STAT 123: Quantitative Finance, Spring 2022 — Prof. Stephen Blyth

Section 7 – Solution

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Section: Monday 10 - 11AM — Office Hour: Monday 11AM - noon

1 Fundamental Theorem of Asset Pricing (FTAP)

FTAP: There are no arbitrage portfolios if and only if for a given positive asset ($N_t > 0$ at time t), there exists a risk-neutral probability distribution Q^* such that the ratios of the price of another asset to N_t are martingale under Q^*, that is

$$\frac{D(t,T)}{N_t} = E_{Q^*}[\frac{D(T,T)}{N_T}|S_t]$$

- N_t is the value of the numeraire at time t. We treat the numeraire as the point of price reference.
- Q^* is the risk-neutral distribution with respect to N_t. E_{Q^*} is the expectation with respect to distribution Q^*. Note: Q^* is with respect to N_t. If we choose a different N_t, we must use a different Q^*.
- S_t is simply an asset, such as a stock.
- $D(t,T)$ is the value at time t of a derivative of S_t that has maturity T.

2 Black-Scholes Model

2.1 Assumption

The Black-Scholes model makes several assumptions about the world:

- There are no arbitrage opportunities in the market.
- There is a risk-free rate r at which we can borrow/lend money.
- We can buy/sell any amount of stock S.
- Stock S pays no dividends.
- The market is frictionless. There are no transaction costs.
- The stock process follows a limiting binomial tree. At each step, stock price changes multiplicatively in one of two directions. With each step infnitessimal in size, the distribution of $\log S_T$ is normal. So S_T follows geometric Brownian motion.

2.2 Black-Scholes formula

The Black-Scholes formula is obtained by applying the Fundamental Theorem using a Log-normal risk-neutral distribution for $S_T|S_t$ with respect to the ZCB numeraire:

Note: the motivation for the Log-normal risk-neutral distribution comes from the asymptotic behavior of binomial tree when N goes to infinity (refer to the derivation of lecture on Wednesday). We firstly have

$$\log S_T = \log S_0 + (r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}W$$

where $W \sim N(0,1)$. Generalizing to start at time t instead of time 0, we obtain the risk-neutral distribution:

$$\log S_T|S_t \sim N\ (logS_t + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t))$$

or equivalently

$$S_T|S_t \sim \text{Lognormal}\ (logS_t + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t))$$

from FTAP with $Z(t,T)$ as the numeraire, then

$$C_K(t,T) = Z(t,T) \cdot E_*[(S_T - K)^+|S_t]$$

Let the distribution of stock price be

$$S_T|S_t \sim \text{Lognormal}\ (logS_t + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t))$$

Applying the risk-neutral distribution to the FTAP equation, the Black-Scholes formula says that, the price of a European call option with strike K, maturity T is:

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2)$$

where

$$d_1 = \frac{\log(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

We can also write its price in terms of the forward price:

$$C_K(t,T) = Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2))$$

Using put-call parity, we can also derive the price of an European put option with the same strike and maturity as the call:

$$P_K(t,T) = Z(t,T)(K\Phi(-d_2) - F(t,T)\Phi(-d_1))$$

2.3 Properties of B-S formula

- As $S_t \to \infty$, $C_K(t,T) \to S_t - KZ(t,T)$

  Note that in this case, it's similar to a forward contract. Intuitively, we definitely exercise the call option.

- As $\sigma \to 0$, $C_K(t,T) \to max(0, S_t - KZ(t,T))$

  If $S_t > KZ(t,T)$, $C_K(t,T) \to S_t - KZ(t,T)$, if $S_t < KZ(t,T)$, $C_K(t,T) \to 0$

- As $\sigma \to \infty$, $C_K(t,T) \to S_t$

2.4 Greeks

- Delta: The delta of an option is how much the value of the option changes when the value of the underlying asset changes. This can be expressed as the partial derivative of the option value with respect to S_t.

  From B-S formula:

  $$\frac{\partial C_K(t,T)}{\partial S_t} = \Phi(d_1) \in [0,1]$$

  For ATMF (at-the-money forward) call option, where $K = F(t,T) = S_t e^{r(T-t)}$, the delta is

  $$\Phi(\frac{\sigma\sqrt{T-t}}{2})$$

  Through put-call parity, we know

  $$\frac{\partial C_K(t,T)}{\partial S_t} - \frac{\partial P_K(t,T)}{\partial S_t} = 1$$

- Vega: The vega is how much the value of an option changes with a change in volatility. It is the partial derivative of the value of the option with respect to σ, since σ represents volatility in Black-Scholes.

  From B-S formula:

  $$\frac{\partial C_K(t,T)}{\partial \sigma} = S_t\sqrt{T-t}\phi(d_1) > 0$$

  Through put-call parity, we know

  $$\frac{\partial C_K(t,T)}{\partial \sigma} = \frac{\partial P_K(t,T)}{\partial \sigma}$$

- Gamma: Gamma is the change in delta as S_t changes, since delta is not constant. Gamma can take on extreme values, depending on how close S_t is to K. Gamma is the second partial derivative of the option value with respect to S_t.

- Theta. Theta measure the time-decay of an option. The more time until maturity, the more valuable the option, since there is the "optionality" of the option. So as t approaches T, the value of the option decays. Theta is expressed as the partial derivative of the option value with respect to T.

3 Exercises

3.1 Numeraire

Assume that $S_0 = 100$ and $S_1 = 150$ in State A and $S_1 = 50$ in State B. Which of the following can be used as a numeraire?

(I) The stock

(II) The money market account.

(III) A ZCB with maturity T = 1.

(IV) A ZCB with maturity T = 2.

(V) A forward contract with maturity T = 1 and delivery price 100.

(VI) A call option with strike 25 and maturity T = 1

(VII) A put option with strike 75 and maturity T = 1

Solution:

All of the above, except V and VII.

3.2 FTAP

Assume that $S_0 = 100$ and $S_1 = 120$ in State A and $S_1 = 80$ in State B. The annually compounded interest rate is 10%.

(a) Use martingale conditions to find the risk-neutral probability of State A occurring associated with (i) the money market account numeraire and (ii) the stock numeraire.

(b) Price a 110-strike call option with maturity T = 1 using both of the above measures, and verify that your answers agree.

Solution:

(a) (i) Using "martingale conditions" means applying FTAP. For the money market account numeraire, we use the stock price in the numerator:

$$\begin{aligned}\frac{S_0}{M_0} &= E_*[\frac{S_1}{M_1}|S_0] \\ \frac{100}{1} &= \frac{120}{1.1}p^* + \frac{80}{1.1}(1-p^*) \\ p^*_M = \frac{3}{4}\end{aligned}$$

(ii) For the stock numeraire, we have to use a different asset price for the numerator. Since we know the interest rate, we also know ZCB prices and the money market account. Out of convenience, let's just use the money market account again:

$$\begin{aligned}\frac{M_0}{S_0} &= E_*[\frac{M_1}{S_1}|S_0] \\ \frac{1}{100} &= \frac{1.1}{120}p^* + \frac{1.1}{80}(1-p^*) \\ p^*_S = \frac{9}{11}\end{aligned}$$

(b) Since we have risk-neutral distributions associated with two different numeraires now, we can now price the numerator on the LHS of FTAP in two different ways. Using the money market account numeraire and its associated risk-neutral probability p^*_M:

$$\begin{aligned}\frac{C_{110}(0,1)}{M_0} &= E_*[\frac{C_{110}(1,1)}{M_1}|S_0] \\ \frac{C_{110}(0,1)}{1} &= \frac{10}{1.1}\frac{3}{4} + \frac{0}{1.1}\frac{1}{4} \\ C_{110}(0,1) = 6.81\end{aligned}$$

Using the stock numeraire and its associated risk-neutral probability p^*_S:

$$\begin{aligned}\frac{C_{110}(0,1)}{S_0} &= E_*[\frac{C_{110}(1,1)}{S_1}|S_0] \\ \frac{C_{110}(0,1)}{100} &= \frac{10}{120}\frac{9}{11} + \frac{0}{80}\frac{2}{11} \\ C_{110}(0,1) = 6.81\end{aligned}$$

Both measures indeed give us the same price for the call option.

3.3 Black-Sholes Formula

We have two independent stocks with prices S_{1t} and S_{2t}. Using Black-Scholes assumptions for the risk-neutral distributions of $S_{1T}|S_{1t}$ and $S_{2T}|S_{2t}$, deduce the price of the 'product call', whose payout is given by:

$$PC_K(T,T) = max(S_{1T}S_{2T} - K, 0)$$

Solution:

Define $e^{Y_1} = S_{1T}$ and $e^{Y_2} = S_{2T}$; $X_T \equiv S_{1T}S_{2T} = e^{Y_1+Y_2}$, and $X_t \equiv S_{1t}S_{2t}$. Since Y_1 and Y_2 are independent:

$$Y_1 + Y_2 \sim N((logS_{1t} + (r - \frac{1}{2}\sigma_1^2)(T-t)) + (logS_{2t} + (r - \frac{1}{2}\sigma_2^2)(T-t)), (\sigma_1^2 + \sigma_2^2)(T-t))$$

and hence:

$$logX_T|X_t \sim N(logS_{1t}S_{2t} + (2r - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))(T-t), (\sigma_1^2 + \sigma_2^2)(T-t))$$

Let's define:

$$c_1 = \frac{log(S_{1t}S_{2t}/K) + (2r + \frac{1}{2}(\sigma_1^2 + \sigma_2^2))(T-t)}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}}$$

$$c_2 = c_1 - \sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}$$

pattern matching from the Black-Scholes call pricing result tells us that:

$$\frac{PC_K(t,T)}{Z(t,T)} = E_*[\frac{(X_T - K)^+}{Z(T,T)}|X_t] = S_{1t}S_{2t}e^{2r(T-t)}\Phi(c_1) - K\Phi(c_2)$$

$$PC_K(t,T) = S_{1t}S_{2t}e^{r(T-t)}\Phi(c_1) - Ke^{-r(T-t)}\Phi(c_2)$$

3.4 Greeks

For each of the following portfolios, state whether the delta and vega are ≥ 0, ≤ 0, $= 0$, or indeterminate. $K_2 > K_1$.

1. short a put
2. long a straddle
3. short a forward
4. short a K_1 put, long a K_2 call
5. Long 1 K_1, K_2, K_3 butterfly, long 1 K_2 call. ($K_2 = \frac{K_1+K_3}{2}$)
6. Long 1 K_1, K_2, K_3 butterfly, long 2 K_2 call. ($K_2 = \frac{K_1+K_3}{2}$)

Solution:

1. delta ≥ 0, vega ≤ 0.
2. delta ?, vega ≥ 0.
3. delta ≤ 0, vega $=0$.
4. delta ≥ 0, vega ?.
5. delta ≥ 0, vega ?.
6. delta ≥ 0, vega ≥ 0.