3. Heat equation in MATLAB

Simple Numerical Method to Solve the Heat Equation

We wish to numerically solve the heat equation

$$\frac{\partial\theta}{\partial t} = \nu\frac{\partial^2\theta}{\partial x^2}, \qquad 0 < x < L,\ t > 0$$

with the boundary conditions $\theta(0,t) = f(t)$ and $\theta(L,t) = g(t)$ and initial condition $\theta(x,0) = h(x)$.

We will use a **forward in time, centered in space** numerical scheme. Let θ_j^i denote the solution at time $i\Delta t$ and position $j\Delta x$.

Then a discrete forward time time derivative is

$$\frac{\partial\theta}{\partial t} = \frac{\theta_j^{i+1} - \theta_j^i}{\Delta t} + (\text{ higher order terms in } \Delta t)$$

and a discrete centered space derivative is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\theta^i_{j+1} - 2\theta^i_j + \theta^i_{j-1}}{\Delta x^2} + (\text{ higher order terms in } (\Delta x)^2)\,.$$

Substituting the discrete time and space derivatives into the heat equation gives

$$\begin{aligned}\frac{\theta^{i+1}_j - \theta^i_j}{\Delta t} &= \nu\frac{\theta^i_{j+1} - 2\theta^i_j + \theta^i_{j-1}}{\Delta x^2} + \text{higher order terms} \\ \theta^{i+1}_j &= \theta^i_j + \frac{\nu\Delta t}{(\Delta x)^2}\left(\theta^i_{j+1} - 2\theta^i_j + \theta^i_{j-1}\right) + \text{higher order terms}\end{aligned}$$

In matrix notation, this becomes

$$\begin{pmatrix}\theta^{i+1}_1 \\ \theta^{i+1}_2 \\ \vdots \\ \theta^{i+1}_{N-1} \\ \theta^{i+1}_N\end{pmatrix} = \begin{pmatrix}1-2r & r & & & \\ r & 1-2r & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & 1-2r & r \\ & & & r & 1-2r\end{pmatrix}\begin{pmatrix}\theta^i_1 \\ \theta^i_2 \\ \vdots \\ \theta^i_{N-1} \\ \theta^i_N\end{pmatrix}, \qquad r = \frac{\nu\Delta t}{\Delta x^2},$$

where at each time step i we impose the boundary conditions $\theta^i_1 = f(i\Delta t)$ and $\theta^i_N = g(i\Delta t)$.

Condition for Numerical Stability

$$\boxed{\frac{\nu\Delta t}{\Delta x^2} \le \frac{1}{2}.}$$

One way to find the condition for numerical stability is to find the eigenvalues of the matrix for this system and find conditions on r so that all of the eigenvalues have magnitude less than 1.

Download the example script

Download the following script to see how to solve the heat equation using MATLAB Online. This example has a dynamic input (it varies in time).

(Use short cut commands for copy and paste: ctrl-c and ctrl-v on windows, and cmd-c, cmd-v on a MAC.)

```
url = 'https://courses.edx.org/asset-v1:MITx+18.03Fx+3T2018+type@asset+block@heatEqn.m';
websave('heatEqn.m',url)
```

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