Functions

CSCI 170 Spring 2021

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https://www.seas.harvard.edu/courses/cs20/Schedule.html

Sets and Functions

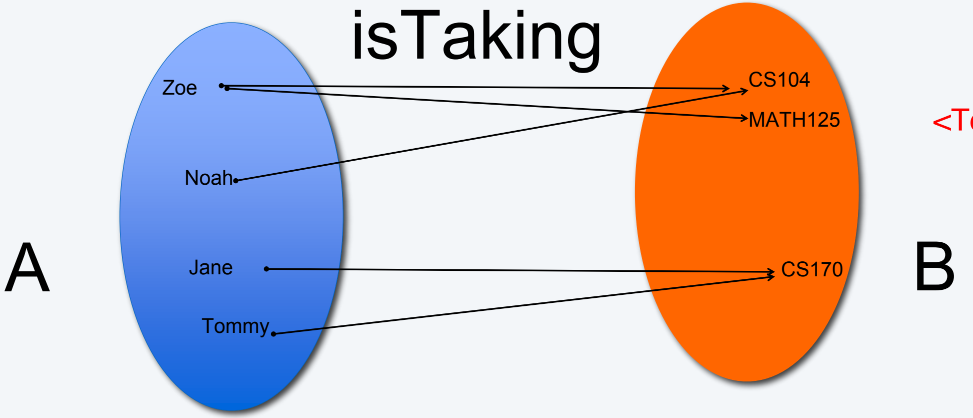
- Set Definitions
- Set Operations
- Sequences
- **Functions**
- Pigeonhole Principle

1.1–1.2

Binary Relations

A **binary relation** on A x B is a subset of A x B.

Example:
Let A = {Zoe, Tommy, Jane, Noah}
B = {CS104, CS170, Math 125}
isTaking = {<Zoe, CS104>, <Zoe, Math 125>, <Tommy, CS170>,<Jane, CS170>, <Noah, CS104>}
isTaking binary relation = {<a, b> : Student a is taking course b}
isTaking is a subset of A x B (i.e. students x courses)



<Tommy, Math125> ∉ isTaking

Inverse of a Binary Relation

A **binary relation** on A x B, R, is a subset of A x B.

The **inverse of the binary relation, R^{-1},** is a subset of B x A:

$\mathbf{R^{-1}} = \{ \langle y,x \rangle \mid \langle x,y \rangle \in R\}$

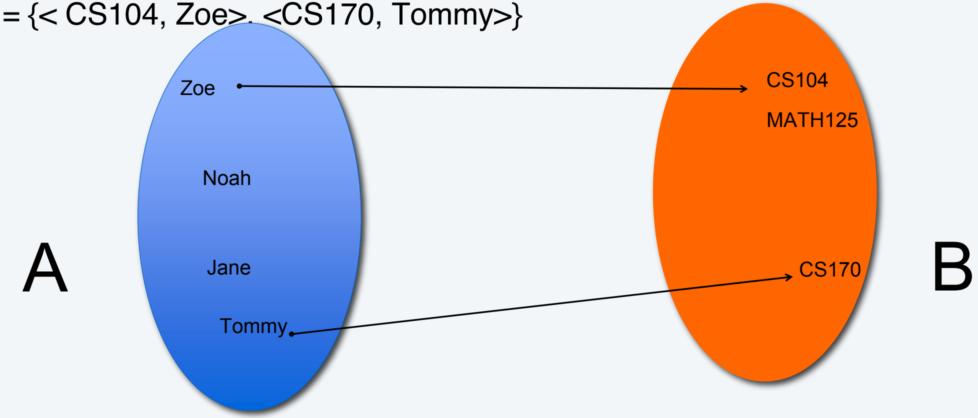
Example:

Let A = {Zoe, Tommy, Jane, Noah}

B = {CS104, CS170, Math 125}

R = {<Zoe, CS104>, <Tommy, CS170>}

R^{-1} = {< CS104, Zoe>, <CS170, Tommy>}



Functions

- A **function, f:** A → B.
- A is the domain and B is the codomain
- Functions are a type of binary relation.
- Each element of A is associated with one element of B.
- a ⟼ b   f(a) = b
- **ONE ARROW OUT OF EACH ELEMENT OF A**

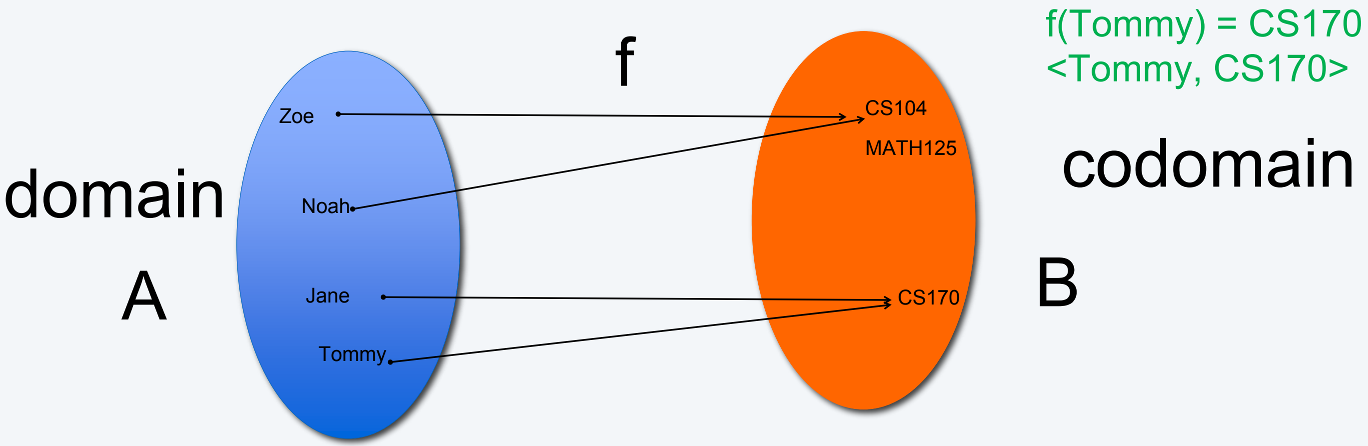
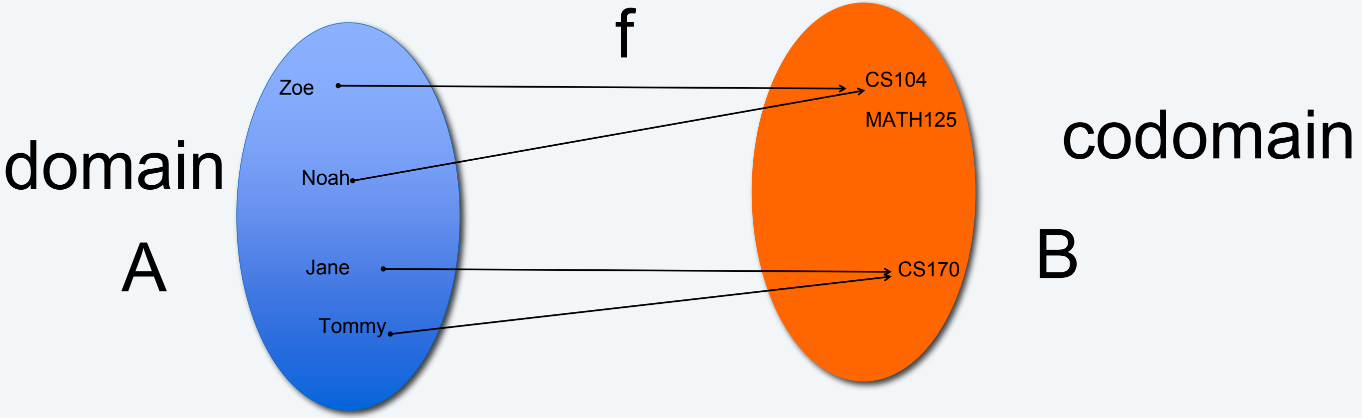


Image of function

- A **function, f:** $A \rightarrow B$.
- The **image of a** is the value of a under the function f, i.e. f(a).
- For a subset of A, S, its image, **f[S]** is the set of images of all of its points:

$f[S] = \{f(x) \mid x \text{ is in } S\} \subseteq B.$

- **Example: f[{Zoe, Noah}] = {CS104}**



Sequences as Functions

- **A sequence is a function f:** N → S.
- For a term of the sequence the notation a_n denotes the image of the integer n.
- Example: 1, 4, 9, 16, 25, …

$a_1, a_2, a_3, a_4, a_5, \dots$

$f(1) = 1 = a_1$

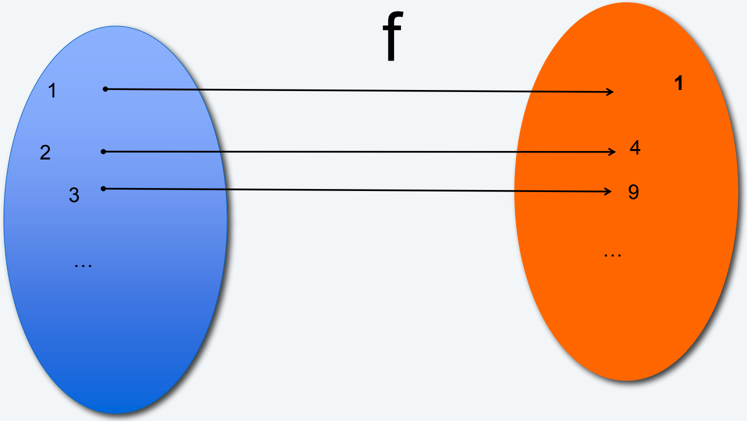
$f(2) = 4 = a_2$

$f(3) = 9 = a_3$

$f(4) = 16 = a_4$

$f(5) = 25 = a_5$

$\vdots$



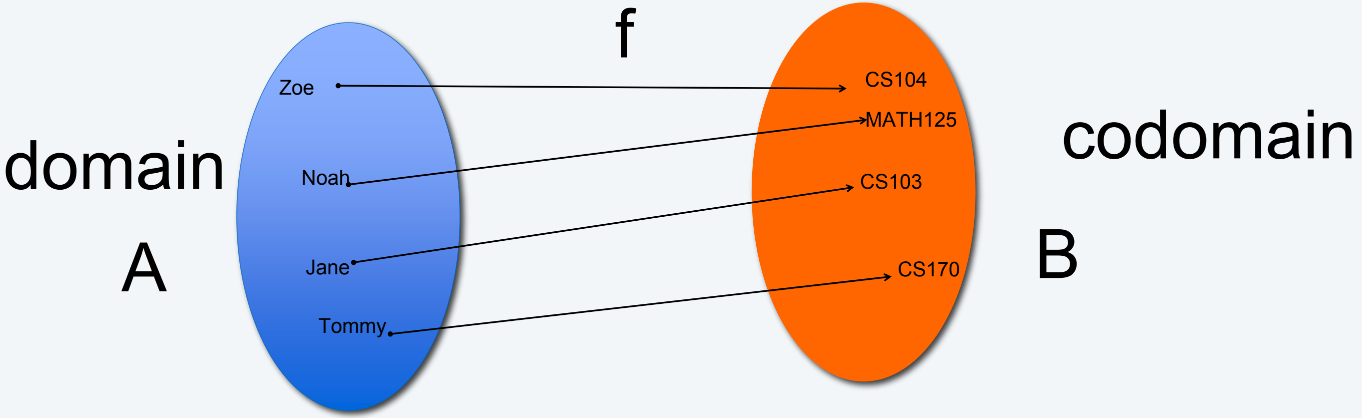
Inverse of a function

A function, f, is a binary relation on A x B.

The **inverse of the function f^{-1}**, is a subset of B x A:

$f^{-1} = \{ <b,a> \mid <a,b> \in f\}$

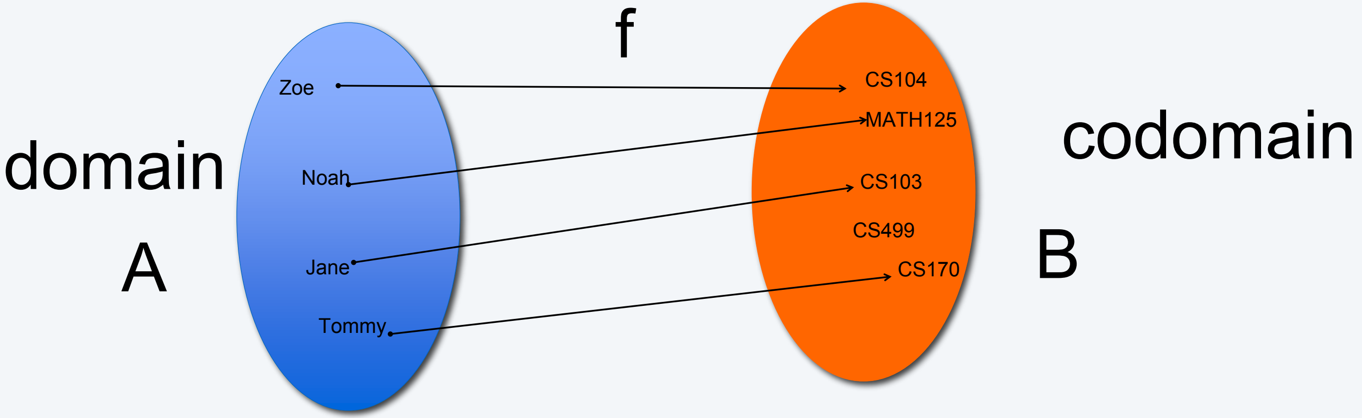
- Example: $f^{-1} = \{ <CS104, Zoe>, <MATH125, Noah>, <CS103, Jane>, <CS170, Tommy>\}$



One-to-One (Injective) Function

- For every element in the codomain, there is at most one element in domain such that f(a) = b.

- **At most one arrow going in**
- To show a function f(x) is one-to-one, show that if f(x) = f(y) then x = y
- To show a function f(x) is not one-to-one, you need only give a counterexample, i.e. distinct x and y such that f(x) = f(y)

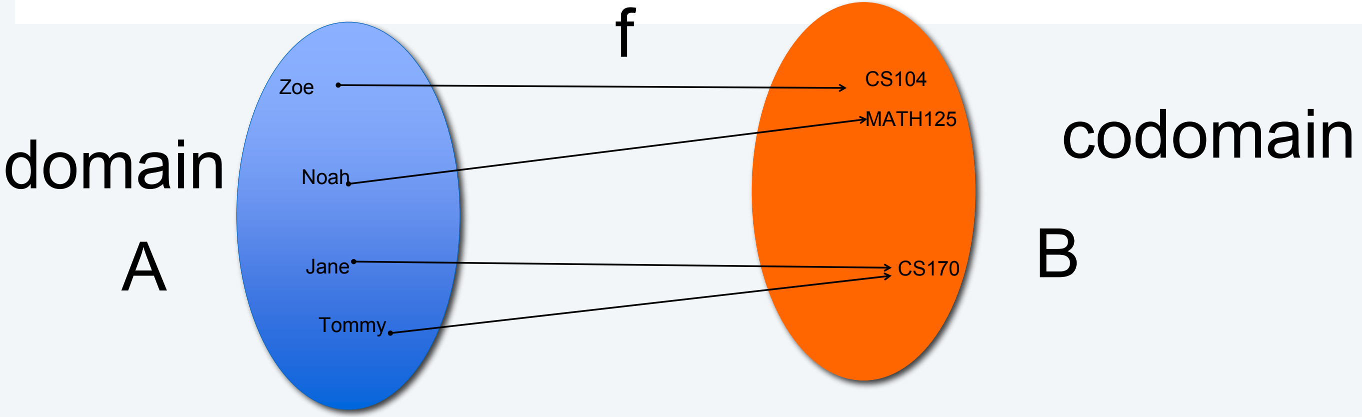


Surjective (Onto) Function

- For every element in the codomain, there is at least one element in domain such that f(a) = b.

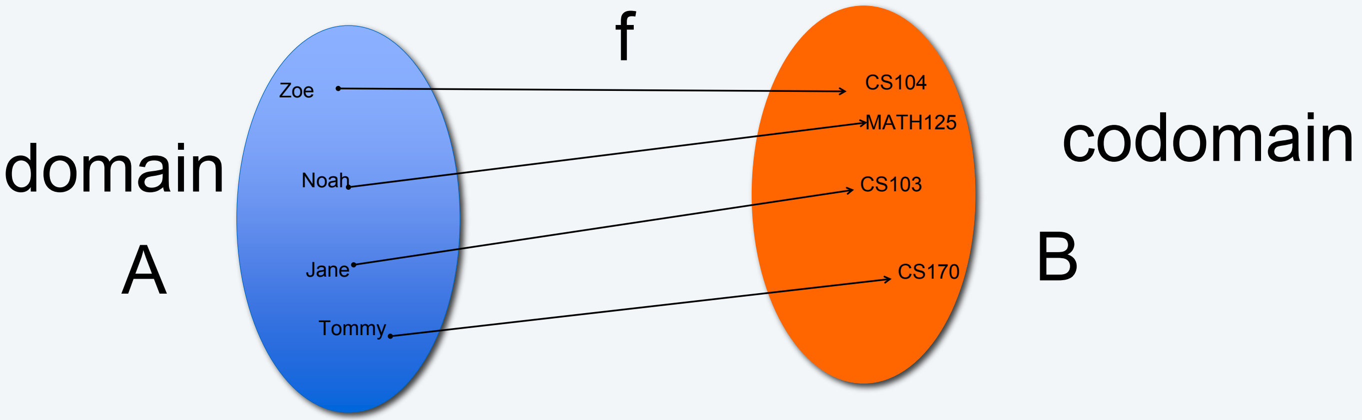
- **At least one arrow going in**

- To show f(a) is onto show ~~the~~ for every b there exists an a such that f(a) = b.
- To show f(a) is not onto, give a counterexample, i.e. show there is a b that has no a mapping to it.



Bijection

- For all elements in the codomain, there exists exactly one element in the domain such that f(a)=b.
- One-to-one and Onto
- **Exactly one arrow out of each element of A and exactly one arrow into each element of B**



Function Practice Problem 1

Consider f: R -> Z where f(x) = $\lfloor x \rfloor$

Recall that the floor of x is the largest integer less than or equal to x.

Is this function injective, surjective, bijective or neither?

i) injective? NO $\lfloor 1.5 \rfloor = \lfloor 1.6 \rfloor = 1$

ii) surjective? Yes for any $b \in Z$ $f(b) = b$

for any $b \in Z$ $b \in R$ maps to itself $(Z \subseteq R)$

iii) Not bijective

Function Practice Problem 2

Consider f: Z-> R where f(x) = $\frac{x}{3}$

Is this function injective, surjective, bijective or neither?

a) injective? yes let $x, y \in \mathbb{Z}$, $x \neq y$ but

$f(x) = f(y) \Rightarrow \frac{x}{3} = \frac{y}{3} \Rightarrow \boxed{x = y}$

$\Rightarrow$ each pt in Z maps to unique pt in codomain under f

b) surjective? NO

$\pi \in \mathbb{R}$ $\pi \notin \mathbb{Q}$ $\Rightarrow$ there is no integer that we can divide by 3 that equals π

c) not bijective

Function Practice Problem 3

Consider f: NxN -> R where f(m,n) = max(m,n)

Is this function injective, surjective, bijective or neither?

a) injective? no $f(5,5)=5$ $f(1,5)=5$ $f(5,4)=5$

b) surjective? NO $1.75 \in R$ or $2.75 \in R$ st they cannot be max of two non-negative integers

c) not bijective

neither

Function Practice Problem 4

Consider f: N -> N where f(n) = n-1 if n is odd and f(n) = n+1 otherwise

Is this function injective, surjective, bijective or neither?

a) injective? yes let $x, y \in N$ $x \neq y$ $f(x) = f(y)$

i) if x, y odd, $f(x) = f(y) \Rightarrow x-1 = y-1 \Rightarrow x = y$

ii) if x, y even, $f(x) = f(y) \Rightarrow x+1 = y+1 \Rightarrow x = y$

b) surjective? yes

i) $b \in N$, b odd $\Rightarrow f(b-1) = b$ because $b-1$ even, $\underline{b-1 \in N}$

ii) $b \in N$, b even $\Rightarrow f(b+1) = b$ because $b+1$ odd $b+1 \in N$

c) bijective