Section 5 – Solution

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1 Binomial Tree and Risk-Neutral Probability

1.1 Binomial Tree

With the binomial tree model, we assume the underlying is a stock whose price at time 0 is S_0. At time 1, there are two possible states: up and down. In the up state, the price of the stock becomes $S_0(1+u)$; in the down state, the price of the stock becomes $S_0(1+d)$. We also assume that the interest rate is r.

Generally, the steps to solve the binomial tree problem:

- Under the binomial tree model, we can replicate an option with λ stocks and μ ZCBs.
- To solve for the replicating portfolio, equate the payoffs of the derivative and the replicating portfolio in both states of the world.
- Then, to price the derivative, discount the replicating portfolio to the present.

1.2 Risk-Neutral Probability

We did all the work in the binomial tree model without specifying the probability that the stock ends up in the up date. In fact, the option's value is independent of that probability. This is a rather counter-intuitive result.

We want to find the risk-neutral probability p^*, the probability under which the present value of the expected option payout is the only possible arbitrage-free option price. For the binomial tree model, if a stock can go up to $S_0(1+u)$ or down to $S_0(1+d)$ and the interest rate is r ,where $d < r < u$,then

$$p^* = \frac{r-d}{u-d}$$

Risk Neutral Pricing – where prices are discounted as risk-neutral expectation. Under the risk-neutral probability p^*, the price of every market instrument is its discounted expected value. (In lecture, we showed it's valid for options, stock and the ZCBs. By linearity of Expectation, risk-neutral pricing must apply to all portfolios because portfolios are essentially linear combination of assets.) That is,

$$V^A(0) = Z(0,1)E_*(V^A(1))$$

Theorem: The binomial tree is arbitrage-free (i.e. There are no arbitrage portfolios) $\Leftrightarrow d < r < u$.

Proof: ($\rightarrow$) Let us assume that $r \geq u$. That means the return of a ZCB is greater than the return of the asset. So we want to sell one asset for S_0 and invest the money at rate r. So at time 1, we have $S_0(1+r) \geq S_0(1+u) > S_0(1+d)$. So after purchasing back the stock at time 1, we have at least 0 and there is a positive probability of having positive cash left over. This is an arbitrage opportunity. So $r < u$. We can similarly prove that $r > d$ by assuming that $r \leq d$.

($\leftarrow$) There are no portfolios consisting of the asset and ZCBs that are arbitrage portfolios. All portfolios with current value of 0 must consist of v/S_0 stocks and $-v(1+r)$ ZCBs, for some $v \in R$. We consider different cases of the value of v and how the stock behaves and we bound $V(1)$ the value of the portfolio at time 1.

- $v = 0$. This case trivially has $V(1) = 0$.
- $v > 0$. When the stock goes down, we must have $V(1) = v(1+d) - v(1+r) < 0$, with the strict inequality. Otherwise, if $V(1) = 0$ when the the stock goes down, then $V(1) > 0$ when the stock goes up with positive probability, and this forms an arbitrage portfolio. So $d < r$.
- $v < 0$. When the stock goes up, we must have $V(1) = v(1+u) - v(1+r) < 0$ ($v < 0$), with the strict inequality. Otherwise, if $V(1) = 0$ when the stock goes up, then $V(1) > 0$ when the stock goes down with positive probability, and this forms an arbitrage portfolio. So $r < u$.

Moreover, based on the theorem, This also means for the risk-neutral probability: $0 < p^* < 1$, i.e. there exists a p^* such that prices are discounted expected values using p^*.

1.3 General Condition

In a binomial tree with n identical time steps, the price at $T = 0$ of a derivative with payout $g(S_n)$) at time n is given by:

$$\frac{1}{(1+r)^n} E_*(g(S_n)) = \frac{1}{(1+r)^n} \sum_{j=0}^{n} \binom{n}{j} p^{*j} (1-p^*)^{n-j} g(S_0(1+u)^j (1+d)^{n-j})$$

2 Exercises

2.1 Binomial Tree for Put Option

(a) Let us say we have a stock such that $S_0 = 110$. At time $T = 1$, in the up state, $S_1 = 135$ and in the down state, $S_1 = 90$. Also $r = 5\%$, compounded annually. We want to find $P_{105}(0,1)$ the fair price of the 105 put.

(b) What is the risk-neutral probability p^*.

Solution:

(a) Let's try to replicate the put option with stock and bond. Long λ amount of stock and μ bond. To replicate the option payoff at maturity, we have: $\lambda S_1 + \mu = (K - S_1)^+$. Considering the two states at T=1:

$$135\lambda + \mu = 0$$
$$90\lambda + \mu = 15$$

solving for the unknowns we have: $\lambda = -\frac{1}{3}, \mu = 45$. Hence $P_{105}(0,1) = 110 \times (-\frac{1}{3}) + 45 \times \frac{1}{1+0.05} = 6.19$.

(b) The present value of the expected payout is

$$Z(0,1)E((K - S_1)^+) = \frac{1}{1.05}(p^* \cdot 0 + (1 - p^*) \cdot 15) = (1 - p^*) \cdot \frac{100}{7}$$

We equate this to the value of the option

$$(1 - p^*) \cdot \frac{100}{7} = \frac{130}{21}$$

$$p^* = \frac{17}{30}$$

2.2 Pricing on binomial tree: European Put Option

A stock has price 100 at t = 0. Each day, it goes up 20% or down 20%, with equal probabilities. The interest rate is 5%. Find the price of a European put option with strike 96 maturing at T = 2, using replicating portfolios.

Solution:

Draw the two-step binomial tree and work backwards from the leaves to the root. First we do time-1 to time-2. If the stock goes up on the first day, then the put is always worthless, so the replicating portfolio in that state is 0 stock, 0 ZCB. If the stock goes down on the first day, then the replicating portfolio satisfies:

$$96\lambda + \mu = 0$$
$$64\lambda + \mu = 32$$

which gives -1 stock and 96 ZCBs as the replicating portfolio; at t = 1, this portfolio has value $96/1.05 - 80 = 11.43$.

Now we go back and do time-0 to time-1. The replicating portfolio satisfies:

$$120\lambda + \mu = 0$$
$$80\lambda + \mu = 11.43$$

which gives -2/7 stock and 240/7 ZCBs as the replicating portfolio; at t = 0, this portfolio has value -2/7*100 + 240/7*1/1.05 = 4.08. So the price of the put is 4.08.

Using risk-neutral probability: The risk-neutral probability is $p^* = (r-d)/(u-d) = 5/8$. The price of the put is the discounted risk-neutral expectation of the put price, so

$$P_{96}(0,2) = \frac{1}{1.05^2}\Big((\frac{5}{8})^2 * 0 + 2 * \frac{5}{8} * \frac{3}{8} * 0 + (\frac{3}{8})^2 * 32\Big) = 4.08$$

2.3 Pricing on binomial tree: American Put Option

A stock has price 100 at t = 0. Each day, it goes up 20% or down 20%, with equal probabilities. The interest rate is 5%. Find the price of an American put option with strike 96 maturing at T = 2.

Solution:
The strategy is to work from left to right and ask yourself at each stage whether you would exercise early. At time 0, we do not want to exercise early. In the up state ($S_1 = 120$), we do not want to exercise early. In the down state ($S_1 = 80$), we do want to exercise early because we can receive 16 today, whereas holding onto the put will only give us a value of 11.43, as proven in the previous example. Thus, the American put option pays off 0 if $S_1 = 120$ and 16 if $S_1 = 80$. The replicating portfolio for this option satisfies:

$$120\lambda + \mu = 0$$
$$80\lambda + \mu = 16$$

which gives -2/5 stock and 48 ZCBs as the replicating portfolio. That has value 5.71 today.