

$$f(x) = \sin(x^2), \quad -1 \leq x \leq 1$$

Midpoint Rule:

n = 10:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{10} \\ &= 0.2\end{aligned}$$

$$\begin{aligned}\int_{-1}^1 f(x) dx &\approx M_n = \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_{10})] \\ &= 0.2 * 3.084391 \\ &= 0.6168782\end{aligned}$$

n = 20:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{20} \\ &= 0.1\end{aligned}$$

$$\begin{aligned}\int_{-1}^1 f(x) dx &\approx M_n = \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_{20})] \\ &= 0.1 * 6.196326 \\ &= 0.6196326\end{aligned}$$

n = 40:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{40} \\ &= 0.05\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f(x) dx &\approx M_n = \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_{40})] \\
&= 0.05 * 12.406225 \\
&= 0.62031125
\end{aligned}$$

Trapezoidal Rule:

n = 10:

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1 - (-1)}{10} \\
&= 0.2
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f(x) dx &\approx T_{10} = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_9) + f(x_{10})] \\
&= \frac{0.2}{2} * 6.278051 \\
&= 0.6278051
\end{aligned}$$

n = 20:

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1 - (-1)}{20} \\
&= 0.1
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f(x) dx &\approx T_{20} = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{19}) + f(x_{20})] \\
&= \frac{0.1}{2} * 12.446832 \\
&= 0.6223416
\end{aligned}$$

n = 40:

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1 - (-1)}{40} \\
&= 0.05
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f(x) dx &\approx T_{40} = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{39}) + f(x_{40})] \\
&= 0.025 * 24.839484 \\
&= 0.6209871
\end{aligned}$$

Simpson's Rule:

n = 10:

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{10} \\
&= 0.2
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f(x) dx &\approx S_{10} = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_8) + 4f(x_9) + f(x_{10})] \\
&= \frac{0.2}{3} * 9.308039 \\
&= 0.620536
\end{aligned}$$

n = 20:

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{20} \\
&= 0.1
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f(x) dx &\approx S_{10} = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_8) + 4f(x_{19}) + f(x_{20})] \\
&= \frac{0.1}{3} * 9.308039 \\
&= 0.620536
\end{aligned}$$

n = 40:

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{40} \\
&= 0.05
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f(x) dx &\approx S_{10} = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_8) + 4f(x_9) + f(x_{10})] \\
&= \frac{0.05}{3} * 9.308055 \\
&= 0.620537
\end{aligned}$$

Error Bounds:

$$\begin{aligned}
|f''(x)| &= |(f'(x))'| \\
&= |(2x \cos(x^2))'| \\
&= |-4x^2 \sin(x^2) + 2 \cos(x^2)| \\
&= 4x^2 |\sin(x^2)| + 2 |\cos(x^2)|
\end{aligned}$$

Because $-1 \leq x \leq 1$, then:

$$\begin{aligned}
|f''(x)| &\leq 4(1)^2(1) + 2(1) \\
&\leq 6
\end{aligned}$$

We have $K = 6$, then for Midpoint Rule:

$$\begin{aligned}
|E_M| &\leq \frac{K(b-a)^3}{24n^2} \\
&\leq \frac{6(2)^3}{24n^2} \\
&\leq \frac{2}{n^2}
\end{aligned}$$

$n = 10$:

$$|E_M| \leq 0.02$$

$n = 20$:

$$|E_M| \leq 0.005$$

$n = 40$:

$$|E_M| \leq 0.000625$$

For Trapezoidal Rule:

$$\begin{aligned}|E_T| &\leq \frac{K(b-a)^3}{12n^2} \\ &\leq \frac{6(2)^3}{12n^2} \\ &\leq \frac{4}{n^2}\end{aligned}$$

n = 10:

$$|E_T| \leq 0.04$$

n = 20:

$$|E_T| \leq 0.01$$

n = 40:

$$|E_T| \leq 0.0025$$

For Simpson's Rule:

$$\begin{aligned}|f^{(4)}(x)| &= |-12\sin(x^2) - 48x^2\cos(x^2) + 16x^4\sin(x^2)| \\ &= 12|\sin(x^2)| + 48x^2|\cos(x^2)| + 16x^4|\sin(x^2)|\end{aligned}$$

Because $-1 \leq x \leq 1$, then:

$$\begin{aligned}|f^{(4)}(x)| &\leq 12(1) + 48(1) + 16(1) \\ &\leq 76\end{aligned}$$

We have $K = 76$, then

$$\begin{aligned}|E_S| &\leq \frac{K(b-a)^5}{180n^4} \\ |E_S| &\leq \frac{76(1-(-1))^5}{180n^4} \\ &\leq \frac{2432}{180n^4}\end{aligned}$$

n = 10:

$$|E_T| \leq 0.001351111$$

n = 20:

$$|E_T| \leq 0.000084444$$

n = 40:

$$|E_T| \leq 0.000005277$$