Questions 1 - 12

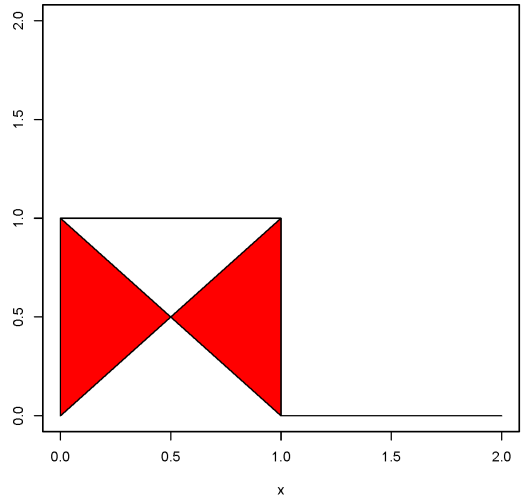
Suppose two sisters, Caroline and Anna, sleep in adjoining rooms. Each has a speaker with which she plays music, and each speaker has a volume dial going from 0 to 1. The joint distribution of the volumes of the two speakers is $f_{XY}(x,y) = c(x+y^2)$ over the unit square, 0 otherwise. (Caroline's volume is denoted by X, Anna's by Y.)

Question 1

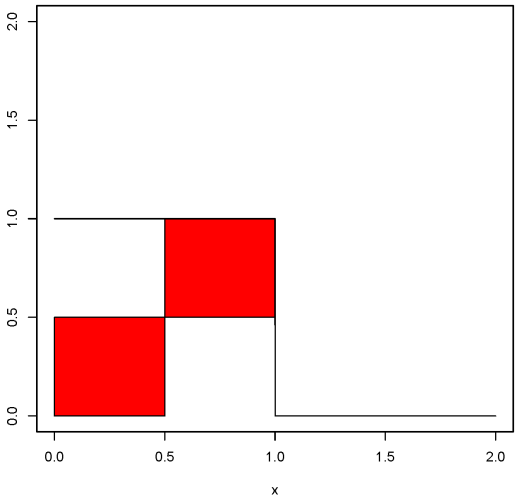
1/1 point (graded)

Which of the following figures represent the domain in which the density function is defined as $f_{XY}(x,y) = c(x+y^2)$?

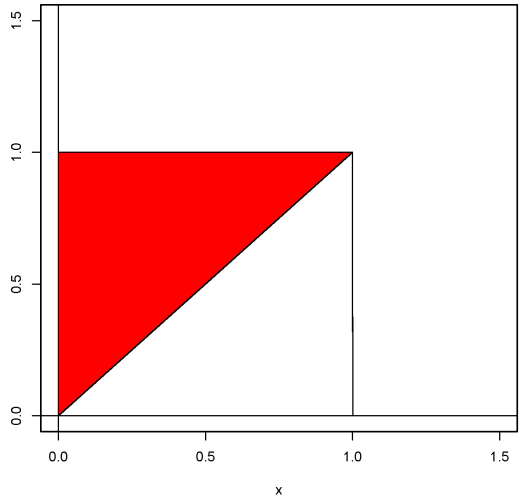
a. The domain is:



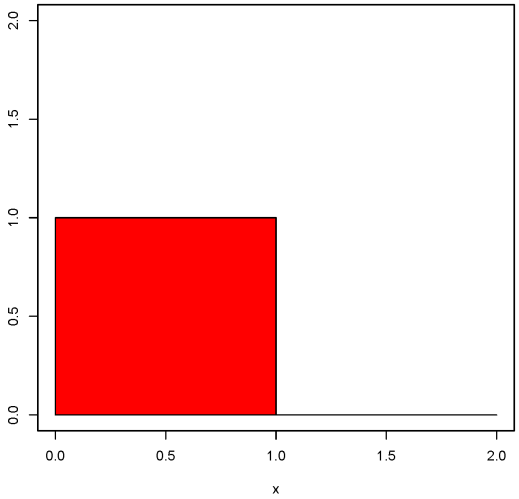
b. The domain is:



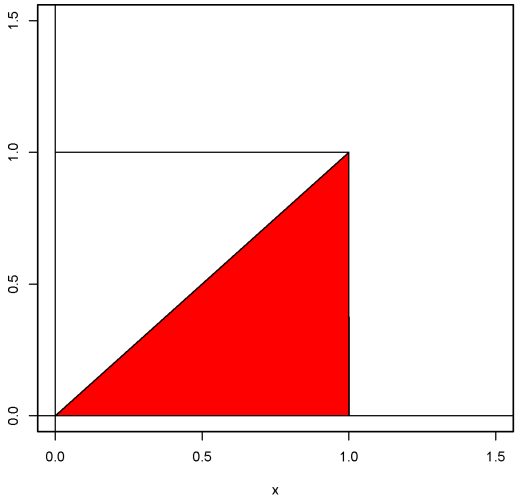
c. The domain is:



d. The domain is:



e. The domain is:



Explanation

As the problem states the domain in which the bivariate density function is defined as $f_{XY}(x,y) = c(x+y^2)$ corresponds to the unit square. The two-dimensional plot of the square is in option (d)

Submit You have used 1 of 2 attempts

✔ Correct (1/1 point)

Question 2

1/1 point (graded)

What does the constant c represent? (Select all that apply)

- [x] a. The constant c is a parameter whose value assures that the joint PDF integrates to 1. ✔
- [x] b. The constant c represents a parameter that changes both the joint PDF and the joint CDF of the random variables X and Y. ✔
- [ ] c. The constant c is an irrelevant parameter in the shape of the joint CDF of the random variables X and Y.
- [ ] d. The constant c is a parameter that helps to infer whether the random variables X and Y are independent.

✔

Explanation

As discussed by Professor Ellison in lecture, the value of the parameter c must assure that the joint PDF of the random variables X and Y integrates to 1. Thus, it affects the shape of both the joint PDF and the joint CDF of these random variables. This implies that it is not irrelevant. It doesn't tell us anything on whether the random variables X and Y are independent, and therefore option (d) is incorrect.

Submit You have used 1 of 2 attempts

✔ Correct (1/1 point)

Question 3

1/1 point (graded)

What is the value of the constant c in this case?

Input the exact value you receive, do not round

6/5 ✔ **Answer:** 6/5

$\frac{6}{5}$

Explanation

$$\int_0^1 \int_0^1 c(x+y^2)\,dy\,dx = \int_0^1 c\left(xy + \frac{y^3}{3}\right)\Big|_0^1 dx$$

$$= \int_0^1 c\left(x + \frac{1}{3}\right)dx = c\left(\frac{x^2}{2} + \frac{1}{3}x\right)\Big|_0^1 = c\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{6}c = 1$$

$$\implies c = \frac{6}{5} = 1.2$$

Submit You have used 1 of 2 attempts

✔ Correct (1/1 point)

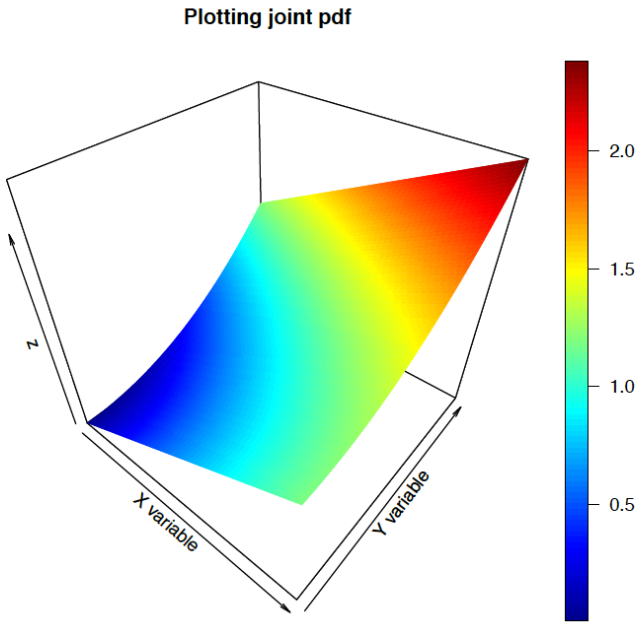
Now we are going to work in R to plot the bivariate PDF. Take a look at the following code in order to create a grid and a 3-dimensional plot of the PDF. Please note that you might need to install the package plot3D.

```
#Preliminaries
#-----------------------------------------------------------
rm(list = ls())
library("utils")
#install.packages('plot3D')
library(plot3D)
setwd("/Users/raz/Dropbox/14.31 edX Building the Course/Problem Sets/PSET 4")

#Creating the vector x and y
M <- mesh(seq(0, 1, length=100), seq(0, 1, length=100))
x <- M$x
y <- M$y
z <- 6/5*(M$x + M$y^2)

#Plotting this pdf
persp3D(x, y, z, xlab = 'X variable', ylab = 'Y variable', xlim = c(0,1), main = 'Plotting joint pdf')
```

Question 4

1/1 point (graded)

The following plot was created by running the code. A student is claiming that this plot is wrong since there are certain regions in which the PDF shows values larger than 1. Is this student correct that there is a mistake and therefore the plot does not correspond to the information given in the problem?



Yes

No ✔

Explanation

No, the student is incorrect - there is no mistake. As Sara has pointed out in the lecture, the pdf can take values larger than one. Since both X and Y are continuous random variables, the important thing is that they jointly integrate to one, which can be assured by having a value of the constant c that

guarantees this.

Submit You have used 1 of 1 attempts

✔ Correct (1/1 point)

Question 5

1/1 point (graded)

Are the volumes of the two speakers independent random variables?

- Yes
- No ✔

Explanation

The volumes are independent if $f_{XY}(x,y)$ can be factored into an X component and a Y component, i.e. $f_{XY}(x,y) = f_X(x)f_Y(y)$. It does not appear to be the case that $f_{XY}(x,y) = \frac{6}{5}\left(x+y^2\right)$ can be factored into an X and Y component. More formally, if the volumes are independent, the conditional distribution of X given Y cannot depend on Y and vice-versa.

Submit You have used 1 of 1 attempts

✔ Correct (1/1 point)

Question 6

1/1 point (graded)

What is the formula for the marginal distribution of Anna's speaker volume?

a. $f_Y(y)$ is given by $\frac{5}{6}\left(\frac{1}{2}+y^2\right)$

b. $f_Y(y)$ is given by $\frac{6}{5}\left(\frac{1}{2}+y^2\right)$ ✔

c. $f_Y(y)$ is given by $\frac{6}{5}\left(\frac{1}{2}+\sqrt{y}\right)$

d. $f_Y(y)$ is given by $\frac{5}{6}\left(\frac{1}{2}+\sqrt{y}\right)$

Explanation

We know that we have to integrate over the complete domain of X for each potential value y that the random variable Y can take. Then, we have that:

$$f_Y(y)=\int_0^1 f_{XY}(x,y)\,dx=\int_0^1 \frac{6}{5}\left(x+y^2\right)dx=\frac{6}{5}\left(\frac{x^2}{2}+xy^2\right)\Big|_0^1=\frac{6}{5}\left(\frac{1}{2}+y^2\right)$$

Submit You have used 1 of 2 attempts

✔ Correct (1/1 point)

Question 7

1/1 point (graded)

What is the conditional distribution of Caroline's volume as a function of Anna's?

a. This is given by $\frac{\left(x+y^2\right)}{\left(\frac{1}{2}+y^2\right)}$ ✔

b.This is given by $\frac{\frac{5}{6}\left(x+y^2\right)}{\frac{6}{5}\left(\frac{1}{2}+y^2\right)}$

c. This is given by $\frac{\left(x+\sqrt{y}\right)}{\left(\frac{1}{2}+y^2\right)}$

d. This is given by $\frac{\frac{6}{5}\left(x+y^2\right)}{\left(\frac{1}{2}+y^2\right)}$

Explanation

We have to use the previous information, then we have that:

$$f_{X|Y}(x|y)=\frac{f_{XY}(x,y)}{f_Y(y)}=\frac{\frac{6}{5}\left(x+y^2\right)}{\frac{6}{5}\left(\frac{1}{2}+y^2\right)}=\frac{\left(x+y^2\right)}{\left(\frac{1}{2}+y^2\right)}$$

Submit You have used 1 of 2 attempts

✔ Correct (1/1 point)

Question 8

1/1 point (graded)

From this conditional distribution can you infer whether Caroline likes Anna's music or not?

Hint: Think whether Caroline's stereo volume is lower when Anna's is higher

- a. Caroline does like Anna's music ✔
- b. Caroline does not like Anna's music

Explanation

It seems they do like each others' music. Suppose Anna goes from having her music off ($y = 0$) to full blast ($y = 1$). Then you can check using the conditional distribution that the probability that Caroline's music volume is high (say between 0.75 and 1) goes down.

Submit You have used 1 of 1 attempts

✔ Correct (1/1 point)

Question 9

1/1 point (graded)

What is the probability that Caroline's volume is less than $\frac{1}{2}$ if Anna's volume is $\frac{1}{2}$?

Please enter your answer as a fraction (e.g. if you obtained 0.5, you would enter 1/2).

1/3 ✔

$\frac{1}{3}$

Submit You have used 2 of 2 attempts

✔ Correct (1/1 point)

Question 10

1/1 point (graded)

Now, what is the marginal distribution of Caroline's speaker volume?

a. It is given by $\frac{5}{6}\left(x+\frac{2}{3}\right)$

b. It is given by $\frac{5}{6}\left(x+\frac{1}{3}\right)$

c. It is given by $\frac{6}{5}\left(x+\frac{2}{3}\right)$

d. It is given by $\frac{6}{5}\left(x+\frac{1}{3}\right)$ ✔

Explanation

We need to calculate $f_X(x)$ which we can do by integrating over all the domain of Y. Then we have that: $f_X(x)=\int_0^1 f_{XY}(x,y)\,dx=\int_0^1 \frac{6}{5}\left(x+y^2\right)dy=\frac{6}{5}\left(xy+\frac{y^3}{3}\right)\Big|_0^1=\frac{6}{5}\left(x+\frac{1}{3}\right)$

Submit You have used 1 of 2 attempts

✔ Correct (1/1 point)

Question 11

1/1 point (graded)

Is there a First Order Stochastic Dominance (FOSD) relationship between the random variables X and Y? (We suggest you compute the CDF's of both variables and plot them in R.)

The distribution of X FOSD the distribution of Y

- The distribution of Y FOSD the distribution of X
- There is no clear relationship ✔

Explanation

If we computed the CDF's, then we know that:

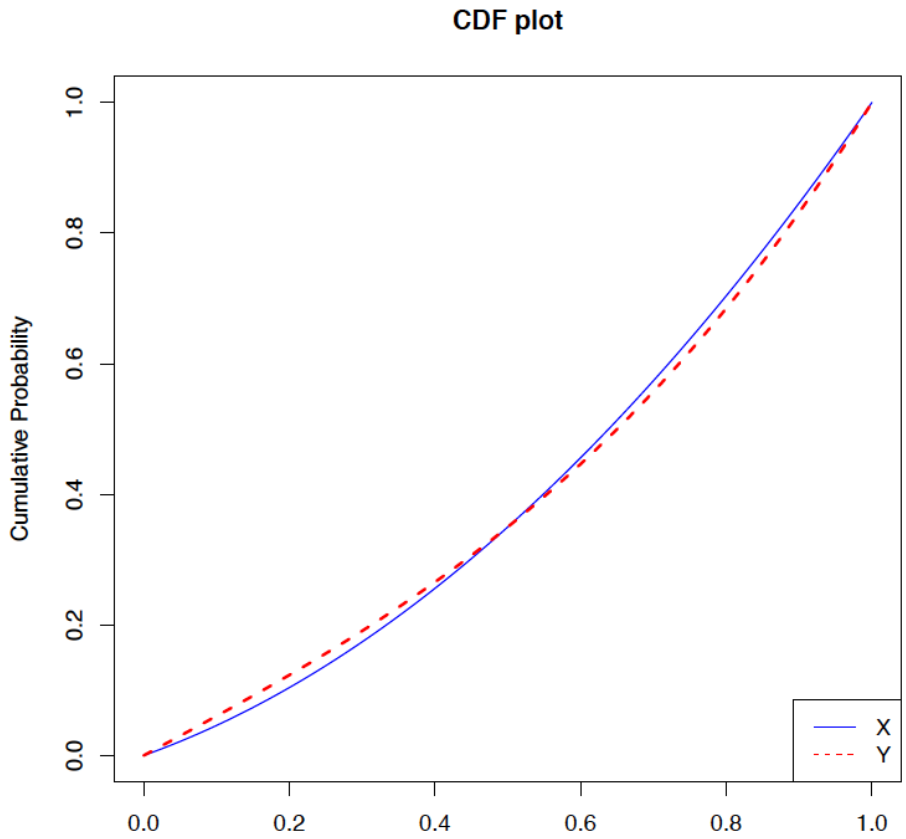
$$P(X \leq x) = \frac{6}{5}\left(\frac{x^2}{2} + \frac{x}{3}\right)$$

$$P(Y \leq y) = \frac{6}{5}\left(\frac{y^3}{3} + \frac{y}{2}\right)$$

We can run the following code in R to plot these functions. Please note that your graph would be saved in the working directory.

```
#Calculating cdf
x <- seq(0, 1, length=1000)
y <- seq(0, 1, length=1000)
cdfy <- 6/5*(1/2*y+y^3/3)
cdfx <- 6/5*(1/3*x+x^2/2)

#Plotting cdf
pdf("cumulative.pdf")
plot(x, cdfx, type = "l", col = "blue", xlab = " ", ylab = "Cumulative Probability", xlim = c(0,1), main = "CDF plot")
lines(y, cdfy, lty=2, col="red", lwd = 2)
legend("bottomright", ncol = 1, legend = c("X", "Y"), lty=c(1,2), col=c("blue", "red"))
dev.off()
```

And the plot is shown by the following figure:



Since it is not the case that one of the CDF's is always below the other, we can't say that there is FOSD between random variables X and Y.

Submit You have used 1 of 1 attempts

✔ Correct (1/1 point)

Question 12

1 point possible (graded)

Can we say that Anna or Caroline prefer higher volumes?

- a. Anna ✖
- b. Caroline
- c. We can't say ✔

Explanation

Because neither first order stochastically dominates the other, we cannot say that one always prefers higher volumes. Anna's volume distribution has "thicker tails" than Caroline's, so Caroline listens to a slightly narrower range of volumes. In other words, at the 25th percentile, Caroline's volume is higher than Anna's, but at the 75th percentile it is lower.

Submit You have used 1 of 1 attempts

✖ Incorrect (0/1 point)