

# Calculus

(Visualizing the Gradient and the Laplacian)



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Interactive Animation: [Visualizing the Gradient and the Laplacian](#)

# Contents

1	Introduction: Scalar Functions	2
2	The Gradient Vector Field	4
3	The Divergence of a Vector Field	9
4	The Laplacian of a Scalar Function	12
5	Guide for Using the Interactive Animation	17

# 1 Introduction: Scalar Functions

In **single-variable calculus**, we typically study functions that take in one real number and output another. If  $f$  is such a function, we represent it as:

$$f : \mathbb{R} \rightarrow \mathbb{R}. \quad (1)$$

To visualize this, we use two axes: one for the input  $x$  and one for the output  $f(x)$ . The result is a **curve** in a 2D plane.

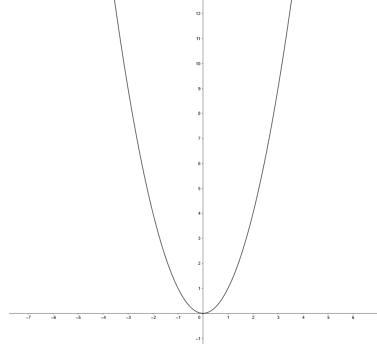


Figure 1: Graph of  $f(x) = x^2$ .

However, we can also consider functions that depend on **two variables** ( $x$  and  $y$ ). In these cases, we study **scalar functions** of the form:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}. \quad (2)$$

This notation tells us that  $f$  takes a point  $(x, y)$  from the **domain**  $\mathbb{R}^2$  (the plane) and assigns it a single real number from the **codomain**  $\mathbb{R}$ .

For instance, we can define a function  $f(x, y) = x^2 + y^2 + 2$ . This function tells us that at the point  $(2, 4)$ , the image is  $f(2, 4) = (2)^2 + (4)^2 + 2 = 4 + 16 + 2 = 22$ .

These functions are very important not only in mathematics, but also in physics, as they can represent many different physical quantities. The function  $f$  could give us at each point the temperature of a plane, the modulus of an electric field in the plane, etc.

## Representing Scalar Functions

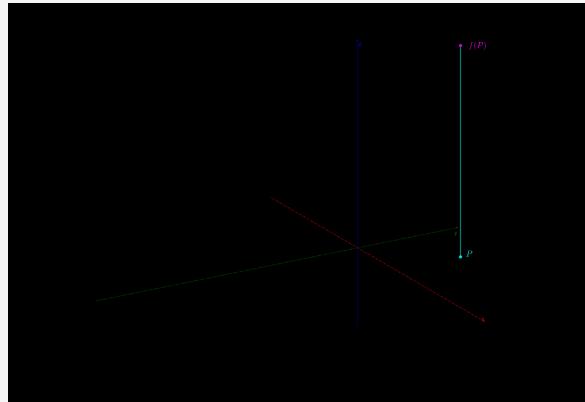
To represent a function  $f(x, y)$  graphically, we need **three dimensions**. We use two horizontal axes to represent the input coordinates  $(x, y)$  and a vertical axis (usually the  $z$ -axis) to represent the value of the function.

1. Choose a point  $P = (x, y)$  in the  $xy$ -plane.
2. Compute the value  $f(x, y)$ .
3. Draw the point  $(x, y, f(x, y))$  in 3D space.
4. Repeat this for all points in the domain to form a **surface**.

### Example 1.1. Representing the Function $f(x, y) = x^2 + y^2 + 2$

We want to visualize the function  $f(x, y) = x^2 + y^2 + 2$ .

1. We choose a point in the plane, for example,  $P = (1, 1)$ .
2. The value of the function at that point is  $f(1, 1) = 1^2 + 1^2 + 2 = 4$ .
3. We plot the point  $(1, 1, 4)$  in 3D space. We can draw a vertical segment of height  $f(1, 1) = 4$  starting from  $P$  to visualize it better.



4. If we perform this process for every point  $(x, y)$ , we obtain a surface known as a **paraboloid**.

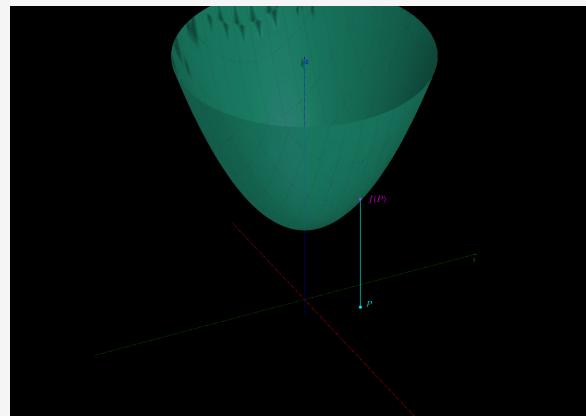


Figure 2: Graphical representation of  $f(x, y) = x^2 + y^2 + 2$  as a 3D surface.

■

### How to Study the Behavior of a Scalar Function

While the 3D surface gives us a global view of the function, it is often difficult to determine exactly how the function is changing just by looking at the graph.

To study the behavior of  $f$  it is useful to think of the graph of  $f$  as if it were a mountain, and we were hiking on it. If you were standing at a point  $P$  on this mountain, you might want to know:

- what direction should I follow to get to the highest point?
- how steep is the path I want to follow?
- am I currently at a peak or a valley?

To answer these questions, we move beyond simple visualization and use two fundamental tools of vector calculus: the **Gradient** and the **Laplacian**.

## 2 The Gradient Vector Field

As we have seen, a scalar function  $f(x, y)$  creates a surface in 3D space. However, to understand how the function changes at any given point  $P = (x, y)$  in the domain, we use the **gradient**.

The **gradient** of a function  $f(x, y)$  is a **vector field** that points in the **direction of the steepest ascent/increase** of the function. It is denoted by  $\vec{\nabla}f$  and is defined as:

$$\vec{\nabla}f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right). \quad (3)$$

While the function  $f$  takes two coordinates and gives a single number (a *scalar*), the gradient takes two coordinates and gives a **vector**. This means that at every point  $(x, y)$  in the plane, there is an associated arrow.

**Example 2.1. The Gradient of  $f(x, y) = x^2 + y^2 + 2$**  Let's consider the function

$$f(x, y) = x^2 + y^2 + 2, \quad (4)$$

then, its gradient is given by

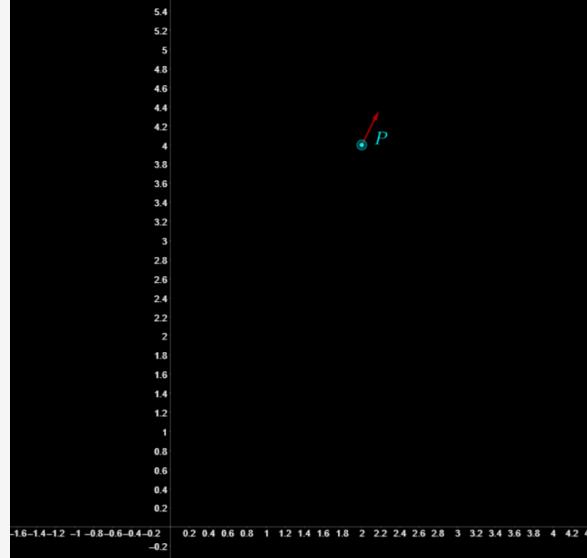
$$\vec{\nabla}f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x, 2y). \quad (5)$$

This vector tells us what direction we should take at any point  $(x, y)$  to reach the highest point.

For instance, if we consider the point  $(1, 2)$ , the gradient is

$$\vec{\nabla}f(1, 2) = (2 \cdot 1, 2 \cdot 2) = (2, 4). \quad (6)$$

We can draw this as a vector coming out of the point  $(1, 2)$  and pointing in the direction  $(2, 4)$



If we do this for many different points, we get the **gradient vector field** of  $f$

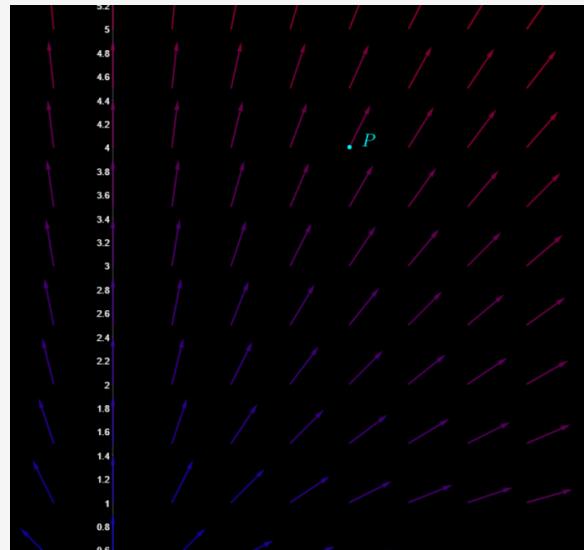


Figure 3: Caption

■

### Visualizing the Gradient: The Vector Field

As we have seen in the example, to visualize the gradient, we draw these vectors directly on the  $xy$ -plane. However, there are two main challenges when trying to create a clear picture:

- 1. The Need for a Grid:** If we were to draw a vector at *every* single point in the plane, the screen would be completely filled with vectors, and we wouldn't be able to see anything.

To solve this, we define a **grid of points** with a specific **separation**. We only compute and draw the gradient at these specific grid intersections.

2. **Normalization and Overlapping:** In many functions, the gradient vectors can become very long, causing them to overlap and create a messy visualization.

To fix this, we **normalize** the vectors so they all have a manageable length, and we use **color coding** to represent their actual magnitude (the norm).

- **Blue vectors** represent a **small norm**, meaning the function is relatively flat in that area.
- **Red vectors** represent a **large norm**, indicating a very steep slope.

### Interpretation of the Gradient

As we have commented, the gradient has a very intuitive interpretation. The arrows always point towards the direction of **steepest ascent**, namely, if we think of the graph of  $f$  as a mountain, the arrows **tell us where to move if we want to reach the top of the mountain**. Similarly, if we go in the direction **opposite to the arrows**, we will reach the lowest points of the mountain.

As a summary:

- **Gradient  $\vec{\nabla}f$ :** The arrows point towards the direction where the mountain (the graph of  $f$ ) is highest.
- **Negative Gradient  $-\vec{\nabla}f$ :** If we change the direction of the arrows given by the gradient  $\vec{\nabla}f$ , we get arrows that **guide us towards the lowest points of the graph**.

#### Example 2.2. Interpretation of the Gradient

Let's consider the function

$$f(x, y) = \sin(x) + \cos(y) + 2. \quad (7)$$

Its graph looks like this:

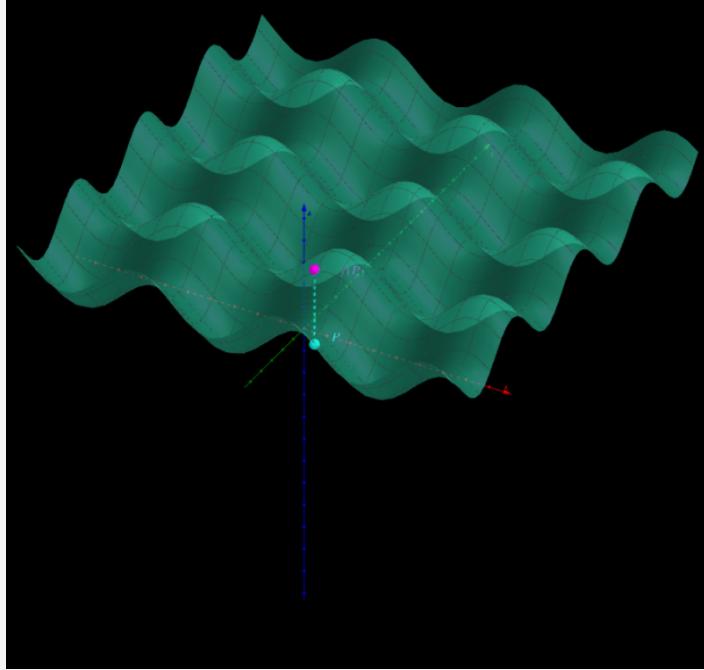
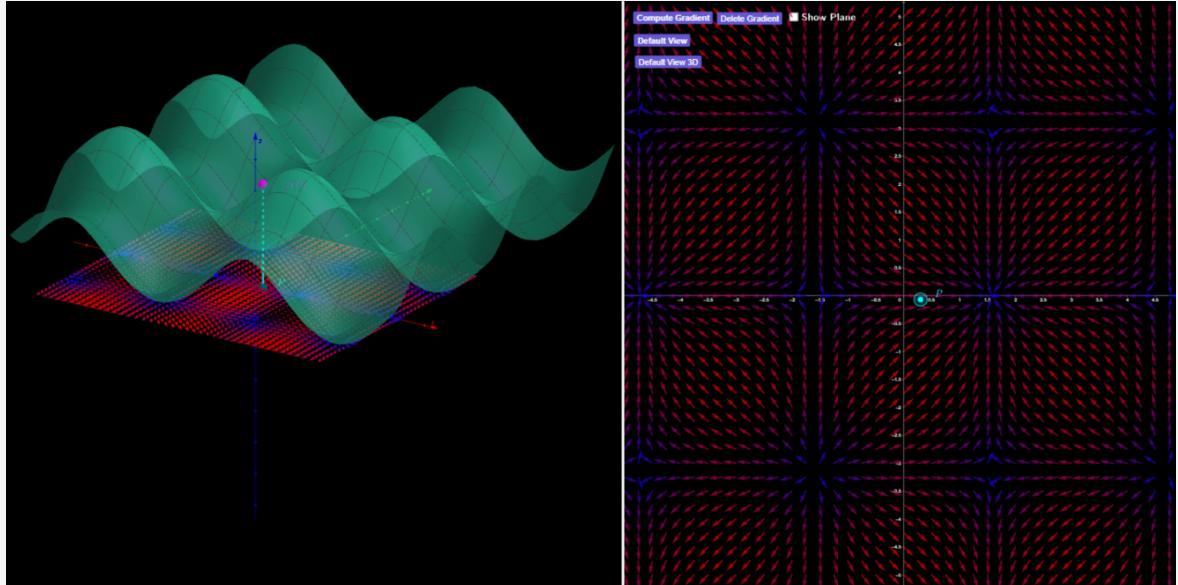


Figure 4: Caption

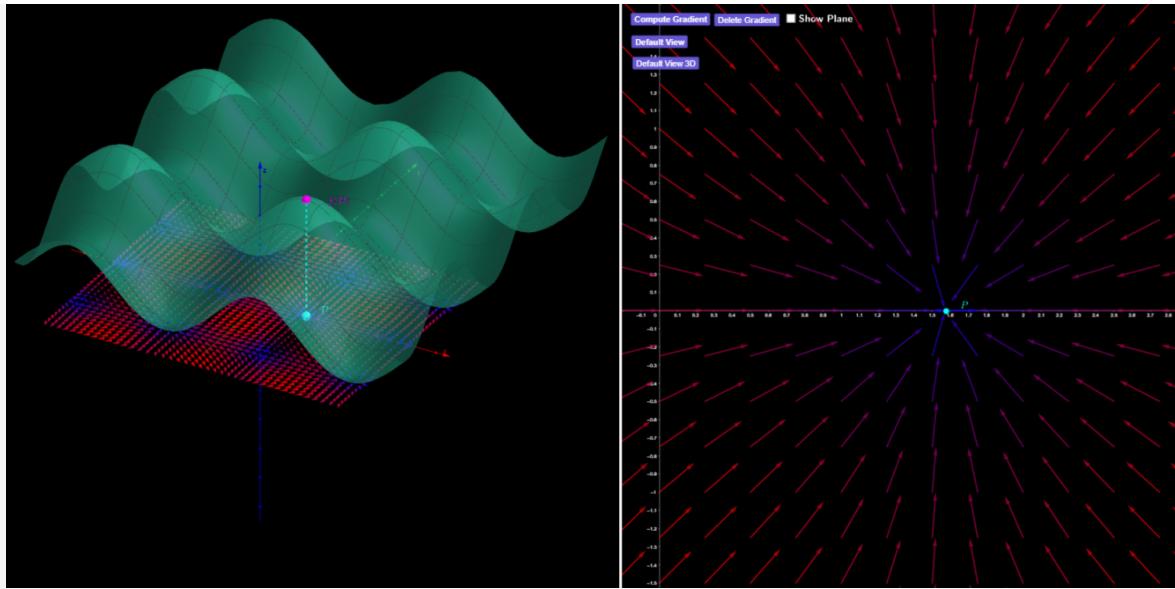
The gradient of the function is given by

$$\vec{\nabla} f(x, y) = (\cos(x), -\sin(y)). \quad (8)$$

If we plot the **gradient vector field**, we get the following:

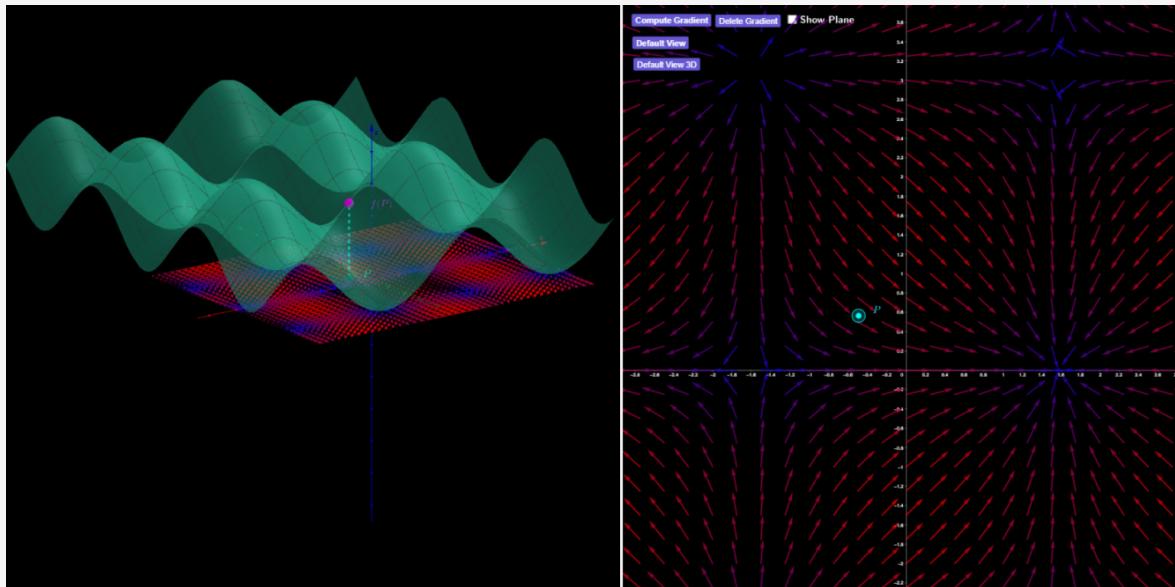


Now, if we move the point  $P$  following the arrows, we reach the following point

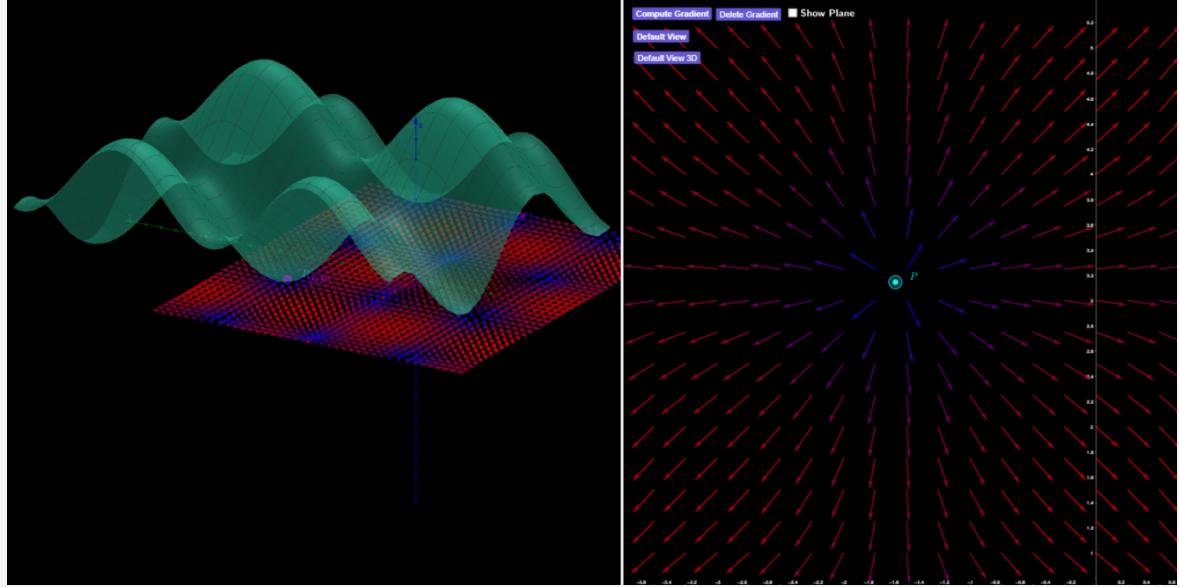


which as we can see, is **one of the peaks of the function.**

If we put  $P$  at some other position



and this time we **go against the arrows**, we reach the following position



which is one of the **valleys** of the function. ■

### 3 The Divergence of a Vector Field

A **vector field**

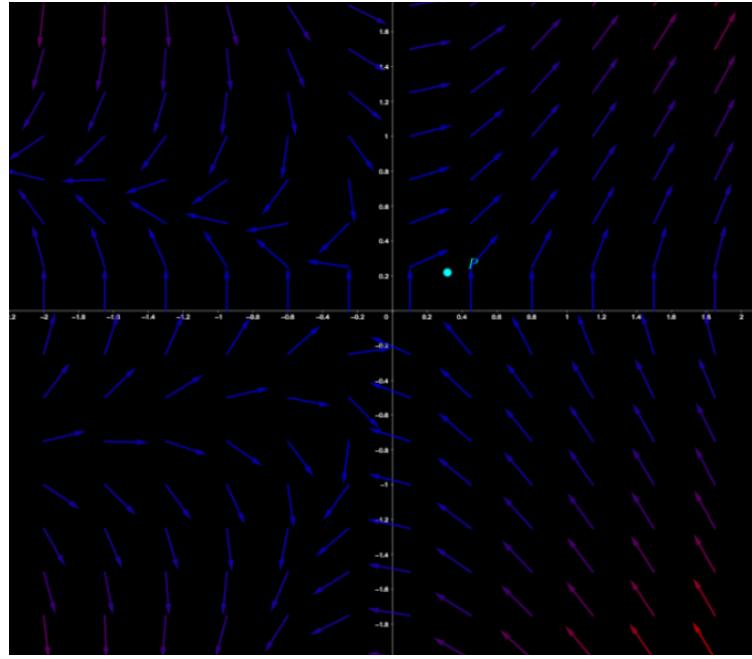
$$\vec{v}(x, y) = (v_x(x, y), v_y(x, y)) \quad (9)$$

is a function which takes in a **point**  $P = (x, y)$  and gives us a **vector**  $\vec{v}(x, y)$ .

For example, we can define a vector field  $v(x, y)$  as

$$\vec{v}(x, y) = (y^3 + 2xy, 3xy^2 + x^2), \quad (10)$$

which can be represented similarly to how we represented the gradient, by drawing the corresponding arrow on top of various points



To analyze **how much the arrows point towards a certain point**, we define the **divergence operator** as follows:

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}. \quad (11)$$

If we consider the vector field defined in Equation 10, the **divergence** is given by

$$\vec{\nabla} \cdot \vec{v}(x, y) = 6xy + 2y. \quad (12)$$

Notice that **the divergence gives us a number**. This number measures **how much the vectors point away from the point  $P = (x, y)$** . You can think of it as a measure of how “unpopular” the point  $P$  is with respect to the vectors.

The divergence of  $\vec{v}$  is also denoted by

$$\vec{\nabla} \cdot \vec{v} = \text{div}(\vec{v}). \quad (13)$$

- **$\text{div}(\vec{v}) > 0$** : If the divergence is **positive**, then, in general, the arrows are **pointing away from the point  $P$** , namely, the point is **unpopular**. We say that the point acts as a **source**.
- **$\text{div}(\vec{v}) < 0$** : If the divergence is **negative**, then, in general, the arrows are **pointing towards the point  $P$** , namely, the point is **popular**. We say that the point acts as a **sink**.
- **$\text{div}(\vec{v}) = 0$** : If the divergence is **exactly 0**, then, the arrows are **pointing away from and towards the point  $P$  in exactly the same amount**, namely, the point is **neither popular nor unpopular**.

### Example 3.1. Divergence of a Vector Field

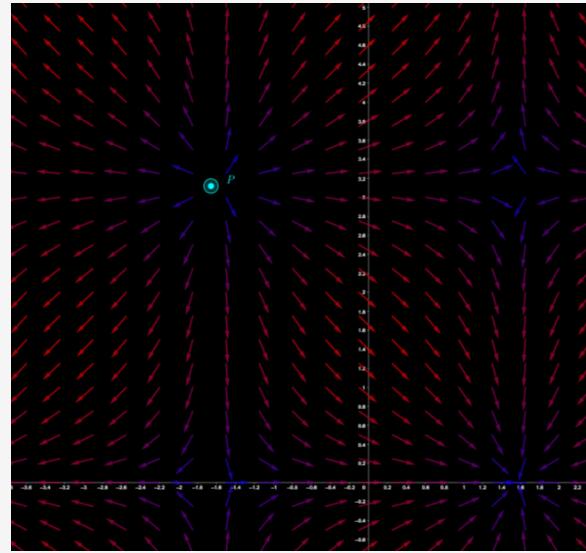
Let's consider the vector field

$$\vec{v}(x, y) = (\cos(x), -\sin(y)). \quad (14)$$

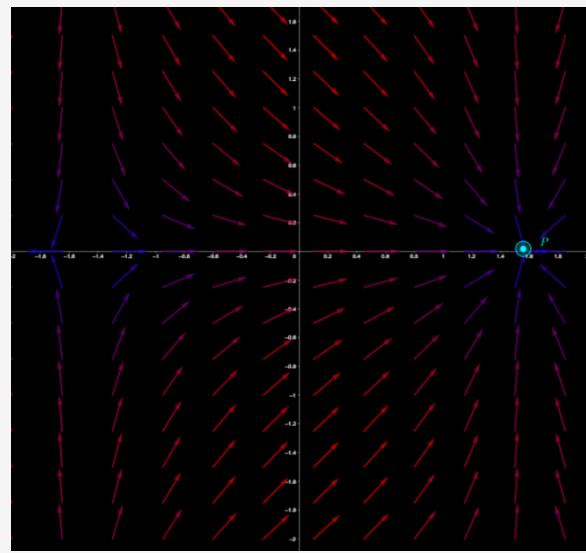
Then, the divergence is given by

$$\operatorname{div}(\vec{v})(x, y) = -\sin(x) - \cos(y). \quad (15)$$

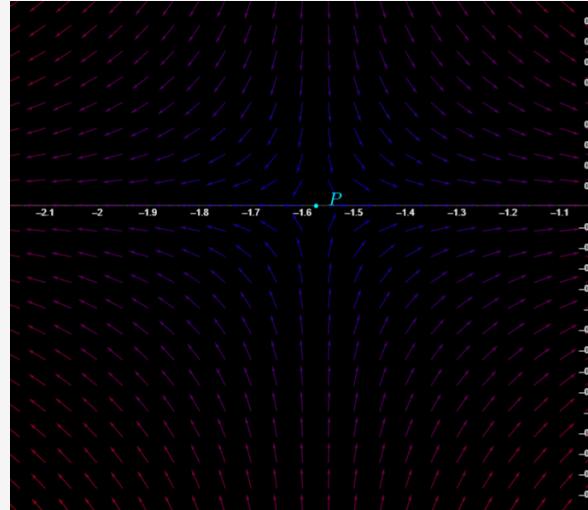
For instance, if we consider the point  $P = (-\frac{\pi}{2}, \pi)$ , the divergence is  $\operatorname{div}(\vec{v}) = 2$ , which is positive, meaning that  $P$  is a source.



On the other hand, if we consider the point  $P = (\frac{\pi}{2}, 0)$ , the divergence  $\operatorname{div}(\vec{v}) = -2$  is negative, so,  $P$  is a sink.



Now, if we consider the point  $P = (-\frac{\pi}{2}, 0)$ , we can see that the divergence is 0.



## 4 The Laplacian of a Scalar Function

Remember that the **gradient** of a function  $f(x, y)$  gives us a **vector at each point**  $P(x, y)$ , namely, **the gradient  $\vec{\nabla}f(x, y)$  is a vector field**.

This means that we can consider the **divergence of  $\vec{\nabla}f$** , which is usually denoted as

$$\nabla^2 f := \vec{\nabla} \cdot (\vec{\nabla} f). \quad (16)$$

If the notation using the symbol “nabla”  $\nabla$  for the divergence and the gradient is confusing, we can just write

$$\nabla^2 f := \text{div}(\text{grad}(f)). \quad (17)$$

This is called the **laplacian of  $f$** , and it gives us a number that tells us **whether we are close to a peak or to a valley**.

Sometimes the notation  $\Delta f$  or  $\text{laplacian}(f)$  are used for the laplacian of  $f$ .

If we use the definition of the gradient and of the divergence, we have that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}. \quad (18)$$

If we combine the interpretations of the gradient and the divergence, we get the following interpretation for the laplacian:

- $\boxed{\nabla^2 f > 0}$ : If the laplacian is **positive**, then, in general, the **arrows of the gradient are pointing away from the point  $P$** . From the interpretation of the gradient, this means that, in general, **the graph of the function gets higher when we move**

**away from  $P$** , namely, the point  $P$  behaves like a valley. The bigger the value of the laplacian, the more the point  $P$  will behave like a valley. So, the points where **the laplacian is biggest** are the **minima/valleys of the function**.

- $\boxed{\nabla^2 f < 0}$ : If the laplacian is **negative**, then, in general, the **arrows of the gradient are pointing towards the point  $P$** . This means that, in general, **the height of the graph of the function decreases when we move away from  $P$** , namely, the point  $P$  behaves like a peak. So, the points where the **laplacian is smallest** are the **maxima/peaks of the function**.
- $\boxed{\nabla^2 f = 0}$ : If the laplacian is **exactly equal to 0**, then, the **arrows of the gradient are pointing away from and towards the point  $P$  in equal amounts**. This means that the point doesn't behave like a valley or like a peak.

#### Example 4.1. Interpretation of the Laplacian

Let's consider the function

$$f(x, y) = \sin(x) + \cos(y) + 2 \quad (19)$$

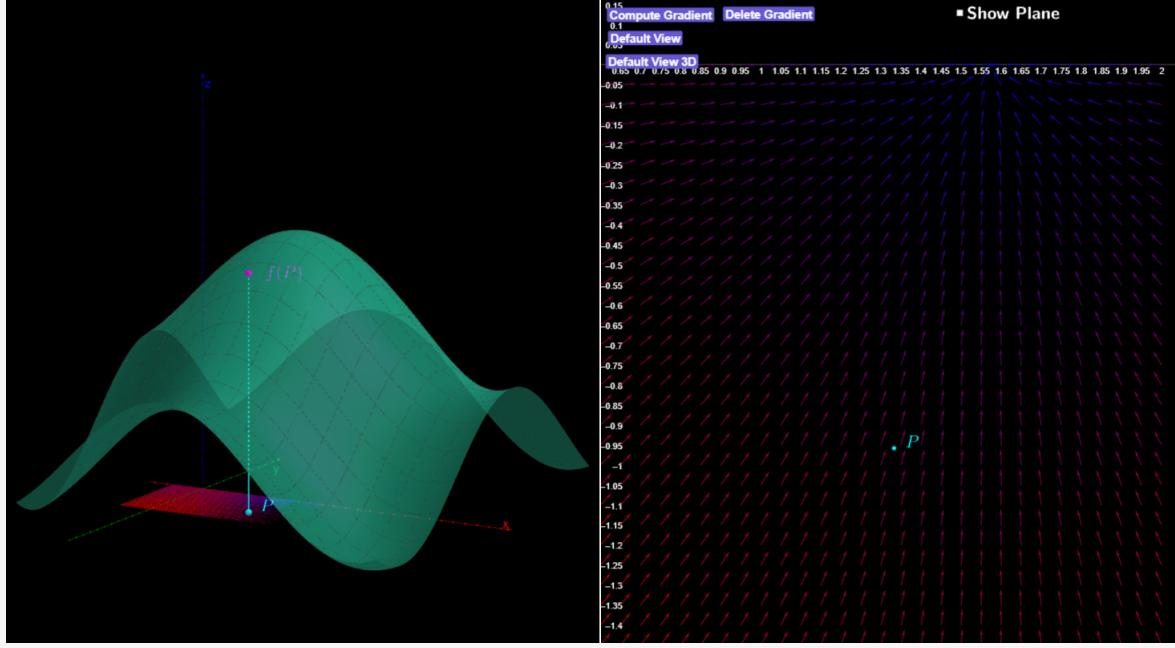
from Example 2.2..

The laplacian of the function is given by

$$\nabla^2 f(x, y) = -\sin(x) - \cos(y). \quad (20)$$

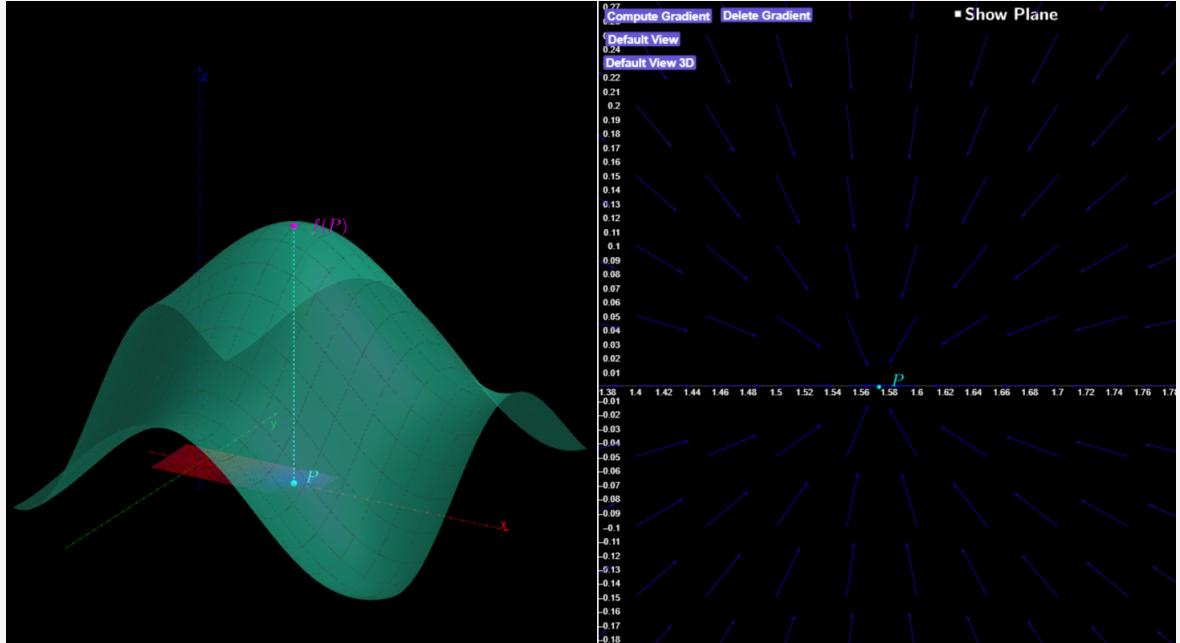
If we consider a point **close to one of the peaks**, the laplacian is negative. For example, the laplacian of the point  $P = (1.33, -0.96)$  is

$$\nabla^2 f(P) = -1.55. \quad (21)$$



If we go exactly to the peak  $P = (\frac{\pi}{2}, 0)$ , we have that the laplacian has the smallest value possible

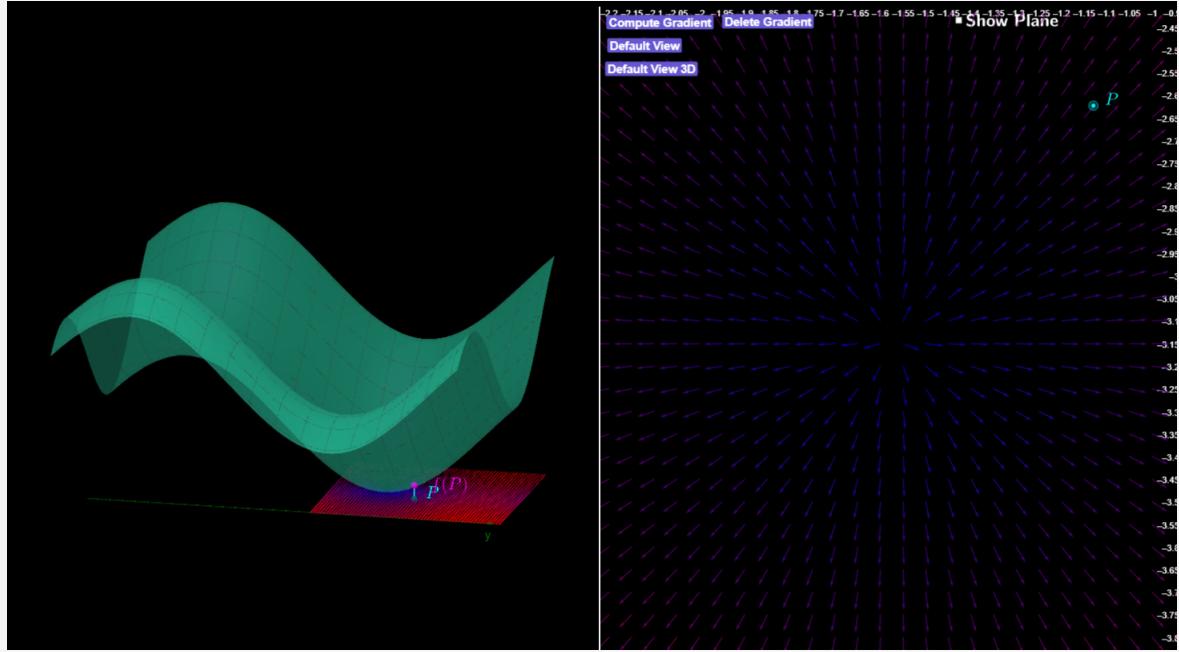
$$\nabla^2 f(P) = -2. \quad (22)$$



Similarly, if we consider a point **close to one of the valleys**, the laplacian is positive.

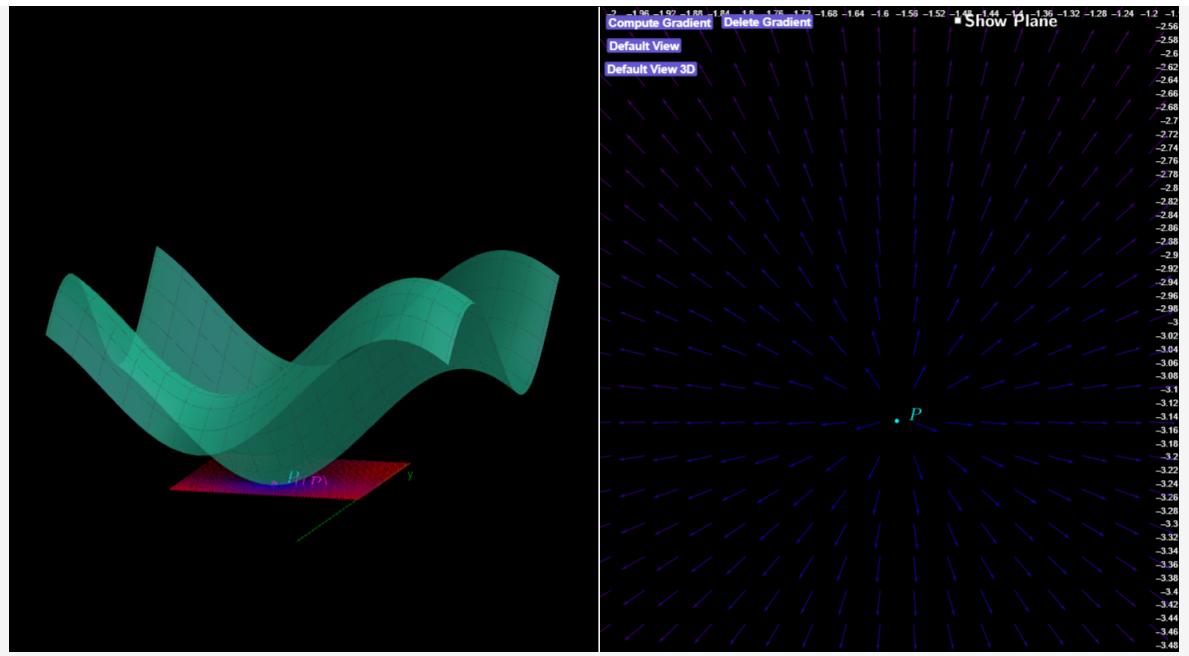
For example, the laplacian of the point  $P = (-1.13, -2.62)$  is

$$\nabla^2 f(P) = 1.77. \quad (23)$$



If we go exactly to the valley  $P = (-\frac{\pi}{2}, -\pi)$ , we have that the laplacian has the biggest value possible

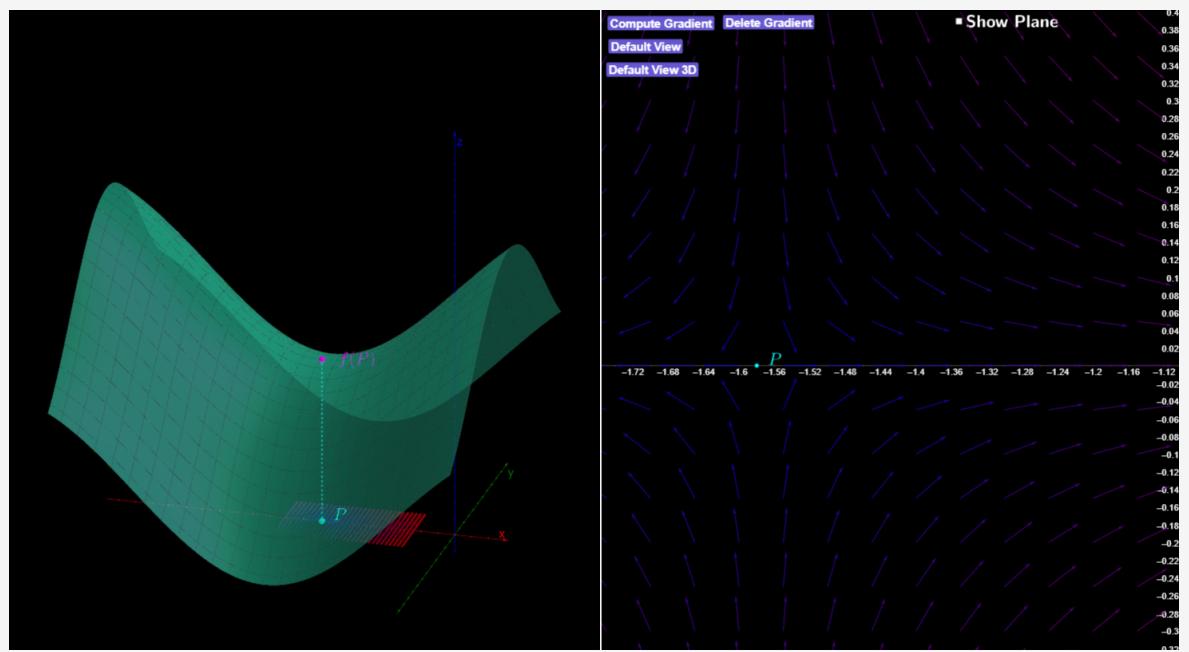
$$\nabla^2 f(P) = 2. \quad (24)$$



Now, let's consider the point  $P = (0, -\frac{\pi}{2})$ , for which the laplacian is 0

$$\nabla^2 f(P) = 0. \quad (25)$$

Then, the function around that point increases in some directions and decreases in other directions in the same amount, the point behaves neither like a valley nor like a peak.

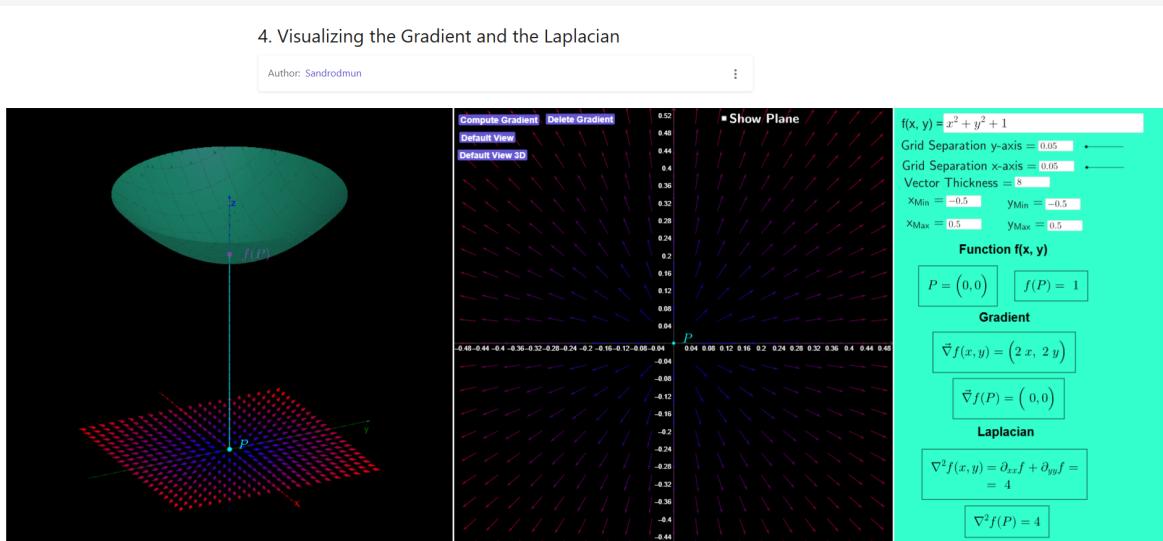


## 5 Guide for Using the Interactive Animation

Now that we have seen the basics of how the gradient and the laplacian work, we will explain how we can play around with these concepts using the following app: [App: Visualizing the Gradient and the Laplacian](#).

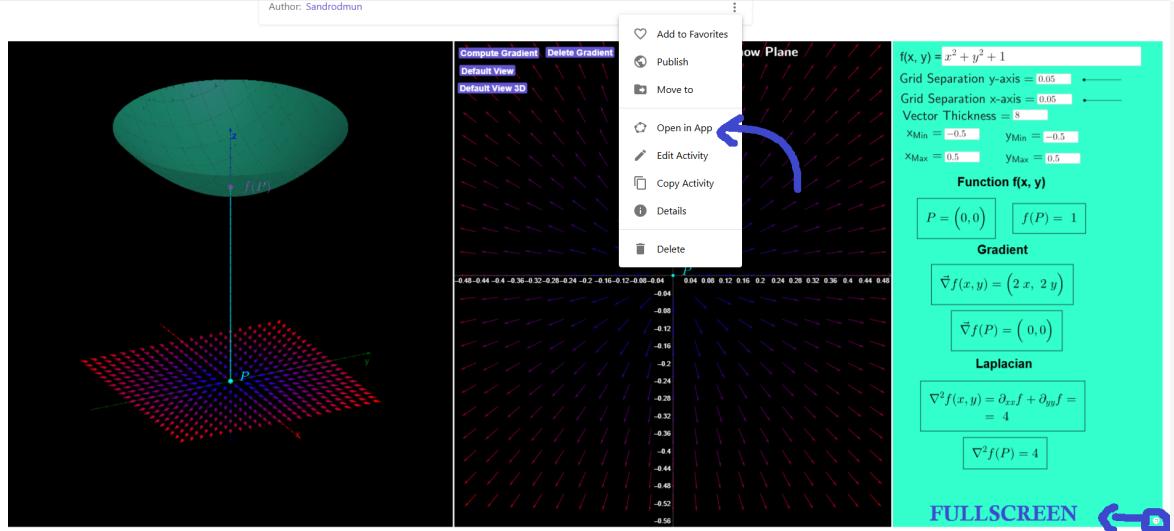
### Guide 5.1. Opening the Applet

- Once we click on the link, we will be taken to the following window:



- There are 2 options:

- Click the **Fullscreen button**:
  - We see the app in fullscreen
  - We will **not** be able to edit things or change the sizes of the Domain and Image planes.
- Click the **3 dots** next to the **author label**, and select **Open in App**:
  - You will be able to edit anything you want
  - You can change the sizes of the windows
  - You can see it fullscreen too
  - You can hide, delete, add, etc., anything you want

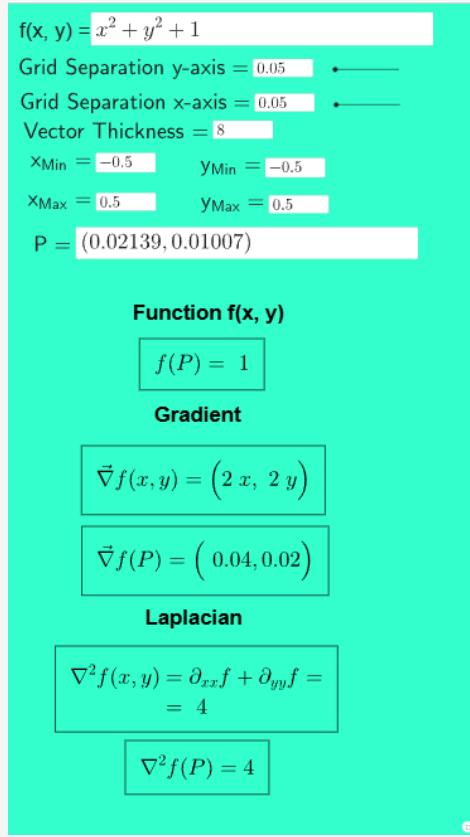


For this applet I recommend using the **Fullscreen** option, as the **Open in App View** might need us to zoom out our browser, given the fact that the applet has very big dimensions.

## Guide 5.2. Choosing a Function

1. The input bars on the right window allow us to choose:
  - the **function**,
  - the **grid separation** for the gradient vector field,
  - the **thickness of the vectors**,
  - the **region where we want to compute the gradient**, and
  - the **point  $P$  where we want to compute  $f(P)$ ,  $\vec{\nabla}f(P)$ , and  $\nabla^2f(P)$** .
2. We can also choose the position of the point  $P$  by dragging it either on the 2D window in the center, or on the 3D window on the left.
3. Of course we can also explore the 2D and 3D views by clicking anywhere other than the point or the function and dragging the mouse.
4. **Default View:** After playing around with the app, you will probably end up in some place far away from the origin, or in some place that isn't useful if we change the function or the point. To fix that, click the **Default View** button, either in the 2D or the 3D view, depending on which view you want to fix.
5. **Show Plane:** This option shows the xy-plane in the 3D view, in case it is needed

for clarity.



### Guide 5.3. Computing the Gradient

To compute the gradient we have to click the **Compute Gradient** button.

The gradient isn't computed immediately after choosing the function, grid separation, etc., to save resources. So, even if we change the function, if we haven't clicked the Compute Gradient button, we will still see the **gradient corresponding to the previous function, not the gradient of the function we have just chosen**.

We can also choose to delete the gradient, whenever we want, but to show it again we have to click the **Compute Gradient** button again. ■