Differential Equations

ASSIGNMENT : 03

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Q : 1 : $2x^2y'' + 5xy' + y = x^2 - x$

First of all we have to find the homogeneous differential equation.

$$2x^2y'' + 5xy' + y = 0$$

$$y = x^m$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)\,x^{m-2}$$

$$2x^2(m(m-1)\,x^{m-2}) + 5x(mx^{m-1}) + x^m = 0$$

$$[2m(m-1) + 5m + 1]\,x^m = 0$$

$$[2m^2 + 3m + 1]\,x^m = 0$$

$$x^m \neq 0$$

$$2m^2 + 3m + 1 = 0$$

$$2m^2 + 2m + m + 1 = 0$$

$$2m(m+1) + 1(m+1) = 0$$

$$(2m+1)(m+1) = 0$$

$$m = -\frac{1}{2},\ m = -1$$

$$y = c_1x^{-1} + c_2\,x^{-\frac{1}{2}}$$

Now, we have to find y_p

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^{-1} & x^{-\frac{1}{2}} \\ -x^{-2} & -\frac{1}{2}x^{-\frac{3}{2}} \end{vmatrix}$$

$$= (x^{-1})\left(\frac{-1}{2}x^{-\frac{3}{2}}\right) - (-x^{-2})(x^{-\frac{1}{2}})$$

$$= \frac{-1}{2}x^{-\frac{5}{2}} + x^{-\frac{5}{2}}$$

$$= +\frac{1}{2}x^{-\frac{5}{2}}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x^{-\frac{1}{2}} \\ f(x) & \frac{-1}{2}x^{-\frac{3}{2}} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x^{-\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2x} & \frac{-1}{2}x^{-\frac{3}{2}} \end{vmatrix}$$

$$= 0 - (x^{-\frac{1}{2}})\left(\frac{1}{2} - + \frac{1}{2x}\right)$$

$$= \frac{-x^{-\frac{1}{2}}}{2} + \frac{1}{2}x^{-\frac{3}{2}} \quad \boxed{\text{Ans}}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2} - \frac{1}{2x} \end{vmatrix}$$

$$= (x^{-1})\left(\frac{1}{2} - \frac{1}{2x}\right)$$

$$= \frac{x^{-1}}{2} - \frac{1}{2}x^{-2}$$

$$u_2' = \frac{W_1}{W}$$

$$= \frac{\frac{1}{2}x^{-1} - \frac{1}{2}x^{-2}}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$= x^{\frac{3}{2}} - x^{\frac{1}{2}}$$

$$u_1' = \frac{W_2}{W}$$

$$= \frac{-\frac{1}{2}x^{-1} + \frac{1}{2}x^{-2}}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$= -x^2 + x$$

$$u_2 = \int u_1' \, dx$$

$$= \int x^{\frac{3}{2}} - x^{\frac{1}{2}} \, dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$u_1 = \int -x^2 + x \, dx$$

$$= \frac{-1}{3} x^3 + \frac{1}{2} x^2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-1}{3} x^3 + \frac{1}{2} x^2\right) \times x^{-1} + \left(\frac{2}{5} x^{\frac{5}{2}} - \frac{2}{3} x^{\frac{3}{2}}\right) \times x^{-\frac{1}{2}}$$

$$= \frac{-1}{3} x^2 + \frac{1}{2} x + \frac{2}{5} x^2 - \frac{2}{3} x$$

$$= \frac{1}{15} \cancel{x^2} x^2 - \frac{1}{6} x$$

$$y = y_c + y_p$$

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$y = c_1 x^{-1} + c_2 x^{4} + \frac{1}{15} x^{2} - \frac{1}{6} x$ (Ans)

② $x^2 y'' + x y' - y = \ln x$

$y'' + \frac{y'}{x} - \frac{y}{x^2} = \frac{\ln x}{x^2} \; f(x)$

First of all we will find the homogeneous equation, so

$y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$x^2 \left(m(m-1) x^{m-2}\right) + x\left(m x^{m-1}\right) - x^m = 0$

$(m^2 - m + m - 1) x^m = 0$

$(m^2 - 1) = 0$

$(m+1)(m-1) = 0$

$m_1 = 1, \; m_2 = -1$

$y = c_1 x + c_2 x^{-1}$

$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}$

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$$= (x)(-x^{-2}) - (x^{-1})(1)$$

$$= -x^{-1} - x^{-1}$$

$$= -\frac{1}{x} - \frac{1}{x}$$

$$= -\frac{2}{x}$$

$$w_1 = \begin{vmatrix} 0 & x^{-1} \\ f(x) & -x^{-2} \end{vmatrix}$$

$$= 0 - (x^{-1})\left(\frac{\ln x}{x^2}\right)$$

$$= -\frac{\ln x}{x^3} \quad \boxed{\text{Ans}}$$

$$w_2 = \begin{vmatrix} x & 0 \\ 1 & f(x) \end{vmatrix}$$

$$= (x)\left(\frac{\ln x}{x^2}\right) - 0$$

$$= \frac{\ln x}{x} \quad \{\text{Ans}\}$$

$$u_1' = \frac{w_1}{w}$$

$$= \frac{\frac{-\ln x}{x^2}}{\frac{-2}{x}}$$

$$= \frac{x}{2} \times \frac{\ln x}{x^3}$$

$$= \frac{\ln x}{2x^3}$$

$$\int u_1' dx = \ln x \int 2x^{-2} dx - \int \left(\frac{d}{dx} \ln x \int 2x^{-2} dx \right) dx$$

$$u_1 = \frac{-\ln x}{2x} - \frac{1}{2x}$$

$$v^2 = \frac{W_2}{W}$$

$$= \frac{\frac{\ln x}{\cancel{x}}}{\frac{-2}{\cancel{x}}}$$

$$= \frac{\ln x}{-2}$$

$$= -\frac{1}{2} \int \ln x \, dx$$

$$= \frac{-1}{2x}$$

$y_p = v_1 y_1 + v_2 y_2$

$= x\left(-\frac{\ln x}{2x} - \frac{1}{2x}\right) + x^{-1}\left(\frac{-1}{2x}\right)$

$= -\ln x$

$y = y_c + y_p$

$= C_1 x + C_2 x^{-1} - \ln x$ $\boxed{\text{Ans}}$

Q3: $x^2 y'' + x y' - y = \frac{1}{x+1}$

$y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$x^2 \left(m(m-1) x^{m-2}\right) + x\left(m x^{m-1}\right) - x^m = 0$

$(m^2 - m + m - 1) x^m = 0$

$(m^2 - 1) x^m = 0$

$x^m \neq 0$

$m_1 = 1, \; m_2 = -1$

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$y_c = c_1 x + c_2 x^{-1}$

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}$$

$= -x^{-1} - (x^{-1})$

$= -2x^{-1}$

$= \dfrac{-2}{x}$

$x^2 y'' + x y' - y = \dfrac{1}{x+1}$

$y'' + \dfrac{y'}{x} - \dfrac{y}{x^2} = \dfrac{1}{x^2(x+1)}$

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ f(x) & -x^{-2} \end{vmatrix}$$

$= \dfrac{-1}{x^2(x+1)}$

$= \dfrac{-1}{x(x+1)}$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 \\ 1 & \dfrac{1}{x^2(x+1)} \end{vmatrix}$$

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$$= \frac{1}{x(x+1)}$$

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-1}{x^3(x+1)} \left(\frac{-x}{2}\right)$$

$$u_1 = \int \frac{1}{2x^2(x+1)}\, dx$$

Using Partial fraction,

$$\frac{1}{2x^2(x+1)} = \frac{-1}{2x} + \frac{1}{2x^2} + \frac{1}{2(x+1)}$$

$$u_2' = \frac{W_2}{W}$$

$$= \frac{-1}{2(x+1)}$$

$$u_2 = \frac{-1}{2} \ln(x+1)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left[\frac{-1}{2} \ln x - \frac{1}{2x} + \frac{1}{2} \ln(x+1)\right] . x + \left[\frac{-1}{2} \ln(x+1)\right] .$$

x^{-1}

$$y_p = -\frac{1}{2} - \frac{1}{2} x \ln x + \frac{1}{2} x \ln(x+1) - \frac{1}{2x} \ln(x+1)$$

$$y = y_c + y_p$$

$$y = c_1 x + c_2 x^{-1} - \frac{1}{2} - \frac{1}{2} x \ln x + \frac{1}{2} x \ln(x+1)$$
$$- \frac{1}{2x} \ln(x+1) \quad \boxed{\text{Ans}}$$

Question : 4

$$x^2 y'' - 2xy' + 2y = x^x e^x$$

$$y'' - \frac{2xy}{x^2} + \frac{2y}{x^2} = \frac{x^x e^x}{x^2}$$

$$y'' - 2xy + \frac{2y}{x^2} = x^2 e^x$$

$$y = x^m$$

$$y' = m x^{m-1}$$

$$y'' = m(m-1) x^{m-2}$$

$$x^2 \left(m(m-1) x^{m-2}\right) - 2x \left(m x^{m-1}\right) + 2x^m = x^x e^x$$

$$m^2 - m - 3m + 1 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, \ m = 2$$

$$y = c_1 x^1 + c_2 x^2$$

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2$$

$$W_1 = \begin{vmatrix} 0 & x^2 \\ f(x) & 2x \end{vmatrix}$$

$$= -x^2 (x^2 e^x)$$

$$= -x^4 e^x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & f(x) \end{vmatrix}$$

$$= x^3 e^x$$

$$U_1' = \frac{W_1}{W}$$

$$= \frac{-x^4 e^x}{x^2}$$

$$= -x^2 e^x$$

$$u_2' = \frac{W_2}{W}$$

$$= \frac{x^3 e^x}{x^2} = x e^x$$

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$$\int u_1' \, dx = -\int x^2 e^x \, dx$$

$$= -x^2 + 2xe^x - \int 2e^x dx$$

$$= -(x^2 - 2x + 2)e^x$$

$$\int u_2' \, dx = \int x e^x \, dx$$

$$= xe^x - \int e^x dx$$

$$= (x-1)e^x$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -(x^2 - 2x + 2)e^x . x + (x-1)e^x . x^2$$

$$= e^x (x^2 - 2x)$$

$$y = y_c + y_p$$

$$= c_1 x + c_2 x^2 + e^x (x^2 - 2x)$$ Ans

Q6 : $x^2 y'' + 10xy' + 8y = x^2$

$$y'' + \frac{10y'}{x^2} + \frac{8y}{x^2} = 1$$

$$y = x^m$$

$$y' = mx^{m-1}$$

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$y'' = m(m-1)x^{m-2}$

$x^2(m(m-1)x^{m-2}) + 10x(mx^{m-1}) + 8x^m = 0$

$m^2 - m + 10m + 8 = 0$

$m^2 + 9m + 8 = 0$

$m(m+8) + 1(m+8) = 0$

$(m+1)(m+8) = 0$

$m = -1 \quad , m = -8$

$y_c = c_1 x^{-1} + c_2 x^{-8}$

$$W = \begin{vmatrix} x^{-1} & x^{-8} \\ -x^{-2} & -8x^{-4} \end{vmatrix}$$

$(-x^{-2})(x^{-8}) - (x^{-1})(-8x^{-4})$

$= -x^{-10} + 8x^{-10}$

$= 7x^{-10}$

$$W_1 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & f(x) \end{vmatrix}$$

$= x^{-1}(1)$

$= x^{-1}$

$$W_2 = \begin{vmatrix} 0 & x^{-8} \\ f(x) & -8x^{-9} \end{vmatrix}$$

$= x^{-8}(1)$

$= x^{-8}$

$$v_1' = \frac{W_1}{W}$$

$$= \frac{x^{-1}}{7x^{-10}}$$

$$= \frac{x^9}{7}$$

$$u_2' = \frac{W_2}{W}$$

$$= \frac{x^{-8}}{7x^{-10}}$$

$$= \frac{x^2}{7}$$

$$\int v_1' = \int \frac{x^9}{7}\, dx$$

$$= \frac{1}{7} \int x^9\, dx$$

$$= \frac{1}{7} \frac{x^{10}}{10}$$

$$= \frac{x^{10}}{70}$$

$$\int u_2' \, dx = \int \frac{W_2}{W} \, dx$$

$$= \int \frac{x^2}{7} \, dx$$

$$= \frac{1}{7} \int x^2 \, dx$$

$$= \frac{1}{7} \left(\frac{x^3}{3} \right)$$

$$= \frac{x^3}{21}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{x^{10}}{70} (x^{-1}) + \frac{x^3}{21} (x^{-8})$$

$$y_p = \frac{x^9}{70} + \frac{x^{-5}}{21}$$

$$y = y_c + y_p$$

$$c_1 x^{-1} + c_2 x^{-8} + \frac{x^9}{70} + \frac{x^{-5}}{21}$$ Ans

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Question : 5

$4x^2y'' + y = 0, \; y(-1) = 2, \; y'(-1) = 4$

Sol:- $y = x^m$

$y' = mx^{m-1}$

$y'' = (m-1)m x^{m-2}$

~~$4x^2(m(m-1)$~~

$4x^2(m(m-1)x^{m-2}) + x^m = 0$

$x^m(4(m(m-1)) + 1) = 0$

$x^m(4m^2 - 4m + 1) = 0$

Thus the auxillary equation is,

$4m^2 - 4m + 1 = 0$

$$m = \frac{+4 \pm \sqrt{(-4)^2 - 4(4)(1)}}{2(4)}$$

$$= \frac{4 \pm \sqrt{16 - 6}}{8}$$

$$m = \frac{1}{2}$$

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$y = c_1 x^{\frac{1}{2}} + c_2 x^{\frac{1}{2}} \ln x$

$y(-1) = 2$

$2 = c_1 \left(\frac{-1}{2}\right) + c_2 \frac{-1}{2} \ln(-1)$

$2 = c_1 \left(\frac{-1}{2}\right) + c_2 (0)$

So,

$y = -4x^{\frac{1}{2}} + c_2 x^{\frac{1}{2}}$

Now taking derivative we have,

$y' = -4 \frac{d}{dx}\left(x^{\frac{1}{2}}\right) + c_2 \frac{d}{dx} x^{\frac{1}{2}}$

$= -4 \left(\frac{1}{2}\right) x^{\frac{1}{2}-1} + c_2 \left(\frac{1}{2} x^{\frac{1}{2}-1}\right)$

$= -2x^{-\frac{1}{2}} + c_2 \left(\frac{1}{2\sqrt{x}}\right)$

$y'(1) = -4$

$\frac{c_1}{2} + c_2(0 + 1) = -4$

$\frac{2}{2} + c_2 = -4$

$1 + c_2 = -4$

$c_2 = -5$

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$$y = -2(-x)^{\frac{1}{2}} - 5(-x)^{\frac{1}{2}} \ln(-x), \quad x < 0$$

END.