Hyperplane based Classification: Perceptron and (Intro to) Support Vector Machines

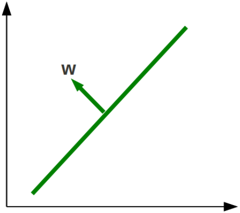
Piyush Rai

CS5350/6350: Machine Learning

September 8, 2011

Hyperplane

- Separates a D-dimensional space into two half-spaces



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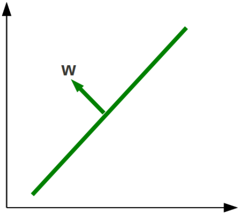

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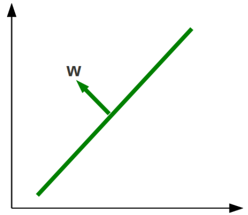

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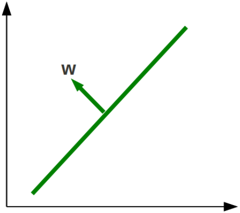

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- $\mathbf{w}$ is orthogonal to any vector lying on the hyperplane
- Assumption: The hyperplane passes through origin. If not,
  - have a *bias* term b; we will then need both $\mathbf{w}$ and b to define it
  - $b > 0$ means moving it parallely along $\mathbf{w}$ ($b < 0$ means in opposite direction)

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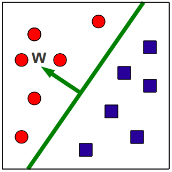
- Linear Classifiers: Represent the decision boundary by a hyperplane $\mathbf{w}$



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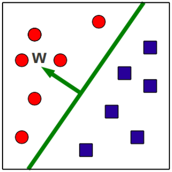

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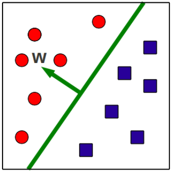

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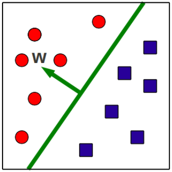

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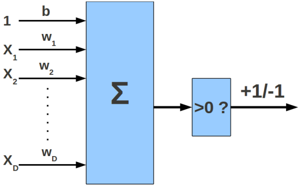

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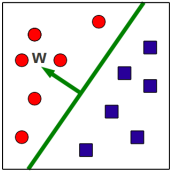

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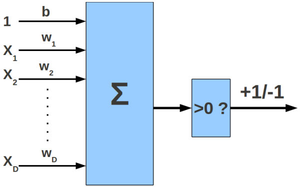

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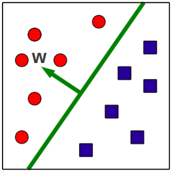

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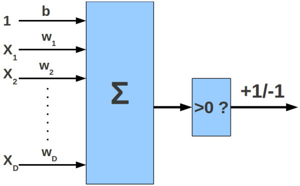
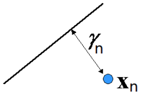


- **Question:** What about the points **x** for which $\mathbf{w}^T\mathbf{x} + b = 0$?
- **Goal:** To learn the hyperplane $(\mathbf{w}, b)$ using the training data

Concept of Margins

- Geometric margin γ_n of an example $\mathbf{x}_n$ is its distance from the hyperplane

$$\gamma_n = \frac{\mathbf{w}^T \mathbf{x}_n + b}{||\mathbf{w}||}$$



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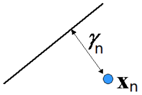

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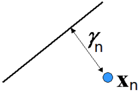

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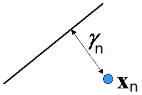

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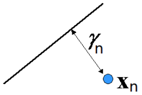

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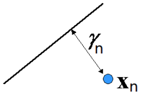

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  - large margin $\Rightarrow$ high confidence

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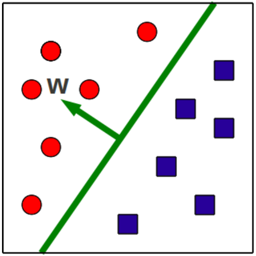

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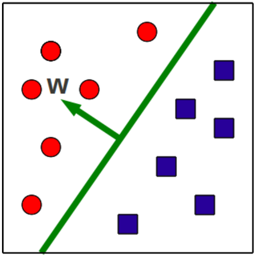

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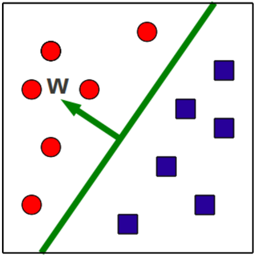

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  - .. or use a *combination* of multiple perceptrons (Neural Networks)

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    - Often batch problems can be solved using online learning!

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- Given: Sequence of N training examples $\{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)\}$
- Initialize: $\mathbf{w} = [0, \dots, 0], b = 0$
- Repeat until convergence:
  - For $n = 1, \dots, N$
    - if $sign(\mathbf{w}^T\mathbf{x}_n + b) \neq y_n$ (i.e., mistake is made)

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  - Completed one pass over the data (each example seen once)

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The Perceptron Algorithm: Formally

- Given: Sequence of N training examples $\{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)\}$
- Initialize: $\mathbf{w} = [0, \ldots, 0], b = 0$
- Repeat until convergence:
  - For $n = 1, \ldots, N$
    - if $sign(\mathbf{w}^T\mathbf{x}_n + b) \neq y_n$ (i.e., mistake is made)

$$\begin{aligned} \mathbf{w} &= \mathbf{w} + y_n\mathbf{x}_n \\ b &= b + y_n \end{aligned}$$

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- Note: $sign(\mathbf{w}^T\mathbf{x}_n + b) \neq y_n$ is equivalent to $y_n(\mathbf{w}^T\mathbf{x}_n + b) < 0$

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- Let's look at a misclassified positive example ($y_n = +1$)
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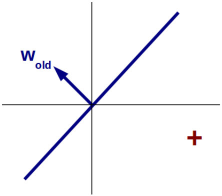
$$\mathbf{w}_{new} = \mathbf{w}_{old} + \mathbf{x}$$



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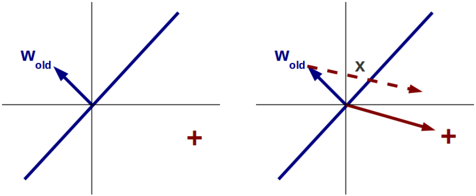

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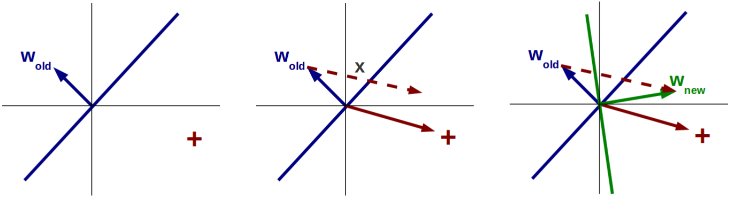

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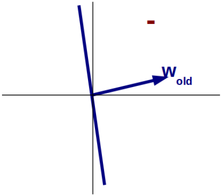

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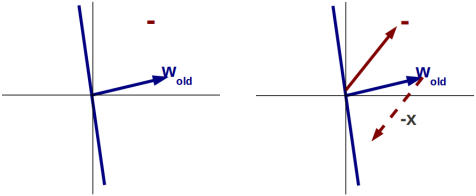

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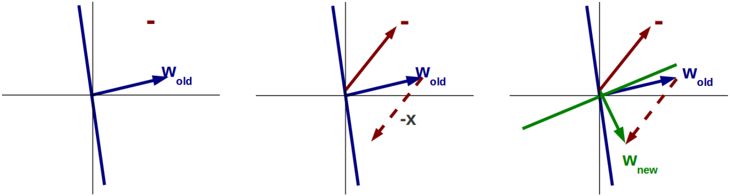

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- Using (1), (2), and $||\mathbf{w}_*|| = 1$, we get $k\gamma < \mathbf{w}_{k+1}^T \mathbf{w}_* \leq ||\mathbf{w}_{k+1}|| \leq R\sqrt{k}$

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- Repeating iteratively k times, we get $\mathbf{w}_{k+1}^T \mathbf{w}_* > k\gamma$ (1)
- $||\mathbf{w}_{k+1}||^2 = ||\mathbf{w}_k||^2 + 2y_n \mathbf{w}_k^T \mathbf{x}_n + ||\mathbf{x}||^2 \leq ||\mathbf{w}_k||^2 + R^2$ (since $y_n \mathbf{w}_k^T \mathbf{x}_n \leq 0$)
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- Using (1), (2), and $||\mathbf{w}_*|| = 1$, we get $k\gamma < \mathbf{w}_{k+1}^T \mathbf{w}_* \leq ||\mathbf{w}_{k+1}|| \leq R\sqrt{k}$

$$\boxed{k \leq R^2/\gamma^2}$$

Convergence of Perceptron

Theorem (Block & Novikoff): If the training data is linearly separable with margin γ by a unit norm hyperplane $\mathbf{w}_*$ ($||\mathbf{w}_*|| = 1$) with $b = 0$, then perceptron converges after R^2/γ^2 mistakes during training (assuming $||\mathbf{x}|| < R$ for all $\mathbf{x}$).
Proof:

- Margin of $\mathbf{w}_*$ on any *arbitrary example* $(\mathbf{x}_n, y_n)$: $\frac{y_n \mathbf{w}_*^T \mathbf{x}_n}{||\mathbf{w}_*||} = y_n \mathbf{w}_*^T \mathbf{x}_n \geq \gamma$
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Nice Thing: Convergence rate does not depend on the number of training examples N or the data dimensionality D. Depends only on the margin!!!

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- The Perceptron loss function (without any regularization on $\mathbf{w}$):

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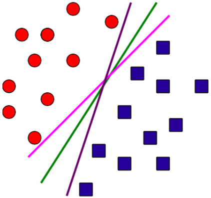

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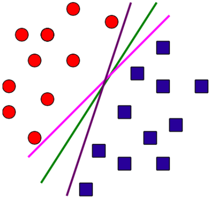

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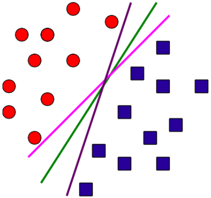

- Intuitively, we want the hyperplane having the maximum margin
- Large margin leads to good generalization on the test data
  - We will see this formally when we cover Learning Theory

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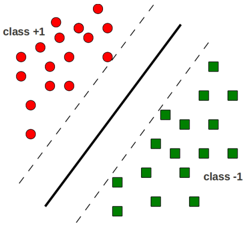
- *Additionally* uses the Maximum Margin Principle
  - Finds the hyperplane with *maximum separation margin* on the training data



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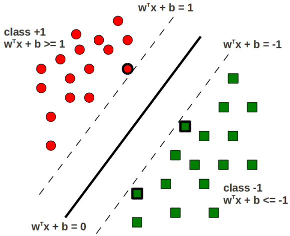

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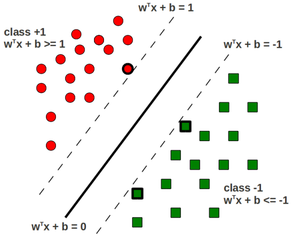

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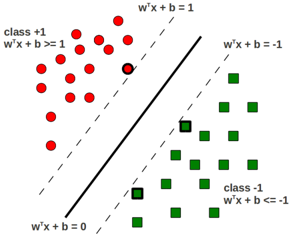

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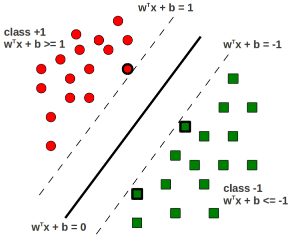

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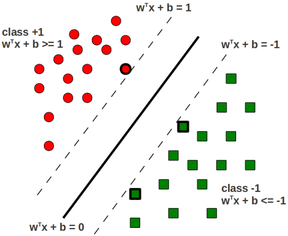

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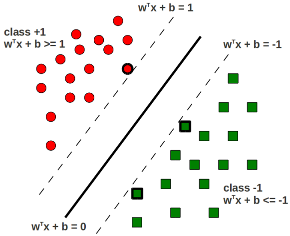

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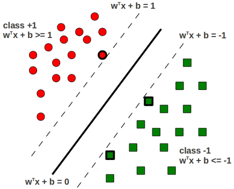

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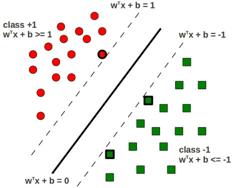

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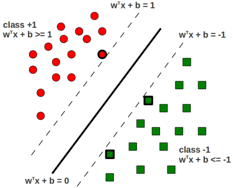

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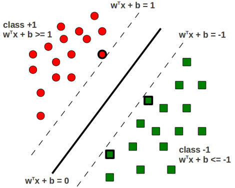

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- This is a Quadratic Program (QP) with N linear inequality constraints

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- Large margin $\Rightarrow$ small $||\mathbf{w}||$
- Small $||\mathbf{w}||$ $\Rightarrow$ regularized/simple solutions (w_i's don't become too large)
- Simple solutions $\Rightarrow$ good generalization on test data

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- Large margin $\Rightarrow$ small $||\mathbf{w}||$
- Small $||\mathbf{w}||$ $\Rightarrow$ regularized/simple solutions (w_i's don't become too large)
- Simple solutions $\Rightarrow$ good generalization on test data
- Want to see an even more formal justification? :-)

Large Margin = Good Generalization

- Large margins intuitively mean good generalization
- We can give a slightly more formal justification to this
- Recall: Margin $\gamma = \frac{1}{||\mathbf{w}||}$
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- Simple solutions $\Rightarrow$ good generalization on test data
- Want to see an even more formal justification? :-)
  - Wait until we cover Learning Theory!

Solving the SVM Optimization Problem

- Our optimization problem is:

$$\boxed{\begin{array}{ll} \text{Minimize} \;\; f(\mathbf{w}, b) = \dfrac{||\mathbf{w}||^2}{2} & \\ \text{subject to} \;\;\; 1 \leq y_n(\mathbf{w}^T\mathbf{x}_n + b), & \qquad n = 1, \ldots, N \end{array}}$$

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- Introducing Lagrange Multipliers α_n $(n = \{1, \dots, N\})$, one for each constraint, leads to the **Lagrangian**:

$$\boxed{\begin{array}{l} \text{Minimize} \quad L(\mathbf{w}, b, \alpha) = \dfrac{||\mathbf{w}||^2}{2} + \displaystyle\sum_{n=1}^{N} \alpha_n \{1 - y_n(\mathbf{w}^T \mathbf{x}_n + b)\} \\ \text{subject to} \quad \alpha_n \ge 0; \quad n = 1, \dots, N \end{array}}$$

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- We can now solve this Lagrangian
  - i.e., optimize $L(\mathbf{w}, b, \alpha)$ w.r.t. $\mathbf{w}$, b, and α
  - .. making use of the Lagrangian Duality theory..

Next class..

- Solving the SVM optimization problem
- Allowing misclassified training examples (**non-zero loss**)
- Introduction to kernel methods (nonlinear SVMs)