Section 3 – Solution

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Section: Monday 10 - 11AM Office Hour: Monday 11AM - noon

1 Derivatives

A *derivative contract* is simply a financial contract between two counter-parties and whose value is derived from the value of an underlying asset or variable. For *financial derivative contract*, the underlying variable is financial asset price, or interest rate, etc. A lot of this course is learning about different derivatives and how to find their current value.

1.1 Forward ZCB

- Definition: A forward ZCB is a forward contract with maturity T_1 on a ZCB with maturity T_2. In other words, it's a forward contract where the underlying asset is a ZCB maturing at T_2, not a stock.
- Price: The fair price K that gives the forward zero value at time t is

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}.$$

- Proof by replication: Portfolio A: 1 ZCB with maturity T_2, Portfolio B: 1 forward contract with maturity T_1 with delivery price K, plus K ZCBs with maturity T_1.

1.2 Forward Interest Rates

- Definition: The forward rate at time t for period T_1 to T_2, denoted f_{12}, is the rate agreed on at t where one can borrow/lend from T_1 to T_2.
- "Price": f_{12} satisfies

$$e^{r_2(T_2-t)} = e^{r_1(T_1-t)} e^{f_{12}(T_2-T_1)}$$

then

$$f_{12} = \frac{r_2(T_2-t) - r_1(T_1-t)}{(T_2-T_1)}$$

or

$$(1+r_2)^{T_2-t} = (1+r_1)^{T_1-t}(1+f_{12})^{T_2-T_1}$$

where r_i is interest rate for period t to T_i, $i = 1, 2$.

The equation comes from (1) deposit 1 at r_2 until T_2, (2) deposit 1 at r_1 until T_1, and agree at t to deposit $e^{r_1(T_1-t)}$ at f_{12} from T_1 to T_2.

- Proof by no arbitrage: Suppose $f_{12} > \frac{r_2(T_2-t)-r_1(T_1-t)}{(T_2-T_1)}$, we could create an arbitrage opportunity through (1) borrow 1 until T_2 at rate r_2, (2) deposit 1 until T_1 at rate r_1, (3) agree to deposit $e^{r_1(T_1-t)}$ from T_1 to T_2 at rate f_{12}.

1.3 LIBOR

- Libor: London InterBank Offered Rate is the rate at which banks borrow and lend to each other. This rate is defined by its accrual factor (α) and the date at which the rate is set. At current time $t \leq T$, rate $L_t[t, t+\alpha]$ represents the libor rate between t and $t+\alpha$ set at t.
- The Libor rate $L_T[T, T+\alpha]$ for a future date $T > t$ is a random variable
- Banks can deposit (or borrow) N at time t, and receive (or pay back) $N(1+\alpha L_t[t, t+\alpha])$ at time $t+\alpha$.
- For 3 month Libor, $\alpha = 0.25$. If 6 month Libor rate is 4%, I can deposit 1 and receive 1.02 after 6 months.

1.4 Forward Rate Agreement (FRA)

- Definition: In a FRA with maturity T and delivery price K, the buyer agrees to pay αK at time $T+\alpha$ and receive the random amount $\alpha L_T[T, T+\alpha]$ at time $T+\alpha$. Note that $L_T[T, T+\alpha]$ fixes (crystallizes into a value) at time T, but the cashflows occur at time $T+\alpha$. FRA has payout at $T+\alpha$: $\alpha(L_T[T, T+\alpha] - K)$.
- "Price": $L_t[T, T+\alpha]$, the forward libor rate, is the "fair" K that gives the FRA zero value at time t. It satisfies:
$$Z(t, T+\alpha) = Z(t, T)\frac{1}{1+\alpha L_t[T, T+\alpha]}$$
equivalently:
$$L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}$$
- Proof by replication: Portfolio A: 1 FRA, Portfolio B: long 1 ZCB with maturity T, short $(1+\alpha K)$ ZCBs with maturity $T+\alpha$. Both worth $\alpha(L_T[T, T+\alpha] - K)$ at $T+\alpha$.

1.5 Futures

- Differences between futures contracts and forward contracts.
Attributes of forwards
  - Custom-built
  - Traded over-the-counter
  - Contract specifies either cash or physical settlement
  - Don't involve variational margin: the payout of $S_T - K$ at T.

Attributes of futures

- Standardized in maturity, quantity, and quality
  - Traded on exchanges
  - Contract type specifies either cash or physical settlement?but actual delivery is rare, even if contract specifies physical settlement
  - Require variational margin: the payouts are $\Phi(i,T) - \Phi(i-1,T)$ at the end of every day i.

- Futures Convexity Correction: $\Phi(t,T) - F(t,T) \propto Cov(S_T, M_T)$

1.6 Options

- A **European Call option** with strike K and exercise date T on an asset is the right to buy the asset for K at time T. Payoff at time T: $(S_T - K)^+$.

- A **European Put option** with strike K and exercise date T on an asset is the right to sell the asset for K at time T. Payoff at time T: $(K - S_T)^+$.

- A **straddle** is a call plus a put of the same maturity. Payoff at time T: $|S_T - K|$.

- At time t, a call option with strike K and maturity T is:
  at-the-money if $S_t = K$
  in-the-money if $S_t > K$
  out-the-money if $S_t < K$

2 Exercises

2.1 Forward Rates and ZCB

(1) The 3-year rate is 3%, compounded quarterly. The 5-year rate is 5%, compounded quarterly rate. Find the forward rate from time $T = 3$ to $T = 5$.

(2) Suppose that currently at time $t = 0$, we have that $Z(0,1) = 0.8$ and $Z(0,2) = 0.7$. What is the value of $Z(1,2)$?

Solution:

(1)
$$(1 + \frac{0.03}{4})^{12}(1 + \frac{r}{4})^8 = (1 + \frac{0.05}{4})^{20}$$
And we could get $r = 0.0801863$.

(2) This is a trick question. $Z(1,2)$ is a random variable, whose value we do not know. The value of $Z(1,2)$ will depend on the interest rate from time $t = 1$ to $t = 2$.

Note that the forward rate is not what the interest rate will be, but rather the rate that we can borrow at in the future that is agreed upon right now between two parties.

2.2 Forward Rate Agreement

Suppose $L_t[T, T+\alpha] > \frac{Z(t,T)-Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}$. Show there exists an arbitrage opportunity that we could make money.

Solution:

The general idea is that the fixed rate L_t is too high relative to its fair value, so we want to enter into a FRA and receive fixed (pay floating).

1. At time t: sell FRA with delivery $L_t[T, T+\alpha]$, paying floating. FRA can be entered at no cost. Go long 1 ZCB with maturity T and short $\frac{Z(t,T)}{Z(t,T+\alpha)}$ ZCB with maturity $T+\alpha$. Hence we have hold a portfolio with no cost. [Recall that when we prove by replication in lecture, we did: Portfolio A: 1 FRA, Portfolio B: long 1 ZCB with maturity T, short $(1+\alpha K)$ ZCBs with maturity $T+\alpha$. Both worth $\alpha(L_T[T, T+\alpha] - K)$ at $T+\alpha$.]

2. At time T, the ZCB of maturity T gives us \$ 1. Put it in the bank.

3. At time $T+\alpha$, the \$ 1 invested at T has grown to $1 + \alpha L_T[T, T+\alpha]$. Need to pay \$ 1 for the ZCB of maturity $T+\alpha$. The FRA payout is $\alpha(L_t[T, T+\alpha] - L_T[T, T+\alpha])$. The borrowing we need to payback is $\frac{Z(t,T)-Z(t,T+\alpha)}{Z(t,T+\alpha)}$. Hence the total gain from the portfolio is:

$$
\begin{aligned}
&1 + \alpha L_T[T, T+\alpha] - 1 + \alpha(L_t[T, T+\alpha] - L_T[T, T+\alpha]) - \frac{Z(t,T) - Z(t,T+\alpha)}{Z(t,T+\alpha)} \\
&= \alpha L_t[T, T+\alpha] - \frac{Z(t,T) - Z(t,T+\alpha)}{Z(t,T+\alpha)} \\
&> 0
\end{aligned}
$$

2.3 LIBOR

Suppose two companies are offered the following interest rates for a one-year loan, with same principal, compounding (annual).

	Floating	Fixed
A	Libor+0.1%	11%
B	Libor+0.7%	12%

Suppose that Company A wants to pay floating rates, while Company B wants to pay fixed rates. Can you design a FRA such that both party improves and is same attractive to both parties? In what condition such improvement can not be achieved?

Solution:

If Company A pays floating rates with its own offered rate, while Company B pays fixed rates with its own offered rate, the sum of the rates they need to pay to bank is LIBOR+12.1%. If Company A instead borrows at fixed rate from the bank, while B borrows at floating, then the sum of the rates should be 11% + LIBOR + 0.7% = LIBOR + 11.7%. So it will be cheaper if Company A pays fixed rate to the bank, while Company B pays floating rate to the bank. The total saving will be 0.4% times the principal. We can use FRA between the two companies in order to let them have their preferred type of interest. There will be a 0.4% rate decrease in total, so both companies should pay 0.2% less than the original offered rate that they prefer. So They can enter a FRA as following:

- A pays LIBOR
- B pays a fixed rate of 11.1%

Then the net payment for Company A is: 11%+Libor-11.1% = Libor - 0.1%. Net payment for Company B is: Libor + 0.7% +11.1% - Libor = 11.8%. Both parties are 0.2% better off.

In order to make the FRA attractive, we need that the companies can benefit by exchange their interest rate type with the bank. So we need that the difference of floating rates should be small than the difference of fixed rate, which makes fixed rate relatively cheaper for Company A.

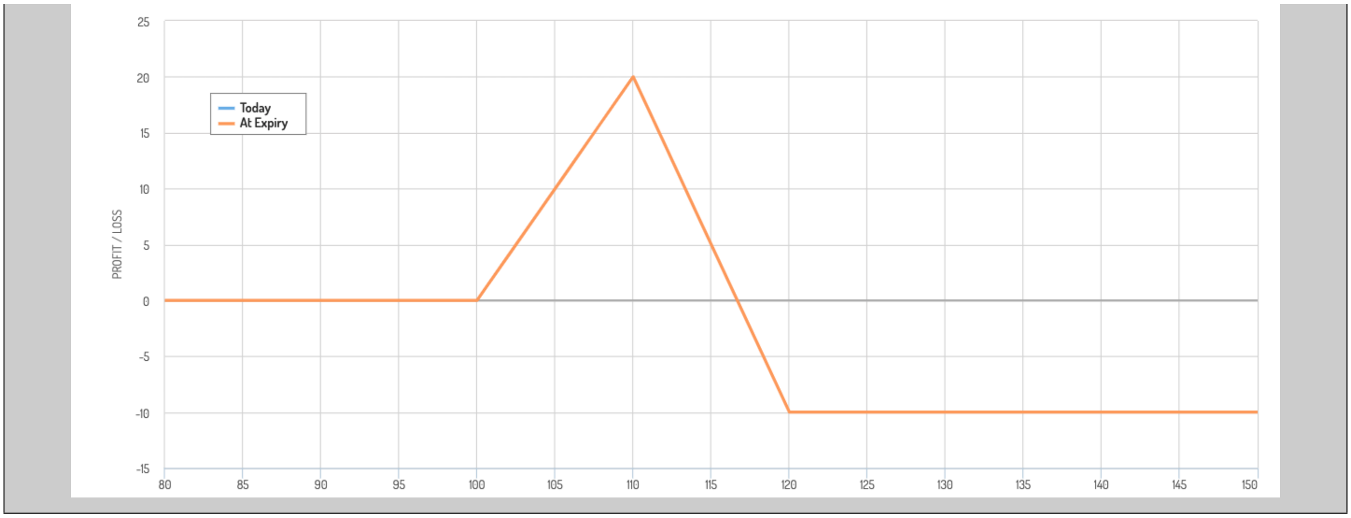
2.4 Option

Draw the payout $g(S_T)$ of an "injured" call butterfly, which consists of:

- +2 call options with strike 100
- -5 call options with strike 110
- +3 call options with strike 120

all with maturity T.

Solution:



25
20
15
10
5
0
-5
-10
-15
PROFIT / LOSS
Today
At Expiry
80
85
90
95
100
105
110
115
120
125
130
135
140
145
150