STAT 123: Quantitative Finance, Spring 2022	Prof. Stephen Blyth

Section 1 – Solution

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Section: Monday 10 - 11AM Office Hour: Monday 11AM - noon

1 Probability Review

Here are some concepts that mentioned in the first lecture that would be used in the future:

- **Fundamental Bridge.** Let A be an event, and I_A be the indicator for event A. Then
$$E[I_A] = P(A)$$
- **Normal Distribution.** The Normal distribution will be used quite often. Familiarize yourself with the PDF and CDF of the standard Normal. Also, for $Z \sim N(0,1)$ the standard Normal, and $W \sim N(\mu, \sigma^2)$, we have
$$\frac{W-\mu}{\sigma} \equiv Z$$
This is useful for standardizing to the standard Normal.
- **Conditional Expectation and Adam's Law.** Refamiliarize yourself with conditional expectation. Remember that a conditional expectation conditioned on random variable X will be a function of X. Also, remember that
$$E[f(X) \cdot Y \mid X] = f(X) \cdot E[Y \mid X]$$
An important property is Adam's Law
$$E[E[Y \mid X]] = E[Y]$$
- **LOTUS.** Law of the Unconscious Statistician. For X a discrete and continuous r.v., respectively:
$$E[g(X)] = \sum_x g(x)P(X=x)$$
$$E[g(X)] = \int_x g(x)f(x)dx$$
- **Log-Normal Distribution.** For $Y \sim \text{LogN}(\mu, \sigma^2)$, then
$$Y \equiv e^Z \quad \text{for} \quad Z \sim N(\mu, \sigma^2)$$
In particular,
$$E(Y) = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$
Hint: You'll need to use LOTUS and complete the square in the exponent inside the integral during the proof.

- **Central Limit Theorem (CLT)**

  Lindeberg–Lévy CLT

  Suppose $\{X_1, \ldots, X_n\}$ is a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$. Then as n approaches infinity, the random variables $\sqrt{n}(\bar{X}_n - \mu)$ converge in distribution to a normal $\mathcal{N}(0, \sigma^2)$

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$$

2 Interest Rates

In this course, assume that interest rates are non-negative unless explicitly specified.

- Compounding frequency: if you invest \$1 at a rate of r_m, compounded m times per year, you have

$$\left(1 + \frac{r_m}{m}\right)^{mT}$$

  after T years.

- Converting between discrete and continuous compounding: if continuous compounding rate is r, the equivalent rate r_m with compounding frequency m satisfies

$$e^{rT} = \left(1 + \frac{r_m}{m}\right)^{mT}$$

3 Numeraire

3.1 Zero Coupon Bond (ZCB)

A zero coupon bond (ZCB) with maturity T is an asset that is worth 1 at time T (and pays nothing else). We denote $Z(t,T)$ as the value at current time t of a ZCB with maturity T. By definition, $Z(T,T) = 1$. The value of $Z(t,T)$ depends on how interest is compounded.

If r is the constant interest rate, and the interest is compounded continuously, then

$$Z(t,T) = e^{-r(T-t)}$$

If r_A is the constant interest rate, and interest is compounded annually, then

$$Z(t,T) = \frac{1}{(1 + r_A)^{T-t}}$$

Hint: The equations could be proved through replication. Consider the following assets at time t, (1) 1 ZCB at time T, (2) $e^{-r(T-t)}$ of cash, deposit at rate r. The two assets are worth the same at time T, then they worth the same at time t.

Throughout this course, we will use the ZCB to represent the current value of 1 at time T. If something is worth 1 at time T, then at time t, it should be worth $Z(t,T)$.

3.2 Money Market Account

The close cousin of ZCBs are Money Market Accounts, which is the value at time t of 1 that was invested at time 0 at rate r. Trivially, $M_0 = 1$.
In general, we have that

$$M_t = \frac{1}{Z(0,t)}$$

3.3 Annuity

An annuity pays value C at times $T_1, \ldots, T_n$. The current value of an annuity is simply the current value of all the future payments of C

$$V = C \sum_{i=1}^{n} Z(t, T_i)$$

4 Stock and Bonds

4.1 Stock

A security giving partial ownership of a company. It may pay a dividend. In this course we will use q to denote the dividend rate, or, the percentage of the stock price the dividend represents. The stock price at time t written as S_t. If we are currently at time t, then we call S_t the spot price.

4.2 Fixed Rate Bond

A contract defined by coupon c and notional N that pays the bondholder cN at a particular frequency until maturity, when N is repaid in full. A floating rate bond will be covered extensively later in the course, and differs from the fixed rate bond in its coupon payments to the bondholder.

5 Derivatives

A *derivative contract* is simply a financial contract between two counter-parties and whose value is derived from the value of an underlying asset or variable. For *financial derivative contract*, the underlying variable is financial asset price, or interest rate, etc.
A lot of this course is learning about different derivatives and how to find their current value.

5.1 Forward

A forward is an agreement between two counter-parties to exchange an asset (e.g. a stock) at time T for fixed price K. T is the *maturity* of the forward, and K is the *delivery (strike) price*. The counter-party that has agreed to buy the underlying (e.g. a stock) is long the forward. The counter-party that has agreed to sell the underlying (e.g. a stock) is short the forward.

Let S_t denote the value of the underlying asset at time t. Then at time T, the value of long the forward is just $V_K(T,T) = S_T - K$. Note that S_T is a random variable.

If the underlying does not pay any income or give any dividends, then the current value of the forward is $S_t - KZ(t,T)$. We generally assume that the interest rate is constant r and is continuously compounded. So the current value is

$$S_t - Ke^{-r(T-t)}$$

The forward price is the value of K such that the value of the forward at time t is zero, $V_K(t,T) = 0$. It is denoted $F(t,T)$. From the above, we get that

$$F(t,T) = S_t e^{r(T-t)} = \frac{S_t}{Z(t,T)}$$

The forward price will change depending on if the underlying has dividends or generates income. Generally, we assume that dividends affect the interest rate r, and income affects the current underlying price S_t.

6 Exercises

6.1 ZCBs and Money Market

The interest rate is $r = 10\%$, compounded annually.

(a) Find $Z(0,3)$ and M_3.

> **Solution:**
>
> $$Z(0,3) = \frac{1}{(1+0.10)^3} \approx 0.751, \quad M_3 = (1+0.10)^3 = 1.331$$

(b) What is the value of the semi-anually compounded interest rate?

> **Solution:**
> We should be indifferent to investing 1 at either interest rate. So
>
> $$(1+0.10)^t = \left(1 + \frac{r'}{2}\right)^{2t}$$
>
> $$\Rightarrow \quad r' = 2\left(\sqrt{1.10} - 1\right) \approx 0.0976 = 9.76\%$$
>
> As a sanity check, the annual rate is higher than the equivalent semi-annual rate. Does that make sense intuitively? (Yes!)

(c) Assume non-negative interest rates which are not necessarily constant. Cross out the values that are unknown (random variables) at time 0. Rank the remaining values.

$$Z(0,1), Z(1,1), Z(1,2), Z(0,10), Z(1,10)$$

Solution:
Random variables at time 0: $Z(1,2), Z(1,10)$

Rank the known values: $Z(0,10) \leq Z(0,1) \leq Z(1,1) = 1$

Note that we can also conclude that $Z(1,10) \leq Z(1,2)$

6.2 Annuity Formula

Recall that at time t the value V of an annuity which pays coupons C at times $T_1, ..., T_n$ is $V = C\sum_{i=1}^{n} Z(t, T_i)$. If the coupons are paid annually for M years and the annually compounded interest rate is fixed at r, then derive V in terms of C and r.

Solution:

$$\begin{aligned} V &= C\sum_{i=1}^{M} \frac{1}{(1+r)^i} \\ &= C\Big(\sum_{i=0}^{\infty} \frac{1}{(1+r)^i}\Big)\Big(\frac{1}{1+r} - \frac{1}{(1+r)^{M+1}}\Big) \\ &= C\Big(\frac{1}{1-\frac{1}{1+r}}\Big)\Big(\frac{(1+r)^M - 1}{(1+r)^{M+1}}\Big) \\ &= \frac{C}{r}\Big(1 - \frac{1}{(1+r)^M}\Big) \end{aligned}$$

6.3 Mortgages as Annuities

A traditional fixed rate mortgage is an asset which pays a fixed cashflow C monthly for N years, where the coupon is chosen such that a loan of size P is repaid in full with interest once the mortgage comes to term.

(a) Assuming the interest rate is fixed at r, derive an expression for monthly payment C.

(b) If $N = 30$ and $r = 0.05$, what fraction of the principal is outstanding after ten years?

Solution:

1. As suggested in the problem name, a mortgage is just an annuity with present value P, the size of the loan. We can use this fact and the result from Ex 2 to solve for the monthly payment C in terms of the other variables. Note that the cashflows are monthly while interest rates are always quoted annually; we must remember to account for this by normalizing r by 12 in the expressions below. The monthly payment C is that which satisfies

$$P = \frac{12C}{r}\Big(1 - \frac{1}{(1+\frac{r}{12})^{12N}}\Big)$$

$$C = \frac{rP}{12\Big(1 - \frac{1}{(1+\frac{r}{12})^{12N}}\Big)}$$

2. We can compute the desired quantity by subtracting from one the present value of the amount of the loan paid off after ten years. Using the annuity formula, the first ten years of monthly payments C has present value V_1, where $V_1 = C\sum_{i=1}^{120} \frac{1}{(1+\frac{r}{12})^i}$. Substituting C, N and r into V_1 we get $V_1 = 0.506P$. This is the fraction of original principal paid off, so the fraction of original principal outstanding is $1 - V_1/P = 0.494$. In present value, roughly half the principal is paid off after ten years.