$$\frac{\partial}{\partial y} f(x,\xi) = x^2 \cos(\xi x) = \frac{f(x,y_2) - f(x,y_1)}{y_2 - y_1}$$

or

$$|f(x,y_2) - f(x,y_1)| = |x^2 \cos(\xi x)|\ |y_2 - y_1|$$
$$\leq 4|y_2 - y_1|.$$

Thus, f satisfies a Lipschitz condition in the variable y with Lipschitz constant 4. Moreover, f(x,y) is continuous for $0 < x < 2$ and $-\infty < y < \infty$. The Theorem 12.1 implies that the given initial value problem has a unique solution.

EXERCISE 12.1

1. Use Theorem 12.1 to show that

$$\frac{dy}{dx} = y \cos x,\ 0 \leq x \leq 1,\ y(0) = 1$$

has a unique solution. Find the solution.

2. Use Theorem 12.1 to show that

$$\frac{dy}{dx} = \frac{2}{x} y + x^2 e^x,\ 1 \leq x \leq 2,\ y(1) = 0$$

has a unique solution. Find the solution also.

12.2 EULER'S METHOD

The simplest method for the numerical solution of a first order ordinary differential equation is Euler's method. It is not very accurate and is seldom used in practice. A first-order ordinary differential equation may be written as

$$\frac{dy}{dx} = f(x,y). \tag{12.2}$$

The solution of (12.2) may be written symbolically as

$$y = \phi(x). \tag{12.3}$$

The graph of (12.3) in xy-plane is a curve; and since a smooth curve is practically straight for a small distance from any point

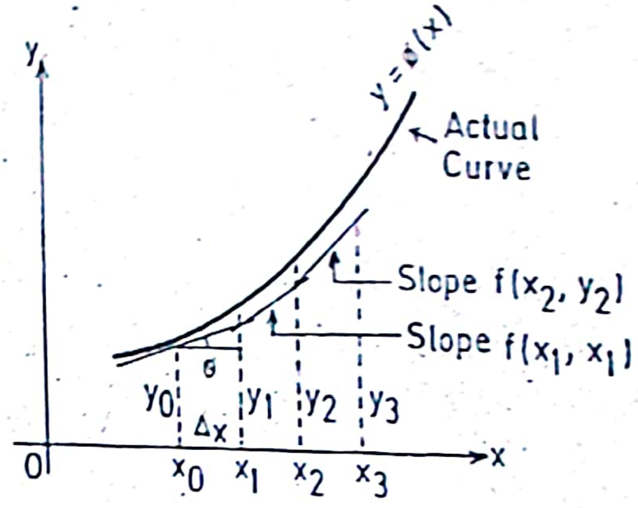

Fig. 12.1 Euler's method

on it, we get from Fig. 12.1 the approximate relation

$$\Delta y = \Delta x \tan \theta = \Delta x \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$$

$$= \Delta x\, f(x_0, y_0)$$

Thus

$$y_1 = y_0 + \Delta y \tag{12.4}$$

or

$$y_1 = y_0 + \Delta x\, f(x_0, y_0).$$

If we use h in place of Δx then the above equation becomes

$$y_1 = y_0 + h\, f(x_0, y_0).$$

Taking $x_i = x_0 + ih$; $i = 0, 1, 2, \ldots, n + 1$

we get

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$y_3 = y_2 + h\, f(x_2, y_2)$$

and in general

$$y_{n+1} = y_n + h\, f(x_n, y_n). \tag{12.5}$$

This is known as Euler's method. The calculation work is continued, interval by interval, along the x-axis from the starting point x_0 to the required finishing point. When Euler's method is applied repeatedly on several intervals in sequence, the numerical solution traces out a polygonal path with sides of slopes $f(x_n, y_n)$, $n = 0, 1, \ldots, n - 1$.

EXAMPLE 12.2: Find an approximate solution of the initial-value problem

$$y' = x - y, \quad 0 \le x \le 0.5, \quad y(0) = 1$$

taking $h = 0.1$.

SOLUTION: Using Euler's method we have

$$y_1 = y_0 + hf(x_0, y_0), \qquad f(x,y) = x - y$$
$$= 1 + 0.1(0 - 1) = 1 - 0.1 = 0.9$$

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 0.9 + 0.1(0.1 - 0.9) = 0.9 - 0.08 = 0.82$$

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 0.82 + 0.1(0.2 - 0.82) = 0.82 - 0.062 = 0.758$$

$$y_4 = y_3 + hf(x_3, y_3)$$
$$= 0.758 + 0.1(0.3 - 0.758) = 0.7122$$

$$y_5 = y_4 + hf(x_4, y_4)$$
$$= 0.7122 + 0.1(0.4 - 0.7122) = 0.68098.$$

The exact solution of the given problem is $y = 2e^{-x} + x - 1$. The approximate values are compared with actual values and error in each case is also evaluated as shown in the following table:

Table 12.1 Results of Euler's Method

n	x_n	y_n App. value	$y(x_n)$ True value	$y(x_n)-y_n$ Error
0	0	1.0	1.0	0.0
1	0.1	0.9	0.9097	0.0097
2	0.2	0.82	0.8375	0.0175
3	0.3	0.758	0.7816	0.0236
4	0.4	0.7122	0.7406	0.0284
5	0.5	0.6810	0.7131	0.0321

a) Improved Euler's method.
b) Modified Euler's method.

4. Solve the differential equation

$$\frac{dy}{dx} = 2y + x^2e^x, \; 0 \le x \le 0.3, \; y(0) = -2,$$

using
a) Improved Euler's method,
b) Modified Euler's method,
with $h = 0.1$.

12.5 RUNGE-KUTTA METHODS

Runge-Kutta methods are also based on (12.4) but use a weighted average of several estimates of Δy to give a more accurate approximation. The second order Runge-Kutta method will be developed in detail. The derivation of the higher order is complicated but can be derived in a similar way.

Suppose we are solving an equation of the form

$$\frac{dy}{dx} = f(x,y).$$

Let Δy be a weighted average of two derivative evaluations k_1, and k_2 on the interval $x_n \le x \le x_{n+1}$, that is,

$$\Delta y = ak_1 + bk_2$$

where

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + rh, y_n + sk_1) \qquad (12.8)$$

and then

$$y_{n+1} = y_n + \Delta y = y_n + ak_1 + bk_2 \qquad (12.9)$$

where a, b, r, s are constants chosen so that (12.9) agrees as closely as possible with Taylor's series

$$y_{n+1} = y(x_n+h) = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \ldots$$

or $$y_{n+1} = y_n + hf(x_n,y_n) + \frac{h^2}{2!}f'(x_n,y_n) + \frac{h^3}{3!}f''(x_n,y_n) + \ldots \qquad (12.10)$$

where we have used the differential equation

$$y' = f(x,y) \text{ or } y'_n = f(x_n, y_n)$$

and so
$$y''_n = \frac{df}{dx}(x_n, y_n) = f'(x_n, y_n), \quad y'''_n = f''(x_n, y_n).$$

Using the chain rule

$$\frac{df}{dx} = f_x(x_n, y_n) + f_y(x_n, y_n)\frac{dy}{dx}$$
$$= f_x(x_n, y_n) + f_y(x_n, y_n)\, f(x_n, y_n).$$

Similarly,

$$y''' = \frac{d}{dx}[f_x(x_n, y_n) + f_y(x_n, y_n) f(x_n, y_n)]$$
$$y''' = f_{xx}(x_n, y_n) + 2f_{xy}(x_n, y_n) f(x_n, y_n) + f_x(x_n, y_n) f_y(x_n, y_n)$$
$$+ f_{yy}(x_n, y_n) f^2(x_n, y_n) + f_y^2(x_n, y_n) f(x_n, y_n).$$

Substituting these values in (12.10), we obtain

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2}[f_x(x_n, y_n) + f_y(x_n, y_n) f(x_n, y_n)] + \ldots \tag{12.11}$$

Expanding k_2 in a Taylor's series for a function of two variables (see Appendix A) gives

$$k_2 = h[f(x_n, y_n) + rh\, f_x(x_n, y_n) + sk_1 f_y(x_n, y_n) + \ldots]$$
$$= h[f(x_n, y_n) + h(r\, f_x(x_n, y_n) + s\, f(x_n, y_n) f_y(x_n, y_n)) + \ldots]$$

Substituting the values of k_1 and k_2 in (12.9), we get

$$y_{n+1} = y_n + ahf(x_n, y_n) + bh[f(x_n, y_n) + h(rf_x(x_n, y_n) + sf(x_n, y_n) f_y(x_n, y_n))] + \ldots$$

$$y_{n+1} = y_n + h(a+b) f(x_n, y_n) + h^2[br\, f_x(x_n, y_n) + sb\, f_y(x_n, y_n) f(x_n, y_n)] + \ldots \tag{12.12}$$

Comparing (12.11) and (12.12) we obtain

$$a + b = 1,$$
$$br = 1/2,$$
$$bs = 1/2.$$

Because the four unknowns have to satisfy only three equations, the value of one of them can be chosen arbitrarily provided the resulting equations have a solution. This shows that there are infinite number of second-order Runge-Kutta methods each of which is $O(h^2)$. The two common choices are $b = \frac{1}{2}$ and $b = 1$.

For $b = \frac{1}{2}$, $a = \frac{1}{2}$, $r = s = 1$. Then (12.9) becomes

$$y_{n+1} = y_n + \frac{h}{2}\,[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

which is the **improved Euler** method.

For $b = 1$, $a = 0$, $r = s = 1/2$. Then (12.9) becomes

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2}\, f(x_n, y_n)\right)$$

which is the **midpoint** method.

If we choose $b = 3/4$ then $a = 1/4$, $r=s=2/3$. In this case (12.9) becomes

$$y_{n+1} = y_n + \frac{1}{4}\,(k_1 + 3k_2)$$

or

$$y_{n+1} = y_n + \frac{h}{4}\left(f(x_n, y_n) + 3f(x_n + \frac{2}{3}h, y_n + \frac{2}{3}h\, f(x_n, y_n))\right).$$

This is known as the **Heun's method.**

When Heun's method is applied to an initial-value problem

$$y' = x^2 + y^2,\quad y(0) = 0 \text{ with } h = 0.1,$$

it calculates $y(1) \approx 0.349640$, whereas the true value is $y(1)=0.350232$. Euler's method calculates $y(1) \approx 0.292542$. However, Eulers's requires only 10 times calculation of the function $f(x,y)$, whereas Heun's method requires 20 times. At $h = 0.05$, Euler's method also requires 20 times computation of $f(x,y)$ and gives the approximation $y(1) \approx 0.3202117$, which still has about 50 times the error of the Heun's method.

The higher-order Runge-Kutta methods are derived in similar way. In this case

$$\Delta y = ak_1 + bk_2 + ck_3$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + qh,\ y_n + qk_1)$$
$$k_3 = hf(x_n + rh,\ y_n + sk_2 + (r - s)k_1).$$

Then third-order Runge-Kutta methods are given by

$$y_{n+1} = y_n + ak_1 + bk_2 + ck_3. \qquad (12.13)$$

We first expand k_2 and k_3 about (x_n, y_n) in Taylor's series as function of two variables. Coefficients of like powers of h through the h^3 terms in (12.10) and (12.13) are compared to get a formula with a local error $O(h^4)$. The four equations in six unknowns obtained in this way are

$$a + b + c = 1$$
$$bq + cr = 1/2$$
$$bq^2 + cr^2 = 1/3$$
$$cqs = 1/6.$$

Two out of six constants a,b,c,q,r and s are arbitrary. The third-order Runge-Kutta method with constants selected by Kutta is

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{h}{6}(k_1 + 4k_2 + k_3)\\ k_1 &= f(x_n, y_n)\\ k_2 &= f(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_1)\\ k_3 &= f(x_n + h,\ y_n + 2hk_2 - hk_1) \end{aligned}\right\} \qquad (12.14)$$

If $\frac{dy}{dx} = f(x)$, i.e., dy/dx is a function of x only, then Runge-Kutta method reduces to Simpson's 1/3 rule, Eq.(11.9). In this case

$$k_1 = f(x_n),$$
$$k_2 = f(x_n + \frac{h}{2}),$$
$$k_3 = f(x_n + h),$$

and therefore

$$\Delta y = \frac{h}{6}\left[f(x_n) + 4f(x_n + \frac{h}{2}) + f(x_n + h)\right]$$

$$\Delta y = \frac{h/2}{3}\left[f(x_n) + 4f(x_n + \frac{h}{2}) + f(x_n + h)\right]$$

which is the same result as would be obtained by applying Simpson's 1/3 rule on the interval $[x_n, x_n+h]$ if we take two equal subintervals of width $h/2$.

The fourth-order Runge-Kutta methods are most widely used in computer solutions to differential equations. Comparing coefficients of like power of h through the h^4 terms gives a set of eleven equations in thirteen unknowns. The eleven equations can be solved with two unknowns chosen arbitrarily. Several fourth-order algorithms are used. The most commonly used fourth-order method is

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \\ k_3 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \\ k_4 &= hf(x_n + h, y_n + k_3). \end{aligned}\right\} \quad (12.15)$$

The local error of the fourth-order Runge-Kutta method is $O(h^5)$. We note that (12.15) again reduces to Simpson's 1/3 rule if $f(x,y)$ is a function of x only.

EXAMPLE 12.5: Solve the initial-value problem of example 12.2 using fourth-order Runge-Kutta method.

SOLUTION: We have

$$k_1 = hf(x_0, y_0) = 0.1\ f(0,1) = 0.1\ (0-1) = -0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1\ f(0.05,\ 1-0.05)$$
$$= 0.1f(0.05,\ 0.95) = 0.1\ (0.05-0.95) = -0.09$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1\ f(0.05,\ 1-0.045)$$
$$= 0.1\ f(0.05,\ 0.955) = 0.1(0.05-0.955) = -0.0905$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\ f(0.1,\ 0.9095)$$
$$= 0.1(0.1 - 0.9095) = 0.1(-0.8095) = -0.08095.$$

Therefore $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[-0.1-0.18 - 0.1810-0.08095] = 0.90968.$

Similarly, for the next step

$$k_1 = hf(x_1, y_1) = 0.1\ f(0.1,\ 0.90968) = 0.1(0.1-0.90968)$$
$$= -\ 0.08097.$$

$$k_2 = hf(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}) = 0.1\ f(0.15,\ 0.86920) = -0.07192$$

$$k_3 = hf(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}) = 0.1\ f(0.15,\ 0.87372 = -0.07237$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3) = 0.1\ f(0.2,\ 0.83731) = -0.06373$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.90968 + \frac{1}{6}[-0.08097+2(-0.07192)+2(-0.07237)-0.06373]$$
$$= 0.83747.$$

Now for the third step we find

$$k_1 = hf(x_2, y_2) = 0.1\ f(0.2,\ 0.83747) = -0.06375$$

$$k_2 = hf(x_2 + \frac{h}{2},\ y_2 + \frac{k_1}{2}) = 0.1\ f(0.25,\ 0.80560) = -0.05556$$

$$k_3 = hf(x_2 + \frac{h}{2},\ y_2 + \frac{k_2}{2}) = 0.1\ f(0.25,\ 0.80969) = -0.05597$$

$$k_4 = hf(x_2 + h,\ y_2 + k_3) = 0.1\ f(0.3,\ 0.7815) = -0.04815$$

$$y_3 = y_2 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.78164.$$

Continuing this process, we get $y_4 = 0.74064$, $y_5 = 0.71306$.

We note that these values agree with the true values.