Midterm Review – Practice Questions & Solutions

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Section: Monday 10 - 11AM Office Hour: Monday 11AM - noon

1 Properties of Call / Put Options (Sample Midterm 2 Question 1)

(a) Assume no arbitrage. By considering a portfolio of an amount of ZCB and a call option prove that the value at time $t < T$ of a call option with strike K on a stock that pays no dividends satisfies

$$(S_t - KZ(t,T))^+ \leq C_K(t,T)$$

(b) Hence, prove that if $t \leq T_1 \leq T_2$, $C_K(t,T_2) \geq C_K(t,T_1)$. Hint, consider the case $t = T_1$.

(c) Does the same result hold for puts? That is: prove or find a counterexample to the statement $P_K(t,T_2) \geq P_K(t,T_1)$ for $t \leq T_1 \leq T_2$.

Solution:

(a) portfolio A: long one call option, long K ZCBs with maturity T

portfolio B: long one stock

At T, if $S_T - K \geq 0$ the call is exercised (with value $S_T - K$) and portfolio A has value S_T. If $S_T < K$, the call expires worthless and portfolio A has value K at T

thus portfolio A has value at T equal to $\max\{S_T, K\} \geq S_T$, which is the value of portfolio B at time T. therefore, by the monotonicity theorem,

$$C_K(t,T) + KZ(t,T) \geq S_t, \quad C_K(t,T) \geq S_t - KZ(t,T) \quad ①$$

Besides, since the payout of call option at T is $\max\{S_t - K, 0\} \geq 0$

then $C_K(t,T) \geq 0$ otherwise if $C_K(t,T) < 0$, violate no-arbitrage

Combined with ①, we can get $C_K(t,T) \geq \max\{S_t - KZ(t,T), 0\}$

(b) when $t = T_1$, $C_K(t,T_2) = C_K(T_1,T_2)$ $C_K(t,T_1) = C_K(T_1,T_1)$

by definition $C_K(T_1,T_1)$ means the option to buy the asset at time T_1

so $C_K(T_1,T_1) = \max\{S_{T_1} - K, 0\}$

from (a), we can get $C_K(T_1,T_2) \geq \max\{S_{T_1} - KZ(T_1,T_2), 0\} \geq \max\{S_{T_1} - K, 0\} = C_K(T_1,T_1)$

(because $Z(T_1,T_2) \leq 1$)

then, by monotonicity theorem,

$C_K(t,T_2) \geq C_K(t,T_1)$ holds for $t \leq T_1$

(C) the same result doesn't hold for puts.

find a counter example : payout for the put option with maturity T_2 at T_2 is $\max(K-S_{T_2}, 0)$; payout for the put with maturity T_1 at T_1 is $\max(K-S_{T_1}, 0)$

suppose $K > S_{T_1}$ then the payout is $K-S_{T_1}$ at T_1. after obtaining $K-S_{T_1}$ cash at T_1, we can reinvest until T_2. and get $\frac{K-S_{T_1}}{Z(T_1,T_2)}$ at time T_2

if $\frac{K-S_{T_1}}{Z(T_1,T_2)} > K-S_{T_2}$ (for instance, $S_{T_1} < S_{T_2}$)

besides. $K-S_{T_1} > 0$. so $\frac{K-S_{T_1}}{Z(T_1,T_2)} > 0$. then $\frac{K-S_{T_1}}{Z(T_1,T_2)} > \max(K-S_{T_2}, 0)$

by monotonicity theorem. at $t \leq T_2$ we can get $P_K(t,T_2)$ $P_K(t,T_1)$

thus finding a counterexample for $P_K(t,T_2) \geq P_K(t,T_1)$

2 Ranking Problem in Options (Sample Midterm 2 Question 4)

Consider the following 10 option portfolios. Let $K_{i+1} = K_i + \beta$ for $i = 1, 2$ for a constant $\beta > 0$:

a) K_1 call

b) K_1 put

c) K_3, K_2 put spread

d) K_1, K_2, K_3 call butterfly (Number of K_1 call is 1.)

e) K_1 call that knocks out if $S_T \geq K_2$

f) Digital call with strike K_1 and payout β

g) K_1, K_3 digital call spread both with payout β

h) Digital put with strike K_3 and payout β that knocks out if $S_T \leq K1$

i) A portfolio of $+1$ K_1 call and -2 K_2 calls, all of which knock out if $S_t < K_3$ for any $0 \leq t \leq T$

j) A portfolio of $+1$ K_1 call and -2 K_2 calls, all of which knock out if $S_T > K_3$

For each pair, choose the most appropriate relationship between prices at time $t \leq T$ out of $=, \geq, \leq$, and ?, where ? means the relationship is indeterminate.

- a b

- b c
- c d
- d e
- d j
- g h
- c h
- d g
- b i
- b j

Solution:

- a ? b
- b ? c
- c $\geq$ d
- d $\geq$ e
- d = j
- g = h
- c ? h
- d $\leq$ g
- b $\geq$ i
- b ? j

3 Annuity Formula

Recall that at time t the value V of an annuity which pays coupons C at times $T_1, ..., T_n$ is $V = C\sum_{i=1}^{n} Z(t, T_i)$. If the coupons are paid annually for M years and the annually compounded interest rate is fixed at r, then derive V in terms of C and r

Solution:

$$
\begin{aligned}
V &= C\sum_{i=1}^{M}\frac{1}{(1+r)^i} \\
&= C\Big(\sum_{i=0}^{\infty}\frac{1}{(1+r)^i}\Big)\Big(\frac{1}{1+r}-\frac{1}{(1+r)^{M+1}}\Big) \\
&= C\Big(\frac{1}{1-\frac{1}{1+r}}\Big)\Big(\frac{(1+r)^M-1}{(1+r)^{M+1}}\Big) \\
&= \frac{C}{r}\Big(1-\frac{1}{(1+r)^M}\Big)
\end{aligned}
$$

4 Forward Contract Value

If I went long a forward contract at the fair price and the stock doesn't move, then at time T I will MAKE/LOSE money (circle one). Explain. Assume no income/dividends, and positive interest rate.

Solution:

LOSE. The value of the forward contract with delivery $F(t_0, T)$ at time T is: $V_{F(t_0,T)}(T,T) = (F(T,T) - F(t_0,T))e^{-r(T-T)} = S_T - S_{t_0}e^{r(T-t_0)}$.

Under the assumption that the stock price doesn't move (so $S_T = S_{t_0}$), the value above is negative, so I lost money. The above equation also shows that in order to break even on the forward contract, the stock must grow by at least the risk-free interest rate.

5 Terms Comparison

Comparing Terms For Each Pair – choose the relationship ($\leq, \geq, =, ?$) that best describes the connection between the two terms at time t. Assume $t < T_1 < T_2 < T_3$ and non-negative interest rates.

(a) $Z(t, T_3) \quad Z(T_2, T_3)$

(b) $V_{F(T_1,T_2)}(T_1, T_2) \quad Z(T_1, T_2)$

(c) $F(t, T_1, T_1) \quad F(t, T_2, T_2)$

(d) $L_t[T_1, T_2] \quad L_T[T_1, T_2]$

Solution:

(a) $Z(t, T_3)$? $Z(T_2, T_3)$ because the LHS is a constant and the RHS is an unknown random variable. Even though t to T_3 is a longer time period than T_2 to T_3, interest rates may change (we can only know the interest rate from t to some future time, or the forward rate from one future time to another). $Z(T_2, T_3)$ involves the future spot interest rate from T_2 to T_3 (which is distinct from the forward rate).

(b) $V_{F(T_1,T_2)}(T_1, T_2) \leq Z(T_1, T_2)$ because the LHS is equal to 0 by definition and the RHS is a ZCB price, which is between 0 and 1 inclusive. Note that even though the RHS is a random variable, we still know its bounds, so we can determine this relationship.

(c) $F(t, T_1, T_1) = F(t, T_2, T_2)$ because both sides are equal to 1 by definition.

(d) $L_t[T_1, T_2]$? $L_T[T_1, T_2]$ because the LHS is a constant forward libor rate and the RHS is an unknown future spot libor rate.

6 Forward Rates

(a) The one-year and two-year zero rates are 1% and 2% respectively. What is the one-year forward one-year rate (that is, f_{11})? Assume all rates are annually compounded.

(b) If the two-year forward one-year rate (f_{21}) is 3%, what is the three-year zero rate?

Solution:

(a) $(1+1\%)(1+f_{12}) = (1+2\%)^2$ $\quad f_{12} = 3.01\%$

(b) Suppose three-year zero rate is x

then $(1+2\%)^2(1+f_{23}) = (1+x)^3$ $\quad (1+2\%)^2(1+3\%) = (1+x)^3$ $\quad x = 2.33\%$

7 Forwards and Carry

(a) Use arbitrage arguments involving two forward contracts with maturity T to prove that

$$V_K(t, T) = (F(t, T) - K)e^{-r(T-t)}$$

(b) Verify that $V_K(T, T)$ equals the payout of a forward contract with delivery price K. For an asset that pays no income, substitute the expression for its forward price into the above equation and give an intuitive explanation for the resulting expression.

(c) Suppose at time t_0 you go short a forward contract with maturity T (and with delivery price equal to the forward price). At time t, $t_0 < t < T$, suppose both the price of the asset and interest rates are unchanged. How much money have you made or lost? (This is sometimes called the carry of the trade.) How does your answer change if the asset pays dividends at constant rate q?

Solution:

(a) suppose $V_K(t,T) > (F(t,T)-K)e^{-r(T-t)}$

at time t, we go long a forward contract with delivery price $F(t,T)$ (at no cost)

and go short a forward contract with delivery price K. For the latter we receive

$V_K(t,T)$ at t, which is invested at rate r

at maturity T, the payout of the two forward contract is

$(S_T - F(t,T)) + (K - S_T) = -(F(t,T)-K)$

therefore, the value of portfolio at time T is $V_K(t,T)e^{r(T-t)} - (F(t,T)-K) > 0$

under no arbitrage, there is no way for one's gains to outpace market gains without taking on more risk. so it's impossible for $V_K(t,T) > (F(t,T)-K)e^{-r(T-t)}$

suppose $V_K(t,T) < (F(t,T)-K)e^{-r(T-t)}$

we go long a forward contract with delivery price K, paying $V_K(t,T)$ at t to do so

and short a forward contract with delivery price $F(t,T)$ for no cost

at maturity T, the payout of the two forward contract is

$-(S_T - F(t,T)) - (K - S_T) = (F(t,T)-K)$

therefore, the value of portfolio at time T is $-V_K(t,T)e^{r(T-t)} + (F(t,T)-K) > 0$

under no arbitrage, there is no way for one's gains to outpace market gains without taking on more risk. so it's impossible for $V_K(t,T) < (F(t,T)-K)e^{-r(T-t)}$

so $V_K(t,T) = (F(t,T)-K)e^{-r(T-t)}$

(b) from (a). we get $V_K(T,T) = (F(T,T) - K)\, e^{-r(T-T)} = F(T,T) - K = S_T - K$

the value of a forward at maturity is just the payoff of a forward at time T.

we buy a certain stock with price K. the value is just stock price minus K

(c) since the delivery price equals to the forward price. at time t_0 we have

$V_K(t_0,T) = (F(t_0,T) - K)\, e^{-r(T-t_0)} = 0 \qquad F(t_0,T) = K = S_{t_0} e^{r(T-t_0)}$

at time t: $V_K(t,T) = (F(t,T) - K)\, e^{-r(T-t)} = [S_t e^{r(T-t)} - S_{t_0} e^{r(T-t_0)}]\, e^{-r(T-t)}$

$= S_{t_0}(1 - e^{r(t-t_0)})$

since $t_0 < t < T$, $e^{r(t-t_0)} > 1$ so $V_K(t,T) < 0$

since we short this forward contract, we make money with the amount $S_{t_0}(e^{r(t-t_0)} - 1)$

If the asset pays dividend at constant rate q. then

$V_K(t,T) = -Ke^{-r(T-t)} + S_t e^{-q(T-t)}$ since $F(t,T) = S_t e^{(r-q)(T-t)}$

$V_K(t,T) = (F(t,T) - K)\, e^{-r(T-t)}$ and $F(t_0,T) = S_{t_0} e^{(r-q)(T-t_0)}$

so $V_K(t,T) = [S_{t_0} e^{(r-q)(T-t)} - S_{t_0} e^{(r-q)(T-t_0)}]\, e^{-r(T-t)}$

since $t_0 < t < T$. $T - t < T - t_0$ if $r > q$. then we still make money with the amount of $S_{t_0}[e^{(r-q)(T-t_0)} - e^{(r-q)(T-t)}]\, e^{-r(T-t)}$

if $r < q$. then we lose money with the amount of $S_{t_0}[e^{(r-q)(T-t)} - S_{t_0} e^{(r-q)(T-t_0)}]\, e^{-r(T-t)}$

if $r = q$ then we don't lose or make money, the profit is 0

8 Binomial Tree and Risk-Neutral Probability

(a) Let us say we have a stock such that $S_0 = 110$. At time $T = 1$, in the up state, $S_1 = 135$ and in the down state, $S_1 = 90$. Also $r = 5\%$, compounded annually. We want to find $P_{105}(0, 1)$ the fair price of the 105 put.

(b) What is the risk-neutral probability p^*.

Solution:

(a) Let's try to replicate the put option with stock and bond. Long λ amount of stock and μ bond. To replicate the option payoff at maturity, we have: $\lambda S_1 + \mu = (K - S_1)^+$. Considering the two states at T=1:

$$135\lambda + \mu = 0$$
$$90\lambda + \mu = 15$$

solving for the unknowns we have: $\lambda = -\frac{1}{3}, \mu = 45$. Hence $P_{105}(0,1) = 110\times(-\frac{1}{3})+45\times\frac{1}{1+0.05} = 6.19$.

(b) The present value of the expected payout is

$$Z(0,1)E((K - S_1)^+) = \frac{1}{1.05}(p^* \cdot 0 + (1 - p^*) \cdot 15) = (1 - p^*) \cdot \frac{100}{7}$$

We equate this to the value of the option

$$(1 - p^*) \cdot \frac{100}{7} = \frac{130}{21}$$

$$p^* = \frac{17}{30}$$

9 Butterflies, Condors and Call Ladders (IQF Chapter 7, Exercise 3)

(a) Recall that a call butterfly with strikes $(K_1, K_1+\beta, K_1+2\beta)$, for some fixed $\beta > 0$, is a portfolio consisting of +1 K_1 call, +1 $(K_1+2\beta)$ call and –2 $(K_1+\beta)$ calls. Using put-call parity or otherwise, restate the call butterfly as a portfolio consisting solely of puts.

(b) A call condor is a portfolio consisting of +1 K call, –1 $(K + \beta)$ call, –1 $(K + 2\beta)$ call and +1 $(K + 3\beta)$ call. Draw the payout of the condor, and express the condor as a portfolio consisting solely of call butterflies.

(c) A call ladder consists of +1 K call, –1 $(K+\beta)$ call and –1 $(K+2\beta)$ call. What relationships hold between the prices at time $t \leq T$ of the call ladder, butterfly and condor with common maturity T?

Solution:

(a) for K_1 call: put-call parity: $C_{K_1}(t,T) - P_{K_1}(t,T) = V_{K_1}(t,T)$

for $K_1+2\beta$ call: put-call parity: $C_{K_1+2\beta}(t,T) - P_{K_1+2\beta}(t,T) = V_{K_1+2\beta}(t,T)$

for $K_1+\beta$ call: put-call parity: $C_{K_1+\beta}(t,T) - P_{K_1+\beta}(t,T) = V_{K_1+\beta}(t,T)$

put-call parity states that long one call and short one put equals long one forward contract

since $V_{K_1}(t,T) + V_{K_1+2\beta}(t,T) - 2V_{K_1+\beta}(t,T)$

$= S_t - K_1 Z(t,T) + S_t - (K_1+2\beta) Z(t,T) - 2S_t + 2(K_1+\beta) Z(t,T) = 0$

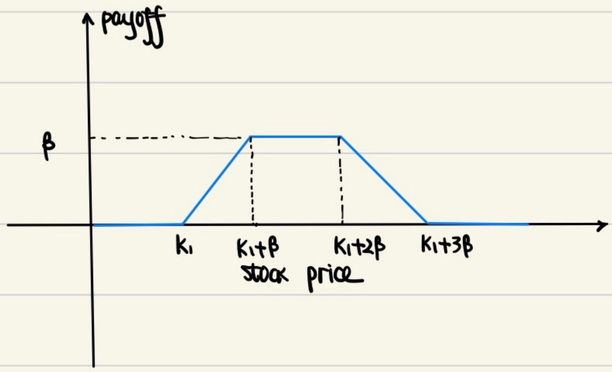
so the position for forward contract in the call butterfly cancel out.

$C_{K_1}(t,T) + C_{K_1+2\beta}(t,T) - 2C_{K_1+\beta}(t,T) = P_{K_1}(t,T) + P_{K_1+2\beta}(t,T) - 2P_{K_1+\beta}(t,T)$

the call butterfly can be regarded as a portfolio consisting solely of puts

which are long 1 K_1 put, long 1 $(K_1+2\beta)$ put. short 2 $(K_1+\beta)$ puts

(b) payout of condor = $\begin{cases} 0 & S_T < K \\ S_T - K & K \le S_T < K+\beta \\ (S_T-K)-(S_T-K-\beta) = \beta & K+\beta \le S_T < K+2\beta \\ (S_T-K)-(S_T-K-\beta)-(S_T-K-2\beta) = -S_T+K+3\beta & K+2\beta \le S_T < K+3\beta \\ (S_T-K)-(S_T-K-\beta)-(S_T-K-2\beta)+(S_T-K-3\beta) = 0 & S_T \geq K+3\beta \end{cases}$



this condor can be expressed as the portfolio: long a call butterfly with strikes $(K_1, K_1+\beta, K_1+2\beta)$ and long a call butterfly with strikes $(K_1+\beta, K_1+2\beta, K_1+3\beta)$

long a call butterfly with strikes $(K_1, K_1+\beta, K_1+2\beta)$ contains long +1 K_1 call, short 2 $(K_1+\beta)$ call. long +1 $(K_1+2\beta)$ call.

long a call butterfly with strikes $(K_1+\beta, K_1+2\beta, K_1+3\beta)$ contains long +1 $(K_1+\beta)$ call. short 2 $(K_1+2\beta)$ call. long +1 $(K_1+3\beta)$ call.

so the portfolio of the two call butterfly contains long +1 K_1 call, short 1 $(K_1+\beta)$ call short 1 $(K_1+2\beta)$ call. long +1 $(K_1+3\beta)$ call. which is the same as condor

(c)

	payout of call ladder	payout of call butterfly	payout of condor
$S_T < K$	0	0	0
$K \le S_T < K+\beta$	$S_T - K$	$S_T - K$	$S_T - K$
$K+\beta \le S_T < K+2\beta$	β	$-S_T + K + 2\beta$	β
$K+2\beta \le S_T < K+3\beta$	$-S_T + K + 3\beta$	0	$-S_T + K + 3\beta$
$S_T \ge K+3\beta$	$-S_T + K + 3\beta$	0	0

when $S_T < K+\beta$ payout of ladder = payout of butterfly = payout of condor

$K+\beta \le S_T < K+2\beta$: $0 < -S_T + K + 2\beta \le \beta$

payout of ladder = payout of condor $\ge$ payout of butterfly

$K+2\beta \le S_T < K+3\beta$ $0 < -S_T + K + 3\beta \le \beta$

payout of ladder = payout of condor $\ge$ payout of butterfly

$S_T \ge K+3\beta$ $-S_T + K + 3\beta \le 0$

payout of ladder $\le$ payout of condor = payout of butterfly

at T. we can get $V^{condor}(T, w_i) \ge V^{ladder}(T, w_i)$ for all i

$V^{condor}(T, w_i) \ge V^{butterfly}(T, w_i)$ for all i

according to monotonicity theorem. $V^{condor}(t) \ge V^{ladder}(t)$ and $V^{condor}(t) \ge V^{butterfly}(t)$

and the relationship between $V^{ladder}(t)$ and $V^{butterfly}(t)$ is uncertain