15. Type 2 Error and Power of a Statistical Test

Type 2 Error and Power of a Statistical Test

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Testing the Support of a Uniform Variable: Type 2 Error of a Test

1/1 point (graded)

As on the previous page, let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Unif}\,[0, \theta]$ for an unknown parameter θ and we designed the statistical test

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$$\psi_n = \mathbf{1}(\max_{1 \le i \le n} X_i > 1/2)$$

to decide between the null and alternative hypotheses

$$\begin{aligned} H_0 : \theta &\le 1/2 \\ H_1 : \theta &> 1/2. \end{aligned}$$

Recall from lecture that the **type 2 error (rate)** of a test ψ_n is the **function**

$$\begin{aligned} \beta_{\psi_n} : \Theta_1 &\to \mathbb{R} \\ \theta &\mapsto \mathbf{P}_\theta(\psi_n = 0) \end{aligned}$$

where $\mathbf{P}_\theta(\psi_n = 0)$ is the probability of the event $\psi_n = 0$ under the probability distribution $\mathbf{P}_\theta$ when $\theta \in \Theta_1$, i.e. the probability of not rejecting H_0 when H_1 is true. In this example, the region Θ_1 defining the alternative hypothesis is $(1/2, \infty)$, and $\mathbf{P}_\theta = \mathrm{Unif}[0, \theta]$.

Evaluate $\mathbf{P}_\theta(\psi_n = 0) = \mathbf{P}_\theta\left(\max_{1 \le i \le n} X_i \le 1/2\right)$ at $\theta = 1/2$, the boundary between Θ_0 and Θ_1.

$\mathbf{P}_{\theta=1/2}\left(\max_{1 \le i \le n} X_i \le 1/2\right) =$ 1 ✔ **Answer:** 1

Solution:

$$\begin{aligned} \beta_{\psi_n}(1/2) &= \mathbf{P}_{1/2}(\max_{1 \le i \le n} X_i < 1/2) \\ &= \mathbf{P}_{1/2}(X_1 < 1/2) \ldots \mathbf{P}_{1/2}(X_n < 1/2) \\ &= 1 \times 1 \ldots \times 1 = 1 \end{aligned}$$

where we applied independence of the X_i's in the second line.

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Answers are displayed within the problem

Testing the Support of a Uniform Variable: Type 2 Error of a Test Continued

3/3 points (graded)

As above, let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Unif}[0, \theta]$ for an unknown parameter θ and we designed the statistical test

$$\psi_n = \mathbf{1}(\max_{1 \leq i \leq n} X_i > 1/2)$$

to decide between the null and alternative hypotheses

$$\begin{aligned} H_0 &: \theta \ \leq 1/2 \\ H_1 &: \theta \ > 1/2. \end{aligned}$$

Recall from lecture that the **type 2 error** of a test ψ_n is the **function**

$$\begin{aligned} \beta_{\psi_n} : \Theta_1 &\to [0, 1] \\ \theta &\mapsto \mathbf{P}_\theta\left(\psi_n = 0\right) \end{aligned}$$

where $\mathbf{P}_\theta\left(\psi_n = 0\right)$ is the probability of the event $\psi_n = 0$ under the probability distribution $\mathbf{P}_\theta$ when $\theta \in \Theta_1$, i.e. the probability of not rejecting H_0 when H_1 is true.

In this example, $\Theta_1 = (1/2, \infty)$, and $\mathbf{P}_\theta = \mathrm{Unif}[0, \theta]$.

What is $\beta_{\psi_n}(\theta)$?

$\beta_{\psi_n}(\theta) =$ (1/2/theta)^n ✔ **Answer:** (1/(2*theta))^n

$\left(\frac{1}{2 \cdot \theta}\right)^n$

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Find $\lim_{\theta \to 1/2} \beta_{\psi_n}(\theta)$.

$\lim_{\theta \to 1/2} \beta_{\psi_n}(\theta) =$ 1 ✔ **Answer:** 1

1

Find $\lim_{\theta \to \infty} \beta_{\psi_n}(\theta)$.

$\lim_{\theta \to \infty} \beta_{\psi_n}(\theta) =$ 0 ✔ **Answer:** 0

0

STANDARD NOTATION

Solution:

For any $\theta \in \Theta_1 = [1/2, \infty)$,

$$\begin{aligned} \beta_{\psi_n}(\theta) = \mathbf{P}_\theta(\psi_n = 0) &= \mathbf{P}_\theta\left(\max_{1 \le i \le n} X_i < 1/2\right) \\ &= \mathbf{P}_\theta(X_1 < 1/2) \ldots \mathbf{P}_\theta(X_n < 1/2) = \left(\frac{1/2}{\theta}\right)^n. \end{aligned}$$

As $\theta \to 1/2$,

$$\beta_{\psi_n}(\theta) \to \left(\frac{1/2}{1/2}\right)^n = 1.$$

As $\theta \to \infty$,

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$$\beta_{\psi_n}(\theta) = \left(\frac{1/2}{\theta}\right)^n \to 0.$$

Remark: This test is rather extreme example in that it minimizes type-1 error while maximizing the type-2 error. In general, we want to design tests so that the type-1 and type-2 error are both controlled. These types of trade-offs are crucial to consider in the context of hypothesis testing.

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Testing the Support of a Uniform Variable: : Power of a Test

1/1 point (graded)

The **power** of the test ψ_n is defined to be

$$\pi_{\psi_n} = \inf_{\theta\in\Theta_1}\left(1-\beta_{\psi_n}(\theta)\right).$$

Continuing from the problem above, what is the power π_{ψ_n}?

$\pi_{\psi_n} =$ 0 ✔ **Answer:** 0

Solution:

A priori we have that

$$\pi_{\psi_n} = \inf_{\theta\in[1/2,\infty)}\left(1-P_\theta\left(\psi_n=0\right)\right) = \inf_{\theta\in[1/2,\infty)} P_\theta\left(\psi_n=1\right) \geq 0].$$

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Moreover, we computed above that $\beta_{\psi_n}(1/2) = P_{0.5}[\psi_n = 0] = 1$. Thus,

$$\pi_{\psi_n} = 0.$$

Remark: The power of a test is the largest lower bound on the probability that if H_1 is true, that indeed H_0 is rejected in favor of H_1. In this example, as $\theta \in \Theta_1$ approaches the boundary $1/2$, the probability of rejecting H_0 decreases and approaches 0.

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Answers are displayed within the problem

Testing the Support of a Uniform Variable: Graphing the errors

1/1 point (graded)

As above, let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Unif}[0, \theta]$ for an unknown parameter θ and we designed the statistical test

$$\psi_n = \mathbf{1}(\max_{1 \le i \le n} X_i > 1/2)$$

to decide between the null and alternative hypotheses

$$\begin{aligned} H_0 &: \theta \ \le 1/2 \\ H_1 &: \theta \ > 1/2. \end{aligned}$$

Let $\alpha_{\psi_n}(\theta)$ and $\beta_{\psi_n}(\theta)$ denote the type 1 and type 2 errors respectively.

On the graph below, do the following:

- Place a vertical line at the boundary of Θ_0 and Θ_1 using the **boundary tool**.

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- Sketch the graph of $\mathbf{P}_\theta\left(\psi_n = 1\right)$ as a function of θ using the **probabilty of rejecting null** tool.
- Sketch the graph of the type 1 error $\alpha_{\psi_n}\left(\theta\right)$ on the **correct domain** using the **type 1 error** tool.
- Sketch the graph of the type 2 error $\beta_{\psi_n}\left(\theta\right)$ on the **correct domain** using the **type 2 error** tool.

Note: To use the spline tool for sketching the graphs, click on point on the graph, and the tool will connect these points with a smooth curve.

For each curve, you will be graded on its domain, its limiting values, its value on the boundary between Θ_0 and Θ_1, and its shape and continuity.

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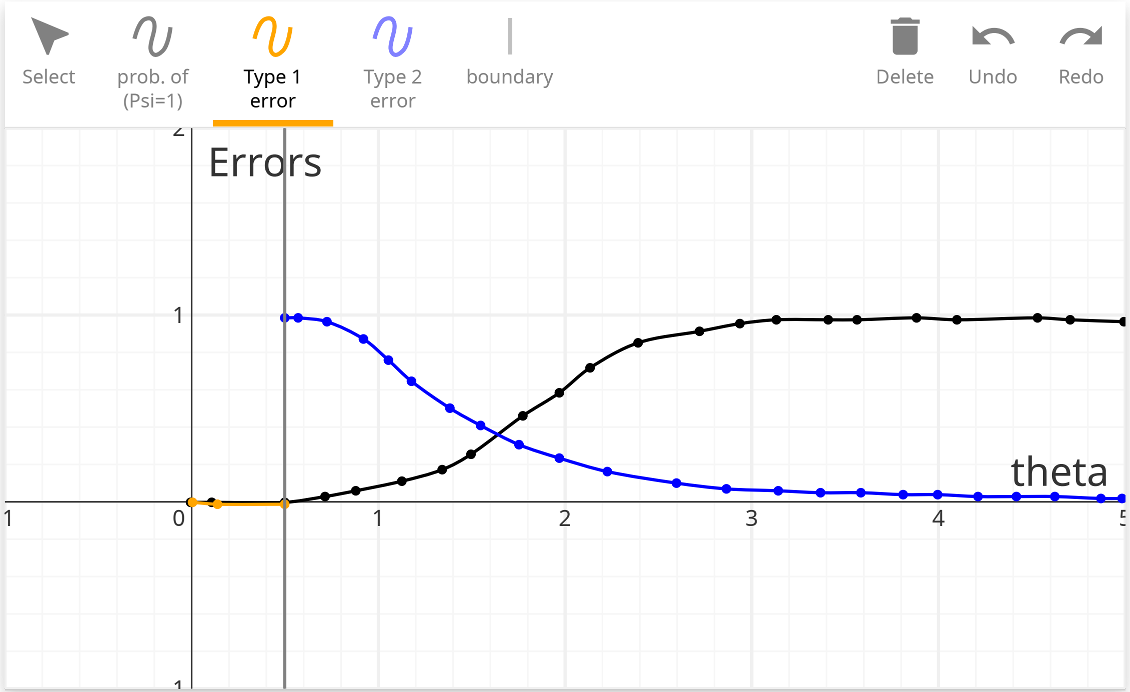

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Answer: See solution.

Good job on the graph of the probability of rejecting the null! Good job on the graph of the type 1 error! Good job on the graph of the type 2 error!

Solution:

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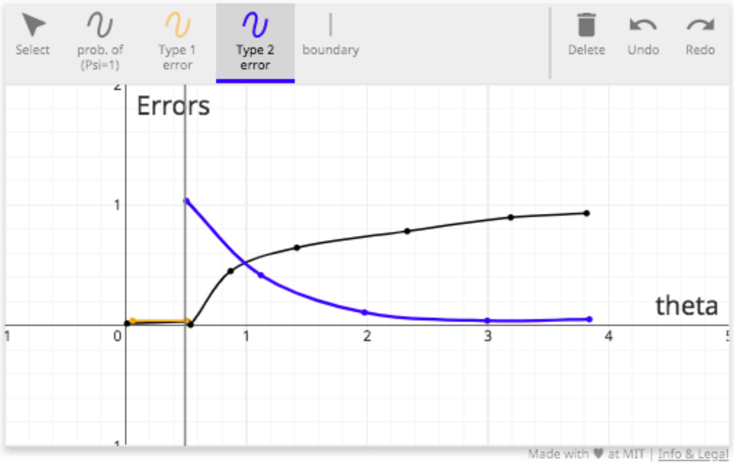

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