Linear Algebra

(Change of Basis in $\mathbb{R}^2$)

Sandro Rodríguez Muñoz

February 17, 2026

YouTube Channel: Sandrodmun

Interactive Animation: Change of Basis in R^2

Contents

1 Introduction: What is a Basis? 2

2 Changing Bases 4

3 The Change of Basis Matrix 4

4 Guide for Using the Applet 6

1 Introduction: What is a Basis?

In linear algebra, a **basis** for a vector space is a set of **linearly independent vectors** that **span** the entire space. This means that **any** vector $\vec{v}$ in the can be written uniquely as a linear combination of these basis vectors.

Let's consider the example of the vector space $\mathbb{R}^2$ consisting of **vectors in the plane starting at the origin**.

Then, a set $\mathcal{B}_u = \{\vec{u}_A, \vec{u}_B\}$ is a basis for $\mathbb{R}^2$ iff:

- **Linear Independence:** Neither of the vectors can be written as a multiple of the other, namely,

$$\nexists k \in \mathbb{R} \text{ such that: } \vec{u}_A = k\vec{u}_B. \tag{1}$$

  This just means that none of the vectors are not parallel.

- **Spanning:** Every vector $\vec{v} \in \mathbb{R}^2$ can be expressed as $\vec{v} = c_1\vec{u}_A + c_2\vec{u}_B$ for some unique scalars c_1, c_2. We usually write this as

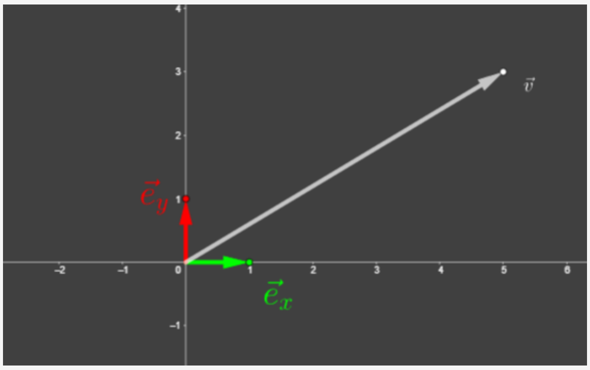
$$\vec{v} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_{\mathcal{B}_u}. \tag{2}$$

These scalars $c_1, c_2 \in \mathbb{R}$ are called the **coordinates** of $\vec{v}$ in the basis $\mathcal{B}_u$.

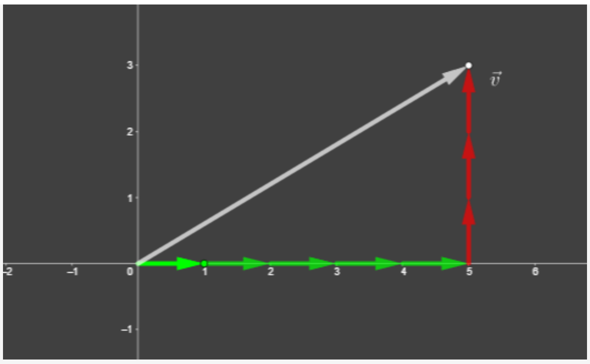
Example 1.1. The Canonical Basis vs. Alternative Bases

The most common way we represent vectors in $\mathbb{R}^2$ is using the **Canonical Basis** $\mathcal{B}_e = \{\vec{e}_x, \vec{e}_y\}$, where $\vec{e}_x$ is the vector that goes **1 unit to the right**, and $\vec{e}_y$ is the vector that goes **1 unit up**.

Let's consider some vector $\vec{v}$ in this basis:



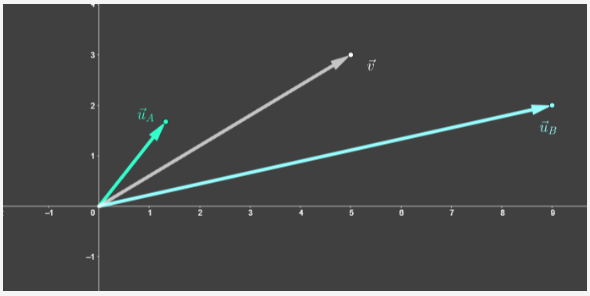
As we can see, this vector is obtained by stacking 5 $\vec{e}_x$ vectors, and 3 $\vec{e}_y$ vectors:



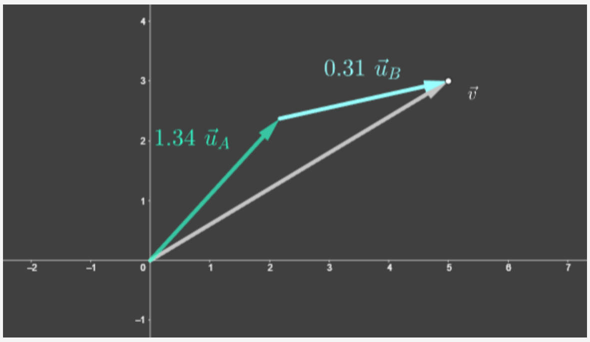
So, the **components of $\vec{v}$ in the $\mathcal{B}_e$ basis** are

$$\vec{v} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}_{\mathcal{B}_e} . \tag{3}$$

Now, if we consider another basis $\mathcal{B}_u = \{\vec{u}_A, \vec{u}_B\}$



we will get different coordinates for the vector $\vec{v}$. We need to stack **1.34 times the vector $\vec{u}_A$** and **0.31 times the vector $\vec{u}_B$**



So, the coordinates of $\vec{v}$ in this basis are

$$\vec{v} = \begin{pmatrix} 1.374 \\ 0.31 \end{pmatrix}_{\mathcal{B}_u}. \tag{4}$$

■

2 Changing Bases

In the previous section, we saw that the same geometric object, the vector $\vec{v}$, can be described by different sets of coordinates depending on the basis we choose. The process of translating these coordinates from one basis to another is called a **change of basis**.

Suppose we have two different bases for $\mathbb{R}^2$:

- $\mathcal{B}_u = \{\vec{u}_A, \vec{u}_B\}$
- $\mathcal{B}_w = \{\vec{w}_A, \vec{w}_B\}$

Let's say that we know the coordinates of a vector $\vec{v}$ in the $\mathcal{B}_u$ basis

$$\vec{v} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_{\mathcal{B}_u}. \tag{5}$$

Then, we want to find the **coordinates of $\vec{v}$ in the $\mathcal{B}_w$ basis**

$$\vec{v} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}_{\mathcal{B}_w}. \tag{6}$$

To find them, all we have to do is to compute the **change of basis matrix** $M_{w \leftarrow u}$, which **translates coordinates from $\mathcal{B}_u$ to $\mathcal{B}_w$**, and multiply by the coordiantes in $\mathcal{B}_u$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}_{\mathcal{B}_w} = M_{w \leftarrow u} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_{\mathcal{B}_u}. \tag{7}$$

So, all we have to do to change bases if to compute the **change of basis matrix**, which we will explain in the next section.

3 The Change of Basis Matrix

To find the specific form of the matrix $M_{w \leftarrow u}$, we must look at the relationship between the two bases. By definition, if $\vec{v}$ has coordinates in $\mathcal{B}_u$, then:

$$\vec{v} = c_1 \vec{u}_A + c_2 \vec{u}_B. \tag{8}$$

Our goal is to express this same vector $\vec{v}$ in terms of $\vec{w}_A$ and $\vec{w}_B$. To do this, we represent the "old" basis vectors $\vec{u}_A$ and $\vec{u}_B$ using the "new" basis $\mathcal{B}_w$:

$$\vec{u}_A = \begin{pmatrix} p_1^A \\ p_2^A \end{pmatrix}_{\mathcal{B}_w} \quad \text{and} \quad \vec{u}_B = \begin{pmatrix} p_1^B \\ p_2^B \end{pmatrix}_{\mathcal{B}_w}. \tag{9}$$

Substituting these into the equation for $\vec{v}$, we get:

$$\vec{v} = c_1(p_1^A \vec{w}_A + p_2^A \vec{w}_B) + c_2(p_1^B \vec{w}_A + p_2^B \vec{w}_B). \tag{10}$$

Rearranging the terms to group $\vec{w}_A$ and $\vec{w}_B$ together:

$$\vec{v} = (p_1^A c_1 + p_1^B c_2)\vec{w}_A + (p_2^A c_1 + p_2^B c_2)\vec{w}_B. \tag{11}$$

This gives us the coordinates of $\vec{v}$ in the basis $\mathcal{B}_w$

$$\vec{v} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}_{\mathcal{B}_w} = \begin{pmatrix} p_1^A c_1 + p_1^B c_2 \\ p_2^A c_1 + p_2^B c_2 \end{pmatrix}_{\mathcal{B}_w}. \tag{12}$$

This can be written as

$$\vec{v} = \begin{pmatrix} p_1^A & p_1^B \\ p_2^A & p_2^B \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_{\mathcal{B}_u}. \tag{13}$$

Therefore, the **change of basis matrix** is constructed by placing the coordinates of the old basis vectors into the columns of a matrix:

$$\boxed{M_{w\leftarrow u} = \begin{pmatrix} | & | \\ \vec{u}_A & \vec{u}_B \\ | & | \end{pmatrix}_{\mathcal{B}_w} = \begin{pmatrix} p_1^A & p_1^B \\ p_2^A & p_2^B \end{pmatrix}}. \tag{14}$$

The Change of Basis Process

To change the coordinates of any vector from $\mathcal{B}_u$ to $\mathcal{B}_w$, follow these three steps:

1. **Find the coordinates** of the original basis vectors ($\vec{u}_A$ and $\vec{u}_B$) in terms of the target basis ($\mathcal{B}_w$).
2. **Construct the matrix** $M_{w\leftarrow u}$ by using those coordinates as the columns of the matrix.
3. **Multiply** the matrix by the coordinates of $\vec{v}$ in the original basis to find the new coordinates.

Example 3.1. Example: Calculating New Coordinates

Let $\mathcal{B}_w = \{\vec{w}_A, \vec{w}_B\}$ be our target basis. Suppose the vectors of $\mathcal{B}_u$ are expressed in $\mathcal{B}_w$ as:

$$\vec{u}_A = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathcal{B}_w} \quad \text{and} \quad \vec{u}_B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathcal{B}_w}. \tag{15}$$

Then the change of basis matrix is:

$$M_{w\leftarrow u} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}. \tag{16}$$

If a vector $\vec{v}$ has coordinates $\begin{pmatrix} 3 \\ 2 \end{pmatrix}_{\mathcal{B}_u}$, its coordinates in $\mathcal{B}_w$ are:

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}_{\mathcal{B}_w} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}_{\mathcal{B}_w}. \tag{17}$$

■

Invertibility and Linear Independence

The matrix $M_{w\leftarrow u}$ is **nonsingular** (invertible) if and only if its columns are linearly independent. Since the columns are the coordinates of $\vec{u}_A$ and $\vec{u}_B$, and these vectors form a basis, they are by definition linearly independent. If they were parallel, the matrix would have a determinant of zero, meaning that we would lose a dimension and could not uniquely describe the plane.

Because the matrix is invertible, we can always move back to the original basis using the **inverse change of basis matrix**:

$$M_{u\leftarrow w} = (M_{w\leftarrow u})^{-1} \tag{18}$$

This inverse matrix allows us to translate coordinates from $\mathcal{B}_w$ back to $\mathcal{B}_u$ using the same multiplication logic:

$$\boxed{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_{\mathcal{B}_u} = M_{u\leftarrow w} \cdot \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}_{\mathcal{B}_w}}. \tag{19}$$

The elements of the matrix $M_{u\leftarrow w}$ are given by

$$\boxed{M_{w\leftarrow u} = \frac{1}{p_1^A p_2^B - p_1^B p_2^A} \begin{pmatrix} p_2^B & -p_1^B \\ -p_2^A & p_1^A \end{pmatrix}}. \tag{20}$$

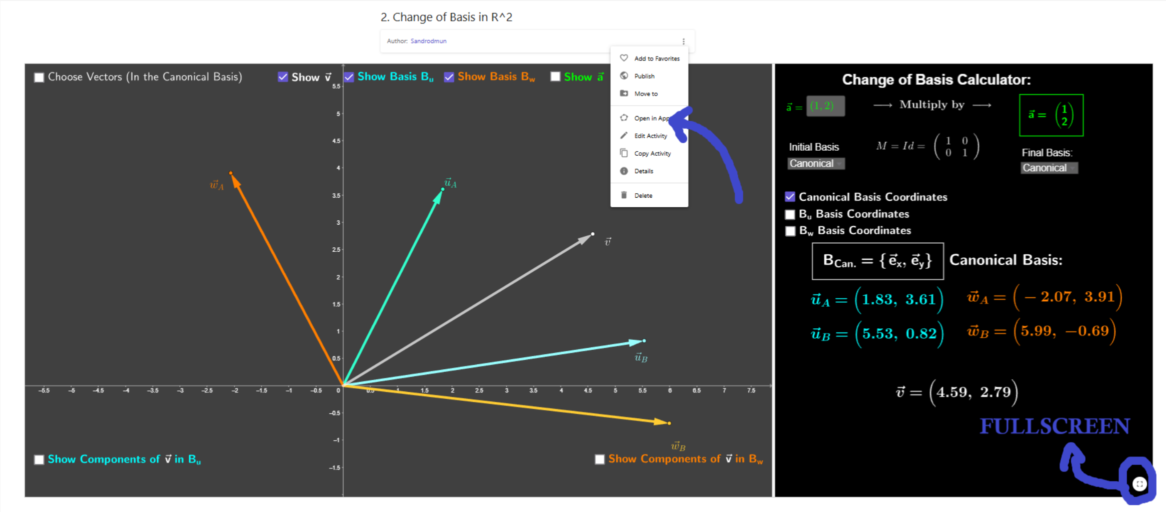
4 Guide for Using the Applet

Now that we have established the mathematical framework for basis transformations, we will explain how to use the following app to visualize these changes dynamically: App: Change of Basis in $\mathbb{R}^2$.

The applet is designed with two distinct environments: a geometric plane on the left to see the vectors, and a numerical calculator on the right to handle the coordinates and matrices.

Guide 4.1. Opening the Applet

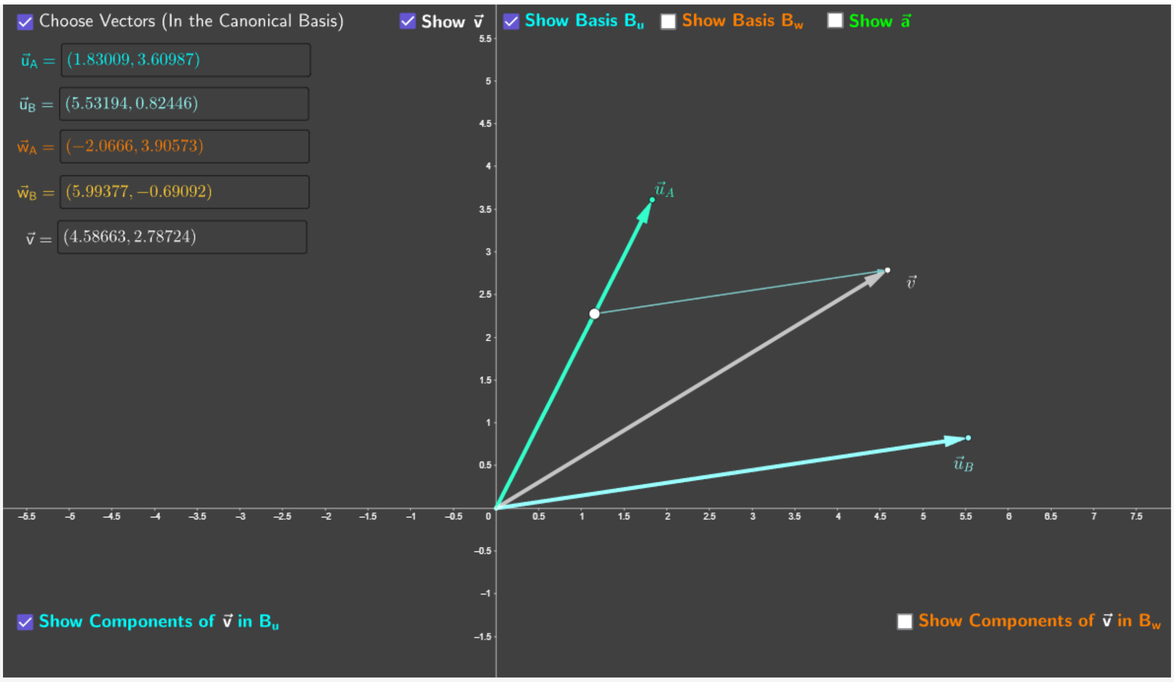
1. Once you click on the link, you will see the main interface.
2. For this specific applet, it is **highly recommended** to click the **3 dots** in the top right corner and select **Open in App**.
   - This applet uses multiple windows and input boxes that require the flexibility of the App View to be resized.
   - You will be able to adjust the divider between the training panel and the graph deck.



■

Guide 4.2. The Plane View (Left Window) The left window shows the visual representation of the vectors.

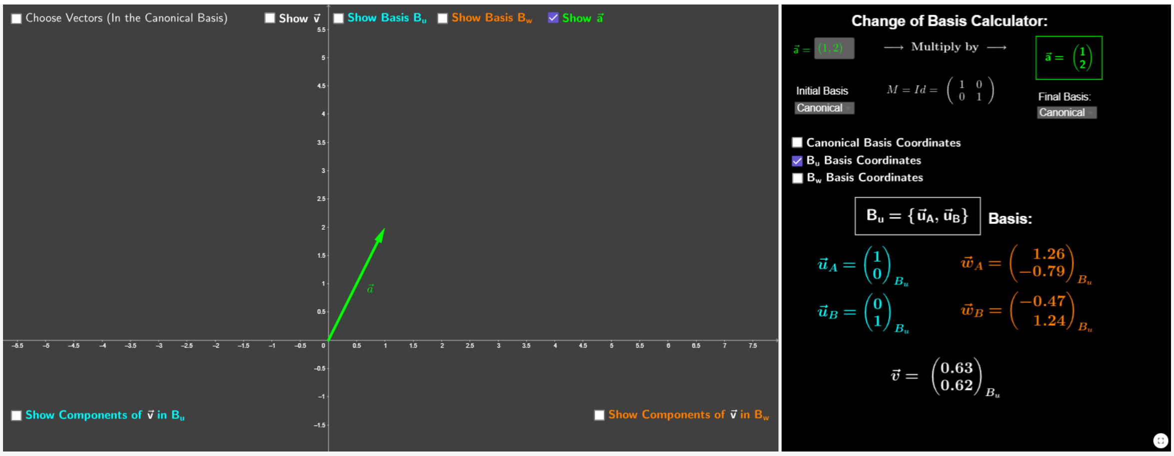
- **Modifying Vectors:** You can change the values of $\vec{v}$, $\vec{u}_A$, $\vec{u}_B$, $\vec{w}_A$, and $\vec{w}_B$ by dragging their heads directly on the screen, or by using the input boxes for specific coordinates by clicking the **Choose Vectors** checkbox.
- **Toggling Visibility:** Use the checkboxes to show or hide the basis vectors.
- **Show Components:** If you want to see how a vector is projected onto the basis, click the **Show Components** checkboxes. This will show the "shadows" of $\vec{v}$ along the basis vectors.



∎

Guide 4.3. The Calculator (Right Window) The right window functions as a dedicated coordinate translator.

- **Choose Basis:** We can choose which basis we want to consider by clicking the checkboxes. We can see the coordinates of all the vectors in the **Canonical basis**, the $\mathcal{B}_u$ **basis**, or the $\mathcal{B}_w$ **basis**.
- **Custom Vector Calculator:** Use the input box for the vector $\vec{a}$ and select its source basis. The app will show the **change of basis matrix** from the **initial basis** to the **final basis**, and it will compute the coordinates of $\vec{a}$ in the final basis. You can also show this vector in the left window.



Guide 4.4. Special Cases to Explore

- **The Identity Matrix:** Notice that when changing from a basis to itself, the change of basis matrix is the identity.
- **Singularity:** Try to make $\vec{u}_A$ and $\vec{u}_B$ parallel. You will see that the coordinates become undefined as the matrix becomes non-invertible and the "grid" collapses into a single line.