DESIGN AND ANALYSIS OF ALGORITHMS

CS 4120/5120

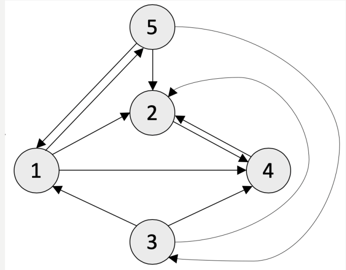
GRAPH REPRESENTATION

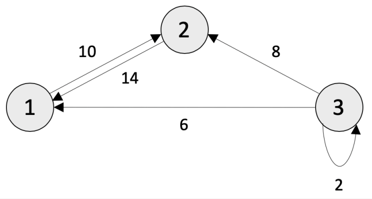
AGENDA

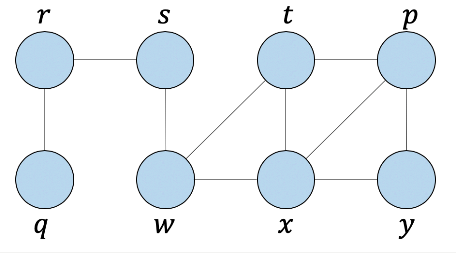
- Implementation of graphs
  - Adjacency list
  - Adjacency matrix

GRAPH
DIRECTED AND UNDIRECTED

- Computer science and graph theories
  - A graph, denoted by G is represented by $G = (V, E)$, where
    - V is the **set** of vertices (nodes), and
    - E is the **set** of edges
  - Note that $G = (V, E)$ can represent both ***directed*** and ***undirected*** graphs.
- Examples of directed (left two) and undirected graph



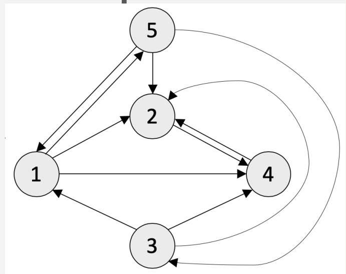


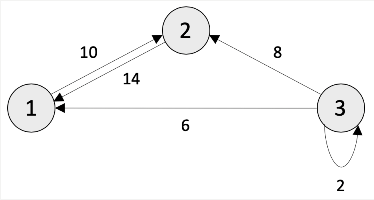


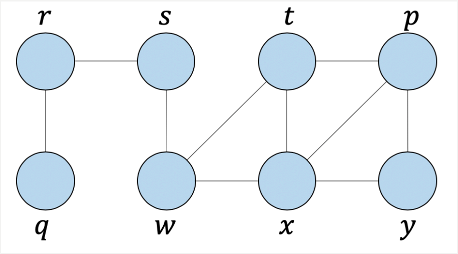
GRAPH
WEIGHTED AND UNWEIGHTED

- In addition to directed and undirected graph, we also have ***weighted*** and ***unweighted*** graphs.
- The edges in a ***weighted*** graph have their own weights (or distances between the two adjacent vertices)
- The edges in an **unweighted** graph may be thought of as weighing 1.

- Examples of weighted (middle) and unweighted graph







GRAPHS IN COMPUTER WORLD

- Some problems can be modeled as a graph related problem.
- We can then solve these problems by developing a graph algorithm.
- Need a way to represent graphs in programming languages

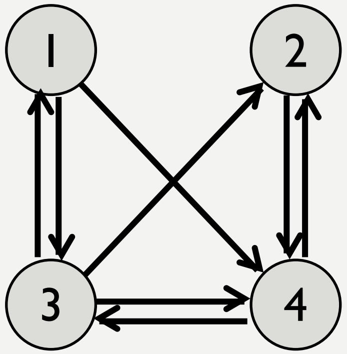
- In the field of ***computer science***, a graph can be represented in the following ways
  - Adjacency list
  - Adjacency matrix
- We shall take a look at both representations.

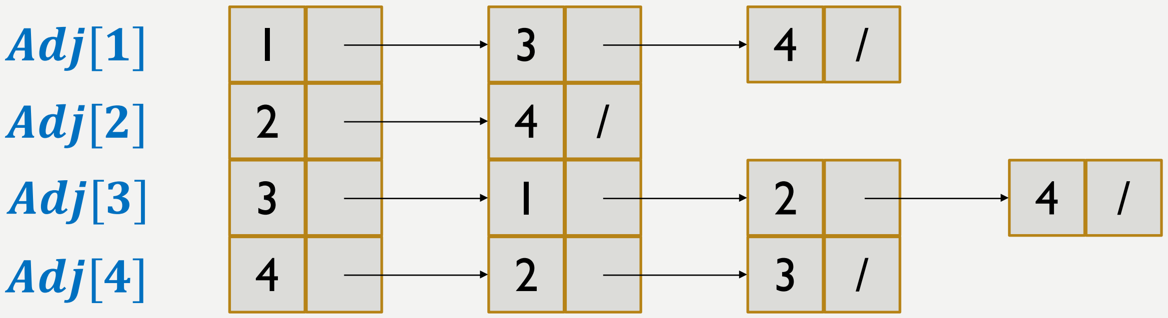
THE ADJACENCY-LIST REPRESENTATION

- Given a graph $G = (V, E)$, the adjacency list consists of **an array adj** of $|V|$ lists, one for each vertex in V.
- For each $u \in V$, the adjacency list $\boldsymbol{Adj[u]}$ **contains all the vertices $\boldsymbol{v}$** such that **there is an edge $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}$.**
  - $Adj[u]$ consists of all the vertices adjacent to u in G.
  - In pseudocode, we treat Adj as an attribute of the graph, i.e.,
    - $G.Adj[u]$ represents the adjacent vertices to u in graph G.
- $\boldsymbol{Adj[u]}$ can be implemented by ***linked list***.

THE ADJACENCY-LIST REPRESENTATION VISUALIZATION

- Consider the graph (left) and its corresponding adjacency-list representation (right)
  - The representation contains an array of ***linked lists***.
    - Each element $\boldsymbol{Adj}[\boldsymbol{i}]$ is a linked list of the vertices adjacent to vertex $\boldsymbol{i}$.





THE ADJACENCY-LIST REPRESENTATION PROPERTIES

- If G is a ***directed*** **graph**, the sum of the lengths of all the adjacency lists is $|E|$.
  - An edge of the form (u, v) is represented by having v appear in $Adj[u]$.

- If G is **undirected** **graph**, the sum of the lengths of all the adjacency lists is $2|E|$,
  - If (u, v) is an **undirected** edge, then u appears in v's adjacency list and vice versa.

THE ADJACENCY-LIST REPRESENTATION WITH WEIGHT

- If G is a ***weighted*** **graph**, meaning that each edge has an associated **weight**, typically given by a **weight function** $w: E \rightarrow \mathbb{R}$, where $\mathbb{R}$ denotes **real numbers**.

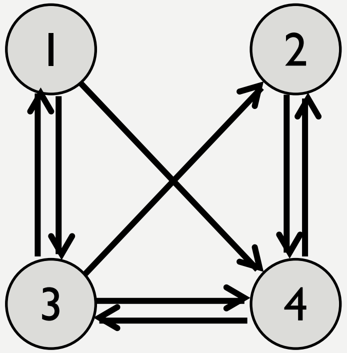
- Example
  - Let $G = (V, E)$ be a weighted graph with weight function w.
  - We simply store the weight $w(u, v)$ of the edge $(u, v) \in E$ with vertex v in u's adjacency list.

THE ADJACENCY-MATRIX REPRESENTATION

- Given a graph $G = (V, E)$.
  - We assume that **vertices are numbered as $1, 2, 3, \dots, |V|$** in some arbitrary manner.
  - The adjacency matrix consists of a $|V| \times |V|$ matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & if\ (i,j) \in E, \\ 0 & otherwise. \end{cases}$$

- Example



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

THE ADJACENCY-MATRIX REPRESENTATION PROPERTIES

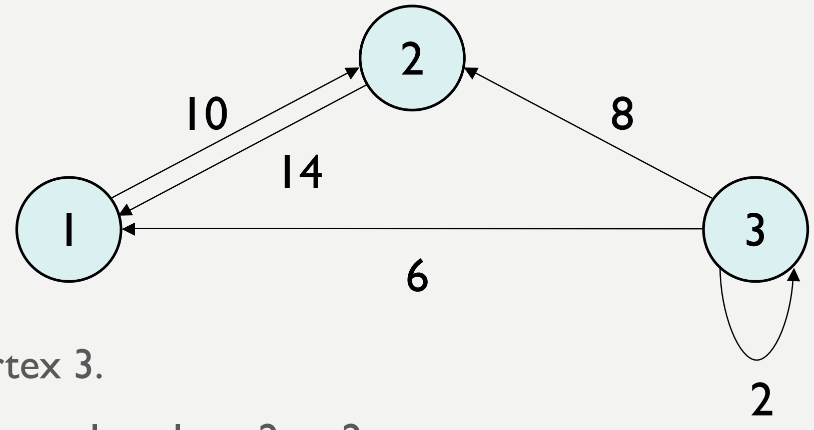
- The adjacency matrix of a graph **requires $\Theta(|V|^2)$ memory**, *independent of the number of edges* in the graph.

- The adjacency matrix of an **undirected** graph is its own transpose: $\boldsymbol{A} = \boldsymbol{A}^{T}$.
- If an edge does not exist, we can store a $\mathtt{NIL}$ value as its corresponding matrix entry, though for many problems it is convenient to use a value such as 0 or ∞.
  - By convention
    - we use ∞ for an entry $[u, v]$ to indicate v is ***unreachable*** from u.
    - we use 0 for an entry $[u, u]$ (diagonal entries) when there is no cyclic edge that comes back to vertex u.

THE ADJACENCY-MATRIX REPRESENTATION WITH WEIGHT

- If $G = (V, E)$ is a ***weighted*** graph with edge weight function w, we can simply store the weight $w(u, v)$ of the edge $(u, v) \in E$ as the entry in row u and column v of the the adjacency **matrix**.
- Example
  - The adjacency-matrix of the weighted directed graph is

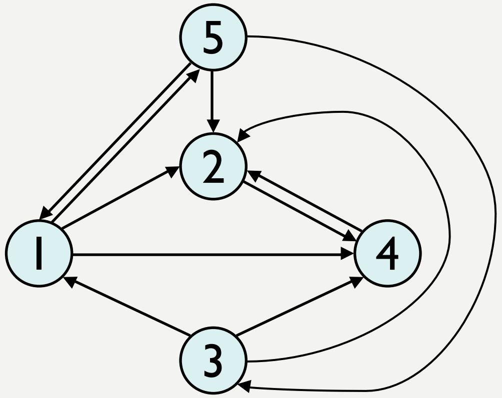
$$G = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 10 & \infty \\ 14 & 0 & \infty \\ 6 & 8 & 2 \end{bmatrix} \end{array}$$



    - Entry $\boldsymbol{G_{13}} = \infty$ means there is no edge from vertex 1 to vertex 3.
    - Entry $\boldsymbol{G_{11}} = \boldsymbol{G_{22}} = \mathbf{0}$ means there is no cyclic edge from vertex 1 to 1, or 2 to 2.

GRAPH REPRESENTATION
PRACTICE #1

- Show the adjacency-list representation of the graph. (A sample is given in the table.)
  - Order the adjacency lists in increasing order of the key of the vertex.
  - Within a linked-list, order the adjacent vertexes in increasing order of the key of the vertex.
  - A $\rightarrow$ should be used to indicate the "link" in a linked-list.
  - A cell should contain an element of array adj.
- $|E| =$ ____________.
- The sum of lengths of the adjacency lists is __________________.



$1 \rightarrow 2 \rightarrow 4 \rightarrow 5$

GRAPH REPRESENTATION
PRACTICE #1

- Show the adjacency-list representation of the graph. (A sample is given in the table.)
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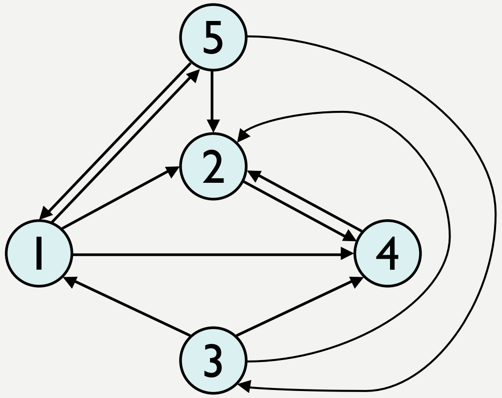
  - Within a linked-list, order the adjacent vertexes in increasing order of the key of the vertex.
  - A $\rightarrow$ should be used to indicate the "link" in a linked-list.
  - A cell should contain an element of array adj.
- $|E| =$ $\underline{\;11\;\;\;\;\;\;\;\;\;\;\;}$.
- The sum of lengths of the adjacency lists is $\underline{\;3+1+3+1+3=11\;}$.

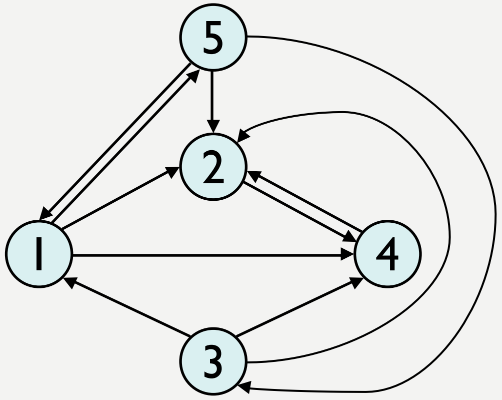


$1 \rightarrow 2 \rightarrow 4 \rightarrow 5$
$2 \rightarrow 4$
$3 \rightarrow 1 \rightarrow 2 \rightarrow 4$
$4 \rightarrow 2$
$5 \rightarrow 1 \rightarrow 2 \rightarrow 3$

GRAPH REPRESENTATION
PRACTICE #2

- Show the adjacency-matrix representation of the graph.

$$G = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \left[\begin{matrix} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{matrix} \right] \end{array}$$

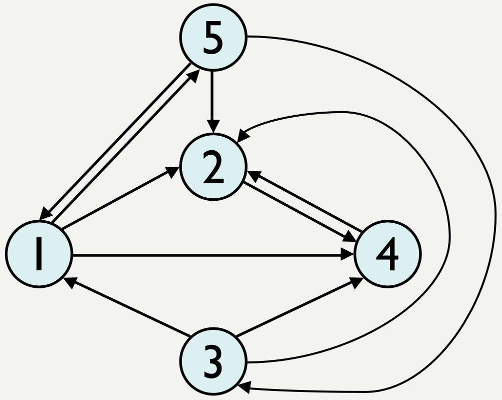


- $|E| =$ ____________. The number of entries $[i, j] \neq 0$ and $[i, j] \neq \infty$ is ________________.

GRAPH REPRESENTATION
PRACTICE #2

- Show the adjacency-matrix representation of the graph.

$$G = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & \infty & 1 & 1 \\ \infty & 0 & \infty & 1 & \infty \\ 1 & 1 & 0 & 1 & \infty \\ \infty & 1 & \infty & 0 & \infty \\ 1 & 1 & 1 & \infty & 0 \end{bmatrix} \end{array}$$



- $|E| =$ ____11____. The number of entries $[i, j] \neq 0$ and $[i, j] \neq \infty$ is ____11____.

ADJACENCY-LIST VS. ADJACENCY-MATRIX

- Determine to which representation the row is referring to.

Representation	Time for searching an edge	Good for sparse/dense matrices	Space requirement
Adjacency-list	$Not\ O(1)$	Sparse	$O(\|V\| + \|E\|)$
Adjacency-matrix	$O(1)$	Dense	$O(\|V\|^2)$

  – **Note**
    - **Sparse** graphs: those for which $|E|$ is much less than $|V|^2$.
    - **Dense** graphs: those for which $|E|$ is close to $|V|^2$.

NEXT UP
BREADTH-FIRST SEARCH

REFERENCE