STAT 123: Quantitative Finance, Spring 2022 — Prof. Stephen Blyth

Section 10 – Solution

Xin Zeng (xinzeng@fas.harvard.edu) — 18th April 2022

Section: Monday 10 - 11AM — Office Hour: Monday 11AM - noon

1 Bermudan Swaptions

- Definition: A T_0 into $T_n - T_0$ Bermudan payer swaption with strike K gives the buyer the option at times $T_0, T_1, ..., T_{n-1}$ to enter into a swap from that time until T_n, paying fixed rate K. If the buyer exercises at T_i, she is locked into a swap from T_i to T_n, with no further optionality. If the buyer does not exercise at T_i, she can decide again at T_{i+1} whether to exercise.
- Pricing a Bermudan swaption is complex, but we can still put bounds on the value of a swaption. Let $C_K(t, T_0, T_n)$ be the value at time t of a K-strike T_0 by T_n cap, $\Psi_K(t, T_i, T_n)$ be the value at time t of a K-strike T_j into $T_n - T_j$ European payer swaption. Then, the bounds on price:
  - $B_K(t, T_0, T_n) \leq C_K(t, T_0, T_n)$
  - $B_K(t, T_0, T_n) \geq \underset{0 \leq i \leq n-1}{max} \Psi_K(t, T_i, T_n)$
  - $B_K(t, T_0, T_n) \leq \sum_{i=0}^{n-1} \Psi_K(t, T_i, T_n)$
- If exercise, we get $(y_{T_i}[T_i, T_n] - K)P_{T_i}[T_i, T_n]$, if not exercise, we get $B_K(T_j, T_{j+1}, T_n)$. However, $B_K(T_j, T_{j+1}, T_n)$ is unknown. The Do-Not-Exercise criteria:
  - Do not exercise at T_i if $y_{T_i}[T_i, T_n] < K$
  - Do not exercise at T_i if $(y_{T_i}[T_i, T_n] - K)P_{T_i}[T_i, T_n] < \underset{i+1 \leq j \leq n+1}{max} \Psi_K(T_i, T_j, T_n)$
  - Do not exercise at T_i if $y_{T_i}[T_i, T_j] < K$ for any j: $i + 1 \leq j \leq n$

2 Cancellable Swaps / Bermudan Cancellable Swaps

- European Cancellable Swap: the party is in a swap and pays fixed K and receives libor from $T = T_0$ to T_n, but at a single time T_j , the party has the option to cancel the swap. If the option is exercised, no more swap payments are made after T_j.

  We can construct a European cancellable swap from a swap and a European swaption in several ways:

  1) enter into swap paying K from T to T_n, plus long a T_j into $T_n - T_j$ European receiver swaption

  2) enter into swap paying K from T to T_j , plus long a T_j into $T_n - T_j$ European payer swaption

- Bermudan Cancellable Swaps: A T_n noncall T_j Bernumdan cancellable swap is a swap from T_0 to T_n where the party who pays fixed has the right to cancel at $T_j, T_{j+1}, \cdots, T_{n-1}$.
- Results
  - K{6nc2 Berm q} $\geq$ K{5nc2 Berm q} $\geq$ K{5nc2 Berm s/a } $\geq$ K{ 5nc3 Berm s/a } $\geq$ K{5nc3 Euro} $\geq$ five-year swap rate
  - K{5nc3 Euro} $\geq$ three-year swap rate
  - K{5nc2 Berm s/a } $\geq$ K{5nc2 Euro} $\geq$ two-year swap rate

3 Exercises

3.1 Bounds on Bermudan Swaption

Prove the given bounds on the value of a Bermudan swaption

- $B_K(t, T_0, T_n) \leq C_K(t, T_0, T_n)$
- $B_K(t, T_0, T_n) \geq \underset{0 \leq i \leq n-1}{max} \Psi_K(t, T_i, T_n)$
- $B_K(t, T_0, T_n) \leq \sum_{i=0}^{n-1} \Psi_K(t, T_i, T_n)$

Solution:

- $B_K(t, T_0, T_n) \leq C_K(t, T_0, T_n)$

  Assume that the cap is worth less than the Bermudan. Then we can sell the Bermudan and buy the cap. We have positive cash leftover, which we can invest. If the counterparty exercises the Bermudan at T_j, then we choose to exercise the cap at $T_j, ..., T_{n-1}$, to cancel out all the Bermudan payments. This guarantees that we have a positive value in the portfolio at the end. This is arbitrage.

- $B_K(t, T_0, T_n) \geq \underset{0 \leq i \leq n-1}{max} \Psi_K(t, T_i, T_n)$

  Assume otherwise, $B_K(t, T_0, T_n) < \underset{0 \leq i \leq n-1}{max} \Psi_K(t, T_i, T_n)$ for some i. Then we can sell T_i into $T_n - T_j$ European payer swaption, and then buy the Bermudan and invest the positive leftover cash. If the counterparty exercises the European payer swaption, then you can exercise the Bermudan. Then all we are left with is the returns on the initial positive cash. This is arbitrage.

- $B_K(t, T_0, T_n) \leq \sum_{i=0}^{n-1} \Psi_K(t, T_i, T_n)$.

  Assume that the sum of the European payer swaptions is less than the Bermudan. Then we can sell the Bermudan and buy all of the European payer swaptions. We will have positive cash leftover, which we can invest at the risk-free rate. If the counterparty exercises the Bermudan at T_j , then we exercise the T_i into $T_n - T_j$ payer swaption to cancel out the Bermudan payments. This cancels out all the swaption payments, leaving us with the positive value from the cash investment. This is arbitrage.

3.2 Bermudan Cancellable Swaps

Rank the following portfolios. Assume annual exercises, annual cancellations, and annual payment dates throughout.

I 5-nc-2 European cancellable swap

II 5-nc-2 Bermudan cancellable swap

III swap from 0 to 3

IV swap from 0 to 5, plus 2-into-3 Bermudan receiver swaption

V swap from 0 to 2, plus 2-into-3 Bermudan payer swaption

VI 5-nc-4 Bermudan cancellable swap

VII 6-nc-4 Bermudan cancellable swap

I II

III II

VI II

VI VII

II IV

I V

Solution:

I $\leq$ II

III $\leq$ II

VI $\leq$ II

VI $\leq$ VII

II = IV

$I \leq V$