STAT 123: Quantitative Finance, Spring 2022 — Prof. Stephen Blyth

Midterm Review – Concepts Summary

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Section: Monday 10 - 11AM — Office Hour: Monday 11AM - noon

Good luck with your midterm!

General Tips

- Exam questions (or their sub-parts!) will not necessarily proceed from easy to difficult. Wisely allocate your time during the exam.
- Make yourself familiar with homework questions and sample midterm questions (i.e. start with questions written by Professor Blyth as they usually have a distinct style). If you're running short on time, try to focus on key concepts.

1 Probability Review

- **Fundamental Bridge.** Let A be an event, and I_A be the indicator for event A. Then

$$E[I_A] = P(A)$$

- **Normal Distribution.** The Normal distribution will be used quite often. Familiarize yourself with the PDF and CDF of the standard Normal. Also, for $Z \sim N(0,1)$ the standard Normal, and $W \sim N(\mu, \sigma^2)$, we have

$$\frac{W-\mu}{\sigma} \equiv Z$$

  This is useful for standardizing to the standard Normal.

- **LOTUS.** Law of the Unconscious Statistician. For X a discrete and continuous r.v., respectively:

$$E[g(X)] = \sum_x g(x)P(X=x)$$
$$E[g(X)] = \int_x g(x)f(x)dx$$

- **Log-Normal Distribution.** For $Y \sim \mathrm{LogN}(\mu, \sigma^2)$, then

$$Y \equiv e^Z \quad \text{for} \quad Z \sim N(\mu, \sigma^2)$$

  In particular,

$$E(Y) = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

2 Interest Rates

In this course, assume that interest rates are non-negative unless explicitly specified.

- Compounding frequency: if you invest \$1 at a rate of r_m, compounded m times per year, you have

$$\left(1+\frac{r_m}{m}\right)^{mT}$$

after T years.

- Converting between discrete and continuous compounding: if continuous compounding rate is r, the equivalent rate r_m with compounding frequency m satisfies

$$e^{rT}=\left(1+\frac{r_m}{m}\right)^{mT}$$

3 Numeraire

3.1 Zero Coupon Bond (ZCB)

A zero coupon bond (ZCB) with maturity T is an asset that is worth 1 at time T (and pays nothing else). We denote $Z(t,T)$ as the value at current time t of a ZCB with maturity T. By definition, $Z(T,T)=1$. The value of $Z(t,T)$ depends on how interest is compounded.

If r is the constant interest rate, and the interest is compounded continuously, then

$$Z(t,T)=e^{-r(T-t)}$$

If r_A is the constant interest rate, and interest is compounded annually, then

$$Z(t,T)=\frac{1}{(1+r_A)^{T-t}}$$

Hint: The equations could be proved through replication. Consider the following assets at time t, (1) 1 ZCB at time T, (2) $e^{-r(T-t)}$ of cash, deposit at rate r. The two assets are worth the same at time T, then they worth the same at time t.

Throughout this course, we will use the ZCB to represent the current value of 1 at time T. If something is worth 1 at time T, then at time t, it should be worth $Z(t,T)$.

3.2 Money Market Account

The close cousin of ZCBs are Money Market Accounts, which is the value at time t of 1 that was invested at time 0 at rate r. Trivially, $M_0=1$.
In general, we have that

$$M_t=\frac{1}{Z(0,t)}$$

3.3 Annuity

An annuity pays value C at times $T_1, \dots, T_n$. The current value of an annuity is simply the current value of all the future payments of C

$$V = C \sum_{i=1}^{n} Z(t, T_i)$$

4 Stock and Bonds

4.1 Stock

A security giving partial ownership of a company. It may pay a dividend. In this course we will use q to denote the dividend rate, or, the percentage of the stock price the dividend represents. The stock price at time t written as S_t. If we are currently at time t, then we call S_t the spot price.

4.2 Fixed Rate Bond

A contract defined by coupon c and notional N that pays the bondholder cN at a particular frequency until maturity, when N is repaid in full. A floating rate bond will be covered extensively later in the course, and differs from the fixed rate bond in its coupon payments to the bondholder.

5 Derivatives

A *derivative contract* is simply a financial contract between two counter-parties and whose value is derived from the value of an underlying asset or variable. For *financial derivative contract*, the underlying variable is financial asset price, or interest rate, etc.

5.1 Forward

A forward is an agreement between two counter-parties to exchange an asset (e.g. a stock) at time T for fixed price K. T is the *maturity* of the forward, and K is the *delivery (strike) price*. The counter-party that has agreed to buy the underlying (e.g. a stock) is long the forward. The counter-party that has agreed to sell the underlying (e.g. a stock) is short the forward.

Let S_t denote the value of the underlying asset at time t. Then at time T, the value of long the forward is just $V_K(T,T) = S_T - K$. Note that S_T is a random variable.

If the underlying does not pay any income or give any dividends, then the current value of the forward is $S_t - KZ(t,T)$. We generally assume that the interest rate is constant r and is continuously compounded. So the current value is

$$S_t - Ke^{-r(T-t)}$$

The forward price is the value of K such that the value of the forward at time t is zero, $V_K(t,T) = 0$. It is denoted $F(t,T)$. From the above, we get that

$$F(t,T) = S_t e^{r(T-t)} = \frac{S_t}{Z(t,T)}$$

The forward price will change depending on if the underlying has dividends or generates income. Generally, we assume that dividends affect the interest rate r, and income affects the current underlying price S_t.

Let t be the current time, and $T > t$ be a future time. Let r be a continuous rate.

Values of Derivatives			
Symbol	Derivative	Value at t	Payoff at T
$V_K(t,T)$	forward with strike K (no matter on asset without dividends, with dividends q, or on asset generate income I)	$(F(t,T) - K)e^{-r(T-t)}$	$S_T - K$
$V_K(t,T)$	forward with strike K (no matter on asset without dividends, with dividends q, or on asset generate income I)	$(F(t,T) - K)e^{-r(T-t)}$	$S_T - K$
$V_K(t,T)$	forward with strike K on foreign exchange	$(F(t,T) - K)e^{-r_{\$}(T-t)}$	$X_T - K$

For $V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}$ regardless of asset, this could be proved by arbitrage. Assume $V_K(t,T) < (F(t,T) - K)e^{-r(T-t)}$. Then, at t, construct portfolio through: 1) go short 1 forward with delivery price $F(t,T)$ with no cost, 2) go long 1 forward with delivery price K, paying $V_K(t,T)$ to do. Then at time t, the portfolios {-1 forward with strike price K, +1 forward contract with strike price K and $-V_K(t,T)$} has value of 0. At T, the value of the portfolio is $[F(t,T) - S_T] + [S_T - K] - V_K(t,T)e^{r(T-t)} > 0$. The process for assuming $V_K(t,T) > (F(t,T) - K)e^{-r(T-t)}$ is similar.

At t, go long 1 forward with delivery price $F(t,T)$ with no cost. At T_1, go short 1 forward with delivery price $F(T_1,T)$, also with no cost. Then at T_1, we know we could get $F(T_1,T) - F(t,T)$ in T, so, the value of the two trades at T_1 is $(F(T_1,T) - F(t,T))e^{-r(T-T_1)}$. This is another way to get $V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}$.

Prices, Rates, and Other Values			
Symbol	Description	Value	Comments
$F(t,T)$	forward price on asset without dividends	$S_te^{r(T-t)}$	value of K such that a forward has value 0 at t
$F(t,T)$	forward price on asset with dividends q	$S_te^{(r-q)(T-t)}$	q the rate of dividends
$F(t,T)$	forward price on asset generating income I	$(S_t - I)e^{r(T-t)}$	I the value of income at time t
$F(t,T)$	forward price on foreign exchange	$X_te^{(r_{\$}-r_f)(T-t)}$	$r_{\$}, r_f$ the riskless rates of the currencies

Note that $F(t,T)$ does not depend on distribution of S_T. For two stocks with same spot price S_t, same $F(t,T)$.

5.2 Forward ZCB

- Definition: A forward ZCB is a forward contract with maturity T_1 on a ZCB with maturity T_2. In other words, it's a forward contract where the underlying asset is a ZCB maturing at T_2, not a stock.
- Price: The fair price K that gives the forward zero value at time t is

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}.$$

- Proof by replication: Portfolio A: 1 ZCB with maturity T_2, Portfolio B: 1 forward contract with maturity T_1 with delivery price K, plus K ZCBs with maturity T_1.

5.3 Forward Interest Rates

- Definition: The forward rate at time t for period T_1 to T_2, denoted f_{12}, is the rate agreed on at t where one can borrow/lend from T_1 to T_2.
- "Price": f_{12} satisfies

$$e^{r_2(T_2-t)} = e^{r_1(T_1-t)}e^{f_{12}(T_2-T_1)}$$

then

$$f_{12} = \frac{r_2(T_2-t) - r_1(T_1-t)}{(T_2-T_1)}$$

or

$$(1+r_2)^{T_2-t} = (1+r_1)^{T_1-t}(1+f_{12})^{T_2-T_1}$$

where r_i is interest rate for period t to T_i, $i = 1, 2$.

The equation comes from (1) deposit 1 at r_2 until T_2, (2) deposit 1 at r_1 until T_1, and agree at t to deposit $e^{r_1(T_1-t)}$ at f_{12} from T_1 to T_2.

- Proof by no arbitrage: Suppose $f_{12} > \frac{r_2(T_2-t)-r_1(T_1-t)}{(T_2-T_1)}$, we could create an arbitrage opportunity through (1) borrow 1 until T_2 at rate r_2, (2) deposit 1 until T_1 at rate r_1, (3) agree to deposit $e^{r_1(T_1-t)}$ from T_1 to T_2 at rate f_{12}.

5.4 LIBOR

- Libor: London InterBank Offered Rate is the rate at which banks borrow and lend to each other. This rate is defined by its accrual factor (α) and the date at which the rate is set. At current time $t \leq T$, rate $L_t[t, t+\alpha]$ represents the libor rate between t and $t+\alpha$ set at t.
- The Libor rate $L_T[T, T+\alpha]$ for a future date $T > t$ is a random variable
- Banks can deposit (or borrow) N at time t, and receive (or pay back) $N(1+\alpha L_t[t, t+\alpha])$ at time $t+\alpha$.
- For 3 month Libor, $\alpha = 0.25$. If 6 month Libor rate is 4%, I can deposit 1 and receive 1.02 after 6 months.

5.5 Forward Rate Agreement (FRA)

- Definition: In a FRA with maturity T and delivery price K, the buyer agrees to pay αK at time $T+\alpha$ and receive the random amount $\alpha L_T[T, T+\alpha]$ at time $T+\alpha$. Note that $L_T[T, T+\alpha]$ fixes (crystallizes into a value) at time T, but the cashflows occur at time $T+\alpha$. FRA has payout at $T+\alpha$: $\alpha(L_T[T, T+\alpha] - K)$.

- "Price": $L_t[T, T+\alpha]$, the forward libor rate, is the "fair" K that gives the FRA zero value at time t. It satisfies:

$$Z(t, T+\alpha) = Z(t, T)\frac{1}{1+\alpha L_t[T, T+\alpha]}$$

  equivalently:

$$L_t[T, T+\alpha] = \frac{Z(t, T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}$$

- Proof by replication: Portfolio A: 1 FRA, Portfolio B: long 1 ZCB with maturity T, short $(1+\alpha K)$ ZCBs with maturity $T+\alpha$. Both worth $\alpha(L_T[T, T+\alpha] - K)$ at $T+\alpha$.

5.6 Futures

- Differences between futures contracts and forward contracts.
  Attributes of forwards
  - Custom-built
  - Traded over-the-counter
  - Contract specifies either cash or physical settlement
  - Don't involve variational margin: the payout of $S_T - K$ at T.

  Attributes of futures
  - Standardized in maturity, quantity, and quality
  - Traded on exchanges
  - Contract type specifies either cash or physical settlement?but actual delivery is rare, even if contract specifies physical settlement
  - Require variational margin: the payouts are $\Phi(i, T) - \Phi(i-1, T)$ at the end of every day i.

- Futures Convexity Correction: $\Phi(t, T) - F(t, T) \propto Cov(S_T, M_T)$

5.7 Options

- A **European Call option** with strike K and exercise date T on an asset is the right to buy the asset for K at time T. Payoff at time T: $(S_T - K)^+$.

- A **European Put option** with strike K and exercise date T on an asset is the right to sell the asset for K at time T. Payoff at time T: $(K - S_T)^+$.

- A **straddle** is a call plus a put of the same maturity. Payoff at time T: $|S_T - K|$.
- A **call spread** is long a call with strike price K_1 plus short a call with strike price K_2, $K_2 > K_1$, both with the same maturity T.
- A **put spread** is similar to a call spread, but with puts. It is a portfolio consisting of long a put with strike K_2 and short a put with strike $K_1 < K_2$. Note that it is long the option with the higher strike, not the lower strike.
- A **digital call option** on stock with strike price K and maturity T has value at T, 1 if $S_T > K$ and 0 otherwise.
- At time t, a call option with strike K and maturity T is:
  at-the-money if $S_t = K$
  in-the-money if $S_t > K$
  out-the-money if $S_t < K$

- The European call price on a non-dividend paying stock satisfies:

$$(S_t - KZ(t,T))^+ \leq C_K(t,T) \leq S_t$$

- Some Properties of European Call options:
  1. $C_K(t,T) \geq 0$
  2. $C_K(t,T) \leq S_t$
  3. $C_K(t,T) \geq S_t - KZ(t,T)$ [This could be proved through Monotonicity Theorem with Portfolio A be 1 stock, and Portfolio B be 1 call option and K ZCBs.]
  4. $C_{K_1}(t,T) \geq C_{K_2}(t,T)$ for $K_1 \leq K_2$
  5. $C_{K+\Delta K}(t,T) \leq C_K(t,T) \leq C_{K+\Delta K}(t,T) + \Delta K Z(t,T)$ [This could be proved through the construction of call spread.]
  6. $C_K(t,T)$ is a convex function of K: Let $K_1 < K_2$, $\lambda \in (0,1)$, and let $K^* = \lambda K_1 + (1-\lambda)K_2$. Then we have:

     $$C_{K^*}(t,T) \leq \lambda C_{K_1}(t,T) + (1-\lambda)C_{K_2}(t,T)$$

     [This could be proved through the construction of call butterflies, portfolio λK_1 call, $(1-\lambda)K_2$ call, $-1K^*$ call.]
  7. Put-Call Parity: $C_K(t,T) - P_K(t,T) = V_K(t,T)$

6 Replication and No-Arbitrage

6.1 Proof by Replication

Proofs by replication involve creating two portfolios such that at time T the portfolios have the same value if we do not add or remove anything from the portfolios. Then it follows that at current time t the portfolios must also have the same value; otherwise, there would be an opportunity for arbitrage.

In general, a proof by replication for a derivative X generally proceeds as follows:

1. Create two portfolios A and B comprised of derivatives with known value and one containing X.
2. Show that if we do not add or remove anything from the portfolios, then they have the same value at time T.
3. The portfolios now must have the same value at time t. Use this to find the present value of X.

6.2 Proof by No-Arbitrage

Proofs by No-Arbitrage accomplish this by showing that if the derivative currently had any other value, then there would be arbitrage opportunities.

A proof by no-arbitrage to show that $V_X(t) = x$ generally proceeds as follows:

1. Assume that $V_X(t) > x$. Create an arbitrage portfolio with X.
2. Now assume that $V_X(t) < x$. Create an arbitrage portfolio with X. This portfolio is usually the "opposite" of the portfolio you created in (1.)
3. Because of the No-Arbitrage Principle, X must have current value $V_X(t) = x$.

6.3 No-Arbitrage Principle

- A portfolio is a linear combination of asset. A self-funding portfolio allows addition of zero-cost ("at market" trade).
- A portfolio is an **arbitrage portfolio** if it has non-positive value at time t, and has certainly non-negative and possibly positive value at time $T > t$.
- **Assumption of no-arbitrage** - there do not exist any arbitrage portfolios.
- **Monotonicity Theorem** - Assume no-arbitrage. If portfolio A and B are such that $V^A(T, w_i) \geq V^B(T, w_i)$ for all i, then $V^A(t) \geq V^B(t)$. If in addition $V^A(T, w_j) > V^B(T, w_j)$ for some j with $P(\{w_j\}) > 0$, then $V^A(T, w_i) > V^B(T, w_i)$.
- **Corollary to Monotonicity Theorem** If $V^A(T, w_i) = V^B(T, w_i)$ for all i, then $V^A(t) = V^B(t)$.

7 Binomial Tree and Risk-Neutral Probability

7.1 Binomial Tree

With the binomial tree model, we assume the underlying is a stock whose price at time 0 is S_0. At time 1, there are two possible states: up and down. In the up state, the price of the stock becomes $S_0(1 + u)$; in the down state, the price of the stock becomes $S_0(1 + d)$. We also assume that the interest rate is r.

Generally, the steps to solve the binomial tree problem:

- Under the binomial tree model, we can replicate an option with λ stocks and μ ZCBs.
- To solve for the replicating portfolio, equate the payoffs of the derivative and the replicating portfolio in both states of the world.
- Then, to price the derivative, discount the replicating portfolio to the present.

7.2 Risk-Neutral Probability

We did all the work in the binomial tree model without specifying the probability that the stock ends up in the up date. In fact, the option's value is independent of that probability. This is a rather counter-intuitive result.

We want to find the risk-neutral probability p^*, the probability under which the present value of the expected option payout is the only possible arbitrage-free option price. For the binomial tree model, if a stock can go up to $S_0(1+u)$ or down to $S_0(1+d)$ and the interest rate is r ,where $d < r < u$,then

$$p^* = \frac{r-d}{u-d}$$

Risk Neutral Pricing – where prices are discounted as risk-neutral expectation. Under the risk-neutral probability p^*, the price of every market instrument is its discounted expected value. (In lecture, we showed it's valid for options, stock and the ZCBs. By linearity of Expectation, risk-neutral pricing must apply to all portfolios because portfolios are essentially linear combination of assets.) That is,

$$V^A(0) = Z(0,1)E_*(V^A(1))$$

Theorem: The binomial tree is arbitrage-free (i.e. There are no arbitrage portfolios) $d < r < u$.

Proof: ($\rightarrow$) Let us assume that $r \geq u$. That means the return of a ZCB is greater than the return of the asset. So we want to sell one asset for S_0 and invest the money at rate r. So at time 1, we have $S_0(1+r) \geq S_0(1+u) > S_0(1+d)$. So after purchasing back the stock at time 1, we have at least 0 and there is a positive probability of having positive cash left over. This is an arbitrage opportunity. So $r < u$. We can similarly prove that $r > d$ by assuming that $r \leq d$.

($\leftarrow$) There are no portfolios consisting of the asset and ZCBs that are arbitrage portfolios. All portfolios with current value of 0 must consist of v/S_0 stocks and $-v(1+r)$ ZCBs, for some $v \in R$. We consider different cases of the value of v and how the stock behaves and we bound $V(1)$ the value of the portfolio at time 1.

- $v = 0$. This case trivially has $V(1) = 0$.
- $v > 0$. When the stock goes down, we must have $V(1) = v(1+d) - v(1+r) < 0$, with the strict inequality. Otherwise, if $V(1) = 0$ when the the stock goes down, then $V(1) > 0$ when the stock goes up with positive probability, and this forms an arbitrage portfolio. So $d < r$.
- $v < 0$. When the stock goes up, we must have $V(1) = v(1+u) - v(1+r) < 0$ ($v < 0$), with the strict inequality. Otherwise, if $V(1) = 0$ when the stock goes up, then $V(1) > 0$ when the stock goes down with positive probability, and this forms an arbitrage portfolio. So $r < u$.

Moreover, based on the theorem, This also means for the risk-neutral probability: $0 < p^* < 1$, i.e. there exists a p^* such that prices are discounted expected values using p^*.