In practice we select two formulae with comparable order for local truncation error. One of these formulae is selected from (12.18) and the other is selected from (12.19). The formula selected from (12.18) is used to predict the value y_{n+1} and is called the predictor. The formula selected from (12.19) is then used to correct the value y_{n+1} and is called the corrector.

12.6.1 MILNE'S METHOD

A predictor-corrector method, known as Milne's method, is obtained from (12.18) and (12.19) by retaining three terms inside parentheses and setting k =3 in (12.18, whereas k = 1 in (12.19). The resulting formulae are:

Predictor:

$$y_{n+1} = y_{n-3} + h(4 - 4\nabla + \frac{8}{3}\nabla^2)f_n \tag{12.20}$$

or

$$y_{n+1} = y_{n-3} + \frac{4}{3}h(2f_n - f_{n-1} + 2f_{n-2}). \tag{12.20'}$$

corrector:

$$y_{n+1} = y_{n-1} + 2h(1 - \nabla + \frac{1}{6}\nabla^2)f_{n+1} \tag{12.21}$$

or

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}). \tag{12.21'}$$

The truncation errors of (12.20) and (12.21) are the first neglected terms in the series. But due to particular values chosen for k, the next terms are zero. In fact, the truncation errors of each formula are proportional to $h\nabla^4 f$. Using $\nabla \approx hD$, this is equivalent to h^5D^4f. Thus the local truncation error of each formula is of order h^5 whereas the global errors are of order h^4.

12.6.2 ADAMS-MOULTON METHOD

Another predictor-corrector method, known as Adams-Moulton method, is obtained from (12.18) and (12.19) by retaining four terms from each of these formulae and setting k = 0. The resulting formulae are:

Predictor:

$$y_{n+1} = y_n + h(f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{9}{24} f_n) \quad (12.22)$$

or

$$y_{n+1} = y_n + \frac{h}{24}\{55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\}. \quad (12.22')$$

Corrector:

$$y_{n+1} = y_n + h(f_{n+1} - \frac{1}{2}\nabla f_{n+1} - \frac{1}{12}\nabla^2 f_{n+1} - \frac{1}{24} f_{n+1}\} \quad (12.23)$$

or

$$y_{n+1} = y_n + \frac{h}{24}\{9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\}. \quad (12.23')$$

As in Milne's method, the local truncation errors of these formulae are of order h^5 and the global errors are of order h^4. Like the Runge-Kutta method, it is one of the reliable and most widely used procedures for the numerical solution of ordinary differential equations.

Predictor-corrector methods have one drawback that they are not self-starting. For example, if we use (12.20'), then we require the values of y at x_{n-3}, x_{n-2}, x_{n-1} and x_n. These values, speaking generally, are not provided in the initial conditions of the problem. Starting values, therefore, must be obtained by some other method. The modified Euler's method which is a single-step method, for example, can be used to obtain (x_1,y_1), (x_2,y_2), (x_3,y_3). However, for a given mesh size it is relatively inaccurate and, as a consequence, the further values calculated using any higher order predictor-corrector method would be based on poor quality starting data having an inherited error. To overcome this a smaller step size must be used while finding the starting values.

EXAMPLE 12.6: Solve the initial-value problem of example 12.2 using

a) Milne's method.
b) Adams-Moulton method.

SOLUTION: We have already computed the values y_1, y_2, y_3 and so these values can be taken as starting values of these methods. The best values computed numerically so far are due to the fourth-order Runge-Kutta method. These values are:

$y_0 = 1$, $y_1 = 0.90968$, $y_2 = 0.83747$ and $y_3 = 0.78164$.

a) Milne's method:

$$y_4^p = y_0 + \frac{4}{3}h\left[2f_3 - f_2 + 2f_1\right]$$

$$= 1 + \frac{0.4}{3}\left[2\,f(x_3,y_3) - f(x_2,y_2) + 2\,f(x_1,y_1)\right]$$

$$= 1 + \frac{0.4}{3}\left[2(0.3-0.78164)-(0.2-0.83747)+2(0.1-0.90968)\right]$$

$$= 1 - 0.25936 = 0.74075.$$

$$y_4^c = y_2 + \frac{h}{3}\left[f_4 + 4f_3 + f_2\right]$$

$$= y_2 + \frac{h}{3}\left[f(x_4,y_4^p) + 4\,f(x_3,y_3) + f(x_2,y_2)\right]$$

$$=0.83747+\frac{0.1}{3}\left[(0.4-0.74075)+4(.3-0.78164)+(.2-0.83747)\right]$$

$$= 0.74064.$$

Similarly,

$$y_5^p = y_1 + \frac{4}{3}h\left[2f_4 - f_3 + 2f_2\right]$$

$$= 0.90968+\frac{0.4}{3}\left[2(.4-0.74064)-(.3-0.78164)+2(.2-0.83747)\right]$$

$$= 0.90968 - 0.19661 = 0.71307.$$

$$y_5^c = y_3 + \frac{h}{3}\left[f_5 + 4f_4 + f_3\right]$$

$$= 0.78164+\frac{0.1}{3}\left[(0.5-0.71307)+4(.4-0.74064)+(.3-0.78164)\right]$$

$$= 0.78164 - 0.06858 = 0.71306.$$

b) **Adams-Moulton method:**

$$y_4^p = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= y_3 + \frac{0.1}{24}\left[55(0.3-0.78164)-59(.2-0.83747)+37(.1-0.90968)+9\right]$$

$$= 0.78164 - 0.04099 = 0.74065.$$

$$y_4^c = y_3 + \frac{h}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right]$$

$$= y_3 + \frac{0.1}{24}\left[9(-0.34065)+19(-0.48164)-5(-0.63747)+(-0.80968)\right]$$

$$= 0.78164 - 0.04100 = 0.74064.$$

$$y_5^p = y_4 + \frac{h}{24}\left[55f_4 - 59f_3 + 37f_2 - 9f_1\right]$$

$$= y_4 + \frac{0.1}{24}\left[55(-0.34064)-59(-0.48164)+37(-0.63747)+9(.80968)\right]$$

$$= 0.74064 - 0.02757 = 0.71307.$$

$$y_5^c = y_4 + \frac{h}{24}\left[9f_5 + 19f_4 - 5f_3 + f_2\right]$$

$$= y_4 + \frac{h}{24}\left[9(-0.21307)+19(-0.34064)-5(-0.48164)-0.63747\right]$$

$$= 0.74064 - 0.02758 = 0.71306.$$

The corresponding results given by analytical solution of the given differential equation correct to 5 decimal places are

$$y(x_4) = y(0.4) = 0.74064 \text{ and } y(x_5) = y(0.5) = 0.71306.$$

We note that values of Milne's and Adams-Moulton methods agree with the true values.

However, if we take $y_1 = 0.9095, y_2=0.8372, y_3=0.7813$, the values computed by using modified Euler's method, which are less accurate as compared to those obtained by using fourth-order Runge-Kutta method, as starting values for Milne's and Adams-Moulton methods we obtain y_4, y_5 which are tabulatd below. These values are less accurate as compared to previous ones.

Table 12.3 Comparison of Results of Milne & Adams-Moulton Methods

n	x_n	$y(x_n)$ True value	Milne's method y_n^p			Adams-Moulton method y_n^p		
			y_n^p	y_n^c	$y(x_n)-y_n^c$ Error	y_n^p	y_n^c	Error
4	0.4	.74064	.74075	.74043	0.00021	.74035	.74034	.0003
5	0.5	.71306	.71297	.71277	0.00029	.71279	.71279	.00027

The above discussion shows that the accuracy of the results of predictor-corrector methods is based on the quality of starting values.