Exercise: Shannon Information Content

(3/3 points)

I have an integer in mind, uniformly distributed between 0 and 127. You can keep guessing what my number is until you get it right (and each time you guess, I tell you whether you got it right). Each time you guess a number wrong, you discard that number so as to not guess it again.

For example, if you guess wrong the first time, then the Shannon information content of guessing wrong is

$$\log_2 \frac{1}{\frac{127}{128}} = \log_2 \frac{128}{127} = 0.0113\ldots \text{ bits.}$$

If you guess wrong the second time, then the Shannon information content of the second guess is

$$\log_2 \frac{1}{\frac{126}{127}} = \log_2 \frac{127}{126} = 0.0114\ldots \text{ bits.}$$

If you guess right on the third time, then the Shannon information content of the third guess is

$$\log_2 \frac{1}{\frac{1}{126}} = \log_2 \frac{126}{1} = 6.9772\ldots \text{ bits.}$$

- For the above example where the third guess is right, what is the sum of the Shannon information content of the three guesses? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

7

✔ **Answer:** 7

- Suppose you guessed right only after 5 tries. What is the sum of the Shannon information content of these 5 guesses? (Please be precise with at least 3 decimal places, unless of course the answer doesn't need that many decimal places. You could also put a fraction.)

7

✔ **Answer:** 7

- Suppose you guess right after k tries ($k \in \{1, 2, \ldots, 128\}$). If you sum up the Shannon information content for all the guesses up and including the one in which you guess right, does this total number of bits depend on k? (While we aren't asking for you to justify your answer, we encourage you to be able to do so! For example, if your answer is "Yes" then you should be able to come up with two specific cases for two different number of tries before guessing right that yield different number of total bits, and if your answer is "No" then you should be able to show why the total number of bits gained is always the same.)

○ Yes

No ✔

Solution:

- For the above example where the third guess is right, what is the sum of the Shannon information content of the three guesses?

  Solution:

$$\begin{aligned}
&\log_2 \frac{1}{\frac{127}{128}} + \log_2 \frac{1}{\frac{126}{127}} + \log_2 \frac{1}{\frac{1}{126}} \\
&= \log_2 \frac{128}{127} + \log_2 \frac{127}{126} + \log_2 126 \\
&= \log_2 128 - \log_2 127 + \log_2 127 - \log_2 126 + \log_2 126 \\
&= \log_2 128 \\
&= \boxed{7 \text{ bits}}.
\end{aligned}$$

- Suppose you guessed right only after 5 tries. What is the sum of the Shannon information content of these 5 guesses?

  Solution:

$$\begin{aligned}
&\log_2 \frac{1}{\frac{127}{128}} + \log_2 \frac{1}{\frac{126}{127}} + \log_2 \frac{1}{\frac{125}{126}} + \log_2 \frac{1}{\frac{124}{125}} + \log_2 \frac{1}{\frac{1}{124}} \\
&= \log_2 \frac{128}{127} + \log_2 \frac{127}{126} + \log_2 \frac{126}{125} + \log_2 \frac{125}{124} + \log_2 124 \\
&= \log_2 128 - \log_2 127 + \log_2 127 - \log_2 126 + \log_2 126 \\
&\quad - \log_2 125 + \log_2 125 - \log_2 124 + \log_2 124 \\
&= \log_2 128 \\
&= \boxed{7 \text{ bits}}.
\end{aligned}$$

- Suppose you guess right after k tries ($k \in \{1, 2, \dots, 128\}$). If you sum up the Shannon information content for all the guesses up and including the one in which you guess right, does this total number of bits depend on k?

  Solution: The answer is no. The previous two parts provide a clue: the sum we're computing is a telescoping sum where all the terms cancel out except for the first one: $\log_2 128 = 7$.

  In general, if we guess right after k tries, then the total amount of information “learned" is:

$$
\begin{aligned}
&\left[\sum_{i=1}^{k-1} \log_2 \frac{1}{\frac{128-i}{128-(i-1)}}\right] + \log_2(128-(k-1)) \\
&= \left[\sum_{i=1}^{k-1} \log_2 \frac{128-(i-1)}{128-i}\right] + \log_2(128-(k-1)) \\
&= \left[\sum_{i=1}^{k-1} \log_2(128-(i-1)) - \log_2(128-i)\right] + \log_2(128-(k-1)) \\
&= \log_2 128 - \log_2 127 + \log_2 127 - \cdots \\
&\quad - \log_2(128-(k-1)) + \log_2(128-(k-1)) \\
&= \log_2 128 \\
&= 7 \text{ bits.}
\end{aligned}
$$

Put another way, 7 bits of information are needed before you know the number, and with wrong guesses, you learn very few bits of information (although as the number of possibilities shrinks, wrong guesses provide more and more bits of information).

You have used 1 of 5 submissions