The actual error has been reduced to - 0.26570. We subdivide each of the above subintervals and use Simpson's rule with h = 1/2, giving

$$\int_0^4 e^x dx = \int_0^1 e^x dx + \int_1^2 e^x dx + \int_2^3 e^x dx + \int_3^4 e^x dx$$

$$\approx \frac{1}{6}\left[e^0 + 4e^{1/2} + e\right] + \frac{1}{6}\left[e + 4e^{3/2} + e^2\right] +$$

$$\frac{1}{6}\left[e^2 + 4e^{5/2} + e^3\right] + \frac{1}{6}\left[e^3 + 4e^{7/2} + e^4\right]$$

$$= \frac{1}{6}\left[e^0 + 4e^{1/2} + 2e + 4e^{3/2} + 2e^2 + 4e^{5/2} + 2e^3 + 4e^{7/2} + e^4\right]$$

$$= 53.61622.$$

The actual error for this approximation is - 0.01807.

Integration formulae resulting from interval subdivision and repeated application of low-order formula are called **composite integration formulae.**

11.6.1 COMPOSITE TRAPEZOIDAL RULE

Let $f \in C^2[a,b]$. Subdivide the interval of integration $[a,b]$ into n equal subintervals $[x_0,x_1]$, $[x_1,x_2],\ldots,[x_{n-1},x_n]$ and then use the trapezoidal rule on each subinterval (see Fig. 11.3). With $h = \frac{b-a}{n}$ and $x_i = a + ih$, $i = 0, 1, \ldots, n$,

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

$$= \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{h}{2}\Big[f(x_0)+f(x_1)\Big] + \frac{h}{2}\Big[f(x_1)+f(x_2)\Big] + \ldots + \frac{h}{2}\Big[f(x_{n-1})+f(x_n)\Big]$$

$$- \frac{h^3}{12}\sum_{i=1}^{n} f''(\xi_i),\quad \xi_i \in (x_{i-1},x_i)$$

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0)+f(x_n)+2\sum_{i=1}^{n-1} f(x_i)\right] - \frac{h^3}{12}\sum_{i=1}^{n} f''(\bar{\xi}_i),\ \bar{\xi}_i \in (x_{i-1}, x_i)$$

Since f'' is continuous on $[a,b]$, there exists a maximum value

$$M = \max_{x\in[a,b]} f''(x) \text{ and a minimum value } m = \min_{x\in[a,b]} f''(x).$$

Thus
$$nm \le \sum_{i=1}^{n} f''(\bar{\xi}_i) \le nM.$$

It then follows by the intermediate value theorem (Appendix A) that there exists a point $\xi\in[a,b]$ such that

$$\sum_{i=1}^{n} f''(\bar{\xi}_i) = nf''(\xi).$$

Hence we can write the above formula

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right] - \frac{h^3}{12}nf''(\xi),\ \xi\in[a,b]$$

or

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0)+f(x_n)+2\sum_{i=1}^{n-1} f(x_i)\right] - \frac{(b-a)}{12}h^2 f''(\xi),\ \xi\in[a,b]. \tag{11.14}$$

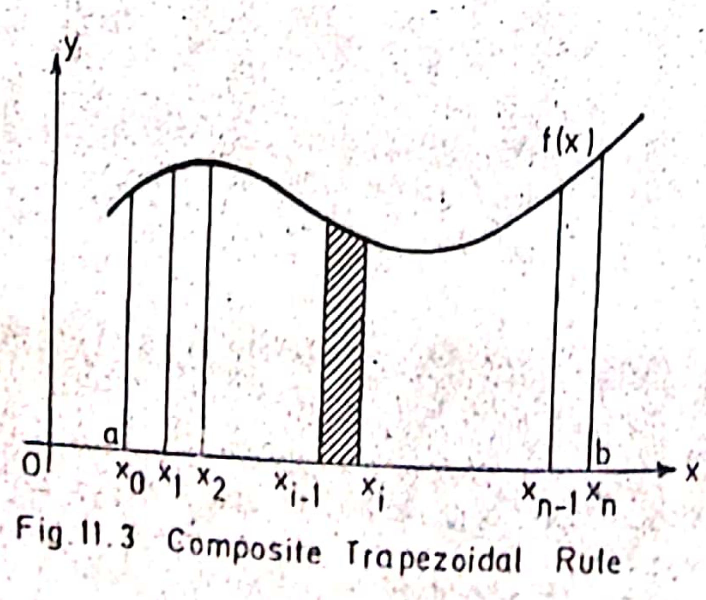

Fig. 11.3 Composite Trapezoidal Rule

In terms of n, the number of application of the rule (the number of subintervals), and a and b, the limits of integration, the composite trapezoidal rules is

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a)+f(b)+2\sum_{i=1}^{n-1} f\left(a+\frac{b-a}{n}i\right)\right] - \frac{(b-a)^3}{12n^2}f''(\xi),\ \xi\in[a,b]. \tag{11.15}$$

The error for the composite trapezoidal rule is proportional to $1/n^2$. Therefore, if we double the number of applications, the error will decrease roughly by a factor of four [$f''(\xi)$ will usually be different for two different values of n].

11.6.2 COMPOSITE SIMPSON'S RULE

Let $f\in C^4[a,b]$. Subdivide the interval [a,b] into 2m equal sub-intervals. With $h = (b-a)/2m$, and $x_i = a_0+ih$, $i=0,1,\ldots,2m$, apply Simpson's rule (11.10) to each sub-interval

$$[x_0,x_2],[x_2,x_4],\ldots,[x_{2m-2},x_{2m}]$$

of length 2h (see Fig. 11.4), then

$$\int_a^b f(x)dx = \int_{x_0}^{x_{2m}} f(x)dx$$

$$= \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{2m-2}}^{x_{2m}} f(x)dx$$

$$= \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] + \frac{h}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]$$

$$+\ldots+\frac{h}{3}\left[f(x_{2m-2}) + 4f(x_{2m-1}) + f(x_{2m})\right]$$

$$-\frac{h^5}{90}\sum_{i=1}^{m} f^{(4)}(\xi_i),\ \xi_i\in(x_{2i-2},x_{2i})\ \text{for } i=1,2,\ldots,m.$$

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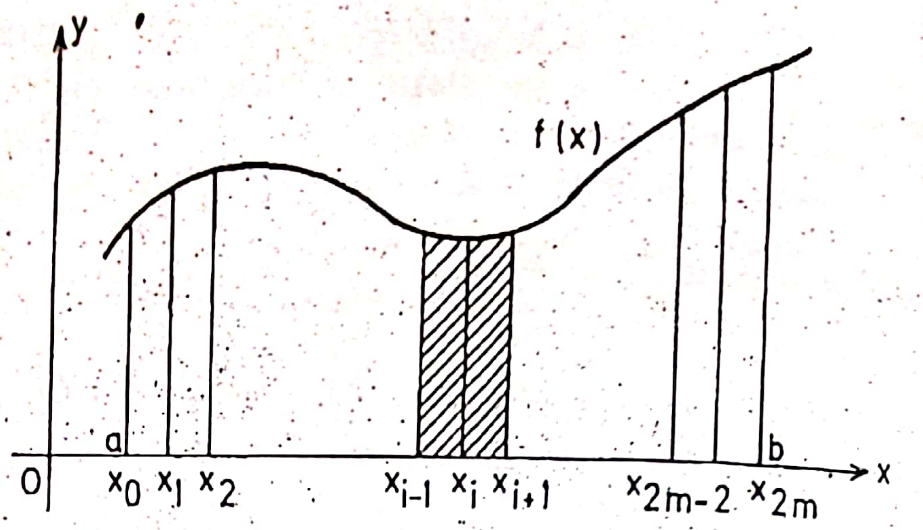

Fig. 11.4 Composite Simpson's Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\left\{f(x_1) + f(x_3) + \ldots + f(x_{2m-1})\right\} + 2\left\{f(x_2) + f(x_4) + \ldots + f(x_{2m-2})\right\} + f(x_{2m})\right] - \frac{h^5}{90}\sum_{i=1}^{m} f^{(4)}(\xi_i),\ \xi_i \in \left(x_{2i-2}, x_{2i}\right) \text{ for } i = 1,2,\ldots,m$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_{2m})\right] - \frac{h^5}{90}\sum_{i=1}^{m} f^{(4)}(\xi_i),\ \xi_i \in (x_{2i-2}, x_{2i}),\ i = 1,2,\ldots,m.$$

Since $f \in C^4[a,b]$,

$$\min_{x\in[a,b]} f^{(4)}(x) \le f^{(4)}(\xi_i) \le \max_{x\in[a,b]} f^{(4)}(x)$$

so

$$m \min_{x\in[a,b]} f^{(4)}(x) \le \sum_{i=1}^{m} f^{(4)}(\xi_i) \le m \max_{x\in[a,b]} f^{(4)}(x)$$

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or

$$\min_{x\in[a,b]} f^{(4)}(x) \le \frac{1}{m}\sum_{i=1}^{m} f^{(4)}(\bar{\xi}_i) \le \max_{x\in[a,b]} f^{(4)}(x).$$

By the intermediate value theorem (Appendix A), there exists a point $\xi \in [a,b]$ such that

$$\sum_{i=1}^{m} f^{(4)}(\bar{\xi}_i) = m\, f^{(4)}(\xi).$$

Thus,

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_{2m})\right] - \frac{h^5}{90}\, m f^{(4)}(\xi), \quad \xi\in[a,b].$$

Since

$$h = \frac{b-a}{2m}, \quad f(x_0) = f(a), \quad f(x_{2m}) = f(b),$$

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_{2m})\right] - \frac{(b-a)}{180} h^4 f^{(4)}(\xi), \quad \xi\in[a,b]. \tag{11.16}$$

In terms of a, b, and m, (11.16) is given by

$$\int_a^b f(x)dx = \frac{(b-a)}{6m}\left[f(a) + 4\sum_{i=1}^{m} f\left(a + \left(\frac{b-a}{2m}\right)(2i-1)\right) + 2\sum_{i=1}^{m-1} f\left(a + \frac{b-a}{m}\, i\right) + f(b)\right] - \frac{(b-a)^5}{2880m^4} f^{(4)}(\xi), \quad \xi\in[a,b]. \tag{11.17}$$

Composite formulae similar to those of (11.14) and (11.16) can be generated for any of the low-order integration formulae.

If the maximum admissible error $\varepsilon > 0$ is given, then, denoting

$$M_4 = \max|f^{(4)}(x)|,$$

we will have, for determining the spacing h, the inequality

$$(b-a)\frac{h^4}{180}M_4 < \varepsilon.$$

Hence $h < \left(\frac{180\varepsilon}{(b-a)M_4}\right)^{1/4}$. Thus h is of the order $(\varepsilon)^{1/4}$.

EXAMPLE 11.5: Apply the composite Simpson's rule to the integral

$$\int_1^{1.30} \sqrt{x}\, dx \quad \text{taking } 2m = 6.$$

SOLUTION: We have $h = \frac{1.30-1}{6} = .05.$

Now, $\int_1^{1.30} \sqrt{x}\, dx = \frac{2}{3}\left[(1.3)^{3/2} - 1\right] = .32149$ to five decimal places.

Using the composite Simpson's rule, we obtain

$$\int_1^{1.3} \sqrt{x}\, dx = \frac{.05}{3}\left[f(1) + 4\left\{f(1.05) + f(1.15) + f(1.25)\right\} + 2\left\{f(1.10) + f(1.20)\right\} + f(1.30)\right]$$

$$\int_1^{1.30} \sqrt{x}\, dx = \frac{.05}{3}\left[1.0 + 4(1.02470 + 1.07238 + 1.11803) + 2(1.04881 + 1.09544) + 1.14017\right] = 0.32149$$

which is correct to five decimal places.