4. Extension to more variables

Bookmark this page

Problem Set B due Sep 13, 2021 20:30 IST Completed

Apply

4

1.0/1 point (graded)
Find the maximum area of a triangle whose vertices lie on a circle of radius r.

3*sqrt(3)*r^2/4 **Answer:** 3*sqrt(3)/4*r^2

? INPUT HELP

Solution:

The hardest part is setting up coordinates for this problem so that we can find formulas easily. If we have three points on a circle (x_1, y_1), (x_2, y_2), and (x_3, y_3), we can think of forming a triangle between the two vectors $\langle x_1 - x_3, y_1 - y_3\rangle$ and $\langle x_2 - x_3, y_2 - y_3\rangle$.

The first step is to choose one point that we can use as (x_3, y_3). One way to do this is to use a circle that is not centered at the origin, but instead passes through the origin so we can use the origin as one of our points. One way to do this is to center the circle at the point $(r, 0)$. The formula for this circle is

$$(x-r)^2 + y^2 = r^2 \tag{4.307}$$

and one particularly nice point on this circle is the origin $(0,0)$. So we choose $(x_3, y_3) = (0,0)$, which has already simplified the expression for the two vectors that describe the triangle.

We now need a formula for the area of the triangle formed by the vectors $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$.

Let's think of $\langle x_1, y_1\rangle$ as the base of the triangle. To find the height, of the triangle, we need to find the length of the component of $\langle x_2, y_2\rangle$ that points in the same direction as the vector that is normal to $\langle x_1, y_1\rangle$.

1. A vector normal to $\langle x_1, y_1\rangle$ is $\langle -y_1, x_1\rangle$.

2. The height is the magnitude of the component of $\langle x_2, y_2\rangle$ in the direction $\langle -y_1, x_1\rangle$ is given by

$$\langle x_2, y_2\rangle \cdot \frac{\langle -y_1, x_1\rangle}{\sqrt{y_1^2 + x_1^2}} = \frac{x_1 y_2 - x_2 y_1}{\sqrt{y_1^2 + x_1^2}} \tag{4.308}$$

3. The area of the triangle is the magnitude of base times the height, over 2. This is

$$\frac{1}{2}\underbrace{\sqrt{x_1^2 + y_1^2}}_{\text{base}} \cdot \underbrace{\frac{x_1 y_2 - x_2 y_1}{\sqrt{y_1^2 + x_1^2}}}_{\text{height}} = \frac{1}{2}(x_1 y_2 - x_2 y_1) \tag{4.309}$$

Therefore we are trying to maximize the function of 4 variables

$$f(x_1, y_1, x_2, y_2) = \frac{1}{2}(x_1 y_2 - x_2 y_1).$$

Calculator Hide Notes

Subject to the constraint

$$\begin{aligned} (x_1 - r)^2 + y_1^2 &= r^2 && (4.311) \\ (x_2 - r)^2 + y_2^2 &= r^2 && (4.312) \end{aligned}$$

Note that we can turn this into one function with 4 variables by adding them together: the level curve is $g(x_1, y_1, x_2, y_2) = 2r^2$ where $g(x_1, y_1, x_2, y_2) = (x_1 - r)^2 + y_1^2 + (x_2 - r)^2 + y_2^2$.

Let's solve the Lagrange multiplier problem!

$$\begin{aligned} \nabla f &= \frac{1}{2}\langle y_2, -x_2, -y_1, x_1\rangle && (4.313) \\ \nabla g &= \langle 2(x_1 - r), 2y_1, 2(x_2 - r), 2y_2\rangle && (4.314) \end{aligned}$$

The Lagrange multiplier problem asks us to solve a system of 4 equations. We omit the common constant multiples by absorbing them into the Lagrange multiplier λ.

$$\begin{aligned} y_2 &= \lambda(x_1 - r) && (4.315) \\ -x_2 &= \lambda y_1 && (4.316) \\ -y_1 &= \lambda(x_2 - r) && (4.317) \\ x_1 &= \lambda y_2 && (4.318) \end{aligned}$$

Note that we get two expression for λ:

$$\begin{aligned} \lambda &= \frac{-x_2}{y_1} && (4.319) \\ &= \frac{x_1}{y_2} && (4.320) \end{aligned}$$

Plug in these values of λ to eliminate λ and get expressions for y_2 in terms of x_1;

$$\begin{aligned} y_2 &= \lambda(x_1 - r) && (4.321) \\ &= \frac{x_1}{y_2}(x_1 - r) && (4.322) \\ y_2^2 &= x_1^2 - rx_1 && (4.323) \end{aligned}$$

and y_1 in terms of x_2.

$$\begin{aligned} -y_1 &= \lambda(x_2 - r) && (4.324) \\ &= \frac{-x_2}{y_1}(x_2 - r) && (4.325) \\ -y_1^2 &= -x_2^2 + rx_2 && (4.326) \end{aligned}$$

Now we can plug in these expressions for y_1^2 and y_2^2 into our two constraint equations to get a system of two equations in two unknowns x_1 and x_2.

$$\begin{aligned} (x_1 - r)^2 + y_1^2 &= r^2 \text{ substitute for } y_1 && (4.327) \\ (x_1 - r)^2 + (x_2^2 - rx_2) &= r^2 \text{ expand and simplify} && (4.328) \\ x_1^2 - 2rx_1 + x_2^2 - rx_2 &= 0 && (4.329) \\ (x_2 - r)^2 + y_2^2 &= r^2 \text{ substitute for } y_2 && (4.330) \\ (x_2 - r)^2 + x_1^2 - rx_1 &= r^2 \text{ expand and simplify} && (4.331) \\ x_2^2 - 2rx_2 + x_1^2 - rx_1 &= 0 && (4.332) \end{aligned}$$

Calculator Hide Notes

Subtracting (4.332) from (4.329) we get

$$-rx_1 + rx_2 = 0 \qquad \implies x_1 = x_2 \tag{4.333}$$

Now we can replace x_2 by x_1 in (4.326), plug back into our constraint equation (4.311) to solve for x_1.

$$\begin{aligned}
y_1^2 &= x_2^2 - rx_2 && (4.334)\\
&= x_1^2 - rx_1 && (4.335)\\
(x_1 - r)^2 + y_1^2 &= r^2 && (4.336)\\
(x_1 - r)^2 + x_1^2 - rx_1 &= r^2 && (4.337)\\
x_1^2 - 2rx_1 + r^2 + x_1^2 - rx_1 &= r^2 && (4.338)\\
2x_1^2 - 3rx_1 &= 0 && (4.339)\\
x_1\left(2x_1 - 3r\right) &= 0 && (4.340)
\end{aligned}$$

Therefore since we cannot choose x_1 equal to zero since one point on our triangle is already at the point $(0, 0)$ on our circle, we must have $x_1 = x_2 = \dfrac{3r}{2}$.

Plugging back into (4.311) to solve for y we get

$$\begin{aligned}
\left(\frac{3r}{2} - r\right)^2 + y^2 &= r^2 && (4.341)\\
\frac{r^2}{4} + y^2 &= r^2 && (4.342)\\
y^2 &= \frac{3}{4}r^2 && (4.343)\\
y &= \pm\frac{\sqrt{3}}{2}r && (4.344)
\end{aligned}$$

Note that there are two possible y values and these give us the y_1 and y_2.

Finally, we plug into our formula for the area to get

$$\frac{1}{2}\left(\frac{3r}{2}\frac{\sqrt{3}}{2}r - \frac{3r}{2}\frac{\left(-\sqrt{3}\right)}{2}r\right) = \frac{3\sqrt{3}}{4}r^2$$

Submit You have used 1 of 5 attempts

Answers are displayed within the problem

4. Extension to more variables

Hide Discussion

Topic: Unit 3: Optimization / 4. Extension to more variables

Add a Post

Show all posts by recent activity

A far simpler way using basic geometry and trig. 11
Hi! I found the official solution to be quite complex. I found a simpler method which assumes that the triangle is an equilateral one, w...

The number of constraints 4
Looking at the provided solutions, aren't there suppose to be two contraints in this problem? Was it correct to merge them into a sin...

My solution (sum of areas of three triangles)
(TA redacts answer) [solution][1] [1]: https://edxuploads.s3.amazonaws.com/1631045545648999302657.i...

Calculator Hide Notes

- [Another approach using cosine and sine functions](#) 6
- [STAFF] Problem is not shown; error message.
  The problem is not shown, and the browser gives the following error message: "Could not format HTML for problem. Contact course ... 5
- Another brutal solution, demanding CAS calculator 2
- [STAFF] Alternate simpler solution 8
- Another simple method using tools at hand 12
- Solving geometry problem using calculus
  First of all, thank you so much for including a geometry problem into this course. It's a really challenging question for me, but I learne... 2
- Use symmetry!
  There is no privileged orientation for the triangle, so your first clue is that the triangle must by symmetrical. So some basic trig is all ... 1
- Simple Solution
  It's easier to solve this problem by minimizing the region outside the inscribed triangle. That is, by determining the area contained wi... 1
- Area of triangle from cross product determinant formula 2
- [Staff] PSB Question 4: EDx Status code: -9 error
  Hello. You asked to let you know if this error occurs. It just did here. As advised I resubmitted my answer a few seconds later and on ... 2
  Community TA

< Previous

Next Up: Rate this unit >
1 min + 1 activity

edX

About
Affiliates
edX for Business
Open edX
Careers
News

Legal

Terms of Service & Honor Code

Calculator Hide Notes

Terms of Service & Honor Code
Privacy Policy
Accessibility Policy
Trademark Policy
Sitemap

Connect

Blog
Contact Us
Help Center
Media Kit
Donate

Calculator Hide Notes