Predicate Logic

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1.1-1.2

Predicate Logic

- **Truth of Statements in Predicate Logic**
- Nested Quantifiers: Translating between English and Predicate Logic

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Quantificational Logic

Extends propositional logic

Formulas include constants, functions, parentheses, logical operators, variables, **predicates, and quantifiers.**

A **predicate** is a statement with at least one variable. The variables must have a universe of possible values specified.

A variable with a quantifier is called **bound** and variable without a quantifier is called **free.**

In order to determine the truth value of a statement all variables must be **bound** or specific values must be given to free variables.

Precedence of quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x (P(x) \vee Q(x))$.

Let x be in universe of integers

Let $P(x)$: x is even $Q(x)$: x is odd

$(\underline{\forall x P(x)}) \vee Q(\underline{x})$

$\forall x \left(P(x) \vee Q(x)\right)$

Determining the truth of a statement in quantificational logic

To determine the truth of a statement, an interpretation for a statement must include the following details:
1) A universe of the values for the variables
2) For each predicate specification of which values in the universe of values are true
3) For each function and constants, specification of their values in the universe

Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?

a) Q(Denver, Colorado)
b) Q(Detroit, Michigan)
c) Q(~~Massachusetts,~~ Boston) Boston, MA
d) Q(New York, New York)

universe of x will be cities
universe of y will be states

a) T
b) F Lansing
c) T (as written originally false)
d) F (Albany)

Determining the true of a statement in quantificational logic

Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$ **b)** $Q(-1)$ **c)** $Q(1)$
d) $\exists x Q(x)$ **e)** $\forall x Q(x)$ **f)** $\exists x \neg Q(x)$
g) $\forall x \neg Q(x)$

a) $0+1 > 2 \cdot 0$ T

b) $(-1+1) > 2 \cdot -1$ ✓ T

c) $(1+1) > (2 \cdot 1)$ F

d) 0 is such an integer T
$\exists x \, Q(x)$

g) $\forall x \neg Q(x)$ F

e) $\forall x \, Q(x)$ F
$1+1$ is not greater than $2 \cdot 1$

f) $\exists x \neg Q(x)$ T
1 is such an x

Negating Nested Quantifiers and determining truth of statement

Negate: $\neg[\exists x : \forall y : \exists z : P(x,y,z)]$

Let the universe of all the variables be the integers and let P(x,y,z) be x+y = z.

$\forall x \ \exists y \ \forall z : \neg P(x,y,z)$ F

for any integer x there exists an integer y st for any integer z $x+y \neq z$

$\exists x \ \forall y \ \exists z \ P(x,y,z)$ T

There exist an integer x for any integer y there exist an integer z st $x+y=z$.

Logical equivalences of formulas

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

A statements involving quantifiers and predicates or **formulas** are **equivalent** if and only if they are true under the same interpretation.

A **model of a formula** is interpretation for which the statement if true. A **satisfiable formula** has a **model.**

Example. P(x) : $\neg[\forall x\, R(x)]$
Q(x): $\exists x \neg R(x)$

let x be integers
R(x) be x is even

Theorem in Predicate Logic

A quantified statement or formula is a **theorem** or **valid formula** if and only if it is true for every possible interpretation.

Example of a theorem: $\forall x \in S(\neg P(x) \vee P(x))$ tautology

Is the following a theorem?

$$[\forall x \in S, \neg P(x)] \vee [\forall x \in S, P(x)]$$

let x be integers

P(x): x is prime

Any integer is not prime

counterexample is 3.

or

any integer is prime.

counterexample is 4.

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Translating from English to predicate logic

English Statement: The product of any pair of real numbers with opposite signs is negative.

Quantificational Logic Formula:

$$\forall x \forall y((x > 0) \land (y < 0) \rightarrow (xy < 0)),$$

universe of x and y are real numbers

for any x for any y "pair of real numbers"

if x is positive and y is negative "with opposite signs"

then $xy < 0$. "product is negative".

Translating from English statements to quantificational logic.

Let f be a function from A to B.
1) f is surjective.
2) f is injective.

1) $\forall b \in B \ \exists a \in A \quad f(a) = b.$

2) i) $\forall a_1, a_2 \in A \quad f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

ii) $\forall a_1, a_2 \in A \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

Translating from formulas to English

Let the domain of the variables be the real numbers. Let's translate the following formula into English and determine its truth.

$$\forall x \exists y \, (x < y)$$

for any real x there exist a real y
x is less than y.

for any real number there's
a number greater than it.

TRUE.

Translating from formulas to English

Let the domain of the variables be the real numbers. Let's translate the following formula into English and determine its truth.

$$\exists x \forall y \, (x \geq y)$$

There exists a real x for any real y x is greater than or equal to y.

There exists a greatest real number. FALSE.

Translating from formulas to English

Let the domain of the variables be the real numbers. Let's translate the following formula into English and determine its truth.

$p \to q \equiv \neg p \vee q$

$$\forall x \forall y\, \big(((x \geq 0) \wedge (y \geq 0)) \to (xy \ \geq 0)\big)$$

For any real x and any real y, if x is non-negative and y is non-negative then xy greater or equal to zero.

Any pair of non-negative real numbers has a non-negative product. TRUE.

$\neg(\neg p \vee q) \equiv p \wedge \neg q$

Translating from formulas to English

Let the domain of the variables be the real numbers. Let's translate the following formula into English and determine its truth.

$$\exists x \exists y\, \big(((x \geq 0) \wedge (y \geq 0)) \wedge (xy < 0)\big)$$

There exist real numbers x and y st x is non-negative and non-negative and xy is negative.

There exists a pair of non-negative real numbers with a negative product.

FALSE

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