STAT 123: Quantitative Finance, Spring 2022	Prof. Stephen Blyth

Section 4 – Solution

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Section: Monday 10 - 11AM Office Hour: Monday 11AM - noon

1 No-Arbitrage Principle

- A portfolio is a linear combination of asset. A self-funding portfolio allows addition of zero-cost ("at market" trade).
- A portfolio is an **arbitrage portfolio** if it has non-positive value at time t, and has certainly non-negative and possibly positive value at time $T > t$.
- **Assumption of no-arbitrage** - there do not exist any arbitrage portfolios.
- **Monotonicity Theorem** - Assume no-arbitrage. If portfolio A and B are such that $V^A(T, w_i) \geq V^B(T, w_i)$ for all i, then $V^A(t) \geq V^B(t)$. If in addition $V^A(T, w_j) \geq V^B(T, w_j)$ for some j with $P(\{w_j\}) > 0$, then $V^A(T, w_i) > V^B(T, w_i)$.
- **Corollary to Monotonicity Theorem** If $V^A(T, w_i) = V^B(T, w_i)$ for all i, then $V^A(t) = V^B(t)$.

2 Derivatives

A *derivative contract* is simply a financial contract between two counter-parties and whose value is derived from the value of an underlying asset or variable. For *financial derivative contract*, the underlying variable is financial asset price, or interest rate, etc. A lot of this course is learning about different derivatives and how to find their current value.

2.1 Options

- A **European Call option** with strike K and exercise date T on an asset is the right to buy the asset for K at time T. Payoff at time T: $(S_T - K)^+$.
- A **European Put option** with strike K and exercise date T on an asset is the right to sell the asset for K at time T. Payoff at time T: $(K - S_T)^+$.
- A **straddle** is a call plus a put of the same maturity. Payoff at time T: $|S_T - K|$.
- A **call spread** is long a call with strike price K_1 plus short a call with strike price K_2, $K_2 > K_1$, both with the same maturity T.
- A **put spread** is similar to a call spread, but with puts. It is a portfolio consisting of long a put with strike K_2 and short a put with strike $K_1 < K_2$. Note that it is long the option with the higher strike, not the lower strike.

- A **digital call option** on stock with strike price K and maturity T has value at T, 1 if $S_T > K$ and 0 otherwise.

- At time t, a call option with strike K and maturity T is:
at-the-money if $S_t = K$
in-the-money if $S_t > K$
out-the-money if $S_t < K$

- The European call price on a non-dividend paying stock satisfies:

$$(S_t - KZ(t,T))^+ \leq C_K(t,T) \leq S_t$$

- Some Properties of European Call options:

1. $C_K(t,T) \geq 0$
2. $C_K(t,T) \leq S_t$
3. $C_K(t,T) \geq S_t - KZ(t,T)$ [This could be proved through Monotonicity Theorem with Portfolio A be 1 stock, and Portfolio B be 1 call option and K ZCBs.]
4. $C_{K_1}(t,T) \geq C_{K_2}(t,T)$ for $K_1 \leq K_2$
5. $C_{K+\Delta K}(t,T) \leq C_K(t,T) \leq C_{K+\Delta K}(t,T) + \Delta K Z(t,T)$ [This could be proved through the construction of call spread.]
6. $C_K(t,T)$ is a convex function of K: Let $K_1 < K_2$, $\lambda \in (0,1)$, and let $K^* = \lambda K_1 + (1-\lambda)K_2$. Then we have:

$$C_{K^*}(t,T) \leq \lambda C_{K_1}(t,T) + (1-\lambda)C_{K_2}(t,T)$$

[This could be proved through the construction of call butterflies, portfolio λK_1 call, $(1-\lambda)K_2$ call, $-1K^*$ call.]
7. Put-Call Parity: $C_K(t,T) - P_K(t,T) = V_K(t,T)$

3 Exercises

3.1 No-Arbitrage Portfolios

A portfolio currently has value 10 at time 0. We know that the portfolio is guaranteed to have value 11 at time 1.

(a) Is this an arbitrage portfolio? Does this portfolio violate the no-arbitrage principle?
(b) What if we know that the interest rate was 10%? What if we knew that the interest rate was 8%?

Solution:

(a) This is not an arbitrage portfolio, since at current time it has a positive value. We cannot say if this portfolio violates the arbitrage principle or not without more information.

(b) If interest rate is 10%, then the growth of value from 10 to 11 is just what we expect the growth to be if we simply invested the money at 10%. So the portfolio does not violate the no-arbitrage principle.

If there interest is only 8%, then the guaranteed growth of 1 exceeds what we could get from investing. So this portfolio violate the no-arbitrage principle. We can borrow 10 at rate 8% to go long this portfolio at time 0. Note however that even when the interest rate is 8%, the portfolio is still not an arbitrage portfolio. Not all portfolios that violate the arbitrage principle are arbitrage portfolios.

3.2 Payout Graph for Options

With the aid of payout graphs show that:

- $C_{K_1}(t,T) \geq C_{K_2}(t,T) \geq 0$ for $K_1 < K_2$
- $0 \leq P_{K_1}(t,T) \leq P_{K_2}(t,T)$ for $K_1 < K_2$
- Put-Call Parity: $C_K(t,T) - P_K(t,T) = V_K(t,T)$
- bound on a call spread: $C_{K_1}(t,T) - C_{K_2}(t,T) \leq Z(t,T)(K_2 - K_1)$, where $K_1 < K_2$

*Please do not rely on picture proofs in exams or problem sets unless explicitly requested.

Solution:

3.3 No-Arbitrage Proof for Put-Call Parity

Let $C_K(0,T)$, $P_K(0,T)$ be the current price of a European call and put option. Each contract is for one unit of the same underlying stock. Suppose D is the present value of the dividend generated by the underlying stock during the life of the options, and it is paid at some time t between 0 and T. Assume continuously compounded interest rate r. Prove that:

$$C_K(0,T) - P_K(0,T) + D = S_0 - Ke^{-rT}$$

Solution:
Consider the following two portfolios:

A: long one call and invest $Ke^{-rT} + D$ amount of cash.
B: long one put and one stock.

Payout for portfolio A: $max(S_T - K, 0) + K + De^{rT} = max(S_T, K) + De^{rT}$. Payout for portfolio B: $max(K - S_T, 0) + S_T + De^{rT} = max(S_T, K) + De^{rT}$.

By the corollary of monotonicity theorem, we deduce that:
$C_K(0,T) + (Ke^{-rT} + D = P_K(), T) + S_0$
i.e. $C_K(0,T) - P_K(0,T) + D = S_0 - Ke^{-rT}$