CHAPTER 4

INTERPOLATION WITH UNEQUALLY SPACED DATA

4.1 INTRODUCTION

Suppose we are given the values of a function $f(x)$ at certain points $x_0, x_1, \ldots, x_n$ and we want to estimate $f(\alpha)$ where α is any point in the interval $[x_0, x_n]$. We assume that the function can be approximated by a polynomial. This is called polynomial interpolation.

There are a number of interpolation formulae, most of which have certain advantages over the others in certain situations, but no one of which is preferable to all others in all respects. In the general case, when the abscissas are not equally spaced, the use of divided differences is convenient.

4.2 LAGRANGE'S FORMULA

We assume that the polynomial

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is such that $P(x_r) = y_r$, $r = 0, 1, 2, \ldots, n$. Thus the curves representing graphs of the two functions $y = f(x)$ and $y = P(x)$ pass through the same n+1 points (x_0, y_0), $(x_1, y_1), \ldots, (x_n, y_n)$. If the function $f(x)$ has continuous derivatives of all orders, it is reasonable to suppose that $f(x)$ and $P(x)$ will be fairly close at other points of the interval $[x_0, x_n]$. To determine a_0, a_1, $\ldots, a_n$ we have to solve the system of equations:

$$\left.\begin{array}{l} a_0 + a_1x_0 + a_2x_0^2 + \ldots + a_nx_0^n = y_0, \\ a_0 + a_1x_1 + a_2x_1^2 + \ldots + a_nx_1^n = y_1, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_0 + a_1x_n + a_2x_n^2 + \ldots + a_nx_n^n = y_n. \end{array}\right\} \quad (4.1)$$

The determinant of the system is

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix}$$

Since $x_0, x_1, \ldots, x_n$ are all distinct, this determinant is not zero and the system (4.1) has a unique solution. However one or more, but not all, of a_n may turn out to be zero. Thus we have shown that a polynomial of degree $\leq n$ exists such that the curve representing $y = P(x)$ passes through the given n+1 points. Also this polynomial is unique. Let us define

$$L_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}.$$

We see that

$$L_k(x_r) = \begin{cases} 0, & \text{if } r \neq k, \\ 1, & \text{if } r = k. \end{cases} \quad (4.2)$$

If we write

$$P(x) = L_0(x)y_0 + L_1(x)y_1 + \ldots + L_n(x)y_n, \quad (4.3)$$

we see that P(x) is a polynomial of degree $\leq n$ (since $L_0(x), L_1(x)$, etc. are polynomials of degree n and coefficients of x^n, x^{n-1} etc.

may add up to zero), also

$$P(x_0) = y_0,\ P(x_1) = y_1, \ldots,\ P(x_n) = y_n.$$

Thus $P(x)$ is the required polynomial. Equation (4.3) is called the **Lagrange's interpolation formula.** Note that, although we proved the existence of such a polynomial by a different method, we seldom use the determinant method to actually evaluate the constants $a_0, a_1, \ldots, a_n$. Later we shall discuss other methods for finding the interpolation polynomial. Since such a polynomial is unique, it does not matter how we find it. The result is always the same.

EXAMPLE 4.1: Let values of $y = f(x)$ be as given in the following table. Find a polynomial which interpolates the given data

x	y
-1	-8
1	-2
2	4
3	28

SOLUTION: Here $x_0 = -1,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,$

$$y_0 = -8,\ y_1 = -2,\ y_2 = 4,\ y_3 = 28.$$

Thus
$$L_0(x) = \frac{(x-1)(x-2)(x-3)}{(-1-1)(-1-2)(-1-3)},$$

$$L_1(x) = \frac{(x-(-1))(x-2)(x-3)}{(1-(-1))(1-2)(1-3)} \text{ etc.}$$

Here

$$P(x) = \frac{(x-1)(x-2)(x-3)}{(-2)(-3)(-4)}(-8) + \frac{(x+1)(x-2)(x-3)}{2(-1)(-2)}(-2)$$

$$+ \frac{(x-(-1))(x-1)(x-3)}{(2-(-1))(2-1)(2-3)}(4) + \frac{(x-(-1))(x-1)(x-2)}{(3-(-1))(3-1)(3-2)}(28)$$

$$= (x^3 - 6x^2 + 11x - 6)(1/3) + (x^3 - 4x^2 + x + 6)(-1/2)$$
$$+ (x^3 - 3x^2 - x + 3)(-4/3) + (x^3 - 2x^2 - x + 2)(7/2)$$

$$= 2x^3 - 3x^2 + x - 2.$$