Name: M. Sherjeel Akhtar.

Roll no: 20F-0101

Section: BS (CS) - 2D.

Q # 01 $y'' + 3y' + 2y = \sin e^{x}$.

$y'' + 3y' + 2y = \sin e^{x}$,

The associated homogenous equation is

$y'' + 3y' + 2y = 0$

And the auxillary equation is.

$m^2 + 3m + 2 = 0$.

$m^2 + m + 2m + 2 = 0$.

$m(m+1) + 2(m+1) = 0$.

$(m+1)(m+2) = 0$.

$m+1 = 0$ $\quad$ $m+2 = 0$

$m_1 = -1$ $\quad$ $m_2 = -2$.

where m_1 and m_2 are real and distinct, so according to general solution,

$y_c = c_1 y_1 + c_2 y_2$.

Put $y = e^{mx}$,

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

And put $m_1 = -1$, $m_2 = -2$.

$y_c = c_1 e^{-x} + c_2 e^{-2x}$.

Now, we will find the particular solution

$$y_p = u_1(x)y_1 + u_2(x)y_2.$$

Since,

$$y_1 = e^{-x}, \quad y_2 = e^{-2x}, \text{ then}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$W = (e^{-x}) \times (-2e^{-2x}) - (e^{-2x}) \times (-e^{-x}).$$

$$W = -2e^{-3x} + e^{-3x}.$$

$$W = -e^{-3x}.$$

Now, we will find W_1 & W_2, so

$$W_1 \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

And $f(x) = \sin e^x$ with the help of standard form.

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin e^x & -2e^{-2x} \end{vmatrix}.$$

$$W_1 = 0 - e^{-2x} \times (\sin e^x).$$

$$= -e^{-2x} \sin e^x$$

And

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^x & \sin e^x \end{vmatrix}$$

$$W_2 = e^{-x} \times (\sin e^x) - 0.$$
$$\Rightarrow W_2 = e^{-x} \sin e^x.$$

Now, we will find the value of u_1' & u_2'

$$u_1' = \frac{W_1}{W}.$$
$$= \frac{-e^{-2x} \sin e^x}{-e^{-3x}}.$$
$$= -e^{3x} \times -e^{-2x} \sin e^x.$$
$$= e^x \sin e^x.$$

And,

$$u_2' = \frac{W_2}{W}.$$
$$= \frac{e^{-x} \sin e^x}{-e^{-3x}}.$$
$$= -e^{+3x} \times e^{-x} \sin e^x.$$
$$= -e^{2x} \sin e^x.$$

But we need the value of u_1 & u_2 so we will take Integral on b/s.

$$u_1 = \int e^x \sin e^x \, dx.$$

let $v = e^x$

$$dv = e^x dx$$
$$u_1 = \int \sin v \, dv.$$
$$u_1 = -\cos v$$
$$u_1 = -\cos e^x.$$

Now, $u_2 = -\int e^{2x} \sin e^x \, dx.$

$$u_2 = -\int e^x \cdot e^x \sin e^x \, dx.$$

let , $e^x = s$

$$ds = e^x dx.$$

$$u_2 = -\int s \sin s \, ds.$$

Now, we will ~~find~~ use integration by parts etc. i.e,

$V = s$, $dv = ds$, and let $dn = \sin s \, ds$, then $n = -\cos s$, so,

$$u_2 = -(V \times n - \int n \, dv) \qquad (\because \int AB \, dx = A \int B \, dx - \int (\int B \, dx) \cdot A' \, dx)$$

$$u_2 = -(-s \cos s - \int -\cos s \, ds).$$

$$= -(-s \cos s + \sin s)$$

$$= s \cos s - \sin s$$

$$u_2 = e^x \cos e^x - \sin e^x.$$

Now, putting the values in eq.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = (-\cos e^x) \times (e^{-x}) + (e^x \cos e^x - \sin e^x) \times (e^{-2x}).$$

$$= -e^{-x} \cos e^x + e^{-x} \cos e^x - e^{-2x} \sin e^x.$$

$$= -e^{-2x} \sin e^x.$$

Now, putting the values in general solution,

$$y = y_c + y_p.$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x \quad \text{Ans.}}$$

Q #03

$$3y'' - 6y' + 6y = e^x \sec x.$$

$$3y'' - 6y' + 6y = e^x \sec x.$$

Dividing both sides with 3 to obtain standard form:

$$y'' - 2y' + 2y = \frac{e^x \sec x}{3}.$$

The associated homogenous equation is:

$$y'' - 2y' + 2y = 0.$$

And the auxillary equation is -

$$m^2 - 2m + 2 = 0.$$

Now, we will find the roots i.e,

$$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}.$$

$$m = \frac{\cancel{2} \pm \sqrt{-4}}{\cancel{2}} \Rightarrow \frac{2 \pm 2\sqrt{-1}}{2} \Rightarrow \frac{\cancel{2}\,1 + \sqrt{-1}}{\cancel{2}}$$

$$m = 1 \pm i.$$

So,

$$m_1 = 1 \quad \& \quad m_2 = 1 - i$$

where m_1 & m_2 are conjugate & complex roots.

So, according to general solution

$$y_c = c_1 y_1 = c_2 y_2$$

Put $y = e^{mx}$.

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

And put $m_1 = 1 + i$ & $m_2 = 1 - i$

$$y_c = c_1 e^{(1+i)x} + c_2 e^{(1-i)x}.$$

$$y_c = c_1 e^x e^{ix} + c_2 e^x e^{-ix} \rightarrow (1)$$

Now, from Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ , so,}$$

$$e^{ix} = \cos x + i\sin x \ \&$$

$$e^{-ix} = \cos x - i\sin x.$$

Putting the values in ①.

$$y_c = C_1 e^x (\cos x + i\sin x) + C_2 e^x (\cos x - i\sin x).$$

$$= C_1 e^x \cos x + C_1 e^x i\sin x + C_2 e^x \cos x - C_2 e^x \text{-} i\sin x$$

$$= C_1 e^x \cos x + C_2 e^x \cos x + C_1 e^x i\sin x - C_2 e^x i\sin x.$$

$$y_c = C_1 e^x \cos x + C_2 e^x \cos x + C_1 e^x i\sin x - C_2 e^x i\sin x.$$

$$y_c = (C_1 + C_2) e^x \cos x + i(C_1 - C_2) e^x \sin x.$$

$$y_c = C_1 e^x \cos x + C_2 e^x \sin x$$

Now, we will find the particular solution

$$y_p = u_1 y_1 + u_2 y_2..$$

Since,

$$y_1 = e^x \cos x.$$

$$y_2 = e^x \sin x, \text{ then,}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x \end{vmatrix}$$

$$W = (e^x \cos x)\ (e^x [\cos x + \sin x]) - (e^x \sin x)(e^x \cos x - \sin x)).$$

$$W = e^{2x} \cos^2 x + e^{2x} \cancel{\sin x \cos x} - e^{2x} \cancel{\sin x \cos x} + e^{2x} \sin^2 x.$$

$W = e^{2x}(\cos^2 x + \sin^2 x)$.

$W = e^{2x}(\cos^2 x + \sin^2 x)$
($\because \sin^2\theta + \cos^2\theta = 1$).

$W = e^{2x}(1)$

$W = e^{2x}$.

Now, we will find w_1 & w_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

where $f(x) = \frac{e^x \sec x}{3}$, with the help of standard form.

$$W = \begin{vmatrix} 0 & e^x \sin x \\ \frac{e^x \sec x}{3} & e^x(\cos x + \sin x) \end{vmatrix}$$

$$W_1 = 0 - (e^x \sin x)\left(\frac{e^x \sec x}{3}\right).$$

$$W_1 = 0 - \frac{1}{3} e^{2x} \sin x \times \frac{1}{\cos x}.$$

$$W_1 = -\frac{1}{3} e^{2x} \tan x.$$

And

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x(\cos x - \sin x) & \frac{e^x \sec x}{3} \end{vmatrix}$$

$$W_2 = (e^x \cos x)\left(\frac{e^x \sec x}{3}\right) - 0$$

$$W_2 = \frac{1}{3} e^{2x} \cancel{\cos x} \times \frac{1}{\cancel{\cos x}}$$

$$W_2 = \frac{1}{3} e^{2x}.$$

Now, we will find the value of u_1' & u_2', so

$$u_1' = \frac{W_1}{W},$$

$$u_1' = \frac{-\frac{1}{3} e^{2x} \tan x}{e^{2x}}.$$

$$u_1' = -\frac{1}{3} \tan x.$$

And,

$$u_2' = \frac{W_2}{W}.$$

$$= \frac{\frac{1}{3} e^{2x}}{e^{2x}}, \quad u_2' = \frac{1}{3}.$$

But, we need the value of u_1 & u_2, so we will take Integral on b.s.

$$u_1 = \int -\frac{1}{3} \tan x \, dx.$$

$$u_1 = \frac{1}{3} \int \frac{-\sin x}{\cos x} dx.$$

$$u_1 = \frac{1}{3} \ln(\cos x).$$

And,

$$u_2 = \int \frac{1}{3} dx$$

$$u_2 = \frac{1}{3} \int 1 \, dx.$$

$$u_2 = \frac{1}{3} x.$$

Now, put values in eq.

$$y_p = u_1 y_1 + u_2 y_2,$$

$$y_p = \left(\frac{1}{3}\ln\cos x\right) \times \left(e^x \cos x\right) + \frac{1}{3} x \times (e^x \sin x).$$

$$y_p = \frac{1}{3} e^x \cos x \ln(\cos x) + \frac{1}{3} x e^x \sin x$$

Now, put the values in general solution,

$$y = y_c + y_p,$$

$$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{3} e^x \cos x \ln \cos x + \frac{1}{3} x e^x \sin x \quad \text{Ans.}$$

Q # 04

$$4y'' - y = x e^{\frac{1}{2}x}, \quad y(0)=0, \; y'(0)=0$$

$$4y'' - y = x e^{\frac{1}{2}x}.$$

$\div$ ing b.S by 4 to get standard form,

$$y'' - \frac{1}{4} y = \frac{x e^{\frac{1}{2}x}}{4}$$

The associated homogenous equation is.

$$y'' - \frac{1}{4}y = 0.$$

And the auxillary eq is.

$$m^2 - \frac{1}{4} = 0.$$

$$\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right) = 0.$$

$m_1 = \frac{1}{2}$, $m_2 = \frac{-1}{2}$,

Where m_1 and m_2 are real and distinct, so according to general solution,

$$y_c = c_1 y_1 + c_2 y_2$$

Put, $y = e^{m_1}$,

$$y_2 = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

And put $m_1 = \frac{1}{2}$ & $m_2 = \frac{-1}{2}$,

$$y_c = c_1 e^{x/2} + c_2 e^{-x/2}.$$

Now, we will find particular solution.

$$y_p = u_1 y_1 + u_2 y_2$$

Since,

$$y_1 = e^{x/2} \quad , \quad y_2 = e^{-x/2}.$$

then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{x/2} & e^{-x/2} \\ \frac{1}{2} e^{x/2} & -\frac{1}{2} e^{-x/2} \end{vmatrix}$$

$$= (e^{x/2}) \times \left(-\frac{1}{2} e^{-x/2}\right) - (e^{-x/2}) \times \left(\frac{1}{2} e^{x/2}\right).$$

$$= -\frac{1}{2} e^{0} - \frac{1}{2} e^{0}$$

$$\boxed{W = -1}$$

Now, we will find w_1 & w_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

And $f(x) = \frac{xe^{x/2}}{4}$, with the help of standard form.

$$W_1 = \begin{vmatrix} 0 & e^{-x/2} \\ \frac{xe^{x/2}}{4} & -\frac{1}{2}e^{-x/2} \end{vmatrix}$$

$$W_1 = 0 - (e^{-x/2}) \times \left(\frac{xe^{x/2}}{4}\right).$$

$$= -\frac{xe^0}{4},$$

$$= -\frac{x}{4}.$$

And,

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{x/2} & 0 \\ \frac{1}{2}e^{x/2} & xe^{x/2}/4 \end{vmatrix}$$

$$W_2 = (e^{x/2}) \times \left(\frac{xe^{x/2}}{4}\right) - 0.$$

$$= \frac{xe^x}{4}.$$

Now, we will find the value of u_1' & u_2' as,

$$u_1' = \frac{W_1}{W}$$

$$= \frac{x/4}{1}$$

$$u_1' = \frac{x}{4}$$

And

$$u_2' = \frac{W_2}{W}$$

$$= \frac{xe^x/4}{-1}$$

$$u_2' = \frac{-xe^x}{4}$$

But we need the value of u_1 & u_2, so we will take integral on B.S

$$u_1 = \int \frac{x}{4}\, dx$$

$$u_1 = \frac{1}{4}\int x\, dx$$

$$u_1 = \frac{1}{8} x^2$$

And,

$$u_2 = \int -\frac{xe^x}{4}\, dx$$

$$u_2 = \frac{-1}{4}\int xe^x\, dx$$

Now, we will ~~find~~ use integration by parts i.e,

let $s = x$, then we have $ds = dx$ & if we set $dv = e^x$, then we have $v = e^x$, so

$$u_2 = \frac{-1}{4}\left(s \times v - \int v\, ds\right) \qquad \left(\because \int AB\,dx = A\int B\,dx - \int\left(\int B\,dx\right) A'\,dx\right)$$

$u_2 = -\frac{1}{4}\left(xe^x - \int e^x\right)$

$u_2 = -\frac{1}{4}(xe^x - e^x)$.

$u_2 = -\frac{1}{4}xe^x + \frac{1}{4}e^x$.

Now, putting the values in eq.

$y_p = u_1y_1 + u_2y_2$

$y_p = \left(\frac{1}{8}x^2\right) x (e^{x/2}) + \left(-\frac{1}{4}xe^x + \frac{1}{4}e^x\right) x (e^{-x/2})$

$y_p = \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}$.

Now, putting the values in general soln.

$y = y_c + y_p$.

$y = C_1e^{x/2} + C_2e^{-x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}$

$y = \left(C_1 + \frac{1}{4}\right)e^{x/2} + C_2e^{-x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2}$

$y = C_1e^{x/2} + C_2e^{-x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2} \rightarrow (a)$

Now, we will apply the point $(x,y) = (0$ in above eq.

$0 = C_1e^0 + C_2e^0 + 0 + 0$

$C_1 + C_2 = 0 \rightarrow (1)$.

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Now, we will take the first of derivative of general solution:

$$y' = \frac{1}{2} c_1 e^{x/2} - \frac{1}{2} c_2 e^{-x/2} + \frac{1}{4} x e^{x/2} + \frac{1}{16} x^2 e^{x/2} - \frac{1}{4} e^{x/2}$$
$$- \frac{1}{8} x e^{x/2}.$$

$$y' = \frac{1}{2} c_1 e^{x/2} - \frac{1}{2} c_2 e^{-x/2} + \frac{1}{16} x^2 e^{x/2} + \frac{1}{8} x e^{x/2} - \frac{1}{4} e^{x/2}$$

Now, we will apply the point $(x, y') = (0,0)$ in above equation.

$$0 = \frac{1}{2} c_1 e^0 - \frac{1}{2} c_2 e^0 + 0 + 0 - \frac{1}{4} e^0.$$

$$\frac{1}{2} c_1 - \frac{1}{2} c_2 = \frac{1}{4} \rightarrow (2).$$

Solving eq (1) & (2) we get.

$$c_1 = \frac{1}{4} \quad \& \quad c_2 = \frac{3}{4}.$$

Putting the value of c_1 & c_2 in eq (A).

$$y = \frac{1}{4} e^{x/2} + \frac{3}{4} e^{-x/2} + \frac{1}{8} x^2 e^{x/2} - \frac{1}{4} x e^{-x/2} \text{ Ans}$$

Q # 5

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x} \; ; \; y(0) = 1, \; y'(0) = 0.$$

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x}.$$

The associated homogenous equation is

$y'' + 2y' - 8y = 0.$

And the auxillary equation is,

$$m^2 + 2m - 8 = 0.$$
$$m^2 + 4m - 2m - 8 = 0.$$
$$m(m+4) - 2(m+4) = 0$$
$$(m-2)(m+4) = 0.$$
$$m_1 = 2, \quad m_2 = -4$$

where m_1 and m_2 are real and distinct, so according to general solution,

$$y_c = C_1 y_1 + C_2 y_2$$

put $y = e^{mx}$

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

And put $m_1 = 2$ & $m_2 = -4.$

$$y_c = c_1 e^{2x} + c_2 e^{-4x}$$

Now, we will find particular soln.

$$y_p = u_1 y_1 + u_2 y_2.$$

Since,

$$y_1 = e^{2x}.$$
$$y_2 = e^{-4x}$$

then,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix}.$$

$$W = (e^{2x})(-4e^{-4x}) - (e^{-4x})(2e^{2x}).$$
$$= -4e^{-2x} - 2e^{-2x}$$
$$= -6e^{-2x}.$$

Now, we fill find w_1 & w_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}.$$

$$W_1 = \begin{vmatrix} 0 & e^{-4x} \\ 2e^{-2x} - e^{-x} & -4e^{-4x} \end{vmatrix}.$$

$f(x) = 2e^{-2x} - e^{-x}$, with the help of standard from,

$$w_1 = 0 - (e^{-4x}) \times (2e^{-2x} - e^{-x})$$
~~$w_1 = 0 - (e$~~
$$w_1 = -2e^{-6x} + e^{-5x}.$$

And

$$W_2 \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{-2x} - e^{-x} \end{vmatrix}$$

$$W_2 = (e^{2x})(2e^{-2x} - e^{-x}) - 0.$$
$$= \cancel{2e} \quad 2 - e^{+x}$$

Now, we will find the value of u_1' & u_2' So,

$$u_1' = \frac{W_1}{W}.$$

$$= \frac{-2e^{-6x} + e^{-5x}}{-6e^{-2x}}.$$

$$u_2' = \frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x}.$$

And,

$$u_2' = \frac{W_2}{W},$$

$$= \frac{2e^{x}}{-6e^{-2x}}$$

$$u_2' = -\frac{1}{3}e^{2x} + \frac{1}{6}e^{3x}.$$

But we need the value of u_1 & u_2, So, we will take integral on B/S.

$$u_1 = \int\left(\frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x}\right)dx.$$

$$u_1 = \int \frac{1}{3}e^{-4x}\,dx - \int \frac{1}{6}e^{-3x}\,dx.$$

$$u_1 = \int \frac{1}{3}e^{-4x}\,dx - \int \frac{1}{6}e^{-3x}\,dx.$$

$$u_1 = \frac{1}{3}e^{-4x} \times -\frac{1}{4} - \frac{1}{6}e^{-3x} \times -\frac{1}{3}.$$

$$u_1 = -\frac{1}{2}e^{-4x} + \frac{1}{18}e^{-3x}$$

And,

$$u_2 = \int\left(-\frac{1}{3}e^{2x} + \frac{1}{6}e^{3x}\right)dx.$$

$$u_2 = -\int \frac{1}{3} e^{2x} dx + \int \frac{1}{6} e^{3x} dx.$$

$$u_2 = -\frac{1}{3} e^{2x} \times \frac{1}{2} + \frac{1}{6} e^{3x} \times \frac{1}{3}.$$

$$u_2 = -\frac{1}{6} e^{2x} + \frac{1}{18} e^{3x}.$$

Now, putting the values in eq-

$$y_p = u_1 y_1 + u_2 y_2.$$

$$y_p = \left(-\frac{1}{12} e^{-4x} + \frac{1}{18} e^{-3x}\right) \cdot (e^{2x}) + \left(-\frac{1}{6} e^{2x} + \frac{1}{18} e^{3x}\right)(e^{-4x}).$$

$$y_p = -\frac{1}{12} e^{-2x} + \frac{1}{18} e^{-x} - \frac{1}{6} e^{-2x} + \frac{1}{8} e^{-x}.$$

$$y_p = -\frac{1}{4} e^{-2x} + \frac{1}{9} e^{-x}.$$

Now, putting values in general solution.

$$y = y_c + y_p.$$

$$y_g = C_1 e^{2x} + C_2 e^{-4x} - \frac{1}{4} e^{-2x} + \frac{1}{9} e^{-x} \rightarrow (A).$$

Now, we will apply the point $(x, y) = (0, 1)$ in eq (A).

$$1 = C_1 e^0 + C_2 e^0 - \frac{1}{4} e^0 + \frac{1}{9} e^0.$$

$$C_1 + C_2 = \frac{41}{36} \rightarrow (1)$$

Now, we will take the first derivative general solution.

$$y' = 2c_1 e^{2x} - 4c_2 e^{-4x} + \frac{1}{2} e^{-2x} + \frac{1}{9} e^{-x} \rightarrow (B)$$

Now, we will apply the point $(x, y') = (0, 0)$ in eq (B).

$$0 = 2c_1 e^{0} - 4c_2 e^{0} + \frac{1}{2} e^{0} - \frac{1}{9} e^{0}.$$

$$2c_1 - 4c_2 = -\frac{7}{18} \rightarrow (2)$$

Solving eq (1) and (2) we get

$$C_1 = \frac{25}{36} \quad \& \quad C_2 = \frac{4}{9}.$$

Putting values of C_1 & C_2 in eq (A).

$$y = \frac{25}{36} e^{2x} + \frac{4}{9} e^{-4x} - \frac{1}{4} e^{-2x} + \frac{1}{9} e^{-x} \text{ Ans}$$

Q # 02.

$$y'' + 2y' + y = e^{-t} \ln t.$$

$$y'' + 2y' + y = e^{-t} \ln t.$$

The associated homogenous equation is

$$y'' + 2y' + y = 0$$

And the auxillary eq. is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$(m+1)(m+1) = 0$$

$$m_1 = m_2 = 1.$$

where, m_1 & m_2 are repeated real roots, so acc. to general solution.

$$y_c = c_1 y_1 + c_2 y_2.$$

Put

$$y_1 = e^{m_1 x} \quad \& \quad y_2 = x e^{m_2 x}.$$

$$y_c = c_1 e^{m_1 t} + c_2 t e^{m_2 t}.$$

Put m_1 & $m_2 = -1$.

$$y_c = c_1 e^{-t} + c_2 t e^{-t}.$$

Now, we will find the particular soln,

$$y_p = u_1 y_1 + u_2 y_2.$$

Since,

$$y_1 = e^{-t} \quad y_2 = t e^{-t}, \text{ then}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix}$$

$$W = e^{-2t} - \cancel{te^{-2t}} + \cancel{te^{-2t}}$$

$$W = e^{-2t}.$$

Now, we will find u_1 & u_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

And $f(x) = e^{-t} \ln t$, with the help of standard form,

$$W_1 = \begin{vmatrix} 0 & te^{-t} \\ e^{-t}\ln t & e^{-t} - te^{-t} \end{vmatrix}$$

$$W_1 = 0 - (e^{-t}\ln t)(te^{-t}).$$
$$= -te^{-2t}\ln t.$$

And,
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{-t} & 0 \\ e^{-t} & e^{-t}\ln t \end{vmatrix}$$

$$W_2 = (e^{-t})(e^{-t}\ln t) - 0.$$
$$= e^{-2t}\ln t.$$

Now, we will find the value of u_2' & u_1', so,
$$u_1' = \frac{W_1}{W}.$$

$$= \frac{-t\cancel{e^{-2t}}\ln t}{\cancel{e^{-2t}}}$$

$$u_1' = -t\ln t$$

And
$$u_2' = \frac{W_2}{W}.$$

But,

we

$$= \frac{\cancel{e^{-2t}}\ \ln t}{\cancel{e^{-2t}}}$$

$$\boxed{u_1' = -t \ln t}$$

And,

$$u_2' = \frac{W_2}{W}$$

$$= \frac{\cancel{e^{-2t}}\ \ln t}{\cancel{e^{-2t}}}$$

$$\boxed{u_2' = \ln t}$$

But, we need the value of u_1 & u_2, so we will take integral on b.s.

$$u_1 = \int -t \ln t \; dt$$

$$u_1 = - \int t \ln t \cdot dt$$

Now, we will use Integration by parts.

$$u_1 = \ln t \int t dt + \int (\textstyle\int t dt) \cdot \frac{d}{dt} \ln t \cdot dt$$

$$u_1 = -\ln t \cdot \frac{t^2}{2} + \int \frac{1}{t} \cdot \frac{t^2}{2} dt$$

$$u_1 = -\frac{t^2 \ln t}{2} + \int \frac{t}{2} dt$$

$$u_1 = -\frac{t^2 \ln t}{2} + \frac{t^2}{4}$$

Now,

$$u_2 = \int \ln t \; dt$$

$$u_2 = \int \ln t \times 1 \; dt$$

Now, we will use Integration by parts.

$$u_2 = \ln t \int dt - \int \left\{ \left(\int dt\right) \cdot \frac{d}{dt}(\ln t) \cdot dt \right\}$$

$$u_2 = \ln t \cdot t - \int \frac{1}{t} \, dt$$

$$u_2 = t\ln t - t.$$

Now, putting values in eq.

$$y_p = u_1 y_1 + u_2 y_2.$$

$$y_p = \left(\frac{-t^2 \ln t}{2} + \frac{t^2}{4}\right) \cdot (e^{-t}) + (t\ln t - t)(te^{-t}).$$

$$y_p = \left(\frac{-t^2 \ln t}{2} + \frac{t^2}{4}\right) \cdot (e^{-t}) + (t^2 \ln t - t)(e^{-t}).$$

$$y_p = \left(\frac{-t \ln t}{2} + \frac{t^2}{4} + t^2 \ln t - t^2\right) \cdot (e^{-t}).$$

$$y_p = \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4}\right) \cdot (e^{-t}).$$

Now, put values in general solution.

$$y = y_c + y_p.$$

$$y = C_1 e^{-t} + C_2 t e^{-t} + \left(\frac{t^2 \ln t^2}{2} - \frac{3t^2}{4}\right) \cdot (e^{-t})$$ Ans