Optional Mathematics

Grade 10

Government of Nepal
Ministry of Education, Science and Technology
Curriculum Development Centre
Sanothimi, Bhaktpur
Nepal

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Preface

The curriculum and curricular materials have been developed and revised on a regular basis with the aim of making education objective-oriented, practical, relevant and job oriented. It is necessary to instill the feelings of nationalism, national integrity and democratic spirit in students and equip them with morality, discipline and self-reliance, creativity and thoughtfulness. It is essential to develop in them the linguistic and mathematical skills, knowledge of science, information and communication technology, environment, health and population and life skills. it is also necessary to bring in them the feeling of preserving and promoting arts and aesthetics, humanistic norms, values and ideals. It has become the need of the present time to make them aware of respect for ethnicity, gender, disabilities, languages, religions, cultures, regional diversity, human rights and social values so as to make them capable of playing the role of responsible citizens. This textbook for grade nine students as an optional mathematics has been developed in line with the Secondary Level Optional Mathematics Curriculum, 2074 so as to strengthen mathematical knowledge, skill and thinking on the students. It is finalized by incorporating recommendations and feedback obtained through workshops, seminars and interaction programmes.

The textbook is written by Prof. Dr. Siddhi Prasad Koirala, Mr. Ramesh Prasad Awasthi and Mr. Shakti Prasad Acharya. In Bringing out the textbook in this form, the contribution of the Director General of CDC Dr. Lekha Nath Poudel is highly acknowledged. Similarly, the contribution of Prof. Dr. Ram Man Shrestha, Mr. Laxmi Narayan Yadav, Mr. Baikuntha Prasad Khanal, Mr. Krishna Prasad Pokharel, Mr. Anirudra Prasad Neupane, Ms. Goma Shrestha, Mr. Rajkumar Mathema is also remarkable. The subject matter of the book was edited by Dr. Dipendra Gurung, Mr. Jagannath Adhikari and Mr. Nara Hari Acharya. The language of the book was edited by Mr. Nabin Kumar Khadka. The layout of this book was designed by Mr. Jayaram Kuikel. CDC extends sincere thanks to all those who have contributed to developing this textbook.

This book contains various mathematical concepts and exercises which will help the learners to achieve the competency and learning outcomes set in the curriculum. Efforts have been made to make this textbook as activity-oriented, interesting and learner centered as possible. The teachers, students and all other stakeholders are expected to make constructive comments and suggestions to make it a more useful textbook.

2076 Curriculum Development Centre
Sanothimi, Bhaktapur

Table of Content

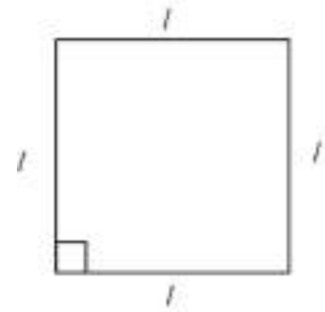
Unit 1

Algebra

1.1 Function

1.1.0 Review

Study the square given and answer the following questions:



(a) What is the area function 'f' in terms of side (l)?

(b) What is the perimeter function 'g' in terms of side (l)?

(c) Are functions 'f' and 'g' one to one?

(d) Are functions 'f' and 'g' onto?

Study the following information and relate them with function.

(a) To each person, there corresponds an annual income.

(b) To each square, there corresponds an area.

(c) To each number, there corresponds its cube.

Each (a), (b) and (c) are examples of function.

Take an example of function and represent it in different forms. Discuss about the graph among your friends.

Graphs provide a means of displaying, interpreting and analyzing data in a visual form. To graph an equation is to make a drawing that represents the solutions of that equation. When we draw the graph of an equation, we must remember the following points:

(a) Calculate solutions and list the ordered pairs in a table.

(b) Use graph paper and scale the axes.

(c) Plot the ordered pairs, look for patterns and complete the graph with the equation being graphed.

1.1.1.(a): Algebraic function and it's graph

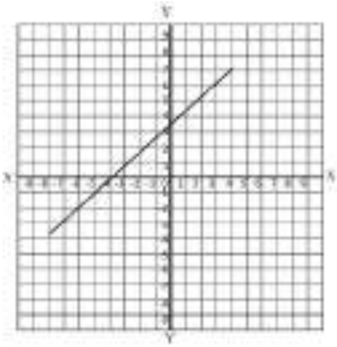
Study the following functions and think about their graphical representation:

$f(x) = x + 5$, $f(x) = x^2$, $f(x) = x^3$

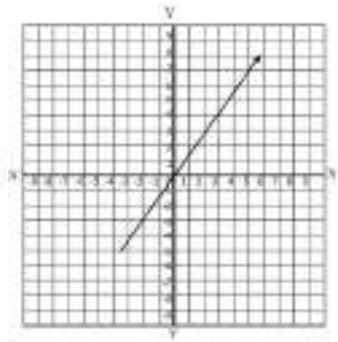
The algebraic equation consisting of the least one variable is called the algebraic function.

A function f is **linear function** if it can be written as $f(x) = mx + c$, where m and c are constant.

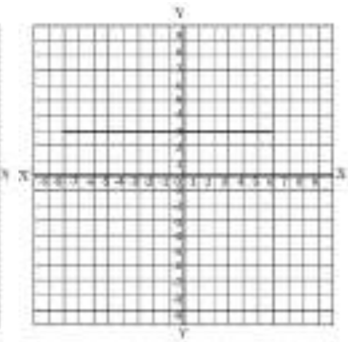
If m = 0, the function is **constant function and written as** $f(x) = c$. If m = 1 and c = 0, the function is the **identity function written as** $f(x) = x$.



Linear Fuction



Identity function



Constant function

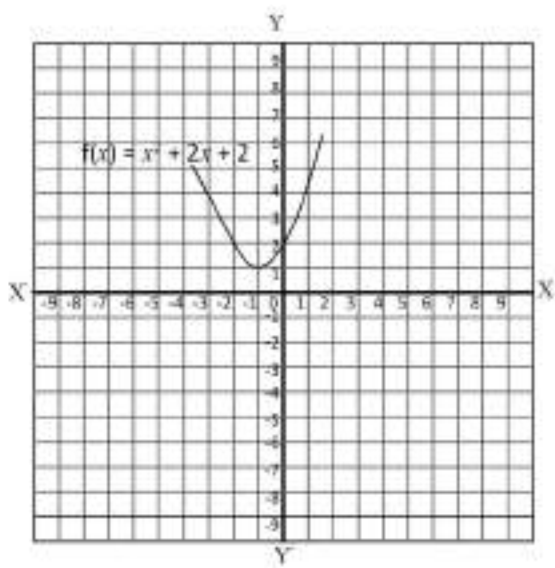
functions like f(x) = ax^2 + bx + c, a ≠ 0 and $f(x)$ = ax^3 + bx^2 + cx + d, a ≠ 0 are examples of Non-linear functions.

f(x) = ax^2 + bx + c, a ≠ 0 is quadratic function where a, b, and c are real numbers. The graph of quadratic function is a parabola.

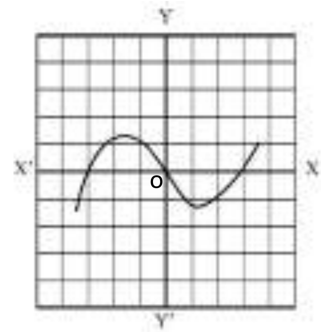
The vertex form of $f(x)$ = ax^2 + bx + c, a ≠ 0 is $f(x)$ = a$(x$-h$)^2$ + k

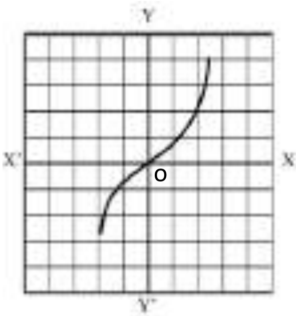
For example, $f(x) = x^2 + 2x + 2 = (x-(-1))^2 + 1$. The vertex of the parabola is (h, k) = (-1, 1). When, x = 0, the curve meets y-axis at (0, 2).

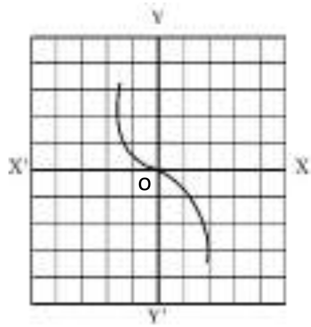
x = h is the axis of parabola about which the curve is symmetric. If a > 0, the curve turns upward from vertex and it turns downwards for a < 0.



The function defined as
$f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$ is
called cubic function, where a, b, c and d are real numbers.
The nature of curve in graph is as shown at the right.
When b, c, d are non-zero, the curve meets X-axis at
three different points.



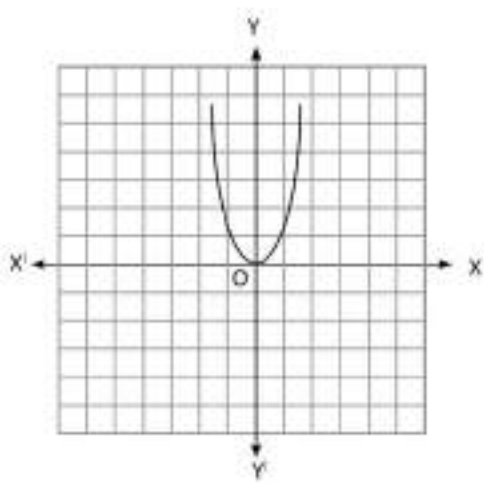




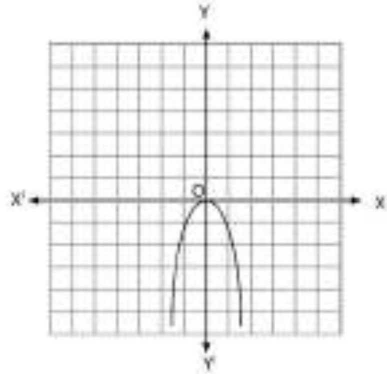
If each of b, c, d is zero, then the curve passes through the origin.

Example 1

(a) When we draw the graph of $y = 2x^2$, we can take different values of x and find their corresponding y-values. Representing the ordered pairs (x, y) in graph, we can find the shape of curve in graph.



(b) When we draw the graph of $y = -2x^2$, we follow the same steps as we do in (a) and give the shape of curve in graph.



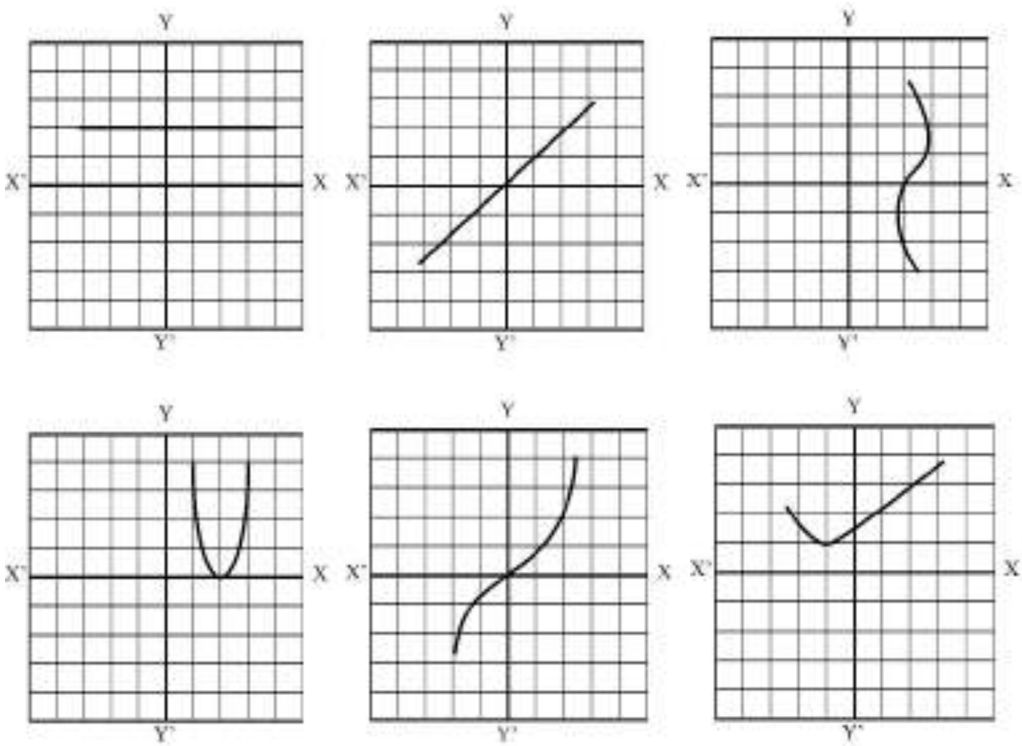
Exercise: 1.1.1 (a)

1. (a) Define linear function with example.

   (b) What is the coordinates of vertex of $f(x) = ax^2 + bx + c$, $a \neq 0$

   (c) Identify the identity function: $f(x) = 5$ and $f(x) = x$.

2. Study the following graphs and identify their nature as identity, constant, quadratic and cubic function.



3. Draw the graph of the following function

   (a) y = $x + 2$ (b) y = 6 (c) y = x^2 (d) y = $-x^2$

   (e) y = x^3

4. Pemba estimates the minimum ideal weight of a woman, in pounds is to multiply her height, in inches by 4 and subtract 130. Let y = minimum ideal weight and x = height.

   (a) Express y as a linear function of x.

   (b) Find the minimum ideal weight of a woman whose height is 62 inches.

   (c) Draw the graph of height and weight.

5. Investigate the nature of graph showing linear, quadratic and cubic function in our daily life. Make a report and present it in classroom.

1.1.1 (b): Trigonometric function

Discuss about trigonometric ratios of any angle in your classroom. Also, tell any nine relations among trigonometric ratios of any magnitude.

The function which relates angles and their measurement to the real number is called trigonometric function. It associates an angle with the definite real number. We know sin $(x + 2\pi)$ = sin$x$, cos $(x + 2\pi)$ = cosx and tan$(x + \pi)$ = tanx. So, if a function $f(x + k) = f(x)$ for positive value of k. k is called period for $f(x)$.

In case of sin$x$ and cos$x$, the period is taken as 2π, but for tanx, it is taken as π (Why?) Trigonometric functions are said to be non-algebraic or transcendental functions.

(i) **Graph of sin$x$ or sine – graph:** Let us take the values of x differing 90° and corresponding values of y for y = sin$x$. The maximum and minimum values of sin$x$ are 1 and -1 respectively.

x:	-360°	-270°	-180°	-90°	0°	90°	180°	270°	360°
sin$x$:	0	1	0	-1	0	1	0	-1	0

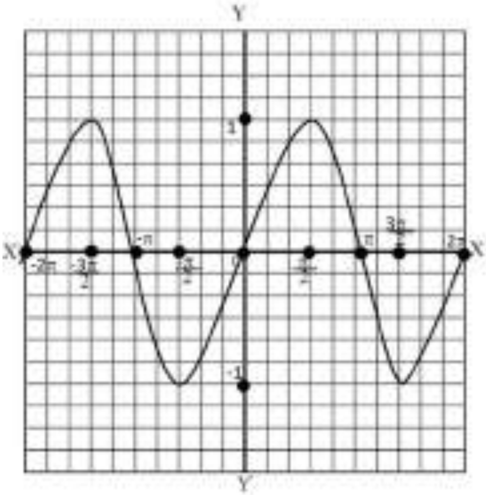
y = sin$x$

Domain: $-360° \le x° \le 360°$

$-2\pi \le x \le 2\pi$

Range: $-1 \le y \le 1$

Period: 2π



(ii) Graph of cos$x$ or cosine – graph

Let us take the values of x differing 90° and corresponding values of y for y = cos$x$. The maximum and minimum values of cos$x$ are 1 and -1 respectively.

x	-360°	-270°	-180°	-90°	0°	90°	180°	270°	360°
cos$x$	1	0	-1	0	1	0	-1	0	1

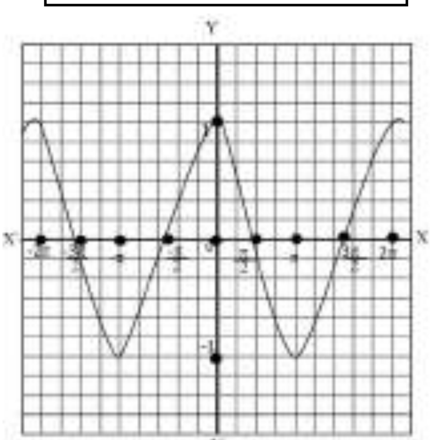
y = cos$x$

Domain: $-360° \leq x° \leq 360°$

$-2\pi \leq x \leq 2\pi$

Range: $-1 \leq y \leq 1$

Period: 2π



(iii) Graph of tan$x$ or tangent - graph

Let us take the values of x differing 90° and the corresponding values of y.
For y = tan$x$.

As x passes through – 360° to 360°, the values of tan$x$ suddenly changes from very large positive and negative values. The line parallel to the y-axis corresponding to the odd multiples of 90° are continuously approached by the graph on either side, but never actually meet. Such lines are called asymptote to the curve.

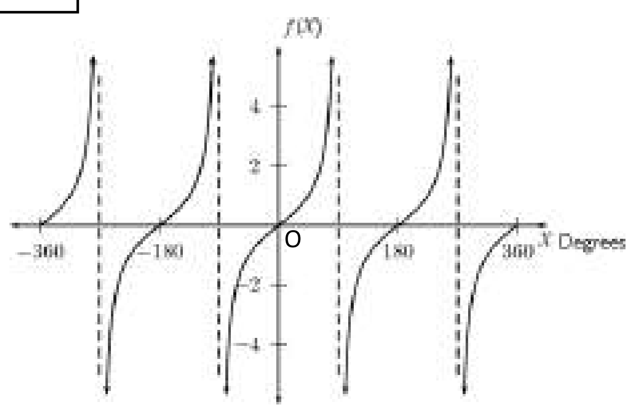
x	-360°	-270°	-180°	-90°	0°	90°	180°	270°	360°
tan$x$	0	Not defined	0	Not defined	0	Not defined	0	No defined	0

y = tan$x$

range: $-\infty < x < \infty$

domain: $-\infty < x < \infty$

period: π



Note: For least value of $x = (2\pi)$, $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$.
So, period of sine and cosine is 2π. But $\tan(x + \pi) = \tan x$, so period of tangent is π.

Exercise: 1.1.1. (b)

1. Write the maximum and minimum values of the following:

   (a) $f(x) = \sin x$ (b) $f(x) = \cos x$ (c) $f(x) = \tan x$

2. Write the period of the following:

   (a) $f(x) = \sin x$ (b) $f(x) = \cos x$ (c) $f(x) = \tan x$

3. Draw the graph of

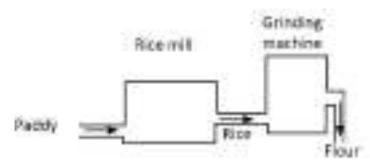
   (a) $f(x) = \sin x\ (-\pi \le x \le \pi)$ (b) $f(x) = \cos x\ (o \le x \le 2\pi)$

   (c) $f(x) = \tan x\ (0 \le x \le \pi)$ (d) $f(x) - 2\sin x\ (-\pi/2 < x < \pi/2)$

4. Study the topic 'sound' in physics of your science book and find the nature of longitudinal wave. Relate this concept with trigonometric function.

1.1.2 Composition of function

Study the following situations:



(i) When paddy is kept into the rice mill, rice comes out.

(ii) When the rice is kept in grinding machine, flour comes out from grinding machine.

(iii) The composite machine produces flour from paddy.

How can we link this situation in mathematical form? Discuss.

Again, Let us suppose two real-valued function, f = {(1,5), (2, 6), (3, 7), (4, 8)} and g = {(5, 25), (6, 36), (8, 64), (7, 49)}. Now,

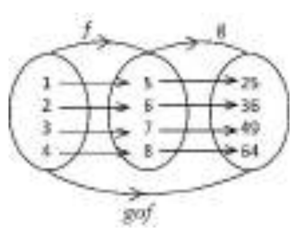
Answer the following questions:

(i) What is the domain set of function g? (ii) What is the range set of function f?

(iii) What is the range set of function g? (iv) What is the domain set of function f?

(v) What is the relation between range of 'f' and domain of 'g'?

(vi) Can we show this situation in combined form given as below?



Let, f and g are two real-valued functions. The composition of f and g, is defined as

- $(f\circ g)(x) = f(g(x))$: Where x is in the domain of g and $g(x)$ is in the domain of f.
- $(g\circ f)(x) = g(f(x))$: Where x is in the domain of f and $f(x)$ is in the domain of g.

Note: (i) 'g∘f' is read as composite of f and g.

(ii) 'f∘g' is read as composite of g and f.

(iii) 'g∘f' and 'f∘g' cannot be defined when (domain of g) $\cap$ (Range of f) = ϕ and (domain of f) $\cap$ (range of g) = ϕ

(iv) for 'g∘f' range of f is subset of domain of 'g' and domain of 'f' = domain of g∘f.

(v) for 'f∘g', range of g is subset of domain of 'f' and domain of 'g' = domain of f∘g.

Example 1

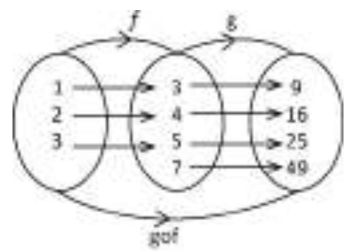
In the adjoining mapping diagram given at right,

(i) Write the range of function f.

(ii) Write the domain of function g.

(iii) Write 'g∘f' in ordered pair form.

(iv) Does 'f∘g' exist? Give reason.



Solution:

The given mapping diagram helps us in visualizing the composite function g o f.

(i) range of f = {3, 4, 5} $\subseteq$ domain of g.

(ii) domain of g = {3, 4, 5, 7}

(iii) g o f = {(1, 9), (2, 16), (3, 25)}

(iv) f o g does not exist because (range of g) $\cap$ (domain of f) = ϕ.

Example 2

Given f and g are two real valued functions defined as f(x) = $\sqrt{x}$ and g(x) = x – 1. Answer the following questions from given information:

(i) domain of f (ii) (g∘f)(x) (iii) domain of g (iv) (f∘g)(x)

Solution: Here,

f(x) = $\sqrt{x}$. the negative real number does not satisfy the function. So,

(i) domain of f is $0 \leq x < \infty$ [i.e. real numbers greater than or equal to 0.)

Again, (ii) (g∘f)(x) = g(f(x)) = g($\sqrt{x}$) $= \sqrt{x} - 1$

(iii) Domain of g is the set of all real numbers each of the real number satisfy g(x).
So, domain of g = $-\infty < x < \infty$.

(iv) $(f\circ g)(x) = f(g(x)) = f(x-1) = \sqrt{x-1}$

Example 3

Let $f(x) = 3 + 2x$ for all $x \in R$ and $g(x) = x^2 + 1$ for all $x \in R$, then find the formula for the following functions:

(a) $(f\circ f)(x)$ (b) $(g\circ f)(x)$ (c) $(f\circ g)(x)$ (d) $(g\circ g)(x)$

Solution: Here,

$f(x) = 3 + 2x$ for all $x \in R$, $g(x) = x^2 + 1$ for all $x \in R$

Now, (a) $(f\circ f)(x) = f(f(x) = f(3 + 2x) = 3 + 2(3 + 2x) = 3 + 6 + 4x = 4x + 9$

(b) $(g\circ f)(x) = g(f(x)) = g(3 + 2x) = (3 + 2x)^2 + 1 = 9 + 12x + 4x^2 + 1 = 4x^2 + 12x + 10$

(c) $(f\circ g)(x) = f(g(x)) = f(x^2 + 1) = 3 + 2(x^2 + 1) = 3 + 2x^2 + 2 = 2x^2 + 5$

(d) $(g\circ g)(x) = g(g(x)) = g(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 1 + 1 = x^4 + 2x^2 + 2$

Example 4

If $f(x) = x^2 + 1$ and $g(x) = x - 3$, find

(a) $(f\circ g)(x)$ (b) $g(f(x))$ (c) $f(g(2))$ (d) $(g\circ f)(3)$

Solution: Here,

$f(x) = x^2 + 1$, $g(x) = x - 3$,

(a) $(f\circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2 + 1 = x^2 - 6x + 9 + 1 = x^2 - 6x + 10$

(b) $g(f(x)) = g(x^2 + 1) = x^2 + 1 - 3 = x^2 - 2$

(c) $f(g(2)) = 2^2 - 6 \text{ x } 2 + 10 = 4 - 12 + 10 = 2$

(d) $g(f(3)) = 3^2 - 2 = 9 - 2 = 7$

Exercise 1.1.2

1. (a) Let f: A$\rightarrow$ B such that f(x) = y and g: B$\rightarrow$ C such that g(y) = Z. Name the function defined from A to C.

   (b) Write the difference between $(f\circ g)(x)$ and $(g\circ f)(x)$.

   (c) Define composition function of f and g.

2. Given f and g are two real-valued functions defined as below. Find (i) domain of f (ii) domain of g (iii) domain of $(f\circ g)$ (iv) range of $(g\circ f)$ in each of the following if exist.

(a) f = {(3, 4), (5, 6), (9, 10)} and g = {(4, 16), (6, 36), (10, 100)}

(b) f = {(1, 2), (2, 3), (3, 4)} and g = {(2, 3), (3, 4), (4, 5)}

3. Given that $f(x) = 2x - 5$ and $g(x) = x^2 - 2x + 6$, calculate:

   (a) $(f \circ g)(x)$ and $(g \circ f)(x)$ (b) $(f \circ g)(5)$ and $(g \circ f)(4)$

   (c) $(g \circ g)(2)$ and $(f \circ f)(9)$ (d) $(f \circ g)(-4)$ and $(g \circ f)(-4)$

4. (a) If $f(x) = x$, $g(x) = x + 1$ and $h(x) = 2x - 1$ then find $f(g \circ h)(x))$ and $(g \circ (h \circ f)(x))$

   (b) Given that $f(x) = 2x - 3$, $g(x) = x^3 + 2$ and $h(x) = x^2 - 2x + 3$, find $(f \circ (g \circ h)(x))$ and $((h \circ f) \circ g(x))$. (Taking composition of two functions as a single function).

5. If f and g are linear functions, what can you say about the domain of $(f \circ g)$ and $(g \circ f)$? Explain.

6. Dolma determines the domain of $f \circ g$ by examining only the formula for $(f \circ g)(x)$. Is her approach valid? Why or why not?

7. Write yourself any two real-valued function. Find their composition.

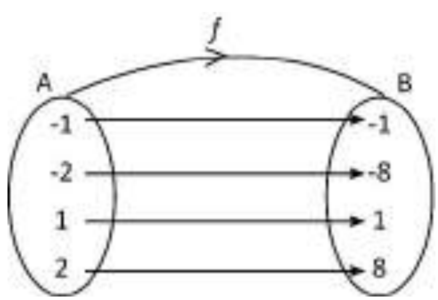
8. A stone is thrown into a pond, creating a circular ripple that spreads over the pond in such a way that the radius is increasing at the rate of 3 ft/sec.

   (a) Find a function r(t) for the radius in terms of t.

   (b) Find a function A(r) for the area of the ripple in terms of the radius r.

   (c) Find $(A \circ r)(t)$. Explain the meaning of this function.

1.1.3 Inverse of a function



Study the function 'f' given in mapping diagram and answer the following:

(a) What is the domain of f? (b) What is the range of f?

(c) Is the functions 'f' one to one and onto (d) What is the formula for 'f'?

Let, f: A→ B be a one to one and onto function then a function g: B→ A such that for each $x \in B$, $g(x) \in A$ is called inverse of f. It is denoted by $g = f^{-1}$.

In the above mapping diagram, f^{-1} exists because it satisfies the following conditions.

If f^{-1} exists, the domain and range of f are range and domain of f^{-1} respectively. If y is the image of x under f then x is image of y under f^{-1}. In the above mapping diagram,

$f(-1) = -1$ if and only if $f^{-1}(-1) = -1$

$f(-2) = -8$ if and only if $f^{-1}(-8) = -2$

$f(1) = 1$ if and only if $f^{-1}(1) = 1$

$f(2) = 8$ if and only if $f^{-1}(8) = 2$

Again, $f(x) = x^3$ if and only if $f^{-1}(x) = \sqrt[3]{x}$

$f = \{(-1, -1), (-2, -8), (1, 1), (2, 8)\}$

$f^{-1} = \{(-1, -1), (-8, -2), (1, 1), (8, 2)\}$

Given function $f: A \rightarrow B$, the inverse image of an element $y \in B$ with respect to f is denoted by $f^{-1}(y)$. That is $f(x) = y$ if and only if $x = f^{-1}(y)$. It is read as 'f inverse of y'.

$f: A \rightarrow B$ such that $f(x) = \{y: x \in A\}$

$f^{-1}: B \rightarrow A$ such that $f^{-1}(y) = \{x \in A : y = f(x)\}$

Example 1

Let $f: R \rightarrow R$ is a function such that $f(1) = 2$, $f(5) = 10$, $f(-4) = -8$ and $f(-3) = -6$

(i) Write function 'f' in ordered pair form.

(ii) Represent 'f' in mapping diagram.

(iii) Represent 'f^{-1}' in ordered pair form.

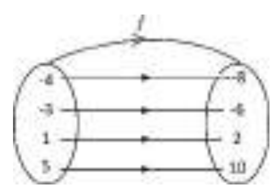
(iv) Represent 'f^{-1}' in mapping diagram.

Solution: Here,

f is function, defined from set of real numbers to set of real number. Certain elements of real numbers are given, So,

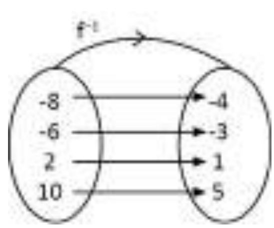
(i) $f = \{(1, 2), (5, 10), (-4, -8), (-3, -6)\}$

(ii)



(iii) $f^{-1} = \{(2, 1), (10, 5), (-8, -4), (-6, -3)\}$

(iv)



Example 2

Let f: R $\rightarrow$ R be defined by f(x) = 2x + 3 then find

(i) $f^{-1}(x)$ (ii) $f^{-1}(-4)$ (iii) $f^{-1}\left(\frac{1}{4}\right)$

Solution:

(i) Let, f: R $\rightarrow$ R such that f(x) = y, (x, y) $\in$ R x R

$f(x) = y$ if and only if $f^{-1}(y) = x$

Now, (i) y = f(x)

or, y = 2x + 3

or, y – 3 = 2x

or, $\frac{y-3}{2} = x$

or, $\frac{y-3}{2} = f^{-1}(y)$

or, $\frac{x-3}{2} = f^{-1}(x)$

$\therefore f^{-1}(x) = \frac{x-3}{2}$

Alternatively

f(x) = y = 2x + 3

y = 2x + 3

Interchanging the places of x and y we get

or, x = 2y + 3

or, x – 3 = 2y

or, $y = \frac{(x-3)}{2}$

$\therefore f^{-1}(x) = \frac{(x-3)}{2}$

(ii) $f^{-1}(-4) = \frac{1}{2}(-4-3)$

$= \frac{1}{2}(-7)$

$= -\frac{7}{2}$

(iii) $f^{-1}(\frac{1}{4}) = \frac{1}{2}\left(\frac{1}{4} - 3\right)$

$= \frac{1}{2}\left(\frac{1-12}{4}\right)$

$= \frac{1}{2}\left(\frac{-11}{4}\right)$

$= \frac{-11}{8}$

Example 3

Let f: R –{3} $\rightarrow$ R is defined as $f(x) = \frac{2x+5}{x-3}$. Find (i) $f^{-1}(x)$ (ii) $(f^{-1}of)(x)$

Solution

Here, domain of f is set of real numbers except 3.

(i) Let, y = f(x) i.e. x = $f^{-1}(y)$

Now, $f(x) = \frac{2x+5}{x-3}$

or, $y = \frac{2x+5}{x-3}$

or, $xy - 3y = 2x + 5$

or, $xy - 2x = 5 + 3y$

or, $x(y - 2) = 5 + 3y$

or, $x = \frac{5+3y}{y-2}$

or, $f^{-1}(y) = \frac{5+3y}{y-2}$

or, $f^{-1}(x) = \frac{5+3x}{x-2}$

(ii) $(f^{-1}\circ f)(x)$

$= f^{-1}(f(x))$

$= f^{-1}\left(\frac{2x+5}{x-3}\right)$

$= \frac{5+3\frac{(2x+5)}{x-3}}{\frac{2x+5}{x-3}-2}$

$= \frac{\frac{5(x-3)+3(2x+5)}{x-3}}{\frac{2x+5-2(x-3)}{x-3}}$

$= \frac{5x-15+6x+15}{2x+5-2x+6}$

$= \frac{11x}{11} = x$

Note: $(f^{-1}of)(x) = (fof^{-1})(x)$

Example 4

Let f and g be real valued functions, defined as f = {(x, ax + 9)} and g(x) = 3x + 8. If $f^{-1}(10) = g^{-1}(11)$, find the value of a.

Solution:

Here, $f(x) = ax + 9$

Let, $y_1 = ax + 9$ [$f(x) = y_1$, i.e. $x = f^{-1}(y_1)$]

or, $y_1 - 9 = ax$

or, $\frac{y_1-9}{a} = x$

or, $\frac{y_1-9}{a}$ = $f^{-1}(y_1)$

or, $\frac{x-9}{a}$ = $f^{-1}(x)$

Again,

$y_2 = g(x)$ [$g(x) = y_2$ i.e. $x = g^{-1}(y_2)$

or, $y_2 = 3x + 8$

or, $y_2 - 8 = 3x$

or, $\frac{y_2-8}{3} = x$

or, $y_2 - 8 = g^{-1}(y_2)$

or, $x - 8 = g^{-1}(x)$

Now, $f^{-1}(10) = \frac{10-9}{a} = \frac{1}{a}$

$g^{-1}(11) = 11 - 8 = 3$

but, $f^{-1}(10) = g^{-1}(11)$

or, $\frac{1}{a} = 3$

or, $a = \frac{1}{3}$

$\therefore a = \frac{1}{3}$

Exercise 1.1.3

1. (a) Define inverse of function, f: R → R.

   (b) What is the relation between composition of a function and its inverse.

2. Represent the following functions in mapping diagram and find their inverse.

   (a) f = {(1, 2), (2, 3), (4, 5)} (b) g = {(1, 4), (2, 5), (3, 6)}

   (c) h = {(-2, 4), (-3, 9), (-6, 36)}

3. If *f* is the real-valued function, find

   (a) $f^{-1}(x)$ (b) $f^{-1}(6)$ (c) $f^{-1}\left(\frac{1}{4}\right)$

   (d) $f^{-1}(-2)$ in each of the following:

   (i) $f(x) = 3x + 1$ (ii) $f(x) = 2x - 5$

   (iii) $f(x) = \frac{x+1}{2}$ (iv) $f = \{(x, \frac{x-2}{x+2}), x \neq -2\}$

4. If $f(x) = x + 1$, $g(x) = 2x$, find

   (a) $(f \circ g^{-1})(x)$ (b) $(g \circ f^{-1})(x)$. f and g are real-valued functions.

5. (a) If $f(x) = 3x - 7$, $g(x) = \frac{x+2}{5}$ and $(g^{-1} \circ f)(x) = f(x)$, find the value of x. f and g are real-valued functions.

   (b) f is a real-valued function defined as $f(x) = 3x + a$. If $(f \circ f)(6) = 10$ then find the values of a and $f^{-1}(4)$.

6. Write the formula of volume and surface area of sphere in terms of radius. Find the functional relation and write their inverse.

1. 2. Polynomials

1.2.0 Review

Study the following algebraic expressions and answer the following questions in group.

(i) $4x + 5y$ (ii) $3y^3 - \frac{1}{2}y^2 + 5y - 7$ (ii) $7y^5 - 2y^{3/2} + 7\sqrt{y} - 6$

(iii) $3x^3 + 4x^2y + 7xy^2 + 9y^3$ (iv) $15xy^{3/2} - 12x^2y^{1/2}$ (v) $x^3 + x^2 - x + 6$

- Which are examples of polynomials and why?
- Which one is polynomial in one variable?
- What are the main characteristics of polynomial that you study in previous grades?

A polynomial in one variable is any expression of the type $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where n is non-negative integers and a_n, a_{n-1} … a_0 are real numbers, called coefficients a_nx^n is called the leading term of the polynomial. 'n' is degree of the polynomial. If $a_n \neq 0$.

In the above examples (i), (ii), (iv), (v) are examples of polynomials because they have non-negative integer in power of each term. (vi) is polynomial in one variable.

1.2.1 Division of Polynomials

Let us take two examples:

$x^2 - 3x + 2 = (x - 1)(x - 2)$

or, $(x^2 - 3x + 2) \div (x - 1) = (x - 2)$

Similarly,

$x^5 + 2x^3 + 4x^2 + 5 = (x^2 + 2)(x^3 + 4) - 3$

or, $(x^5 + 2x^3 + 4x^2 + 5) \div (x^2 + 2) = x^3 + 4 - \frac{3}{x^2+2}$

In the above examples, we divide one polynomial by another, we obtain a quotient and a remainder. If the remainder is 0 as in first example, then the divisor is a factor of the dividend. If the remainder is not 0, as in second examples, then the divisor is not a factor of the dividend.

When we divide a polynomial P(x) by a divisor D(x), we get quotient Q(x) and remainder R(x). The quotient Q(x) must have degree less than that of the dividend P(x).

Remainder R(x) must either be 0 or have degree less than that of the divisor D(x).

We can check the division by algorithm P(x) = D(x). Q(x) + R(x).

Example 1

Divide $x^2 - 5x + 6$ by $x - 2$ and find quotient and remainder.

Solution: Here,

We know, $x^2 - 5x + 6 \quad = x^2 - 3x - 2x + 6$

$= x(x - 3) -2(x - 3)$

$= (x - 3)(x - 2)$

Now, $(x^2 - 5x + 6) \div (x - 2) \quad = \frac{(x-3)(x-2)}{(x-2)} = (x - 3)$

Hence the quotient is (x-3) and remainder is 0.

Alternatively:

$$\begin{array}{r|l|l} x-2 & x^2-5x+6 & x-3 \\ & x^2-2x & \\ & - \quad + & \\ \hline & -3x \;+6 & \\ & -3x \;+6 & \\ & + \quad\; - & \\ \hline & \quad 0 & \end{array}$$

$\therefore$ Quotient = $(x - 3)$ and remainder = 0.

Steps for division:

- Arrange divisor and dividend in division form.
- First term of dividend is product of x and first term of divisor.
- Since division is repeated subtraction, we change the sign in second row.

Example 2

Divide $x^3 + 6x^2 - 25x + 18$ by $(x + 9)$ using long division method. Also write the quotient and remainder.

Solution: Here,

Writing the dividend and divisor in long division method, we get:

$$
\begin{array}{r l}
 & x^2 \quad - 3x \quad + 2 \\
(x+9) & \overline{)\, x^3 + 6x^2 - 25x + 18} \\
 & \underline{\underset{-}{x^3} \; \underset{-}{+ 9x^2}} \\
 & \quad -3x^2 - 25x + 18 \\
 & \quad \underline{\underset{+}{-3x^2} \; \underset{+}{-} \; 27x} \\
 & \qquad\qquad 2x + 18 \\
 & \qquad\qquad \underline{\underset{-}{}2x \underset{-}{+} 18} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$\therefore$ Quotient = $(x^2 - 3x + 2)$ and remainder = 0.

Example 3

Divide p(y) = $10y^3 + 3y^2 - 6y - 3$ by d(y) = y – 4 using long division method. Also find the quotient and remainder.

Solution

Writing the division and divisor in long division method,

We get,

$$
\begin{array}{r l}
 & 10y^2 + 43y + 166 \\
y - 4 & \overline{)\, 10y^3 + 3y^2 - 6y - 3} \\
 & \underline{\underset{-}{10y^3} \underset{+}{-} 40y^2} \\
 & \qquad 43y^2 - 6y - 3 \\
 & \qquad \underline{\underset{-}{}43y^2 \underset{+}{-} 172y} \\
 & \qquad\qquad 166y - 3 \\
 & \qquad\qquad \underline{\underset{-}{166y} \underset{+}{-} 664} \\
 & \qquad\qquad\quad 661
\end{array}
$$

$\therefore$ Quotient Q(y) = $10y^2 + 43y + 166$ and remainder (R) = 661

Exercise 1.2.1

1. (a) Define polynomial of one variable.

   (b) If p(x), q(x), d(x) and r(x) represent polynomial, quotient, divisor and remainder respectively. Write the relation among them.

2. Divide using long division method and find quotient and remainder in each of the following:

   (a) $x^2 - 10x + 21 \div (x - 3)$ (b) $x^3 + 2x^2 - 5x - 6 \div (x + 1)$

(c) $x^3 - 8 \div (x - 2)$ (d) $x^3 + 9x^2 + 27x + 27 \div (x + 3)$

3. Divide using long division method and find quotient and remainder.

   (a) $x^3 + 2x^2 - 5x - 7 \div (x + 1)$ (b) $x^3 - 10x^2 + 16x + 26 \div (x - 5)$

   (c) $2x^4 + 5x^2 - 3x - 7 \div (2x - 1)$ (d) $y^5 + y^3 - y \div (3\text{-}y)$

4. For the function $f(y) = y^3 - y^2 - 17y - 15$, use long division to determine whether each of the following is a factor of f(y) or not.

   (a) y + 1 (b) y + 3 (c) y + 5 (d) y – 1 (e) y – 5

5. For the polynomial $p(x) = x^4 - 6x^3 + x - 2$ and divisor $d(x) = x - 1$, use long division to find the quotient $Q(x)$ and the remainder $R(x)$ when $P(x)$ is divided by $d(x)$. Express $p(x)$. In the form of $d(x)$. $Q(x) + R(x)$. Write your finding.

Synthetic division

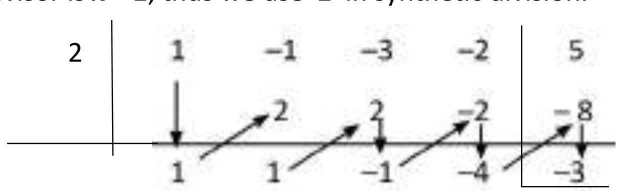
When the divisor is in the form of x – a, we can simplify using addition rather than subtraction. When the procedure is finished, the entire algorithm is known as synthetic division.

Example 1

Divide $x^4 - x^3 - 3x^2 - 2x + 5$ by $x - 2$ by synthetic division.

Solution:

Here, the devisor is $x - 2$, thus we use '2' in synthetic division.



- We 'bring down' the 1 [coefficient of x^4]. Then multiply it by 2 to get 2 and add to get 1.
- We then multiply 1 by 2 to get 2, add, and so on.

- The number -3 is the remainder. The others numbers 1, 1, -1, -4 are coefficients of the quotient, $x^3 + x^2 - x - 4$. In this case, the degree of the quotient is 1 less than the degree of the dividend and the degree of the divisor is 1.

Example 2

Divide $8x^3 - 1$ by $2x - 1$, using synthetic divisions method.

Solution:

When the divisor is not in the form of x – a, we first make it in the form of x – a, taking coefficient of x common. i.e. $2x - 1 = 2\left(x - \frac{1}{2}\right)$.

We write coefficient as zero for missing term in the dividend. Now, dividing by synthetic division method.

$$
\begin{array}{c|ccc|c}
\frac{1}{2} & 8 & 0 & 0 & -1 \\
 & \downarrow & 4 & 2 & 1 \\
\hline
 & 8 & 4 & 2 & 0
\end{array}
$$

So, quotient $= \frac{1}{2}(8x^2 + 4x + 2)$

$= 4x^2 + 2x + 1$

Remainder = 0

Since, divisor is $2\left(x - \frac{1}{2}\right)$, so quotient is $\frac{1}{2}(8x^2 + 4x + 2)$ (why?)

Exercise 1.2.2

1. (a) What is the degree of quotient when the degree of polynomial is 'n' in synthetic division?

   (b) What should be the expression of division in synthetic division?

2. Use synthetic division and divide in each of the following:

   (a) $x^3 + 8$ by $x - 2$

   (b) $2x^4 + 7x^3 + x - 12$ by $(x + 3)$

   (c) $4x^3 - 3x^2 + x + 9$ by $x - 2$

   (d) $2x^3 + 7x^2 - 8$ by $(x + 3)$

   (e) $8x^3 + 4x^2 + 6x - 7$ by $2x - 1$

1.2.3 (A) Some theorems on polynomials

Does $x - 2$ exactly divide $x^2 + 4x + 4$?

Divide and write the conclusion.

We can use 'Remainder theorem' for finding remainder and 'factor theorem' for finding factors of the polynomial.

(a) Remainder theorem: If a number 'a' is substituted for x in the polynomials f(x), the result f(a) is the remainder that would be obtained by dividing f(x) by x –a. That is f(x) = (x – a) Q(x) + R(a)

Proof: We know, $f(x)$ = d(x) x Q(x) + R(x)

or, f(x) = (x – a) Q(x) + R(x)

or, f(a) = (a – a) Q(a) + R(a)

or, f(a) = R(a)

$\therefore$ Remainder is f(a)

If the divisor is in the form of ax ± b, the remainder is $f\left(\pm\frac{b}{a}\right)$.

Example 1

If $f(x) = 2x^3 - 3x^2 + 4x + 7$ is divided by $x - 1$, find the remainder using remainder theorem.

Solution

Here, polynomial, $f(x) = 2x^3 - 3x^2 + 4x + 7$

divisor (x – a) = x – 1

by reminder theorem, remainder = f(a) = f(1)

So, f(1) $= 2 \times 1^3 - 3 \times 1^2 + 4 \times 1 + 7$

$= 2 \times 1 - 3 \times 1 + 4 \times 1 + 7$

$= 2 - 3 + 4 + 7$

$= 13 - 3 = 10.$

$\therefore$ Remainder (R) = 10

Example 2

Use remainder theorem and find the remainder: $(4x^3 + 2x^2 - 4x + 3) \div (2x + 3)$

Solution:

Here, let polynomial $f(x) = 4x^3 + 2x^2 - 4x + 3$

Divisor d(x) = 2x + 3 = $2\left(x+\frac{3}{2}\right) = 2\left(x-\left(\frac{-3}{2}\right)\right)$

By remainder theorem;

Remainder = f(a) = f(-3/2)

Now, f(-3/2) $= 4\times\left(\frac{-3}{2}\right)^3 + 2\times\left(\frac{-3}{2}\right)^2 - 4\left(\frac{-3}{2}\right) + 3$

$= 4\times\left(\frac{-27}{8}\right) + 2\times\frac{9}{4} + 4\times\frac{3}{2} + 3$

$= \frac{-27}{2} + \frac{9}{2} + \frac{12}{2} + 3$

$= \frac{-27+9+12+6}{2}$

$= \frac{0}{2} = 0$

∴ Remainder (R) = 0

Example 3

If $2x^4 + 2x^2 - 2x$ + k is divided by $x - 2$, the remainder is 5, find the value of k.

Solution:

Here, let f(x) = $2x^4 + 3x^2 - 2x$ + k

Divisor (x – a) = x – 2

By remainder theorem,

Remainder = f(a)

= f(2)

= 2 x $(2)^4$ + 3 x $(2)^2$ – 2 x 2 + k

= 32 + 12 – 4 + k

= 40 + k

By the question

Remainder (R) = f(2) = 5

or, 40 + k = 5

or, k = 5 - 40

k = –35

Hence, k = -35

Exercise 1.2.3 (A)

1. (a) State remainder theorem

   (b) If $ax + b$ ($a \neq 0$) divides f(x), what is the remainder?

2. Use remainder theorem and find the remainder in each of the following:

   (a) $(2x^3 - 5x^2 + x - 5) \div (x - 2)$

   (b) $(4x^3 + 7x^2 - 3x + 2) \div (x + 2)$

   (c) $(x^4 - 3x^2 + 15) \div (x - 1)$

   (d) $(x^5 + x^3 + 20) \div (2x - 1)$

   (e) $7x^4 - 6x^3 + 8x^2 - 10x + 9 \div (3x - 9)$

   (f) $6x^4 - 4x^3 + 6x^2 + 8x + 10 \div (2x + 3)$

   (g) $9x^5 - 7x^2 + 12x + 10 \div (3x + 1)$

3. (a) If $x^4 + 2x^2 - 4x + k$ is divided by $x - 2$, the remainder is 4, find the value of k, using remainder theorem.

   (b) If $x^3 - 9x^2 + (k + 1)x - 8$ is divided by $x - 5$, the remainder is 6, find the value of k, using remainder theorem.

   (c) If $x^3 - ax^2 + 8x + 11$ and $3x^3 - ax^2 + 7ax + 13$, both are divided by $(x - 1)$, remainder is same, find the value of a.

   (d) If $(x - 2)$ divides the polynomials $4x^3 + 2x^2 + kx + 5$ and $kx^2 + 5x + 4$ to get the same remainder, find the value of k.

4. Take a polynomial function. Take any three linear divisors in the form of $(x + a)$, $(ax + b)$ and $(ax - b)$. Use remainder theorem and find the remainder.

1.2.3 (B) Factor theorem

Let f(x) be a polynomial of degree n(n > 1) such that f(a) = 0, then $(x - a)$ is a factor of $f(x)$.

Proof: If we divide f(x) by $x - a$, we obtain a quotient and remainder using algorithm, $f(x) = (x - a).\ Q(x) + f(a)$

If f(a) = 0, we have $f(x) = (x - a).\ Q(x)$

So, $(x - a)$ is factor of $f(x)$.

Conversely: Let $(x - a)$ is a factor of $f(x)$ then remainder f(a) = 0, where $f(x)$ is a polynomial of degree n.

The factor theorem is used in factoring polynomials and hence, is solving polynomial and finding factors or zeros of polynomial function.

Example 1

Show that (x – 3) is a factor of $x^3 - 6x^2 + 11x - 6$.

Solution: Here,

Let, f(x) = $x^3 - 6x^2 + 11x - 6$ be the given polynomial. By factor theorem, (x – a) is factor of a polynomial f(x) if f(a) = 0.

Therefore, in order to prove that x – 3 is a factor of f(x), it is sufficient to show that f(3) = 0

Now, f(x) = $x^3 - 6x^2 - 11x - 6$

f(3) = $3^3 - 6 \times 3^2 + 11 \times 3 - 6$

= 27 – 54 + 33 – 6

= 60 – 60

= 0

Since, f(3) = 0, Hence, this shows that (x-3) is a factor of f(x).

Example 2

Find the value of a such that (x – 4) is a factor of $5x^2 - 7x^2 - ax - 28$.

Solution

Let, f(x) = $5x^3 - 7x^2 - ax - 28$ be the given polynomial. If (x – 4) is a factor of f(x), then f(4) = 0

or, $5 \times 4^3 - 7 \times 4^2 - a \times 4 - 28 = 0$

or, 5 x 64 – 7 x 16 – 4a – 28 = 0

or, 320 – 112 – 4a – 28 = 0

or, 180 – 4a = 0

or, 4a = 180

or, a = $\frac{180}{4}$ = 45

Hence, a = 45

Example 3

What must be added in f(x) = $3x^3 + 2x^2 + 5x + 6$, so that the result is exactly divided by $x - 1$?

Solution

Here, f(x) = $3x^3 + 2x^2 + 5x + 6$

When f(x) is divided by $x - 1$, the remainder f(1) = 0. It is only possible when we add certain number in f(x).

Let us add k in f(x), such that $3x^3 + 2x^2 + 5x + 6 + k$ is exactly divided by $x - 1$. Now, by factor theorem

$3 \times 1^2 + 2 \times 1^2 + 5 \times 1 + 6 + k = 0$

or, $3 + 2 + 5 + 6 + k = 0$

or, $16 + k = 0$

or, $k = -16$

$\therefore$ -16 is added in f(x).

Example 4

Using factor theorem, factorize the polynomial $x^3 + 6x^2 + 11x + 6$.

Solution

Let f(x) = $x^3 + 6x^2 + 11x + 6$. The constant term f(x) is equal to 6, and possible factors of 6 are ±1, ±2, ±3, ±6 putting x = 1 in f(x), w have

f(1) = $1^3 + 6 \times 1^2 + 11 \times 1 + 6 = 1 + 6 + 11 + 6 = 24$

Since, the remainder is non-zero so, (x-1) is not a factor of f(x).

Again, putting x = -1 in f(x), we have

f(-1) = $(-1)^3 + 6 \times (-1)^2 + 11 \times (-1) + 6$

$= -1 + 6 - 11 + 6$

$= 0$

$\therefore$ $x + 1$ is a factor of f(x)

Now using synthetic division, we have

$$
\begin{array}{r|rrr|r}
-1 & 1 & 6 & 11 & 6 \\
 & \downarrow & -1 & -5 & -6 \\
\hline
 & 1 & 5 & 6 & \boxed{0}
\end{array}
$$

$\therefore$ Quotient = $x^2 + 5x + 6$

$= x^2 + 3x + 2x + 6$

$= x(x+3) + 2(x+3) = (x+3)(x+2)$

Hence the factors of f(x) are (x+1) x Q(x) = (x + 1) (x + 2) (x + 3)

Example 5

Using factor theorem, factorize the polynomial

f(x) = (x + 5) (x – 2) – 3(x + 18)

Solution

Here, $f(x) = (x + 5)(x - 2) - 3(x + 18)$

$= x^2 + 5x - 2x - 10 - 3x - 54$

$= x^2 - 64$

The possible factors of -64 are ±1, ±2, ±4, ±8, ±16, ±32, ±64

Putting x = 1 in f(x), $f(1) = 1^2 - 64 = -63 \neq 0$

(x – 1) is not factor of $x^2 - 64$

Putting x = 8 in f(x), we have

$f(8) = 8^2 - 64 = 64 - 64 = 0$

So, (x – 8) is factor of f(x)

Using synthetic division

8	1	0	-64
		8	64
	1	8	0

∴ Quotient = x + 8

Now, f(x) = (x – 8) Q(x)

= (x – 8) (x + 8)

Hence, the factor of f(x) are (x – 8) and (x + 8)

Example 6

Solve for x, using factor theorem $6x^3 - 13x^2 + x + 2 = 0$

Solution

Let, the polynomial $6x^3 - 13x^2 + x + 2$ be f(x)

Now, f(x) = 0

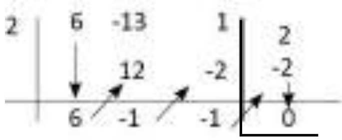
The possible factors of constant term in f(x) are ±1, ±2.

When x = 1, $f(1) = 6 \times 1^3 - 13 \times 1^2 + 1 + 2 = 6 - 13 + 3 \neq 0$

when x = 2, f(2) = 6 x 2^3 – 13 x 2^2 + 2 + 2 = 48 – 52 + 4 = 0

So, (x – 2) is a factor of f(x).

Now, by using synthetic division, we have



Now, quotient, Q(x) $= 6x^2 - x - 1$

$= 6x^2 - 3x + 2x - 1$

$= 3x(2x - 1) + 1(2x - 1)$

$= (2x - 1)(3x + 1)$

Hence, f(x) $= (x - 2) \times Q(x)$

$= (x - 2)(2x - 1)(3x + 1)$

But, f(x) = 0

or, $(x - 2)(2x - 1)(3x + 1) = 0$

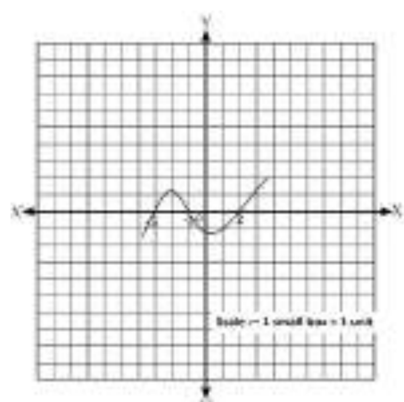
Either $x - 2 = 0$ or, $2x - 1 = 0$ or, $3x + 1 = 0$

or, $x = 2$ or, $x = \frac{1}{2}$ or, $x = -\frac{1}{3}$

Hence, the roots/zeros of $6x^3 - 13x^2 + x + 2$ are $-\frac{1}{3}, \frac{1}{2}$ and 2.

Example 7

Observe the given graph of f(x) and find the values of x when the graph meet X – axis and write f(x).



Solution:

Here, the graph of f(x) meet X- axis at -3, -1 and 2.

So, the polynomial has roots x = -3, x = -1 and x = 2

Now f(x) = $(x + 3)(x + 1)(x - 2) = x^3 + 2x^2 - 5x - 6$

Exercise 1.2.3 (B)

1. (a) State factor theorem.

   (b) If $(x - a)$ is a factor of $x^n - a^n$, what is the degree of quotient.

2. In each of the following, use factor theorem to find whether g(x) is a factor of polynomial f(x) or not?

   (a) $f(x) = x^3 + 9x^2 + 27x + 27$; $g(x) = x + 3$

   (b) $f(x) = x^3 + x^2 + 27x + 27$, $g(x) = x + 3$

   (c) $f(x) = x^3 + 6x^2 + 7x + 9$, $g(x) = x - 2$

   (d) $f(x) = 3x^3 + x^2 - 20x + 12$, $g(x) = 3x - 2$

   (e) $f(x) = 8x^3 - 4x^2 + 7x + 9$; $g(x) = 2x + 1$

3. (a) Find the value of k, if $x + 3$ is a factor of $3x^2 + kx + 6$

   (b) Find the value of k, if $x + 1$ is a factor of $x^3 - kx^2 - 3x - 6$

   (c) Find the value of m, for which $2x^4 - 4x^3 + mx^2 + 2x + 1$ is exactly divisible by $1 - 2x$.

4. Factorize the following by using factor theorem.

   (a) $2x^3 + 3x^2 - 3x - 2$ (b) $x^3 + 2x^2 - x - 2$

   (c) $y^3 - 6y^2 + 3y + 10$ (d) $x^3 + 13x^2 + 32x + 20$

   (e) $2x^3 + x^2 - 2x - 1$ (f) $x^3 - 23x^2 + 142x - 120$

   (g) $(x - 1)(2x^2 + 15x + 15) - 21$

5. Use factor theorem and solve for x.

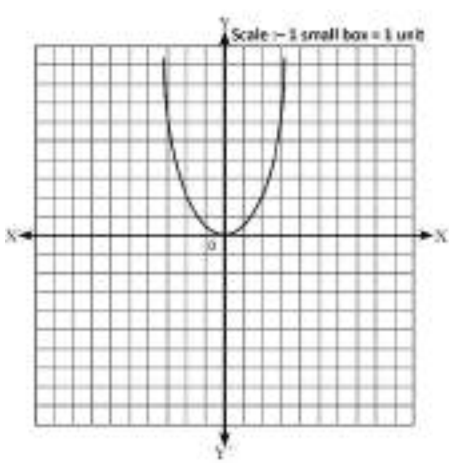
   (a) $x^3 - 4x^2 - 7x + 10 = 0$ (d) $x^3 - 3x^2 - 9x - 5 = 0$

   (b) $x^3 + 4x^2 + x - 6 = 0$ (e) $x^3 - 3x^2 - 10x + 24 = 0$

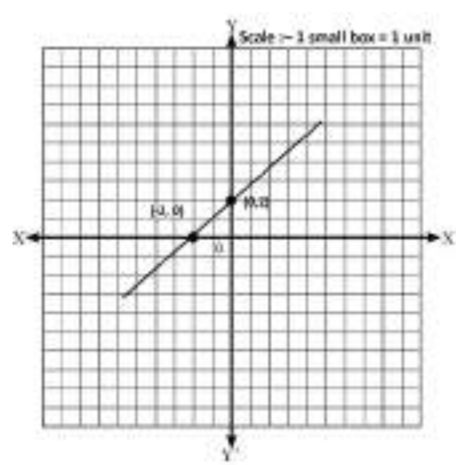
   (c) $3x^3 - x^2 - 3x + 1 = 0$ (f) $y^3 + 11y = 6y^2 + 6$

6. The graph of $f(x)$ is given. Write the roots of $f(x)$ and express $f(x)$ in standard form.

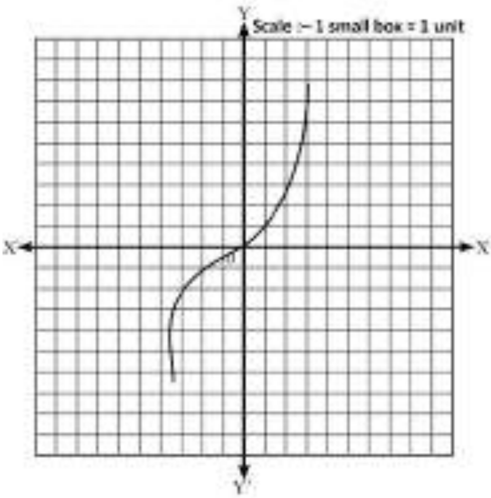
   (a)



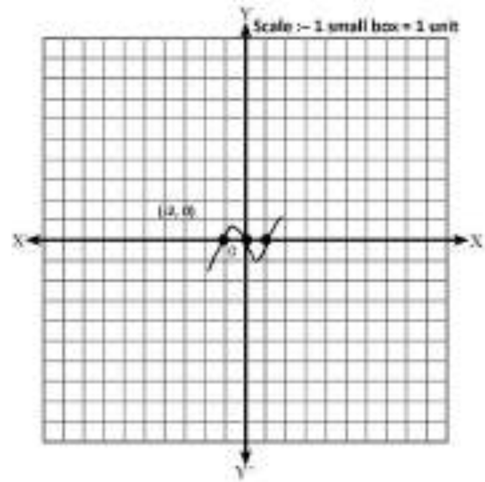
   (b)



(c)



(d)



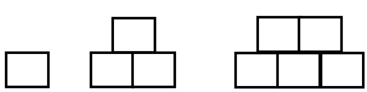
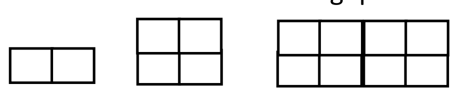
7. Factorize and solve f(x = $x^3 - 3x^2 - 6x + 8$. Give rough sketch of f(x) in graph.

8. Investigate the use of polynomial in our daily life. Write the relation between roots (zeros) of polynomial and polynomial function, with suitable example.

1.3 Sequence and series

1.3.0 Review

Observe the following pattern of the figures and discuss on the following questions.

(i)  (ii) 

(a) Can you draw two more figure in the same pattern of fig (i) and (ii)?

(b) Count the blocks and write the numbers in each figure.

(c) Observe the numbers and discuss, how they are increasing.

(d) Can you find the relation how they are increasing?

Sequences: A set of numbers which is formed under a definite mathematical rule is called a sequence. From above patterns

1, 3, 5, …, …, and, 2, 4, 8, ….. are the example of sequence.

Series: The sum of the terms of a sequence is called a series. The sum of the terms of the series is denoted by S_n. For example

(i) 1 + 3 + 5 + …n

(ii) 2 + 4 + 8 + … n

1.3.1 Arithmetic Sequence

Consider the following sequence of numbers

(i) 1, 2, 3, 4, …,…,…

(ii) 100, 70, 40, 10, …,…,…

(iii) -3, -2, -1, 0, …,…,…

Each of the number in the sequence is called a term. Can you write the next term in each of the above sequence. If so how will you write it? Let us observe and write the rule.

In (i) each term is 1 more than the term preceding it.

In (ii) each term is 30 less than the preceding term.

In (iii) each term is obtained by adding 1 to the term preceding it.

In all the sequence above, we see that the successive terms are obtained by adding a fixed number to the preceding terms.

Definition

A sequence is said to be an arithmetic sequence or arithmetic progression if the difference between a term and its preceding term is constant throughout the whole sequence. It is denoted by AP.

The constant difference obtained by subtracting a term from its succeeding term and is called the common difference. In above example (i) 1, 2, 3, 4, ………………, the common difference = 2 – 1 = 3 – 2 = 4 – 3 = 1

Similarly, the common difference of example (ii) and (iii) are – 30, 1 respectively. The common difference of an arithmetic sequence is denoted by d.

A series corresponding to any arithmetic sequence is known as the arithmetic series associated with the given arithmetic sequence. Hence, 1 + 2 + 3 + ….. is an arithmetic series associated with the arithmetic sequence 1, 2, 3, 4 ……

Note: Common difference can be positive or negative.

1.3.2. General term or n^{th} term of AP

If a = first term and d = common difference of an arithmetic progression (AP), then the terms of progression are

a, a + d, a + 2d, a + 3d, …

If t_1, t_2, t_3, t_4, …,…,…., t_n be the first, second, third, fourth, …….. n^{th} term of an AP, then

$t_1 = a = a + 0 = a + (1 – 1)d$

$t_2 = a + d = a + (2 – 1)d$

$t_3 = a + 2d = a + (3 – 1)d$

$t_4 = a + 3d = a + (4 – 1)d$

……………………

……………………

$t_n = a + (n – 1)d$

Now, to know about an AP.

What is the minimum information that you need?

Is it enough to know the first term? or is it enough to know only the common difference?

We need to know both the first term (a) and the common difference (d).

List of formulae
1. General term or n^{th} term $(t_n) = a + (n – 1)d$
2. Common difference $(d) = t_2 – t_1 = t_3 – t_2 = t_n – t_{n-1}$

Example 1

Write the common difference of the given arithmetic sequence 20, 25, 30, 35, ……………

Solution: Here,

The given sequence is

20, 25, 30, 35, ……………..

Common difference $(d) = t_2 – t_1$

$= 25 – 20$

$d = 5$

$\therefore$ Common difference (d) = 5

Example 2

Find the 20^{th} term of the AP 2, 7, 12, ….

Solution, Here

The given terms of AP are 2, 7, 12

First term (a) = 2

Common difference $(d) = t_2 – t_1 = 7 – 2 = 5$

To find: 20^{th} terms (t_{20})

By formula, $t_n = a + (n - 1)d$

$t_5 = 50 + (5 - 1)d$

$t_{20} = 2 + (20 - 1)5$

$= 2 + 19 \times 5$

$= 97$

∴ The 20^{th} term of the given AP is 97.

Example 3

If the first and fifth term of an AP are 50 and 30 respectively, find its common difference.

Solution: Here,

First term (a) = 50

Fifth term (t_5) = 30

No. of term (n) = 5

To find: common difference (d)

By formula, tn = a+ (n – 1)d

or, 30 = 50 + (5 – 1)d

or, 30 – 50 = 4d

or, $\frac{-20}{4} = d$

or, d = -5

∴ The common difference (d) = -5

Example 4

If 21, 18, 15, … is an AS. Find value of n for $t_n = -81$ and $t_n = 0$ and justify.

Solution: Here,

The given AP is 21, 18, 15, …, -81

First term (a) = 21

Common difference (d) = $t_2 - t_1$ = 18 – 21 = -3

Last term (t_n) or (l) = -81

To find: number of terms (n)

By formula, $t_n = a + (n - 1)d$

or, $-81 = 21 + (n - 1)(-3)$

or, $-81 - 21 = (n - 1)(-3)$

or, $\frac{-102}{-3} = n - 1$

or, $34 + 1 = n$

or, $n = 35$

Therefore, the 35^{th} term of the given AP is -81.

Again, If there is any n for which $t_n = 0$ then $t_n = a + (n - 1)d$

or, $0 = 21 + (n - 1)(-3)$

or, $\frac{-21}{-3} = n - 1$

$n = 7 + 1 = 8$

So, eight term is 0.

Since, the sequence is in decreasing order with common difference-3, so its eighth term is zero.

Example 5

Determine the AP whose 3^{rd} term is 5 and the 7^{th} term is 9.

Solution: Here,

Third term $(t_3) = 5$

Seventh term $(t_7) = 9$

To find: An AP

By formula,

$t_n = a + (n - 1)d$

or, $t_3 = a + (3 - 1)d$

$5 = a + 2d$

or, $a = 5 - 2d$ ………………. (i)

Similarly, $t_7 = a + 6d$

or, $9 = 5 - 2d + 6d$ [∵ From equation (i)]

or, $9 - 5 = 4d$

or, $\frac{4}{4} = d$

or, $d = 1$

Substituting d = 1 in equation (i), we get

a = 5 – 2d = 5 – 2 x 1 = 3

Second term (t_2) = a + d = 3 + 1 = 4

Third term (t_3) = a + 2d = 3 + 2 x 1 = 5

∴ The required A.P is 3, 4, 5, ….

Example 6

Check whether 301 is a term of the arithmetic sequence 5, 11, 17, 23, … or not.

Solution: Here,

The given sequence is 5, 11, 17, 23, ….

First term (a) = 5

Common difference (d) = $t_2 - t_1$ = 11 – 5 = 6

Last term (t_n) = 301

By formula, t_n = a + (n – 1)d

or, 301 = 5 + (n – 1)6

or, 301 = 5 + 6n – 6

or, 301 + 1 = 6n

n = $\frac{302}{6} = \frac{151}{3}$

But n should be a positive integer (why?). So 301 is not a term of the given sequence.

Example 7

If k + 1, K + 5 and 3k + 1 are in AP, find the value of K and its three terms.

Solution: Here,

K + 1, K + 5 and 3k + 1 are in AP

To find:

(i) The value of K (ii) Three terms

Now, K + 5 – (K + 1) = 3k + 1 – (K + 5)

or, K + 5 – K – 1 = 3k + 1 – k – 5

or, 4 = 2k – 4

or, k = 8/ 2 = 4

∴ k = 4

$\therefore$ Again, three terms are

k + 1 = 4 + 1 = 5

k + 5 = 4 + 5 = 9

3k + 1 = 3 x 4 + 1 = 13

Exercise 1.3.1

1. a. Define sequence and series.

   b. What do you mean by arithmetic sequence?

   c. Define common difference in arithmetic sequence.

2. Determine with reason, which of the following are in arithmetic progression

   (a) 3, 7, 11, 15, ….

   (b) 10, 6, 2, -2, …

   (c) 0, -4, -8, -12, …

   (d) 5 + 11 + 15 + 23 + …

   (e) $\frac{-3}{2}, 2, \frac{-5}{2}, -3, \ldots$

   (f) $a, \frac{a+b}{2}, b, \frac{3b-a}{2}$

3. From the following arithmetic sequences, find

   (i) First term

   (ii) Common difference

   (iii) the general term (t_n)

   (iv) the next two terms.

   (a) -1, -3, -5, -7, …

   (b) 2, 6, 10, 14, …

   (c) 7, 17, 27, 37

   (d) $\frac{5}{4}, 1, \frac{3}{4}, \frac{2}{4}$

4. (a) Write the formula for n^{th} terms of an AP

   (b) Is 7, 12, 17, 22, … an arithmetic sequence? Write with reasons.

   (c) In $t_n = a + (n - 1)d$, what does 'a' represent?

5. (a) In an arithmetic sequence, 2^{nd} term is 8 and the common difference is 5, what is the third term? Write it.

   (b) In an arithmetic sequence, the tenth term is 120 and the common difference is 8. What is the ninth term?

6. (a) In an arithmetic sequence, 14^{th} term is 150 and the 13^{th} term is 139 then find the common difference.

   (b) In an AP 6^{th} term is -20 and the 7^{th} term is -25 then find the common difference.

7. (a) In an AP, 2^{nd} term and 3^{rd} term are 10 and 14 respectively. Find the value of 1^{st} term.

   (b) In an arithmetic sequence, common difference is 7 and the 2^{nd} term is 5, what is the first term?

8. (a) The n^{th} term of an AP is 6n + 11. Find the common difference.

   (b) The n^{th} term of an AP is 5n – 2. Find the common difference.

9. (a) If the first term and the common difference of an AP are 3 and 5 respectively, find the 5^{th} term.

   (b) Find the 11^{th} term of arithmetic sequence when the first term and the common difference are 20 and 5 respectively.

10. (a) Determine the first term of an AP whose common difference is -8 and 10^{th} term is 240.

    (b) What is the first term of an arithmetic sequence whose common difference is 4 and tenth term is 40.

11. (a) What is the common difference of an arithmetic sequence whose first term is 150 and 12^{th} term is 40?

    (b) Find the common difference of an arithmetic sequence when first term and fifth term are 9 and 17 respectively.

12. Find the number of terms of following AP's

    (a) 7, 13, 19, ….., 205 (b) 18, $15\frac{1}{2}, 13, \dots, -47$

    (c) First term = 15, common difference = 10, last term = 115

13. (a) Is 44 a term of the arithmetic sequence 11, 14, 17 … ?

    (b) Does the Arithmetic sequence 11, 8, 5, 2, …., contain -150?

14. (a) If $2x + 3$, $x + 11$ and 8x + 3 are in AP, find the value of x.

    (b) If 8b + 4, 6b – 2 and 2b + 7 are first three terms of an AP, find the value of b. Also find first three terms.

15. (a) If the 11^{th} term and 16^{th} term of an arithmetic sequence are 38 and 73 respectively, find

    (i) first term and common difference (ii) 31^{st} term

    (iii) An arithmetic sequence

    (b) If the 3^{rd} and 9^{th} terms of an AP are 4 and -8 respectively find

    (i) first term and common difference.

(ii) Which term of an AP is zero.

(iii) Is 10 a term of an AP? Write with reason

16. (a) An AP consists of 47 terms whose fifth term is 16 and the last term is 100. Find the 20^{th} term.

    (b) An AP consists of 25 terms whose 3^{rd} term is 18 and the last term value is 128. Find the 15^{th} terms.

17. (a) In an AP whose third term is 16 and 7^{th} term exceeds the 5^{th} term by 12, then show that 11^{th} term is 64.

    (b) If 7 times the 7^{th} term of an AP is equal to 11 times its 11^{th} term then show that the 18^{th} term is zero.

18. (a) For what value of n, are the n^{th} terms of two AP's. 63, 65, 67, … and 3,10, 17, … equal?

    (b) If the n^{th} term of the AP 1, 5, 9, 13, … and n^{th} term of an AP 43, 46, 49, … are equal then, find the value of 'n'.

19. The sum of the 4^{th} and 8^{th} terms of an AP is 24 and the sum of the 6^{th} and 10^{th} term is 44. Find first three terms of the AP.

20. (a) A man takes a job in 2019 at an annual salary of Rs. 50,000. He receives an annual increase of Rs. 2000. In which year will his income reach Rs . 70,000?

    (b) After knee surgery, your trainer tell you to return to your jogging program slowly. He suggests for 12 minutes each for the first week. Each week thereafter, he suggests you increase that time by 6 minutes. How many weeks will it be before you are up to jogging 60 minutes per day. Find it.

1.3.2 Arithmetic mean

The arithmetic sequences are

(i) 5, 10, 15 (ii) 4, 8, 12, 16, 20

Discuss in groups about the terms of each sequence.

In (i) 10 is arithmetic mean and

In (ii) 8, 12 and 16 are called arithmetic means. Thus, the terms between first term and last term of an arithmetic progression are called arithmetic means. It is denoted by AM.

Arithmetic mean between two numbers

Let, m be the arithmetic mean between two numbers a and b, then

a m, b are in AP then

or, $m - a = b - m$ $[\therefore t_2 - t_1 = t_3 - t_2]$

or, $m + m = a + b$

or, $2m = a + b$

$\therefore m = \frac{a+b}{2}$

Hence, one AM between a and b is $\frac{a+b}{2}$

n arithmetic mean between the two numbers a and b

Let, $m_1, m_2, m_3, \ldots, m_n$ be the n arithmetic means between a and b. Then a, $m_1, m_2, m_3, \ldots m_n$, b be an arithmetic sequence.

If d be the common difference, then number of terms = number of means + 2 = n + 2

Last term = n^{th} term $(t_n) = b$

First term (a) = a

By formula, $t_n = a + (n-1)d$

or, $b = a + (n + 2 - 1)d$

or, $b - a = (n + 1)d$

or, $d = \frac{b-a}{n+1}$ [where n = no. of means]

The arithmetic means are

First mean $(m_1) = t_2 = a + d = a + \frac{b-a}{n+1}$

2^{nd} mean $(m_2) = t_3 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right)$

3^{rd} mean $(m_3) = t_4 = a + 3d = a + 3.\left(\frac{b-a}{n+1}\right)$

……………………………………

…………………………………………

n^{th} mean $(m_n) = t_{n+1} = a + nd = a + n\left(\frac{b-a}{n+1}\right)$

List of important formulae

1. An arithmetic mean (AM) = $\frac{a+b}{2}$
2. Common difference (d) = $\frac{b-a}{n+1}$
3. First mean $(m_1) = a + d$
4. 2^{nd} mean $(m_2) = a + 2d$ and so on.

Example 1

Find the arithmetic mean between $(m - n)^2$ and $(m + n)^2$.

Solution: Here,

First term (a) = $(m - n)^2$

Last term (b) = $(m + n)^2$

To find: An AM

By formula, A.M = $\frac{a+b}{2} = \frac{(m-n)^2+(m+n)^2}{2}$

$$= \frac{m^2-2mn+n^2+m^2+2mn+n^2}{2}$$

$$= \frac{2m^2+2n^2}{2} = \frac{2(m^2+n^2)}{2}$$

Hence, A.M $= m^2 + n^2$

Example 2

Find the 13th terms of an AP whose 12th term and 14th terms are -7 and 23 respectively.

Solution: Here,

12th term (t_{12}) = -7

14th term (t_{14}) = 23

To find: 13th term (t_{13})

We know that, 13th term is arithmetic mean of 12th term and 14th term

$\therefore$ By formula, 13th term = $\frac{12^{th}\text{ term}+14^{th}\text{term}}{2}$

$$= \frac{-7+23}{2} = \frac{16}{2}$$

$$= 8$$

$\therefore$ 13th term (t_{12}) = 8

Example 3

Find the value of p, q and r if 3, p, q, r, 27 are in A.P

Solution: Here,

The given AP is 3, p, q, r, 27

To find: The value of p, q and r

First term (a) = 3

Last term (b) = 27

number of mean (n) = 3

By formula, common difference (d) = $\frac{b-a}{n+1} = \frac{27-3}{3+1} = \frac{24}{4} = 6$

then, p = m_1 = a + d = 3 + 6 = 9

q = m_2 = a + 2d = 3 + 2 x 6 = 15

r = m_3 = a + 3d = 3 + 3 x 6 = 21

Hence, p = 9, q = 15 and r = 21

Example 4

Insert 5 AM's between -6 and 54

Solution

Let, m_1, m_2, m_3, m_4 and m_5 be 5 AM's between -6 and 54.

$\therefore$ -6, m_1, m_2, m_3, m_4, m_5, 54 are in AP

Now, First term (a) = -6

Last term (b) = 54

No. of mean (n) = 5

To find: 5 AM's (i.e. m_1, m_2, m_3, m_4 and m_5)

By formula, common difference (d) = $\frac{\mathrm{b-a}}{\mathrm{n+1}}$ = $\frac{54+6}{5+1}$ = $\frac{60}{6}$ = 10

Again,

First mean (m_1) = a + d = -6 + 10 = 4

Second mean (m_2) = a + 2d = -6 + 2 x 10 = 14

Third mean (m_3) = a + 3d = -6 + 3 x 10 = 24

Fourth mean (m_4) = a + 4d = -6 + 4 x 10 = 34

Fifth mean (m_5) = a + 5d = -6 + 5 x 10 = 44

Example 5

If n arithmetic means are inserted between 20 and 5. If the ratio of the third mean and the last mean is 31:13, find the value of n.

Solution:

Let, m_1, m_2, m_3 … m_n be the n AM's between 20 and 5.

Frist term (a) = 20

Last term (b) = 5

$m_3 : m_n = 31 : 13$

To find: The value of 'n'

By formula, common difference (d) = $\frac{b-a}{n+1} = \frac{5-20}{n+1} = \frac{-15}{n+1}$

and

Third mean (m_3) = a + 3d = 20 + $3\left(\frac{-15}{n+1}\right)$

$$= \frac{20(n+1)-45}{n+1}$$

$$= \frac{20n+20-45}{n+1}$$

$$= \frac{20n-25}{n+1}$$

n^{th} mean (m_n) = a + nd = 20 + n $\left(\frac{-15}{n+1}\right)$

$$= \frac{20n+20-15n}{n+1}$$

$$= \frac{5n+20}{n+1}$$

By the question

$$\frac{m_3}{m_n} = \frac{31}{13}$$

or, $\frac{\frac{20n-25}{n+1}}{\frac{5n+20}{n+1}} = \frac{31}{13}$

or, $\frac{5(4n-5)}{5(n+4)} = \frac{31}{13}$

or, 52n – 65 = 31n + 124

or, 21n = 189

n = 9

∴ Required number of mean (n) = 9

Exercise 1.3.2

1. (a) Define arithmetic mean.

   (b) What does 'n' represent in formula, d = $\frac{b-a}{n+1}$?

(c) Write the meaning of 'b' in formula d = $\frac{b-a}{n+1}$.

2. (a) State an arithmetic mean between -20 and 20.

(b) Find the arithmetic mean between (a + b) and (a – b).

3. (a) If the arithmetic mean between x and $2x$ is 3, find the value of x.

(b) If the arithmetic mean between y – 1 and 2y is 7, find the value of y.

4. (a) In an AP, 4^{th} mean is 12 and the ratio of first mean to the 4^{th} mean is $\frac{2}{3}$. Find the first mean.

(b) Write the number of terms which is equal to 5^{th} mean.

5. (a) In an AP, 10^{th} and 12^{th} terms are 100 and 120 respectively, find the 11^{th} term.

(b) If 10, x, 20 are in AP, find x.

6. (a) Find the value of x and y if 10, x, y, 25 are in AP.

(b) If -13, p, q, r, 7 are in AP, find the values of p, q and r.

7. (a) If l, m and n are in AP, show that m = $\frac{l+n}{2}$.

(b) If the arithmetic mean between two number is 50 and the second number is 60, find the first number.

(c) Two numbers are in the ratio of 2:3. If their AM is 20, find the numbers.

8. (a) Insert two arithmetic means between 4 and 22.

(b) Insert two arithmetic means between 100 and 10.

9. (a) Insert 5 arithmetic means between 10 and 70.

(b) Insert 6 arithmetic means between 11 and 39.

10. (a) There are 'n' arithmetic means between 15 and 45. If the third mean is 30, find the value of n. Also, find the ratio of 3^{rd} mean to fifth mean.

(b) Some arithmetic means are inserted between 9 and 33. If the third mean is 21, find the number of means and the values of the remaining means.

11. (a) If 'n' arithmetic means are inserted between 5 and 35. The ratio of second and last mean is 1:4, find the value of n.

(b) There are 6 AM's between a and b. If the 2^{nd} mean and 5^{th} means are 11 and 23 respectively, then find the values of a and b.

1.3.3 Sum of first n terms of an arithmetic series

Let, an arithmetic sequence be

1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and its corresponding series is

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 and the sum of series is 55.

Hence, the result obtained by adding the terms of an AP is known as sum of arithmetic series. Sum of n terms is denoted by S_n.

Now, $S_{10} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$ .... (i) and in reverse order is

$S_{10} = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$ .... (ii)

Adding (i) and (ii) we get $2S_{10} = 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11$

$2S_{10} = 11 \times 10$

$S_{10} = \frac{11\times10}{2} = 55$

$\therefore S_{10} = 55$

Similarly, a be the first term, d be the common difference, n be the number of terms, l be the last terms and S_n be the sum of the n terms of an arithmetic series (AS), then

$S_n = a + (a + d) + (a + 2d) + \ldots\ldots\ldots + (l - 2d) + (l - d) + l$ .... (iii)

Writing in the reverse order, we have

$S_n = l + (l - d) + (l - 2d) + \ldots\ldots\ldots + (a + 2d) + (a + d) + a$ .... (iv)

Adding equation (iii) and (iv) we get

$2S_n = (a + l) + (a + l) + (a + l) + \ldots\ldots\ldots + (a + l) + (a + l) + (a + l)$

or, $2S_n = n(a+l)$

or, $S_n = \frac{n}{2}(a + l)$

We know that, Last term $(l) = a + (n - 1)d$

$S_n = \frac{n}{2}[a + a + (n - 1)d]$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

Important formulae

(i) $S_n = \frac{n}{2}(a + l)$

(ii) $S_n = \frac{n}{2}(2a + (n - 1)d]$

Note: To solve the problem easily, we can consider

(i) Three terms of an A.P be $a - d, a, a + d$

(ii) Four terms in an AP be a – 3d, a – d, a + d, a + 3d

Sum of first 'n' natural numbers

Let, 1 2, 3, 4, ……………, n be the set of 'n' natural numbers. It is an arithmetic progression with first term (a) = 1 and the common difference (d) = 1

We have,

$S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}[2 \times 1 + (n-1).1]$

$= \frac{n}{2}(2 + n - 1)$

$= \frac{n}{2}(n + 1)$

Therefore, the sum of first 'n' natural numbers (S_n) = $\frac{n}{2}$ (n + 1)

Example 1

Find the sum of 1 + 2 + 3 + ……… + 20

Solution: Here,

The given series is 1 + 2 + 4 + ………… + 20

First term (a) = 1,

common difference (d) = 2 – 1 = 1

no. of terms (n) = 20

By formula, $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2 \times 1 + (20-1).1]$

= 10 (2 + 10)

= 10 x 21 = 210

$\therefore S_{20} = 210$

Sum of First n Even Natural Number

The first n even numbers are 2, 4, 6, … 2n

Number S_n = 2 + 4 + 6 + …. + 2n

First term (a) = 2, d = 2

Sum of first n terms be S_n

By formula,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$= \frac{n}{2}[2 \times 2 + (n - 1)2]$

$= \frac{n}{2}(4 + 2n - 2)$

$= \frac{n}{2}(2n + 2)$

$= \frac{n}{2} \times 2(n + 1)$

$\therefore S_n = n(n + 1) = n^2 + n$

Sum of first 'n' odd natural numbers

The first n odd natural numbers are 1, 3, 5, ..., (2n – 1)

Let, Sn = 1 + 3 + 5 + ………………….. + (2n – 1)

First terms (a) = 1

Common difference (d) = 3 – 1 = 2

Number of terms (n) = n

Sum of 'n' terms be S_n

By formula,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$= \frac{n}{2}[2 \times 1 + (n - 1)2]$

$= \frac{n}{2}(2 + 2n - 2)$

$= \frac{n}{2} \times 2n = n^2$

Example 2

Find the sum of the first 10 terms of the AP: 2, 7, 12, …

Solution: Here,

The given AP is 2, 7, 12, …

To find: The sum of first 10 terms (S_{10})

First term (a) = 2

Common difference d) = 7 – 2 = 5

no. of terms (n) = 10

By formula,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$= \frac{10}{2}[2 \times 2 + (10 - 1)5]$

$= 5[4 + 45]$

$= 5 \times 49 = 245$

Hence, Sum of first 10 terms is 245.

Example 3

Find the sum of the series: 34 + 32 + 30 + …… + 10

Solution: Here,

The given series is 34 + 32 + 30 + ………… + 10

First term (a) = 34, Last term (l) = 10

Common difference (d) = $t_2 - t_1 = t_3 - t_2$

$= 32 - 34 = 30 - 32$

$= -2 = -2$

$\therefore$ It is an arithmetic series.

To find: Sum of series (Sn)

Now, $t_n = l = a + (n - 1)d$

or, $10 = 34 + (n - 1)(-2)$

or, $10 - 34 = -2n + 2$

or, $-24 = -2n + 2$

or, $2n = 24 + 2$

or, $n = \frac{26}{2} = 13$

$\therefore$ n = 12

Again, by formula,

$S_n = \frac{n}{2}[a + l]$

$= \frac{13}{2}[34 + 10]$

$= \frac{13}{2} \times 44 = 286$

$\therefore S_{13} = 286$

Example 4

How many terms of the AP 24, 21, 81, … must be taken so that their sum is 78?

Solution: The given AP is 24, 21, 18, …

Sum of n terms (S_n) = 78

To find: number of terms (n)

First terms (a) = 24

Common difference (d) = $t_2 - t_1 = 21 - 24 = -3$

By formula,

$S_n = \frac{n}{2}[2 \times 24 + (n-1)(-3)]$

or, $78 = \frac{n}{2}[48 - 3n + 3]$

or, $78 = \frac{n}{2}[51 - 3n)$

or, $156 = 51n - 3n^2$

or, $3n^2 - 51n + 156 = 0$

or, $-3(n^2 - 17n + 52) = 0$

or, $n^2 - (13 + 4)n + 52 = 0$

or, $n^2 - 13n - 4n + 52 = 0$

or, $n(n-13) - 4(n-13) = 0$

or, $(n-13)(n-4) = 0$

Either, $n - 13 = 0$ or, $n - 4 = 0$

$n = 13$ $n = 4$

Since, both values of n are positive. So the number of terms is either 4 or 13.

Note: (i) In this case the sum of the 1st four terms = the sum of the first 13 terms = 78.

Example 5

Find the common difference of an arithmetic series whose first term is 2 and the sum of the 10 terms is 120

Solution: Here

First term (a) = 2

Sum of the first 10 terms (S_{10}) = 120

To find: common difference (d)

By formula,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

or, $S_{10} = \frac{10}{2}[2 \times 2 + (10 - 1)d]$

or, $120 = 5\,(4 + 9d)$

or, $\frac{120}{5} = 4 + 9d$

or, $24 - 4 = 9d$

or $d = \frac{20}{9}$

Hence, common difference (d) = $\frac{20}{9}$.

Example 6

Find the sum of all natural numbers less than 100 which are exactly divisible by 6.

Solution: Here,

The natural numbers less than 100 and exactly divisible by 6 are 6, 12, 18, ……. 96

Now, first term (a) = 6

Common difference, (d) = 6

Last term (l) = 96

To find: Sum (S_n)

By formula, $l = a + (n - 1)d$

or, $96 = 6 + (n - 1)6$

or, $96 - 6 = (n - 1)6$

or, $\frac{90}{6} = n - 1$

or, $15 + 1 = n$

$n = 16$

Again, by formula

$S_n = \frac{n}{2}(a + l)$

or, $S_{16} = \frac{16}{2}(6 + 96)$

or, $S_{16} = 8 \times 102$

or, $S_{16} = 816$

Hence, the required sum is 816.

Example 7

If the sum of first seven term of an arithmetic series is 14 and the sum of the first ten terms is 125, then find the fourth term of the series.

Solution: Here,

Sum of the 1st seven terms of the series in A.P. (S_7) = 14

Sum of the 1st ten terms (S_{10}) = 125

To find: Fourth term (t_4) .

By formula,

$S_n = \frac{n}{2}[2a + (n - 1)d]$

or, $S_7 = \frac{7}{2}[2a + (7 - 1)d$

or, $14 = \frac{7}{2} \times 2 (a + 3d)$

$a = 2 - 3d$ ………… (i)

Similarly, $S_{10} = 125$

or, $\frac{10}{2}[2a + (10 - 1)d] = 125$

or, $5[2a + 9d] = 125$

or, $2a + 9d = 25$

$a = \frac{25-9d}{2}$ ……….. (ii)

Equating equation (i) and (ii), we get

$2 - 3d = \frac{25-9d}{2}$

or, $4 - 6d = 25 - 9d$

or, $9d - 6d = 25 - 4$

or, $3d = 21$

$d = 7$

From equation (i) $a = 2 - 3d = 2 - 3 \times 7 = -19$

Again, fourth term (t_4) = $a + 3d = -19 + 3 \times 7 = 2$

$\therefore$ Fourth term (t_4) = 2

Example 8

The sum of three consecutive terms in an arithmetic series is 18 and their product is 192, find these three terms.

Solution: Here

Let, three consecutive terms in AP be a – d, a, a + d

To find: Three terms in AP.

From first condition, a – d + a + a + d = 18

or, 3a = 18

or, a = 6

From 2nd condition: (a – d) x a x (a + d) = 192

or, (6 – d) x 6 x (6 + d) = 192

or, $36 - d^2 = \frac{192}{6}$

or, $36 - d^2 = 32$

or, $-d^2 = 32 - 36$

or, $-d^2 = -4$

$d = \pm\sqrt{4} = \pm 2$

(i) When a = 6 and d = 2 then three terms are

a – d = 6 – 2 = 4

a = 6

a + d = 6 + 2 = 8

(ii) When a = 6 and d = -2 then three terms are

a – d = 6 + 2 = 8

a = 6

a + d = 6 + (-2) = 4

Hence, three terms of an AP are 4, 6, 8 or 8, 6, 4.

Exercise 1.3.3

1. (a) Write the formula for finding the sum of first 'n' terms in arithmetic series whose first and the last terms are given.

   (b) If sum of first 10 terms of arithmetic series is 80 and the sum of 1^{st} 9 terms of the same series is 72. Find the 10^{th} terms.

2. (a) What is the sum of first 5 odd natural numbers?

   (b) What is the sum of first 5 even natural numbers?

3. Find the sum of the following series:

   (a) -37 – 33 - 29, ………….. to 12 terms

   (b) 5 + 8 + 11 + 14 + ………. to 20 terms

   (c) 7 + $10\frac{1}{2}$ + 14 + ………… + 84

   (d) -5 + (-8) + (-11) + ……….. + (-230)

4. Find the sum of the series:

   (a) $\sum_{n=1}^{10}(2n+1)$ (b) $\sum_{n=2}^{15}(n+2)$

5. (a) The first terms and the common difference of an arithmetic series are 2 and 8 respectively. Find the sum of first 10 terms.

   (b) If the common difference and the sum of 1^{st} 9 terms of an AP are 5 and 75 respectively, find the first terms.

6. (a) Find the common difference of an AS whose first terms is 3 and the sum of first 8 terms is 192.

   (b) How many terms of the series 9 + 6 + 3 + … + … must be taken so that the sum of the series is zero?

7. (a) Find the sum of all numbers from 50 to 150 which are exactly divisible by 9.

   (b) Find the sum of all two digit numbers in AP which are multiple of 5.

   (c) Find the sum of the odd numbers between 0 and 50.

8. (a) The first term of an AP is 5, the last term is 45 and the sum is 400. Find the numbers of terms.

   (b) The first term of an AP is -9, the last term is 47 and the sum is 152. Find the number of terms.

9. (a) How many terms of the series 20 + 18 + 16 + …… must be taken so that the sum of the series may be 110? Explain the double answer.

   (b) If the 3^{rd} term and 11^{th} term of an AP are 18 and 50 respectively, find

   (i) First terms and the common difference

   (ii) Arithmetic series (iii) Sum of first 20 terms.

10. (a) If the 2^{nd} term and 12^{th} term of an AP are 20 and 50 respectively, find

   (i) First terms and the common difference

   (ii) Sum of first 25^{th} terms.

(b) If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289:

(i) Find the first term and the common difference

(ii) Find the arithmetic series

(iii) Find the sum of the first 20 terms.

11. (a) The sum of three terms of an AP is 30 and their product is 840. Find the three terms.

(b) The sum of three terms of an AP is 12 and the sum of their squares is 56. Find the three terms.

12. A sum of Rs. 700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is Rs. 20 less than its preceding prize, find?

(i) The first prize (ii) The 2^{nd} prize

(iii) The 3^{rd} prize (iv) Seventh prize

Geometric sequence (geometric progression) (GP)

Let us consider the following sequence:

i) 2, 4, 8, 16, 32, … ii) 81, 27, 9, 3, …

a) How are the numbers arranged in both sequences?

b) Is the difference between a term and its preceding term constant in both sequences?

c) What relation do you see in any two consecutive of terms in each of the above sequence?

d) What characteristics are found in both sequences?

In (i) sequence the numbers are increased in the multiple of 2, similarly, in (ii) the numbers are decreased by multiple of $\frac{1}{3}$.

Therefore, from above two sequences, the numbers are increased or decreased by a constant number called common ratio and such sequences are called geometric sequences or geometric progression. (GP)

Hence, a sequence or series of numbers that increased or decreased with a constant ratio is called a geometric sequence or series. The constant ratio and it is denoted by r.

$\therefore$ Common ratio (r) = $\frac{\text{a term}}{\text{preceding term}}$

Or, Common ratio (r) = $\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3}$

$\therefore$ From the above sequences (I) and (ii), the common ratios are 2 and $\frac{1}{3}$ respectively.

1.3.4 General term (n^{th} term) of a geometric sequence

If 'a' be the first term and r be the common ratio of a geometric sequence, then the terms of the sequences are

a, ar, ar^2, ar^3, …

If, t_1, t_2, t_3, … t_n be the first, second, third, …, n^{th} terms of a geometric sequence respectively, then

First term (t_1) = a = ar^{1-1}

Second term (t_2) = ar = ar^{2-1}

Third term (t_3) = ar^2 = ar^{3-1}

…………………..

…………………

n^{th} term (t_n) = ar^{n-1}

Note: If we have first term and common ratio then we can find any term of a geometric progression.

Example 1

Find the 10th term of a sequence: 3, 9, 27, …

Solution: Here,
the given sequence is 3, 9, 27, …

To find: 10th term (t_{10})
Now, $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

or, $\frac{9}{3} = \frac{27}{3}$

or, 3 = 3

So, the given sequence forms a GP
First term (a) = 3, common ratio (r) = 3
By formula, $t_n = ar^{n-1}$
$t_{10} = 3 \times 3^{10-1}$
$t_{10} = 59,049$

Hence, 10th term $t_{10} = 59,049$

Example 2
What is the common ratio of a geometric sequence whose first term is 48 and 4th term is 6?
Solution: Here,
First term (a) = 48
4th term (t_4) = 6
Common ratio (r) = ?
By formula,

$$t_n = ar^{n-1}$$

$$or, t_4 = 48 \times r^{4-1}$$

$$or, \frac{6}{48} = r^3$$

$$or, \frac{1}{8} = r^3$$

$$or, \left(\frac{1}{2}\right)^3 = (r)^3$$

$$r = \frac{1}{2}$$

Hence, Common ratio (r) = $\frac{1}{2}$

Example 3

The 3rd term and 7th term of GP are 8 and 128 respectively. Find
i) Common ratio and first term.
ii) Geometric sequence
iii) 15th term

Solution: Here,
3rd term $(t_3) = 8$
7th term $(t_7) = 128$
To find:
(i) Common ratio (r) and first term (a) (ii) Geometric sequence (iii) 15th term (t_{15})
By formula,

$t_n = ar^{n-1}$

Now, $t_3 = 8$, $t_3 = ar^{3-1}$

$or, ar^{3-1} = 8$

$or, ar^2 = 8$ .... (i)

Similarly, $ar^{7-1} = 128$

$ar^6 = 128$ .... (ii)

Dividing equation (ii) by (i), we get

$\frac{ar^6}{ar^2} = \frac{128}{8}$

$or, r^4 = 16$

$or, r^4 = (\pm 2)^4$

$r = \pm 2$

Taking r = 2 then from equation (i) [Since, GP is in increasing order.]

$a(2)^2 = 8$

$a = \frac{8}{4} = 2$

$\therefore$ First term (a) = 2
Second term $(t_2) = ar = 2 \times 2 = 4$
$\therefore$ GP is 2, 4, 8, ... and 15th term $(t_{15}) = ar^{14} = 2 \times (2)^{14} = 32{,}768$

Exercise 1.3.4

1. a) Define geometric sequence with an example.

   b) If a, ar, ar^2, … be the first, second and third term of a geometric sequence, write the sixth term.

2. Which of the following are geometric sequences?

   a) -1, 6, -36, 316, … b) -1, 1, 4, 8, … c) 4, 16, 36, 64, …

   d) -3, -15, -75, -375, … e) -2, -4, -8, -16, … d) 1, -5, 25, -125, …

3. From the following geometric sequences, find the common ratio and next two terms.

   a) 2, 1, $\frac{1}{2}$, … b) 5, -10, 20, … c) 4, 12, 36, … d) a, ab, ab^2 …

4. a) Find the first term of a geometric sequence, whose sixth term is 729 and the common ratio 3.

   b) Determine the first term of geometric sequence, whose 4th term is -8 and the common ratio is $-\frac{1}{2}$.

5. a) If the first term of a GP is 4 and the 5th term is 64, find the common ratio.

   b) How many terms are there in the geometric sequence 3, 12, 48, …, 192?

   c) Find the number of terms in a sequence, 1, 5, 25, ..., 3125.

6. a) Find the value of x, if x, $x + 4$, and $x + 6$ are consecutive terms of geometric sequence.

   b) Find the value of p, if 2p, 2p + 3, and 2p + 9 are three consecutive terms of a geometric sequence. Also find the common ratio.

7. a) The third term of a geometric progression is -108 and the sixth term is 32. Find
   i) The common ratio and the first term.

   ii) Geometric sequence iii) 10th term

   b) The second and fifth term of a geometric sequence are 750 and -6 respectively, Find
   i) Common ratio and the first term

   ii) Geometric Sequence iii) 8th term

8. a) The fourth term of G.P is square of its second term and the first term is -3. Determine the 8th term of GP.

b) In a geometric sequence, 2^{nd} term is 9 and 27 times of 8^{th} term is equal to 5^{th} term, then find the 12^{th} term.

c) The first term of a geometric sequence is 1. The ninth term exceeds the fifth term by 240. Find the possible values for the eighth term.

9. The first three terms of a geometric sequence are (k + 4), k and (2k – 15) respectively, where k is positive constant. (i) show that k = 12 (ii) Find the common ratio (iii) 10^{th} term.

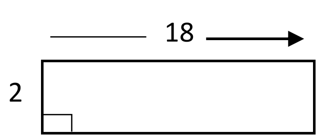
10. Suppose, your parents income is Rs. 50, 0000 per month. They save Rs. 2,000 in the first week of the new year, if they double the amount they save every week, after that, how much will they save in the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} week of the year? Complete the following table and explain in classroom with reason.

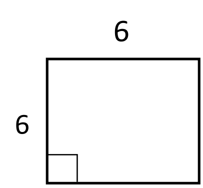
Week No.	1^{st} week	2^{nd} week	3^{rd} week	4^{th} week	5^{th} week
Amount Saved					

1.3.5 Geometric mean

Consider a geometric sequence 2, 6, 18, where 6 is called geometric mean. We can write it, geometric mean of 2 and 18 = $\sqrt{2 \times 18} = 6$

Also, it is like the area which is same





Area (A) = 2 x 18

Area = 6 x 6

Thus, the term between the first and the last term of a GP are known as the geometric means.

Geometric mean between two numbers a and b

Let, GM be the Geometric Mean between a and b then a, GM, b from a GP then, by definition,

$$\frac{\mathrm{GM}}{\mathrm{a}} = \frac{\mathrm{b}}{\mathrm{GM}}$$

$\text{or}, \mathrm{GM}^2 = \mathrm{ab}$

$\text{or}, \mathrm{GM} = \sqrt{\mathrm{ab}}$

Hence, GM between a and b = $\sqrt{ab}$

GM's between two numbers a and b

Let, $m_1, m_2, m_3, \ldots m_n$ be the 'n' geometric means between a and b then, a, $m_1, m_2, m_3, \ldots m_n$, b forms a GP

Now, First term (a) = a
Last term (t_n) = b
number of terms = no. of mean + 2 = n + 2
By formula, $t_n = ar^{n-1}$

$\text{or}, \mathrm{b} = \mathrm{ar}^{\mathrm{n}+2-1}$

$\text{or}, \frac{\mathrm{b}}{\mathrm{a}} = \mathrm{r}^{\mathrm{n}+1}$

Raising power $\frac{1}{n+1}$ on both sides, we get

or, $\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^{\frac{1}{\mathrm{n}+1}} = (\mathrm{r}^{\mathrm{n}+1})^{\frac{1}{\mathrm{n}+1}}$

$\boxed{\mathrm{r} = \left(\frac{\mathrm{b}}{\mathrm{a}}\right)^{\frac{1}{\mathrm{n}+1}}}$ Where, n represents number of means.

Now, First mean (m_1) = second term (t_2) = $ar = a.\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

2^{nd} mean (m_2) = Third term (t_3) = $ar^2 = a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

3^{rd} mean (m_3) = fourth term (t_4) = $ar^3 = a.\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$

………………………………………………………………………………

n^{th} mean (m_n) = $(n+1)^{th}$ term (t_{n+1}) = $ar^n = a.\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

Relation between arithmetic and geometric means of two numbers

Let, a and b be two positive numbers. Let, AM and GM be the arithmetic mean and geometric mean between a and b respectively. Then,
We have,

$\mathrm{AM} = \frac{\mathrm{a+b}}{2}$

And $\mathrm{GM} = \sqrt{\mathrm{ab}}$

Now,

$$\mathrm{AM} - \mathrm{GM} = \frac{\mathrm{a+b}}{2} - \sqrt{\mathrm{ab}}$$

$$= \frac{\left(\mathrm{a+b}-2\sqrt{\mathrm{ab}}\right)}{2}$$

$$= \frac{\left(\sqrt{\mathrm{a}}\right)^2+\left(\sqrt{\mathrm{b}}\right)^2-2\sqrt{\mathrm{a}}\,\sqrt{\mathrm{b}}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}$$

or, AM – GM ≥ 0 [∵ Square of any two quantities is greater than or equal to zero.]

Hence, AM ≥ GM

Case I – When a = b then, AM = GM

Case II – When a ≥ b or b ≥ a then AM ≥ GM

Example 1

Find the geometric mean between 5 and 20

Solution: Here,

First term (a) = 5

Last term (b) = 20

To find: GM

By formula, $\mathrm{GM} = \sqrt{\mathrm{ab}} = \sqrt{5 \times 20} = \sqrt{100} = 10$

Hence, G.M = 10

Example 2

Find the value of p and q if $\frac{1}{8}$, p, q, 8 are in G.P

Solution: Here,

The given GP is $\frac{1}{8}$, p, q, 8

To find: The value of p and q

Now, First term (a) = $\frac{1}{8}$

Last term (b) = 8

no. of means (n) = 2

By formula, common ratio (r)

$$= \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = \left(\frac{8}{\frac{1}{8}}\right)^{\frac{1}{2+1}} = (64)^{\frac{1}{3}} = (4^3)^{\frac{1}{3}} = 4$$

$\therefore$ 1st mean (m_1) $= p = ar = \frac{1}{8} \times 4 = \frac{1}{2}$

2nd mean (m_2) $= q = ar^2 = \frac{1}{8}(4)^2 = 2$

Example 3

Find the number of geometric means inserted between 1 and 64 in which the ratio of first mean to the last mean is 1:16

Solution: Here,
Let, $m_1, m_2, m_3 \ldots m_n$ be the n GM's between 1 and 64

$\therefore$ 1, $m_1, m_2, m_3, \ldots m_n$, 64 form a GP

Since, $\frac{1^{st}\text{mean}(m_1)}{\text{last mean }(m_n)} = \frac{1}{16}$

To find: number of GM (n)

Now, First term (a) = 1

Last term (b) = 64

By formula, common ratio (r) = $\left(\frac{b}{a}\right)^{\frac{1}{1+1}} = \left(\frac{64}{1}\right)^{\frac{1}{n+1}} = (64)^{\frac{1}{n+1}}$

$or, r = (64)^{\frac{1}{n+1}}$ …………… (i)
and, $\frac{m_1}{m_n} = \frac{1}{16}$
$or, \frac{ar}{ar^n} = \frac{1}{16}$
$\therefore r^{1-n} = \frac{1}{16}$
Raising power $\frac{1}{1-n}$ on both sides we get
$r = \left(\frac{1}{16}\right)^{\frac{1}{1-n}}$ … (ii)
Equating equation (i) and (ii) we get

$$(64)^{\frac{1}{n+1}} = \left(\frac{1}{16}\right)^{\frac{1}{1-n}}$$

$$\text{or}, (4)^{\frac{3}{n+1}} = \left(\frac{16}{1}\right)^{-\frac{1}{-(n-1)}}$$

$$\text{or}, 4^{\frac{3}{n+1}} = 4^{\frac{2}{n-1}}$$

$$\text{or,} \frac{3}{n+1} = \frac{2}{n-1}$$

$\text{or,} 3n - 3 = 2n + 2$

$\therefore n = 5$

Therefore, there are 5 GM's between 1 and 16.

Exercise 1.3.5

1. a) Define geometric mean with an example.
   b) Determine a geometric mean between p and q.
   c) Find a geometric mean between $\frac{1}{10}$ and 10.
2. a) Find 10th term of a GP whose 9th term is $\frac{1}{5}$ and 11th term is 125.
   b) The Geometric mean between $\frac{1}{9}$ and b is 2. Find the value of b.
   c) There are 3 GM's in a GP in which first term is 10 and common ratio is $\frac{1}{2}$ then find first mean only.
3. a) Find AM and GM between 4 and 16. Also write their difference.
   b) The arithmetic mean of a and 24 is 15. Find the Geometric Mean (GM).
4. a) Insert 2 GM's between 6 and 48
   b) Insert 3 GM's between 5 and 80
   c) Find the values of x and y when $\frac{1}{2}$, x, 2, y are in GP
5. a) Insert 7 GM's between $\frac{1}{16}$ and 16 and find the difference between 5th mean and first mean.
   b) Insert 6 GM's between $\frac{1}{2}$ and 64 and compare 1st mean and 6th mean.
6. a) There are 4 geometric means between 4 and q. If 2nd mean is 36, find the value of q and the remaining other means.
   b) Some geometric means are inserted between 5 and 80. Find the number of means between two numbers if the third mean is 40. Also find the remaining means.
7. a) The AM between two positive numbers is 50 and GM is 40. Find the numbers.
   b) The AM between two natural numbers is 45 and GM is 27. Find the numbers.
8. a) There are n geometric means between $\frac{1}{81}$ and 81. If the ratio of 3rd mean to the last mean is 1:81. Find the value of n.
   b) There are n geometric means between $\frac{27}{16}$ and $\frac{8}{81}$. If the ratio of (n-1)th mean to the 2nd mean is 8:27. Find the value of n.

1.3.6 The sum of n terms of a geometric series

Let us discuss in the given sequence, 5, 15, 45, 135, 405, 1215.
Is it geometric sequence?
Can you write the sequence into corresponding series?
The corresponding series is 5 + 15 + 45 +135 + 405 + 1215
What is the sum of series?
Is its sum is 1820?

The sum obtained by adding the terms of a GP is known as the sum of geometric series. In a geometric series, the sum of n terms is denoted by s_n

Again, Let a = first term, r = common ratio, t_n or l = last term and n = number of terms of a geometric series. If s_n be the sum of n terms of a GS, then
$s_n = t_1 + t_2 + t_3 + t_4 + \ldots + t_n$
or, $s_n = a + ar + ar^2 + \ldots + ar^{n-1}$ ....(i)

Multiplying each term by 'r', we have,
$rs_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$ .... (ii)

Subtracting equation (ii) from equation (i) we get,

$$\text{or, } s_n - rs_n = a + ar + ar^2 + \cdots ar^{n-1} - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n)$$
$$\text{or, } s_n(1-r) = a + ar + ar^2 + \cdots + ar^{n-1} - ar - ar^2 - ar^3 - \cdots - ar^{n-1} - ar^n$$
$$\text{or, } s_n(1-r) = a - ar^n$$
$$\text{or, } s_n(1-r) = a(1-r^n)$$

$\therefore s_n = \frac{a(1-r^n)}{1-r}$ if r < 1

Also, $s_n = \frac{a(r^n-1)}{r-1}$ if r > 1

S_n can also be written as follows:

$$\therefore s_n = \frac{a(r^n-1)}{r-1}$$
$$= \frac{ar^n-a}{r-1}$$
$$= \frac{ar^{n-1}\times r-a}{r-1}$$

$\therefore s_n = \frac{lr-a}{r-1}$ $[\text{where, } l = ar^{n-1}]$

And $s_n = \frac{a-lr}{1-r}$ if r < 1

Note: to solve the problem easily, we can suppose
i) Three terms in GP be $\frac{a}{r}$, a, ar
ii) Four terms in GP be $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3
iii) Five terms in GP be $\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2

Example 1

Find the sum of the following series: 36 + 12 + 4 + …… up to 6 terms.
Solution: Here,
The given series is 36 + 12 + 4 + … to 6 terms
First term (a) = 36
Common ratio (r) $= \frac{12}{36} = \frac{1}{3}$
∴ It is geometric series
no. of terms (n) = 6
Sum of 6 terms (s_6) = ?
By formula,

$$s_n = \frac{a(1-r^n)}{1-r} \qquad\qquad r < 1$$

$$= \frac{36\left\{1-\left(\frac{1}{3}\right)^6\right\}}{1-\frac{1}{3}} = \frac{36\left(1-\frac{1}{729}\right)}{\frac{3-1}{3}} = \frac{36\times728\times3}{729\times2} = \frac{1456}{27} = 53\,\frac{25}{27}$$

Example 2

If the first and last term of a GP are 7 and 448 respectively and the sum of the series is 889, find the common ratio.

Solution : Here,
First term (a) = 7
Last term (l) = 448
Sum of n terms (s_n) = 889
To find: Common ratio (r)
By formula,

$$\mathrm{s_n} = \frac{\mathrm{lr-a}}{\mathrm{r-1}}$$

$$\text{or, } 889 = \frac{448\mathrm{r}-7}{\mathrm{r}-1}$$

or, 889r – 889 = 448r – 7

or, 889r – 448r = 889 – 7

or, 441r = 882

$$\text{or}, \mathrm{r} = \frac{882}{441} = 2$$

Hence, the common ratio (r) = 2

Example 3

If the sum of first two terms of a GP is 6 and the sum of the first four terms is 30. Find the sum of first 10 terms of the series.

Solutions: Here,

Sum of first two terms of a GP (s_2) = 6

Sum of first four terms of a GP (s_4) =30

Sum of 1st 10 terms (s_{10}) = ?

By formula,

$$s_n = \frac{a(r^n - 1)}{r-1}$$

$$or, s_2 = \frac{a(r^2-1)}{r-1}$$

$$or, 6 = \frac{a(r^2-1)}{r-1} \quad \text{………………… (i)}$$

Similarly, $s_4 = \frac{a(r^4-1)}{r-1}$

$$or, 30 = \frac{a\{(r^2)^2-(1)^2\}}{(r-1)}$$

$$or, 30 = \frac{a(r^2+1)(r^2-1)}{r-1} \quad \text{………… (ii)}$$

Dividing equation (ii) by equation (i), we get

$$\frac{30}{6} = \frac{\frac{a(r^2+1)(r^2-1)}{r-1}}{\frac{a(r^2-1)}{r-1}}$$

$$\text{or}, 5 = \mathrm{r}^2 + 1$$

$$\text{or}, 5 - 1 = \mathrm{r}^2$$

$$\mathrm{r}^2 = 4$$

$$\mathrm{r} = \pm 2$$

Case I) When r = 2 then from equation (i)

$$6 = \frac{a(2^2-1)}{2-1}$$

$$or, 6 = \frac{a(3)}{1}$$

$$\therefore a = 2$$

Then, $S_{10} = \frac{2(2^{10}-1)}{2-1}$ = 2 x 1023 = 2046

Case II) When r = –2 then from equation (i)

$$6 = \frac{a(\{(-2)^2-1)\}}{-2-1}$$

$$or, 6 = \frac{a\times 3}{-3}$$

$$\therefore a = -6$$

Then S_{10} is given by

$$s_{10} = -\frac{6\{(-2)^{10}-1\}}{-2-1} = -\frac{6(1023)}{-3} = 2046$$

∴ The sum of 1st 10 terms is 2046

Example 4

The sum of three numbers in geometric progression is 52 and the product of these three numbers is 1728. Find the numbers.

Solution

Let the three numbers in GP be $\frac{a}{r}$, a, ar

From first condition:

$$\frac{a}{r} + a + ar = 52$$

$$or, a\left(\frac{1}{r} + 1 + r\right) = 52$$

$$or, \frac{a(1+r+r^2)}{r} = 52$$

$a(1 + r + r^2) = 52r$ .... (i)

From second condition:

$$\frac{a}{r} \times a \times ar = 1728$$

$$or, a^3 = 1728$$

$$a = \sqrt[3]{1728} = 12$$

Substituting a = 12 in equation (i) we get

$$12(1 + r + r^2) = 52r$$

$or, 3(1 + r + r^2) = 13r$

$or, 3 + 3r + 3r^2 - 13r = 0$

$or, 3r^2 - 10r + 3 = 0$

$or, 3r^2 - (9 + 1)r + 3 = 0$

$or, 3r^2 - 9r - r + 3 = 0$

$or, 3r(r - 3) - 2(r - 3) = 0$

$or, (r - 3)(3r - 1) = 0$

Either, $r - 3 = 0 \Rightarrow r = 3$

or, $3r - 1 = 0 \Rightarrow r = \frac{1}{3}$

Case I: When a = 12 and r = 3 then three numbers are

$\frac{a}{r} = \frac{12}{3} = 4$

$a = 12$

$ar = 12 \times 3 = 36$

Case II When a = 12 and r = $\frac{1}{3}$ then three numbers are

$\frac{a}{r} = \frac{12}{\frac{1}{3}} = 12 \times 3 = 36$

$a = 12$

$ar = 12 \times \frac{1}{3} = 4$

Hence, the required numbers are 4, 12, 36 or 36, 12, 4

Exercise 1.3.6

1. a) If $lr - a = 90$ and $r - 1 = 9$, find s_n.
   b) Write the sum of first 5 terms of the geometric series if a = 1 and r = 2.
2. a) If $500 = \frac{1893 \times 3 - a}{3 - 1}$, Find the value of a.
   b) In a geometric series 2 + 4 + 8 + 16 + 32 + 64 + 128. What is the common ratio?
   c) Write the first term and last term of the given GP: 81 + 27 + 9 + 3 + 1 + $\frac{1}{3}$
3. a) Find the sum of the following geometric series:

   i) 24 + 12 + 6 + … to + 10 terms
   ii) 3 – 6 + 12 - … to 7 terms
   iii) 27 + 18 + 12 + … to 6 terms
   iv) $\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} +$… to 8 terms
   (v) 3 + 6 + 18 … + 4374
   vi) 3 + 6 + 12 + … + 768

4. a) The first term of a GP is 1, the sum of the third and fifth term is 90. Find the common ratio.
   b) If the sum of first two terms is 10 and the first term is 2. Find the common ratio.
5. a) The sum of 4^{th} terms of a series in GP whose common ratio is 2 is 255. Find the first term of a GP.
   b) The sum of a series in GP whose common ratio 4 is 1364, and the last term is 1024. Find the first term.
   a) If the first term and common ratio of a geometric series are $\frac{1}{3}$ and 3 respectively. Find the sum of 1^{st} 6 terms.
6. a) How many terms are in a GP where a = 2, r = 3 and the sum is 728?
   b) How many terms of the series 5 + 10 + 20 + … must be taken so that the sum becomes 315?
7. a) Find the sum of first 8 terms of a GP whose 3^{rd} and 7^{th} terms are 8 and 128 respectively.
   b) In a GP, 4^{th} term and 7^{th} terms are 27 and 729 respectively. Find
      i) common ratio and first term   ii) Geometric series
      iii) Sum of first eight terms
8. a) In a GP the sum of 1^{st} two terms is 18 and the sum of first 4 terms is 90. Find
      i) Common ratio and first term   ii) Geometric Series
      iii) Sum of first 6 terms
   b) In a GP the sum of 1^{st} three terms is $\frac{7}{4}$ and the sum of 1^{st} six terms is $\frac{63}{4}$. Find the sum of 1^{st} 8^{th} terms
9. a) In a GP the sum of three numbers is 28 and their product is 512. Find the numbers.
   b) The sum of three consecutive terms of GP 13 and their product is 27. Find the three consecutive terms.
10. a) The sum of three numbers in GP is 56. If 1, 7, 21 are subtracted from the numbers respectively which form the consecutive term of an A.P. Find the original numbers.
    b) The sum of three numbers in AP is 15. If 1 and 5 are respectively added to 2^{nd} and 3^{rd} numbers then the first number together with these two are in GP find the original numbers.
11. The IQ scores of 10 students in a test are as the rule that the score of second student is the double of score of first student. The score of third student is double the score of second and so on. If the score of the first student is 2, find the score of 10^{th} students. Also find total score of all 10 students.

1.4 Linear programming

1.4.0 Review

The general form of a linear equation of two variables x and y is ax + by + c = 0. A relation represented by ax + by + c > 0, ax + by + c < 0, or ax + by + c ≥ 0 or ax + by + c ≤ 0 is known as the linear inequality in two variables x and y.

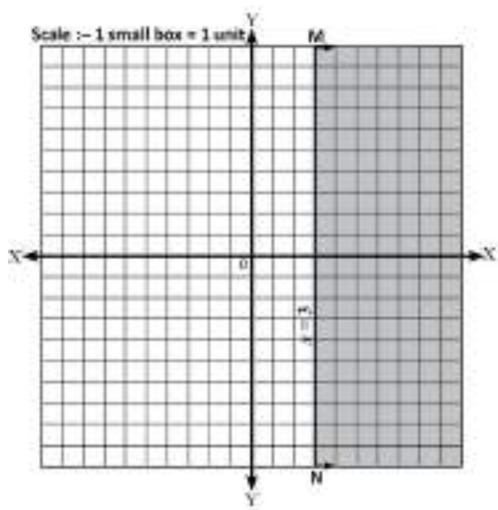
Let us consider one example; $x \geq 3$.

i) How many variable are there in the above inequality?

ii) What do you mean by sign $\geq$?

The associated equation of $x \geq 3$ is $x = 3$ is a straight line (say MN) parallel to y-axis at a distance of 3 units from x-axis.

From the graph, we observe that the line $x = 3$ i.e. MN divides the whole plane into two parts, one on the right side of MN each point of which x-coordinate is greater than 3 and other on the left of MN each point of which x-coordinate less than 3. Each point on MN has x-coordinate 3. The line $x = 3$ is said to be the boundary line.

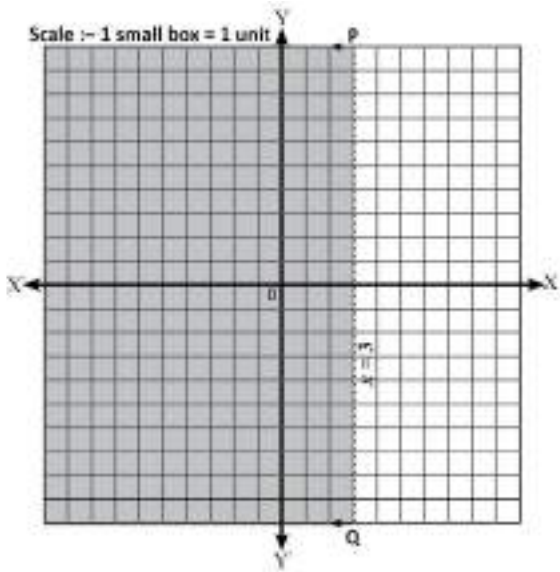


Hence, the graph of $x \geq 3$ is shown in the plane region on the right of MN including MN. Hence, MN must be drawn by the solid line because the line MN is also included on the graph.

Let's take another inequality

$x < 3$

The graph of $x < 3$ will be the plane region on the left of PQ (not containing PQ as the inequality does not contain equality sign as well. Here, we have drawn broken line to indicate that the line PQ is not included on the graph.



When we put $x = 0$, $y = 0$ in $x < 3$, we have $0 < 3$ which is true, so its graph is a plane region containing the origin (0, 0).

1.4.1. Graph of inequalities in two variables

Let us consider $2x + 3y \geq 6$

What are the two variables? Let's discuss.

Now,

i) The corresponding equation of the given inequality is $2x + 3y = 6$ which is the boundary line of the given inequality.

ii) The table from equation is

x	0	3	6
y	2	0	-2

∴ The boundary line passes through the points (0, 2), (3, 0) and (6, -2). It divides the plane region into two parts

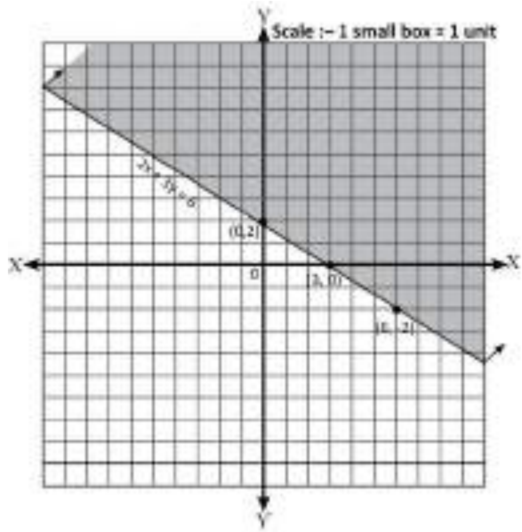
iii) Plot the points (0, 2), (3, 0) and (6, -2). Join the points by the solid line (not by dotted line).

(iv) For the graph of $2x + 3y \geq 6$, we use the test point (0, 0), put $x = 0$, $y = 0$ in the given in equation and if (0, 0) satisfies the given in equation, then the graph of the given in equation is the plane region containing the origin. But if (0, 0) does not satisfy the given in equation, the graph is the plane region not containing the origin.

Taking testing point (0, 0), put $x = 0$, $y = 0$ in $2x + 3y \leq 6$

i.e. $0 + 0 \geq 6$ $0 \geq 6$ which is false

$\therefore$ The graph of $2x + 3y \geq 6$ is the plane region not containing the origin.



Example 1

Draw the graph of $2x - 3y < 6$

Solution: Here,

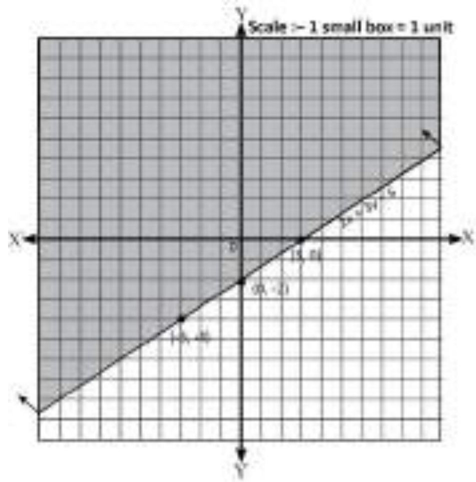
The corresponding equation of given inequality is $2x - 3y = 6$ which is the boundary line

or, $2x = 6 + 3y$

or, $x = \frac{6+3y}{2}$ .... (i)

Table from the equation (i)

x	3	0	-3
y	0	-2	-4



$\therefore$ The boundary line passes through the points (3, 0), (0,-2) and (-3,-4). Plot the points (3, 0), (0, -2) and (-3, -4). Join the points by a dotted line.

For the graph, taking testing point (0, 0), put $x = 0$, $y = 0$ in $2x - 3y < 6$

I.e. $0 - 0 < 6$ or, $0 < 6$ which is true.

Hence, the graph of $2x - 3y < 6$ is the plane region containing the origin but not boundary line.

Note: If the corresponding equation of a given inequation passes through the origin, then the test point should be different from (0, 0) i.e. (1, 0) or (0, 1) etc.

1.4.2 System of Linear Inequalities

A set of two or more linear inequalities having a common solution region (set) is said to be the system of linear inequalities. In this system, the plane regions determined by the set of inequalities are shown in the same graph.

Example 2

Draw the graph of $x - 2y \geq 4$ and $2x + y \leq 8$

Solution

The given in equations are

$x - 2y \geq 4$ and $2x + y \leq 8$

The corresponding equation of given inequations are

$x - 2y = 4$ ………………… (i)

$2x + y = 8$ ………………… (ii)

From equation (i), $x = 4 + 2y$
table from equation (i)

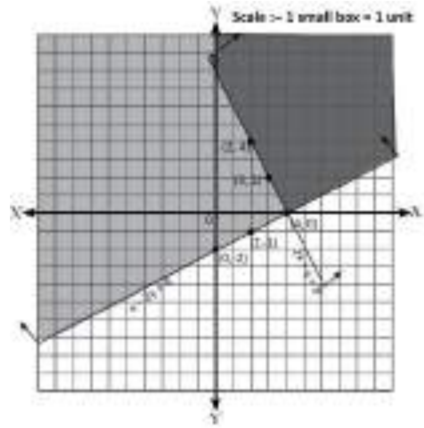
x	4	0	2
y	0	-2	-1

Similarly from equation (ii)

$y = 8 - 2x$

Table from equation (ii)

x	4	3	2
y	0	2	4



One boundary line passes through the points (4, 0), (0, -2) and (2, -1). Plot the points and join these points by solid line. Similarly, another boundary line passes through the points (4, 0) (3, 2) and (2, 4). Plot these points and join these points by another solid line.

Taking testing point (0, 0) in $x - 2y \geq 4$, put x = 0 and y = 0, $0 - 2 \times 0 \geq 4 \Rightarrow 0 \geq 4$ which is false.

Hence, the graph of x-2y ≥ 4 is the plain region does not contain the origin. Similarly, taking testing point (0, 0) in $2x+y \leq 8$, Put x = 0, y = 0, then $2 \times 0 + 0 \leq 8 = 0 \leq 8$ which is true.

Hence, the graph of $2x+y \leq 8$ is the plane region containing the origin.

∴ The intersection part of the shaded region gives the required solution set of the given system of inequalities.

1.4.3 Linear programming

Most of the business and economic activities may have various problem of planning due to limited resources. In order to achieve the business goal i.e. minimizing the cost of production and maximize the profit from the optimum use of available limited resources, linear programming is used.

Linear programming is a mathematical technique of finding the maximum or minimum value of the objective function satisfying the given condition. The problem which has object of finding maximum or minimum value satisfying all the given condition is called linear programming (L.P) problem. To define linear programming, we need some basic definitions:

(i) **Decision variables**: The non-negative independent variables involving in the L.P problem are called decision variables. For example: in $2x + 3y = 7$, x and y are decision variable.

(ii) **Objective function**: The linear function whose value is to be maximized or minimized (optimized) is called an objective function.

(iii) **Constraints**: - The conditions satisfied by the decision variables are called constraints. For example: if x and y be the number of first two kinds of articles produced, then $x + y \geq 1000$; $x \geq 0$ and $y \geq 0$ are the constraints

(iv) **Feasible region (convex polygonal region)**: A closed plane region bounded by the intersection of finite number of boundary lines is known as feasible region.

(v) **Feasible solution**: The values of decision variables x and y involved in objective function satisfying all the given condition is known as feasible solution.

Maximum or the minimum value of an objective function will always occur at the vertex of feasible region.

Example 3

Find the maximum and the minimum values of the objective function (F) = $4x - y$ subject to $2x + 3y \geq 6$, $2x - 3y \leq 6$ and $y \leq 2$

Solution: - Here,

The given constraints are

$2x + 3y \geq 6$, $2x - 3y \leq 6$ and $y \leq 2$

The objective function (F) = $4x - y$

To find: The maximum and the minimum value.

The corresponding equation of given constraints are

$2x+3y = 6$ .... (i)

$2x-3y = 6$ .... (ii)

and $y = 2$ .... (iii)

From equation (i),

$3y = 6 - 2x$

or, $y = \frac{6-2x}{3}$

Table from equation (i)

x	0	3	6
y	2	0	-2

Similarly, from equation (ii)

$2x = 6+3y$

or, $x = \frac{6+2y}{3}$

Table form equation (ii)

x	3	0	-3
y	0	-2	-4

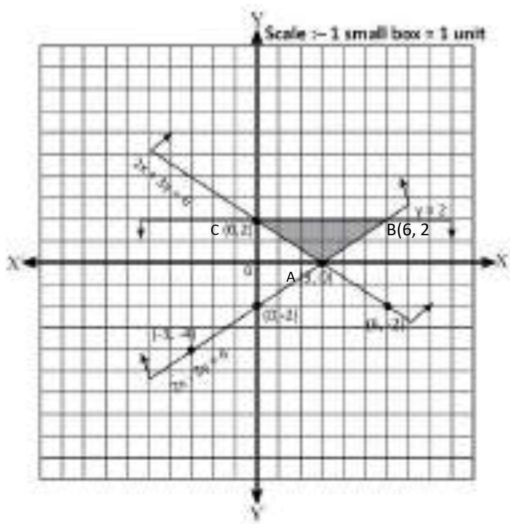
The Boundary line (i) passes through the point (0, 2), (3, 0) and (6, -2). Draw a solid line through these points in graph. Similarly, another boundary line (ii) passes through the points (3, 0), (0, -2) and (-3, -4). Draw a solid line through these points in graph.

Taking common testing point (0, 0) in both inequality $2x + 3y \geq 6$ and $2x - 3y \leq 6$

Now, $x = 0$, and y = 0 then $2x + 3y \geq 6 = 2 \times 0 + 3 \times 0 \geq 6 = 0 \geq 6$ which is false.

So, the graph of $2x + 3y \geq 6$ is the plane region does not contain the origin.
Similarly, taking $x = 0$ and y = 0 in (ii) then $2x - 3y \leq 6 \Rightarrow 2 \times 0 - 3 \times 0 \leq 6 = 0 \leq 6$

which is true. So the graph of $2x-3y \leq 6$ is the plain region containing the origin.



From equation (iii), y = 2 is a straight line parallel to x-axis lying at a distance of 2 units above from x-axis. The graph of $y \leq 2$ is the lower half plane from y = 2 including the boundary line.

From the graph, ΔABC is the feasible region. The vertices of the feasible region are A(3, 0), B(6, 2) and C(0, 2)

Now, the feasible solution is

Vertices	Objective function $F = 4x - y$	Remarks
A (3, 0)	$F = 4 \times 3 - 0 = 12$	
B (6, 2)	$F = 4 \times 6 - 2 = 22$	22 (max)
C (0, 2)	$F = 4 \times 0 - 2 = -2$	-2 (min)

∴ Maximum value of F = 22 at the vertex B(6, 2) i.e. when $x = 6$ and y = 2 and the minimum value of F = -2 at the vertex C(0, 2) i.e. when $x = 0$ and y = 2

Example 4

Maximize and minimize Z = $2x + y$ under the constraints, $x + y \leq 6$, $x - y \leq 4$, $x \geq 0$, $y \geq 0$

Solution: Here,

The given constraints are,

$x + y \leq 6$, $x - y \leq 4$, $x \geq 0$, and $y \geq 0$

The objective function (Z) = $2x+y$

To find: maximum and minimum value

The corresponding equation of the given constraints are:

$x + y = 6$ .... (i)

$x - y = 4$ .... (ii)

$x = 0$ .... (iii)

$y = 0$ .... (iv)

From the equation (i) $y = 6 - x$

Table from equation (i)

x	0	6	3
y	6	0	3

From equation (ii) $y = x - 4$

Table from equation (ii)

x	0	4	2
y	-4	0	-2

The boundary line (i) passes through (0, 6), (6, 0) and (3, 3). Plot the points and join them by the solid line. Similarly, the boundary line (ii) passes through (0, -4), (4, 0) and (2, -2). Plot the points and join them by the solid line.

Taking common testing point (1, 1) in the given inequalities

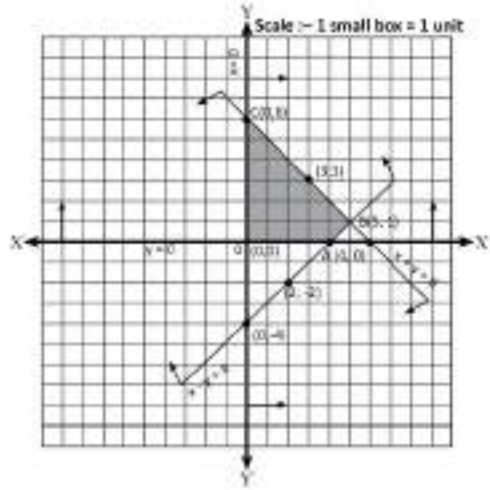
Now, $x = 1$ and $y = 1$ in $x + y \leq 6$

$1 + 1 \leq 6 = 2 \leq 6$ which is true. So the graph of $x + y \leq 6$ in the plane region containing the testing point (1, 1). Similarly, taking $x = 1$ and $y = 1$ in $x - y \leq 4$

1-1 ≤ 4 = 0 ≤ 4 which is true. So the graph of x – y ≤ 4 in the plane region containing the origin.

From the equation (iii) and (iv) x = 0 and y = 0 are the y-axis and x-axis respectively. $x \geq$ 0 is the right half plane containing the y-axis and y ≥ 0 is the upper half plane containing the x-axis.

From the graph quadrilateral OABC is the feasible region. The vertices of the feasible region are O(0, 0), A(4, 0), B(5, 1) and C(0, 6).



Hence the feasible solution is

Vertices	Objective function Z = 2x + y	Remarks
O (0, 0)	Z = 2 x 0 + 0 = 0	0 (min)
A (4, 0)	Z = 2 x 4 + 0 = 8	
B (5, 1)	Z = 2 x 5 + 1 = 11	11 (Max)
C (0, 6)	Z = 2 x 0 + 6 = 6	

∴ Maximum value of Z = 11 at the vertex B(5, 1) and minimum value of Z = 0 at the vertex O (0, 0)

Example 5

In the given diagram the coordinates of A, B, and C are (2, 0), (6, 0) and (1, 4) respectively. The shaded region inside the ΔABC is represented by inequalities. Write down the equations of these inequalities and also calculate the minimum value of 2x+3y from the values which satisfy all the three inequalities.

Solution: Here,

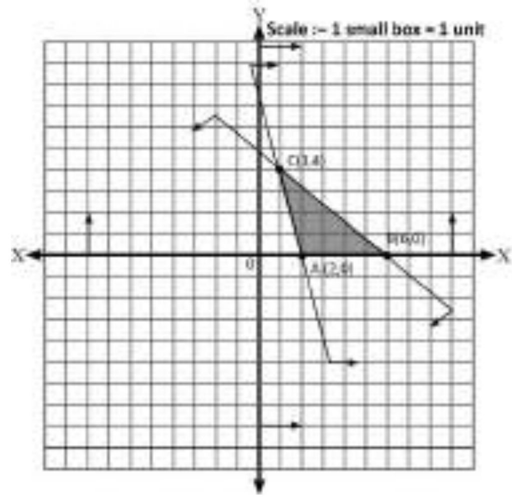
ΔABC is a feasible region and their coordinates are A(2, 0), B(6, 0) and C(1, 4)

The objective function F = 2x + 3y

To find: (i) Equation of inequalities (ii) The minimum value

For the line BC,

The equation of line BC passing through B(6, 0) and C(1, 4)



$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Or, $y - 0 = \frac{4-0}{1-6}(x - 6)$

Or, $y = \frac{4}{-5}(x - 6)$

Or, $4x - 24 = -5y$

Or, $4x + 5y = 24$

Since, the half plane with the boundary line BC contains origin. So the inequality of BC is $4x + 5y \leq 24$

Similarly, for AC

The equation of line AC Passing through A(2, 0) and C(1,4) is

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Or, $y - 0 = \frac{4-0}{1-2}(x - 2)$

Or, $y - 0 = \frac{4}{-1}(x - 2)$

Or, $y - 0 = -4x + 8$

Or, $4x + y = 8$

Since the half plane with the boundary line AC doesn't contain the origin, the inequality of AC is $4x + y \geq 8$.

The equation of AB means the equation of x-axis is y = 0 and the equation of y-axis is $x = 0$.

The shaded region lies in the first quadrant only. So the inequality of AB is $y \geq 0$ and inequality of y-axis is $x \geq 0$

$\therefore$ The inequality equations are

$4x + 5y \leq 24$, $4x + y \geq 8$, $x \geq 0$, $y \geq 0$

Again for the minimum value,

Vertices	Objective function (F) = $2x+3y$	Remarks
A (2, 0)	F= 2 x 2 +3 x 0 = 4	4 (Min)
B (6, 0)	F = 2 x 6 + 3 x 0 = 12	
C (1, 4)	F = 2 x 1 + 3 x 4 = 14	

$\therefore$ Minimum value of F = 4 at the vertex A(2, 0)

Exercise 1.4.1

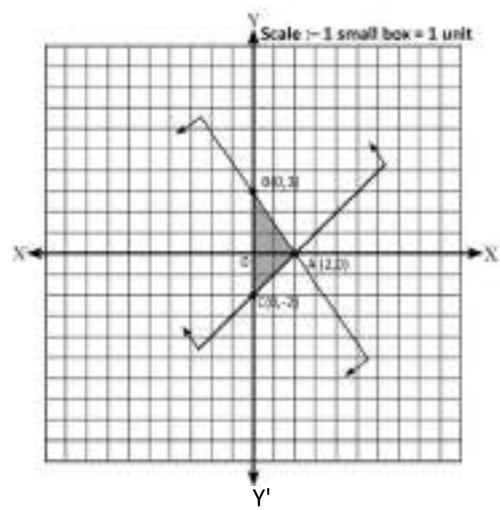
1. a) Define boundary line with an example.
   b) In which condition the boundary line is dotted line?
2. a) Define constraints with an example.
   b) What do you mean by objective function? Also write an example.
3. a) In an objective function (F) = $5x - 2y$, one vertex of feasible region is (5, 2) then find the value of objective function.
   b) From the given inequality $4x + 3y \geq 10$, Write the boundary line equation.
4. a) From the adjoining figure,

   i) What is called the shaded region ΔABC?

   ii) Find the equation of AB.

   c) Draw the graph of $x \geq 0$.

   d) Draw the graph of $y \leq 0$.



5. Where does the solution set lies for the inequalities $x \geq 0$ and $y \geq 0$?
6. a) Draw the graph of the following inequalities.

   i) $x \geq y$ ii) $x + 2y \leq 8$ iii) $x \geq -5$ iv) $y \geq 2x$

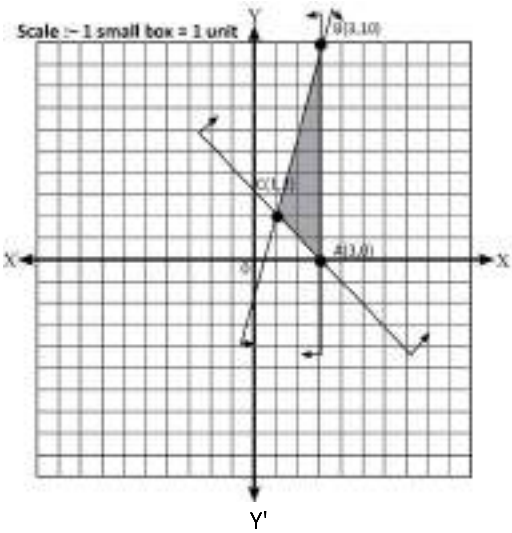
b) Draw the graph of the following inequalities and shade the common solution set.

i) $2x + 2y \geq 6$ and $y \geq 0$ ii) $2x + y \geq 6$ and $x \geq 2$

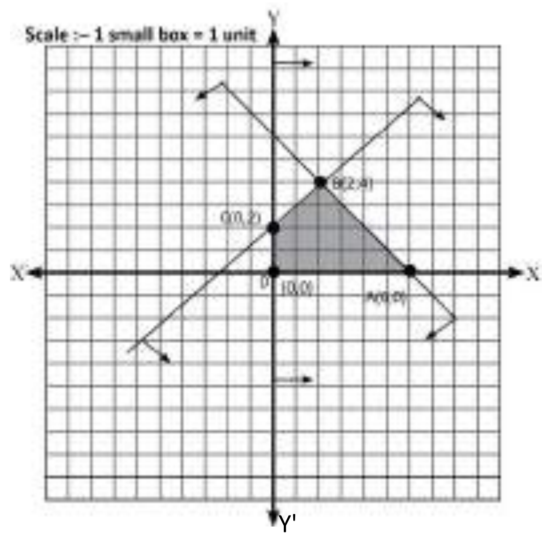
iii) $x + y \leq 2$ and $x \leq 0$ iv) $x - 3y \leq 6$ and $y \leq 3$

7. a) Find the maximum and minimum value of the function Z = 3x + 5y for each of the following feasible region,

i)



ii)



8. Draw the graphs of the following inequalities and find the feasible region. Also find the vertices of the feasible region.

i) $x + y \leq 3$
$x \geq 2$
$y \leq 1$

ii) $x - 2y \geq 4$
$2x + y \leq 8$
$y \geq -1$

iii) $2x + y \geq 4$
$3x + 4y \leq 12$
$x \geq 0, y \geq 0$

iv) $2y \geq x - 1$
$x + y \leq 4$
$x \geq 0, y \geq 0$

9. Find the maximum values of the following objective functions with the given constraints.

i) $p = 14x + 16y$ subject to
$3x + 2y \leq 12$
$7x + 5y \leq 28$
$x \geq 0, y \geq 0$

ii) $z = 6x + 10y + 20$ subject to
$3x + 5y \leq 15$
$5x + 2y \leq 10$
$x \geq 0$ and $y \geq 0$

iii) $F = 6x + 5y$ subject to
$x + y \leq 6$
$x - y \geq -2$
$x \geq 0, y \geq 0$

iv) $Q = 3x + 2y$ subject to
$x + y \geq 0$
$x - y \leq 0$
$y \leq 2, x \geq -1$

10. Find the minimum values of the following objective functions with the given constraints.

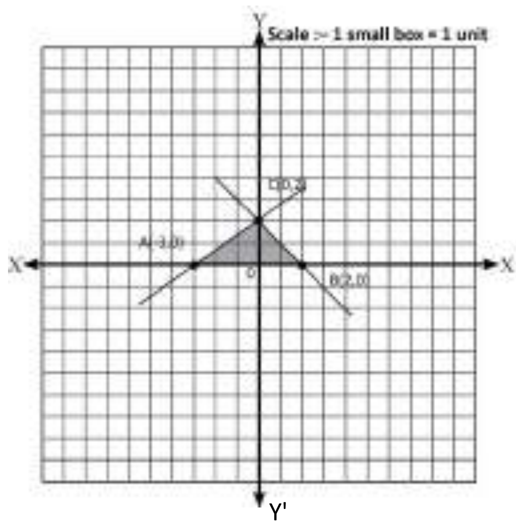
i) $2x + 4y \le 8$
$3x + y \le 3$
$x \ge 0, y \ge 0$
objective function (F) = $5x + 4y$

ii) Objective function (M) = $x + y$ subject to
$3x + 4y \le 21$
$2x + y \ge 4$
$x \ge 0, y \ge 0$

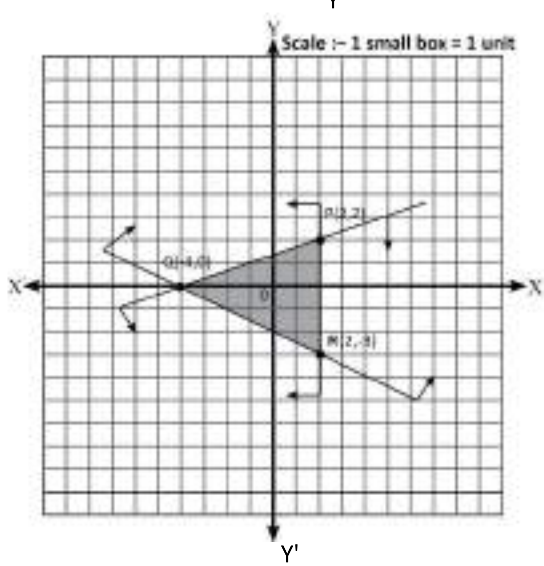
iii) Objective function (L) = $3x + 5y$
subject to $2x + y \le 6$
$x + y \ge 3$
$x \ge 0, y \ge 0$

iv) Objective function (F) = $6x + 9y$
Subject to
$2x + y \le 9$
$y \ge x, x \ge 1$

11. In the adjoining figure, the coordinates of A, B, C are (-3, 0), (2, 0) and (0, 2) respectively. Find the inequalities represented by the shaded region and also calculate the maximum value of $3x + 4y$ which satisfy all three inequalities.



12. Study and discuss, in the adjoining figure, find the inequalities which represent the boundaries of shaded region as solution set and find the maximum and minimum value of Z = $3x + 5y$. Prepare a report and present it in your class room.



1.5 Quadratic equations and graph

1.5.0 Review

Let us consider, an equation and discuss on it

y = 2x + 3

(i) What are x and y called?

(ii) What are the maximum degree of x and y?

(iii) What has the degree of constant term?

(iv) Write the name of given equation.

Can you draw the graph of the given equation? If it is, let's discuss on its shape.

Again, let us consider another equation

$x^2 + 3x + 2 = 0$

(i) Write the degree of the equation.

(ii) What is the variable in that equation?

(iii) Can you write the roots of the equation $x^2 + 3x + 2 = 0$?

Discuss above questions in different groups and write the conclusion

1.5.1 Graph of quadratic function

The general form of quadratic function is y = ax^2 + bx + c where, a, b and c are called coefficient of x^2, coefficient of x and constant term respectively. The graph of quadratic function is called parabola.

a) **Graph of the quadratic function (y = ax^2)**

i) y = ax^2 where a = 1

to draw the graph, let's find some values of x and y

x	0	±1	±2	±3	±4
y	0	1	4	9	16

From the table, plotting the pair of points in a graph and joined them freely

ii) y = x^2 where a = -1

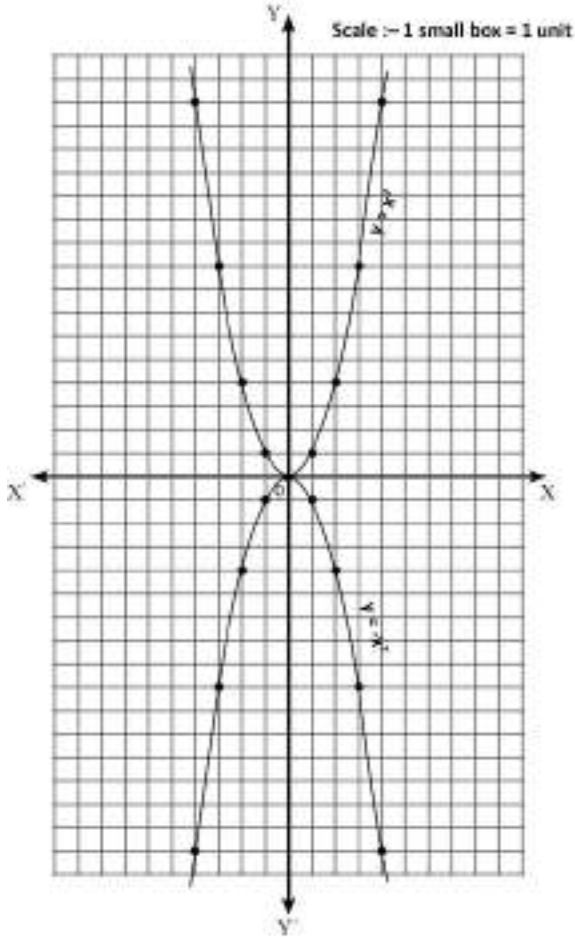
To draw the graph, let's find some values of x and y

x	0	±1	±2	±3	±4
y	0	-1	-4	-9	-16

Plotting the pair of points in a graph and joined them freely.

From graphs, we can say that, for different values of a, we get different curves of the same nature known as the parabola with same turning point origin known as the vertex of parabola. Also, we can find the following information:

i) The parabola turns upward for a > 0 and turns downward for a < 0 .
ii) Each parabola is symmetrical about y-axis i.e. y-axis divides each parabola into two identical parts.
iii) Greater value of 'a' numerically narrower will be the faces of parabola and lesser the value of 'a' numerically wider will be the faces of parabola.



b) **Graph of the quadratic function y = ax^2 + bx + c**

the given quadratic function is

Or, $y = ax^2 + bx + c$

Or, $y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ [∴ Taking a common]

Or, $y = a\left\{(x)^2 + 2.x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right\}$

Or, $y = a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right\}$

Or, $y = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right\}$

Or, $y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$

Which is in the form of y = a (x - h)^2 + k Where h = $\frac{-b}{2a}$, k = $\frac{4ac-b^2}{4a}$

∴ The vertex of parabola = (h, k) = $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

When k = 0, the equation of parabola takes the form y = a(x – h)^2 where the vertex = (h, 0).

Conclusion:

i) The graph of y = ax^2 + bx + c is called parabola is symmetrical to a line parallel to y-axis.

ii) The equation of line of symmetry is x = $-\frac{b}{2a}$

Example 1

Draw the graph of y = x^2 +2x – 8. Also find the equation of line of symmetry.

Solution: here,

The given quadratic equation is

y = x^2 +2x -8 ………… (i)

Now, comparing equation (i) with y = ax^2 + bx + c we get

a = 1, b = 2 and c = -8

Now, x - coordinate of the vertex of parabola

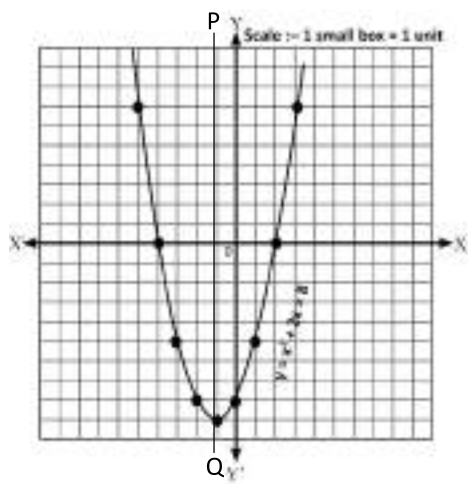
$(x) = -\frac{b}{2a} = \frac{-(2a)}{2xa} = -1$

∴y-coordinate of the vertex of parabola

(y) = (-1)^2 + 2(-1) – 8

= 1 -2 - 8

= -9



∴ Vertex or turning point of the parabola (h, k) = (-1, -9)

Table from equation (i)

x	-1	0	1	2	-2	3	-3	-4	-5
Y	-9	-8	-5	0	-8	7	-5	0	7

Plotting the pair of points

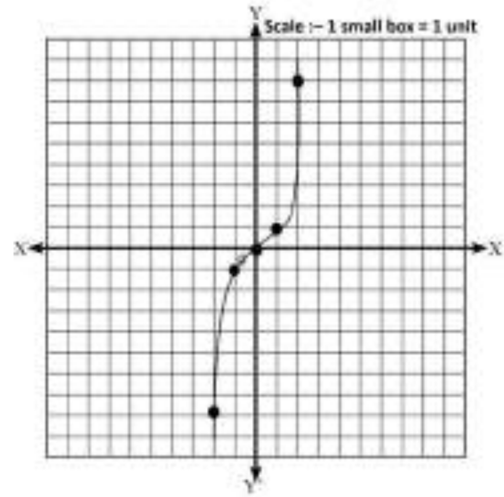
(-1, -9), (0, -8), (1, -5), (2, 0), (-2, -8), (3, 7), (-3,-5), (-4,0), (-5,7). Join these points freely. The Parabola meets the x-axis at two points (2, 0) and (-4, 0).

Again, PQ is the line of symmetry and its equation is x = -1.

Note: For the quadratic function y = ax^2 + bx + c the turning point not at the origin.

1.5.2 Graph of a cubic function

The general form of cubic function is defined by $y = ax^3 + bx^2 + cx + d$ where, a, b, c and d are constants and $a \neq 0$. The simplest form of a cubic function passing through origin is $y = ax^3$. Different values of 'a' will give different curves passing through the origin with similar nature. Now $y = ax^3$ when a = 1 then discuss on its graph



$y = x^3$ …………… (i)

Table from equation (i)

x	0	1	2	-1	-2
y	0	1	8	-1	-8

Plotting the pair of points (0, 0), (1, 1), (2, 8), (-1, -1), (-2, -8) in a graph and joined them freely.

The curve line passes through the points of 1st quadrant, origin and 3rd quadrant points when coefficient of x^3 (a) is positive.

Example 2

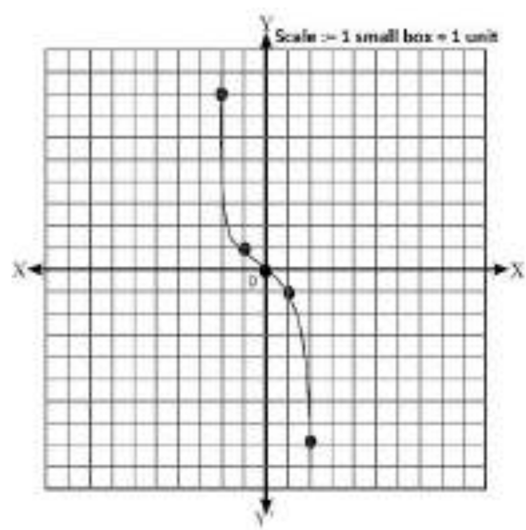
Draw the graph of $y = -x^3$ where a = -1 also, write the nature of graph.

Solution: Here,

The given cubic function is

$y = -x^3$

Table from equation (i)



x	0	1	-1	2	-2
y	0	-1	1	-8	8

Plotting the pair of points (0, 0), (1, -1), (-1, 1), (2, -8) and (-2, 8) in a graph and joined them freely

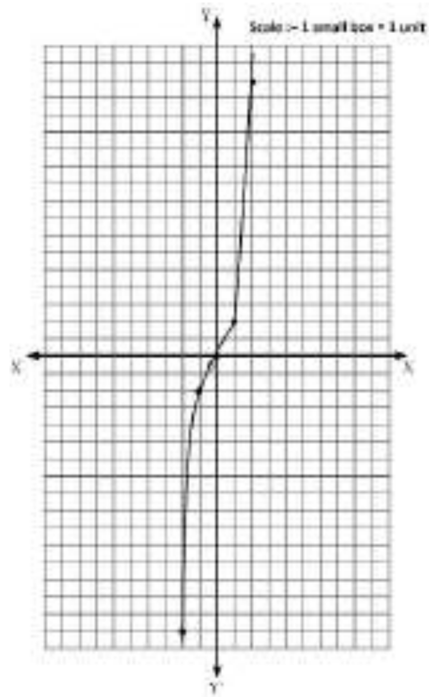
Again, the curve line passes through the points of 2nd quadrant, origin and 4th quadrant points when coefficient of x^3 (a) is negative

Example 3

Draw the graph of y = $2x^3$, also write the value of 'a'.

Solution: Here,
The given cubic equation is
y = $2x^3$ ………………….. (i)
Table from the equation

x	0	1	-1	2	-2
y	0	2	-2	16	-16



Plotting the pair of points
(0, 0), (1, 2), (-1, -2), (2, 16), (-2, -16) in a graph and joined them freely

The value of a = 2

Let us discuss on the following equations and draw the graph of
(i) $y = -2x^3$

(ii) $y = \frac{1}{2}x^3$

(iii) $y = -\frac{1}{2}x^3$

1.5.3 Solution of quadratic equation and linear equation

Let us consider two equations:
y = $x^2 - 5x + 6$ and y = 2

Let us discuss on the nature of the graph of both equations and draw the graph
Now, y = $x^2 - 5x + 6$ ………… (i)
y = 2 ………………….. (ii)

Comparing equation (i) with y = $ax^2 + bx + c$, we get a = 1, b = -5, c = 6

X-coordinate of the vertex of parabola (x) = $\frac{-b}{2a} = \frac{(-5)}{2x1} = \frac{5}{2} = 2.5$

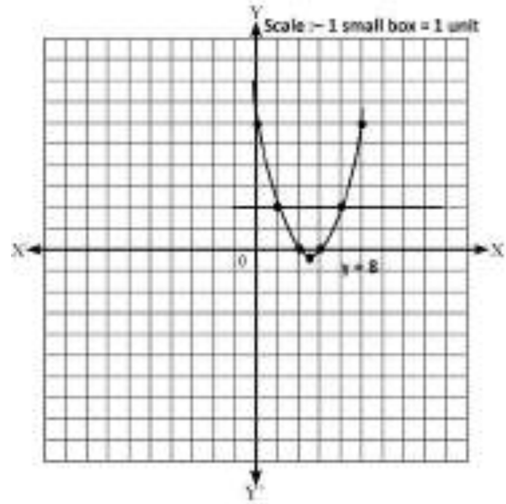
Y-coordinate of the vertex of parabola (y)
= $\left(\frac{5}{2}\right)^2 - 5.\frac{5}{2} + 6$ = $\frac{25}{4} - \frac{25}{2} + 6$ = $\frac{25-50+24}{4}$ = $\frac{-1}{4} = -0.25$

∴ The vertex of parabola (h, k) = (2.5, - 0.25) = $\left(\frac{5}{2}, \frac{-1}{4}\right)$

Table from equation (i)

x	2.5	0	1	2	3	4	5
y	-0.25	6	2	0	0	2	6

Plotting the pair of points (2.5, -0.25), (0, 6), (1, 2), (2, 0), (3, 0), (4, 2), (5, 6) in a graph and joined them freely



Again, From equation (ii) y = 2 represent a straight line parallel to y-axis which is 2 units above the X-axis.

From the graph, the intersection points of the parabola and the straight lines are (1, 2) and (4, 2)
The solutions are $x = 1$, $y = 2$ and $x = 4$, $y = 2$
Hence, $x = 1, 4$ and $y = 2$

By substitution method
the given equations are
$y = x^2 - 5x + 6$ ………………….. (i)
and $y = 2$ ………………………. (ii)

Substituting y = 2 in equation (i), we get
or, $2 = x^2 - 5x + 6$
or, $x^2 - 5x + 6 - 2 = 0$
or, $x^2 - 5x + 4 = 0$
or, $x^2 - (4 + 1)x + 4 = 0$
or, $x^2 - 4x - x + 4 = 0$
or, $x (x - 4) - 1 (x - 4) = 0$
or, $(x - 4) (x - 1) = 0$
Either, $x - 4 = 0$
$x = 4$
or, $x - 1 = 0$
$x = 1$
$\therefore x = 4, 1$ and $y = 2$

Note: *When vertex of parabola (x-coordinate or y coordinate) is in fraction especially their denominator is 4 then take a scale as 8 small boxes = 1 unit so that it is easy to draw the graph.*

Example 1

Solve: y = $x^2 - 3x + 5$ and y = $2x + 1$ by graphical method and substitution method.

Solution: Here,

The given equations are:

y = $x^2 - 3x + 5$ .... (i)

y = $2x + 1$ .... (ii)

Comparing equation (i) with y = $ax^2 + bx + c$, we get

a = 1, b = -3, c = 5

Now, x-coordinate of the vertex of parabola (x) = $\frac{-b}{2a} = -\frac{(-3)}{2} = \frac{3}{2} = 1.5$

And y-coordinate of the vertex of parabola (y) = $\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 5$

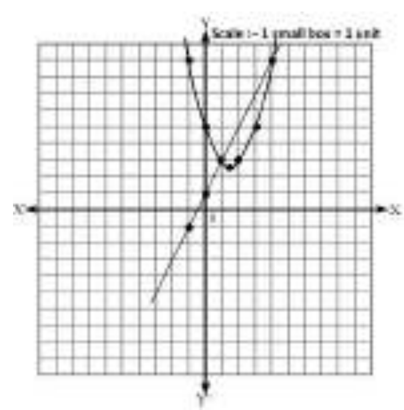
$= \frac{9}{4} - \frac{9}{2} + 5 = \frac{9-18+20}{4} = \frac{11}{4} = 2.75$

∴ The vertex of parabola (h, k) = $\left(\frac{3}{2}, \frac{11}{4}\right)$ = (1.5, 2. 75)

Table from equation (i)

x	1.5	0	1	-1	2	3	4
y	2.75	5	3	9	3	5	9

Plotting the pair of points (1.5, 2.75), (0, 5), (1, 3), (-1, 9), (2, 3), (3, 5) (4, 9) in a graph and joined them freely



Again,
From equation (ii)
y = $2x + 1$

Table for equation (ii)

x	0	-1	1
y	1	-1	3

Plotting the pair of points (0, 1), (-1, -1), (1, 3) in the same graph and draw the straight line
The intersection points of the parabola and a straight line are (1, 3) and (4, 9)

$\therefore$ The solutions are $x = 1$, y = 3, and $x = 4$, y = 9
Hence, (1, 3) and (4, 9) are solution set

By substitution method

The given equations are
$y = x^2 - 3x + 5$ .... (i)
$y = 2x + 1$ .... (ii)

Substituting $y = 2x + 1$ from equation (ii) in equation (i), we get

or, $2x + 1 = x^2 - 3x + 5$

or, $x^2 - 3x + 5 - 2x - 1 = 0$

or, $x^2 - 5x + 4 = 0$

or, $x^2 - (4 + 1)x + 4 = 0$

or, $x^2 - 4x - x + 4 = 0$

or, $x(x - 4) - 1(x - 4) = 0$

or, $(x - 4)(x - 1) = 0$

Either, $x - 4 = 0$
$x = 4$

or, $x - 1 = 0$
$x = 1$

When $x = 4$ then $y = 2 \times 4 + 1 = 9$
When $x = 1$ then $y = 2 \times 1 + 1 = 3$
Hence, the solutions are $x = 4$, y = 9, and $x = 1$, y = 3

Example 2

Solve the equation graphically: $x^2 + 2x - 3 = 0$.

Solution: Here,
The given equation is

$x^2 + 2x - 3 = 0$

or, $x^2 = 3 - 2x$

Let, $y = x^2 = 3 - 2x$

Taking, $y = x^2$ .... (i)

And $y = 3 - 2x$ .... (ii)

From equation (i) $y = x^2$ is in the form of $y = ax^2$ where (a = 1) so its turning point (vertex) is always origin (0, 0).

Table from equation (i)

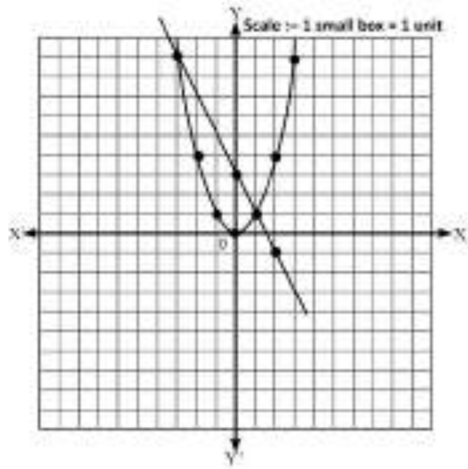
x	0	1	-1	2	-2	3	-3
y	0	1	1	4	4	9	9

Plot the pair of points (0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4), (3, 9) (-3, 9) in a graph and joined them freely

Again, from equation (ii)
y = 3 – 2x

Table from equation (ii)

X	0	1	2
Y	3	1	-1



Plotting the pair of points (0, 3), (1, 1), (2, -1) in the same graph and draw the straight line. From the graph, the intersection points of parabola and the straight line are (1, 1) and (-3, 9)

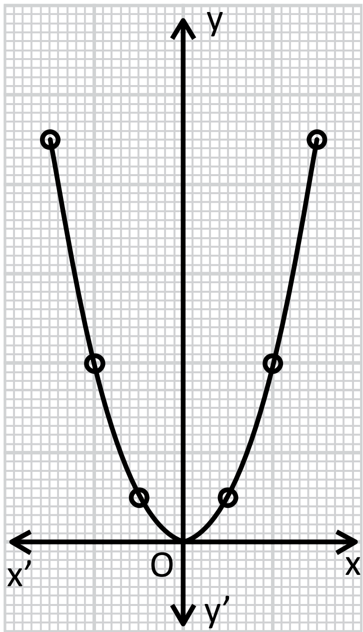
∴ x = 1, -3

Exercise 1.5.1

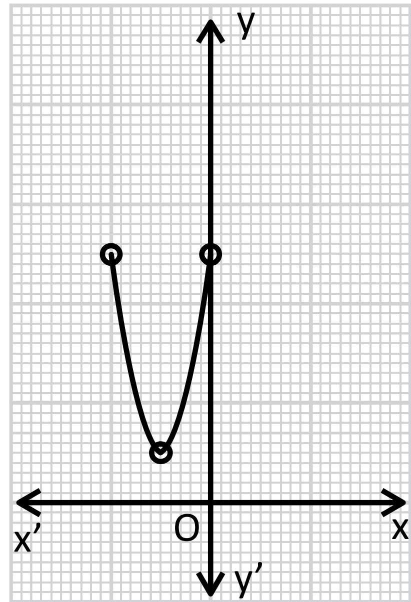
1. a) Define vertex of parabola.
   b) In a quadratic equation ax^2 + bx + c = 0, what a, b and c are called?
2. a) Define line of symmetry in parabolic curve.
   b) Write the equation of line of symmetry in the equation y = x^2.
   c) Write the equation of line of symmetry in the equation y = ax^2 + bx + c.

3. Write the vertex and the equation of line of symmetry of the following graph:
Scale: 5 small boxes = 1 unit

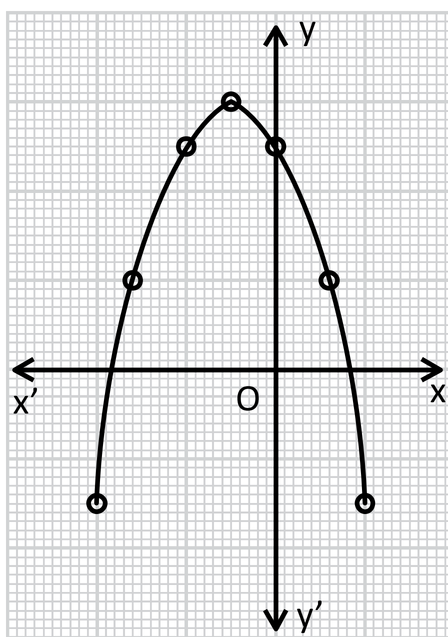
i)



ii)



(iii)



4. a) Write the nature of the graph of $y = \frac{1}{2}\ x^2$.

   b) Write the nature of the graph of $y = -\ x^3$.

5. a) Draw the graph of the following equation (function):

   i) $y = -2x^2$ ii) $y = -\frac{1}{2}x^2$ iii) $y = \frac{1}{2}x^2$ iv) $y = 3x^2$

   b) Draw the graph of following functions:

   i) $y = 2x^2$ ii) $y = 3x^3$ iii) $y = -3x^3$

6. Find the vertex of the following equations:

   i) $y = 4x^2 + 8x + 5$ ii) $y = x^2 - 6x$ iii) $x^2 = 2y$

   iv) $y = x^2 + 3x + 2$ v) $y = x^2 - 6x + 5$

7. Draw the graph of the following function:

   i) $y = x^2 + 2x - 5$ ii) $y = 3x^2 - 2$ iii) $y = x^2 + 4x - 1$

8. Solve the following equations using graphical method as well as substitution method:

   i) $y = x^2$ and $y = 3 - 2x$ ii) $x^2 = 2y$ and $y = x$

   iii) $y = x^2 - 2x$ and $y = x - 2$ iv) $y = x^2 + 3x - 10$ and $x = y$

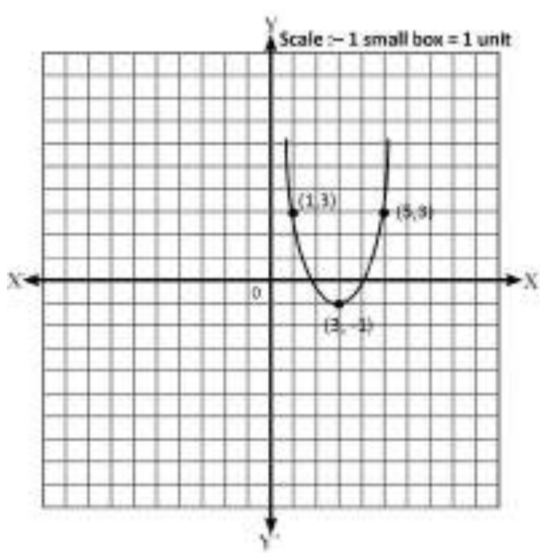
   v) $y = x^2 + 4x - 7$ and $y = 2x + 1$ vi) $y = x^2 + 8x - 6$ and $y = 4 - x$

9. Solve the following equations by graphical method:

   i) $x^2 + 2x - 3 = 0$ ii) $x^2 - 5x + 6 = 0$ iii) $x^2 - 2x - 15 = 0$

   iv) $3x^2 + 5x + 2 = 0$ v) $2x^2 - 7x + 3 = 0$ vi) $x^2 + 6x + 5 = 0$

10. Observe the adjoining graph, write the vertex of parabola and the equation direction of the opening of parabola and the equation of parabola. Also list the steps to find its equation, prepare a report and present it in your class.



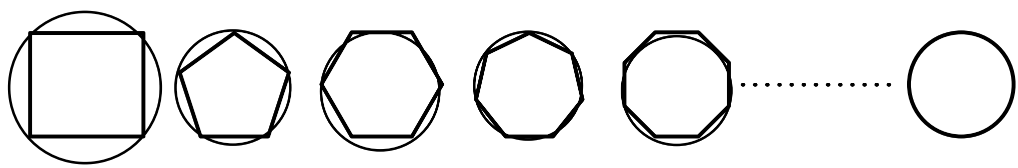
Unit 2

Continuity

2. 0 Review

Limit of a Function

a. Observe the following figures.



Here, let n denotes the number of sides of the polygon. If the sides of polygon increases indefinitely i.e. $n \to \infty$, then the polygon takes the form of a circle. Thus, we write $\lim_{n\to\infty} polygon \ = circle$.

b. For the function $y = f(x) = x^2$, observe the following chart:

x	1.9	1.99	1.999	1.9999
$y = f(x) = x^2$	3.61	3.9601	3.996001	3.999600011
x	2.0001	2.001	2.01	2.1
$y = f(x) = x^2$	4.00040001	4.004001	4.0401	4.41

As x approaches to 2 from left, i.e. $x \to 2^-$ then $f(x) = x^2$ approaches 4. We write $\lim_{x\to 2^-} f(x) = 4$. This is known as left hand limit. Similarly, as x approaches to 2 from right i.e. $x \to 2^+$ then $f(x) = x^2$ approaches to 4. We write $\lim_{x\to 2^+} f(x) = 4$. This is known as right hand limit. In such case, we say the limit of the function exists and has the value 4. We write simply,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 4$$

Thus, the existence of limit if any function depends upon the fact that left hand limit and right hand limit of the function exist and is equal.

i.e. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

It can be summarized that let f(x) be defined about x_0 except possibly at $x_o$itself. If f(x) gets arbitrarily close to L for all sufficiently close to x_o, we say that f(x) approaches the limit L as approaches x_o.

We write $\lim_{x\to x_o} f(x) = L$.

c. Consider the function $f(x) = \frac{x^2-1}{x-1}$. Finding the limiting value of f(x) as $x \to 1$. Here, x = 1 makes f(x) undefined, since $f(x) = \frac{0}{0}$. Therefore, we try to find the limiting value of f(x) when x is sufficiently close to 1 but not equal to 1.

putting x = 0.99

f(0.99) = $\frac{(0.99)^2-1}{0.99-1}$

= $\frac{-0.0199}{-0.01}$

= 1.99

≈ 2

putting x = 1.01

f(1.01) = $\frac{(1.01)^2-1}{1.01-1}$

= 2.01

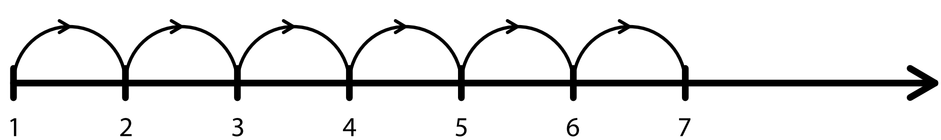
≈ 2

So, $\lim\limits_{x \to 1} \frac{x^2-1}{x-1}$ = 2

2.1 Investigation of continuity in different sets of numbers

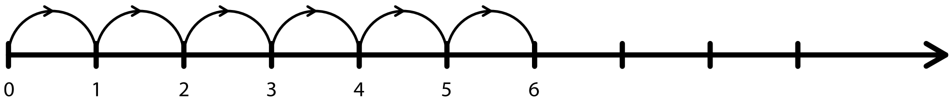
We know, some set of numbers are natural numbers, whole numbers integers, fractions, rational numbers, irrational numbers and real numbers. Here we shall discuss about continuity and discontinuity in their order in real number line. In real number line, numbers are extended in a line from – ∞ to ∞.

a. Observe the number line with Natural number below.



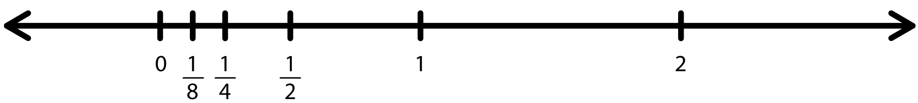
It is obvious that we can find natural numbers between 2 and 6. These are 3, 4, 5. Similarly, we can find natural numbers between 10 and 20. Obviously there are 11, 12, 13, 14, 15, 16, 17, 18, 19. But in number line there is no natural number between 4 and 5. 4.5 is also the number in number line but not natural number. Same property exists in case of integers also. There is no continuity in the set of integers. The set of integers does not hold a property of continuity.

b. The set of whole numbers also holds the same property of discontinuity as that of the set of natural numbers. The set of whole numbers is discontinuous as can be seen below:



c. Consider the set of rational numbers Q, which have no common factors (or in lowest form). $Q = \{x | x = \frac{\mathrm{p}}{\mathrm{q}} \text{ where p and q are integers but q} \neq 0\}$ All the numbers in number line are not only rational numbers. There are other numbers which are irrationals. Rational numbers and irrational numbers form the set of real numbers.

Graphically:



Therefore, the set of rational numbers Q holds the property of discontinuity in increasing or decreasing order.

d. Continuity in set of real numbers

The set of real numbers is represented by the points in the real number line. There is one-one correspondence between the real numbers and the points on the number line. Between any two real numbers on the line there correspond real numbers represented by the points between given two points. This establishes the fact that the set of real numbers is dense. Hence, the real number line is continuous.

Exercise 2.1

1. Write the following sets in set builder or tabular form.

   (a) natural number (b) whole number (c) integers

   (d) rational numbers (e) irrational numbers (f) real numbers

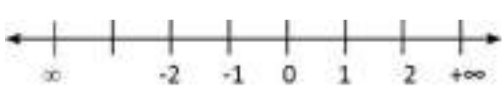
2. (a) Show the first five natural number in a number line.

   (b) Write the natural numbers from 20 to 30.

   (c) Write the rational numbers from -5 to 5.

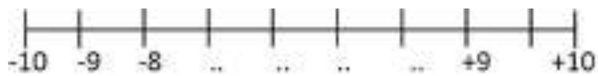
   (d) Form a set of whole numbers with first ten elements. Present them in a number line.

3. Study the following number line and examine the continuity or discontinuity in the set of numbers.

   (a) 

(i) Set of natural numbers (ii) set of integers (iii) set of real numbers

(b)



(i) set of integers (ii) set of rational numbers

(iii) set of whole number (iv) set of irrational numbers

4. Observe the following and find the continuity or discontinuity in their situation:

   (a) The movement of frog from one place to other

   (b) The movement of crocodile from one place to other

   (c) The height of a plant from first Sunday to second Sunday of the same month

   (d) The weight of a person

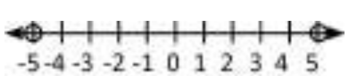
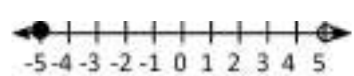
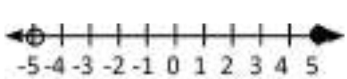
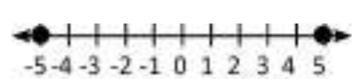
   (e) The flow of water in a river

   (f) The number of presence of students in your class for a week

5. Collect any three examples of continuity and three examples of discontinuity which can be shown in scale. Present your findings in classroom

2.2 Investigation of continuity and discontinuity in graphs

Let us observe the following graphs for inequalities:

(a)  What is the inequality?	(b)  What is the inequality?
(c)  What is the inequality?	(d)  What is the inequality?

Here,

(a) The inequality for solution region is $-5 < x < 5$. -5 and 5 both do not belong to the solution region. It is written as $(-5, 5) = \{x: -5 < x < 5\}$

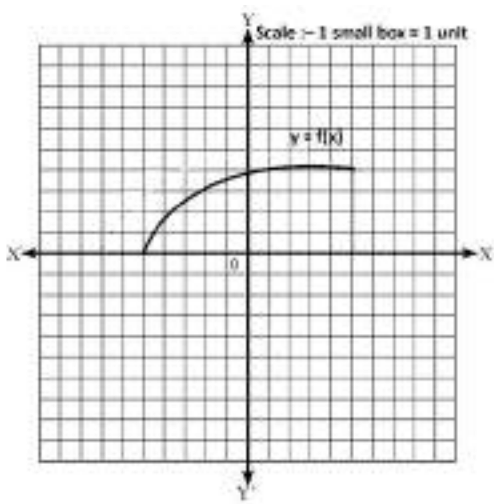
(b) The inequality for solution region is $-5 \leq x < 5$. The end point -5 belongs to the solution region but 5 does not belong to the region.

(c) The inequality for solution region is $-5 < x \leq 5$. The end point -5 does not belong to the region but 5 belongs to the region.

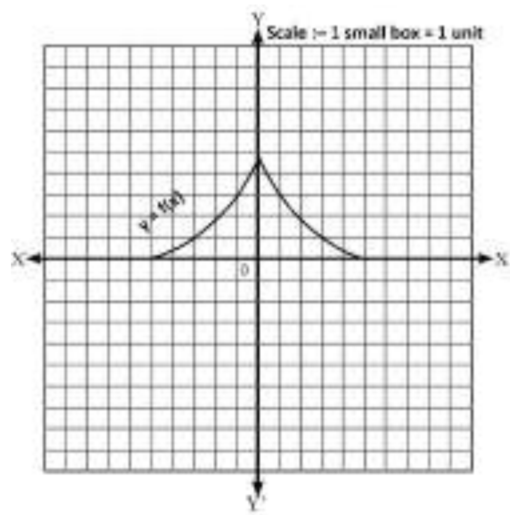
(d) The inequality for solution region is -5 and 5 both belong to the region.

Again observe the following graphs from -5 to +5.

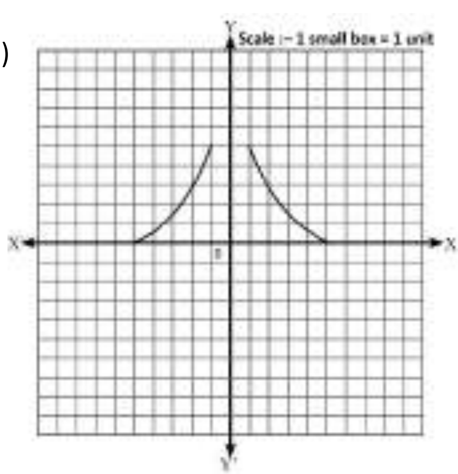
(a)



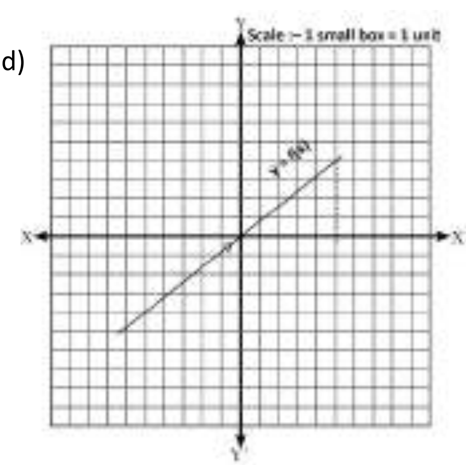
(b)



(c)



(d)



If we discuss about continuity and discontinuity of the above function at $x = 0$, we get the following results:

(a) The function is defined from -5 to +5 in X-axis. At $x = 0$ there is no break, jump, gap or hole. So the function is continuous at x = 0.

(b) The function is defined from -5 to +5 in X-axis At $x = 0$, the curve has no break or no gap, so the function representing by the curve is continuous at x = 0.

(c) The function representing by the curve is defined from $x = -5$ to $x = 5$. At $x = 0$ the graph has a break so the function representing the curve is discontinuous at $x = 0$.

(d) The function representing by the straight line is defined from $x = –5$ to $x = 5$. The graph has break/gap at $x = 0$, so it is discontinuous at $x = 0$.

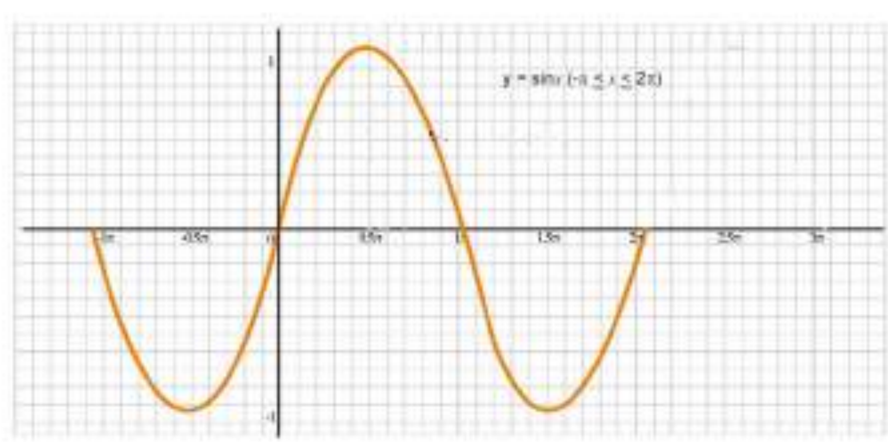
Let y = f(x) be a function defined from $x = a$ to $x = b$. The function f(x) is said to be continuous at $x = c$ if its graph has no 'break', 'jump', 'gap' or 'hole' at $x = c$ otherwise, a function y = f(x) is said to be discontinuous at $x = c$.

Example 1

Draw the graph of y = sin$x$ when $-180° \leq x \leq 360°$ and discuss about its continuity at $x = -90°$ and $x = 180°$.

Solution

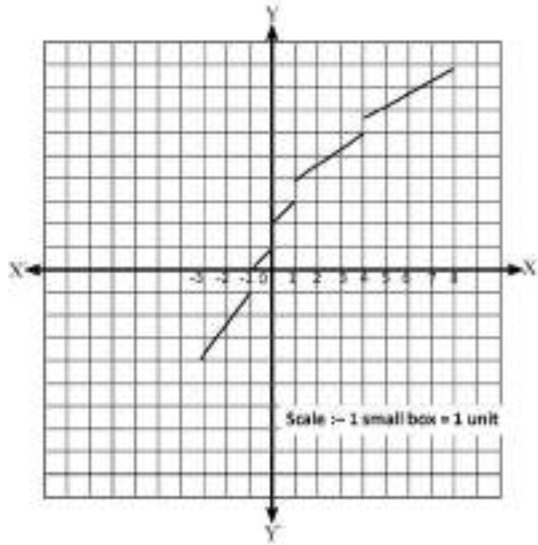
To draw the sine graph as in chapter one, list the value of sin$x$ corresponding to x as following in the difference of 90°. Let 10 small division along horizontal axis represents 90° and along vertical axis 10 small divisions represent 1 unit. We get (-180° and along vertical axis small 10 small divisions represent unit. We get (-180°, 0) (-90°, -1), (0°, 0) (90°, 1), (180°, 0), (270°, -1) …., (360°, 0) coordinates to plot on the graphs paper. Join these coordinates by free hand. We get the graphs as below.



At x = -90° and x = 180°, the graph has no break and no jump, so the function is continuous at x = -90° and x = 180°

Example 2

Examine the continuity or discontinuity of the graph defined from x = -3 to x = 8.



Solution

The function is defined from x = -3 to x = 8
in the graph. The graph has different steps (i.e. 5 steps.)

The graphs is discontinuous at x = -1, x = 0, x = 1 and x = 4, But it is piece wise continuous for $-3 \le x < -1$, $-1 < x < 0$, $o < x < 1$, $1 < x < 4$ and $4 < x \le 8$

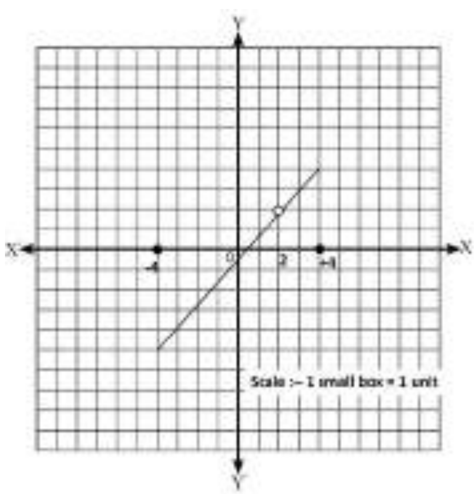
Exercise 2.2

1. From the following graphs, find:

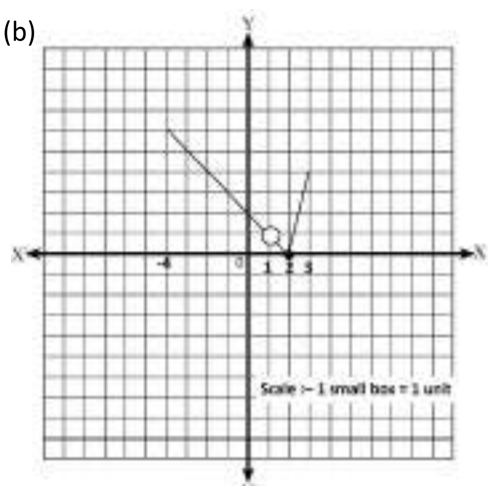
   (a) domain of the function (b) Point of discontinuity

   (c) one point of continuity for the following graphs of function.

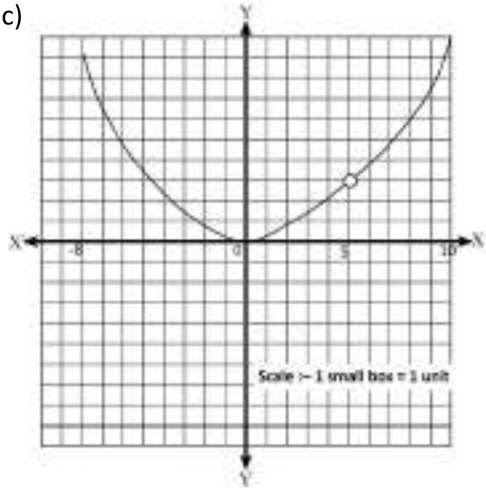
(a)



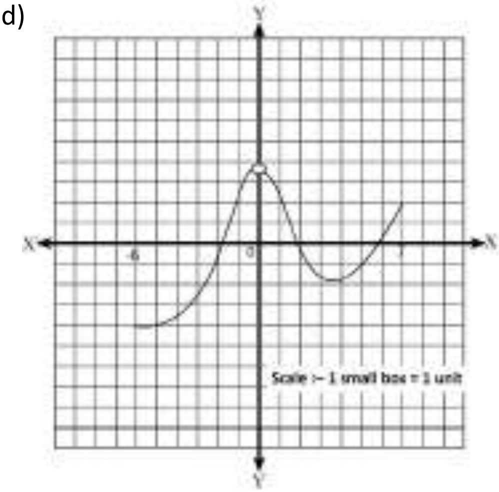
(b)



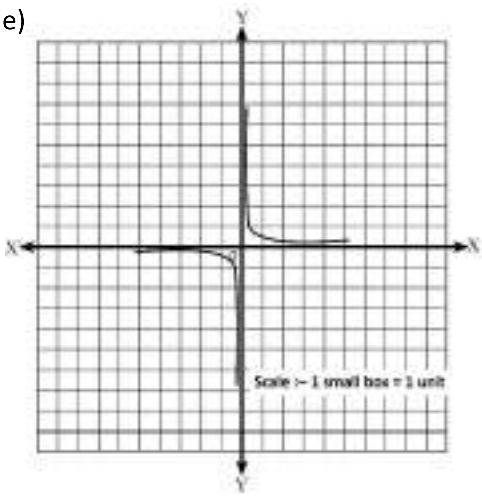
(c)



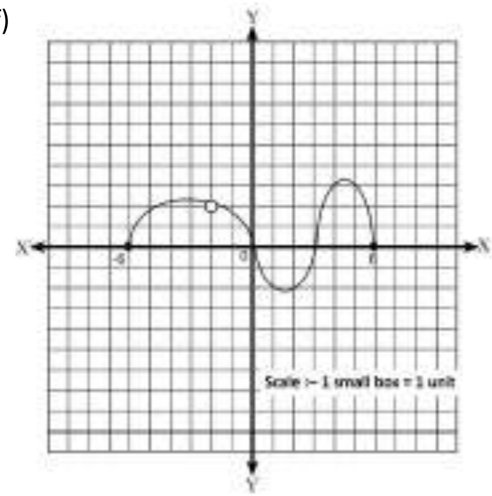
(d)



(e)



(f)



2. Draw graph of the following functions and discuss about continuity at different points at most 3 points.

   (a) y = x + 2($-4 \leq x \leq 5$) (b) y = x^2($-6 \leq x \leq 6$)

   (c) y = x^3 ($-10 \leq x \leq 10$) (d) y = cos$x$($-180° \leq x \leq 360°$)

3. Collect the different daily life examples of continuity and discontinuity and make a short report.

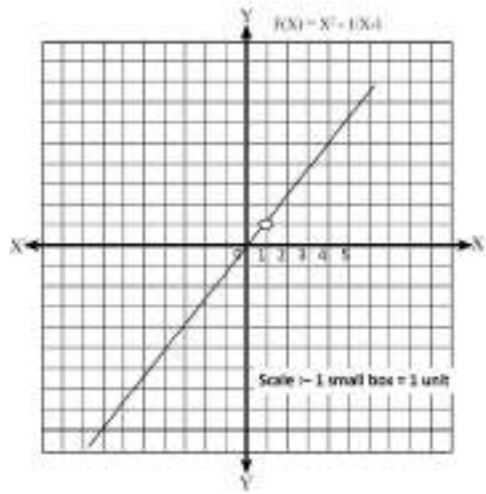
2.3 Symbolic representation of continuity

Let us suppose a function $f(x) = \frac{x^2-1}{x-1}$ defined for $-\infty < x < \infty$. Also take different situations at x = 2 and x = 1

At x = 2, f(2) = $\frac{2^2-1}{2-1}$ = 3

for x < 2, let, x = 1.99, f(1.99) = $\frac{(1.99)^2-1}{1.99-1} = \frac{3.9601-1}{0.99}$ = 2.99 = 3 (nearly)

for x > 2, let x = 2.01(say), f(2.01) = $\frac{(2.01)^2-1}{2.01-1} = \frac{4.0401-1}{1.01}$ = 3.01 = 3 (App.)



At x = 2, $\lim_{x\to 2^-} f(x)$ = 2.99 = 3 (App.)

$\lim_{x\to 2^+} f(x)$ = 3.01 = 3 (App.)

f(2) = 3

So, the function has no jump, hole, break at x = 2. the function is continuous at x = 2

At x = 1, f(1) = $\frac{(1)^2-1}{1-1} = \frac{0}{0}$ (does not exist)

There is gap at x = 1

But for x < 1 and x > 1 let us take x = 0.99 and x = 1.01, we get f(0.99) = 1.99 = 2 (nearly) and f(1.01) = 2.01 = 2(nearly)

$\lim_{x\to 2^-} f(x)$ = $\lim_{x\to 2^+} f(x)$ = 2

The function is discontinuous at x = 1

Let y = f(x), be a function defined at x = a, f(x) is said to be continuous at x = a if $\lim_{x\to 2} -f(x)$ = $\lim_{x\to a^+} f(x)$ = f(a). $\lim_{x\to a^-} f(x)$ is read as when x is nearly a from left to the value of f(x) is nearly approaches to f(a). $\lim_{x\to a^+} +f(x)$ is read as when x is nearly approaches a from right, f(x) is nearly approaches f(a).

Example 1

1. A function f(x) is defined as follows:

   f(x) = 3x + 5 for 0 $\leq x$ < 1 9x – 1 for $x \geq$ 1

   Examine the continuity at x = 1

   Solution:

   for x = 1, f(x) = 9x – 1

   Now, f(1) = 9 x 1 – 1 = 9 – 1 = 8

   for x > 1, f(x) = 9x – 1

   let us take x = 1.01 (nearly x = 1)

   f(1.01) = 9 x (1.01) – 1 = 9.09 – 1 = 8.09 = 8 (nearly)

   since, $\lim_{x\to1^-} f(x) = f(1)$ = $\lim_{x\to1^+} f(x)$, so the function is continuous at x = 1

Exercise 2.2

1. a. Explain $\lim_{x\to2^-} f(x)$, $\lim_{x\to2^+} f(x)$ and f(2) in words.

   b. If f(x) = x + 2, what is f(2)?

   c. If f(x) = 2x – 1, what is $\lim_{x\to2^-} f(x)$? (take x = 0.99)

   d. If f(x) = 3x + 1, What is $\lim_{x\to2^+} f(x)$ (take x = 1.01)

2. If f(x) = 3x + 2,

   a. find f(2.001), f(2.0001), f(1.999), f(1.9999)

   b. find $\lim_{x\to2^-} f(x)$, $\lim_{x\to2^+} f(x)\ and\ f(2)$

   c. Is it continuous at x = 2?

3. If f(x) = $\frac{x^2-9}{x-3}$, find:

   a. f(2.999) and f(3.001)

   b. f(2.999) and f(3.001) equal after approximation

   c. Discuss about continuity at x = 3

4. Examine the continuity at points mentioned below:

a. $f(x) = \begin{cases} 2 - x^2, \text{ for } x \leq 2, \\ x - 4, \text{ for } x > 2 \end{cases}$ at $x = 2$

b. $f(x) = \begin{cases} x, \text{ for } x < 0 \\ 0, \text{ for } x = 0, \\ x^2, \text{ for } x > 0 \end{cases}$ at $x = 0$

c. $f(x) = \begin{cases} \frac{2}{5-x} \text{ for } x \leq 3, \\ 5 - x \text{ for } x > 3 \end{cases}$ at $x = 3$

5. Take a quadratic function and test its continuity at particular point.

Unit 3

Matrix

3.0 Review

Observe the following table:

Articles/Shops	A	B
Pen	Rs. 40	Rs. 50
Copy	Rs. 35	Rs. 30
Bag	Rs. 400	Rs. 450

If we interchange the position of shops and arties, what will change in matrix? Discuss.

The above table shows the price of three articles in two shops. The profit from each unit of pen, copy and bag are Rs 4. Rs. 6 and Rs. 50 respectively.

a) Write the information in the above table as a 3x2 matrix A.

b) Write a row matrix B that represents the profit, per units of each type of product.

c) Find the product of B and A.

d) State what the elements of BA represent?

The product of matrix can be written as $(4 \quad 6 \quad 50)\begin{pmatrix} 40 & 50 \\ 35 & 30 \\ 400 & 450 \end{pmatrix}$

Answer the above questions.

7.1 Determinant of a Matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} \text{pen} & \text{book} \\ \text{copy} & \text{eraser} \end{bmatrix}, C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, D = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

i) What kinds of matrix are these?

ii) What is the order of each matrix?

iii) Corresponding to each matrix, is there a number?

Determinant is a function which associates each square matrix with a number. The determinants of above matrices are denoted by det (A), det (B), det (C), det (D) or |A|, |B|, |C|, |D| 'Δ' is the single notation for determinant of any square matrix.

i.e. If A be a square matrix, its determinant is denoted by det (A) or |A| and is a number associated to that matrix A.

A matrix whose determinant is Zero is said to be singular. Let A = [a_{11}] be a square matrix of order 1x1 then det (A) is a_{11}, If a_{11} = -5 then det (A) = -5 .
If a_{11} = 5, det (A) = 5.

[The determinant of 1x1 matrix is simply the element of the matrix.]

if, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The determinant of A is denoted by $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}a_{22} - a_{21}a_{12}$

Product of elements leading diagonal – product of elements of secondary diagonal.

Let us take two matrices A = $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Now, determinant of A = $\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$ = 2 x 2 – 1 x 0 = 4 – 0 = 4

and determinant of B = $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ = 1 x 4 – 2 x 2 = 4 – 4 = 0

A is non-singular matrix but B is singular matrix.

Example 1

Evaluate

a) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ b) $\begin{vmatrix} -2 & -\sqrt{5} \\ -\sqrt{5} & 3 \end{vmatrix}$ c) $\begin{vmatrix} x & 4 \\ x & x^2 \end{vmatrix}$

Solution

a) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 4 = 10 - 12 = -2$

b) $\begin{vmatrix} -2 & -\sqrt{5} \\ -\sqrt{5} & 3 \end{vmatrix} = (-2) \times 3 - (-\sqrt{5})(-\sqrt{5}) = -6 - 5 = -11$

c) $\begin{vmatrix} x & 4 \\ x & x^2 \end{vmatrix} = x^2.x - 4x = x^3 - 4x$

Example 2

If $A = \begin{pmatrix} 2 & 1 \\ -3 & -4 \end{pmatrix}$ then find |A|

Solution

Here, $A = \begin{pmatrix} 2 & 1 \\ -3 & -4 \end{pmatrix}$

Now, |A| = $\begin{vmatrix} 2 & 1 \\ -3 & -4 \end{vmatrix} = 2 \times (-4) - 1 \times (-3) = -8 + 3 = -5$

Example 3

Solve for x

$$\begin{vmatrix} x-1 & x \\ x^2+1 & x^2+x+1 \end{vmatrix} = 0$$

Solution

Here, $\begin{vmatrix} x-1 & x \\ x^2+1 & x^2+x+1 \end{vmatrix} = 0$

or, $(x-1)(x^2+x+1) - x(x^2+1) = 0$

or, $x^3 - 1 - (x^3 + x) = 0$

or, $-1 - x = 0$

or, $-1 = x$

$\therefore\ x = -1$

Example 4

If I is the identity matrix of order 2 x 2 and $A = \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}$, find the determinant of 4A + 3I

Solution: Here,

$$\mathrm{A} = \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix} \text{ and } \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\text{Now, } 4\mathrm{A} + 3\mathrm{I} &= 4\begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 8 & 16 \\ 12 & 16 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 8+3 & 16+0 \\ 12+0 & 16+3 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 12 & 19 \end{pmatrix} \\
|4\mathrm{A} + 3\mathrm{I}| &= \begin{vmatrix} 11 & 16 \\ 12 & 19 \end{vmatrix} \\
&= 11 \times 19 - 16 \times 12 \\
&= 209 - 192 \\
&= 17
\end{aligned}$$

Exercise 3.1

1. a) Define determinant of a square matrix.
   b) What do you mean by singular matrix?

2. a) Evaluate

   i) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ ii) $\begin{vmatrix} 5 & 3 \\ -1 & -4 \end{vmatrix}$ iii) $\begin{vmatrix} \sqrt{5} & -\sqrt{2} \\ \sqrt{2} & \sqrt{5} \end{vmatrix}$ iv) $\begin{vmatrix} y^2 & -2 \\ y & -3 \end{vmatrix}$

   b) If $A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$, then, find |A| c) If $B = \begin{pmatrix} -a & -b \\ b & a \end{pmatrix}$, then find |B|

d) If $A = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$, then find |A|

e) If $A = \begin{pmatrix} cos50° & sin10° \\ sin50° & cos10° \end{pmatrix}$, then find |A|

3. Solve for x

a) $\begin{vmatrix} x & 5 \\ -4 & x \end{vmatrix} = 24$ b) $\begin{vmatrix} 4x & 2 \\ 6 & 3 \end{vmatrix} = 12$

c) $\begin{vmatrix} x+3 & 4 \\ x-3 & 5 \end{vmatrix} = 7$ d) $\begin{vmatrix} x-1 & x-2 \\ x & x-3 \end{vmatrix} = 0$

4. If I is the identity Matrix of order 2 x 2, find the following:

a) If $\mathrm{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, then $|2\mathrm{A} + 3\mathrm{I}|$

b) If $\mathrm{A} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, then $|\mathrm{A}^2 + 2\mathrm{A} + 4\mathrm{I}|$

c) B= $\begin{pmatrix} 4 & 2 \\ 1 & -2 \end{pmatrix}$, then $|\mathrm{B}^2 - 2\mathrm{B} + 3\mathrm{I}|$

5. If $\mathrm{A} = \begin{pmatrix} \mathrm{m}+2 & -3 \\ \mathrm{m}+5 & -4 \end{pmatrix}$ and |A| = 3m – 1 then find $A^2 + 3A$

6. Construct a matrix A of order 2 x 2 whose element $a_{ij} = 3i + 2j$ and find $|3A^2+2A|$

7. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ verify that |AB| = |A| |B|

3.2 Inverse of a matrix

1. What is the product of a and $\frac{1}{a}$?
2. If $A = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$ *and* $B = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$, Find AB and BA. What conclusion do you get from AB and BA? Discuss in the class.

[For an n x n matrix A, if there is a matrix B for which AB = I = BA, then B is the inverse of A. B is written as A^{-1} and A is written as B^{-1}

Let, $\mathrm{A} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ and $\mathrm{A}^{-1} = \begin{bmatrix} \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} \end{bmatrix}$

$\mathrm{AA}^{-1} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} \begin{bmatrix} \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} \end{bmatrix} = \begin{bmatrix} \mathrm{ae+bg} & \mathrm{af+bh} \\ \mathrm{ce+dg} & \mathrm{cf+dh} \end{bmatrix}$

We know, $\mathrm{AA}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, ae + bg = 1 …….. (i)

ce + dg = 0 ……………(ii)

ace + bcg = c   [Equation (i) x C – equation (ii) x a]

ace + adg = 0

(-)   (-)   (-)

bcg – adg = c

or, g(bc – ad) = c

or, g = $\frac{c}{bc-ad}$

or, g = $\frac{-c}{ad-bc}$

[Equation (i) × d – equation (ii) × b]

Again, $d[ae + bg = 1] = ade + bdg = d$ ………… (i)

$b[ce + dg = 0] = bce + bdg = 0$ ………….. (ii)

$$or, (ad - bc)e = d$$

$$or, e = \frac{d}{ad - bc}$$

Again, af + bh = 0 ……….. (iii)

cf + dh = 1 ………...(iv)

solving (iii) and (iv) we get,

c[ab + bh =0] = acf + bch = 0   [Equation (iii) × c – equation (iv) × b]

a[cf + dh =1] = acf + adh = a

$$(bc - ad)h = -a$$
$$(ad - bc)h = a$$
$$or, h = \frac{a}{ad - bc}$$

$d[af + bh = 0] \to adf + bdh = 0$ [Equation (iii) × d – equation (iv) × b]

$b[cf + dh = 1] \to bcf + bdh = b$

$$(ad - bc)f = -b$$

$$or, f = -\frac{b}{ad - bc}$$

$$Now, A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$So, A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: 1. We can find the inverse of 2 x 2 matrix by interchanging the elements of leading diagonal, changing the sign of elements of secondary diagonal and dividing new matrix by determinant of given matrix.

2. At exists of and only if |A| ≠ 0, ie A is non-singular.

Let A and B are two non-singular matrices, then

i) $A . A^{-1} = A^{-1} . A = I$ ii) $(AB)^{-1} = B^{-1} . A^{-1}$ iii) $(A^{-1}) = A$ iv) $(A^T)^{-1} = (A^{-1})^T$

Example 1

If $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, does A^{-1} exist?

Solution: Here,

$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \times 5 - 3 \times 4 = 10 - 12 = -2$

Since $|A| \neq 0$, so, A^{-1} exists

Example 2

For what value of y, the matrix $\begin{pmatrix} y-3 & y-1 \\ y & 2(y-1) \end{pmatrix}$ does not have its inverse?

solution: Here,

Let $A = \begin{pmatrix} y-3 & y-1 \\ y & 2(y-1) \end{pmatrix}$

$|A| = \begin{vmatrix} y-3 & y-1 \\ y & 2(y-1) \end{vmatrix} = 2(y-1)(y-3) - y(y-1)$

$= 2(y^2 - y - 3y + 3) - (y^2 - y) = 2y^2 - 8y + 6 - y^2 + y = y^2 + 7y + 6$

if A^{-1} does not exist, then $|A| = 0$

or, $y^2 - 7y + 6 = 0$

or, $y^2 - 6y - y + 6 = 0$

or, $y(y-6) - 1(y-6) = 0$

or, $(y-6)(y-1) = 0$

either, $y - 6 = 0$ or $y - 1 = 0$

either, $y = 6$ or $y = 1$

for $y = 6$ or $y = 1$, the matrix $\begin{pmatrix} y-3 & y-1 \\ y & 2(y-1) \end{pmatrix}$ does not have its inverse.

Example 3

If $A = \begin{pmatrix} \frac{1}{3} & 2 \\ 2 & 6 \end{pmatrix}$ then find A^{-1} if exist

Solution: Here, $A = \begin{pmatrix} \frac{1}{3} & 2 \\ 2 & 6 \end{pmatrix}$

$|A| = \begin{vmatrix} \frac{1}{3} & 2 \\ 2 & 6 \end{vmatrix} = \frac{1}{3} \times 6 - 2 \times 2 = 2 - 4 = -2 \neq 0$

Hence A^{-1} exists.

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be the inverse matrix of A. i.e. $A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Now, $AA^{-1} = I = A^{-1}A$

or, $\begin{pmatrix} \frac{1}{3} & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

or, $\begin{pmatrix} \frac{1}{3}a_{11} + 2a_{21} & \frac{1}{3}a_{12} + 2a_{22} \\ 2a_{11} + 6a_{21} & 2a_{12} + 6a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Now, $\frac{1}{3}a_{11} + 2a_{21} = 1$................ (i) [equating the corresponding element]

and $2a_{11} + 6a_{21} = 0$ .... (ii)

Multiplying equation (i) by 3 and subtracting equation (ii), we get

$$\begin{array}{l} a_{11} \;\; + 6a_{21} = 3 \\ \underset{-}{2a_{11}} \underset{-}{+} 6a_{21} \underset{-}{=} 0 \\ \hline \quad -a_{11} = 3 \end{array}$$

or, $a_{11} = -3$

Again, multiplying equation (ii) by $\frac{1}{2}$ and subtracting from $a_{11} + 6a_{21} = 3$

$$\begin{array}{l} a_{11} + 6a_{21} = 3 \\ \underset{-}{a_{11}} \underset{-}{+} 3a_{21} \underset{-}{=} 0 \\ \hline \quad 3a_{21} = 3 \end{array}$$

or, $a_{21} = 1$

Again,

$\frac{1}{3}\, a_{12} + 2a_{22} = 0$...........(iii)

$2a_{12} + 6a_{22} = 1$........(iv)

by 3 × equation (iii) $-\frac{1}{2}$ equation (iv)

$$\begin{array}{l} a_{12} + 6a_{22} = 0 \\ \underset{-}{a_{12}} + \underset{-}{3a_{22}} \underset{-}{=} \frac{1}{2} \\ \hline 3a_{22} = -\frac{1}{2} \end{array}$$

$$or, a_{22} = -\frac{1}{6}$$

by 3 × equation (iii) – equation (iv) we get,

$$\begin{array}{l} a_{12} + 6a_{22} = 0 \\ \underset{-}{2a_{12}} + \underset{-}{6a_{22}} \underset{-}{=} 1 \\ \hline -a_{12} = -1 \end{array}$$

$a_{12} = 1$

Hence, $A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -\frac{1}{6} \end{pmatrix}$

Alternatively:

$$|A| = \begin{vmatrix} \frac{1}{3} & 2 \\ 2 & 6 \end{vmatrix} = \frac{1}{3} \times 6 - 2 \times 2 = -2$$

A^{-1} exists

$$A = \begin{pmatrix} \frac{1}{3} & 2 \\ 2 & 6 \end{pmatrix}$$

Interchanging the elements of leading diagonal and changing sign of secondary diagonal, we get. adj. of A = $\begin{pmatrix} 6 & -2 \\ -2 & \frac{1}{3} \end{pmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ (adj. of A)

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 6 & -2 \\ -2 & \frac{1}{3} \end{pmatrix} = \frac{1}{(-2)}\begin{pmatrix} 6 & -2 \\ -2 & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{-1}{2}\right) \times 6 & \left(\frac{-1}{2}\right) \times (-2) \\ \left(\frac{-1}{2}\right) \times (-2) & \left(\frac{-1}{2}\right) \times \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & \frac{-1}{6} \end{pmatrix}$$

Example 4

If$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}$ find A^{-1} and B^{-1}. Verify that $(AB)^{-1} = B^{-1}.A^{-1}$

Solution: Here,

$$\mathrm{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}, |\mathrm{A}| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \neq 0$$

So A^{-1} exists

$$\mathrm{B} = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}$$

$$|\mathrm{B}| = \begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} = 15 - 4 = 11 \neq 0$$

So, B^{-1} exists

Now,

$$\mathrm{B}^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -1 \\ -4 & 3 \end{pmatrix}$$

$$\mathrm{A}^{-1} = \frac{1}{-2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

$$\mathrm{B}^{-1}\mathrm{A}^{-1} = \frac{1}{-22}\begin{pmatrix} 5 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

$$= \frac{1}{-22}\begin{pmatrix} 5 \times 5 + (-1) \times (-4) & 5 \times (-3) + (-1) \times 2 \\ (-4) \times 5 + 3 \times (-4) & (-4) \times (-3) + 3 \times 2 \end{pmatrix}$$

$$= \frac{1}{-22}\begin{pmatrix} 25 + 4 & -15 - 2 \\ -20 - 12 & 12 + 6 \end{pmatrix}$$

$$= -\frac{1}{22}\begin{pmatrix} 29 & -17 \\ -32 & 18 \end{pmatrix}$$

$$\mathrm{AB} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 3 + 3 \times 4 & 2 \times 1 + 3 \times 5 \\ 4 \times 3 + 5 \times 4 & 4 \times 1 + 5 \times 5 \end{pmatrix}$$

$$= \begin{pmatrix} 6 + 12 & 2 + 15 \\ 12 + 20 & 4 + 25 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 17 \\ 32 & 29 \end{pmatrix}$$

$$|\mathrm{AB}| = \begin{vmatrix} 18 & 17 \\ 32 & 29 \end{vmatrix}$$

$$= 18 \times 29 - 17 \times 32$$

$$= 522 - 544$$

$= -22$

$$(AB)^{-1} = \frac{1}{|AB|}\begin{pmatrix} 29 & -17 \\ -32 & 18 \end{pmatrix}$$

$$= \frac{1}{-22}\begin{pmatrix} 29 & -17 \\ -32 & 18 \end{pmatrix}$$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$ proved.

Exercise 3.2

1. a) Define inverse of a matrix.

   b) Under which condition does A^{-1} exist for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$?

2. Does A^{-1} exist? Give reason.

   a) $A = \begin{pmatrix} -8 & 6 \\ -4 & 3 \end{pmatrix}$ b) $A = \begin{pmatrix} \tan A & \text{Sec}A \\ \text{Sec}A & \tan A \end{pmatrix}$

   c) $A = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}$; $a \neq 0, b \neq 0$

3. Find the inverse of each of the following matrices, if exist.

   a) $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$ b) $B = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix}$ c) $C = \begin{bmatrix} -2 & -5 \\ -3 & -8 \end{bmatrix}$

4. a) If $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ then find $(PQ)^{-1}$

   b) If $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ find $(NM)^{-1}$

5. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ then find $A^2 + AA^{-1} + 2I$, where I is the 2x2 identify matrix.

6. (a) If A = $\begin{pmatrix} 2 & 6 \\ 3 & 10 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ show that $(AB)^{-1} = B^{-1}A^{-1}$.

   (b) If A = $\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ and B = $\begin{pmatrix} 7 & 3 \\ 14 & 6 \end{pmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$.

3.3 Solutions of system of linear equations by using matrix method

Let us take a matrix equation:

$$\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

When we multiply,

$2x - 3y = 1$ …………… (i)

x + y = 2 ………………. (ii)
equation (i) and (ii) are simultaneous equation in x and y
The above equations can be written as AX = B
Where $A = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$
We have,
AX = B
$A^{-1}(AX) = A^{-1}B$ [Multiplying by A^{-1} on both sides]
or, $(A^{-1}A)X = A^{-1}B$ [By associative law]
or, $IX = A^{-1}B$ [$\therefore A^{-1}A = I$]
or, $X = A^{-1}B$ [$\therefore IX = X$]
Since, matrix multiplication is not commutative (AB ≠ BA) in general, case must be taken to multiply on the left by A^{-1}

Example 1

Solve the given system of equations by matrix method: x = 2, x + y = 7

Solution

Here, x + 0.y = 2……. (i)
x + y = 7 ………….. (ii)
Writing equation (i) and (ii) in matrix form, we have,
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$
A X = B
or, $X = A^{-1}B$
Now, $A^{-1} = \frac{1}{|A|}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$
So, $X = A^{-1}B$
$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 0 \times 7 \\ (-1) \times 2 + 1 \times 7 \end{pmatrix} = \begin{pmatrix} 2 + 0 \\ -2 + 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$
$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Example 2

Solve the following equation by using matrix method 2x + 5 = 4 (y + 1) – 1; 3x + 4 = 5 (y + 1) – 3
Solution: Here,
2x + 5 = 4(y + 1) – 1
or, 2x + 5 = 4y + 4 – 1
or, 2x – 4y = -2 ………….. (i)
Again, 3x + 4 = 5(y + 1) – 3
or, 3x + 4 = 5y +5 – 3

or, $3x - 5y = -2$ ………… (ii)

Writing equation (i) and (ii) in matrix form, we have

$$\begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

or, Ax = B

Let, $A = \begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix}, \mathrm{X} = \begin{pmatrix} x \\ y \end{pmatrix} B = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

or, $x = A^{-1}B$

$|A| = \begin{vmatrix} 2 & -4 \\ 3 & -5 \end{vmatrix}$ = (-5) x 2 – (-4) x 3 = -10 + 12 = 2 (A^{-1} Exists)

so, $A^{-1} = \frac{1}{2}\begin{pmatrix} -5 & 4 \\ -3 & 2 \end{pmatrix}$

Now, $x = A^{-1}B$

$$x = \frac{1}{2}\begin{pmatrix} -5 & 4 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (5) \times (-2) + 4 \times (-2) \\ (-3) \times (-2) + 2 \times (-2) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 10 - 8 \\ 6 - 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times 2 \\ \frac{1}{2} \times 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Comparing corresponding components of equal matrices, we get x = 1 and y = 1

Example 3

Solve: $\frac{2x+4}{5} = y = \frac{40-3x}{4}$

Solution: Here,

$y = \frac{2x+4}{5}$

or, $y = \frac{2}{5}x + \frac{4}{5}$

or, $\frac{2}{5}x - y = -\frac{4}{5}$ …….. (i)

$y = \frac{40 - 3x}{4}$

or, $y = \frac{40}{4} - \frac{3}{4}x$

or, $\frac{3}{4}x + y = 10$ ……….(ii)

writing equation (i) and (ii) in matrix form we have,

$$\begin{pmatrix} \frac{2}{5} & -1 \\ \frac{3}{4} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-4}{5} \\ 10 \end{pmatrix}$$

Let, AX = B

X = A^{-1}B ……………. (iii)
for A^{-1}

$$|A| = \begin{vmatrix} \frac{2}{5} & -1 \\ \frac{3}{4} & 1 \end{vmatrix} = \frac{2}{5} \times (1) + \frac{3}{4} = \frac{2}{5} + \frac{3}{4} = \frac{8+15}{20} = \frac{23}{20}$$

$$A^{-1} = 1 \div \frac{23}{20} \begin{pmatrix} 1 & 1 \\ -\frac{3}{4} & \frac{2}{5} \end{pmatrix} = \frac{20}{23} \begin{pmatrix} 1 & 1 \\ -\frac{3}{4} & \frac{2}{5} \end{pmatrix}$$

so, from equation (iii)

X = A^{-1}B

$$or, \text{X} = \frac{20}{23} \begin{pmatrix} 1 & 1 \\ -\frac{3}{4} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ 10 \end{pmatrix}$$

$$= \frac{20}{23} \begin{pmatrix} 1 \times \left(-\frac{4}{5}\right) + 1 \times 10 \\ \left(\frac{-3}{4}\right) \times \left(\frac{-4}{5}\right) + \frac{2}{5} \times 10 \end{pmatrix}$$

$$= \frac{20}{23} \begin{pmatrix} \frac{-4+50}{5} \\ \frac{3+20}{5} \end{pmatrix} = \frac{20}{23} \begin{pmatrix} \frac{46}{5} \\ \frac{23}{5} \end{pmatrix} = \begin{pmatrix} \frac{20}{23} \times \frac{46}{5} \\ \frac{20}{23} \times \frac{23}{5} \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$\therefore$ x = 8 and y = 4

Example 4

The sum of two numbers is 20 and their difference is 4.

a) Write the equation in matrix form.

b) Solve them by matrix method.

Solution

Let the numbers be x and y (x>y)

by question: x + y = 20 ………. (i)

x – y = 4 ……… (ii)

Writing equation (i) and (ii) in matrix form

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 4 \end{pmatrix}$$

Let, AX = B

or, X = A^{-1}B ……… (iii)

For A^{-1} : $|A| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$= 1 \times (-1) - 1 \times 1 = -1 - 1 = -2 \neq 0$ (A^{-1} exists)

Now,

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

Now, X = $A^{-1}B$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 20 \\ 4 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} (-1)\times 20 + (-1)\times 4 \\ (-1)\times 20 + 1 \times 4 \end{pmatrix}$$

$$= \frac{1}{-2}\begin{pmatrix} -20 - 4 \\ -20 + 4 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -24 \\ -16 \end{pmatrix}$$

$$= \begin{pmatrix} \left(-\frac{1}{2}\right) \times (-24) \\ \left(-\frac{1}{2}\right) \times (-16) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

∴ The required numbers are 12 and 8

Exercise 3.3

1. a) If $AA^{-1} = A^{-1}A = I$ and AX = B, Write X in terms of A^{-1} and B.
   b) Write the matrix form of
      $2x + y = 4$
      $3x + 2y = 7$

2. Solve the following system of linear equations using matrix method:
   (a) $x + y = 5$, $x - y = 1$

   (b) $3x + 5y = 3$, $4x + 3y = 4$

   (c) $\frac{3x+5y}{8} = \frac{5x-2y}{3} = 3$

   (d) $\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   (e) $\begin{pmatrix} 8 & 5 \\ 7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   (f) $\frac{1}{x} + \frac{1}{2y} = 8, \quad \frac{1}{2x} - \frac{1}{y} + 1 = 0$

   (g) $\frac{5}{y} = \frac{1}{x} - 1, \quad \frac{5}{y} = \frac{2}{x} - 4$

   h) $4(x - 1) + 5(y + 2) = 10$, $5(x - 1) - 3(y + 2) + 6 = 0$

   (i) $5x + 7y = 31xy$

   (j) $7x + 5y = 29xy$

3. Write equations and solve the following by matrix method:

   a) The cost of a pen and copy is 120. The cost of 2 copy is Rs. 100.

   b) The sum of present ages of a father and Son is 53 years. After 2 years, the age of father will be 47 years.

4. Write yourself two simultaneous equations in (x, y) related to your daily life and solve them by matrix method.

3.4 Cramer's rule

Let, two linear equations in x and y are,

$a_1x + b_1y + c_1 = 0$ ………… (i)
$a_2x + b_2y + c_2 = 0$ ………… (ii)
Multiplying equation (i) by b_2 and equation (ii) by b_1 and subtracting we get

$a_1b_2x + b_1b_2y + b_2c_1 = 0$

$a_2b_1x + b_1b_2y + b_1c_2 = 0$

$- \quad - \quad -$

$a_1b_2x - a_2b_1x + (b_2c_1 - b_1c_1) = 0$

or, $(a_1b_2 - a_2b_1)x = -(b_2c_1 - b_{-1}c_2)$

$$or, x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

Again, multiplying equation (i) by a_2 and equation (ii) by a_1 and subtracting equation (ii) from (i). We get,

$a_1a_2x + a_2b_1y + a_2c_1 = 0$

$a_1a_2x + a_1b_2y + a_1c_2 = 0$

$- \quad - \quad -$

$(a_2b_1 - a_1b_2)y + (a_2c_1 - a_1c_2) = 0$

$(a_2b_1 - a_1b_2)y = -(a_2c_1 - a_1c_2)$

or, $(a_2b_1 - a_1b_2)y = a_1c_2 - a_2c_1$

or, $y = \frac{a_1c_2 - a_2c_1}{a_2b_1 - a_1b_2}$

If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$

The above solution becomes

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

and $y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

i.e. $x = \frac{D_x}{D}, y = \frac{D_y}{D}$

Where $D_x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ and D = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

The method of solving system of linear equations by using determinant is known a Cramer's rule.

Note:

(i) Grabriel Cramer is a Swiss mathematician who introduced technique to solve a system of linear equations using determinant.

(ii) If D = 0, $D_x \neq 0$, $D_y \neq 0$, the system of equations has no solutions, because $0 \cdot x \neq 0$, $0 \cdot y \neq 0$.

(iii) We always write the equation in the form $a_1x + \mathrm{b}_1\mathrm{y} = \mathrm{c}$ and $a_2x + b_2y = c_2$ i.e. for finding $D_x(D_1)$, substitute first column of D by $\begin{pmatrix} \mathrm{c}_1 \\ \mathrm{c}_2 \end{pmatrix}$ and for finding D_y (D_2), substitute the second column by $\begin{pmatrix} \mathrm{c}_1 \\ \mathrm{c}_2 \end{pmatrix}$.

Example 1

Solve 2(x -1) = y and 3 (x – 1) = 4y using Cramer's rule

Solution

Here, 2(x – 1) = y

or, 2x – y = 2 ………… (i)

3(x – 1) = 4y

or, 3x – 4y = 3 ………… (ii)

Now, arranging the coefficients and constant term

Coefficient of x	Coefficient of y	Constant term
2	-1	2
3	-4	3

Now, $D = \begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix}$ = 2 x (-4) – 3 x (-1) = - 8 + 3 = - 5

$D_x = \begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix} = -5$

$D_y = \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 0$

$\therefore x = \frac{D_x}{D} = \frac{-5}{-5} = 1$

$\therefore y = \frac{D_y}{D} = \frac{0}{-5} = 0$

$\therefore (x, y) = (1,0)$

Example 2

Solve the following system of linear equations using Cramer's rule.

$\frac{2}{3}x + y = 1 \frac{1}{2}x + y = 1$

Solution

Coefficient of x	Coefficient of y	Constant term
$\frac{2}{3}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$

$$D = \begin{vmatrix} \frac{2}{3} & 1 \\ \frac{1}{2} & 1 \end{vmatrix}$$

$$= \frac{2}{3} \times 1 - 1 \times \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6} \ (\neq 0)$$

$$D_x = \begin{vmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{vmatrix}$$

$$= 1 \times 1 - 1 \times \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$D_y = \begin{vmatrix} \frac{2}{3} & 1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{2}{3} \times \frac{1}{2} - 1 \times \frac{1}{2} = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$$

$$Now, x = \frac{D_X}{D} = \frac{\frac{1}{2}}{\frac{1}{6}} = \frac{6}{2} = 3$$

$$x = \frac{D_y}{D} = \frac{\frac{-1}{6}}{\frac{1}{6}} = -1$$

$\therefore$ (x, y) = (3, 1)

Exercise 3.4

1. In the equation $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, according to Crammer's rule,

   a) Write D in determinant form.

   b) Write D_x in determinant form.

   c) Write D_y in determinant form.

   d) What is necessary condition for obtaining unique solution?

2. Solve the following system of linear equations using Cramer's rule:

   a) $2x - y = 3$
   $y + 3x = 7$

   b) $y = 2x$
   $x - \frac{3}{2}y + 1 = 0$

   c) $\frac{4}{x} + \frac{5}{y} = 28$
   $\frac{7}{x} + \frac{3}{y} = 67$

   d) $4(x - 1) + 5(y + 2) = 10$
   $5(x - 1) - 3(y + 2) + 6 = 0$

   e) $3xy - 10y = 6x$
   $5xy + 3x = 21y$

   f) $3y + 4x = 2xy$ and
   $18y - 4x = 5xy$

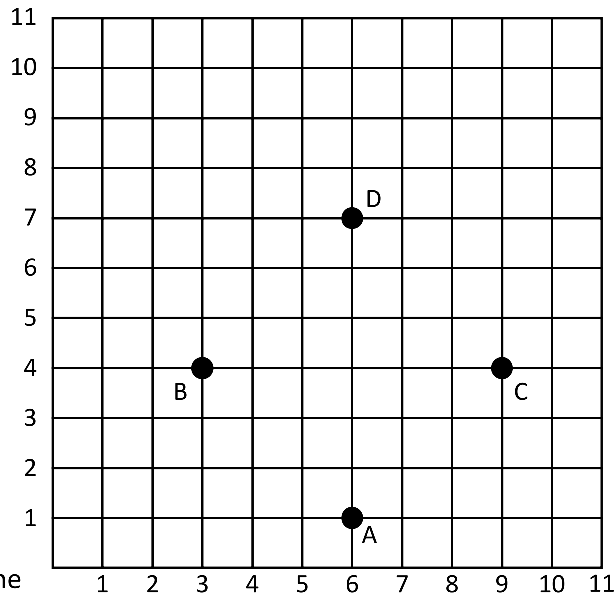
3. Ask the price of any five daily uses goods. Make two different system of equation a and b in terms of x and y. Solve these equations by Cramer's rule and present your findings in the classroom.

Unit 4

Coordinate Geometry

4.0 Review

In a class four friends are seated at the points A, B, C and D as shown in graph. Sarita and Bimal walk into the class and after observing for a few minute Sarita asked following questions to Bimal.



Discuss about these answers in group and write conclusions.

1. Find the equation of AB, BC, CD and AD
2. Find the slope of each of the lines AB, BC, CD and AD.

The slope of a line parallel to X-axis is 0. The slope of Y-axis is not defined. The slope is independent of the sense of the line segment (i.e. slope of AB = slope of BA). The equation of a straight line is the relation between x and y, which satisfies the co-ordinates of each and every point on the line and not by those of any other point. The equation of a line with slope m and passing through a point (x_1, y_1) is given by

$y - y_1 = m(x - x_1)$ and the equation of the line through the two given points (x_1, y_1) and (x_2, y_2) is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

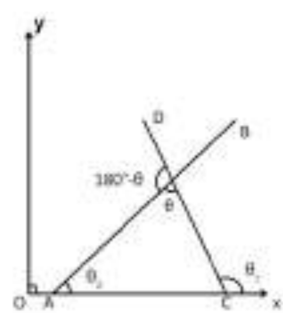
4.1 Angle between two straight lines

Let AB and CD be two straight lines with inclination θ_2 and θ_1 respectively. Answer the following questions:

i. What is the slope (m_1) of CD in terms of θ_1?

ii. What is the slope (m_2) of AB in terms of θ_2?

iii. What is the relation between θ, θ_1, θ_2?



As we know θ_1 and θ_2 are angles made by the lines with positive direction, so

$m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$

Again, we have $\theta_1 = \theta + \theta_2$ [exterior units]

or, $\theta = \theta_1 - \theta_2$

taking tangents on both sides,

$\tan \theta = \tan (\theta_1 - \theta_2)$

or, $\tan \theta = \frac{\tan \theta_1 - \tan \theta_2}{1 + tan\theta_1 \,.\, tan\theta_2}$

or, $\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$or, \tan(180° - \theta) = -tan\theta = -\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

$\therefore tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

Remember:

(i) 'By the angle between two lines' we mean 'the acute or obtuse angle between the lines'

(ii) If $\frac{m_1 - m_2}{1 + m_1 m_2}$ is positive, then θ is acute angle. If $\frac{m_1 - m_2}{1 + m_1 m_2}$ is negative, θ is obtuse angle.

(iii) The complete angle formula is used when the angle is given.

(iv) If the lines are parallel, θ = 0° or 180° then $m_1 = m_2$

(v) If $m_1 = m_2$, the lines are coincident to each other and if $m_1 m_2 = -1$, the lines are perpendicular to each other.

Example 1

Find the angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

Solution:

Slope of the lines $a_1x + b_1y + c_1 = 0$ is

$$m_1 = -\frac{\text{coefficient of x}}{\text{coefficient of y}} = -\frac{a_1}{b_1} \text{ (say)}$$

Slope of the line $a_2x + b_2y + c_2 = 0$ is $m_2 = -\frac{\text{coefficient of x}}{\text{coefficient of y}} = -\frac{a_2}{b_2}$ (say)

If θ be the angle between the lines then,

$$\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \pm\left[\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \frac{a_1}{b_1}.\frac{a_2}{b_2}}\right] = \pm\left[\frac{\frac{a_2b_1 - a_1b_2}{b_1b_2}}{\frac{b_1b_2 + a_1a_2}{b_1b_2}}\right] = \pm\left(\frac{a_2b_1 - a_1b_2}{b_1b_2}\right) \times \left(\frac{b_1b_2}{b_1b_2 + a_1a_2}\right)$$

$$= \pm\left[\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right]$$

$$\therefore\ \theta = \tan^{-1}\left(\pm\left(\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right)\right)$$

Example 2

Find the acute angle between the lines.: $\sqrt{3}x - y + 8 = 0\ and\ y + 10 = 0$

Solution

Slope of the line $\sqrt{3}x - y + 8 = 0$ is $m_1 = \sqrt{3}$ (Since, y = $\sqrt{3}x$ + 8)

Slope of the line y + 10 = 0 is

m_2 = 0. If the θ be the angle between the lines then,

$$tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$$

$$or, tan\theta = \pm\left(\frac{\sqrt{3} - 0}{1 + \sqrt{3} \times 0}\right)$$

$$= \pm\left(\sqrt{3}\right)$$

Taking (+ve) sign:

$$\tan\theta = \sqrt{3}$$

$$\tan\theta = \tan 60°$$

$$\text{or}, \theta = 60°$$

∴ Required acute angle is 60°

Example 3

Find the obtuse angle between the lines 2x – y + 3 = 0 and x – 3y + 2 = 0

Solution

Slope of line 2x – y + 3 = 0 is

$$m_1 = -\frac{\text{coefficient of } x}{\text{coefficient of y}} = -\left(\frac{2}{-1}\right) = 2$$

Slope of line $x - 3y + 2 = 0$

$$m_2 = -\frac{\text{coefficient of } x}{\text{coefficient of y}} = -\left(\frac{1}{-3}\right) = \frac{1}{3}$$

If θ is the obtuse angle between the lines, then

$$tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$or,\ tan\theta = \pm\left(\frac{2 - \frac{1}{3}}{1 + 2 \times \frac{1}{3}}\right)$$

$$or,\ tan\theta = \pm\left(\frac{\frac{6-1}{3}}{\frac{3+2}{3}}\right)$$

$$or,\ tan\theta = \pm\left(\frac{5}{3} \times \frac{3}{5}\right)$$

$$or,\ tan\theta = \pm(1)$$

For obtuse angle take $tan\theta = -1$

$$or,\ tan\theta = tan135°$$

$$or,\ \theta = 135°,$$

∴ Obtuse angle between the lines is 135°

Example 4

If the lines $ax + 5y - 16 = 0$ and $6x + 10y - 9 = 0$ are perpendicular to each-other, find the value of a.

Solution

Slope of the line $ax+5y-16=0$ is

$$m_1 = -\frac{\text{coefficient of } x}{\text{coefficient of y}} = -\left(\frac{a}{5}\right) = \frac{-a}{5}$$

Slope of the line $6x + 10y - 9 = 0$

$$m_2 = -\frac{\text{coefficient of } x}{\text{coefficient of y}} = -\left(\frac{6}{10}\right) = \frac{-3}{5}$$

Since the lines are perpendicular to each-other, so

$$m_1 m_2 = -1$$

$$or, \quad \frac{-a}{5} \times \frac{-3}{5} = -1$$

$$or, \quad \frac{3a}{25} = -1$$

$$or, \quad 3a = -25$$

$$\therefore\ a = -\frac{25}{3}$$

Example 5

Find the equation of the line which passes through the point(3,4) and parallel to the line 3x + 4y – 12 = 0

Solution

Slope of the line 3x + 4y – 12 = 0 is

m_1 = - $\frac{\text{coefficient of } x}{\text{coefficient of y}} = -\frac{3}{4}$

Slope of the parallel line to the given line

$$m = -\frac{3}{4}\ [\because\ m_1 = m_2]$$

The line passes through the point,

$$(x_1, y_1) = (3, 4)$$

We know, equation of the line:

$$y - y_1 = m(x - x_1)$$

or, $y - 4 = -\frac{3}{4}(x - 3)$

or, $4(y - 4) = -3(x - 3)$

or, $4y - 16 = -3x + 9$

or, $3x + 4y - 16 - 9 = 0$

or, $3x + 4y - 25 = 0$

$\therefore$ The required equation of line is 3x + 4y – 25 = 0

Example 6

Find the equation of the lines which passes through (2 , 4) and make angle 60° with the line $y = -\sqrt{3}x + 2$

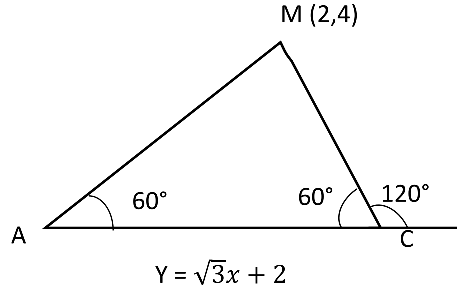
Solution

Let M(2, 4) be the point. MA and MC be the lines which make 60°. With the line y = $-\sqrt{3x}+2$

Slope of the given line = $-\sqrt{3} = \mathrm{m}_2\ (\mathrm{say})$

Slope of unknown lines be m_1 = (m) say.

If θ be the angle between the lines, then



$$or, \quad \tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$or, \quad \tan 60° = \pm\left(\frac{m - (-\sqrt{3})}{1 + m \times (-\sqrt{3})}\right)$$

$$or, \quad \sqrt{3} = \left(\pm\frac{m + \sqrt{3}}{1 - \sqrt{3}m}\right)$$

Taking (+ve) sign:

$$\sqrt{3} = \frac{m + \sqrt{3}}{1 - \sqrt{3}m}$$

$$or, \quad \sqrt{3} - 3m = m + \sqrt{3}$$

$$or, \quad -4m = 0$$

$$or, \quad m = 0 \qquad [Let\ (x_1, y_1) = (2,4)]$$

Hence, equation of the line is $y - y_1 = m(x - x_1)$

$$or, \quad y - 4 = 0(x - 2)$$

$$or, \quad y - 4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Again, taking (-ve) sign:

$$\sqrt{3} = -\frac{m + \sqrt{3}}{1 - \sqrt{3}\,m}$$

$$or, \quad \sqrt{3} - 3m = -m - \sqrt{3}$$

$$or, \quad -2m = -2\sqrt{3}$$

$$or, \quad m = \sqrt{3}$$

Again, equation of the line passing through the point M(2, 4)

$$y - y_1 = m(x - x_1)$$

$or,\quad y - 4 = \sqrt{3}(x - 2)$

$or,\quad y - 4 = \sqrt{3}x - 2\sqrt{3}$

$or,\quad \sqrt{3}x - y + (4 - 2\sqrt{3}) = 0$

$\therefore$ required equation of the lines are

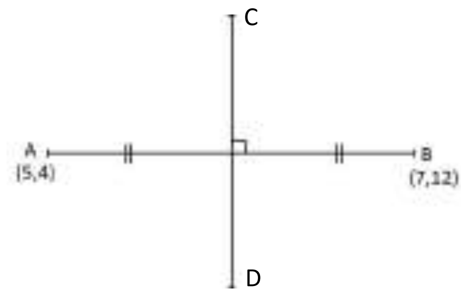
$y - 4 = 0$ and $\sqrt{3}x - y + (4 - 2\sqrt{3}) = 0$

Example 7

Find the equation of the perpendicular bisector of the line joining the points (5 , 4) and (7 , 12).

Solution

Let, A(5,4) and B(7,12). CD is the perpendicular bisector of AB.



Mid-points of AB $= \left(\frac{5+7}{2}, \frac{4+12}{2}\right)$

$= (6,8)$

$= (x_1, y_1)$ *say*

Slope of AB = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{12-4}{7-5}$ = 4

Slope of the perpendicular bisector of AB $= -\frac{1}{4}$ [Since $m_1 \times m_2 = -1$]

Since, perpendicular bisector passes through mid-point, $(x_1, y_1) = (6,8)$

So, equation of the perpendicular bisector is

$y - y_1 = m(x - x_1)$

$or,\quad y - 8 = -\frac{1}{4}(x - 6)$

$or,\quad 4y - 32 = -x + 6$

$or,\quad 4y + x - 32 - 6 = 0$

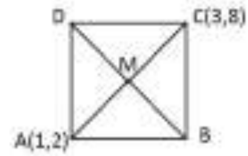
$or,\quad x + 4y - 38 = 0$

$\therefore$ Required equation is $x + 4y - 38 = 0$

Example 8

(1,2) and (3,8) are pair of opposite vertices of a square. Find the equations of the diagonals of the square.

Solution



Let A(1,2) and C(3,8) be the opposite vertices of square ABCD

Now, M be the point of intersection of diagonals AC and BD.

$$\text{So, Co}-\text{ordinates of M} = \left(\frac{3+1}{2}, \frac{8+2}{2}\right)$$

= (2,5)

Equation of diagonal AC is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$or, \quad y - 2 = \frac{8-2}{3-1}(x-1)$$

$$or, \quad y - 2 = \frac{6}{2}(x-1)$$

$$or, \quad y - 2 = 3(x-1)$$

$$or, \quad y - 2 = 3x - 3$$

$$or, \quad 3x - y - 1 = 0$$

Slope of AC = $\frac{y_2-y_1}{x_2-x_1} = \frac{8-2}{3-1} = \frac{6}{2} = 3$

Since, AC ⊥ BD, so, slope of $BD = -\frac{1}{3}$ [∵ $m_1m_2 = -1$]

BD passes through m(2, 5)

So, equation of BD is

$$y - 5 = -\frac{1}{3}(x-2)$$

$$or, \quad (y-5) \times 3 = -1 \times (x-2)$$

$$or, \quad 3y - 15 = -x + 2$$

$$or, \quad x + 3y - 17 = 0$$

∴ Equation of diagonals of the square are $3x - y - 1 = 0$ and $x + 3y - 17 = 0$

Exercise: 4.1

1. If θ be the angle between two straight lines with slopes m_1 and m_2 then
   a. Write the expression for $\tan\theta$.
   b. Write the relation between m_1 and m_2 when $\theta = 0°$ or $180°$.
   c. Write the relation between m_1 and m_2 when $\theta = 90°$.
2. If β be the angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$,
   a. Express $\tan\beta$ in terms of a_1, a_2, b_1 and b_2.
   b. Identify the relation when both lines are parallel.
   c. Identify the relation when both of the lines are perpendicular to each-other.
3. Find the acute angle between the lines given as below:
   a. $2x - y + 7 = 0$ and $x - 3y + 6 = 0$
   b. $3x - 2y - 5 = 0$ and $4x + y - 7 = 0$
   c. $\sqrt{3}x - y + 6 = 0$ and $y + 3 = 0$
4. Find the obtuse angle between the lines given as below:
   a. $x - 2y + 1 = 0$ and $x + 3y - 2 = 0$
   b. $x - \sqrt{3}y - 5 = 0$ and $\sqrt{3}x - y - 6 = 0$
   c. $x - 5y - 3 = 0$ and $x - 3y - 4 = 0$
5. a. Show that the straight lines (i) $5x + 12y + 13 = 0$ and $12x - 5y - 18 = 0$ (ii) $27x - 18y + 25 = 0$ and $2x + 3y + 7 = 0$ are perpendicular to each-other.
   b. Show that the straight lines (i) $x - 2y + 3 = 0$ and $2x - 4y + 9 = 0$ (ii) $x + 2y - 9 = 0$ and $2x + 4y + 5 = 0$ are parallel to each other.
   c. If the lines $3y + kx - 8 = 0$ and $2x - 3y - 11 = 0$ are parallel to each-other, find the value of k.
   d. If the lines $ax - y - 7 = 0$ and $3y + x - 9 = 0$ are perpendicular to each-other, find the value of a.
   e. If the lines $ax - 5y - 2 = 0$ and $4x - 2y + 3 = 0$ are perpendicular, find the value of a.
6. a. Find the equation of the line passes through (1, 2) and parallel to $3x - 4y - 12=0$.

b. Find the equation of the line passes through (–3, –2) and parallel to $3x + 4y - 11 = 0$

c. Find the equation of the line passes through the point (2, 3) and perpendicular to $5x - 4y + 3 = 0$

d. Find the equation of the line passes through the point (-1, 3) and perpendicular to the line $5x + 7y + 18 = 0$.

7. a. Find the equations of the lines passes through the point (2, 3) and make angle 45° with $x - 3y - 2 = 0$

   b. Find the equations of straight lines passing through the point (1,-4) and inclined at 45° to the straight line $2x + 3y + 5 = 0$

   c. Find the equations of straight line passing through the point of intersection of $2x - 3y + 1 = 0$ and $x + y - 2 = 0$ and make 45° with $x + 2y - 5 = 0$

8. a. Find the equation of perpendicular bisector of the line segment joining the points (2, 3) and (4, -1).

   b. Find the equation of perpendicular bisectors of the triangle having vertices A(8, -10), B(7, -3) and C(0, -4).

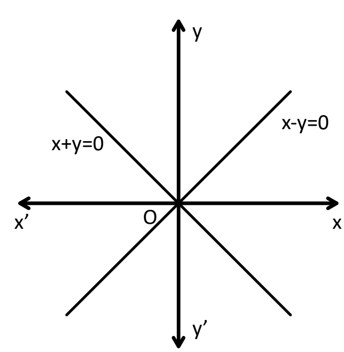
   c. Two opposite corners of a square are (3, 2) and (3, 6). Find the equations of diagonals of square.

9. Draw a rectangle in graph paper. Find the equations of diagonals. Also find the angle between each diagonal and sides of rectangle. Present your result in classroom.

10. Find out a parallelogram shape in your surroundings, trace it and find the co-ordinates of its vertices. Find the angle between their diagonals.

4.2 Equation of pair of straight lines

Consider the equation $x - y = 0$ and $x + y = 0$. Multiply both of the equation. Find the power of x and y.



Both of the equations $x - y = 0$ and $x + y = 0$ pass through the origin. The combined equation of the lines $x - y = 0$ and $x + y = 0$ is $x^2 - y^2 = 0$. The equation $x^2 - y^2 = 0$ represents a pair of straight lines passing through the origin. The equation $x^2 - y^2 = 0$ contains only second degree terms.

Similarly, let us consider two equations y – 2x = 0 and y – 3x = 0. Multiplying both the equations, we get $y^2 - 5xy + 6x^2 = 0$.

Write the degree of each of the terms in $y^2 - 5xy + 6x^2 = 0$. An equation like $x^2 - y^2 = 0$ and $y^2 - 5xy + 6x^2 = 0$ containing only second degree terms in x and y is called a homogenous equation of second degree in x and y.

Let $y - m_1x = 0$ and $y - m_2x = 0$ be the two straight lines obtained from $ax^2 + 2hxy + by^2 = 0$.

Multiplying $y - m_1x = 0$ and $y - m_2x = 0$, we get $y^2 - (m_1 + m_2)xy + m_1m_2 = 0$ ……….(i)

Similarly, $ax^2 + 2hxy + by^2 = 0$ can be written as

$$\frac{a}{b}x^2 + \frac{2h}{b}xy + y^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(\mathrm{ii})$$

Comparing equation (i) and (ii) we have,

$$m_1 + m_2 = -\frac{2h}{b} \ and \ \ m_1m_2 = \frac{a}{b}$$

We know,

$$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$

$$or, m_1 - m_2 = \pm\sqrt{(m_1 + m_2)^2 - 4m_1m_2}$$

$$= \pm\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}} = \pm\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}} = \pm\frac{2}{b}\sqrt{h^2 - ab}$$

$$1 + m_1m_2 = 1 + \frac{a}{b} = \frac{a+b}{b}$$

and $$\frac{m_1 - m_2}{1 + m_1m_2} = \pm\frac{\frac{2}{b}\sqrt{(h^2) - ab}}{\frac{a+b}{b}} = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

If θ be the angle between the lines

$$tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

$$or, \qquad \theta = \tan^{-1}\left(\pm\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

When $a + b = 0, \theta = 90°$

i.e. both lines are perpendicular to each-other, if coefficient of x^2 + coefficient of $y^2 = 0$.

Again,

when $\mathrm{h}^2 - \mathrm{ab} = 0$ $(\mathrm{h}^2 = \mathrm{ab})$

$\theta = 0° \; or \; 180°$

i.e. both lines are parallel to each other if coefficient of xy = product of coefficient of x^2 and y^2.

We can conclude that the angle between the lines represented by $ax^2+2hxy+by^2=0$ is

$$\tan^{-1}\left(\pm\frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

Again consider the equation,

$$ax^2 + 2hxy + by^2 = 0$$

Case I: when b=0,

$$ax^2 + 2hxy = 0$$

$$or, \qquad x(ax + 2hy) = 0$$

$$or, \qquad x = 0 \; and \; ax + 2hy = 0$$

Both of the equations pass through origin.

Case II: if $b \neq 0$

$$ax^2 + 2hxy + by^2 = 0$$

$$or, \qquad \frac{a}{b}x^2 + \frac{2h}{b}xy + y^2 = 0$$

$$or, \qquad \frac{a}{b} + \frac{2h}{b}\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = 0$$

$$or, \qquad \frac{y}{x} = -\frac{2h}{b} \pm \sqrt{\left(\frac{2h}{b}\right)^2 - 4\frac{a}{b}}$$

$$or, \qquad \frac{y}{x} = -\frac{2h}{b} \pm \frac{2\sqrt{h^2 - ab}}{b}$$

$$or, \qquad y = -\frac{2\left(h \pm \sqrt{h^2 - ab}\right)}{b}x$$

$$or, \qquad y = -\frac{2\left(h + \sqrt{h^2 - ab}\right)}{b}x \qquad \text{and}$$

$$or, \qquad y = -\frac{2\left(h - \sqrt{h^2 - ab}\right)}{b}x \qquad \text{pass through the origin.}$$

Note: Let $a_1x + b_1y + c_1 = 0$ and $a_2 x + b_2y + c_2 = 0$ be any two intersecting straight lines, then the equation $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ represent the pair of straight lines passing through the intersection of the given lines. It's general form can be written as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Hence, we conclude that, a homogenous equation of the second degree in x and y represent a pair of lines passing through the origin.

Example 1

Find the single equation for the pair of lines represented by $3x + 2y = 0$ and $2x - 3y = 0$.

Solution: Here,

Pair of straight lines are,

$3x + 2y = 0$ and $2x - 3y = 0$

Multiplying both of the equations, we get

$$(3x + 2y)(2x - 3y) = 0$$

$$or, \qquad 3x(2x - 3y) + 2y(2x - 3y) = 0$$

$$or, \qquad 6x^2 - 9xy + 4xy - 6y^2 = 0$$

$$or, \qquad 6x^2 - 5xy - 6y^2 = 0$$

∴ The required equation is $6x^2 - 5xy - 6y^2 = 0$

Example 2

Find the separate equations of the pair of lines represented by $9x^2 - 24xy + 16y^2 = 0$

Solution: Here,

$9x^2 - 24xy + 16y^2 = 0$

or, $9x^2 - 12xy - 12xy + 16y^2 = 0$

or, $3x(3x - 4y) - 4y(3x - 4y) = 0$

or, $(3x - 4y)(3x - 4y) = 0$

∴ Required equations are $3x - 4y = 0$ and $3x - 4y = 0$

Example 3

Find the separate equations represented by the lines $y^2\sin^2\theta - 2xy + x^2(1 + \cos^2\theta) = 0$

Solution: Here,

$y^2\sin^2\theta - 2xy + x^2(1 + \cos^2\theta) = 0$

$$or, \quad y^2 \sin^2 \theta - 2xy + x^2 + x^2 \cos^2 \theta = 0$$

$$or, \quad y^2(1 - \cos^2 \theta) - 2xy + x^2 + x^2 \cos^2 \theta = 0$$

$$or, \quad y^2 - y^2 \cos^2 \theta - 2xy + x^2 + x^2 \cos^2 \theta = 0$$

$$or, \quad (x^2 - 2xy + y^2) + (x^2 - y^2)cos^2\theta = 0$$

$$or, \quad (x - y)^2 + (x - y)(x + y) \cos^2 \theta = 0$$

$$or, \quad (x - y)[(x - y) + (x + y) \cos^2 \theta)] = 0$$

$$or, \quad (x - y)\big(x(1 + \cos^2 \theta) - y(1 - \cos^2 \theta)\big) = 0$$

$$or, \quad (x - y)(x(1 + \cos^2 \theta) - ysin^2\theta) = 0$$

∴ The required equations are $x - y = 0$ and $x(1 + \cos^2 \theta) - ysin^2\theta = 0$

Example 4

Find the angle between the lines represented by $3x^2 - 4xy + y^2 = 0$

Solution: Here,

$x^2 - 4xy + y^2 = 0$

Comparing $3x^2 - 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$

We have, a = 3, h = –2, b = 1

Again, θ be the angle between pair of lines.

$$tan\theta = \pm\frac{2\sqrt{\mathrm{h}^2 - ab}}{a + b}$$

$$or, tan\theta = \pm\frac{2\sqrt{(-2)^2 - 3 \times 1}}{3 + 1}$$

$$or, tan\theta = \pm\frac{2\sqrt{4 - 3}}{4}$$

$$or, tan\theta = \pm\frac{2}{4}$$

$$or, tan\theta = \pm\frac{1}{2}$$

$$or, tan\theta = \pm\tan(26.56°)$$

$$or, tan\theta = tan26.56°; \theta = 26.56°$$

$tan\theta = \tan(180° - 26.56°)\,; \theta = 153.43°$

∴The required angle is 26.56° or 153.43°.

Example 5

Prove that the lines represented by $4x^2 - 12xy + 9y^2 = 0$ are coincident

Solution: Here,

$4x^2 - 12xy + 9y^2 = 0$

Comparing $4x^2 - 12xy + 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$

We have, a = 4, b = 9 and h = -6

Now, $h^2 - ab = (-6)^2 - 4 \times 6$

$= 36 - 36 = 0$

Hence, the lines represented by $4x^2 - 12xy + 9y^2 = 0$ are coincident.

Example 6

If the equation $(3a + 2)x^2 - 48xy + a^2y^2 = 0$ represents two perpendicular lines, find the value of a.

Solution: Here,

$(3a + 2)x^2 - 48xy + a^2y^2 = 0$, by comparing with $ax^2 + 2hxy + by^2 = 0$

coefficient of $x^2 = (3a + 2)$

coefficient of $y^2 = a^2$

We know, coefficient of x^2 + coefficient of $y^2 = 0$ for two perpendicular lines.

So, $3a + 2 + a^2 = 0$

$or, a^2 + 2a + a + 2 = 0$

$or, a(a + 2) + 1(a + 2) = 0$

$or, (a + 2)(a + 1) = 0$

$or, a = -1 \; and \; a = -2$

∴ Required values of a are -1 and -2

Example 7

Find the single equation of two lines passing through the origin and perpendicular to the lines represented by $2x^2 + 3xy - 2y^2 = 0$

Solution: Here,

$2x^2 + 3xy - 2y^2 = 0$

$or, 2x^2 + 4xy - xy - 2y^2 = 0$

$or, 2x(x + 2y) - y(x + 2y) = 0$

$or, x + 2y = 0 \text{ and } 2x - y = 0$

$or, y = -\frac{1}{2}x \text{ and } y = 2x \ldots\ldots\ldots.(i)$

Perpendicular lines to the line represented in (i) are

$y = 2x \text{ and } y = -\frac{1}{2}x \; [\because m_1 \times m_2 = -1]$

Now, $y - 2x = 0$ and $2y + x = 0$

$or, (y - 2x)(2y + x) = 0$

$or, 2y^2 - 4xy + xy - 2x^2 = 0$

$or, 2y^2 - 3xy - 2x^2 = 0$ is the single equation which passes through the origin.

Hence, required equation is $2y^2 - 3xy - 2x^2 = 0$.

Exercise 4.2

1. a. Write general form of homogeneous equation of second degree in x and y.

   b. Write the condition of pair of coincident lines represented by $ax^2 + 2hxy + by^2 = 0$

   c. Write the condition of pair of perpendicular lines represented by $ax^2 + 2hxy + by^2 = 0$

   d. If θ be the angle between the lines $ax^2 + 2hxy + by^2 = 0$ express tanθ in terms of a, b and h.

2. Find the single equation of line containing the following pair of straight line:

   a. $x - 5y = 0,\; x - y = 0$ b. $2x + 3y = 0,\; 3x - y = 0$

   c. $x = 2y,\; y = -\frac{1}{2}x$ d. $x = y\sin\theta,\; y = x\cos\theta$

3. Find the two straight lines represented by following equation of second degree:

   a. $3x^2 + 7xy + 2y^2 = 0$

   b. $6x^2 + 5xy + y^2 = 0$

c. $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$

d. $x^2 - 2xycosec\theta + y^2 = 0$

e. $x^2 - 2xysec\theta + y^2 = 0$

f. $ab(x^2 - y^2) + (a^2 - b^2)xy = 0$

g. $(x - y)^2 - (x - y) = 0$

h. $y^2 \cos^2 \theta - 2xy + (1 + \sin^2 \theta)x^2 = 0$

4. Find the angle between the lines represented by:

a. $\sqrt{3}x^2 - 4xy - \sqrt{3}y^2 = 0$

b. $2x^2 - 7xy + 3y^2 = 0$

c. $3x^2 + 7xy + 2y^2 = 0$

d. $x^2 + 2xyseca + y^2 = 0$

5. Prove that the lines represented by the following equations are coincident:

a. $x^2 - 4xy + 4y^2 = 0$

b. $3x^2 - 12xy + 12y^2 = 0$

c. $kx^2 + 2xy\sqrt{kh} + hy^2 = 0$

6. Find the value of k if the equations represent perpendicular lines:

a. $x^2 - 5xy + ky^2 = 0$

b. $k^2x^2 - 5xy - 9y^2 = 0$

c. $(k + 5)x^2 - 5xy - (3k - 1)y^2 = 0$

d. The lines represented by $kx^2 + 6xy + 9y^2 = 0$ are coincident, find the value of K.

e. The lines represented by $3x^2 + 2kxy + 12 = 0$, are coincident, find the value of k.

7. a. Find the equation of the straight line through the origin and at right angles to the line $x^2 - 5xy + 4y^2 = 0$.

b. Find the equation of the straight line through the origin and at right angles to the lines $x^2 + 3xy + 2y^2 = 0$.

c. Find the single equation of the two lines perpendicular to the lines $x^2 - y^2 = 0$ and passes through the origin.

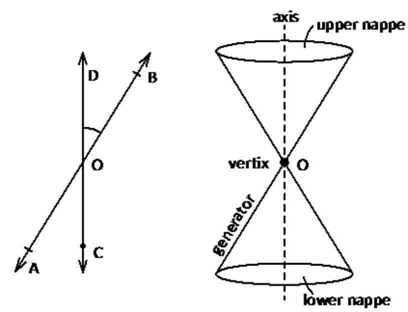
d. Take examples of two straight lines passes through origin, multiply and get single equation. Also find the angle between them. Share the solution with friends.

e. Discuss about the shape of homogenous pair of second degree equation in x and y.

4.3 Conic sections

What plane shapes may you get if you cut a carrot by a knife with different angles?

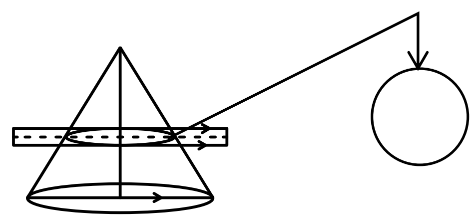
Right circular cone:



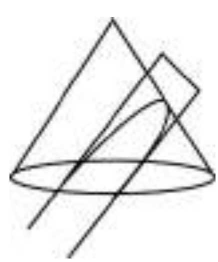
Let AB and CD be two fixed lines intersecting at the point O at an angle (0° < θ < 90°). If the plane containing these two lines AB and CD is rotated about the line CD, then the surface generated by AB is called right circular cone. O is vertex, CD is vertical axis, θ is vertical angle and AB is generator of the cone.

Take the solid cone and cut them by sharp knife as shown below.

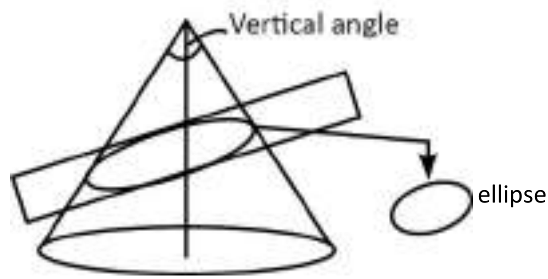
(i) Cut the cone parallel to base. The section in between cone and plane is said to be circle.



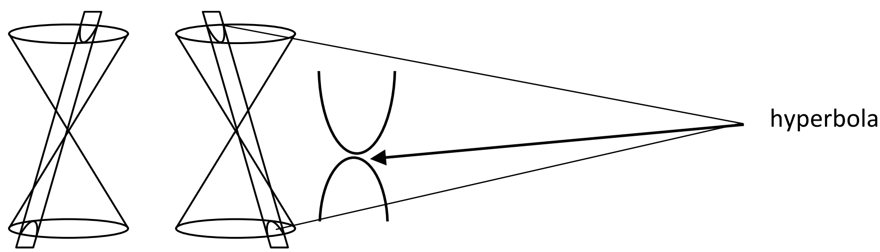
(ii) Cut the cone parallel to slant edge of the cone. The section in between cone and plane is said to be parabola.



(iii) Cut the cone in such a way that the angle between plane and axis of cone is greater than vertical angle of cone. But less than 90°. the section in between cone and the plane is said to be ellipse.

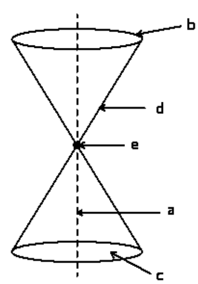


(iv) Intersect double circular cone parallel to edge of the cone (slant edge). The pair of parabola or the sections formed between double cone and plane is said to be hyperbola.



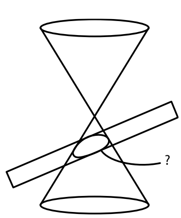
Exercise: 4.3

1. Name the different parts of cone indicated by a, b, c, d and e.

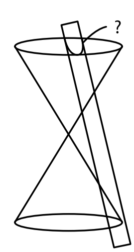


2. Name the conic section from following figures.

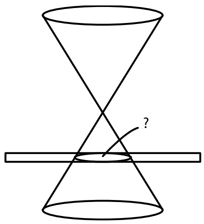
a)



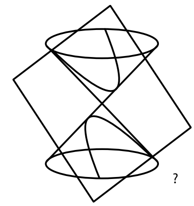
b)



c)



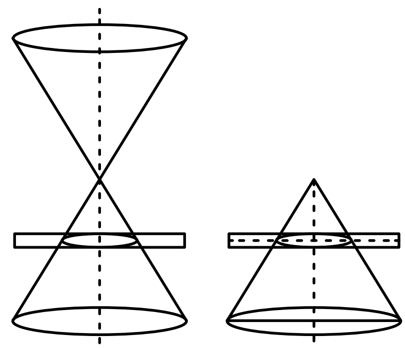
d)



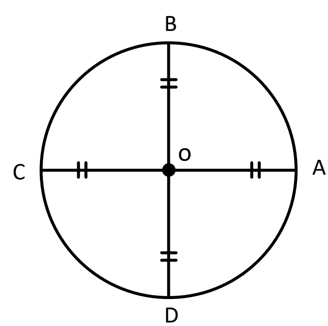
3. Take a solid circular double cone of either local made or from market. Cut it in different positions and name the conic section obtained.

4.4 Circle

A figure formed by the intersection of a plane and a circular cone is conic section. As in figure when we intersect circular cone parallel to the base we get a conic section, which is known as circle.



Take a thumb pin and rubber band. Fix the thumb pin at the position O and rotate the rubber band equal distance. The positions A, B, C and D are obtained. ABCD is circle and O is centre of the circle. Circle is the locus of a point which moves so that its distance from a fixed point is always constant.



The fixed point is called the centre and the constant distance is the radius of the circle.

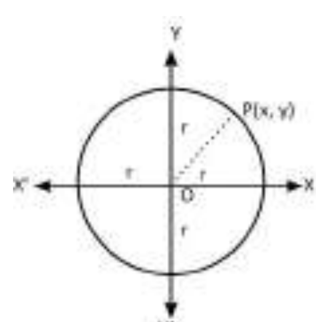
a. Equation of circle with centre O (0, 0) and radius 'r' units

Let any point P(x, y) on the circumference of circle.

Now, $\mathrm{OP} = \mathrm{r}, \quad \mathrm{OP}^2 = \mathrm{r}^2$

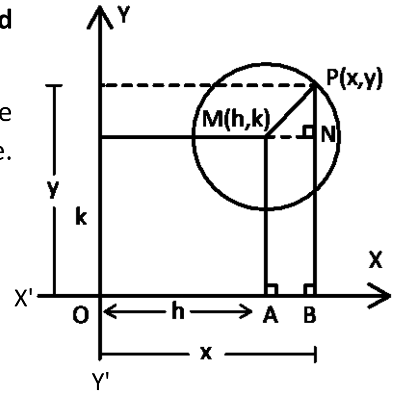
or, $(\mathrm{x} - 0)^2 + (\mathrm{y} - 0)^2 = \mathrm{r}^2$

or, $\mathrm{x}^2 + \mathrm{y}^2 = \mathrm{r}^2$



Hence, the required equation of circle is $x^2 + y^2 = r^2$.

b. Equation of the circle with centre (h, k) and Radius 'r' units



Let M(h, k) be the centre of circle and P(x, y) be any point on the circle 'r' be the radius of circle. Draw MA $\perp$ OX, PB$\perp$OX and MN$\perp$PB.

Now, MN = AB = OB – OA = x – h

PN = PB – BN = PB – MN = y – k

MP = r.

We Know, in right angled $\triangle$PNM,

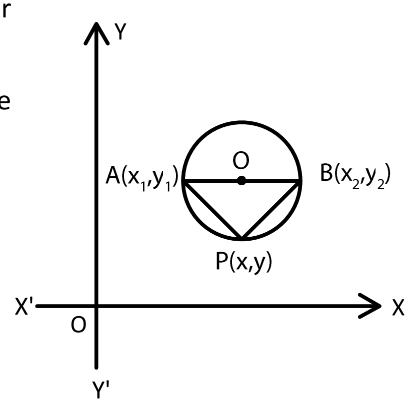
$MP^2 = MN^2 + PN^2$

or, $r^2 = (x-h)^2 + (y-k)^2$

or, $(x - h)^2 + (y - k)^2 = r^2$

Which is the required equation of the circle.

c. Equation of circle with end points of a diameter (diameter form)



Let A(x_1, y_1) and B(x_2, y_2) be ends of diameter of the circle. P(x, y) be any point on circle.

Join AP and BP, Such that $\measuredangle \mathrm{APB} = 90°$ (angle at semi-circle)

Now, (Slope of PA) × (Slope of PB) = -1

$$or, \quad \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

$$or, \quad (y-y_1)(y-y_2) = -1(x-x_1)(x-x_2)$$

$$or, \quad (y-y_1)(y-y_2) + (x-x_1)(x-x_2) = 0$$

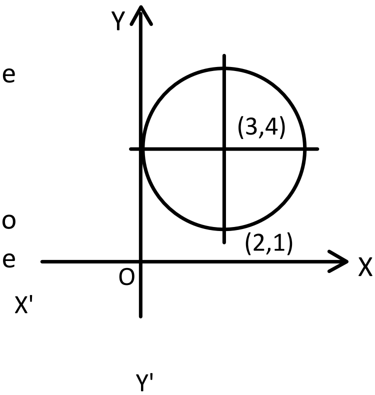
$\therefore$ $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Which is the required equation of the circle.

Note: The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ can be written as $(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$. It's center is (-g, -f) and radius is $\sqrt{g^2 + f^2 - c}$.

Example 1

From the given figure, find the centre and radius of the circle.



Solution:

In the figure, point (3,4) is equidistant from the circle. so centre of circle is (3,4). Again, centre is 3 units far from the y-axis. So, Radius of the circle is 3 units.

Example 2

Find the centre and radius of the circle from the following:

$a.\ (x-2)^2 + y^2 = 4$

$b.\ x^2 + 4x + y^2 - 2y - 4 = 0$

Solution: Here,

a. $(x-2)^2 + y^2 = 4$

$or, (x-2)^2 + (y-0)^2 = 2^2 \dots\dots..(i)$

Comparing equation (i) with $(x-h)^2 + (y-k)^2 = r^2$

We get, centre of the circle (h,k)= (2,0)

Radius of the circle (r) =2 units.

b. $x^2 + 4x + y^2 - 2y - 4 = 0$

$or, (x^2 + 4x + 4) + (y^2 - 2y + 1) - 4 = 5$ [Adding 5 on both sides.]

$or, (x+2)^2 + (y-1)^2 = 9$

$or, (x+2)^2 + (y-1)^2 = 3^2 \dots\dots..(i)$

Comparing equation (i) with$(x-h)^2 + (y-k)^2 = r^2$, we get centre (h,k)= (-2,1)

Radius (r) = 3 units

Example 3

Find the equation of the circle as given conditions below.

a. Centre at the origin and radius 4 units

b. Centre at (-4, 5) and radius 6 units

Solution: Here,

a. Centre of the circle = (0, 0)

Radius of the circle (r) = 4 units.

We know, equation of the circle is

$x^2 + y^2 = r^2$

or, $x^2 + y^2 = 4^2$

or, $x^2 + y^2 - 16 = 0$

b. Centre of the circle (h, k) = (-4, 5)

Radius of the circle (r) = 6 units

We know, equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$

$\therefore (x + 4)^2 + (y - 5)^2 = 6^2$

$or, x^2 + 8x + 16 + y^2 - 10y + 25 = 36$

$or, x^2 + y^2 + 8x - 10y + 41 - 36 = 0$

$\therefore x^2 + y^2 + 8x - 10y + 5 = 0$ is the required equation of circle.

Example 4

Find the equation of the circle having ends points of the diameter (4, -1) and (3, -4)

Solution:

Let end points of diameter are $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (3, -4)$

We know, equation of the circle is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

or, $(x - 4)(x - 3) + (y + 1)(y + 4) = 0$

or, $x^2 - 7x + 12 + y^2 + 5y + 4 = 0$

$x^2 + y^2 - 7x + 5y + 16 = 0$ is the required equation of circle.

Example 5

Find the equation of the circle passes through the pints (1, 1), (4, 4) and (5, 1)

Solution: Here,

Let the centre of the circle be M(h, k) and A(1, 1), B(4 ,4) and C(5, 1) are the points on the circle.

We know, $\mathrm{MA}^2 = \mathrm{MB}^2$

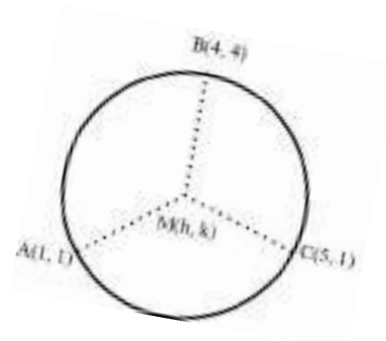
or, $(\mathrm{h}-1)^2+(\mathrm{k}-1)^2=(\mathrm{h}-4)^2+(\mathrm{k}-4)^2$

or, $\mathrm{h}^2-2\mathrm{h}+1+\mathrm{k}^2-2\mathrm{k}+1=\mathrm{h}^2-8\mathrm{h}+16+\mathrm{k}^2-8\mathrm{k}+16$

or, $6\mathrm{h}+6\mathrm{k}=30$

or, $\mathrm{h}+\mathrm{k}=5$

or, $\mathrm{h}=5-\mathrm{k}\ldots\ldots\ldots\ldots.(\mathrm{i})$



Again, $\mathrm{MA}^2=\mathrm{MC}^2$

or, $(\mathrm{h}-1)^2+(\mathrm{k}-1)^2$
$=(\mathrm{h}-5)^2+(\mathrm{k}-1)^2$

or, $\mathrm{h}^2-2\mathrm{h}+1=\mathrm{h}^2-10\mathrm{h}+25$

or, $8\mathrm{h}=24$

or, $\mathrm{h}=3$

From equation (i) $3=5-\mathrm{k}$
or, $\mathrm{k}=2$
$\therefore$ Centre of the circle $(\mathrm{h},\mathrm{k})=(3,2)$
$\mathrm{MA}^2 \quad =(3-1)^2+(2-1)^2$
$=4+1=5$

Radius of circle = $\sqrt{5}$ units

Equation of circle is $(x-h)^2+(y-k)^2=r^2$

$or, \quad (x-3)^2+(y-2)^2=5$
$or, \quad x^2-6x+9+y^2-4y+4=5$

$or,\ x^2+y^2-6x-4y+8=0$ is the required equation of circle.

Note: If circle passes through four points, find the equation of the circle from any three points and satisfy the equation by fourth point. Four points are said to be con-cyclic.

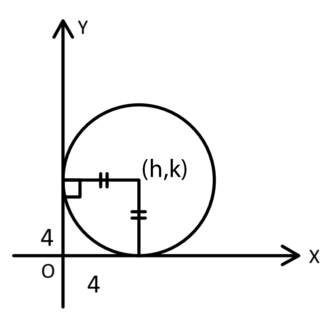
Example 6

Find the equation of such a circle that touches both the axes in the first quadrant at distance of 4 units from the origin.

Solution: Here,

Let (h, k) be the centre of the circle. Since it touches both the axes, so (h, k) = (4, 4)

Radius = 4 units



Now, Equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$

$$or, \quad (x-4)^2 + (y-4)^2 = 4^2$$
$$or, \quad x^2 - 8x + 16 + y^2 - 8y + 16 = 16$$

$$or, \quad x^2 + y^2 - 8x - 8y + 16 = 0$$

Hence, the required equation of the circle is $x^2 + y^2 - 8x - 8y + 16 = 0$

Example 7

Find the equation of a circle whose centre lies on intersection of x + y = 5 and x – y = 1 and passes through point (-4,-3).

Solution:

Solving x + y = 5 and x – y = 1

Adding x + y = 5 and x – y = 1

We get 2x = 6

or, x = 3

Substituting x = 3 in x + y = 5, we get

or, 3 + y = 5 ⇒ y = 5 – 3 = 2

So, centre of the circle (h, k) = (3, 2)

Since, it passes through the point (-4,-3)

$so, r^2 = (3+4)^2 + (2+3)^2$

$= 7^2 + 5^2$

$= 49 + 25$

$= 74$

Now, equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$or, \quad (x-3)^2 + (y-2)^2 = 74$$
$$or, \quad x^2 - 6x + 9 + y^2 - 4y + 4 = 74$$

$$or, \quad x^2 + y^2 - 6x - 4y - 61 = 0$$

Exercise 4.4

1. a. What is the equation of circle having centre (0,0) and radius ‘a’ units?

   b. Write the equation of circle whose radius is ‘r’ units and centre is (h, k).

c. (x_1, y_1) and (x_2, y_2) are the end points of a diameter of a circle. What is the equation of circle in terms of (x_1, y_1) and (x_2, y_2)?

2. Find the centre and radius of the circle from the following equations of the circle:

   a. $x^2 + y^2 = 9$

   b. $(x - 2)^2 + y^2 = 25$

   c. $(x + 1)^2 - (y - 3)^2 = 16$

   d. $x^2 - 2x + y^2 - 5 = 0$

   e. $x^2 + y^2 - 4x - 6y - 23 = 0$

   f. $x^2 + y^2 - 8x + 2y - 24 = 0$

3. Find the equation of circle in each case:

   a. Centre at origin, radius 5 units

   b. Centre at origin, radius $1\frac{1}{2}$ units

   c. Centre at (3, 4), radius 3 units

   d. Centre at (0, -4), radius 4 units

   e. Centre at (1, 2), radius 5 units

   f. Centre at (-1, -3), radius 6 units

4. Find the centre, radius and equation of circle, if the end points of a diameter are given as:

   a. (2, 0) and (0, 6)

   b. (-1, 2) and (7, -4)

   c. $\left(\frac{7}{3}, \frac{3}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$

5. a. Find the equation of circle passing through the points (1, 2), (3, 1) and (-3,-1).

   b. Find the equation of circle passing through the points (2, -2), (6, 6) and (5, 7).

   c. If (3, 3), (6, 4), (7, 1) and (4, 6) are the vertices of a cyclic quadrilateral, find the equation of circle.

   d. Prove that the points (1, 0), (2, -7), (8, 1) and (9, -6) are con-cyclic.

6. Find the equation of circles in each of the following cases:

   a. Touches both axes, radius 5 units and centre in first quadrant.

   b. Touches x-axis, centre at (-3, 4)

   c. Touches y-axis, centre at (-4, -5)

7. Draw a circle of diameter PQ, where P (8, 1) and Q (8, 11). Take a point R (3, 6) on the circle.

   a. Calculate the distance PQ, PR and QR.

b. Is $(\mathrm{PQ})^2 = (\mathrm{PR})^2 + (\mathrm{QR})^2$?

c. What conclusion do you get?

d. Also find the equation of the circles.

8. The Vertices of a triangle are A (0, 1), B (-2, 0) and C (1, 0).

   a. Find the equation of sides of the triangle.

   b. Find the equation of the circle drawn on that sides of the triangle lying in the first quadrant as diameter.

9. Find the equation of the circle passing through the points (5,3) and (-2, 2) and the centre lies on the line $x + 3y + 1 = 0$

10. Make a circular piece of paper. Trace this circle in a graph paper. Find the centre, radius and equation of the circle. Present your findings in classroom.

Unit 5

Trigonometry

5.0 Review

If A and B are two different angles, what are their compound angles? How can we find their Trigonometric ratios of Sine, Cosine, tangent and cotangent? Discuss in a group and compare with the following results.

$$\sin(\mathrm{A}+\mathrm{B}) = \sin\mathrm{A}\cos\mathrm{B} + \cos\mathrm{A}\sin\mathrm{B} \qquad \sin(\mathrm{A}-\mathrm{B}) = \sin\mathrm{A}\cos\mathrm{B} - \cos\mathrm{A}\sin\mathrm{B}$$

$$\cos(\mathrm{A}+\mathrm{B}) = \cos\mathrm{A}\cos\mathrm{B} - \sin\mathrm{A}\sin\mathrm{B} \qquad \cos(\mathrm{A}-\mathrm{B}) = \cos\mathrm{A}\cos\mathrm{B} + \sin\mathrm{A}\sin\mathrm{B}$$

$$\tan(\mathrm{A}+\mathrm{B}) = \frac{\tan\mathrm{A} + \tan\mathrm{B}}{1 - \tan\mathrm{A}\tan\mathrm{B}} \qquad \tan(\mathrm{A}-\mathrm{B}) = \frac{\tan\mathrm{A} - \tan\mathrm{B}}{1 + \tan\mathrm{A}\tan\mathrm{B}}$$

$$\cot(\mathrm{A}+\mathrm{B}) = \frac{\cot\mathrm{A}\cot\mathrm{B} - 1}{\cot\mathrm{B} + \cot\mathrm{A}} \qquad \cot(\mathrm{A}-\mathrm{B}) = \frac{\cot\mathrm{A}\cot\mathrm{B} + 1}{\cot\mathrm{B} - \cot\mathrm{A}}$$

The Trigonometric value of standard angles.

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined
cosec	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined
cot	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

5.1.1 Trigonometric ratios of multiple angles

What are the multiple angles of an angle A? Discuss in groups.

a. Trigonometric ratios of multiple angle 2A

With the help of compound angle, we can find the trigonometric ratios of Sine, Cosine, tangent and cotangent of the angle 2A:

We know that,

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Now, sin2A = sin(A + A)

= sinAcosA + cosAsinA

= sinAcosA + sinAcosA

= 2sinAcosA

$\therefore \sin 2A = 2\sin A \cos A \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

Similarly,

$\cos 2A \quad = \cos(A+A)$

$= \cos A \cos A - \sin A \sin A$

= $\cos^2 A - \sin^2 A$

$\therefore \cos 2A = \cos^2 A - \sin^2 A \ldots\ldots\ldots\ldots\ldots.(ii)$

Also,

$\cos 2A \quad = \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$\therefore \cos 2A = 2\cos^2 A - 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(iii)$

$\cos 2A = 1 - \sin^2 A - \sin^2 A$

$\therefore \cos 2A = 1 - 2\sin^2 A \ldots\ldots\ldots\ldots\ldots\ldots\ldots(iv)$

$$\tan 2A = \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1 - \tan^2 A}$$

$$\therefore \tan 2A = \frac{2\tan A}{1 - \tan^2 A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(v)$$

$$\cot 2A = \cot(A+A) = \frac{\cot A . \cot A - 1}{\cot A + \cot A} = \frac{\cot^2 A - 1}{2\cot A}$$

$$\therefore \cot 2A = \frac{\cot^2 A - 1}{2\cot A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(vi)$$

Also, we can express Sin2A and Cos2A in terms of tanA.

We have,

$\sin 2A = 2\sin A \cos A$

$= 2\frac{\operatorname{Sin}A}{\operatorname{Cos}A} \; . \; \cos^2 A$ [Multiplying numerator and denominator by cosA]

$$= 2\tan A \times \frac{1}{\operatorname{Sec}^2 A}$$

$$\therefore \sin 2A = \frac{2\tan A}{1+\tan^2 A} \dots\dots\dots\dots\dots\dots\dots\dots\dots.. (vii)$$

and, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \cos^2 A - \sin^2 A \times \frac{\operatorname{Cos}^2 A}{\operatorname{Cos}^2 A}$$ [In 2nd term multiplying numerator and denominator by $\cos^2 A$]

$$= \cos^2 A \left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$= \cos^2 A(1 - \tan^2 A)$$

$$= \frac{1-\tan^2 A}{\operatorname{Sec}^2 A}$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} \text{ .... (viii)}$$

From above relation (iii) also we can write

$$2\cos^2 A = 1 + \cos 2A \text{ .... (ix)}$$

or, $$\cos^2 A = \frac{1+\cos 2A}{2} \dots\dots\dots\dots.. (x)$$

Similarly, from above equation (iv), we can write

$$2\sin^2 A = 1 - \cos 2A \dots\dots\dots\dots\dots (xi)$$

$$\sin^2 A = \frac{1-\cos 2A}{2} \dots\dots\dots\dots\dots.. (xii)$$

Dividing equation (xii) by equation (x), we get

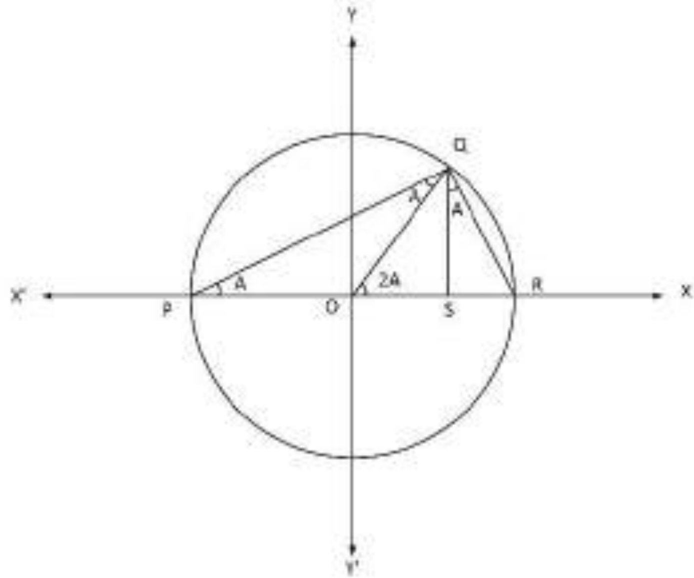
$$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A} \dots\dots\dots\dots\dots\dots\dots.. (xiii)$$

and $$\cot^2 A = \frac{1+\cos 2A}{1-\cos 2A} \dots\dots\dots\dots\dots\dots.. (xiv)$$

Can you express sin2A and cos2A in terms of cotA? Try it.

Geometrical method

Let, O be the centre of the unit circle PQRS. PR is a diameter, which lies on X-axis. The end points P and R of diameter PR are joined with the point Q. We know that, ∡ PQR = 90° so △PQR is a right angled triangle. Draw QS ⊥ PR. Let, ∡QPR = A then ∡PQO = A, ∡RQS = A and ∡QOS = 2A, why?



In right angled △QOS

$$\sin 2A = \frac{p}{h} = \frac{QS}{OQ}$$

$$= \frac{2QS}{2OQ}$$

$$= \frac{2QS}{PR} \text{ [why } 2OQ = PR?]$$

$$= 2\frac{QS}{PQ} \cdot \frac{PQ}{PR}$$

$$\therefore \sin 2A = 2\sin A \,.\, \cos A$$

Similarly,

$$\cos 2A = \frac{b}{h} = \frac{OS}{OQ} = \frac{2OS}{2OQ}$$

$$= \frac{(PO + OS) - (PO - OS)}{PR}$$

$$= \frac{(PO + OS) - (RO - OS)}{PR}$$

$$= \frac{PS - SR}{PR}$$

$$= \frac{PS}{PR} - \frac{SR}{PR}$$

$$= \frac{PS}{PQ} \cdot \frac{PQ}{PR} - \frac{SR}{QR} \cdot \frac{QR}{PR}$$

$$= \cos A.\cos A - \sin A.\sin A$$

$$\therefore \cos 2A = \cos^2 A - \sin^2 A$$

And

$$\tan 2A = \frac{p}{b} = \frac{QS}{OS} = \frac{2QS}{2OS}$$

$$= \frac{2QS}{(PO + OS) - (PO - OS)}$$

$$= \frac{2QS}{(PO + OS) - (RO - OS)}$$

$$= \frac{\frac{2QS}{PS}}{\frac{PS}{PS} - \frac{SR}{PS}}$$

$$= \frac{2\frac{QS}{PS}}{1 - \frac{SR}{QS}\cdot\frac{QS}{PS}}$$

$$= \frac{2\frac{QS}{PS}}{1 - \frac{SR}{QS}\cdot\frac{QS}{PS}}$$

$$= \frac{2\tan A}{1 - \tan A.\tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

$$\therefore \tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

b. Trigonometric ratios of multiple angle 3A

Can you write the trigonometric ratios of sin(A+B), cos(A+B) and tan(A+B)? Also we have just discussed above on sin2A, cos2A and tan2A results.

$$\sin 3A = \sin(2A + A)$$

$$= \sin 2A.\cos A + \cos 2A.\sin A$$

$$= 2\sin A\cos A.\cos A + (1 - 2\sin^2 A).\sin A$$

$$= 2\sin A\cos^2 A + \sin A - 2\sin^3 A$$

$$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$$

$$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$$

$\therefore \sin 3A = 3\sin A - 4\sin^3 A \ldots\ldots\ldots\ldots\ldots\ldots.(i)$

Similarly,

$\cos 3A = \cos(2A + A)$

$$= \cos 2A . \cos A - \sin 2A. \sin A$$

$$= (2\cos^2 A - 1)\cos A - 2\sin A\cos A. \sin A)$$

$$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$$

$$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A). \cos A$$

$$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$$

$\therefore \cos 3A = 4\cos^3 A - 3\cos A \ldots\ldots\ldots\ldots\ldots\ldots (i)$

And, $\tan 3A = \tan(2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A. \tan A}$$

$$= \frac{\frac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2\tan A}{1 - \tan^2 A} . \tan A}$$

$$= \frac{\frac{2\tan A + \tan A(1 - \tan^2 A)}{1 - \tan^2 A}}{\frac{1 - \tan^2 A - 2\tan^2 A}{1 - \tan^2 A}}$$

$$= \frac{2\tan A + \tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$\therefore \tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

From relation (i) $\quad \sin 3A = 3\sin A - 4\sin^3 A$

or, $\quad 4\sin^3 A = 3\sin A - \sin 3A \ldots\ldots\ldots\ldots\ldots (iv)$

or, $\quad \sin^3 A = \dfrac{3\sin A - \sin 3A}{4} \ldots\ldots\ldots\ldots\ldots (v)$

From relation (ii) $\quad \cos 3A = 4\cos^3 A - 3\cos A$

or, $\quad 4\cos^3 A = \cos 3A + 3\cos A \ldots\ldots\ldots\ldots\ldots (vi)$

or, $\quad \cos^3 A = \dfrac{\cos 3A + 3\cos A}{4} \ldots\ldots\ldots\ldots\ldots.. (vii)$

Discuss, what is the result of cot3A? Find it.

Example 1

If $\sin A = \dfrac{5}{13}$ then find the value of $\sin 2A, \cos 2A$ and $\tan 2A.$

Solution: Here,

$$\sin A = \frac{5}{13}$$

We know that,

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

Now, by formula

i) $\sin 2A = 2\sin A \cdot \cos A = 2 \times \dfrac{5}{13} \times \dfrac{12}{13} = \dfrac{120}{169}$

ii) $\cos 2A = 1 - 2\sin^2 A = 1 - 2\left(\dfrac{5}{13}\right)^2 = 1 - \dfrac{2 \cdot 25}{169} = \dfrac{169 - 50}{169} = \dfrac{119}{169}$

iii) $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{\frac{120}{169}}{\frac{119}{169}} = \dfrac{120}{119}$

Example 2

If $\sin\theta = \dfrac{1}{2}$ then find the value of $\sin 3\theta, \cos 3\theta$ and $\tan 3\theta$

Solution: Here, $\sin\theta = \frac{1}{2}$

By formula,

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$= 3 \times \frac{1}{2} - 4 \times \left(\frac{1}{2}\right)^3 = \frac{3}{2} - 4 \times \frac{1}{8} = \frac{3}{2} - \frac{1}{2}$$

$$\therefore \sin 3\theta = 1$$

$$\text{Cos}\theta = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Now, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4\left(\frac{\sqrt{3}}{2}\right)^3 - 3\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$

and, $\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{1}{0} = \infty$

Example 3

Prove that: $cotA = \pm\sqrt{\frac{1+cos2A}{1-cos2A}}$

Solution: Here,

$$2\cos^2 A = 1 + \cos 2A$$

or, $\cos^2 A = \frac{1+\cos 2A}{2}$

or, $\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}} \ldots\ldots\ldots\ldots\ldots(\mathrm{i})$

Similarly, we have,

$$\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}} \ldots\ldots\ldots\ldots..(\mathrm{ii})$$

Dividing equation (i) by (ii) we get

$$\frac{\cos A}{\sin A} = \frac{\left(\pm\sqrt{\frac{1+\cos 2A}{2}}\right)}{\pm\sqrt{\frac{1-\cos 2A}{2}}} = \pm\frac{\sqrt{1+\cos 2A}}{\sqrt{1-\cos 2A}}$$

$$\therefore \cot A = \pm\sqrt{\frac{1+\cos 2A}{1-\cos 2A}}$$

Example 4

Prove that: $\frac{1-cos2A+sin2A}{1+cos2A+sin2A} = \tan A$

Solution: Here,

LHS $= \frac{1-cos2A+sin2A}{1+cos2A+sin2A} = \frac{2sin^2A+2sinA\,cosA}{2cos^2A+2sinAcosA} = \frac{2sinA(sinA+cosA)}{2cosA(cosA+sinA)}$

$$= tanA = R.H.S$$

Example 5

Prove that: $\tan\left(\frac{\pi^c}{4}+\theta\right)=\frac{cos2\theta}{1-sin2\theta}$

Solution: Here,

LHS $\tan\left(\frac{\pi^c}{4}+\theta\right)=\tan(45°+\theta)$

$$=\frac{tan45°+tan\theta}{1-tan45°\cdot tan\theta}=\frac{1+\frac{sin\theta}{cos\theta}}{1-1\cdot\frac{sin\theta}{cos\theta}}$$

$$=\frac{cos\theta+sin\theta}{cos\theta-sin\theta}=\frac{(cos\theta+sin\theta)}{cos\theta-sin\theta}\times\frac{cos\theta-sin\theta}{cos\theta-sin\theta}$$

$$=\frac{cos^2\theta-sin^2\theta}{(cos\theta-sin\theta)^2}=\frac{cos2\theta}{cos^2\theta-2sin\theta cos\theta+sin^2\theta}$$

$$=\frac{cos2\theta}{1-sin2\theta}=R.H.S.$$

Example 6

Prove that: $\sqrt{2+\sqrt{2+\sqrt{2+2\cos8\theta}}}=2\cos\theta$

Solution: Here,

LHS $=\sqrt{2+\sqrt{2+\sqrt{2+2cos8\theta}}}=\sqrt{2+\sqrt{2+\sqrt{2+2cos2(4\theta)}}}$

$$=\sqrt{2+\sqrt{2+\sqrt{2(1+2cos^24\theta-1)}}}=\sqrt{2+\sqrt{2+2cos2(2\theta)}}$$

$$=\sqrt{2+\sqrt{2(1+2cos^22\theta-1)}}=\sqrt{2+\sqrt{4cos^22\theta}}=\sqrt{2+2cos2\theta}$$

$$=\sqrt{2(1+2cos^2\theta-1)}=\sqrt{4cos^2\theta}$$

$$=2cos\theta=R.H.S.$$

Example 7

Prove that: $\cos^6\theta+\sin^6\theta=\frac{1}{4}(1+3\cos^22\theta)$

Solution: Here,

LHS $cos^6\theta+sin^6\theta=(cos^2\theta)^3+(sin^2\theta)^3$

$$= (cos^2\theta + sin^2\theta)\{(cos^2\theta)^2 - sin^2\theta\ cos^2\theta + (sin^2\theta)^2\}$$

$$= 1\ \{(cos^2\theta + sin^2\theta)^2 - 2sin^2\theta cos^2\theta - sin^2\theta cos^2\theta\}$$

$$= 1 - 3sin^2\theta cos^2\theta = 1 - \frac{3}{4} \times 4sin^2\theta cos^2\theta$$

$$= 1 - \frac{3}{4}\ (2sin\theta cos\theta)^2 = \ 1 - \frac{3}{4}\ (sin2\theta)^2$$

$$= 1 - \frac{3}{4}\ sin^2 2\theta = 1 - \frac{3}{4}(1 - cos^2 2\theta)$$

$$= \frac{4 - 3 + 3cos^2 2\theta}{4} = \frac{1}{4}(1 + 3cos^2 2\theta) = R.H.S.$$

Exercise 5.1

1. a. Define multiple angle with an example.

   b. Write cos2A in terms of cosA and sinA.

2. a. Write sin2A in terms of tanA

   b. Write $\tan^2$ A in terms of cos2A

   c. Write sin3A in terms of sinA

   d. Write tan3θ in terms of tanA.

3. a. If sinA = $\frac{3}{5}$, find the value of cos2A

   b. If sin2A $= \frac{24}{25}$ and sinA $= \frac{4}{5}$, then find the value of cosA

   c. If $sinA\ = \frac{4}{5}$ find the value of sin2A.

   d. If $cos\theta\ = \frac{12}{13}$ find the value of sin2θ and cos2θ

   e. If $tan\theta\ = \frac{3}{4}$ find the value of sin2θ and tan2θ.

   f. If $sin\alpha = \frac{1}{2}$ find the value of sin3α and cos3α.

   g. If $cos\alpha\ = \frac{\sqrt{3}}{2}$ find the value of sin3α and cos3α.

   h. If tan $\beta = \frac{1}{2}$ find the value of tan3β.

4. a. If cos2A $= \frac{7}{25}$ then Show that $sinA = \frac{3}{4}$.

   b. If cos2A $= -\frac{1}{2}$ then show that $cosA = \frac{1}{2}$.

5. Prove that:

a. $sinA = \pm\sqrt{\dfrac{1 - cos2A}{2}}$ b. $cosA = \pm\sqrt{\dfrac{1 + cos2A}{2}}$

c. $tanA = \pm\sqrt{\dfrac{1 - cos2A}{1 + cos2A}}$ d. $sec2A = \dfrac{cot^2A + 1}{cot^2A - 1}$

6. Prove the following identities:

a. $\dfrac{sin2A}{1 + cos2A} = tanA$ b. $\dfrac{1 - cos2A}{sin2A} = tanA$

c. $\dfrac{sin2A}{1 - cos2A} = cotA$ d. $\dfrac{1 - cos2\theta}{1 + cos2\theta} = \tan^2\theta$

e. $\dfrac{1 - tan\alpha}{1 + tan\alpha} = \dfrac{1 - sin2\alpha}{cos2\alpha}$ f. $\dfrac{cos2\theta}{1 + sin2\theta} = \dfrac{1 - tan\theta}{1 + tan\theta}$

g. $\dfrac{sin\theta + sin2\theta}{1 + cos\theta + cos2\theta} = tan\theta$ h. $\dfrac{1 + cos\beta + cos2\beta}{sin\beta + sin2\beta} = cot\beta$

i. $\dfrac{1 - sin2\alpha}{cos2\alpha} = \dfrac{1 - tan\alpha}{1 + tan\alpha} = \tan(45° - \alpha)$ j. $tan\theta + cot\theta = 2cosec2\theta$

k. $cosec2A - cot2A = tanA$

7. Prove the following identities:

a. $\tan(45° + \theta) = Sec2\theta + tan2\theta$ b. $1 - sin2A = 2Sin^2(45° - A)$

c. $2cos^2(45° - \theta) = 1 + sin2\theta$

d. $cos^2(45° - A) - sin^2(45° - A) = sin2A$

e. $\tan(A + 45°) - \tan(A - 45°) = \dfrac{2(1 + \tan^2 A)}{1 - \tan^2 A}$

f. $\dfrac{1 + \tan^2(45° - \theta)}{1 - \tan^2(45° - \theta)} = Cosec2\theta$

8. a. If $cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$ then prove that : $cos2\theta = \frac{1}{2}\left(a^2 + \frac{1}{a^2}\right)$

b. If $sin\theta = \frac{1}{2}\left(b + \frac{1}{b}\right)$ then prove that $cos2\theta = -\frac{1}{2}\left(b^2 + \frac{1}{b^2}\right)$

c. If $sin\beta = \frac{1}{2}\left(k + \frac{1}{k}\right)$ then prove that $sin3\beta = -\frac{1}{2}\left(k^3 + \frac{1}{k^3}\right)$

9. Prove the following:

   a. $(\cos 2\mathrm{A} - \cos 2\mathrm{B})^2 + (\sin 2\mathrm{A} + \sin 2\mathrm{B})^2 = 4\sin^2(\mathrm{A} + \mathrm{B})$

   b. $(\sin 2\mathrm{A} - \sin 2\mathrm{B})^2 + (\cos 2\mathrm{A} + \cos 2\mathrm{B})^2 = 4\cos^2(\mathrm{A} + \mathrm{B})$

10. Prove the following:

    a. $cos^2\theta + sin^2\theta \,.\, cos2\alpha = cos^2\alpha + sin^2\alpha \,.\, cos2\theta$

    b. $(1 + cos2\theta + sin2\theta)^2 = 4cos^2\theta(1 + sin2\theta)$

    c. $(2cosA + 1)(2cosA - 1)(2cos2A - 1) = 1 + 2cos4A$

    d. $1 + cos8\theta = (2cos4\theta - 1)(2cos2A - 1)(2cos\theta - 1)(2cos\theta + 1)$

11. Prove the following:

    $a.$ $cos^6\theta - sin^6\theta = \frac{1}{4}(cos^3 2\theta + 3cos2\theta)$

    $b.$ $cos^6\theta + sin^6\theta = \frac{1}{8}(5 + 3cos4\theta)$

    $c.$ $cos^8\theta + sin^8\theta = 1 - sin2\theta + \frac{1}{8}Sin^4 2\theta$

    $d.$ $\frac{1}{sin10} - \frac{\sqrt{3}}{cos10} = 4$

    e. $sec40° + \sqrt{3}cosec40° = 4$

    $f.$ $\sqrt{3}cosec20° - sec20° = 4$

12. Prove the following:

    $a.$ $\frac{1}{sin2A} + \frac{cos4A}{sin4A} = cotA - cosec4A$

    $b.$ $cot8A + cosec4A = cot2A - cosec8A$

    $c.$ $\frac{sec4\theta - 1}{sec2\theta - 1} = tan4\theta \,.\, Cot\theta$

    $d.$ $\frac{sec8A - 1}{sec4A - 1} = \frac{tan8A}{tan2A}$

    $e.$ $tanA + 2tan2A + 4tan4A + 8cot8A = cotA$

13. Prove the following:

    $a.\ 4(cos^3 10° + sin^3 20°) = 3(cos10° + sin20°)$

    $b.\ sin^3\theta\ .cos3\theta + cos^3\theta\ .sin3\theta = \frac{3}{4} sin4\theta$

    $c.\ cos^3 A.cos3A + sin^3 A.sin3A = cos^3 2A$

    $d.\ tanA + \tan\left(\frac{\pi^c}{3} + A\right) - \tan\left(\frac{\pi^c}{3} - A\right) = 3tan3A$

14. If 2tanA = 3tanB then prove that

    $i)\ \tan(A + B) = \frac{5sin2B}{5cos2B - 1}$ $\qquad ii)\ \tan(A - B) = \frac{sin2B}{5 - cos2B}$

15. Prove the following:

    $a.\quad sin^4 A = \frac{1}{8}(3 - 4cos2A + cos4A)$

    $b.\quad cos^4 A = \frac{3}{8} + \frac{1}{2}cos2A + \frac{1}{8}cos4A$

    $c.\quad sin5\theta = 16sin^5\theta - 20sin^3\theta + 5sin\theta$

    $d.\quad cos5A = 16cos^5 A - 20cos^3 A + 5cosA$

16. With the help of multiple angles relation of Sine and Cosine, find the value of sin18°, sin36° and sin54°. By using these values, find the values of cos18°, cos36° and cos54°. Also, find the value of tan18°, tan36° and tan54°. Share your result to your friend and prepare combine report.

5.2 Trigonometric ratios of submultiple angles

What are the submultiple angles of an angle A? Are $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}, \dots\dots\dots\dots.\frac{A}{n},\ n \in N$ submultiple angle of A? Discuss about it. Prepare the list of formula of trigonometric ratios of sine, cosine, and tangent of multiple angles 2A and 3A. With the help of them, we can find the trigonometric ratios of submultiple angles of A in $\frac{A}{2}\, and\, \frac{A}{3}$.

We know that,

$$sinA = sin2\cdot\left(\frac{A}{2}\right)$$

$$sinA = 2sin\frac{A}{2}\ \cdot cos\frac{A}{2} \dots\dots\dots\dots\dots\dots\dots\dots (i)$$

Similarly,

$$cosA = cos2\left(\frac{A}{2}\right) = cos^2\frac{A}{2} - sin^2\frac{A}{2} \dots\dots\dots\dots\dots\dots..(ii)$$

Again,

$$cosA = cos2\left(\frac{A}{2}\right) = 2cos^2\frac{A}{2} - 1 \dots\dots\dots\dots\dots.(iii)$$

$$cosA = cos2\ \left(\frac{A}{2}\right) = 1 - 2sin^2\frac{A}{2} \dots\dots\dots\dots\dots..(iv)$$

$$tanA = tan2\ \left(\frac{A}{2}\right) = \frac{2tan\frac{\mathrm{A}}{2}}{1 - \tan^2\frac{A}{2}} \dots\dots\dots\dots\dots\dots..(v)$$

Expression of sinA and cosA in terms of tan $\left(\frac{A}{2}\right)$

We Know that

$$sinA = sin2\left(\frac{A}{2}\right) = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}} \dots\dots\dots\dots\dots\dots\dots..(vi)$$

And

$$cosA = cos2\ \left(\frac{A}{2}\right) = \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}} \dots\dots\dots\dots\dots..(vii)$$

Write sinA, cosA and tanA in terms of $\sin\frac{\mathrm{A}}{3}$, $\cos\frac{\mathrm{A}}{3}$ and $\tan\frac{\mathrm{A}}{3}$ respectively.

We have,

$$sinA = sin3\ \left(\frac{A}{3}\right) = 3sin\frac{A}{3} - 4sin^3\frac{A}{3} \dots\dots\dots\dots\dots..(viii)$$

$$cosA = cos3\left(\frac{A}{3}\right) = 4cos^3\frac{A}{3} - 3cos\frac{A}{3} \dots\dots\dots\dots..(ix)$$

$$tanA = tan3\ \left(\frac{A}{3}\right) = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1 - 3\tan^2\frac{A}{3}} \dots\dots\dots\dots\dots\dots(x)$$

With the help of compound angles relations also, we can establish the above relations.

Now,

$$sinA = sin\left(\frac{A}{2}+\frac{A}{2}\right) = sin\frac{A}{2} \cdot cos\frac{A}{2} + cos\frac{A}{2} \cdot sin\frac{A}{2}$$

$$= sin\frac{A}{2}cos\frac{A}{2} + sin\frac{A}{2} \cdot cos\frac{A}{2}$$

$$\therefore sinA = 2sin\frac{A}{2} \cdot cos\frac{A}{2}$$

Similarly,

$$cosA = cos\left(\frac{A}{2}+\frac{A}{2}\right) = cos\frac{A}{2} \cdot cos\frac{A}{2} - sin\frac{A}{2} \cdot sin\frac{A}{2} = cos^2\frac{A}{2} - sin^2\frac{A}{2}$$

Again,

$$tanA = \tan\left(\frac{A}{2}+\frac{A}{2}\right) = \frac{\tan\frac{A}{2}+\tan\frac{A}{2}}{1-\tan\frac{A}{2}\cdot\tan\frac{A}{2}} = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

$$\therefore tanA = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

For $\frac{A}{3}$

$$osA = cos\left(\frac{2A}{3}+\frac{A}{3}\right) = cos\frac{2A}{3} \cdot cos\frac{A}{3} - sin\frac{2A}{3} \cdot sin\frac{A}{3}$$

$$= cos2\left(\frac{A}{3}\right).cos\frac{A}{3} - sin2\left(\frac{A}{3}\right) \cdot sin\frac{A}{3}$$

$$= \left(2cos^2\frac{A}{3} - 1\right)cos\frac{A}{3} - 2sin\frac{A}{3} \cdot cos\frac{A}{3} \cdot sin\frac{A}{3}$$

$$= 2cos^3\frac{A}{3} - cos\frac{A}{3} - 2sin^2\frac{A}{3} \cdot cos\frac{A}{3}$$

$$= 2cos^3\frac{A}{3} - cos\frac{A}{3} - 2\left(1 - cos^2\frac{A}{3}\right) \cdot cos\frac{A}{3}$$

$$= 2cos^3\frac{A}{3} - cos\frac{\mathrm{A}}{3} - 2cos\frac{A}{3} + 2cos^3\frac{A}{3}$$

$$\therefore cosA = 4cos^3\frac{A}{3} - 3cos\frac{A}{3}$$

Similarly, derive the remaining relation and discuss for your correctness.

5.2.1 Comparative study of trigonometric ratios of multiple and submultiple angles

	Multiple Angle		Submultiple Angle
1	$sin2A = 2sinA\ cosA$	1	$sinA = 2sin\frac{A}{2}.cos\frac{A}{2}$
2	$sin2A = \frac{2tanA}{1+\tan^2 A}$	2	$sinA = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$
3	$sin2A = \frac{2cotA}{1+\cot^2 A}$	3	$sinA = \frac{2cot\frac{A}{2}}{1+cot^2\frac{A}{2}}$
4	$cos2A = cos^2A - sin^2A$	4	$cosA = cos^2\frac{A}{2} - sin^2\frac{A}{2}$
5	$cos2A = 2cos^2A - 1$	5	$cosA = 2cos^2\frac{A}{2} - 1$
6	$cos2A = 1 - 2sin^2A$	6	$cosA = 1 - 2sin^2\frac{A}{2}$
7	$cos2A = \frac{1-tan^2A}{1+\tan^2 A}$	7	$cosA = \frac{1-tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$
8	$cos2A = \frac{cot^2A-1}{cot^2A+1}$	8	$cosA = \frac{cot^2\frac{A}{2}-1}{cot^2\frac{A}{2}+1}$
9	$2cos^2A = 1 + cos2A$	9	$2cos^2\frac{A}{2} = 1 + cosA$
10	$2sin^2A = 1 - cos2A$	10	$2sin^2\frac{A}{2} = 1 - cosA$
11	$\tan^2 A = \frac{1-cos2A}{1+cos2A}$	11	$\tan^2\frac{A}{2} = \frac{1-cosA}{1+cosA}$

12	$cot^2A = \dfrac{1+cos2A}{1-cos2A}$	12	$cot^2\dfrac{A}{2} = \dfrac{1+cosA}{1-cosA}$
13	$tan2A = \dfrac{2tanA}{1-\tan^2 A}$	13	$tanA = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$
14	$cot2A = \dfrac{cot^2A-1}{2cotA}$	14	$cotA = \dfrac{cot^2\frac{A}{2}-1}{2\cot\frac{A}{2}}$
15	$sin3A = 3sinA - 4sin^3A$	15	$sinA = 3sin\dfrac{A}{3} - 4sin^3\dfrac{A}{3}$
16	$sin^3A = \dfrac{3sinA - sin3A}{4}$	16	$sin^3\dfrac{A}{3} = \dfrac{3sin\frac{A}{3} - sinA}{4}$
17	$cos3A = 4cos^3A - 3cosA$	17	$cosA = 4cos^3\dfrac{A}{3} - 3cos\dfrac{A}{3}$
18	$cos^3A = \dfrac{cos3A - 3cosA}{4}$	18	$cos^3\dfrac{A}{3} = \dfrac{cosA - 3cos\frac{A}{3}}{4}$
19	$tan3A = \dfrac{3tanA - \tan^3 A}{1-3\tan^2 A}$	19	$tanA = \dfrac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}$
20	$cot3A = \dfrac{cot^3A - 3cotA}{3cot^2A - 1}$	20	$cotA = \dfrac{cos^3\frac{A}{3} - 3cot\frac{A}{3}}{3cot^2\frac{A}{3} - 1}$

Example 1

If $cos\frac{\theta}{2} = \frac{3}{5}$, find the value of sinθ, cosθ and tanθ.

Solution: Here,

$$cos\frac{\theta}{2} = \frac{3}{5}$$

$$sin\frac{\theta}{2} = \sqrt{1 - Cos^2\frac{\theta}{2}} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

Now,

$$i)\ sin\theta = 2Sin\frac{\theta}{2} \cdot Cos\frac{\theta}{2} = 2 \cdot \frac{4}{5} \times \frac{3}{5} = \frac{24}{25}$$

$$ii)\ cos\theta = 2Cos^2\frac{\theta}{2} - 1 = 2 \times \left(\frac{3}{5}\right)^2 - 1 = 2 \times \frac{9}{25} - 1 = -\frac{7}{25}$$

$$iii)\ tan\theta = \frac{sin\theta}{cos\theta} = \frac{\frac{24}{25}}{-\frac{7}{25}} = -\frac{24}{7}$$

Example 2

If $sin\frac{\alpha}{3} = \frac{4}{5}$, find the value of $Sin\alpha$.

Solution: Here,

$$sin\frac{\alpha}{3} = \frac{4}{5}$$

By formula,

$$sin\alpha = 3sin\frac{\alpha}{3} - 4sin^3\frac{\alpha}{3} = 3 \times \frac{4}{5} - 4 \cdot \left(\frac{4}{5}\right)^3 = \frac{12}{5} - 4 \times \frac{64}{125}$$

$$= \frac{25 \times 12 - 256}{125} = \frac{44}{125}$$

Example 3

If $\cos 30° = \frac{\sqrt{3}}{2}$, show that $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Solution: Here,

Let, $A = 30°$

We know that, $\cos A = 1 - 2\,sin^2\frac{A}{2}$

or, $\cos 30° = 1 - 2sin^2\frac{30^0}{2}$

or, $\frac{\sqrt{3}}{2} = 1 - 2sin^2 15°$

or, $2sin^2 15° = 1 - \frac{\sqrt{3}}{2}$

or, $\quad sin^2 15° = \frac{2-\sqrt{3}}{4}$

or, $\quad sin^2 15° = \frac{2-\sqrt{3}}{4} \times \frac{2}{2}$ $\qquad$ [∵ Multiplying numerator and denominator by 2]

or, $\quad sin^2 15° = \frac{4-2\sqrt{3}}{8}$

or, $\quad sin^2 15° = \frac{3+1-2\sqrt{3}}{8}$

or, $\quad sin^2 15° = \frac{(\sqrt{3})^2+(1)^2-2\sqrt{3}.1}{8}$

or, $\quad sin^2 15° = \frac{(\sqrt{3}-1)^2}{8}$

or, $\quad sin 15° = \pm\sqrt{\frac{(\sqrt{3}-1)^2}{8}}$

or, $\quad sin 15° = \pm\frac{\sqrt{3}-1}{2\sqrt{2}}$

Since, $sin 15°$ lies in first quadrant where the value of $\sin 15°$ is +ve .

$$\therefore \sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example 4

Prove that: $\frac{\sin\theta+\sin\frac{\theta}{2}}{1+\cos\theta+\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$

Solution: Here,

$$\text{LHS} = \frac{\sin\theta+\sin\frac{\theta}{2}}{1+\cos\theta+\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{2\cdot cos^2\frac{\theta}{2}+\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}(2\cos\frac{\theta}{2}+1)}{\cos\frac{\theta}{2}(2\cos\frac{\theta}{2}+1)}$$

$$= \tan\frac{\theta}{2} \quad = \text{RHS.}$$

Example 5

Prove that: $\quad \sec\left(\frac{\pi^c}{4}+\frac{A}{2}\right)\cdot\sec\left(\frac{\pi^c}{4}-\frac{A}{2}\right) = 2\sec A$

Solution: Here,

LHS $\quad = \sec\left(\frac{\pi^c}{4}+\frac{A}{2}\right)\cdot\sec\left(\frac{\pi}{4}-\frac{A}{2}\right)$

$$= \frac{1}{\cos\left(45° + \frac{A}{2}\right)} \cdot \frac{1}{\cos(45° - \frac{A}{2})} \qquad \left[\because \frac{\pi^c}{4} = 45°\right]$$

$$= \frac{1}{(\cos 45° \cdot \cos\frac{A}{2} - \sin 45° \cdot \sin\frac{A}{2})(\cos 45° \cdot \cos\frac{A}{2} + \sin 45° \cdot \sin\frac{A}{2})}$$

$$= \frac{1}{(\frac{1}{\sqrt{2}}\cos\frac{A}{2} - \frac{1}{\sqrt{2}}\sin\frac{A}{2})(\frac{1}{\sqrt{2}}\cos\frac{A}{2} + \frac{1}{\sqrt{2}}\sin\frac{A}{2})}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}(\cos\frac{A}{2} - \sin\frac{A}{2}) \cdot \frac{1}{\sqrt{2}}(\cos\frac{A}{2} + \sin\frac{A}{2})} = \frac{1}{\frac{1}{2}(\cos^2\frac{A}{2} - \sin^2\frac{A}{2})}$$

$$= \frac{2}{\cos A} = 2\sec A = \text{RHS.}$$

Exercise 5.2

1. a) Define submultiple angle with an example.

   b) Write $\sin A$, $\cos A$ *and* $\tan A$ in terms of $\frac{A}{2}$.

   c) Write $\sin A$ in terms of $\sin\frac{A}{3}$ and $\cos A$ in terms of $\cos\frac{A}{3}$.

2. a) If $\sin\frac{\beta}{2} = \frac{1}{7}$ and $\cos\frac{\beta}{2} = \frac{7}{9}$, find the value of $\sin\beta$.

   b) If $\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$, find the value of $\cos\theta$.

   c) If $\cos\frac{\theta}{3} = \frac{1}{2}$, find the value of $\cos\theta$.

3. a) If $\sin\frac{A}{2} = \frac{1}{2}$, find the value of $\sin A$, $\cos A$ and $\tan A$.

   b) If $\cos\frac{\theta}{2} = \frac{4}{5}$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

   c) If $\tan\frac{\theta}{2} = \frac{4}{3}$, find the value of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

4. a) If $\cos\frac{\gamma}{3} = \frac{1}{2}$, find the value of $\cos\gamma$ and $\sin\gamma$.

   b) If $\sin\frac{B}{3} = \frac{1}{2}$, find the value of $\sin B$.

   c) If $\tan\frac{A}{3} = 1$, find the value of $\tan A$.

5. a) If $\sin\frac{\alpha}{2} = \frac{1}{2}(a + \frac{1}{a})$, show that $\cos\alpha = -\frac{1}{2}(a^2 + \frac{1}{a^2})$.

b) If $\cos\frac{\theta}{2} = \frac{1}{2}\left(a + \frac{1}{a}\right)$, show that $\cos\theta = \frac{1}{2}(a^2 + \frac{1}{a^2})$.

c) If $\sin\frac{\alpha}{3} = \frac{1}{2}(a + \frac{1}{a})$, show that $\sin\alpha = -\frac{1}{2}(a^3 + \frac{1}{a^3})$.

d) If $\cos\frac{\phi}{3} = \frac{1}{2}(a + \frac{1}{b})$, show that $\cos\emptyset - \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right) = 0$

6. a) If $\cos 30° = \frac{\sqrt{3}}{2}$, prove that:

$i.\ \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ $\qquad$ $ii.\ \tan 15° = 2 - \sqrt{3}$

b) If $\cos 45° = \frac{1}{\sqrt{2}}$, show that

i. $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2-\sqrt{2}}$ $\qquad$ ii. $\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{2+\sqrt{2}}$

iii. $\tan 22\frac{1}{2}° = \sqrt{3-2\sqrt{2}}$

7. Prove that:

a) $\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$ $\qquad$ b) $\frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2}$

c) $\frac{1-\cos\alpha}{1+\cos\alpha} = \tan^2\frac{\alpha}{2}$ $\qquad$ d) $\frac{1+\cos\beta}{1-cos\beta} = \cot^2\frac{\beta}{2}$

e) $1 + \sin\theta = (\sin\frac{\theta}{2} + \cos\frac{\theta}{2})^2$ $\qquad$ f) $1 - \sin A = (\sin\frac{A}{2} - \cos\frac{A}{2})^2$

g) $\frac{1+\cos\theta+\sin\theta}{1-\cos\theta+\sin\theta} = \cot\frac{\theta}{2}$ $\qquad$ h) $\frac{1-\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta} = \tan\frac{\theta}{2}$

i) $\frac{\sin^3\frac{\theta}{2}+\cos^3\frac{\theta}{2}}{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}} = 1 - \frac{1}{2}\sin\theta$ $\qquad$ j) $\frac{2\sin A+\sin 2A}{2\sin A-\sin 2A} = \cot^2\frac{A}{2}$

k) $\frac{\sin 2\alpha}{1+\cos 2\alpha} \cdot \frac{\cos\alpha}{1+\cos\alpha} = \tan\frac{\alpha}{2}$ $\qquad$ l) $\frac{\sin\beta+\sin\frac{\beta}{2}}{1+\cos\beta+\cos\frac{\beta}{2}} = \tan\frac{\beta}{2}$

j) $\frac{\sin\frac{\theta}{2}-\sqrt{1+\sin\theta}}{\sin\frac{\theta}{2}-\sqrt{1+\sin\theta}} = \tan\frac{\theta}{2}$

8. a) $\frac{2\tan\left(\frac{\pi^c}{4}-\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\pi^c}{4}-\frac{\theta}{2}\right)} = \cos\theta$ $\qquad$ b) $1 - 2\sin^2\left(\frac{\pi^c}{4} - \frac{A}{2}\right) = \sin A$

c) $\frac{1-\tan^2\left(\frac{\pi^c}{4}-\frac{\theta}{4}\right)}{1+\tan^2\left(\frac{\pi^c}{4}-\frac{\theta}{4}\right)} = \sin\frac{\theta}{2}$ d) $\cos^2\left(\frac{\pi^c}{4}-\frac{\theta}{4}\right) - \sin^2\left(\frac{\pi^c}{4}-\frac{\theta}{4}\right) = \sin\frac{\theta}{2}$

9. Prove that:

a) $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\frac{A-B}{2}$

b) $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2\frac{A-B}{2}$

10. Prove that:

a) $\tan\left(45° - \frac{A}{2}\right) = \sqrt{\frac{1-\sin A}{1+\sin A}} = \frac{\cos A}{1+\sin A}$

b) $\tan\left(45° + \frac{A}{2}\right) = \sec A + \tan A$

c) $\tan\left(\frac{\pi^c}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi^c}{4} - \frac{\theta}{2}\right) = 2\sec\theta$

d) $\cot\left(\frac{\theta}{2} + 45°\right) - \tan\left(\frac{\theta}{2} - 45°\right) = \frac{2\cos\theta}{1+\sin\theta}$

11. Prove that:

a) $\frac{\sin 2A}{1+\cos 2A} \cdot \frac{\cos A}{1+\cos A} = \tan\frac{A}{2}$

b) $\frac{1+\cos 2\theta}{\sin 2\theta} \cdot \frac{1+\cos\theta}{\cos\theta} = \cot\frac{\theta}{2}$

c) $(1 + \sin\frac{\pi^c}{8})(1 + \sin\frac{3\pi^c}{8})(1 - \sin\frac{5\pi^c}{8})(1 - \sin\frac{7\pi^c}{8}) = \frac{1}{8}$

d) $(\cos^6\frac{\theta}{4} - \sin^6\frac{\theta}{4}) = \cos\frac{\theta}{2}(1 - \frac{1}{4}\sin\frac{\theta}{2})$

12. If $2\tan\frac{A}{2} = 3\tan\frac{B}{2}$ then prove that $\cos A = \frac{13\cos B-5}{12-5\cos B}$

Transformation of trigonometric formula

The sum or difference form of Sine and Cosine ratios can be transformed into the product form and the product form can be transformed into the sum or difference form. Let's discuss about it.

i. Transformation of products into sum or difference

We have, the compound angles formulas of Trigonometric ratios as

$\sin A \cdot \cos B + \cos A \cdot \sin B = \sin(A + B)$……………….(i)

$\sin A \cdot \cos B - \cos A \cdot \sin B = \sin(A - B)$…………………(ii)

$\cos A \cdot \cos B - \sin A \cdot \sin B = \cos(A + B)$…………………(iii)

$\cos A \cdot \cos B + \sin A \cdot \sin B = \cos(A - B)$…………………(iv)

Now, adding equation (i) and (ii) we get,

$\sin A \cdot \cos B + \cos A \cdot \sin B + \sin A \cdot \cos B - \cos A \cdot \sin B = \sin(A + B) + \sin(A - B)$

$2\sin A \cdot \cos B = \sin(A + B) + \sin(A - B)$ ……………………….(v)

Subtracting equation (ii) from equation (i), we get

$\sin A \cdot \cos B + \cos A \cdot \sin B - (\sin A \cdot \cos B - \cos A \cdot \sin B) = \sin(A + B) - \sin(A - B)$

or, $\sin A \cdot \cos B + \cos A \cdot \sin B - \sin A \cdot \cos B + \cos A \cdot \sin B = \sin(A + B) - \sin(A - B)$

or, $2 \cos A \cdot \sin B = \sin(A + B) - \sin(A - B)$……………………….(vi)

Similarly, adding equation (iii) and (iv), we get

$\cos A \cdot \cos B - \sin A \cdot \sin B + \cos A \cdot \cos B + \sin A \cdot \sin B = \cos(A + B) + \cos(A - B)$

or, $2 \cos A \cdot \cos B = \cos(A + B) + \cos A - B)$……………….(vii)

Again, subtracting equation (iii) from (iv), we get

$\cos A \cdot \cos B + \sin A \cdot \sin B - (\cos A \cdot \cos B - \sin A \cdot \sin B) = \cos(A - B) - \cos(A + B)$

or, $\cos A \cdot \cos B + \sin A \cdot \sin B - \cos A \cdot \cos B + \sin A \cdot \sin B = \cos(A - B) - \cos(A + B)$

or, $2 \sin A \cdot \sin B = \cos(A - B) - \cos(A + B)$……………………(viii)

ii. Transformation of sum or difference into product

From above relation (v), (vi), (vii) and (viii), we have

$\sin(A + B) + \sin(A - B) = 2 \sin A \cdot \cos B$……………………………………(a)

$\sin(A + B) - \sin(A - B) = 2 \cos A \cdot \sin B$……………………………………(b)

$\cos(A + B) + \cos(A - B) = 2 \cos A \cdot \cos B$…………………………………(c)

$\cos(A - B) - \cos(A + B) = 2 \sin A \cdot \sin B$……………………………………(d)

Let, $A + B = C$ and $A - B = D$ then, adding them we get

$2A = C + D$

$\therefore A = \frac{C+D}{2}$

Again, subtracting A – B = D from A + B = C, we get

$2B = C - D$

$\therefore\ B = \frac{C-D}{2}$

Substituting the values of A, B, (A + B) and (A – B) in equation (a), (b), (c) & (d), we get

$\sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$ ....(ix)

$\sin C - \sin D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$ ....(x)

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$ ....(xi)

$\cos D - \cos C = 2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$ ....(xii)

Technique to remember the above formulae.

Remember!

S + S = 2SC

S – S = 2CS

C + C = 2CC

Remember!

S = sin

C = cosine

Example 1

Express the following products into sum or difference:

a) $\sin 25°\cdot\cos 15°$ b) $\cos 4A\cdot\cos 5A$ c) $2\sin 42°\cdot\sin 22°$

Solution:

a) Here,

$= \sin 25°\cdot\cos 15°$

$= \frac{1}{2}(2\sin 25°\cdot\cos 15°)$

$= \frac{1}{2}[\sin(25° + 15°) + \sin(25° - 15°)]$

$= \frac{1}{2}(sin 40° + sin 10°)$

b) Here,

$= \cos 4A \cdot \cos 5A$

$= \frac{1}{2}(2\cos 5A \cos 4A)$

$= \frac{1}{2}[\cos(5A + 4A) + \cos(5A - 4A)$

$= \frac{1}{2}(cos9A + cosA)$

c) Here,

$= 2sin42° \cdot sin22°$

$= \cos(42° - 22°) - \cos(42° + 22°)$

$= cos20° - cos64°$

Example 2

Express the following sum or difference into product form.

a) $\cos 7\theta - \cos 5\theta$ b) $\sin 25° - \sin 11°$

Solution: Here,

a) $cos7\theta - cos5\theta$

$$= 2\sin\frac{7\theta + 5\theta}{2} . \sin\frac{5\theta - 7\theta}{2}$$

$= 2sin6\theta \cdot \sin(-\theta)$

$= -2sin6\theta \cdot sin\theta$

b) $sin25° - sin11°$

$$= 2cos\frac{25° + 11°}{2} \cdot \sin\frac{25° - 11°}{2}$$

$= 2cos18° \cdot sin7°$

Example 3

Prove that: $sin\alpha + \sin(\alpha + 120°) + \sin(\alpha + 240°) = 0$

Solution: Here,

LHS = $\sin\alpha + \sin(\alpha + 120°) + \sin(\alpha + 240°)$

$$= \sin\alpha + 2\sin\frac{\alpha+120°+\alpha+240°}{2}\cos\frac{(\alpha+120°-\alpha-240°)}{2}$$

$$= \sin\alpha + 2\sin(180° + \alpha)\,.\cos(-60°)$$

$$= \sin\alpha + 2(-\sin\alpha)\cdot\cos60° \quad [\because \sin(180° + \alpha) = -\sin\alpha \text{ and } \cos(-\theta) = \cos\theta]$$

$$= \sin\alpha - 2\sin\alpha\frac{1}{2}$$

$$= \sin\alpha - \sin\alpha$$

$$= 0 = \text{RHS.}$$

Example 4

Prove that: $\cos\theta + \cos3\theta + \cos5\theta + \cos7\theta = 4\cos\theta\cdot\cos2\theta\cdot\cos4\theta$

Solution: Here,

Technique: Arrange the pair of angles with sum is 8θ.

LHS $= \cos\theta + \cos3\theta + \cos5\theta + \cos7\theta$

$$= (\cos5\theta + \cos3\theta) + \cos7\theta + \cos\theta)$$

$$= 2\cos\frac{5\theta + 3\theta}{2}\cdot\cos\frac{5\theta - 3\theta}{2} + 2\cos\frac{7\theta + \theta}{2}\cdot\cos\frac{7\theta - \theta}{2}$$

$$= 2\cos4\theta.\cos\theta + 2\cos4\theta.\cos3\theta$$

$$= 2\cos4\theta(\cos3\theta + \cos\theta)$$

$$= 2\cos4\theta\left(2\cos\frac{3\theta + \theta}{2}\cdot\cos\frac{3\theta - \theta}{2}\right)$$

$$= 4\cos\theta\cdot\cos2\theta\cdot\cos4\theta = \text{RHS}$$

Example 5

Prove that: $\dfrac{\sin2\theta+\sin3\theta+\sin4\theta}{\cos2\theta+\cos3\theta+\cos4\theta} = \tan3\theta$

Solution: Here,

LHS : $= \dfrac{\sin2\theta+\sin3\theta+\sin4\theta}{\cos2\theta+\cos3\theta+\cos4\theta}$

$$= \frac{\sin3\theta + (\text{Sin}4\theta + \sin2\theta)}{\cos3\theta + (\cos4\theta + \cos2\theta)}$$

$$= \frac{\sin3\theta+2\sin\frac{4\theta+2\theta}{2}\cdot\cos\frac{4\theta-2\theta}{2}}{\cos3\theta+2\cos\frac{4\theta+2\theta}{2}\cdot\cos\frac{4\theta-2\theta}{2}}$$

$$= \frac{\sin3\theta + 2\sin3\theta\cdot\cos\theta}{\cos3\theta + 2\cos3\theta\cdot\cos\theta}$$

$$= \frac{\sin3\theta(1 + 2\cos\theta)}{\cos3\theta(1 + 2\cos\theta)} = \tan3\theta = \text{RHS}.$$

Example 6

Prove that: $sin\theta.\sin\left(\frac{\pi^c}{3} + \theta\right).\sin\left(\frac{\pi^c}{3} - \theta\right) = \frac{1}{4}sin3\theta$

Solution: Here,

LHS: $= sin\theta\cdot\sin\left(\frac{\pi^c}{3} + \theta\right)\cdot\sin\left(\frac{\pi^c}{3} - \theta\right)$

$$= \frac{1}{2}sin\theta\cdot[2\sin(60° + \theta)\cdot\sin(60° - \theta)]$$

$$\left[\because \text{Multiplying numerator and denominator by 2 and } \frac{\pi^c}{3} = 60°\right]$$

$$= \frac{1}{2}sin\theta\ [\cos(60° + \theta - 60° + \theta)\cdot - cos(60° + \theta + 60° - \theta)\ ]$$

$$= \frac{1}{2}sin\theta[cos2\theta - cos120°]$$

$$= \frac{1}{2}sin\theta\left[cos2\theta + \frac{1}{2}\right]$$

$$= \frac{1}{2}cos2\theta\cdot sin\theta + \frac{1}{4}sin\theta$$

$$= \frac{1}{4}(2\cos2\theta\cdot\sin\theta) + \frac{1}{4}sin\theta$$

$$= \frac{1}{4}[\sin(2\theta + \theta) - \sin(2\theta - \theta)] + \frac{1}{4}sin\theta$$

$$= \frac{1}{4}[sin3\theta - sin\theta] + \frac{1}{4}\sin\theta$$

$$= \frac{1}{4}sin3\theta - \frac{1}{4}sin\theta + \frac{1}{4}sin\theta$$

$$= \frac{1}{4}sin3\theta = RHS.$$

Example 7

Prove that: $sin20°\cdot sin30°\cdot sin40°\cdot sin80° = \frac{\sqrt{3}}{16}$

Solution: Here,

LHS = $sin20°\cdot sin30°\cdot sin40°\cdot sin80°$

$$= sin20°\cdot\frac{1}{2}\cdot\frac{1}{2}(2sin80°\cdot sin40°)$$

$$= \frac{1}{4}sin20°[\cos(80° - 40°) - \cos(80° + 40°)]$$

$$= \frac{1}{4}sin20°[cos40° - cos120°]$$

$$= \frac{1}{4}sin20°\left[cos40° + \frac{1}{2}\right]$$

$$= \frac{1}{4}cos40°\cdot sin20° + \frac{1}{8}\sin 20°$$

$$= \frac{1}{8}(2cos40°\cdot sin20°) + \frac{1}{8}sin20°$$

$$= \frac{1}{8}[\sin(40° + 20°) - \sin(40° - 20°) + \frac{1}{8}sin20°$$

$$= \frac{1}{8}[sin60° - sin20°] + \frac{1}{8}sin20°$$

$$= \frac{1}{8}\left[\frac{\sqrt{3}}{2} - sin20°\right] + \frac{1}{8}sin20°$$

$$= \frac{\sqrt{3}}{16} - \frac{1}{8}sin20° + \frac{1}{8}sin20°$$

$$= \frac{\sqrt{3}}{16} = RHS.$$

Example 8

Proved that: $\frac{\cos^2 A - \sin^2 B}{sinAcosA + SinBcosB}$ = cot(A+B)

Solution: Here,

LHS $= \frac{\cos^2 \text{A} - \sin^2 \text{B}}{\text{sinAcosA + SinBcosB}}$

$$= \frac{2\cos^2 \text{A} - 2\sin^2\text{B}}{\text{2sinAcosA + 2sinBcosB}} \quad [\because \text{Multiplying numerator and denominator by 2}]$$

$$= \frac{1 + \text{cos2A} - (1 - \text{cos2B})}{\text{sin2A + sin2B}}$$

$$= \frac{1 + \cos 2A - 1 + \cos 2B}{2\sin\frac{2A+2B}{2} \cdot \cos\frac{2A-2B}{2}}$$

$$= \frac{\cos 2A + \cos 2B}{2\sin(A+B) \cdot \cos(A-B)}$$

$$= \frac{2\cos\frac{2A+2B}{2} \cdot \cos\frac{2A-2B}{2}}{2\sin(A+B) \cdot \cos(A-B)}$$

$$= \frac{2\cos(A+B) \cdot \cos(A-B)}{2\sin(A+B) \cdot \cos(A-B)}$$

$$= \frac{\cos(A+B)}{\sin(A+B)}$$

= cot (A + B) = RHS.

Example 9

Prove that: $\cos^2\theta \cdot \sin^3\theta = \frac{1}{16}(2sin\theta - sin5\theta + sin3\theta)$

Solution: Here,

RHS = $\frac{1}{16}(2sin\theta - sin5\theta + sin3\theta)$

$$= \frac{1}{16}[2sin\theta - (sin5\theta - sin3\theta)]$$

$$= \frac{1}{16}\left[2sin\theta - 2cos\frac{5\theta + 3\theta}{2} \cdot sin\frac{5\theta - 3\theta}{2}\right]$$

$$= \frac{1}{16}[2sin\theta - 2cos4\theta \cdot sin\theta]$$

$$= \frac{1}{16} \times 2sin\theta[1 - cos2(2\theta)]$$

$$= \frac{1}{8}sin\theta[1 - 1 + 2sin^2 2\theta]$$

$$= \frac{1}{8}sin\theta \times 2(sin2\theta)^2$$

$$= \frac{1}{4}sin\theta \cdot (2sin\theta\ cos\theta)^2$$

$$= \frac{1}{4} sin\theta \cdot 4sin^2\theta\ cos^2\theta$$

$$= cos^2\theta \cdot sin^3\theta\ =\ RHS$$

Example 10

Prove that: $2\cos\frac{12\pi^c}{13} \cdot \cos\frac{4\pi^c}{13} - \cos\frac{10\pi^c}{13} - \cos\frac{8\pi^c}{13} = 0$

Solution: Here,

LHS = $2\cos\frac{12\pi^c}{13} \cdot \cos\frac{4\pi^c}{13} - \cos\frac{10\pi^c}{13} - \cos\frac{8\pi^c}{13}$

$= \cos\left(\frac{12\pi^c}{13} + \frac{4\pi^c}{13}\right) + \cos\left(\frac{12\pi}{13} + \frac{4\pi}{13}\right) - \cos\frac{10\pi^c}{13} - \cos\frac{8\pi^c}{13}$

$= \cos\frac{16\pi^c}{13} + \cos\frac{8\pi^c}{13} - \cos\frac{10\pi^c}{13} - \cos\frac{8\pi^c}{13}$

$= \cos\frac{16\pi^c}{13} - \cos\frac{10\pi^c}{13}$

$= 2\sin\left(\frac{\frac{10\pi^c}{13} + \frac{16\pi^c}{13}}{2}\right) \cdot \sin\frac{\frac{10\pi^c}{13} - \frac{16\pi^c}{13}}{2}$

$= 2\sin\frac{26\pi^c}{26} \cdot \text{Sin}\left(\frac{-6\pi^c}{26}\right)$

$= 2\sin\pi^c \text{Sin}\left(\frac{-3\pi^c}{13}\right)$

$= 2 \times 0 \times \sin\left(\frac{-3\pi^c}{13}\right)$

= 0 = RHS.

Exercise 5.3

1. a. Express 2sinA sosB into the sum or difference of Sine or Cosine.
   b. Express 2sosA cosB into the sum or difference of Cosine.
   c. Reduce 2sin$x$ siny into the sum or difference of Cosine.
2. a. Express sinc + sinD into the product form of Sine and Cosine.
   b. Express $cos\alpha - cos\beta$ into the product form of Sine.
3. a. If $sin(A + B) = \frac{1}{4}\ and\ sin(A - B) = \frac{3}{4}$, find the value of 2sinA.cosB
   b. If $cos(A - B) = \frac{2}{3}\ and\ cos(A + B) = \frac{1}{3}$, find the value of 2cosA.cosB

4. Find the value of:

a. $cos70° - cos40°$ b. $cos15° - cos75°$

c. $cos105° + cos15°$ d. $sin75° sin15°$

e. $4sin105° sin15°$ f. $cos15°\ cos105°$

5. Express the following products into sum or difference form:

a. $sin61° \cdot sin43°$ b. $sin36° \cdot sin24°$

c. $cos140° \cdot sin40°$ d. $2sin48° \cdot cos32°$

e. $2sin5\theta \cdot cos2\theta$ f. $sin9\theta \cdot cos7\theta$

g. $2cos11\theta \cdot cos3\theta$ h. $2sin7\theta \cdot . sin3\theta$

6. Express the following sum or difference into product form:

a. $cos65° + cos25°$ b. $sin46° - sin20°$

c. $cos70° + cos30°$ d. $sin7\theta - sin3\theta$

e. $sin11\theta + sin3\theta$ f. $cos15\alpha + cos5\alpha$

7. Prove the following identities:

a. $sin5A + sin7A = 2sin6A \cdot cosA$

b. $cosA + cos7A = 2cos4A \cdot . cos3A$

c. $\frac{cosB - cosA}{cosA + cosB} = \tan\frac{A+B}{2} \cdot \tan\frac{A-B}{2}$

d. $\frac{sinA + sinB}{sinA - sinB} = \tan\frac{A+B}{2} \cdot cot\frac{A-B}{2}$

e. $sin4\theta \cdot cos2\theta + cos3\theta \cdot sin\theta = sin5\theta \cdot cos\theta$

f. $sin\frac{11\alpha}{4} \cdot sin\frac{\alpha}{4} + sin\frac{7\alpha}{4} \cdot sin\frac{3\alpha}{4} = sin2\alpha \cdot sin\alpha$

g. $\frac{cos10° - sin10°}{cos10° + sin10°} = tan35°$

h. $\frac{cos75° + cos15°}{sin75° - sin15°} = \sqrt{3}$

8. Prove that:

a. $sin75° + sin15° = \sqrt{\frac{3}{2}}$ b. $sin50° + sin70° = \sqrt{3}cos10°$

c. $cos40° - sin40° = \sqrt{2}sin5°$ d. $cos40° + sin40° = \sqrt{2}cos5°$

e. $sin10° + cos40° cos20°$ f. $cos56° + sin86° = \sqrt{3}cos26°$

9. Prove that:

a. $2cos(45° + \theta)\cdot cos(45° - \theta) = cos2\theta$

b. $cos(45° + \theta)\cdot sin(45° - \theta) = \frac{1}{2}(1 - sin2\theta)$

c. $sin15°\cdot cos105° = \frac{\sqrt{3}}{4}$

d. $cos45°\cdot cos15° = \frac{\sqrt{3}+1}{4}$

e. $tan70° - tan20° = 2tan50°$

f. $tan65° - tan25° = 2tan40°$

10. Prove that:

a. $\frac{\sin 2\theta + \sin 3\theta + \sin 5\theta + \sin 6\theta}{\cos 2\theta + \cos 3\theta + \cos 5\theta + \cos 6\theta} = \tan 4\theta$

b. $\frac{\sin\beta + \sin 2\beta + \sin 4\beta + \sin 5\beta}{\cos\beta + \cos 2\beta + \cos 4\beta + \cos 5\beta} = \tan 3\beta$

c. $\frac{\sin 5\phi + \sin\phi - \sin 3\phi - \sin 7\phi}{\cos\phi - \cos 3\phi - \cos 5\phi + \cos 7\phi} = \cot 2\phi$

d. $\frac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + \cos 2\theta + \cos\theta} = \tan 2\theta$

e. $\frac{\cos\theta - \cos 2\theta + \cos 3\theta}{\sin\theta - \sin 2\theta + \sin 3\theta} = \cot 2\theta$

f. $\frac{\sin 25° + \sin 20° + \sin 10° + \sin 5°}{\cos 5° + \cos 10° + \cos 20° + \cos 25°} = \tan 15°$

g. $\frac{1 - \cos 10° + \cos 40° - \cos 50°}{1 + \cos 10° - \cos 40° - \cos 50°} = \cot 20° \tan 5°$

11. Prove that:

a. $\frac{sin2A\cdot sinA + sin6A\cdot sin3A}{cos2A\cdot sinA + cos6A\cdot sin3A} = tan5A$

b. $cot2\theta = \frac{cos5\theta\cdot sin2\theta - cos4\theta\cdot sin3\theta}{sin5\theta\cdot sin2\theta - cos4\theta\cdot cos3\theta}$

c. $\frac{sin18°\cdot cos24° - cos12°\cdot sin6°}{sin24°\cdot sin6° + cos36°\cdot cos6°} = tan12°$

d. $\dfrac{sin5°·cos10° + sin15°·cos30°}{sin5° ·sin10° + sin15°·sin30°} = cot25°$

12. Prove that:

a. $cos\theta·cos(60° - \theta)·\, cos(60° + \theta) = \dfrac{1}{4}cos3\theta$

b. $sec\left(\dfrac{\pi^c}{4} + \dfrac{\alpha}{2}\right) \times\ sec\left(\dfrac{\pi^c}{4} - \dfrac{\alpha}{2}\right) = 2sec\alpha$

c. $cosec(45° + \theta) \cdot cosec(45° - \theta) = 2sec2\theta$

d. $cos^2\alpha + cos^2(\alpha + 120°) + cos^2(\alpha - 120°) = \dfrac{3}{2}$

13. Prove that:

a. $\sin 10° \sin 50° \sin 70° = \dfrac{1}{8}$

b. $8\cos 20° \cos 40° \cos 80° = 1$

c. $\cos 40° \cos 80° \cos 160° = -\dfrac{1}{8}$

d. $\sin 20°· \sin 30°· \sin 40°· \sin 80° = \dfrac{\sqrt{3}}{16}$

e. $\sin 10°· \sin 50°· \sin 60°·\sin 70° = \dfrac{\sqrt{3}}{16}$

f. $\cos 20°· \cos 40°· \cos 60°· \cos 80° = \dfrac{1}{16}$

g. $\cos 10°. \cos 30°· \cos 50°· \cos 70° = \dfrac{3}{16}$

h. $\cos 12°· \cos 24°· \cos 48°· \cos 84° = \dfrac{1}{16}$

14. Prove that:

a. $\dfrac{\sin^2\theta - \sin^2\alpha}{\sin\theta·\cos\theta - \sin\alpha·\cos\alpha} = \tan(\theta + \alpha)$

b. $\dfrac{\cos^2 A - \cos^2 B}{\sin A·\cos A - \sin B·\cos B} = \tan(B - A)$

c. $\dfrac{\sin 130° - \sin 140°}{\cos 220° + \cos 310°} = -1$

d. $\dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cdot\cos\alpha + \sin\beta\cdot\cos\beta} = \tan(\alpha - \beta)$

15. Prove that:

a. $\cos^3 x\cdot\sin^2 x = \dfrac{1}{16}(2\cos x - \cos 3x - \cos 5x)$

b. $\cos^4\theta\cdot\sin^2\theta = \dfrac{1}{32}(2 + \cos 2\theta - \cos 6\theta - 2\cos 4\theta)$

c. $16\cos^3 y = 16\cos^5 y + 2\cos y - \cos 3y - \cos 5y$

d. $2\cos\dfrac{\pi^c}{13}\cdot\cos\dfrac{9\pi^c}{13} + \cos\dfrac{3\pi^c}{13} + \cos\dfrac{5\pi^c}{13} = 0$

16. a. $If\ Sin(\alpha+\beta) = k\sin(\alpha-\beta)\,, Prove\ that\ (k-1)tan\alpha = (k+1)tan\beta$

b. $cos\theta = cos\alpha\cdot Cos\beta$, Prove that $\tan\dfrac{\theta+\alpha}{2}\cdot tan\dfrac{\theta-\alpha}{2} = \tan^2\dfrac{\beta}{2}$

5.4 Conditional trigonometric identities

Let us consider the following two identities:

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) sinA = cosB

Is identity (i) is true for every value of θ? Discuss about it.

Is identity (ii) is true for any values of A and B?

Certainly, it is not true. It is true, only when $A + B = \frac{\pi^c}{2}$. It means, any values of A and B if their sum is $\frac{\pi^c}{2}$ or 90°. So that identity (ii) is called conditional Trigonometric identity.

∴ The trigonometric identities which are true only for certain given condition is known as conditional trigonometric identities. In a triangle ABC, Sum of interior angles of triangle is 180° or π^c.

i.e. $A + B + C = \pi^c$ or $180°$

What are the relations that can be formed on A + B + C = π^c? Discuss

We have,

i. $A + B + C = \pi^c$

$A + B = \pi^c - C,\qquad B + C = \pi^c - A,\qquad C + A = \pi^c - B$

ii. $A + B + C = \pi^c$

Dividing both sides by 2, we get

$$\frac{A + B + C}{2} = \frac{\pi^c}{2}$$

a. $\left(\frac{A}{2} + \frac{B}{2}\right) = \left(\frac{\pi^c}{2} - \frac{C}{2}\right)$

b. $\left(\frac{B}{2} + \frac{C}{2}\right) = \left(\frac{\pi^c}{2} - \frac{A}{2}\right)$

c. $\left(\frac{C}{2} + \frac{A}{2}\right) = \left(\frac{\pi^c}{2} - \frac{B}{2}\right)$

iii. $A + B + C = \pi^c$

Multiplying both sides by 2, we get

$$2(A + B + C) = 2\pi^c$$

a. $2A + 2B = 2\pi^c - 2C$

b. $2B + 2C = 2\pi^c - 2A$

c. $2C + 2A = 2\pi^c - 2B$

For the above relations formed on (i), (ii) and (iii) make a list of relations with the Trigonometric Ratios Sine, Cosine and tangent and discuss in class.

Example 1

If $A + B + C = \pi^c$, Prove that; $\operatorname{Cot}\frac{A}{2} + \operatorname{Cot}\frac{B}{2} + \operatorname{Cot}\frac{C}{2} = \operatorname{Cot}\frac{A}{2} \cdot \operatorname{Cot}\frac{B}{2} \cdot \operatorname{Cot}\frac{C}{2}$

Solution: Here,

$$A + B + C = \pi^c$$

or, $\frac{A}{2} + \frac{B}{2} = \frac{\pi^c}{2} - \frac{C}{2}$ [$\because$ Dividing both sides by 2]

Taking Trigonometric Ratio cotangent on both sides, we get

$$\cot\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{\pi^c}{2} - \frac{C}{2}\right)$$

or, $\dfrac{\cot\frac{A}{2} \cdot \cot\frac{B}{2} - 1}{\cot\frac{B}{2} + \cot\frac{A}{2}} = \tan\frac{C}{2}$

or, $$\frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\frac{1}{\cot\frac{C}{2}}$$

or, $$\cot\frac{A}{2}+\cot\frac{B}{2}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}-\cot\frac{C}{2}$$

or, $$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$$

Example 2

If A + B + C = π^c then prove that: cos2A – cos2B – cos2C = 4cosA sinB sinC – 1

Solution: Here,

$A + B + C = \pi^c$

or, $A + B = \pi^c - C$

Taking, sine ratio on both sides, we get $\text{Sin}\,(A+B) = \text{Sin}\,(\pi^c - C) = \text{SinC}$

LHS = $\cos 2A - \cos 2B - \cos 2C$

$$= 2\sin\frac{2A+2B}{2}\cdot\sin\frac{2B-2A}{2} - \cos 2C$$

$$= 2\sin\,(A+B)\ \text{Sin} - (A-B) - \cos 2C$$

$$= -2\sin C\,\sin(A-B) - 1 + 2\sin^2 C$$

$$= 2\sin C\,[-\sin(A-B) + \sin C] - 1$$

$$= 2\sin C\,[-\sin(A-B) + \sin(A+B)] - 1$$

$$= 2\sin C\,[-\sin A\cos B + \cos A\sin B + \sin A\cos B + \cos A\sin B] - 1$$

$$= 2\sin C\,[2\cos A\sin B] - 1$$

$$= 4\cos A\,\sin B\,\sin C - 1 = \text{RHS}$$

Example 3

If A + B + C = 180°, Prove that: sinA – sinB + sinC = $4\sin\frac{A}{2}\,.\cos\frac{B}{2}\,.\sin\frac{C}{2}$

Solution: Here

A + B + C = 180°

or, A + C = 180° – B [∵The term contain angle B is –ve so doing it minus from 180°]

Dividing both sides by 2, we get

$$\frac{A}{2}+\frac{C}{2}=90^\circ-\frac{B}{2}$$

Operating Sin and Cos ratios on both sides, we get

$$sin\ \left(\frac{A}{2}+\frac{C}{2}\right) = sin\ \left(90° - \frac{B}{2}\right) = cos\frac{B}{2}$$

$$cos\ \left(\frac{A}{2}+\frac{C}{2}\right) = cos\ \left(90° - \frac{B}{2}\right) = sin\frac{B}{2}$$

LHS $= sinA - sinB + sinC$

$$= 2sin\frac{A+C}{2}\cdot cos\frac{A-C}{2} - sinB$$

$$= 2\ cos\frac{B}{2}\cdot cos\frac{A-C}{2} - 2sin\frac{B}{2}\cdot Cos\frac{B}{2}$$

$$= 2\ cos\frac{B}{2}\left[\ cos\ \left(\frac{A}{2}-\frac{C}{2}\right) - cos\left(\frac{A}{2}+\frac{C}{2}\right)\right]$$

$$= 2cos\frac{B}{2}\left[2sin\frac{A}{2}\cdot sin\frac{C}{2}\right]$$

$= 4sin\frac{A}{2}\cdot cos\frac{B}{2}\cdot sin\frac{C}{2}$ = RHS

Example 4

If A + B + C = π^c then prove that: $\cos^2A - \cos^2B - \cos^2C$ = 2cosA ·sinB ·sinC -1

Solution: Here,

$$\mathrm{A + B + C} = \Pi^{\mathrm{C}}$$

$$\mathrm{A + B} = \Pi^{\mathrm{C}} - \mathrm{C}$$

Taking, sine on both sides, we get

$$sin(A+B) = sin(\pi^c - C) = sinC$$

LHS

$$cos^2A - cos^2B - cos^2C$$

$$\frac{1+cos2A}{2} - \frac{1+cos2B}{2} - cos^2C$$

$$= \frac{1}{2}[1 + cos2A - 1 - cos2B] - cos^2C$$

$$= \frac{1}{2}[cos2A - cos2B] - cos^2C$$

$$= \frac{1}{2}\left[2sin\frac{2B+2A}{2}\cdot sin\frac{2B-2A}{2}\right] - cos^2C$$

$= sinC \cdot sin(B - A) - 1 + sin^2C$

$= sinC\ [-sin(A - B) + sinC] - 1 \qquad [\because sin(-\theta) = -sin\theta]$

$= sinC\ [-sin(A - B) + sin(A + B)] - 1$

$= sinC\ [-sinA\ cosB + cosA\ sinB + sinA\ cosB + cosA\ sinB] - 1$

$= 2cosA\ sinB\ sinC - 1 \quad RHS$

Example 5

If A + B + C = 180° then prove that: $sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2} = 1 - 2sin\frac{A}{2} \cdot sin\frac{B}{2} \cdot sin\frac{C}{2}$

Solution: Here,

$\mathrm{A + B + C = 180°}$

$or, \quad A + B = 180° - C$

$or, \quad \left(\frac{A}{2} + \frac{B}{2}\right) = 90° - \frac{C}{2} \qquad [\because \text{Dividing both sides by 2 }]$

Operating, cosine ratio on both sides, we get

$cos\left(\frac{A}{2} + \frac{B}{2}\right) = cos\left(90° - \frac{C}{2}\right) = sin\frac{C}{2}$

LHS

$= sin^2\frac{A}{2} + sin^2\frac{B}{2} + sin^2\frac{C}{2}$

$= \frac{1 - cosA}{2} + \frac{1 - cosB}{2} + sin^2\frac{C}{2}$

$= \frac{1}{2}[1 - cosA + 1 - cosB] + sin^2\frac{C}{2}$

$= \frac{1}{2} \times 2 - \frac{1}{2}\ (cosA + cosB) + sin^2\frac{C}{2}$

$= 1 - \frac{1}{2} \times 2\ cos\frac{A + B}{2} \cdot cos\frac{A - B}{2} + sin^2\frac{C}{2}$

$= 1 - sin\frac{C}{2} \cdot cos\left(\frac{A}{2} - \frac{B}{2}\right) + sin^2\frac{C}{2}$

$= 1 - sin\frac{C}{2}\left[cos\ \left(\frac{A}{2} - \frac{B}{2}\right) - sin\frac{C}{2}\right]$

$= 1 - sin\frac{C}{2}\left[cos\ \left(\frac{A}{2} - \frac{B}{2}\right) - cos\left(\frac{A}{2} + \frac{B}{2}\right)\right]$

$$= 1 - \sin\frac{C}{2}\left[2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\right] \quad [\because \cos(A-B) - \cos(A+B) = 2\sin A\sin B]$$

$$= 1 - 2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2} \quad = \quad \text{RHS}$$

Example 6

If $A + B + C = \pi^c$ then prove that

$$cos\frac{A}{2} + cos\frac{B}{2} + cos\frac{C}{2} = 4cos\left(\frac{\pi^c - C}{4}\right)\cdot cos\left(\frac{\pi^C - B}{4}\right)\cdot cos\left(\frac{\pi^c - A}{4}\right)$$

$$= 4cos\left(\frac{B+C}{4}\right)\ cos\left(\frac{C+A}{4}\right)\ cos\left(\frac{A+B}{4}\right)$$

Solution: Here,

$A + B + C = \pi^c$

$A + B = \pi^c - C, \qquad B + C = \pi^c - A, \qquad C + A = \pi^c - B$

LHS =

First expression

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

$$= 2cos\left(\frac{\frac{A}{2}+\frac{B}{2}}{2}\right)\cdot cos\left(\frac{\frac{A}{2}-\frac{B}{2}}{2}\right) + cos\frac{C}{2} + cos\frac{\pi^c}{2} \qquad \left[\because cos\frac{\pi^c}{2} = 0\right]$$

$$= 2cos\left(\frac{A+B}{4}\right)\cdot cos\left(\frac{A-B}{2}\right) + 2cos\left(\frac{\frac{C}{2}+\frac{\pi^c}{2}}{2}\right)\cdot cos\left(\frac{\frac{C}{2}-\frac{\pi^c}{2}}{2}\right)$$

$$= 2cos\left(\frac{A+B}{4}\right)\cdot cos\left(\frac{A-B}{4}\right) + 2cos\left(\frac{C+\pi^c}{4}\right)\cdot cos\left(\frac{\pi^c - C}{4}\right) \left[\because \cos\left(\frac{C-\pi^c}{4}\right) = \cos\frac{\pi^c - c}{4}\right]$$

$$= 2cos\left(\frac{\pi^c - C}{4}\right)\cdot cos\left(\frac{A-B}{4}\right) + 2cos\left(\frac{C+\pi^c}{4}\right)\cdot cos\left(\frac{\pi^c - C}{4}\right)$$

$$= 2cos\left(\frac{\pi^c - C}{4}\right)\left[cos\left(\frac{A-B}{4}\right) + cos\left(\frac{\pi^c + C}{4}\right)\right]$$

$$= 2cos\left(\frac{\pi^c - C}{4}\right)\left[2cos\left(\frac{\frac{A-B}{4}+\frac{\pi^c+C}{4}}{2}\right)\cdot cos\left(\frac{\frac{A-B}{4}-\frac{\pi^c+C}{4}}{2}\right)\right]$$

$$= 2cos\frac{\pi^c - C}{4}\left[2cos\left(\frac{A-B+\pi^c+C}{8}\right)\ cos\left(\frac{A-B-\pi^c-C}{8}\right)\right]$$

$= 2cos\frac{\pi^c - C}{4}\left[2cos\left(\frac{A - B + A + B + C + C}{8}\right) \cdot cos\left(\frac{A - B - A - B - C - C}{8}\right)\right]$

$= 4cos\frac{\pi^c - C}{4} \cdot cos\left(2\frac{A+C}{8}\right) \cdot cos\left(2\frac{B+C}{8}\right)$ [∵ cos(−θ) = cosθ]

$= 4cos\left(\frac{\pi^c - C}{4}\right) \cdot cos\left(\frac{\pi^c - B}{4}\right) \cdot cos\left(\frac{\pi^c - A}{4}\right)$ (Second expression)

$= 4cos\frac{A+B}{4}\ cos\frac{C+A}{4}\ cos\frac{B+C}{4}$

$= 4cos\frac{B+C}{4} \cdot cos\frac{C+A}{4} \cdot cos\frac{A+B}{4}$ (Third expression)

Exercise 5.4

1. a. Define conditional trigonometric identities with example.

   b. What is the true condition for the identity tanA = CotB?

   c. Write any three relations which can be formed from A + B + C = π^c.

2. If $A + B + C = \pi^c$, prove that:

   a. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

   b. $\tan\frac{A}{2} \cdot \tan\frac{B}{2} + \tan\frac{B}{2} \cdot \tan\frac{C}{2} + \tan\frac{C}{2} \cdot \tan\frac{A}{2} = 1$

   c. $\tan 2A + \tan 2B + \tan 2C = \tan 2A\,.\tan 2B\,.\tan 2C$

   d. $\cot A \cot B + \cot B \cot C + \cot C \cot A - 1 = 0$

   e. $\cot 2A \cdot \cot 2B + \cot 2B \cdot \cot 2C + \cot 2C \cdot \cot 2A = 1$

3. If A, B, and C are the vertices of ΔABC then prove that:

   a. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

   b. $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

   c. $\sin A - \sin B - \sin C = -4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

   d. $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \sin\frac{C}{2}$

   e. $-\cos A + \cos B + \cos C = 4\sin\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2} - 1$

4. If A + B + C = π^c, prove that:

   a. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

   b. $\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C$

c. $\sin 2A - \sin 2B - \sin 2C = -4\sin A \cos B \cos C$

d. $\cos 2A - \cos 2B + \cos 2C = 1 - 4\sin A \cos B \sin C$

e. $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$

5. If A+B+C = π^c, prove that:

a. $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4 \sin A \sin B \sin C$

b. $\cos(B + C - A) + \cos(C + A - B) + \cos(A + B - C) = 4\cos A \cos B \cos C + 1$

c. $\dfrac{\cos A}{\sin B \sin C} + \dfrac{\cos B}{\sin C \sin A} + \dfrac{\cos C}{\sin A \sin B} = 2$

d. $\dfrac{\sin A}{\cos B \cos C} + \dfrac{\sin B}{\cos C \cos A} + \dfrac{\sin C}{\cos A \cos B} = 2\tan A\,.\tan B.\tan C$

6. If A + B + C = π^c, prove that:

a. $\cos^2 A + \mathrm{Cos}^2 B + \mathrm{Cos}^2 C = 1 - 2\mathrm{Cos}A\,\mathrm{Cos}B\,\mathrm{Cos}C$

b. $\cos^2 A + \mathrm{Cos}^2 B - \mathrm{Sin}^2 C = -2\mathrm{Cos}A\,\mathrm{Cos}B\,\mathrm{Cos}C$

c. $\sin^2 A + \mathrm{Sin}^2 B + \mathrm{Sin}^2 C = 2(1 + \mathrm{Cos}A\,\mathrm{Cos}B\,\mathrm{Cos}C)$

d. $\sin^2 A - \mathrm{Sin}^2 B - \mathrm{Sin}^2 C = -2\cos A\,\mathrm{Sin}\,B \sin C$

e. $\sin^2 A - \mathrm{Sin}^2 B + \mathrm{Sin}^2 C = 2\mathrm{Sin}A\,\mathrm{Cos}B\,\mathrm{Sin}C$

7. If $\alpha + \beta + \gamma = 180°$, prove that:

a. $\sin^2\dfrac{\alpha}{2} - \sin^2\dfrac{\beta}{2} + \sin^2\dfrac{\gamma}{2} = 1 - 2\cos\dfrac{\alpha}{2} \cdot \sin\dfrac{\beta}{2} \cdot \cos\dfrac{\gamma}{2}$

b. $\sin^2\dfrac{\alpha}{2} + \sin^2\dfrac{\beta}{2} - \sin^2\dfrac{\gamma}{2} = 1 - 2\cos\dfrac{\alpha}{2}\,\cos\dfrac{\beta}{2}\,\sin\dfrac{\gamma}{2}$

c. $-\cos^2\dfrac{\alpha}{2} + \cos^2\dfrac{\beta}{2} + \cos^2\dfrac{\gamma}{2} = 2\mathrm{Sin}\dfrac{\alpha}{2} \cdot \mathrm{Cos}\dfrac{\beta}{2} \cdot \mathrm{Cos}\dfrac{\gamma}{2}$

d. $\mathrm{Cos}^2\dfrac{\alpha}{2} + \mathrm{Cos}^2\dfrac{\beta}{2} - \mathrm{Cos}^2\dfrac{\gamma}{2} = 2\mathrm{sos}\dfrac{\alpha}{2} \cdot \cos\dfrac{\beta}{2} \cdot \sin\dfrac{\gamma}{2}$

8. If A + B +C = π^c, proves that:

a. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{4\cos\frac{A}{2}\,.\cos\frac{B}{2}\,.\cos\frac{C}{2}} = 8\sin\dfrac{A}{2} \cdot \sin\dfrac{B}{2} \cdot \sin\dfrac{C}{2}$

b. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\dfrac{A}{2} \cdot \sin\dfrac{B}{2} \cdot \sin\dfrac{C}{2}$

c. $\dfrac{\sin A + \sin B - \sin C}{\sin A \cos B} = \sec\dfrac{A}{2} \cdot \sec\dfrac{B}{2} \cdot \cos\dfrac{C}{2}$

d. $\dfrac{\sin^2 A + \sin^2 B - \sin^2 C}{\sin A \sin B \sin C} = 2\cot C$

9. If X + Y + Z = 180°, prove that:

a. sinX.cosYcosZ + sinYcosZcosX + sinZcosXcosY = sinXsinYsinZ

b. $\cos X \sin Y \sin Z + \cos Y \sin Z \sin X + \cos Z \sin X \operatorname{Sin} Y - \cos X \cos Y \cos Z = 1$

10. If $A + B + C = \pi^C$, prove that:

$$a.\ \ sin\frac{A}{2} + Sin\frac{B}{2} + Sin\frac{C}{2} = 1 + 4Sin\left(\frac{A+B}{4}\right) \cdot Sin\left(\frac{B+C}{4}\right) \cdot Sin\left(\frac{C+A}{4}\right)$$

$$b.\ \ cos\frac{A}{2} - cos\frac{B}{2} + cos\frac{C}{2} = 4cos\frac{\pi^c + A}{4} \cdot cos\frac{\pi^c - B}{4} \cdot cos\frac{\pi^c + C}{4}$$

$$c.\ \ sinA + sinB + sinC = 4sin\frac{B+C}{2} \cdot sin\frac{C+A}{2} \cdot sin\frac{A+B}{2}$$

$$d.\ \ cosA + cosB + cosC = 1 + 4cos\frac{\pi^c - A}{2} \cdot cos\frac{\pi^c - B}{2} \cdot cos\frac{\pi^c - C}{2}$$

5.5 Trigonometric equations

Before defining a trigonometric equation and its solution, let us observe the following relation:

i. $sin\ \theta = \frac{1}{2}$

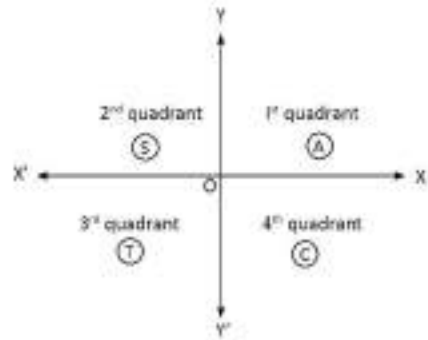
ii. $sin\ \theta + cos\ \theta = 1$

iii. $2cos^2 x + sin\ x = 2$

Can we say all the above equations are trigonometric equations? Why? Is there any difference between equation (i) and (ii)? Discuss in your group.

∴ An equation containing the trigonometric ratios of an unknown angle, is known as a trigonometric equation. The value of an unknown angle which satisfies the given trigonometric equation is known as its root or solution.

Let us observe the following figure.



- What are the angles range in each quadrant?
- Which trigonometric ratios are positive in 1st quadrant?
- Which trigonometric ratios are positive and which are negative in 2nd quadrant?
- Similarly, discuss in 3rd quadrant and 4th quadrant.

The standard angles 0°, 30°, 45°, 60° lies in 1st quadrant. What are their values with Sine, cosine and tangent? Write them in tabular form.

Steps to find the angles

1. At first we determine the quadrant where the angle falls for this use the CAST rule.
2. Find the least positive angle of trigonometric function in the first quadrant for the given relation.

   For example:

   i. If $\cos\theta = \frac{\sqrt{3}}{2}$ the least positive angle in the first quadrant is 30°.

   ii. If $\tan\theta = \sqrt{3}$, the least positive angle in the first quadrant for $\tan\theta = \sqrt{3}$ is 60°.

   iii. If $\sin\theta = -1$ the least positive angle in the first quadrant for $\sin\theta$ is 90°.
3. When 'θ' is the least positive angle in the first quadrant then

   i) The angle in the second quadrant $=180^{\circ} - \theta$

   ii) The angle in the 3rd quadrant $= 180^{\circ} + \theta$

   iii) The angle in the fourth quadrant $= 360^{\circ} - \theta$
4. To find the angle more than 360° we add or subtract the least positive angles of 1st quadrant in even multiple of 90°.

   What are the smallest and greatest values of $\sin\theta$ and $\cos\theta$? Discuss.

   If $\alpha \le \sin\theta \le \beta$ then what are the values of α and β. Are $\alpha = -1$ and $\beta = 1$? Find it.

Example 1

Solve: $\sqrt{3}\tan\theta = 3$ $(0° \leq \theta \leq 90°)$

Solution: Here,

$\sqrt{3}\tan\theta = 3$

or, $\tan\theta = \frac{3}{\sqrt{3}}$

or, $\tan\theta = \sqrt{3}$

Since $\tan\theta$ is positive, so it lies in 1st and 3rd quadrants

Now, in 1st quadrant, $\tan\theta = \tan 60^{\circ}$

$\theta = 60^{\circ}$

But, in third quadrant it is out of range

Hence, $\theta = 60^{\circ}$

Example 2

Solve for $\boldsymbol{\theta}$: $4cos^2\theta - 1 = 0$ $(0° \leq \theta \leq 90°)$

Solution: Here,

$4cos^2\theta - 1 = 0$

or, $4cos^2\theta = 1$

or, $cos^2\theta = \frac{1}{4}$

or, $\cos\theta = \pm\sqrt{\frac{1}{4}}$

or, $\cos\theta = \pm\frac{1}{2}$

Taking positive sign, $\cos\theta = \frac{1}{2}$

or, $\cos\theta = \cos 60°$

$\theta = 60°$

Taking negative sign, $\cos\theta = -\frac{1}{2}$

$\cos\theta = \cos(180° - 60°),\ \cos(360° - 60°)$

$\theta = 120°, 300°$ but $0° \leq \theta \leq 90°)$

Hence, $\theta = 60°$

Example 3

Solve for x: $(8\sin x + 4)(2\cos x + 1) = 0 \ \ (0° \le x \le 180°)$

Solution: Here,

$(8\sin x + 4)(2\cos x + 1) = 0$

Either, $8\sin x + 4 = 0$ ……… (i) or, $2\cos x + 1 = 0$ ……… (ii)

From equation (i), $8\sin x = -4$

$\sin x = -\frac{4}{8}$

or, $\sin x = -\frac{1}{2}$

Least positive angle (P.A.) $= 30°$. Since the value sin$x$ is (-ve), so it lies in 3rd and 4th quadrant.

$\therefore \sin x = \sin(180° + 30°), \sin(360° - 30°)$

$x = 210°, 330°$

But, $(0° \le x \le 180°)$

From equation (ii) $2\cos x + 1 = 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$

Least positive angle $= 60°$, since, $\cos x$ is (-ve) so, it lies in 2nd and 3rd quadrant.

$\therefore$ cos$x$ = cos(180° – 60°), cos (180° + 60°)

$x = 120°, 240°$

But $x = 240°$ (out of range)

Hence, $x = 120°$

Example 4

Solve: $tan^2x - 3\sec x + 3 = 0 \qquad (0° \le \theta \le 360°)$

Solution: Here,

$tan^2x - 3\sec x + 3 = 0$

or, $sec^2x - 1 - 3\sec x + 3 = 0$

or, $sec^2x - 3\sec x + 2 = 0$

or, $sec^2x - (2+1)\sec x + 2 = 0$

or, $sec^2x - 2\sec x - \sec x + 2 = 0$

or, $\sec x(\sec x - 2) - 1(\sec x - 2) = 0$

or, $(\sec x - 2)(\sec x - 1) = 0$

Either, $\sec x - 1 = 0$ …………. (i)

or, $\sec x - 2 = 0$ …………. (ii)

From equation (i) $\quad \sec x = 1$

or, $\quad \cos x = 1$

Since $\cos x$ is positive, so it lies in 1st and 4th quadrant

Least Positive angle (P.A) $= 0°$

$\therefore \cos x = \cos 0°, \cos(360° - 0°)$

$x = 0°, 360°$

From, equation (ii) $\quad \sec x - 2 = 0$

or, $\quad \sec x = 2$

$\cos x = \frac{1}{2}$

Least Positive angle = 60°. Since $\cos x$ is (+ve), so it lies in 1st and 4th quadrant.

$\therefore$ cos$x$ = cos60°, cos(360° – 60°)

Hence, x = 0°, 60°, 300°, 360°

Example 5

Solve: $\sqrt{3}\sin\alpha - \cos\alpha = \sqrt{2} \qquad (0° \le \alpha \le 360°)$

Solution: Here,

$\sqrt{3}\sin\alpha - \cos\alpha = \sqrt{2}$ ……………………..(i)

Now,

$\sqrt{(Coeff.of\ \sin\alpha)^2 + (Coeff.of\ \cos\alpha)^2}$

$= \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= \sqrt{3+1}$

$= 2$

Dividing equation (i) on both sides by 2, we get

or, $\frac{\sqrt{3}}{2}\sin\alpha - \frac{1}{2}\cos\alpha = \frac{\sqrt{2}}{2}$

or, $\sin\alpha.\cos 30° - \cos\alpha.\sin 30° = \frac{1}{\sqrt{2}}$

or, $\sin(\alpha - 30°) = \frac{1}{\sqrt{2}}$

Least Positive angle (P.A.) = 45°, since sin is (+ve) so it lies in 1st and 2nd quadrant.

Now,

In 1st quadrant: $\sin(\alpha - 30°) = \sin 45°$

or, $\alpha - 30° = 45°$

or, $\alpha = 45° + 30°$

$\alpha = 75°$

In 2nd quadrant: $\sin(\alpha - 30°) = \sin(180° - 45°)$

$\alpha - 30° = 135°$

$\alpha = 135° + 30°$

$\alpha = 165°$

Hence, $\alpha = 75°, 165°$

Example 6

Solve: $\cos 3\theta - \sin\theta - \cos 5\theta = 0$ $(0° \le \theta \le 180°)$

Solution: Here,

$\cos 3\theta - \sin\theta - \cos 5\theta = 0$

or, $(\cos 3\theta - \cos 5\theta) - \sin\theta = 0$

or, $2\sin\frac{5\theta+3\theta}{2}\cdot\sin\frac{5\theta-3\theta}{2} - \sin\theta = 0$

or, $2\sin 4\theta\cdot\sin\theta - \sin\theta = 0$

or, $\sin\theta(2\sin 4\theta - 1) = 0$

Either, $\sin\theta = 0$ ....(i)

or, $2\sin 4\theta - 1 = 0$ ....(ii)

From equation (i), $\sin\theta = 0$

$\sin\theta = \sin 0°,\ \sin 180°$

$\theta = 0°, 180°$

From equation (ii), $2\sin 4\theta - 1 = 0$

or, $\sin 4\theta = \frac{1}{2}$

$\text{Least Positive angle} = 30°$

$\text{Since, Sin is (+ve) so it lies in 1st and 2nd quadrant.}$

In 1st quadrant, $\sin 4\theta = \sin 30°$

$4\theta = 30°$

$\theta = \frac{30}{4} \qquad = 7.5°$

In 2nd quadrant, $\sin 4\theta = \sin(180° - 30°)$

$4\theta = 180° - 30°$

$4\theta = 150°$

$\theta = \frac{150°}{4} \qquad = 37.5°$

In 5th quadrant, $\sin 4\theta = \sin(4 \times 90° + 30°)$

or, $4\theta = 390°$

or, $\theta = 97.5°$

In 6th quadrant, $\sin 4\theta = \sin(6 \times 90° - 30°)$

or, $4\theta = 510°$

$\theta = 127.5°$

Hence, $\theta = 0°, 7.5°, 37.5°, 97.5°, 127.5°, 180°$

Example 7

Find the minimum value of (C + D), if tanC + tanD = 2 and cosC.cosD = $\frac{1}{2}$

Solution: Here,

tanC + tanD = 2

or, $\frac{sinC}{cosC} + \frac{sinD}{cosD}$ = 2

or, sinC·cosD + cosC·sinD = 2cosC·cosD

or, sin(C + D) = 2cosC·cosD … (i)

and 2cosC·cosD = 1 ………. (ii)

From equation (i) and (ii), ,we get

sin(C + D) = 1

Least positive angle = 90°

sin(C + D) = sing90°

C + D = 90°

∴ The minimum value of (C + D) = 90°

Exercise 5.5

1. a) Define trigonometric equation with example.

   b) What do you mean by root (solution) of the given trigonometric equation?

2. a) If $\sec\theta = -2$, what is the least positive angle in 1st quadrant.

   b) How to find the angle in 4th quadrant, if the least positive angle(θ) is given?

   c) What are the minimum and maximum values of $\sin\theta$ and $\cos\theta$?

3. Solve: $(0° \le \theta \le 90°)$

   a) $\sin\theta = \frac{\sqrt{3}}{2}$ b) $\cos\theta = \frac{1}{2}$ c) $\sqrt{3}\cot\theta = 1$ d) $\tan\theta - 1 = 0$

   e) $2\sin\theta - 1 = 0$ f) $\sin\theta = 1$ g) $\cos\theta - \frac{1}{\sqrt{2}} = 0$ h) $\sec\theta = 2$

4. Solve : $(0° \le \theta \le 180°)$

   a) $2cos\theta + 1 = 0$ b) $\sqrt{2}sec\theta + 2 = 0$ c) $2sin\theta - \sqrt{3} = 0$

   d) $3cot^2\theta - \sqrt{3} = 0$ e) $\sqrt{3}cosec\theta - 2 = 0$ f) $\sqrt{3}tan\theta + 1 = 0$

5. Solve : $(0° \le \alpha, \theta \le 180°)$

   a) $2\sin^2\alpha - 1 = 0$ b) $4\sin\alpha = 3\text{cosec}\alpha$ c) $\tan^2\alpha - 1 = 2$

   d) $3cot^2\theta - \sqrt{3} = 0$ e) $4\sin^2\alpha = \tan^2 60°$ f) $4\cos^2\alpha - 1 = \tan 0°$

6. Solve: $(0° \le \theta \le 180°)$

   a) $2\cos^2\theta = -\sqrt{3}cos\theta$ b) $2\cos^2\theta = 3sin\theta$ c) $cosec\theta - 2sin\theta = 1$

   d) $\cos^2\frac{\theta}{2} - cos\frac{\theta}{2} + \frac{1}{4} = 0$ e) $cos\theta(2sin\theta - 1) = 0$ f) $sin2\theta = sin\theta$

   g) $sin3\theta = cos6\theta$ h) $cot5\theta = tan\theta$

7. Solve:- $(0° \leq x \leq 360°)$

a) $3\sin^2 x + 4cosx = 4$

b) $\cos^2 x = 3sin^2x + 4cosx$

c) $tanx + cotx = 2$

d) $tanx - sinx = 0$

e) $secx. tanx = \sqrt{2}$

f) $\cot^2 x + cosec^2x = 3$

g) $\left(1-\sqrt{3}\right)tanx + 1 + \sqrt{3} = \sqrt{3}sec^2\theta$

i) $\cot^2 x + \left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)cotx + 1 = 0$

j) $\tan^2 x + \left(1-\sqrt{3}\right)tanx - \sqrt{3} = 0$

k) $2sinx + cotx - cosecx = 0$

8. Solve: $(0° \leq \theta \leq 360°)$

a) $\sqrt{3}Sin\theta + cos\theta = 1$

b) $Sin\theta + \sqrt{3}cos\theta = 1$

c) $sin\theta + cos\theta = \sqrt{2}$

d) $cos\theta + \frac{1}{\sqrt{3}}sin\theta = 1$

e) $sin\theta + cos\theta = \frac{1}{\sqrt{2}}$

f) $cosx + \sqrt{3}Sinx = 2$

g) $tan\theta + \sqrt{3}sec\theta = \sqrt{3}$

h) $cosec\theta + cot\theta = \sqrt{3}$

i) $\sqrt{3}tan\theta + 1 = sec\theta$

9. Solve:- $(0° \leq \theta \leq 180°)$

a) $sin4\theta + sin2\theta = 0$

b) $sin3\theta + sin2\theta = sin\theta$

c) $cos3\theta + cos\theta = cos2\theta$

d) $cos\theta + cos3\theta = -cos5\theta$

e) $cos3\theta + cos\theta = 2cos\theta$

f) $sin2\theta + sin4\theta = cos\theta + cos3\theta$

g) $cos\theta + sin\theta = cos2\theta + sin2\theta$

10. Solve : $\frac{\sqrt{3}}{sin2\alpha} + \frac{1}{cos2\alpha} = 4$ $[0° \leq \alpha \leq 90°]$

11. a) If $2sinxsiny = \frac{\sqrt{3}}{2}, \frac{1}{tanx} + \frac{1}{tany} = 2$, find the minimum value of x + y.

b) If $2cosxsiny = \frac{1}{\sqrt{2}}(x > y), tanx + coty = 2, find\ x - y$. [o° ≤ (x – y) ≤ 360°]

5.6 Height and distance

Let's observe the following figures and discuss:

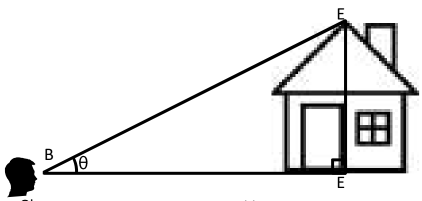


Figure (I)

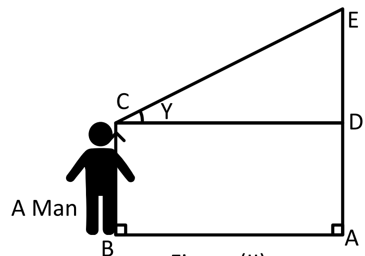


Figure (II)

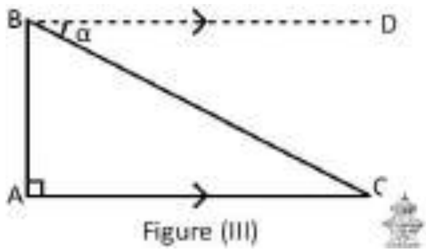


Figure (III)

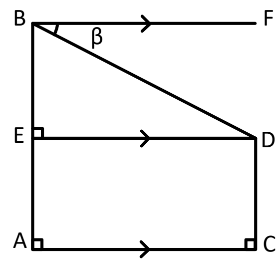


Figure (IV)

i) What do θ, γ, α *and* β stand for?

ii) What do AB and AC called in figure (iii)?

iii) Are there any similarity between θ, γ, α *and* β ?

iv) What is the relation between AB and CD in Figure (II)?

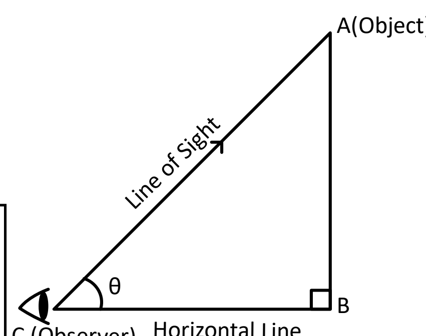
v) What is the relation between $\angle FBD$ and $\angle BDE$ in figure (IV)?

From above figures, θ *and* γ are called angle of elevation and α *and* β are called angle of depression.

Let's define angle of elevation and angle of depression

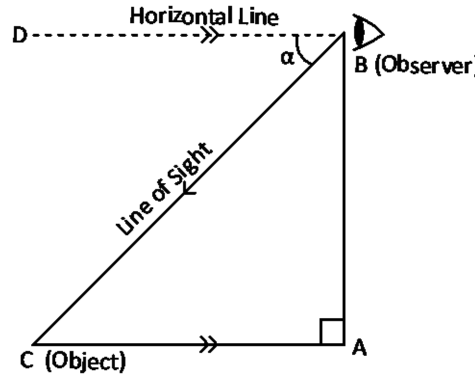
Angle of elevation: - In the adjoining figure, C be the position of observer and A be the position of an object. CA be the line of sight or line of observation. BC be the horizontal line through the observation point C. Then $\angle ACB = \theta$ is said to be angle of elevation.

When an observer observes an object lying above the horizontal line (eye level), the angle formed by the line of sight with the horizontal line is called an angle of elevation. Angle of elevation is also called altitude of A.



Angle of depression

In the adjoining figure, 'B' be the position of an eye of an observer and 'C' be the position of an object. BC be the line of sight or line of observation. BD be the horizontal line through D, which is parallel to the horizontal line AC. C lies below the position of an eye, then ∠DBC is said to be the angle of depression.



When an observer observes an object lying below the horizontal line (eye level), the angle formed by the line of sight with the horizontal line parallel to the ground is called an angle of depression.

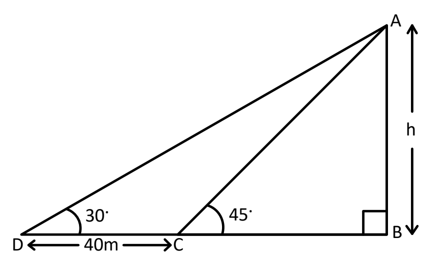
When the actual measurement of the height of an object or distance between two points (object) is not easy or even not possible, as an application of the trigonometry, a technique is used to find them with the help of the angle/angles subtended at a point by the object/objects whose distance or height is to be determined. The instruments like theodolites or clinometer are used to measure the angle. This method is mostly used in surveying, map making, aviation and astronomy etc.

Example 1

Find the height of a building, when it is found that on walking towards it 40m in a horizontal line through its base the angular elevation of its top changes from $30°$ to $45°$.

Solution: Here,

Let AC = h be the height of building. Let, D and C be the position of two points such that CD = 40m. The angle of elevation from point C and D to the points A are $45°$ and $30°$ respectively.



$$\therefore \angle ACB = 45° \text{and} \ \angle ADB = 30°$$

To find: Height of building (AB)

Now, in right angled triangle ABC,

$\tan 45° = \dfrac{AB}{BC}$

Or, $1 = \dfrac{h}{BC}$

BC = h ………………………… (i)

Similarly, in rt. Angled ΔABD,

$\tan 30^\circ = \frac{\text{AB}}{\text{BD}}$

or, $\frac{1}{\sqrt{3}} = \frac{h}{(DC+BC)}$

or, $\frac{1}{\sqrt{3}} = \frac{h}{40m+h}$ $[\because From\ (i)]$

or, $\sqrt{3}h = 40m + h$

or, $\sqrt{3}h - h = 40m$

or, $h(\sqrt{3} - 1) = 40m$

$$h = \frac{40m}{(\sqrt{3} - 1)} = 54.64m$$

Hence, the height of building is 54.64m.

Example 2

From the top of a house 200m high, the angles of depression of two rested cars are observed as 60° and 45° respectively. Find the distance between the two cards, if,

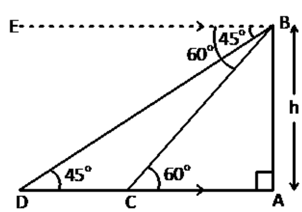
i. The cars are on the same side of house.

ii. The cars are on the opposite sides of the house.

Solution: Here,

i) Let, AB = 200m be the height of house. Let, C and D be the position of two rest cars on the same side.

From B drawn $\text{BE} \parallel \text{AD}$



$\therefore \angle EBD = \angle BDA = 45^\circ$

$and\ \angle EBC = \angle BCA = 60^\circ$

$Let, CD = x$

In rt. Angled $\angle ABC$

$$tan60^\circ = \frac{AB}{AC}$$

or, $\sqrt{3} = \frac{200m}{AC}$

or, $AC = \frac{200m}{\sqrt{3}} = 115.47m$

Again, in rt. angled ΔABD,

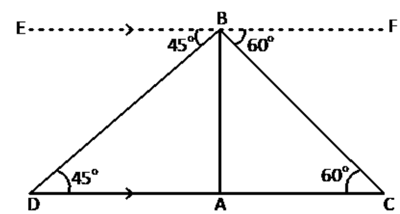
$$tan45° = \frac{AB}{AD}$$

$$\text{or, } 1 = \frac{200m}{AD}$$

or, $\mathrm{AD} = 200\mathrm{m}$

∴ The distance between the cars (CD) = AD – AC = 200m – 115.47m = 84.52m

ii) Let, AB = 200m be the height of house. Let, C and D be the position of two rest cars on the opposite side of house. Drawn, EF ∥ DC through B.



∴ $\angle\mathrm{EBD} = \angle\mathrm{BDA} = 45°$ and $\angle\mathrm{FBC} = \angle\mathrm{BCA} = 60°$

Since, AC = 115.47m [calculation from (i)]

AD = 200m

When the cars are on the opposite sides of house, the distance between the two cars (CD) = AD + AC = 200m + 115.47m = 315.47m

Example 3

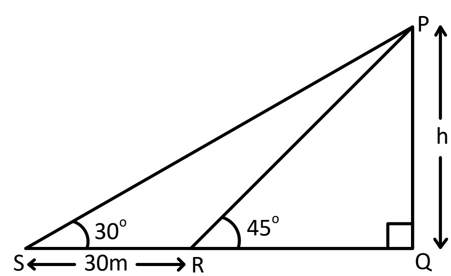
The length of the shadow of a tower standing on level place is found to be 30m longer when the sun's altitude is $30°$, then when it was $45°$. Prove that the height of tower is $15(\sqrt{3}+1)m$

Solution: Here,

Let, PQ = h be the height of tower. Let, R and S be the two points such that

RS = 30m, $\angle\mathrm{PSQ} = 30°$ and $\angle\mathrm{PRQ} = 45°$.

To Prove: Height of tower(PQ) = $15(\sqrt{3}+1)m$



In rt. angled ΔPQR

$$tan45° = \frac{\mathrm{PQ}}{\mathrm{QR}}$$

or, $1 = \frac{\mathrm{h}}{\mathrm{QR}}$

QR = h ………………………………………….(i)

Similarly, In rt. angled ΔPQS

$$\tan 30° = \frac{PQ}{SQ}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{SR + RQ}$$

or, $\frac{1}{\sqrt{3}} = \frac{h}{30m+h}$ [$\because$ from equation (i)]

or, $\sqrt{3}h = 30m + h$

or, $\sqrt{3}h - h = 30m$

or, $h(\sqrt{3} - 1) = 30m$

or, $h = \frac{30m}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$$h = \frac{30(\sqrt{3} + 1)}{3 - 1}$$

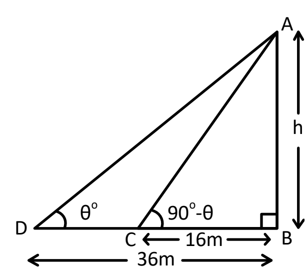
or, $h = 15(\sqrt{3} + 1)m$

Hence, it is proved that height of tower is $15(\sqrt{3} + 1)m$.

Example 4

The angles of elevation of the top of a tower observed from the distances of 36m and 16m from the foot of the tower are found to be complementary. Find the height of tower.

Solution: Here,



Let, AB = h be the height of a tower. C and D be the position of the points which are at a distance of 16m and 36m respectively from the foot of tower.

Let, $\angle$ ADB $= \theta$ then $\angle$ACB $= 90° - \theta$

BC = 16m and BD = 36m

To find: Height of tower (AB) = h

In right angled ΔABC,

$$\tan(90° - \theta) = \frac{AB}{BC}$$

or, $\cot\theta = \frac{h}{16m}$

or, $\frac{1}{tan\theta} = \frac{h}{16m}$ …………………………………………………………….(i) or, $tan\theta = \frac{16m}{h}$

Again, in right angled ΔABD,

$$tan\theta = \frac{AB}{BD}$$

$$tan\theta = \frac{h}{36m} \text{ ....(ii)}$$

Equating equation (i) and (ii), we get,

$$\frac{16\text{m}}{\text{h}} = \frac{\text{h}}{36\text{m}}$$

or, $h^2 = 16m \times 36m = 576m^2$

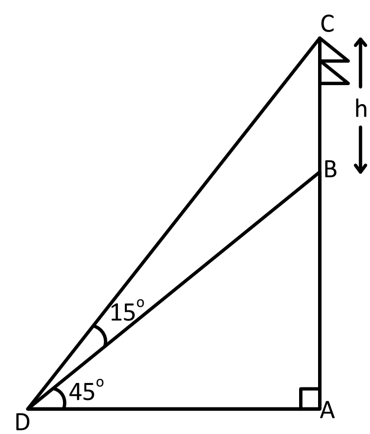
or, h = 24m.

Hence, the height of tower is 24m.

Example 5

A flagstaff of height 7m stands on the top of a tower. The angle subtended by the tower and the flagstaff at a point on the ground are $45°\ and\ 15°$ respectively. Find the height of tower.

Solution: Here,



Let, AB=h be the height of tower and BC =7m be the height of flagstaff. AB and BC subtend angles at the point D are $\angle BDA = 45°\ and\ \angle CDB = 15°$.

Now, $\angle CDA = 15° + 45° = 60°$

To find: Height of tower (AB) = h

In right angled $\Delta\text{ABD}, \tan 45° = \frac{\text{AB}}{\text{AD}}$

or, $1 = \frac{h}{AD}$

or, AD = h ....(i)

Similarly, in right angled $\Delta ACD, tan60° = \frac{AC}{AD}$

or, $\sqrt{3} = \frac{BC+AB}{AD}$

or, $\sqrt{3} = \frac{7m+h}{h}$ $[\because From\ (i)$

or, $\sqrt{3}h - h = 7m$

or, $h\left(\sqrt{3} - 1\right) = 7m$

or, $h = \frac{7m}{\sqrt{3}-1} = 9.56m$

Hence, the height of tower (AB) = h = 9.56m.

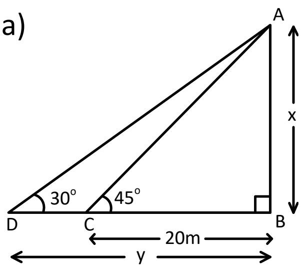
Exercise 5.6

1. a) Define angle of elevation. Illustrate it with figure.

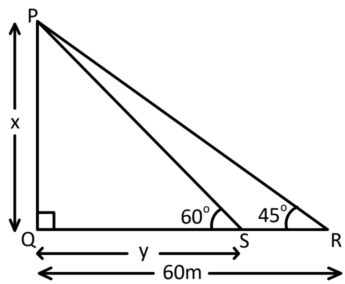
   b) Define angle of depression. Illustrate it with figure.

2. From the given figures, find the values of x, y and θ:

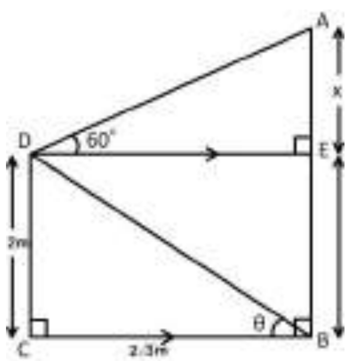
a)



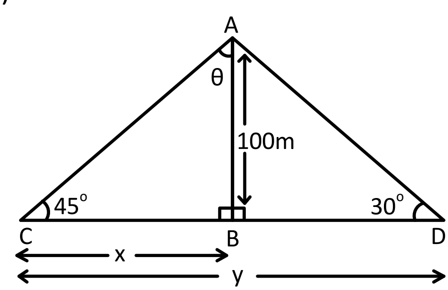
b)



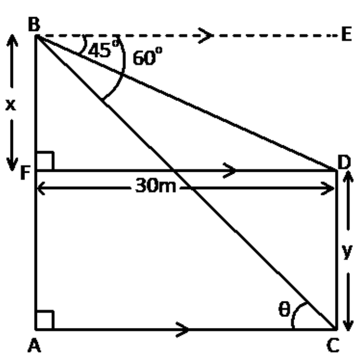
c)



d)



e)



3. a) The angle of elevation of a tower from a point was 60°. From a point on walking 300 meter away from the point it was found $30°$. Find the height of the tower.

   b) The angle of elevation of the top of a house from a point on the ground was observed to be $60°$ on walking 60 m away from that point it was found to be 30°. If the house and these points are in the same line of the same plain, find the height of house.

   c) The elevation of the top of a tower of height 60m from two places on the same horizontal line due west of it are $60°$ and $45°$. Find the distance between the two places.

4. a) From the top of a tower 100 meters high the measures of the angles of depression of two object due east of the tower are found to be 45° and 60° Find the distance between the objects.

   b) From the top of 21m high cliff, the angles of depression of the top and the bottom of a tower are observed to be $45° and\ 60°$ respectively. Find the height of tower.

   c) From the top of a tower 192 meter high the angle of depression of two vehicles on a road at the same level as the base of tower and on the same side of it are $x°$ and $y°$ when $\tan x° = \frac{3}{4}$ and $\tan y° = \frac{1}{3}$. Calculate the distance between them.

5. a) The shadow of a tower standing on a level ground is found to be 40m longer when the sun's altitude is $30°$ then when it is $60°$. Find the height of the tower.

   b) The shadow of a tower on the level ground increases in length by 'x' meter when the sun's altitude is $30°$ then when it is $45°$. If the height of the tower is 25 meter, find the value of x.

6. a) The angles of elevation of the top of a tower as observed from the distance of 4m and 16m from the foot of the tower are complementary. Find the height of tower.

   b) Two lamp posts are 200m apart and height of one is double of the other. From the midpoint of the line joining their feet an observer finds the angle of elevation of their tops to be complementary. Find the height of the both post.

7. a) A flagstaff stands on the top of a post 20m high. From a point on the ground the angles of elevation of the top and bottom of the flagstaff are found to be $60°$ and $45°$ respectively. Find the length of the flagstaff.

   b) From a point P on the ground the angle of elevation of the top of a 10m tall building is $30°$. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from P is $45°$. Find the length of the flagstaff and the distance of the building from the point P.

8. a) A vertical pole is divided at a point in the ratio of 1:9 from the base. If both parts of the pole subtend equal angles at a point 20m from the foot of the pole, find the height of the pole.

   b) AB is a vertical tower with its top A. C is a point on AB such that AC:CB = 13:5. If the parts AC and CB subtend equal angles at a point on the ground which is at a distance of 30 meters from the foot of the tower. Find the height of the tower.

9. a) An aeroplane flying horizontally 900m above the ground is observed at an elevation of $30°$. After 10 seconds the elevation is observed to be $45°$. Find the speed of the aeroplane in km/hr.

   b) Two poles of equal heights are standing opposite each other on either side of the road, which is 80m wide. From a point between them on the road, the angles of elevation of the top of the poles are $60° and\ 30°$ respectively. Find the height of the poles and the distance of the point from the poles.

10. a) A ladder 10m long reaches a point 10m below the top of a vertical flagstaff. From the foot of the ladder, the elevation of the flagstaff is $60°$. Find the height of the flagstaff.

    b) From the top of a 7m high building, the angle of elevation of the top of a cable tower is $60°$ and the angle of depression of its foot is $45°$. Determine the height of tower.

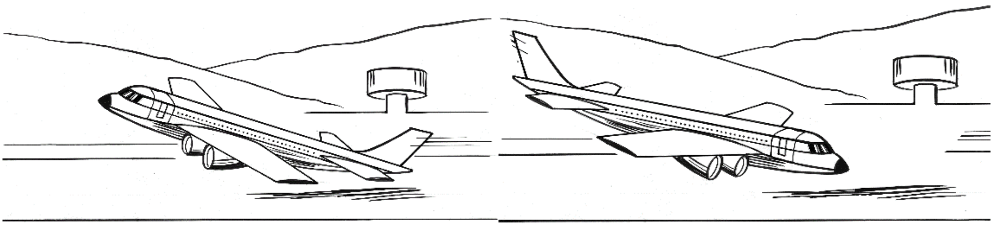
    c) The angle of depression and elevation of the top of a pole 25m high observed from the top and bottom of the tower are $60° and\ 30°$ respectively. Find the height of tower.

11. Make the clinometer by different group of students. From a point of the ground measure the distance between the point and the base of building. What is the height of your school building? Find it. Discuss in your calculation.

Unit 6

Vector

6.0 Review



- What is the direction of aeroplane when it is flying and landing?
- If $\overrightarrow{OA} = (2,3)$ and $\overrightarrow{OB}$ = (5, 2), then find the value of
  - i) $\overrightarrow{OA} + \overrightarrow{OB}$ in row form
  - ii) $\overrightarrow{OA} - \overrightarrow{OB}$ in column form
- What does $\vec{i}$ and $\vec{j}$ represent?
- What are the types of vectors? Write one example of each.
- What are the important roles of vectors in our daily life? Let us discuss with some examples.

In a vector there are two types of products. They are dot product and cross product. Dot product is a scalar quantity but cross product is a vector quantity. In this grade we will study only scalar product.

When an aeroplane covers 20km distance in air but it doesn't cover that distance on the ground, the perpendicular drawn from the positon of an aeroplane to the ground is called the projection of an aeroplane to the ground.

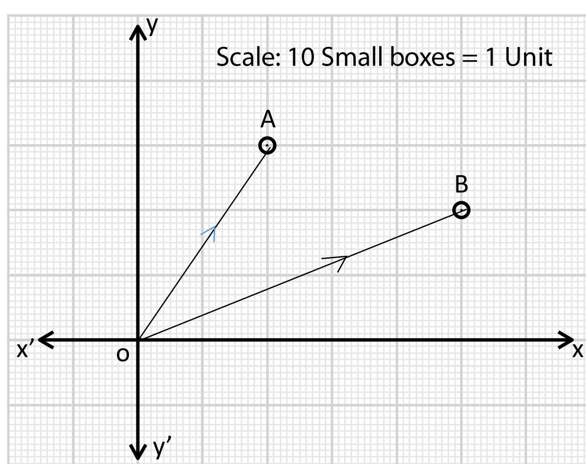
6.1 Scalar or dot product of two vectors

Study the given graph and answer the following questions:

i) What are the coordinates of A and B?

ii) What are the position vectors of A and B?

iii) Multiply X-coordinates of the points A and B and Y coordinates of the points A and B separately.



iv) What is the sum of the product of the x-coordinates and y-coordinates of the points A and B?

v) Can we show the result obtained from (iv) in the same graph?

Again, Discuss on the following questions:

(a) What is the difference between zero (0) and origin (0, 0)?
(b) Is the product of two numbers zero?
(c) Can we find the coordinates of the origin (0, 0) by adding and subtracting of two vectors?
(d) Is the product of two vectors is 0 (0, 0) or not?

∴ The dot product of two vectors gives the result in scalar form so it is known as scalar product.

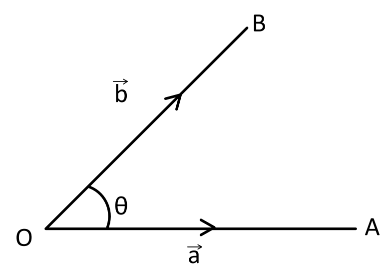
If $\vec{a}$ and $\vec{b}$ are two vectors and θ is the angle between them, then dot product is defined as

$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|cos\theta$

$= abcos\theta \qquad (0° \leq \theta \leq \pi^c)$

Where, $a = |\vec{a}|$ and $b = |\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively.

Thus, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|cos\theta$



∴The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is defined as the product of the magnitudes of two vectors multiplied by the cosine of the angle between them.

6.1.1 Geometrical interpretation of a scalar product.

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let θ be the angle between the two vectors $\vec{a}$ and $\vec{b}$.

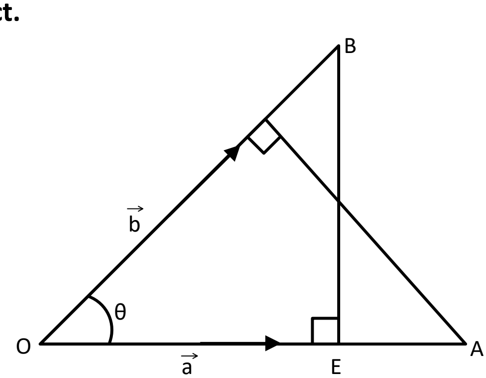
From A and B draw AD and BE perpendiculars to OB and OA respectively.

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|cos\theta$ (where $0° \leq \theta \leq \pi^c$)

$= abcos\theta$

$= (OA)(OB\ cos\theta)$

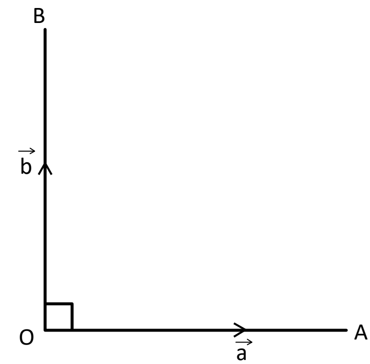
$= OA \times OE$



$\therefore\ \vec{a}\cdot\vec{b} = magninude\ of\ \vec{a} \times projection\ of\ \vec{b}\ on\ \vec{a}$

Similarly, $\vec{a}\cdot\vec{b} = magnitude\ of\ \vec{b} \times projection\ of\ \vec{a}\ on\ \vec{b}$.

Hence, the scalar product of two vectors is a product of the magnitude of one of the vectors and the projection of the second vector on the first.



When two vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other,

then the angle between them $(\theta) = 90°$

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|cos\theta$

$= |\vec{a}||\vec{b}|cos90°$

$\therefore\ \vec{a}.\vec{b} = 0$

Thus, when two vectors are perpendicular to each other, their scalar product is zero. Conversely,

When $\vec{a}.\vec{b} = 0$, then

$|\vec{a}||\vec{b}|cos\theta = 0$

or, $cos\theta = 0$

$cos\theta = \cos 90°$

$\theta = 90°$

Thus, if the dot product of two vectors is zero, they are perpendicular (orthogonal) to each other.

6.1.2 Scalar product of unit vectors $\vec{\imath}$ and $\vec{\jmath}$.

Here let $\vec{\imath}$ in the unit vector along X-axis and $\vec{\jmath}$ represents the unit vector Y-axis then,

The angle between $\vec{\imath}$ and $\vec{\imath}$ is 0° and that between $\vec{\imath}$ and $\vec{\jmath}$ is 90°.

i) $\vec{\imath}\cdot\vec{\imath} = |\vec{\imath}||\vec{\imath}|cos0° = (1)\cdot(1)\cdot(1) = 1$

ii) $\vec{\jmath}\cdot\vec{\jmath} = |\vec{\jmath}||\vec{\jmath}|cos0° = (1)\cdot(1)\cdot(1) = 1$

iii) $\vec{\imath}\cdot\vec{\jmath} = |\vec{\imath}||\vec{\jmath}|cos90° = (1)\cdot(1)\cdot(0) = 0$

iv) $\vec{\jmath}\cdot\vec{\imath} = |\vec{\jmath}||\vec{\imath}|cos90° = (1)\cdot(1)\cdot(0) = 0$

6.1.3 Scalar product of two vectors in terms of their components

Let, $\vec{a} = (x_1, y_1)$ and $\vec{b} = (x_2, y_2)$ be two given vectors.

Writing $\vec{a}$ and $\vec{b}$ in form of $x\vec{i} + y\vec{j}$.

$\therefore \vec{a} = x_1\vec{i} + y_1\vec{j}$ and $\vec{b} = x_2\vec{i} + y_2\vec{j}$

$$Now, \vec{a}.\vec{b} = (x_1\vec{i} + y_1\vec{j}).(x_2\vec{i} + y_2\vec{j})$$
$$= x_1x_2(\vec{i}.\vec{i}) + x_1y_2(\vec{i}.\vec{j}) + x_2y_1(\vec{j}.\vec{i}) + y_1y_2\vec{j}.\vec{j}$$
$$= x_1x_2(1) + x_1y_2(0) + x_2y_1(0) + y_1y_2(1)$$
$$= x_1x_2 + 0 + 0 + y_1y_2$$
$$= x_1x_2 + y_1y_2$$

$\therefore \vec{a}.\vec{b} = x_1x_2 + y_1y_2$ which is the scalar product of two vectors in terms of their respective component.

Also, we can write,

$$cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{x_1x_2+y_1y_2}{ab} \quad \text{where, } |\vec{a}| = a \quad \text{and} \quad |\vec{b}| = b$$

6.1.4 Length of a vector

Let, $\vec{a} = (x_1, y_1), then$

$$\vec{a}.\vec{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = x_1x_1 + y_1y_1 = x_1^2 + y_1^2$$

$$\vec{a}^2 = \vec{a}.\vec{a} = x_1^2 + y_1^2$$

$$\therefore \vec{a}^2 = a^2 = \vec{a}.\vec{a} = x_1^2 + y_1^2 = |\vec{a}|^2 \text{ [But } \vec{a} \neq a]$$

6.1.5 Properties of a scalar product

If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors then scalar product satisfies the following properties:

i) Commutative property: $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

ii) Distributive property :- $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

6.7 Some Algebraic Relations

i) $(\vec{a} + \vec{b})^2 = \vec{a}^2 + 2\vec{a}\cdot\vec{b} + \vec{b}^2 = a^2 + 2\vec{a}\vec{b} + b^2$ $[\because \vec{a}^2 = a^2 = |a|^2]$

ii) $(\vec{a} - \vec{b})^2 = a^2 - 2\overrightarrow{\vec{a}\cdot b} + b^2$

iii) $(\vec{a}+\vec{b})(\vec{a}-\vec{b}) = a^2 - b^2$

Note:

i) if $\theta = 0°$ then the value of $\vec{a}\cdot\vec{b}$ $is\ maximum.$

ii) If $\theta = 180°$ then the value of $\vec{a}\cdot\vec{b}$ is minimum.

Example 1

If $\vec{m} = \binom{-5}{2}$ and $\vec{n} = \binom{2}{7}$, find the angle between $\vec{m}$ and $\vec{n}$. Also, find $\vec{m}^2$ and $\vec{n}^2$.

Solution: Here,

Let, $\vec{m} = \binom{-5}{2} = \binom{x_1}{y_1}$ and $\vec{n} = \binom{2}{7} = \binom{x_2}{y_2}$

To find: i) Angle between $\vec{m}$ and $\vec{n}$. ii) $\vec{m}^{\ 2}$ and $\vec{n}^{\ 2}$

Now, By formula, $\vec{m}.\vec{n} = x_1x_2 + y_1y_2 = -5 \times 2 + 2 \times 7 = 4$

$|\vec{m}| = \sqrt{x_1^2 + y_1^2} = \sqrt{(-5)^2 + (2)^2} = \sqrt{25+4} = \sqrt{29}$ units.

$|\vec{n}| = \sqrt{x_2^2 + y_2^2} = \sqrt{2^2 + 7^2} = \sqrt{4+49} = \sqrt{53}$ units.

Let, θ be the angle between $\vec{m}$ and $\vec{n}$.

By formula, $cos\theta = \frac{\vec{m}.\vec{n}}{|\vec{m}||\vec{n}|} = \frac{4}{\sqrt{29}.\sqrt{53}}$

Or, $\theta = \cos^{-1}\left(\frac{4}{\sqrt{29}.\sqrt{53}}\right) = 84.14°$

ii) $\vec{m}^{\ 2}\ and\ \vec{n}^{\ 2}$

Since, $|\vec{m}| = \sqrt{29}$

We have, $\vec{m}^2 = |\vec{m}|^2 = \left(\sqrt{29}\right)^2 = 29$

And $\vec{n}^2 = |\vec{n}|^2 = \left(\sqrt{53}\right)^2 = 53$

Example 2

If $\vec{a} = 6\vec{\imath} + 2\vec{\jmath}$ and $\vec{b} = \vec{\imath} + k\vec{\jmath}$ are perpendicular vectors, Find the value of k.

Solution: Here,

$\vec{a} = 6\vec{\imath} + 2\vec{\jmath} = (6, 2) = (x_1, y_1)$

$\vec{b} = \vec{\imath} + k\vec{\jmath} = (1, k) = (x_2, y_2)$

To find: The value of k,

Since, $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then,

$\vec{a}\cdot\vec{b} = 0$

Or, $x_1x_2 + y_1y_2 = 0$

Or, $6 \times 1 + 2 \times k = 0$

Or, $2k = -6$

$k = -\left(\frac{6}{2}\right) = -3$

Hence, $\mathrm{k} = -3$

Example 3

If A(2, 4), B(2, 2) and C(4, 2) are three points, prove that AB is perpendicular to BC.

Solution: Here,

The given points are

A(2, 4), B(2, 2) and C(4, 2)

To prove: $AB \perp BC$

Now, $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1) = (2-2, 2-4) = (0, -2) = \binom{0}{-2}$

Similarly, $\overrightarrow{BC} = (4-2, 2-2) = (2,0) = \binom{2}{0}$

Now, $\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{BC}} = \binom{0}{-2}.\binom{2}{0} = 0 \times 2 + \ -2 \times 0 = 0 + 0 = 0$

Since, $\overrightarrow{AB}\cdot\overrightarrow{BC} = 0$, hence, it is proved that $AB \perp BC$.

Example 4

If $\vec{a} + 2\vec{b}$ and $5\vec{\mathrm{a}} - 4\vec{\mathrm{b}}$ are orthogonal to each other and $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a}$ and $\vec{b}$.

Solution: Here,

Let, $\vec{p} = \vec{a} + 2\vec{b}$ and $\vec{\mathrm{q}} = 5\vec{a} - 4\vec{b}$ are orthogonal to each other.

$|\vec{a}| = 1, |\vec{b}| = 1$

To find: The angle between $\vec{a}$ and $\vec{b}$.

Since, $\vec{p}$ and $\vec{q}$ are orthogonal so $\vec{p}.\vec{q} = 0$

or, $(\vec{a} + 2\vec{b})(5\vec{a} - 4\vec{b}) = 0$

or, $\vec{a}(5\vec{a} - 4\vec{b}) + 2\vec{b}(5\vec{a} - 4\vec{b}) = 0$

or, $5\vec{a}^2 - 4\vec{a}\cdot\vec{b} + 10\vec{a}\cdot\vec{b} - 8\vec{b}^{\ 2} = 0$

or, $5|\vec{a}|^2 + 6\vec{a}\cdot\vec{b} - 8\left|\vec{b}\right|^2 = 0$

or, $5 \times (1)^2 + 6\vec{a}\cdot\vec{b} - 8(1)^2 = 0$

or, $5 + 6\vec{a}.\vec{b} - 8 = 0$

or, $6\vec{a}\cdot\vec{b} = 3$

or, $\vec{a}\cdot\vec{b} = \frac{3}{6} = \frac{1}{2}$

Let, $'\theta'$ be the angle between $\vec{a}$ and $\vec{b}$ then by formula,

$$cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\left|\vec{b}\right|} = \frac{\left(\frac{1}{2}\right)}{1 \times 1}$$

or, $cos\theta = \frac{1}{2}$

or, $cos\theta = cos60°$

or, $\theta = 60°$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $60°$.

Example 5

If $\vec{a} + \vec{b} + \vec{c} = O(0,0), |\vec{a}| = 6, \left|\vec{b}\right| = 3\sqrt{2}$ and $|\vec{c}| = 3\sqrt{2}$ units then find the angle between $\vec{a}$ and $\vec{b}$.

Solution: Here,

$\vec{a} + \vec{b} + \vec{c} = O(0,0), |\vec{a}| = 6, \left|\vec{b}\right| = 3\sqrt{2}$ and $|\vec{c}| = 3\sqrt{2}$

To find: Angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a} + \vec{b} + \vec{c} = (0,0)$

Or, $\vec{a} + \vec{b} + \vec{c} = 0$

Or, $\vec{a} + \vec{b} = -\vec{c}$

Squaring on both sides, we get

$$\left(\vec{a} + \vec{b}\right)^2 = (-\vec{c})^2$$

or, $\vec{a} + 2\vec{a}\vec{b} + \vec{b}^2 = \vec{c}^2$

or, $|\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{c}|^2$

or, $(6)^2 + 2\vec{a}.\vec{b} + (3\sqrt{2})^2 = (3\sqrt{2})^2$

or, $36 + 2\vec{a}.\vec{b} + 18 = 18$

or, $2\vec{a}.\vec{b} = -36$

or, $\vec{a}.\vec{b} = -18$

Let, θ be the angle between $\vec{a}$ and $\vec{b}$ then,

$$cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = -\frac{18}{6\times3\sqrt{2}}$$

or, $cos\theta = -\frac{1}{\sqrt{2}}$

or, $cos\theta = cos135°$

or, $\theta = 135°$

Hence, angle between $\vec{a}$ and $\vec{b}$ is 135°.

Exercise 6.1

1. a) Define scalar product.

   b) If $\vec{a}.\vec{b} = 0$, what is the relation between $\vec{a}$ and $\vec{b}$?

2. a) If $\vec{a} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$, what is $\vec{a}.\vec{b}$?

   b) What should be the angle between $\vec{p}$ and $\vec{q}$ to obtain the maximum value of $\vec{p}.\vec{q}$?

3. a) What should be the angle between $\vec{a}$ and $\vec{b}$ to obtain the minimum value of $\vec{a}.\vec{b}$?

   b) If $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, find $\vec{a}.\vec{a}$.

4. a) If $\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $\vec{a}.\vec{b}$.

   b) If $\overrightarrow{OA} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} \sqrt{3} \\ -\sqrt{3} \end{pmatrix}$, find $\overrightarrow{OA}.\overrightarrow{OB}$.

   c) If $\vec{p} = 7\vec{i} + 2\vec{j}$ and $\vec{q} = 5\vec{i} - 8\vec{j}$, find $\vec{p}.\vec{q}$.

5. If A(-2, 1), B(-1, -3), C(3, -2) and D(2, 2) then,

   i) find $\overrightarrow{\mathrm{AB}}, \overrightarrow{\mathrm{BC}}, \overrightarrow{\mathrm{CD}}, \overrightarrow{\mathrm{DA}}$, and $\overrightarrow{\mathrm{BD}}$ in form of $\mathrm{x}\vec{\mathrm{i}} + \mathrm{y}\vec{\mathrm{j}}$.

   ii) find $\overrightarrow{\mathrm{AB}}.\overrightarrow{\mathrm{BD}}$.

   iii) find $\overrightarrow{\mathrm{AC}}^2$ and $\overrightarrow{\mathrm{CD}}^2$.

6. a) Calculate the dot product of $\vec{U} = (-4, -9)$ and $\vec{V} = (-1, 2)$. Do the vectors form an acute angle, right angle or obtuse angle?

   b) If $\vec{a} = \binom{2}{1}, \vec{b} = \binom{2}{-3}$ and $\vec{c} = \binom{0}{-2}$, find

   i) $\vec{a}\cdot\vec{b}$ *ii)* $\vec{a}\cdot\vec{c}$ *iii)* $\vec{b}\cdot\vec{c}$

   iv) a^2 *v)* b^2 *vi)* c^2

7. a) Find the angle between the two vectors with the following values:

   i) $|\vec{p}| = 21, \quad |\vec{q}| = 2$ and $\vec{p}\cdot\vec{q} = 21$

   ii) $|\vec{m}| = 10, |\vec{n}| = 20$ and $\vec{m}\cdot\vec{n} = -100\sqrt{3}$

   b) Find the angle between the following pair of vectors:

   i) $\vec{a} = 2\vec{\imath} - \vec{\jmath}$ and $\vec{b} = \vec{\imath} + 2j$ ii) $\vec{p} = 3\vec{\imath} + 2\vec{\jmath}$ and $\vec{q} = 6\vec{\imath} + 4\vec{\jmath}$

   iii) $\vec{a} = \binom{2}{-3}$ and $\vec{b} = \binom{3}{1}$ iv) $\overrightarrow{OA} = \binom{1}{3}$ and $\overrightarrow{OB} = \binom{2}{1}$

   c) i) If $|\overrightarrow{OC}| = 4, |\overrightarrow{OD}| = 6$ and $\overrightarrow{OC}.\overrightarrow{OD} = 12$, find $\angle COD$.

   ii) If $\overrightarrow{AB} = (\sqrt{3}, 1)$ and $\overrightarrow{AC} = \binom{\sqrt{3}}{3\sqrt{3}}$, find $\angle$ BAC.

8. a) Prove that the following pair of vectors are orthogonal to each other:

   i) $\vec{p} = 4\vec{\imath}$ and $\vec{q} = 3\vec{\jmath}$ ii) $\vec{m} = -6\vec{\imath} + 2\vec{\jmath}$ and $\vec{n} = \vec{\imath} + 3\vec{\jmath}$

   b) i) For what value of 'x' vectors $2\vec{\imath} - 3\vec{\jmath}$ and $x\vec{\imath} - 2\vec{\jmath}$ are perpendicular to each other?

   ii) If $\vec{p} = \binom{2}{a}$ and $\vec{q} = \binom{3}{-1}$ are perpendiculars to each other, what is the value of 'a'?

   iii) If $\vec{a} = \binom{-5}{3}$ and $\vec{b} = \binom{m}{m+2}$ are perpendicular to each other, find the value of 'm' .

9. a) i) If $|\overrightarrow{OA}| = 6, \overrightarrow{OA}.\overrightarrow{OB} = 24$ and $\angle AOB = 60°$, find the value of $|\overrightarrow{OB}|$

   ii) If $|\overrightarrow{OP}| = 4, \angle OPQ = 150°$ and $\overrightarrow{OP}.\overrightarrow{OQ} = 14\sqrt{3}$, find the value of $|\overrightarrow{OQ}|$

iii) If $|\overrightarrow{AB}| = 12, |\overrightarrow{AC}| = 8$ and $\angle BAC = 60°$, find the value of $\overrightarrow{AB}.\overrightarrow{AC}$.

10. a) i) If $5\vec{a} + 3\vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other and $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a}$ *and* $\vec{b}$.

ii) If $3\vec{m} - \vec{n}$ and $\vec{m} - 5\vec{n}$ are orthogonal vectors and $\vec{m}$ and $\vec{n}$ are unit vectors, find the angle between $\vec{m}$ and $\vec{n}$.

b) i) If $\vec{p} + \vec{q} + \vec{r} = O(0,0), |\vec{p}| = 3, |\vec{q}| = 5$ and $|\vec{r}| = 4$, find the angle between $\vec{p}$ and $\vec{r}$.

ii) If $\vec{a} + \vec{b} + \vec{c} = O(0,0), |\vec{a}| = 6, |\vec{b}| = 7$ and $|\vec{c}| = \sqrt{127}$, find the angle between $\vec{a}$ and $\vec{b}$.

iii) If $\vec{x}$ and $\vec{y}$ are perpendicular to each other then prove that $(\vec{x} + \vec{y})^2 = (\vec{x} - \vec{y})^2$

11. Draw and equilateral triangle PQR (any size) in a graph then,

i) Find $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{PR}, \overrightarrow{QP}$ and $\overrightarrow{RQ}, \overrightarrow{PR}$.

ii) Find the midpoint 'M' of $\overrightarrow{QR}$.

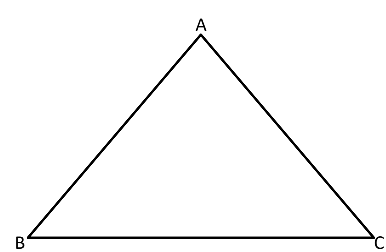
iii) Find the dot product of $\overrightarrow{QR}$ and $\overrightarrow{PM}$. Also, write the relation between $\overrightarrow{QR}$ and $\overrightarrow{PM}$.

6.2 Vector Geometry

6.2.0 Review

Let us discuss the following questions.

In the adjoining triangle ABC,



i) Is AB +BC = AC?

ii) Is $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$?

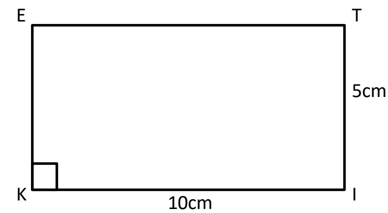
iii) Are there any differences between (i) and (ii)?

Similarly, in a rectangle KITE,

i) Find the product of KI and KE.

ii) What is the scalar product of $\overrightarrow{\mathrm{KI}}$ and $\overrightarrow{KE}$?

iii) What is the difference between $\mathrm{KI} \times \mathrm{KE}$ and $\overrightarrow{\mathrm{KI}}.\overrightarrow{\mathrm{KE}}$? Discuss with each other.

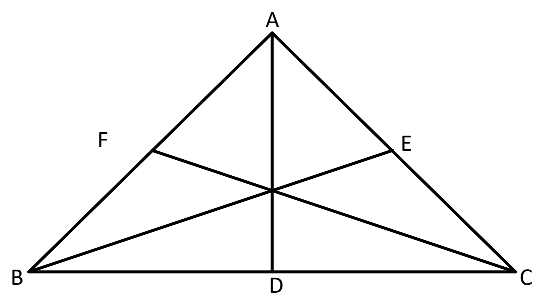


We can establish and prove different properties of geometry by the help of vectors. Such a study is called vector geometry.

6.2.1 (a) Mid-Point Theorem

If D, E and F be the midpoints of sides BC, CA and AB respectively of ΔABC. Express $\overrightarrow{AD}, \overrightarrow{BE}$ and $\overrightarrow{CF}$ in terms of $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CA}$.

Solution:



Given: In $\Delta ABC, D, E$ and F are the midpoints of BC, CA and AB respectively.

To Express: $\overrightarrow{\mathrm{AD}}, \overrightarrow{\mathrm{BE}}$ and $\overrightarrow{\mathrm{CF}}$ in terms of $\overrightarrow{\mathrm{AB}}, \overrightarrow{\mathrm{BC}}$ and $\overrightarrow{\mathrm{CA}}$.

Proof: In $\Delta\mathrm{ABD}$, by triangle law of vector addition,

$$\overrightarrow{\mathrm{AD}} = \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BD}} \dots\dots\dots\dots\dots\dots\dots\dots. \quad \text{(i)}$$

Similarly, In $\Delta\mathrm{ADC}$,

$$\overrightarrow{\mathrm{AD}} = \overrightarrow{\mathrm{AC}} + \overrightarrow{\mathrm{CD}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \text{(ii)}$$

Adding equation (i) and (ii) we get

$$\overrightarrow{\mathrm{AD}} + \overrightarrow{\mathrm{AD}} = \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BD}} + \overrightarrow{\mathrm{AC}} + \overrightarrow{\mathrm{CD}}$$

Or, $2\overrightarrow{AD} = \left(\overrightarrow{AB} + \overrightarrow{AC}\right) + \left(\overrightarrow{BD} - \overrightarrow{DC}\right)$

Or, $2\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BD} - \overrightarrow{BD}$ $\quad [\because \overrightarrow{BD} = \overrightarrow{DC}]$

Or, $2\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} + 0$

Or, $2\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$

Or, $\overrightarrow{\mathrm{AD}} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$

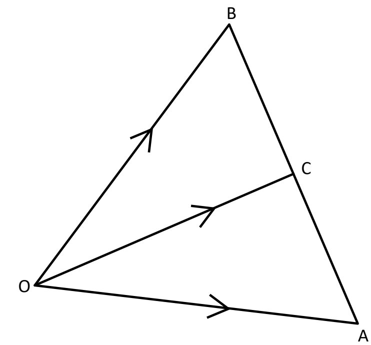
$$\overrightarrow{AD} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$$

Similarly,

$$\overrightarrow{BE} = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{BC}\right) and\ \overrightarrow{CF} = \frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{CB}\right)$$

Again,

$A(x_1, y_1)$ and $B(x_2, y_2)$ be the two end points of line segment AB. C is the mid-point of AB then the position vector of the point C.

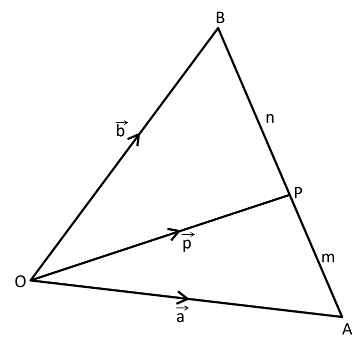


$$\overrightarrow{OC} = \left(\frac{x_1+x_2}{2}, \frac{(y_1+y_2)}{2}\right)$$

b) Section Formula

i) Internal division theorem

Statement:- If $\vec{a}$ and $\vec{b}$ be the position vectors of the point A and B respectively and $\vec{p}$ be the position vector of the point p which divides AB in the ratio m:n internally, then, $\vec{p} = \frac{m\vec{b}+n\vec{a}}{m+n}$



Solution:

Given: $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OP} = \vec{p}$ where P divides AB internally in the ratio m:n

i.e $\mathrm{AP:PB = m:n}$

To prove: $\vec{P} = \frac{m\vec{b}+n\vec{a}}{m+n}$

Proof: In ΔOAP, by Δ law of vector addition

$$\overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OP}$$

Or, $\overrightarrow{AP} = \vec{p} - \vec{a}$ ………………………………..(i)

Similarly, in ΔOPA, $\overrightarrow{OP} + \overrightarrow{PB} = \overrightarrow{OB}$

Or, $\overrightarrow{PB} = \vec{b} - \vec{p}$ ……………………………………………(ii)

Since, $\frac{AP}{PB} = \frac{m}{n}$

Or, $n\overrightarrow{AP} = m\overrightarrow{PB}$

Or, $n(\vec{p} - \vec{a}) = m\left(\vec{b} - \vec{p}\right)$ $[\because \text{From equation (i)\& (ii)}]$

Or, $\mathrm{n\vec{p} - n\vec{a} = m\vec{b} - m\vec{p}}$

Or, $\quad m\vec{p} + n\vec{p} = m\vec{b} + n\vec{a}$

Or, $\quad \vec{p}(m+n) = m\vec{b} + n\vec{a}$

Or, $\quad \vec{p} = \frac{m\vec{b}+n\vec{a}}{m+n}$ is the required relation.

ii) External division theorem

If $\vec{a}$ and $\vec{b}$ be the position vectors of the point A and B respectively and $\vec{p}$ be the position vector of the point P which divides AB in the ratio of m:n externally then $\vec{p} = \frac{m\vec{b}-n\vec{a}}{m-n}$.

Solution: Here,

Given, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OP} = \vec{p}$ where P divides AB externally in the ratio of m:n

i.e. $AP:BP = m:n$

To prove: $\vec{p} = \frac{m\vec{b}-n\vec{a}}{m-n}$

Proof: In ΔOAP, by Δ law of vector addition

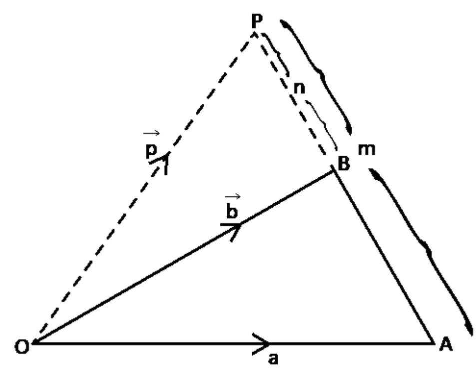
$\overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OP}$

or, $\quad \overrightarrow{AP} = \vec{p} - \vec{a}$ ....(i)

Similarly, in ΔOBP,

$\overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OP}$

$\overrightarrow{BP} = \vec{p} - \vec{b}$ ....(ii)



Since, $\frac{AP}{BP} = \frac{m}{n}$

or, $\quad n\overrightarrow{AP} = m\overrightarrow{BP}$

or, $\quad n(\vec{p} - \vec{a}) = m(\vec{p} - \vec{b})$ [∵ From (i) and (ii)]

or, $\quad n\vec{p} - n\vec{a} = m\vec{p} - m\vec{b}$

or, $\quad m\vec{b} - n\vec{a} = m\vec{p} - n\vec{p}$

or, $\quad m\vec{b} - n\vec{a} = \vec{p}(m-n)$

or, $\quad \vec{p} = \frac{m\vec{b}-n\vec{a}}{m-n}$

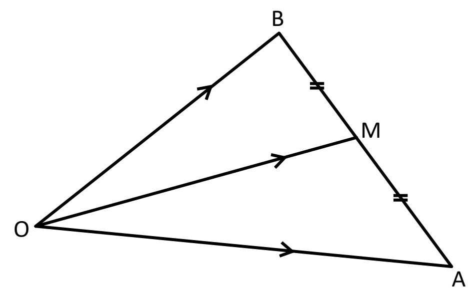
$\therefore \vec{p} = \frac{m\vec{b}-n\vec{a}}{m-n}$ is the required relation.

Example 1

If the position vector of the midpoint of the line segment AB is $3\vec{i} + \vec{j}$ and the position vector of A is $5\vec{i} + 4\vec{j}$ then find the position vector of point B.

Solution: Here,

Let, M be the midpoint of AB



$$\overrightarrow{OA} = 5\vec{i} + 4\vec{j}$$

$$\overrightarrow{OM} = 3\vec{i} + \vec{j}$$

To find: $\overrightarrow{OB}$

By using midpoint theorem

$\overrightarrow{OM} = \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$

or, $3\vec{i} + \vec{j} = \frac{5\vec{i}+4\vec{j}+\overrightarrow{OB}}{2}$

or, $6\vec{i} + 2\vec{j} = 5\vec{i} + 4\vec{j} + \overrightarrow{OB}$

or, $6\vec{i} - 5\vec{i} + 2\vec{j} - 4\vec{j} = \overrightarrow{OB}$

or, $\overrightarrow{OB} = \vec{i} - 2\vec{j}$

∴ The position vector of point B is $\vec{i} - 2\vec{j}$.

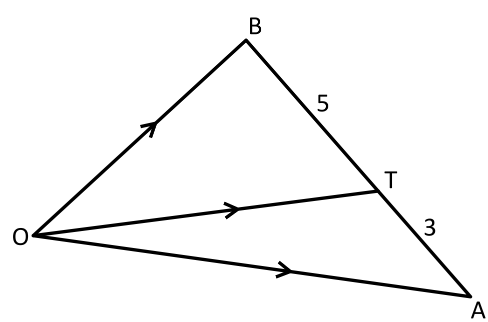
Example 2

The position vectors of the points A and B are $3\vec{i} - \vec{j}$ and $4\vec{i} - 7\vec{j}$ respectively. Find the position vector of T and U.

i) If T divides AB internally in the ratio of 3:5

ii) If U divides AB externally in the ratio of 2:1

Solution: Here,



i) $\overrightarrow{OA} = 3\vec{i} - \vec{j}$, $\overrightarrow{OB} = 4\vec{i} - 7\vec{j}$. T divides AB internally

i.e. $AT{:}TB = m{:}n = 3{:}5$

To find: $\overrightarrow{OT}$

By using internal section theorem

$$\overrightarrow{OT} = \frac{m\overrightarrow{OB} + n\overrightarrow{OA}}{m+n}$$

$$= \frac{3(4\vec{\imath}-7\vec{\jmath}) + 5(3\vec{\imath}-\vec{\jmath})}{3+5}$$

$$= \frac{12\vec{\imath}-21\vec{\jmath} + 15\vec{\imath}-5\vec{\jmath}}{8}$$

$$= \frac{27\vec{\imath}-26\vec{\jmath}}{8}$$

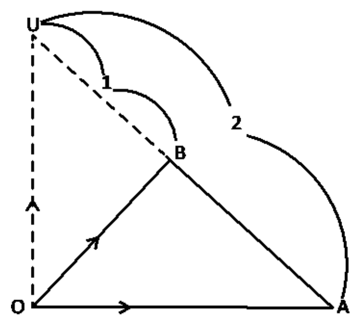
$$= \frac{27\vec{\imath}}{8} - \frac{26\vec{\jmath}}{8}$$

∴ The position vector of the point $(\overrightarrow{OT}) = \frac{27}{8}\vec{\imath} - \frac{13}{4}\vec{\jmath}$.

ii) Here, $\overrightarrow{OA} = 3\vec{\imath} - \vec{\jmath}$, $\overrightarrow{OB} = 4\vec{\imath} - 7\vec{\jmath}$. U divides AB externally in the ratio of 2:1

∴ $AU: BU = m: n = 2: 1$

To find: $\overrightarrow{OU}$



By using external section theorem

$$\overrightarrow{OU} = \frac{m\overrightarrow{OB} - n\overrightarrow{OA}}{m - n}$$

$$= \frac{2(4\vec{\imath}-7\vec{\jmath})-1(3\vec{\imath}-\vec{\jmath})}{2-1}$$

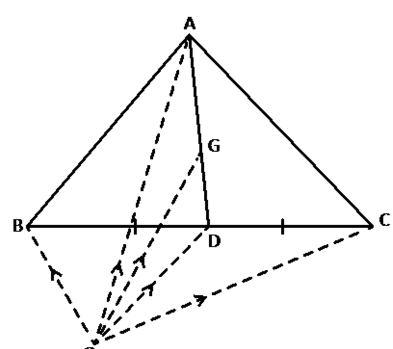
$$= \frac{8\vec{\imath}-14\vec{\jmath}-3\vec{\imath}+\vec{\jmath}}{1}$$

$$= 5\vec{\imath} - 13\vec{\jmath}$$

∴ The position vector of the point $U(\overrightarrow{OU}) = 5\vec{\imath} - 13\vec{\jmath}$.

Example 3

The position vector of the vertices of ΔABC are $\overrightarrow{OA} = -\vec{\imath} + \vec{\jmath}$, $\overrightarrow{OB} = 5\vec{\imath} - \vec{\jmath}$ and $\overrightarrow{OC} = 2\vec{\imath} + 5\vec{\jmath}$ respectively. Find the position vector of its centroid.



Solution: Here,

In ΔABC, let, G be the centroid. $\overrightarrow{OA} = -\vec{\imath} + \vec{\jmath}$, $\overrightarrow{OB} = 5\vec{\imath} - \vec{\jmath}$ and $\overrightarrow{OC} = 2\vec{\imath} + 5\vec{\jmath}$

To find: $\overrightarrow{OG}$

By using centroid theorem (formula)

$$\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$$

$$= \frac{1}{3}(-\vec{\imath} + \vec{\jmath} + 5\vec{\imath} - \vec{\jmath} + 2\vec{\imath} + 5\vec{\jmath})$$

$$= \frac{1}{3}(6\vec{i} + 5\vec{j})$$

$$\overrightarrow{OG} = \left(\frac{6\vec{i}}{3} + \frac{5\vec{j}}{3}\right) = 2\vec{i} + \frac{5}{3}\vec{j}$$

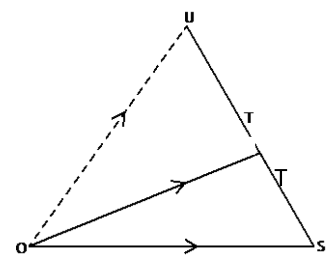
∴ The position vector of centroid of ΔABC is $2\vec{i} + \frac{5}{3}\vec{j}$.

[Note: We don't have to prove centroid formula but we have to use in problem solving.]

Exercise 6.2

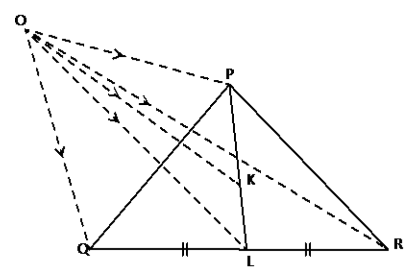
1. a. Write the statement of the midpoint theorem.

   b. The point Q divides the line segment PR internally in the ratio of $m_1 : m_2$. Write the position vector of the point Q in terms of the position vector of P and R.



2. a. In the adjoining figure, $SU:TU = m_1 : m_2$ then find $\overrightarrow{OU}$ in terms of $\overrightarrow{OS}$ and $\overrightarrow{OT}$.

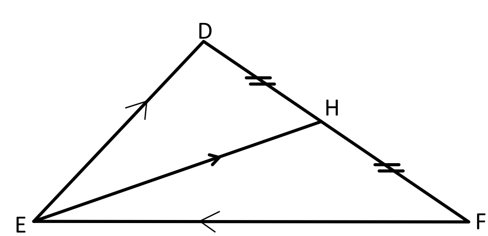
   b. In the given figure, L is the midpoint of QR, K divides PL in the ratio of 2:1 then write $\overrightarrow{OK}$ in terms of the position vector of the points P, Q and R.



3. a. If the position vectors of A and B are $5\vec{i} - 6\vec{j}$ and $3\vec{i} + 2\vec{j}$ respectively, find the position vectors of the midpoint of AB.

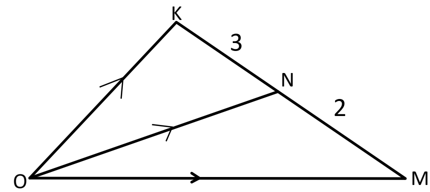
   b. If the position vectors of P and Q are $-2\vec{i} - 3\vec{j}$ and $3\vec{i} + 4\vec{j}$ respectively, find the position vector of the midpoint R of PQ.

   c. In the given figure, $DH = FH$, $\overrightarrow{ED} = 7\vec{i} + 3\vec{j}$ and $\overrightarrow{FE} = 3\vec{i} - 5\vec{j}$ find $\overrightarrow{EH}$.



4. a. The position vectors of M and N are $4\vec{i} + 6\vec{j}$ and $-\vec{i} + 3\vec{j}$ respectively. Find the position vector of the point Q which divides MN internally in the ratio of 3:2.

b. In the adjoining figure, the point N divides MK in the ratio 2:3. If $\overrightarrow{ON} = -3\vec{i} - 8\vec{j}$ and $\overrightarrow{OK} = 2\vec{i} + 5\vec{j}$. Find $\overrightarrow{OM}$.



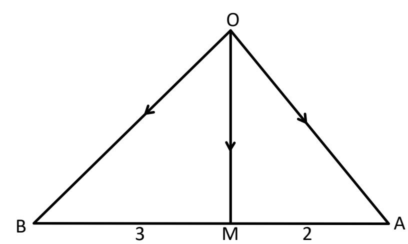
5. a. If the position vectors of the points A and B are $5\vec{i} + 2\vec{j}$ and $3\vec{i} + 6\vec{j}$ respectively, find the position vector of the point T which divides AB externally in the ratio of 5:2.

b. A and B are two points with coordinates (-4, 8) and (3, 7) respectively. Find the position vector of Q which divides AB externally in the ratio of 4:3.

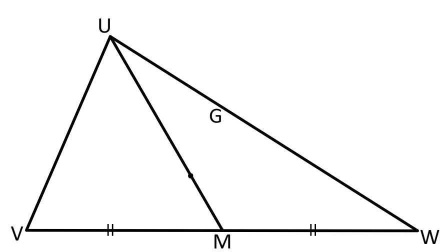
6. a. A(-1, 1), B(5, -1) and C(2, 5) are the three vertices of ΔABC. Find the position vector of the centroid of ΔABC.

b. In a ΔABC, $\overrightarrow{OA} = 3\vec{i} - 5\vec{j}$, $\overrightarrow{OB} = -7\vec{i} + 4\vec{j}$ and the position vector of the centroid G is $2\vec{i} + \vec{j}$. Find the position vector of C.

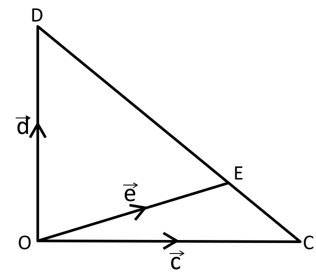
7. a. In the given figure, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OM} = \vec{m}$ and M divides BA in the ratio of 3:2, then prove that $\vec{m} = \frac{1}{5}(3\vec{a} + 2\vec{b})$.



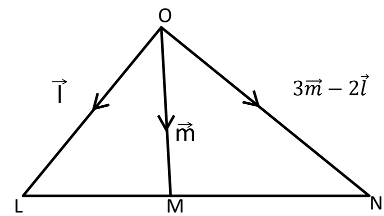
b. In the figure, UM is the median where the position vectors of U and M are $3\vec{i} - 2\vec{j}$ and $-3\vec{i} - 4\vec{j}$ respectively. Find the position vector of centroid G.



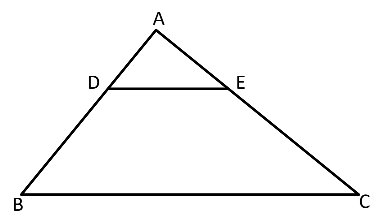
8. a. In the given figure, if $\overrightarrow{CE} = \frac{1}{4}\overrightarrow{CD}$, then prove that: $\vec{e} = \frac{1}{4}(3\vec{c} + \vec{d})$.



b. In the given figure $\overrightarrow{OL} = \vec{l}$, $\overrightarrow{OM} = \vec{m}$ and $\overrightarrow{ON} = 3\vec{m} - 2\vec{l}$ then prove that $\overrightarrow{LM} = \frac{1}{3}\overrightarrow{LN}$.

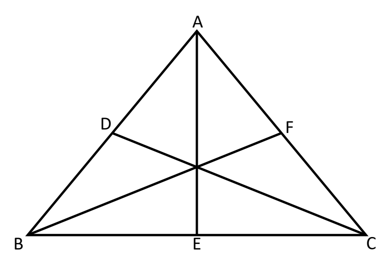


9. In ΔABC, D and E divides AB and AC in the ratio of 1:2 respectively. Prove that $\overrightarrow{DE} = \frac{1}{3}\overrightarrow{BC}$.

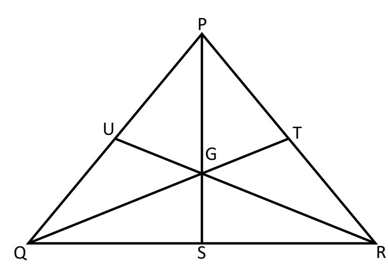


10. a. In ΔABC, D, E and F are the midpoints of sides AB, BC and AC respectively.

Prove that: $\overrightarrow{AE} + \overrightarrow{BF} + \overrightarrow{CD} = (0, 0)$.



b. In ΔPQR, S, T and U are the midpoints of sides QR, PR and PQ respectively. Prove that: $\overrightarrow{GP} + \overrightarrow{GQ} + \overrightarrow{GR} = (0, 0)$.



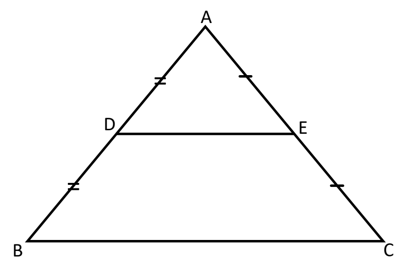
11. Do we use triangle law of vector addition in vectors midpoint theorem and section formula? If use it, how? Explain it.

6.3 Theorems related to triangles

By using triangle law of vector addition and the scalar product we can prove some theorems of triangle like in Geometry.

Theorem 1

Statement: The line segment joining the midpoints of any two sides of triangle is parallel to third side and half of it.



Solution:

Given: In ΔABC, D and E are the midpoints of AB and AC respectively.

To prove: $DE = \frac{1}{2}BC$ and $DE \parallel BC$

Proof: In ΔABC, by triangle law of vector addition

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} \qquad \text{.......................................(i)}$$

Similarly, in ΔADE,

$$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE}$$

or, $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$ [∵ D and E are the midpoints of sides AB and AC respectively]

or, $\overrightarrow{DE} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$

or, $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$ [∵ From (i)]

$$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$$

Again, by the definition of parallel vectors

$\overrightarrow{DE} = K\overrightarrow{BC}$ where $K = \frac{1}{2}$

and $|\overrightarrow{DE}| = \frac{1}{2}|\overrightarrow{BC}|$

$\therefore$ $\overrightarrow{DE}$ is parallel to $\overrightarrow{BC}$.

Hence, it is proved that the line segment joining the mid points of any two sides of a triangle is parallel to third side and half of it.

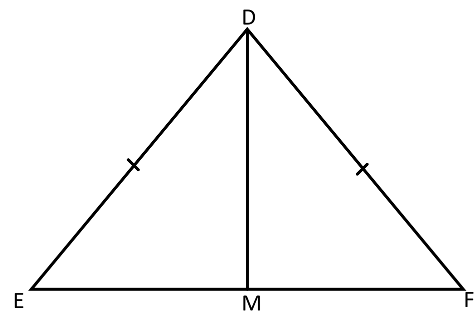
Theorem 2

Statement: The line segment joining the vertex and the midpoint of the base of an isosceles triangle is perpendicular to the base.

OR

The median of an isosceles triangle is perpendicular to the base.

Solution:



Given: In ΔDEF, DE=DF, M is the midpoint of the base EF i.e. EM=MF.

To prove: $DM \perp EF$

Proof: In ΔDEF, for median DM by using midpoint theorem

$$\overrightarrow{DM} = \frac{1}{2}(\overrightarrow{DE} + \overrightarrow{DF}) \text{ ....(i)}$$

Again, in ΔDEF, by using triangle law of vector addition

$$\overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{DF}$$

or, $\overrightarrow{EF} = \overrightarrow{DF} - \overrightarrow{DE}$ ....(ii)

The dot product of equation (i) and (ii), we get

$$\overrightarrow{DM}.\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{DE} + \overrightarrow{DF}\right)\cdot(\overrightarrow{DF} - \overrightarrow{DE})$$

$$= \frac{1}{2}(\overrightarrow{DF} + \overrightarrow{DE})(\overrightarrow{DF} - \overrightarrow{DE})$$

$$= \frac{1}{2}((\overrightarrow{DF})^2 - (\overrightarrow{DE})^2)$$

$$= \frac{1}{2}(DF^2 - DE^2)$$

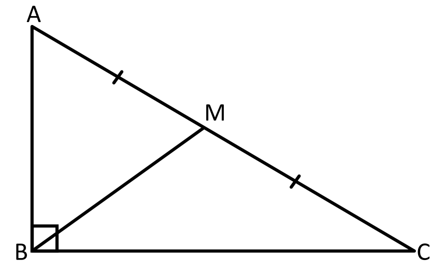
$$= \frac{1}{2}(DF^2 - DF^2) \quad [\because DE = DF]$$

$$= \frac{1}{2} \times 0$$

$$= 0$$

Since, $\overrightarrow{DM}\cdot\overrightarrow{EF} = 0$, hence, it is proved that the median of an isosceles triangle is perpendicular to the base.

Theorem 3



Statement: The middle point of hypotenuse of a right angled triangle is equidistant from its vertices.

Given: In right angled triangle ABC, $\angle ABC = 90°$. AC is hypotenuse and M is the midpoint of AC.
i.e. $AM = MC$.

To Prove: $\mathrm{AM} = \mathrm{MC} = \mathrm{BM}$

Proof: In ΔABM, by triangle law of vector addition

$$\overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AM}$$

or, $\overrightarrow{AB} = \overrightarrow{AM} - \overrightarrow{BM}$ ……………………..(i)

Similarly, in ΔBMC,

$$\overrightarrow{BC} + \overrightarrow{CM} = \overrightarrow{BM}$$

or, $\overrightarrow{BC} = \overrightarrow{BM} - \overrightarrow{CM}$ …………………….(ii)

we have, $\overrightarrow{\mathrm{AB}}.\overrightarrow{\mathrm{BC}} = 0$ $[\because \angle \mathrm{ABC} = 90°]$

or, $\left(\overrightarrow{\mathrm{AM}} - \overrightarrow{\mathrm{BM}}\right) \cdot \left(\overrightarrow{\mathrm{BM}} - \overrightarrow{\mathrm{CM}}\right) = 0$ [∵ From eq^{n} (i) and (ii)]

or, $\left(\overrightarrow{\mathrm{AM}} - \overrightarrow{\mathrm{BM}}\right) \cdot \left(\overrightarrow{\mathrm{BM}} + \overrightarrow{\mathrm{MC}}\right) = 0$ $[\because -\overrightarrow{\mathrm{CM}} = \overrightarrow{\mathrm{MC}}]$

or, $\left(\overrightarrow{\mathrm{AM}} - \overrightarrow{\mathrm{BM}}\right) \cdot \left(\overrightarrow{\mathrm{MC}} + \overrightarrow{\mathrm{BM}}\right) = 0$

or, $\left(\overrightarrow{\mathrm{AM}} - \overrightarrow{\mathrm{BM}}\right) \cdot \left(\overrightarrow{\mathrm{AM}} + \overrightarrow{\mathrm{BM}}\right) = 0$ $[\because \overrightarrow{\mathrm{AM}} = \overrightarrow{\mathrm{MC}}]$

or, $\overrightarrow{\mathrm{AM}}^2 - \overrightarrow{\mathrm{BM}}^2 = 0$

or, $\mathrm{AM}^2 - \mathrm{BM}^2 = 0$

or, $\mathrm{AM}^2 = \mathrm{BM}^2$

or, $\mathrm{AM} = \mathrm{BM}$

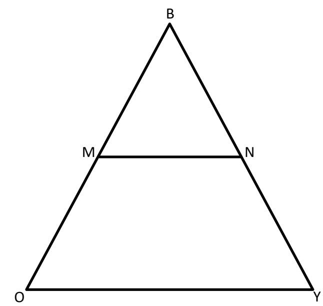
$\text{Since, } \mathrm{AM} = \mathrm{MC}$

$\therefore \mathrm{AM} = \mathrm{MC} = \mathrm{BM}$

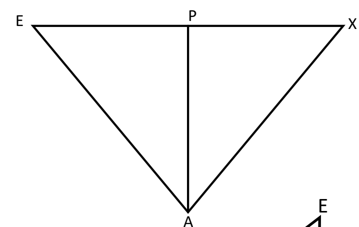
Hence, it is proved that the middle point of hypotenuse of a right angled triangle is equidistant from its vertices.

Exercise 6.3

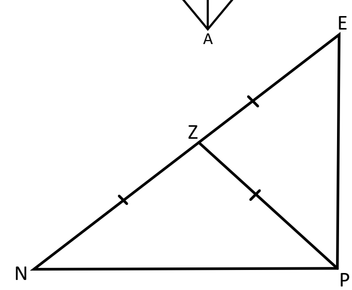
1. a) In ΔBOY, M and N are the midpoints of sides BO and BY respectively. Write the relation between MN and OY.



b) In the given ΔAXE, AE = AX and EP = XP. Write the relation between AP and EX.



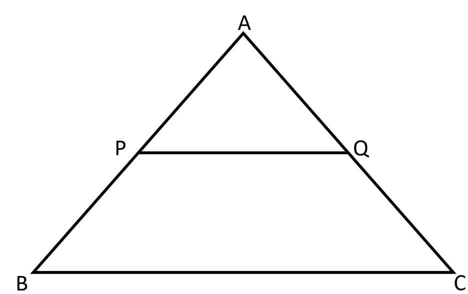
c) In the given ΔPEN, $PZ = NZ = EZ$. Write the relation between EP and NP.



2. a) In the given figure, P and Q are the middle points of AB and AC respectively of the ΔABC. Prove by vector method that

i) $BC = 2PQ$

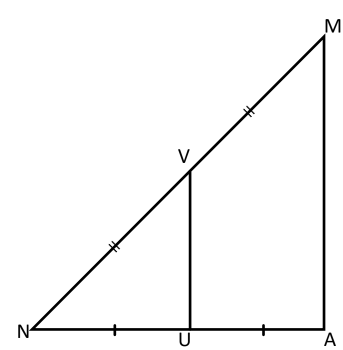
ii) $PQ \parallel BC$



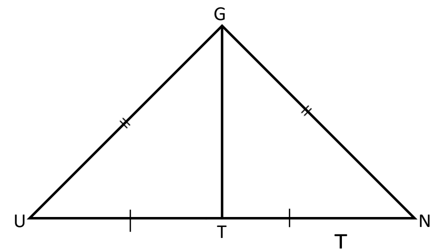
b) In the given ΔMAN, UV is a line segment joining the midpoints of sides AN and MN then prove by vector method that:

i) $\frac{1}{2}AM = UV$

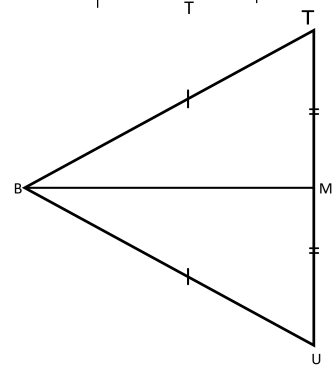
ii) $AM \parallel UV$



3. a) In the given ΔGUN, $\mathrm{GU} = \mathrm{GN}$ and UT =TN then prove by vector method that : $GT \perp UN$.

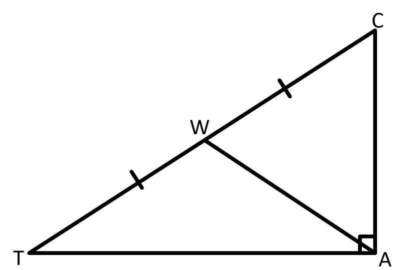


b) In the given ΔBUT, $\mathrm{BT} = \mathrm{BU}$ and TM = UM, then prove by vector method that $\angle\mathrm{BMT} = \angle\mathrm{BMU} = 90°$.

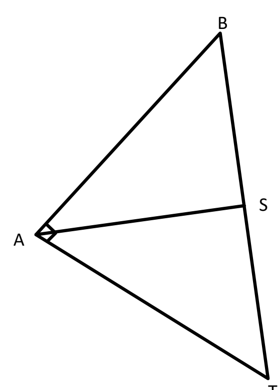


4. a) In the given triangle CAT, $\angle CAT = 90°$, TW = CW, prove by vector method that:

CW = TW = AW



b) In a right angled ΔBAT, $\angle BAT = 90°$, S is the midpoint of BT. Prove by vector method that TS = BS.

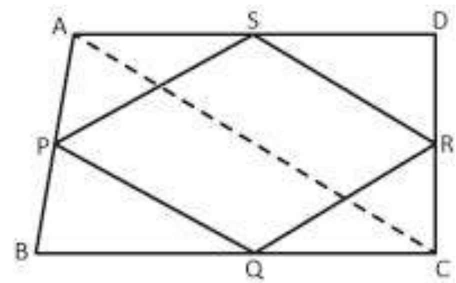


6.4 Theorems on Quadrilateral and Semi-circle

Theorem 4

Statement: The lines joining the middle points of the sides of a quadrilateral taken in order is a parallelogram.

Solution:



Given: In quadrilateral ABCD, P, Q, R and S are the midpoints of sides AB, BC, CD and DA respectively

To Prove: PQRS is a parallelogram

Construction: joined A and C

In ΔABC, by triangle law of vector addition, we get

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

or, $2\overrightarrow{PB} + 2\overrightarrow{BQ} = \overrightarrow{AC}$ [∵ Being P and Q are the midpoints of sides AB and BC respectively.]

or, $2(\overrightarrow{PB} + \overrightarrow{BQ}) = \overrightarrow{AC}$

or, $\overrightarrow{PB} + \overrightarrow{BQ} = \frac{1}{2}\overrightarrow{AC}$

or, $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$ ....(i) [∵ In ΔPBQ by Δ law]

and By the definition of parallel vectors

$\overrightarrow{\mathrm{PQ}} = \mathrm{K}\overrightarrow{\mathrm{AC}}$ where, $K = \frac{1}{2}$

∴ $\overrightarrow{\mathrm{PQ}} \parallel \overrightarrow{\mathrm{AC}}$ ....(ii)

Similarly, in ΔACD, by Δ law

$$\overrightarrow{\mathrm{AD}} + \overrightarrow{\mathrm{DC}} = \overrightarrow{\mathrm{AC}}$$

or, $2\overrightarrow{SD} + 2\overrightarrow{DR} = \overrightarrow{AC}$ [∵ Being S and R be the midpoints of sides AD and DC respectively]

or, $2(\overrightarrow{SD} + \overrightarrow{DR}) = \overrightarrow{AC}$

or, $\overrightarrow{SD} + \overrightarrow{DR} = \frac{1}{2}\overrightarrow{AC}$

or, $\overrightarrow{SR} = \frac{1}{2}\overrightarrow{AC}$ ....(iii) [In ΔSDR, by Δ by law] and $\overrightarrow{SR} \parallel \overrightarrow{AC}$ ....(iv)

From equation (i), (ii), (iii) and (iv), we get

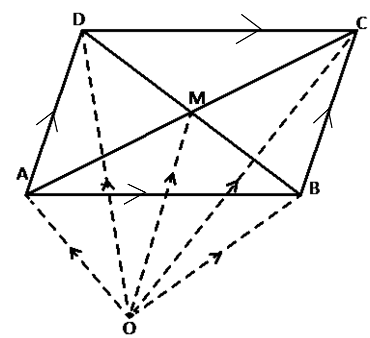
$$\overrightarrow{PQ} = \overrightarrow{SR} \text{ and } \overrightarrow{PQ} \parallel \overrightarrow{SR}$$

And $\overrightarrow{PS} = \overrightarrow{QR}$ and $\overrightarrow{PS} \parallel \overrightarrow{QR}$ [∵ Line segments joining the end points of same side of equal and parallel line segments are always equal and parallel.]

∴ PQRS is a parallelogram [∵ Being opposite sides equal and parallel]

Hence, it is proved that the lines joining the midpoint of the sides of a quadrilateral taken in order is a parallelogram.

Theorem 5



Statement: The diagonals of a parallelogram bisect to each other.

Solution:

Given: ABCD is a parallelogram. AC and BD are its diagonals. O be the origin M is the midpoint of BD.

To Prove: Diagonals AC and BD bisect to each other.

Construction: Joined OA, OB, OC, OD and OM.

Proof: In ΔOBD, by midpoint theorem

$$\overrightarrow{OM} = \frac{\overrightarrow{OB}+\overrightarrow{OD}}{2} \text{ ....(i)}$$

Again, in ΔOAD, by Δ law of vector addition

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} \text{ ....(ii)}$$

From equation (i) and (ii), we get

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{AD})$$

$= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{BC})$ [∵ Being opposite sides of parallelogram $\overrightarrow{AD} = \overrightarrow{BC}$]

$= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$ [∵ In ΔOBC by Δ law of vector addition]

$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$ ....(iii)

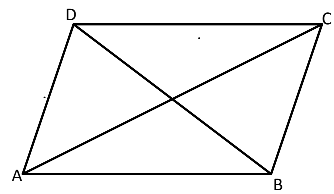
∴ M is the midpoint of AC [∵ From (iii) by midpoint theorem]

Both diagonals AC and BD have a common midpoint M. Hence, it is proved that diagonals of parallelogram bisect to each other.

Theorem 6

Statement: The diagonals of a rhombus bisect each other at right angle.

Solution:



Given: In a rhombus ABCD, AC and BD are the diagonals.

To Prove: ABCD is a rhombus.

We know that, rhombus is also a parallelogram and the diagonals of the parallelogram bisect each other.

So, in rhombus also, diagonals bisect each other

Again, in ΔABC, by Δ law of vector addition

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \qquad \text{.........................(i)}$$

Similarly, in ΔDAB, by Δ law of vector addition

$$\overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB} \qquad \text{...........................(ii)}$$

Now, taking dot product of (i) and (ii), we get

$$(\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}})(\overrightarrow{\mathrm{DA}} + \overrightarrow{\mathrm{AB}}) = \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{DB}}$$

or, $(\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}}) \cdot (\overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{AD}}) = \overrightarrow{\mathrm{AC}}. \overrightarrow{\mathrm{DB}}$

or, $(\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}}) \cdot (\overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{BC}}) = \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{DB}}$ [$\because \overrightarrow{\mathrm{AD}} = \overrightarrow{\mathrm{BC}}$]

or, $\mathrm{AB}^2 - \mathrm{BC}^2 = \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{DB}}$

or, $\mathrm{AB}^2 - \mathrm{BC}^2 = \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{DB}}$

or, $\mathrm{AB}^2 - \mathrm{AB}^2 = \overrightarrow{\mathrm{AC}} \cdot \overrightarrow{\mathrm{DB}}$ [∵ Being sides of rhombus AB = BC]

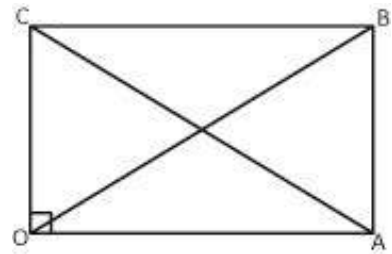
or, $\overrightarrow{\mathrm{AC}}. \overrightarrow{\mathrm{DB}} = 0$

Since, the dot product of two vectors is zero, they are at 90°. So, $AC \perp DB$.

Hence, it is proved that the diagonals of a rhombus bisect each other at 90°.

Theorem 7

Statement: Diagonals of a rectangle are equal.



Solution:

Given: OABC is a rectangle. OB and AC are two diagonals.

To Prove: $OB = AC$

Proof: In ΔOAB, by Δ law of vector addition.

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

Squaring on both sides, we get

$$(\overrightarrow{OB})^2 = \left(\overrightarrow{OA} + \overrightarrow{AB}\right)^2$$

or, $\overrightarrow{OB}^2 = \overrightarrow{OA}^2 + 2\overrightarrow{OA}\cdot\overrightarrow{AB} + \overrightarrow{AB}^2$

or, $OB^2 = OA^2 + 2 \times 0 + AB^2$ $[\because \angle OAB = 90°]$

or, $OB^2 = OA^2 + AB^2$ ....(i)

Similarly, in ΔOAC,

$$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$$

or, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

Squaring on both sides, we get

$$\left(\overrightarrow{CA}\right)^2 = \left(\overrightarrow{OA} - \overrightarrow{OC}\right)^2$$

or, $\overrightarrow{CA}^2 = \overrightarrow{OA}^2 - 2.\overrightarrow{OA}\cdot\overrightarrow{OC} + \overrightarrow{OC}^2$

or, $CA^2 = OA^2 - 2 \times 0 + OC^2$ $[\because \angle AOC = 90°]$

or, $CA^2 = OA^2 + AB^2$ $[\because OC = AB]$

or, $CA^2 = OA^2 + AB^2$ ....(ii)

From equation (i) and (ii), we get

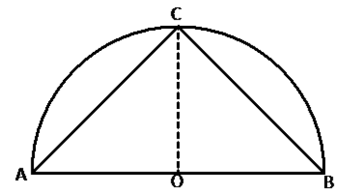
$$OB^2 = CA^2$$

or, $OB = CA$

Hence, it is proved that, the diagonals of a rectangle are equal.

Theorem 8

Statement: Angle at the circumference in a semicircle is a right angle.



Solution:

Given: O is the centre of semi-circle ABC. C is a point on the circumference and ∠ACB is circumference angle.

To Prove: $\angle ACB = 90°$

Construction: Joined O and C

We have, $|\overrightarrow{AO}| = |\overrightarrow{OC}| = |\overrightarrow{OB}|$ [∵ Radii of the same semi-circle]

In ΔAOC, by Δ law of rector addition

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} \qquad \text{……………………….(i)}$$

Similarly, in ΔBOC, by Δ law

$$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$$

$$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} \qquad \text{………………………….(ii)}$$

Taking dot product of (i) and (ii), we get

$$\overrightarrow{AC}.\overrightarrow{CB} = (\overrightarrow{AO} + \overrightarrow{OC})\cdot(\overrightarrow{OB} - \overrightarrow{OC})$$

$$= (\overrightarrow{AO} + \overrightarrow{OC})\cdot(\overrightarrow{AO} - \overrightarrow{OC}) \quad [\because\ \overrightarrow{AO} = \overrightarrow{OB}]$$

$$= \overrightarrow{AO}^2 - \overrightarrow{OC}^2$$

$$= |\overrightarrow{AO}|^2 - |\overrightarrow{OC}|^2$$

$$= AO^2 - OC^2$$

$= AO^2 - AO^2$ [∵ AO = OC, being radii of a semi-circle]

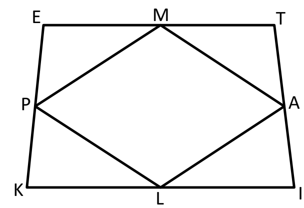
= 0

Since, $\overrightarrow{\mathrm{AC}}.\overrightarrow{\mathrm{CB}} = 0$, they are at $90°$. So, $\overrightarrow{\mathrm{AC}} \perp \overrightarrow{\mathrm{CB}}$

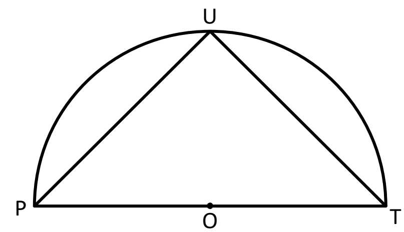
Hence, it is proved that the angle at the circumference in a semi-circle is a right angle.

Exercise 6.4

1. a) In the given figure, L, A, M and P are the midpoints of sides KI, IT, TE and EK respectively of quadrilateral KITE. What type of quadrilateral LAMP is it?

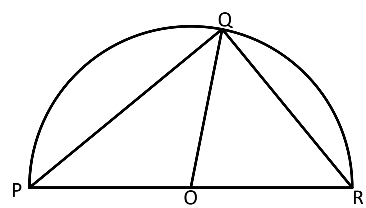


   b) In the figure, O is the centre of semi-circle PUT. PT is a diameter. Write the value of $\angle PUT$.

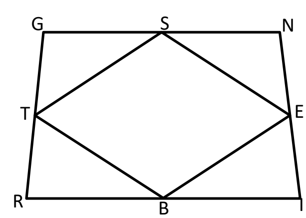


2. a) In a rhombus ABCD, AC and BD are two diagonals. What is the value of $\overrightarrow{AC}.\overrightarrow{BD}$?

   b) The given figure is a semi-circle with center O. Prove that: $(\overrightarrow{QO} + \overrightarrow{OP}).(\overrightarrow{QO} + \overrightarrow{OR}) = 0$



3. a) In the given figure, B, E, S and T are the midpoints of RI, IN, NG and GR respectively. Prove by vector method that BEST is a parallelogram.

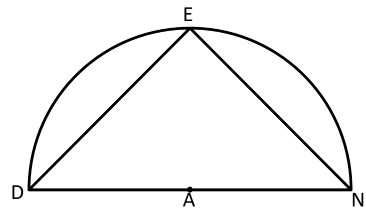


   b) Prove by vector method that the diagonals of a rectangle ROSE are equal.

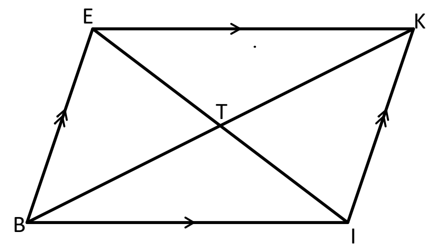
4. a) Prove that the diagonals of a parallelogram TAPE bisect to each other.

b) Prove that the diagonals of a rhombus HOME bisect to each other at right angle.

5. a) In the given figure, A is the centre of semi-circle DEN. DN is a diameter. Prove that DE is perpendicular to EN.



b) In the given figure, BIKE is a parallelogram. BK and EI are two diagonals, then prove that ET=TI and BT=TK.



6. What is the difference between theorems which are proved in vector geometry and geometry? Make a short report with examples and present in the class.

Unit 7

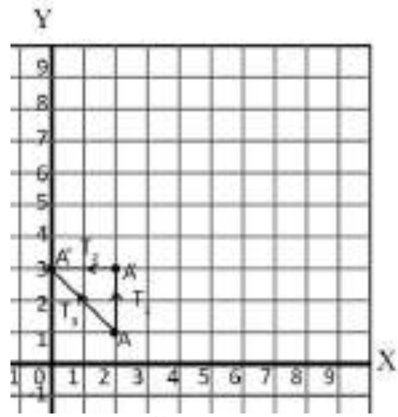
Transformation

7.0 Review

Write about reflection, rotation, translation, enlargement of a geometric figure. Discuss in small group of students and list down formulae about transformation that you have learned in previous classes.

7.1 Composition of transformation/combined transformation

From the adjoining graph, write the co-ordinate of points A, A', A'' and B, B', B'' in your exercise book. Observe the single transformations that changes position A to A″ and B to B″. Discuss about the situation that is found in the graph.



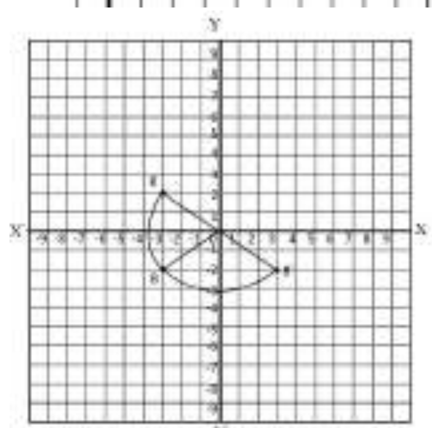
Here, $A(2,1) \xrightarrow{T_1\binom{0}{2}} A'\ (2,3)$

$A'(2,3) \xrightarrow{T_2\binom{-2}{0}} A''\ (0,3)$

$A(2,1) \xrightarrow{T_3\binom{-2}{2}} A''\ (0,3)$

$T_1 + T_2 = \binom{0}{2} + \binom{-2}{0} = \binom{-2}{2} = T_3$

$\text{So}, T_1 + T_2 = T_3 = T_1 O T_2$



In combination of transformations, one form of transformation can be combined with another form of transformation.

$So, T_1 + T_2 = T_3 = T_1 O T_2$

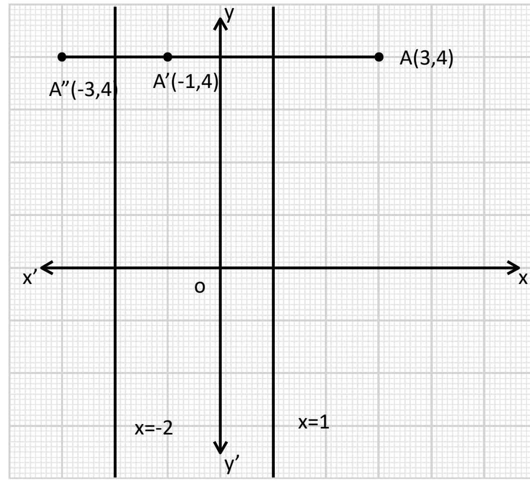
Similarly, B changes its position to B and B′ changes its position to B″. The corresponding transformations are as:

$B(-3,2) \xrightarrow{reflection} B'(-3,-2)$

$B'(-3,2) \xrightarrow{reflection} B''(3,-2)$

But, B” (3, -2) is obtained after rotation of B through $180°$ about the origin.

Again, let us reflect the point A(3, 4) about x = 1 and then about x = -3



In the graph when A(3, 4) is reflected about x = 1, the image A'(-1, 4) is obtained. Again, when we reflects A'(-1,4) about x = -2 we get the new image point A''(-3, 4).

The distance between x = 1 and x = -2 is 3 units.

When A$\binom{3}{4}$ is translated by $\binom{-6}{0}$ then we get

$A(3,4) \xrightarrow{T\binom{-6}{0}} A''(-3,4)$

Hence, reflection about x = 1 followed by reflection about x = -2 is equivalent to translation by $\binom{-6}{0}$, we find x = 1 and x = -2 are parallel lines.

So, if the axis of reflections are parallel, a reflection followed by another reflection is equivalent to the translation.

Example 1

Let the reflection in y-axis be r_1 and reflection in the line x = 2 be r_2, find the images under following transformations.

a) $r_1 o r_2$ (-2, 3) b) $r_2 o r_1$ (0, 5)

Solution: We know,

$r_1(x, y) \longrightarrow (-x, y)$

$r_2(x, y) \longrightarrow (4 - x, y)$

So, $r_1 o r_2$ (-2, 3) = $r_1(r_2(-2,3))$

= $r_1(4 + 2, 3)$

= r_1 (6, 3)

=(-6, 3)

Note: It is equivalent to $(-2,3) \xrightarrow{\binom{-4}{0}} (-6,3)$

$\therefore$ $r_2 o r_1$ = T$\binom{-4}{3}$

b) $r_2 o r_1(0, 5) = r_2(r_1(0, 5))$

= r_2 (0, 5)

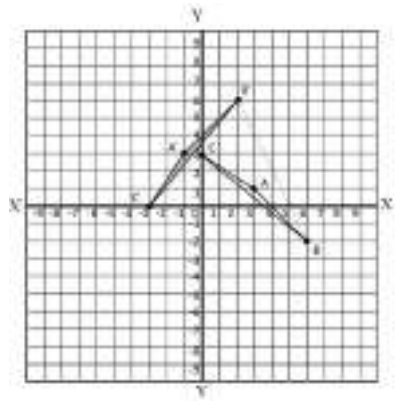
= (4 – 0, 5)

= (4, 5)

Example 2

Draw a ΔABC having vertices A(3, 1), B(6, -2) and C(0, 3) in a graph. Find the single transformation or transformation equivalent to composition of reflection on X-axis followed by reflection about Y-axis i.e. ($r_y \circ r_x$). Transform the ΔABC by $r_y \circ r_x$ and represent the image A'B'C' in same graph.

Solution:



$(r_y \circ r_x)(a, b) = r_y(r_x(a, b))$

$= r_y\ (a – b)$

$= (–a, –b)$

$\therefore$ (a, b) $\longrightarrow$ (–a, –b) is equivalent to rotation about (0, 0) through +180°

$$\mathrm{A}(3,1) \xrightarrow{[+180°,(0,0)]} \mathrm{A}'(-3,-1)$$

$$\mathrm{B}(6,-2) \xrightarrow{[+180°,(0,0)]} \mathrm{B}'(-6,-2)$$

$$\mathrm{C}(0,3) \xrightarrow{[+180°,(0,0)]} \mathrm{C}'(0,-3)$$

Example 3

Let E denotes the enlargement with centre at origin and scale factor 2 and R denotes the reflection through y + x = 0. Find a) (E∘R) (2, 5) b)(R∘E) (-3,4)

Solution:

We know, $p(x,y) \xrightarrow{E[(0,0),2]} P'(2x,2y)$

$$p(x,y) \xrightarrow{R[x+y]\,=\,0} P'(-y,-x)$$

a) (E∘R) (2, 5) = E(R(2,5))
= E(-5, -2)
= (-10, -4)

b) (R∘E) (-3, 4) = R(E(-3, 4))
= R(-6, 8)
= (-8, 6)

Example 4

A(-2, 0), B(0, 4) and C(3, 0) are vertices of ΔABC. $T\binom{-3}{1}$ and R[+90°, (0, 0)] are two transformations, find RoT(ΔABC). Represent the object and image in same graph.

Solution: we know,

(R∘T) (x, y)

= R(T(x,y))

= R$\begin{pmatrix} x-3 \\ y+1 \end{pmatrix}$ [Since, $T\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is given]

= [–(y + 1), x – 3)]

Now,

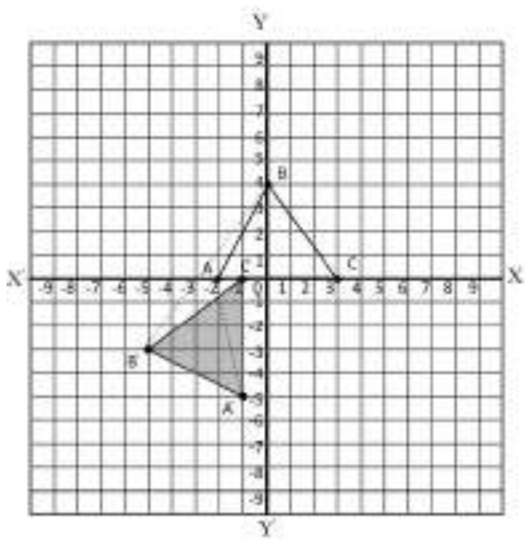
(R∘T) (–2, 0) = (-(0 + 1), -2 -3) = (-1, -5)

(R∘T) (0, 4) = (-(4 + 1), 0-3) = (-5, -3)

(R∘T) (3, 0) = (-(0 + 1), 3 – 3) = (-1, 0)

∴ A' (-1, -5), B'(-5, -3) and C' (-1, 0) are the images of A(2, 4) B(3, 3) and C(1, 5) under RoT

Representing the object and image figure in same graph, we get the following:



Exercise 7.1

1. a) Define composition of two transformations.

   b) If T$_1$ $\begin{pmatrix} a \\ b \end{pmatrix}$ and T$_2\begin{pmatrix} c \\ d \end{pmatrix}$ are two translations then find T$_1$∘T$_2$ (x, y).

   c) If the axis of two reflections are parallel to each other, which transformation is equivalent to the two reflections?

2. Let, R$_1$ represents the reflection on x-axis.

   R$_2$ represents the reflection about x + y = 0.

   r$_1$ represents the rotation through +90^o about origin.

r_2 represents the rotation through 180° about origin.
E_1: represents the enlargement about (0, 0) with scale factor 2.
E_2: [(0, 0), -3]

Compute the following transformations:

a) $(R_1 \circ R_2)$ (1, 2)   b) $(R_2 \circ R_1)$ (-2, 3)
c) $(r_1 \circ R_2)$ (-4, 3)   d) $(R_2 \circ r_2)$ (8, 9)
e) $E_1 \circ R_1)$ (1, 2)   f) $(E_1 \circ E_2)$ (2, 4)
g) $r_1 \circ E_1)$ (-3, 2)   h) $(E_2 \circ r_2)$ (-6, 2)

3. a) A(1, 1), B(3, 5) and C(5, -1) are the vertices of ΔABC. r_1 is reflection about the line x = 2 and r_2 is reflection about the line x = -1. Find the single transformation equivalent to $r_1 \circ r_2$. Transform ΔABC by $r_1 \circ r_2$. Represent the object and image triangles in same graph.

   b) A(1, 2), B(4, -1) and C(2, 5) are vertices of ΔABC. r_1 and r_2 are the reflection about the line x = 1 and y = -1 find the transformation equivalent to $r_1 \circ r_2$. Transform ΔABC by $r_1 \circ r_2$ represent the object and image triangles in the same graph.

   c) P(2, 2), Q(1, -1) and R(3,0) are vertices of ΔPQR. r_1 is reflection about x-axis and r_2 is rotation about origin through +90°. Find the rule that is equivalent to $r_2 \circ r_1$. Transform ΔPQR by $r_2 \circ r_1$. Represent the object and image in same graph.

   d) If R_1 is the rotation through + 90° about origin and R_2 is the rotation through -270° about origin then find the transformation equivalent to $R_1 \circ R_2$. Transform ΔXYZ with vertices X(1, 2), Y(-2, 3), Z(2, 5) by $R_1 \circ R_2$. Represent the object triangle and image triangle in same graph.

4. a) A(2, 0), B(0, 2) and C(-3,0) are vertices of ΔABC. E_1 [(0, 0), 2] and E_2 [(0, 0), $\frac{3}{2}$] are two enlargements. Transform ΔABC by $E_1 \circ E_2$. Represent the object and image triangles in same graph.

   b) A(1, 3), B(1, 6) C(3, 5) and D(4, 2) are the vertices of quadrilateral E[(0, 0), 2] and T$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ are two transformations. Find the rule related to $E \circ T$. Transform ABCD by $E \circ T$. Represent the objects and the image in same graph.

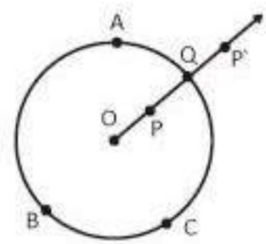
   c) AB is a line segment with end-points A(1, 2) and B(2, 1). It is enlarged by $E_2 \circ E_1$ where E_2 [(0, 0), $2\frac{1}{2}$] and E_1 [(0, 0), 2] are two enlargements. Transform AB by $E_2 \circ E_1$

5. If R_1 is the reflection about $x - 3 = 0$ and R_2 is the reflection about $y + x = 0$, show that $R_2 \circ R_1$ and $R_1 \circ R_2$ gives rotation. Are $R_1 \circ R_2$ and $R_2 \circ R_1$, give same result, if not why? Give reason.
6. Write the different situations of combined transformation (composition of transformation) in brief. Verify these situations taking suitable example.
7. Investigate the different situations of combined transformations in our daily life and make a report.

7.2 Inversion transformation and inversion circle

In the figure, O is centre of circle ABC. Q is any point on circle. P and P' are interior and exterior points on circle. Measure, OP, OP'. P, Q and P' lie on same line. Examine whether OP x OP' = OQ x OQ or not.

Take yourself a suitable radius. (eg. OQ = 6cm, OP = 3cm, OP' = 12cm)



In the above figure, for any point P, different from centre point (O), the inverse point P with respect to the circle ABC is unique. It satisfies the condition, OP x OP' = (radius)^2

i) The circle is said to be circle of inversion.
ii) The point O is called the centre of inversion.
iii) r or OQ is called the radius of inversion.
iv) The point O, P and P' are collinear
v) If the radius of circle is '1' unit
(OP) × (OP') = 1
or, OP = $\frac{1}{OP'}$

7.2.1 Characteristics or features of inversion

i) To each point of the plane except centre, these corresponds on inverse point.

ii) If any point lies on the circumference of the circle, its inversion point also lies on the circumference

iii) If P is inside the circle, inversion point of P lies outside the circle.

iv) The Point and its inversion point can be always interchanged.

v) If P' is image of P then P is image of P'.

7.2.2 Equations of Inversion Point

(i) Let O(0, 0) be the centre of circle 'r' be the radius of circle. Equation of circle is $x^2 + y^2 = r^2$

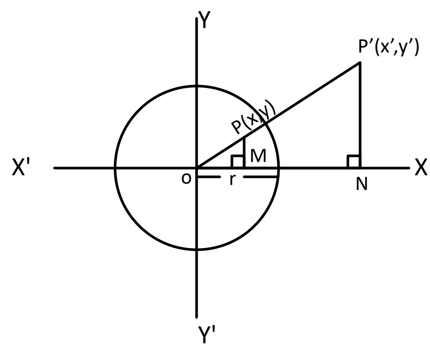
In the figure, ΔOMP and ΔONP' are similar

OP = $\sqrt{x^2 + y^2}$ and OP' = $\sqrt{(x')^2 + (y')^2}$

By definition, OP x OP' = r^2

$\therefore \left[\sqrt{x^2+y^2}.\sqrt{(x')^2+(y')^2} = r^2\right]$

And $\frac{\text{PM}}{\text{P'N}} = \frac{\text{OM}}{\text{ON}} = \frac{\text{OP}}{\text{OP'}}$ [$\because$ ratio of corresponding sides of similar triangles]



$or, \frac{y}{y'} = \frac{x}{x'} = \frac{\sqrt{x^2+y^2}}{\sqrt{(x')^2+(y')^2}} \times \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

$or, \frac{y}{y'} = \frac{x}{x'} = \frac{x^2+y^2}{OP' \times OP}$

$or, \frac{y}{y'} = \frac{x}{x'} = \frac{(x^2+y^2)}{r^2}$

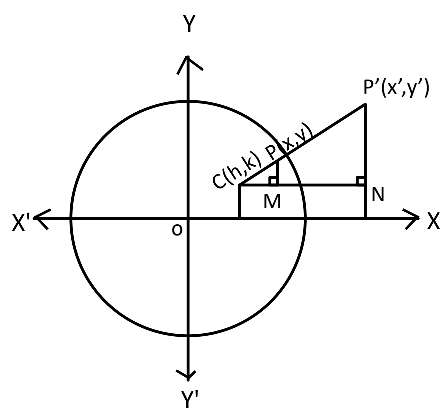
$or, \frac{y'}{y} = \frac{r^2}{x^2+y^2} \, and \, \frac{x'}{x} = \frac{r^2}{x^2+y^2}$

$or, y' = \frac{x^2 y}{x^2+y^2} \; and \; x' = \frac{r^2 x}{x^2+y^2}$

Image point (x', y') with respect to C(O, r) is $\left(\frac{r^2x}{x^2+y^2}, \frac{r^2y}{x^2+y^2}\right)$

(ii) Again, let C(h, k) be the centre of circle and r be the radius of circle.

We know, $(x-h)^2 + (y-k)^2 = r^2$ is the equation of circle



P(x, y) and P' (x', y') be the two points such that P' is inversion point of P about the circle

CP = $\sqrt{(x-h)^2 + (y-k)^2}$,

CP' = $\sqrt{(x'-h)^2 + (y'-k)^2}$

By definition CP × CP' = r^2

C, P, P' are collinear and ΔCPM and ΔCP'N are similar

$\frac{\text{CP'}}{\text{CP}} = \frac{\text{P'N}}{\text{PM}} = \frac{\text{CN}}{\text{CM}}$

$$\text{or}, \frac{\text{P}'\text{N}}{\text{PM}} = \frac{\text{CN}}{\text{CM}} = \frac{\text{CP}'}{\text{CP}} \times \frac{\text{CP}}{\text{CP}}$$

$$or, \frac{Y'-k}{y-k} = \frac{x'-h}{x-h} = \frac{r^2}{(x-h)^2+(y-k)^2}$$

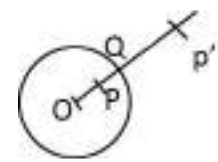
$$or, y' - k = \frac{r^2(y-k)}{(x-h)^2+(y-k)^2}, y' = \frac{r^2(y-k)}{(x-h)^2+(y-k)^2} + k$$

$$x' - h = \frac{r^2(x-h)}{(x-h)^2+(y-k)^2}, x' = \frac{r^2(x-h)}{(x-h)^2+(y-k)^2} + h$$

So, image point $(x', y') = \left(\frac{r^2(x-h)}{(x-h)^2+(y-k)^2} + h, \frac{r^2(y-k)}{(x-h)^2+(y-k)^2} + k\right)$

Example 1

In the figure, O is centre of circle, OP = 4 units, OQ = 8 units and OP' = 16 units, write the relation between P and P'.



Solution: Here,

OP = 4, OQ = 8, OP' = 16 units

We have, OP × OP' = OQ^2

4 × 16 = 8^2

64 = 64

Which is like OP × OP' = r^2. So P' is inversion point of P and vice-versa with respect to the circle.

Example 2

Find the inverse image of the point (4, 5) with respect to the circle $x^2 + y^2 = 100$

Solutions: Here,

Centre of the circle = (0, 0) and radius of the circle (r) = 10 units
object point P(x, y) = (4, 5)
inversion point P'(x', y') = ?
We know that,

$$(x', y') = \left(\frac{r^2x}{x^2+y^2}, \frac{r^2y}{x^2+y^2}\right)$$

$$= \left(\frac{(10)^2\times4}{4^2+5^2}, \frac{(10)^2\times5}{4^2+5^2}\right)$$

$$= \left(\frac{100\times4}{41}, \frac{100\times5}{41}\right)$$

$$= \left(\frac{400}{41}, \frac{500}{41}\right)$$

Hence, the inverse of the point (4, 5) with respect to given circle is

$(x', y') = \left(\frac{400}{41}, \frac{500}{41}\right)$

Example 3

Find the inverse image of the point (3, 4) about the circle $(x - 2)^2 + (y - 2)^2 = 36$

Solution: Here,

Centre of circle (h, k) = (2, 2) and radius of the circle (r) = 6 units

object point (x, y) = (3, 4) and inversion point (x', y')

We know that

$$(x', y') = \left(\frac{r^2(x-h)}{(x-h)^2+(y-k)^2} + h, \frac{r^2(y-k)}{(x-h)^2+(y-k)^2} + k\right)$$

$$or, (x', y') = \left(\frac{36(3-2)}{(3-2)^2+(4-2)^2} + 2, \frac{36(4-2)}{(3-2)^2+(4-2)^2} + 2\right)$$

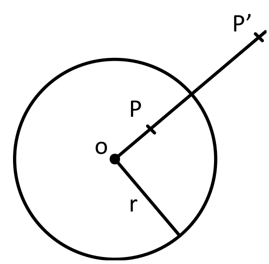
$$= \left(\frac{36\times1}{1+4} + 2, \frac{36\times2}{1+4} + 2\right)$$

$$= \left(\frac{36}{5} + 2, \frac{72}{5} + 2\right) = \left(\frac{46}{5}, \frac{82}{5}\right)$$

Hence, $\left(\frac{46}{5}, \frac{82}{5}\right)$ is the required image of the point (3, 4) about the given circle.

Exercise 7.2

1. In the figure, O(0, 0) is the centre of the circle and r be the radius of the circle
   a) Write the relation between OP, OP' and r [P' is inversion point of P]
   b) If P(x, y) is given, Find P' (x',y') in terms of x, y and r.



2. Complete the following table after calculation

S.N.	Point	Equation of inversion circle	Inversion Point
(a)	(2, 3)	$x^2 + y^2 = 1$	?
(b)	(-1, 2)	$x^2 + y^2 = 4$	?
(c)	(0, 4)	$x^2 + y^2 = 9$	?
(d)	(2, -3)	$x^2 + y^2 = 16$	?
(e)	(1, 5)	$x^2 + y^2 = 25$	?
(f)	(4, 3)	$x^2 + y^2 = 64$	?
(g)	(-1, -3)	$(x-2)^2 + (y + 1)^2 = 16$	?
(h)	(2, -5)	$x^2 + y^2 - 2x - 6y + 6 = 0$	?

3. a) Find the inverse of point (6, 10) with respect to the circle $x^2 + y^2 = 64$.

   b) Find the inverse image of the point (4, 6) in respect to the circle $x^2 + 2x + y^2 = 8$.

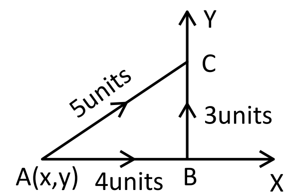
   c) Find the inverse image of point K(4, -3) with respect to the circles:

   i) $x^2 + y^2 + 6x – 8y – 11 = 0$ ii) $x^2 + y^2 – 4x – 6y – 23 = 0$

4. Write two equations of circle having centre at origin and other than origin use these circles to find the images of inversion point of A (4, 5) and B(-6, 7).

7.3 Transformation Using Matrices

a) Transformation using 2 x 1 matrix



In the right figure, A to C can be moved as 4 units horizontal and then 3 units vertical. If the position of A is (x, y), write down the position of B and C in column matrix

i) The column matrix for point A is $\binom{x}{y}$

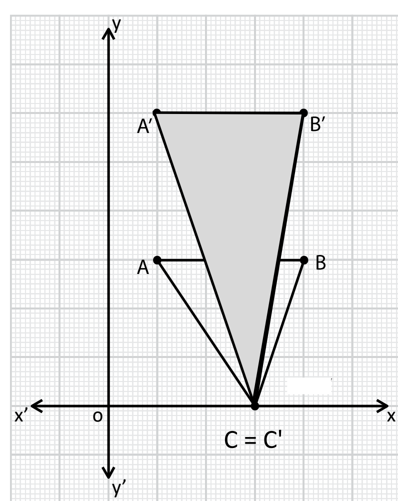
ii) The column matrix for point B is $\binom{x+4}{y} = \binom{x}{y} + \binom{4}{0}$

iii) The column matrix for point C is $\binom{x+4}{y+3} = \binom{x}{y} + \binom{4}{3}$

So, the 2 x 1 matrix that transform point A to point B is $\binom{4}{0}$ and the transformation matrix that transform point A to C is $\binom{4}{3}$

[If T$\binom{a}{b}$ be the 2 x 1 translation vector, then P(x, y) will be translated into (x + a, y + b)]

b) Transformation using 2 x 2 matrix



In the figure shown in a graph, write the coordinate of A, B, C, A', B' and C' represent the ΔABC and ΔA'B'C' in matrix form.

Which matrix can pre-multiply the matrix of ABC to get image ΔA'B'C'? Discuss.

Here, the matrix for ΔABC

$=\begin{pmatrix} 1 & 4 & 3 \\ 3 & 3 & 0 \end{pmatrix}$ and that for ΔA'B'C' is $\begin{pmatrix} 1 & 4 & 3 \\ 6 & 6 & 0 \end{pmatrix}$

Let us suppose a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which pre-multiply the matrix of ΔABC and result the matrix of $\Delta A^I B^I C^I$

Now, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 4 & 3 \\ 3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 6 & 0 \end{pmatrix}$

or, $\begin{pmatrix} a+3b & 4a+3b & 3a \\ c+3d & 4c+3d & 3c \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 6 & 0 \end{pmatrix}$

Equating the corresponding elements of two equal matrices, we have,

3a = 3; a = 1

3c = 0; c = 0

a + 3b = 1

or, 1 + 3b = 1; b = 0

or, c + 3d = 6

0 + 3d = 6; d = 2

Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

So, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 3 \\ 3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 3 & 0 \end{pmatrix}$

Coordinate of geometrical figures can be represented by a matrix. The matrix is called coordinate matrix or object matrix. Similarly, the coordinates of image figure is represented by image matrix. And the 2 x 2 matrix is said to be transformation matrix.

$(\text{Transformation matrix})_{2\times2}\ (\text{object matrix})_{2\times n} = (\text{image matrix})_{2\times n}$

Where n is the number of vertices of object figure and image figure each.

We can express each of the formulae of reflection, rotation and enlargement in matrix multiplication form.

e.g. $P(x,y) \xrightarrow{\text{reflection in x−axis}} P^i(x,-y)$

Now, for image: $x = 1.x + 0.y$

$$-y = 0.x + (-1)y$$

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$(\text{Image matrix})_{2\times1} = (\text{Transformation matrix})_{2\times2}\ (\text{object matrix})_{2\times1}$

Alternatively,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$or, \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

Or, $ax + by = x$; $x(a - 1) + by = 0$

$cx + dy = -y$; $cx + y(d + 1) = 0$

Now, a – 1 = 0; b = 0

c = 0; d + 1 = 0 gives $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Similarly, we can perform the other translation and obtain the following result

Matrix	Result	Geometric transformation
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	(x, y) →(x, -y)	Reflection in X-axis
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	(x, y)→(-x, y)	Reflection in Y-axis
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	(x, y)→(y, x)	Reflection in the y = x
$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	(x, y) →(-y, –x)	Reflection in the line y = –x
$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	(x, y)→(-x, -y)	Rotation through 180° about the origin
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	(x, y)→(-y, x)	Anti-clock wise rotation through 90° about origin
$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	(x, y)→(y, -x)	Clock wise rotation through 90° about origin
$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$	(x, y)→(mx, my)	Enlargement with scale factor, and centre at origin

Example 1

A(4, 5) and B(6, 7) are end-points of line segment AB. Translate A and B using matrix $\binom{-4}{3}$

Solution: Here,

A and B has matrix $\binom{4}{5}$ and $\binom{6}{7}$ respectively.

We know, $\binom{x}{y} \xrightarrow{T\binom{a}{b}} \binom{x+a}{y+b}$

So, $\binom{4}{5} \xrightarrow{T\binom{-4}{3}} \binom{-4+4}{3+5} = \binom{0}{8}$

$\binom{6}{7} \xrightarrow{T\binom{-4}{3}} \binom{-4+6}{3+7} = \binom{2}{10}$

Hence, the required image points are A'(0, 8) and B'(2, 10).

Example 2

A(1, 1), B(3, 2), C(3, 5) and D(1, 4) are the vertices of a parallelogram ABCD. Find the co-ordinates of the vertices of the images of parallelogram ABCD under the transformation by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$

Solution: Here,

object matrix $= \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 2 & 5 & 4 \end{pmatrix}$, I

transformation matrix $= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$

image matrix = ?

We know, Image matrix = $(\text{transformation matrix}) \times (\text{object matrix})$

or, Image matrix $= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 2 & 5 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 2 \times 1 + 0 \times 1 & 2 \times 3 + 0 \times 2 & 2 \times 3 + 0 \times 5 & 2 \times 1 + 0 \times 4 \\ 0 \times 1 + (-1) \times 1 & 0 \times 3 + (-1) \times 2 & 0 \times 3 + (-1) \times 5 & 0 \times 1 + (-1) \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & 6 & 2 \\ -1 & -2 & -5 & -4 \end{pmatrix}_{2\times4}$$

$\therefore$ Required image points are $A'(2, -1), B'(6, -2), C'(6, -5)$ and $D'(2 - 4)$

Example 3

Find the transformation matrix in which a unit square $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ is transformed into the parallelogram $\begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$

Solution: Here,

Let, the transformation matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

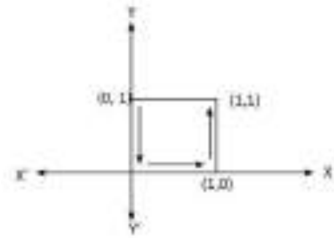
We know that, transformation matrix × object matrix = Image matrix

By the question $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$

or, $\begin{pmatrix} a\times 0+b\times 0 & a\times 1+b\times 0 & a\times 1+b\times 1 & a\times 0+b\times 1 \\ c\times 0+d\times 0 & c\times 1+d\times 0 & c\times 1+d\times 1 & c\times 0+d\times 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$

$\begin{pmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{pmatrix} = \begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$



Equating the corresponding elements, we get

a = 3, c = 1, b = 1, d = 2

∴ Required transformation matrix is $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.

Example 4

Find the matrix for the transformation given by the relation $(x, y) \rightarrow (x+y, x-y)$

Solution: Here,

Let, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the transformation matrix,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2} \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}_{2\times 1}$

or, $\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$

Equating the corresponding elements

$or, \quad ax+by = x+y \dots\dots\dots\dots (i)$

$or, \quad cx+dy = x-y \dots\dots\dots\dots (ii)$

From (i) and (ii)

$ax - x + by - y = 0$

$or, \quad (a-1)x + (b-1)y = 0$

$cx - x + dy - y = 0$

$or, \quad (c-1)x + (d+1)y = 0$

Both conditions can be satisfied if and only if coefficient of x and y equals to zero in each case.

So, a – 1 = 0 ⇒ a = 1

b – 1 = 0 ⇒ b = 1

c – 1 = 0 ⇒ c = 1

d + 1 = 0; d = –1

Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_{2\times2}$

Example 5

The vertices of $\Delta\mathrm{ABC}$ are A(2, 3), B(4, 5) and C(6, 2). If E_1 = [(0, 0), 2] and E_2 = [(0, 0), 2] then find the co-ordinates of images of $\Delta\mathrm{ABC}$ under $E_1 \circ E_2$ using matrix method.

Solution: Here,

$\mathrm{E_1} = [(0,0), 2]$ and $\mathrm{E_2} = [(0,0), 2\,]$

We know $E = E_1 o E_2 = E_2 o E_1 = [(0,0), 2 \times 2] = [(0,0), 4]$

The matrix represented by [(0, 0), 4] is $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Now,

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \times 2 + 0 \times 3 & 4 \times 4 + 0 \times 5 & 4 \times 6 + 0 \times 2 \\ 0 \times 2 + 4 \times 3 & 0 \times 4 + 4 \times 5 & 0 \times 6 + 4 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 16 & 24 \\ 12 & 20 & 8 \end{bmatrix}$$

Hence, the co-ordinates of images are $\mathrm{A'}(8, 12), \mathrm{B'}(16, 20)$ and $\mathrm{C'}(24, 8)$.

Exercise 7.3

1. Write the 2 × 2 matrix related to following transformations:
   a. Reflection in X-axis
   b. Reflection in the line x + y = 0
   c. Rotation through +90° about the origin
   d. Rotation through 180° about the origin
   e. Enlargement about (0,0) and scale factor 2
2. Find the image of following points:
   a. A(2, 3) by $T_1 \circ T_2$, where $\mathrm{T_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathrm{T_2} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$
   b. P(-4, 5) by $T_1 \circ T_2$, where $\mathrm{T_1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\mathrm{T_2} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

3.a. A(1, 0), B(2, 2) and C(0, 3) are vertices of $\Delta\mathrm{ABC}$. $\Delta\mathrm{ABC}$ is transformed by $\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$. Find the images of $\Delta\mathrm{ABC}$.

b. A(2, 0), B(4, 4), C(2, 5) and D(0, 4) are vertices of kite. Find the co-ordinates of images of kite under the transformation by matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

c. O(0, 0), P(6, 2), Q(8, 6) and R(2, 4) are vertices of parallelogram OPQR. Find the images of OPQR when it is transformed by the matrix $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

4.a. Find the transformation matrix which transforms a square $\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ into the quadrilateral $\begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 \end{pmatrix}$.

b. A quadrilateral ABCD with vertices A(0, 3), B(1, 1), C(3, 2) and D(2, 4) is mapped into the quadrilateral A'(3, 0), B'(1, -1), C'(2, -3) and D'(4, -2). Find the 2×2 transformation matrix.

c. Find the 2×2 transformation matrix which transform $\Delta\mathrm{ABC}$ with vertices A(1, 3), B(4, 3), C(3, 0) into $\Delta\mathrm{A'B'C'}$ with vertices A'(1, 6), B'(4, 6) and C'(3, 0).

5.a. Find the transformation matrix which transformed rectangle $\begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ into unit square.

b. Find the transformation matrix which transform a quadrilateral $\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ into the quadrilateral $\begin{pmatrix} 0 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 \end{pmatrix}$.

c. Find the transformation matrix which transform the quadrilateral $\begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$ into $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix}$.

6. Find the 2x2 matrix which transform the points are indicated below

a. $\mathrm{A}(x, y) \rightarrow \mathrm{A'}(-x, -y)$ b. $\mathrm{B}(x, y) \rightarrow \mathrm{B'}(3x, 3y)$

c. $\mathrm{C}(x, y) \rightarrow \mathrm{C'}(x, -2y)$

7.a. Show by matrix method that a reflection about the line y – x = 0 followed by the rotation about origin through + 90° is a reflection in x = 0. Discuss about the order of matrix multiplication.

b. Write any one difference between transformation using 2 x 1 matrix and 2 x 2 matrix. Show by using matrix method that reflection on the line about X-axis followed by the Y-axis is the rotation about the origin through 180°.

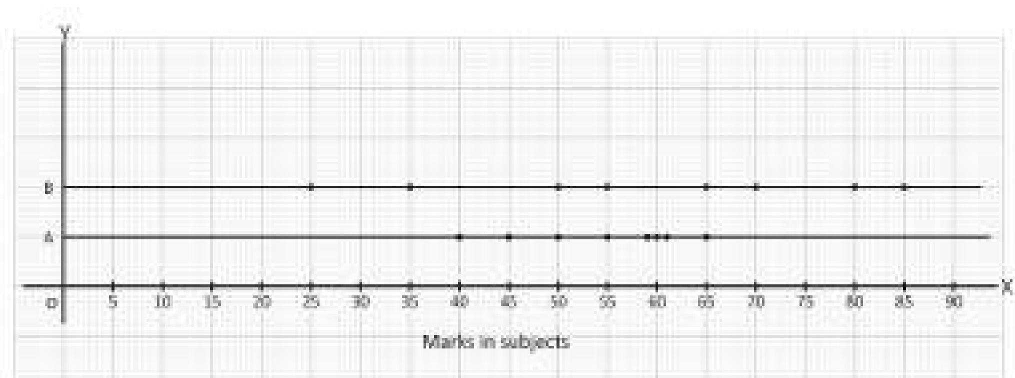
Unit 8 Statistics

8.0 Review

The marks obtained by two students of grade 9 in eight subjects are given below:

Subject	Students	
	A	B
English	40	50
Nepali	50	65
C. Maths	65	80
Science	60	35
Social Studies	58	55
Population	62	70
Opt. Maths	55	85
Computer Science	46	25

Representing their marks in a graph,



Study the above graph and answer the following questions:

i. What is the total scores of student A and student B? Also, find their average marks.
ii. Whose scores are more Scattered?
iii. Whose achievement is better? And why?
iv. What are the methods to measure the consistency and variability of the data? Which one is the best among them?
v. How can we compare the obtained marks of these two students?

In a data, measures of central tendency gives the idea about the concentration of the items around the central value. Dispersion means scatterness, variability, deviation or fluctuation etc. But the average (i.e. mean, median and mode) is not sufficient for clear information about the distribution. We study measure of dispersion which shows the scattering of data. It tells the variation of the data from one another and gives a clear idea about the distribution of the data. The measure of dispersion shows the homogeneity or the heterogeneity of the distribution of the observations.

Can you get an idea about the distribution if we get to know about the dispersion of the observations from one another, within and between the data? The main idea about the measure of dispersion is to get to know how the data are spread. It shows how much the data vary from their average value.

Classification of measures of dispersions

The measure of dispersion is categorized as:

a. An absolute measure of dispersion
   - The measures which expresses the scattering of observation in terms of distance i.e. range, quartile deviation.
   - The measures which expresses the variations in terms of the average deviation of the observations like mean deviation and standard deviation.

b. A relative measure of dispersion

   We use a relative measure of dispersion comparing distribution of two or more data set and for unit free comparison. They are the coefficient of range, the coefficient of quartile deviation, the coefficient of mean deviation, the coefficient of standard deviation and the coefficient of variation. Here, we will study to find quartile deviation, mean deviation and standard deviation of the continuous series only.

8.1 Quartile deviation (Semi-interquartile range)

The marks obtained by grade 9 students in mathematics are given belowM

Marks Obtained	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of Students	4	5	10	15	8	7	1

What are the values of quartiles from the above data? What do these values represents?

The values of quartiles means first quartiles (Q_1), second quartiles (Q_2) and third quartile (Q_3). To find the quartiles of continuous data, the data are tabulating below in ascending order:

Marks Obtained	No. of Students (f)	Cumulative frequency (C f)
20-30	4	4
30-40	5	9
40-50	10	19
50-60	15	34
60-70	8	42
70-80	7	49
80-90	1	50
	$\sum f = N = 50$	

Total number of students (N) =50

The position of 1st quartile = $\left(\frac{N}{4}\right)^{th}$ item $= \left(\frac{50}{4}\right)^{th}$ item $= 12.5^{th}$ item

In c.f column 19 is just greater than 12.5. So its corresponding class is 40-50. To find Q_1, the formula is

$$Q_1 = l + \frac{\frac{N}{4} - Cf}{f} \times i \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (i)$$

Where, l = lower limit of the Q_1 Class

C.f = cumulative frequency of just before Q_1 class

f = frequency of the Q_1 class

i = width of class interval

From the above table l = 40, f = 10, C.f =9, i = 10

From equation (i),

$$Q_1 = 40 + \frac{12.5 - 9}{10} \times 10 = 40 + 3.5 = 43.5$$

Again, the position of third quartile = $3\left(\frac{N}{4}\right)^{th}$ item $= 3 \times 12.5^{th}$ item $= 37.5^{th}$ item

In C.f Column, 42 is just greater than 37.5 so its corresponding class is 60-70. The formula to find the Q_3 value of Q_3 is

$$Q_3 = l + \frac{3\left(\frac{N}{4}\right) - Cf}{f} \times i \ldots \ldots \ldots \ldots \ldots . . (iii)$$

Where. l = lower limit of the Q_3 Class

Cf = cumulative frequency of just before Q_3 class

f = frequency of the Q_3 class

i = width of class interval

From the above table, l = 60, C.f = 34, f = 8, and i = 10

from the equation (iii),

$$Q_3 = 60 + \frac{37.5 - 34}{8} \times 10 = 60 + \frac{3.5 \times 10}{8} = 64.375$$

$\therefore$ The quartiles of the given data are Q_1 = 43.5 and Q_3 = 64.375. Half of the difference between upper quartile (Q_3) and lover quartile (Q_1) is called quartile deviation. It is invented by Galton.

$$\therefore \text{Quartile deviation(QD)} = \frac{\mathrm{Q}_3 - \mathrm{Q}_1}{2}$$

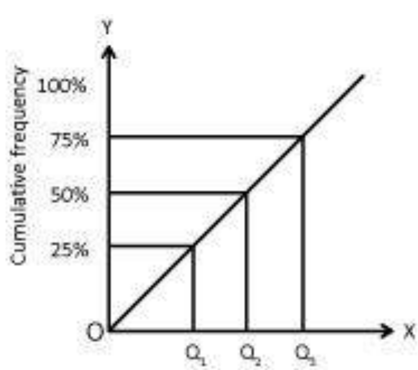
$$= \frac{64.375 - 43.5}{2} = \frac{20.875}{2} = 10.437$$

Again, the relative measure based on lower and upper quartile known as coefficient of quartile deviation. It is given by

Coefficient of Quartile deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{64.375 - 43.5}{64.375 + 43.5} = \frac{20.875}{107.875}$ = 0.193

Hence, Coefficient of QD = 0.193

Can we show the relationship of Q_1, Q_2, Q_3 by graph? Discuss on it.



- The lower quartile (Q_1) means, the value of $\left(\frac{N}{4}\right)^{th}$ item or the value of the 25% of the total frequency of the distribution.
- The middle quartile (2nd quartile) (Q_2) means the value of $2\left(\frac{N}{4}\right)^{th}$ item or the value of 50% of the total frequency of the distribution.
- The upper quartile (Q_3) means the value of $3\left(\frac{N}{4}\right)^{th}$ item or the value of 75% of the total frequency of the distribution.

$\therefore$ Semi-interquartile range or Quartile deviation $= \frac{Q_3 - Q_1}{2}$ and coefficient of QD $= \frac{Q_3 - Q_1}{Q_3 + Q_1}$

Example 1

Find the quartile deviation and the coefficient of quartile deviation from the following data:

Marks	20-30	30-40	40-50	50-60	60-70	70-80
No. of Students	4	5	2	4	3	2

Solution: Here,

Tabulating the given data in ascending order for the calculation of QD.

Marks	No. of Students (f)	Cumulative frequency (Cf)
20-30	4	4
30-40	5	9
40-50	2	11
50-60	4	15
60-70	3	18
70-80	2	20
	$\sum f = N = 20$	

The position of first quartile $= \left(\frac{N}{4}\right)^{th}$ item $= \left(\frac{20}{4}\right)^{th}$ item $= 5^{th}$ item

In Cf column, 9 is just greater than 5 so its corresponding class is 30-40.

$\therefore$ l = 30, f = 5, Cf = 4, i = 10

By formula,

$$Q_1 = l + \frac{\frac{N}{4} - Cf}{f} \times i = 30 + \frac{5-4}{5} \times 10 = 30 + 2 = 32$$

Again, the position of third quartile $= 3\left(\frac{N}{4}\right)^{th}$ item $= 3 \times 5^{th}$ item $= 15^{th}$ item

In Cf Column, 15^{th} item corresponding class is 50-60.

$\therefore$ l = 50, f = 4, Cf = 11, i = 10

Now,

$$Q_3 = l + \frac{\frac{3N}{4} - Cf}{f} \times i = 50 + \frac{15-11}{4} \times 10 = 50 + 10 = 60$$

Now, quartile deviation (QD) $= \frac{Q_3 - Q_1}{2} = \frac{60-32}{2} = \frac{28}{2} = 14$

And Coefficient of QD $= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{60-32}{60+32} = \frac{28}{92} = 0.304$

Exercise 8.1

1. a. What is dispersion?

   b. Write the various measures of dispersion.

2. a. Define quartile deviation and write the formula to calculate quartile deviation.

   b. What do you mean by coefficient of quartile deviation?

   c. Write the difference between quartile deviation and the coefficient of quartile deviation.

3. a. In a Continuous data, the first quartile and third quartile are 40 and 60 respectively, find the quartile deviation.

   b. In a continuous series, the lower quartile is 25 and its quartile deviation is 10, find the upper quartile.

4. a. In a continuous series, the coefficient of quartile deviation is $\frac{1}{2}$ and its upper quartile is 60, find its first quartile.

b. In a continuous series, the coefficient of quartile deviation is $\frac{3}{7}$ and its first quartile is 20, find its upper quartile.

5. Find the quartile deviation and its coefficient from the following data:

a.

Marks obtained	60-65	65-70	70-75	75-80	80-85	85-90
No. of Students	7	5	8	4	3	3

b.

Weight (kg)	20-30	30-40	40-50	50-60	60-70	70-80
No. of persons	5	15	10	8	6	2

c.

Class interval	20-30	30-40	40-50	50-60	60-70
Frequency	8	16	4	4	3

d.

Size	4-8	8-12	12-16	16-20	20-24	24-28	28-32	32-36	36-40
Frequency	6	10	18	30	15	12	10	6	5

e.

Height (in Inches)	60-62	62-64	64-66	66-68	68-70	70-72
No. of students	4	6	8	12	7	2

f

Expenditure	$0 \le x < 10$	$10 \le x < 20$	$20 \le x < 30$	$30 \le x < 40$	$40 \le x < 50$
No. of Workers	5	15	10	8	6

6. The following are the marks obtained by class 9 students in their internal examination. Taking class interval of (10-20) as first class, prepare a frequency distribution table and find the quartile deviation. Also find its coefficient from the following data:

a. 22, 25, 46, 34, 57, 69, 44, 36, 12, 27, 50, 36, 35, 62, 46, 52, 54, 61, 66, 55, 29, 39, 40, 33, 14, 41, 25, 20, 16.

b. 21, 45, 60, 57, 15, 41, 48, 50, 34, 29, 56, 40, 14, 62, 28, 70, 22, 30, 38, 74, 13, 47, 20, 53, 64, 34, 75, 66.

8.2 Mean deviation

We know that range depends on the largest and smallest value of the distribution. Quartile deviation depends on first quartile and third quartile. They are not based on all the observations and they do not measure the scatteredness of the items from the average value. Thus, they are not considered as good measure of dispersion. But mean deviation measures the variation of each observation of the total distribution from the average.

Mean deviation is defined as the average of the absolute values of the deviation of each item from mean, median or mode. It is also known as average deviation. Mean deviation calculated from mean is called mean deviation from mean. Similarly mean deviation from median is known as mean deviation from median. Mean deviation from mean is considered more reliable.

Mean deviation of continuous series

Let, $m_1, m_2, m_3, \ldots, m_n$ be the middle values of the corresponding classes $x_1, x_2, x_3, \ldots, x_n$ and $f_1, f_2, f_3, \ldots, f_n$ be their respective frequencies.

i) $\text{Mean deviation from mean} = \frac{\sum f|D|}{N}$

Where, $D = m - \overline{X}$, $\overline{X} = \text{mean of the class}$, m = mid value

f = frequency of the corresponding term, N = ∑f = total of the frequency

ii) $\text{Mean deviation from median} = \frac{\sum f|D|}{N}$

Where, $D = m - M_d$, M_d = median value of the class N = ∑f = total of the frequency

Mean deviation is an absolute measure. So to compare two or more series having different units, the relative measure corresponding to mean deviation is used, which is called coefficient of mean deviation. Coefficient of mean deviation is calculated from mean and median.

i) Coefficient of MD from mean $= \frac{\text{M.D from mean}}{\text{Mean }(\overline{X})}$

ii) Coefficient of MD from median $= \frac{\text{M.D from median}}{\text{Median (Md)}}$

Steps to compute mean deviation

i. Calculate the value of mean or median.
ii. Take deviations from the mean or median of each term.
iii. Ignore the negative signs of the deviation.
iv. Apply the formula to get average deviation from mean or median.

Example 1

Find the mean deviation from mean and also calculate its coefficient:

Marks	0-10	10-20	20-30	30-40	40-50
No. of Students	3	5	7	3	4

Solution: Here,
Tabulating the given data in ascending order for the calculation of mean deviation and it coefficient from mean

Marks	No. of Students (f)	Mid value (m)	fm	$m - \bar{X} = D$	|D|	F|D|
0-10	3	5	15	-20	20	60
10-20	5	15	75	-10	10	50
20-30	7	25	175	0	0	0
30-40	3	35	105	10	10	30
40-50	4	45	180	20	20	80
	$\sum f = N = 22$		$\sum fm = 550$			$\sum f\|D\| = 220$

$$\text{Mean } (\bar{X}) = \frac{\sum fm}{N} = \frac{550}{22} = 25$$

Now, mean deviation from mean = $\frac{\sum f|D|}{N} = \frac{220}{22} = 10$

Again, Coefficient of MD from mean = $\frac{MD}{\bar{X}} = \frac{10}{25} = 0.4$

Example 2

Calculate the mean deviation from median and its coefficient from the following data:

Weight (in kg)	10-20	20-30	30-40	40-50	50-60	60-70
No. of People	6	8	11	14	8	3

Solution: Here,
Tabulating the given data in ascending order for the calculation of mean deviation and its coefficient from median.

Weight	No. of People (f)	Cf	Mid-value (m)	$m - m_d = D$	|D|	f|D|
10-20	6	6	15	-25	25	150
20-30	8	14	25	-15	15	120
30-40	11	25	35	-5	5	55

40-50	14	39	45	5	5	70
50-60	8	47	55	15	15	120
60-70	3	50	65	25	25	75
	$\sum f = N = 50$					$\sum f\|D\| = 590$

Now, the position of median = $\left(\frac{N}{2}\right)^{th}$ item $= \left(\frac{50}{2}\right)^{th}$ item $= 25^{th}$ item

In Cf column, the corresponding class of 25^{th} item is 30-40

∴ median class is 30-40

Now, l = 30, f = 11, Cf = 14, i = 10

$$\therefore \text{Median (Md)} = l + \frac{\frac{\text{N}}{2} - \text{Cf}}{\text{f}} \times \text{i} = 30 + \frac{25 - 14}{11} \times 10 = 30 + \frac{11}{11} \times 10 = 40$$

Now, mean deviation from median = $\frac{\sum f|D|}{N} = \frac{590}{50} = 11.8$

Then coefficient of MD = $\frac{\text{MD}}{\text{Md}} = \frac{11.8}{40} = 0.295$

Exercise 8.2

1. a. Define mean deviation.

   b. What do you mean by coefficient of mean deviation?

2. a. Write the formula to find the mean deviation (MD) from mean.

   b. Write the formula to calculate mean deviation and its coefficient from median.

   c. What are the methods to find the mean deviation and its coefficient? Which one is the best? Write with reason.

3. a. In a continuous series, $\sum f$m = 1000, N = 50 and $\sum f|m - \overline{\text{X}}| = 308$ then find the mean deviation from mean and its coefficient.

   b. In a continuous series, $\sum f|m - \overline{\text{X}}| = 680$, mean deviation (MD) =17 find $\sum f$.

4. a. In a continuous series, median (M_d) = 40, $\sum f = 50$ and $\sum f|\text{m} - \text{Md}|$ = 530 then find the mean deviation from median and its coefficient.

   b. In a continuous series, coefficient of mean deviation is 0.5 and median = 40 then find the mean deviation (MD).

5. Find the mean deviation from mean and also calculate its coefficient from the following data:

a.

Marks	0-10	10-20	20-30	30-40	40-50
No. of Students	5	8	15	16	6

b.

Ages (in yrs.)	0-4	4-8	8-12	12-16	16-20	20-24
No. of Students	7	7	10	15	7	6

c.

Class interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No. of Students	7	12	18	28	16	14	8

d.

Daily Income (Rs.)	100-200	200-300	300-400	400-500	500-600	600-700	700-800
No. of Workers	4	6	10	20	10	6	4

6. Find the mean deviation from median and also calculate its coefficient from the following data:

a.

Weight (in kg)	20-25	25-30	30-35	35-40	40-45	45-50
No. of Students	7	3	6	4	8	2

b.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No. of Students	5	3	7	5	10	3	2

7. The following distribution gives the marks of 30 students in a certain examination:

Marks	20-30	30-40	40-50	50-60	60-70	70-80
No. of Students	7	12	18	28	16	14

Find the mean deviation from median and also find its coefficient.

8. Find the mean deviation from mean and its coefficient from the following distribution:

Class interval	$5 \le x < 10$	$10 \le x < 15$	$15 \le x < 20$	$20 \le x < 25$	$25 \le x < 30$
Frequency	7	4	5	6	3

9. In a survey of 28 students, the following are the marks obtained in a certain examination:

48	56	28	38	20	75	45
50	40	70	74	53	66	57
34	14	22	13	64	60	15
29	62	30	47	34	21	41

Construct a frequency distribution table taking 10-20 as first class interval. Then calculate i) Mean deviation from mean and its coefficient ii) Mean deviation from median and its coefficient.

10. a. Find the mean deviation from the mean of the following frequency distribution. Also, find its coefficient. The people having ages less than 70 years and above 10 years are participating in a survey:

Ages	No. of People
Less than 20	3
Less than 30	7
Less than 40	15
Less than 50	20
Less than 60	26
Less than 70	30

b. The students who obtained marks more than 0 and less than 48 are participating in a survey. The following frequency distribution is the marks obtained by 50 students in a certain class examination:

Marks	No. of Students
More than 0	50
More than 8	42
More than 16	35
More than 24	30
More than 32	18
More than 40	6

i. Find the mean deviation from mean and its coefficient.
ii. Find the mean deviation from median and its coefficient.

8.3 Standard deviation

We can find the dispersion of above data from various methods. But, Standard deviation is the most popular and useful measure of dispersion used in practice. It removes the drawback presented in other measure of dispersion. It gives the uniform, correct and stable result.

A Standard deviation (SD) is defined as the positive square root of mean of the square of deviation from the arithmetic mean. It is also known as root mean square deviation. It is denoted by Greek letter 'σ ' (read as sigma). It was first introduced by Karl Pearson in 1823.

Standard deviation is the best measure of dispersion because

i. Its value is based on all the observation.

ii. The deviation of each item is taken from mean.

iii. All algebraic sign are also considered.

The smaller value of Standard deviation (SD) means a high degree of uniformity of the observation as well as homogeneity of the series.

Calculation of Standard deviation from continuous series

We can find the standard deviation for continuous series by following methods:

a. Actual mean method
b. Assumed mean method (Short cut method)
c. Step deviation method
d. Direct method

a. Actual mean method

Let, $m_1, m_2, m_3, \ldots, m_n$ *be the mid values of the corresponding classes* $X_1, X_2, X_3, \ldots, X_n$ and $f_1, f_2, f_3, \ldots \ldots \ldots f_n$ be their respective frequencies and $\bar{X}$ be the actual mean then

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{\sum f(m-\bar{X})^2}{N}} = \sqrt{\frac{\sum f d^2}{N}}$$

where $\mathrm{d} = m - \bar{X}$ and $\mathrm{N} = \sum f$

S

Steps to be used in Actual mean method:

i. Find the midpoint of each class interval
ii. Find actual mean ($\bar{X}$) by formula $\bar{X} = \frac{\sum fm}{N}$
iii. Find the difference between each mid value and mean ($\bar{X}$) i.e. $d = (m - \bar{X})$
iv. Find the square of d i.e d^2
v. Multiply each value of d^2 by corresponding 'f' and denoted by fd^2.
vi. Find $\sum fd^2$
vii. Use formula,

$$\mathrm{SD}(\sigma) = \sqrt{\frac{\sum fd^2}{N}}$$

b. Assumed mean method (Short cut method)

This method is also called short cut method. To calculate Standard deviation, actual mean method is difficult and takes more time. In assumed mean method, by supposing a number as mean we will calculate standard deviation.

Let, $m_1, m_2, m_3, \dots \dots \dots \dots m_n$ be the mid value of the class intervals, $f_1, f_2, f_3 \dots \dots \dots f_n$ be their corresponding frequencies and 'A' be the assumed mean then.

$$\text{Standard deviation (SD) or } (\sigma) = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} \qquad \text{where } \mathrm{d = m - A}$$

Steps to be used in assumed mean method:

i. Consider as assumed mean 'A' from mid value.
ii. Take deviation of each item from 'A' and denoted it by d.
iii. Multiply each deviation 'd' by the corresponding frequency f and denoted by them by fd. Find $\sum$fd.
iv. Find the square of each value of d and denote it by d^2
v. Multiply each values of d^2 by the corresponding frequency (f) and denote them by fd^2.
vi. Find $\sum fd^2$.
vii. Use formula

$$\mathrm{SD}\ (\sigma) = \sqrt{\frac{\sum fd^2}{\mathrm{N}} - \left(\frac{\sum fd}{\mathrm{N}}\right)^2}$$

c. Step-deviation method

Let, $m_1, m_2, m_3, \dots, m_n$ be the mid value of the class intervals, $f_1, f_2, f_3, \dots, f_n$ be their corresponding frequencies and 'A' be the assumed mean then.

$$\text{Standard deviation (S.D) or } (\sigma) = \sqrt{\frac{\sum f d'^2}{N} - \left(\frac{\sum f d'}{N}\right)^2} \times i$$

$$\text{where, d} = \text{m} - \text{A}, \qquad d' = \frac{d}{C} = \frac{m - A}{C}, \qquad i = \text{Class interval}$$

Steps to be used in step-deviation method:

i. Find the middle value of each class interval and denote it by 'm'
ii. Take consider an assumed mean 'A'
iii. Take deviation of each item from A and denoted it by d.
iv. Each of the deviation d is divided by class interval 'i' and denote it by d'
v. Multiply each 'd' by corresponding f and denoted by fd'
vi. Find the square of d' and denote it by d'^2
vii. Multiply each of d'^2 by corresponding frequency 'f'. denote this by fd'^2
viii. Use formula

$$SD\ (\sigma) = \sqrt{\frac{\sum f d'^2}{N} - \left(\frac{\sum f d'}{N}\right)^2} \times i$$

d. Direct method

Let, $m_1, m_2, m_3, \dots, m_n$ be the mid values of the corresponding classes $X_1, X_2, X_3, \dots, X_n$ and $f_1, f_2, f_3, \dots, f_n$ be their corresponding frequencies then

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{\sum f m^2}{N} - \left(\frac{\sum f m}{N}\right)^2}$$

Example 1

Find the standard deviation and its coefficient from the following data:

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	3	5	4	3	2	3

By

i. Actual mean method

ii. Direct method

iii. Assumed mean method

iv. Step-deviation method

Solution: Here,

Tabulating the given data in ascending order for the calculation of SD and its coefficient

i. By actual mean method

Marks	No. of students (f)	Mid value (m)	fm	$d = m - \bar{X}$	d²	fd²
0-10	3	5	15	-22.5	506.25	1518.75
10-20	5	15	75	-12.5	156.25	781.25
20-30	4	25	100	-2.5	6.25	25
30-40	3	35	105	7.5	56.25	168.75
40-50	2	45	90	17.5	306.25	612.50
50-60	3	55	165	27.5	756.25	2268.75
	$\sum f = N = 20$		$\sum fm = 550$			$\sum fd^2 = 5375$

Now,

$$\text{mean}(\overline{\text{X}}) = \frac{\sum fm}{N} = \frac{550}{20} = 27.5$$

By formula,

$$\text{SD } (\sigma) = \sqrt{\frac{\sum fd^2}{N}} = \sqrt{\frac{5375}{20}} = \sqrt{268.75} = 16.39$$

ii. By direct method

Marks	No. of students (f)	Mid value (m)	fm	m^2	fm^2
0-10	3	5	15	25	75
10-20	5	15	75	225	1125
20-30	4	25	100	625	2500
30-40	3	35	105	1225	3675
40-50	2	45	90	2025	4050
50-60	3	55	165	3025	9075
	$\sum f = N = 20$		$\sum fm = 550$		$\sum fm^2 = 20500$

By formula,

$$SD\ (\sigma) = \sqrt{\frac{\sum fm^2}{N} - \left(\frac{\sum fm}{N}\right)^2} = \sqrt{\frac{20500}{20} - \left(\frac{550}{20}\right)^2} = \sqrt{1025 - 756.25}$$

$$= \sqrt{268.75}$$

$\therefore \sigma = 16.39$

iii. By assumed mean method

Marks	No. of students (f)	Mid value (m)	d = m – A	fd	d^2	fd^2
0-10	3	5	-30	-90	900	2700
10-20	5	15	-20	-100	400	2000
20-30	4	25	-10	-40	100	400
30-40	3	35	0	0	0	0
40-50	2	45	10	20	100	200
50-60	3	55	20	60	400	1200
	$\sum f = N = 20$			$\sum fd = -150$		$\sum fd^2 = 6500$

Now,

Let, assumed mean (A) = 35

By formula,

$$SD\ (\sigma) = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} = \sqrt{\frac{6500}{200} - \left(-\frac{150}{20}\right)^2} = \sqrt{325 - 56.25} = \sqrt{268.75}$$

$\therefore \sigma = 16.69$

iv. By step deviation method

Marks	No. of students (f)	Mid value (m)	d = m – A	$d' = \frac{d}{i}$	fd'	d'^{2}	fd'^{2}
0-10	3	5	-30	-3	-9	9	27
10-20	5	15	-20	-2	-10	4	20
20-30	4	25	-10	-1	-4	1	4
30-40	3	35	0	0	0	0	0
40-50	2	45	10	1	2	1	2
50-60	3	55	20	2	6	4	12
	$\sum f = N = 20$				$\sum fd' = -15$		$\sum fd'^2 = 65$

Let, assumed mean (A) = 35, class interval (i) = 10

By formula,

$$\text{SD } (\sigma) = \sqrt{\frac{\sum fd'^2}{N} - \left(\frac{\sum fd'}{N}\right)^2} \times i = \sqrt{\frac{65}{20} - \left(-\frac{15}{20}\right)^2} \times 10$$

$$= \sqrt{3.25 - 0.5625} \times 10 = \sqrt{2.6975} \times 10 = 1.6393 \times 10 = 16.39$$

And for the coefficient of standard deviation,

$$\text{Actual mean } (\overline{\text{X}}) = \text{A} + \frac{\sum \text{fd}'}{\text{N}} \times \text{i} = 35 + \left(-\frac{15}{20}\right) \times 10 = 35 - 7.5 = 27.5$$

$$\text{the coefficient of SD} = \frac{\sigma}{\overline{\text{X}}} = \frac{16.39}{27.5} = 0.596$$

Note:

1. $\text{Actual mean } (\overline{\text{X}}) = \text{A} + \frac{\sum \text{fd}}{\text{N}}$

2. $\text{Actual mean } (\overline{\text{X}}) = \text{A} + \frac{\sum \text{fd}}{\text{N}} \times \text{i}$

Coefficient of variation

It is purely a number and independent of unit. If the coefficient of standard deviation is multiplied by 100, it is known as coefficient of variation. Thus coefficient of variation denoted by C.V is defined by

$$\mathrm{CV} = \frac{\text{standard deviation}}{\text{mean}} \times 100\%$$

$$\text{or, CV} = \frac{\sigma}{\overline{\mathrm{X}}} \times 100\%$$

Greater the coefficient of variation, greater will be variation and less will be the consistency or uniformity. Less the coefficient of variation, greater will be the consistency or uniformity or more homogenous or more stable. Hence, to compare the two given distribution, we use the coefficient of variation.

Example 1

The following table gives the marks obtained by 20 students in a certain examination:

Marks obtained	30-40	40-50	50-60	60-70	70-80
No. of students	2	3	6	5	4

Find the arithmetic mean, standard deviation, coefficient of standard deviation and coefficient of variation.

Solution: Here,

Tabulating the given data in ascending order for the calculation of standard deviation and its coefficient.

Marks obtained	No. of students (f)	Mid value (m)	fm	$d = m - \bar{X}$	d^2	fd^2
30-40	2	35	70	-23	529	1058
40-50	3	45	135	-13	169	507
50-60	6	55	330	-3	9	54
60-70	5	65	325	7	49	245
70-80	4	75	300	17	289	1156
	$\sum f = N = 20$		$\sum fm = 1160$			$\sum fd^2 = 3020$

i. mean $(\bar{X}) = \frac{\sum fm}{N} = \frac{1160}{20} = 58$

ii. Standard deviation $(\sigma) = \sqrt{\frac{\sum f d^2}{N}} = \sqrt{\frac{3020}{20}} = \sqrt{151} = 12.29$

iii. Coefficient of SD $= \frac{\sigma}{\bar{X}} = \frac{12.29}{58} = 0.21$

iv. Coefficient of variation (CV) = $\frac{\sigma}{\bar{X}} \times 100\% = 0.21 \times 100\% = 21\%$

Example 2

An analysis of monthly wages paid to the works in two firm A and B belonging to the same industry given the following results:

	A	B
Average monthly wage	Rs. 586	Rs. 575
Standard deviation of distribution of wage	Rs. 9	Rs. 10

i. Examine which firm A or B has greater variability in wage distribution

ii. Which firm has more homogeneity? Give Reason.

Solution: Here,

For Firm A

$\bar{X} = \text{Rs.} 586 \qquad \sigma = \text{Rs.} 9 \qquad \text{C.V} = ?$

We have,

$$CV = \frac{\sigma}{\bar{X}} \times 100\% = \frac{\text{Rs.} 9}{\text{Rs.} 586} \times 100\% = 1.53\%$$

For Firm B

$\bar{X} = \text{Rs.} 575, \qquad \sigma = \text{Rs.} 10 \qquad \text{C.V} = ?$

$$CV = \frac{\sigma}{\bar{X}} \times 100\% = \frac{\text{Rs.} 10}{\text{Rs.} 575} \times 100\% = 1.73\%$$

i. Since, the CV for the firm B is higher, so the variability of wage distribution is firm B is greater.

ii. The CV for the firm A is less then B, so the firm A has more homogeneity.

Note:

Less cv
- more uniformity
- more consistency
- more stability
- more homogenity

more CV
- more variability
- more scatterness

Exercise 8.3

1. a. Define standard deviation.

   b. What is coefficient of standard deviation? Also, write its formula.

   c. Define coefficient of variation. Also, write its formula.

2. a. Write the difference between standard deviation and mean deviation.

   b. Write the difference between coefficient of standard deviation and the coefficient of variation.

3. a. In a continuous series, $\sum f(m-\bar{X})^2 = 484,\ \mathrm{N} = 24\ \ and\ \bar{\mathrm{X}}$ = 25 find the standard deviation and its coefficient.

   b. In a continuous series $\sum fd = 0,\ \sum fd^2 = 848,\ N = 100,$ assumed mean (A) = 12 then find the standard deviation and its coefficient.

4. a. If $\sum fd'^2 = 125,\ \sum fd' = -4,\ N = 20,\ C = 10,$ find SD (σ)

   b. What is the meaning a low coefficient of variation?

5. Find the standard deviation and its coefficient from the following data:

   a.

Class interval	0-4	4-8	8-12	12-16	16-20	20-24
Frequency	7	7	10	15	7	6

   b.

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	8	12	20	40	12	8

   c.

Daily Wages (Rs.)	100-125	125-150	150-175	175-200	200-225
No. of workers	75	57	81	19	12

6. a. The following table gives the marks obtained by 40 students of class 10 in a unit test of Opt. Mathematics. Calculate the standard deviation and coefficient of variation.

Marks obtained	30-40	40-50	50-60	60-70	70-80	80-90
No. of Students	4	8	10	16	6	6

b. Calculate the coefficient of standard deviation and coefficient of variation from the following data:

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No. of students	10	15	25	25	10	10	5

7. a. The following table gives per day salary of 50 high school teachers:

Salary	1000-1100	1100-1200	1200-1300	1300-1400	1400-1500	1500-1600
No. of teachers	10	15	5	8	10	2

i. Find the average salary of teachers.

ii. Find the standard deviation of the above data.

iii. Find the coefficient of variation of the data.

b. The following distribution gives the marks of 30 students in a certain examination:

Marks	20-30	30-40	40-50	50-60	60-70	70-80
No. of students	3	5	6	8	4	4

Find the standard deviation and coefficient of variation by

i. Actual mean method

ii. Assumed mean method

iii. Step deviation method

8. a. A purchasing agent obtained samples of incandescent lamps from two suppliers. He had the samples tested in his own laboratory for length of life with the following results:

Lengths of life (hrs)		700-900	900-1100	1100-1300	1300-1500
Sample of company	A	10	16	26	8
	B	3	42	12	3

i. Which company's bulb gives a higher average life?

ii. Find the standard deviation of both companies.

iii. Which company's lamps are more uniform?

b. An analysis of marks obtained by boys students and girls students are given below:

No. of students	Boys	Girls
Average marks	63	54
Standard deviation of marks	9	6

i. Which group has greater variability in marks obtained?

ii. Which group has more uniform marks obtained?

9. The weight (in kg) of 20 people are given below. Construct a frequency distribution table taking (20-30) as the first class interval. Also, find the standard deviation and the coefficient of variations.

59,	71,	45,	44,	35,	21,	29,	49,	42,	37,
58,	69,	55,	39,	79,	50,	65,	52,	60,	64

10. Construct a frequency distribution table taking (0-10) as the first class interval. Then, find the standard deviation and its coefficient.

25,	24,	45,	28,	33,	10,	20,	5,	11,	30,
25,	40,	15,	31,	2,	29,	23,	17,	14,	26,
30,	13,	41,	35,	7					

11. Collect the data related to your locality and find coefficient of variation of the data. Present your finding in your class.

Answer

Note: Remaining answers: show to your subject teacher

Exercise 1.1.1(a)

2.(a) constant function (b) Identify function (c) cubic function (d) quadratic function

(e) cubic function (f) quadratic function

4.(a) y = 4x – 130 (b) 118 pounds

Exercise 1.1.1. (b)

1(a) +1, -1 (b) +1, -1 (c) No maximum, no minimum values

2(a) 3π b() 2π (c) π

Exercise 1.1.2

2.(a) (i) {3, 5, 9} (ii) {4, 6, 10} (iii) does not exists (iv) {16, 36, 100}

(b) (i) {1, 2, 3} (ii) {2, 3 4} (iii) does not exist (iv) {3, 4, 5}

3.(a) $2x^2 - 4x + 7$, $4x^2 - 24x + 41$ (b) 37, 9 (c) 30, 21 (d) 55, 201

4.(a) 2x and 2x (b) $2(x^2 - 2x + 3)^3 + 1$, $4(x^2 + 2)^2 - 16(x^2 + 2) + 18$

8(a) 3t (b) πr^2 (c) $(A\circ r)(t) = 9\pi r^2$

Exercise 1.1.3

2(a) $f^{-1} = \{(2,1), (3,2), (5,4)\}$ (b) $g^{-1} = \{(4, 1), (5, 2), (6, 3)\}$

(c) $h^{-1} = \{(4, -2), (9, -3) (36, -6)\}$

3(a) (i) $\frac{1}{3}(x + 5)$ (ii) $\frac{5}{3}$ (iii) $-\frac{1}{4}$ (iv) -1

(b) (i) $\frac{1}{2}(x + 5)$ (ii) $\frac{11}{2}$ (iii) $\frac{9}{8}$ (iv) $\frac{3}{2}$

(c) (i) 2x – 1 (ii) 11 (iii) $-\frac{1}{2}$ (iv) -5

(d (i) $\frac{2(1+x)}{(1-x)}$ (ii) $\frac{14}{5}$ (iii) $\frac{10}{3}$ (iv) $\frac{-2}{3}$

4.(a) $\frac{1}{2}(x + 2)$ (b) $2(x-1)$ 5.(a) $\frac{5}{2}$ (b) -11, 5

6. Volume: $f(x) = \frac{4}{3}\pi x^3$, $f^{-1}(x) = \left(\frac{3x}{4\pi}\right)^{\frac{1}{3}}$

Surface area: $f(x) = 4px^2$, $f^{-1}(x) = \pm\sqrt{\frac{x}{4\pi}}$

Exercise 1.2.1

2. (a) $x - 7$; 0 (b) $x^2 + x - 6$; 0 (c) $x^2 + 2x + 4$; 0 (d) $x^2 + 6x + 9$; 0

3.(a) $x^2 + x - 6$; -1 (b) $x^2 - 15x + 91$; -429

(c) $2x^3 + x^2 + \frac{11}{2}x - \frac{1}{4}$; $\frac{-57}{8}$ (d) $y^4 + 3y^3 + 10y^2 + 30y + 89$; 267

4(a) yes (b) yes (c) No (d) No (e) No

Exercise 1.2.2

2.(a) quotient = $x^2 - 2x + 4$, remainder = 0 (b) quotient = $2x^3 + x^2 - 3x + 10$

(c) quotient = $4x^2 - 11x + 23$, remainder = -37 (d) quotient = $2x^2 + x - 3$, remainder = 1

(e) quotient = $8x^2 + 8x + 10$, remainder = -2

Exercise 1.2.3(A)

2(a) – 7 (b) 4 (c) 3 (d) $\frac{645}{32}$ (e) 456 (f) $\frac{-283}{8}$ (g) $\frac{680}{81}$

3(a) – 12 (b) $\frac{11}{5}$ (c) $\frac{-6}{7}$ (d) $\frac{31}{2}$

Exercise 1.2.3 (B)

2(a) yes (b) No (c) No (d) No (e) No

3(a) 11 (b) -4 (c) $\frac{-13}{2}$

4.(a) $(x - 1)(x + 2)(2x + 1)$ (b) $(x - 1)(x + 1)(x + 2)$

(c) $(y + 1)(y - 2)(y - 5)$ (d) $(x + 1)(x + 10)(x + 2)$

(e) $(x - 1)(x + 1)(2x + 1)$ (f) $(x - 1)(x - 10)(x - 12)$

(g) $(x + 2)(2x - 3)(x + 6)$

5.(a) 1, 5, -2 (b) -3, 2, 1 (c) -1, 1, $\frac{1}{3}$ (d) -1, -1,5 (e) -3, 2, 4 (f) 1, 1, 6

Exercise 1.3.1

3.a) -1, -2, 1 – 2n, -9, -11 (b) 2, 4, 4x – 2, 18, 22

(c) 7, 10, 10n – 3, 47, 57 (d) $\frac{5}{4}, \frac{-1}{4}, \frac{6-n}{4}, \frac{1}{4}$, 0

4.(a) $t_n = a + (n-1)\,d$ (b) yes, common difference is constant (c) first term

5.a) 13 (b) 112 6.(a) 11 (b) -5 7.(a) 6 (b) – 2

8.(a) 6 (b) 5 9.(a) 23 (b) 70 10.(a) 312 (b) 4

11.(a) -10 (b) 2 12(a) 34 (b) 27 (c) 11

13.(a) yes (b) No 14.(a) 2 (b) b = $\frac{15}{2}$, 60, 43, 22

15)(a) (i) -32, 7 (ii) 178 (iii) -32, -25, -18, …..

(b) (i) 8, -2 (ii) 5 (iii) No, because, no of terms is zero

16. a) 46 (b) 78 18) a) 13 (b) 22

19) -13, -8, -3 20.(a) 2030 (b) 9 weeks

Exercise 1.3.2

1.a) no of mean (b) last term 2.(a) 0 (b) a

3.(a) 2 (b) 5 4.(a) 8 b) t_6 5.(a) 110 (b) 15

6.a) 15, 20 (b) -8, -3, 2 7.(b) 40 (c) 16, 24 8.(a) 10, 16
(b) 70, 40

9.(a) 20, 30, 40, 50, 60 (b) 15, 19, 23, 27, 31, 35

10.(a) 5, 3:4 (b) n = 5, 13, 17, 25, 29 11.(a) 17 (b) 3, 31

Exercise 1.3.3

1.(a) $S_n = \frac{n}{2}[2a + (n-1)d]$ (b) 8 2(a) 25 (b) 30 3.(a) -180 (b) 670

(c) $\frac{2093}{2}$ (d) -8930 4.(a) 120 (b) 147

5.(a) 380 (b) $\frac{-35}{3}$ 6.(a) d = 6 (b) 7

7.(a) 1089 (b) 945 (c) 625 8.(a) 6 (b) 8 9.(a) n = 10 or 11 (b)(i) 10, 4 (ii) 10 + 14 + 18 + …. (iii) 960

10)(a) (i) 17, 3 (ii) 1325

(b) (i) 1, 2 (ii) 1 + 3 + 5 + … (iii) 400

11.(a) 6, 10, 14 or 14, 10, 6 (b) 2, 4, 6 or 6, 4 2

12.(i) Rs. 160 (ii) 140 (iii) 120 (iv) 40

Exercise 1.3.4

1, 2 and 3 consult with your teacher

4.(a) 3 (b) 64 (5) (a) 2 (b) 4 (c) 6 6.(a) -8 (b) $\frac{3}{2}$, 2

7.(a) (i) $\frac{-2}{3}$, -81 (ii) -81, 54, -36, ……… (iii) $\frac{-512}{243}$

(b)(i) $\frac{-1}{5}$, -3750 (ii) -3750, 750, -150, …. (iii) $\frac{6}{125}$

8.(a) 6561 (b) $\frac{1}{6561}$ (c) ±2 9.(ii) r = $\frac{3}{4}$ (iii) $\frac{19683}{262144}$

Exercise 1.3.5

1.(b) $\sqrt{pq}$ (c) 1

2.(a) 5 (b) 36 (c) 5 3.(a) 10, 8, 2 (b) 12

4.(a) 12, 24 (b) 10, 20, 40 (c) 1 4

5.(a) $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, 1, 2, 4, 8, $\frac{15}{18}$ (b) 1, 2, 4, 8, 16 and 31

6.(a) q = 972, 12, 108, 324 (b) n = 3, 10 , 20

7.(a) 80, 20 or 20, 80 (b) 81, 9 or 9, 81

8.(a) 7 (b) 6

Exercise 1.3.6

1.(a) 10 (b) 31 2.(a) 4679 (b) r = 2 (c) 81, $\frac{1}{3}$

3.(i) $\frac{3069}{64}$ (ii) 129 (iii) $\frac{665}{9}$ (iv) $\frac{255\sqrt{2}}{128}$ (v) 8745 (vi) 1533

4.(a) ±3 (b) ±2 5.(a) 17 (b) 4 (c) $\frac{634}{3}$

6.(a) 6 (b) 6 7.(a) 510 (b)(i) 3, 1 (ii) 3 + 9 + 27 + ...

(iii) 3280 8.(a)(i) 2, 6 (ii) 6 + 12 + 24 + ... (iii) 378 (b) $\frac{255}{4}$

9.(a) 4, 8, 16 or 16, 8, 4 (b) 1, 3, 9 or 9, 3, 1

10.(a) 32, 16, 8 (b) 3, 5, 7 or 12, 5, -2

Exercise 1.4.1

1, 2, 3, 4, 5 and 6 show the subject teacher

7.(a) Max. value = 59 at B(3, 10)

Min. value = 9 at A(3, 0)

(b) Max. value = 26 at B(2,4)

Min. value = 0 at 0(0, 0)

8.(i) (2, 0), (3, 0), (2, 1) (ii) 2, -1), (4, 0), (9/2 , 1)

(iii) (2, 0), (4, 0), (4/5, 12/5) (iv) (0,0), (1, 0), (3, 1), (0, 4)

9.(i) Max = 96 (ii) Max = 51 at (1, 5/2) (iii) 36 at (6, 4) (iv) 10 at (2, 2)

10(i) 0 at (0, 0) (ii) 2 at (2, 0) (ii) 9 at (3, 0) (iv) 15 at (1, 1)

11. Max = 8, Y $\geq$ 0, x + y = 2, 2x – 3y $\geq$ –6

12) Max = 16, Min = –12, x – 3y > -4, x + 2y $\geq$ –4, $x \leq 2$

Exercise 1.5.1

Show 1 and 2 to the subject teacher.

3.(i) (0,0), x = 0 (ii) (–1, 1), x = –1 (ii) (–1, 6) x = –1

4.(a) Turning upwards from origin

(b) A curve line from 2nd quadrant to fourth quadrant through origin.

6.(a) (–1, 1) (ii) (6, 0) (iii) (0, 0) (iv) $\left(\frac{-3}{2}, \frac{-1}{4}\right)$ (v) (3, –4)

8. (i) x = –3, 1 (ii) x = 0, 2 (iii) x = 1, 2

y = 9, 1 y = 0, 2 y = -1, 0

(iv) x = 0.3, –2.3 (v) x = –4, 2 (vi) x = 10 , –1

y = 0.3, –2.3 y = –7, 5 y = –6, 5

9.(i) –3, 1 (ii) 2, 3 (iii) 5, –3 (iv) –1, $\frac{-2}{3}$

(v) 3, $\frac{1}{2}$ (vi) –5, –1

Exercise 2.1

1. (a) N = {1, 2, 3, ….} (b) W = {0, 1, 2, 3, ……..} (c) I = {……, -2, -1, 0, 1, 2, ……}

   (d) Q = {x : x = 1/q, p and q are integers and q ≠ 0}

   (e) Q = {x : x ∉ Q} (f) R = {x : -a < x <q}

   2, 3, 4 and 5 show to your teacher

Exercise 2.2

1(a) $-4 \leq x \leq 4$, at $x = 2$ discontinuous.

(b) $-4 \leq x \leq 3$, at $x = 1$ discontinuous, at remaining points continuous.

(c) $-8 < x < 10$, at $x = 5$ discontinuous at remaining points continuous.

(d) $6 \leq x \leq 7$, at $x = 0$, discontinuous, at remaining points continuous.

(e) $-\infty < x < \infty$, at $x = 0$ discontinuous at remaining points continuous.

(f) $-6 \leq x \leq 6$, at $x = -2$ discontinuous, at remaining points continuous.

2, 3 show to your teacher.

Exercise 3.1

2.(a) (i) -2 (ii) -17 (iii) 3 (iv) $-3y^2 + 2y$

(b) -14 (c) $b^2 - a^2$ (d) 1 (e) $\frac{1}{2}$

3.(a) ±2 (b) 2 (c) -20 (d) 1.5

4.(a) 45 (b) 228 (c) 185 5) $\begin{bmatrix} 7 & -9 \\ 21 & -7 \end{bmatrix}$ 6) -168

Exercise 3.2

2.(a) |A| = 0, does not exist (b) |A|≠ 0, exists (c) |A| ≠ 0, exists

4.(a) $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$ 5) $\begin{bmatrix} 19 & 27 \\ 36 & 68 \end{bmatrix}$

Exercise 3.3

2(a) (3, 2) (b) (1, 0) (c) (3, 3) (d) 3, 7) (e) (1/5, 3/5)

(f) (-1,2) [correction in q. no. 2]

(g) (1/5, 3/5) (f) (-1,2) [correction in q. no. 2]

(g) $\left(\frac{1}{6}, \frac{1}{4}\right)$ (h) $\left(\frac{1}{3}, \frac{5}{2}\right)$ 3(a) Pen: Rs 70, copy: Rs 50 (b) 49 years, 4 years

Exercise: 3.4

2(a) (2,1) (b) $\left(\frac{7}{2}, \frac{21}{2}\right)$ (c) $\left(\frac{23}{251}, \frac{-23}{72}\right)$ (d) (1, 0) (e) $\left(\frac{32}{7}, \frac{96}{13}\right)$

Exercise 4.1

1. Show to your teacher

2.(a) $\tan\theta = \pm\left(\frac{a_2b_1 - a_1b_2}{a_1a_2b_1b_2}\right)$ (b) $a_2b_1 = a_1b_2$ (c) $a_1a_2 + b_1b_2 = 0$

3.(a) 45° (b) 47.72° (c) 60° 4.(a) 135° (b) 150° (c) 172.87°

5.(c) -2 (d) 3 (e) $\frac{-5}{2}$

6.(a) $3x - 4y + 5 = 0$ (b) $3x + 4y + 17 = 0$ (c) $4x + 54y - 23 = 0$

(d) $7x - 5y + 22 = 0$

7(a) $x - y + 1 = 0$, $x + y - 5 = 0$ (b) $x - 5y - 21 = 0$, $5x - y + 1 = 0$

(c) $x - y + 1 = 0$, $x + y - 3 = 0$

8(a) $x - 2y - 1 = 0$ (b) $x - 7y - 53 = 0$, $7x + y - 21 = 0$

(c) $x - 3 = 0$, $y - 4 = 0$

Exercise 4.2

2.(a) $x^2 - 6xy + 5y^2 = 0$ (b) $6x^2 - 7xy - 3y^2 = 0$ (c) $x^2 - 4y^2 = 0$

(d) $x^2\cos\theta - (1 + \sin\theta\cos\theta)xy + y^2\sin\theta = 0$

3.(a) $x + 2y = 0$, $3x + y = 0$ (b) $2x + y = 0$, $3x + y = 0$

(c) $x - \sqrt{3}y = 0$, $\sqrt{3}x - y = 0$

(d) $y = (\operatorname{cosec}\theta + \tan\theta)x$; $y = (\operatorname{cosec}\theta - \cot\theta)x$ (e) $y = (\sec\theta + \tan\theta)x$; $y = (\sec\theta - \tan\theta)x$

(f) $bx + ay = 0$; $ax - by = 0$ (g) $x - y = 0$; $x - y - 1 = 0$

(h) $(y - x) = 0$, $(y + x)\cos^2\theta - 2x = 0$

4.(a) 90° (b) 45°, 135° (c) 45°, 135° (d) 2, 180° - α

6(a) – 1 (b) ±3 (c) – 1 (d) 4 (e) ±6

7(a) $4x^2 + 5xy + y^2 = 0$ (b) $2x^2 - 3xy + y^2 = 0$ (c) $y^2 - x^2 = 0$

Exercise 4.3

Show to your teacher

Exercise 4.4

2(a) (0,0), 3 (b) (2,0), 5 (c) (-1, 3), 4 (d) (1, 0) , $\sqrt{6}$ (e) (2, 3) 6

(f) (4, -1), $\sqrt{41}$ 3.(a) $x^2 + y^2 = 25$

(b) $2x^2 + 2y^2 = 9$ (c) $x^2 + y^2 - 6x - 8y- + 16 = 0$ (d) $x^2 + y^2 + 8y = 0$

(e) $x^2 + y^2 - 2x - 4y - 20 = 0$

(f) $x^2 + y^2 + 2x + 6y - 26 = 0$

4.(a) (1,3), $\sqrt{10}$, $x^2 + y^2 - 3x - 6y = 0$ (b) (3,-1), 5, $x^2 + y^2 - 3x - 6y$

(c) $\left(\frac{11}{12}, \frac{7}{4}\right)$, $\frac{\sqrt{149}}{6\sqrt{2}}$, $12x^2 + 12y^2 - 22x - 24y - 5 = 0$

5.(a) $x^2 + y^2 - x + 3y - 10 = 0$ (b) $x^2 + y^2 - 4x - 6y - 12 = 0$

(c) $x^2 + y^2 - 10x - 4y + 24 = 0$

6.(a) $x^2 + y^2 - 10x - 10y + 25 = 0$ (b) $x^2 + y^2 + 6x - 8y + 9 = 0$

(c) $x^2 + y^2 + 8x + 10y + 25 = 0$

7, 8 show to your teacher

9) $x^2 + y^2 - 4x + 2y - 20 = 0$ 10) show to your teacher

Exercise 5.1

1.(b) $\cos 2A = 2\cos^2 A - 1$ $\cos 2A = 1 - 2\sin^2 A$

2.a) $\frac{2tanA}{1+tan^2A}$ (b) $\frac{1-cos2A}{1+cos2A}$ (c) Sina3A = 3SinA – $4\sin^3 A$

(d) tan3θ = $\frac{3tan\theta - tan^3\theta}{1-3tan^2A}$

3(a) $\frac{7}{25}$ (b) $\frac{3}{5}$ (c) $\frac{24}{25}$ (d) $\frac{120}{169}, \frac{119}{169}$ (e) $\frac{24}{25}$ (f) 1, $\frac{-9\sqrt{3}}{16}$

(g) 1, 0 (h) $\frac{11}{2}$

Exercise 5.2

2.(a) $\frac{2}{9}$ (b) 0 (c) = 1

3.(a) $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$ (b) $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$ (c) $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$

4.(a) -1, 0 (b) 1 (c) –1

Exercise 5.3

1 and 2 show to your subject teacher

3.(a) 1 (b) 1 4.(a) –2sin55°.sin15° b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{4}$ (e) 1 (f) $\frac{1}{4}$

5.(a) $\frac{1}{2}$[cos18° – cos104°] (b) $\frac{1}{2}$[cos12° - $\frac{1}{2}$] (c) $\frac{-1}{2}$sin100°

(d) sin80° + sin16° (e) sin7θ + sin3θ (f) $\frac{1}{2}[\sin 16\theta + \sin 2\theta]$

(g) cos14θ + cos8θ (h) cos4θ – cos10θ

6.(a) $\sqrt{2}\cos 20°$ (b) 2cos33°sin13° (c) 2cos50°.cos20°

(d) 2cos5θ.sin2θ (e) 2sin7θ.cos4θ (f) 2cos10α.cos5α

Exercise 5.5

1 show to your subject teacher.

2.(a) 60° (b) 360° – θ (c) Max. value = 1, Min. value = 0

3.(a) θ = 60° (b) θ = 60° (c) θ = 60° (d) θ = 45° (e) θ = 30°

(f) θ = 90° (g) θ = 45° (h) θ = 60°

4.(a) θ = 120° (b) θ = 135° (c) θ = 60°, 120° (d) θ = 60°

(e) $\theta = 60°, 120$ (f) $\theta = 150°$

5.(a) $\alpha = 45°, 135°$ (b) $\alpha = 60°, 120°$ (c) $\alpha = 60°, 120°$

(d) $\theta = 60°, 120°$ (e) $\alpha = 60°, 120°$ (f) $\theta = 60°, 120°$

6.(a) $\theta = 90°, 150°$ (b) $\theta = 30°, 150°$ (c) $\theta = 30°, 150°$ (d) $\theta = 30°$

(e) $\theta = 30°, 90°, 150°$ (f) $\theta = 0°, 60°, 180°$ (g) $\theta = 10°, 50°, 90°, 130°, 170°$

(h) $\theta = 15°, 45°, 75°, 105°, 135°, 165°$

7.(a) $0°, \cos^{-1}\left(\frac{1}{3}\right), 360°$ (b) $120°, 240°$ (c) $45°, 225°$

(d) $0°, 180°, 360°$ (e) $45°, 135°$ (f) $45°, 135°, 225°, 315°$

(g) $30°, 135°, 210°, 315°$ (h) $120°, 150°, 300°, 330$

(i) $60°, 135°, 240°, 315°$ (j) $0°, 60°, 300°, 360°$

8.(a) $0°, 120°, 360°$ (b) $90°, 330°$ (c) $45°$ (d) $0°, 60°, 360°$

(e) $105°, 345°$ (f) $60°$ (g) $300°, 360°$ (h) $60°, 180°$

(i) $0°, 120°, 360°$

9.(a) $0°, 60°, 90°, 120°, 180°$ (b) $0°, 60°, 180°$ (c) $45°, 60°, 135°, 300°$

(d) $30°, 60°, 90°$ (e) $0°, 180°$ (f) $18°, 90°$ (g) $0°, 90°$ 10) $20°, 30°, 80°$

11.(a) $60°$ (b) $45°, 315°$

Exercise 5.6

1 and 2 show to your teacher

3.(a) $150\sqrt{3}$m (b) $30\sqrt{3}$m (c) 25.35 m

4.(a) 42.26m (b) 8.87m (c) 320m

5.(a) 38.97m (b) 18.30m

6.(a) 8m (b) $50\sqrt{2}$m, $100\sqrt{2}$m

7.(a) 14.64 m (b) 7.32m, $10\sqrt{3}$m

8.(a) 178.9m (b) 288m

9.(a) 136.96km/hr (b) $29\sqrt{3}$m, it is 20m from on pole and 60 m form another

10. (a) 15 m (b) 19.12m (c) 100m

Exercise 6.1

1, 2 and 3 show to your teacher

4.(a) 7 (b) 3-$\sqrt{3}$ (c) 19

5.(i) $\overrightarrow{AB} = \vec{i} - 4\vec{j}$, $\overrightarrow{BC} = 4\vec{i} + \vec{j}$, $\overrightarrow{CD} = -\vec{i} + 4\vec{j}$, $\overrightarrow{DA} = 4\vec{i} + \vec{j}$, $\overrightarrow{BD} = 3\vec{i} + 5\vec{j}$

(ii) -17 (iii) 34, 17

6.(a) -14, obtuse (b) (i) 1 (ii) -2 (iii) 6 (iv) 5 (v) 13 (vi) 4

7.(a) (i) 60° (ii) 150° (b) (i) 90° (ii) 0° (iii) 74.74° (iv) 45°

(c) (i) 60° (ii) 41.56°

8.(b) (i) –3 (ii) 6 (iii) 3 9.(a) (i) 8 (ii) 7 (iii) 48

10.(a)(i) 0° (ii) 60° (b)(i) 90° (ii) $\cos^{-1}\left(\frac{1}{4}\right)$

Exercise 6.2

1 and 2 show to your teacher.

3.(a) $4\vec{i} - 2\vec{j}$ (b) $\frac{1}{2}(\vec{i} + \vec{j})$ (c) $2\vec{i} + 4j$

4.(a) $\frac{1}{5}(11\vec{i} + 21\vec{j})$ (b) $\frac{1}{3}(-19\vec{i} - 50\vec{j})$

5.(a) $\frac{1}{3}(5\vec{i} + 26\vec{j})$ (b) $24\vec{i} + 4\vec{j}$ 6.(a) $2\vec{i} + \frac{5}{3}\vec{j}$ (b) $10\vec{i} + 4\vec{j}$

7.(b) $\overline{-}\vec{i} - \frac{10}{3}\vec{j}$

Exercise 7.1

1. show to your teacher.

2.(a) (-2, 1) (b) (3, 2) (c) (-4, - 3) (d) 9m 8) (e) (2, –4)

(f) (-12, -24) (g) (-4, -6) (h) (6, 18)

3, 4, 5, 6, 7 show to your teacher.

Exercise 7.2

2.(a) $\left(\frac{2}{13}, \frac{3}{13}\right)$ (b) $\left(\frac{-4}{5}, \frac{8}{5}\right)$ (c) $\left(0, \frac{9}{4}\right)$ (d) $\left(\frac{32}{13}, \frac{-48}{13}\right)$

(e) $\left(\frac{25}{26}, \frac{125}{26}\right)$ (f) $\left(\frac{256}{25}, \frac{192}{25}\right)$ (g) $\left(\frac{-22}{13}, \frac{45}{13}\right)$ (h) $\left(\frac{69}{65}, \frac{163}{65}\right)$

3.(a) $\left(\frac{48}{17}, \frac{80}{17}\right)$ (b) $\left(\frac{-25}{61}, \frac{54}{61}\right)$ (c) $\left(\frac{-75}{49}, \frac{142}{49}\right)$

4. Show to your teacher.

Exercise 7.3

1. Show to your teacher.

2.(a) A' (7, 7) (b) p'(-1, 9)

3.(a) A'(2, 0), B'(4, 2), C'(-3, 3) (b) A'(2, 0), B'(12, 8), C'(9, 10), D'(4, 8)

(c) O(0,0), P'(2, 14), Q'(8, 16), R'(-6, 8)

4.(a) $\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

5.(a) $\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

6.(a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

7 and 8 show to your teacher.

Exercise 8.1

1 and 2 show to your teacher.

3.(a) 10 (b) 45 4.(a) 20 (b) 50

5.(a) 6.31, 0.09 (b) 10.65kg, 0.23 (c) 7.39, 0.19 (d) 5.5, 0.273

(e) 1.0225 inches, 0.015 (f) 9.875, 0.41

6.(a) 13.21, 0.327 (b) 16, 0.36

Exercise 8.2

3.(a) 6.16, 0.308 (b) 40 4.(a) 10.6, 0.265 (b) 20

5.(a) 9.44, 0.35 (b) 5.08, 0.423 (c) 12.82, 0.36 (d) 113.3, 0.25

6.(a) 727, 0.213

7. 11.53, 0.21

8. 10.08, 0.30 9.(i) 15.81, 0.35 (ii) 15.85, 0.36

10. Show to your teacher.

Exercise 8.3

3.(a) 4.49, 0.17 (b) 2.91, 0.242

4.(a) 24.91 (b) greater will be the consistency

5.(a) 6.05 (b) 12.96, 0.41 (c) 28.35

6.(a) 11.23, 23% (b) 0.51, 51.41%

7.(a)(i) 1248.12 (ii) 155.55 (iii) 12.46%

(b)(i) 14.98, 29.57% (ii) 14.985, 29.57% (iii) 14.98, 29.57%

8.(a)(i) A (ii) 184.27, 124.49 (iii) B

(b) (i) Boys (ii) Girls

9 and 10 show to your teacher.