Calculus

(Visualizing the Gradient and the Laplacian)

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Interactive Animation: **Visualizing the Gradient and the Laplacian**

Contents

1 Introduction: Scalar Functions

In **single-variable calculus**, we typically study functions that take in one real number and output another. If f is such a function, we represent it as:

$$f : \mathbb{R} \to \mathbb{R}. \tag{1}$$

To visualize this, we use two axes: one for the input x and one for the output $f(x)$. The result is a **curve** in a 2D plane.

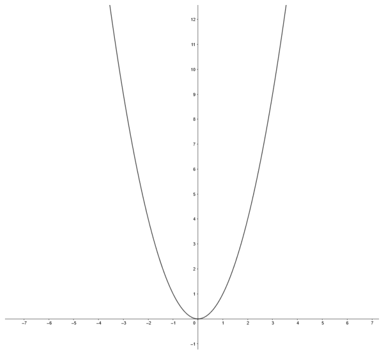

Figure 1: Graph of $f(x) = x^2$.

However, we can also conside functions that depend on **two variables** (x and y). In these cases, we study **scalar functions** of the form:

$$f : \mathbb{R}^2 \to \mathbb{R}. \tag{2}$$

This notation tells us that f takes a point (x, y) from the **domain** $\mathbb{R}^2$ (the plane) and assigns it a single real number from the **codomain** $\mathbb{R}$.

For instance, we can define a function $f(x, y) = x^2 + y^2 + 2$. This function tells us that at the point $(2, 4)$, the image is $f(2, 4) = (2)^2 + (4)^2 + 2 = 4 + 16 + 2 = 22$.

These functions are very important not only in mathematics, but also in physics, as they can represent many different physical quantities. The function f could give us at each point the temperature of a plane, the modulus of an electric field in the plane, etc.

Representing Scalar Functions

To represent a function $f(x, y)$ graphically, we need **three dimensions**. We use two horizontal axes to represent the input coordinates (x, y) and a vertical axis (usually the z-axis) to represent the value of the function.

1. Choose a point $P = (x, y)$ in the xy-plane.
2. Compute the value $f(x, y)$.
3. Draw the point $(x, y, f(x, y))$ in 3D space.
4. Repeat this for all points in the domain to form a **surface**.

Example 1.1. Representing the Function $f(x, y) = x^2 + y^2 + 2$

We want to visualize the function $f(x, y) = x^2 + y^2 + 2$.

1. We choose a point in the plane, for example, $P = (1, 1)$.

`Images/intro_1.png`

2. The value of the function at that point is $f(1, 1) = 1^2 + 1^2 + 2 = 4$.
3. We plot the point $(1, 1, 4)$ in 3D space. We can draw a vertical segment of height $f(1, 1) = 4$ starting from P to visualize it better.

Images/intro_2.png

4. If we perform this process for every point (x, y), we obtain a surface known as a **paraboloid**.

Images/intro_3.png

Figure 2: Graphical representation of $f(x, y) = x^2 + y^2 + 2$ as a 3D surface.

■

How to Study the Behavior of a Scalar Function

While the 3D surface gives us a global view of the function, it is often difficult to determine exactly how the function is changing just by looking at the graph.

To study the behavior of f it is useful to think of the graph of f as if it were a mountain, and we were hiking on it. If you were standing at a point P on this mountain, you might want to know:

- **what direction should I follow to get to the highest point?**
- **how steep is the path I want to follow?**
- **am I currently at a peak or a valley?**

To answer these questions, we move beyond simple visualization and use two fundamental tools of vector calculus: the **Gradient** and the **Laplacian**.

2 The Gradient Vector Field

As we have seen, a scalar function $f(x, y)$ creates a surface in 3D space. However, to understand how the function changes at any given point $P = (x, y)$ in the domain, we use the **gradient**.

The **gradient** of a function $f(x, y)$ is a **vector field** that points in the **direction of the steepest ascent/increase** of the function. It is denoted by $\vec{\nabla} f$ and is defined as:

$$\vec{\nabla} f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) . \tag{3}$$

While the function f takes two coordinates and gives a single number (a *scalar*), the gradient takes two coordinates and gives a **vector**. This means that at every point (x, y) in the plane, there is an associated arrow.

Example 2.1. The Gradient of $f(x, y) = x^2 + y^2 + 2$ Let's consider the function

$$f(x, y) = x^2 + y^2 + 2 , \tag{4}$$

then, its gradient is given by

$$\vec{\nabla} f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x, 2y). \tag{5}$$

This vector tells us what direction we should take at any point (x, y) to reach the highest point.

For instance, if we consider the point $(1, 2)$, the gradient is

$$\vec{\nabla} f(1, 2) = (2 \cdot 1, 2 \cdot 2) = (2, 4). \tag{6}$$

We can draw this as a vector coming out of the point $(1, 2)$ and pointing in the direction $(2, 4)$

`Images/10.png`

If we do this for many different points, we get the **gradient vector field of** f

`Images/11.png`

Figure 3: Caption

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Visualizing the Gradient: The Vector Field

As we have seen in the example, to visualize the gradient, we draw these vectors directly on the xy-plane. However, there are two main challenges when trying to create a clear picture:

1. **The Need for a Grid:** If we were to draw a vector at *every* single point in the plane, the screen would be completely filled with vectors, and we wouldn't be able to see

anything.

`Images/12.png`

To solve this, we define a **grid of points** with a specific **separation**. We only compute and draw the gradient at these specific grid intersections.

`Images/12.png`

2. **Normalization and Overlapping:** In many functions, the gradient vectors can become very long, causing them to overlap and create a messy visualization.

Images/12.png

To fix this, we **normalize** the vectors so they all have a manageable length, and we use **color coding** to represent their actual magnitude (the norm).

Images/12.png

- **Blue vectors** represent a **small norm**, meaning the function is relatively flat in that area.
- **Red vectors** represent a **large norm**, indicating a very steep slope.

Interpretation of the Gradient

As we have commented, the gradient has a very intuitive interpretation. The arrows always point towards the direction of **steepest ascent**, namely, if we think of the graph of f as a mountain, the arrows **tell us where to move if we want to reach the top of the mountain**. Similarly, if we go in the direction **opposite to the arrows**, we will reach the lowest points of the mountain.

As a summary:

- **Gradient $\vec{\nabla} f$:** The arrows point towards the direction where the mountain (the graph of f) is highest.
- **Negative Gradient $-\vec{\nabla} f$:** If we change the direction of the arrows given by the gradient $\vec{\nabla} f$, we get arrows that **guide us towards the lowest points of the graph**.

Example 2.2. Interpretation of the Gradient

Let's consider the function

$$f(x, y) = sin(x) + cos(y) + 2. \tag{7}$$

Its graph looks like this:

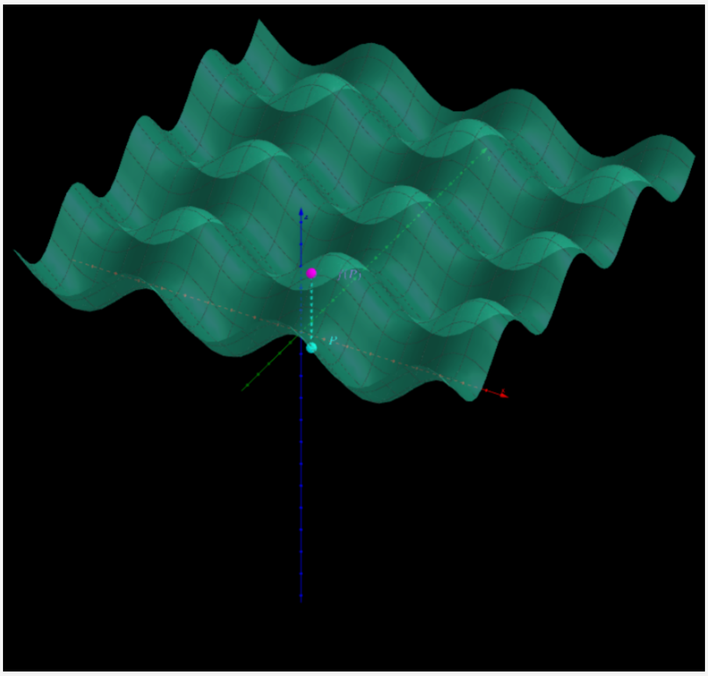
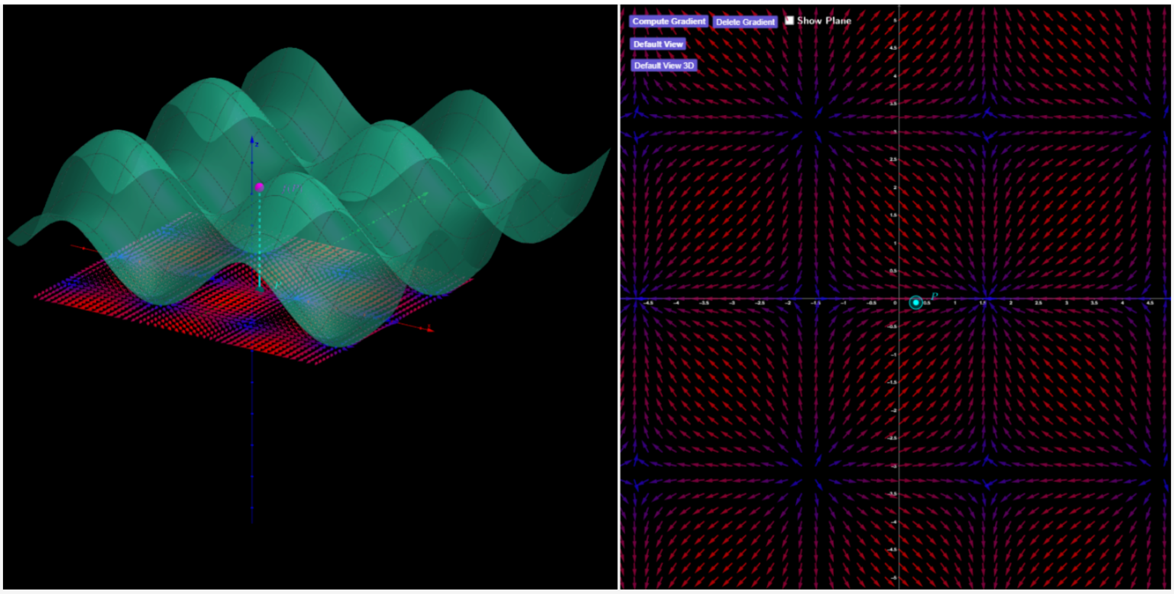


Figure 4: Caption

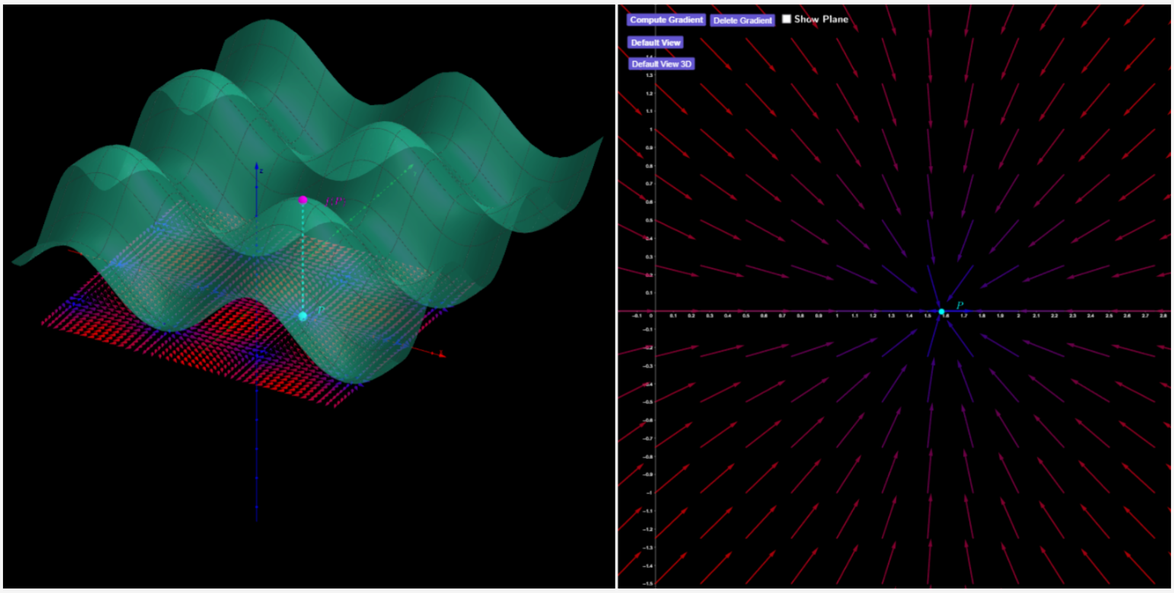
The gradient of the function is given by

$$\vec{\nabla} f(x, y) = \big(cos(x), -sin(y)\big)\,. \tag{8}$$

If we plot the **gradient vector field**, we get the following:

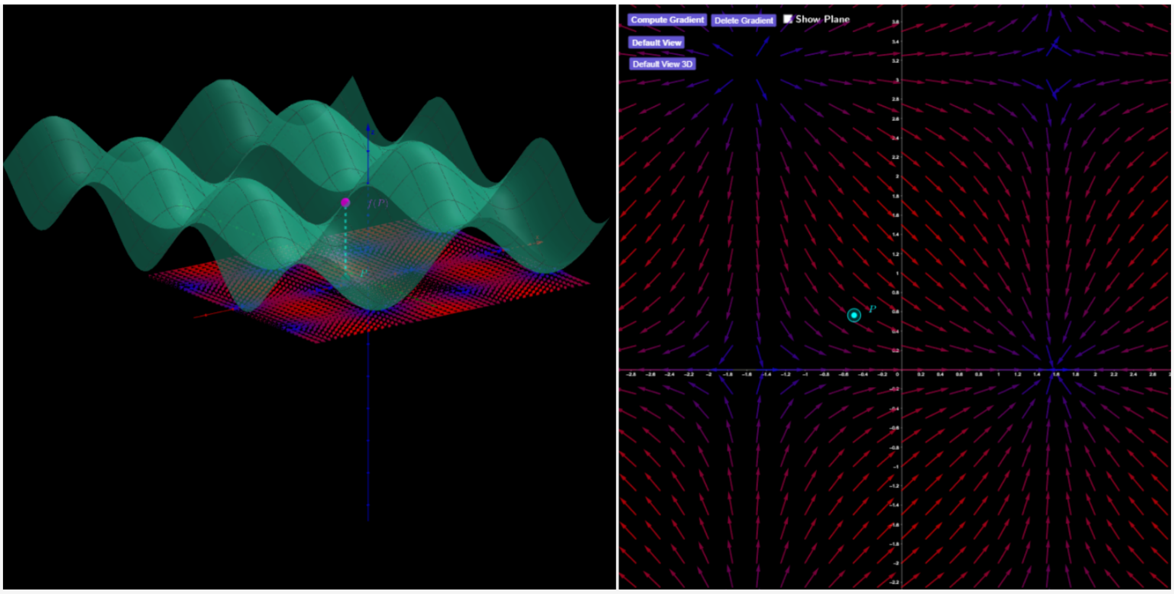


Now, if we move the point P following the arrows, we reach the following point

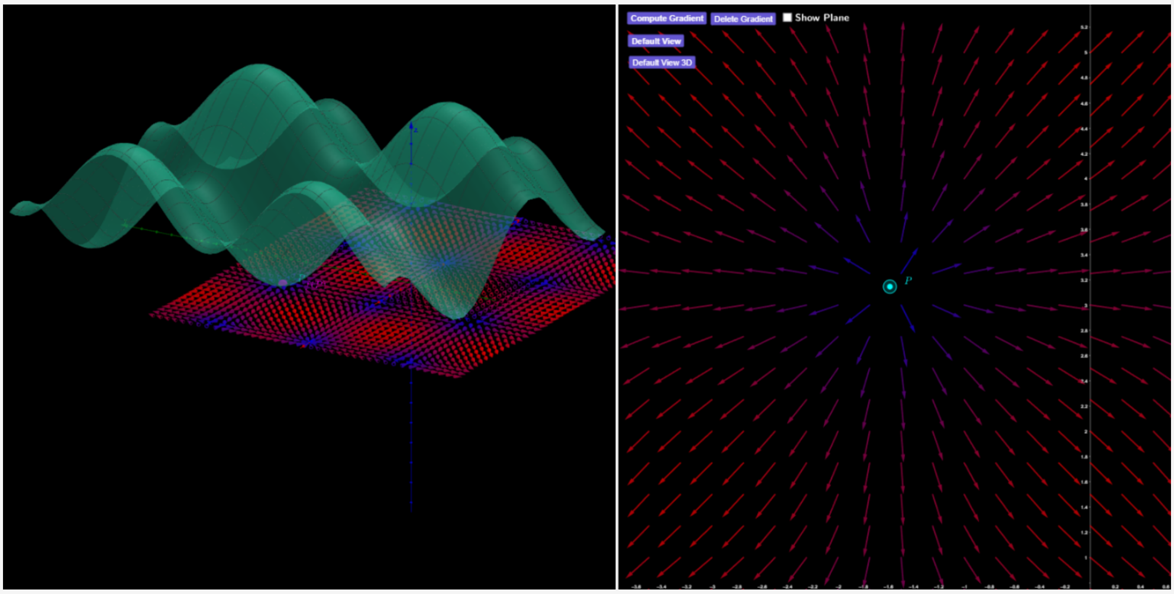


which as we can see, is **one of the peaks of the function**.

If we put P at some other position



and this time we **go against the arrows**, we reach the following position



which is one of the **valleys of the function**. ■

3 The Divergence of a Vector Field

A **vector field**

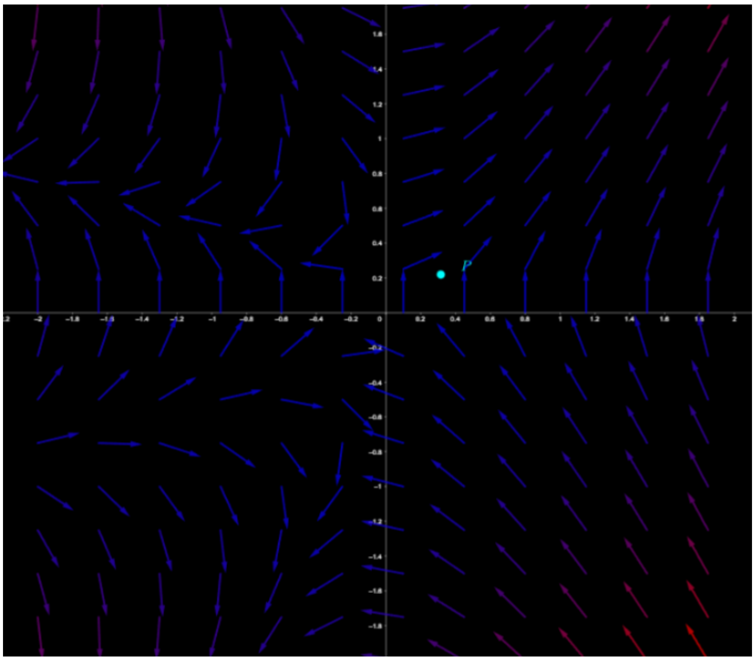
$$\vec{v}(x, y) = \Big(v_x(x, y), v_y(x, y)\Big) \tag{9}$$

is a function which takes in **a point** $P = (x, y)$ and gives us a **vector** $\vec{v}(x, y)$.

For example, we can define a vector field $v(x, y)$ as

$$\vec{v}(x, y) = (y^3 + 2xy, 3xy^2 + x^2)\,, \tag{10}$$

which can be represented similarly to how we represented the gradient, by drawing the corresponding arrow on top of various points



To analyze **how much the arrows point towards a certain point**, we define the **divergence operator** as follows:

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}. \tag{11}$$

If we consider the vector field defined in Equation 10, the **divergence** is given by

$$\vec{\nabla} \cdot \vec{v}(x, y) = 6xy + 2y. \tag{12}$$

Notice that **the divergence gives us a number**. This number measures **how much the vectors point away the point** $P = (x, y)$. You can think of it as a measure of how **"unpopular" the point** P **is with respect to the vectors**.

The divergence of $\vec{v}$ is also denoted by

$$\vec{\nabla} \cdot \vec{v} = \mathrm{div}(\vec{v}). \tag{13}$$

- $\boxed{\mathbf{div}(\vec{v}) > 0}$**:** If the divergence is **positive**, then, in general, the arrows are **pointing away from the point** P, namely, the point is **unpopular**. We say that the point acts as a **source**.
- $\boxed{\mathbf{div}(\vec{v}) < 0}$**:** If the divergence is **negative**, then, in general, the arrows are **pointing towards the point** P, namely, the point is **popular**. We say that the point acts as a sink.

- $\boxed{\textbf{div}(\vec{v}) = 0}$**:** If the divergence is **exaclty** 0, then, the arrows are **pointing away from and towards the point** P **in exactly the same amount**, namely, the point is **neither popular nor unpopular**.

Example 3.1. Divergence of a Vector Field

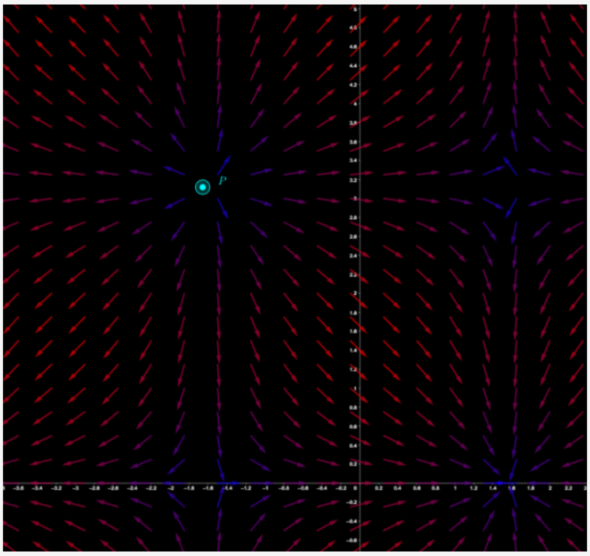
Let's consider the vector field

$$\vec{v}(x, y) = (cos(x), -sin(y)). \tag{14}$$

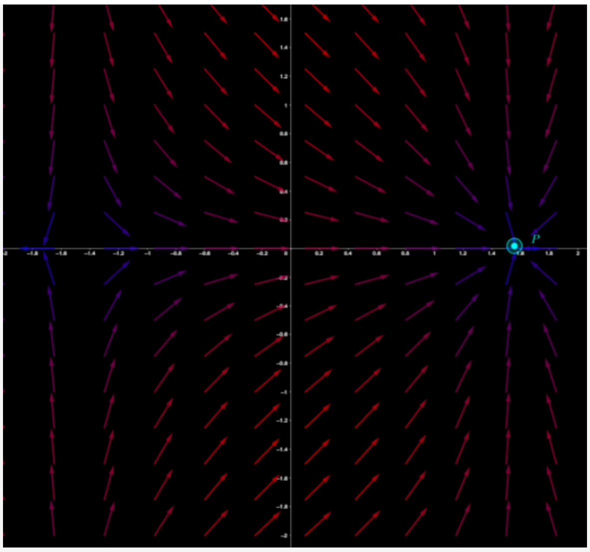
Then, the divergence is given by

$$\text{div}(\vec{v})(x, y) = -sin(x) - cos(y). \tag{15}$$

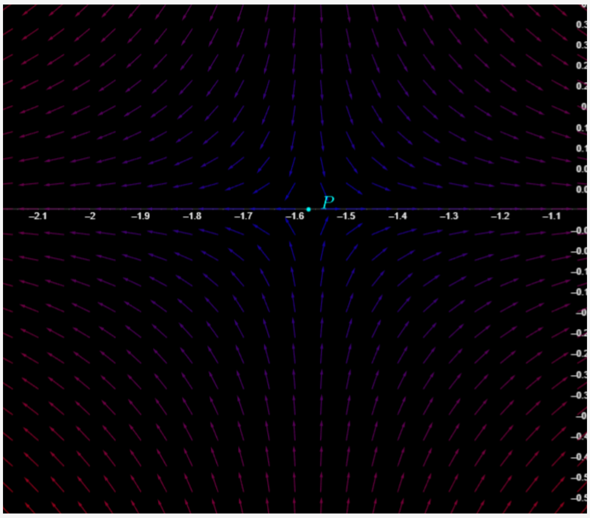
For instance, if we consider the point $P = (-\frac{\pi}{2}, \pi)$, the divergence is $\text{div}(\vec{v}) = 2$, which is positive, meaning that P is a source.



On the other hand, if we consider the point $P = (\frac{\pi}{2}, 0)$, the divergence $\text{div}(\vec{v}) = -2$ is negative, so, P is a sink.



Now, if we consider the point $P = (-\frac{\pi}{2}, 0)$, we can see that the divergence is 0.



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4 The Laplacian of a Scalar Function

Remember that the **gradient** of a function $f(x, y)$ gives us a **vector at each point** $P(x, y)$, namely, **the gradient** $\vec{\nabla} f(x, y)$ **is a vector field**.

This means that we can consider the **divergence of** $\vec{\nabla} f$, which is usually denoted as

$$\nabla^2 f := \vec{\nabla} \cdot \left(\vec{\nabla} f\right). \tag{16}$$

If the notation using the symbol "nabla" ∇ for the divergence and the gradient is confusing, we can just write

$$\nabla^2 f := \mathrm{div}\Big(\mathrm{grad}(f)\Big). \tag{17}$$

This is called the **laplacian of** f, and it gices us a number that tells us **whether we are close to a peak or to a valley**.

Sometimes the notation Δf or laplacian(f) are used for the laplacian of f.

If we use the definition of the gradient and of the divergence, we have that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}. \tag{18}$$

If we combine the interpretations of the gradient and the divergence, we get the following interpretation for the laplacian:

- $\boxed{\nabla^2 f > 0}$**:** If the laplacian is **positive**, then, in general, the **arrows of the gradient** are **pointing away from the point** P. From the interpretation of the gradient, this means that, in general, **the graph of the function gets higher when we move away from** P, namely, the point P behaves like a valley. The bigger the value of the laplacian, the more the point P will behave like a valley. So, the points where **the laplacian is smallest** are the **mininma/valleys of the function**.
- $\boxed{\nabla^2 f < 0}$**:** If the laplacian is **negative**, then, in general, the **arrows of the gradient** are **pointing towards the point** P. This means that, in general, **the height of the graph of the function decreases when we move away from** P, namely, the point P behaves like a peak. So, the points where the **laplacian is biggest** are the **maxima/peaks of the function**.
- $\boxed{\nabla^2 f = 0}$**:** If the laplacian is **exactly equal to** 0, then, the **arrows of the gradient** are **pointing away from and towards the point** P **in equal amounts**. This means that the point doesn't behave like a valley or like a peak.

Example 4.1. Interpretation of the Laplacian

Let's consider the function

$$f(x, y) = sin(x) + cos(y) + 2 \tag{19}$$

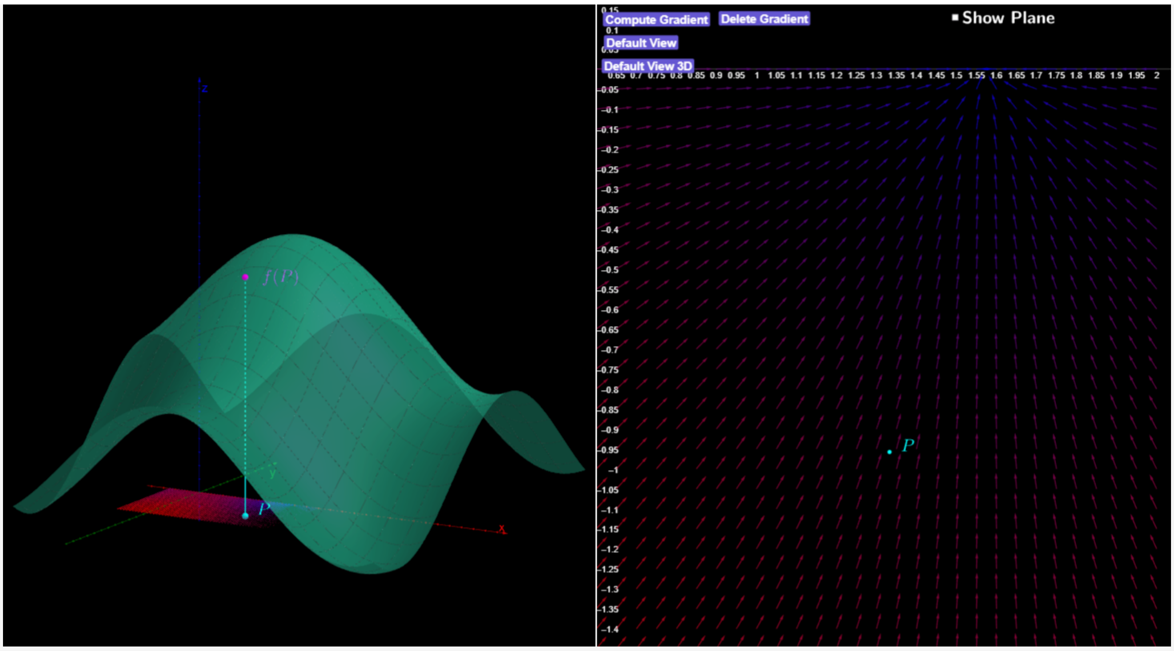
from Example 2.2..

The laplacian of the function is given by

$$\nabla^2 f(x, y) = -sin(x) - cos(y). \tag{20}$$

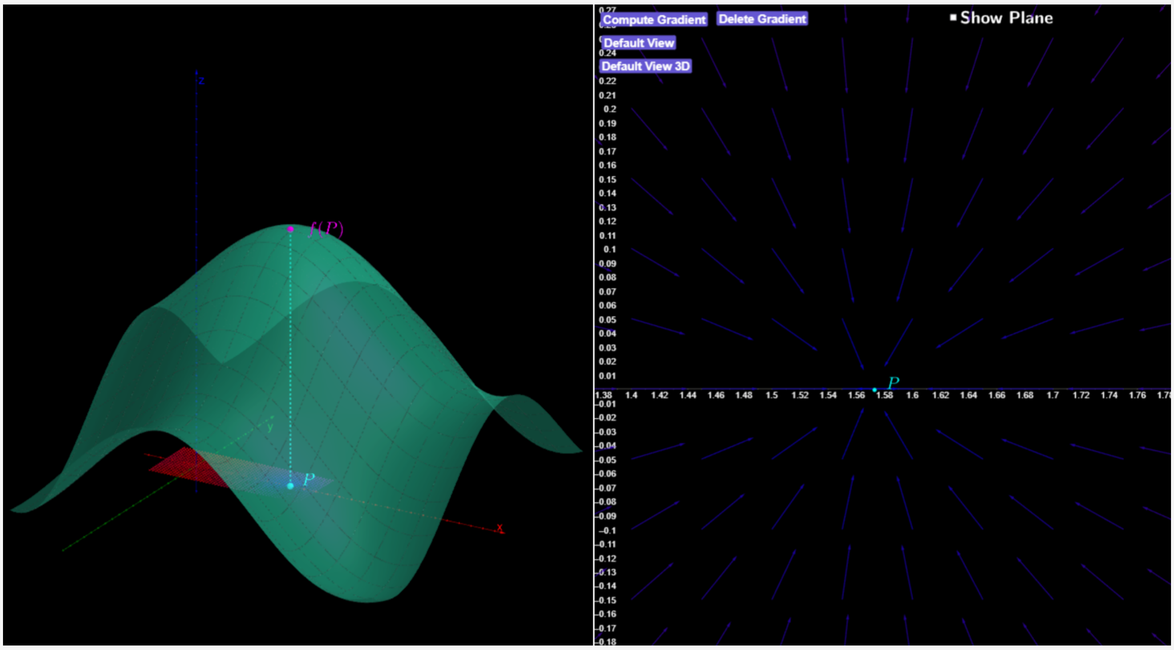
If we consider a point **close to one of the peaks**, the laplacian is negative. For example, the laplacian of the point $P = (1.33, -0.96)$ is

$$\nabla^2 f(P) = -1.55. \tag{21}$$



If we go exactly to the peak $P = (\frac{\pi}{2}, 0)$, we have that the laplacian has the smallest value possible

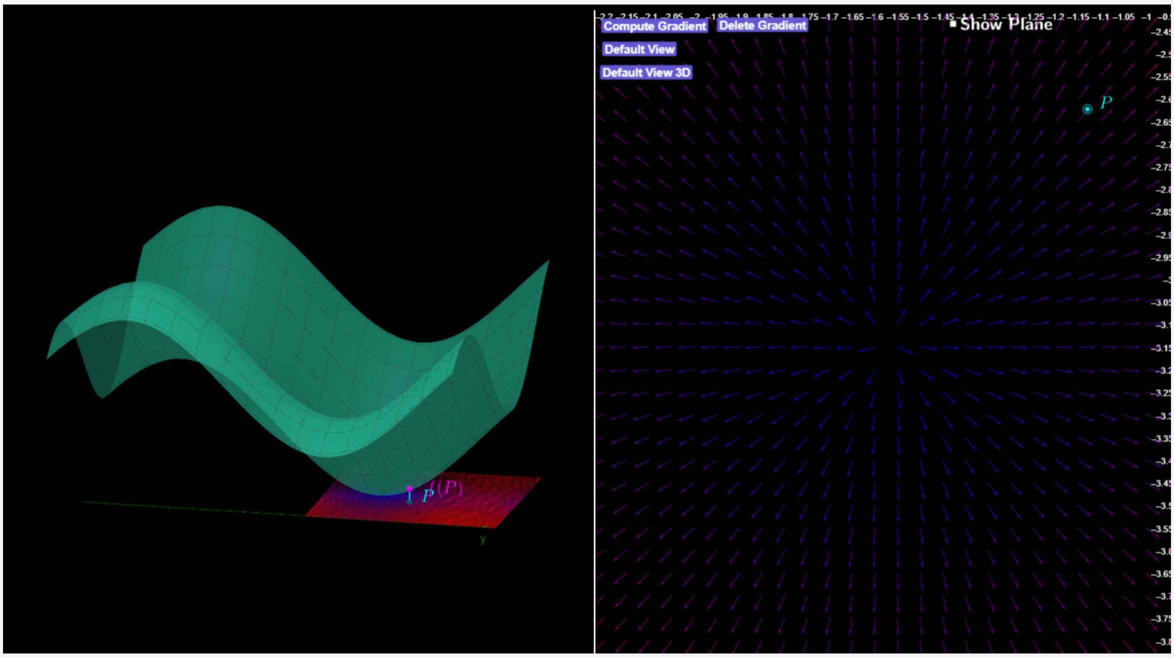
$$\nabla^2 f(P) = -2. \tag{22}$$



Similarly, if we consider a point **close to one of the valleys**, the laplacian is positive.

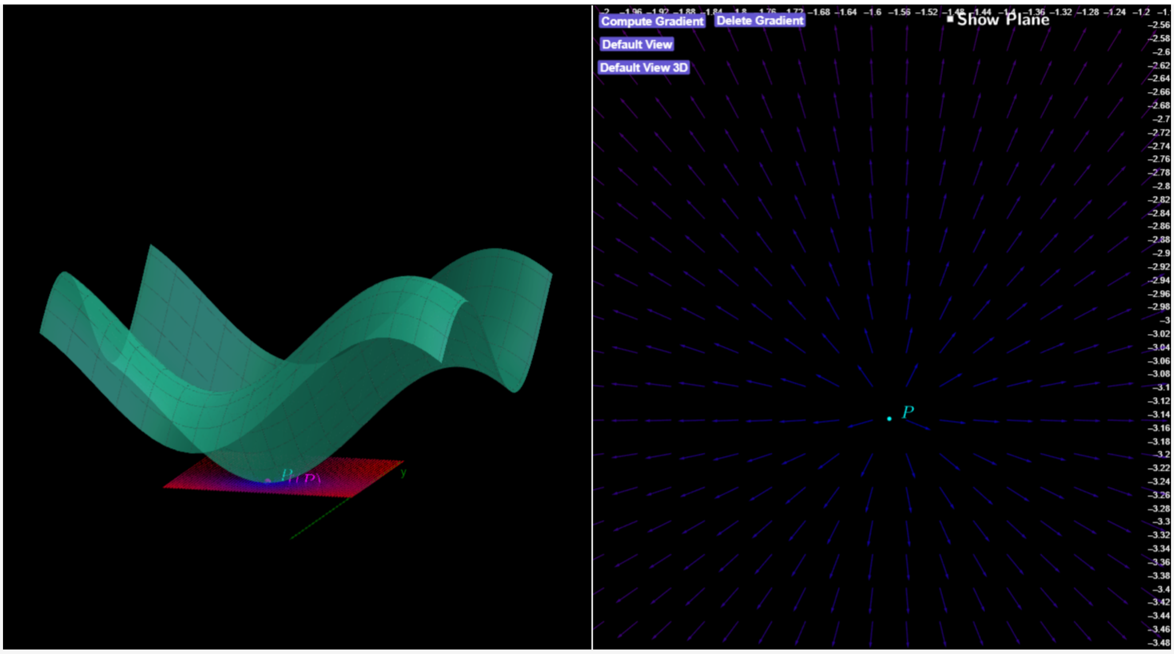
For example, the laplacian of the point $P = (-1.13, -2.62)$ is

$$\nabla^2 f(P) = 1.77. \tag{23}$$



If we go exactly to the valley $P = (-\frac{\pi}{2}, -\pi)$, we have that the laplacian has the biggest value possible

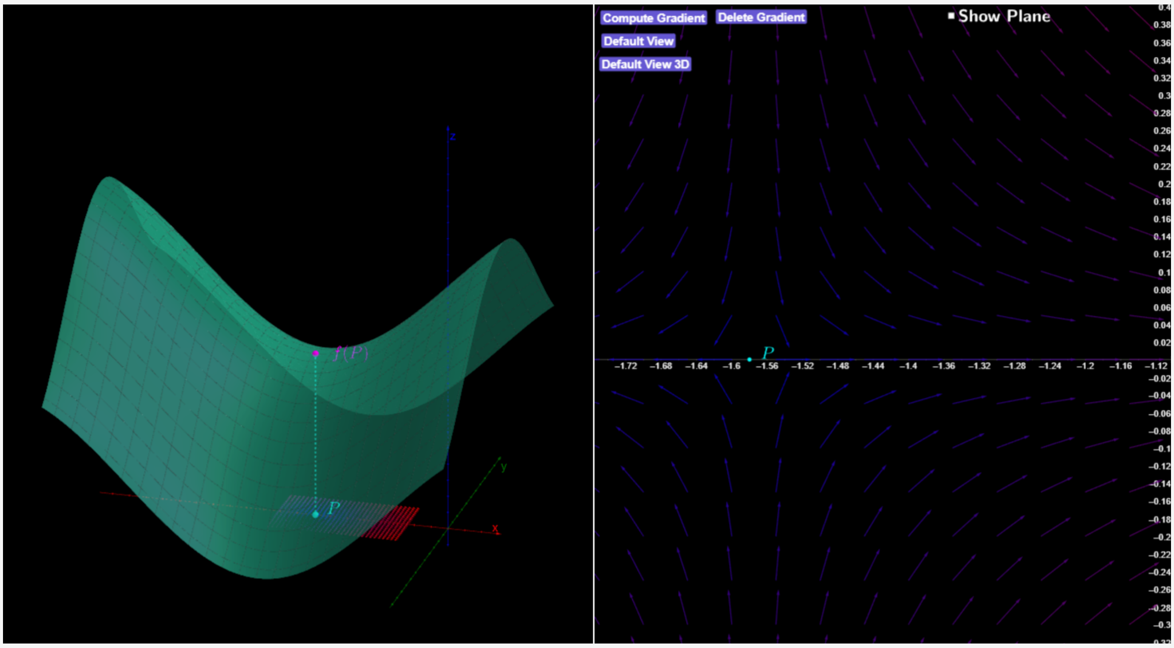
$$\nabla^2 f(P) = 2. \tag{24}$$



Now, let's consider the point $P = (0, -\frac{\pi}{2})$, for which the laplacian is 0

$$\nabla^2 f(P) = 0. \tag{25}$$

Then, the function around that point increases in smoe directions and decreases in other directions in the same amount, the point behaves neither like a valley nor like a peak.



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