Introduction to computational thinking and programming for CFD (13251)

Dr. rer. nat. Marten Klein

Chair of Numerical Fluid and Gas Dynamics, BTU Cottbus-Senftenberg

Sheet 6

Goals

- Ordinary differential equations (ODEs)
- Lorenz system and Lorenz attractor
- Numerical solution algorithm (explicit Euler method)
- Advanced plotting: sublots and 3-D plots
- Data management (lists, arrays, separation of data generation and plotting)
- Data IO (simple save and load tools)

Problem statement

Turbulent flows are generally unsteady. They may be statistically stationary, but even then, the flow variables are fluctuating giving rise to variable bulk quantities that may change in an unpredictable manner. Such fluctuations are common in fluid dynamical system and known as **chaotic dynamics**.

An example of such a flow system is **thermal convection**, which denotes the flow driven by buoyancy forces. Applications of this idealized, canonical flow problem are found in the Atmosphere (Hadley circulation) and the gravity-driven circulation in self-convecting heating systems. The basic experiment is known as the **Bénard experiment**, which is given by a fluid layer of finite thickness that is confined between a bottom heating and an upper cooling plate. The system exhibits the state at rest with only molecular conductive heat transfer, periodic states of well-organized large-scale convection, and nonperiodic states characterized by turbulent flow.

Lorenz[1] (1963) was able to mathematically model the flow capturing the flow regimes based on the simplified Navier–Stokes equations leaving three nondimensional **dynamical variables**, here denoted as $x(t)$, $y(t)$, and $z(t)$. These variables represent integral measures of the momentary total kinetic energy, the dissipation rate, and the heat transfer, as functions of nondimensional time t. The corresponding **Lorenz system** is given by the following three coupled

[1] E. N. Lorenz, "Deterministic nonperiodic flow", *J. Atmos. Sci.* (1963) **20**:130.

ordinary differential equations (ODEs):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = s\,(y - x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (r - z)\,x - y, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz. \tag{1}$$

Here b, r, s are freely selectable, nondimensional **control parameters** that are related to the thickness of the fluid layer in relation to the smallest dissipative flow scales, the buoyancy forcing (Rayleigh number), and the ratio of molecular energy and momentum diffusion (Prandtl number) in the fluid, respectively.

The system exhibits intricate nonperiodic solutions. The most widely known solution is the **Lorenz attractor**, which is obtained for the following control parameter values and initial conditions (denoted by time index superscript 0):

$$\begin{aligned} b &= 8/3 & r &= 28 & s &= 10 \\ x^0 &= -8 & y^0 &= -1 & z^0 &= 33 \end{aligned}$$

Tasks

1. Discretize the Lorenz system (1) with an explicit Euler method. *Hint:* You can discretize the derivatives with a forward difference between time levels t^{n+1} and t^n. Alternatively, integrate over the time step Δt and discretize with the lower-sum quadrature rule.

2. Develop an update algorithm that solves (integrates) the Lorenz system. *Hint:* Solve for the unknown ‘future’ nodal value at time level t^{n+1} in each of the ODEs.

3. Implement the solution algorithm. *Hint:* Take your code from the previous exercise on integration and think how to extend that rather than starting from scratch.

4. Integrate the Lorenz system over the time interval $0 \leq t \leq 40$ using the constant time step size $\Delta t = 10^{-3}$.

5. Plot the following 2-D sections:

   (a) x vs. z

   (b) y vs. z

   (c) $x + y$ vs. $x - y$

   Hint: Generate a figure with multiple subplots by following this tutorial: https://matplotlib.org/3.1.0/gallery/subplots_axes_and_figures/subplots_demo.html

6. ($*$) Use matplotlib’s `mplot3d` plotting interface in order to generate a 3-D visualization of the Lorenz attractor. Follow the *parametric curve* example from this tutorial: https://matplotlib.org/2.0.2/mpl_toolkits/mplot3d/tutorial.html. For better visibility, use the coordinates $x + y$, $x - y$, and z as shown in Fig. 1.

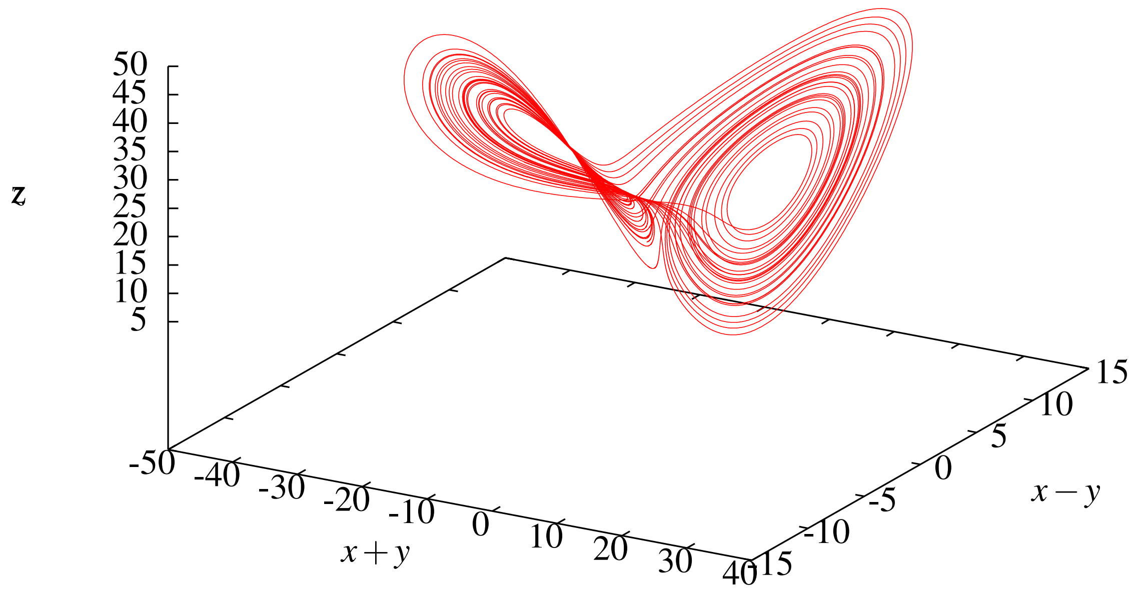

Figure 1: 3-D visualization of the Lorenz attractor using the outlined configuration.