Sets and Functions

- Set Definitions
- Set Operations
- **Sequences**
- Functions
- Pigeonhole Principle

1.1–1.2

Sequences

- Sequences (or lists or tuples) are **ordered** collections of objects.
- Sequence notation may give objects using the same variables, but different subscripts: $x_1, x_2, x_3, \ldots$
- If two objects called an ordered pair, <a, b>.
- If n-objects, called an n-tuple.

Example:
Let's consider the ordered pairs of latitude and longitude for some locations: <latitude, longitude>

Los Angeles: <34.0522, -118.2437>

Paris: <48, 2>

Boston: <42, -71>

Antartica: <-71, 42>

Let's consider some ordered triples or 3-tuples of <temperature, humidity, location>:

<86 F, 52%, Los Angeles>

<71 F, 84%, Boston>

Cartesian Product

Cartesian product of sets A and B, A x B = {<a,b> : a ∈ A and b ∈ B}

That is the set of all ordered pairs such that the first element is from set A and the second from set B.

Example: Let’s consider how we may order drinks at a café. We have choice of temperature = {hot, iced} and beverage = {coffee, tea}

If we order first giving the temperature then beverage, that is temperature x beverage:

Temperature x beverage = { <hot, coffee>,<hot,tea>, <iced, coffee>, <iced, tea>}

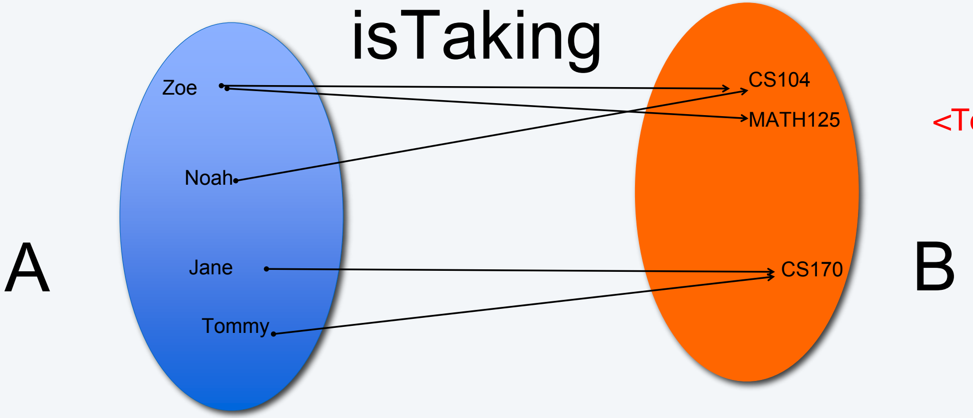
If we order first giving the beverage then temperature, that is beverage x temperature:

Beverage x temperature = { <coffee,hot>, <tea,hot>, <coffee, iced>, <tea,iced>}

Binary Relations

A **binary relation** on A x B is a subset of A x B.

Example:
Let A = {Zoe, Tommy, Jane, Noah}
B = {CS104, CS170, Math 125}
isTaking = {<Zoe, CS104>, <Zoe, Math 125>, <Tommy, CS170>,<Jane, CS170>, <Noah, CS104>}
isTaking binary relation = {<a, b> : Student a is taking course b}
isTaking is a subset of A x B (i.e. students x courses)



<Tommy, Math125> $\notin$ isTaking

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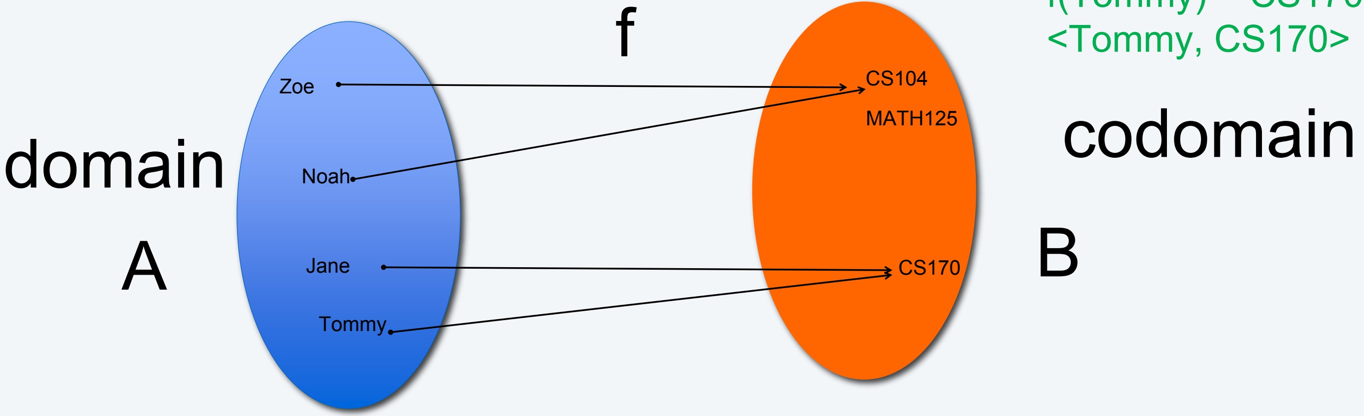
1.1–1.2

Functions

- A **function, f:** A → B.
- A is the domain and B is the codomain
- Functions are a type of binary relation.
- Each element of A is associated with one element of B.
- a $\longmapsto$ b   f(a) = b
- **ONE ARROW OUT OF EACH ELEMENT OF A**

f(Tommy) = CS170
<Tommy, CS170>



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- **Pigeonhole Principle**

1.1–1.2

Pigeonhole Principle

"*If there are more pigeons than pigeonholes and every pigeon goes into a pigeonhole, then some pigeonhole must contain more than one pigeon in it.*"

Let $f : A \to B,$ where A and B are finite sets and $|A| > |B|$. Then there exist distinct elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.

$a_1 \neq a_2$

A is set of 5 mice

B is set of 3 teacups

f maps mice to teacups

Image Source: http://artesanogalleria.com/charming-tails-teacup-mice/

[From Lewis & Zax text cited]

Pigeonhole Principle Application Example

Claim: In any group of 8 people, at least two were born on the same day of the week

What are the “pigeons” and what are the “pigeonholes”?

A = the set of people, B = {Sun, … Sat}, $f(a)$ = the day of the week on which a was born

Since $|A| = 8 > |B| = 7$, by PHP, at least two people in A were born on the same day of the week

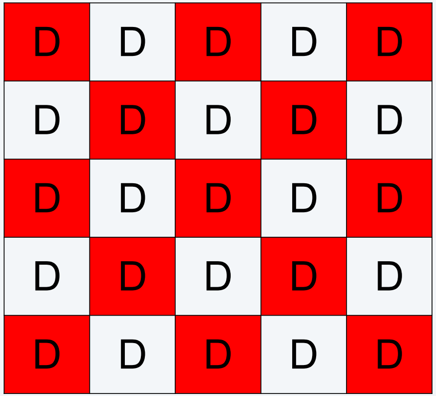
Image Source: http://artesanogalleria.com/charming-tails-teacup-mice/

Pigeonhole Principle Application Example

- Suppose each pigeonhole has one bird
- Every bird moves to an adjacent square (up, down, left or right).
- Show that no matter how this is done, some pigeonhole winds up with at least 2 birds.

$$A = \text{ birds on red squares}$$
$$B = \text{ gray squares}$$
$$f(a) = \text{ the square } a \text{ moves to}$$
$$|A| = 13, |B| = 12$$

Since IAI = 13 > IBI = 12, by PHP, at least two birds will move to the same gray square.



Pigeonhole Principle Practice Problem 1

N people are at a party. Show that ~~any~~ at least two guests must have the same number of friends at the party. In this example, friendship is symmetric. If Alice is friends with Bob, Bob is friends with Alice.

Photo by Glenn Han on Unsplash

Let A be party guests $|A| = N$

B range of the number of friends a guest has at the party $|B| = N-1$

$f: A \to B$ $f(a) = \#$ of friends at party

$B = \underline{\{0, \dots, n-2\}}$ or $B = \underline{\{1, \dots, n-1\}}$

Since $|A| = N > |B| = N-1$

by PHP, at least 2 guests have the same number of friends at the party.

Pigeonhole Principle Practice Problem 2

Let S = {3,4,5,6,7,8,9,10,11,12}. Show that any subset of six distinct integers from S must contain at least two integers whose sum is 15.

Let A = subset of six distinct integers from S, $|A| = 6$

$B = \{\{3,12\},\{4,11\},\{5,10\},\{6,9\},\{7,8\}\}$ $|B| = 5$

$f: A \to B$ $f(a) = X$ st $X \in B$ and $a \in X$

Since $|A| = 6 > |B| = 5$ by PHP at least two integers in A must be in the same subset of S in B and ~~by~~ construction must sum to 15.

Pigeonhole Principle Practice Problem 3

Show that in any set of 9 positive integers at least two share all of their prime factors less than or equal to 5. $\rightarrow \{2,3,5\}$

Let A be set of 9 positive integers

$B = \mathcal{P}(\{2,3,5\}) = \{\emptyset, \{2\}, \{3\}, \{5\}, \{2,3\}, \{3,5\}, \{2,5\}, \{2,3,5\}\}$

$|B| = 2^3 = 8$

$f: A \to B$ $f(a)$: its subset of prime factors less than or equal to 5

$f(7) = \emptyset$

$f(30) = \{2,3,5\}$

Since $|A| = 9 > |B| = 8$

by PHP at least two integers in A map to the same point/element in B which is a subset of prime factors less than or equal to 5.

Extended Pigeonhole Principle

If n pigeons are placed into k pigeonholes, then there is at least one pigeonhole containing at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

N pigeons = 5 mice

Pigeonholes = 3 teacups

$\left\lceil \frac{n}{k} \right\rceil$ is the ceiling function: It is the smallest integer ~~great~~ greater than or equal to its argument.

Image Source: http://artesanogalleria.com/charming-tails-teacup-mice/

Pigeonhole Principle Practice Problem 4

Twenty-five people go to daily yoga classes at the same gym, which offers 8 classes every day.
Each attendee wears either a blue, red, or green shirt to class.
Show that on a given day, there is at least one class where two people are wearing the same color shirt.

let $n = 25$ people $k = 8$ classes

$\Rightarrow$ at least 1 class with at least

$\left\lceil \frac{25}{8} \right\rceil = 4$ students attending.

$\rightarrow$ 4 students in class

holes will be 3 shirt colors $f(\text{student}) = \text{shirt color}$

$\Rightarrow$ at least 2 students in at least 1 class are wearing the same color shirt.

Photo by Annie Spratt on Unsplash

Extended Pigeonhole Principle Practice Problem

An MLB baseball card collector only collects baseball card for players from teams from the NL West. There are 5 teams in the NL West. What is the minimum number of cards that must be in the collection to guarantee that there are at least 100 cards from the same NL West team?

Pigeons are cards n

holes are NL West teams $k=5$

f : cards $\to$ teams $f(\text{player}) = \text{team}$ on card

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{n}{5} \right\rceil = 100$$

if $n = 500$ too many, but satisfies

$n = 99 * 5 = 495$ too few

$\Rightarrow$ $n = (99*5)+1 = \underline{496}$

WEST	W	L	%	GB
LAD	88	47	.652	-
ARI	67	66	.504	20.0
SF	65	67	.492	21.5
SD	61	71	.462	25.5
COL	59	75	.440	28.5