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CS 4620
A5-Splines (written portion)

1. a. A curve segment specified with a polynomial of degree n requires n + 1 control points. The original cubic Bezier required 4 control points, but the modified quintic requires 6.

b. We construct a matrix $A$ such that $f(u) = \vec{u}AP$, where $\vec{u} = [1, u, u^2, u^3, u^4, u^5]$ and P is a matrix where rows correspond to control points.

We take the matrix $[a_0, a_1, a_2, a_3, a_4, a_5]^T = AP$.

We can write $f(u) = \vec{a_0} + \vec{a_1}u + \vec{a_2}u^2 + \vec{a_3}u^3 + \vec{a_4}u^4 + \vec{a_5}u^5$

Then $f(0) = \vec{a_0}$, and $f(1) = \vec{a_0} + \vec{a_1} + \vec{a_2} + \vec{a_3} + \vec{a_4} + \vec{a_5}$.

Also, $f(0.5) = \vec{a_0} + (0.5)\vec{a_1} + (0.25)\vec{a_2} + (0.125)\vec{a_3} + (0.0625)\vec{a_4} + (0.03125)\vec{a_5}$

In the general case, $f'(u) = \vec{a_1} + 2\vec{a_2}u + 3\vec{a_3}u^2 + 4\vec{a_4}u^3 + 5\vec{a_5}u^4$

So, specifically, $f'(0) = \vec{a_1}$, and $f'(1) = \vec{a_1} + 2\vec{a_2} + 3\vec{a_3} + 4\vec{a_4} + 5\vec{a_5}$

The second derivative in the general case is $f''(u) = 2\vec{a_2} + 6\vec{a_3}u + 12\vec{a_4}u^2 + 20\vec{a_5}u^3$

So, specifically $f''(0) = 2\vec{a_2}$.

Now that we have all of our constraints listed, we can substitute control points for the listed $f(u), f'(u), f''(u)$.

We have a matrix equation:

$$\begin{bmatrix} f(0) \\ f(1) \\ f'(0) \\ f'(1) \\ f''(0) \\ f(0.5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} \end{bmatrix} \begin{bmatrix} \vec{a_0} \\ \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \\ \vec{a_4} \\ \vec{a_5} \end{bmatrix}$$

We also need a matrix to convert the function values into our control points. We already have an equation:

$$\begin{bmatrix} f(0) \\ f(1) \\ f'(0) \\ f'(1) \\ f''(0) \\ f(0.5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{p_0} \\ \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{bmatrix}$$

Now we combine equations and solve for the coefficients of u, our a-vectors.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} \end{bmatrix} \begin{bmatrix} \vec{a_0} \\ \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \\ \vec{a_4} \\ \vec{a_5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{p_0} \\ \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{bmatrix}$$

$$\begin{bmatrix} \vec{a_0} \\ \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \\ \vec{a_4} \\ \vec{a_5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{p_0} \\ \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{bmatrix}$$

$$\begin{bmatrix} \vec{a_0} \\ \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \\ \vec{a_4} \\ \vec{a_5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -26 & -6 & -11 & 1 & -2 & 32 \\ 47 & 17 & 18 & -3 & \frac{5}{2} & -64 \\ -22 & -10 & -8 & 2 & 1 & 32 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{p_0} \\ \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{bmatrix}$$

$$\begin{bmatrix} \vec{a_0} \\ \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \\ \vec{a_4} \\ \vec{a_5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 11 & -33 & -3 & -3 & -4 & 32 \\ -12 & 54 & 9 & 8 & 5 & -64 \\ 0 & 24 & -6 & -4 & 2 & 32 \end{bmatrix} \begin{bmatrix} \vec{p_0} \\ \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{bmatrix}$$

$$f(u) = \begin{bmatrix} 1 \\ u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix}^T \begin{bmatrix} \vec{a_0} \\ \vec{a_1} \\ \vec{a_2} \\ \vec{a_3} \\ \vec{a_4} \\ \vec{a_5} \end{bmatrix} = \begin{bmatrix} 1 \\ u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 11 & -33 & -3 & -3 & -4 & 32 \\ -12 & 54 & 9 & 8 & 5 & -64 \\ 0 & 24 & -6 & -4 & 2 & 32 \end{bmatrix} \begin{bmatrix} \vec{p_0} \\ \vec{p_1} \\ \vec{p_2} \\ \vec{p_3} \\ \vec{p_4} \\ \vec{p_5} \end{bmatrix}$$

So we finalize our matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 11 & -33 & -3 & -3 & -4 & 32 \\ -12 & 54 & 9 & 8 & 5 & -64 \\ 0 & 24 & -6 & -4 & 2 & 32 \end{bmatrix}$$

c. [Insert Graph Here]

2. Find a matrix $M$ such that if $P_{Bez} = MP_{Bsp}$, then $f_{Bez}(u) = f_{Bsp}(u) \forall u$.

$$f_{Bsp}(u) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$f_{Bez}(u) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

So given that we have $f_{Bsp}(u) = \vec{u}\frac{1}{6}M_{Bsp}P_{Bsp}$ and $f_{Bez}(u) = \vec{u}M_{Bez}P_{Bez}$, we substitute as $f_{Bez}(u) = f_{Bsp}(u) = \vec{u}M_{Bez}MP_{Bsp}$

Now we can isolate $M_{Bez}M = \frac{1}{6}M_{Bsp}$ and solve for $M = M_{Bez}^{-1}\frac{1}{6}M_{Bsp}$

$$M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$M = \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$M = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix}$$