Existence and Stability of Travelling Waves

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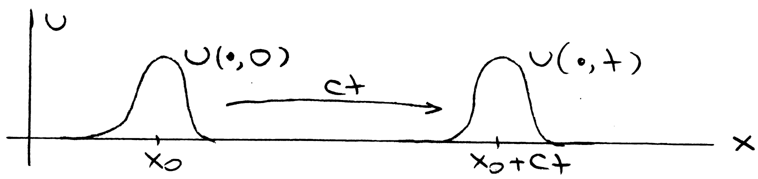
§1 Introduction

Travelling waves: $u(x,t) = \underbrace{u_*(x-ct)}$

constant velocity c | same shape & profile

x position
t time
c speed



Applications:

- nonlinear optics (electric field)
- nerve impulses (voltage & ion concentrations)
- chemical waves (concentrations)
- flame fronts (temperature & concentrations)
- buckled structures (displacement)
- fluids (surface elevation)

Often modelled by partial differential equations (PDEs):

- Reaction-diffusion eqns. $u_t = D u_{xx} + f(u)$
- Hamiltonian PDEs:
  - Korteweg-de Vries $u_t + u_{xxx} + (u^2)_x = 0$
  - nonlinear Schrödinger $i u_t + u_{xx} + |u|^2 u = 0$ $(u \in \mathbb{C})$
- Conservation Laws $u_t = f(u)_x$ or $u_t = \underbrace{u_{xx}}_{\text{viscous}} + f(u)_x$
- Fourth-order PDEs $u_t + u_{xxxx} = f(u, u_x, u_{xx})$
- Lattice equations $\partial_t u_n = \alpha(u_{n+1} - 2u_n + u_{n-1}) + f(u_n)$ $(n \in \mathbb{Z})$
- Multidimensional PDEs $u_t = D\Delta u + f(u)$

  with $x \in \mathbb{R} \times \Omega$ and $\Omega \subset \mathbb{R}^d$

Concentrate on reaction-diffusion equations

(1) $\quad u_t = D u_{xx} + f(u) \qquad x \in \mathbb{R},\ t>0,\ u \in \mathbb{R}^n$

$D = \operatorname{diag}(d_j)$ with $d_j > 0$

Travelling wave : $\quad u(x,t) = u_*(\underbrace{x-ct}_{})$

$\xi = x - ct$ comoving frame

$\longrightarrow$ substitute into (1) to get

(2) $\quad \underbrace{-c u_\xi = D u_{\xi\xi} + f(u)}_{} \qquad \xi \in \mathbb{R},\ c \in \mathbb{R}$

2^{nd}-order ordinary differential equation (ODE) for profile u_* travelling with speed c

Alternatively, use $(x,t) \to (\xi,t) = (x-ct, t)$ which transforms (1) via $u(x,t) = \tilde{u}(x-ct,t) = \tilde{u}(\xi,t)$ into

(3) $\quad u_t = D u_{\xi\xi} + c u_\xi + f(u) \qquad \xi \in \mathbb{R},\ t>0$

Travelling wave of (1) with speed c_* $\longleftrightarrow$ ODE solution of (2) for $c = c_*$ $\longleftrightarrow$ Stationary solution of (3) for $c = c_*$

offer two complementary viewpoints that can be exploited

$\longrightarrow$

- PDEs on $\mathbb{R}$: travelling waves correspond to ODE solutions
- PDEs on $\mathbb{R}\times\Omega$:

$$u_t = D(u_{xx} + \Delta_y u) + f(u) \qquad y \in \Omega \qquad \text{(PDE)}$$

$$D u_{xx} + c u_x + D \Delta_y u + f(u) = 0 \qquad \text{(TW)}$$

travelling-wave equation is still a PDE

- Lattice equations :

$$\dot{u}_n = \alpha(u_{n+1} - 2u_n + u_{n-1}) + f(u_n) \qquad (n \in \mathbb{Z})$$

travelling wave : $\quad u_n(t) = \varphi(n - ct)$

$$-c\varphi_\xi(\xi) = \alpha\big(\varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1)\big) + f(\varphi(\xi))$$

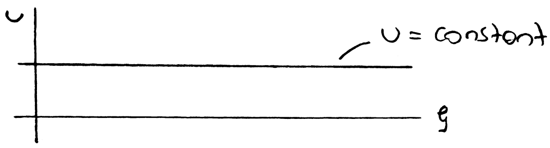
travelling-wave equation is an advanced-retarded functional differential eqn.

„Interesting" travelling waves:

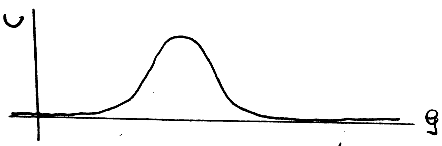
PDE

equilibria of

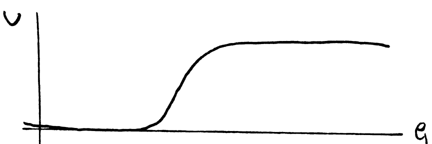
$$u_t = D u_{\xi\xi} + c u_\xi + f(u)$$



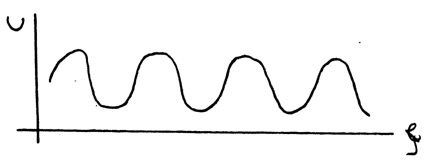
homogeneous rest state



pulse (localised wave)



front connecting two different homogeneous rest states

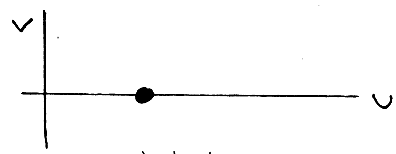


wave train

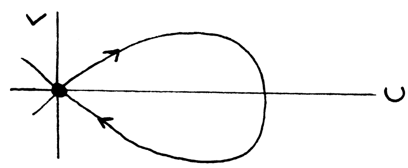
ODE

solution of

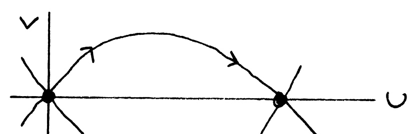
$$\begin{pmatrix} u \\ v \end{pmatrix}_\xi = \begin{pmatrix} v \\ -D^{-1}[cv + f(u)] \end{pmatrix}$$



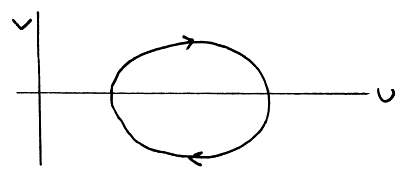
equilibrium



homoclinic orbit



heteroclinic orbit



periodic orbit

Of interest to us : Connection between PDE and ODE

→ 1) What does PDE stability of a wave tell us about its properties as a solution to the ODE?

2) What can the ODE tell us about PDE stability?

3) Bifurcations of waves :
Should we analyse them using (PDE) or (ODE)?

4) Can these ideas be applied to multidimensional PDEs and to lattice equations?

Stability of travelling waves u_* with speed c_*:

→ equilibria of PDE (3) for $c = c_*$

→ Linearise right-hand side of (3) to get linear operator

$$\mathcal{L}_* = D\partial_{\xi\xi} + c_*\partial_\xi + f_u(u_*(\xi))$$

Function spaces for admissible perturbations:

(i) $X = L^2(\mathbb{R}, \mathbb{R}^n)$ square-integrable functions

$\|u\|_{L^2} = \left(\int_{\mathbb{R}} |u(x)|^2 \, dx\right)^{1/2}$ (Hilbert space)

(ii) $X = C^0_{unif}(\mathbb{R}, \mathbb{R}^n)$ bounded, uniformly continuous functions

$\|u\|_{C^0} = \sup_{x\in\mathbb{R}} |u(x)|$ (Banach space: no scalar product!)

Spectral stability: calculate spectrum of $\mathcal{L}_* : X \to X$ defined on domain $\mathcal{D}(\mathcal{L}_*) \subset X$.

- $\lambda \in \rho(\mathcal{L}_*)$ resolvent set iff $\exists K : \forall h \in X \; \exists! u \in X : (\mathcal{L}_* - \lambda)u = h$ and $\|u\|_X \leq K\|h\|_X$
- Spectrum of $\mathcal{L}_*$: $\text{spec}(\mathcal{L}_*) := \mathbb{C} \setminus \rho(\mathcal{L}_*)$
- λ is an eigenvalue of $\mathcal{L}_*$ iff $\exists u \in X \setminus \{0\} : \mathcal{L}_* u = \lambda u$
- We write

$$\text{spec}(\mathcal{L}_*) = \Sigma_{pt} \,\dot\cup\, \Sigma_{ess}$$

essential spectrum: defined as $\text{spec}(\mathcal{L}_*) \setminus \Sigma_{pt}$

point spectrum: defined as the union of all isolated eigenvalues with finite multiplicity

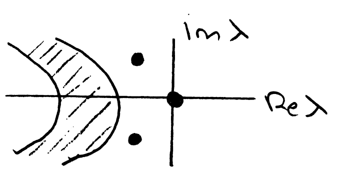
Example $\mathcal{L} : \ell^\infty \to \ell^\infty$, $(a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots)$

→ $\lambda = 0$ is not an eigenvalue but $0 \in \text{spec}(\mathcal{L})$

since $\mathcal{L}u = (1, 0, 0, \dots)$ does not have a solution in ℓ^∞.

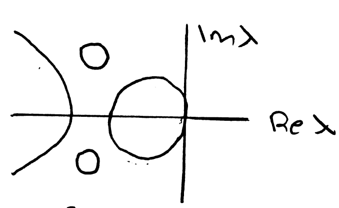
Expectation

pulses, fronts:



Spectra (point & essential)

wave trains:



Spectra (only essential)

Example If $u_*' \neq 0$ and $u_*' \in X$, then $0 \in \operatorname{spec}(\mathcal{L}_*)$

Proof The travelling wave $u_*(\xi)$ satisfies

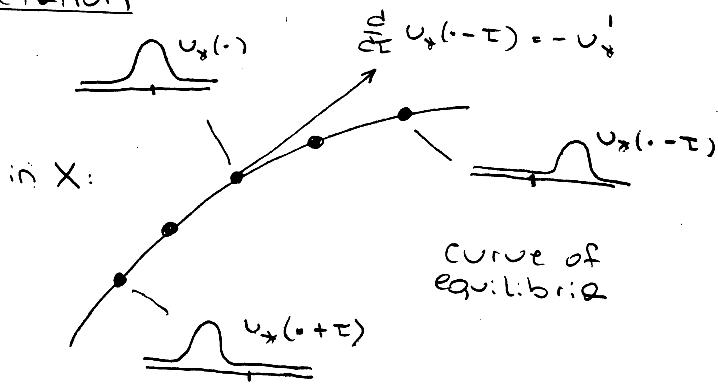
$$D u_*''(\xi) + c_* u_*'(\xi) + f(u_*(\xi)) = 0 \qquad \forall \xi \in \mathbb{R}$$

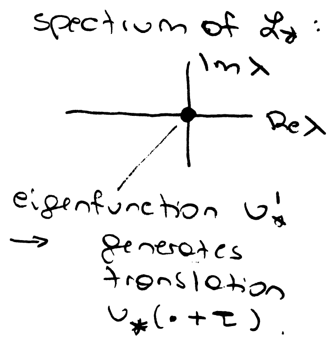
Taking a further derivative $\frac{d}{d\xi}$, we find

$$D u_*'''(\xi) + c_* u_*''(\xi) + f_u(u_*(\xi))\, u_*'(\xi) \qquad \forall \xi \in \mathbb{R}$$

and therefore $\mathcal{L}_* u_*' = 0$ $\square$

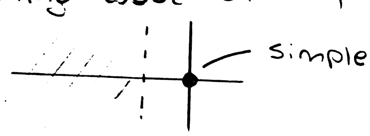
<u>Interpretation</u>



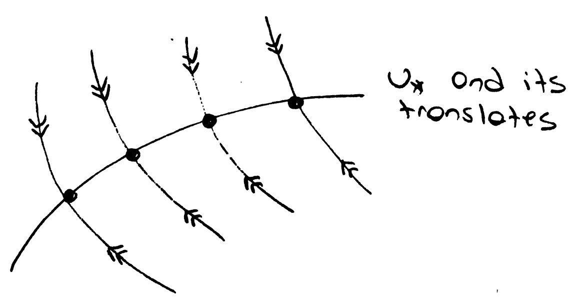


<u>Nonlinear stability</u>

Theorem: If u_* is a travelling wave of $u_t = D u_{xx} + f(u)$ on X with spectrum



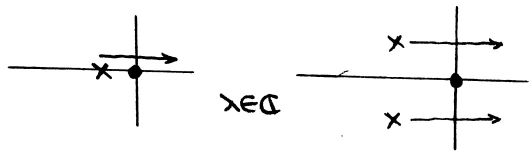
, then u_* is nonlinearly asymptotically stable: $\forall \varepsilon_* > 0 \; \exists \delta_* > 0$: $\forall u_0 \in X$ with $\| u_0 - u_* \|_X < \delta_*$, we have $\| u(\cdot, t) - u_* \|_X < \varepsilon_*$ $\forall t \geq 0$ and $\exists\, \xi_* \in \mathbb{R}$ with $\| u(\cdot, t) - u_*(\cdot - \xi_*) \|_X \to 0$ exponentially as $t \to \infty$.



Remark: Theorem is also true if some of the diffusion coefficients vanish or for higher-order PDEs.

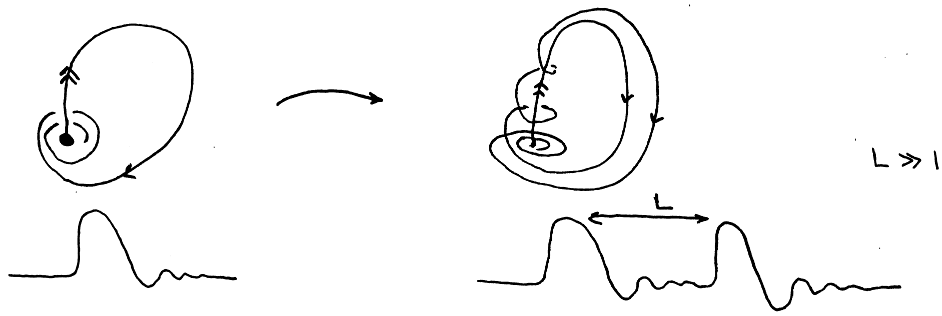
Bifurcations of travelling waves:

(i) Saddle-node and Hopf:

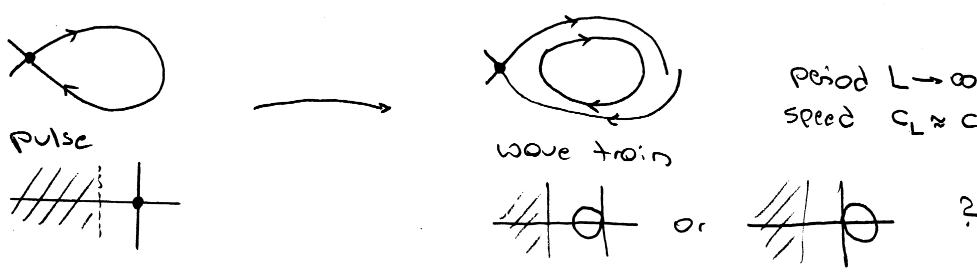


as a parameter μ is varied

(ii) multi-pulses:

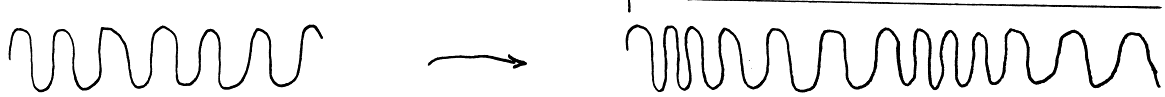


(iii) wave trains



period $L \to \infty$
speed $c_L \approx c_*$

Modulation equations



wave train

(wave number $q = 2\pi/$spatial period)

modulated wave train:

wave number q varies on scale that is large compared to spatial period

$\longrightarrow$ expect $q = q(x,t)$: derive equation for q ?

§2 Spectra

Recall $\quad \mathcal{L}_* = D\partial_{xx} + c_*\partial_x + f_u(u_*(x)) \quad$ on $X = L^2$ or $X = C^0_{unif}$

$\lambda \in \mathbb{C} \setminus \operatorname{spec}\mathcal{L}_*$ iff $\exists K > 0 : \forall h \in X \; \exists!\, u \in X : (\mathcal{L}_* - \lambda)u = h, \; \|u\|_X \le K\|h\|_X$

§2.1 Homogeneous rest states $\quad u_*(x) = u_0$ constant

(4) $\quad u_t = Du_{xx} + c_* u_x + f_u(u_0)\,u \qquad$ constant coefficients

$u(x,t) = e^{\lambda t + \nu x} v_0 \qquad$ for some $\nu \in \mathbb{C}$, $v_0 \in \mathbb{C}^n \setminus \{0\}$

$\rightarrow \quad \lambda v_0 = [D\nu^2 + c_*\nu + f_u(u_0)]\, v_0$

$\rightarrow \quad d(\lambda,\nu) := \det[D\nu^2 + c_*\nu + f_u(u_0) - \lambda] = 0 \qquad$ necessary & sufficient for having an eigenmode $e^{\lambda t + \nu x} v_0$

$\llcorner$ linear dispersion relation

intuition: $\quad$ want $e^{\lambda t + ikx} v_0$ for $k \in \mathbb{R}$ to satisfy (4)

$\rightarrow \quad \operatorname{spec}\mathcal{L}_* = \{\lambda \in \mathbb{C} \,/\, d(\lambda, ik) = \det(-k^2 D + ikc_* + f_u(u_0) - \lambda) = 0$ for some $k \in \mathbb{R}\}$

<u>Theorem</u> Assume that $u_*(x) = u_0$, then

$\operatorname{spec}\mathcal{L}_* = \{\lambda \in \mathbb{C} \,/\, d(\lambda, ik) = 0$ for some $k \in \mathbb{R}\}$ on L^2 and C^0_{unif}

<u>Proof</u> Fix $\lambda \in \mathbb{C}$.

(i) Assume $d(\lambda, ik) \neq 0 \;\; \forall k \in \mathbb{R}$. Pick $h \in X$ and consider

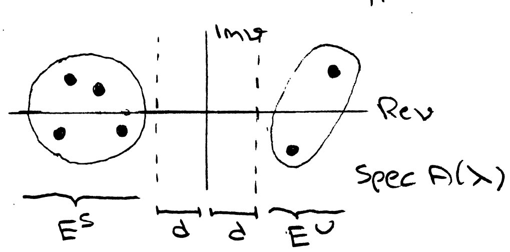
$Du_{xx} + c_* u_x + f_u(u_0)\,u = h(x) \qquad$ constant coefficients

which we write as

(5) $$\begin{pmatrix} u \\ v \end{pmatrix}_x = \underbrace{\begin{pmatrix} 0 & 1 \\ D^{-1}[\lambda - f_u(u_0)] & -c_* D^{-1} \end{pmatrix}}_{=:\, A(\lambda)} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ h(x) \end{pmatrix}$$

<u>Key</u> $\quad \det[A(\lambda) - \nu] = d(\lambda,\nu)\,\dfrac{1}{\det D}$

$\rightarrow \quad A(\lambda)$ is hyperbolic since $d(\lambda, ik) \neq 0 \;\; \forall k \in \mathbb{R}$



2n spatial eigenvalues in $\operatorname{spec}(A(\lambda))$ for each $\lambda \in \mathbb{C}$.

$\rightarrow$ generalized stable and unstable eigenspaces with spectral projections $P^s(\lambda)$ and $P^u(\lambda)$

$$E^s(\lambda) \oplus E^u(\lambda) = \mathbb{C}^{2n}, \qquad P^u(\lambda) + P^s(\lambda) = 1_{\mathbb{C}^{2n}}$$

$$e^{A(\lambda)x} = e^{A(\lambda)x} P^s(\lambda) + e^{A(\lambda)x} P^u(\lambda) \quad \text{with}$$

$$\begin{cases} \| e^{A(\lambda)x} P^s(\lambda) \| \le \tilde{K} e^{-d|x|} & x \ge 0 \\ \| e^{A(\lambda)x} P^u(\lambda) \| \le \tilde{K} e^{-d|x|} & x \le 0 \end{cases}$$

→ Unique bounded solution of (5) is given by

$$\begin{pmatrix} u \\ v \end{pmatrix}(x) = \int_{-\infty}^{x} e^{A(\lambda)(x-y)} P^s(\lambda) \begin{pmatrix} 0 \\ h(y) \end{pmatrix} dy + \int_{\infty}^{x} e^{A(\lambda)(x-y)} P^u(\lambda) \begin{pmatrix} 0 \\ h(y) \end{pmatrix} dy$$

$$=: \int_{\mathbb{R}} G(x-y) \begin{pmatrix} 0 \\ h(y) \end{pmatrix} dy \qquad \text{with } \|G(x)\| \le \tilde{K} e^{-d|x|}$$

Using Young's inequality

$$\| G * H \|_{L^p} \le \| G \|_{L^1} \| H \|_{L^p} \quad \text{for } 1 \le p \le \infty$$

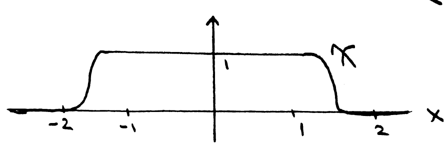
we find

$$\|u\|_{C^0} \le \frac{2\tilde{K}}{d} \|h\|_{C^0}, \qquad \|u\|_{L^2} \le \frac{2\tilde{K}}{d} \|h\|_{L^2}.$$

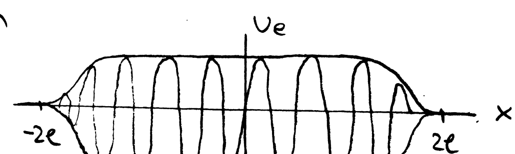
(ii) Assume $d(\lambda, ik_0) = 0$ for some $k_0 \in \mathbb{R}$.

We shall prove that $\exists\, u_\ell \in X : \|(\mathcal{L}_* - \lambda) u_\ell \|_X \le \frac{1}{\ell} \|u_\ell\|_X$ as $\ell \to \infty$

which shows that $\lambda \in \operatorname{spec} \mathcal{L}_*$ $\left(\text{otherwise, } \exists K : \|u_\ell\|_X \le K \|(\mathcal{L}_* - \lambda) u_\ell\|_X \le \frac{K}{\ell} \|u_\ell\|_X \;\lightning \right)$



cutoff function



Set $u_\ell(x) = \chi\left(\frac{x}{\ell}\right) e^{ik_0 x} v_0$

where $v_0 \in \mathbb{C}^n \setminus \{0\} : (-k_0^2 D + ik_0 c_* + f_u(u_0) - \lambda) v_0 = 0$

$$\to \quad \|u_\ell\|^2_{L^2} \ge 2\ell\, |v_0|^2_\infty \qquad \left(|v_0|_\infty = \max_j |v_0^{(j)}| \right)$$

$$\|(\mathcal{L}_* - \lambda) u_\ell \|^2_{L^2} \le \left\| \left[D\left(\frac{2ik_0}{\ell} \chi' + \frac{1}{\ell^2} \chi'' \right) + \frac{c_*}{\ell} \chi' \right] e^{ik_0 x} v_0 \right\|^2_{L^2} \le \frac{\text{const.}}{\ell}$$

□ 3]

<u>Conclusion</u> u_0 homogeneous rest state : $A(\lambda) = \begin{pmatrix} 0 & 1 \\ D^{-1}[\lambda - f_u(u_0)] & -c_* D^{-1} \end{pmatrix}$

$$\underbrace{\lambda \in \operatorname{spec} \mathcal{L}_*}_{\text{temporal PDE spectrum}} \iff \underbrace{\operatorname{spec} A(\lambda)}_{\text{spatial ODE spectrum}} \cap i\mathbb{R} \ne \emptyset$$

<u>Note</u> travelling-wave ODE $Du_{xx} + c_* u_x + f(u) = 0$ or

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} v \\ -D^{-1}[c_* v + f(u)] \end{pmatrix}$$

Linearize about equilibrium $(u,v) = (u_0, 0)$ → $A(0) = \begin{pmatrix} 0 & 1 \\ -D^{-1} f_u(u_0) & -c_* D^{-1} \end{pmatrix}$

Properties of spec $\mathcal{L}_*$

$$u_t = \underbrace{D u_{xx} + c_* u_x + f_u(u_0)\, u}_{= \mathcal{L}_* u}$$

$u(x,t) = e^{\lambda t + \nu x} v_0$ soln for some $v_0 \neq 0$ <=>

$d(\lambda,\nu) = \det\left[D\nu^2 + c_*\nu + f_u(u_0) - \lambda \right] = 0$

$$\begin{aligned} \mathrm{spec}(\mathcal{L}_*) &= \{ \lambda \mid d(\lambda, ik) = 0 \quad \text{for some } k \in \mathbb{R} \} \\ &= \{ \lambda \mid \mathrm{spec}\, A(\lambda) \cap i\mathbb{R} \neq \emptyset \} \end{aligned}$$

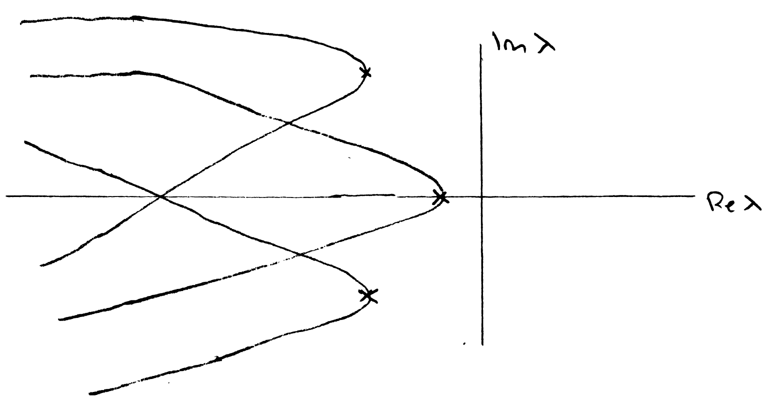
$$A(\lambda) = \begin{pmatrix} 0 & 1 \\ D^{-1}(\lambda - f_u(u_0)) & -c_* D^{-1} \end{pmatrix}$$

$A(0)$ ODE Linearization about $(u_0, 0)$

Structure of spec $\mathcal{L}_*$

- all eigenvalues of $f_u(u_0)$ lie in spec $\mathcal{L}_*$: set $k=0$
- if $d(\lambda_0, ik_0) = 0$, $d_\lambda(\lambda_0, ik_0) \neq 0$, then $\exists$ curve $\lambda_*(ik)$ defined for $k \approx k_0$ with $\lambda_*(ik_0) = \lambda_0$ so that $\lambda_*(ik) \in \mathrm{spec}(\mathcal{L}_*)$ for all k
- As $|k| \to \infty$, we have $\mathrm{Re}\,\lambda \to -\infty$.

Assume that $d(\lambda, ik) = 0$ implies $d_\lambda(\lambda, ik) \neq 0$, then



$n=3$

$\times$ eigenvalues of $f_u(u_0)$

Genericity

- Do we expect to encounter points (λ_0, ik_0) with $d(\lambda_0, ik_0) = d_\lambda(\lambda_0, ik_0) = 0$?

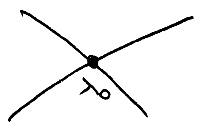
$$\begin{pmatrix} d(\lambda,\nu) \\ d_\lambda(\lambda,\nu) \end{pmatrix} = 0$$

: 2 eqns for 2 unknowns $\longrightarrow$ expect finite set of solns (λ_j, ν_j) (true when $d_i \neq d_j$ $\forall i \neq j$)

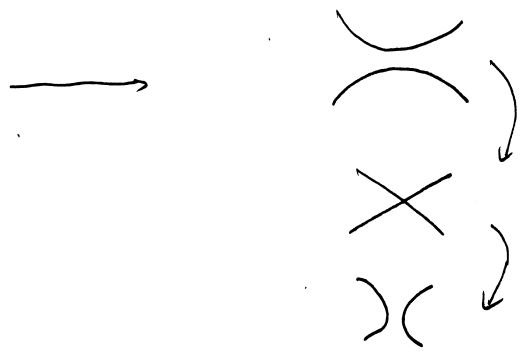
However, „typically", we have $\mathrm{Re}\,\nu_j \neq 0$ $\forall j$

=> We do not expect to encounter points where the implicit function theorem fails when continuing λ in k.

- If $d(\lambda_0, ik_0) = d_\lambda(\lambda_0, ik_0) = 0$ and $d_{\lambda\lambda}(\lambda_0, ik_0) \neq 0$, then



structurally unstable



as a system parameter is varied

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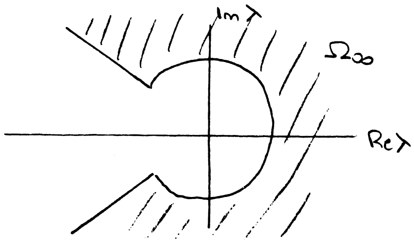
Implications

(i) Spectrum lies in sector:



$\lambda = \frac{e^{i\varphi}}{\varepsilon^2}$ where $0 < \varepsilon \ll 1$, $|\varphi| \le \pi - \delta$ ($\delta > 0$ fixed)

$\nu \in \operatorname{spec} A(\lambda)$ iff

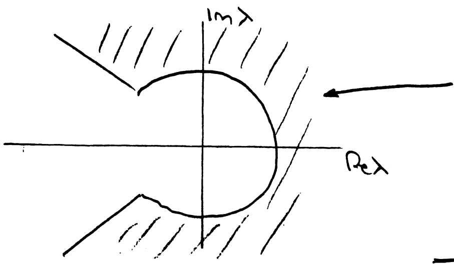
$\det [D\nu^2 + c_* \nu + f_u(u_0) - \lambda] = 0$

$\rightarrow \quad \det\left(\nu^2 D + \nu c_* + f_u(u_0) - \frac{e^{i\varphi}}{\varepsilon^2}\right) \overset{\nu = \frac{\tilde{\nu}}{\varepsilon}}{=\!=} \frac{1}{\varepsilon^{2n}} \det\left(\tilde{\nu}^2 D + \tilde{\nu} c_* \varepsilon + \varepsilon^2 f_u(u_0) - e^{i\varphi}\right) = 0$

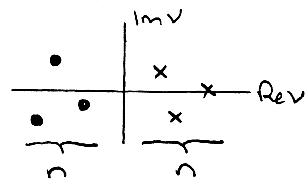
$\varepsilon = 0$: $\quad \det(\tilde{\nu} D - e^{i\varphi}) = 0 \rightarrow \quad \tilde{\nu}_j^0 = \pm \sqrt{\frac{e^{i\varphi}}{d_j}}$

$\varepsilon > 0$: eigenvalues $\tilde{\nu}_j = \tilde{\nu}_j^0 + O(\varepsilon)$: hyperbolic

not purely imaginary



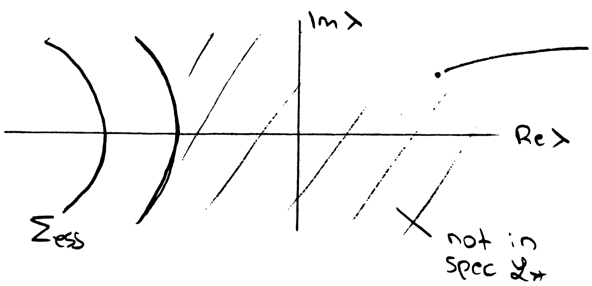
Spec $A(\lambda)$

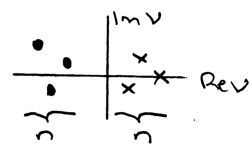


n stable, n unstable } spatial eigenvalues

$\rightarrow$ numbers can change only if $\operatorname{spec} A(\lambda) \cap i\mathbb{R} \neq \emptyset$
ie when $\lambda \in \operatorname{spec} \mathcal{L}_*$

$\rightarrow$ if u_0 is PDE stable, then

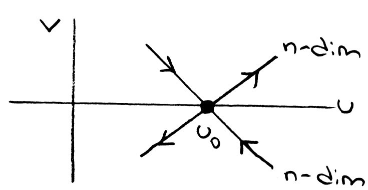




spec $A(\lambda)$

applied to $A(0)$ = ODE linearization about $\binom{u_0}{0}$:

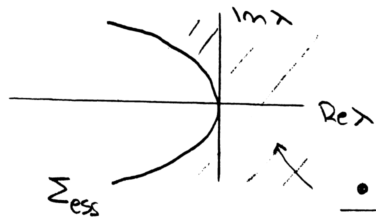
$\binom{u_0}{0}$ saddle with n-dimensional stable and unstable manifolds



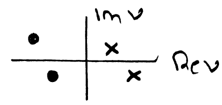
provided u_0 is PDE stable

" PDE stability $\Rightarrow$ ODE hyperbolicity with n:n split of spatial eigenvalues "

(ii) Suppose

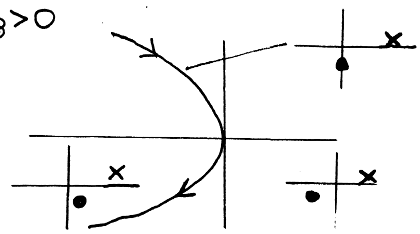


Curve $\lambda = \lambda(ik)$ for $k \in \mathbb{R}$
with $\lambda(0) = 0$, $\lambda'(0) =: -c_g \neq 0$.

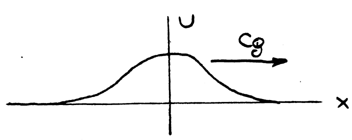


$$\lambda(\nu) = \lambda(0) + \lambda'(0)\,\nu + O(|\nu|^2) = -c_g \nu + O(|\nu|^2)$$

① $c_g > 0$



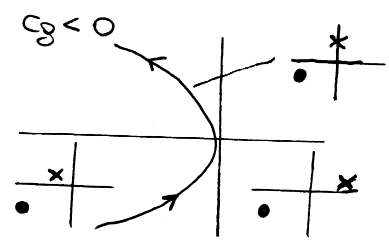
$\lambda(ik) = -ik c_g$



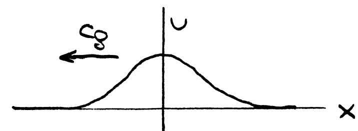
as t increases

$u_t = \mathcal{L}_* u$

② $c_g < 0$



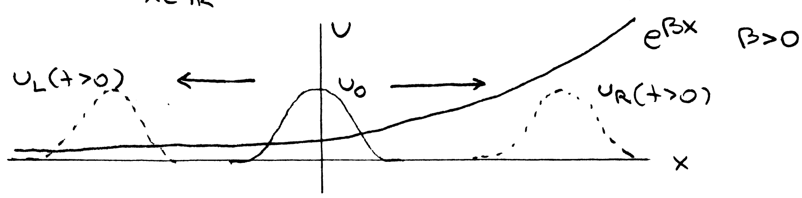
$\lambda(ik) = ik|c_g|$



as t increases

to validate this claim, introduce weighted norm

$$\|u\|_\beta := \sup_{x\in\mathbb{R}} e^{\beta x} |u(x)| \qquad \text{for } \beta \in \mathbb{R} \text{ fixed}$$



$u_t = \mathcal{L}_* u$

$\beta > 0$: $\begin{cases} \|u_L(\cdot,t)\|_\beta & \text{decreases} \\ \|u_R(\cdot,t)\|_\beta & \text{increases} \end{cases}$

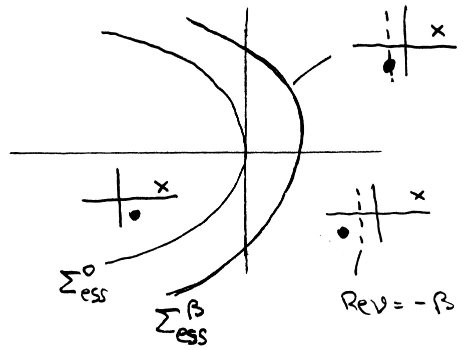
→ compute spectra in

$$X_\beta = \{ u \mid \|u\|_\beta < \infty \}$$
$$= \{ u = e^{-\beta x} v \mid \|v\|_{\infty} < \infty \}$$

→ $\Sigma^\beta_{ess} = \{ \lambda \in \mathbb{C} \mid d(\lambda, -\beta + ik) = 0 \text{ for some } k \in \mathbb{R} \}$
$= \{ \lambda \in \mathbb{C} \mid \operatorname{spec} A(\lambda) \cap \{ \operatorname{Re} \nu = -\beta \} \neq \emptyset \}$.

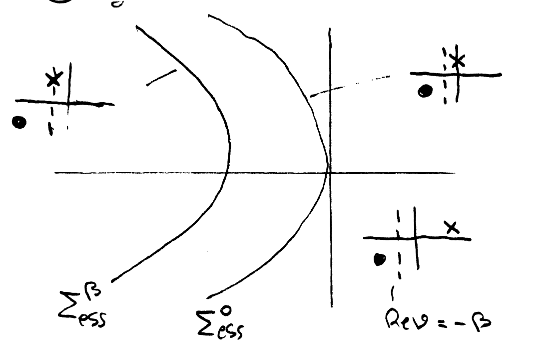
Continue with $0<\beta\ll 1$:

① $c_g>0$



unstable in $\|\cdot\|_\beta$

② $c_g<0$



stable in $\|\cdot\|_\beta$

and analogously with $\beta<0$.

(iii) Absolute spectrum

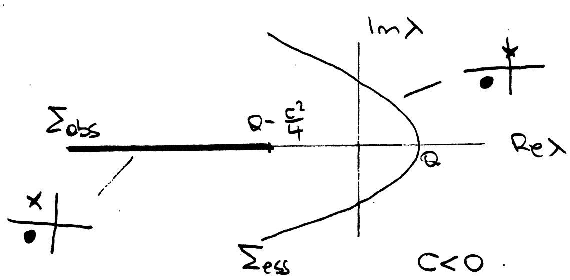
Consider $\mathcal{L}_* = \partial_{xx} + c\partial_x + a$ for $u\in\mathbb{R}$

on $(-L,L)$ with $u(\pm L)=0$

Need to solve $\begin{pmatrix} u \\ v\end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \lambda-a & -c\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix} = A(\lambda)\begin{pmatrix} u \\ v\end{pmatrix}$

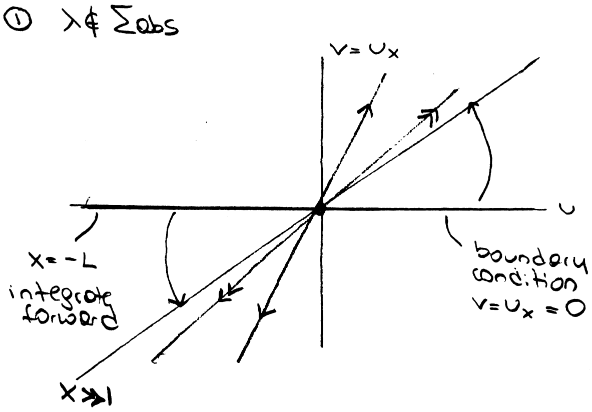
$$\Sigma_{ess} = \{\lambda = -k^2+ikc+a \;;\; k\in\mathbb{R}\}$$

$$\Sigma_{abs} = \{\lambda = -k^2+a-\tfrac{c^2}{4} \;;\; k\in\mathbb{R}\} = \{\lambda \mid \text{eigenvalues of } A(\lambda) \text{ have the same real part}\}$$

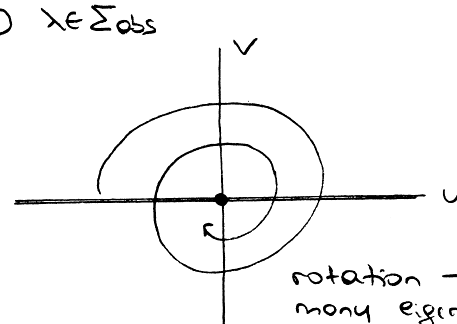


$c<0$.

① $\lambda\notin\Sigma_{abs}$



② $\lambda\in\Sigma_{abs}$



rotation → many eigenvalues depending on precise value of L.

§2.2 Wave trains $\quad u_*(x) = u_*(x + \frac{2\pi}{q}) \quad \forall x \in \mathbb{R}$

Spectra of wave trains :

$$(\mathcal{L}_* - \lambda) u = D u_{xx} + c_* u_x + \underbrace{f_u(u_*(x))}_{\frac{2\pi}{q}\text{-periodic coefficients}} u - \lambda u = 0$$

becomes the ODE

$$(6) \qquad \begin{pmatrix} u \\ v \end{pmatrix}_x = A(x;\lambda) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ D^{-1}[\lambda - f_u(u_*(x))] & -c_* D^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

with $A(x + \frac{2\pi}{q}; \lambda) = A(x;\lambda) \quad \forall x$.

General solution of (6) $\longrightarrow$ Floquet theory :

$\exists\, R(x,\lambda)$: $\frac{2\pi}{q}$-periodic in x with $R(0,\lambda) = 1$, and $B(\lambda)$:

$\begin{pmatrix} u \\ v \end{pmatrix}(x) = R(x,\lambda)\, e^{B(\lambda)x} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$: general solution of (6)

Theorem $\quad$ Assume that $u_*(x + \frac{2\pi}{q}) = u_*(x) \;\forall x$, then

$\operatorname{spec} \mathcal{L}_* = \{ \lambda \in \mathbb{C} \mid \det(B(\lambda) - ik) = 0 \text{ for some } k \in \mathbb{R} \}$

Proof $\quad$ Same proof as for homogeneous rest states :

Replace $e^{A(\lambda)x}$ by $R(x,\lambda) e^{B(\lambda)x}$ and use eigenspaces of $B(\lambda)$ instead of $A(\lambda)$ $\square$

6)

Implications

(i) Eigenmodes are of the form $u(x) = u_{per}(x)\, e^{ikx}$ with $k \in \mathbb{R}$

where $u_{per}(x + \frac{2\pi}{q}) = u_{per}(x) \;\forall x$ depends on k $\longrightarrow$

$$\operatorname{spec} \mathcal{L}_* = \bigcup_{k \in [0,q)} \underbrace{\operatorname{spec} \mathcal{L}_*(ik)}_{\substack{\text{computed with periodic boundary} \\ \text{conditions on } (0, \frac{2\pi}{q})}}$$

where $\mathcal{L}_*(\nu) = D(\partial_x + \nu)^2 + c_*(\partial_x + \nu) + f_u(u_*(x))$

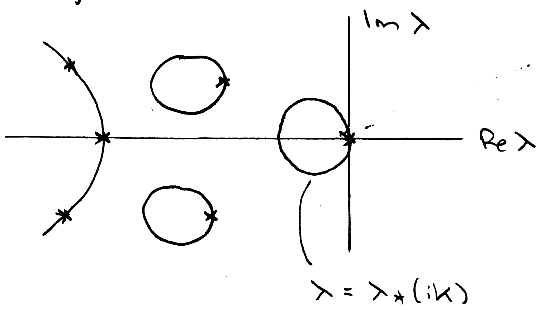
Proof $\quad$ Use $u(x) = v(x)\, e^{ikx}$ with $v(x + \frac{2\pi}{q}) = v(x) \;\forall x$

$\longrightarrow$ gives eigenvalue problem $\mathcal{L}_*(\nu) v = \lambda v$ for v $\square$

(ii) Statements for $\begin{cases} \text{sectoriality of spectrum} \\ \text{weighted norms} \end{cases}$

are also true for wave trains upon using $B(\lambda)$ instead of $A(\lambda)$

In particular,



$k=0$, $u_{per} = u_*'$ } always lies in spectrum

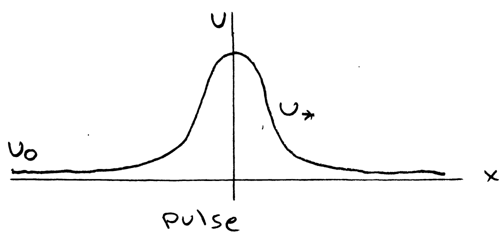
$*$: $k=0$.

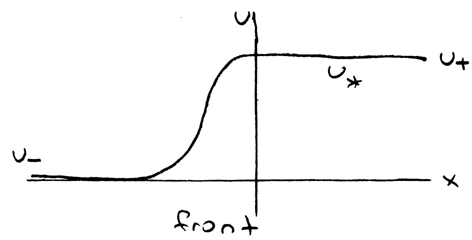
$c_g = -\left.\dfrac{d\lambda}{d\nu}\right|_{\nu=0}$ group velocity of wave train

$\underbrace{\hspace{8cm}}_{\text{direction of transport}}$

§2.3 Pulses and fronts

$u_*(x) \longrightarrow u_\pm$ as $x \to \pm\infty$



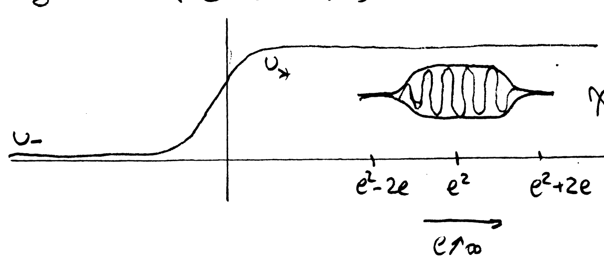


<u>Notation</u> $\mathcal{L}[u] = D\partial_{xx} + c_*\partial_x + f(u(x))$ Operator associated with wave u

$\mathcal{L}_* = \mathcal{L}[u_*]$ pulse or front

<u>Theorem</u> If $\lambda \in \operatorname{spec}\mathcal{L}[u_+] \cup \operatorname{spec}\mathcal{L}[u_-]$, then $\lambda \in \operatorname{spec}\mathcal{L}_*$

<u>Proof</u> E.g. $\lambda \in \operatorname{spec}\mathcal{L}[u_+]$, then



$\chi\left(\frac{x-e^2}{e}\right) e^{ik_+x} v_+$ approximate eigenfunction

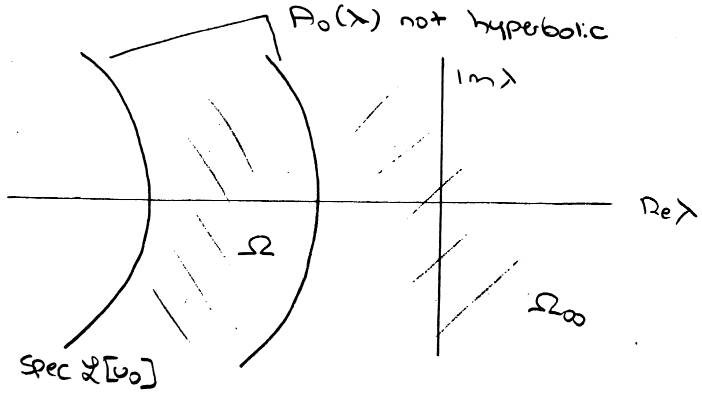
$\square$

<u>Pulses</u> Seek eigenvalues : $(\mathcal{L}_* - \lambda)u = 0$ ie bounded solution of

$$(7) \qquad \begin{pmatrix} u \\ v \end{pmatrix}_x = \underbrace{\begin{pmatrix} 0 & 1 \\ D^{-1}[\lambda - f_u(u_*(x))] & -c_*D^{-1} \end{pmatrix}}_{=:\, A(x,\lambda)} \begin{pmatrix} u \\ v \end{pmatrix}$$

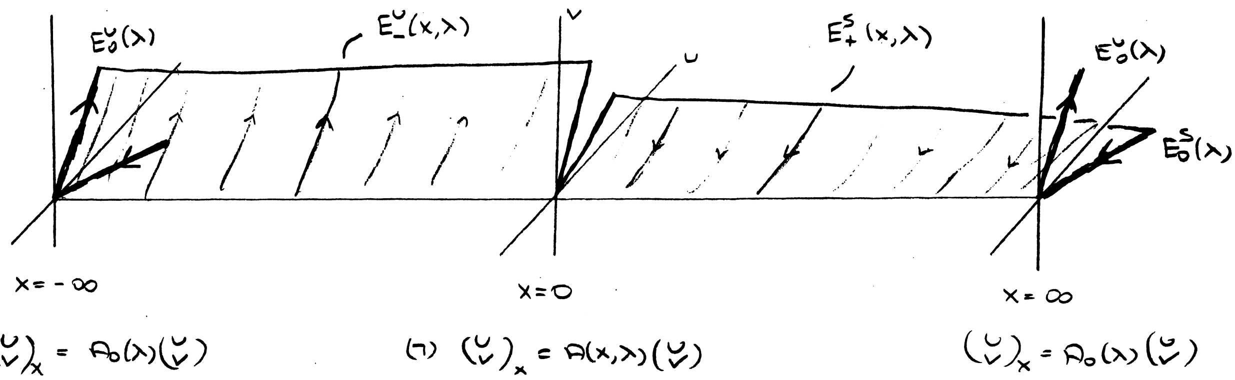
and $A(x,\lambda) \to A_0(\lambda)$ as $|x| \to \infty$.

where $A_0(\lambda) = \begin{pmatrix} 0 & 1 \\ D^{-1}[\lambda - f_u(u_0)] & -c_*D^{-1} \end{pmatrix}$ Corresponds to asymptotic rest state u_0



Fix connected component Ω of $\mathbb{C} \setminus \operatorname{spec} \mathcal{L}[u_0]$

$\rightarrow$ $A_0(\lambda)$ is hyperbolic $\forall \lambda \in \Omega$:



$$E_+^s(x_0,\lambda) = \left\{ \binom{u_0}{v_0} \in \mathbb{C}^{2n} \,\Big|\, \binom{u}{v}(x) \to 0 \text{ as } x \to \infty \text{ where } \binom{u}{v}(x_0) = \binom{u_0}{v_0} \right\}$$

$$E_-^u(x_0,\lambda) = \left\{ \binom{u_0}{v_0} \in \mathbb{C}^{2n} \,\Big|\, \binom{u}{v}(x) \to 0 \text{ as } x \to -\infty \text{ where } \binom{u}{v}(x_0) = \binom{u_0}{v_0} \right\}$$

<u>Note</u> Due to hyperbolicity of the asymptotic equation, the only bounded solutions of (7) are, in fact, exponentially decaying as $|x| \to \infty$ and can therefore be found by seeking nontrivial intersections

$$E_-^u(0,\lambda) \cap E_+^s(0,\lambda) \neq \{0\} \tag{8}$$

<u>Evans function</u> Consider $\lambda \in \Omega$, then $E_-^u(0,\lambda)$ and $E_+^s(0,\lambda)$ depend analytically on λ.

$\rightarrow$ Choose analytic bases $\{v_j^u(\lambda)\}_{j=1,\dots,k}$ and $\{v_j^s(\lambda)\}_{j=1,\dots,n-k}$ of $E_-^u(0,\lambda)$ and $E_+^s(0,\lambda)$, respectively.

$\rightarrow$ Evans function :

$$\mathcal{D}(\lambda) = \det\left[v_1^u(\lambda), \dots, v_k^u(\lambda), v_1^s(\lambda), \dots, v_{n-k}^s(\lambda)\right]$$

analytic in $\lambda \in \Omega$.

Theorem Let Ω be a connected component of $\mathbb{C} \setminus \operatorname{spec} \mathcal{L}[U_0]$, then

(9) $\mathcal{D}(\lambda) = 0 \iff E_-^u(0,\lambda) \cap E_+^s(0,\lambda) \neq \{0\} \iff \lambda$ is an eigenvalue

Furthermore, either

(i) $\Omega \cap \operatorname{spec} \mathcal{L}_*$ is a discrete set of isolated eigenvalues with finite multiplicity: these eigenvalues correspond to roots of $\mathcal{D}(\lambda)$ and

(10) Order of roots of $\mathcal{D}$ = PDE multiplicity of eigenvalues

or else

(ii) $\Omega \subset \operatorname{spec} \mathcal{L}_*$ and each $\lambda \in \Omega$ is an eigenvalue : $\mathcal{D}(\lambda) \equiv 0$ in Ω

For Ω_∞, option (i) is the only possibility. 7

Remark Case (ii) is ungeneric, and I do not know of any PDE example where it occurs.

Idea of Proof We already proved (9). If $\mathcal{D}(\lambda) \equiv 0$ in Ω, then (ii) occurs. Otherwise, $\mathcal{D}(\lambda)$ has only a discrete set of roots, with finite order. I will not prove (10) $\longrightarrow$ see 9 for a sketch of proof.

It remains to show that $\lambda \notin \operatorname{spec} \mathcal{L}_*$ when $\mathcal{D}(\lambda) \neq 0$. In this case, $E_-^u(x,\lambda) \oplus E_+^s(x,\lambda) = \mathbb{C}^{2n}$ $\forall x \in \mathbb{R}$. I claim that

$$(\mathcal{L}_* - \lambda) u = h$$

has a unique solution u with $\|u\|_{\mathbb{X}} \leq K \|h\|_X$:

Indeed, we have an x-dependent splitting into stable and unstable direction, given by $E_+^s(x,\lambda)$ and $E_-^u(x,\lambda)$, for $x \in \mathbb{R}$ $\longrightarrow$

$$\begin{pmatrix} u \\ v \end{pmatrix}(x) = \int_{-\infty}^{x} \underbrace{\Phi(x,y)}_{\substack{\text{evolution of} \\ \binom{u}{v}_x = A(x,\lambda)\binom{u}{v}}} P^s(y,\lambda) \begin{pmatrix} 0 \\ h(y) \end{pmatrix} dy + \int_{\infty}^{x} \Phi(x,y) P^u(y,\lambda) \begin{pmatrix} 0 \\ h(y) \end{pmatrix} dy$$

Projections associated with $E_-^u(y,\lambda) \oplus E_+^s(y,\lambda) = \mathbb{C}^{2n}$

$\square$

Fronts

$$\begin{pmatrix}u\\v\end{pmatrix}_x = A(x,\lambda)\begin{pmatrix}u\\v\end{pmatrix} \quad \text{and}$$

$A(x,\lambda) \to A_\pm(\lambda)$ as $x \to \pm\infty$

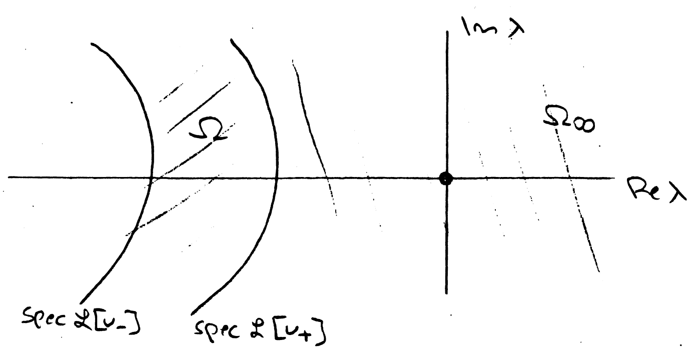
associated with rest states $u_\pm$ at $x = \pm\infty$.

Fix connected component Ω of $\mathbb{C} \setminus \operatorname{spec} \mathcal{L}[u_-] \cup \operatorname{spec} \mathcal{L}[u_+]$,

For $\lambda \in \Omega$, $A_+(\lambda)$ and $A_-(\lambda)$ are hyperbolic $\to$

Morse index: $i_\pm(\lambda) = \dim E^u_\pm(\lambda)$: dimension of unstable eigenspace of $A_\pm(\lambda)$.

$\to$
- $i_\pm(\lambda)$ is constant for $\lambda \in \Omega$
- $\lambda \in \Omega_\infty$ implies $i_+(\lambda) = i_-(\lambda) = n$.



(i) $i_+(\lambda) = i_-(\lambda) \quad \forall \lambda \in \Omega$ $\longrightarrow$ same situation as for pulses

Evans function well defined and analytic in λ $\to$

- either discrete set of isolated eigenvalues
- or else $\Omega \subset \operatorname{spec} \mathcal{L}_*$ because $\mathcal{D}(\lambda) \equiv 0$.

(ii) $i_-(\lambda) > i_+(\lambda) \quad \forall \lambda \in \Omega$ $\Rightarrow$ $\Omega \subset \operatorname{spec} \mathcal{L}_*$ consists entirely of eigenvalues:

$$\begin{cases} \dim E^u_-(0,\lambda) = \dim E^u_-(\lambda) = i_-(\lambda) \\ \dim E^s_+(0,\lambda) = \dim E^s_+(\lambda) = 2n - i_+(\lambda) \end{cases}$$

sum of dimensions $- 2n = i_-(\lambda) + (2n - i_+(\lambda)) - 2n$

$= i_-(\lambda) - i_+(\lambda) > 0$

$\to$ subspaces have intersection of dimension at least $i_-(\lambda) - i_+(\lambda) > 0$

$\to$ $E^u_-(0,\lambda) \cap E^s_+(0,\lambda) \neq \{0\}$

$\to$ $\dim N(\mathcal{L}_* - \lambda) = \dim \{u \in X \,/\, (\mathcal{L}_* - \lambda)u = 0\} \geq i_-(\lambda) - i_+(\lambda)$

(iii) $i_-(\lambda) < i_+(\lambda) \quad \forall \lambda \in \Omega \quad \Rightarrow \quad \Omega \subset \operatorname{spec} \mathcal{L}_*$.

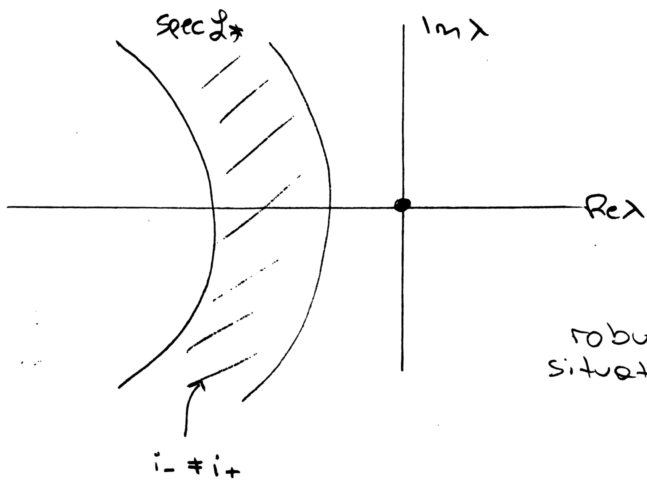
In fact, $(\mathcal{L}_* - \lambda)$ is not onto, and we have

$$\operatorname{codim} R(\mathcal{L}_* - \lambda) = \operatorname{codim} \{ h \in X \,/\, \exists u \in X : (\mathcal{L}_* - \lambda) u = h \}$$
$$\geq i_+(\lambda) - i_-(\lambda)$$

The reason is that

$$\dim E^u_-(0,\lambda) + \dim E^s_+(0,\lambda) = 2n + i_-(\lambda) - i_+(\lambda) < 2n$$

$\longrightarrow$ do not have splitting into stable and unstable subspaces for $x \in \mathbb{R}$



robust situation !

§2.4 Comments

• Spectrum in sector:

$$D u_{xx} + c_* u_x + (f_u(u_*(x)) - \lambda)\, u = 0$$

$$\lambda = \frac{e^{i\varphi}}{\varepsilon^2} \quad \longrightarrow$$

$$D u_{xx} + c_* u_x + \left(f_u(u_*(x)) - \frac{e^{i\varphi}}{\varepsilon^2}\right) u = 0$$

$x = \varepsilon y$ so that $\frac{d}{dx} \to \frac{1}{\varepsilon}\frac{d}{dy}$:

$$\frac{1}{\varepsilon^2} D u_{yy} + \frac{c_*}{\varepsilon} u_y + \left(f_u(u_*(\varepsilon y)) - \frac{e^{i\varphi}}{\varepsilon^2}\right) u = 0$$

$$\underbrace{D u_{yy} - e^{i\varphi} u}_{\text{constant-coefficients}} + \underbrace{\varepsilon c_* u_y + \varepsilon^2 f_u(u_*(\varepsilon y))\, u}_{\text{small as } \varepsilon \to 0} = 0$$

hyperbolic $\rightarrow$ $\begin{pmatrix} u \\ v \end{pmatrix}_y = \underbrace{\begin{pmatrix} 0 & 1 \\ D^{-1} e^{i\varphi} & 0 \end{pmatrix}}_{\text{hyperbolic}} \begin{pmatrix} u \\ v \end{pmatrix}$

$\longrightarrow$ no bounded nontrivial solutions for $0 < \varepsilon \ll 1$.

- „Order of roots of $\mathcal{D}(\lambda)$ = algebraic PDE multiplicity":

idea
$$\left.\begin{aligned}(\mathcal{L}_* - \lambda)\, u_0 &= 0\\ (\mathcal{L}_* - \lambda)\, u_1 &= u_0\\ &\dots\\ (\mathcal{L}_* - \lambda)\, u_j &= u_{j-1}\end{aligned}\right\downarrow \quad \text{Jordan chain at } \lambda = \lambda_*.$$

Suppose $(\mathcal{L}_* - \lambda_*)\, u_0 = 0$ has nontrivial soln. u_0

We need to see whether $(\mathcal{L}_* - \lambda_*)\, u_1 = u_0$ has a soln. u_1.

→ Solve
$$\left.\begin{aligned}(\mathcal{L}_* - \lambda)\, u_0 &= 0\\ (\mathcal{L}_* - \lambda)\, u_1 &= u_0\end{aligned}\right\}$$
for $u_0(\lambda)$, $u_1(\lambda)$, for $\lambda \approx \lambda_*$ seperately on $\mathbb{R}^+$ and $\mathbb{R}^-$ and look at jumps at $x = 0$

Key
$$\frac{d}{d\lambda}\left[(\mathcal{L}_* - \lambda)\, u_0(\lambda) = 0\right] \quad \text{gives}$$
$$(\mathcal{L}_* - \lambda)\, \underbrace{\partial_\lambda u_0(\lambda)}_{\text{should be } u_1} = u_0(\lambda)$$

→ derivatives of ODE solutions with respect to λ are candidates for Jordan-chain solutions

→ this allows us (after much more algebra) to relate derivatives of $\mathcal{D}(\lambda)$ to the length of Jordan chains.

§3 Outlook and Guide to Literature

Nonlinear stability

(i) „Spectral stability => Linear stability"

We wish to see whether $\operatorname{spec} \mathcal{L}_* \subset \{\operatorname{Re}\lambda < 0\}$ implies that solutions to

$$u_t = \mathcal{L}_* u \quad , \quad u(t) = e^{\mathcal{L}_* t} u_0$$

decay to zero.

<u>Spectral Mapping Theorem</u> Under appropriate assumptions on $\mathcal{L}_*$,

$$e^{\operatorname{spec}(\mathcal{L}_*) t} = \operatorname{spec}\left(e^{\mathcal{L}_* t}\right)$$

[Pazy: §2.2], [Lunardi: Co. 2.3.7].

(ii) „spectral (in)stability => nonlinear (in)stability"

dissipative systems (reaction-diffusion, even-order PDEs):

nonlinear stability: [Henry: Thm 6.2.1], [Henry: §5.1 exc.6]

nonlinear instability: [Henry: Thm. 5.1.5]

e^0- PDEs (some diffusion coefficients vanish)

[Bates & Jones: Dynamics Reported, 1989]

Hamiltonian PDEs:

[Grillakis, Shatah, Strauss: J. Funct. Anal. 1987+1990]

Merle, Weinstein

Conservation Laws:

[Zumbrun: In „Handbook of mathematical fluid dynamics III" Elsevier, 2004]

Spectra [Sandstede]

[Henry: §5.4 + appendix to §5.4]

Multidimensional PDEs and Lattice equations

[Volpert³]

Evans function: [Deng & Nii: JDE 225 (2006)]
[Sandstede & Scheel: Math. Nachr. 232 (2001)]

Exponential dichotomies:
[Mallet-Paret & Verduyn-Lunel: JDE to appear]
[Härterich, S., Scheel: Indiana Univ. Math. J. 51 (2002)]

Bifurcations

n-pulses [Sandstede: §5.2]

Center manifolds [Henry: Thm. 6.2.1]

References: Stability and bifurcations of travelling waves

- D. Henry. *Geometric theory of semilinear parabolic equations.* Lecture Notes **84**. Springer, 1981, 1989.
- A. Lunardi. *Analytic semigroups and optimal regularity in parabolic systems.* Birkhäuser 1995.
- A. Pazy. *Semigroups of linear operators and applications to partial differential operators.* Springer, 1983.
- A.I. Volpert, V.I. Volpert and V.I. Volpert. *Traveling wave solutions of parabolic systems.* AMS Monographs **140**, 1994. [Free download from AMS].
- B. Sandstede. "Stability of travelling waves". In: *Handbook of Dynamical Systems II* (B Fiedler, ed.). North-Holland (2002) 983-1055.