STAT 123: Quantitative Finance, Spring 2022 — Prof. Stephen Blyth

Final Review – Practice Questions & Solutions

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Final Review Section: April 29 1 - 3 PM

1 Relating the Interest Rate Derivatives

Using words and equations, write down any relationships you can think of between the following derivatives:

FRAs, Caplets/Caps and Floorlets/Floors, Libor-in-Arrears, Swaps, Swaptions, Cancellable Swaps

Hint: Consider put-call parity involving caps and floors, European payer and receiver swaptions.

Solution:

a) A caplet is a "call" on libor because its payout is $\alpha(L_T - K)^+$ at time $T + \alpha$, and a FRA has payout at $T + \alpha$: $\alpha(L_T[T, T + \alpha] - K)$. A floorlet is a "put" on libor because its payout is $\alpha(K - L_T)^+$ at time $T + \alpha$.

b) Note that in a cap, the long counter-party exercises the caplet at each T_i if $L_{T_i}[T_i, T_{i+1}] > K$. And in a floor, the long counter-party exercises the floorlet at each T_i if $L_{T_i}[T_i, T_{i+1}] < K$. Thus, if we are long a cap and short a floor, then we are always paying fixed K and receiving libor at each time period (either we exercise the caplet, or the counterparty exercises the floorlet). This is equivalent to a swap, so we have the following put-call parity for caps and floors.

$$\boxed{\text{cap} - \text{floor} = \text{swap}}$$

A related result is found by examining the cashflows at each time T_i:

$$\boxed{\text{caplet} - \text{floorlet} = \text{FRA}}$$

c) Note that the European payer swaption with exercise date T_0 is only exercised if $Y_{T_0}[T_0, T_n] > K$. And the European receiver swaption with exercise date T_0 is only exercised if $Y_{T_0}[T_0, T_n] < K$. Thus, if we are long a European payer swaption with exercise date T_0 and short a corresponding European receiver swaption, then one or the other is always exercised, so we always pay fixed K and receive libor. This is equivalent to a swap, so we have the following put-call parity for European payer and receiver swaptions.

$$\boxed{\text{European payer swaption} - \text{European receiver swaption} = \text{swap}}$$

d) We can construct a European cancellable swap from a swap and a European swaption in several ways:

1) enter into swap paying K from T to T_n, plus long a T_j into $T_n - T_j$ European receiver swaption

2) enter into swap paying K from T to T_j , plus long a T_j into $T_n - T_j$ European payer swaption

2 Volatility (Sample Final Question 3)

For each of the following positions A-N, determine whether:
I. HMC is long volatility (ie the position has positive vega and thus increases in value if volatility increases);
II HMC is short volatility (ie the position has negative vega);
III There is no volatility exposure in the trade;
IV The volatility exposure is indeterminate with the information given.
Glossary: 3mL means 3-month libor; 1mL 1-month libor; Options in A-G all have maturity T on a stock that pays no dividends, and $K_1 < K_2$.

- A. HMC is long a K-strike call
- B. HMC is long a K-strike call and long a K-strike put
- C. HMC is short a K-strike call and long a K-strike put
- D. HMC is long a K_1, K_2 call spread
- E. HMC is long a $K_1, (K_1 + K_2)/2, K_2$ call butterfly
- F. HMC is short a K_1-strike call which knocks out if $S_t > K_2$ at any time
- G. HMC is short a K_2-strike call which knocks out if $S_t > K_1$ at any time
- H. HMC is long a forward contract on a stock
- J. HMC is long a futures contract on a stock
- K. HMC is long a futures contract on a fixed rate bond
- L. HMC pays 3mL q, receives 3mL in arrears q, for 10yrs
- N. HMC pays 1mL q, receives 3mL q, for 10yrs

Solution:

A I
B I
C III
D IV
E IV
F IV
G III
H III
J IV
K IV. K is indeterminate because increasing interest rates can potentially cause bond prices to be positively correlated with short-dated interest rates (when they are negative). Most of the time, bond prices are negatively correlated and have negative vega, but they can have positive vega when positively correlated.

L I. What this trade means is that is that at each period $t-1$, we know what we're going to be paying at time t, but we don't know what we're going to be receiving – libor in arrears means we're receiving the rate from t to $t+1$ at time t (which is only known at t), whereas libor just means we're receiving the rate from $t-1$ to t at time t. Given interest rates are non-negative, there's a cap on how much downside is associated with interest rates, which means that in general, if we're receiving an interest rate payment, we're long vega, because more volatility increases our chances of upside.

N I. So the 1mL q refers to getting 1-month libor every quarter, whereas 3mL q refers to getting 3-month libor every quarter. So for example, let's say a quarter ends in March, the 1-month libor would be the libor rate from March 1 to March 31, whereas the 3-month libor would be the libor rate from January to March. In this case, we're long vega, because we want volatility to increase so that the 3-month libor increases. in general, a longer time period means more exposure to volatility, and because a 3-month has more exposure to volatility, we gain more upside from volatility in our 3-month than we lose in the volatility in our 1-month.

3 Terms Comparison

Comparing Terms For Each Pair – choose the relationship ($\leq, \geq, =, ?$) that best describes the connection between the two terms at time t. Assume $t < T_1 < T_2 < T_3$ and non-negative interest rates.

(a) $Z(t, T_3) \quad Z(T_2, T_3)$

(b) $V_{F(T_1,T_2)}(T_1, T_2) \quad Z(T_1, T_2)$

(c) $F(t, T_1, T_1) \quad F(t, T_2, T_2)$

(d) $L_t[T_1, T_2]$ $L_T[T_1, T_2]$

(e) $P_t[T_1, T_2]$ $P_t[T_1, T_3]$

Solution:

(a) $Z(t, T_3)$? $Z(T_2, T_3)$ because the LHS is a constant and the RHS is an unknown random variable. Even though t to T_3 is a longer time period than T_2 to T_3, interest rates may change (we can only know the interest rate from t to some future time, or the forward rate from one future time to another). $Z(T_2, T_3)$ involves the future spot interest rate from T_2 to T_3 (which is distinct from the forward rate).

(b) $V_{F(T_1,T_2)}(T_1, T_2) \leq Z(T_1, T_2)$ because the LHS is equal to 0 by definition and the RHS is a ZCB price, which is between 0 and 1 inclusive. Note that even though the RHS is a random variable, we still know its bounds, so we can determine this relationship.

(c) $F(t, T_1, T_1) = F(t, T_2, T_2)$ because both sides are equal to 1 by definition.

(d) $L_t[T_1, T_2]$? $L_T[T_1, T_2]$ because the LHS is a constant forward libor rate and the RHS is an unknown future spot libor rate.

(e) $P_t[T_1, T_2] \leq P_t[T_1, T_3]$ because $P_t[T_1, T_2] = \alpha Z(t, T_2)$ and $P_t[T_1, T_3] = \alpha Z(t, T_2) + \alpha Z(t, T_3)$

4 Caplet in Arrears

A caplet in arrears has payout $\alpha(L_T - K)^+$ at time T. Show that the price of the caplet in arrears at t can be expressed as:

- the price of X number of regular caplets with strike K. (Please determine X)
- the price of a "square-everything caplet" whose payout at $T + \alpha$ is $\alpha^2(L_T^2 - K^2)$ if $L_T > K$ and 0 otherwise.

Solution:
At time $T + \alpha$, the payout of caplet in arrears is:

$$\alpha(L_T - K)(1 + \alpha L_T)I(L_T > K)$$

The payout of a portfolio of the described in the bulleted list at time $T + \alpha$ is:

$$\begin{aligned} &X\alpha(L_T - K)I(L_T > K) + \alpha^2(L_T^2 - K^2)I(L_T > K) \\ =&\alpha(L_T - K)[X + \alpha L_T + \alpha K] \end{aligned}$$

Equating the two, we get $X = 1 - \alpha K$. And by monotonicity theorem, we know the price of caplet in arrears must equal the price of the portfolio above.

5 Bermudan Cancellable Swaps

Rank the following portfolios. Assume annual exercises, annual cancellations, and annual payment dates throughout.

I 5-nc-2 European cancellable swap

II 5-nc-2 Bermudan cancellable swap

III swap from 0 to 3

IV swap from 0 to 5, plus 2-into-3 Bermudan receiver swaption

V swap from 0 to 2, plus 2-into-3 Bermudan payer swaption

VI 5-nc-4 Bermudan cancellable swap

VII 6-nc-4 Bermudan cancellable swap

I II

III II

VI II

VI VII

II IV

I V

Solution:

I $\leq$ II

III $\leq$ II

VI $\leq$ II

VI $\leq$ VII

II = IV

I $\leq$ V

6 Binomial Tree and Risk-Neutral Probability

(a) Let us say we have a stock such that $S_0 = 110$. At time $T = 1$, in the up state, $S_1 = 135$ and in the down state, $S_1 = 90$. Also $r = 5\%$, compounded annually. We want to find $P_{105}(0,1)$ the fair price of the 105 put.

(b) What is the risk-neutral probability p^*.

Solution:

(a) Let's try to replicate the put option with stock and bond. Long λ amount of stock and μ bond. To replicate the option payoff at maturity, we have: $\lambda S_1 + \mu = (K - S_1)^+$. Considering the two states at T=1:

$$\begin{aligned}135\lambda + \mu &= 0 \\ 90\lambda + \mu &= 15\end{aligned}$$

solving for the unknowns we have: $\lambda = -\frac{1}{3}, \mu = 45$. Hence $P_{105}(0,1) = 110 \times (-\frac{1}{3}) + 45 \times \frac{1}{1+0.05} = 6.19$.

(b) The present value of the expected payout is

$$Z(0,1)E((K - S_1)^+) = \frac{1}{1.05}(p^* \cdot 0 + (1 - p^*) \cdot 15) = (1 - p^*) \cdot \frac{100}{7}$$

We equate this to the value of the option

$$(1 - p^*) \cdot \frac{100}{7} = \frac{130}{21}$$

$$p^* = \frac{17}{30}$$

7 Swap Value (IQF Chapter 4, Exercise 1 (b))

By expressing a swap as a difference between a floating rate bond and fixed rate bond, prove that, for a given K, the value of swap $V_K^{SW}(t)$ is bounded, that is, there exists finite l and u independent of interest rates such that $l \leq V_K^{SW}(t) \leq u$. For $t = T_0 = 0$ (a spot starting swap), $T_n = n$ and frequency $\alpha = 1$, find bounds in terms of n and K.

Solution:

Express the swap as a difference between a floating rate bond and fixed rate bond.

the cashflows associated with a fixed rate bond with coupon rate $C=K$ correspond almost exactly to the fixed payments of the swap, except that the fixed leg of a swap contract does not include the final payment of one dollar at time T_n.

Likewise. note that the cashflows associated with a floating rate bond correspond almost exactly to the floating payments of the swap, except that the floating leg of a swap contract does not include the final payment of one dollar at time T_0. If an investor is short the fixed rate bond and long the floating rate bond. however, the fixed and floating payments associated with each bond correspond exactly to the payments of a long swap position. and the final payments of one dollar at time T_n cancel out. Therefore. we can replicate a long swap position by going short a fixed rate bond and long a floating rate bond. $V_K^{SW}(t) = B^{FL}(t) - B_K^{FXD}(t)$

since $B^{FL}(t) = Z(t,T_0)$ we know $0 \leq B^{FL}(t) \leq 1$, since $B_K^{FXD}(t) = K\sum_{i=1}^{n} \alpha Z(t,T_i) + Z(t,T_n)$ we know $0 \leq B_K^{FXD}(t) \leq K\alpha n+1$ since both $B^{FL}(t)$ and $B_K^{FXD}(t)$ are bounded. and the bounds are independent of interest rates. so there exists finite l and u independent of interest rates such that $l \leq V_K^{SW}(t) \leq u$

$B^{FL}(t) = Z(t,T_0)$ when $t = T_0 = 0$ $B^{FL}(t) = 1$ if $r \to \infty$ if $r \to 0$

$B_K^{FXD}(t) = K\sum_{i=1}^{n} \alpha Z(t,T_i) + Z(t,T_n) \leq K\alpha n + 1$ then $0 \leq B_K^{FXD}(t) \leq K\alpha n+1$

so we can get $-K\alpha n \leq V_K^{SW} \leq 1$ when $\alpha = 1$ $-Kn \leq V_K^{SW} \leq 1$

then the upper bound is 1. and the lower bound is $-Kn$