5. A worked example

Problem 5.1 Find the general solution $(x(t), y(t), z(t))$ to the system

$$\begin{aligned}\dot{x} &= 2x\\ \dot{y} &= -6x+8y+3z\\ \dot{z} &= 18x-18y-7z.\end{aligned}$$

Solution:

In matrix form, this is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0\\ -6 & 8 & 3\\ 18 & -18 & -7\end{pmatrix}$.

Step 1. Find the eigenvalues. To do this, compute

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \begin{vmatrix} \lambda-2 & 0 & 0\\ 6 & \lambda-8 & -3\\ -18 & 18 & \lambda+7\end{vmatrix}.$$

Use Laplace expansion along the first row to get

$$(\lambda-2)\big((\lambda-8)(\lambda+7)-18(-3)\big) = (\lambda-2)(\lambda^2-\lambda-2) = (\lambda-2)(\lambda-2)(\lambda+1),$$

so the eigenvalues are $2, 2, -1$.

Step 2. Find a basis of each eigenspace and write down the exponential solutions.

Eigenspace of $\lambda = 2$: This is the nullspace of

$$2\mathbf{I} - \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 6 & -6 & -3 \\ -18 & 18 & 9 \end{pmatrix}.$$

Converting to row-echelon form gives

$$\begin{pmatrix} 6 & -6 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which corresponds to the single equation

$$-6x + 6y + 3z = 0.$$

Solve by back-substitution: $z = c_1$, $y = c_2$, $x = y + z/2 = c_2 + c_1/2$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

so $\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ form a basis for the eigenspace at 2. (We were lucky here that the number of basis eigenvectors is as large as the multiplicity of the eigenvalue, so that the eigenspace of 2 was not deficient.)

The exponential solutions built from these eigenvectors are:

$$e^{2t}\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}, \quad e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Eigenspace of $\lambda = -1$: This is the nullspace of

$$-\mathbf{I} - \mathbf{A} = \begin{pmatrix} -3 & 0 & 0 \\ 6 & -9 & -3 \\ -18 & 18 & 6 \end{pmatrix}$$

Converting to row-echelon form gives

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -9 & -3 \\ 0 & 0 & 0 \end{pmatrix},$$

which corresponds to the system

$$\begin{aligned} 3x &= 0 \\ 9y + 3z &= 0. \end{aligned}$$

Back-substitution leads to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix},$$

so $\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ by itself is a basis for the eigenspace at -1.

The exponential solution built from this eigenvector is

$$e^{-t}\begin{pmatrix}0\\-1\\3\end{pmatrix}.$$

Steps 3. Check whether there are enough independent solutions and write the general solution.

We have three independent solutions,

$$e^{2t}\begin{pmatrix}1/2\\0\\1\end{pmatrix},\quad e^{2t}\begin{pmatrix}1\\1\\0\end{pmatrix},\quad e^{-t}\begin{pmatrix}0\\-1\\3\end{pmatrix},$$

so they form a basis of all solutions. The general solution is

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=c_1e^{2t}\begin{pmatrix}1/2\\0\\1\end{pmatrix}+c_2e^{2t}\begin{pmatrix}1\\1\\0\end{pmatrix}+c_3e^{-t}\begin{pmatrix}0\\-1\\3\end{pmatrix}.$$

If there were initial conditions, we could solve for c_1, c_2, c_3 to get a specific solution.

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Problem 5.2

Find the general solution to the system $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ with $\mathbf{A}=\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}$. Recall that we have found (in the last lecture) the eigenvalues and eigenspace of $\mathbf{A}$ to be

$$\begin{array}{lcl}\text{Eigenvalue} & & \text{Corresponding eigenspace}\\ \lambda=0 & ; & \text{Span}\begin{pmatrix}1\\0\\-1\end{pmatrix}\end{array}$$

$$\lambda = 1 \quad ; \quad \text{Span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 2 \quad ; \quad \text{Span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Solution

The eigenvalues are all distinct, so we automatically have enough independent eigenvectors. The exponential solutions $\mathbf{v}e^{\lambda t}$ are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t},$$

and the general solution is

$$\mathbf{x}(t) \quad = \quad c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

(If there were initial conditions, we could solve for c_1, c_2, c_3 to get a specific solution.)

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Linearly connected tanks

4/4 points (graded)

You found in a previous problem the system of DE that describes the fluid flow between 3 linearly connected tanks.

Let us consider the simpler case in which 3 tanks are connected (but there is no direct pipe between tanks 1 and 3).

The system of DE is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \text{where } \mathbf{x} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The general solution of this system is

$$\mathbf{x}(t) = = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \mathbf{v}_2 + c_3 e^{\lambda_3 t} \mathbf{v}_3.$$

where $\lambda_1, \lambda_2, \lambda_3$ are scalars that are not necessarily distinct, and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^3$.

Find $\lambda_1, \lambda_2,$ and λ_3.

(The order of the eigenvalues does not matter.)

$\lambda_1 =$ 0 ✔ **Answer:** 0

0

$\lambda_2 =$ -1 ✔ **Answer:** -1

-1

$\lambda_3 =$ -3 ✔ **Answer:** -3

-3

What is the long term behavior of the system?

- ○ The heights of fluid in the three tanks tend to the ratio $1:2:1$
- ○ The heights of fluid in the three tanks tend to the ratio $1:0:1$
- ◉ The heights of fluid in the three tanks tend to the same level ✔

Solution:

The unknowns $\lambda_1, \lambda_2, \lambda_3$ and $\mathbf{v}_1, \mathbf{v}_2\ \mathbf{v}_3$ in the given expression for the general solution are the eigenvalues and corresponding eigenvectors of $\mathbf{A}$.

To find the eigenvalues, we need to find the roots of the characteristic polynomial:

$$\begin{aligned}\det(\lambda\mathbf{I}-\mathbf{A}) &= \begin{vmatrix}\lambda+1 & -1 & 0\\ -1 & \lambda+2 & -1\\ 0 & -1 & \lambda-1\end{vmatrix}\\ &= (\lambda+1)\left((\lambda+2)(\lambda+1)-1\right)+(-1)(\lambda+1)\\ &= (\lambda+1)\left((\lambda+2)(\lambda+1)-2\right)\\ &= (\lambda+1)\left(\lambda(\lambda+3)\right).\end{aligned}$$

This gives the eigenvalues: $0, -1, -3$. Since these are all distinct, the corresponding eigenvectors will form a basis of all possible solutions. The general solution is of the form

$$\mathbf{x}(t) \;=\; c_1\mathbf{v}_1+c_2\mathbf{v}_2e^{-t}+c_3\mathbf{v}_3e^{-3t}.$$

Regardless of initial conditions, the solution will tend to the constant solution $c_1\mathbf{v}_1$.

To find the constant solution $\mathbf{v}_1$, we need to find $\mathrm{NS}(\mathbf{A})$. However, from physical intuition, we know that $h_1=h_2=h_3$ is a constant solution, since when the fluid heights are the same in the 3 tanks, there will be no flow. Therefore, we can just verify that $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is in $\mathrm{NS}(\mathbf{A})$. Indeed:

$$\begin{pmatrix}-1 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Therefore, the long term behavior predicted by this the solution is that the heights in the 3 tanks will become the same, and this is consistent with physical experience.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

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