Uniformly Distributed Modulo 1 Sampling For Computer Graphics

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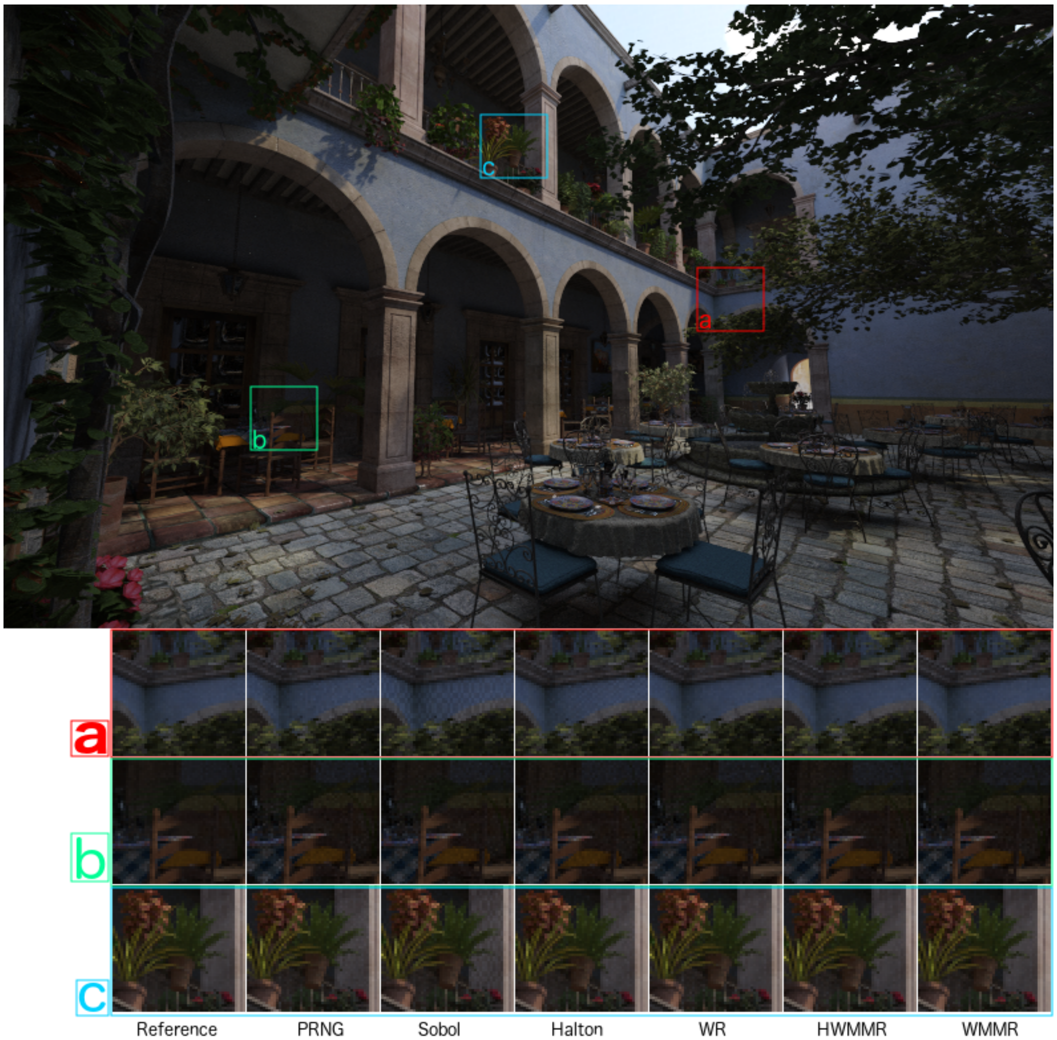

Figure 1: *San Miguel Path Tracing Reference rendered at 16384spp and the comparisons at 4096spp.*

Abstract

Random Numbers play a very fundamental role for Sampling in Computer Graphics Imagery (CGI). Since the advent of Computer Graphics the type of Random Numbers used were mainly Pseudo Random Number Generator(PRNG) and Quasi Random Number Generator(QRNG). These methods were based upon Rational Numbers. In this paper Random numbers based upon the theory of Uniformly Distributed Modulo 1(UDM1) are introduced for computer graphics. The UDM1 Sequences that will be used for sampling will be based on Weyl's Criterion. These sequences are also Deterministic in nature like the QRNG but are based upon Irrational Numbers instead of Rational one's. Being Deterministic in nature UDM1 provide faster convergence for monte carlo integral applications. Unlike the PRNG and QRNG , they are quite simple to generate, making them a drop in replacement.

The main contributions of this paper will be the following:

1. Formal Introduction of UDM1 for Sampling in Computer Graphics.
2. Multi-dimensional sampling scheme based on Weyl/Kronecker Sequences for Monte Carlo Path Tracing. The only other previous ones with demonstrated applications were PRNG and QMC Sequences based on Halton and Sobol .
3. A New UDM1 Point Set based on the combination of Equally Partitioned Interval Set and PRNG. This point set when paired with a Weyl sequence gives state of the art results in 2d sampling for the application discussed in 4.2.

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4. New Modified Weyl/Kronecker Sequences based on metallic means discussed in Section (5). This gives competitive and in some cases state of the art results for Multi dimensional sampling for Path Tracing.

Keywords: Sampling, Rendering, Global Illumination, Uniformly Distributed Modulo 1, Real Analysis, Number Theory, Weyl Sequences, Kronecker Sequences, Richtmyer Sequences, Golden Ratio, Metallic Means Ratio

Concepts: •**Computing methodologies** → **Computer graphics;** *Rendering; Ray tracing;*

Code and Assets used in this paper are mentioned in the Acknowledgment section (7) and corresponding copyright is owned by them.

1 Introduction

Random Number Sampling in CGI are used for simulation, procedural pattern generation , multi dimensional monte carlo integration application for rendering techniques and many more. Since the use of Random Numbers is quite vast in CGI, only some important example application will be presented in this paper. The examples that will be demonstarted will be noise generation in 2d and monte carlo rendering methods such as Ambient Occlusion, and most importantly Path Tracing.

The core idea behind the theory of UDM1 lies on the use of irrational numbers. So before getting into the details of the theory of UDM1 we will discuss related work which were based upon irrational numbers and then we will layout the mathematical background upon which the applications will take place. And the following will be the outline for the remaining sections.

• Related Work (2).
• Background(UDM1) (3).
• Applications (4).
• Limitations, Discussions, and Future Work (6).
• Conclusion (7).

2 Related Work

There has been quite significant use of irrational numbers previously for sampling. The most interesting among them is the Golden Ratio that has been demonstrated to be quite efficient in radial MRI. The next sections touches upon those topics.

2.1 Golden Ratio

Golden Ratio is an irrational number which is mathematically expressed as $\varphi = \dfrac{1+\sqrt{5}}{2} = 1.6180339887\ldots$. It can be approximated by the ratio of two consecutive Fibonacci numbers and thats how they are related.

2.1.1 Medical Imaging

For Time Resolved MRI [Winkelmann et al. 2007] they investigated a novel sampling strategy that facilitates and supports the usage of radial MRI in dynamic or contrast studies.They used a constant azimuthal profile spacing (111.246), based on the Golden Ratio as optimal for image reconstruction from an arbitrary number of profiles in radial MRI. This angle caused the radial lines to be very evenly placed with time. A similar pattern can be found in nature: leaves of plants often grow in this particular order to ensure as little overlap as possible with previously grown leaves.They found out that while a uniform profile distribution with a constant angle increment is optimal for a fixed and determined number of profiles, a profile distribution based on the Golden Ratio proved to be an appropriate solution for an arbitrary number of profiles. They utilized it for two applications: for dynamic cardiac imaging and multiple contrast reconstruction from one single inversion recovery shot of the brain by k-space filling.

Later [Wundrak et al. 2015] found out that even though this profile order is advantageous for various real-time imaging methods, in combination with balanced SSFP sequences the large azimuthal angle increment may lead to strong image artifacts, due to varying eddy currents introduced by the rapidly switching gradient scheme. Instead they used a generalized Fibonacci sequence, and introduced a new sequence of smaller irrational angles (49.759..., 32.039..., 27.198..., 23.628..., ...) . The subsequent profile orders guarantee the same sampling efficiency as the golden angle if at least a minimum number of radial profiles is used for reconstruction. They applied the angular increments for dynamic imaging of the heart and the temporomandibular joint. They showed that for balanced SSFP sequences, trajectories using the smaller golden angle surrogates strongly reduce the image artifacts, while the free retrospective choice of the construction window is maintained. This was an example of 1 dimensional sampling.

2.1.2 Spherical Integrals for Rendering

Using Golden Point Sets [Schretter et al. 2012] as Low Discrepancy Samples proved quite superior to PRNG samples and competitive to Halton and Blue Noise Point Sequences. Ambient Occlusion Rendering and 2d Image Density sampling was used as examples to prove the results. Mathematically this is an example of 2 dimensional sampling.
Fibonacci point sets [Marques et al. 2013] were used directly for spherical sampling unlike the trivial way of lifting point sets from the unit square to the unit sphere through an equal-area transform. Comparing the quality of Fibonacci point sets for estimating the illumination integral with that of state-of-the-art QMC compliant point set distributions such as blue noise, Larcher-Pillicshammer point sets and the popular Sobol(0,2)-sequence it was evident that it consistently outperformed the other methods and that the improvement is, in general, remarkable in terms of RMSE value and percentage of rays saved for the same RMSE quality.

2.1.3 Data Engineering

For allocating uniform multidimensional data among parallel disks, a new declustering scheme [Chen et al. 2003] was proposed which aimed at reducing disk access time for range queries was based on Golden Ratio Sequences for two dimensional and Kronecker Sequences for higher dimensions.

2.2 Kronecker/Weyl Sequences

2.2.1 Numerical Methods

The generation of appropriate parellel and high-quality quasi-random sequences(low-discrepancy sequences) is crucial to quasi-Monte carlo based Numerical methods. Kronecker sequence are well known to be one of the special types of low-discrepancy sequences along with the advantage of their implementation due to its definition via the fractional parts of multiples of irrationals. In the paper [Chi 2013] the original Kronecker sequence was modified since it suffered from correlations for different dimensions with a generalized golden ratio. They presented a new algorithm for finding the modified Kronecker sequence by choices of special irrationals. Numerical testing with the modifed sequences was shown to be empirically superior to the other widely used quasirandom sequences.

A class of uniform pseudorandom number generators based on Weyl sequences was proposed [Holian et al. 1994] for modelling and simulations on massively parallel computers. They called it Nested Weyl sequence(NWS) and Shuffled Nested Weyl sequence(SNWS). The algorithm to generate them is very simple, non-recursive, and is easily transported to serial or vector computers. A novel class of random number generators based on Weyl sequence [Heng et al. 2005] for the numerical solution of Stochastic Differential Equations(SDE) was propsed. A large amount of numerical and statistical tests showed that the new type of RNGs proposed were better in uniformity and competitive in "randomness", which is very important for the accuracy of numerical solutions to SDEs.

2.2.2 Monte Carlo Integration

To reduce randomness drastically in Monte Carlo(MC) integration, a pairwise independent sampling instead of i.i.d. samples called Random Weyl Sampling(RWS) was proposed [Sugita and Takanobu 2000]. The reduction in randomness is quite drastic. For examples, let a random variable W be a function of 500 tosses of a coin. To integrate W numerically, i.i.d.-sampling with 10^7 samples requires 500 x 10^7 = 5 x 10^9 random bits, while RWS with the same sample size requires only [500 + $\log_2 10^7$] x 2 = 1048 random bits. Also both methods have a same mean square error. The algorithm for DRWS is simple and works very fast, even though the pseudo-random generators, the source of randomness might be slow. A cryptographically secure pseudo-random generator for DRWS was used to obtain the most reliable numerical integration method for complicated functions. Later the author proposed a pairwise independent sampling that is applicable even if the length of random bits to generate a sample may vary and called it dynamic random Weyl sampling(DRWS) [Sugita 2003]. in essence DRWS is applicable whenever i.i.d.-sampling is applicable, and it is faster and much more reliable than the i.i.d.-sampling.

3 Background (UDM1)

The theory of UDM1 has been quite popularly adopted after Weyl published his paper on 1916. After a century it has found application in the fields of Cryptography, Networking, Statistical Simulation. In its simplest form the Theory of Uniform Distribution Modulo 1(UDM1) is concerned with the Distribution of Fractional Parts of Real Numbers in the Unit Interval(0,1). Historically the seeds of this theory was deeply rooted in Diophantine Approximations. But the development of this theory was started by Hermann Weyl . His work was primarily intended as improvement of Kronecker's Approximation Theorem. And since Weyl the theory has gone beyond its initial framework and has been elaborated by many mathematicians. As of today it has found its presence in Number Theory, Functional Analysis,Topological Algebra and, so on. A detailed and thorough treatment of this subject can be found in the book of Uniform Distribution of Sequences [Kuipers and Niederreiter 1974] .

Although there has been various formulations for the theory , but we are going to describe the Weyl's Criterion for UDM1. Then the Weyl's Criterion will be applied to Kronecker's sequences to prove that they are UDM1 . Kronecker's Sequences are of the form $\boldsymbol{x_n} = \boldsymbol{n\alpha}$ where $\boldsymbol{\alpha}$ is an Irrational Number and $\boldsymbol{n}$ = 1, 2, 3, ... Once we have the theoretical basis set up for these sequences, their use as a source for random numbers for some computer graphics problems will be demonstrated.

3.1 Uniform Distribution and Weyl's Criterion

For a Real Number $\boldsymbol{x}$, let $[\boldsymbol{x}]$ denote the Integral part of $\boldsymbol{x}$, that is the greatest Integer $\leq \boldsymbol{x}$; Let $\{\boldsymbol{x}\} = \boldsymbol{x} - [\boldsymbol{x}]$ be the fractional part of $\boldsymbol{x}$, or the residue of $\boldsymbol{x}$ modulo 1. Note that the fractional part of any Real Number is contained in the Unit Interval I = [0,1).

3.2 Definition

Definition 3.1 (Uniform Distribution)**.** *A Sequence* $\{x_n\}$ *is said to be* Uniformly Distributed *in* $[0, 1)$ *if* $\forall\ a, b \in [0, 1)$*,* $a < b$*,*

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : x_n \in (a, b)\}}{N} = b - a$$

This is the fundamental principle in the Theory of Uniformly Distributed Modulo 1 and was discovered by Weyl in 1916. The condition means that the proportion of the sequence $\{x_n\}$ lying in [a,b] converges to b - a, the length of the interval as $N \to \infty$.
For example, if a Sequence is Uniformly Distributed in [0, 4], then the interval [0.3, 0.7] occupies $\frac{1}{10}$ of the length of the interval [0,4] as N becomes large, the proportion of the first $\boldsymbol{n}$ members of the sequence which fall between 0.3 and 0.7 is equally likely to fall in anywhere in its range. In fact this idea gave birth to the Discrepancy Theory about Point Set's.

The generalisation to Multi-dimensions for Uniform Distribution of Sequences in $\mathbb{R}^k$ is similar as defined below.

Definition 3.2 (Uniform Distribution for Multi-Dimension)**.** *A sequence* $(x_n^k) = (\{x_n^1\}, \cdots, \{x_n^k\}) \in \mathbb{R}^k$ *is uniformly distributed mod 1 if, for each choice of k intervals* $[a_1, b_1], \cdots, [a_k, b_k] \subset [0, 1)$*, we have that*

$$(1) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} \chi_{[a_i,b_i]}(\{x_n^i\}) \to \prod_{i=1}^{k} (b_i - a_i),$$

as $\boldsymbol{n} \to \boldsymbol{\infty}$.

Theorem 3.3 (Weyl's Theorem)**.** *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is uniformly distributed in* $[0, 1)$ *if and only if for every real-valued continous function f defined on* $[0, 1]$*.*

$$(2) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx$$

holds.

Theorem 3.4 (Weyl's Theorem for Multi-Dimension)**.** *A sequence* $(x_n^k) = (\{x_n^1\}, \cdots, \{x_n^k\}) \in \mathbb{R}^k$ *is uniformly distributed in* $[0,1)^k$ *if and only if for every real-valued continous function f defined on* $[0,1)^k$.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f((x_n^k)) = \int_{[0,1]^k} f(x^k)dx^k \tag{3}$$

holds.

The above theorem leads to the idea of Monte-Carlo integration, where integrals are numerically approximated by sampling the function over a sequence of random variables equidistributed within $[0,1)^s$.
Two more general results can be found as corollaries of the previous theorem.The first one is a generalisation to Riemann-integrable functions, while the second one to complex-valued continous function.
Corollary 3.5. *A sequence* $\{x_n^k\}$ *is equidistributed in* $[0,1)^k$ *if and only if for every Riemann-integrable* $f : [0,1)^k$ *Equation (3) holds.*

Corollary 3.6. *A sequence* $\{x_n^k\}$ *is equidistributed in* $[0,1)^k$ *if and only if for every complex-valued continous function* $f : [0,1)^k \to \mathbb{C}$ *defined on* $\mathbb{R}^k$ *with period 1 Equation (2) holds.*
.
Theorem 3.7 (Weyl's Criterion)**.** *A sequence* $\{x_n\}$ *is equidistributed in* $[0,1)$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i l x_n} = 0 \; \forall \; l \; \in \mathbb{Z} - \{0\} \tag{4}$$

Theorem 3.8 (Weyl's Criterion For Multi Dimension)**.** *A sequence* (x_n^k) *is equidistributed in* $[0,1)^k$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i(l_1 x_n^1 + \cdots + l_k x_n^k)} \to 0, \tag{5}$$

as $n \to \infty$, $\forall \; l = (l_1, \cdots, l_k) \; \in \mathbb{Z}^k - \{0\}$

3.3 Application of Weyl's Criterion

Weyl's Criterion gives us the necessary tools to test whether a sequence is UDM1 and will be applied to Kronecker's Sequences. The following corollary is one of the most well known.

Corollary 3.9. *Fix* $\alpha \in \mathbb{R} - \mathbb{C}$*, and let* $x_n = \{n\alpha\}$*, where* $\{s\}$ *denotes the fractional parts of s. Fix any* $l \neq 0$*. Then we have*

(6)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i l x_n} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i l \{n\alpha\}}$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i l n\alpha} = \frac{e^{2\pi i l\alpha}}{N} \cdot \frac{1 - e^{2\pi i l\alpha N}}{1 - e^{2\pi i l\alpha}}$$

where the last equality is the sum of the geometric series.

As $1 - e^{2\pi i l\alpha}$ is a nonzero constant since α is an irrational , we see that the above tends to 0 as N $\to \infty$. Therefore by Weyl's Criterion $\{x_n\}$ is uniformly distributed mod 1 in [0,1).
The sequence takes the name of Kronecker since this results refines a theorem due to Kronecker showing that the points $e^{in\alpha}$ are dense in the unit circle, whenever α is an irrational multiple of π (Kronecker's approximation theorem).

Corollary 3.10 (MultiDimension)**.** *The proof is essentialy the same as in the case* $k = 1$ *and we shall apply that result to the sequence* $(x_n^k) = (\{x_n^1\}, \cdots, \{x_n^k\}) = (n\alpha_1, \cdots, n\alpha_k) \in \mathbb{R}^k$.

Now suppose that the numbers $\alpha_1, \cdots, \alpha_k$ *are rationally independent and if* $r_1, \cdots, r_k$ *are rational numbers such that*
$r_1\alpha_1 + \cdots + r_k\alpha_k + r = 0$*, then*
$r_1 = \cdots = r_k = r = 0$.

In particular, for

$$\ell = (l_1, \cdots, l_k) \in \mathbb{Z}^k \setminus \{0\} \text{ and } n \to \mathbb{N},$$

$$\ell_1 n\alpha_1 + \cdots + \ell_k n\alpha_k \notin \mathbb{Z},$$

so that

$$e^{2\pi i(l_1 n\alpha_1 + \cdots + l_k n\alpha_n)} \neq 1$$

Therefore we have
(7)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i l_k (x_n^k)}$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i(l_1 x_n^1 + \cdots + l_k x_n^k)}$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i(l_1 n\alpha_1 + \cdots + l_k n\alpha_n)}$$
$$= \frac{e^{2\pi i(l_1\alpha_1 + \cdots + l_k\alpha_n)}}{N} \cdot \frac{1 - e^{2\pi i N(l_1\alpha_1 + \cdots + l_k\alpha_n)}}{1 - e^{2\pi i(l_1\alpha_1 + \cdots + l_k\alpha_n)}}$$

Now $1 - e^{2\pi i(l_1\alpha_1 + \cdots + l_k\alpha_n)}$ is a non zero constant since $(\alpha_1, \cdots, \alpha_n)$ are all irrationals, the above tends to 0 as N $\to \infty$ Therefore by Weyl's Criterion, the sequence $(n\alpha_1, \cdots, n\alpha_k)$ is uniformly distribted mod 1 in $[0,1)^k$.
So in general any irrational number could be used to generate these sequences. Richtymer proposed the sequence of square root of prime numbers for multi dimensional sequence generation as they are all irrationals and independent over the rational and thus obey weyl's criterion. Being square root of prime numbers they can also be easily represented upto certain precision since truly irrational numbers cannot be represented in computers. Further in this paper we will refer them as WeylRichtmyer(WR) sequences.

4 Applications

The applications that will be presented will be as such

1. Noise generation in 2D with examples of procedural pattern generation using them.
2. Ambient Occlusion(2D monte carlo integration).
3. Path Tracing(MultiDimensional monte carlo integration).

4.1 Noise Generation

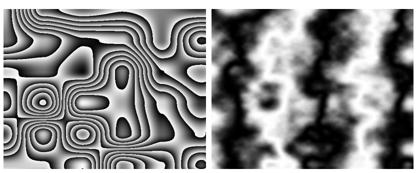
Use of noise functions based on Perlin's seminal paper [Perlin 1985] is at the heart of giving naturalistic visual complexity to CGI.

The system proposed by him are used to create very convincing representations of fire, clouds, stars, water, marble, wood, rock, soap films, crystal etc. The core random number generator used to create these noise functions was based on drand48() which is based on PRNG. As proposed in this paper we can use UDM1 sequences instead of PRNG to faithfully recreate the noise functions in different dimensions. This in my knowledge is the first time where deterministic sequences are used for the generation of noise functions as shown in the following figures. The source code to create the example images shown is also available.

(a) *White Noise* **(b)** *Value Noise* **(c)** *FractalNoise*

Figure 2: *Noise based on UDM1*

And shown in **Figure 3** are some patterns generated using the above generated noise.



(a) *Wood* **(b)** *Marble*

Figure 3: *Procedural Patterns*

4.2 Ambient Occlusion

Ambient Occlusion is a shading and rendering technique used to calculate how exposed each point in a scene is to ambient lighting. This technique is a global method, meaning that the illumination at each point is a function of other geometry in the scene but is a very crude approximation to full global illumination. The appearance achieved by ambient occlusion is similar to how an object might appear on an overcast day. This technique became quite popular in CGI and the authors of this technique [Landis 2002] received a Scientific and Technical Academy Award.
The method to calculate occlusion $\boldsymbol{A_{\bar{p}}}$ at a point $\boldsymbol{\bar{p}}$ on a surface with normal $\boldsymbol{\hat{n}}$ is by integrating the visibilty function over the hemisphere $\boldsymbol{\Omega}$ with respect to projected solid angle:

$$\boldsymbol{A_{\bar{p}} = \frac{1}{\pi}\int_{\Omega} V_{\bar{p},\hat{w}}(\hat{n}.\hat{w})d\omega}$$

where $\boldsymbol{V_{\bar{p},\hat{w}}}$ is the visibility function at $\boldsymbol{\bar{p}}$, defined to be zero if $\boldsymbol{\bar{p}}$ is occluded in the direction $\boldsymbol{\hat{p}}$ and one otherwise, and $\boldsymbol{d\omega}$ is the infinitesimal solid angle step of the integration variable $\boldsymbol{\hat{w}}$.
The method to solve the above integral is through Monte Carlo Sampling based on PRNG and QRNG. And now we can also use 2 dimensional Weyl/Kronecker sequences for solving the integral discussed earlier in 3.3. In the next sub sections we will discuss the type of irrational points that will be used and comparison between them and previous methods.

4.2.1 Weyl 2D Sequences

Weyl 2d sequence have the general form as given below.
$(x_n^2) = (\{x_n^1\},\{x_n^2\}) = (n\alpha_1, n\alpha_2)$ is the nth point of the sequence and α_1 and α_2 are the irrationals.
A variety of pairs of irrationals can be used to generate the 2D sequence mentioned above. The ones that will be tested in this paper are as.

1. WeylRichtmyer(WR) $(\sqrt{2}, \sqrt{13})$.
2. WeylExponential(WE) (e, e^2). where e is the base of the natural logarithm.
3. WeylGoldenRatio(WGR) $(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{3}}{2})$.

Since these sequences are also Low Discrepancy Sequences and Deterministic, they suffer from co-relations, which can be elevated by giving them a unique random start for each pixel sampled.

4.2.2 UDM1 Point Set

A combination of Equally Partitioned Interval Set and prng given as

$$\boldsymbol{x_n = \{\frac{n}{K} + prng\}}, \text{ where } \boldsymbol{\{s\}} \text{ denotes the fractional parts of } \boldsymbol{s} \text{ and K is the total number of points}$$

is a new UDM1 based point set that is being introduced in this paper. This Point Set when paired with a Weyl Sequence based on Richtmyer gives state of the art results in terms of PSNR for the application discussed in this section. The irrational number used for the Weyl sequence was the square root of the prime number 137. Moving further we would refer this pair as EquallyPartitionedIntervalSetsWeylRichtmyer(EPISWR).

4.2.3 Comparisons

Now 2D sampling is the most commonly used dimensions in CGI, so we will cover it in more detail. For comparisons, PRNG, Correlated Multi-Jittered Sampling(CRMJS) [Kensler 2013], Golden Ratio Point Sets(GRPS) [Schretter et al. 2012] , Hammersley , LarcherPilichshammer(LP) 2D Halton and Sobol Sequences [Kollig and Keller 2002] were used. . Looking at the visualization for 256 samples in **Figure** (5) for the 2D points of the various methods mentioned before it is evident and clear of what to expect from each of them when used for sampling. It should also be noted that Golden Point Sets are a subset of UDM1, since it uses the irrational $\boldsymbol{\varphi}$ to generate the Low Discrepancy Point Set. Also UDM1 sequences and point sets don't require shuffling/scrambling unlike the Golden Point Sets and Co-related Multi Jittered Sampling which makes them trivial to generate . The images in figure **Figure** (6) show the type of points used along with PSNR as measured against a reference image in **Figure** (4) rendered at 8192 spp. The results show that, EPISWR beats all other sequences in terms of visual quality and PSNR and it is free from any kind of artefacts , biasing, and co-relations .It can also be observed that Halton and Sobol are the poorest performers and also suffer from co-relations.

5 Path Traced Rendering

Path Tracing is a Monte Carlo method of solving the Rendering equation [Kajiya 1986] in CGI. This is an integral equation which generalizes a variety of known rendering algorithms and captures the full light transport in a scene and faithfully simulates many effects such as soft shadows, depth of field, motion blur, caustics,

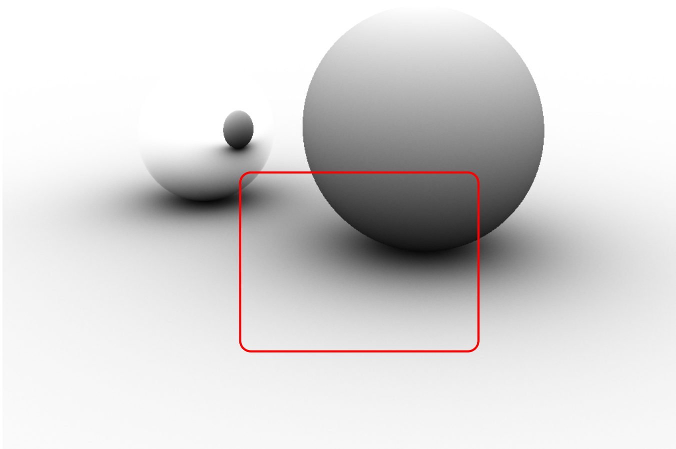


Figure 4: *Ambient Occlusion GroundTruth (8192spp)*

ambient occlusion and indirect lighting.. The equation is shown below.

(8)

$$\boldsymbol{L_0(x,\omega,\lambda,t) = L_e(x,\omega,\lambda,t)}$$
$$\boldsymbol{+ \int_{\omega} f_r(x,\omega',\omega,\lambda,t) L_i(x,\omega',\omega,\lambda,t)(-\omega'.n)d\omega'}$$

It states that the outgoing radiance from one surface point to another is simply the sum of emitted radiance and the reflected radiance from all other surfaces.

It can be seen that to solve the above equation we need to solve the integral on the right side, which is again solved using Monte Carlo methods. The type of random numbers that are generally used to solve this multi dimensional integral are based on PRNG and QRNG(Halton and Sobol) . As discussed in 3.3 the Multi Dimensional Weyl/Kronecker Sequences can now be used as a new source of random numbers. In the next sub sections we will discuss the type of irrational sequences that will be used and comparison between them and previous methods.

5.1 Weyl Multi Dimensional Sequences

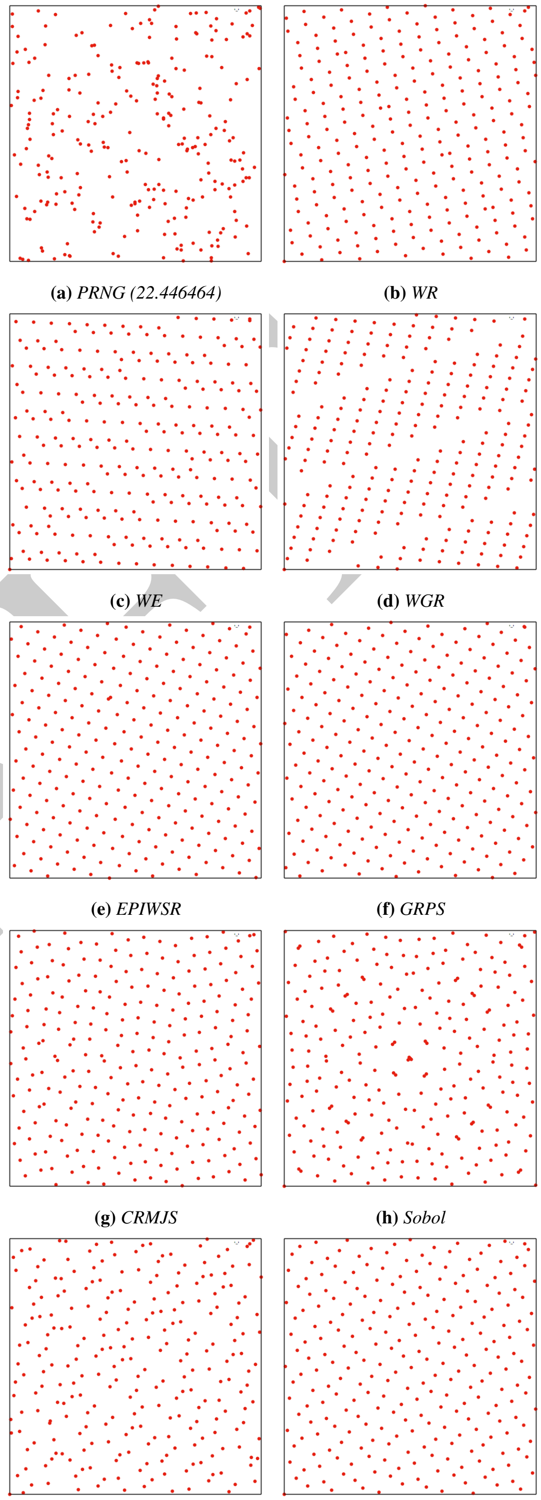
Multi dimensional Weyl/Kronecker sequence have the general form as given below.

$(x_n^k) = (\{x_n^1\}, \{x_n^2\}, \cdots, \{x_n^k\}) = (n\alpha_1, n\alpha_2, \cdots, n\alpha_k)$ is the nth point of the sequence and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are the sequence of irrationals.

We will be testing three types of Irrational sequence for the above sequences. They are given below.

1. WeylRichtmyer (WR).
   This sequence is based on the square root of primes. We will be using the first 20 prime numbers, which will be sufficient for our testing.

2. Weyl Metallic Means Ratio (WMMR).
   As discussed in [Chi 2013], they used special choice of irrationals for the generalized Fibonacci sequence such that the square of a prime number + 4 is again another prime number . e.g. 3x3 + 4 = 11. Direcly using the suggested method did not give quite interesting results. Instead choosing pair of irrationals based on their choices and which also had pairwise



(a) *PRNG (22.446464)* **(b)** *WR*

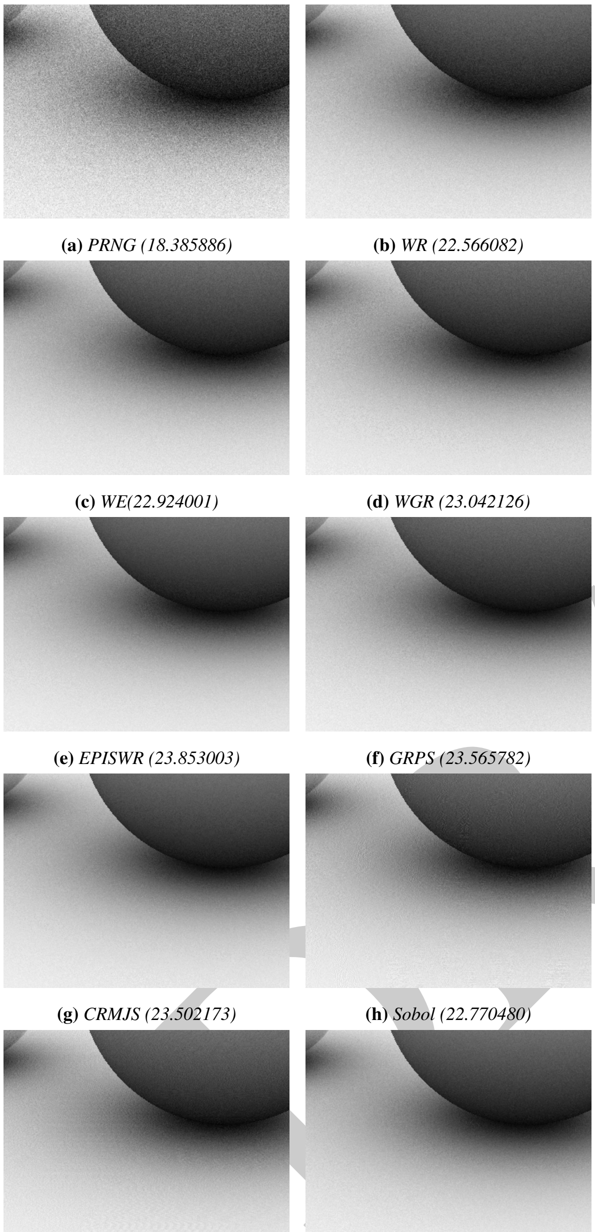
(c) *WE* **(d)** *WGR*

(e) *EPIWSR* **(f)** *GRPS*

(g) *CRMJS* **(h)** *Sobol*

(i) *Halton* **(j)** *LP*

Figure 5: *2D Sampling Visualization(256samples) used for 2D Monte Carlo Integrals.*



(a) *PRNG (18.385886)* **(b)** *WR (22.566082)*

(c) *WE(22.924001)* **(d)** *WGR (23.042126)*

(e) *EPISWR (23.853003)* **(f)** *GRPS (23.565782)*

(g) *CRMJS (23.502173)* **(h)** *Sobol (22.770480)*

(i) *Halton (22.446464)* **(j)** *LP (23.472503)*

Figure 6: *Ambient Occlusion(64spp) as example for 2D Monte Carlo Integrals.*

Figure 7: *Path Tracing GroundTruth for San Miguel (32768spp)*

good 2D projection proved to be quite efficient. The primes used are given below.
$\{\sqrt{13}, \sqrt{173}, \sqrt{17}, \sqrt{293}, \sqrt{47}, \sqrt{2213}, \sqrt{67}, \sqrt{4493}, \sqrt{97}, \sqrt{9413}, \sqrt{167}, \sqrt{27893}, \sqrt{193}, \sqrt{37253}, \sqrt{277}, \sqrt{76733}, \sqrt{307}, \sqrt{94253}, \sqrt{317}, \sqrt{100493}, \sqrt{487}, \sqrt{237173}\}$.

3. Hybrid Weyl Metallic Means Ratio(HWMMR)
This sequence is a combination of the following irrational numbers as given below and performs slightly better than the above two.
$\{\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{3}}{2}, e, e^2, \sqrt{13}, \sqrt{173}, \sqrt{17}, \sqrt{293}, \sqrt{47}, \sqrt{2213}, \sqrt{67}, \sqrt{4493}, \sqrt{97}, \sqrt{9413}, \sqrt{167}, \sqrt{27893}, \sqrt{193}, \sqrt{37253}, \sqrt{277}, \sqrt{76733}, \sqrt{307}, \sqrt{94253}\}$.

5.2 Comparisons

The tests were done by adding a new sampler inside the pbrt-v3 framework [Pharr et al. 2016]. The 3d scenes used for the tests are San Miguel and the Crown.
For comparison we have used PRNG, Halton and Sobol sequences. For the San Miguel scene the Weyl sequences gave the best PSNR without any co-relation patterns or biasing, especially HWMMR. The PSNR for Halton was competitive, but for Sobol was very less comparatively and both of them suffered from co-relation patterns as shown in **Figure** (8). For the Crown scene Sobol gave the best PSNR and the rest except PRNG were competitive as shown in **Figure** (10).

6 Limitation, Discussion, and Future Work

As noted before UDM1 sequences like other Low Discrepancy sequences such as Halton and Sobol suffer from co-relations. To get rid of it we have to take help of PRNG , which makes it dependent on them. For simplicity the drand48() PRNG was used in this paper. But in fact even PRNG type random numbers can also be generated using UDM1 as discussed in [Holian et al. 1994], and thereby removing any kind of dependence on them entirely .

Since the core of UDM1 depends on irrational numbers, it might be difficult to get good irrational numbers for sampling in general thereby making it's use hard and tricky. It would be interesting to delve into the mathematics of UDM1 deeply and find methods to ensure irrational numbers which are good for sampling.

It would be interesting to try UDM1 sampling for other rendering algorithms such as Bi-directional Path Tracing, Metropolis Sampling, VCM and see how they perform against the random number methods used in them.

(a) *PRNG (25.509749)*

(b) *Halton (26.980754)*

(c) *Sobol (25.815170)*

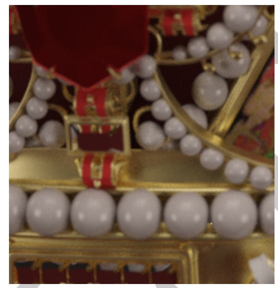
(d) *WR (27.238521)*

(e) *WMMR (27.355182)*

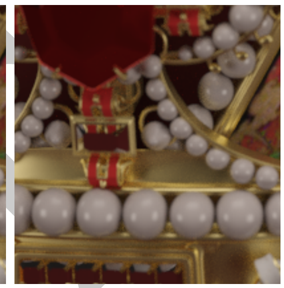
(f) *HWMMR (27.362858)*

Figure 8: *Path Tracing(4096spp) as example for Multi Dimensional Monte Carlo Integrals.*

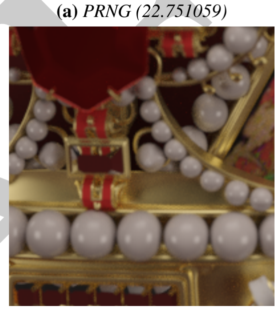
Figure 9: *Path Tracing GroundTruth for Crown (16384spp)*



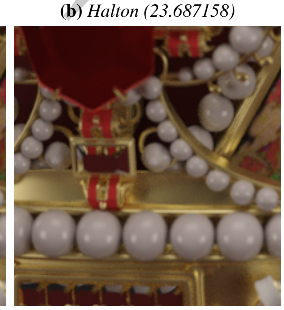
(a) *PRNG (22.751059)*



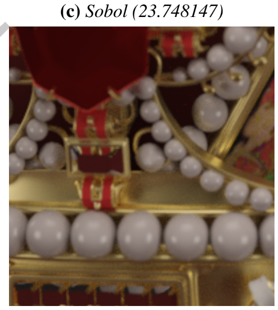
(b) *Halton (23.687158)*



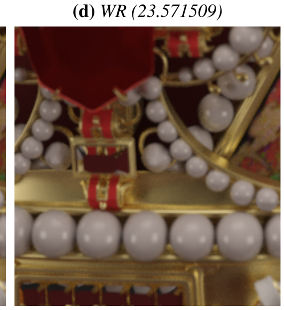
(c) *Sobol (23.748147)*



(d) *WR (23.571509)*



(e) *WMMR (23.553944)*



(f) *HWMMR (23.677106)*

Figure 10: *Path Tracing(4096spp) as example for Multi Dimensional Monte Carlo Integrals.*

Due to the simplicity of generating UDM1 sequences for sampling, it can be useful to parallelize them on current CPU and GPU architecture and compare them to other random number generators in terms of speed , generation, and implementation for CGI.

The UDM1 sequences that were discussed and tested in this paper were based on Weyl/Kronecker sequences. As a future continuation of this paper random numbers based on Weyl's irrational polynomials of certain kinds would be interesting to explore upon.

7 Conclusion

This paper showed that random numbers based on UDM1 can be competitive and in some cases with use of specially arranged Weyl sequences can outperform the state-of-the-art QMC techniques out there for path tracing. They can be used for a variety of purposes as discussed in the applications section in (4) .UDM1 is backed up by elaborate mathematical basis [Kuipers and Niederreiter 1974] with continued research into the subject and shows the beauty of Applied Real Analysis for Sampling in Computer Graphics .The simplicity and beauty of generating and using UDM1 numbers will appeal to the Computer Graphics community and open up new paths to explore upon in the future.

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References

CHEN, C.-M., BHATIA, R., AND SINHA., R. K. 2003. Multidimensional declustering schemes using golden ratio and kronecker sequences. *IEEE Transactions on Knowledge and Data Engineering 15*, 3 (March), 659–670.

CHI, H. 2013. Generation of parallel modified kronecker sequences. *Monte Carlo Methods and Applications 19*, 3 (December), 261–271.

HENG, L., QINGHUA, L., AND FENGSHAN, B. 2005. A class of random number generators based on weyl sequences. *Applied Mathematics-A Journal of Chinese Universities 20*, 4 (December), 483–490.

HOLIAN, B. L., PERCUS, O. E., WARNOCK, T. T., AND WHITLOCK, P. A. 1994. Pseudorandom number generator for massively parallel molecular-dynamics simulations. *Phys. Rev. E 50*, 2 (August), 1607–1615.

KAJIYA, J. T. 1986. The rendering equation. *ACM SIGGRAPH Computer Graphics 20*, 4 (August), 143–150.

KENSLER, A. 2013. Correlated multi-jittered sampling. *Pixar Technical Memo 13*, 01 (March).

KOLLIG, T., AND KELLER, A. 2002. Efficient multidimensional sampling. *Computer Graphics Forum. 21*, 3 (September), 557–563.

KUIPERS, L., AND NIEDERREITER, H. 1974. *Uniform Distribution of Sequences*. Wiley, May.

LANDIS, H., 2002. Global illumination in production. ACM SIGGRAPH 2002 Course #16 Notes, July.

MARQUES, R., BOUVILLE, C., RIBARDIERE, M., SANTOS, L. P., AND BOUATOUCH, K. 2013. Spherical fibonacci point sets for illumination integrals. *Computer Graphics Forum. 32*, 8 (December), 134–143.

PERLIN, K. 1985. An image synthesizer. *Computer Graphics 19*, 3, 287–296.

PHARR, M., HUMPHREYS, G., AND JAKOB, W. 2016. *Physically Based Rendering:From Theory to Implementation*, vol. 3. Morgan Kaufmann Publishers Inc, November.

SCHRETTER, C., KOBBELT, L., AND DEHAYEL, P.-O. 2012. Golden ratio sequences for low-discrepancy sampling. *Journal of Graphics Tools. 16*, 2 (January), 95–104.

SUGITA, H., AND TAKANOBU, S. 2000. Random weyl sampling for robust numerical integration of complicated functions. *Monte Carlo Methods and Applications 6*, 1 (January), 27–48.

SUGITA, H. 2003. Dynamic random weyl sampling for drastic reduction of randomness in monte carlo integration. *Mathematics and Computers in Simulation 62*, 3-6 (March), 529–537.

WINKELMANN, S., SCHAEFFTER, T., KOEHLER, T., EGGERS, H., AND DOESSEL, O. 2007. An optimal radial profile order based on the golden ratio for time resolved mri. *IEEE Transactions on Medical Imaging. 26*, 1 (January), 68–76.

WOJCIECHOWSKI, K. W. 1999. Pseudorandom number generators based on the weyl sequence. *Computational Methods in Science and Technology 5*, 1, 81–85.

WUNDRAK, S., PAUL, J., ULRICI, J., HELL, E., AND RASCHE, V. 2015. A small surrogate for the golden angle in time-resolved radial mri based on generalized fibonacci sequences. *IEEE Transactions on Medical Imaging. 34*, 6 (June), 1262–1269.