1.2 Sample Spaces and Pebble World

Unit 1: Probability and Counting

Adapted from Blitzstein-Hwang Chapter 1.

The mathematical framework for probability is built around *sets*. Imagine that an experiment is performed, resulting in one out of a set of possible outcomes. Before the experiment is performed, it is unknown which outcome will be the result; after, the result "crystallizes" into the actual outcome.

> **Definition 1.2.1 (Sample space and event).**
>
> The *sample space* S of an experiment is the set of all possible outcomes of the experiment. An *event* A is a subset of the sample space S, and we say that A *occurred* if the actual outcome is in A.

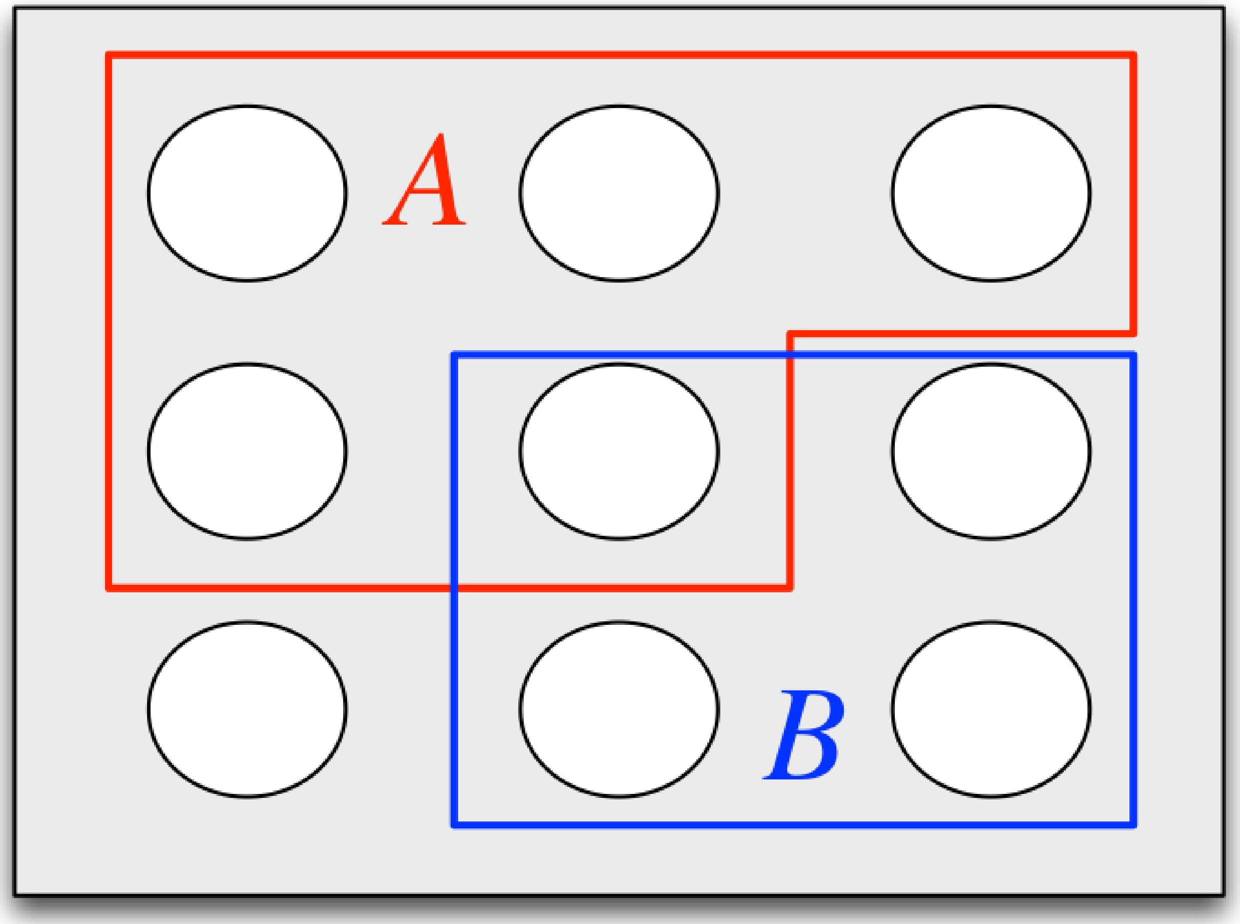

Figure 1.2.2: A sample space as Pebble World, with two events A and B spotlighted.

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Image Description

The sample space of an experiment can be finite or infinite. When the sample space is finite, we can visualize it as *Pebble World*, as shown in Figure 1.2.2. Each pebble represents an outcome, and an event is a set of pebbles. Performing the experiment amounts to randomly selecting one pebble.

Set theory is very useful in probability, since it provides a rich language for expressing and working with events. Set operations, especially unions, intersections, and complements, make it easy to build new events in terms of already-defined events.

> **Note**
>
> This course sometimes uses set theory notation. See the Set Theory Dictionary below if you need a refresher on set theory symbols, such as $\cup$ for union.

For example, let S be the sample space of an experiment and let $A, B \subseteq S$ be events. Then the union $A \cup B$ is the event that occurs if and only if at least one of A and B occurs, the intersection $A \cap B$ is the event that occurs if and only if both A and B occur, and the complement A^c is the event that occurs if and only if A does not occur. We also have *De Morgan's laws*:

$$(A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c,$$

since saying that it is *not* the case that at least one of A and B occur is the same as saying that A does not occur and B does not occur, and saying that it is *not* the case that both occur is the same as saying that at least one does not occur. Analogous results hold for unions and intersections of more than two events.

In the example shown in Figure 1.2.2, A is a set of 5 pebbles, B is a set of 4 pebbles, $A \cup B$ consists of the 8 pebbles in A or B (including the pebble that is in both), $A \cap B$ consists of the pebble that is in both A and B, and A^c consists of the 4 pebbles that are not in A.

Example 1.2.3 (Coin flips).

A coin is flipped 10 times. Writing Heads as H and Tails as T, a possible outcome (pebble) is $HHHTHHTTHT$, and the sample space is the set of all possible strings of length 10 of H's and T's. We can (and will) encode H as 1 and T as 0, so that an outcome is a sequence $(s_1, \ldots, s_{10})$ with $s_j \in \{0, 1\}$, and the sample space is the set of all such sequences. Now let's look at some events:

1. Let A_1 be the event that the first flip is Heads. As a set,

$$A_1 = \{(1, s_2, \ldots, s_{10}) : s_j \in \{0, 1\} \text{ for } 2 \leq j \leq 10\}.$$

This is a subset of the sample space, so it is indeed an event; saying that A_1 occurs is the same thing as saying that the first flip is Heads. Similarly, let A_j be the event that the jth flip is Heads for $j = 2, 3, \ldots, 10$.

2. Let B be the event that at least one flip was Heads. As a set,

$$B = \bigcup_{j=1}^{10} A_j.$$

3. Let C be the event that all the flips were Heads. As a set,

$$C = \bigcap_{j=1}^{10} A_j.$$

4. Let D be the event that there were at least two consecutive Heads. As a set,

$$D = \bigcup_{j=1}^{9} (A_j \cap A_{j+1}).$$

A Set Theory Dictionary

Let S be a sample space and s_{actual} be the actual outcome of the experiment (the pebble that ends up getting chosen when the experiment is performed). A mini-dictionary for converting between English and sets is shown below. For example, for events A and B, the English statement A implies B says that whenever the event A occurs, the event B also occurs; in terms of sets, this translates into saying that A is a subset of B.

Events and occurrences	
English	**Sets**
sample space	S
the empty set (which is an impossible event)	$\emptyset$
s is a possible outcome	$s \in S$
A is an event	$A \subseteq S$
A occurred	$s_{\text{actual}} \in A$
something must happen	$s_{\text{actual}} \in S$

New events from old events	
English	**Sets**
A or B (inclusive)	$A \cup B$
A and B	$A \cap B$
not A	A^c
A or B, but not both	$(A \cap B^c) \cup (A^c \cap B)$
at least one of $A_1, \dots, A_n$	$A_1 \cup \dots \cup A_n$ (also written as $\cup_{j=1}^{n} A_j$)
all of $A_1, \dots, A_n$	$A_1 \cap \dots \cap A_n$ (also written as $\cap_{j=1}^{n} A_j$)

Relationships between events	
English	**Sets**
A implies B	$A \subseteq B$
A and B are mutually exclusive	$A \cap B = \emptyset$
$A_1, \dots, A_n$ are a partition of S	$A_1 \cup \dots \cup A_n = S, A_i \cap A_j = \emptyset$ for $i \neq j$