THEOREM 5.1: The n-th differences of an n-th degree polynomial are constant and all the higher differences are zero.

5.2 NEWTON'S FORWARD AND BACKWARD DIFFERENCE FORMULAE

Suppose we are given the values $\ldots y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$ of a function for $x = \ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$, where $x_{r+1} - x_r = h (r = \ldots, -2, -1, 0, 1, \ldots)$ and we wish to find $f(x_0 + ph)$ where in general $-1 < p < 1$. Now we have found that

$$P_n(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})f(x_0, x_1, \ldots, x_n)$$

is the polynomial that interpolates the given function at the points (x_r, y_r), $r = 0, 1, \ldots, n$. Now

$$f(x_0,x_1) = \frac{\Delta y_0}{h},$$

$$f(x_0,x_1,x_2) = \frac{\Delta^2 y_0}{2!h^2}$$

$$\vdots$$

$$f(x_0,x_1,\ldots,x_n) = \frac{\Delta^n y_0}{n!h^n}$$

Also if we let $x = x_0 + ph$ then

$$x - x_0 = ph$$

$$x - x_1 = x - x_0 - (x_1 - x_0) = ph - h = h(p - 1)$$

$$x - x_2 = x - x_1 - (x_2 - x_1) = h(p - 1) - h = h(p - 2)$$

$$\vdots$$

$$x - x_n = h(p - n).$$

Substitution in $P_n(x)$ gives

$$y_p \approx P_n(x_0+ph) = y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\ldots+\frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n y_0 \tag{5.1}$$

This is the Newton's forward difference formula for equally spaced data.

ALTERNATIVE DERIVATION:

Using the operator notation we have

$$y_p = E^p y_0 = (1+\Delta)^p y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

which is the same formula as above.

NEWTON'S BACKWARD DIFFERENCE FORMULA:

We have

$$y_p \approx (E^{-1})^{-p} y_0 = (1-\nabla)^{-p} y_0$$

$$= y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \ldots \tag{5.2}$$

This is called Newton's backward difference formula. The two Newton's formulae are usually used for interpolation near the beginning or the end of a table. If we are interested in a value in the middle, we should make use of a formula which uses values symmetrically placed about y_0. We shall discuss such formulae in the next section.

EXAMPLE 5.2: The following table lists values of the function $y = \sin(x^2+1) + \cos x$ at certain points. Estimate $y(-0.4)$ from the given data.

x	-0.5	-0.3	-0.1	0.1	0.3	0.5
y	1.8265671	1.8419634	1.841836	1.841836	1.8419634	1.8265671

SOLUTION: Since the point -0.4 lies near the beginning of the table, Newton's forward difference formula would be appropriate. The difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-0.5	1.8265671					
		0.0153963				
-0.3	1.8419634		-0.0155237			
		-0.0001274		0.0156511		
-0.1	1.8418360		0.0001274		-0.0156511	
		0.0000000		0.0000000		0
0.1	1.8418360		0.0001274		-0.0156511	
		0.0001274		-0.0156511		
0.3	1.8419634		-0.0155237			
		-0.0153963				
0.5	1.8265671					

Now $f(x_0+ph) = y_p \approx y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \dots$

Here $x_0 = -0.5$, $h = 0.2$ and $x = -0.4$.

But we have $x = x_0 + ph$, so

$$-0.4 = -0.5 + p(0.2)$$

or $p = 0.5$.

Let $p = 0.5$ and substitute for y_0, Δy_0, $\Delta^2 y_0$ etc. We have

$$f(-0.4) \approx 1.8378641.$$

Compare this with the exact value of $\sin(x^2+1) + \cos x$ at -0.4 which is $1.8378\ldots$

5.3 THE LOZENGE DIAGRAM

In this section we introduce the lozenge diagram which is very useful in finding interpolation formulae of various kinds. For this purpose we introduce the notation

$$C(p,n) = \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}$$

In this notation Newton's forward difference formula becomes

EXERCISE 5

1. A function $y = f(x)$ is given by the following table.

x	0	1	2	3	4	5	6
$f(x)$	2	3	10	29	66	127	218

By constructing a difference table, determine the order of the most appropri ate polynomial approximation to this data. Hence find

a) $f(1.5)$ Using Newton's forward formula

b) $f(5.5)$ Using Newton's backward formula

c) $f(3.5)$ Using i) Stirling's ii) Bessel's

iii) Everett's formulae.

2. From the following data:

x	0	0.2	0.4	0.6	0.8	1.0	1.0
y	-1.651	0.300	2.178	3.505	3.800	2.582	-0.627

a) approximate $y(0.1)$, $y(0.45)$, $y(1.1)$

b) estimate $y(0.25)$ using Newton's forward formula and Stirling's formula:

3. Construct a difference table for the following data, and hence estimate $y(0.5437)$.

x	0.51	0.52	0.53	0.54	0.55
y	0.5292437	0.5378987	0.5464641	0.5549392	0.5633233
x	0.56	0.57			
y	0.5716157	0.5798158			

4. Find the sum of series

$$S_n = 1^2 + 2^2 + \ldots + n^2$$

by constructing a difference table and then using Newton's forward formula.

5. Find a formula for the function $f(x)$ given by the following table:

x	0	1	2	3	4	5
f(x)	5.2	8.0	10.4	12.4	14.0	15.2

5.5 STIRLING'S INTERPOLATION FORMULA

If we take average of the two Gauss formulae we get

$$y_p \approx y_0 + C(p,1)\,\frac{\delta y_{-1/2} + \delta y_{1/2}}{2} + \frac{C(p+1,2)+C(p,2)}{2}\,\delta^2 y_0$$

$$+\ C(p+1,3)\,\frac{\delta^3 y_{-1/2} + \delta^3 y_{1/2}}{2} + \ldots$$

or

$$y_p \approx y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\,\delta^2 y_0 + \frac{p(p^2-1)}{3!}\,\mu\delta^3 y_0 + \frac{p^2(p^2-1)}{4!}\,\delta^4 y_0 + \ldots \qquad (5.5)$$

This is known as **Stirling's interpolation formula**. Note that this formula can also be obtained from the lozenge diagram by applying rules 3a and 3b [start at y_0 and go horizontally to the right].