STAT 123: Quantitative Finance, Spring 2022 Prof. Stephen Blyth

Section 8 – Solution

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Section: Monday 10 - 11AM Office Hour: Monday 11AM - noon

1 Option Prices and Probability Duality

1.1 Digital Options and Probability

A digital call option can be approximated with a portfolio of European call options. Consider a portfolio that consists of $N(K, K+1/N)$ call spread with maturity T. That is, the portfolio is long N calls with strike K, and short N calls with strike $(K+1/N)$. Then as N approaches infinity, the payout of the portfolio at time T approaches the payout of a digital call option. The price at time t of a digital call option with strike K equals:

$$D(t,T) = \lim_{\lambda\to\infty} \lambda(C_K(t,T) - C_{K+\frac{1}{\lambda}}(t,T)) = -\frac{\partial C_K(t,T)}{\partial K}$$

Recall the Fundamental Bridge: The expectation of an indicator variable is the probability of the event being indicated. So the expected payout of the digital option is

$$E_*[D(T,T)|S_t] = E_*[I_{S_{T-K}}|S_t] = P^*(S_T > K|S_t)$$

We can also apply FTAP

$$\frac{D(t,T)}{Z(t,T)} = E_*[\frac{D(T,T)}{Z(T,T)}|S_t] = E_*[D(T,T)|S_t] = P^*(S_T > K|S_t)$$

Putting it all together, we can relate call prices to the probability distribution of S_T:

$$-\frac{\partial C_K(t,T)}{\partial K} \cdot \frac{1}{Z(t,T)} = P^*(S_T > K|S_t)$$

1.2 Butterflies and Probability

In a similar to how we approximate the derivative of the call value with a call spread, we can approximate the second derivative of a call value with a butterfly spread. The price at time t of a call butterfly defined by:

$$\begin{cases} \lambda & \text{calls w. strike } K-\frac{1}{\lambda} \\ -2\lambda & \text{calls w. strike K} \\ \lambda & \text{calls w. strike } K+\frac{1}{\lambda} \end{cases}$$

equals

$$\lim_{\lambda\to\infty} B_{K,\lambda}(t,T) = \frac{1}{\lambda}\frac{\partial^2 C_K(t,T)}{\partial^2 K}$$

So we can use the value of the butterfly to approximate the PDF of the risk-neutral distribution of $S_T|S_t$.

$$f_{S_T|S_t}(x) = \frac{1}{Z(t,T)} \frac{\partial^2 C_K(t,T)}{\partial^2 K}\Big|_x = \frac{\lambda B_{K,\lambda}(t,T)}{Z(t,T)}$$

2 Exercises

2.1 Digital Option

Assume the typical lognormal distribution for stock price, i.e.

$$S_T|S_t \sim Lognormal(logS_t + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t))$$

derive the Black-Scholes formula for a digital call $DC_K(t,T)$, and hence for a digital put.

Solution:

Let $\mu = logS_t + (r - \frac{\sigma^2}{2})(T-t)$. By fundamental theorem we have

$$\begin{aligned} DC_K(t,T) &= Z(t,T)E_*(I(S_T > K)) \\ &= Z(t,T)\int_{logK}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} exp(-\frac{(y-\mu)^2}{2\sigma^2(T-t)})dy \\ &= Z(t,T)(1 - \Phi(\frac{logK - \mu}{\sigma\sqrt{T-t}})) \\ &= Z(t,T)\Phi(\frac{\mu - logK}{\sigma\sqrt{T-t}}) \\ &= Z(t,T)\Phi(d_2) \end{aligned}$$

Notice that $DC_K(T,T) + DP_K(T,T) = 1$, by monotonicity theorem we have: $DC_K(t,T) + DP_K(t,T) = Z(t,T)$. Hence

$$DP_K(t,T) = Z(t,T)(1 - \Phi(d_2))$$

2.2 Probability Duality

Using result from Exercise 1, and the probability duality formula:

$$f_{S_T|S_t}(x) = \frac{1}{Z(t,T)} \frac{\partial^2 C_K(t,T)}{\partial^2 K}\Big|_x$$

to show that we can recover a lognormal risk-neutral distribution for $S_T|S_t$ from the call prices.

Solution:

Using the result we derived from Ex1 together with the result presented in class, we have:

$$-\frac{\partial C_K(t,T)}{\partial K} = Z(t,T)\Phi(d_2)$$

Together with duality formula we have:

$$\begin{aligned} f_{S_T|S_t}(x) &= \frac{1}{Z(t,T)}\frac{\partial^2 C_K(t,T)}{\partial^2 K}\Big|_x \\ &= -\frac{\partial}{\partial K}\Phi(d_2)\Big|_x = -\phi(d_2)\frac{\partial d_2}{\partial K}\Big|_x \\ &= \frac{1}{x}\frac{1}{\sqrt{2\pi}\sigma^2(T-t)}e^{-\frac{(logx - logS_t - (r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} \end{aligned}$$

which is exactly the pdf of a lognormal distribution.