Graph Theory Introduction

CSCI 170 Spring 2021

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1.1–1.2

Graph Theory Introduction

- **Graph Definitions**
- Paths and Cycles
- Connectivity

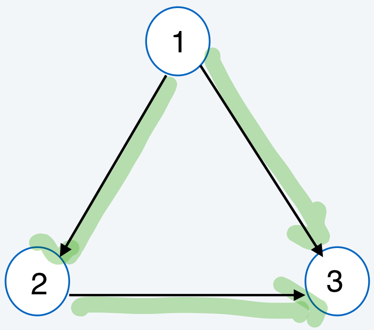
1.1–1.2

What is a graph?

A **graph**, G = (V,E) is a set of **vertices or nodes**, V, and a set of **edges or arcs**, E.

The set of edges is a subset of the Cartesian product of VxV.

Example:
Graph, G = {{1,2,3}, {(1,2), (2,3), (1,3)}}
The vertices are the set {1,2,3}
The edges or arcs are the set {(1,2),(2,3), (1,3)}

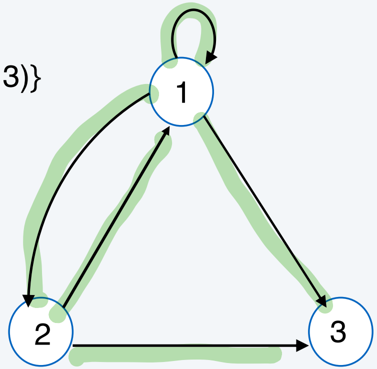


Arcs

If an arc or edge is between a vertex and itself, that is a **self-loop**

A graph is **simple** if it has no self-loops and no multi-edges

Example:
The vertices are the set {1,2,3}
The edges are the set {(1,1), (2,1), (1,2),(2,3), (1,3)}

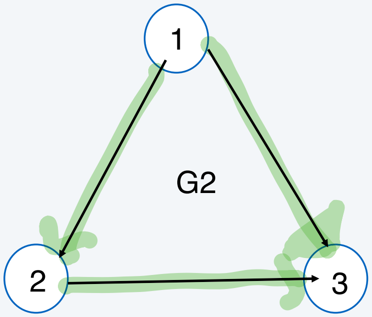


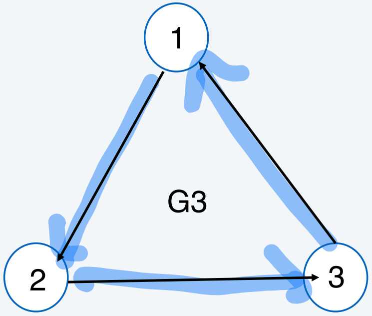
Directed Graphs

Edges have **start** and **end**. The edge (u,v) has start u and end v.

In a **directed** graph, the start (source) and end (sink) do matter. The edge (u,v) is not the same as the edge (v,u) in a directed graph.

Examples:
Directed: G2 = {{1,2,3}, {(1,2), (2,3), (1,3)}}
Directed: G3 = {{1,2,3}, {(1,2), (2,3), (3,1)}}





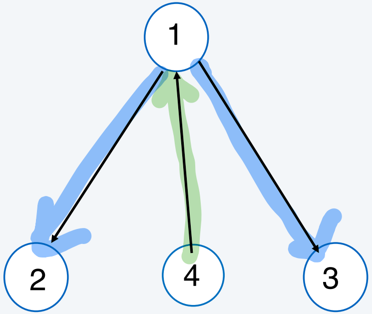
Neighbors

Vertices are called **neighbors** if they share an edge between them. If a graph contains edge, (u,v), vertex u and vertex v are neighbors and are also called **adjacent.**

Edges touching a vertex or are **incident** to it

In a directed graph, the **out-degree** of a vertex is the number of edges for which it is the starting point or source.

The **in-degree** of a vertex is the number of edges for which it is the ending point or sink.



In-Degree(1) = 1 (4,1)

Out-Degree(1) = 2

(1,2) (1,3)

Handshaking Lemma for Directed Graphs

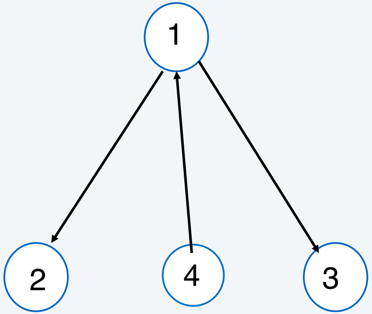
The sum of the out-degrees of all vertices equals the sum of the in-degrees of all vertices and equals the number of edges: $$\sum_{v\in V} out(v) = \sum_{v\in V} in(v) = |E|$$

$V = \{1, 2, 3, 4\}$

$E = \{(1,2)\ (1,3)\ (4,1)\}$

$|E| = 3$

Let's verify this statement on this graph.



$out(1) = 2$
$out(2) = 0$
$out(3) = 0$
$+\ out(4) = 1$
$\overline{\qquad\qquad 3}$

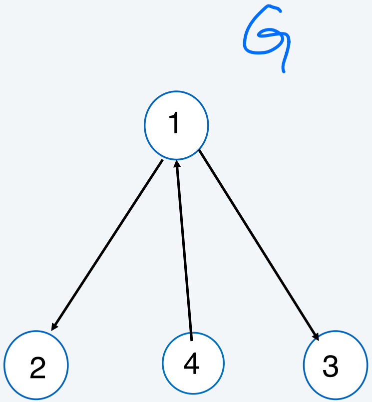
$in(1) = 1$
$in(2) = 1$
$in(3) = 1$
$+\ in(4) = 0$
$\overline{\qquad\qquad 3}$

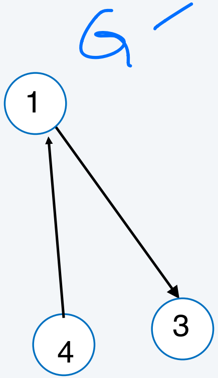
Subgraphs

A graph G'=(V',E') is a **subgraph** of a graph G =(V,E), if its vertices are a subset of V and its edges a subset of E containing only vertices in V', i.e. $V' \subseteq V, E' \subseteq E$

Some ways to create subgraphs: Removing edges, removing vertices, edge contraction

If a vertex is removed, all edges incident to it are also removed.





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- **Paths and Cycles**
- Connectivity

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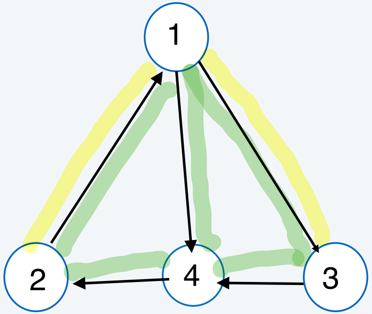
Walks and Paths

A **walk** is a sequence of vertices in the graph that traverse edges in the graph.

A walk is a **path** or **simple path** if all vertices are distinct

The **length** of a walk is the number of edges it traverses.



Example walk: 1,4, 2, 1, 3, 4
Its length is 5.

Example path: 2, 1, 3
Its length is 2.

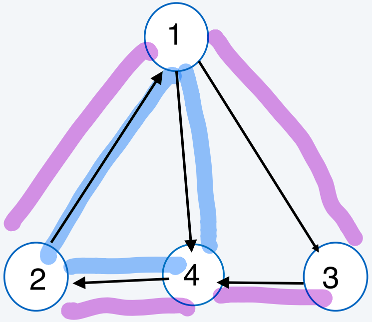
Cycles

A **circuit** is a walk that ends where it begins.

A **cycle** is circuit that only repeats the first and last vertex.

A single vertex is a **trivial** path or cycle of length 0.

A **nontrivial** path or cycle has length greater than zero.



Example trivial cycle: 1
Its length is 0.

Example nontrivial cycle: 1,3, 4, 2,1
Its length is 4.

Example nontrivial cycle: 1, 4, 2, 1
Its length is 3.

Example of circuit: 1, 4, 2, 1, 3, 4, 2, 1
Its length is 7.

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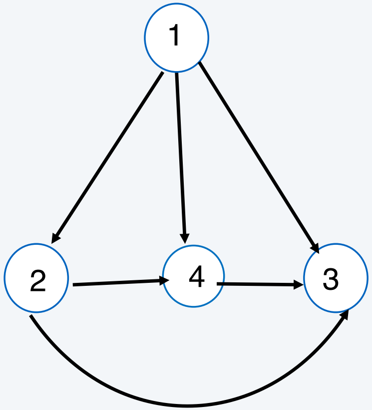
Connectivity in Directed Graphs

A directed graph is **connected** if for every pair of vertices u and v, there exists a path u to v or a path v to u in the graph.

A directed graph is **strongly connected** if for every pair of vertices u and v there exists a path from u to v and v to u.

If there is a walk from vertex v to vertex w, vertex w is **reachable from** vertex v.



Example:

This graph is connected. Why?

(1, 2)
(1, 4)
(1, 3)
(2, 4)
(4, 3)
(2, 3)

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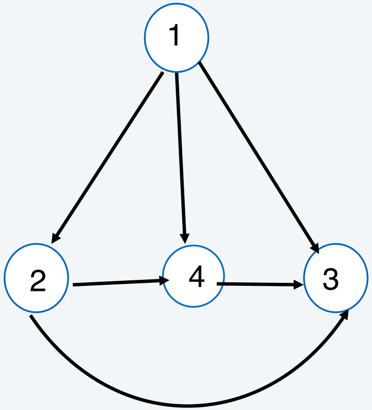


Example:

the vertex 1 has indegree of 0

This graph is not strongly connected. Why?

Directed Acyclic Graphs

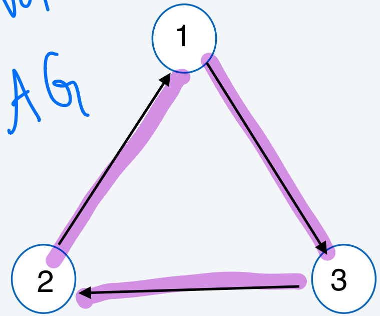
A **Directed Acyclic Graph (DAG)** is a directed graph with no cycles.

Example: Tournament Graph

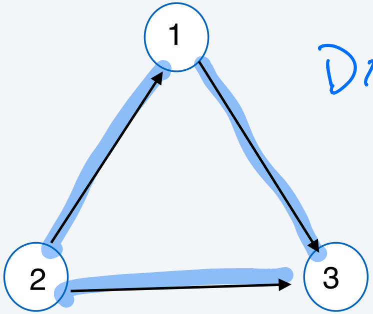
$\{2, 1, 3\}$

$2 < 1 < 3$

Not DAG



DAG



$\langle 2,1 \rangle$ $\langle 2,3 \rangle$ $(1,3)$