent* of m. Consequently, we can set $\lambda = -m \log(q_m)$. For any rational number, $t \in \mathbb{Q}$, we can write $t = \frac{n}{m}$ for some integers n and m.

$$\begin{aligned} P(T \geq t) &= P(T \geq n/m) \\ &= P(\text{int}(mT) \geq n) \\ &= q_m^n \\ &= \left[e^{\log(q_m)}\right]^n \\ &= \left[e^{m \log(q_m)}\right]^{n/m} \\ &= \left[e^{-\lambda}\right]^t = e^{-\lambda t} \end{aligned}$$

To complete the proof for all nonnegative real numbers t observe that we can approximate any irrational number by rational numbers. That is, take an increasing sequence of rationals r_n and a decreasing sequence of rationals t_n so that $\lim r_n = \lim t_n = t$. So $r_n < t < t_n$ for every n and

$$P(T \geq t_n) < P(T \geq t) < P(T \geq r_n).$$

The probabilities on the left and right converge to $e^{-\lambda t}$, so by the Squeeze Theorem from calculus, $P(T \geq t) = e^{-\lambda t}$.