STAT 123: Quantitative Finance, Spring 2022 — Prof. Stephen Blyth

Section 9 – Solution

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Section: Monday 10 - 11AM — Office Hour: Monday 11AM - noon

1 Caplets / Floorlet

1.1 Caplets

- A caplet is a call on libor. A caplet with strike K and maturity T pays

$$\alpha(L_T[T,T+\alpha]-K)^+$$

  at time $T+\alpha$.

  The decision to exercise or not is at time T, while the actual exchange of cashflow is at $T+\alpha$. Hence:

$$C_K(T,T)=\alpha(L_T-K)^+Z(T,T+\alpha)$$

- Price: By the Fundamental Theorem using the forward numeraire $Z(t,T+\alpha)$, we have

$$C_K(t,T)=\alpha Z(t,T+\alpha)E_*[(L_T[T,T+\alpha]-K)^+]$$

- Assuming a Lognormal distribution for $L_T :\equiv L_T[T,T+\alpha]$ with respect to $Z(t,T+\alpha)$,

$$L_T|L_{tT}\sim \text{Lognormal}(logL_{tT}-\frac{1}{2}\sigma^2(T-t),\sigma^2(T-t))$$

  we get

$$C_K(t,T)=\alpha Z(t,T+\alpha)(L_{tT}\Phi(d_1)-K\Phi(d_2))$$

  where

$$d_1=\frac{log(\frac{L_{tT}}{K})+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2=d_1-\sigma\sqrt{T-t}$$

1.2 Floorlet

- A floorlet is a put on libor rate. A floorlet with strike K and maturity T has payout

$$\alpha(K-L_T)^+$$

  at time $T+\alpha$.

1.3 Cap / Floor

A cap consists of consecutive caplets. A cap from T_0 to T_n with $T_{i+1} = T_i + \alpha$ is a series of caplets with expires $T_0, \ldots\ldots, T_{n-1}$ and payout dates $T_1, \ldots\ldots, T_n$. This is called a "T_0 by T_n cap."

Similarly, a floor is a portfolio consisting of consecutive floorlets. A portfolio consisting of a cap and a floor with same strike and dates is a cap-floor straddle, similar to an option straddle.

2 Interest Rate Swaps

- A swap is an agreement between two counterparties to exchange a series of cashflows at agreed dates. A swap has start date T_0, maturity T_n, and payment dates T_i, $i = 1, ..., n$.
  - Fixed stream: αK at times $T_1, ..., T_n$.
  - Floating stream: $\alpha L_{T_0}[T_0, T_1]$ at time T_1, $\alpha L_{T_1}[T_1, T_2]$ at time T_2,..., $\alpha L_{T_{n-1}}[T_{n-1}, T_n]$ at time T_n.
- The value of the fixed leg is:

$$V_K^{FXD}(t) = K \sum_{i=1}^{n} \alpha Z(t, T_i) = K P_t[T_0, T_n]$$

  where $P_t[T_0, T_n]$ is called "pv01" of the swap, the present value of receiving α at each payment dates.

- The value of the floating leg is:

$$V_K^{FL}(t) = \sum_{i=1}^{n} L_t[T_{i-1}, T_i] \alpha Z(t, T_i) = Z(t, T_0) - Z(t, T_n)$$

- Forward swap rate, $y_t[T_0, T_n]$ is the value of the fixed rate K such that the value of the swap at time t is 0:

$$\begin{aligned} y_t[T_0, T_n] &= \frac{\sum_{i=1}^{n} L_t[T_{i-1}, T_i] \alpha Z(t, T_i)}{\sum_{i=1}^{n} \alpha Z(t, T_i)} \\ &= \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]} \end{aligned}$$

- The value $V_K^{SW}(t)$ of a swap that pay fixed rate K in exchange for floating libor is

$$V_K^{SW}(t) = (y_t[T_0, T_n] - K) P_t[T_0, T_n]$$

3 Swaption

- A swaption is an option on a swap:

- Definition: A payer swaption with strike K from T to T_n, called a T *into* $T_n - T$ payer swaption, gives the buyer the right at T to enter into a swap from T to T_n, paying fixed rate K. If the buyer exercises at T, she receives αL_T at $T+\alpha$, $\alpha L_{T+\alpha}$ at $T+2\alpha$, etc., and pays αK at each time point. Discounting back to time T, we have

$$\Psi_K(T,T,T_n) = P_T[T,T_n](y_T[T,T_n] - K)^+$$

- A receiver swaption with strike K from T to T_n gives the buyer the right at T to enter into a swap from T to T_n, receiving fixed rate K.

- Price: By FTAP using the swap numeraire *pv01* $P_t[T,T_n] = \sum_{i=1}^{n} \alpha Z(t,T_i)$, the payer swaption has price

$$\Psi_K(t,T,T_n) = P_t[T,T_n]E_*[(y_T[T,T_n] - K)^+]$$

- Black Formula for swaptions:
  Assuming a Lognormal distribution for $y_T[T,T_n]$ with respect to $pv01$,

$$y_T[T,T_n]|y_t[T,T_n] \sim lognormal(log y_t[T,T_n] - \frac{1}{2}\sigma^2(T-t), \sigma^2(T-t))$$

  we get

$$\Psi_K(t,T,T_n) = P_t[T,T_n](y_t[T,T_n]\Phi(d_1) - K\Phi(d_2))$$

  where

$$d_1 = \frac{log(y_t[T,T_n]/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma(T-t)$$

4 Exercises

4.1 Caplet-Floorlet Parity

(a) Derive an expression for caplet-floorlet parity, along the lines of put-call parity.

(b) Use the Black-Sholes formula and caplet-floorlet parity to come with an expression for a floorlet.

Solution:

4.2 Swap

Propose in detail what you can do if you observe

$$y_t[T_0, T_n] < \frac{Z(t, T_0) - Z(t, T_n)}{\alpha \sum_{i=1}^{n} Z(t, T_i)}$$

Solution:

4.3 Caplet in Binomial Tree

Suppose T=1, $\alpha = 0.25$, $Z(0, T) = 0.969$, $Z(0, T + \alpha) = 0.95$. At T =1, $Z(T, T + \alpha)$ is either 0.9756 or 0.9828. (a) What is the price of a caplet on libor rate $L_T[T, T + \alpha]$ with expiry T = 1 and strike K = 850 base points? ($\frac{1-0.9756}{0.25*0.9756} \approx 0.1$ and $\frac{1-0.9828}{0.25*0.9828} \approx 0.07$).

Solution: