Induction

CSCI 170 Spring 2021

Sandra Batista

1.1-1.2

Induction

- **Induction**
- Strengthening the Inductive Hypothesis
- Strong Induction
- Error Checking

1.1–1.2

Induction

To prove claims of the form: $$\forall n \in N : P(n)$$

We apply proof technique called **induction.**

Examples of claims:

≥ 0

For any integer n, there exists a way to tile a 2xn grid with dominoes.

> 0

For any integer n, 3n+2 is odd.

Induction

To prove claims of the form: $\forall n \in N : P(n)$

We apply proof technique called **induction.**

To prove a claim holds for any natural number:

1. Show the base case(s), i.e. the claim holds for n=0, P(0).
2. Assume the inductive hypothesis. This means to assume that the predicate holds for a specific value, e.g. Assume P(n) holds.
3. For the inductive step, use the inductive hypothesis to show that the predicate holds for the next value.

$$P(n) \Rightarrow P(n+1)$$

Why Induction Works

- Show P(0) directly.
- If we then show P(0)->P(1), then by modus ponens, P(1) holds.
- Now use P(1) holds and show P(1)->P(2), so by modus pones P(2) holds.
- P(2) holds and show P(2)->P(3), so by modus ponens P(3) holds.
- For any n, we can apply this sequence of inferences using the inductive step n times to show that P(n) holds.

Induction Example

$P(3): \sum_{i=0}^{3} 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 1+2+4+8 = 15 = 2^{3+1} - 1 = 16 - 1 = 15$

Exercise: For any n in the natural numbers, show $\forall n \in N \; P(n): \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Proof by Induction:

Base case: $n=0$ $\quad \sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^{0+1} - 1 = 1$ ✓

Inductive hypothesis (IH): Assume $P(n)$ (for some specific n)

Inductive step: Show $P(n) \Rightarrow P(n+1)$

$P(n+1): \sum_{i=0}^{n+1} 2^i = \sum_{i=0}^{n} 2^i + 2^{n+1} = \underbrace{2^{n+1} - 1}_{\text{by IH}} + 2^{n+1} = 2^{n+2} - 1$ ✓

$P(n+1)$ holds

Induction Example

$P(3) = \sum_{i=0}^{3} i = 0+1+2+3 = 6 = \frac{3(4)}{2}$ ✓

$\forall n \in N \; P(n):$

Exercise: For any n in the natural numbers, show $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Proof by Induction:

Base case: $n=0$ $P(0): \sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$

IH: Assume $P(n)$ (for some specific n)

Inductive step: show $P(n) \rightarrow P(n+1)$

$P(n+1): \sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$ ✓

by IH $P(n+1)$ holds

Induction

- Induction
- **Strengthening the Inductive Hypothesis**
- Strong Induction
- Error Checking

1.1–1.2

Strengthening the Inductive Hypothesis

- Sometimes, we may prove a stronger or more specific result in order to prove a statement.
- This is called **strengthening the inductive hypothesis.**

Example: Show for any n greater than or equal to 0, $\sum_{i=0}^{n} i = \Theta(n^2)$

We have already shown the stronger, more specific statement:

For any n greater than or equal to 0, $\sum_{i=0}^{n} i = \dfrac{n(n+1)}{2}$

The previous claim follows since $\dfrac{n(n+1)}{2} \in \Theta(n^2)$

Strengthening Inductive Hypothesis Example

Exercise: Show that for any natural number k greater than or equal to 1, the sum of the first k odd positive integers is a perfect square. A perfect square is an integer squared.

$$\forall\ k \geq 1 \quad P(k): \sum_{i=1}^{k}(2i-1) = x^2 \text{ for some } x \in \mathbb{Z}^{\geq 0}$$

How to start?

$$2\sum_{i=1}^{k} i - \sum_{i=1}^{k} 1$$

$$\cancel{2}\,\frac{k(k+1)}{\cancel{2}} - k = k^2$$

Exercise: Show that for any natural number k greater than or equal to 1, the sum of the first k odd positive integers is a perfect square. A perfect square is an integer squared.

$$\forall k \geq 1, \sum_{i=1}^{k} 2i - 1 = k^2 \; : P(k)$$

Base case: $P(1)$ $k=1$ $\sum_{i=1}^{1}(2i-1) = 2-1 = 1 = 1^2$ ✓

IH: Assume $P(k)$ (for some specific k)

Inductive step: $P(k) \rightarrow P(k+1)$

$$P(k+1): \sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1) = k^2 + 2k + 2 - 1$$

$$= (k+1)^2 \checkmark$$

By IH $P(k+1)$ ✓

Strengthening Inductive Hypothesis Example

Prove that any 2^n by 2^n board with a middle square removed can be tiled with L-shaped tiles for any n greater than or equal to 1.

Strengthen Inductive Hypothesis: Show that any 2^n by 2^n board with any square removed can be tiled with L-shaped tiles for any n greater than or equal to 1.

Our predicate, P(n): A 2^n by 2^n board with any square removed can be tiled with L-shaped tiles

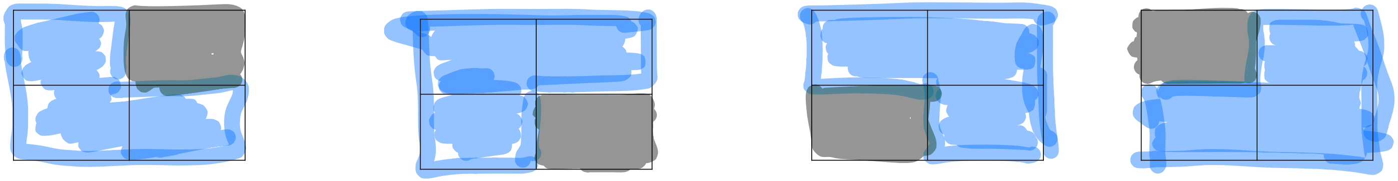
We are showing that P(n) holds for any n greater than or equal to 1.

We will show how to do so constructively in proof.

Strengthening Inductive Hypothesis Example

Strengthen Inductive Hypothesis: Show that any 2^n by 2^n board with any square removed can be tiled with L-shaped tiles for any n greater than or equal to 1.

Proof by Induction:
Base case: n =1



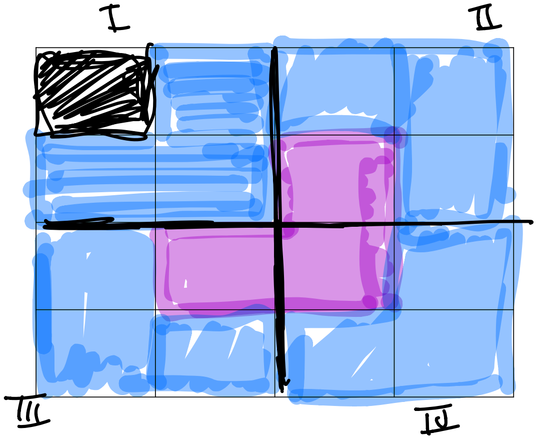
Assume the inductive hypothesis: Assume P(n) holds for some specific value n greater than or equal to 1.
A 2^n by 2^n board with any square removed can be tiled with L-shaped tiles.

Strengthening Inductive Hypothesis Example

$P(1) \rightarrow P(2)$

Strengthen Inductive Hypothesis: Show that any 2^n by 2^n board with <u>any</u> square removed can be tiled with L-shaped tiles for any n greater than or equal to 1.

Before we get to the inductive step, let's consider n = 2. How can we use the base case, n= 1, showing that we can tile a 2x2 board to tile this 4x4 board?



0) each quadrant is 2'x2' board

1) If quadrant I has missing tile we can tile it using base case (P(1)) using L tile.

2) place an L tile in center so quandrants II, III, IV are missing tile.

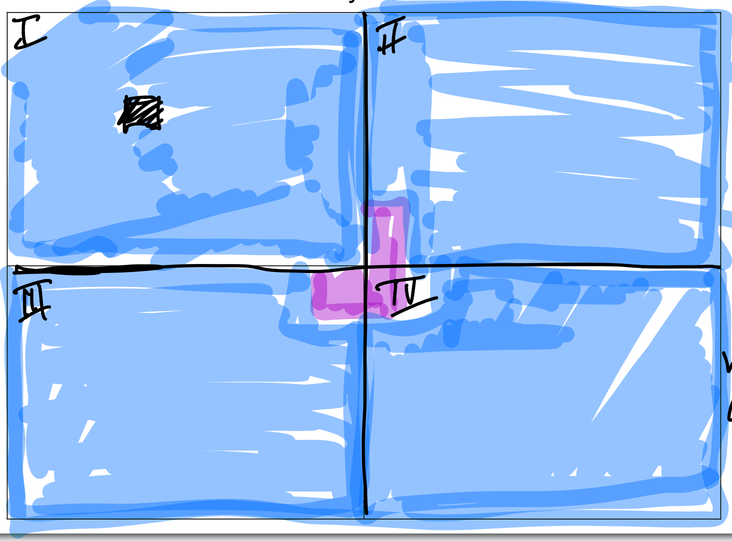
3) apply P(1) L tile remaining quadrants with a single tile

Strengthening Inductive Hypothesis Example

Strengthen Inductive Hypothesis: Show that any 2^n by 2^n board with <u>any</u> square removed can be tiled with L-shaped tiles for any n greater than or equal to 1.

Inductive Step. Assume we can tile a 2^n by 2^n board with any square removed using L-tiles.

Show we can tile a 2^{n+1} by 2^{n+1}



0/ separate the $2^{n+1} \times 2^{n+1}$ board into 4 quadrants, $2^n \times 2^n$ boards

1/ WLOG quadrant I missing tile by IH we can tile $2^n \times 2^n$ board w/ L tiles

2) place an L tile in center of board so that each remaining quadrant is missing a tile

3) by IH we can L-tile the $2^n \times 2^n$ boards that are quadrants II, III, IV

Induction

- Induction
- Strengthening the Inductive Hypothesis
- **Strong Induction**
- Error Checking

1.1–1.2

Is Weak Induction Necessary?

Given how induction works, does it matter if we show the base case and assume for some specific value, n-1, that P(n-1)->P(n)?

We can show through many steps of inference:
P(0)->P(1)->P(2)->P(3)->P(4)->…

What if we could assume many hypotheses at once?

Strong Induction

To prove claims of the form: $\forall n \in N : P(n)$

We apply proof technique called strong **induction.**

To prove a claim holds for any natural number:

1. Show the base case(s), i.e. the claim holds for n=0, P(0).
2. Assume the inductive hypothesis. Assume the predicate holds for predicate holds for all values from the base case to a specific value:

$$P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n)$$

$$\forall 0 \leq k \leq n : P(k)$$

3. For the inductive step, use the inductive hypothesis to show that the predicate holds for the next value.

$$P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)$$

Strong Induction Example

Exercise: Show that for any n greater than or equal to 12, n cents in postage can be formed using only 4-cent and 5-cent stamps.

$\forall n \geq 12 \quad P(n): n = 4x + 5y \quad x, y \in \mathbb{Z}^{\geq 0}$

Proof by Induction:

Base case:

$4¢ \quad 5¢$

$P(12): x = 3 \quad y = 0$

$P(13): x = 2 \quad y = 1$

$P(14): x = 1 \quad y = 2$

$P(15): x = 0 \quad y = 3$

IH: Assume $P(12) \wedge P(13) \wedge P(14) \wedge P(15) \wedge \cdots \wedge P(n)$

$\forall\, 12 \leq k \leq n \; P(k)$ for some specific n

Strong Induction Example

Exercise: Show that for any n greater than or equal to 12, n cents in postage can be formed using only 4-cent and 5-cent stamps.

Proof by Induction: Inductive step Assume IH $(n \geq 15)$

$P(12) \wedge P(13) \wedge P(14) \wedge P(15) \wedge \cdots \wedge P(n) \Rightarrow P(n+1)$

$n+1 = (n-3) + 4$ (use a 4¢ stamp to cover the extra cent but need to cover n-3 cents in stamps)

Since $12 \leq (n-3) \leq n$ by IH $P(n-3)$ holds.

$\Rightarrow n-3 = 4x^* + 5y^*$ $\quad x^*, y^* \in \mathbb{Z}^{\geq 0}$

$\Rightarrow n+1 = 4(x^*+1) + 5y^*$ $\quad$ let $\underline{x = x^*+1}$, $y = y^*$

* (use 1 more 4¢ stamp than was used for n-3)

$\Rightarrow$ by IH $P(n+1)$ holds and claim follows

Strong Induction Example

Recall the Fibonacci Sequence: $f_0=0$, $f_1=1$, $f_2=1$, $f_3=2$, $f_4=3$, $f_5=5$,…

What is the recursive or inductive definition for the Fibonacci sequence?

Base cases: $f_0 = 0 \quad f_1 = 1 \qquad (\text{or } f_1 = 1,\ f_2 = 1)$

Recursive or inductive case: Define the sequence in terms of earlier terms in the same sequence itself

$$f_n = f_{n-1} + f_{n-2}$$

Strong Induction Example

Let's prove the following predicate for the Fibonacci sequence where $a = \frac{1+\sqrt{5}}{2} < 1.7$:

$$\forall n \geq 3 : P(n) = f_n > a^{n-2}$$

[Useful fact: $a^2 = a + 1$.]

Proof by Induction: Base case: $P(3): f_3 = 2 > a^{3-2} < 1.7$ ✓

$P(4): f_4 = 3 > a^{4-2} \simeq 2.89$ ✓

Strong IH: Assume $P(3) \wedge P(4) \wedge \ldots \wedge P(n)$ holds

for some specific $n \geq 4$ $\forall_{3 \leq k \leq n} P(k)$

Strong Induction Example

Let's prove the following predicate for the Fibonacci sequence where $a = \frac{1+\sqrt{5}}{2} < 1.7$:

$$\forall n \geq 3 : P(n) = f_n > a^{n-2}$$

[Useful fact: $a^2 = a + 1$.]

Proof by Induction:

Inductive step

Show $P(3) \wedge P(4) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$, $n \geq 4$

$f_{n+1} = f_n + f_{n-1} > a^{n-2} + a^{n-3}$ ($\cancel{}$ by IH $P(n) \wedge P(n-1)$ hold)

(by $P(n) \wedge P(n-1)$)

$f_{n+1} > a^{n-3}(a+1)$

$f_{n+1} > a^{n-3}(a^2)$

$f_{n+1} > a^{n-1}$ ✓ $P(n+1)$ holds ✓

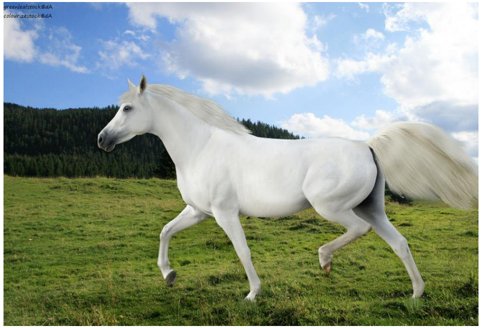
Induction

- Induction
- Strengthening the Inductive Hypothesis
- Strong Induction
- **Error Checking**

1.1–1.2

Error Checking : textbook exercise

Bold claim: Every horse is the same color!



Error Checking

Bold claim: Every horse is the same color!

Induction

- Induction
- Strengthening the Inductive Hypothesis
- Strong Induction
- Error Checking

1.1-1.2