$$f(x) = sin(x^2), \quad -1 \leq x \leq 1$$

Midpoint Rule:

n = 10:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{10} \\ &= 0.2\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx M_n &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{10})] \\ &= 0.2 * 3.084391 \\ &= 0.6168782\end{aligned}$$

n = 20:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{20} \\ &= 0.1\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx M_n &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{20})] \\ &= 0.1 * 6.196326 \\ &= 0.6196326\end{aligned}$$

n = 40:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{40} \\ &= 0.05\end{aligned}$$

$$\int_{-1}^{1} f(x)\,dx \approx M_n = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{40})]$$
$$= 0.05 * 12.406225$$
$$= 0.62031125$$

Trapezoidal Rule:

n = 10:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{10}$$
$$= 0.2$$

$$\int_{-1}^{1} f(x)\,dx \approx T_{10} = \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_9) + f(x_{10})]$$
$$= \frac{0.2}{2} * 6.278051$$
$$= 0.6278051$$

n = 20:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{20}$$
$$= 0.1$$

$$\int_{-1}^{1} f(x)\,dx \approx T_{20} = \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{19}) + f(x_{20})]$$
$$= \frac{0.1}{2} * 12.446832$$
$$= 0.6223416$$

n = 40:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{40}$$
$$= 0.05$$

$$\int_{-1}^{1} f(x)\,dx \approx T_{40} = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{39}) + f(x_{40})\right]$$
$$= 0.025\ * 24.839484$$
$$= 0.6209871$$

Simpson's Rule:

n = 10:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{10}$$
$$= 0.2$$

$$\int_{-1}^{1} f(x)\,dx \approx S_{10} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_8) + 4f(x_9) + f(x_{10})\right]$$
$$= \frac{0.2}{3} * 9.304105423275795$$
$$= 0.620273694885053$$

n = 20:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{20}$$
$$= 0.1$$

$$\int_{-1}^{1} f(x)\,dx \approx S_{10} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_8) + 4f(x_19) + f(x_{20})\right]$$
$$= \frac{0.1}{3} * 9.307807032996628$$
$$= 0.620520468866442$$

n = 40:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{40}$$
$$= 0.05$$

$$
\begin{aligned}
\int_{-1}^{1} f(x)\,dx \approx S_{10} &= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_8) + 4f(x_39) + f(x_{40})\right] \\
&= \frac{0.05}{3} * 9.308034003962234 \\
&= 0.620535600264149
\end{aligned}
$$

Error Bounds:

$$
\begin{aligned}
|f^{"}(x)| &= |(f^{'}(x))^{'}| \\
&= |(2x\,cos(x^2))^{'}| \\
&= |-4x^2\,sin(x^2) + 2\,cos(x^2)| \\
&= 4x^2|sin(x^2)| + 2|cos(x^2)|
\end{aligned}
$$

Because $-1 \leq x \leq 1$, then:

$$|f^{"}(x)| \leq 2$$

We have $K = 6$, then for Midpoint Rule:

$$
\begin{aligned}
|E_M| &\leq \frac{K(b-a)^3}{24n^2} \\
&\leq \frac{2\,(2)^3}{24n^2} \\
&\leq \frac{2}{3n^2}
\end{aligned}
$$

n = 10:

$$|E_M| \leq 0.006666666666666667$$

n = 20:

$$|E_M| \leq 0.0016666666666666668$$

n = 40:

$$|E_M| \leq 0.0004166666666666667$$

For Trapezoidal Rule:

$$\begin{aligned}|E_T| &\le \frac{K(b-a)^3}{12n^2}\\ &\le \frac{2\,(2)^3}{12n^2}\\ &\le \frac{4}{3n^2}\end{aligned}$$

n = 10:

$$|E_T| \le 0.013333333333333334$$

n = 20:

$$|E_T| \le 0.0033333333333333335$$

n = 40:

$$|E_T| \le 0.0008333333333333334$$

For Simpson's Rule:

$$\begin{aligned}|f^{(4)}(x)| &= |-12sin(x^2) - 48x^2cos(x^2) + 16x^4sin(x^2)|\\ &= 12|sin(x^2)| + 48x^2|cos(x^2)| + 16x^4|sin(x^2)|\end{aligned}$$

Because $-1 \le x \le 1$, then:

$$\begin{aligned}|f^{(4)}(x)| &\le 12\,(1) + 48\,(1) + 16\,(1)\\ &\le 76\end{aligned}$$

We have $K = 76$, then

$$\begin{aligned}|E_S| &\le \frac{K(b-a)^5}{180\,n^4}\\ |E_S| &\le \frac{76(1-(-1))^5}{180\,n^4}\\ &\le \frac{2432}{180n^4}\end{aligned}$$

n = 10:

$$|E_T| \le 0.001351111$$

n = 20:

$$|E_T| \leq 0.000084444$$

n = 40:

$$|E_T| \leq 0.000005277$$