The Proof

Now for the main event: we will prove $|\mathbb{N}| < |\mathbb{R}|$.

Our Strategy:

We'll focus our attention on the subset $[0, 1)$ of $\mathbb{R}$, which consists of real numbers greater or equal to 0 but smaller than 1. Proving $|\mathbb{N}| < |\,[0, 1)\,|$ is enough for present purposes because we know that $|\,(0, 1]\,| \leq |\mathbb{R}|$, and it follows from exercise of Section that $|\mathbb{N}| < |\,[0, 1)\,|$ and $|\,(0, 1]\,| \leq |\mathbb{R}|$ together entail $|\mathbb{N}| < |\mathbb{R}|$.

In order to prove $|\mathbb{N}| < |\,[0, 1)\,|$, we'll need to verify each of the following two claims:

1. $|\mathbb{N}| \leq |\,[0, 1)\,|$

2. $|\mathbb{N}| \neq |\,[0, 1)\,|$

The first of these claims is totally straightforward. In fact, we've already looked at a proof of it.

So all we need to prove is $|\mathbb{N}| \neq |\,[0, 1)\,|$. We'll do so by using a technique that is sometimes called *reductio ad absurdum* (from the Latin for "reduction to absurdity"). The basic idea is very simple. Suppose you want to prove not-P. You can proceed by assuming its negation (which is not-not-P, or, equivalently, P), and trying to prove a contradiction. If you succeed, you've learned that P is false (since no truth entails a contradiction), and therefore that not-P is true, which is what you wanted to show.

The proof itself:

The claim we want to prove here is $|\mathbb{N}| \neq |\,[0,1)\,|$. So we'll assume $|\mathbb{N}| = |\,[0,1)\,|$, and use it to prove a contradiction. This will show that $|\mathbb{N}| = |\,[0,1)\,|$ is false, and therefore that $|\mathbb{N}| \neq |\,[0,1)\,|$ is true.

$|\mathbb{N}| = |\,[0,1)\,|$ is the claim that there is a bijection f from $\mathbb{N}$ to $[0,1)$. So let us assume that such an f exists. The first thing to note is that f can be used to make a complete *list* of real numbers between 0 and 1. The zeroth member of our list is $f(0)$, the first member of our list is $f(1)$, and so forth, for each natural number. Since f is a bijection, our list must be complete: every member of $[0,1)$ must occur on the list.

Each real number in $[0,1)$ can be represented by a numeral of the form "$0.\delta_0\delta_1\delta_2\ldots$" where each δ_i is a digit between "0" and "9". And since we are excluding duplicate names, each number in $[0,1)$ corresponds to a unique such numeral. The listing of $[0,1)$ that is induced by f can be represented in the following way:

$$
\begin{array}{lcllllll}
f(0) & = & 0. & \boldsymbol{a_0} & a_1 & a_2 & a_3 & a_4 & \cdots \\
f(1) & = & 0. & b_0 & \boldsymbol{b_1} & b_2 & b_3 & b_4 & \cdots \\
f(2) & = & 0. & c_0 & c_1 & \boldsymbol{c_2} & c_3 & c_4 & \cdots \\
f(3) & = & 0. & d_0 & d_1 & d_2 & \boldsymbol{d_3} & d_4 & \cdots \\
f(4) & = & 0. & e_0 & e_1 & e_2 & e_3 & \boldsymbol{e_4} & \cdots \\
\vdots & & & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

We are assuming the existence of a function f, which induces a complete list of real numbers in $[0,1)$.

I have highlighted certain digits using boldface: the zeorth digit of $f(0)$, the first digit of $f(1)$, and, in general, the nth digit of $f(n)$. These digits form the following sequence:

$$\text{diagonal sequence: } a_0, b_1, c_2, d_3, e_4 \ldots$$

We will now transform this sequence of digits into an "evil twin". Let the **evil sequence** be the result of applying the following transformation η to each digit d in the diagonal sequence:

$$\eta(d) = \begin{cases} ``0", \text{ if } d \neq ``0" \\ ``1", \text{ if } d = ``0" \end{cases}$$

And let the **evil number** be the number whose decimal expansion is "0." followed by the digits in the evil diagonal sequence.

Suppose, for example, that function f is as as follows:

$$\begin{array}{ccccccccc}
f(0) & = & 0. & \mathbf{3} & 5 & 7 & 0 & 1 & \cdots \\
f(1) & = & 0. & 4 & \mathbf{0} & 7 & 3 & 4 & \cdots \\
f(2) & = & 0. & 1 & 0 & \mathbf{1} & 1 & 1 & \cdots \\
f(3) & = & 0. & 6 & 2 & 8 & \mathbf{0} & 9 & \cdots \\
f(4) & = & 0. & 2 & 7 & 7 & 5 & \mathbf{4} & \cdots \\
\vdots & & & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

An example of the list of real numbers in $[0, 1)$ that might be induced by f.

Then we have the following:

$$\begin{array}{rcl} \text{diagonal sequence:} & & 3,\ 0,\ 1,\ 0,\ 4, \ldots \\ \text{evil sequence:} & & 0,\ 1,\ 0,\ 1,\ 0, \ldots \\ \text{evil number:} & 0\ . & 0\ 1\ 0\ 1\ 0\ \ldots \end{array}$$

Regardless of what the diagonal sequence turns out to be, the evil number will always be greater than or equal to 0, and smaller than 1. So it will always be a number in $[0, 1)$.

The climax of our proof is the observation that even though the evil number is in $[0, 1)$, it cannot appear anywhere on our list.

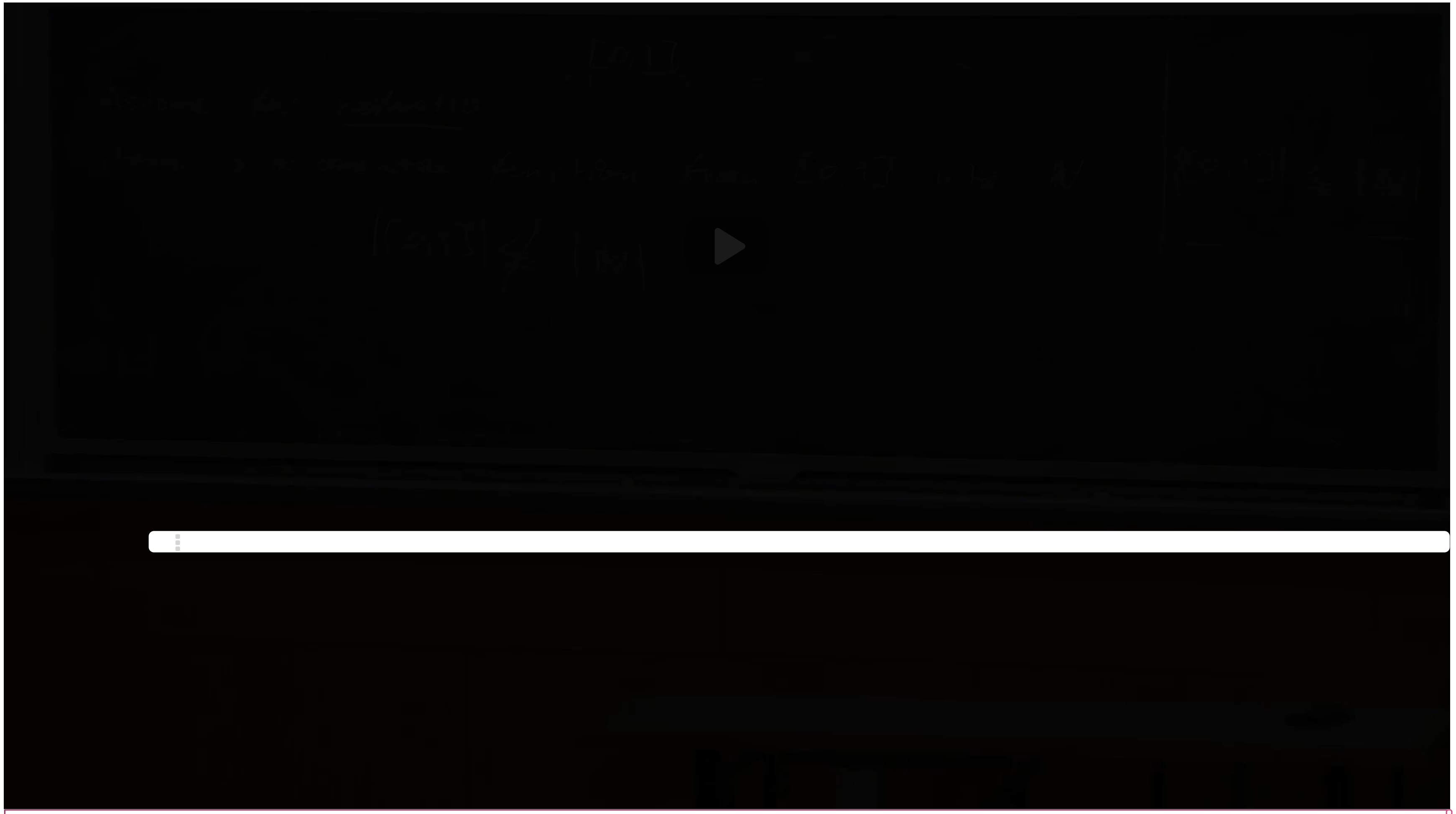
To see this, note that the evil number's zeroth digit differs from the zeroth digit of $f(0)$'s decimal expansion, the evil number's first digit differs from the first digit of $f(1)$'s decimal expansion, and so forth. (In general, the evil number's kth digit differs from the kth digit of $f(k)$'s decimal expansion.) So: *the evil number is distinct from every number on our list*.

We have reached our contradiction.

We began by assuming that $|\mathbb{N}| = |\,[0, 1)\,|$. It follows from this assumption that we can generate a complete list of real numbers in $[0, 1)$. But we have seen that the evil number is an element of $[0, 1)$ that cannot be on that list, contradicting our earlier claim that the list is complete. Since $|\mathbb{N}| = |\,[0, 1)\,|$ entails a contradiction, it is false. So $|\mathbb{N}| \neq |\,[0, 1)\,|$ is true, which is what we wanted to prove. The infinite size of the real numbers is *bigger* than the infinite size of the natural numbers.

Amazing!

Video Review: There are More Reals than Naturals



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Prove $|\mathbb{N}|<|\mathbb{R}|$: Existence of a function f which induces a complete list of reals in [0, 1)?
Would a function f to induce said list not be actually producing an -enumerable- list of reals? What are we actually trying to say when we call it a -complete- list of reals?
6

Diagonal and evil numbers
Could be elaborate on the need for creating 'diagonal' number and 'evil' number? They seem to be abstract concepts!
2

Why can't we append the evil number (or an infinite number of evil numbers) to the list?
I understand that we assume the list to be complete before we generate the evil number, but isn't this the situation as Hilbert's Hotel - specifically, whereby we retain the abili...
4

Why doesn't this map all of the reals from [0,1) to the natural numbers?
6

Real numbers in [0,1) can't be mapped to real numbers in [0,1)?
If we define a function f that maps any real number in [0,1) to itself: 0.a1a2a3a4a5... -> 0.a1a2a3a4a5 0.b1b2b3b4b5... -> 0.b1b2b3b4b5... ... We can yet apply the Cantor's diag...
4