Complex Analysis

(Visualizing Complex Functions)

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February 15, 2026

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Interactive Animation: Visualizing Complex Functions

Contents

1 Introduction: Real vs Complex Functions

In **real analysis**, we study functions that take in a real number and give out another real number. if f is such a function, then, we represent it as follows:

$$f : \mathbb{R} \to \mathbb{R}. \tag{1}$$

This notation tells us that f takes in a number from the **domain** $\mathbb{R}$, and gives out a number from the **codomain** $\mathbb{R}$. The **rule** that tells us which number f gives out can be expressed in many differnt ways. For instance,

$$f(x) = x^2\,, \tag{2}$$

defines a function $f : \mathbb{R} \to \mathbb{R}$ that assigns to each number $x \in \mathbb{R}$, the number $x^2 \in \mathbb{R}$.

In **complex analysis** the situation is very similar, but now, our functions are **rules** that assign a complex number to another complex number. If g is a **complex function**, then, we write:

$$g : \mathbb{C} \to \mathbb{C}. \tag{3}$$

We can define these functions just like we did in the real case. For example, we can define a complex function $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = z + 1. \tag{4}$$

Representing Real Functions

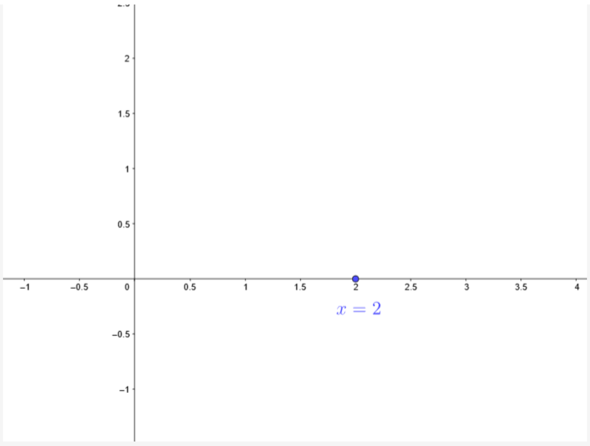
In the **real case**, it is very easy to represent a function f. All we have to do is:

1. Choose some $x \in \mathbb{R}$.
2. Compute $f(x) \in \mathbb{R}$.
3. Draw the point $(x, f(x))$.
4. Do this for every value of $x \in \mathbb{R}$.

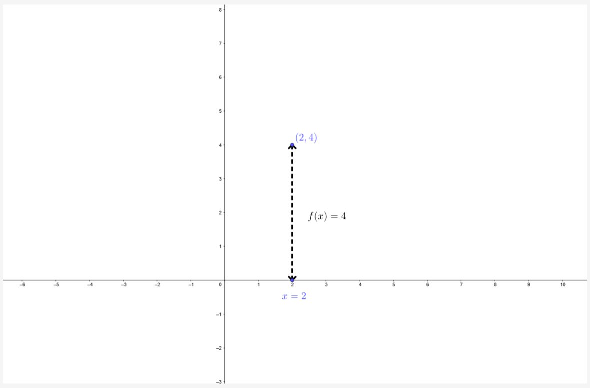
Example 1.1. Representing the Function $f(x) = x^2$

We want to represent the function $f(x) = x^2$.

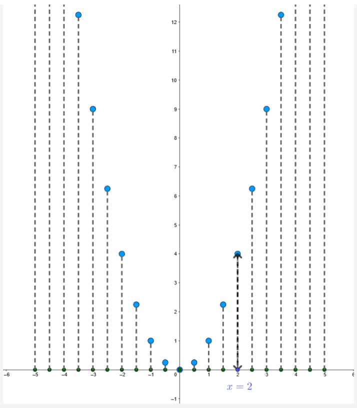
1. We will choose the number $x = 2$.



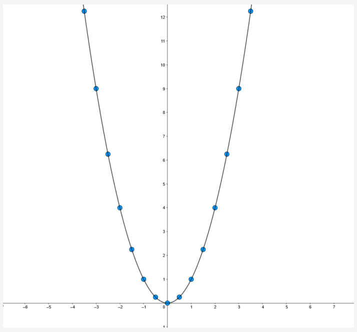
2. The value of the funtion is $f(2) = (2)^2 = 4$.
3. Now, we can draw the point $(x, f(x)) = (2, 4)$.



4. We do the same for many other points $x \in \mathbb{R}$.



We can infer the shape of the function.



To get the exact shape, we have to do this for **every point** $x \in \mathbb{R}$.

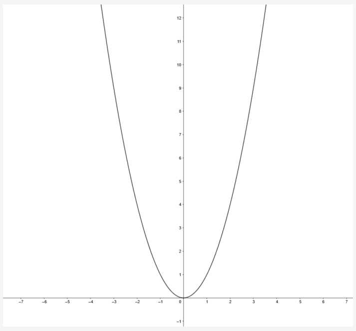


Figure 1: Graphical Representation of the function $f(x) = x^2$.

$\blacksquare$

Representing Complex Functions

When we consider functions $g : \mathbb{C} \to \mathbb{C}$, it is harder to represent them graphically, given the fact that we **need more dimensions**.

Remember, a **real function** $f : \mathbb{R} \to \mathbb{R}$ takes in a **single number** x, and gives out another **single number** $f(x)$. This means that we only need **2 axes to represent the values of** x **and** $f(x)$, namely, the **horizontal axis** can be used to represent the values of x, and the **vertical axis** can be used to represent the values of $f(x)$. With this convention, a function f can be represented easily with graphs like the one in Figure 1.

The problem with **complex functions** is that a complex function $g : \mathbb{C} \to \mathbb{C}$ takes in a **complex number**

$$z = a + i \cdot b = (a, b)\,, \tag{5}$$

which is made up of **2 real numbers** $a, b \in \mathbb{R}$, and gives out **another complex number**

$$g(z) = c + i \cdot d = (c, d)\,, \tag{6}$$

which is made up of **2 other real numbers** $c, d \in \mathbb{R}$. So, to represent g the same way we represented f, we would need **4 axes**, something that is not possible, as we can visualize at most **3 spatial directions**.

This means that we need to **find another way of representing the function** g.

There are many different possibilites for doing this, like **domain coloring**, **plotting two 3D graphs, one for the modulus and another one for the angle**, etc.

However, the way we will do it today will be much simpler, and in my opinion, is better suited for understanding how complex functions actually work.

2 Representing a Complex Function

Let's consider a complex function

$$f : \mathbb{C} \to \mathbb{C}. \tag{7}$$

We know that this type of function takes in a complex number

$$z = a + i \cdot b \in \mathbb{C} \tag{8}$$

and gives out another complex number

$$f(z) = c + i \cdot d \in \mathbb{C}. \tag{9}$$

As we discussed previously, we need **2 axes to represent** z and another **2 axes to represent** $f(z)$, so, we will just put the axes for z on the **left**, and the axes for $f(z)$ on the **right**.

Let's consider the function

$$f(z) = z^2 \tag{10}$$

to see how this representation works.

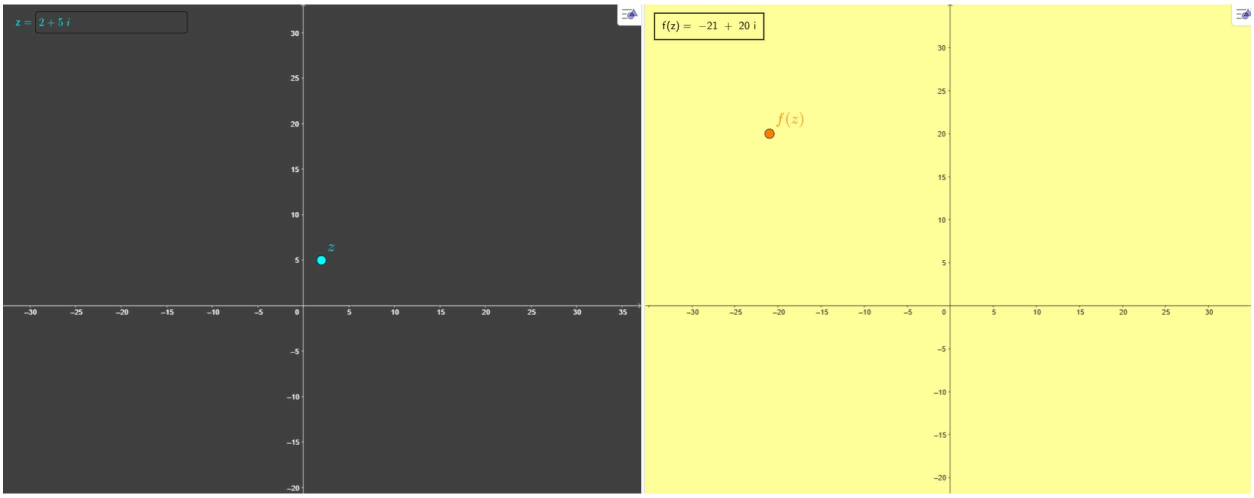


Figure 2: Domain Plane (left, in black) and Image Plane (right, in yellow).

The **black plane** represents the **domain of the function**, namely, the values of z, so, we will call it the **Domain Plane**. Similarly, the **yellow plane** represents the **image of the function**, that is, the values of $f(z)$, so, we will call it the **Image Plane**.

In Figure 2, we can see that the value of z is

$$z = 2 + i \cdot 5 \tag{11}$$

and that the value of $f(z)$ is

$$f(z) = -21 + i \cdot 20. \tag{12}$$

If we want to explore how this function behaves, we will move around the point z and see where the point $f(z)$ ends up.

To make it even more visual, we can **draw some path with the point** z and see **what the path that** $f(z)$ **follows looks like**.

Let's see some examples:

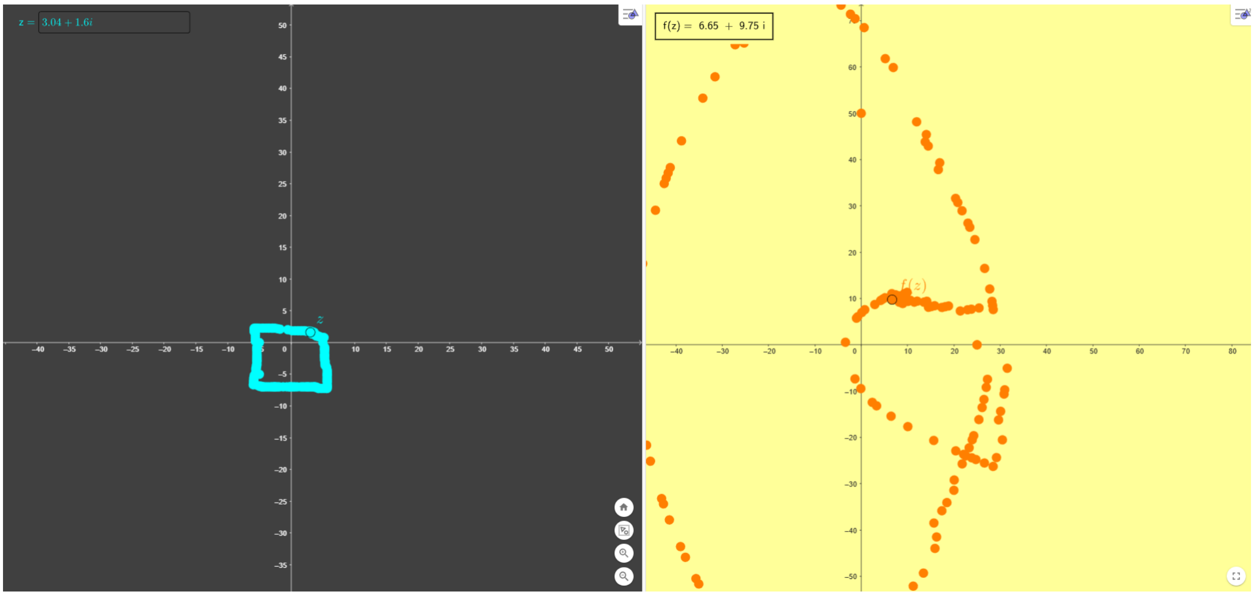


Figure 3: Square Path with the point z.

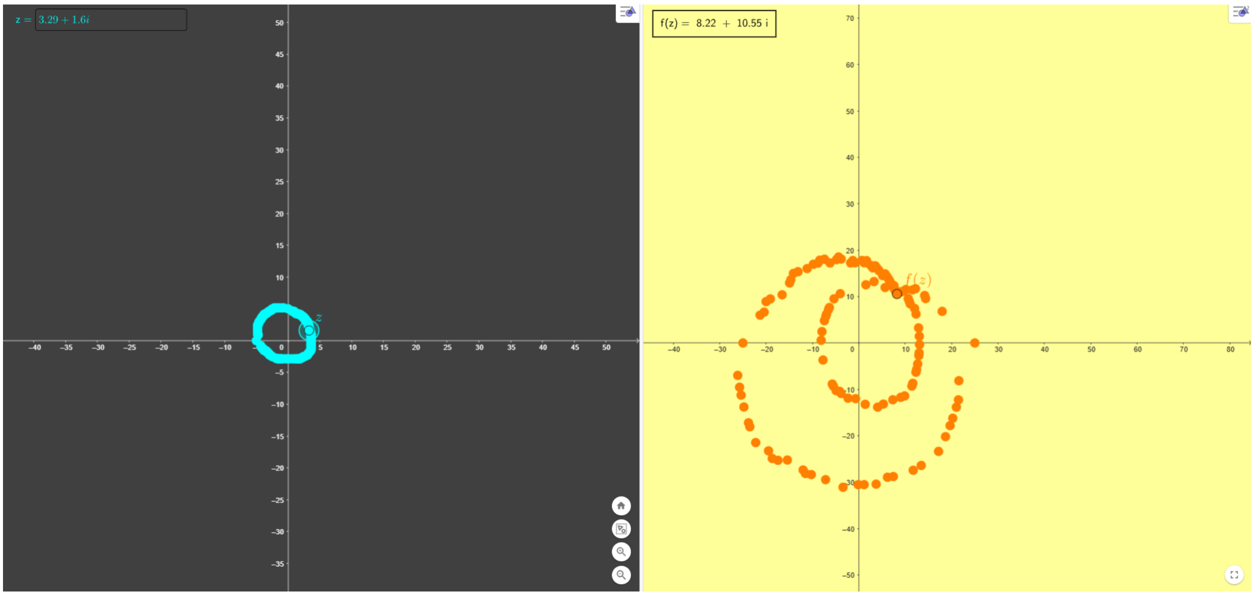


Figure 4: Circular Path with the point z.

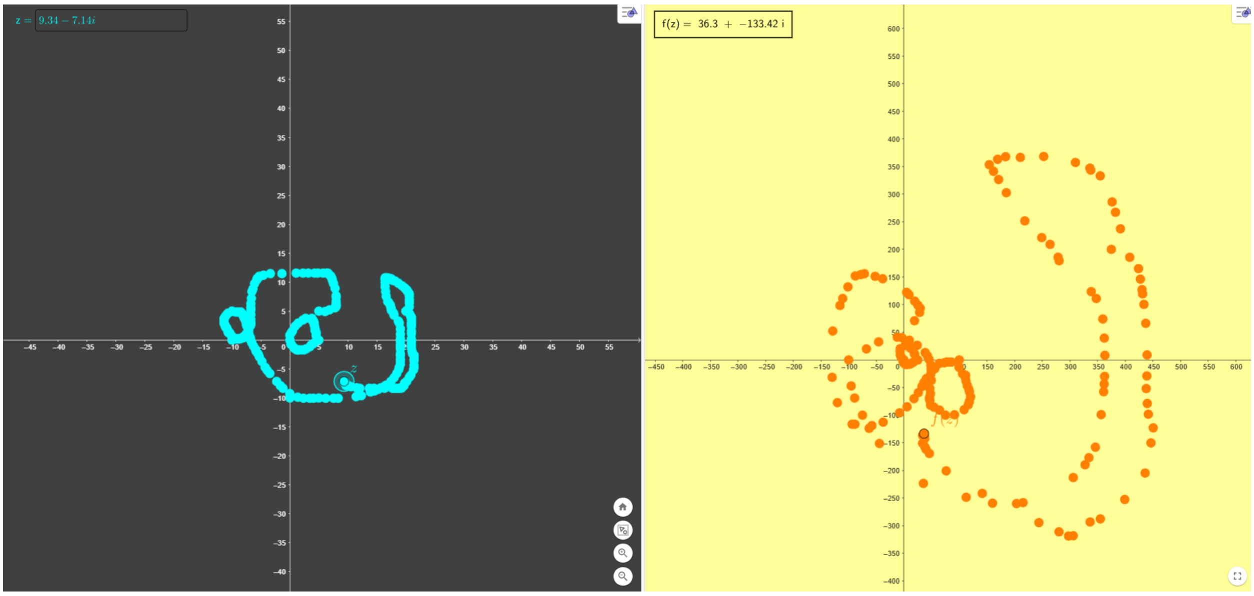
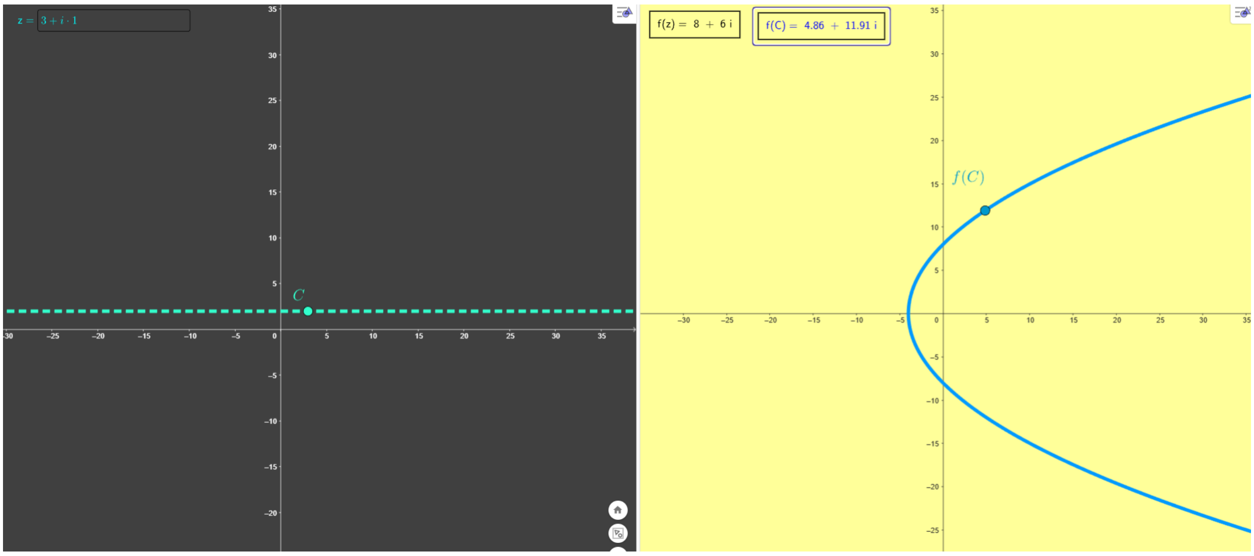


Figure 5: Random Path with the point z.

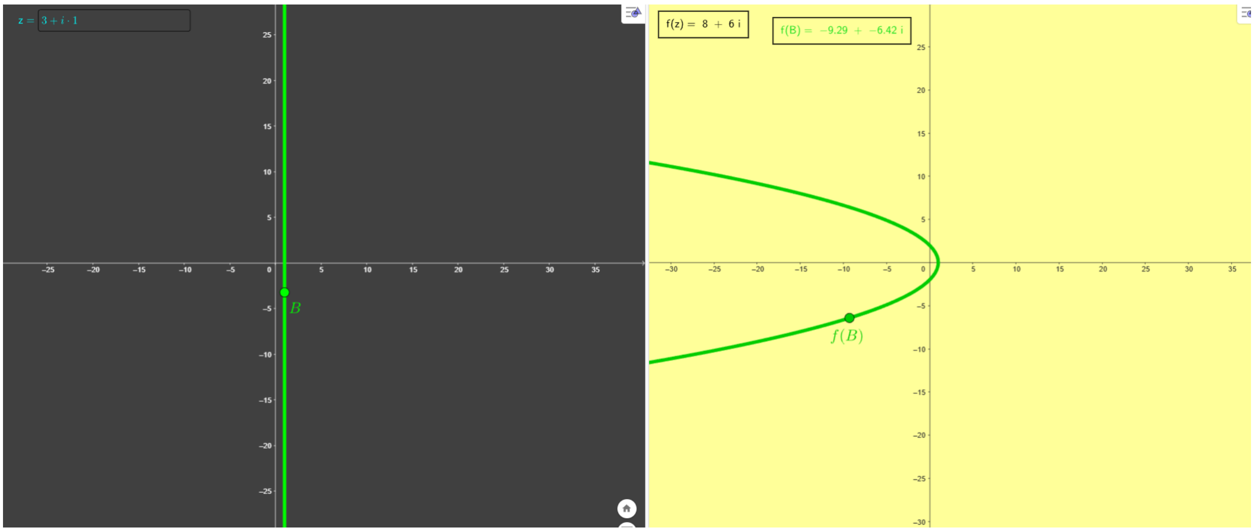
Usually, it is interesting to study **what the horizontal and vertical paths transform to after we apply f.**

To see this, we can draw a **horizontal line** in the **Domain Plane** and see its image in the **Image Plane**



We can slide the point $C \in \mathbb{C}$ along the horizontal line, and see how $f(C)$ slides along the parabola plotted on the Image plane.

Similarly, we can draw a **vertical line** in the **Domain Plane** and see its image in the **Image Plane**



We can also slide the point $B \in \mathbb{C}$ along the vertical line, and see how $f(B)$ slides along the other parabola plotted on the Image plane.

Defining New Paths

If we want to see how some specific path transforms, without having to draw it by hand, we can just define it, and plot its image.

If we have a path in the plane

$$\mathbf{r}(t) = \Big(x(t), y(t)\Big) \tag{13}$$

we can apply f to the path by considering the point

$$z = x(t) + i \cdot y(t)\,, \tag{14}$$

namely,

$$f(\mathbf{r}(t)) = f\Big(x(t) + i \cdot y(t)\Big) \tag{15}$$

For example, if we consider the path

$$\begin{aligned} x(t) &= t^2 \\ y(t) &= t^3 \end{aligned} \tag{16}$$

for every $t \in [t_{Min}, t_{Max}]$, the image is given by

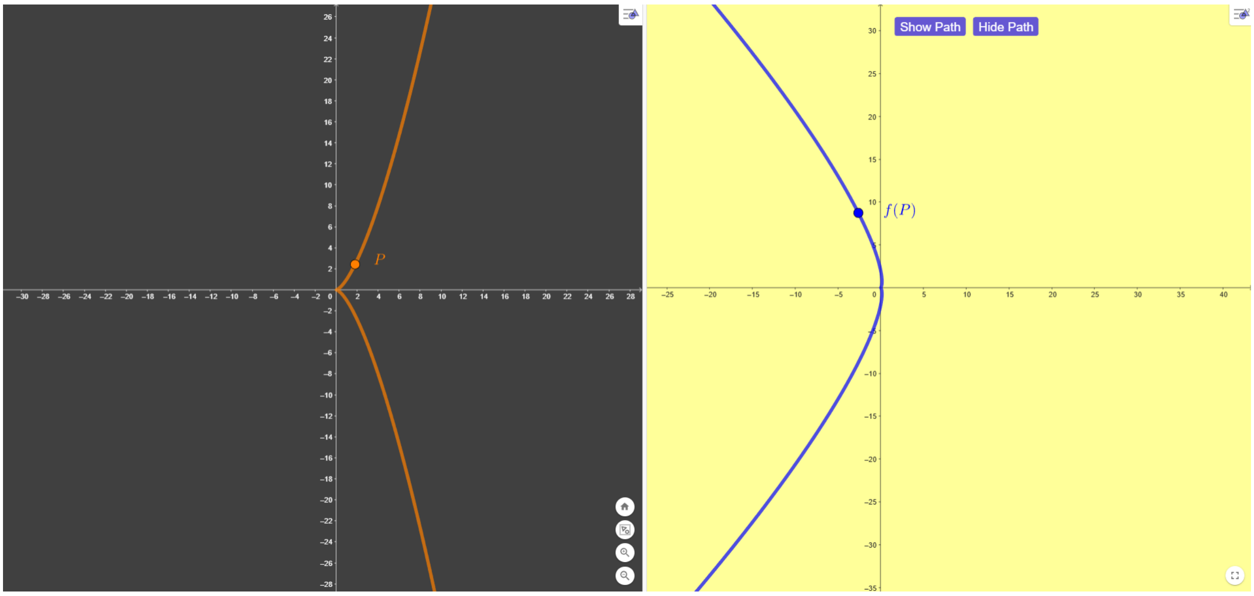
$$f(x(t) + i \cdot y(t)) = f(t^2 + i \cdot t^3) \tag{17}$$

for every $t \in [t_{Min}, t_{Max}]$.

If we compute this for the function $f(z) = z^2$, we get

$$f(z) = z^2 = \Big(t^2 + i \cdot t^3\Big)^2 = (t^4 - t^6) + i \cdot (2t^5). \tag{18}$$

We can represent this graphically as follows:



3 Guide for Using the Interactive Animation

Now that we have seen the basics of how to represent a complex function, we will explain how we can play around with these concepts using the following app: App: Visualizing Complex Functions.

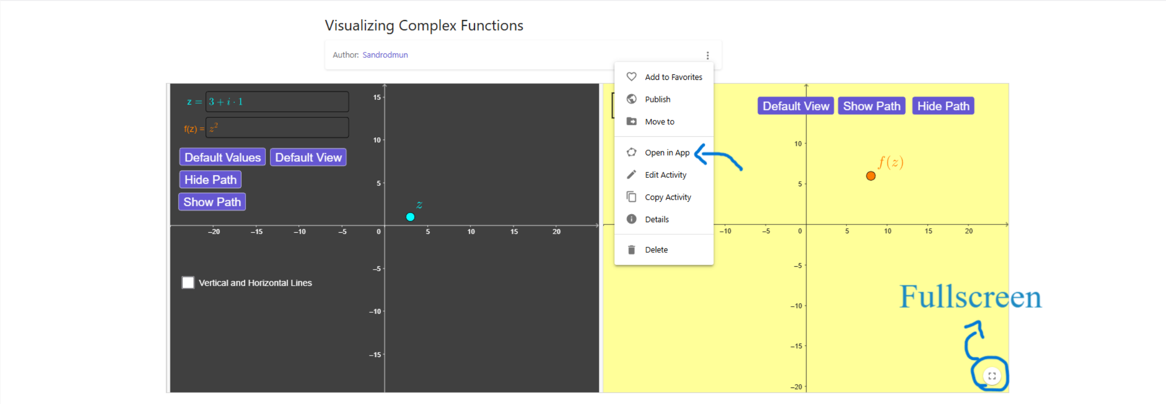
Guide 3.1. Choosing a Function

1. Once we click on the link, we will be taken to the follwing window:

2. There are 2 options:
   - Click the **Fullscreen button**:
     – We see the app in fullscreen

- We will **not** be able to edit things or change the sizes of the Domain and Image planes.

- Click the **3 dots** next to the **author label**, and select **Open in App**:
  - You will be able to edit anything you want
  - You can change the sizes of the windows
  - You can see it fullscreen too
  - You can hide, delete, add, etc., anythnig you want



I recommend choosing **Open in App** always, as it is much more flexible.

3. Now, you can choose the point z either by using the **input box** or by dragging the point.
4. Similarly, you can choose the function usnig the input box (always write $f(z)$ in terms of the variable z, otherwise, the function will not be recognized).
5. **Defaul View:** After playing around with the app, you will probably end up in some place far away from the origin, or in some place that isn't useful if you change the function or the point. To fix that, click the **Default View** button. Each button restores the default view of its corresponding window.

■

Guide 3.2. Paths

If you want to show the path you draw when dragging point z, you can click the **Show Path** button, and if you want to also see what path the point $f(z)$ follows, you can click the **Show Path** button of the Image Plane.

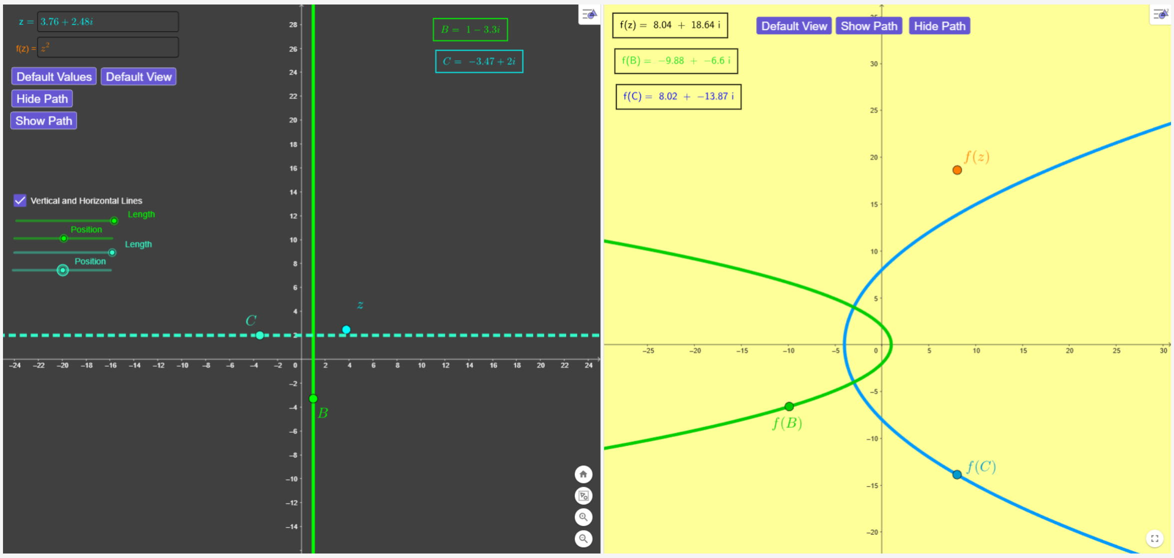
To delete the paths either Zoom in/out in the corresponding window, or click the **Hide**

Path button of the corresponding window.

After clicking the **Hide Path** button, you will no longer see the path, if you want to reactivate it, just click **Show Path** again. ■

Guide 3.3. Horizontal and Vertical Lines

If you click onn **Vertical and Horizontal Lines**, you will get the following:



The **green slider called "Position"** allows us to move the vertical line up and down, and the **green slider called "Length"** allows us to make the horizontal line longer or shorter.

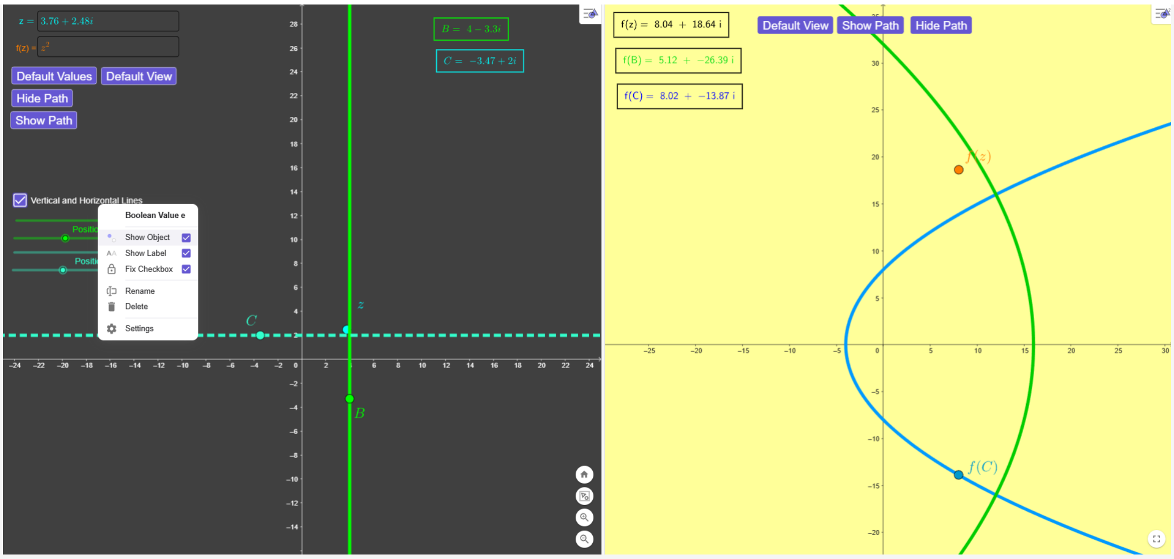
The blue sliders do the same for the horizontal line.

On the top parts of the windows we have the values of the points B and C, and their images $f(B)$ and $f(C)$. ■

Guide 3.4. Hiding Objects

Sometimes the screen gets too crowded, and if there are things that you don't need to see at that time, they are just bothering.

To get rid of something, you just have to click on it, and unckeck the **Show Object** box.



You can do this for buttons, checkboxes, points, input boxes, sliders, curves, axes, etc.

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Guide 3.5. Defining a Path Parametrically

If instead of drawing a path by hand, we prefer to draw it exactly, we can define it parametrically by selecting the **Draw Parametric Path** checkbox.

Then, all we have to do is to select the interval for the parameter $t \in [t_{Min}, t_{Max}]$, and the parametric expressions for $x(t)$ and $y(t)$.

You can drag the point D along the curve to see what its image looks like.