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7.

Setup:

Suppose you observe an i.i.d. sample X_1, \ldots, X_n of a Bernoulli random variable X with unknown parameter $\theta_0 = \mathbf{P}(X=1)$.

We use the Bayesian approach to statistical inference and model the unknown parameter θ_0 as a random variable Θ (defined jointly with X). That is, let the conditional distribution of X given Θ be

$$X|\Theta \sim \mathsf{Ber}\left(\Theta
ight)$$

and let

$$\pi_{\Theta}\left(heta
ight)=3 heta^{2}\mathbf{1}\left(heta\in\left[0,1
ight]
ight)$$

be the prior distribution of Θ .

Let ${\cal P}$ denote the probabilities implied by the Bayesian model.

Useful Facts:

The $\mathsf{Beta}\left(lpha,eta
ight)$ distribution has pdf

$$f\left(t
ight)=rac{t^{lpha-1}(1-t)^{eta-1}}{B\left(lpha,eta
ight)} \quad ext{where} \quad B\left(lpha,eta
ight)=\int_{0}^{1}t^{lpha-1}(1-t)^{eta-1}\;dt,$$

and

$$egin{array}{lll} {
m Mean} & = & rac{lpha}{lpha+eta} \ & {
m Variance} & = & rac{lphaeta}{\left(lpha+eta
ight)^2\left(lpha+eta+1
ight)} \ & {
m Mode} = {
m argmax}f & = & rac{lpha-1}{lpha+eta-2} \ {
m for} \ lpha, eta>1. \end{array}$$

The **mode** is the value of the random variable at which the pdf attains its **maximum**.

Posterior Distribution

3.0/3.0 points (graded)

The posterior distribution $\pi_{\Theta|X_1,...,X_n}$ of Θ given the sample $X_1,...,X_n$ is a Beta distribution $\operatorname{Beta}(a,b)$. Specify the parameters a and b below.

(Enter **barX_n** for \overline{X}_n .)

$$b =$$
 n-n*barX_n +1 \checkmark Answer: n +1 - n*barX_n

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You have used 1 of 3 attempts

• Answers are displayed within the problem

Bayes Estimator and its Limit

2/2 points (graded)

What is the mean $\hat{ heta}_n^{\mathrm{Bayes}}$ of the posterior distribution in terms of \overline{X}_n ?

(Enter \mathbf{barX}_n for \overline{X}_n).

$$\hat{ heta}^{\mathrm{Bayes}} = \mathbb{E}\left[\Theta | \vec{X}_n \right] = \boxed{ (\mathsf{n*barX_n} + 3)/(\mathsf{n+4}) }$$
 \tag{Answer: (3+ n*barX_n)/(n + 4)

The estimator $\hat{ heta}_n^{(\pi)}$ converges in probability to a constant as $n o \infty$? What is this limiting constant?

(Enter in terms of the true parameter θ_0 .)

$$\hat{ heta}_n^{(\pi)} \stackrel{ ext{in } \mathbb{P}}{\longrightarrow} \boxed{ ext{theta_0}}$$
 theta_0

STANDARD NOTATION

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1 Answers are displayed within the problem

MAP Estimator

1/1 point (graded)

What is the maximum a posteriori (MAP) estimator $\hat{\theta}_n^{\text{MAP}}$ of θ_0 in terms of \overline{X}_n ? The maximum a posteriori (MAP) estimator is the value of θ where the posterior distribution is maximum, i.e. $\hat{\theta}^{\text{MAP}} = \arg\max_{\theta} \pi_{\Theta|\vec{X}_n}(\theta)$.

(Enter **barX_n** for \overline{X}_n).

What is the limit in probability of $\hat{ heta}_n^{ ext{MAP}}$ as $n o \infty$? (There is no answer box for this question.)

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MLE

2/2 points (graded)

What is the MLE $\hat{ heta}^{ ext{MLE}}$ of heta in terms of \overline{X}_n ?

(Enter **barX_n** for the mean \overline{X}_n .)

$$\hat{ heta}^{ ext{MLE}} = egin{bmatrix} ext{barX_n} \ & \checkmark ext{ Answer: barX_n} \ & \times ext{ Answer: barX_n} \ & \times$$

The MLE $\hat{ heta}_n^{
m MLE}$ converges in probability as $n o \infty$ to what constant?

$$\hat{ heta}_n^{ ext{MLE}} \stackrel{ ext{in } \mathbb{P}}{\longrightarrow}$$
 theta_0 $m{ heta}_0$

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You have used 1 of 3 attempts

1 Answers are displayed within the problem

Ungraded Questions for Thought

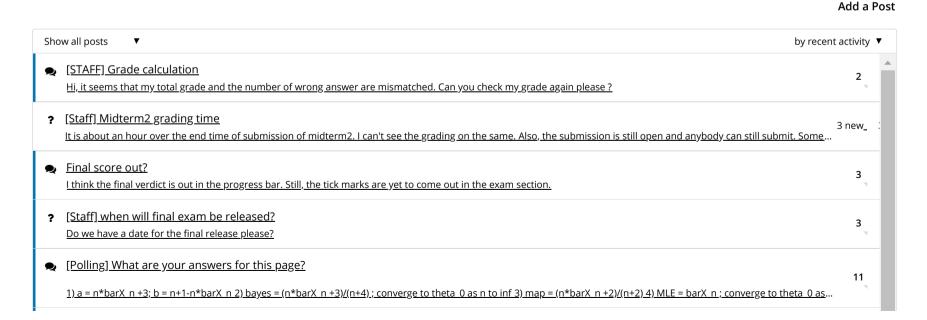
The following questions do not have answer boxes and are for you to explore.

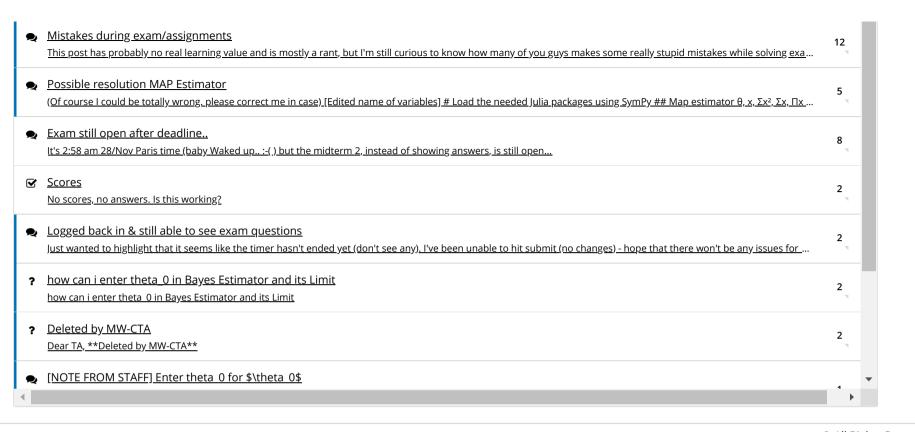
- 1. What is the Fisher information $\mathcal{I}\left(\theta\right)$?
- 2. What is the variance $\mathsf{Var}\left(\Theta|\vec{X}_n\right)$ of the posterior distribution?
- 3. What is the limit of $n {\sf Var}\left(\Theta | \vec{X}_n \right)$ in terms of the Fisher information $\mathcal{I}\left(\theta \right)$?

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