4. Random walk model

Exercises due Nov 10, 2021 17:29 IST Completed

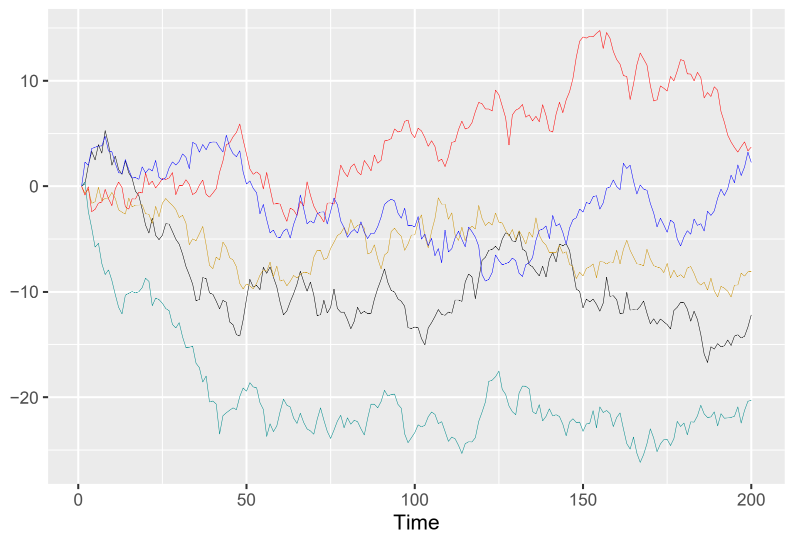
Before investigating the properties and stationarity of the AR (p) model, we consider one very important special case – **the random walk** .

A time series $\{X_t\}_{t\geq 1}$ is a random walk if the value of X_t is obtained from the value of X_{t-1} by adding a random perturbation W_t (white noise) that is independent of (or uncorrelated with) the past history of the series $\{X_s\}_{s<t}$:

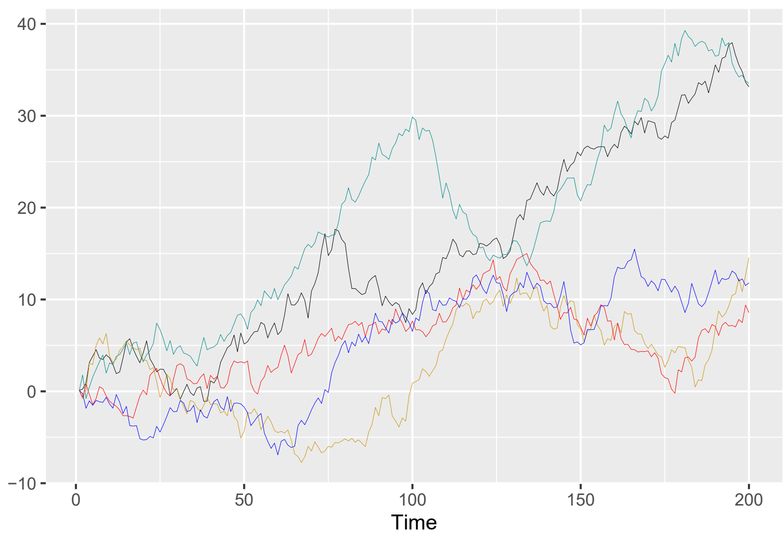
$$X_t = X_{t-1} + W_t,$$

A time series $\{Y_t\}_{t\geq 1}$ is a random walk with drift if it is equal to the sum of a random walk process $\{X_t\}_{t\geq 1}$ (with no drift) with a deterministic linear trend:

$$\begin{aligned} Y_t &= \delta \cdot t + X_t, \\ &= \delta + Y_{t-1} + W_t, \qquad \text{where } Y_{t-1} = \delta\,(t-1) + X_{t-1}. \end{aligned}$$



Random walk



Random Walk with Drift

Coin toss and random walk

5/5 points (graded)

Consider the following random walk model. Let the starting position $X_0 = 0$ be deterministic. At each time t you flip a fair coin and add 1 to X_{t-1} if you see heads H, else add -1 if you see tails T. Suppose the first 9 coin flips result in the sequence HTTTHHHTTT.

Compute the first 10 terms of the time series $\{X_t\}_{t=0}^{9}$. (There is no answer box for this question.)

Find the expected position $\mathbf{E}\left[X_{10}\right]$ at time $t = 10$.

0 ✔ **Answer:** 0

Find the expected position $\mathbf{E}\left[X_{20}\right]$ at time $t = 20$.

0 ✔ **Answer:** 0

Find the variance of the position X_{10} at time $t = 10$.

10 ✔ **Answer:** 10

Find the variance of the position X_{20} at time $t = 20$.

20

✔ **Answer:** 20

Find the forecast $\mathbf{E}[X_{10}|X_9]$.

- ○ 0
- ○ 0.5
- ◉ X_9
- ○ $(X_1 + \ldots + X_9)/9$

✔

Solution:

The series is

$$\{X_0 = 0, X_1 = 1, X_2 = 0, X_3 = -1, X_4 = -2, X_5 = -1, X_6 = 0, X_7 = -1, X_8 = -2, X_9 = -3\}$$

If we let W_t to be the random variable with values ± 1 depending on the outcome of the tth flip of the fair coin (i.e., a Rademacher r.v.), then $X_t = 0 + \sum_{s=1}^{t} W_s$. By linearity of expectations,

$$\mathbf{E}[X_t] = \mathbf{E}[\sum_{s=1}^{t} W_s] = \sum_{s=1}^{t} \mathbf{E}[W_s] = 0.$$

Therefore, $\mathbf{E}[X_{10}] = \mathbf{E}[X_{20}] = 0$. Next, we compute the variance.

$$\begin{aligned}\mathsf{Var}(X_t) &= \mathsf{Var}(\sum_{s=1}^{t} W_s) = \sum_{s=1}^{t} \mathsf{Var}(W_s) + \sum_{t \neq s} \mathsf{Cov}(W_t, W_s) \\ &= t \cdot 1 + 0.\end{aligned}$$

Therefore, $\mathsf{Var}(X_{10}) = 10$ and $\mathsf{Var}(X_{20}) = 20$, so the position of the random walk at $t = 20$ is more uncertain than the position of the random walk at $t = 10$. This means that the random walk is not stationary.

Finally, the (best in the sense of smallest quadratic risk) prediction of X_{10} given the value of X_9 is

$$\begin{aligned}\mathbf{E}[X_{10}|X_9] &= \mathbf{E}\Big[\sum_{s=1}^{10} W_s \mid \sum_{s=1}^{9} W_s\Big] \\ &= \mathbf{E}\Big[\sum_{s=1}^{9} W_s \mid \sum_{s=1}^{9} W_s\Big] + \mathbf{E}\Big[W_{10} \mid \sum_{s=1}^{9} W_s\Big] \\ &= \sum_{s=1}^{9} W_s + 0 = X_9.\end{aligned}$$

In words, the best prediction of the future position of a random walk is the present position of the random walk.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

Biased coin toss and random walk

6/6 points (graded)

Continuing with the setup of the previous problem, suppose now that the coin you flip is biased with $\mathbf{P}(\mathrm{H}) = \frac{3}{4}$ and $\mathbf{P}(\mathrm{T}) = \frac{1}{4}$.

Find the expected position $\mathbf{E}[X_{10}]$ at time $t = 10$.

5

✔ **Answer:** 5

Find the expected position $\mathbf{E}[X_{20}]$ at time $t = 20$.

10

✔ **Answer:** 10

Find the variance of the position X_{10} at time $t = 10$.

15/2

✔ **Answer:** 7.5

Find the variance of the position X_{20} at time $t = 20$.

15

✔ **Answer:** 15

What is the slope of the drift of this random walk?

0.5

✔ **Answer:** 0.5

Find the forecast $\mathbf{E}[X_{10}|X_9]$.

- ○ 0
- ○ 0.5
- ○ X_9
- ○ $(X_1 + \ldots + X_9)/9$
- ● $X_9 + 0.5$
- ○ $(X_1 + \ldots + X_9)/9 + 0.5$

✔

Solution:

Let V_t denote the random variable with values ± 1 depending on the outcome of the tth flip of the biased coin. Then $\mathbf{E}(V_t) = 1 \cdot \frac{3}{4} - 1 \cdot \frac{1}{4} = \frac{1}{2}$ and $\mathrm{Var}(V_t) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.

Continuing with the calculations of the previous problem, $\mathbf{E}[X_t] = \sum_{s=1}^{t} \mathbf{E}[V_s] = t \cdot \frac{1}{2}$. So, $\mathbf{E}[X_{10}] = 5$ and $\mathbf{E}[X_{20}] = 10$.

Next, $\mathrm{Var}(X_t) = \sum_{s=1}^{t} \mathrm{Var}(V_s) = t \cdot \frac{3}{4}$. So, $\mathrm{Var}(X_{10}) = 7.5$ and $\mathrm{Var}(X_{20}) = 15$.

The drift is another name for the trend in the context of a random walk time series. To see that X is a random walk with drift, let $W_t = V_t - \frac{1}{2}$ and note that $\{W_t\}$ is white noise. Then

$$X_t = \sum_{s=1}^{t} V_s = \frac{1}{2}t + \sum_{s=1}^{t} W_s,$$

so the slope of the drift of X is $\frac{1}{2}$, and equal to the bias of the random perturbation at each step.

Finally, the forecast of a future position of the process

$$\mathbf{E}[X_{10}|X_9] = X_9 + \mathbf{E}[W_{10}|X_9] = X_9 + \frac{1}{2}$$

is the present position adjusted for the drift.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

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Calculating variance for biased coin toss and random walk

discussion posted 2 months ago by **kozaronek**

@Staff can you please check whether my submission is graded correctly?

We still don't have to consider correlation here, correct? This means that we can simply calculate the variances at the given time step, which is essentially the variance of a binomial random variable.

I even found this blog post that arrives at the same conclusion, yet my results are graded as incorrect.
https://rstudio-pubs-static.s3.amazonaws.com/133160_a760d80cfd384314bbfd9b40551df822.html

Am I missing something?

This post is visible to everyone.

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4 responses

mtuznik
2 months ago

It's tricky,

Binomial is on a Bernoulli foundation that returns {1, 0}, but our result returns {1, -1}. I used the variance formula:

$Var(X) = \sum(x^2p) - E[X]^2$

$(p - probability)$

(The solution given in the responses is different, but the same in the outcome)

Hint:

if W was Bernoulli, var(X_{10}) = sum of i.i.d. Bernoulli random variables, where var(Bernoulli)=p(1-p). Bernoulli returns {0,1}.

W in the model return {-1,1}, so you need to calculate the variance. It is not p(1-p).

You can use definition of variance in terms of moments: var(X)=E[X^2]-$E[X]^2$

posted 2 months ago by **ababs**

Add a comment

kozaronek
2 months ago

Thanks @ababs & @mtuznik, your hints were very helpful :)

For those who are still a little confused, formulating the variance as such, might be even more clear.

$$Var\,(X_t) = \sum_{i=1}^{t}\left(\sum_{coin}^{2}\left(x_{coin}^2 * p\right) - E[X]^2\right)$$

Where coin = {Head = 1, Tails = -1}

Add a comment

michael_seitz1992
2 months ago

There is also a quick and intuitive way using a Bernoulli random variable: As mentioned before, Bernoulli random variale X returns {0,1}. Using Y = 2*X results in a variable which returns {0,2}. We know that multiplying a constant with a random Variable scales the variance quadratically with the constant: Var(Y) = 2^2*Var(X). Defining Z = Y - 1 results in the variable we are interested in, which returns {-1,1}. However, shifting a variable does not change the variance. Hence: Var(Z) = Var(Y) = ...

Hey Michael. Great way to frame the problem in terms of probabilistic thinking. Thanks for the tip. The fair coin version is called Rademacher distribution.

posted 2 months ago by **jtourkis**

Nice way Mike !

posted 2 months ago by **ncranwell**

Add a comment

ZY-A
2 months ago

Thanks everyone. This was indeed a very helpful post!

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