$$f(x) = sin(x^2), \quad -1 \leq x \leq 1$$

Midpoint Rule:

n = 10:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{10} \\ &= 0.2\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx M_{10} &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{10})] \\ &= 0.2 * 3.084391 \\ &= 0.6168782\end{aligned}$$

n = 20:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{20} \\ &= 0.1\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx M_{20} &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{20})] \\ &= 0.1 * 6.196326 \\ &= 0.6196326\end{aligned}$$

n = 40:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{40} \\ &= 0.05\end{aligned}$$

$$
\begin{aligned}
\int_{-1}^{1} f(x)\,dx \approx M_{40} &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{40})] \\
&= 0.05 * 12.406225 \\
&= 0.62031125
\end{aligned}
$$

Trapezoidal Rule:

n = 10:

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{10} \\
&= 0.2
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} f(x)\,dx \approx T_{10} &= \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_9) + f(x_{10})] \\
&= \frac{0.2}{2} * 6.278051 \\
&= 0.6278051
\end{aligned}
$$

n = 20:

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{20} \\
&= 0.1
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} f(x)\,dx \approx T_{20} &= \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{19}) + f(x_{20})] \\
&= \frac{0.1}{2} * 12.446832 \\
&= 0.6223416
\end{aligned}
$$

n = 40:

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{40} \\
&= 0.05
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} f(x)\,dx \approx T_{40} &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{39}) + f(x_{40})\right] \\
&= 0.025 \ * 24.839484 \\
&= 0.6209871
\end{aligned}
$$

Simpson's Rule:

n = 10:

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{10} \\
&= 0.2
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} f(x)\,dx \approx S_{10} &= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_8) + 4f(x_9) + f(x_{10})\right] \\
&= \frac{0.2}{3} * 9.304105423275795 \\
&= 0.620273694885053
\end{aligned}
$$

n = 20:

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{20} \\
&= 0.1
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} f(x)\,dx \approx S_{20} &= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_{18}) + 4f(x_{19}) + f(x_{20})\right] \\
&= \frac{0.1}{3} * 9.307807032996628 \\
&= 0.620520468866442
\end{aligned}
$$

n = 40:

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{1-(-1)}{40} \\
&= 0.05
\end{aligned}
$$

$$\int_{-1}^{1} f(x)\,dx \approx S_{40} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_{38}) + 4f(x_{39}) + f(x_{40})\right]$$
$$= \frac{0.05}{3} * 9.308034003962234$$
$$= 0.620535600264149$$

Error Bounds:

$$|f^{"}(x)| = |(f^{'}(x))^{'}|$$
$$= |(2x\,cos(x^2))^{'}|$$
$$= |-4x^2\,sin(x^2) + 2\,cos(x^2)|$$
$$= 4x^2|sin(x^2)| + 2|cos(x^2)|$$

Because $-1 \leq x \leq 1$, then:

$$|f^{"}(x)| \leq 2$$

We have $K = 2$, then for Midpoint Rule:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$
$$\leq \frac{2\,(2)^3}{24n^2}$$
$$\leq \frac{2}{3n^2}$$

n = 10:

$$|E_M| \leq 0.006666666666666667$$

n = 20:

$$|E_M| \leq 0.0016666666666666668$$

n = 40:

$$|E_M| \leq 0.0004166666666666667$$

Doubling n will result in dividing $2^2 = 4$

$$0.006666666666666667\,/\,0.0016666666666666668 \approx 4$$
$$0.0016666666666666668\,/\,0.0004166666666666667 \approx 4$$

Error bounds for Midpoint Rule checked.

For Trapezoidal Rule:

$$|f^{"}(x)| \leq 2$$

We have $K = 2$, then for Trapezoidal Rule:

$$\begin{aligned} |E_T| &\leq \frac{K(b-a)^3}{12n^2} \\ &\leq \frac{2\,(2)^3}{12n^2} \\ &\leq \frac{4}{3n^2} \end{aligned}$$

n = 10:

$$|E_T| \leq 0.013333333333333334$$

n = 20:

$$|E_T| \leq 0.0033333333333333335$$

n = 40:

$$|E_T| \leq 0.0008333333333333334$$

Doubling n will result in dividing $2^2 = 4$

$$0.013333333333333334\,/\,0.0033333333333333335 \approx 4$$
$$0.0033333333333333335\,/\,0.0008333333333333334 \approx 4$$

Error bounds for Trapezoidal Rule checked.

For Simpson's Rule:

$$\begin{aligned}|f^{(4)}(x)| &= |-12sin(x^2) - 48x^2cos(x^2) + 16x^4sin(x^2)| \\ &= 12|sin(x^2)| + 48x^2|cos(x^2)| + 16x^4|sin(x^2)|\end{aligned}$$

Because $-1 \leq x \leq 1$, then:

$$|f^{(4)}(x)| \leq 28.4285$$

We have $K = 28.4285$, then

$$\begin{aligned}|E_S| &\leq \frac{K(b-a)^5}{180\,n^4} \\ |E_S| &\leq \frac{28.4285(1-(-1))^5}{180\,n^4} \\ &\leq \frac{909.712}{180n^4}\end{aligned}$$

n = 10:

$$|E_S| \leq 0.0005053955555555555$$

n = 20:

$$|E_S| \leq 0.0000315872222222222$$

n = 40:

$$|E_S| \leq 0.0000019742013888888$$

Doubling n will result in dividing $2^4 = 16$

$$0.0005053955555555555 \,/\, 0.0000315872222222222 \approx 16$$
$$0.0000315872222222222 \,/\, 0.0000019742013888888 \approx 16$$

Error bounds for Simpson's Rule checked.

$\int_{-1}^{1} sin(x^2)\, dx = 0.626537 \rightarrow$ calculated on wolfram alpha

Midpoint Rule:

$$n = 10 \implies |0.6168782 - 0.626537| = 0.0036588 \leq |E_M| = 0.0066666...$$

$$n = 20 \implies |0.6196326 - 0.620537| = 0.0009044 \leq |E_M| = 0.0016666...$$

$$n = 40 \implies |0.62031125 - 0.620537| = 0.00022575 \leq |E_M| = 0.00041666...$$

Trapezoidal Rule:

$$n = 10 \implies |0.6278051 - 0.626537| = 0.0072681 \leq |E_T| = 0.0133333333...$$

$$n = 20 \implies |0.6223416 - 0.620537| = 0.0018046 \leq |E_T| = 0.0033333333...$$

$$n = 40 \implies |0.6209871 - 0.620537| = 0.0.0004501 \leq |E_T| = 0.0008333333...$$

Simpson's Rule:

$$n = 10 \implies |0.620273694885053 - 0.626537| = 0.000263305 \leq |E_S| = 0.00050539555...$$

$$n = 20 \implies |0.620520468866442 - 0.620537| = 0.000016531 \leq |E_S| = 0.000031587222...$$

$$n = 40 \implies |0.620505600264149 - 0.620537| = 0.000000014 \leq |E_S| = 0.00000019420138...$$