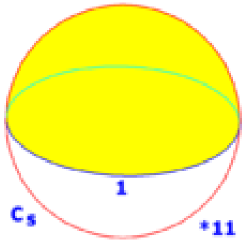
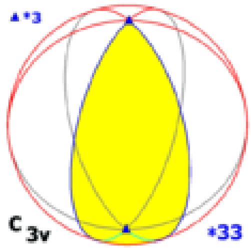
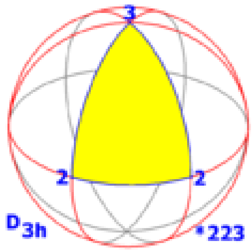
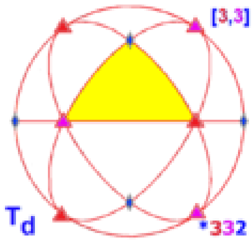
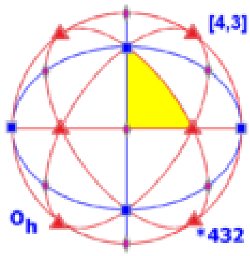
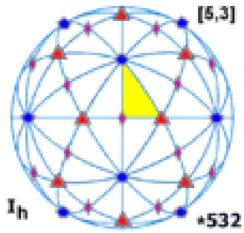
Tetrahedral symmetry

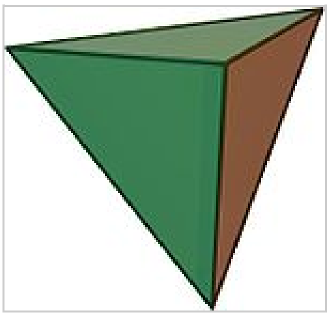
From Wikipedia, the free encyclopedia

Point groups in three dimensions

 Involutional symmetry C_s, (*) [ ] =	 Cyclic symmetry C_{nv}, (*nn) [n] =	 Dihedral symmetry D_{nh}, (*n22) [n,2] =
Polyhedral group, [n,3], (*n32)		
 **Tetrahedral symmetry** T_d, (*332) [3,3] =	 Octahedral symmetry O_h, (*432) [4,3] =	 Icosahedral symmetry I_h, (*532) [5,3] =

A regular tetrahedron has 12 rotational (or orientation-preserving) symmetries, and a symmetry order of 24 including transformations that combine a reflection and a rotation.

The group of all symmetries is isomorphic to the group S_4, symmetric group, as a permutations of four objects, since there is exactly one such symmetry for each permutation of the vertices of the tetrahedron. The set of orientation-preserving symmetries forms a group referred to as the alternating subgroup A_4 of S_4.



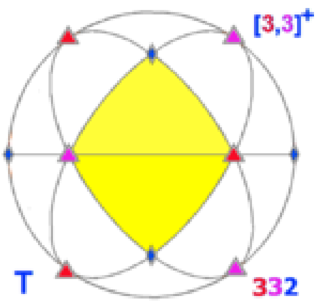
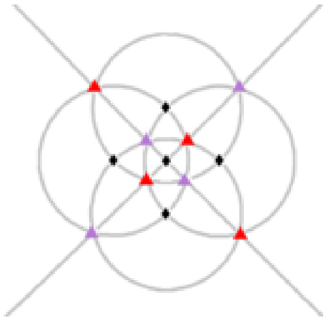
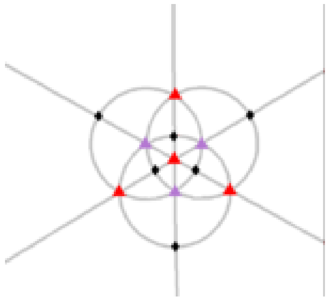
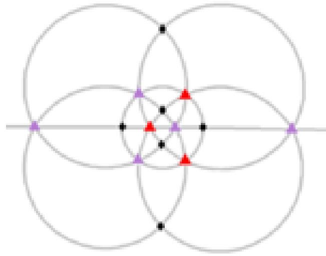
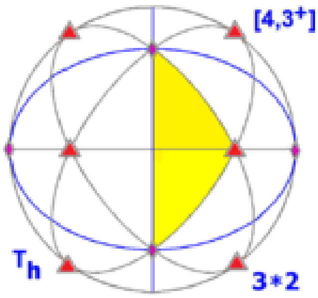
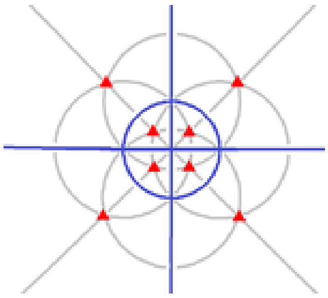
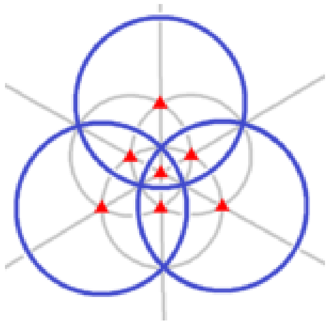
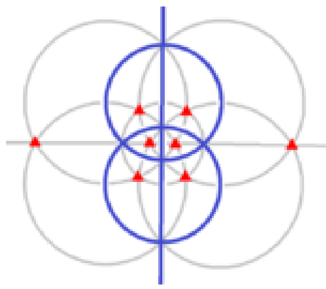
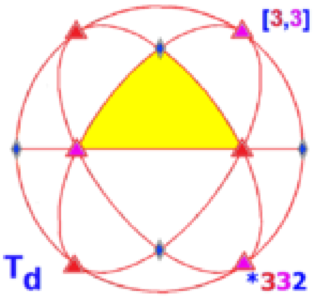
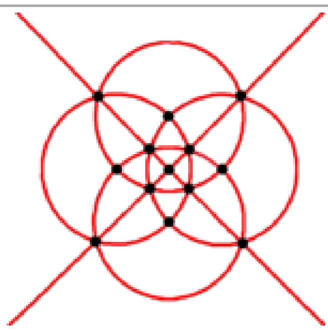
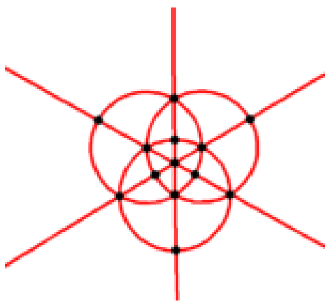
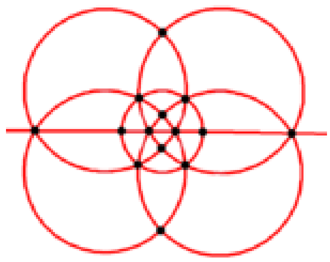
A regular tetrahedron, an example of a solid with full tetrahedral symmetry

Contents

Details

Chiral and **full** (or **achiral) tetrahedral symmetry** and **pyritohedral symmetry** are discrete point symmetries (or equivalently, symmetries on the sphere). They are among the crystallographic point groups of the cubic crystal system.

Seen in stereographic projection the edges of the tetrakis hexahedron form 6 circles (or centrally radial lines) in the plane. Each of these 6 circles represent a mirror line in tetrahedral symmetry. The intersection of these circles meet at order 2 and 3 gyration points.

Orthogonal	Stereographic projections		
	4-fold	3-fold	2-fold
Chiral tetrahedral symmetry, T, (332), [3,3]$^+$ = [1$^+$,4,3$^+$],			
			
Pyritohedral symmetry, T$_h$, (3*2), [4,3$^+$],			
			
Achiral tetrahedral symmetry, T$_d$, (*332), [3,3] = [1$^+$4,3],			
			

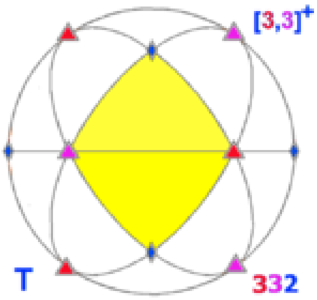
Chiral tetrahedral symmetry

T, **332**, [3,3]$^+$, or **23**, of order 12 - **chiral** or **rotational tetrahedral symmetry**. There are three orthogonal 2-fold rotation axes, like chiral dihedral symmetry D_2 or 222, with in addition four 3-fold axes, centered *between* the three orthogonal directions. This group is isomorphic to A_4, the alternating group on 4 elements; in fact it is the group of even permutations of the four 3-fold axes: e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23).

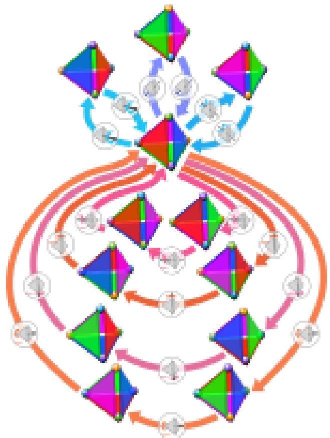
The conjugacy classes of *T* are:

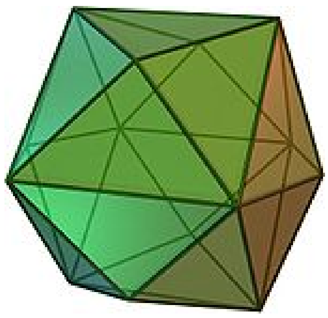
- identity
- 4 × rotation by 120° clockwise (seen from a vertex): (234), (143), (412), (321)
- 4 × rotation by 120° counterclockwise (ditto)
- 3 × rotation by 180°

The rotations by 180°, together with the identity, form a normal subgroup of type Dih_2, with quotient group of type Z_3. The three elements of the latter are the identity, "clockwise rotation", and "anti-clockwise rotation", corresponding to permutations of the three orthogonal 2-fold axes, preserving orientation.



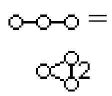
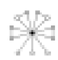
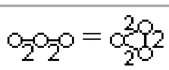
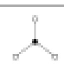
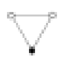
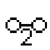
The tetrahedral rotation group *T* with fundamental domain; for the triakis tetrahedron, see below, the latter is one full face


A tetrahedron can be placed in 12 distinct positions by rotation alone. These are illustrated above in the cycle graph format, along with the 180° edge (blue arrows) and 120° vertex (reddish arrows) rotations that permute the tetrahedron through those positions.


In the tetrakis hexahedron one full face is a fundamental domain; other solids with the same symmetry can be obtained by adjusting the orientation of the faces, e.g. flattening selected subsets of faces to combine each subset into one face, or replacing each face by multiple faces, or a curved surface.

A_4 is the smallest group demonstrating that the converse of Lagrange's theorem is not true in general: given a finite group G and a divisor d of $|G|$, there does not necessarily exist a subgroup of G with order d: the group $G = A_4$ has no subgroup of order 6. Although it is a property for the abstract group in general, it is clear from the isometry group of chiral tetrahedral symmetry: because of the chirality the subgroup would have to be C_6 or D_3, but neither applies.

Subgroups of chiral tetrahedral symmetry

Schoe.	Coxeter		Orb.	H-M	Structure	Cycle graph|Cyc	Order	Index
T	$[3,3]^+$		332	23	A_4		12	1
D_2	$[2,2]^+$		222	222	Dih_2		4	3
C_3	$[3]^+$		33	3	Z_3		3	4
C_2	$[2]^+$		22	2	Z_2		2	6
C_1	$[\ ]^+$		11	1	Z_1		1	12

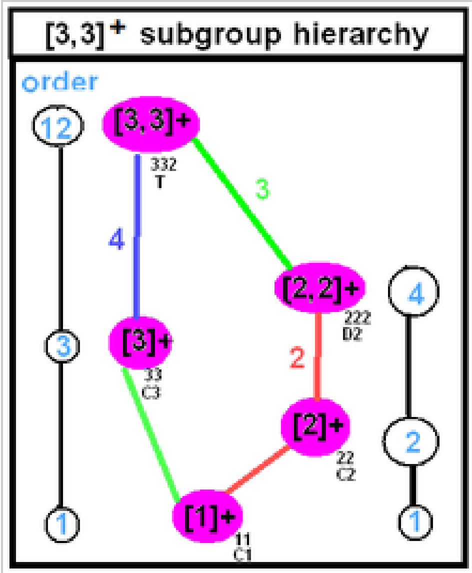
Achiral tetrahedral symmetry

***T_d*, *332**, [3,3] or $\bar{4}3m$, of order 24 - **achiral** or **full tetrahedral symmetry**, also known as the (2,3,3) triangle group. This group has the same rotation axes as *T*, but with six mirror planes, each through two 3-fold axes. The 2-fold axes are now S_4 ($\bar{4}$) axes. T_d and *O* are isomorphic as abstract groups: they both correspond to S_4, the symmetric group on 4 objects. T_d is the union of *T* and the set obtained by combining each element of *O* \ *T* with inversion. See also the isometries of the regular tetrahedron.

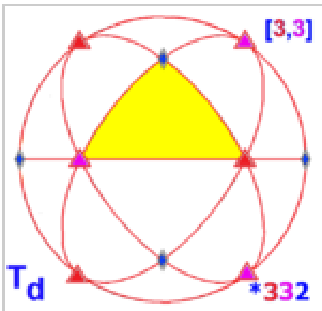
The conjugacy classes of T_d are:

- identity
- 8 × rotation by 120°
- 3 × rotation by 180°
- 6 × reflection in a plane through two rotation axes
- 6 × rotoreflection by 90°

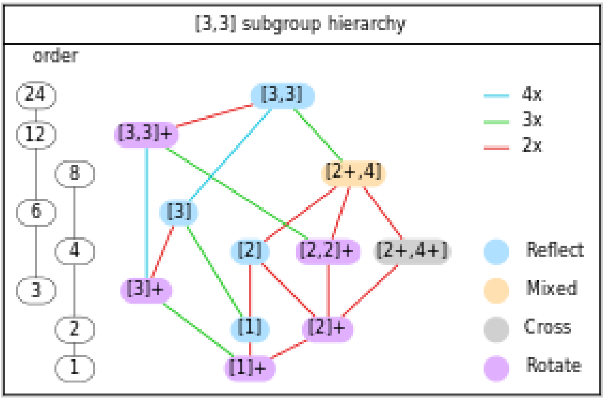
Subgroups of achiral tetrahedral symmetry



Chiral tetrahedral symmetry subgroups



The full tetrahedral group T_d with fundamental domain



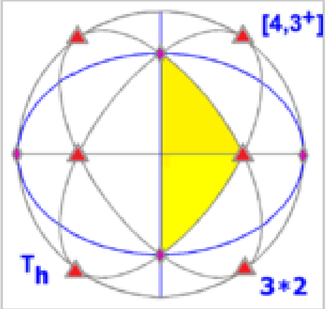
Achiral tetrahedral subgroups

Schoe.	Coxeter		Orb.	H-M	Structure	Cyc	Order	Index
T_d	[3,3]		*332	$\bar{4}3m$	S_4		24	1
C_{3v}	[3]		*33	3m	$Dih_3=S_3$		6	4
C_{2v}	[2]		*22	mm2	Dih_2		4	6
C_s	[ ]		*	$\bar{2}$ or m	Dih_1		2	12
D_{2d}	$[2^+,4]$		2*2	$\bar{4}2m$	Dih_4		8	3
S_4	$[2^+,4^+]$		2×	$\bar{4}$	Z_4		4	6
T	$[3,3]^+$		332	23	A_4		12	2
D_2	$[2,2]^+$		222	222	Dih_2		4	6
C_3	$[3]^+$		33	3	$Z_3=A_3$		3	8
C_2	$[2]^+$		22	2	Z_2		2	12
C_1	$[\ ]^+$		11	1	Z_1		1	24

Pyritohedral symmetry

***T_h*, 3*2**, $[4,3^+]$ or $m\bar{3}$, of order 24 - **pyritohedral symmetry**. This group has the same rotation axes as *T*, with mirror planes through two of the orthogonal directions. The 3-fold axes are now S_6 ($\bar{3}$) axes, and there is inversion symmetry. T_h is isomorphic to $T \times Z_2$: every element of T_h is either an element of *T*, or one combined with inversion. Apart from these two normal subgroups, there is also a normal subgroup D_{2h} (that of a cuboid), of type $Dih_2 \times Z_2 = Z_2 \times Z_2 \times Z_2$. It is the direct product of the normal subgroup of T (see above) with C_i. The quotient group is the same as above: of type Z_3. The three elements of the latter are the identity, "clockwise rotation", and "anti-clockwise rotation", corresponding to permutations of the three orthogonal 2-fold axes, preserving orientation.

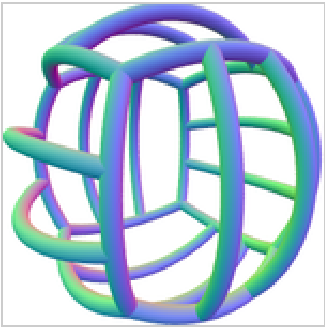
It is the symmetry of a cube with on each face a line segment dividing the face into two equal rectangles, such that the line segments of adjacent faces do not meet at the edge. The symmetries correspond to the even permutations of the body diagonals and the same combined with inversion. It is also the symmetry of a pyritohedron, which is extremely similar to the cube described, with each rectangle replaced by a pentagon with one symmetry axis and 4 equal sides and 1 different side (the one corresponding to the line segment dividing the cube's face); i.e., the cube's faces bulge out at the dividing line and become narrower there. It is a subgroup of the full icosahedral symmetry group (as isometry group, not just as abstract group), with 4 of the 10 3-fold axes.



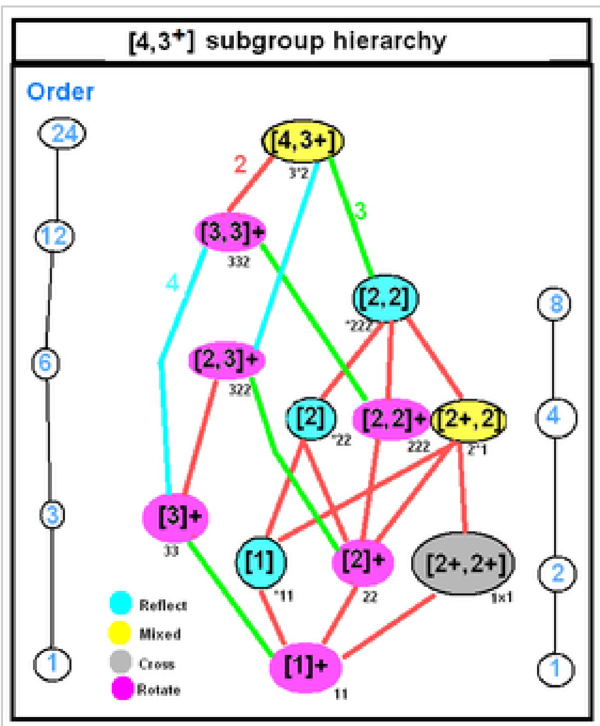
The pyritohedral group T_h with fundamental domain

The conjugacy classes of T_h include those of T, with the two classes of 4 combined, and each with inversion:

- identity
- 8 × rotation by 120°
- 3 × rotation by 180°
- inversion
- 8 × rotoreflection by 60°
- 3 × reflection in a plane


The seams of a volleyball have pyritohedral symmetry

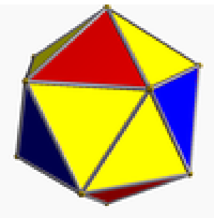
Subgroups of pyritohedral symmetry



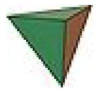
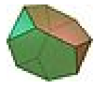
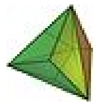
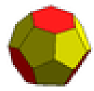
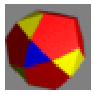
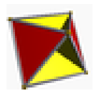
Pyritohedral subgroups

Schoe.	Coxeter		Orb.	H-M	Structure	Cyc	Order	Index
T_h	$[3^+,4]$		3*2	$m\overline{3}$	$A_4 \times 2$		24	1
D_{2h}	$[2,2]$	• • •	*222	mmm	$Dih_2 \times Dih_1$		8	3
C_{2v}	$[2]$	• •	*22	mm2	Dih_2		4	6
C_s	[ ]	•	*	$\overline{2}$ or m	Dih_1		2	12
C_{2h}	$[2^+,2]$		2*	2/m	$Z_2 \times Dih_1$		4	6
S_2	$[2^+,2^+]$		×	$\overline{1}$	2 or Z_2		2	12
T	$[3,3]^+$		332	23	A_4		12	2
D_3	$[2,3]^+$		322	3	Dih_3		6	4
D_2	$[2,2]^+$		222	222	Dih_4		4	6
C_3	$[3]^+$		33	3	Z_3		3	8
C_2	$[2]^+$		22	2	Z_2		2	12
C_1	$[\ ]^+$	○	11	1	Z_1	•	1	24

Solids with chiral tetrahedral symmetry


The Icosahedron colored as a **snub tetrahedron** has chiral symmetry.

Solids with full tetrahedral symmetry

Class	Name	Picture	Faces	Edges	Vertices
Platonic solid	tetrahedron		4	6	4
Archimedean solid	truncated tetrahedron		8	18	12
Catalan solid	triakis tetrahedron		12	18	8
Near-miss Johnson solid	Truncated triakis tetrahedron		16	42	28
	Tetrated dodecahedron		28	54	28
Uniform star polyhedron	Tetrahemihexahedron		7	12	6

See also

- octahedral symmetry
- icosahedral symmetry
- binary tetrahedral group

References

- Peter R. Cromwell, *Polyhedra* (1997), p. 295
- *The Symmetries of Things* 2008, John H. Conway, Heidi Burgiel, Chaim Goodman-Strass, ISBN 978-1-56881-220-5
- **Kaleidoscopes: Selected Writings of H.S.M. Coxeter**, edited by F. Arthur Sherk, Peter McMullen, Anthony C. Thompson, Asia Ivic Weiss, Wiley-Interscience Publication, 1995, ISBN 978-0-471-01003-6 [1] (http://www.wiley.com/WileyCDA/WileyTitle/productCd-0471010030.html)
- N.W. Johnson: *Geometries and Transformations*, (2015) Chapter 11: Finite symmetry groups

External links

- Weisstein, Eric W., "Tetrahedral group" (http://mathworld.wolfram.com/TetrahedralGroup.html), *MathWorld*.

Categories: Finite groups | Rotational symmetry