STAT 123: Quantitative Finance, Spring 2022 | Prof. Stephen Blyth

Section 2 – Solution

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Section: Monday 10 - 11AM | Office Hour: Monday 11AM - noon

1 Derivatives

A *derivative contract* is simply a financial contract between two counter-parties and whose value is derived from the value of an underlying asset or variable. For *financial derivative contract*, the underlying variable is financial asset price, or interest rate, etc. A lot of this course is learning about different derivatives and how to find their current value.

1.1 Forward

A forward is an agreement between two counter-parties to exchange an asset (e.g. a stock) at time T for fixed price K. T is the *maturity* of the forward, and K is the *delivery (strike) price.* The counter-party that has agreed to buy the underlying (e.g. a stock) is long the forward. The counter-party that has agreed to sell the underlying (e.g. a stock) is short the forward.

Let S_t denote the value of the underlying asset at time t. Then at time T, the value of long the forward is just $V_K(T,T) = S_T - K$. Note that S_T is a random variable.

If the underlying does not pay any income or give any dividends, then the current value of the forward is $S_t - KZ(t,T)$. We generally assume that the interest rate is constant r and is continuously compounded. So the current value is

$$S_t - Ke^{-r(T-t)}$$

The forward price is the value of K such that the value of the forward at time t is zero, $V_K(t,T) = 0$. It is denoted $F(t,T)$. From the above, we get that

$$F(t,T) = S_t e^{r(T-t)} = \frac{S_t}{Z(t,T)}$$

Let t be the current time, and $T > t$ be a future time. Let r be a continuous rate.

Values of Derivatives			
Symbol	Derivative	Value at t	Payoff at T
$V_K(t,T)$	forward with strike K (no matter on asset without dividends, with dividends q, or on asset generate income I)	$(F(t,T)-K)e^{-r(T-t)}$	$S_T - K$
$V_K(t,T)$	forward with strike K (no matter on asset without dividends, with dividends q, or on asset generate income I)	$(F(t,T)-K)e^{-r(T-t)}$	$S_T - K$
$V_K(t,T)$	forward with strike K on foreign exchange	$(F(t,T)-K)e^{-r_\$(T-t)}$	$S_T - K$

For $V_K(t,T) = (F(t,T)-K)e^{-r(T-t)}$ regardless of asset, this could be proved by arbitrage. Assume $V_K(t,T) < (F(t,T)-K)e^{-r(T-t)}$. Then, at t, construct portfolio through: 1) go short 1 forward with delivery price $F(t,T)$ with no cost, 2) go long 1 forward with delivery price K, paying $V_K(t,T)$ to do. Then at time t, the portfolios {-1 forward with strike price K, +1 forward contract with strike price K and $-V_K(t,T)$} has value of 0. At T, the value of the portfolio is $[F(t,T)-S_T] + [S_T - K] - V_K(t,T)e^{r(T-t)} > 0$. The process for assuming $V_K(t,T) > (F(t,T)-K)e^{-r(T-t)}$ is similar.

At t, go long 1 forward with delivery price $F(t,T)$ with no cost. At T_1, go short 1 forward with delivery price $F(T_1,T)$, also with no cost. Then at T_1, we know we could get $F(T_1,T)-F(t,T)$ in T, so, the value of the two trades at T_1 is $(F(T_1,T)-F(t,T))e^{-r(T-T_1)}$. This is another way to get $V_K(t,T) = (F(t,T)-K)e^{-r(T-t)}$.

Prices, Rates, and Other Values			
Symbol	Description	Value	Comments
$F(t,T)$	forward price on asset without dividends	$S_t e^{r(T-t)}$	value of K such that a forward has value 0 at t
$F(t,T)$	forward price on asset with dividends q	$S_t e^{(r-q)(T-t)}$	q the rate of dividends
$F(t,T)$	forward price on asset generating income I	$(S_t - I)e^{r(T-t)}$	I the value of income at time t
$F(t,T)$	forward price on foreign exchange	$X_t e^{(r_\$ - r_f)(T-t)}$	$r_\$, r_f$ the riskless rates of the currencies

Note that $F(t,T)$ does not depend on distribution of S_T. For two stocks with same spot price S_t, same $F(t,T)$.

[**Just for Career Interest**] Delta One. Delta one products are financial derivatives that have no optionality and as such have a delta of (or very close to) one – meaning that for a given instantaneous move in the price of the underlying asset there is expected to be an identical move in the price of the derivatives.

$$\frac{\partial V_K(t,T)}{\partial S_t} = 1$$
$$\frac{\partial S_t}{\partial S_t} = 1$$

Delta one trading desks are either part of the equity finance or equity derivatives divisions of most major investment banks. They generate most revenue through a variety of strategies related to the various delta one products as well as related activities, such as dividend trading, equity financing and equity index arbitrage.

In practice, how do delta one team operate, especially in the Asia market, and what are the advantages?

- Short Selling? No! Generally, we do synthetic short selling!
- Scale Effect! We do inventory optimization!
- Other Profit? Yes! We do equity index arbitrage!

2 Proofs by Replication and by No-Arbitrage

We will run into many derivatives whose values at future time T we know in terms of the prices of other derivitives. But we do not know their values at current time t. There are two ways of finding and proving the current values of these derivatives: proof by replication and proof by no-arbitrage. Both stem from the *no-arbitrage principle*: there cannot be any free money.

Let the unknown derivative be X. Let us denote the value of X at time T as $V_X(T)$. We treat $V_X(T)$ as a value we know, in terms of the prices at time T. (Eg. $V_X(T) = S_T - K$ for X a forward with strike K and maturity T.) We want to find $V_X(t)$ the value of X at current time t.

2.1 Proof by Replication

Proofs by replication involve creating two portfolios such that at time T the portfolios have the same value if we do not add or remove anything from the portfolios. Then it follows that at current time t the portfolios must also have the same value; otherwise, there would be an opportunity for arbitrage.

In general, a proof by replication for a derivative X generally proceeds as follows:

1. Create two portfolios A and B comprised of derivatives with known value and one containing X.
2. Show that if we do not add or remove anything from the portfolios, then they have the same value at time T.
3. The portfolios now must have the same value at time t. Use this to find the present value of X.

2.2 Proof by No-Arbitrage

We again want to prove the current value of a derivative. Proofs by No-Arbitrage accomplish this by showing that if the derivative currently had any other value, then there would be arbitrage opportunities.

A proof by no-arbitrage to show that $V_X(t) = x$ generally proceeds as follows:

1. Assume that $V_X(t) > x$. Create an arbitrage portfolio with X.
2. Now assume that $V_X(t) < x$. Create an arbitrage portfolio with X. This portfolio is usually the "opposite" of the portfolio you created in (1.)
3. Because of the No-Arbitrage Principle, X must have current value $V_X(t) = x$.

3 Exercises

3.1 Forward Terminology

For each of the following, indicate whether the quantity is a constant or a random variable. If the quantity is a constant, then express it in terms of known values. If the quantity is a random variable, then indicate when we will observe it, and express it in terms of values we will know at that time. Assume the current time is t, where $t < T_1 < T_2$.

(1) $F(t, T_2)$, (2) $F(T_2, T_2)$, (3) $F(T_1, T_2)$, (4) $V_K(t, T_2)$, (5) $V_K(T_2, T_2)$, (6) $V_K(T_1, T_2)$, (7) $V_{F(T_1,T_2)}(T_1, T_2)$

Solution:

(1) $F(t, T_2) = \frac{S_t}{Z(t,T_2)}$, constant because it's known at t.

(2) $F(T_2, T_2) = S_{T_2}$, random variable.

(3) $F(T_1, T_2) = \frac{S_{T_1}}{Z(T_1,T_2)}$, random variable, unknown until T_1.

(4) $V_K(t, T_2) = (\frac{S_t}{Z(t,T_2)} - K)Z(t, T_2)$, known at t.

(5) $V_K(T_2, T_2) = S_{T_2} - K$, random variable.

(6) $V_K(T_1, T_2) = (\frac{S_{T_1}}{Z(T_1,T_2)} - K)Z(T_1, T_2)$, random variable, unknown until T_1.

(7) $V_{F(T_1,T_2)}(T_1, T_2) = 0$ by definition, constant.

3.2 Forwards on Assets with Cost

If I go long an orange juice forward, I agree to buy 15,000 pounds of orange juice at the delivery price K at time T. I can also buy 15,000 pounds of orange juice now (at time t), paying S_t today. But I'll have to rent a really really big refrigerator between now and T, and that's an upfront cost of R (for "refrigerator"). If the interest rate is r, what is the fair price K at which I can enter into the forward at no cost? Give a replication proof, and a proof by no-arbitrage.

Solution:

1. Intuitively, by thinking of the storage cost as negative income, we can plug $I = -R$ into the income equation to get:

$$F(t,T) = (S_t + R)e^{r(T-t)}$$

without any additional work.

2. Proof by Replication:

	Portfolio	time t	time T
A:	15000 lb OJ, borrow R cash and pay storage cost	$S_t + R - R = S_t$	$S_T - Re^{r(T-t)}$
B:	long 1 forward with delivery K and $Ke^{-r(T-t)} - R$ cash	$V_K(t,T) + Ke^{-r(T-t)} - R$	$S_T - K + K - Re^{r(T-t)} = S_T - Re^{r(T-t)}$

Since the two portfolios are worth the same at time T, they must also be worth the same at time t:

$$S_t = V_K(t,T) + Ke^{-r(T-t)} - R$$

To solve for the fair price K (aka the forward price), set $V_K(t,T) = 0$. This gives

$$K = (S_t + R)e^{r(T-t)}$$

3. Proof by No Arbitrage Argument:

Suppose $F(t,T) > (S_t + R)e^{r(T-t)}$: Short 1 forward contract with delivery price $F(t,T)$, borrow $S_t + R$ cash to buy 15000lb OJ and pay storage cost. This is a 0 cost portfolio. At time T: sell the 15000lb OJ receive $F(t,T)$, payback $(S_t + R)e^{r(T-t)}$. Your net gain is $F(t,T) - (S_t + R)e^{r(T-t)} > 0$. Hence there is arbitrage opportunity.

Similarly if $F(t,T) < (S_t + R)e^{r(T-t)}$: Long 1 forward contract with delivery price $F(t,T)$, sell 15000lb OJ to get S_t, and together with the saving from storage cost, invest at rate r. At time T: Pay $F(t,T)$ to receive the 15000lb OJ. Your net gain is $(S_t + R)e^{r(T-t)} - F(t,T) > 0$. Hence there is arbitrage opportunity.

Therefore we must have $F(t,T) = (S_t + R)e^{r(T-t)}$.

3.3 Creating an Arbitrage Opportunity

You have the ability to trade at no cost a forward with maturity T and strike K on an underlying stock with value S_t. The stock pays dividends at rate q. The riskless rate is r compounded continuously.

Suppose that $K > S_t e^{(r-q)(T-t)}$. Find an arbitrage opportunity.

Solution:

At time t, you should go short the forward at no cost. You also borrow $S_t e^{-q(T-t)}$ cash to purchase $e^{-q(T-t)}$ stocks. The dividends received from the stocks are always re-invested to buy more stock. (So effectively, the amount of stock you have grows continuously at rate q.)

At time T, you have 1 stock, which you sell to the counterparty of the forward and receive K cash. You also must pay $S_t e^{(r-q)(T-t)}$ to pay off your loan.

At the end, after fulfilling all oblitations, you are left with $K - S_t e^{(r-q)(T-t)} > 0$ cash, which is arbitrage since we started with 0 cash.

3.4 Forward on Stock with Income and Dividends.

A stock pays out income I at time t and also pays out dividends at rate q continuously. The riskless rate is r compounded continuously. Find the forward price for a forward on the stock with maturity T.

Solution:

Let portfolio A consist of $e^{-q(T-t)}$ units of the stock, and $-Ie^{-q(T-t)}$ cash. The dividends from the stock are continually re-invested in more stock. Let portfolio B consist of long one forward contract with delivery K, plus $Ke^{-r(T-t)}$ cash.

At time T, portfolio A has value $S_T - Ie^{(r-q)(T-t)} + Ie^{(r-q)(T-t)} = S_T$. The value of portfolio B is also S_T after fulfilling the terms of the forward.

So at t the values of the portfolios are equal.

$$(S_t - I)e^{-q(T-t)} = V_K(t,T) + Ke^{-r(T-t)}$$

Setting $V_K(t,T) = 0$, we solve for K:

$$F(t,T) = (S_t - I)e^{(r-q)(T-t)}$$

3.5 Forward Price on Foreign Exchange

Recall that the forward price at time t for one unit of foreign currency and maturity T is given by

$$F(t,T) = X_t e^{(r_\$ - r_f)(T-t)}$$

- Prove the above result using a no arbitrage argument.
- Suppose the one year zero rate in the US is $r_\$ = 0.0015$ and in the UK is $r_f = 0.0050$. In the US, suppose contracts for sterling one year forward are currently trading at 1.60. What is the implied currency exchange rate, expressed as dollars per sterling?
- Suppose that the actual exchange rate is 1.58 dollars per sterling. What would you do?

Solution:

- Suppose we find that $F(t,T) < X_t e^{(r_\$ - r_f)(T-t)}$. Then the forward $F(t,T)$ is too cheap and the exchange rate X_t is too high. As is suggested in the strategies for no arbitrage proofs at the beginning of the handout, we should go long the cheaper asset (the forward) and go short the expensive asset (the currency). The precise sequence of steps to perform is as follows:

  - At t, go long the forward contract $F(t,T)$. (How many? See below.)
  - At t, borrow one unit of foreign currency at rate r_f.
  - At t, convert the 1 unit of foreign currency at the spot rate X_t into $\$X_t$ and invest at rate $r_\$$.
  - At T, we have $X_t e^{r_\$(T-t)}$ in cash dollars. Since we agreed to buy pounds at the forward exchange rate $F(t,T)$, we redeem our $X_t e^{r_\$(T-t)}$ of dollars for $\frac{X_t e^{r_\$(T-t)}}{F(t,T)}$ units of foreign currency.(this is the number of forwards needed above).
  - At T , use the proceeds from above to pay our debt of $e^{r_f(T-t)}$ units of foreign currency.

  The postion at time t was entered into with 0 cost, my net gain at time T is:

$$\begin{aligned}
\frac{X_t e^{r_\$(T-t)}}{F(t,T)} - e^{r_f(T-t)} &> \frac{X_t e^{r_\$(T-t)}}{X_t e^{(r_\$ - r_f)(T-t)}} - e^{r_f(T-t)} \\
&= e^{r_f(T-t)} - e^{r_f(T-t)} \\
&= 0
\end{aligned}$$

  Hence arbitrage opportunity exists. You must have $F(t,T) \geq X_t e^{(r_\$ - r_f)(T-t)}$. Similar argument can be established under the case $F(t,T) > X_t e^{(r_\$ - r_f)(T-t)}$.

- Starting with the formula for the FX forward price from above, we have

$$\begin{aligned}
F(t,T) &= X_t e^{(r_\$ - r_f)(T-t)} \\
1.60 &= X_0 e^{(0.0015 - 0.0050)(1-0)} \\
X_0 &= 1.606
\end{aligned}$$

- The actual exchange rate that we observe is not what the forward price would suggest. In particular, if the actual exchange rate is \$1.58 for 1 unit of foreign currency, then the forward price should be $1.58e^{(0.0015-0.0050)(1-0)} = 1.574$, instead of what we observe, namely 1.60. That is, the forward price is too expensive. To profit from this mispricing, an arbitrageur should go long the cheap asset (pounds sterling) and go short the expensive asset (the forward contract).