Calculus

(Derivative Trainer)

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Interactive Animation: **Derivative Trainer**

Contents

1 Introduction: What is a Derivative?

In **single-variable calculus**, we study functions that take in a real number and give out another real number. If f is such a function, we represent it as:

$$f : \mathbb{R} \to \mathbb{R}. \tag{1}$$

As we have seen, we can visualize these functions using a 2D plane, where the **horizontal axis** represents the input x and the **vertical axis** represents the output $f(x)$.

However, just knowing the value of a function $f(x)$ at a specific point is not always enough. In mathematics and physics, we often need to know **how fast** the function is changing.

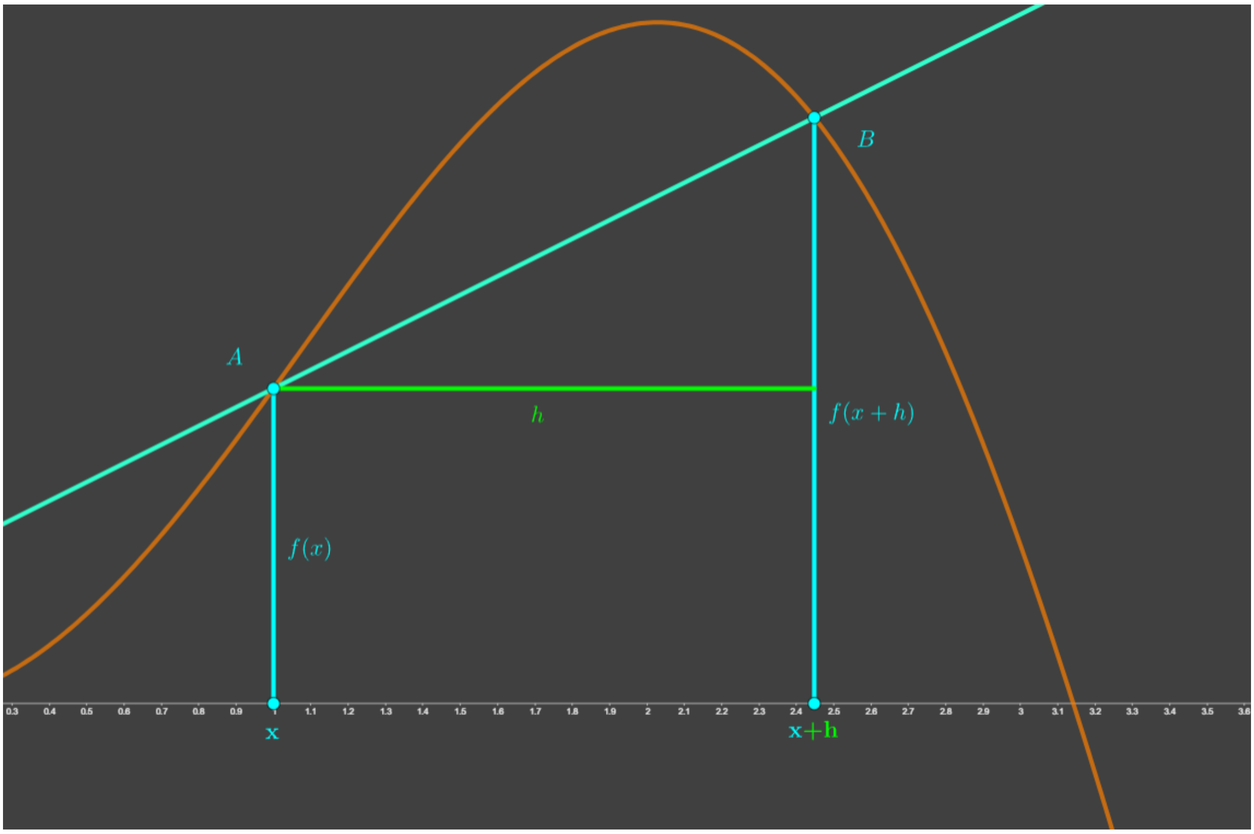
For instance, if $f(t)$ represents the position of a car at a time t, then knowing $f(t)$ only tells us where the car is. If we want to know its **velocity**, we need to find out how its position changes as time passes.

The Rate of Change

To measure how much a function $f(x)$ changes, we can look at two different points, x and $x + h$, where h is a small step forward. The **average rate of change** between these two points is given by:

$$\text{Average Rate of Change} = \frac{f(x+h) - f(x)}{h}. \tag{2}$$

Geometrically, this is exactly the **slope of the secant line** that passes through the points $A = (x, f(x))$ and $B = (x + h, f(x + h))$.



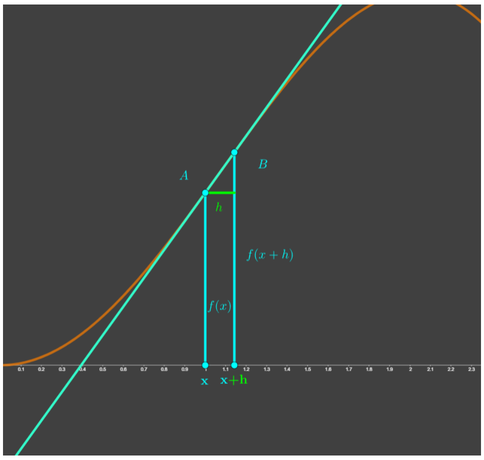
The Derivative

To find the **instantaneous rate of change** exactly at the point x, we need to make our step h as small as possible. We do this by taking the **limit** as h approaches 0:

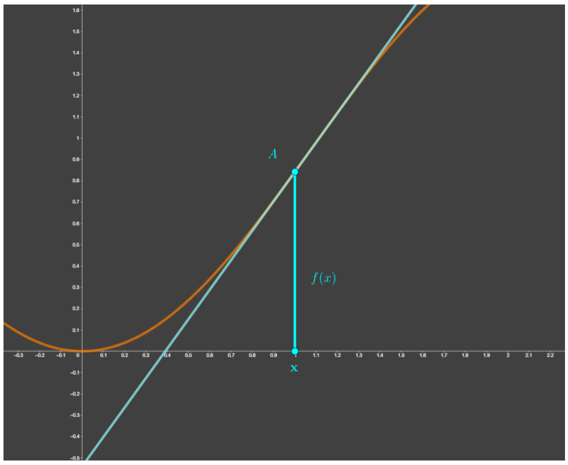
$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}. \tag{3}$$

This limit is called the **derivative** of $f(x)$. It is commonly denoted as $f'(x)$ or $\frac{df}{dx}$.

Geometrically, as h shrinks to 0, the secant line becomes the **tangent line** to the graph at the point x. Therefore, the derivative $f'(x)$ gives us the **slope of the tangent line** at any given point.



(a) Secant Line when h is small.

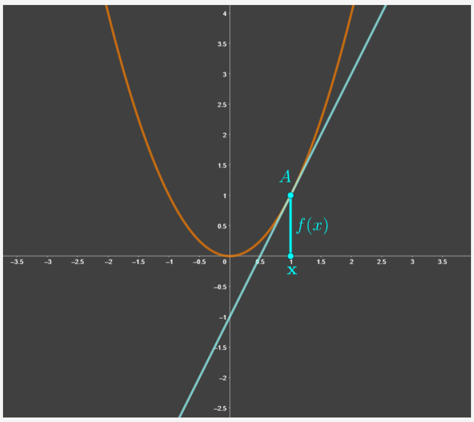


(b) Tangent Line at x.

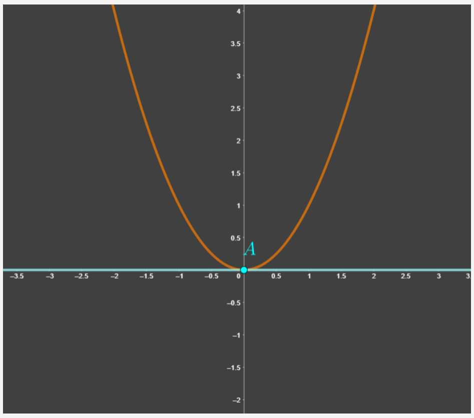
Example 1.1. Visualizing the Derivative of $f(x) = x^2$

We want to understand the derivative of the function $f(x) = x^2$ geometrically.

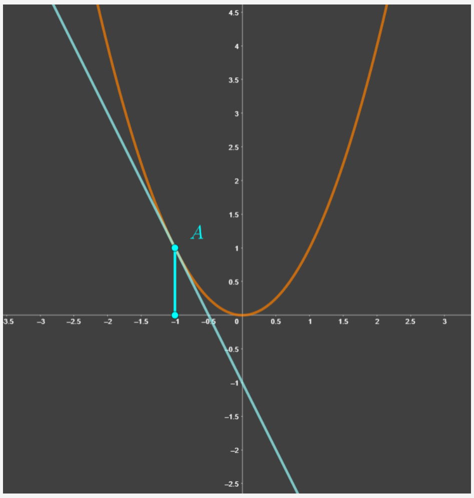
1. We choose a point, for example, $x = 1$. The value of the function is $f(1) = 1^2 = 1$.
2. We draw the **tangent line** to the graph at the point $(1, 1)$.



3. If we measure the slope of this tangent line, we see that it goes up 2 units for every 1 unit it moves to the right. So, the slope is 2. This means $f'(1) = 2$.
4. Let's do this for another point, say $x = 0$. The tangent line at $(0, 0)$ is perfectly flat.



5. Since a flat line has a slope of 0, we know that $f'(0) = 0$.
6. Now, let's look at $x = -1$. The tangent line at $(-1, 1)$ is pointing downwards.



7. The slope here is -2. This means $f'(-1) = -2$.
8. If we compute the slope of the tangent line for **every point** $x \in \mathbb{R}$ and plot these slopes as a new graph, we get a straight line passing through the origin.

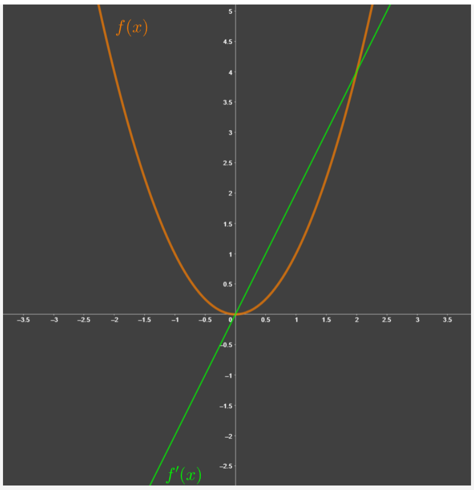


Figure 2: The graph of $f(x) = x^2$ (orange) and its derivative $f'(x) = 2x$ (green).

This suggests visually that the derivative of $f(x) = x^2$ is the new function $f'(x) = 2x$.

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Example 1.2. Computing the Derivative of $f(x) = x^2$

The geometric arguments we gave in the previous example are nice visually, but they aren't very formal, because, **how do we know that the lines we showed are exactly the tangent lines?**, or that **they had exactly the slope we claimed they did?**

If we want to actually prove that the derivative of $f(x) = x^2$ is $f'(x) = 2x$, we have to be a bit more precise.

The proper way to do it is to consider the definition of the derivative

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \tag{4}$$

If we substitute the function f, we get

$$\begin{aligned} f'(x) &= \lim_{h \longrightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \longrightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \longrightarrow 0} \frac{2xh + h^2}{h} = \\ &= \lim_{h \longrightarrow 0} \frac{(2x+h) \cdot h}{h} = \lim_{h \longrightarrow 0} 2x + h = 2x. \end{aligned} \tag{5}$$

So, ,the derivative of $f(x)$ is indeed

$$f'(x) = 2x. \tag{6}$$

$\blacksquare$

Why Do We Need Rules?

While we can find the derivative of any function by using the limit definition, doing so for very complex functions can take a huge amount of time.

Imagine trying to compute the limit for a function like $f(x) = \sin(x^3) \cdot e^{2x}$. It would be a nightmare!

Because of this, mathematicians have developed a series of **differentiation rules** that allow us to calculate derivatives algebraically, bypassing the limit entirely. In the next section, we will review these rules, which are the main focus of our interactive Derivative Trainer.

2 Differentiation Rules

As we discussed at the end of the previous section, computing the limit definition of the derivative every single time is impractical. Instead, we use **differentiation rules** that act as shortcuts for finding the derivative of almost any function we encounter.

Basic Derivatives

First, we need to memorize the derivatives of the most fundamental functions. These are the building blocks we will use to differentiate more complex expressions.

Function $f(x)$	**Derivative** $f'(x)$
C (any constant)	0
x^n (Power Rule)	nx^{n-1}
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

Table 1: Derivatives of basic functions.

In addition to these basic functions, the derivative is **linear**. This means that if you have a constant multiplying a function, or if you are adding two functions together, the rules are very simple:

- **Constant Multiple Rule:** $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$.
- **Sum/Difference Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.

The Product Rule

When a function is the multiplication of two other functions, we cannot simply multiply their derivatives. Instead, we must apply the **Product Rule**:

$$\frac{d}{dx}\Big[f(x)\cdot g(x)\Big] = f'(x)\cdot g(x) + f(x)\cdot g'(x). \tag{7}$$

A helpful way to remember this is: *"The derivative of the first times the second, plus the first times the derivative of the second."*

Example 2.1. Using the Product Rule Let's compute the derivative of $h(x) = x^2\sin(x)$.

Here, our function is a product of $f(x) = x^2$ and $g(x) = \sin(x)$.

1. The derivative of the first part is $f'(x) = 2x$.
2. The derivative of the second part is $g'(x) = \cos(x)$.
3. Applying the product rule:

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (2x)(\sin(x)) + (x^2)(\cos(x)) = \\ &= 2x(x) + x^2 \cdot cos(x). \end{aligned} \tag{8}$$

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The Quotient Rule

When a function is a fraction where both the numerator and denominator depend on x, we use the **Quotient Rule**:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}. \tag{9}$$

Notice that the numerator of the quotient rule is very similar to the product rule, but with a **minus** sign instead of a plus.

Example 2.2. Using the Quotient Rule Let's differentiate $h(x) = \frac{e^x}{x^3}$.

Let the numerator be $f(x) = e^x$ and the denominator be $g(x) = x^3$.

1. We compute $f'(x) = e^x$.
2. We compute $g'(x) = 3x^2$.

3. Applying the quotient rule:

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{e^x \cdot x^3 - e^x \cdot 3x^2}{(x^3)^2} = \frac{x^2 e^x (x-3)}{x^6} = \frac{e^x(x-3)}{x^4}. \end{aligned} \tag{10}$$

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The Chain Rule

The most important and frequently used rule in calculus is the **Chain Rule**. It tells us how to differentiate **composite functions**, namely, functions that are placed inside other functions.

If we have a function $h(x) = f(g(x))$, the Chain Rule states:

$$\frac{d}{dx}\Big[f(g(x))\Big] = f'(g(x)) \cdot g'(x). \tag{11}$$

This means we take the derivative of the **outside** function f, leaving the **inside** function $g(x)$ exactly as it is, and then we **multiply** everything by the derivative of the inside function $g'(x)$.

Example 2.3. Chain Rule Let's consider the function

$$h(x) = sin(x^2). \tag{12}$$

Then, if we call

$$f(x) = sin(x) \quad \text{and} \quad g(x) = x^2, \tag{13}$$

we have that

$$h(x) = f(g(x)). \tag{14}$$

The derivatives of $f(x)$ and $g(x)$ are

$$f'(x) = cos(x) \quad \text{and} \quad g'(x) = 2x. \tag{15}$$

So, using the Chain Rule, we have that the derivative of $h(x)$ is

$$h'(x) = f'(g(x)) \cdot g'(x) = cos(x^2) \cdot 2x. \tag{16}$$

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General Basic Functions (Chain Rule Applied)

By applying the Chain Rule to our table of basic functions, we get a much more powerful set of general rules. Here, $u(x)$ represents any function of x:

General Function	General Derivative
$(u(x))^n$	$n(u(x))^{n-1} \cdot u'(x)$
$e^{u(x)}$	$e^{u(x)} \cdot u'(x)$
$\ln\big(u(x)\big)$	$\frac{u'(x)}{u(x)}$
$\sin\big(u(x)\big)$	$\cos\big(u(x)\big) \cdot u'(x)$
$\cos\big(u(x)\big)$	$-\sin\big(u(x)\big) \cdot u'(x)$

Table 2: General rules using the Chain Rule.

Guide to Computing the Derivative

When faced with a complicated function, it is easy to get lost. Follow these steps to systematically apply the Chain Rule:

1. **Identify the outermost function.** Ask yourself: if I were to calculate this function for a specific value of x, what is the very last mathematical operation I would perform? That is your outer function.
2. **Apply the differentiation rule for that outer function.** Do not change the inner function yet! Just write it exactly as it was.
3. **Multiply by the derivative of the inner function.**
4. **Repeat if necessary.** If the inner function is also a composite function, you will have to apply the chain rule again, creating a "chain" of multiplications.

Example 2.4. Using the Chain Rule Let's find the derivative of $h(x) = \sin\big(x^2 + 3x\big)$.

1. **Identify the outermost function:** The last operation is the sine function. The inner function is $u(x) = x^2 + 3x$.
2. **Apply the rule for the outer function:** The derivative of $\sin(u)$ is $\cos(u)$. So, we write $\cos\big(x^2 + 3x\big)$. Notice we kept the inside exactly the same.
3. **Multiply by the derivative of the inner function:** The derivative of x^2+3x is $2x + 3$.
4. **Combine them:**

$$h'(x) = \cos\big(x^2 + 3x\big) \cdot (2x + 3). \tag{17}$$

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Example 2.5. Combining the Chain Rule and the Quotient Rule Let's tackle a more challenging function: $h(x) = \left(\frac{2x}{x+1}\right)^3$.

1. **Identify the outermost function:** The outermost operation is raising everything to the power of 3. Our inner function is the fraction $u(x) = \frac{2x}{x+1}$.
2. **Apply the rule for the outer function:** Using the generalized power rule, we bring down the 3 and subtract 1 from the exponent: $3\left(\frac{2x}{x+1}\right)^2$.
3. **Multiply by the derivative of the inner function:** We now need the derivative of $\frac{2x}{x+1}$. Because this is a fraction, we must use the **Quotient Rule**:

$$u'(x) = \frac{(2)(x+1) - (2x)(1)}{(x+1)^2} = \frac{2x + 2 - 2x}{(x+1)^2} = \frac{2}{(x+1)^2}. \tag{18}$$

4. **Combine them:**

$$\begin{aligned} h'(x) &= 3\left(\frac{2x}{x+1}\right)^2 \cdot \frac{2}{(x+1)^2} \\ &= 3\left(\frac{4x^2}{(x+1)^2}\right) \cdot \frac{2}{(x+1)^2} = \frac{24x^2}{(x+1)^4}. \end{aligned} \tag{19}$$

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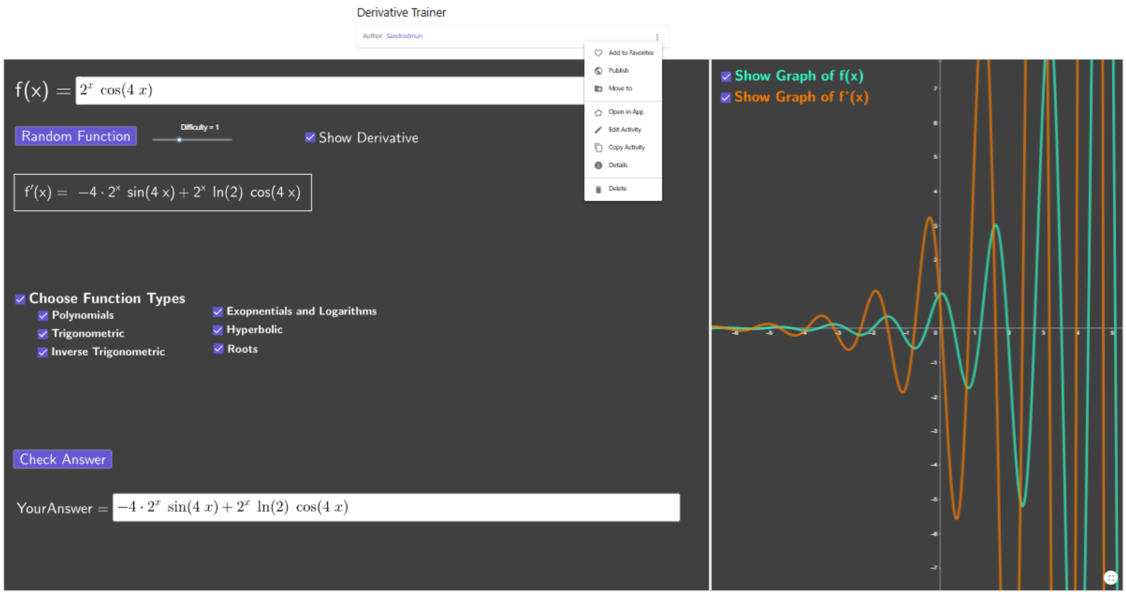
3 Guide for Using the Interactive Animation

Now that we have reviewed the algebraic rules for differentiation, we will explain how to use the following app to practice these concepts: App: Derivative Trainer.

Guide 3.1. Opening the Applet

1. Once you click on the link, you will see the main interface.
2. For this specific applet, it is **highly recommended** to click the **3 dots** in the top right corner and select **Open in App**.
   - This applet uses multiple windows and input boxes that require the flexibility of the App View to be resized.
   - You will be able to adjust the divider between the training panel and the graph deck.



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Guide 3.2. Configuring the Training Session The panel on the left serves as your control center. Here you can customize your practice:

1. **Select Function Types:** Use the checkboxes to choose which functions you want the generator to use (Polynomials, Trigonometric, Exponential, Logarithmic, etc.).
2. **Set Difficulty:** Use the slider to increase the complexity. Higher levels will include more complex Chain Rule applications and combinations of Product and Quotient rules.
3. **Generate:** Click the **Random Function** button to generate a random problem based on your settings.
4. **Manual Input:** If you want to practice a specific function from your homework, you can also type it directly into the $f(x)$ input box.

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Guide 3.3. Checking Your Results Once you have calculated the derivative on paper:

- **Inputting the Answer:** Type your result into the **Check Answer** input box at the bottom.

- **Instant Feedback:** The app will display a message indicating if your answer is correct. It is programmed to recognize mathematically equivalent expressions (e.g., it knows that $2x$ is the same as $x + x$).
- **Show Derivative:** If you cannot find your error, toggle the **Show Derivative** checkbox to see the correct symbolic derivative.

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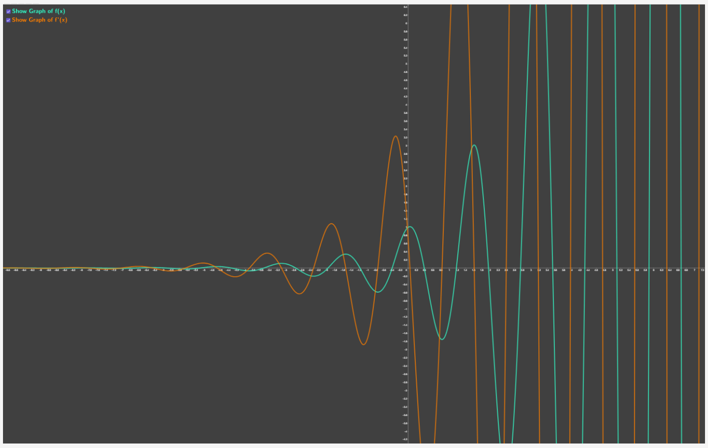
Guide 3.4. Graphical Visualization The window on the right allows you to connect the algebra to the geometry of the curves:

- **Show $f(x)$:** Plots the original function.
- **Show $f'(x)$:** Plots the derivative. You can observe how the derivative is positive when $f(x)$ is increasing, negative when it is decreasing, and 0 when the function has a maximum or a minimum.



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