edX

Abbreviations

Our language L is constructed on the basis of a very restricted set of symbols. But that doesn't mean that we're not allowed to define new notation within L. Recall that we earlier introduced "3" as an abbreviation for "$((1+1)+1)$". In doing so, we didn't really add new symbols to our language. All we did is introduce a notational *shortcut* to make it easier for us to keep track of certain strings of symbols on our original list.

Although abbreviations are avoidable in principle, they are very hard to do without in practice. In the remainder of this section I'll mention a few additional abbreviations that will be useful in proving Gödel's Theorem.

Here are three additions to our logical vocabulary:

The Existential Quantifier Symbol, $\exists$
L can be used to express existential statements of the form "there exists a number such that…". This is because of a happy equivalence between "there exists a number such that so-and-so" and "it is not the case that every number is such that it is not the case that so-and-so" is true. (For example, "there exists a prime number" is equivalent to "not every number is not prime".) Accordingly, we can introduce "$\exists x_i$" (read "there exists a number x_i such that') as an abbreviation for "$\neg\forall x_i \neg$".

The Disjunction Symbol, $\vee$
L can be used to express disjunctive statements of the form "either A is the case or B is the case (or both)". This is because of a happy equivalence between "either A is the case or B is the case (or both)" and "it is not the case that (not-A and not-B)". (For example, "every number is even or odd (or both)" is equivalent to "every number is not such that it is both not even and not odd".) Accordingly, we can introduce "$A \vee B$" (read "A or B (or both)") as an abbreviation for "$\neg(\neg A \,\&\, \neg B)$".

The Conditional Symbol, $\supset$
L can be used to express conditional statements of the form "if A then B". This is because of a happy equivalence (within mathematical contexts) between "if A then B" and "either

not-A or B". (For instance, "if a is even, then a is divisible by two" is equivalent to "either a is not even or a is divisible by two".) Accordingly, we can introduce "$A \supset B$" (read "if A then B") as an abbreviation for "$\neg A \vee B$".

We can also use syntactic abbreviations to enrich our arithmetical vocabulary. Suppose, for example, that we wish to introduce the less-than symbol "$<$". Again, we can do so by taking advantage of a happy equivalence. In general, if a and b are natural numbers, a is smaller than b if and only if $b = a + c$, for some natural number c distinct from 0. So "$x_i < x_j$" can be defined as follows:

$$x_i < x_j \ \leftrightarrow_{df} \ \exists x_k \left((x_j = x_i + x_k) \ \& \neg (x_k = 0) \right)$$

Two observations:

1. The symbol "$\leftrightarrow_{df}$" is not part of L. I use it to indicate that the expression to its left is to be treated as a syntactic abbreviation for the expression to its right.

2. You'll notice that I've used letters rather than numbers as variable-indices. These letters should be replaced by numbers when the abbreviation is used as part of a sentence. Any numbers will do, as long as they are distinct from one another, and as long as they do not already occur as indices in the context where the abbreviation is embedded.

Let me mention an additional example. Suppose we want to enrich L with an expression "Prime(x_i)" which is true of x_i if and only if x_i is a prime number. In general, a is prime if and only if: (i) a is greater than 1, and (ii) a is only be the product of b and c if $a = b$ or $a = c$. So "$\text{Prime}\,(x_i)$" can be defined as follows:

$$\text{Prime}\,(x_i) \ \leftrightarrow_{df} \ (1 < x_i) \ \& \ \forall x_j \forall x_k \left((x_i = x_j \times x_k) \supset (x_i = x_j \vee x_i = x_k) \right)$$

I'll ask you to introduce further abbreviations of this kind in the exercises below.

Video Review: Defining Things in L

right.

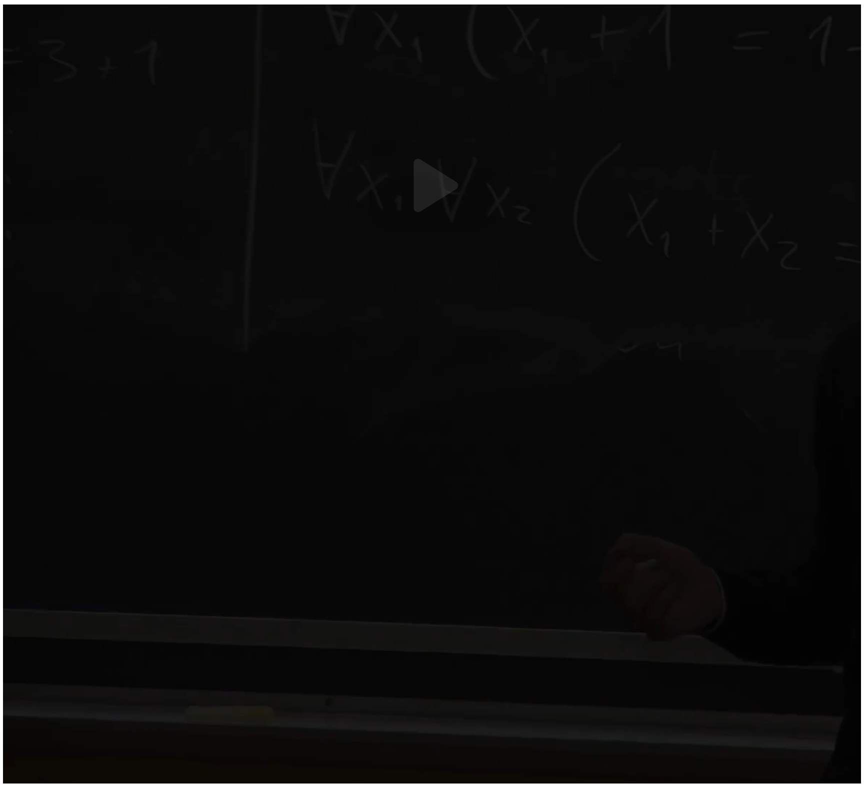
So basically this only works when

you're dealing with mathematical truths, which are, for example,

necessary.

But things can easily go wrong if you

try to formalize the natural language 'if then'



using method.

End of transcript. Skip to the start.

5:34 / 5:34 1.50x

Video

Download video file

Transcripts

Download SubRip (.srt) file

Download Text (.txt) file

Problem 1

5/5 points (ungraded)

The problems below all have at least one answer, but they may have more than one. Make sure you check all right answers. (You may avail yourself of any abbreviations previously defined.)

Introduce an abbreviation "$x_i|x_j$", by finding a formula of L that is true of x_i and x_j if and only if x_i divides x_j, with no remainder.

- [ ] $x_i|x_j \leftrightarrow_{df} (x_j + x_i = 0)$
- [x] $x_i|x_j \leftrightarrow_{df} \exists x_k \, (x_k \times x_i = x_j)$

- [ ] $x_i|x_j \leftrightarrow_{df} \exists x_k\,(x_k \times x_i = x_j + x_k)$

Introduce an abbreviation "$x_i \leq x_j$", by finding a formula of L that is true of x_i and x_j if and only if x_i is smaller than or equal to x_j.

- [ ] $x_i \leq x_j \leftrightarrow_{df} (x_i < x_j \;\&\; x_i = x_j)$
- [ ] $x_i \leq x_j \leftrightarrow_{df} (x_i < x_j \supset x_i = x_j)$
- [x] $x_i \leq x_j \leftrightarrow_{df} (x_i < x_j \vee x_i = x_j)$

Introduce an abbreviation "$\text{Even}\,(x_i)$", by finding a formula of L that is true of x_i if and only if x_i is an even natural number.

- [x] $\text{Even}\,(x_i) \leftrightarrow_{df} \exists x_j\,(x_i = 2 \times x_j)$
- [ ] $\text{Even}\,(x_i) \leftrightarrow_{df} x_i|2$
- [x] $\text{Even}\,(x_i) \leftrightarrow_{df} 2|x_i$

Introduce an abbreviation "$\text{Odd}\,(x_i)$", by finding a formula of L that is true of x_i if and only if x_i is an uneven natural number.

- [x] $\text{Odd}\,(x_i) \leftrightarrow_{df} \neg\,(2|x_i)$
- [x] $\text{Odd}\,(x_i) \leftrightarrow_{df} \exists x_j\,(\text{Even}\,(x_j)\;\&\; x_i = (x_j + 1))$
- [ ] $\text{Odd}\,(x_i) \leftrightarrow_{df} \neg\exists x_j\,(\text{Even}\,(x_j))$

For a given formula $\phi(x_i)$, find an abbreviation "$\exists! x_i (\phi(x_i))$" by finding a sentence of L which is true if and only if there is exactly one number n such that $\phi(n)$ is the case.

- [x] $\exists! x_i (\phi(x_i)) \leftrightarrow_{df} \exists x_i (\phi(x_i)) \ \& \ \forall x_j \forall x_k ((\phi(x_j) \& \phi(x_k)) \supset x_j = x_k)$
- [x] $\exists! x_i (\phi(x_i)) \leftrightarrow_{df} \exists x_i (\phi(x_i) \ \& \ \forall x_j (\phi(x_j) \supset x_j = x_i))$
- [ ] $\exists! x_i (\phi(x_i)) \leftrightarrow_{df} \exists x_i (\phi(x_i)) \ \& \ \neg \exists x_j (\phi(x_j))$

✔

Submit

Answers are displayed within the problem

Problem 2

3/3 points (ungraded)

Express each of the following claims in L. As before, problems all have at least one answer, but they may have more than one. Make sure you check all right answers. (You may avail yourself of any abbreviations previously defined.)

There is at least one prime number.

- [x] $\exists x_1 (\text{Prime}(x_1))$
- [ ] $\exists x_1 (x_1 | x_1 \ \& \ \neg(2 | x_1))$
- [x] $\exists x_1 (1 < x_1 \ \& \ \forall x_2 (x_2 | x_1 \supset (x_2 = 1 \lor x_2 = x_1)))$

✔

Every number is even or odd.

- [x] $\forall x_1 (\text{Even}(x_1) \lor \text{Odd}(x_1))$

- [x] $\forall x_1 \, (2|x_1 \vee \neg \, (2|x_1))$
- [ ] $\forall x_1 \, (2|x_1 \vee x_1|2)$

✔

Every even number greater than two is the sum of two primes. (Goldbach's Conjecture)

- [ ] $\exists x_1((\text{Even}\,(x_1) \;\&\; 2 < x_1) \;\supset\; \forall x_2 \forall x_3 \, (\text{Prime}\,(x_2) \;\&\; \text{Prime}\,(x_3) \;\&\; x_1 = x_2 + x_3)$
- [x] $\forall x_1((\text{Even}\,(x_1) \;\&\; 2 < x_1) \;\supset\; \exists x_2 \exists x_3 \, (\text{Prime}\,(x_2) \;\&\; \text{Prime}\,(x_3) \;\&\; x_1 = x_2 + x_3)$
- [ ] $\forall x_1((\text{Even}\,(x_1) \;\&\; 2 < x_1) \;\supset\; \forall x_2 \forall x_3 \, (\text{Prime}\,(x_2) \;\&\; \text{Prime}\,(x_3) \;\&\; x_1 = x_2 + x_3)$

✔

Submit

Answers are displayed within the problem

Discussion

Topic: Week 10 / Abbreviations

Hide Discussion

Add a Post

Show all posts   by recent activity

? Existential quantifier
Never mind on my previous question about the existential quantifier; he addresses it in this lecture.   1