9. Let $L_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be linear transformations with

$$L_A\left(\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\right) = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, L_A\left(\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, L_A\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let A be the matrix that represents linear transformation L_A. Compute

$A^{-1} =$

(Hint: it is not necessary to compute A!)

Answer: (Notice that we are happy if you just write down the answer by looking at the question. But here is the reasoning behind the answer. In the video, a different reasoning is given that you may find more intuitive.)

Let X have the property that $XA = I$ (the identity), making X the inverse of A. Let L_X be the linear transformation represented by matrix X. This means that for any vector z

$$L_X(L_A(z)) = z$$

because of the relationship between matrix multiplication and composition of linear transformations.

Now, we would like to compute the columns of X:

$$X = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}.$$

We remember that

$$x_0 = L_X(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}), x_1 = L_X(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}), \text{ and } x_2 = L_X(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}).$$

Let's focus on x_0:

$$x_0 = L_X(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = L_X(L_A(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix})) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Similarly, we notice that we can just make the vectors that are mapped by L_A to the unit basis vectors the columns of the inverse matrix:

$$A^{-1} = X = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$