Dynamics of modulated wave trains

§1 Existence of wave trains

Consider

(1) $\quad u_t = D u_{xx} + f(u) \qquad x \in \mathbb{R}, \; u \in \mathbb{R}^n$

with $D = \operatorname{diag}(d_j) > 0$, $f: \mathbb{R}^n \to \mathbb{R}^n$ smooth.

Wave trains are solutions $u(x,t)$ of (1) of the form

$$u(x,t) = u_0(k_0 x - \omega_0 t) \qquad \text{for some } k_0, \omega_0 \neq 0$$

where $u_0(\theta)$ is 2π-periodic. Substituting this ansatz into (1), we see that $u_0(\theta)$ satisfies the ordinary differential equation

(2) $\quad D k_0^2 u_{\theta\theta} + \omega_0 u_\theta + f(u) = 0, \qquad u(\theta + 2\pi) = u(\theta) \;\; \forall \theta$

Two approaches to gain insight into wave trains:

(i) Functional-analytic viewpoint

(ii) Dynamical-systems viewpoint

(i) Consider

$$\mathcal{F}: C^2_{per}(0,2\pi) \times \mathbb{R} \times \mathbb{R} \to C^0_{per}(0,2\pi)$$
$$(u, k, \omega) \longmapsto D k^2 u_{\theta\theta} + \omega u_\theta + f(u)$$

then $\mathcal{F}(u_0, k_0, \omega_0) = 0$ with linearization

$$\mathcal{L}_0 u = \mathcal{F}_u(u_0, k_0, \omega_0)\, u = D k_0^2 u_{\theta\theta} + \omega_0 u_\theta + f_u(u_0)\, u$$

Note that $\mathcal{L}_0 u_0' = 0$: differentiate $D k_0^2 u_0'' + \omega_0 u_0' + f(u_0) = 0$ in θ.

<u>(H1)</u> $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_0$:

If $\mathcal{L}_0 u = 0$, then $\exists a \in \mathbb{R}$ with $u = a u_0'$, and $u_0' \notin R_g(\mathcal{L}_0)$

<u>Lemma</u> The linear operator $C^2_{per} \times \mathbb{R} \to C^0_{per}$, $(u, \omega) \longmapsto \mathcal{L}_0 u + \omega u_0'$ is onto

<u>Proof</u> Will be an exercise problem

Thus, we can now solve by the implicit function theorem uniquely (up to spatial translations) for (v, ω) as functions of the wave number k for k near k_0:

Wave trains come in a one-parameter family

$$\begin{cases} u_0(kx - \omega_{nL}(k)t;\, k), & u_0(\theta; k) \text{ is } 2\pi\text{-periodic in } \theta \\ \omega = \omega_{nL}(k) & : \text{ nonlinear dispersion relation} \end{cases}$$

We define

$$\begin{cases} c_p = \dfrac{\omega_{nL}(k)}{k} = \dfrac{\omega}{k} & : \text{ phase velocity} \\ c_g = \dfrac{d\omega_{nL}}{dk}(k) & : \text{ group velocity} \end{cases}$$

(ii) Alternatively, seek wave trains in the form

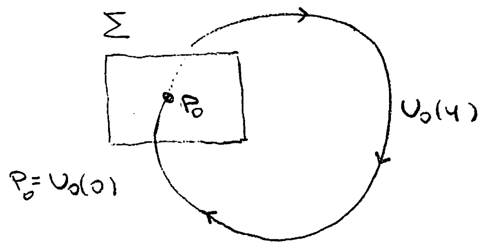
$$u(x,t) = \tilde{v}_0(x - c_0 t), \qquad \tilde{v}_0(y) \text{ is periodic in } y \text{ with period } \frac{2\pi}{k_0}$$

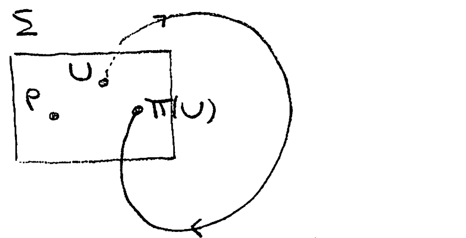
then $\tilde{v}_0(y)$ satisfies the ODE

$$D u_{yy} + c_0 u_y + f(u) = 0$$

or, equivalently, $U_0 = (\tilde{v}_0, \tilde{v}_0')$ is a periodic orbit of

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} v \\ -D^{-1}(c_0 v + f(u)) \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{2n}$$





Poincare map Π (smooth)

Periodic orbits with minimal period close to $2\pi/k_0$ are in one-to-one correspondence with fixed points of Π. If $1 - \Pi_U(P_0)$ is invertible, then $\exists!$ fixed point of Π near P_0 for each c near c_0:

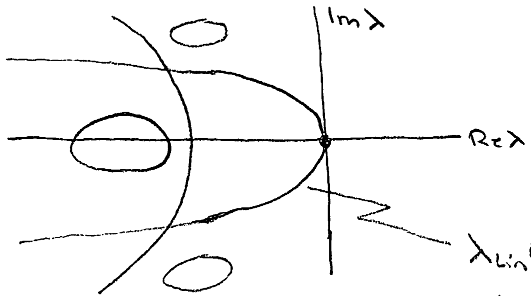
Wave trains come in one-parameter families

$$\underbrace{\tilde{v}_0(x - ct}_{\text{period } 2\pi/k(c)};\, c) \quad \text{with wave number } k = k(c) \text{ near } k_0$$

and frequency $\omega = c\,k(c)$ near ω_0 for c near c_0.

outline of stability lecture

- stability for ODEs
- comoving coordinate frame
- linearized operator $\mathcal{L}$, definition of spectrum
- goal: assume (H1), then best possible case is as follows:



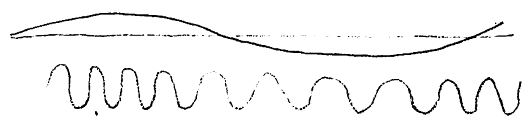
$\lambda_{Lin}(i\gamma) = (c_p - c_g)\, i\gamma - d\gamma^2 + O(\gamma^3)$ with solution

$v(\theta,t) = e^{\lambda_{Lin}(i\gamma)t}\, e^{i\gamma\theta/k_0} \underbrace{\left[u_0'(\theta) + O(\gamma)\right]}_{2\pi\text{-periodic}}$

of $v_t = \mathcal{L}v$

$$\rightsquigarrow \quad u_0(\theta) + \varepsilon \cos\left(\tfrac{\gamma\theta}{k_0}\right) u_0'(\theta) = u_0\left(\theta + \varepsilon\cos\left(\tfrac{\gamma\theta}{k_0}\right)\right) + O(\varepsilon^2)$$

$$\overset{t=0}{=} \quad u_0(k_0 x + \varepsilon \cos(\gamma x)) + O(\varepsilon^2)$$



$k_0 - \varepsilon\gamma \sin\gamma x$

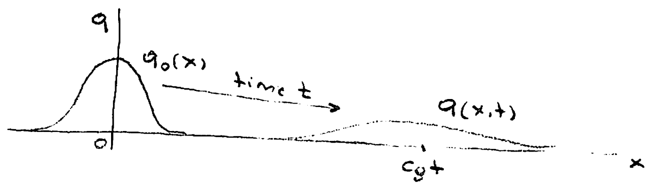
$u_0 + \varepsilon\, v(\cdot, 0)$

} wave number modulation

- nonlinear stability: difficult
- group velocity: modulations are described by $e^{\lambda t} e^{i\gamma\theta/k_0}$ where $\lambda = (c_p - c_g)i\gamma - d\gamma^2 + \dots$

  $q_t = d\, q_{\theta\theta} + (c_p - c_g) q_\theta$: $q(\theta,t) = e^{\lambda t} e^{i\gamma\theta/k_0}$ gives the same linear dispersion relation

  go back to original frame to get $q_t = d\, q_{xx} - c_g q_x$



- solve $(\mathcal{L} - \lambda) v = h$ using Floquet theory: state theorem.
- reduce existence of eigenfunctions to 2π-periodic problem.
- solve 2π-periodic problem: $\lambda u_0' = \mathcal{L}_0 v + O(\gamma + \lambda v)$

  which we solve by implicit function theorem

§2 Spectral stability of wave trains

Stability for ODEs:

- $u_t = f(u)$ with equilibrium u_0 so that $f(u_0)=0$
- linearize about u_0 to get $v_t = Av$ with $A = f_u(u_0)$
- spectrum of A yields linear stability information via $v(t) = e^{At}v_0$
- if $\mathrm{Re}\,\mathrm{spec}(A) < 0$, then nonlinear stability of u_0 follows

Recall

(1) $\quad u_t = Du_{xx} + f(u)$

We write $u(x,t) = \tilde{u}(k_0x - \omega_0 t, t)$ and (upon omitting the tilde) arrive at

(3) $\quad u_t = Dk_0^2 u_{\theta\theta} + \omega_0 u_\theta + f(u), \qquad \theta \in \mathbb{R},$

which has the spatially periodic steady state $u_0(\theta)$: wave train in comoving frame $\theta = k_0x - \omega_0 t$. Linearizing (3) about u_0 gives

(4) $\quad v_t = Dk_0^2 v_{\theta\theta} + \omega_0 v_\theta + f_u(u_0(\theta))\, v =: \mathcal{L}v, \qquad \theta \in \mathbb{R}.$

We focus on spectral stability of $\mathcal{L}$. To this end, we consider $\mathcal{L}$ as an operator on $L^2(\mathbb{R})$ or on $\mathcal{C}^0_{\mathrm{unif}}(\mathbb{R})$ with domain $H^2(\mathbb{R})$ and $\mathcal{C}^2_{\mathrm{unif}}(\mathbb{R})$, respectively.

- We say that λ is in the resolvent set $\rho(\mathcal{L}) \subset \mathbb{C}$ if
  $\exists\, C : \forall h \in X \ \exists!\, u \in D(\mathcal{L}) : (\mathcal{L}-\lambda)u = h$, and $\|u\|_X \le C\|h\|_X$.
  (ie $\mathcal{L}$ has a bounded inverse)
- The spectrum of $\mathcal{L}$ is defined as $\mathrm{spec}\,\mathcal{L} = \mathbb{C} \setminus \rho(\mathcal{L})$
- We say that $\lambda \in \mathrm{spec}\,\mathcal{L}$ is an eigenvalue if $\exists\, u \in D(\mathcal{L})\setminus\{0\}$ with $\mathcal{L}u = \lambda u$.

Hence, consider $(\mathcal{L}-\lambda)v = h$ for given $h \in \mathcal{C}^0_{\mathrm{unif}}(\mathbb{R})$, that is

$$Dk_0^2 v_{\theta\theta} + \omega_0 v_\theta + f_u(u_0(\theta))\,v - \lambda v = h(\theta)$$

or

$$V_\theta = A(\theta;\lambda)V + \begin{pmatrix} 0 \\ h(\theta) \end{pmatrix} \quad \text{where} \quad A(\theta,\lambda) = \underbrace{\begin{pmatrix} 0 & \frac{1}{k_0^2}D^{-1} \\ \lambda - f_u(u_0(\theta)) & -\frac{\omega_0}{k_0^2}D^{-1} \end{pmatrix}}_{2\pi\text{-periodic coefficients}}$$

Floquet theory implies that the system $V_\Theta = A(\Theta;\lambda)V$ has a fundamental matrix solution $\Phi(\Theta;\lambda)$ of the form

$$\Phi(\Theta;\lambda) = R(\Theta;\lambda)\, e^{B(\lambda)\Theta}, \qquad \Theta \in \mathbb{R}$$

$$\Phi(0;\lambda) = \mathrm{id}$$

where $R(0) = \mathrm{id}$, $R(\Theta + 2\pi) = R(\Theta)$ $\forall \Theta$, and R, B are analytic in λ locally in λ.

<u>Theorem</u> On both $X = L^2(\mathbb{R})$, $C^0_{unif}(\mathbb{R})$, we have

$$\begin{aligned} \operatorname{spec} \mathcal{L} &= \{\lambda \in \mathbb{C} : \operatorname{spec} B(\lambda) \cap i\mathbb{R} \neq \emptyset\} \\ &= \{\lambda \in \mathbb{C} : v_t = Dk_0^2 v_{\Theta\Theta} + \omega_0 v_\Theta + f_u(u_0(\Theta))v \text{ has a solution} \\ &\qquad \text{of the form } v(\Theta,t) = e^{\lambda t} e^{i\gamma\Theta/k_0} v_0(\Theta) \text{ with } \gamma \in \mathbb{R} \\ &\qquad \text{and } v_0 \text{ is } 2\pi\text{-periodic and not identical zero}\}. \end{aligned}$$

<u>Proof</u> Focus on $X = C^0_{unif}(\mathbb{R})$:

„$\supset$" Clear because $e^{i\gamma\Theta/k_0} v_0(\Theta) \in X$ lies in the null space of $\mathcal{L} - \lambda$

„$\subset$" We show that $\operatorname{spec} B(\lambda) \cap i\mathbb{R} = \emptyset$ implies $\lambda \in \rho(\mathcal{L})$:

Let P^s, P^u be the stable, unstable projections, respectively, of $B(\lambda)$, then $\|e^{B(\lambda)\Theta}P^s\| + \|e^{-B(\lambda)\Theta}P^u\| \le Ce^{-\alpha\Theta}$ for $\Theta \ge 0$.

The unique bounded solution of $(\mathcal{L} - \lambda)v = h$ is then obtained via

$$V(\Theta) = \int_{-\infty}^{\infty} G(\Theta,\tilde{\Theta}) \begin{pmatrix} 0 \\ h(\tilde{\Theta}) \end{pmatrix} d\tilde{\Theta}, \qquad G(\Theta,\tilde{\Theta}) = \begin{cases} \Phi(\Theta,\lambda) P^s \Phi(\tilde{\Theta},\lambda)^{-1} & \Theta \ge \tilde{\Theta} \\ -\Phi(\Theta,\lambda) P^u \Phi(\tilde{\Theta},\lambda)^{-1} & \Theta < \tilde{\Theta} \end{cases} \qquad \square$$

Thus, to understand the structure of the spectrum of $\mathcal{L}$, it suffices to find values of λ so that

$$v_t = Dk_0^2 v_{\Theta\Theta} + \omega_0 v_\Theta + f_u(u_0(\Theta))\, v$$

has a solution of the form $v(\Theta,t) = e^{\lambda t} e^{i\gamma\Theta/k_0} v_0(\Theta)$ with $v_0 \not\equiv 0$ 2π-periodic in its argument.

More generally, consider

$$v(\theta,t) = e^{\lambda t} e^{\nu\theta/k_0} v_0(\theta)$$

then substitution gives the equation

(5) $$\lambda v_0 = Dk_0^2 \left(\partial_\theta + \frac{\nu}{k_0}\right)^2 v_0 + \omega_0 \left(\partial_\theta + \frac{\nu}{k_0}\right) v_0 + f_u(u_0(\theta)) v_0 =: \mathcal{L}_\nu v_0$$

where $v_0 \in C^2_{per}(0,2\pi)$.

Setting $\lambda = 0$, we see that (5) has the solution $\nu = 0$, $v_0 = u_0'$. If $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_0$ (ie if (H1) holds), then $\mathcal{L}_\nu$ has a unique eigenvalue $\lambda(\nu)$ near $\lambda = 0$ for each ν near zero. We denote this function by $\lambda_{Lin}(\nu)$ and refer to it as the linear dispersion relation.

<u>Lemma</u> Assume (H1) holds; then there are analytic functions $\lambda_{Lin}(\nu)$ and $v_0(\theta;\nu) \in C^2_{2\pi}$ with $\lambda_{Lin}(0) = 0$ and $v_0(\theta;0) = u_0'(\theta)$ such that (5) with λ,ν near zero has a unique solution $\lambda = \lambda_{Lin}(\nu)$ and $v_0 = v_0(\theta;\nu)$.

<u>Proof</u> as in exercise problem: we need to solve

$$\left(1 - \Phi(2\pi;0)\right) V_0 = \int_0^{2\pi} \Phi(2\pi;0)\, \Phi(\theta;0)^{-1} \left[\nu(\dots) + \lambda \begin{pmatrix} 0 \\ u_0' \end{pmatrix}\right] d\theta + O(\nu^2 + \lambda^2)$$

With $W_0 \perp Rg\left(1 - \Phi(2\pi;0)\right)$, this equation can be solved iff $\left\langle W_0, \int_0^{2\pi} \dots \right\rangle = \lambda \langle w_0, u_0' \rangle + O(\nu + \lambda^2) = 0$, which can be solved for λ.

$\square$

Note that

$$V(\theta,t) = e^{\lambda_{Lin}(\nu)t}\, e^{\nu\theta/k_0}\, v_0(\theta;\nu)$$

becomes

$$V(x,t) = e^{\lambda_{Lin}(\nu)t}\, e^{\nu(k_0 x-\omega_0 t)/k_0}\, v_0(k_0 x-\omega_0 t;\nu)$$

$$= e^{(\lambda_{Lin}(\nu)-c_p\nu)t}\, e^{\nu x}\, v_0(k_0x-\omega_0 t;\nu)$$

in the original coordinates (x,t). We define

$$\tilde\lambda_{Lin}(\nu) := \lambda_{Lin}(\nu) - c_p\nu$$: linear dispersion relation in original reference frame

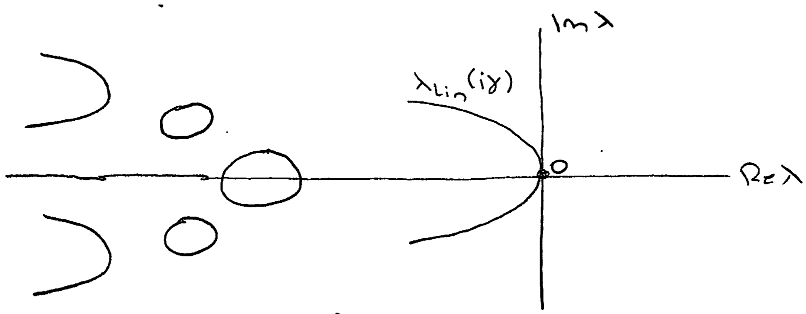
We have

$$\frac{d\tilde\lambda_{Lin}}{d\nu}(0) = \frac{d\lambda_{Lin}}{d\nu}(0) - c_p = -c_g$$ linear = nonlinear group velocity.

From now on, we assume

(H2) (i) $\tilde\lambda_{Lin}(\nu) = -c_g\nu + \frac{\tilde\lambda''_{Lin}(0)}{2}\nu^2 + O(\nu^3)$ with $\tilde\lambda''_{Lin}(0)>0$

(ii) $\mathrm{Re}\left[\mathrm{spec}\,\mathcal{L}\setminus\{\lambda_{Lin}(i\gamma) : |\gamma|\le\delta_0,\ \gamma\in\mathbb{R}\}\right] \le -\delta_1 < 0$



spec $\mathcal{L}$
with (H2)
satisfied

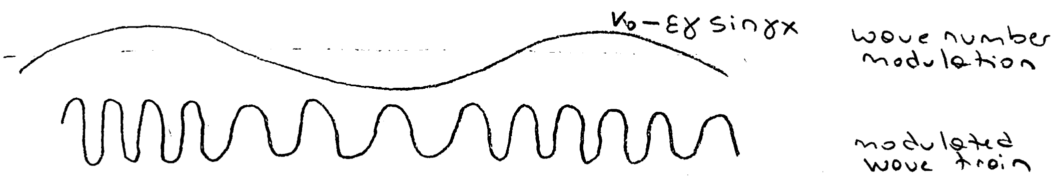
Comments and remarks :

(i) Note that $v_0(\theta;\nu) = u_0'(\theta) + O(\nu)$: 2π-periodic functions

Hence $$V(x,t) = e^{\tilde\lambda_{Lin}(i\gamma)t}\, e^{i\gamma x}\left[u_0'(k_0x-\omega_0 t;k_0) + O(\gamma)\right]$$

and $$u_0(k_0x;k_0) + \varepsilon V(x,0) = u_0(k_0x;k_0) + \varepsilon\cos(\gamma x)\left[u_0'(k_0x;k_0) + O(\gamma)\right]$$

$$\approx u_0(k_0x+\varepsilon\cos\gamma x;k_0) + O(\varepsilon(\varepsilon+\gamma))$$

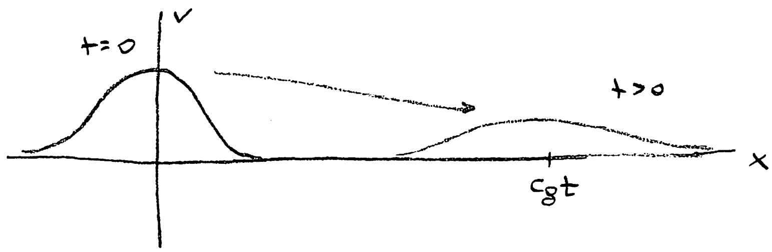


(ii) $\check{\lambda}_{Lin}(\nu) \approx -c_g \nu + \frac{\check{\lambda}''_{Lin}(0)}{2} \nu^2$

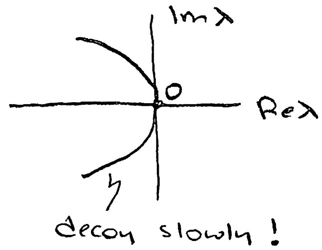
$\uparrow$ temporal frequency, ν spatial wavenumber

$\rightarrow$ suggests that wave number $q(x,t)$ satisfies

(6) $q_t = -c_g q_x + \frac{\check{\lambda}''_{Lin}(0)}{2} q_{xx}$: advection - diffusion



(iii) Nonlinear stability : nontrivial!



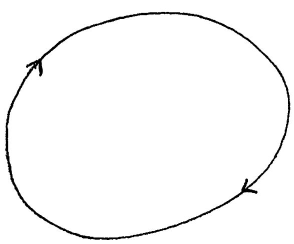
In fact, if we believe that (6) captures small perturbations, then it is tempting to add nonlinear terms to address nonlinear stability :

eg $v_t = v_{xx} - c_g v_x + v^2$ $\longrightarrow$ Solutions to positive localized initial data blow up in finite time!

(iv) Relation between temporal stability (H2) and spatial dynamics :

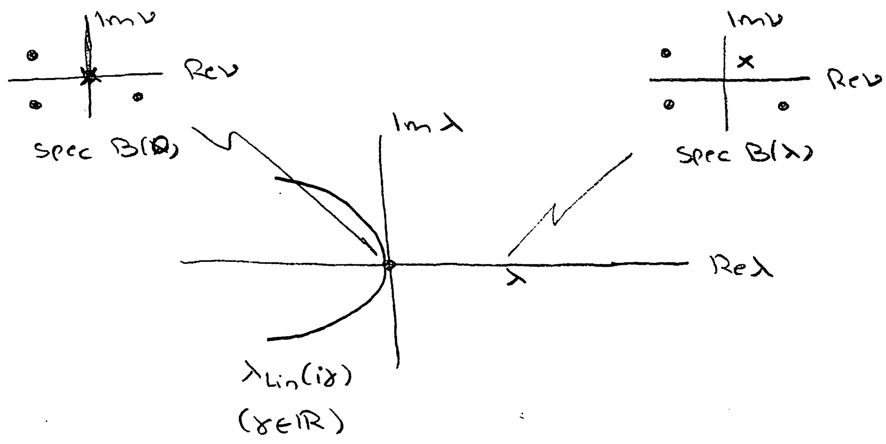
(*) $V_\theta = A(\theta;\lambda) V$ has fundamental solution $\Phi(\theta;\lambda) = R(\theta;\lambda) e^{B(\lambda)\theta}$

$\lambda = 0$: Equation (*) is the ODE linearization about periodic orbit U_0



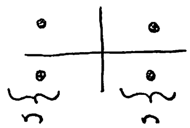
$U_0(\theta)$

2π- periodic



Since $V \in \mathbb{R}^{2n}$, spec $B(\lambda)$ contains $2n$ eigenvalues (the Floquet exponents) that correspond to linear growth or decay of orbits of $V_\theta = A(\theta;\lambda)V$.

<u>Claim</u> For $\lambda \gg 1$, we have



spec $B(\lambda)$

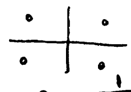
(recall that spec $B(\lambda) \cap i\mathbb{R} = \emptyset$ for $\mathrm{Re}\,\lambda > 0$ by (H2)).

<u>Proof</u>

$$\lambda v = D k_0^2 v_{\theta\theta} + \omega_0 v_\theta + f_u(u_0(\theta))\, v$$

$y = \sqrt{\lambda}\,\theta$ gives

$$\underbrace{v = D k_0^2 v_{yy}} + \underbrace{\frac{1}{\sqrt{\lambda}}\,\omega_0 v_y + \frac{1}{\lambda} f_u(u_0(y/\sqrt{\lambda}))\, v}_{= O(1/\sqrt{\lambda})\, v}$$



$\nu^2 = \frac{1}{d_j k_0^2}$ with $d_j > 0$.

plus perturbation analysis

(construct stable and unstable manifolds of linear system
$\rightarrow$ both n-dimensional)

$\square$

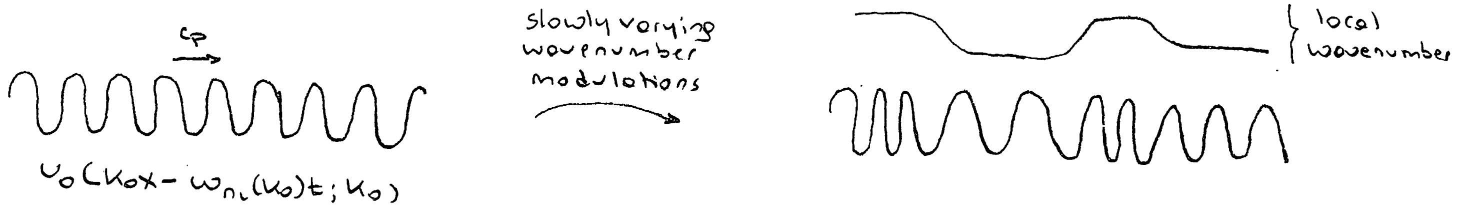
§3 Dynamics of wave-number modulations

(1) $\quad u_t = D u_{xx} + f(u), \qquad x \in \mathbb{R}, \quad u \in \mathbb{R}^n$

One-parameter family $u^0(kx - \omega_{nl}(k)t\,;k)$ of wave trains that are parametrized by k near $k_0 \neq 0$ and satisfy

(2) $\quad D k^2 u^0_{\theta\theta} + \omega_{nl}(k)\, u^0_\theta + f(u^0) = 0, \qquad \theta \in \mathbb{R}, \quad k \text{ near } k_0.$

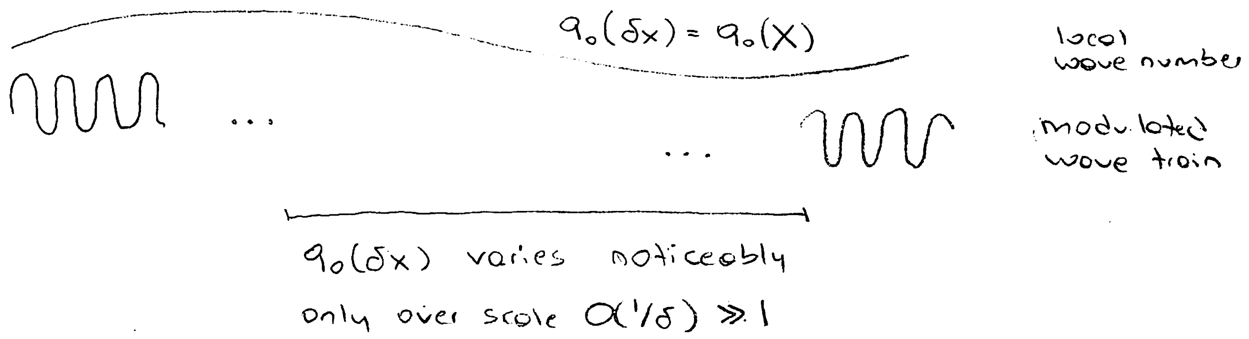
Fix $k = k_0$:



Consider solutions to (1) of the form

(3) $\quad u(x,t) = u_0\big(\,k_0 x - \omega_{nl}(k_0)t + \underbrace{\varphi(x,t)}_{\text{phase } \varphi}\;;\; k_0 + \underbrace{\varphi_x(x,t)}_{\text{wavenumber } \varphi_x}\big)$

The ansatz will likely be preserved only when modulations vary slowly in space and time compared with the spatial and temporal scales of the wave trains set by k_0 and $\omega_0 = \omega_{nl}(k_0)$, respectively. Thus, introduce a small dimensionless parameter δ with $0 < \delta \ll 1$ that describes the spatial scale, via the new coordinate $X = \delta x$, over which phase and wave number modulations are introduced:



It remains to find / determine the time scale and the allowed size of phase or wave number modulations in terms of δ : these scales are linked via the nonlinearity, and the remaining freedom is to set the time over which we wish to track modulations.

(i) Hyperbolic scaling : $X = \delta x \,,\; T = \delta t$

Set $\varphi(x,t) = \frac{1}{\delta}\Phi(\delta x, \delta t)$ phase

$\varphi_x(x,t) = \Phi_X(\delta x, \delta t)$ wave number

with $\Phi_X(X,T) = q(X,T)$ finite but sufficiently small. Note that a large constant phase shift only shifts the wave train and is therefore not dangerous : changes of these phase shifts (ie φ_x) should be small though. Thus, we consider solutions of the form

$$u(x,t) = u^0\left(k_0 x - \omega_{nl}(k_0)t + \tfrac{1}{\delta}\Phi(\delta x, \delta t);\; k_0 + \Phi_X(\delta x, \delta t)\right)$$

Substitution into $u_t = D u_{xx} + f(u)$ gives

$$\begin{aligned}
(-\omega_{nl}(k_0) + \Phi_T(X,T))\, u^0_\theta + O(\delta) &= D\,\partial_x\left((k_0 + \Phi_X)\, u^0_\theta + O(\delta)\right) + f(u^0) \\
&= D\,(k_0 + \Phi_X(X,T))^2\, u^0_{\theta\theta} + O(\delta) + f(u^0)
\end{aligned}$$

Let $\delta \to 0$ while fixing $(X,T) = (\delta x, \delta t)$ and the argument $\theta = k_0 x - \omega_{nl}(k_0)t + \frac{1}{\delta}\Phi(X,T)$ of u^0 to get formally that

$$D\,(k_0 + \Phi_X(X,T))^2\, u^0_{\theta\theta}(\theta) + \left(\omega_{nl}(k_0) - \Phi_T(X,T)\right) u^0_\theta(\theta) + f(u^0(\theta)) = 0.$$

Comparison with (2) shows that

$$k = k_0 + \Phi_X(X,T) \quad \text{and} \quad \omega_{nl}(k) = \omega_{nl}(k_0) - \Phi_T(X,T).$$

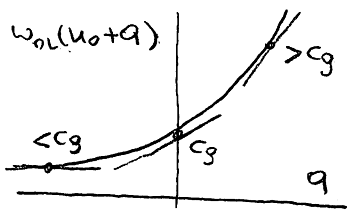
Hence, the phase $\Phi(X,T)$ satisfies the PDE

$$\Phi_T = \omega_{nl}(k_0) - \omega_{nl}(k_0 + \Phi_X) \qquad X \in \mathbb{R},\ T>0$$

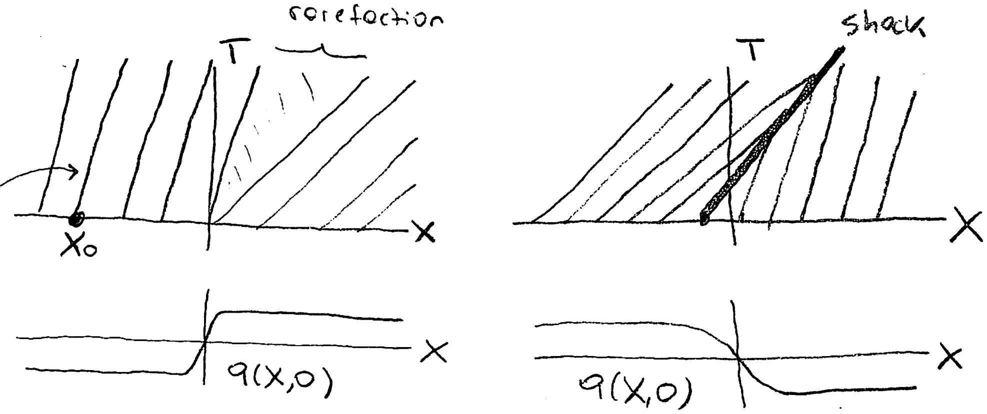
and the local wave number $q(X,T) := \Phi_X(X,T)$ satisfies the conservation law

$$q_T + \omega_{nl}(k_0+q)_X = 0 \qquad \text{or} \qquad q_T + \underbrace{\omega_{nl}'(k_0+q)}_{\approx\, \omega_{nl}'(k_0) = c_g\ !}\, q_X = 0$$

provided $|q| \ll 1$.



$\omega_{nl}''(k_0)>0$
(assume this sign)



$q(X,T)$ is constant along line
$X = X_0 + \omega_{nl}'(k_0 + q(X_0,0))\,T$
(see exercise session 3)

(ii) Parabolic scaling :

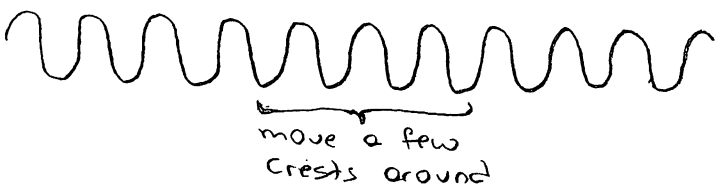
$$\begin{cases} X = \delta(x - c_g t), \qquad T = \delta^2 t & \text{spatio-temporal scales} \\ \varphi(x,t) = \Phi(X,T) & \text{local phase} \\ \varphi_x(x,t) = \delta\,\Phi_X(X,T) = \delta\, q(X,T) & \text{local wave number} \end{cases}$$

More complicated derivation$^{(*)}$ leads to Burgers equation

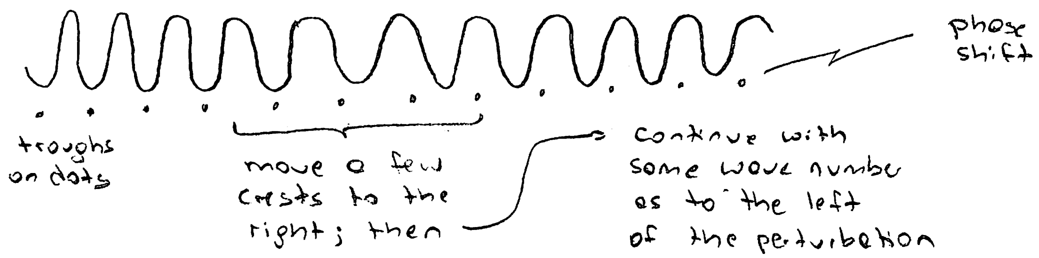
$$(4) \qquad q_T = \frac{\lambda_{lin}''(0)}{2}\, q_{XX} - \frac{\omega_{nl}''(k_0)}{2}\,(q^2)_X$$

((*) but see exercise session 2 for CGL !)

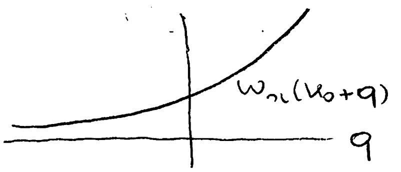
- $\phi(X,0)$ localized ($q(X,0)$ localized with zero mean);
  then $\|q(\cdot;T)\|_{L^\infty(\mathbb{R})} \leq \dfrac{C}{1+T}$ as $T \to \infty$

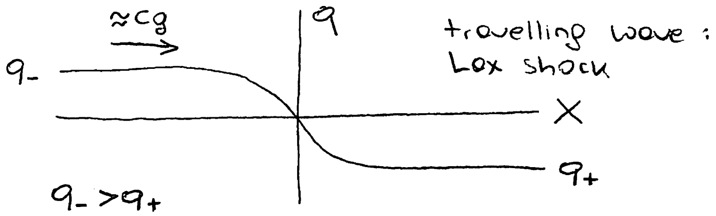


- $q(X,0)$ localized; then $\|q(\cdot,T)\|_{L^\infty(\mathbb{R})} \leq \dfrac{C}{\sqrt{1+T}}$ as $T \to \infty$

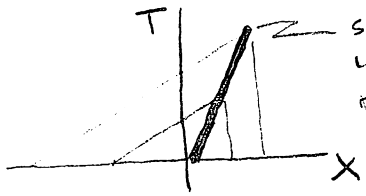


- $q(X,0) \to q_\pm$ as $X \to \pm\infty$ for $q_+ \neq q_-$ close to zero :

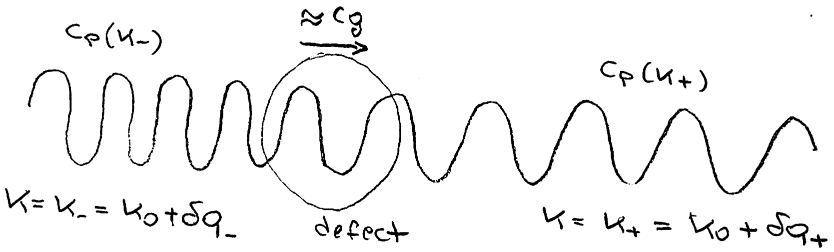




characteristics point
towards the interface



smooth interface between wave trains with wave numbers $k_\pm = k_0 + \delta q_\pm$



Validity : Parabolic scaling $(X,T) = (\delta(x-c_g t), \delta^2 t)$

Theorem Given $T_0 > 0$, $\exists$ $C_0, C_1, \delta_1 > 0$ such that we have :
Pick $0 < \delta < \delta_1$ and a solution $q(X,T)$ of (4) for $0 \le T \le T_0$ such that $\|q(\cdot,0)\| \le C_0$ and $q(X,0) \to q_\pm$ as $X \to \pm\infty$ sufficiently rapidly. Then there is a solution $u(\theta,t)$ of

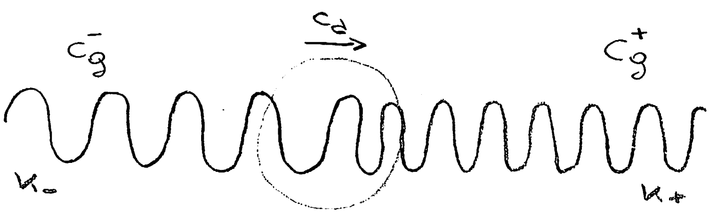
$$u_t = D k_0^2 u_{\theta\theta} + \omega_{nL}(k_0)\, u_\theta + f(u)$$

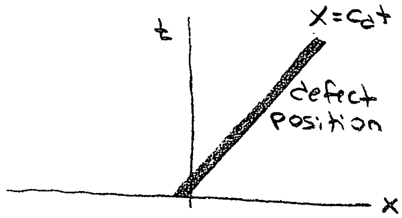
with

$$\sup_{\substack{0 \le t \le T_0/\delta^2 \\ \theta \in \mathbb{R}}} \left| u(\theta,t) - u_0\left[\theta + \mathcal{O}(\delta);\ k_0 + \delta q\left(\delta\left(\frac{\theta}{k_0} + (c_p - c_g)t + \mathcal{O}(\delta)\right);\ \delta^2 t\right) + \mathcal{O}(\delta^2)\right]\right| \le C_1 \delta^2$$

where $\mathcal{O}(\delta)$ are uniformly bounded functions.

§4 Defects and coherent structures





coherent structure or defect :

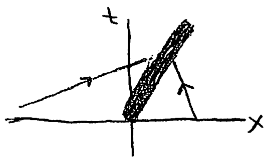
Solutions $u(x,t) = u_*(x - c_* t, t)$ so that

- $u_*(\theta, t)$ is time-periodic
- $u_*(\theta, t) \to$ wave trains as $\theta \to \pm\infty$

then

$$c_* = \begin{cases} \dfrac{\omega(k_+) - \omega(k_-)}{k_+ - k_-} & k_+ \neq k_- \\ c_g(k_0) \text{ or arbitrary} & k_+ = k_- = k_0 \end{cases}$$

Heuristic classification :



$c_g^- > c_* > c_g^+$

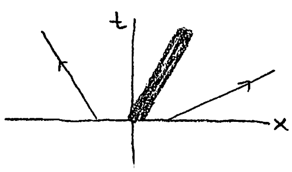
sink

(Lax shock)

$\downarrow$

„passive" :

$\exists$ for range of choices for k_-, k_+



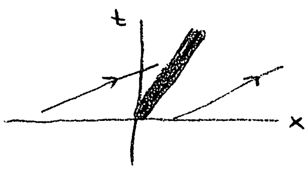
$c_g^- < c_* < c_g^+$

source

$\downarrow$

„active" :

selects k_-, k_+ uniquely

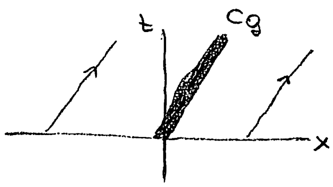


$c_g^- > c_* > c_g^+$

transmission/ homoclon

$\downarrow$

$\exists$ for range of k_-, but k_+ then determined by k_-



$c_g^- = c_* = c_g^+$

contact

$\downarrow$

$\exists$ for range of choices of $k_- = k_+$

$\underbrace{\text{sink, source}}$ van Saarloos & Hohenberg

$\underbrace{\text{transmission}}$ van Hecke

Towards a rigorous classification:

$$u_t + a u u_x = u_{xx} \qquad \text{Burgers equation}$$

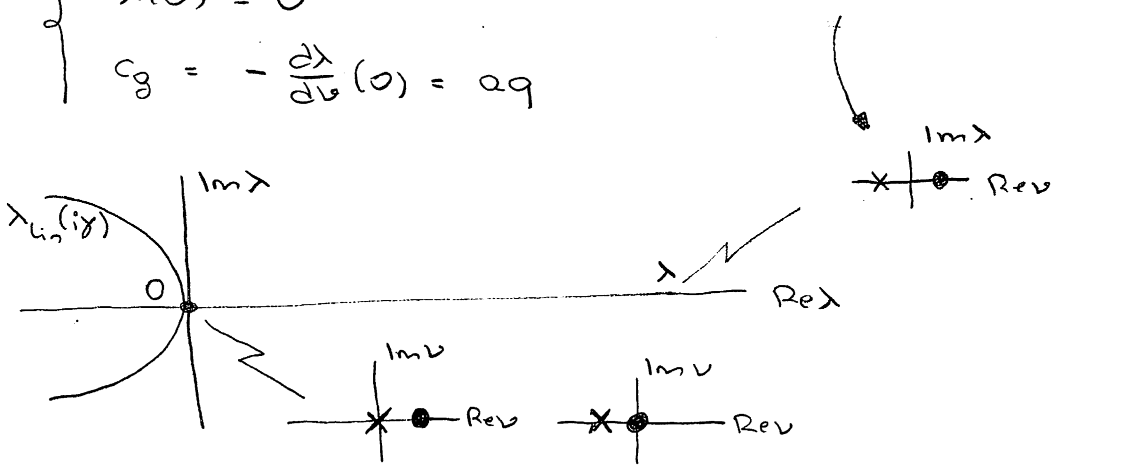
which has a family of homogeneous rest states $u(x,t) = q \in \mathbb{R}$.

Linearization about $u = q$ gives

$$v_t + a q v_x = v_{xx}$$

Spectrum: $v(x,t) = e^{\lambda t} e^{\nu x}$ gives $\underbrace{\lambda = \lambda(\nu) = -aq\nu + \nu^2}$

with
$$\begin{cases} \lambda(0) = 0 \\ c_g = -\dfrac{d\lambda}{d\nu}(0) = aq \end{cases}$$

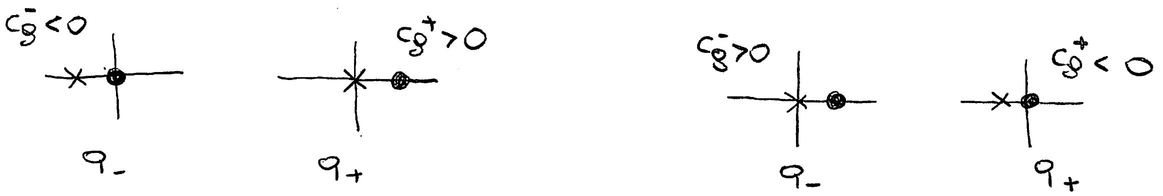


$\lambda(\nu) = -c_g \nu$ $\qquad c_g > 0 \qquad c_g < 0$

Stationary front that connects q_- at $x = -\infty$ to q_+ at $x = \infty$:

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} v \\ a u v \end{pmatrix}$$

Without analysing this equation, note that



cannot work $\qquad$ could work

Exercise session 1

Consider the complex Ginzburg-Landau equation

$$(CGL) \qquad u_t = (1+i\alpha)\,u_{xx} + u - (1+i\beta)\,|u|^2 u$$

with $x \in \mathbb{R}$, $u(x,t) \in \mathbb{C}$ for given numbers $\alpha, \beta \in \mathbb{R}$.

(1) Determine $r, k, \omega \in \mathbb{R}$ so that $u(x,t) = r\,e^{i(kx-\omega t)}$ is a wave-train solution of the CGL.

<u>answer</u> $\quad r = \sqrt{1-k^2}\,, \quad \omega = \omega_{nL}(k) = \beta + (\alpha-\beta)k^2$

(2) Setting $k=0$, we want to describe solutions near the wave train $e^{-i\beta t}$. Thus, write $u(x,t) = (1+r(x,t))\,e^{i(-\beta t + \phi(x,t))}$ and find equations for $(r,\phi)(x,t)$ so that $u(x,t)$ satisfies (CGL).

<u>answer</u> $(CGL)_0$

$$\left\{\begin{aligned} r_t &= r_{xx} - 2r - (\phi_x)^2 - (\phi_x)^2 r - 2\alpha\, r_x \phi_x - \alpha\,\phi_{xx} \\ &\quad - \alpha r\,\phi_{xx} - 3r^2 - r^3 \\ \phi_t &= \phi_{xx} + \frac{\alpha r_{xx}}{1+r} - \alpha(\phi_x)^2 + \frac{2 r_x \phi_x}{1+r} - 2\beta r - \beta r^2 \end{aligned}\right.$$

(3) Assume that $u_0(\theta)$ is a 2π-periodic solution of $Dk_0^2 u_{\theta\theta} + \omega_0 u_\theta + f(u) = 0$ for appropriate constants $k_0, \omega_0 \neq 0$. Let

$$\mathcal{L}_0 : C^2_{per}(0,2\pi) \to C^0_{per}(0,2\pi), \quad v \mapsto Dk_0^2 v_{\theta\theta} + \omega_0 v_\theta + f_u(u_0(\theta))\,v,$$

then $\mathcal{L}_0 u_0' = 0$ (see lecture notes).

<u>Assume</u> (H1) If $\mathcal{L}_0 v = 0$, then $v = a\,u_0'$ for some $a \in \mathbb{R}$;
$\mathcal{L}_0 v = u_0'$ does not have a solution $v \in C^2_{per}(0,2\pi)$.

<u>Prove</u> $\quad \forall h \in C^0_{per}(0,2\pi)$ there are $v \in C^2_{per}(0,2\pi)$ and $\omega \in \mathbb{R}$ with $\mathcal{L}_0 v + \omega\, u_0' = h$.

<u>answer</u> following pages.

Problem Assume that $u_0(\theta)$ is a 2π-periodic solution of

$$D k_0^2 u_{\theta\theta} + \omega_0 u_\theta + f(u) = 0$$

for appropriate constants $k_0, \omega_0 \neq 0$. Let

$$\mathcal{L}_0 : \mathcal{C}^2_{per}(0,2\pi) \longrightarrow \mathcal{C}^0_{per}(0,2\pi)$$
$$v \longmapsto D k_0^2 v_{\theta\theta} + \omega_0 v_\theta + f_u(u_0(\theta))\, v$$

then $\mathcal{L}_0 u_0' = 0$ (see notes). Assume:

(H1) If $\mathcal{L}_0 v = 0$, then $v = a u_0'$ for some $a \in \mathbb{R}$, and $\mathcal{L}_0 v = u_0'$ does not have a solution $v \in \mathcal{C}^2_{per}(0,2\pi)$.

Prove: $\forall h \in \mathcal{C}^0_{per}(0,2\pi)$ there are $v \in \mathcal{C}^2_{per}(0,2\pi)$ and $\omega \in \mathbb{R}$ with $\mathcal{L}_0 v + \omega u_0' = h$.

Solution We want to solve

(1) $D k_0^2 v_{\theta\theta} + \omega_0 v_\theta + f_u(u_0(\theta))\, v + \omega u_0'(\theta) = h(\theta)$

where $v \in \mathcal{C}^2_{per}$ and $\omega \in \mathbb{R}$. We rewrite (1) as a first-order system using $\tilde{v} = D k_0^2 v'$ to get

$$(2) \quad \begin{pmatrix} v \\ \tilde{v} \end{pmatrix}_\theta = \begin{pmatrix} 0 & D^{-1} k_0^{-2} \\ -f_u(u_0(\theta)) & -\omega_0 D^{-1} k_0^{-2} \end{pmatrix} \begin{pmatrix} v \\ \tilde{v} \end{pmatrix} - \omega \begin{pmatrix} 0 \\ u_0'(\theta) \end{pmatrix} + \begin{pmatrix} 0 \\ h(\theta) \end{pmatrix}$$

or, using $V = \begin{pmatrix} v \\ \tilde{v} \end{pmatrix} \in \mathbb{R}^{2n}$

$$(3) \quad V_\theta = A(\theta) V - \omega \begin{pmatrix} 0 \\ u_0'(\theta) \end{pmatrix} + \begin{pmatrix} 0 \\ h(\theta) \end{pmatrix}.$$

<u>Note 1</u> $V(\theta) = \begin{pmatrix} v(\theta) \\ \check{v}(\theta) \end{pmatrix}$ is a 2π-periodic solution of (3)

iff $v \in C^2_{2\pi}$ is a 2π-periodic solution of

$$\mathcal{L}_0 v + \omega u_0' = h(\theta)$$

Inspecting (3), we want to solve a equation of the form

$$(4) \qquad V_\theta = A(\theta) V + H(\theta), \qquad H(\theta) \text{ given.}$$

The solution $V(\theta)$ to the linear homogeneous equation

$$V_\theta = A(\theta) V, \quad V(0) = V_0, \quad V_0 \in \mathbb{R}^{2n} \text{ given}$$

exists and is of the form

$$V(\theta) = \Phi(\theta) V_0$$

for some invertible matrix $\Phi(\theta) \in \mathbb{R}^{2n \times 2n}$. The general solution to (4) is then given by

$$(5) \quad V(\theta) = \underbrace{\Phi(\theta) V_0}_{\substack{\text{general soln} \\ \text{of homogeneous} \\ \text{problem}}} + \underbrace{\int_0^\theta \Phi(\theta) \Phi(\tilde{\theta}) H(\tilde{\theta}) \, d\tilde{\theta}}_{\substack{\text{particular soln of} \\ \text{inhomogeneous problem}}}$$

The formula (5) is often called variation-of-constant formula. Applying (4)-(5) to our equation (3), we find that the solution to (3) is given by

$$(6) \quad V(\theta) = \Phi(\theta) V_0 + \int_0^\theta \Phi(\theta) \Phi(\tilde{\theta}) \left[-\omega \begin{pmatrix} 0 \\ u_0'(\tilde{\theta}) \end{pmatrix} + \begin{pmatrix} 0 \\ h(\tilde{\theta}) \end{pmatrix} \right] d\tilde{\theta}$$

where $V_0 \in \mathbb{R}^{2n}$ and $\omega \in \mathbb{R}$ are arbitrary.

Now recall that $V(\Theta)$ needs to be 2π-periodic, so we need to choose $V_0 \in \mathbb{R}^{2n}$ and $\omega \in \mathbb{R}$ so that

$$V(2\pi) = V(0).$$

Substituting (6), this equation means

$$\Phi(2\pi)V_0 + \int_0^{2\pi} \Phi(2\pi)\Phi(\Theta)^{-1}\left[-\omega\begin{pmatrix}0\\ u_0'(\Theta)\end{pmatrix} + \begin{pmatrix}0\\ h(\Theta)\end{pmatrix}\right] d\Theta = V_0$$

or

(7) $$(\Phi(2\pi) - 1)\,V_0 = \int_0^{2\pi} \Phi(2\pi)\Phi(\Theta)^{-1}\left[\omega\begin{pmatrix}0\\ u_0'(\Theta)\end{pmatrix} - \begin{pmatrix}0\\ h(\Theta)\end{pmatrix}\right] d\Theta.$$

If $\Phi(2\pi) - 1 \in \mathbb{R}^{2n\times 2n}$ were invertible, we could solve (7) uniquely for V_0 for each given ω and h. However, recall that u_0' is a 2π-periodic solution of $\mathcal{L}_0 u_0' = 0$. Note 1 implies that $\hat{V}(\Theta) = \begin{pmatrix} u_0'(\Theta) \\ D k_0^2 u_0''(\Theta)\end{pmatrix}$ is a 2π-periodic solution of $V_\Theta = A(\Theta)V$, hence

$$(\Phi(2\pi) - 1)\,V(0) = \hat{V}(2\pi) - \hat{V}(0) = 0, \qquad \hat{V}(0) \neq 0.$$

Since we assumed that u_0' is the only 2π-periodic solution of $\mathcal{L}_0 V = 0$ (up to scalar multiples), we can conclude that

$$(\Phi(2\pi) - 1)\,V_0 = 0 \iff V_0 = a\hat{V}(0)$$

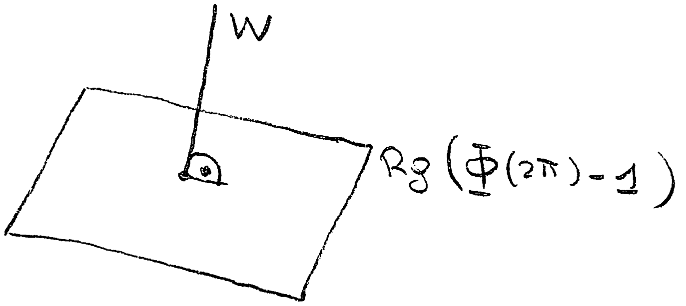
for some $a \in \mathbb{R}$.

Hence, the range

$$Rg(\Phi(2\pi)-1) := \{ H \in \mathbb{R}^{2n} : \exists V_0 \in \mathbb{R}^{2n} \text{ with } (\Phi(2\pi)-1)V_0 = H \}$$

is a $(2n-1)$-dimensional subspace of $\mathbb{R}^{2n}$. Pick $W \in \mathbb{R}^{2n} \setminus \{0\}$ so that $W \perp Rg(\Phi(2\pi)-1)$.

(Note that W is unique up to scalar multiples)



note: $\mathbb{R}W \oplus Rg(\Phi(2\pi)-1) = \mathbb{R}^{2n}$

<u>Note 2</u> $(\Phi(2\pi)-1)V_0 = H$ has a solution $V_0 \in \mathbb{R}^{2n}$

iff $H \in Rg(\Phi(2\pi)-1)$ (by definition of Rg)

iff $\langle W, H \rangle = 0$. (Since $H \perp W$ implies $H \in Rg(\Phi(2\pi)-1)$ by definition of W)

<u>Note 3</u>

Exploiting <u>Note 2</u>, we have shown the following statement:

$$V_\Theta = A(\Theta)V - \omega\begin{pmatrix}0\\u_0'(\Theta)\end{pmatrix} + \begin{pmatrix}0\\h(\Theta)\end{pmatrix}$$

has a 2π-periodic solution $V(\Theta)$

iff $\int_0^{2\pi} \Phi(2\pi)\Phi(\Theta)^{-1}\left[\omega\begin{pmatrix}0\\u_0'(\Theta)\end{pmatrix} - \begin{pmatrix}0\\h(\Theta)\end{pmatrix}\right] d\Theta \in Rg(\Phi(2\pi)-1)$

(by <u>Note 2</u>, and equation (7))

which, in turn, is true iff

$$(8)\quad \left\langle W, \int_0^{2\pi} \Phi(2\pi)\,\Phi(\theta)^{-1}\left[\omega\begin{pmatrix}0\\ u_0'(\theta)\end{pmatrix} - \begin{pmatrix}0\\ h(\theta)\end{pmatrix}\right] d\theta \right\rangle = 0 .$$

Equation (8) can be solved for ω (thus proving our result) iff

$$(9)\quad \left\langle W, \int_0^{2\pi} \Phi(2\pi)\,\Phi(\theta)^{-1}\begin{pmatrix}0\\ u_0'(\theta)\end{pmatrix} d\theta \right\rangle \neq 0 .$$

Thus, to finish our proof, we need to prove that the above scalar product does not vanish.

Assume that the left-hand side of (9) vanishes; then

$$(10)\quad \left\langle W, \int_0^{2\pi} \Phi(2\pi)\,\Phi(\theta)^{-1}\begin{pmatrix}0\\ u_0'(\theta)\end{pmatrix} d\theta \right\rangle = 0$$

and Note 3 implies that

$$V_\theta = A(\theta)V - \begin{pmatrix}0\\ u_0'(\theta)\end{pmatrix} \qquad (\omega = 1,\ h = 0)$$

has a 2π-periodic solution $V(\theta) = \begin{pmatrix}v(\theta)\\ \tilde{v}(\theta)\end{pmatrix}$. Note 1 then shows that $v(\theta)$ is a 2π-periodic solution of

$$\mathcal{L}_0 v + u_0'(\theta) = 0 \qquad (\omega = 1,\ h = 0).$$

However, this contradicts our hypothesis that such a solution does not exist, hence (10) cannot be true. We conclude that (9) is true and our proof is complete! $\square$

Exercise session 2

(1) Prove the following theorem: On $L^2(\mathbb{R})$ and $C^0(\mathbb{R})$, we have

$$\operatorname{spec} \mathcal{L} = \{ \lambda \in \mathbb{C} : \operatorname{spec} B(\lambda) \cap i\mathbb{R} \neq \emptyset \}$$

$$= \{ \lambda \in \mathbb{C} : v_t = D k_0^2 v_{\theta\theta} + \omega_0 v_\theta + f_u(v_0(\theta)) v \text{ has a solution of the form } v(\theta,t) = e^{\lambda t} e^{i\gamma\theta/k_0} v_0(\theta) \text{ for some } \gamma \in \mathbb{R} \text{ and some nonzero } 2\pi\text{-periodic function } v_0(\theta) \}$$

(2) Consider the equation $(CGL)_0$ for (r,ϕ) where $u = (1+r) e^{-i\beta t + i\phi}$ satisfies (CGL).

- Show that $(CGL)_0$ can be written as a PDE for (r,ψ) where $\psi = \phi_x$ is the wave number modulation
- To describe long-wavelength modulations of the wave train $e^{-i\beta t}$, we seek solutions of the form

$$r(x,t) = \delta^2 R(\delta x, \delta^2 t), \qquad \psi(x,t) = \delta \Psi(\delta x, \delta^2 t).$$

Derive the leading-order equations for $R(X,T)$ and $\Psi(X,T)$ in the variables $X = \delta x$, $T = \delta^2 t$ (the heat equation scaling).

<u>answer</u> $R = -\frac{1}{2}\left(\Psi^2 + \alpha \Psi_X\right)$ algebraic equation for R

$\Psi_T = (1+\alpha\beta)\Psi_{XX} + (\beta-\alpha)(\Psi^2)_X$ Burgers equation for Ψ

(3) Temporal stability versus spatial dynamics (if time permits)

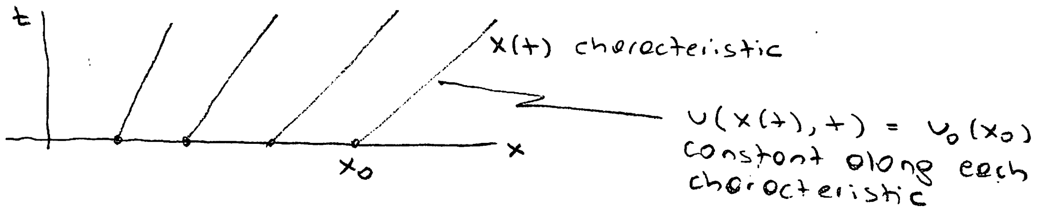
Exercise session 3

(1) Given $f: \mathbb{R} \to \mathbb{R}$ in C^2, we want to solve

$$(*) \quad \begin{cases} u_t + f(u)_x = 0, & x \in \mathbb{R},\ t > 0,\quad u \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$$

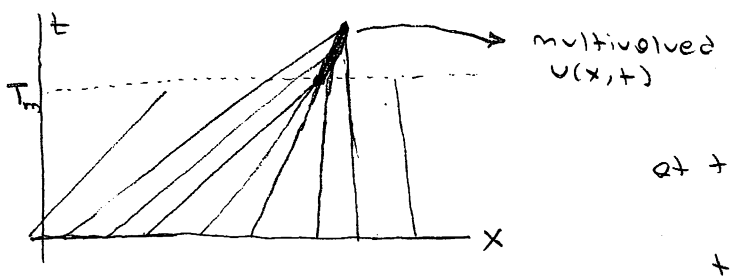
for a given initial condition $u_0 \in C^1$ using the method of characteristics.

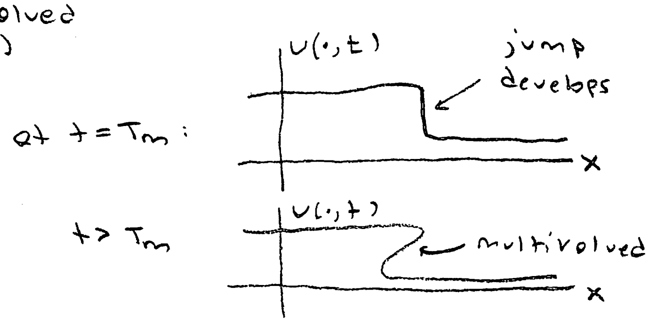
- Assume that $u(x,t)$ is a solution of $(*)$ and that $x(t)$ satisfies the ODE
$$\dot{x}(t) = f'(u(x,t)); \qquad x(0) = x_0.$$
Prove that $\frac{d}{dt} u(x(t),t) = 0 \quad \forall t \geq 0$.
Conclude that $(**)\quad x(t) = x_0 + f'(u_0(x_0))\, t$: Line



- Let $\Phi_t : \mathbb{R} \to \mathbb{R}$, $x_0 \mapsto x(t)$. Assume that $\Phi_t(x_0) = x$ has a unique solution x_0 for each $x \in \mathbb{R}$ and each t with $0 \leq t \leq T$. Prove: $u(x,t) := u_0(\Phi_{-t}(x))$ satisfies $(*)$.
(where $\Phi_{-t}(x) = x_0$ is the inverse).

- In general, we cannot expect that $T = \infty$ is possible:





(2) Consider the heat equation

$$(*) \begin{cases} u_t = u_{xx} & x \in \mathbb{R},\ t > 0,\ u \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$$

whose solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\, e^{-(x-y)^2/4t}\, u_0(y)\, dy \quad =: (G * u_0)(x,t)$$

where

$$G(x,t) := \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$$; heat kernel, Green's function, ...

<u>Facts</u>.

- $\int_{\mathbb{R}} G(x,t)\, dx = 1 \quad \forall\, t > 0$
- $\left.\begin{array}{l} \| (G * u_0)(\cdot, t) \|_{L^r} \le \| G(\cdot,t) \|_{L^p} \, \| u_0 \|_{L^q} \\ \text{for } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 \le r, p, q \le \infty \end{array}\right\}$ Young's inequality

<u>Prove</u>

- $\| u(\cdot, t) \|_{L^p} \le \| u_0 \|_{L^p} \quad \forall\, t \ge 0,\ p \ge 1$
- Let v be the solution of $(*)$ with $v(x,0) = \partial_x u_0(x)$ so that

  $$v(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\, e^{-(x-y)^2/4t}\, \partial_y u_0(y)\, dy$$

  Show that

  $$\| v(\cdot, t) \|_{L^p} \le \underbrace{\frac{C}{\sqrt{t}} \| u_0 \|_{L^p}}_{\to 0 \text{ as } t \to \infty} \quad \text{for } p \ge 1.$$

$\rightarrow$ Solutions to initial data with zero mass decay in contrast to general solutions which do not decay.

$\rightarrow$ Validity of Burgers equation:

$$q_t = q_{xx} + \underset{\substack{\uparrow \\ \text{helps!}}}{\partial_x} (\dots)$$

(3) Cole - Hopf

Let $u(x,t)$ be a solution to Burgers equation

(*) $\quad u_t = u_{xx} + (u^2)_x \,, \qquad x \in \mathbb{R},\ t > 0,\ u \in \mathbb{R}.$

Define $\quad v(x,t) := \exp\left(\int_{-\infty}^{x} u(y,t)\, dy \right)$, where we assume that $u(x,t)$ is sufficiently localized so that the integral exists.

Prove • $v(x,t)$ satisfies $v_t = v_{xx}$.

• $u(x,t)$ can be recovered from $v(x,t)$ via $u = \dfrac{v_x}{v}$.

→ Burgers equation with localized initial data can be solved via the above transformation (which is referred to as Cole - Hopf).

Solutions to exercise session 3:

(1) • Assume that $u(x,t)$ is a solution of $u_t + f(u)_x \overset{①}{=} 0$ with $u(x,0) = u_0(x)$. Furthermore, suppose that $x(t)$ satisfies $\frac{dx}{dt} \overset{②}{=} f'(u(x,t))$ with $x(0) = x_0$, then

$$\frac{d}{dt} u(x(t),t) = u_x \dot{x} + u_t \overset{②}{=} f'(u)u_x + u_t \overset{①}{=} 0 \quad \forall t \geq 0.$$

Hence $u(x(t),t)$ does not depend on t, and we conclude that $u(x(t),t) = u(x(0),0) = u_0(x_0)$ $\forall t \geq 0$, and ② shows that $\frac{dx}{dt}(t) = f'(u(x(t),t)) = f'(u_0(x_0))$ so that $x(t) = x_0 + f'(u_0(x_0))t$. is a line with slope $\frac{1}{f'(u_0(x_0))}$.

• By assumption, $u(x,t) = u_0(\Phi_{-t}(x))$ is well defined and differentiable for all $x \in \mathbb{R}$ and all $0 \leq t \leq T$. We consider $x(t) = \Phi_t(x_0)$, then $u(x(t),t) = u_0(x_0)$ by definition, and differentiation gives

$$u_x(x,t)\, f'(u_0(x_0)) + u_t(x,t) = 0$$

$$③ \quad u_x(x,t)\, f'(u(x,t)) + u_t(x,t) = 0$$

where $x = \Phi_t(x_0)$ and $0 \leq t \leq T$ are arbitrary. Since $\Phi_t : x_0 \mapsto x$ was assumed to be onto, ③ holds for all x, and $u(x,t)$ is a solution as claimed.

(2) • Recall that $\| G(\cdot,t)\|_{L^1} = 1 \quad \forall\, t>0$ since $G(x,t) \geq 0$.

and $\int_{\mathbb{R}} G(x,t)\,dx = 1$. Using Young's inequality

$$\| f * g \|_{L^r} \leq \| f\|_{L^p} \, \| g\|_{L^q}, \qquad 1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q},$$

with $f = G(\cdot,t)$, $g = v_0$, $r=q$ and $p=1$, we ~~get~~

$$\| u(\cdot,t)\|_{L^q} = \| G(\cdot,t) * v_0\|_{L^q} \leq \| G(\cdot,t)\|_{L^1} \, \| v_0\|_{L^q} = \| v_0\|_{L^q}$$

as claimed (upon using p instead of q).

• We have

$$v(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\, e^{-(x-y)^2/4t}\, \partial_y v_0(y)\, dy$$

$$= - \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\, \partial_y \left(e^{-(x-y)^2/4t} \right) v_0(y)\, dy \qquad \left(\begin{array}{c}\text{integration}\\ \text{by parts}\end{array}\right)$$

$$= - \int_{\mathbb{R}} \frac{2(x-y)}{4t\sqrt{4\pi t}}\, e^{-(x-y)^2/4t}\, v_0(y)\, dy$$

$$= \left(\tilde{G}(\cdot,t) * v_0 \right)(x)$$

where $\tilde{G}(x,t) = -\frac{1}{2\sqrt{t}} \cdot \frac{x}{\sqrt{t}}\, \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$. Young's inequality

implies again that

$$\| v(\cdot,t)\|_{L^p} = \| \tilde{G}(\cdot,t) * v_0\|_{L^p} \leq \| \tilde{G}(\cdot,t)\|_{L^1} \, \| v_0\|_{L^p}$$

and

$$\| \tilde{G}(\cdot,t)\|_{L^1} = \int_{\mathbb{R}} \frac{1}{2\sqrt{t}}\, \frac{|x|}{\sqrt{t}}\, e^{-x^2/4t}\, \frac{1}{\sqrt{4\pi t}}\, dx$$

$$\underset{\underline{\underline{\sqrt{t}\,z = x}}}{=} \int_{\mathbb{R}} \frac{1}{2\sqrt{t}}\, |z|\, e^{-z^2/4}\, \frac{1}{\sqrt{4\pi t}}\, \sqrt{t}\, dz$$

$$= \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \frac{1}{4\sqrt{\pi}}\, |z|\, e^{-z^2/4}\, dz \leq \frac{C}{\sqrt{t}}$$

So that

$$\| v(\cdot,t)\|_{L^p} \leq \frac{C}{\sqrt{t}} \| v_0\|_{L^p}, \quad t>0,$$

as claimed.

(3) Let $v(x,t) = e^{\int_{-\infty}^{x} u(y,t)\,dy}$ and calculate :

$$v_t(x,t) = \int_{-\infty}^{x} u_t(y,t)\,dy \quad v(x,t)$$

$$= \int_{-\infty}^{x} \left(u_{yy}(y,t) + (u^2(y,t))_y \right) dy \quad v(x,t) \qquad (u \text{ satisfies Burgers})$$

$$= \left(u_x(x,t) + u^2(x,t) \right) v \qquad (\text{integration by parts})$$

so that $v_t = (u_x + u^2)\, v$. Similarly

$$v_x(x,t) = u(x,t)\, v(x,t)$$

so that

$$v_x = uv$$ and

$$v_{xx} = u_x v + u v_x = u_x v + u\, uv = (u_x + u^2)\, v$$

Thus,

$$v_t = v_{xx}$$

as claimed.

Finally,

$$\log v(x,t) = \int_{-\infty}^{x} u(y,t)\,dy$$

and differentiating in x yields

$$\frac{v_x}{v} = u$$

as claimed (and actually seen above).

References

[1] A. Doelman, B. Sandstede, A. Scheel, G. Schneider.
„The dynamics of modulated wave trains".
Memoirs Amer. Math. Soc. 199 (934) 2009.

[2] B. Sandstede, A. Scheel.
„Defects in oscillatory media: toward a classification".
SIAM J. Appl. Dynam. Syst. 3 (2004) 1-68.

[3] B. Sandstede.
„Stability of travelling waves"
In: Handbook of Dynamical Systems II. Elsevier (2002) 983-1055.

[4] Various notes on my homepage on Evans function, absolute spectra, and dynamics of waves.

---

[5] I.S. Aranson, L. Kramer.
„The world of the complex Ginzburg - Landau equation".
Rev. Mod. Phys. 74 (2002) 99 - 143.

[6] A. Mielke.
„The Ginzburg-Landau equation in its role as a modulation equation"
In: Handbook of Dynamical Systems II, Elsevier (2002) 759 - 834.

[7] W. van Saarloos, P.H. Hohenberg.
„Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations".
Physica D 56 (1992) 303 - 367.