

MITx: 6.041x Introduction to Probability - The Science of Uncertainty



- Unit 0: Overview
- ► Entrance Survey
- Unit 1: Probability models and axioms
- Unit 2: Conditioning and independence
- Unit 3: Counting
- Unit 4: Discrete random variables
- Exam 1
- Unit 5: Continuous random variables

Exam 2 > Exam 2 > Exam 2 vertical5

■ Bookmark

Problem 6: Two measurement instruments

(4/4 points)

Let Θ be an unknown random variable that we wish to estimate. It has a prior distribution with mean $\mathbf 1$ and variance $\mathbf 2$. Let W be a noise term, another unknown random variable with mean $\mathbf 3$ and variance $\mathbf 5$. Assume that $\mathbf \Theta$ and W are independent.

We have two different instruments that we can use to measure Θ . The first instrument yields a measurement of the form $X_1=\Theta+W$, and the second instrument yields a measurement of the form $X_2=2\Theta+3W$. We pick an instrument at random, with each instrument having probability 1/2 of being chosen. Assume that this choice of instrument is independent of everything else. Let X be the measurement that we observe, without knowing which instrument was used.

Give numerical answers for all parts below.

2.

- Unit 6: Further topics on random variables
- Unit 7: Bayesian inference
- **▼** Exam 2

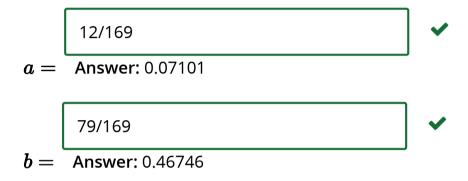
Exam 2

Exam 2 due Apr 20, 2016 at 23:59 UTC

 Unit 8: Limit theorems and classical statistics



3. The LLMS estimator of Θ given X is of the form aX+b. Give the numerical values of a and b.



Answer:

1. Let I be the instrument used to perform the measurement, with $\mathbf{P}(I=1)=\mathbf{P}(I=2)=1/2$.

Using the total expectation theorem, we have

$$egin{align} \mathbf{E}[X] &= \mathbf{E}[X \mid I = 1] \mathbf{P}(I = 1) + \mathbf{E}[X \mid I = 2] \mathbf{P}(I = 2) \ &= rac{1}{2} \cdot (\mathbf{E}[\Theta + W] + \mathbf{E}[2\Theta + 3W]). \ &= rac{1}{2} \cdot (4 + 11) \ \end{aligned}$$

$$=\frac{15}{2}.$$

2. Note that

$$egin{align} \mathbf{E}[\Theta^2] &= ext{var}(\Theta) + (\mathbf{E}[\Theta])^2 = 3, \ \mathbf{E}[W^2] &= ext{var}(W) + (\mathbf{E}[W])^2 = 14. \ \end{aligned}$$

Also, $\mathbf{E}[\Theta W] = \mathbf{E}[\Theta]\mathbf{E}[W]$ by independence.

Using the total expectation theorem again, we obtain

$$egin{aligned} \mathbf{E}[X^2] &= \mathbf{E}[X^2 \mid I = 1] \mathbf{P}(I = 1) + \mathbf{E}[X^2 \mid I = 2] \mathbf{P}(I = 2) \\ &= rac{1}{2} \cdot (\mathbf{E}[(\Theta + W)^2] + \mathbf{E}[(2\Theta + 3W)^2]). \\ &= rac{1}{2} \cdot (5\mathbf{E}[\Theta^2] + 14\mathbf{E}[\Theta W] + 10\mathbf{E}[W^2]) \\ &= rac{1}{2} \cdot (15 + 42 + 140) \\ &= rac{197}{2}. \end{aligned}$$

3. The LLMS estimator of $oldsymbol{\Theta}$ given $oldsymbol{X}$ is

$$\widehat{\Theta}_{LLMS} = \mathbf{E}[\Theta] + rac{\mathrm{cov}(X,\Theta)}{\mathrm{var}(X)}(X - \mathbf{E}[X]).$$

First, ${\bf var}(X)={\bf E}[X^2]-({\bf E}[X])^2=197/2-(15/2)^2=169/4$. Next, the covariance term is computed by first calculating ${\bf E}[X\Theta]$:

$$\begin{aligned} \mathbf{E}[X\Theta] &= \mathbf{E}[X\Theta \mid I=1]\mathbf{P}(I=1) + \mathbf{E}[X\Theta \mid I=2]\mathbf{P}(I=2) \\ &= \mathbf{E}[(\Theta+W)\Theta]\mathbf{P}(I=1) + \mathbf{E}[(2\Theta+3W)\Theta]\mathbf{P}(I=2) \\ &= \frac{1}{2} \cdot (3\mathbf{E}[\Theta^2] + 4\mathbf{E}[\Theta W]) \\ &= \frac{1}{2} \cdot (9+12) \\ &= \frac{21}{2}. \end{aligned}$$

Hence, $\operatorname{cov}(X,\Theta) = 21/2 - (15/2) \cdot 1 = 3$. Therefore,

$$egin{align} \widehat{\Theta}_{LLMS} &= 1 + rac{3}{169/4}igg(X - rac{15}{2}igg) \ &= rac{12}{169}X + rac{79}{169}. \end{split}$$

You have used 1 of 2 submissions

© All Rights Reserved



© edX Inc. All rights reserved except where noted. EdX, Open edX and the edX and Open EdX logos are registered trademarks or trademarks of edX Inc.

















