Rolling Shapes and Some Other Things

Original paper "Roads and Wheels" by Hall and Wagon (1992)

Introduction

I discovered the topic of wheels and rolling in general while watching a video about the brachistochrone problem: finding the fastest path of descent between two points on a vertical plane in an environment that obeys gravitational acceleration. The solution to this problem isn't actually the shortest distance, a straight line, as it doesn't accelerate fast enough. In fact, counter-intuitively, a line that goes straight down and straight across is faster since it accelerates a lot in the beginning. Johann Bernoulli approached the problem with reference to light. When light passes from one medium to another, it always takes the most efficient path where there is an optimum balance between the amount of time spent in the medium where it travels slower in and the time spent in the one where it's faster. The relationship between the incident and refractive angles is called Snell's Law, and it just so happens that the path traced out by a point on a circle's circumference as it rolls (called a cycloid) obeys Snell's Law at every point. Initially I wanted look deeper into this and found a lot of resources explaining this proof and the brachistochrone problem. However, in the video I watched, cycloids were introduced by talking about all kinds of rolling shapes, and that also sparked my interest.

The line traced out by a point on a rolling shape in general is called a roulette. As circles roll, they trace out straight roulettes with their centres, which is why they make good wheels. However, polygons, or rather any shape can produce a straight roulette and roll "smoothly" given that the surface has the right corresponding shape.

I'm equally interested in both, but I chose this topic instead of brachistochrones because unlike the latter, I couldn't find a lot of information about this except for a paper written by Stan Wagon and Leon Hall, and I wanted to use this as an opportunity to actually understand the journal and the problem. The aim of this exploration is to show how a method can be derived to find the corresponding wheel equation given a road function.

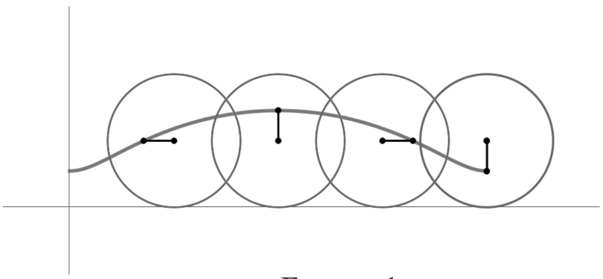

FIGURE 1
Circle tracing out a curtate (tracing point inside the circumference) roulette

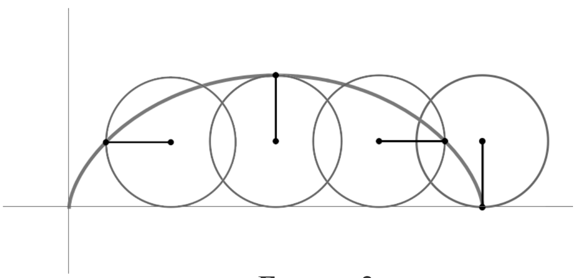

FIGURE 2
Circle tracing out a cycloid

Overview

In Sections 1-3, I will be outlining the mathematical concepts that are related to this problem. Section 4 combines the different aspects to get the general differential equation. Section 5 uses the circle as a base example, and Section 6 gives a more complex one. In Section 6, the actual rolling movement will be modelled using Excel.

1 Polar Form

The most familiar form of notating coordinates to me so far is Cartesian form (x, y), in units horizontally and vertically. However, another way to get to the same point is to simply move in a straight line for r units, at the angle θ to the x-axis (r, θ). Conversion between the two can be done simply with Pythagoras (FIGURE 3).

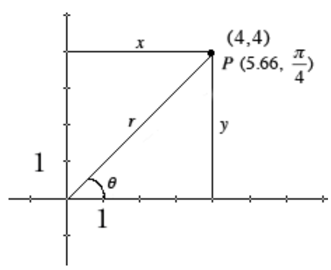

FIGURE 3

The use of polar form is prevalent throughout, as a way of representing wheels mathematically, since they are not one-to-many or many-to-one functions, and thus cannot be represented in Cartesian form.

2 Arc Lengths

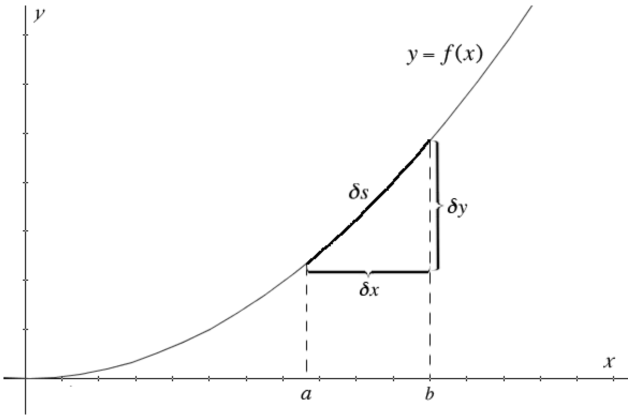

FIGURE 4

Having defined polar form, I'm going to elaborate on arc length of both polar and Cartesian equations. Finding the length of a curve is similar to finding the area under a curve; instead of finding infinitesimally small rectangles and adding them together, infinitesimally small segments of a line are added together.

Let a section of a curve be δs, and $\delta s \to ds$. The arc length can be represented as $\int_a^b ds$ "the summa of all the segments"

To write this out in an integrable form with respect to x, ds can be represented in terms of x and y with Pythagoras (FIGURE 4) $\int_a^b \sqrt{(dx)^2 + (dy)^2}$

To obtain the dx, it can be factored out to give $\int_a^b \sqrt{(dx)^2 (1 + \left(\frac{dy}{dx}\right)^2)}$

Which can be simplified to become $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ This is the formula for arc length (in Cartesian form)

With regards to polar equations, one can simply represent the x and y in terms of r and θ. Lets say $r = f(\theta)$. With Pythagoras (and referencing FIGURE 3), we can see that $x = r\cos\theta$ and $y = r\sin\theta$, and r can be substituted with $f(\theta)$. Substituting those into $\int_a^b \sqrt{(dx)^2 + (dy)^2}$, we can get the expression $\int_a^b \sqrt{(f'(\theta)\cos\theta - f'(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta$.

Simplifying this gives $\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$ This is the arc length formula for polar coordinates.

3 Parameters

Parameters are important when mapping something with respect to time, and can be used when some equations can't be perfectly defined in terms of one or the other. So instead of defining them separately, each component can be defined in terms of a third parameter, most commonly t, such that $x = f(t)$ and $y = g(t)$. So the Cartesian coordinate (x, y) can now be represented as $(f(t),\ g(t))$. A characteristic of parametric equations, while appearing the same as rectifiable ones, is that they have a direction. Parametric equations are always plotted in increasing values of the parameter.

For example, the Cartesian equation for a circle is $x^2 + y^2 = r^2$. If we were to express this in the form of a function, it would be $y = \pm\sqrt{r^2 - x^2}$. However, this actually represents two functions and two components of the circle: the positive side and the negative side. A variable that can be used to define both x and y is θ, where $x = r\cos\theta$ and $y = r\sin\theta$. The circle plotted would have a direction of anticlockwise.

The arc length formula for parametric equations is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. The proof for this formula is quite long and involves more concepts so I'm unable to show it here, but refer to the second link in the sources for the full thing!

4 Rolling

Now that I've introduced the relevant topics, they can all come together to form the basis of the wheel and the road. The wheel is defined by a polar equation $r = r(\theta)$, and the road can be parameterized as $f(t) = (x(t), y(t))$. The rolling motion is defined as $\theta = \theta(t)$: how much (in angles) the wheel rotates as it rolls from $f(0)$ to $f(t)$. $\theta(0)$ is defined as $-\pi/2$, which sets the initial contact point between the road and the wheel beneath the origin.

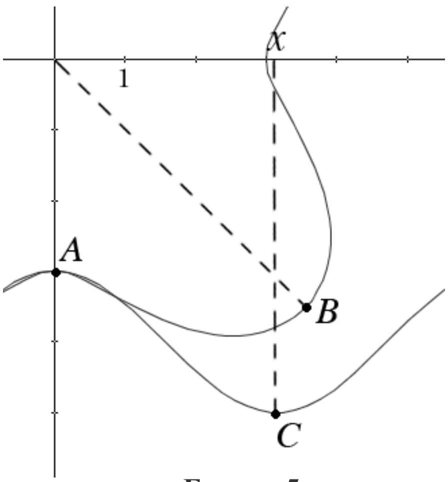

FIGURE 5
Bottom right section of a wheel on a road

As briefly mentioned before in the introduction, a wheel is considered to be rolling "smoothly" if the axle traces out a horizontal line. For simplicity, the axle should remain on the *x*-axis. A way that this can be accomplished is to ensure that the radius of the wheel matches the "depth" of the road (the vertical distance between the *x*-axis and the road function) at any $f(t)$. With reference to FIGURE 5, the length OB needs to equal *x*C. This can be represented mathematically with $r(\theta(t)) = -y(t)$ (***radius condition***), where the left hand side represents the radius parameterised by *t* (the turning motion, and thus the changing radius between the centre and the point of contact with the road), and the right hand side is depth (negative because the *y* value will always be negative, but the final value needs to be positive since it equates the magnitude of the radius).

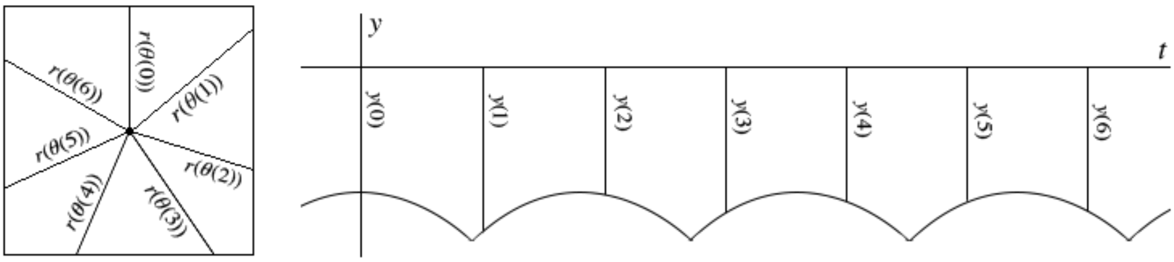

FIGURE 6
A square wheel and its road

FIGURE 6 shows a sort of visualisation of how I thought about the radius condition. In the diagram, $r(\theta(0))$ is equal to the magnitude of $y(0)$, $r(\theta(1))$ is equal to $y(1)$, and so on. A way to think about the creation of the road is the changing radius mapped onto the negative *y*-axis, both with regards to the same *t*. How much the angle of the radius changes with each increase in *t* is defined by the $\theta(t)$ function.

Another way the rolling motion can be considered is that the wheel "unravels" onto the road. Referring to FIGURE 5, this is represented by the arc lengths AB and AC and equating them to each other. Therefore, equating the arc lengths of the parameterised road function and the polar equation of the wheel:

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-\pi/2}^{\theta(t)} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The bounds for the polar equation are as such because the bounds are $\theta(0)$ and $\theta(t)$, where $\theta(0)$ was previously defined as $-\pi/2$ in the beginning of this section.

To simplify this equation, both sides can be differentiated with respect to *t*:

$$\sqrt{\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\frac{d\theta}{dt}$$

And both sides squared and expanded:

$$\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{d\theta}{dt}\right)^2 r^2+\left(\frac{d\theta}{dt}\frac{dr}{d\theta}\right)^2$$

To substitute, the ***radius condition*** $r(\theta(t))=-y(t)$ (found in the beginning of this section) can be differentiated (chain rule) to get $\frac{d\theta}{dt}\frac{dr}{d\theta}$:

$$\frac{d}{dt}(r(\theta(t))=-y(t))\Rightarrow\frac{d\theta}{dt}\frac{dr}{d\theta}=-\frac{dy}{dt}$$

Therefore:

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=r(\theta)^2\left(\frac{d\theta}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$$

To simplify it, use the ***radius condition*** to substitute $-y(t)$ into $r(\theta)$, divide it over, and square root:

$$\frac{d\theta}{dt}=\pm\frac{dx}{dt}\frac{1}{-y(t)}$$

We take the positive component of the derivative, simplify:

$$\frac{d\theta}{dt}=-\frac{dx}{dt}\frac{1}{y(t)}$$

If all the information we had was the rectangular form of the road function, the first step could be to parameterise it to get both the $y(t)$ component and to get the derivative for the $x(t)$ component. After $d\theta/dt$ is gotten it can be integrated to get $\theta(t)$. Using the ***radius condition*** $r(\theta(t))=-y(t)$, this can then give $r(\theta)$. Alternatively, a different way to solve the problem would simply be to use *x* as a parameter from a given road function $y=f(x)$. The ***radius condition*** would then subsequently be $r(\theta(x))=-f(x)$. The differential equation then becomes:

$$\frac{d\theta}{dx}=-\frac{1}{f(x)}$$

This change in parameters, with regards to the model, is just a shift in what variable is used to model the wheel's motion. Instead of using *t*, which represents the passing of time as the wheel moves, *x* is used, which is the position and horizontal displacement of the wheel.

5 Example with a Circle

With the method described above, a rectifiable road of a straight line should yield the polar equation of a circle as the wheel.

Let the road equation be $f(x) = -4$

The next thing that could be done is to parameterize it. There are theoretically infinite ways to parameterise a rectifiable function. The variable *x* is first defined in a certain way in terms of *t*, or any other parameter, and subsequently the *y* is defined. If we were to parameterise $y = x^2 + 5$, the easiest way would be to set $x = t$, and therefore $y = t^2 + 5$. Doing the same to the road equation:

$$x = y$$
$$y = -4$$

The component that is needed next is dx/dt. Differentiating *x* with respect to *t*:

$$\frac{d}{dt}(t) = 1$$

Substituting into the final obtained differential equation in Section 4:

$$(-1)\left(\frac{1}{-4}\right) = \frac{1}{4}$$
$$\frac{d\theta}{dt} = \frac{1}{4}$$

Integrate $d\theta/dt$ in order to be able to use the ***radius condition*** to get $r(\theta)$:

$$\int_0^t \frac{1}{4}dt = \frac{1}{4}t$$
$$\theta(t) = \frac{1}{4}t$$
$$t = 4\theta$$

At this point, we would substitute this into $-y(t)$, but since $-y(t)$ is simply 4, the polar equation is:

$$r(\theta) = 4$$

As expected, the end polar equation is that of a circle with radius 4:

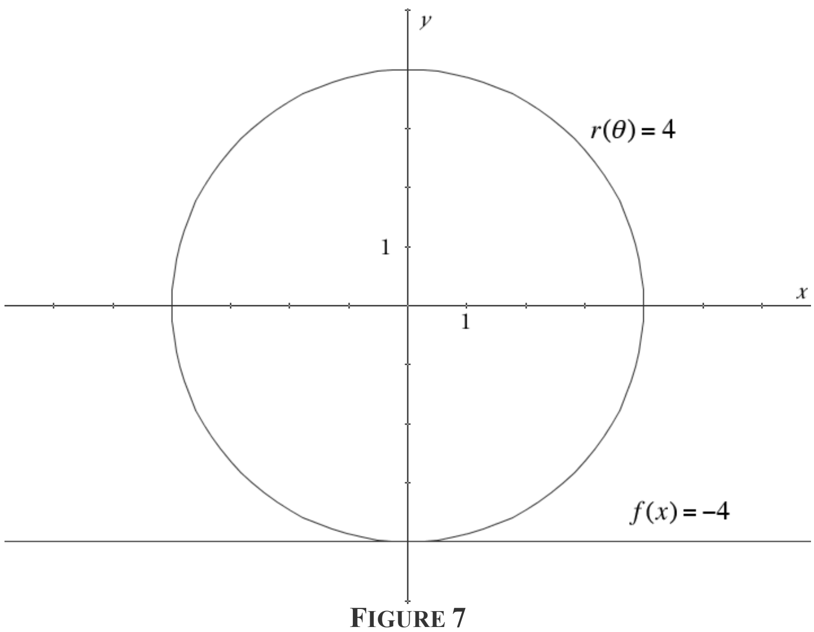

FIGURE 7

6 Further Example

For a circle, referring to the earlier description in Section 4 of it unraveling perfectly onto a road, the same concept can be applied backwards. A road can be "wrapped up" to form a wheel, however, the resulting shape isn't always closed. For example, for a road defined as $f(x) = \cos(x) - a$, a happens to be $\sqrt{n^2+1}$, where $n \in \mathbb{Z}^+$ represents different cases. More specifically, cases where one revolution of the wheel covers n periods of the road function. This is explained in the original paper and would cause quite a tangent if shown here, so please refer to the Hall and Wagon paper for the explanation. This example will use the $n = 4$ case.

Let the road equation be $f(x) = \cos(x) - \sqrt{17}$

Using the differential equation obtained in Section 4, it can be integrated to yield $\theta(x)$. Multiple *dx* to the right hand side and integrate:

$$\int_{\theta(0)}^{\theta(x)} d\theta = \int_0^x -\frac{1}{f(x)}dx$$

$$\theta(x) - \theta(0) = \int_0^x -\frac{1}{f(x)}dx$$

The general form of *a* is used instead of $\sqrt{17}$ to create in a generalisable integral. $\theta(0)$ is defined as $-\pi/2$, therefore:

$$\theta(x) = \int_0^x -\frac{1}{\cos(x) - \sqrt{n^2+1}}dx - \frac{\pi}{2}$$

After spending literal weeks trying to integrate this on my own, I finally used the Wolfram calculator and discovered that it needed a trigonometric substitution called the Weierstrass substitution:

$$\theta(x) = \frac{2\tan^{-1}\left(\dfrac{(\sqrt{n^2+1}+1)\tan\left(\dfrac{x}{2}\right)}{n}\right)}{2} - \frac{\pi}{2}$$

Rearrange to make *x* the subject so that the ***radius condition*** can be solved for $r(\theta)$:

$$x = 2\tan^{-1}\left(\frac{n\tan\left(\dfrac{n\left(\theta+\dfrac{\pi}{2}\right)}{2}\right)}{\sqrt{n^2+1}+1}\right)$$

Simplifying that whole thing:

$$x = 2\tan^{-1}\left(\frac{n\tan\left(\frac{n(2\theta+\pi)}{4}\right)}{\sqrt{n^2+1}+1}\right)$$

And substituting $n = 4$:

$$x = 2\tan^{-1}\left(\frac{4\tan\left(\frac{4(2\theta+\pi)}{4}\right)}{\sqrt{17}+1}\right)$$

Finally, substituting this into $-f(x)$ yields $r(\theta)$:

$$r(\theta) = -\cos\left(2\tan^{-1}\left(\frac{4\tan(2\theta+\pi)}{\sqrt{17}+1}\right)\right)+\sqrt{17}$$

The resulting polar equation, when graphed, yields a reasonable shape of a closed disk with a sinusoidal-esque edge. It has 4 "petals", which is expected since $n = 4$, one full revolution of the shape as it rolls should cover 4 periods of the road:

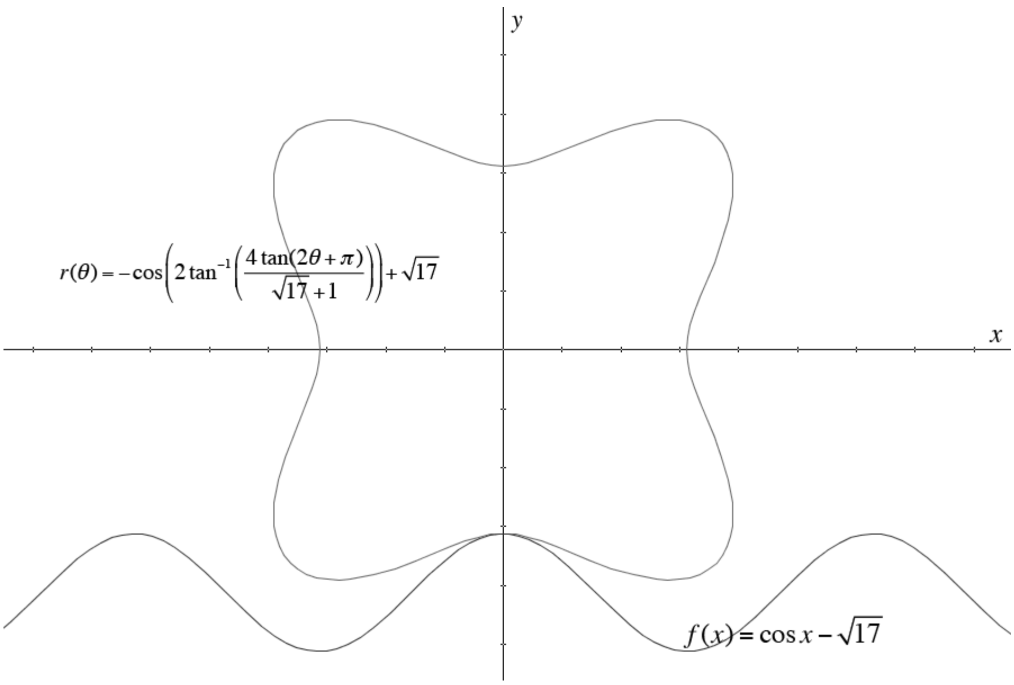

FIGURE 8

It worked!!! I was so incredibly surprised and happy when the function yielded this cool looking result. When the integral wasn't working out I had a lot of doubts about what would come of it but it turned out to be perfect and it's so cool!!

7 Graphing Movement with Excel

Even though I was really happy with the shape of the function, I wasn't fully convinced and wanted to see if I could somehow practically prove that the produced shape would actually roll smoothly. So I wrote an Excel file that could model the wheel's movement by plotting a series of changing Cartesian coordinates.

After the $r(\theta)$ values have been calculated, x and y values can be calculated simply with $x = r\cos(\theta)$ and $y = r\sin(\theta)$, with the interval of θ being $\{\theta | 0 \le \theta \le 2\pi\}$. The turning motion is done by also calculating the x and y values in the same manner, with the new θ being the original subtracted by $\theta(x) - \theta(0)$. It also translated by x. Or:

$$x = r\cos(\theta - \Delta\theta) + \Delta x$$

Where:

$$\Delta\theta = \theta(\Delta x) - \theta(0)$$

The $\Delta\theta$ is subtracted because the shape is turning clockwise, and increasing values of θ turns something anticlockwise. Similarly, the turning motion of y is written as such, without the translation:

$$y = r\sin(\theta - \Delta\theta)$$

With n as a variable, shapes with more "petals" can also be drawn, as shown in FIGURE 10. The x value can be changed to show the movement of the wheel! The full code of the functions used in the cells can be found in Appendix A.

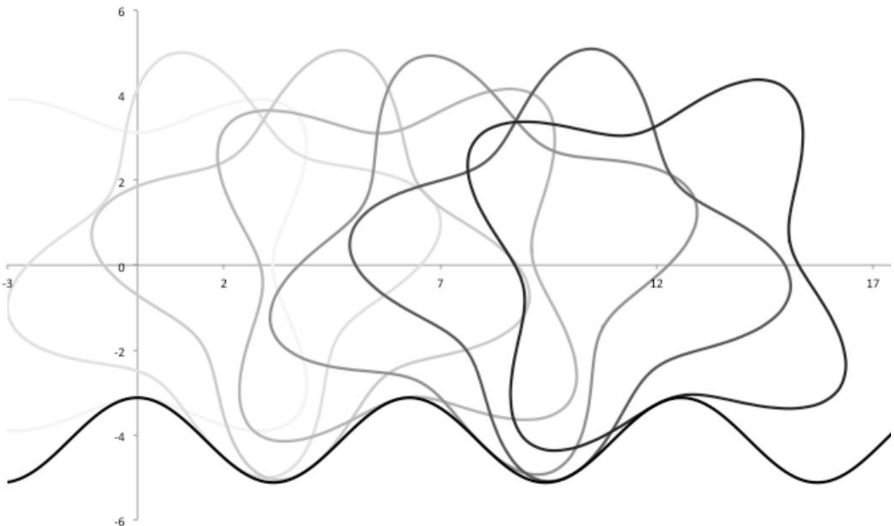

FIGURE 9

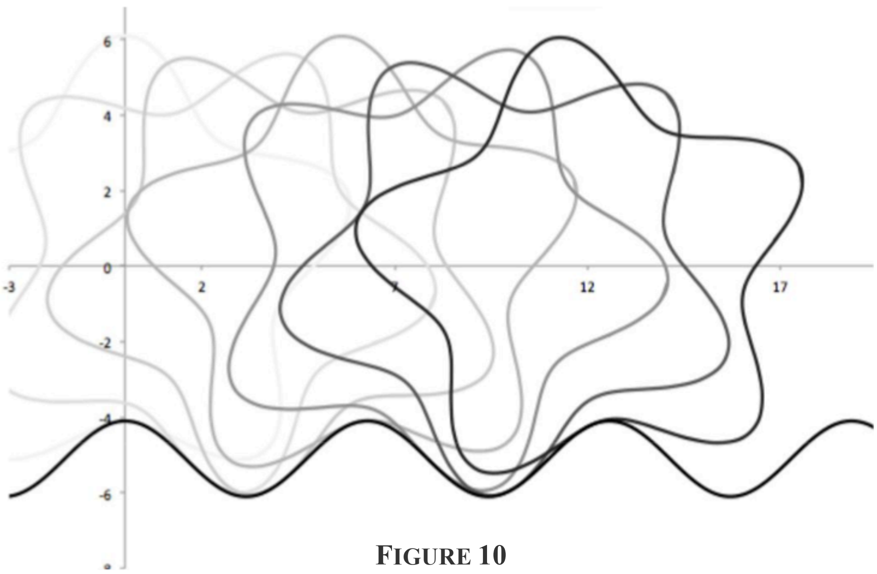

FIGURE 10

Conclusion

There are some limitations and difficulties with this method, one of the more obvious ones being that there are a lot of shapes that are difficult or impossible to graph, an example being polygons. To my knowledge, there is no way to graph them without using piecewise functions, and the roads for those shapes have to be derived from catenaries. The integration has also been very hard for me, and the problem demonstrated in this exploration has been a relatively easy example in comparison with other possibilities such as ellipses, hippopedes and irregular asymmetrical shapes. However, in the original paper there are outlines of the numerous different approaches that can be taken to solve these cases.

There are no direct real life applications that immediately come to mind, but there are some possibilities such as making wheels with special shapes to compensate for rough terrain. Another field that I think this can be applied to is mechanical engineering, as I have come to learn about the Wankel engine. It converts pressure from combusting gases into rotary motion, and the rotor is in the shape of a Reuleaux triangle, which has the special property of having the same width across all sides. This means that it can replace circles for wheels on a vehicle and allow it to move smoothly! The similarities between these shapes give the implication that this topic has potential for expansion with regards to mechanical design.

The process of working through this paper has actually been quite fun! Making the diagrams myself (I created figures 3-10) really solidified my understanding of the theory through editing and drawing the graphs. Also as a computer science student, I'm really glad that I realized the potential of technology in this paper and managed to produce a more visually tangible result.

On a whole, this entire paper has been extremely challenging for me from start to finish, in both the understanding of the theory and the mathematics itself, but was enjoyable as well. Crossing such a previously high barrier to entry has made more complex maths less intimidating and allowed me to fully approach this field of study with less reluctance and a lot more interest.

Works Cited

S. Wagon and L. Hall, Roads and Wheels, *Mathematics Magazine,* Vol. 65, No.5 (Dec, 1992), pp. 283-301

Resources Used

Cartesian equation of star shape - Mathematica Stack Exchange. (2015, June 11). Retrieved from https://mathematica.stackexchange.com/questions/85657/cartesian-equation-of-star-shape

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The Weierstrass Substitution. (n.d.). Retrieved from http://jekyll.math.byuh.edu/courses/m113/handouts/weierstrass.pdf

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Appendix A

Two custom functions were written, one for $r(\theta)$ and one for $\theta(x)$.

```
Public Function theta (n As Double, x As Double) As Double
      Dim chunkWithN As Double, chunkWithTan As Double, inverseTanChunk
            As Double
      chunkWithN = Sqr(n^2+1)+1
      chunkWithTan = Tan(x / 2) / n
      inverseTanChunk = 2 * Atn(chunkWithN * chunkWithTan)
      theta = (inverseTanChunk / n) - (WorksheetFunction.Pi() / 2)
End Function

Public Function rTheta(theta As Double, n As Double) As Double
      Dim theHugeArctanThing As Double, topbit As Double, bottombit
            As Double
      topbit = n * (Tan(n * (2 * theta + WorksheetFunction.Pi()) /
            4))
      bottombit = Sqr(n ^ 2 + 1) + 1
      theHugeArctanThing = 2 * Atn(topbit / bottombit)
      rTheta = -Cos(theHugeArctanThing) + Sqr(n ^ 2 + 1)
End Function
```

	A	B	C	D	E	F	G	H
1	n	5				x	2	
2	x	y	theta	r(theta)	xcoor	ycoor	xcoormove	ycoormove

Above shows the columns used. The n 5, and x 2 are the constants that can be changed. The n value changes the n value for the wheel, and increasing the x moves the wheel forward. The following show the functions below the columns:

x

This column is the domain for the road function, it being $\{x|-\pi \le x \le 7\pi\}$

```
=(cell above)+0.1
```

y

```
=COS(x)-SQRT(n^2+1)
```

theta

As mentioned in Section 7, this dictates the interval of the θ , $\{\theta|0 \le x \le 2\pi\}$

```
=(cell above)+0.1
```

$\pi\pi$

r(theta)

```
=rTheta(theta, n)
```

xcoor

This stands for x coordinate.

```
=r(theta)*COS(theta)
```

ycoor

This stands for y coordinate.

```
=r(theta)*SIN(theta)
```

xcoormove

```
=r(theta)*COS(theta-(theta(n, x)+(PI()/2)))+x
```

ycoormove

```
=r(theta)*SIN(theta-(theta(n, x)+(PI()/2)))
```