STAT 123: Quantitative Finance, Spring 2022 Prof. Stephen Blyth

Section 9 – Solution

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Section: Monday 10 - 11AM Office Hour: Monday 11AM - noon

1 Caplets / Floorlet

1.1 Caplets

- A caplet is a call on libor. A caplet with strike K and maturity T pays

  $$\alpha(L_T[T, T+\alpha] - K)^+$$

  at time $T + \alpha$.

  The decision to exercise or not is at time T, while the actual exchange of cashflow is at $T+\alpha$. Hence:

  $$C_K(T,T) = \alpha(L_T - K)^+ Z(T, T+\alpha)$$

- Price: By the Fundamental Theorem using the forward numeraire $Z(t, T+\alpha)$, we have

  $$C_K(t,T) = \alpha Z(t, T+\alpha) E_*[(L_T[T, T+\alpha] - K)^+]$$

- Assuming a Lognormal distribution for $L_T :\equiv L_T[T, T+\alpha]$ with respect to $Z(t, T+\alpha)$,

  $$L_T | L_{tT} \sim \text{Lognormal}(log L_{tT} - \frac{1}{2}\sigma^2(T-t), \sigma^2(T-t))$$

  we get

  $$C_K(t,T) = \alpha Z(t, T+\alpha)(L_{tT}\Phi(d_1) - K\Phi(d_2))$$

  where

  $$d_1 = \frac{log(\frac{L_{tT}}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
  $$d_2 = d_1 - \sigma\sqrt{T-t}$$

1.2 Floorlet

- A floorlet is a put on libor rate. A floorlet with strike K and maturity T has payout

  $$\alpha(K - L_T)^+$$

  at time $T + \alpha$.

1.3 Cap / Floor

A cap consists of consecutive caplets. A cap from T_0 to T_n with $T_{i+1} = T_i + \alpha$ is a series of caplets with expires $T_0, \ldots\ldots, T_{n-1}$ and payout dates $T_1, \ldots\ldots, T_n$. This is called a "T_0 by T_n cap."

Similarly, a floor is a portfolio consisting of consecutive floorlets. A portfolio consisting of a cap and a floor with same strike and dates is a cap-floor straddle, similar to an option straddle.

2 Interest Rate Swaps

- A swap is an agreement between two counterparties to exchange a series of cashflows at agreed dates. A swap has start date T_0, maturity T_n, and payment dates T_i, $i = 1, ..., n$.
  - Fixed stream: αK at times $T_1, ..., T_n$.
  - Floating stream: $\alpha L_{T_0}[T_0, T_1]$ at time T_1, $\alpha L_{T_1}[T_1, T_2]$ at time T_2,..., $\alpha L_{T_{n-1}}[T_{n-1}, T_n]$ at time T_n.
- The value of the fixed leg is:

$$V_K^{FXD}(t) = K \sum_{i=1}^{n} \alpha Z(t, T_i) = K P_t[T_0, T_n]$$

  where $P_t[T_0, T_n]$ is called "pv01" of the swap, the present value of receiving α at each payment dates.
- The value of the floating leg is:

$$V_K^{FL}(t) = \sum_{i=1}^{n} L_t[T_{i-1}, T_i] \alpha Z(t, T_i) = Z(t, T_0) - Z(t, T_n)$$

- Forward swap rate, $y_t[T_0, T_n]$ is the value of the fixed rate K such that the value of the swap at time t is 0:

$$\begin{aligned} y_t[T_0, T_n] &= \frac{\sum_{i=1}^{n} L_t[T_{i-1}, T_i] \alpha Z(t, T_i)}{\sum_{i=1}^{n} \alpha Z(t, T_i)} \\ &= \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]} \end{aligned}$$

- The value $V_K^{SW}(t)$ of a swap that pay fixed rate K in exchange for floating libor is

$$V_K^{SW}(t) = (y_t[T_0, T_n] - K) P_t[T_0, T_n]$$

3 Swaption

- A swaption is an option on a swap:

– Definition: A payer swaption with strike K from T to T_n, called a T *into* $T_n - T$ payer swaption, gives the buyer the right at T to enter into a swap from T to T_n, paying fixed rate K. If the buyer exercises at T, she receives αL_T at $T + \alpha$, $\alpha L_{T+\alpha}$ at $T + 2\alpha$, etc., and pays αK at each time point. Discounting back to time T, we have

$$\Psi_K(T, T, T_n) = P_T[T, T_n](y_T[T, T_n] - K)^+$$

– A receiver swaption with strike K from T to T_n gives the buyer the right at T to enter into a swap from T to T_n, receiving fixed rate K.

- Price: By FTAP using the swap numeraire *pv01* $P_t[T, T_n] = \sum_{i=1}^{n} \alpha Z(t, T_i)$, the payer swaption has price

$$\Psi_K(t, T, T_n) = P_t[T, T_n]E_*[(y_T[T, T_n] - K)^+]$$

- Black Formula for swaptions:
Assuming a Lognormal distribution for $y_T[T, T_n]$ with respect to $pv01$,

$$y_T[T, T_n]|y_t[T, T_n] \sim lognormal(log y_t[T, T_n] - \frac{1}{2}\sigma^2(T - t), \sigma^2(T - t))$$

we get

$$\Psi_K(t, T, T_n) = P_t[T, T_n](y_t[T, T_n]\Phi(d_1) - K\Phi(d_2))$$

where

$$d_1 = \frac{log(y_t[T, T_n]/K) + \frac{1}{2}\sigma^2(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = d_1 - \sigma(T - t)$$

4 Exercises

4.1 Caplet-Floorlet Parity

(a) Derive an expression for caplet-floorlet parity, along the lines of put-call parity.

(b) Use the Black-Sholes formula and caplet-floorlet parity to come with an expression for a floorlet.

Solution:

(a) Let us consider a portfolio that consists of a long caplet and a short floorlet, both with strike K and payout at $T+\alpha$. This portfolio is equivalent to a FRA with delivery K and payout at $T + \alpha$.

Let $P_K^{AP}(t, T)$ denote the value of the floorlet. (Nothing to do with the word "floorlet" but capitalizing on a put in comparison to a caplet with value $C_K^{AP}(t, T)$.)

Then the parity equation is

$$C_K^{AP}(t,T) - P_K^{AP}(t,T) = V_K(t,T) = \alpha(L_{tT} - K)Z(t,T+\alpha)$$

(b) We have that

$$C_K^{AP}(t,T) - P_K^{AP}(t,T) = V_K(t,T) = \alpha(L_{tT} - K)Z(t,T+\alpha)$$

and

$$C_K^{AP}(t,T) = \alpha Z(t,T+\alpha)(L_{tT}\Phi(d_1) - K\Phi(d_2))$$

where

$$d_1 = \frac{log(\frac{L_{tT}}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Then solving for the floorlet value:

$$P_K^{AP}(t,T) = \alpha Z(t,T+\alpha)(K\Phi(-d_2) - L_{tT}\Phi(-d_1))$$

4.2 Swap

Propose in detail what you can do if you observe

$$y_t[T_0,T_n] < \frac{Z(t,T_0) - Z(t,T_n)}{\alpha\sum_{i=1}^n Z(t,T_i)}$$

Solution:

$y_t[T_0,T_n] < \frac{Z(t,T_0)-Z(t,T_n)}{\alpha\sum_{i=1}^n Z(t,T_i)}$ suggests that the market swap rate is lower than what it should be. We would like to enter a swap contract paying the fixed leg.
Consider the following trading strategy:

- Enter a swap contract from T_0 to T_n, with market swap rate $y_t[T_0,T_n]$. This will cost 0 to enter at time t.
- Long ZCBs maturing from T_1 to T_n, with principal $\alpha y_t[T_0,Tn]$.
  This will cost $\alpha y_t[T_0,Tn]\sum_{i=1}^n Z(t,T_i)$.
- Short 1 unit ZCB maturing at T_0, long 1 unit ZCB maturing at T_n. This costs $-Z(t,T_0)+Z(t,T_n)$.
- At time T_i, $i = 0,1,2,...,n-1$, ZCB matures. Reborrow the principal \$1 in LIBOR rate maturing at T_{i+1}. So at T_{i+1} we will have payment $\$1 + \alpha L_{T_i}[T_i,T_{i+1}]$.

It is easy to show that the portfolio has total payment 0 at T_i, $i = 0,1,2,...,n$. So this portfolio has zero value in the future.

However, the payout of the portfolio at time t is $-\alpha y_t[T_0, T_n] \sum_{i=1}^{n} Z(t, T_i) + Z(t, T_0) - Z(t, T_n) > 0$, as showed by the condition given.
So we form a portfolio which costs nothing in the future but make positive payout now.

(However, to reach a formal arbitrage portfolio, you need to invest the payoff now, $Z(t, T_0) - Z(t, T_n) ? \alpha y_t[T_0, T_n] \sum_{i=1}^{n} Z(t, T_i)$ in ZCBs, maturing in the future. Then this portfolio has zero value now but positive payout in the future with probability 1, which fits the definition of no-arbitrage portfolio.)

4.3 Caplet in Binomial Tree

Suppose T=1, $\alpha = 0.25$, $Z(0, T) = 0.969$, $Z(0, T + \alpha) = 0.95$. At T =1, $Z(T, T + \alpha)$ is either 0.9756 or 0.9828. (a) What is the price of a caplet on libor rate $L_T[T, T + \alpha]$ with expiry T = 1 and strike K = 850 base points? ($\frac{1-0.9756}{0.25*0.9756} \approx 0.1$ and $\frac{1-0.9828}{0.25*0.9828} \approx 0.07$).

Solution:

$L_0[T, T + \alpha] = \frac{Z(0,T) - Z(0,T+\alpha)}{\alpha Z(0,T+\alpha)} = \frac{0.969-0.95}{0.25*0.95} = 800$ bp. Since $L_T[T, T + \alpha] = \frac{1-Z(T,T+\alpha)}{\alpha Z(T,T+\alpha)}$ is either $\frac{1-0.9756}{0.25*0.9756} = 1000$ bp, or $\frac{1-0.9828}{0.25*0.9828} = 700$ bp. We can solve the risk neutral probability $p^* = P(L_T = 1000bp)$ by using the martingale $E[L_T] = L_t$ under the numeraire $\alpha Z(T, T + \alpha)$. Equiting: $1000p^* + 700(1 - p^*) = 800$, we get $p^* = \frac{1}{3}$.

The payout of a caplet is $\alpha(L_T - K)^+$ at $T + \alpha$. So the value of the caplet with K=850 bp at $T + \alpha$ is 150α bp if $L_T = 1000$ bp and 0 o.w. According to the fundamental theorem using $\alpha Z(t, T + \alpha)$ as numeraire:

$$\frac{C_K(0, T)}{\alpha Z(0, T + \alpha)} = p^* \frac{150\alpha}{\alpha Z(T + \alpha, T + \alpha)}$$

and $C_K(0, T) = 50\alpha Z(0, T + \alpha) = 11.875$ bp.