Introduction to computational thinking and programming for CFD (13251)

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Sheet 8

Goals

- Random numbers
- Random sampling from uniform and nonuniform distribution
- Difference between PDF and CDF

Tasks

1. Uniform random number generation.
   (a) Generate $N = 20$ random numbers over the range $0 \leq x \leq 1$ with `rand(N)` from the `numpy.random` module.
   (b) Generate $N = 20$ random numbers over the range $-3 \leq x \leq 5$.
2. Roll of a dice.
   (a) Perform $N = 5{,}000$ dice rolls. Sample $1, 2, \ldots, 6$ from a uniform distribution.
   Hint: Use `randint(lower, upper, N)`, which is described here: https://numpy.org/doc/stable/reference/random/generated/numpy.random.randint.html
   (b) Plot the histogram of the random numbers with `hist(...)` from the `matplotlib.pyplot` module in analogy to Exercise 7.
   (c) Compute the mean and standard deviation of the random numbers.
3. Linear congruential generator (LCG).
   (a) Implement the LCG from Lecture 8.
   (b) Investigate the behavior by varying the seed and the modulus.
   (c) Normalize by m to obtain random numbers in $[0, 1)$. Compare with task 1(a).

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4. Nonuniform random numbers: Preimplemented distribution and CDF inversion.

   (a) Draw $N = 50{,}000$ samples from the exponential distribution

   $$p(x) = \mathrm{e}^{-x},$$

   where $0 \leq x < \infty$. Plot $p(x)$ together with the discrete PDF (normalized histogram) of the random numbers using $M = 100$ bins. *Hint:* See the documentation for call options: `https://numpy.org/doc/stable/reference/random/generated/numpy.random.exponential.html`

   (b) The CDF $c(x)$ corresponding to the exponential PDF reads

   $$r = c(x) = \int_0^x \mathrm{e}^{-x'}\,\mathrm{d}x' = \left[-\mathrm{e}^{-x'}\right]_0^x = 1 - \mathrm{e}^{-x}.$$

   The inverted CDF is hence given by

   $$x = c^{-1}(r) = -\ln(1-r).$$

   Now, sample r uniformly from across $[0,1]$. Then evaluate $x = c^{-1}(r)$. Repeat this for all N. Plot the PDF of generated values x_i together with the reference PDF $p(x)$.

5. $(*)$ Random initial conditions and ensemble statistics for the Lorenz system using the numerical solver from the previous exercises.

   (a) Generate an ensemble of $N = 100$ initial conditions for the Lorenz system,

   $$x_i^0 = x^0 + \xi_i \qquad y_i^0 = y^0 + \eta_i \qquad z_i^0 = z^0 + \zeta_i \qquad \text{for} \qquad i = 1, 2, \ldots, N.$$

   Here, $\xi_i, \eta_i, \zeta_i \in [-0.5, 0.5]$ denote random numbers.

   (b) Numerically solve the Lorenz systems for the ensemble of initial conditions.

   (c) Plot the solutions for all ensemble members. *Hint:* Plot each member with a thin translucent line by calling `plot(x, y, 'b-', lw=1., alpha=0.5)`.

   (d) Ensemble average the simulated time series $x_i(t)$, $y_i(t)$, $z_i(t)$. Is the mean evolving or steady?