Euclid's Elements

Book VII

Definitions:

1 A unit is that by virtue of which each of the things that exist is called one

2 A number is a multitude composed of units. (not one)

3 A number is part of a number, the less of the greater, when it measures the greater

11 A prime number is that which is measured by a unit alone.

12 Numbers prime to one another are those which are measured by a unit alone as a common measure

15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)

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Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.

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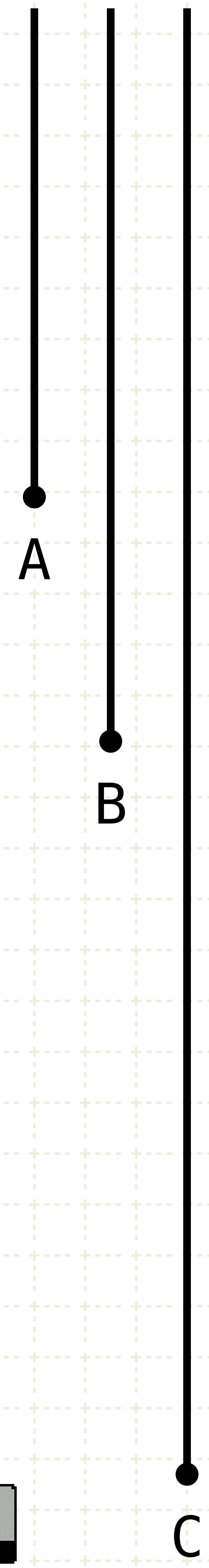


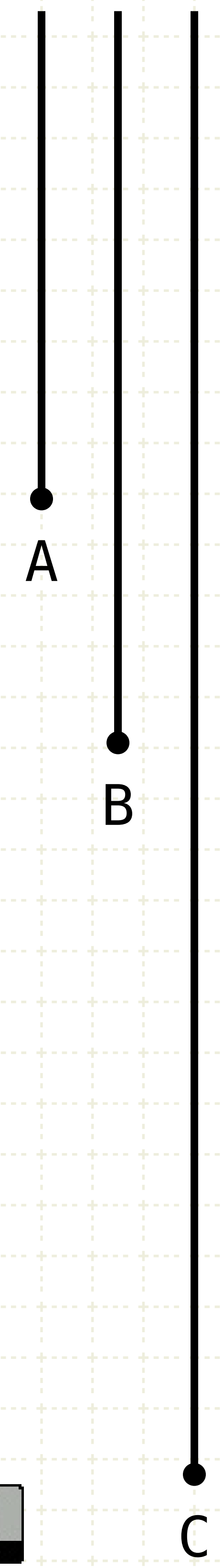
$$S=\{(x,y,z)\mid x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$$

In other words

Given as many numbers as we please, say A,B,C

Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

Find

$(X,Y,Z)\in S$
such that $X\leq x,\ Y\leq y,\ Z\leq z, \forall (x,y,z)\in S$

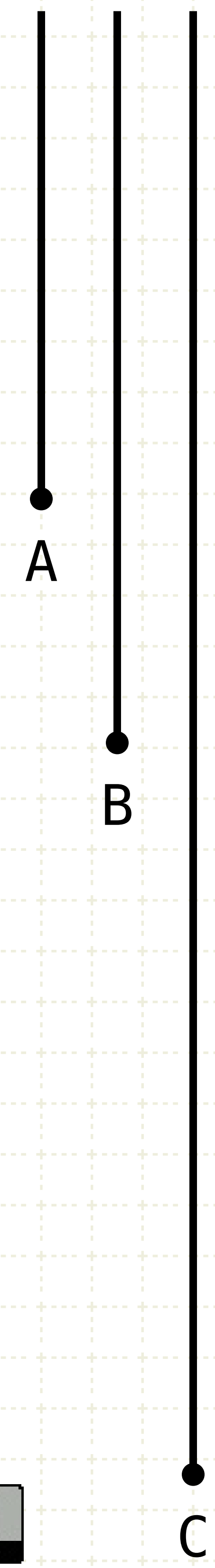
In other words

Given as many numbers as we please, say A,B,C

Find the least numbers X,Y,Z which are in the same ratio as A,B,C

Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



Method

$S=\{(x,y,z)|x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

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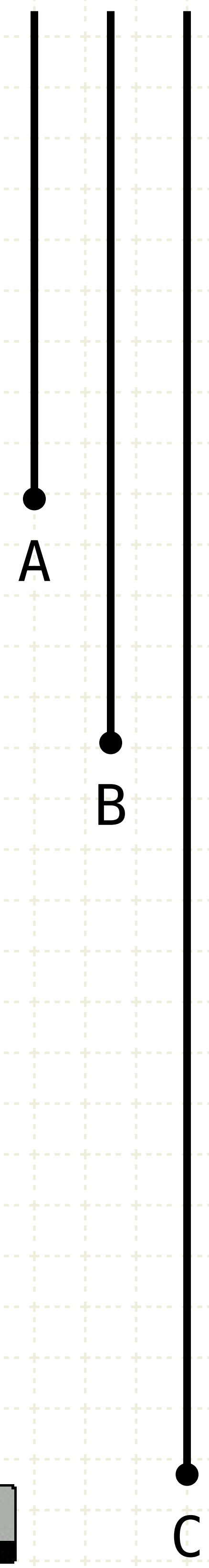

$S=\{(x,y,z) | x,y,z \in \mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = 1$

Method

Let A,B,C be prime to one another

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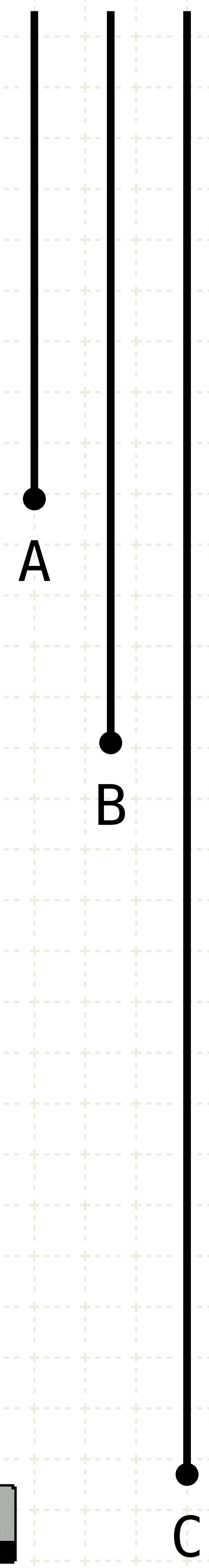


$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = 1$

$A\leq x,\ B\leq y,\ C\leq z,\ \forall(x,y,z)\in S$

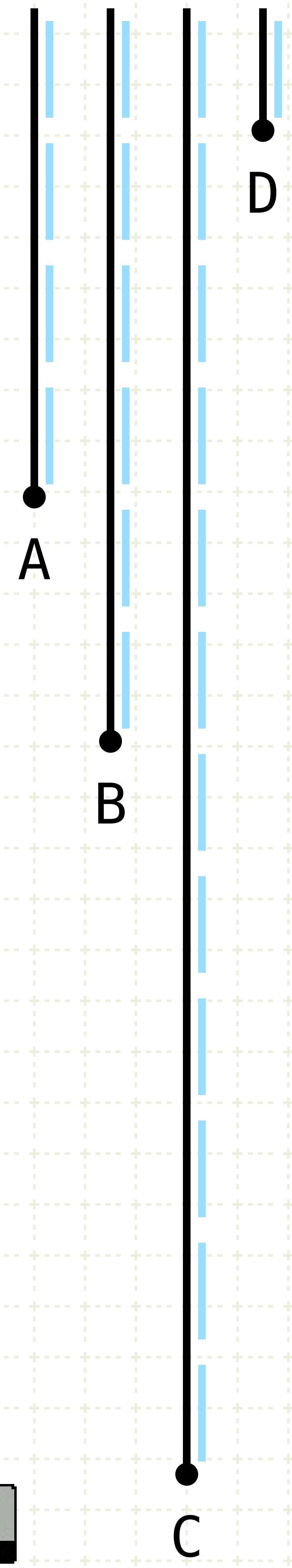
Method

Let A,B,C be prime to one another

Then A,B,C are the least numbers with this ratio (VII·21)

Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z) \mid x,y,z \in \mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

Method

Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)

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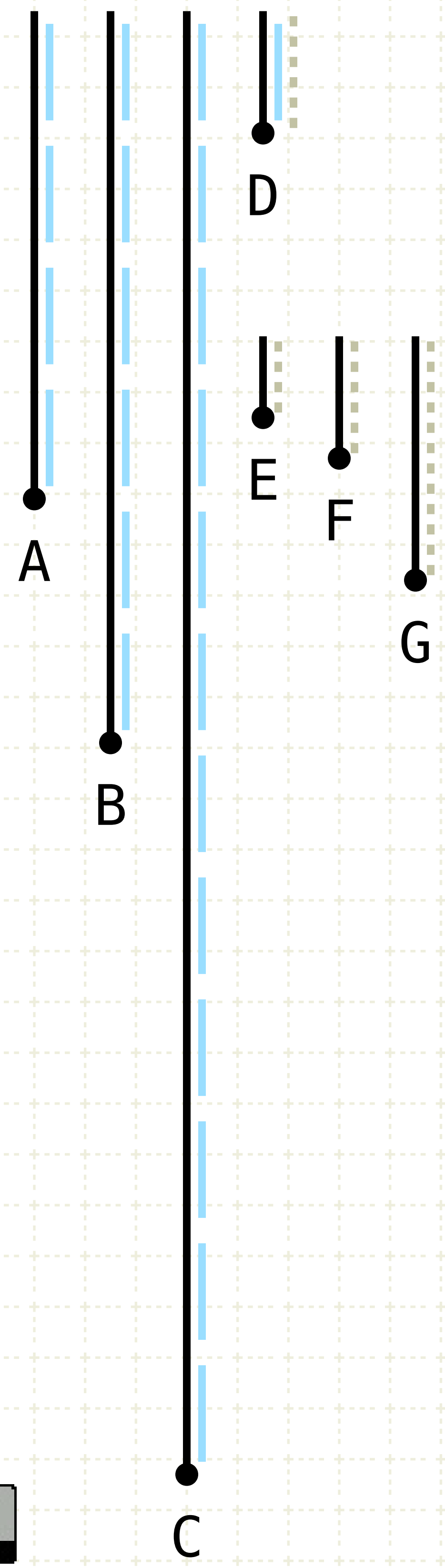

$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = E\cdot D$

$B = F\cdot D$

$C = G\cdot D$

Method

Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)

and, as many times a D measure the number A,B,C, let there be the same number of units in E,F,G respectively

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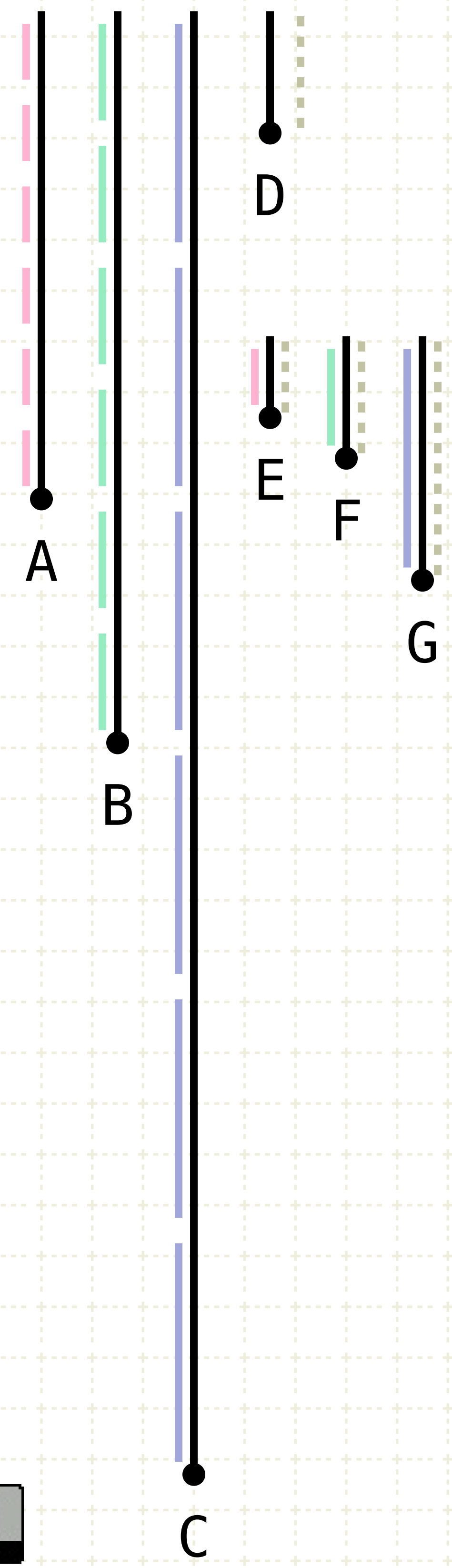

S={(x,y,z)|x,y,z∈ℕ, x:y:z=A:B:C}

gcd(A,B,C) = D

A = E·D

B = F·D

C = G·D

A = D·E

B = D·F

C = D·G

Method

Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)

and, as many times a D measure the number A,B,C, let there be the same number of units in E,F,G respectively

Therefore E,F,G measure A,B,C according to the units in D (VII·16)

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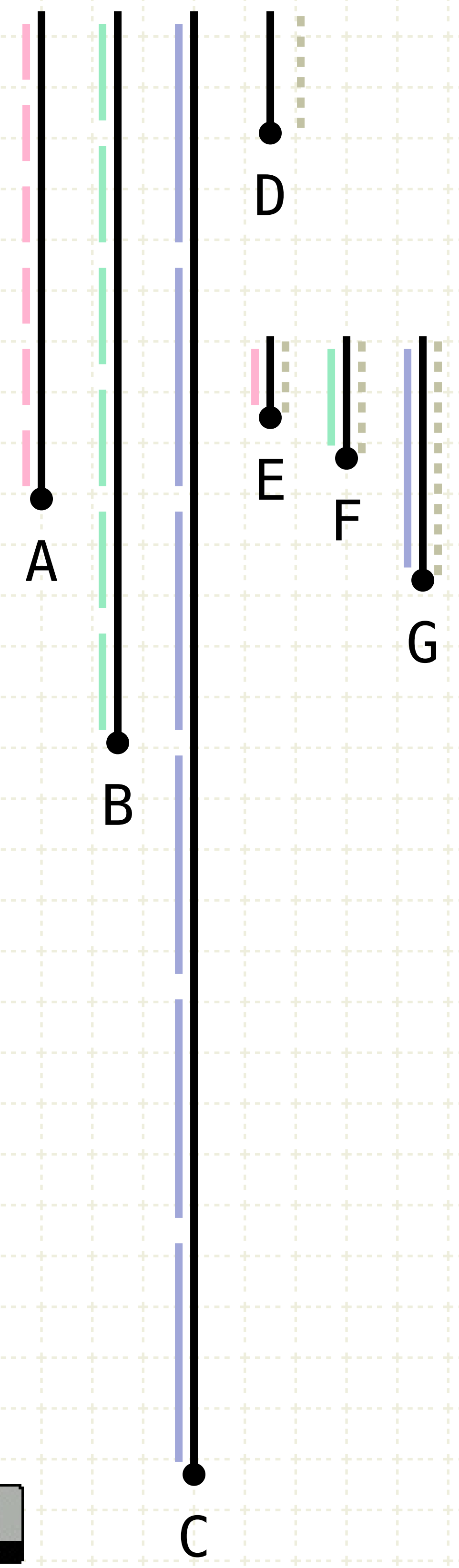

$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = E\cdot D$

$B = F\cdot D$

$C = G\cdot D$

$A = D\cdot E$

$B = D\cdot F$

$C = D\cdot G$

$E:F:G = A:B:C$

Method

Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)

and, as many times a D measure the number A,B,C, let there be the same number of units in E,F,G respectively

Therefore E,F,G measure A,B,C according to the units in D (VII·16)

Therefore E,F,G are in the same ratio as A,B,C (VII.Def.20)

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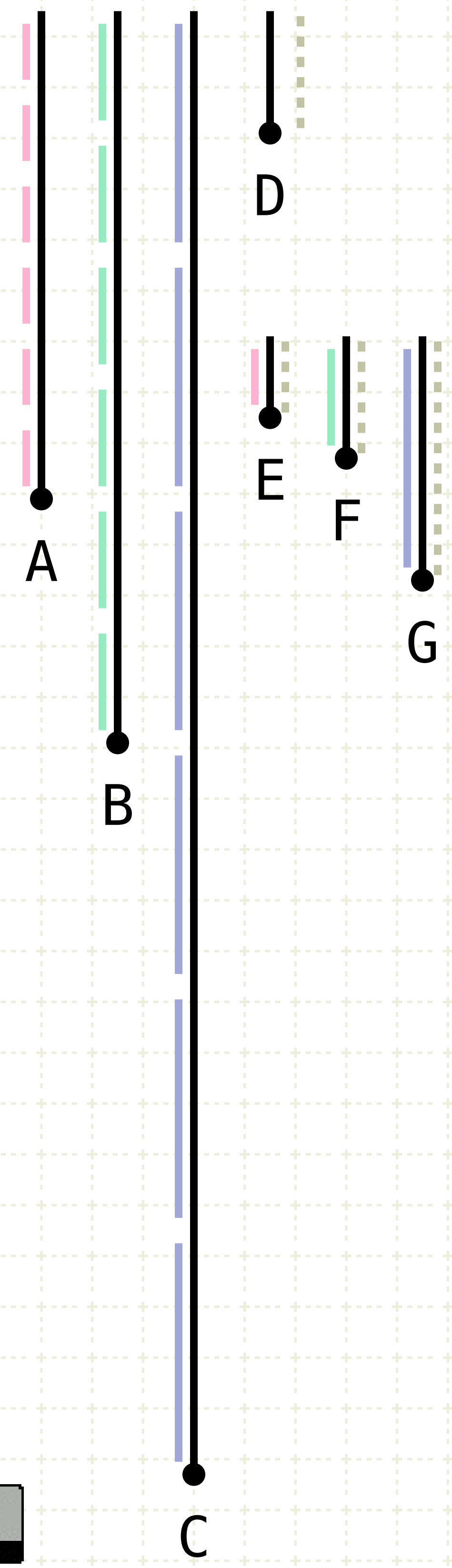

$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = E\cdot D$

$B = F\cdot D$

$C = G\cdot D$

$A = D\cdot E$

$B = D\cdot F$

$C = D\cdot G$

$E:F:G = A:B:C$

$E\le x,\ F\le y,\ G\le z,\ \forall(x,y,z)\in S$

Method

Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)

and, as many times a D measure the number A,B,C, let there be the same number of units in E,F,G respectively

Therefore E,F,G measure A,B,C according to the units in D (VII·16)

Therefore E,F,G are in the same ratio as A,B,C (VII.Def.20)

E,F,G are the smallest numbers that have the ratio of A,B,C

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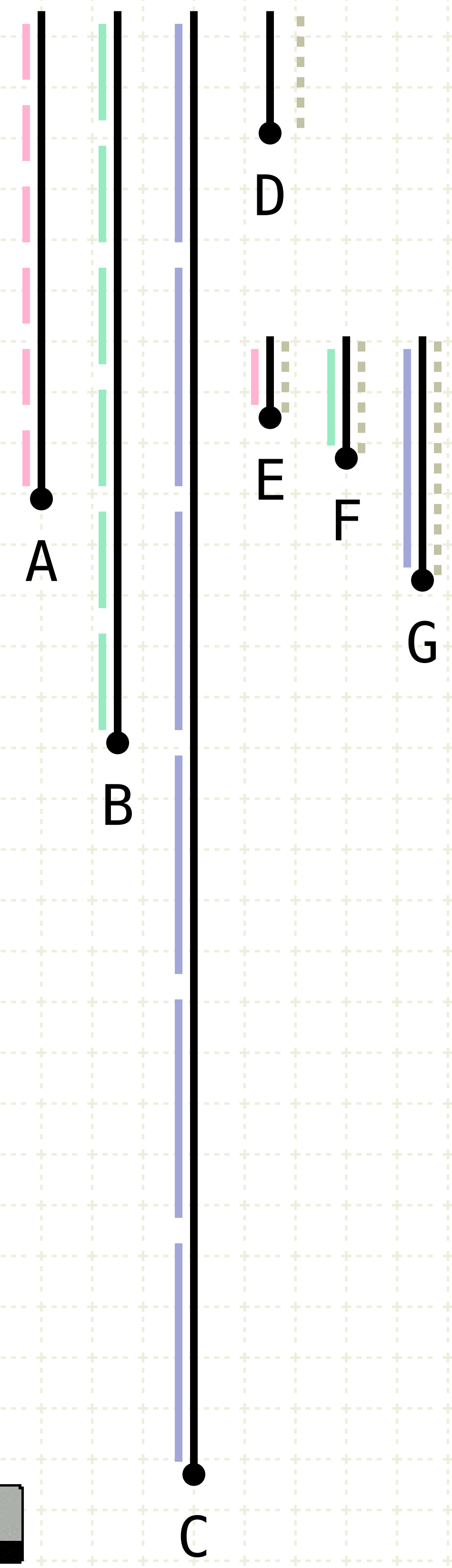

Proof by Contradiction

$S=\{(x,y,z)|x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

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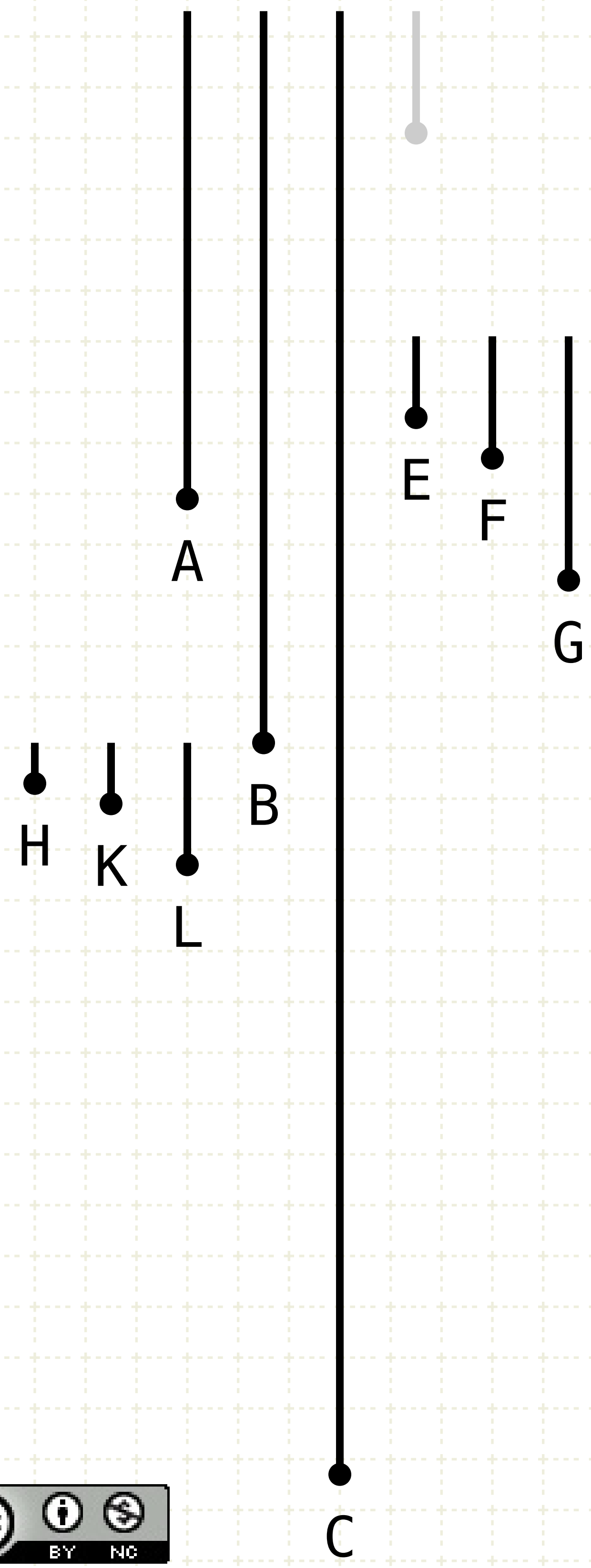

$S=\{(x,y,z)|x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

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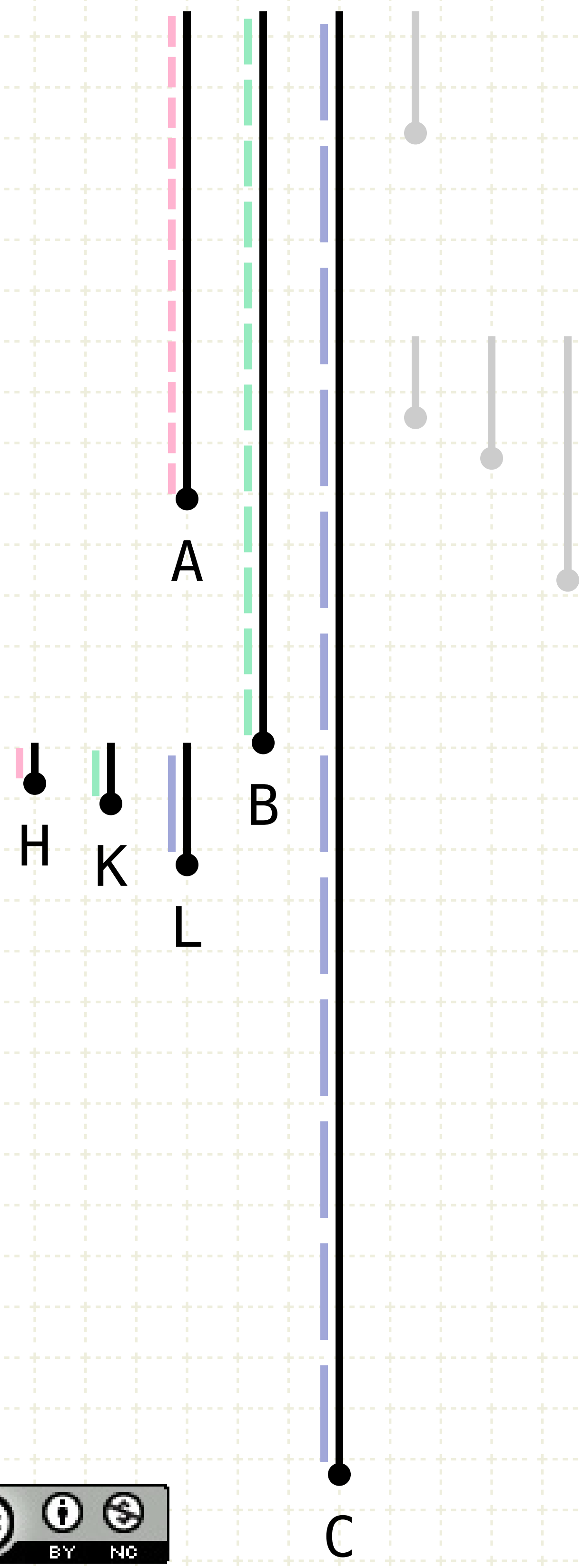
Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

$A = M\cdot H,\ B = M\cdot K,\ C = M\cdot L$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

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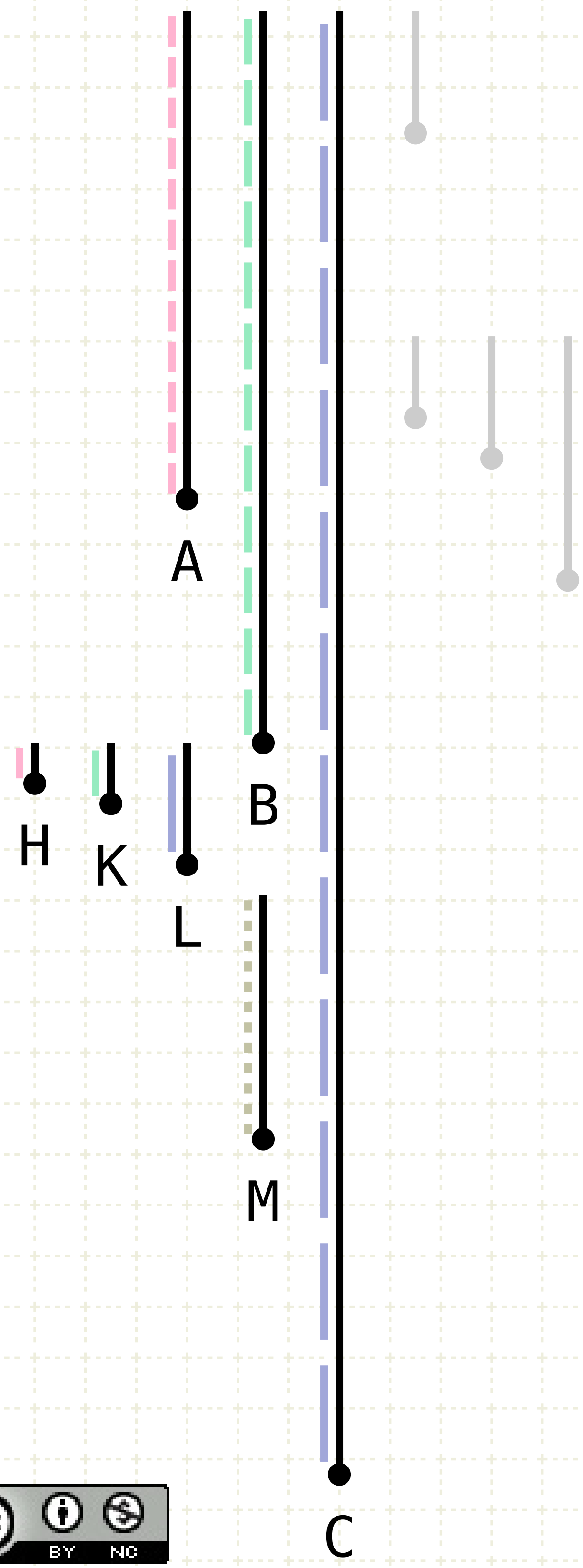

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As many times as H measures A, let M have the same number of units

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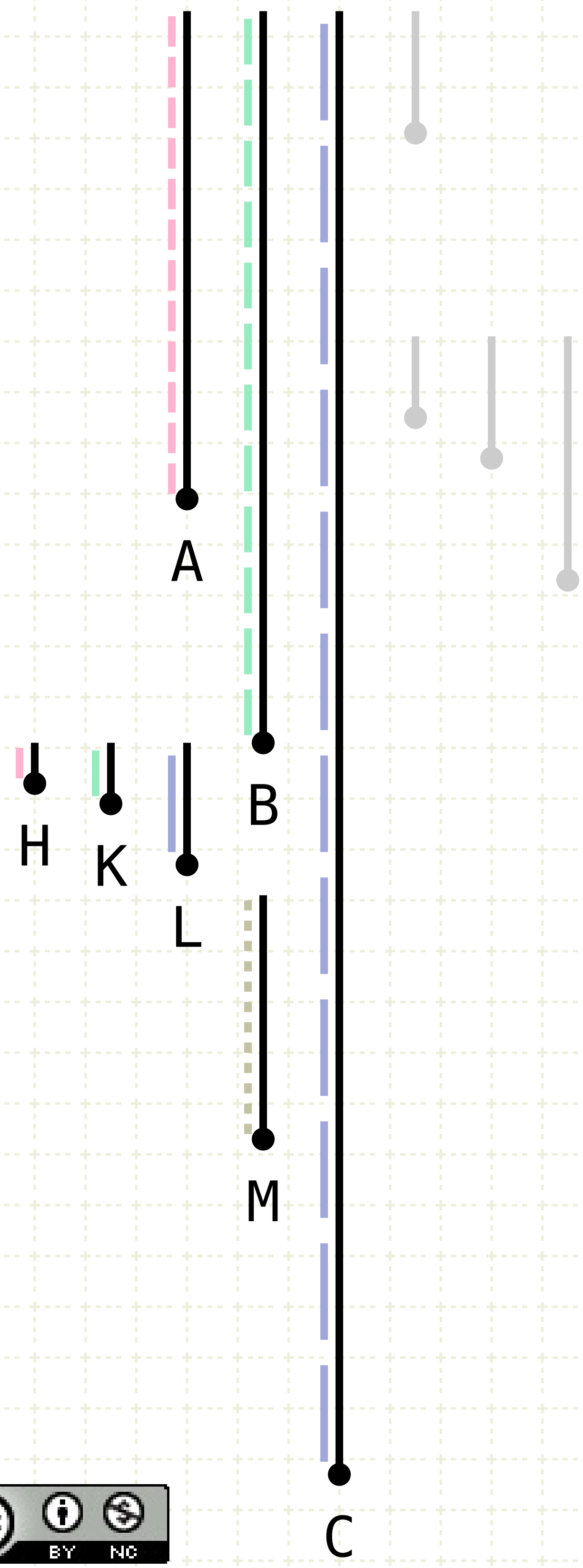

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Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

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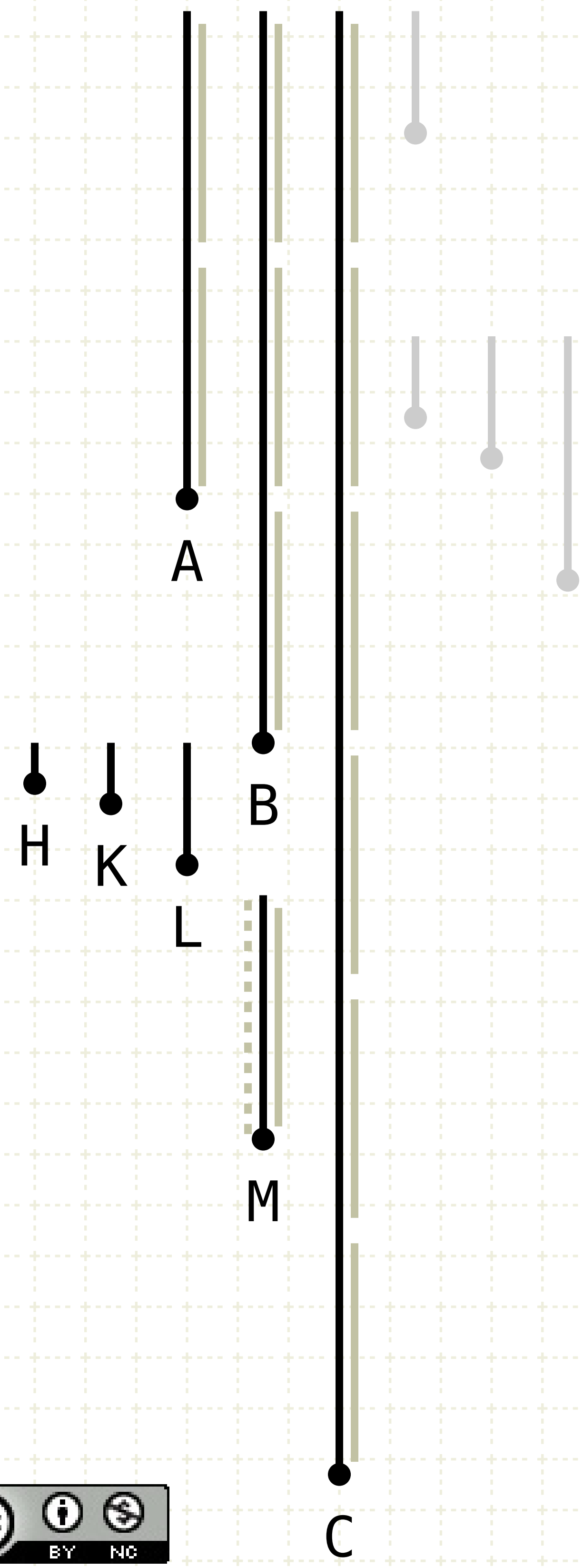

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$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

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$A = H\cdot M,\ B = K\cdot M,\ C = L\cdot M$

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Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

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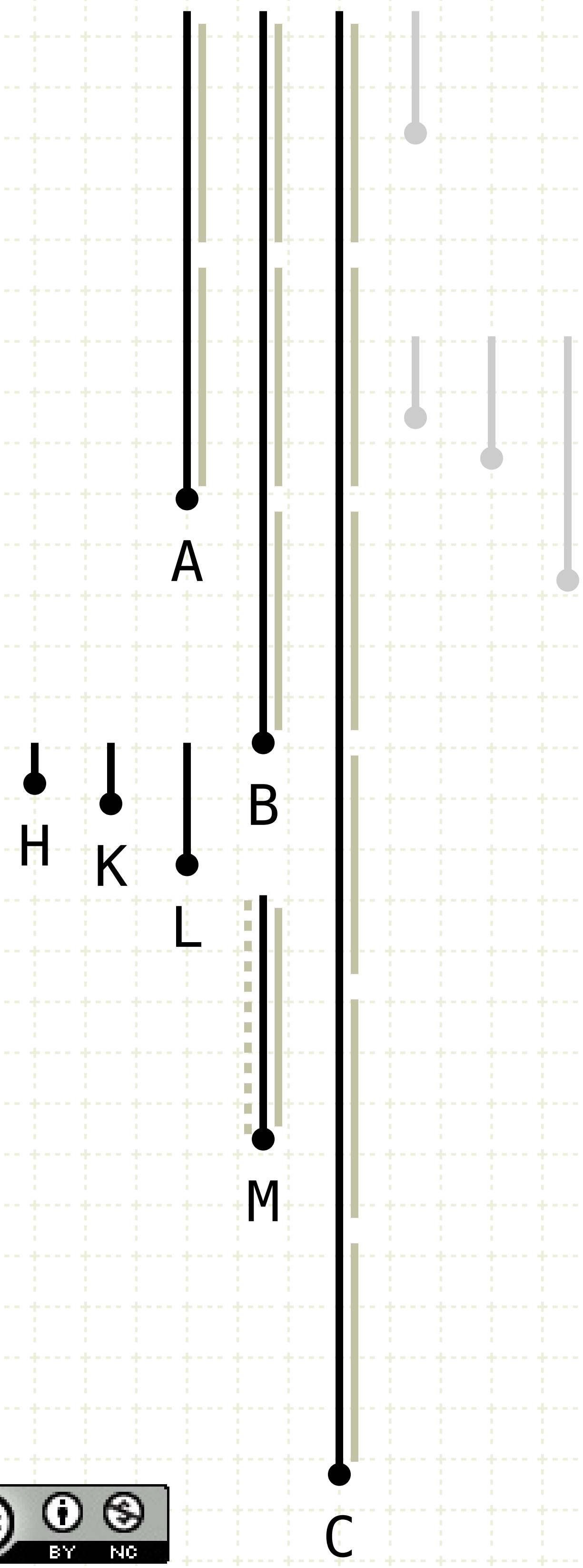

$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

$A = M\cdot H,\ B = M\cdot K,\ C = M\cdot L$

$A = H\cdot M,\ B = K\cdot M,\ C = L\cdot M$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C

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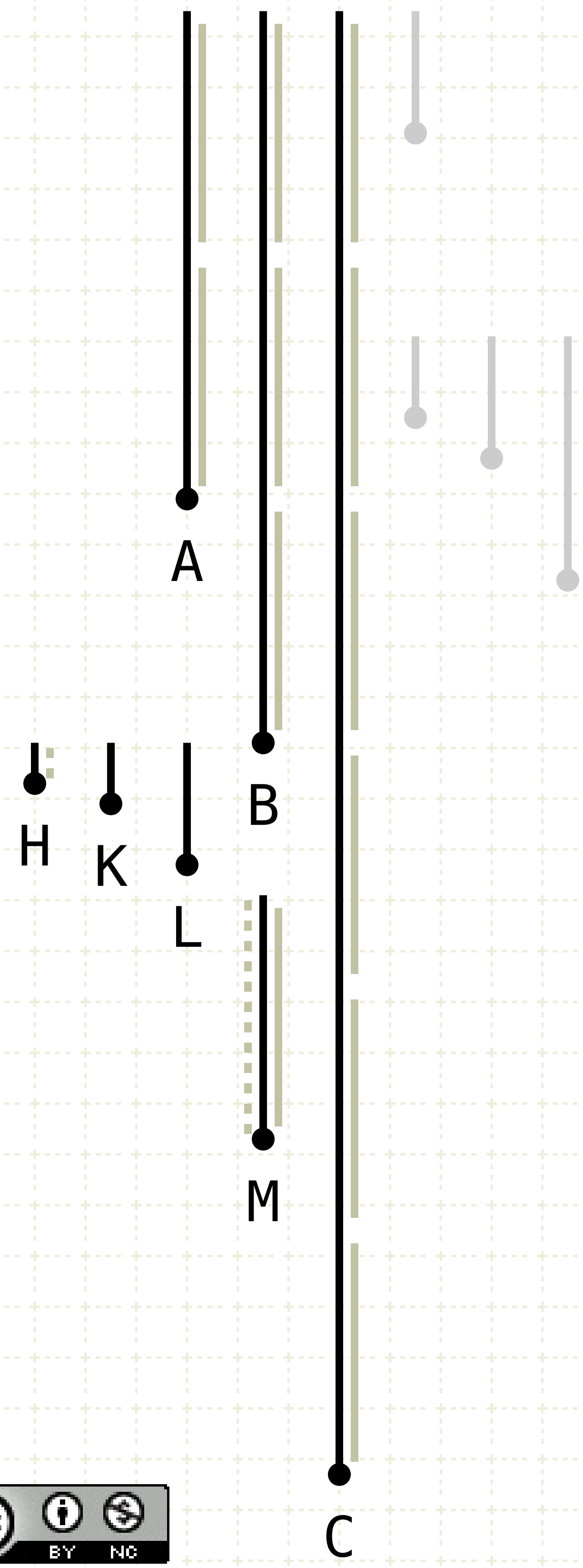
Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

$A = M\cdot H,\ B = M\cdot K,\ C = M\cdot L$

$A = H\cdot M,\ B = K\cdot M,\ C = L\cdot M$

$A = M \times H$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C

Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)

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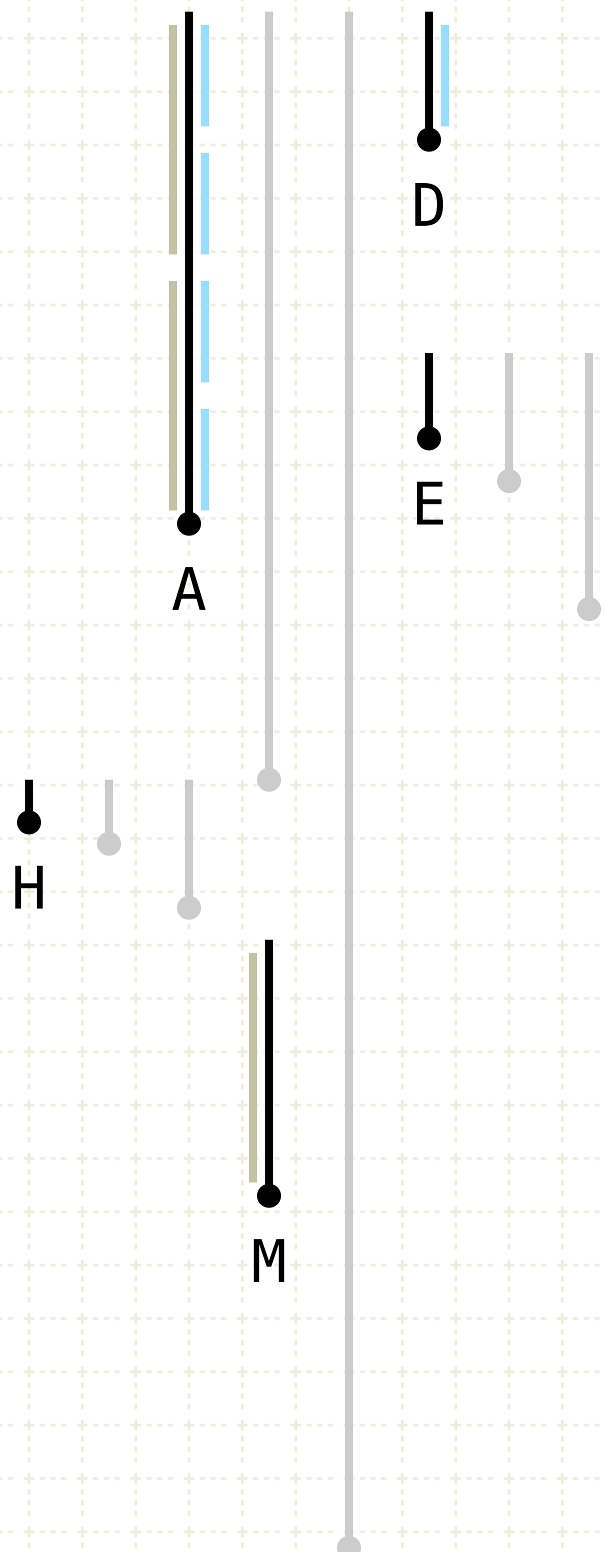
Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

$A = M\cdot H,\ B = M\cdot K,\ C = M\cdot L$

$A = H\cdot M,\ B = K\cdot M,\ C = L\cdot M$

$A = M \times H$

$A = M\times H = E\times D$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

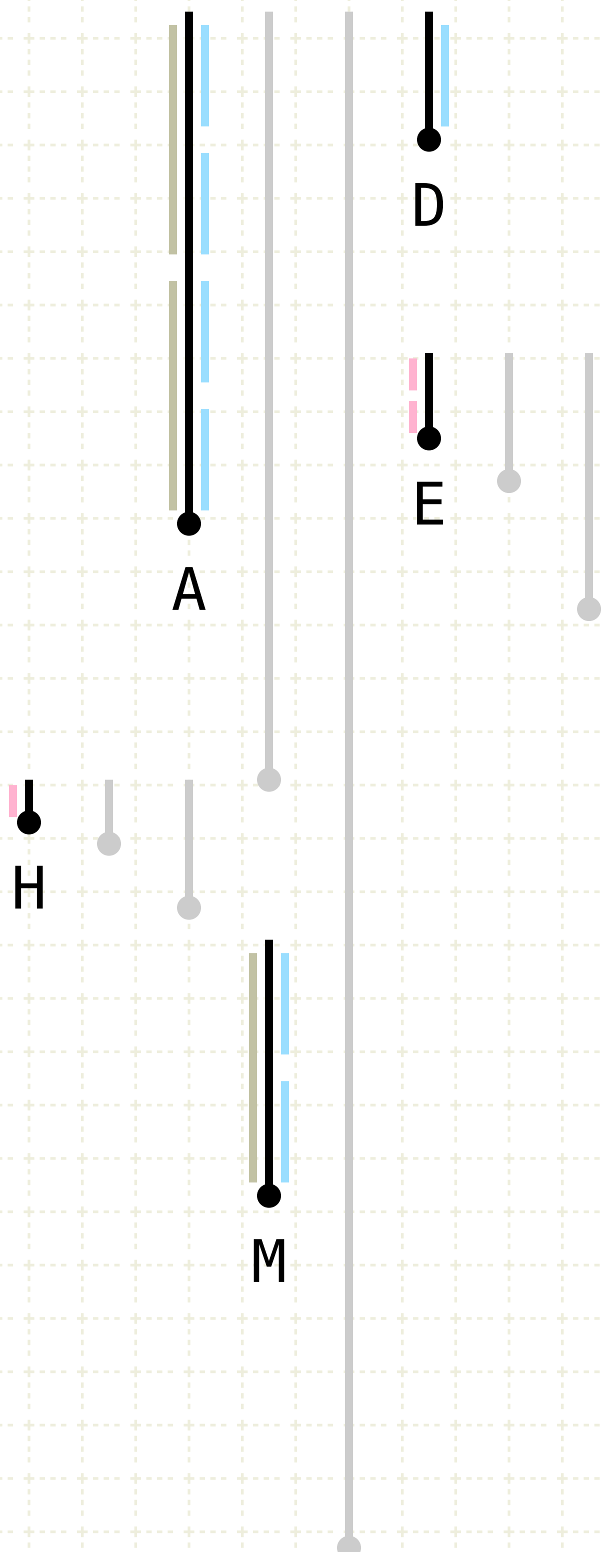
Therefore M measures A,B,C

Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)

Therefore the product of E,D is equal to the product H,M

Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall (x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

$A = M\cdot H,\ B = M\cdot K,\ C = M\cdot L$

$A = H\cdot M,\ B = K\cdot M,\ C = L\cdot M$

$A = M \times H$

$A = M\times H = E\times D$

$E:H = M:D$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C

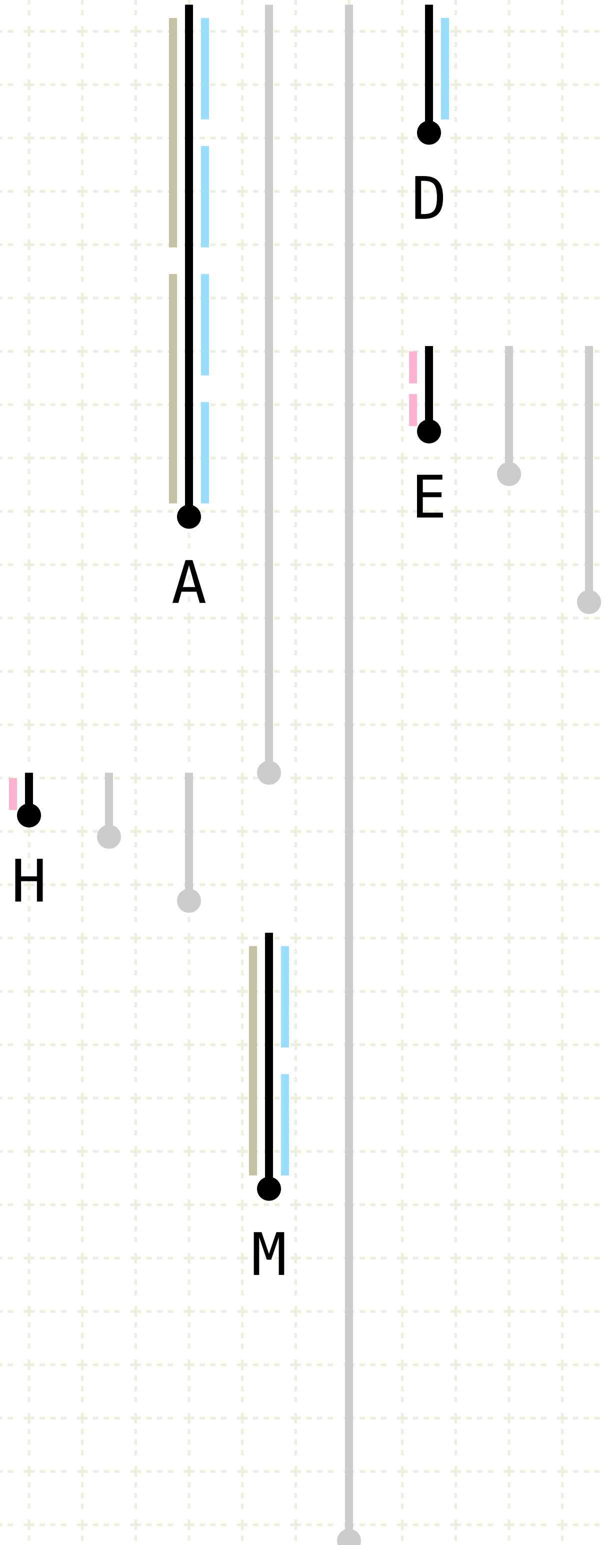
Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)

Therefore the product of E,D is equal to the product H,M

Therefore E is to H as M is to D (VII·19)

Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

$A = M\cdot H,\ B = M\cdot K,\ C = M\cdot L$

$A = H\cdot M,\ B = K\cdot M,\ C = L\cdot M$

$A = M \times H$

$A = M\times H = E\times D$

$E:H = M:D$

$M > D$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C

Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)

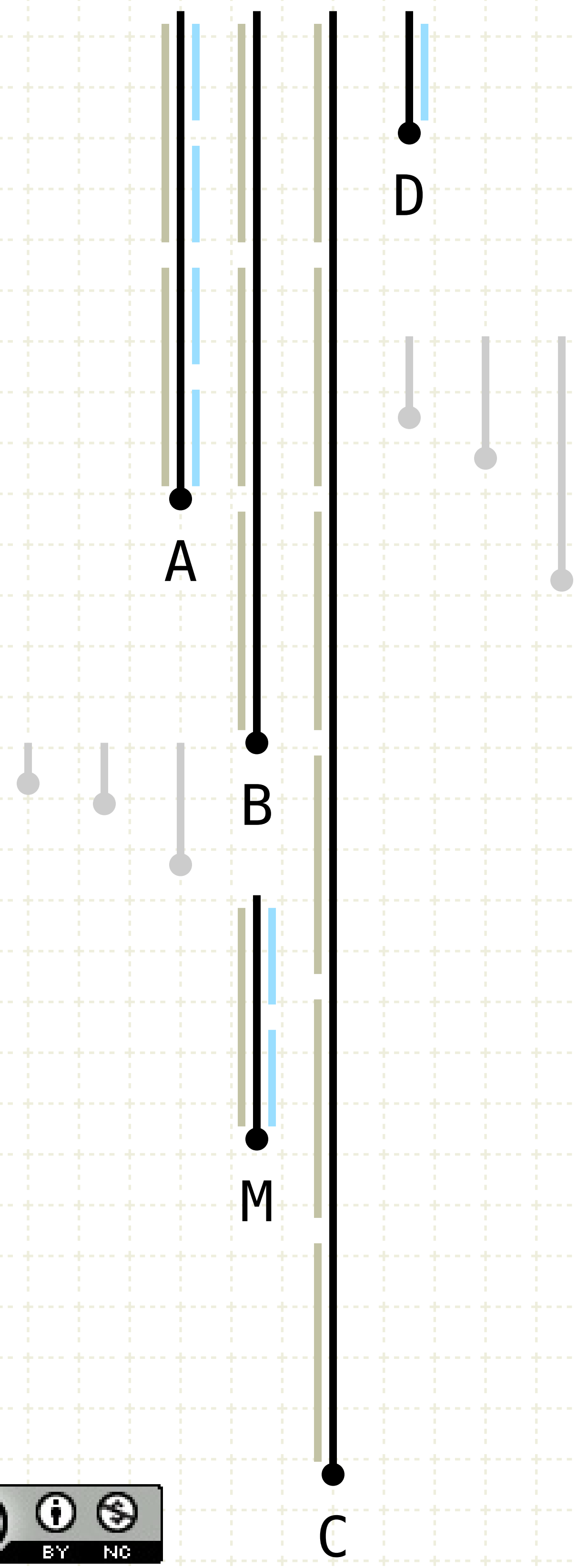
Therefore the product of E,D is equal to the product H,M

Therefore E is to H as M is to D (VII·19)

But E is greater H, therefore M is also greater than D

Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$S=\{(x,y,z)\,|\,x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$

$\gcd(A,B,C) = D$

$A = D\cdot E,\ B = D\cdot F,\ C = D\cdot G$

$E:F:G = A:B:C$

$E\leq x,\ F\leq y,\ G\leq z,\ \forall(x,y,z)\in S$

$H:K:L = E:F:G = A:B:C$

$H < E,\ K < F,\ L < G$

$A = M\cdot H,\ B = M\cdot K,\ C = M\cdot L$

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$A = M \times H$

$A = M\times H = E\times D$

$E:H = M:D$

$M > D$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C

Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)

Therefore the product of E,D is equal to the product H,M

Therefore E is to H as M is to D (VII·19)

But E is greater H, therefore M is also greater than D

And it measures A,B,C

Proposition 33 of Book VII

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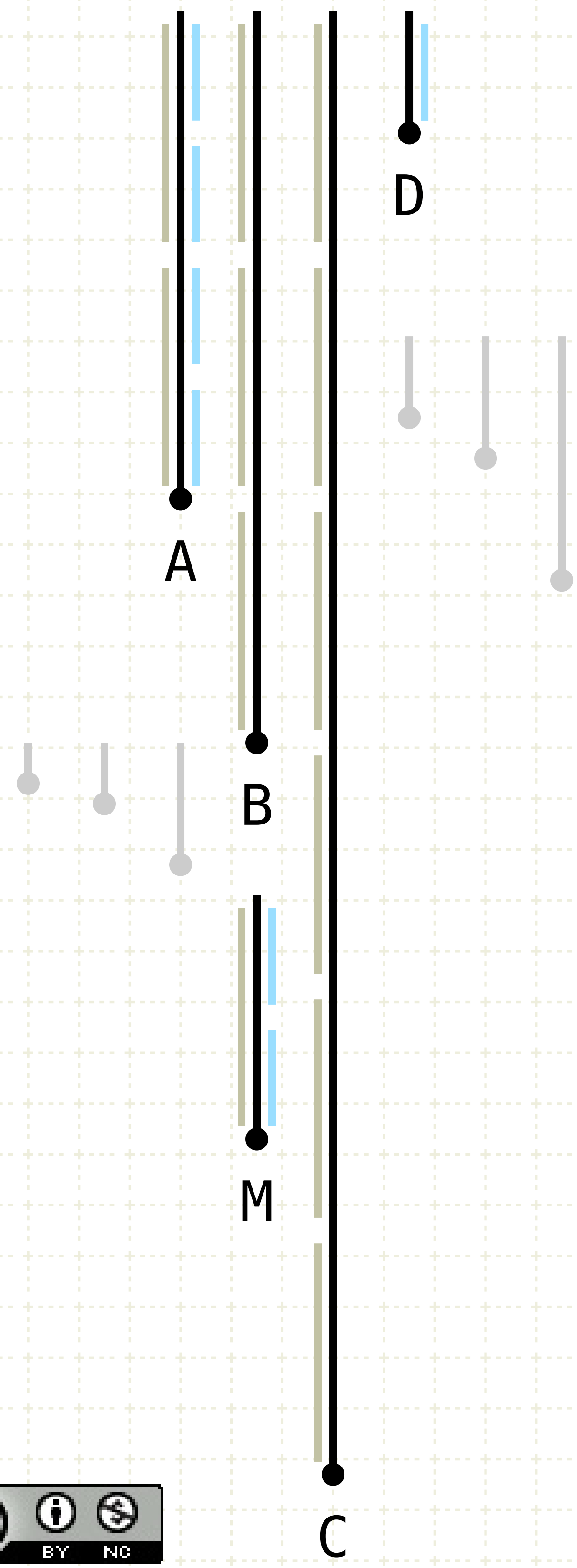

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Which is impossible, because D is the greatest common divisor of A,B,C

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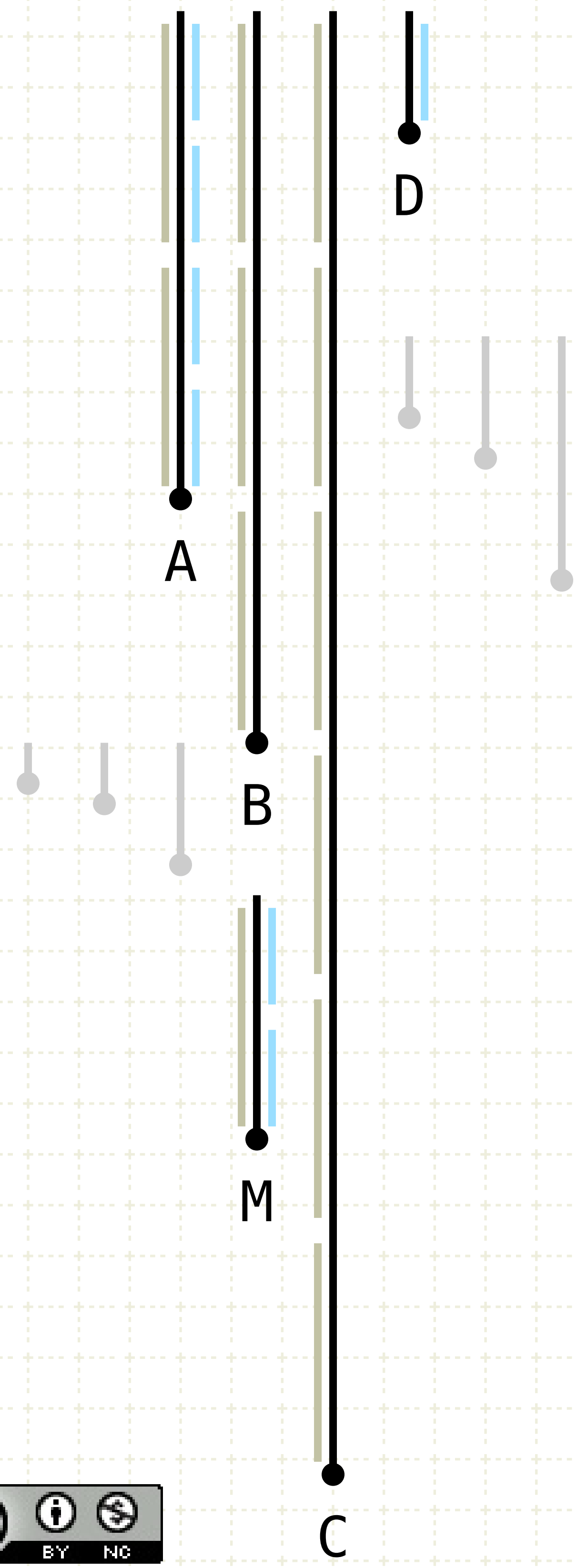

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Therefore, there cannot be any numbers less than E,F,G with the same ratio as A,B,C, which was the original assumption

Proposition 33 of Book VII

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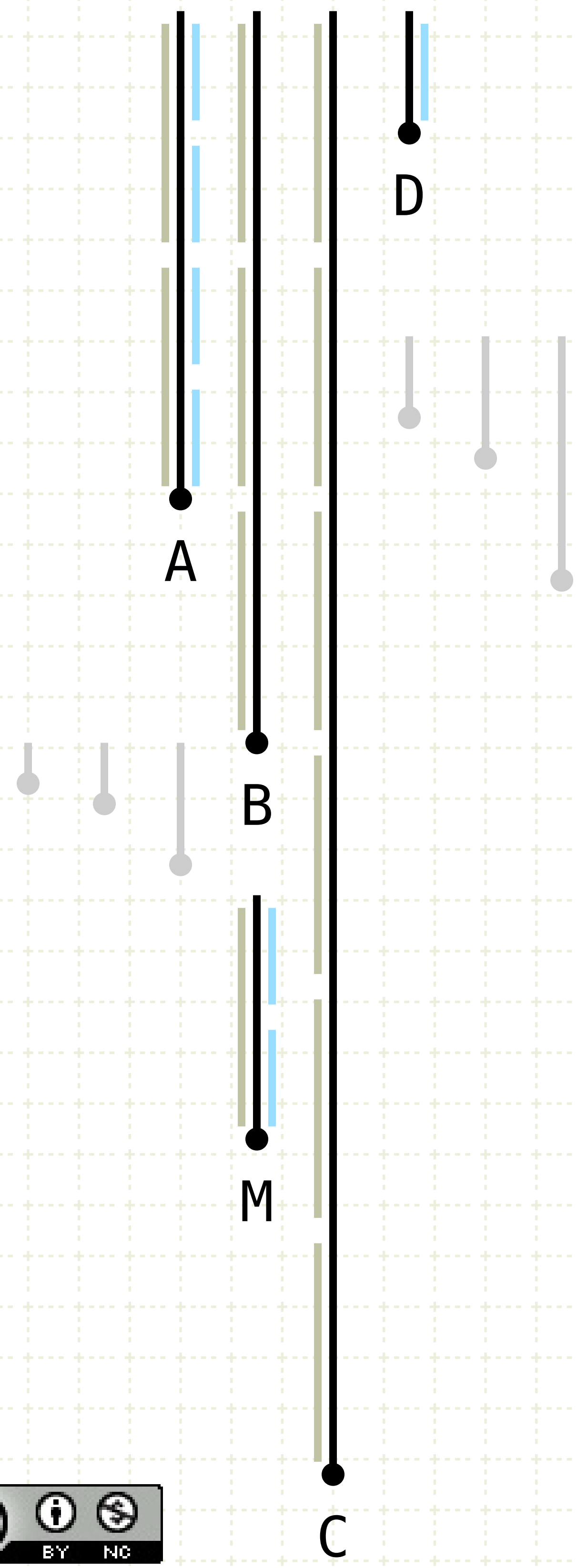

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Thus, E,F,G are the smallest numbers with the ratio A,B,C

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