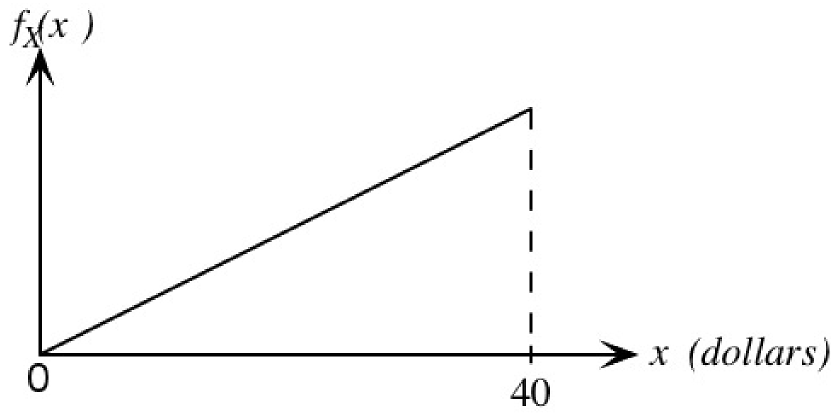
Problem 4: Paul goes to the casino

(7/7 points)

Paul is vacationing in Monte Carlo. On any given night, he takes X dollars to the casino and returns with Y dollars. The random variable X has the PDF shown in the figure. Conditional on $X = x$, the continuous random variable Y is uniformly distributed between zero and $2x$.



1. Determine the joint PDF $f_{X,Y}(x, y)$.

If $0 < x < 40$ and $0 < y < 2x$,

$f_{X,Y}(x,y) =$ 1/1600 ✔ **Answer:** 0.000625

If $y < 0$ or $y > 2x$,

$f_{X,Y}(x,y) =$ 0 ✔ **Answer:** 0

2. On any particular night, Paul makes a profit of $Z = Y - X$ dollars. Find the probability that Paul makes a positive profit (i.e., $\mathbf{P}(Z > 0)$):

1/2 ✔ **Answer:** 0.5

3. Find the PDF of Z. Express your answers in terms of z using standard notation . *Hint:* Start by finding $f_{Z|X}(z\,|\,x)$.

If $0 < z < 40$, $f_Z(z) =$ 1/40-z/1600 ✔ **Answer:** (40-z)/1600

If $-40 < z < 0$, $f_Z(z) =$ 1/40+z/1600 ✔ **Answer:** (40+z)/1600

If $z < -40$ or $z > 40$, $f_Z(z) =$ 0 ✔ **Answer:** 0

4. What is $\mathbf{E}[Z]$?

$\mathbf{E}[Z] =$ 0 Answer: 0

Answer:

1. By the multiplication rule, $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x)$. We have $f_X(x) = ax$, as shown in the figure. Furthermore,

$$1 \;=\; \int_0^{40} ax\,dx \;=\; 800a.$$

Hence, $f_X(x) = x/800$. From the problem statement, $f_{Y|X}(y \mid x) = 1/(2x)$ if $0 \le y \le 2x$. Therefore,

$$f_{X,Y}(x,y) = \begin{cases} 1/1600, & \text{if } 0 \le x \le 40 \text{ and } 0 \le y \le 2x, \\ 0, & \text{otherwise.} \end{cases}$$

2. Paul makes a positive profit if and only if $Y > X$. This occurs with probability

$$\mathbf{P}(Y > X) \;=\; \int_{-\infty}^{\infty} \int_{x}^{\infty} f_{X,Y}(x,y)\,dy\,dx \;=\; \int_0^{40} \int_x^{2x} \frac{1}{1600}\,dy\,dx \;=\; \frac{1}{2}.$$

We could have also arrived at this answer by realizing that for each possible value of X, there is a $1/2$ probability that $Y > X$, and therefore by the total probability theorem,

$$\begin{aligned}\mathbf{P}(Y > X) &= \int_0^{40} \mathbf{P}(Y > X|X = x) f_X(x)\mathrm{d}x \\ &= \int_0^{40} \frac{1}{2} f_X(x)\mathrm{d}x \\ &= \frac{1}{2}.\end{aligned}$$

3. The joint PDF of X and Z satisfies $f_{X,Z}(x,z) = f_X(x)\, f_{Z|X}(z|x)$. Given $X = x$, Y is uniformly distributed on $[0, 2x]$, which implies that $Y - x$ is uniformly distributed on $[-x, x]$. Thus, given $X = x$, $Z = Y - X$ is uniformly distributed on $[-x, x]$. Hence, $f_{Z|X}(z \mid x) = \frac{1}{2x}$ for $-x \le z \le x$.

   Therefore, $f_{X,Z}(x,z) = (x/800) \cdot (1/2x) = 1/1600$ for $0 \le x \le 40$ and $-x \le z \le x$ and is 0 elsewhere. Graphically, on the x-z plane, the joint PDF is a constant $1/1600$ on the triangle with vertices at $(0,0)$, $(40,-40)$, and $(40,40)$.

   To calculate the marginal PDF $f_Z(z)$, we integrate the joint PDF over x. The joint PDF as found above is nonzero only for z between -40 and 40. For z in this range, the joint PDF is nonzero when x is between $|z|$ and 40. Therefore,

$$f_Z(z) \;=\; \int_{-\infty}^{+\infty} f_{X,Z}(x,z)\,dx \;=\; \int_{|z|}^{40} \frac{1}{1600}\,dx \;=\; \frac{40-|z|}{1600}, \;\; \text{if } |z| < 40,$$

   and $f_Z(z) = 0$ if $|z| > 40$.

4.

We observe that $\mathbf{E}[Y|X=x]=x$ for any $x \in [0,40]$. Thus, using the total expectation theorem,

$$\begin{aligned}\mathbf{E}[Y] &= \int_0^{40} \mathbf{E}[Y|X=x] f_X(x)dx \\ &= \int_0^{40} x f_X(x)dx \\ &= \mathbf{E}[X].\end{aligned}$$

We conclude that $\mathbf{E}[Z]=\mathbf{E}[Y]-\mathbf{E}[X]=0$.

You have used 3 of 3 submissions

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