Sets and Functions

CSCI 170 Fall 2021 Lecture 2

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1.1-1.2

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https://www.seas.harvard.edu/courses/cs20/Schedule.html

Sets and Functions

- **Set Definitions**
- Set Operations
- Sequences
- Functions
- Pigeonhole Principle

1.1–1.2

Sets

A set is an unordered collection of distinct objects.

Examples of Sets

The set of first-year college students

$\boldsymbol{Z}$ = the set of integers

$\mathbf{N}$ = the set of nonnegative integers

R = the set of real numbers

{1, 2, 3}

{{1}, {2}, {3}}

{{Z}}

∅ = the empty set $= \{\ \}$

Set Membership, $\in$

Is an object x an element of a set A? Is x in A? $x \in A$?

Example:
Let $A = \{\text{apple}, 10, 1.5\}$

$10 \in A$ “10 is in A”

$15 \notin A$ “15 is not in A”

Elements of set must be distinct and order in which they are given does not matter.

Let $B = \{\text{poppy, sunflower, rose}\} = \{\text{rose, poppy, sunflower}\} = \{\text{sunflower, rose, poppy, rose}\}$

Cardinality: size of set

The cardinality of set A, |A|, is the number of distinct elements in A.

Examples:

l {12,13,14,15,2*6}l = 4

l{1,2,3}l = 3

l**{**{1,2,3}**}**l = 1

|**{N}**| = 1

| ∅ | = 0

|{∅ }| = 1

A set is **finite** if it can be counted using some initial segment of the integers

|{∅, {1}, {2}, {1,2}}| = 4

1 2 3 4

Otherwise **infinite**

{ {a}, {b}, {a} }

Subset: $\subseteq$

$A \subseteq B$: "A is a subset of B" or "A is contained in B"

For any element x, if $x \in A$, then $x \in B$

Examples:

$N \subseteq Z$
$\{7\} \subseteq \{7, \text{“Sunday”}, \pi\}$

$\emptyset \subseteq A$ for any set A
Why? For any element x, if $x \in \emptyset$, then $x \in A$.
Since no element can be in the empty set, this is vacuously true

$A \subseteq A$ for any set A

For proper subset meaning $A \subseteq B$ but $A \neq B$,
write $A \subsetneq B$

Equal: =

Sets A and B are equal, A = B, if and only if $A \subseteq B$ and $B \subseteq A$

If and only if is two statements:

1) If A = B, then $A \subseteq B$ and $B \subseteq A$

2) If $A \subseteq B$ and $B \subseteq A$, then A = B.

For proper subset meaning $A \subseteq B$ but $A \neq B$, write $A \subsetneq B$

Examples:

$\{1,2,3\} \subsetneq \{1, 2,3,4\}$

$\mathbf{N} \subsetneq \boldsymbol{Z}$

Powerset

The powerset of a set A, P(A), is the set of all subsets of A.

Examples:

P({1,2}) = the set of all subsets of {1,2}
= {∅, {1}, {2}, {1,2}}

P($\boldsymbol{Z}$) = the set of all sets of integers ("the power set of the integers")
N ∈ P(Z)

Powerset

The powerset of a set A, P(A), is the set of all subsets of A.

How do we build the powerset of a set A?

For each element in A for each subset in the powerset, let's consider if we will include the element in the subset or not.

We must get all combinations of including and excluding all elements.

An easier way: First generate all subsets of A of size 0, then size 1, then size 2, and so forth until |A|

Let's give it a try with the set A = { a, b , c}: $|A| = 3$

P(A) = {∅ ,{a}, {b}, {c} , {a,b}, {b,c}, {a,c}, { a, b , c}} $= 2^3$

Intuition: What is the size of P(A)? $2^{|A|}$

a	b	c
0	0	0
1	0	0
0	1	0
0	0	1
1	1	0
0	1	1
1	0	1
1	1	1

Powerset Practice Problem

For each of the following sets, write out every member of the entire set:

i) P({7,8,9}) – P({7,9})

$= \{\cancel{\emptyset}, \cancel{\{7\}}, \{8\}, \cancel{\{9\}}, \{7,8\}, \cancel{\{7,9\}}, \{9,8\}, \{7,8,9\}\}$

$\emptyset = \{\}$

ii) P(∅)

$= \{\emptyset\}$

iii) P(P({2}))

$P(\{2\}) = \{\emptyset, \{2\}\}$

$P(P(\{2\})) = \{\emptyset, \{\emptyset\}, \{\{2\}\}, \{\emptyset, \{2\}\}\}$

$|\emptyset| = 0$

$|\{\emptyset\}| = 1$

$\emptyset \neq \{\emptyset\}$

$\{\} \neq \{\{\}\}$

Set Builder

- Let U be the universal set of all possible elements.
- Let P(x) be a predicate or a condition that for all elements x in the universal set U is either true or false.
- Set Abstraction Notation specifies the set of elements of A of which P is true:
  - $\{x \in U\text{: } P(x)\}$ or $\{x \in U \mid P(x)\}$

: or | "such that"

n mod 2

Examples:

Let E be the set of even numbers: P(n) is the condition n % 2 = 0, $28 \in E$ $27 \notin E$

$E = \{n \in Z : n\%2 = 0\}$

Let O be the set of odd numbers: P(n) is the condition n % 2 = 1, $31 \in O$ $30 \notin O$

$O = \{n \in Z \mid n \% 2 = 1\}$

Let P be the set of prime numbers: P(n) is the condition n is prime, $7 \in P$ $28 \notin P$

$P = \{n \in Z : n \text{ is prime}\}$

Sets and Functions

- Set Definitions
- **Set Operations**
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1.1–1.2

Set Union

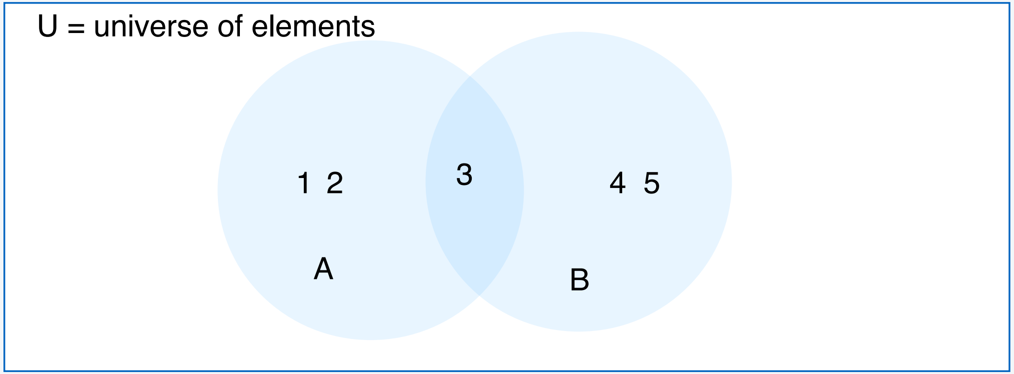
$A \cup B = B \cup A$

$x \in A \cup B$ if and only if $x \in A$ or $x \in B$

$(A \cup B) \cup C = A \cup (B \cup C)$

Example: Let A = {1,2,3} and B = {3, 4,5}

Let's consider the Venn Diagram of the union of A and B, $A \cup B$ = {1,2, 3, 4, 5}:

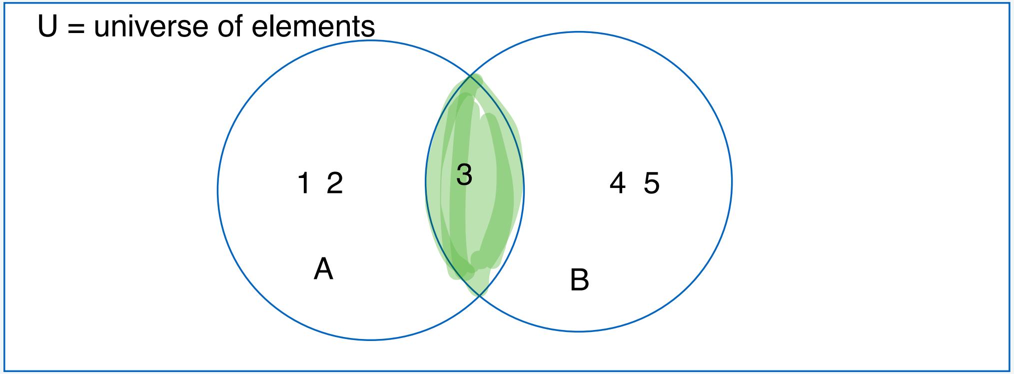


Set Intersection

$x \in A \cap B$ if and only if $x \in A$ and $x \in B$

Example: Let A = {1,2,3} and B = {3, 4,5}

Let's consider the Venn Diagram of the intersection of A and B, $A \cap B = \{3\}$:



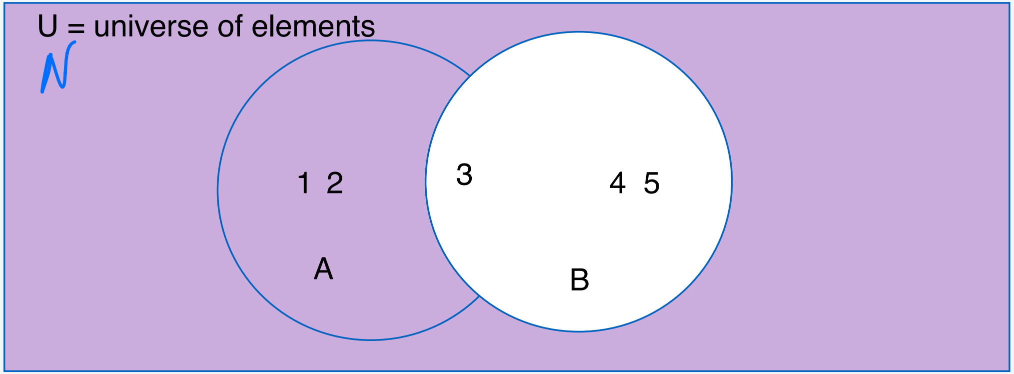
Set Complement

$\overline{B}$ B^c B'

$x \in \overline{B}$ if and only if $x \notin \mathbf{B}$

Example: Let A = {1,2,3} and B = {3, 4,5}

Let's consider the Venn Diagram of the complement of B:



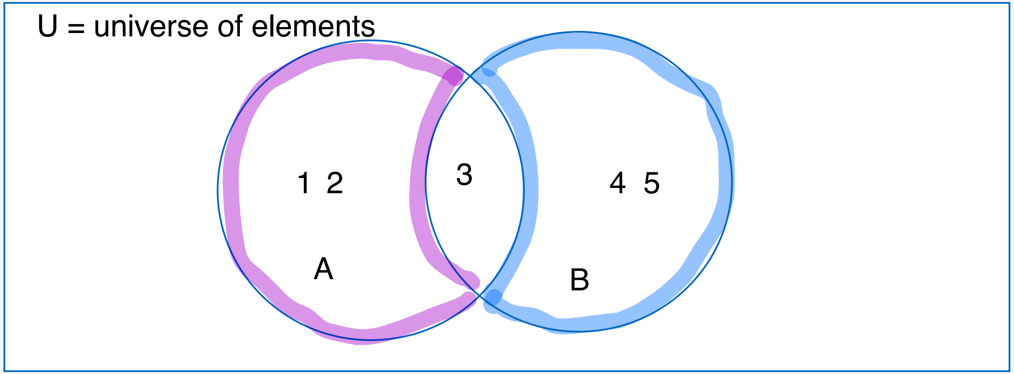
Set Difference

$A \setminus B = A - B = A \cap \overline{B}$

- $x \in$ **A\B** if and only if $x \in A$ and $x \notin B$
- Sometimes written A-B
- This set is the intersection of A and the complement of B

Example: Let A = {1,2,3} and B = {3, 4,5}

Let's consider the Venn Diagram of A\B in purple and B\A in blue:



Set Laws

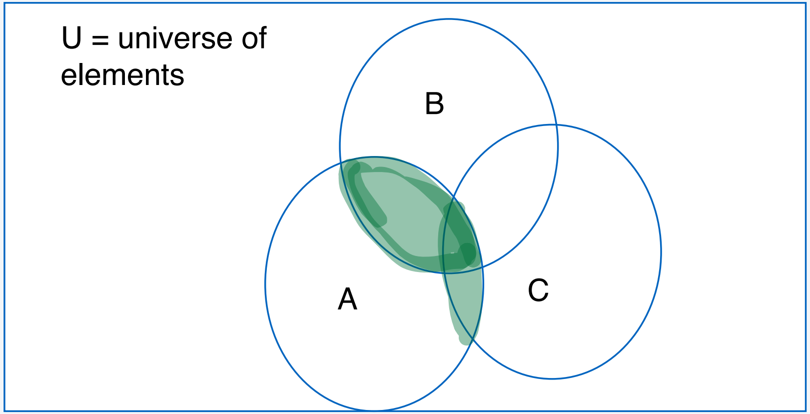
$A \cup (B \cap C)$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

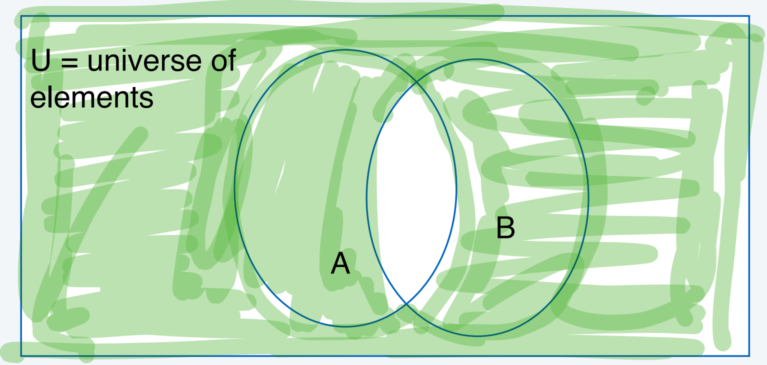
A distributive law for sets: A ∩(B ∪ C) = (A ∩ B) ∪(A ∩ C)

DeMorgan's Law for sets: The complement of intersection of two sets equals the union of their complements.

$\overline{A \cap B} = \bar{A} \cup \bar{B}$



Distributive Law



DeMorgan's Law

Set Difference Practice Problem

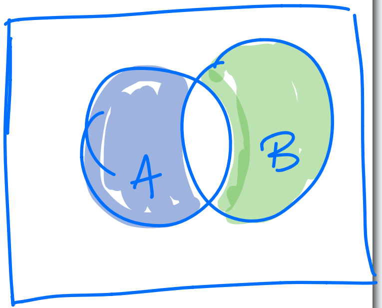
Show (A-B) ∩(B-A) = ∅ using definitions and set operations. Use Venn diagram to develop intuition.

$$(A-B)\cap(B-A) =$$

$$(A\cap\bar{B})\cap(B\cap\bar{A})$$

$$= A\cap(\bar{B}\cap B)\cap\bar{A}$$

$$= A\cap\emptyset\cap\bar{A} = \emptyset$$

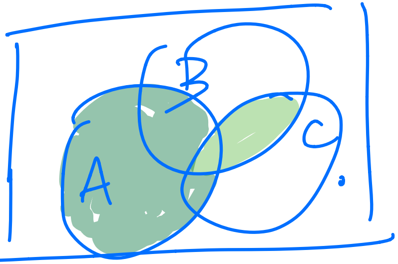


Set Equality Practice Problem

Show $A - (B \cap C) = (A - B) \cup (A - C)$ using set difference, distributive law, and DeMorgan's law. Use Venn diagram to build intuition.

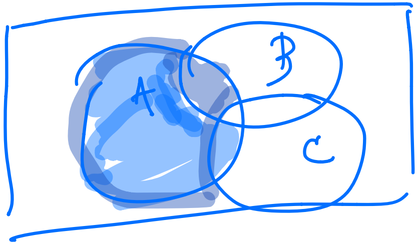
$$A - (B \cap C) = A \cap \left(\overline{B \cap C}\right)$$ set difference

$$= A \cap (\overline{B} \cup \overline{C})$$ de morgan



$$= (A \cap \overline{B}) \cup (A \cap \overline{C})$$ distributive law

$$= (A - B) \cup (A - C)$$ Set difference

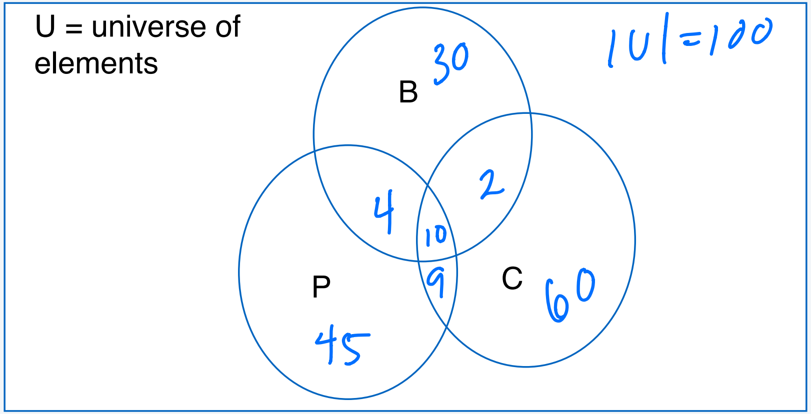


Venn Diagram Practice Problem

100 Students are enrolled in chemistry, physics, and biology. Enrollments: Physics = 45, Chemistry = 60, Biology = 30
10 Students are enrolled in all three and some students are enrolled in exactly 2 of these classes.
i) How many students are enrolled in 2 classes? $= 15$
ii) Of those taking 2 classes: 9 are in physics and chemistry and 4 are in physics and biology. How many are in chemistry and biology?

$$|B \cup P \cup C| = 100$$

$$|P \cap B \cap C| = 10$$



$$
\begin{aligned}
+\ |P| &= 45 - 10 \\
+\ |C| &= 60 - 10 \\
+\ |B| &= 30 - 10 \\
\hline
& 105 \\
-\ & 90 \\
\hline
& 15
\end{aligned}
$$

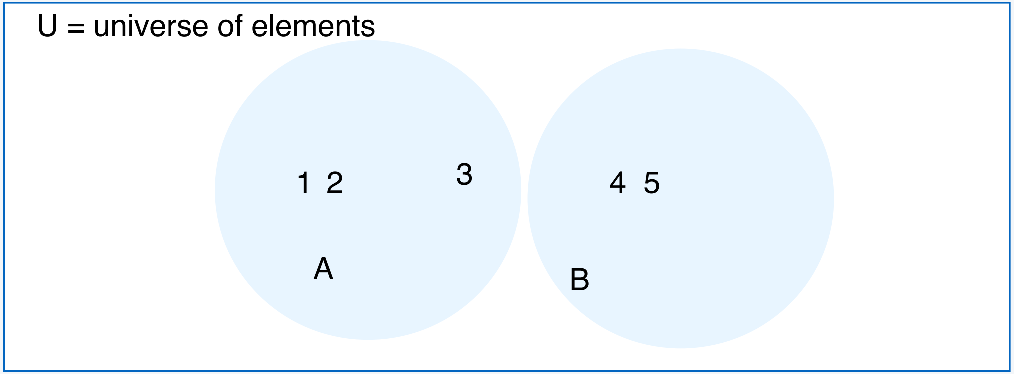
$$|B \cap C - (B \cap C \cap P)| = 2$$

Disjoint Sets

Sets A and B are **disjoint** if $A \cap B = \emptyset$.

Example: Let $A = \{1,2,3\}$ and $B = \{ 4,5\}$

Let’s consider the Venn Diagram of disjoint A and B:

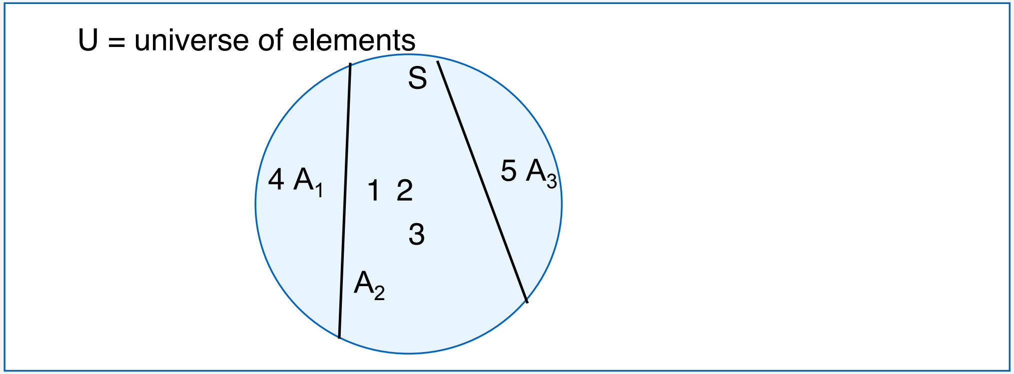


Partition

A partition of a set S is a set of nonempty sets $\{A_1, A_2, A_{3,\ldots,} A_k\}$, k greater than or equal to 1, such that

1) $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = S$
2) For any distinct pair of sets, A_i and A_j are disjoint $(i, j \in \{1,2,3,\ldots,k\})$

Example: Let S = {1,2,3,4,5}, A_1 = {4}, A_2 = {123}, and A_3 = {5}
Let's draw the partition of S using a Venn Diagram:



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1.1–1.2

Sequences

- Sequences (or lists or tuples) are **ordered** collections of objects.
- Sequence notation may give objects using the save variables, but different subscripts: x_1, x_2, x_3,…
- If two objects called an ordered pair, <a, b>.
- If n-objects, called an n-tuple.

Example:
Let’s consider the ordered pairs of latitude and longitude for some locations: <latitude, longitude>

Los Angeles: <34.0522, -118.2437>

Paris: <48, 2>

Boston: <42, -71>

Antartica: <-71, 42>

Let’s consider some ordered triples or 3-tuples of <temperature, humidity, location>:

<86 F, 52%, Los Angeles>

<71 F, 84%, Boston>

Cartesian Product

Cartesian product of sets A and B, A x B = {<a,b> : a ∈ A and b ∈ B}

That is the set of all ordered pairs such that the first element is from set A and the second from set B.

Example: Let’s consider how we may order drinks at a café. We have choice of temperature = {hot, iced} and beverage = {coffee, tea}

If we order first giving the temperature then beverage, that is temperature x beverage:

Temperature x beverage = { <hot, coffee>,<hot,tea>, <iced, coffee>, <iced, tea>}

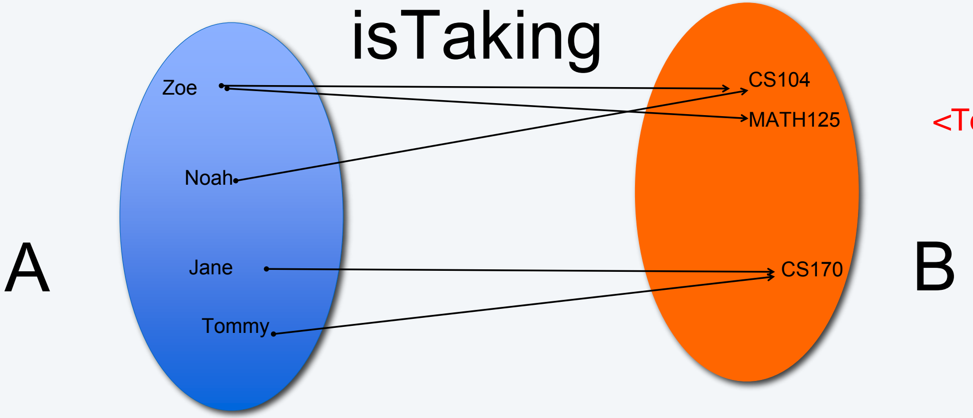
If we order first giving the beverage then temperature, that is beverage x temperature:

Beverage x temperature = { <coffee,hot>, <tea,hot>, <coffee, iced>, <tea,iced>}

Binary Relations

A **binary relation** on A x B is a subset of A x B.

Example:
Let A = {Zoe, Tommy, Jane, Noah}
B = {CS104, CS170, Math 125}
isTaking = {<Zoe, CS104>, <Zoe, Math 125>, <Tommy, CS170>,<Jane, CS170>, <Noah, CS104>}
isTaking binary relation = {<a, b> : Student a is taking course b}
isTaking is a subset of A x B (i.e. students x courses)



<Tommy, Math125> $\notin$ isTaking

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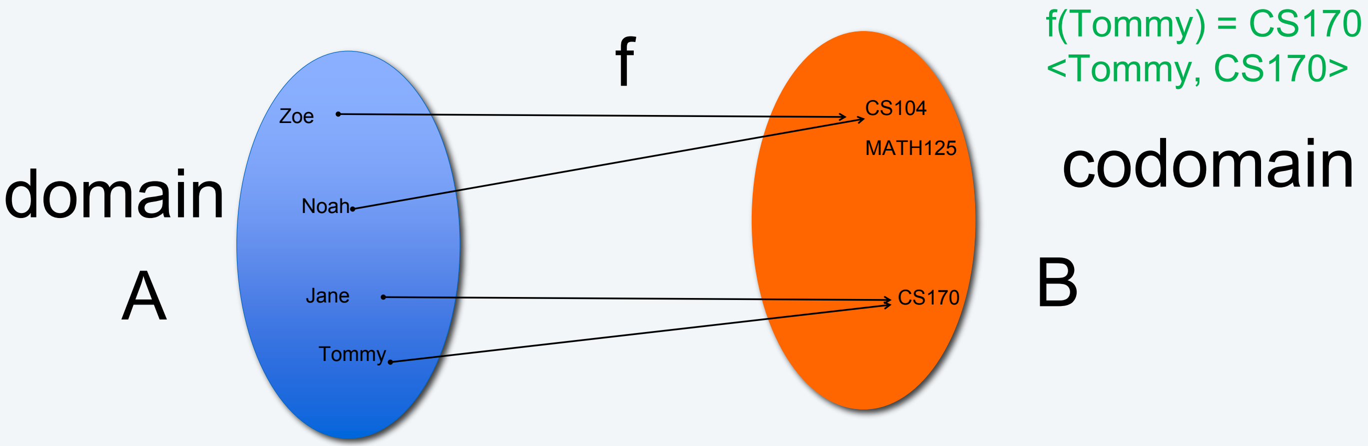
1.1–1.2

Functions

- A **function, f:** A → B.
- A is the domain and B is the codomain
- Functions are a type of binary relation.
- Each element of A is associated with one element of B.
- a ⟼ b f(a) = b
- **ONE ARROW OUT OF EACH ELEMENT OF A**



Sets and Functions

- Set Definitions
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- **Pigeonhole Principle**

1.1–1.2

Pigeonhole Principle

If there are more pigeons than pigeonholes and every pigeon goes into a pigeonhole, then some pigeonhole must contain more than one pigeon in it.

Let $f : A \to B$, where A and B are finite sets and $|A| > |B|$. Then there exist distinct elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.

A is set of 5 mice

B is set of 3 teacups

f maps mice to teacups

Image Source: http://artesanogalleria.com/charming-tails-teacup-mice/

Pigeonhole Principle Application Example

In any group of 8 people, two were born on the same day of the week

What are the "pigeons" and what are the "pigeonholes"?

A = the set of people, *B* = {Sun, … Sat}, *f(a)* = the day of the week on which *a* was born

Since $|A| = 8 > |B| = 7$, by PHP, at least two people in A were born on the same day of the week

Image Source: http://artesanogalleria.com/charming-tails-teacup-mice/

Pigeonhole Principle Application Example

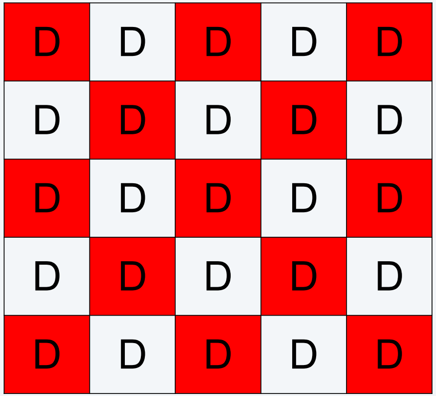
- Suppose each pigeonhole has one bird
- Every bird moves to an adjacent square (up, down, left or right).
- Show that no matter how this is done, some pigeonhole winds up with at least 2 birds.

$A =$ birds on red squares

$B =$ gray squares

$f(a) =$ the square a moves to

$|A| = 13, |B| = 12$



Since |A| = 13 > |B| = 12, by PHP, at least two birds will move to the same gray square.

Pigeonhole Principle Practice Problem 1

N people are at a party. Show that any two guests must have the same number of friends at the party. In this example, friendship is symmetric. If Alice is friends with Bob, Bob is friends with Alice.

Photo by Glenn Han on Unsplash

Pigeonhole Principle Practice Problem 2

Let S = {3,4,5,6,7,8,9,10,11,12}. Show that any subset of six distinct integers from S must contain at least two integers whose sum is 15.

Pigeonhole Principle Practice Problem 3

Show that in any set of 9 positive integers at least two share all of their prime factors less than or equal to 5.

Extended Pigeonhole Principle

If n pigeons are placed into k pigeonholes, then there is at least one pigeonhole containing at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

N pigeons = 5 mice

Pigeonholes = 3 teacups

$\left\lceil \frac{n}{k} \right\rceil$ is the ceiling function: It is the smallest integer great than or equal to its argument.

Image Source: http://artesanogalleria.com/charming-tails-teacup-mice/

Pigeonhole Principle Practice Problem 4

Twenty-five people go to daily yoga classes at the same gym, which offers 8 classes every day.
Each attendee wears either a blue, red, or green shirt to class.
Show that on a given day, there is at least one class where two people are wearing the same color shirt.

Photo by Annie Spratt on Unsplash

Extended Pigeonhole Principle Practice Problem

An MLB baseball card collector only collects baseball card for players from teams from the NL West. There are 5 teams in the NL West. What is the minimum number of cards that must be in the collection to guarantee that there are at least 100 cards from the same NL West team?

WEST	W	L	%	GB
LAD	88	47	.652	-
ARI	67	66	.504	20.0
SF	65	67	.492	21.5
SD	61	71	.462	25.5
COL	59	75	.440	28.5