Problem 6: True or False II

(3/3 points)

Determine whether each of the following statement is true (i.e., always true) or false (i.e., not always true).

1. Let X be a random variable that takes values between 0 and c only, for some $c \geq 0$, so that $\mathbf{P}(0 \leq X \leq c) = 1$. Then, $\mathrm{var}(X) \leq c^2/4$.

   True ▾ ✔ **Answer:** True

2. X and Y are continuous random variables. If $X \sim N(\mu, \sigma^2)$ (i.e., normal with mean μ and variance σ^2), $Y = aX + b$, and $a > 0$, then $Y \sim N(a\mu + b, a\sigma^2)$.

   False ▾ ✔ **Answer:** False

3. The expected value of a non-negative continuous random variable X, which is defined by $\mathbf{E}[X] = \int_0^\infty x f_X(x) dx$, also satisfies $\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > t) \mathrm{d}t$.

   True ▾ ✔ **Answer:** True

Answer:

1. The statement is true. Since $0 \le X \le c$,

$$\begin{aligned}\mathbf{E}[X^2] &= \mathbf{E}[XX] \\ &\le \mathbf{E}[cX] \\ &= c\mathbf{E}[X].\end{aligned}$$

   Therefore,

$$\begin{aligned}\mathrm{var}(X) &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\ &\le c\mathbf{E}[X] - (\mathbf{E}[X])^2 \\ &= c^2\left(\frac{\mathbf{E}[X]}{c}\right) - c^2\left(\frac{\mathbf{E}[X]}{c}\right)^2 \\ &= c^2\left(\frac{\mathbf{E}[X]}{c}\left(1 - \frac{\mathbf{E}[X]}{c}\right)\right) \\ &= c^2[\alpha(1-\alpha)] \\ &\le c^2/4,\end{aligned}$$

   where $\alpha = \mathbf{E}[X]/c$. The last inequality is obtained by noticing that the function $\alpha(1-\alpha)$ is largest at $\alpha = 1/2$, where it takes a value of $1/4$.

2. The statement is false. The correct statement is: $Y \sim N(a\mu + b, a^2\sigma^2)$.

3. The statement is true. By performing an interchange of the order of integration, we obtain

$$\begin{aligned}\int_0^\infty \mathbf{P}(X > t)\mathrm{d}t &= \int_0^\infty \int_t^\infty f_X(x)dx\,dt \\ &= \int_0^\infty \int_0^x f_X(x)dt\,dx \\ &= \int_0^\infty x f_X(x)dx \\ &= \mathbf{E}[X].\end{aligned}$$

This result is analogous to the result for discrete random variables that was shown in the Unit 4 solved problem .

You have used 1 of 1 submissions

DISCUSSION

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