13. Type 1 Error of a Statistical Test

Type 1 Error of a Statistical Test

Errors

- Rejection region of a test ψ:

$$R_\psi = \{x \in E^n : \psi(x) = 1\}.$$

- Type 1 error of a test ψ (rejecting H_0 when it is actually true):

$$\begin{aligned} \alpha_\psi \ : \ \Theta_0 &\to \mathbb{R} \\ \theta &\mapsto \mathbb{P}_\theta[\psi = 1]. \end{aligned}$$

- Type 2 error of a test ψ (not rejecting H_0 although H_1 is actually true):

$$\begin{aligned} \beta_\psi \ : \ \Theta_1 &\to \mathbb{R} \\ \theta &\mapsto \end{aligned}$$

- Power of a test ψ:

$$\pi_\psi = \inf_{\theta \in \Theta_1} (1 - \beta_\psi(\theta)).$$

(Caption will be displayed when you start playing the video.)

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An Analogy to the U.S. Justice System:Type 1 and Type 2 Errors

3/3 points (graded)

In a criminal court in the U.S., the goal is to decide between the following null and alternative hypotheses:

$$\begin{aligned} H_0 &: \text{The defendant is innocent.}\\ H_1 &: \text{The defendant is guilty.} \end{aligned}$$

In the U.S. criminal justice system, the informal principle “innocent until proven guilty" is the status quo, so this is the rationale for the choice of null hypothesis above. While this example is not, strictly speaking, a statistical hypothesis test, it provides some intuition about the meaning of type 1 and type 2 errors.

Suppose we have a defendant X who will be tried by a jury in the U.S. If guilty, X will go to jail, and otherwise is free to go.

In this example, let's say that the jury makes a **type 1 error** if the suspect satisfies H_0 while the jury rules in favor of H_1. Let's say the jury makes a **type 2 error** if the suspect satisfies H_1 while the jury rules in favor of H_0.

If the jury commits a type 1 error, the defendant is...

○ Innocent in reality, and will walk away free.

○ Guilty in reality, and will go to jail.

● Innocent in reality, but still will go to jail.

○ Guilty in reality, but will walk away free.

✓

If the jury commits a type 2 error, the defendant is...

○ Innocent in reality, and will walk away free.

○ Guilty in reality, and will go to jail.

○ Innocent in reality, but still will go to jail.

● Guilty in reality, but will walk away free.

✓

What strategy could the jurors follow if they wanted to never commit a type 2 error?

○ Always acquit– *i.e.*, always decide that the defendant is innocent.

● Always convict– *i.e.*, always decide that the defendant is guilty.

✓

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✓ Correct (3/3 points)

The Threshold for a Statistical Test

1/1 point (graded)

Continuing from problem on the previous page, let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$ where μ is an unknown parameter. You are interested in answering the **question of interest**: "**Does $\mu = 0$?**".

To do so, you construct

- the **null hypothesis** $H_0 \, : \, \mu = 0$;
- the **alternative hypothesis** $H_1 \, : \, \mu \neq 0$.

Motivated by the central limit theorem, you decide to use a test of the form

$$\psi_C = \mathbf{1}\left(\sqrt{n}\,|\overline{X}_n| > C\right)$$

where $C > 0$ is a constant known as the **threshold** that you will choose in designing the test. (In the previous problem, C was chosen to be $0.25.$) On observing the data set, if $\psi = 1,$ you will **reject** $H_0.$ If $\psi = 0,$ then you will **fail to reject** $H_0.$

Suppose that indeed $\mu = 0$. Then $\mathbf{P}\left(\psi_C = 1\right)$, the probability of rejecting H_0, quantifies how likely we are to make the error of rejecting H_0 even though H_0 holds.

Under the assumption that $H_0 : \mu = 0$, for which value of C is $\mathbf{P}\left(\psi_C = 1\right)$ likely the largest?

● $C = 0.01$

○ $C = 0.1$

○ $C = 0.5$

○ $C = 1.0$

✓

Submit You have used 1 of 2 attempts

✓ Correct (1/1 point)

Compute the Type 1 Error

0/1 point (graded)

As above, let $X_1, \ldots, X_n \overset{iid}{\sim} N\left(\mu, 1\right)$ where μ is an unknown parameter. You are interested in answering the **question of interest**: "**Does $\mu = 0$?**".

To do so, you construct

- the **null hypothesis** $H_0 : \mu = 0$;
- the **alternative hypothesis** $H_1 : \mu \neq 0$.

Motivated by the central limit theorem, you decide to use a test of the form

$$\psi_C = \mathbf{1}\left(\sqrt{n}\,|\overline{X}_n| > C\right).$$

Recall from lecture that the **type 1 error** (also known as **type 1 error rate**) of a test ψ is the **function**

$$\begin{aligned}\alpha_\psi : \Theta_0 &\to [0,1]\\ \theta &\mapsto \mathbf{P}_\theta\left(\psi = 1\right)\end{aligned}$$

If you choose the threshold $C = q_{0.05}$, what is the type 1 error α_ψ?
(In this case, since H_0 only consists of one point, the function α_ψ is defined only at one point, and we loosely use the termininology “type 1 error" to mean the value of α_ψ at that point.)

Type 1 Error α_ψ : 0.960122388323255 ✖ **Answer:** 0.1

Solution:

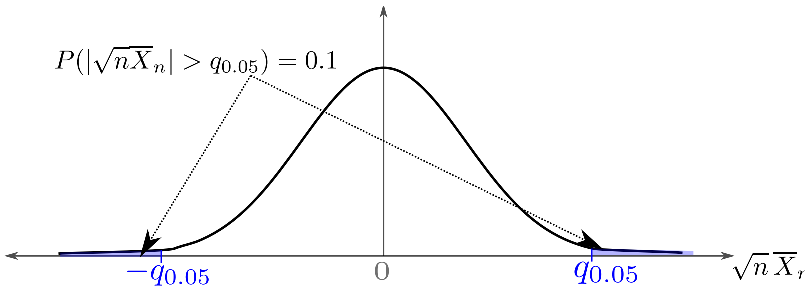
If we assume the null hypothesis $H_0 : \mu = 0$, and since the variance is known to be 1, the CLT gives

$$\sqrt{n}\,\overline{X}_n \sim \mathcal{N}\left(0,1\right) \qquad \text{for large } n.$$

The probability of a type 1 error is

$$\alpha_\psi\left(0\right) = \mathbf{P}_0\left(\psi_C = 1\right) = \mathbf{P}_0\left(\sqrt{n}\,|\overline{X}_n| > q_{0.05}\right) = 0.1.$$

as depicted in the figure below:



If H_0 is true, i.e. $\mu = 0$, then $\sqrt{n}\,\overline{X}_n$ is asymptotically normal.

Hence, the total area of the two shaded regions is $\mathbf{P}_0\left(\psi_C = 1\right) \;=\; \mathbf{P}_0\left(\sqrt{n}\,|\overline{X}_n| > q_{0.05}\right)$, the probability that H_0 is rejected even though it is true.

Submit You have used 3 of 3 attempts

Answers are displayed within the problem

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Getting used to the notation
Even though I'm able to answer the questions, I'm having a hard time getting used to all the notations! 1

☑ Unsure of my understanding of quantiles 5