Predicate Logic

CSCI 170 Spring 2021

Sandra Batista

1.1-1.2

Predicate Logic

- **Precedence of Logical Operators**
- Propositional Equivalences
- Creating Valid Arguments Using Inference
- Predicates and Quantifiers
- Negating Quantifiers

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Precedence of Logical Operators

Logical operators have the following precedence:

1. Negation
2. And
3. Or, Exclusive Or, and Implication
4. Biconditional (if and only if)

To avoid ambiguity, it is best to use parentheses

Logical Operator Precedence Practice Problems

1. $p \wedge q \rightarrow r$

$p = F \quad r = F$

$(\underset{F}{p} \wedge q) \rightarrow r$ $F \rightarrow ?$ True

$\not\equiv$

$p \wedge \underbrace{(q \rightarrow r)}$ $F \wedge ?$ False

2. $p \vee q \wedge r$

$\text{true} = p \vee \underbrace{(q \wedge r)}_{F} = T \vee F$

$\not\equiv$

$\underbrace{(p \vee q) \wedge r} = \text{false}$

$p = T \quad r = F$

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Showing when propositions are not equivalent

To show that two propositions are not equivalent:

1. Show that the truth tables for the propositions are not the same
2. Give a counterexample

Example: Show that the following propositions are **not equivalent**:

$$(p \to q) \to r \quad \not\equiv \quad p \to (q \to r)$$

Can you give a counterexample? A counterexample would be propositions, p, q, and r such that one implication is true and the other is false at the same time.

Showing when propositions are not equivalent

To show that two propositions are not equivalent:

1. Show that the truth tables for the propositions are not the same
2. Give a counterexample

false p=F r=F

Example: Show that the following propositions are **not equivalent**:

1 $(p \to q) \to r$ $\not\equiv$ $p \to (q \to r)$ 2

F true

T → F

Counterexample: Let p = 7 is even

q = 3 is even

r = 5 is even

p = F

q = F

r = F

1 $(p \to q) \to r$

$(F \to F)$

$T \to F$

F

2 $p \to (q \to r)$

$F \to (F \to F) = \text{True}$

Useful Propositional Equivalences

- Contrapositive: $(p \to q) \equiv (\neg q \to \neg p)$.
- Tautology: $p \lor \neg p \equiv \mathbf{T}$.
- Contradiction: $p \land \neg p \equiv \mathbf{F}$.
- Idempotent: $p \lor p \equiv p$. $p \land p \equiv p$
- Implication: $(p \to q) \equiv (\neg p \lor q)$.
- Identity: $p \land \mathbf{T} \equiv p$. $p \lor \mathbf{F} \equiv p$
- Domination: $p \lor \mathbf{T} \equiv \mathbf{T}$. $p \land \mathbf{F} \equiv \mathbf{F}$

Useful Propositional Equivalences

- Double Negation: $\neg(\neg p) \equiv p$.
- Commutative: $p \vee q \equiv q \vee p \quad p \wedge q \equiv q \wedge p$
- Associative: $(p \vee q) \vee r \equiv p \vee (q \vee r)$

  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
- DeMorgan's: $\neg(p \wedge q) \equiv (\neg p \vee \neg q) \quad \neg(p \vee q) \equiv (\neg p \wedge \neg q)$
- Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.
- Exclusive or: $p \oplus q \equiv (p \leftrightarrow \neg q)$.

Logical Equivalence Practice Problem

Show the following using logical equivalences: $p \oplus q \equiv (p \wedge \neg q) \vee (q \wedge \neg p)$

$p \oplus q \equiv (p \leftrightarrow \neg q)$ (exclusive or)

$\equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$ (biconditional)

$\equiv (\neg p \vee \neg q) \wedge (q \vee p)$ (implication, double negation)

$\equiv (\neg p \wedge q) \vee (\neg p \wedge p) \vee (\neg q \wedge q) \vee (\neg q \wedge p)$ (distributive law)

$\equiv (\neg p \wedge q) \vee F \vee F \vee (\neg q \wedge p)$ (contradiction)

$\equiv (p \wedge \neg q) \vee (q \wedge \neg p)$ (identity, commutative law)

notice we fix one proposition and use equivalences to show that it is the same as the other

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Constructing Valid Arguments

- An **argument** is a sequence of statements starting from a set of premises (or assumptions) that uses steps of reasoning from logical equivalences or inference to reach a conclusion.
- An argument is **valid** if and only if the conclusion can be shown true using steps of reasoning if the premises are true.
- An argument is **sound** if and only if the conclusion can be shown true using steps of reason and the premises are true.

Let's express the statements for a valid and sound argument in propositional logic:

Let p = "an argument is valid"
    s = "an argument is sound"
    r = "the premises are true"
    q = "the conclusion can be shown true"

validity: $p \iff (r \to q)$

soundness: $s \iff (r \land q)$

Rules of Inference

- Modus Ponens: $p, p \to q$, then q
- Modus Tollens: $p \to q$, $\neg q$, then $\neg p$
- Hypothetical Syllogism: $p \to q$, $q \to r$, then $p \to r$

Examples:
Modus Ponens: *It is raining and if it is raining, you carry an umbrella, then you carry an umbrella.*

Modus Tollens: *If it is raining, you carry an umbrella, and you are not carrying an umbrella, then it is not raining.*

Hypothetical Syllogism*: If it is sunny, then you go to the beach and if you go to the beach, you swim in the ocean, then if it is sunny, you swim in the ocean.*

Rules of Inference

- Disjunctive Syllogism: $p \vee q$, $\neg p$, then q
- Addition: p, then $p \vee q$
- Simplification: $p \wedge q$, then p

Examples:
Disjunctive Syllogism: *The farmer harvests ripe kale today or the farmer eats winter squash and the farmer did not harvest any kale, so the farmer eats squash.*

Addition: The farmer harvests kale, then the farmer harvests kale or the farmer eats winter squash.

Simplification: *The farmer harvests kale today and the farmer eats winter squash, then the farmer harvests kale today.*

Rules of Inference

- Conjunction: p, q, then $p \wedge q$
- Resolution: $p \vee q$, $\neg p \vee r$, then $q \vee r$

Examples:

Conjunction: *The farmer harvests kale today. The farmer eats squash today. Therefore, the farmer harvests kale and eats squash today.*

Resolution: *The farmer harvests kale today or the farmer eats squash today.*
The farmer does not harvest kale today or the farmer eats oranges today.
Then the farmer eats squash or oranges today.

Showing a rule of inference is a tautology

Show that modus ponens is a tautology.

p,p→q, then q

$$(p \land (p \to q)) \to q \equiv T$$

Proof:

$(p \land (p \to q)) \to q \equiv (p \land (\neg p \lor q)) \to q$ (implication)

$\equiv ((p \land \neg p) \lor (p \land q)) \to q$ (distributive law)

$\equiv (F \lor (p \land q)) \to q$ (contradiction)

$\equiv (p \land q) \to q$ (identity)

$\equiv \neg(p \land q) \lor q$ (implication)

$\equiv \neg p \lor (\neg q \lor q)$ (De Morgan's law, associativity)

$\equiv \neg p \lor T$ (tautology)

$\equiv T$ (domination)

Constructing a valid argument example

Dr. Doe is worried about Patient A's cholesterol and is trying construct a treatment plan.
Premises:
1) If Patient A has lower cholesterol, Patient A will be healthier.
2) If Patient A loses weight, Patient A will have lower cholesterol.
3) If Patient A does not lose weight, Patient A eats at In-N-Out every day.
4) Patient A no longer eats at In-N-Out everyday.
Conclusion: Patient A will be healthier.

How can we show that Dr. Doe's plan is valid?

Constructing a valid argument example

Dr. Doe is worried about Patient A's cholesterol and is trying construct a treatment plan.
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Conclusion: Patient A will be healthier.

First, we can translate the statements into propositional logic:

Let p = Patient A has lower cholesterol.
q = Patient A is healthier.
s = Patient A loses weight.
t = Patient A eats at In-N-Out every day.

Premises:
1) $p \rightarrow q$
2) $s \rightarrow p$
3) $\neg s \rightarrow t$
4) $\neg t$

Conclusion: q

Constructing a valid argument example

Dr. Doe is worried about Patient A's cholesterol and is trying construct a treatment plan.
Premises:
1) If Patient A has lower cholesterol, Patient A will be healthier. $p \to q$
2) If Patient A loses weight, Patient A will have lower cholesterol. $s \to p$
3) If Patient A does not lose weight, Patient A eats at In-N-Out every day. $\neg s \to t$
4) Patient A no longer eats at In-N-Out everyday. $\neg t$

Conclusion: Patient A will be healthier. q

Next use rules of inference to reach conclusion:

Since Patient A does not eat out every day (premise 4) $\neg t$

By Premise 3 and Modus Tollens: $\neg s \to t, \neg t \to \neg(\neg s) \equiv \underline{s}$

We can infer s (Patient A loses weight)

By Premise 1 and 2 and Hypothetical Syllogism: $s \to p, p \to q$ then $\underline{s \to q}$

If Patient A loses weight, Patient A will be healthier.

By the first step and second and Modus Ponens:

s and $s \to q$ then q Patient A will be healthier

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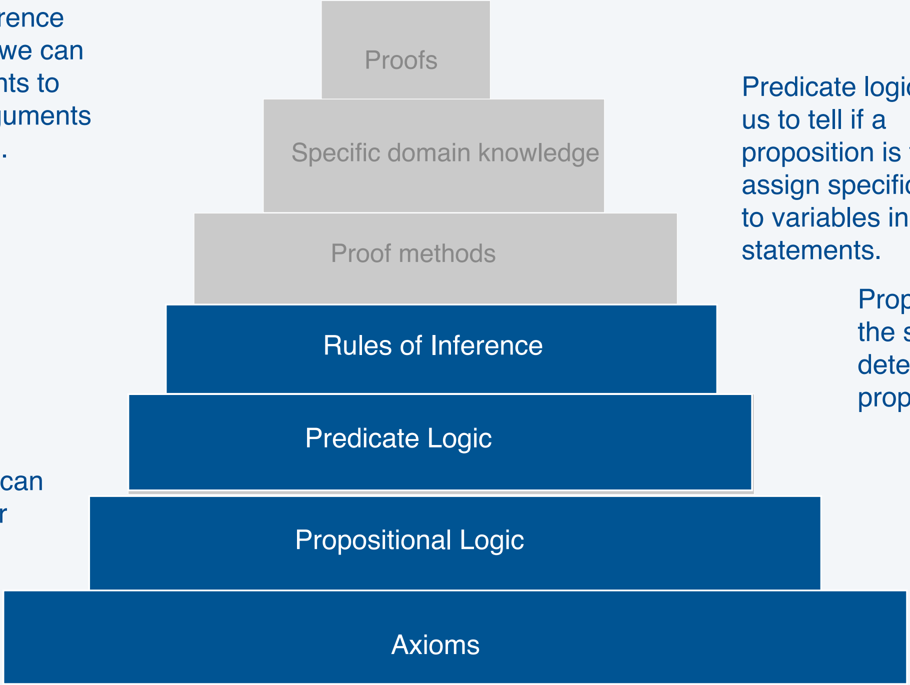
The foundation for writing proofs

Rules of inference instruct how we can use statements to construct arguments that are valid.

Predicates are statements with variables.

Propositions are statements that we can determine as true or false.



Predicate logic helps us to tell if a proposition is true if we assign specific values to variables in statements.

Propositional logic is the steps we use to determine the truth of a proposition

Axioms are assumptions we may use.

Predicates

A **predicate** is a statement with at least one variable that is a proposition when specific values are given for each variable.

The variables must have a universe of possible values specified.

$n \in \mathbb{Z}^{>0}$

positive integers

Examples:

Predicate: isPowerofTwo(n): n = 2^k from some non-negative integer k and where n is a positive integer.

T T F

Propositions: isPowerofTwo(1), isPowerofTwo(8). isPowerofTwo(6).

$2^0 = 1$ $8 = 2^3$ $3 * 2 = 6$

Predicate: isPrime(n): n is prime where n is a positive integer.

Propositions: isPrime(15). isPrime(7).

F T

Universal Quantifier

For all or for any $\forall$

For all x in S, P(x) $\forall x \in S, P(x)$

Example: For any integer, n, n is greater than or equal to 2 and isPrime(n)

$\forall n \in Z, \boxed{n \geq 2 \wedge IsPrime(n)}$

let $n = 15$

$\boxed{15 > 2 \text{ but } 15 \text{ is not prime}}$ F

counterexample

Coding up the universal quantifier

For all or for any $\qquad \forall$

For all x in S, P(x) $\qquad \forall x \in S, P(x)$

Assume the universe for x, S, is finite. Suppose we have function that given a value for x could return true or false for P(x). Could we code up a for loop to check for a universal quantifier?

Pseudocode:

```
for (x ∈ S) {
    if (¬P(x)) return false;
    // counterexample
}
return true;
```

Existential Quantifier

There exists

$$\exists$$

There exists x in S, P(x)

$$\exists x \in S, P(x)$$

Example: There exists an integer, n, greater than or equal to 2, isPrime(n)

$$\underline{\exists n \in Z}, \boxed{n \geq 2 \wedge IsPrime(n)}$$

predicate

Let $n = 3$ $\quad \underline{3 \geq 2} \wedge \underline{3 \text{ is prime}}$

Coding up the existential quantifier

There exists $\exists$

There exists x in S, P(x) $\exists x \in S, P(x)$

Assume the universe for x, S, is finite. Suppose we have function that given a value for x could return true or false for P(x). Could we code up a for loop to check for an existential quantifier?

Pseudocode:

```
for (x ∈ S) {
    if (P(x)) return true;
}
return false;
```

Quantifiers Practice Problem

$n \in \mathbb{Z}^{>0}$

$d \mid n$ (no remainder)

$(n \% d == 0)$

How to write isPrime(n) using quantifiers? An integer n is prime if and only if n is greater than or equal to 2 and its only divisors are 1 and itself.

$$\text{isPrime}(n) \triangleq n \geq 2 \land \forall d \in \mathbb{Z}^{>0} : d \mid n \Rightarrow d = 1 \lor d = n$$

How to write isPowerofTwo(n) using quantifiers?

$$\text{isPowerofTwo}(n) \triangleq \exists k \in \mathbb{Z}^{\geq 0} : n = 2^k$$

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Negating universal quantifiers

To negate a statement containing quantifiers, first negate the quantifier.

To negate $\forall$ change it to a $\exists$

Then remember to negate the predicate also.

$\neg[P(x_1) \wedge P(x_2) \wedge P(x_3) \ldots]$ $\neg P(x_1) \vee \neg P(x_2) \vee \neg P(x_3) \ldots$

$$\neg[\forall x \in S : P(x)] \Leftrightarrow [\exists x \in S : \neg P(x)]$$

$x_1, x_2, x_3 \in S$

Example: Any integer greater than or equal to 2 is prime

$\forall n \in \mathbb{Z}^{\geq 2}$ isPrime(n) False

There exists an integer greater than or equal to 2 that is not prime.

Let $n = 4$

true ✱ $\exists n \in \mathbb{Z}^{\geq 2}$ $\neg$isPrime(n) ✱ 4 is not prime

Negating existential quantifiers

To negate a statement containing quantifiers, first negate the quantifier.

To negate $\exists$ change it to a $\forall$

Then remember to negate the predicate also.

$$\neg[\exists x \in S : P(x)] \Leftrightarrow [\forall x \in S : \neg P(x)]$$

let S be set of all zoos

P(x): Zoo x has no animals

Example: There exists a zoo with no animals.

$\exists x \in S : P(x)$

negate predicate

Every zoo has at least one animal.

$\forall x \in S \ \neg P(x)$

Negating Quantifiers Practice Problem 1

Negate: $\neg[\exists x : \forall y : \exists z : P(x,y,z)]$

let $P(x,y,z)$: $x+y=z$

Remember to negate each quantifier and then the predicate.

$\equiv \forall x : \exists y : \forall z : \neg P(x,y,z)$

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