Chapter 9

Misc topics

Contents (class version)

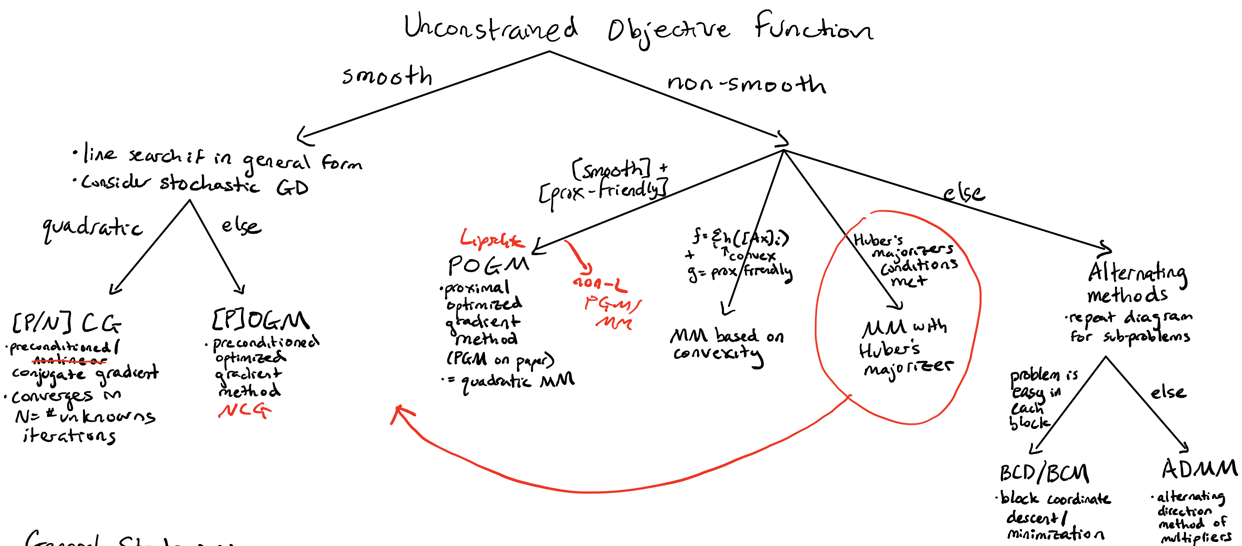
9.0 Review

Final course notes on gdrive.

Two principal kinds of questions.

- **Procedural**. Given $\Psi(\boldsymbol{x})$, given algorithm (by name), then implement/analyze it
  Interesting Exam2 sample questions (by pdf file page)
  - p1: composite with $\|\boldsymbol{x}\|_1$ and box constraint
  - p20: LS + hinge regularizer
  - p24: sigmoid as smooth 0-1 loss
  - p43: prox for leaky ReLU
  - p44: hinge + 0-norm
- **Conceptual**. Given application, then determine $\Psi(\boldsymbol{x})$, choose algorithm, and implement/analyze it
  - p2: $\|\boldsymbol{x}\|_0$ constraint
  - p6: L+S
  - p45: compressed sensing with transform/analysis sparsity

Caroline's review diagram



General Strategies:

- Rewrite as a joint cost function $\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}$
- Write constraints using the characteristic function
- Rewrite to make convex, e.g., $\|x\|_1 = u + v$
  or linear: $\|x\|_\infty < t \Rightarrow -t\mathbf{1} \le x \le t\mathbf{1}$

- Things never appearing here:
  - PSD / PGD → use [P] OGM?
  - FGM → use OGM
  - FPGM → use POGM

Binary classifier design: Review

General cost function for binary classifier design for $\mathrm{sign}(\boldsymbol{v}'\boldsymbol{x})$ with $\boldsymbol{x} \in \mathbb{R}^N$, $\boldsymbol{A} \in \mathbb{R}^{M \times N}$, $\beta \geq 0$:

$$\Psi(\boldsymbol{x}) = f(\boldsymbol{x}) + \beta \|\boldsymbol{x}\|_p^p, \quad f(\boldsymbol{x}) = \mathbf{1}_M' h.(\boldsymbol{A}\boldsymbol{x}), \quad p \in \{1, 2\}$$

Quadratic

$$h(t) = \frac{1}{2}(t-1)^2 \Longrightarrow f(\boldsymbol{x}) =$$

Method for $p = 2$?

A: GD B: CG C: OGM D: POGM E: ADMM **??**

Other options?

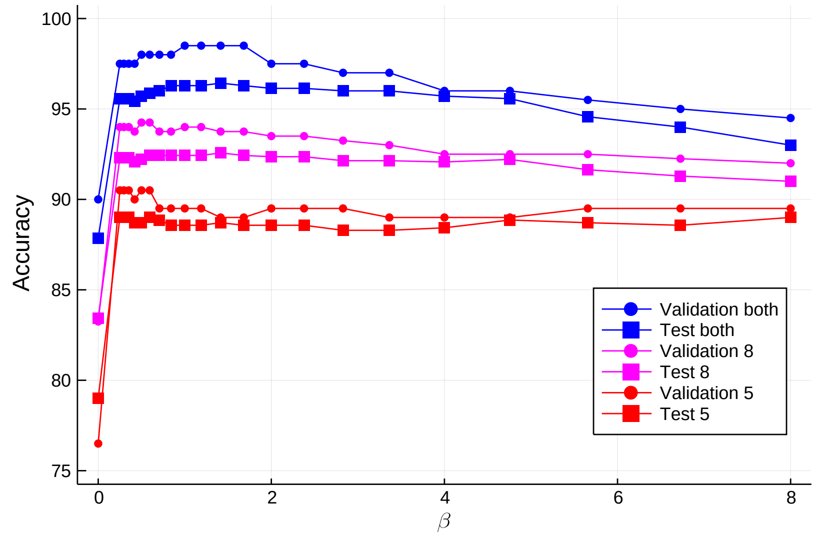
Method for $p = 1$?

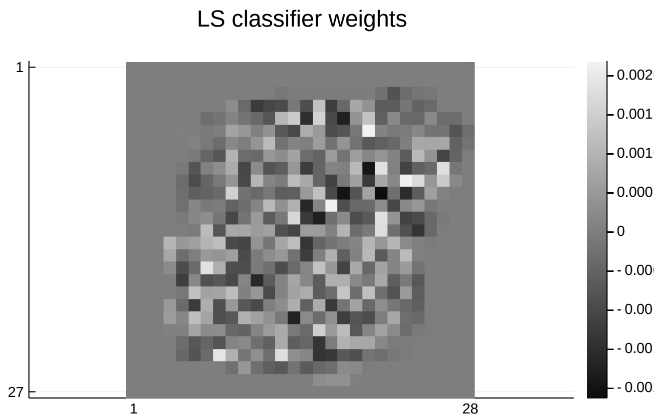
A: GD B: CG C: OGM D: POGM E: ADMM **??**

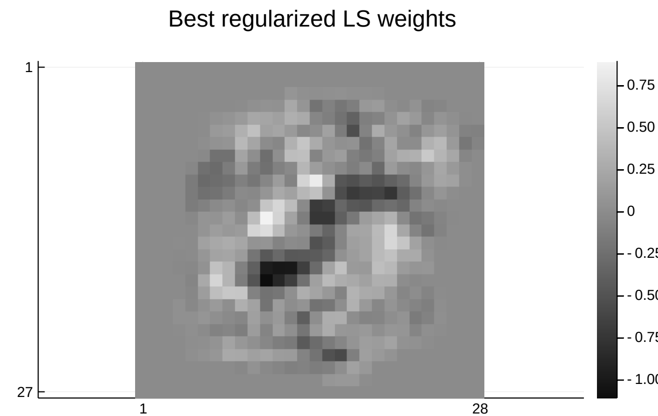
Other options?

Example

Same 5/8 handwritten digit data as in HW




LS classifier weights
1
27
1
28
0.002
0.001
0.001
0.000
0
- 0.00
- 0.00
- 0.00
- 0.00


Best regularized LS weights
1
27
1
28
0.75
0.50
0.25
0
- 0.25
- 0.50
- 0.75
- 1.00

Logistic / Huber hinge

Method for $p = 2$?

A: GD B: CG C: OGM D: POGM E: ADMM $\boxed{\textbf{??}}$

Other options?

Method for $p = 1$?

A: GD B: CG C: OGM D: POGM E: ADMM $\boxed{\textbf{??}}$

Other options?

Hinge

Method for $p = 2$?
A: GD B: CG C: OGM D: POGM E: ADMM $\boxed{??}$

Other options?

Method for $p = 1$?
A: GD B: CG C: OGM D: POGM E: ADMM $\boxed{??}$

Other options?

Sigmoid (smooth version of 0-1) loss

Method for $p = 2$?

A: GD B: CG C: OGM D: POGM E: ADMM $\boxed{\textbf{??}}$

Other options?

Method for $p = 1$?

A: GD B: CG C: OGM D: POGM E: ADMM $\boxed{\textbf{??}}$

Other options?

0-1 loss

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9.1 If time had permitted...

recommender systems

neural networks

parallel computing

semidefinite programming

non-convex regularizers that lead to convex cost functions [9]

relationships of image models and priors for restoration problems [11]

Regularization parameter selection

SURE etc.

CNN training using SURE without ground-truth images [1] [2]

Optimization on manifolds

Minimization subject to constraints like a matrix being unitary (Stiefel manifold) [5–8]

9.2 Towards CNN methods

This section reviews some iterative algorithms based on sparsity models and summarizes how those algorithms provide a foundation for "variational neural networks" when "unrolled."

Review of denoising by soft thresholding

Consider the measurement model $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\varepsilon}$ and the signal model that assumes $\boldsymbol{T}\boldsymbol{x}$ is sparse for some unitary transform $\boldsymbol{T}$. The natural optimization problem for estimating $\boldsymbol{x}$ is

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{x} - \boldsymbol{y}\|_2^2 + \beta \|\boldsymbol{T}\boldsymbol{x}\|_1 .$$

The non-iterative solution is the following **denoising** operation:

$$\hat{\boldsymbol{x}} =$$

The essential ingredients of a **convolutional neural network** (**CNN**) are present in this simple form:

Review of compressed sensing with transform sparsity

For the compressed sensing measurement model $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}$, again assuming $\boldsymbol{T}\boldsymbol{x}$ is sparse for some unitary transform $\boldsymbol{T}$, the natural optimization problem for estimating $\boldsymbol{x}$ is

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 + \beta \|\boldsymbol{T}\boldsymbol{x}\|_1 .$$

In this case there is no closed-form solution and iterative algorithms are needed. The simplest algorithm is the **proximal gradient method** (**PGM**), aka **ISTA**, which is an MM update based on the majorizer

$$\tilde{\boldsymbol{x}}_k = \boldsymbol{x}_k - \frac{1}{L} \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{x}_k - \boldsymbol{y})$$
$$\phi_k(\boldsymbol{x}) = \frac{L}{2} \|\boldsymbol{x} - \tilde{\boldsymbol{x}}_k\|_2^2 + \beta \|\boldsymbol{T}\boldsymbol{x}\|_1 ,$$

where $L = \|\boldsymbol{A}\|_2^2$, for which the minimization step is a **denoising** operation:

$$\boldsymbol{x}_{k+1} = \arg\min_{\boldsymbol{x}} \phi_k(\boldsymbol{x}) =$$

The "unrolled loop" block diagram for this algorithm is the basis for **learned ISTA** (**LISTA**) [3]:

$$\boldsymbol{y} \to \boldsymbol{x}_0 \to \boxed{\text{data}} \to \tilde{\boldsymbol{x}}_0 \to \boxed{\text{denoise}} \to \boldsymbol{x}_1 \to \boxed{\text{data}} \to \tilde{\boldsymbol{x}}_1 \to \boxed{\text{denoise}} \to \boldsymbol{x}_2 \to \boxed{\text{data}} \to \tilde{\boldsymbol{x}}_2 \cdots$$

Denoise options:

Can learn: Early SURE-LET work: [4].

Review of patch transform sparsity

The natural cost function for a patch transform sparse model is

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 + \beta R(\boldsymbol{x}), \quad R(\boldsymbol{x}) = \min_{\boldsymbol{Z}} \sum_{p=1}^{P} \frac{1}{2} \|\boldsymbol{T}\boldsymbol{P}_p\boldsymbol{x} - \boldsymbol{z}_p\|_2^2 + \alpha \|\boldsymbol{z}_p\|_1 .$$

For a BCD approach, the $\boldsymbol{Z}$ update is simple:

$$\boldsymbol{z}_p^{(t+1)} =$$

To better understand the $\boldsymbol{x}$ update, it is helpful to expand the regularizer:

$$\sum_{p=1}^{P} \frac{1}{2} \left\|\boldsymbol{T}\boldsymbol{P}_p\boldsymbol{x} - \boldsymbol{z}_p^{(t+1)}\right\|_2^2 =$$

$$= \frac{1}{2}\boldsymbol{x}'\boldsymbol{H}\boldsymbol{x} - \boldsymbol{x}'\tilde{\boldsymbol{x}}_t + c_1,$$

If $\boldsymbol{T}$ has orthonormal columns (*e.g.*, is unitary), and if the patches have d pixels and are chosen with stride=1 and periodic boundary conditions, then

$$\boldsymbol{H} =$$

Completing the square for the regularizer term yields

$$\ldots = \frac{d}{2}\boldsymbol{x}'\boldsymbol{x} - \boldsymbol{x}'\tilde{\boldsymbol{x}}_t + c_1 = \qquad\qquad \bar{\boldsymbol{x}}_t \triangleq$$

Thus the $\boldsymbol{x}$ update for the **BCD** algorithm is

$$\boldsymbol{x}_{t+1} = \arg\min_{\boldsymbol{x}} \frac{1}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 + \beta \frac{d}{2} \|\boldsymbol{x} - \bar{\boldsymbol{x}}_t\|_2^2 .$$

Here, $\bar{\boldsymbol{x}}_t$ acts like a prior for the update. For some cases there is a closed-form solution (like single-coil Cartesian MRI). Otherwise, one or more iterations are needed.

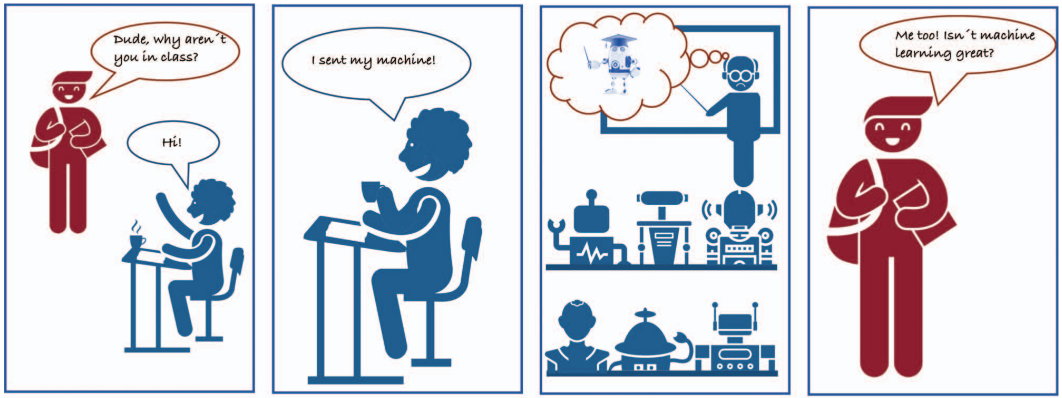
Unrolling the BCD loop in block diagram form:

$$\boldsymbol{y} \to \boldsymbol{x}_0 \to \boxed{\text{denoise}} \to \bar{\boldsymbol{x}}_0 \to \boxed{\text{data}} \to \boldsymbol{x}_1 \to \boxed{\text{denoise}} \to \bar{\boldsymbol{x}}_1 \to \boxed{\text{data}} \to \boldsymbol{x}_2 \to \boxed{\text{denoise}} \to \bar{\boldsymbol{x}}_2 \cdots$$

Denoise options:

Machine-Learning Class

by Robert W. Heath, Jr. and Nuria González-Prelcic



Note: This cartoon was created entirely by real humans, no machine learning was involved with conception or realization to practice.

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