$$f(x) = sin(x^2), \quad -1 \le x \le 1$$

Midpoint Rule:

n = 10:

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{10} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} \int_{-1}^{1} f(x)\,dx \approx M_n &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{10})] \\ &= 0.2 * 3.084391 \\ &= 0.6168782 \end{aligned}$$

n = 20:

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{20} \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} \int_{-1}^{1} f(x)\,dx \approx M_n &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{20})] \\ &= 0.1 * 6.196326 \\ &= 0.6196326 \end{aligned}$$

n = 40:

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{40} \\ &= 0.05 \end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx M_n &= \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + ... + f(\overline{x}_{40})] \\ &= 0.05 * 12.406225 \\ &= 0.62031125\end{aligned}$$

Trapezoidal Rule:

n = 10:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{10} \\ &= 0.2\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx T_{10} &= \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_9) + f(x_{10})] \\ &= \frac{0.2}{2} * 17.670891 \\ &= 1.7670891\end{aligned}$$

n = 20:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{20} \\ &= 0.1\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx \approx T_{20} &= \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{19}) + f(x_{20})] \\ &= \frac{0.1}{2} * 33.658839 \\ &= 1.68294195\end{aligned}$$

n = 40:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{1-(-1)}{40} \\ &= 0.05\end{aligned}$$

$$\int_{-1}^{1} f(x)\,dx \approx T_{40} = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{39}) + f(x_{40})\right]$$
$$= 0.025\ * 67.317679$$
$$= 1.682941975$$

Simpson's Rule:

n = 10:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{10}$$
$$= 0.2$$

$$\int_{-1}^{1} f(x)\,dx \approx S_{10} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_8) + 4f(x_9) + f(x_{10})\right]$$
$$= \frac{0.2}{3} * 15.696044$$
$$= 1.04640293$$

n = 20:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{20}$$
$$= 0.1$$

$$\int_{-1}^{1} f(x)\,dx \approx S_{10} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_8) + 4f(x_1 9) + f(x_{20})\right]$$
$$= \frac{0.1}{3} * 28.150280$$
$$= 0.93834267$$

n = 40:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1-(-1)}{40}$$
$$= 0.05$$

$$\int_{-1}^{1} f(x)\,dx \approx S_{10} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_8) + 4f(x_39) + f(x_{40})\right]$$
$$= \frac{0.05}{3} * 52.990672$$
$$= 0.883177867$$

Error Bounds:

$$\begin{aligned} |f^{"}(x)| &= |(f^{'}(x))^{'}| \\ &= |(2x\,cos(x^2))^{'}| \\ &= |-4x^2\,sin(x^2) + 2\,cos(x^2)| \\ &= 4x^2|sin(x^2)| + 2|cos(x^2)| \end{aligned}$$

Because $-1 \leq x \leq 1$, then:

$$\begin{aligned} |f^{"}(x)| &\leq 4\,(1)^2\,(1) + 2\,(1) \\ &\leq 6 \end{aligned}$$

We have $K = 6$, then for Midpoint Rule:

$$\begin{aligned} |E_M| &\leq \frac{K(b-a)^3}{24n^2} \\ &\leq \frac{6\,(2)^3}{24n^2} \\ &\leq \frac{2}{n^2} \end{aligned}$$

n = 10:

$$|E_M| \leq 0.02$$

n = 20:

$$|E_M| \leq 0.005$$

n = 40:

$$|E_M| \leq 0.000625$$

For Trapezoidal Rule:

$$\begin{aligned}|E_T| &\leq \frac{K(b-a)^3}{12n^2} \\ &\leq \frac{6\,(2)^3}{12n^2} \\ &\leq \frac{4}{n^2}\end{aligned}$$

n = 10:

$$|E_T| \leq 0.04$$

n = 20:

$$|E_T| \leq 0.01$$

n = 40:

$$|E_T| \leq 0.0025$$

For Simpson's Rule:

$$\begin{aligned}|f^{(4)}(x)| &= |-12sin(x^2) - 48x^2cos(x^2) + 16x^4sin(x^2)| \\ &= 12|sin(x^2)| + 48x^2|cos(x^2)| + 16x^4|sin(x^2)|\end{aligned}$$

Because $-1 \leq x \leq 1$, then:

$$\begin{aligned}|f^{(4)}(x)| &\leq 12\,(1) + 48\,(1) + 16\,(1) \\ &\leq 76\end{aligned}$$

We have $K = 76$, then

$$\begin{aligned}|E_S| &\leq \frac{K(b-a)^5}{180\,n^4} \\ |E_S| &\leq \frac{76(1-(-1))^5}{180\,n^4} \\ &\leq \frac{2432}{180n^4}\end{aligned}$$

n = 10:

$$|E_T| \leq 0.001351111$$

n = 20:

$$|E_T| \leq 0.000084444$$

n = 40:

$$|E_T| \leq 0.000005277$$