

# Euclid's Elements

## Book VI

*One can state, without exaggeration, that the observation of and the search for similarities and differences are the basis of all human knowledge.*

**Alfred Nobel**



# Table of Contents, Chapter 6

1	If the height of two triangles are equal, then the ratio of the areas is equal to the ratio of the bases	7	If two triangles have one angle equal to one angle, and the sides about other angles are proportional, and the remaining angles either both less or both not less than a right angle, then triangles will be equiangular	14	In equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional; and vice versa
2	If a line cuts a triangle, parallel to its base, it will cut the sides of the triangle proportionally			15	In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and vice versa
3	If an angle of a triangle is bisected and the straight line cutting the angle also cuts the base, the segments of the base will have the same ratio as the remaining sides of the triangle	8	If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another	16	If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means, and vice versa
4	If two triangles have equal angles, then the sides opposite the equal angles are proportional, as well, the sides of the triangles on either side of the equal angles are also proportional	9	From a given straight line to cut off a given fraction	17	If three straight lines are proportional, the rectangle contained by the extremes is equal to the square on the mean; and vice versa
5	If two triangles have proportional sides, the triangles will be equiangular	10	To cut a given uncut straight line similarly to a given cut straight line	18	On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure
6	If two triangles have one angle equal to one angle and the sides about the equal angles are proportional, then the triangles will be equiangular	11	To two given straight lines to find a third proportional	19	Similar triangles are to one another in the duplicate ratio of the corresponding sides
		12	To three given straight lines to find a fourth proportional		
		13	To two given straight lines to find a mean proportional		



# Table of Contents, Chapter 6

20	Similar polygons are divided into the same number of similar triangles, which have the same ratio as the wholes, and the polygons have duplicate ratios to their corresponding sides	26	If from a parallelogram a similar parallelogram with a common angle is subtracted, it is about the same diameter as the original	31	In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle
21	Figures which are are similar to the same rectilineal figure are also similar to one another	27	Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar to a parallelogram drawn on half the said line, the largest will be one that is drawn on half of the straight line and is similar to the defect	32	If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangle will be in a straight line
22	If four straight lines are proportional, similar rectilineal figures will also be proportional; and vice versa	28	To a given straight line, apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one	33	In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences
23	<b>Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides</b>	29	To a given straight line, apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one		
24	In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another	30	To cut a finite straight line in extreme ratio		
25	To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure				



# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides





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Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides

## Definition: Compound Ratios

Euclid simply uses 'compound ratios' in this proof without previously defining them, so the definition of compound ratio has been retroactively defined as...<sup>1</sup>

<sup>1</sup> Euclid's Elements - all thirteen books complete in one volume, The Thomas L Heath Translation, Dana Densmore, Editor, Green Lion Press (c) 2013, pg xxix



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...If we have a series of magnitudes, A...D for example

$A:D = \text{compound ratio of } A:B, B:C, C:D$

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$A:D = \text{compound ratio of } A:B, B:C, C:D$

If A to B is equal to E to F, B to C equal to G to H, C to D equal J to K, then the ratio of A to D is also equal to the compound ratio of G to H, C to D, and J to K <sup>1</sup>

$A:B = E:F$

$B:C = G:H$

$C:D = J:K$

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$A:D = \text{compound ratio of } E:F, G:H, J:K$

For two or more ratios, if we take antecedent as product of antecedents of the ratios and consequent as product of consequents of the ratios, then the ratio thus formed is called mixed or compound ratio. <sup>2</sup>

$\text{compound ratio of } m:n, p:q = mp:nq$

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$\text{compound ratio of } m:n, p:q = mp:nq$

$\therefore A:D = ABC:BCD$

and  $A:D = EGJ:FHK$

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$$\frac{A}{D} = \frac{A}{D} \times \frac{B \cdot C}{B \cdot C} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D}$$



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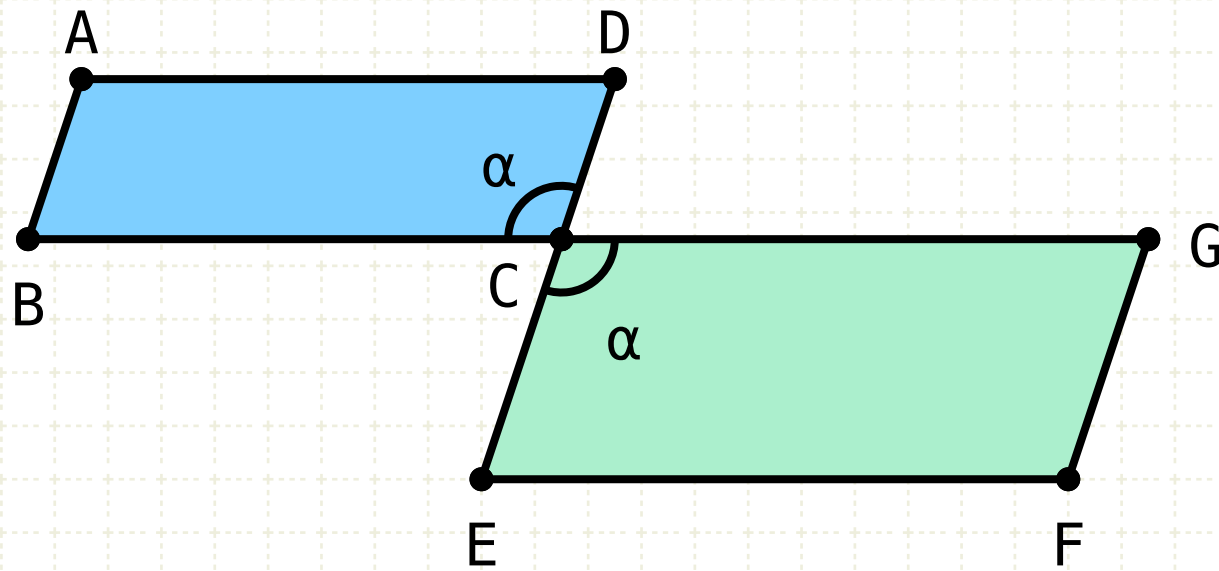
$$\frac{A}{D} = \frac{A}{D} \times \frac{B \cdot C}{B \cdot C} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{E}{F} \times \frac{G}{H} \times \frac{J}{K} = \frac{E \cdot G \cdot J}{F \cdot H \cdot K}$$





## Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



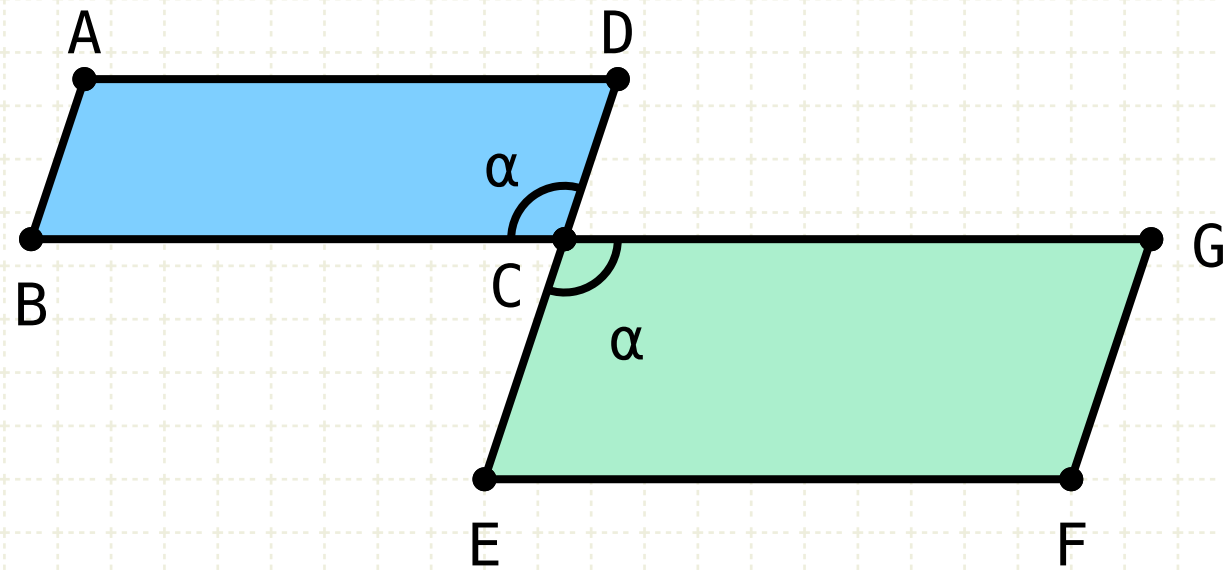
### In other words

The areas of equiangular parallelograms have ratios that equal the multiplication of the ratios of their sides (compounded ratio)

$$\square AC : \square CF = (BC \cdot CD) : (CG \cdot CE)$$

# Proposition 23 of Book VI

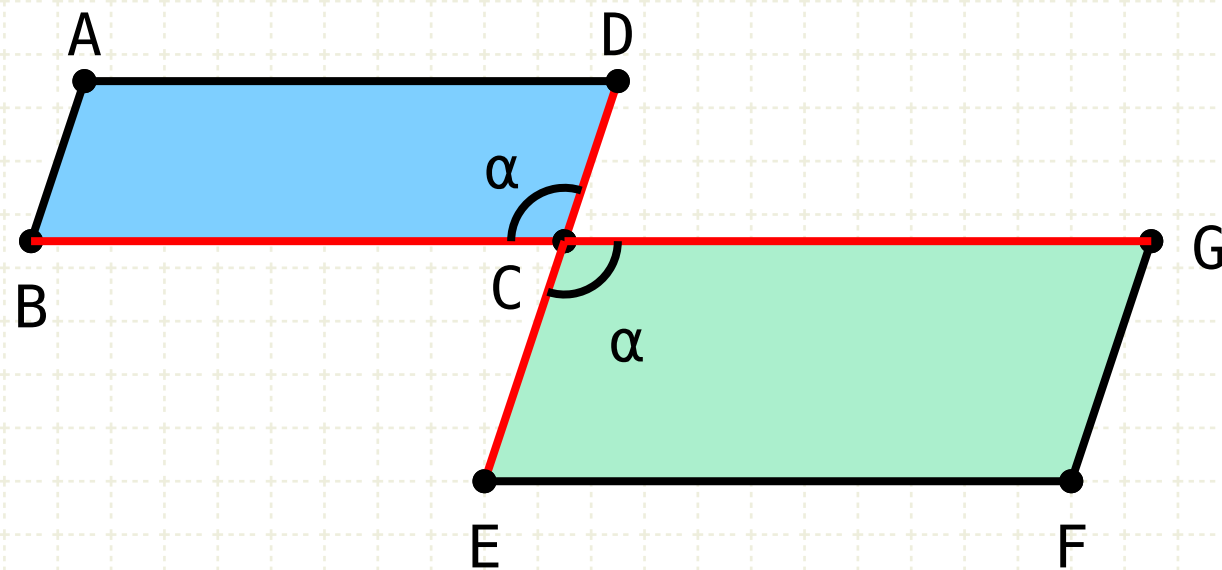
Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



## Proof

## Proposition 23 of Book VI

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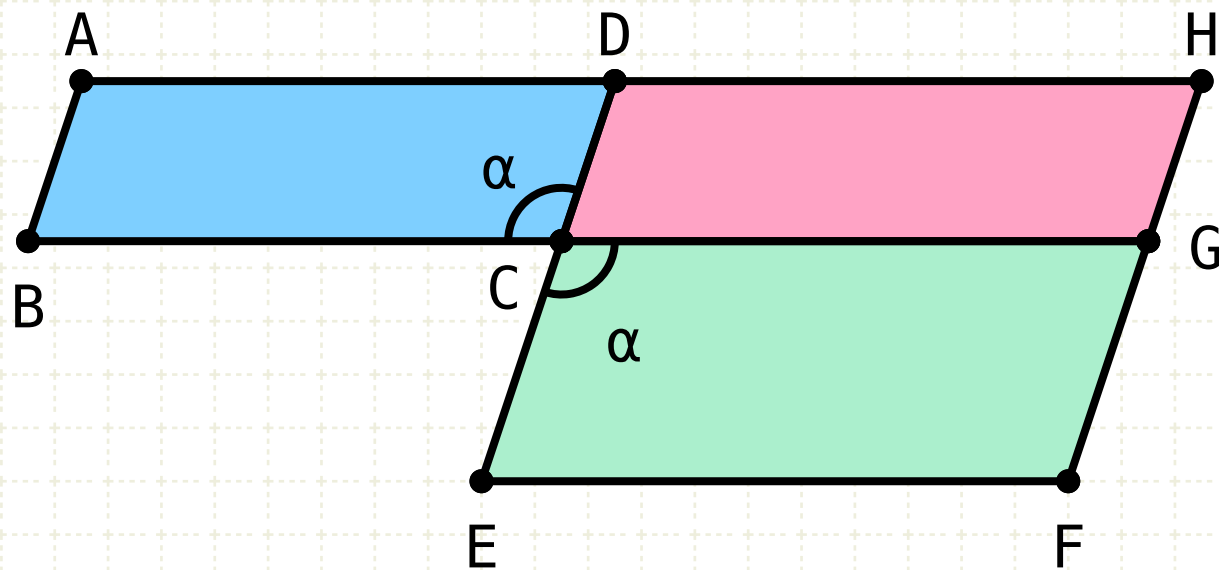


### Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

## Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



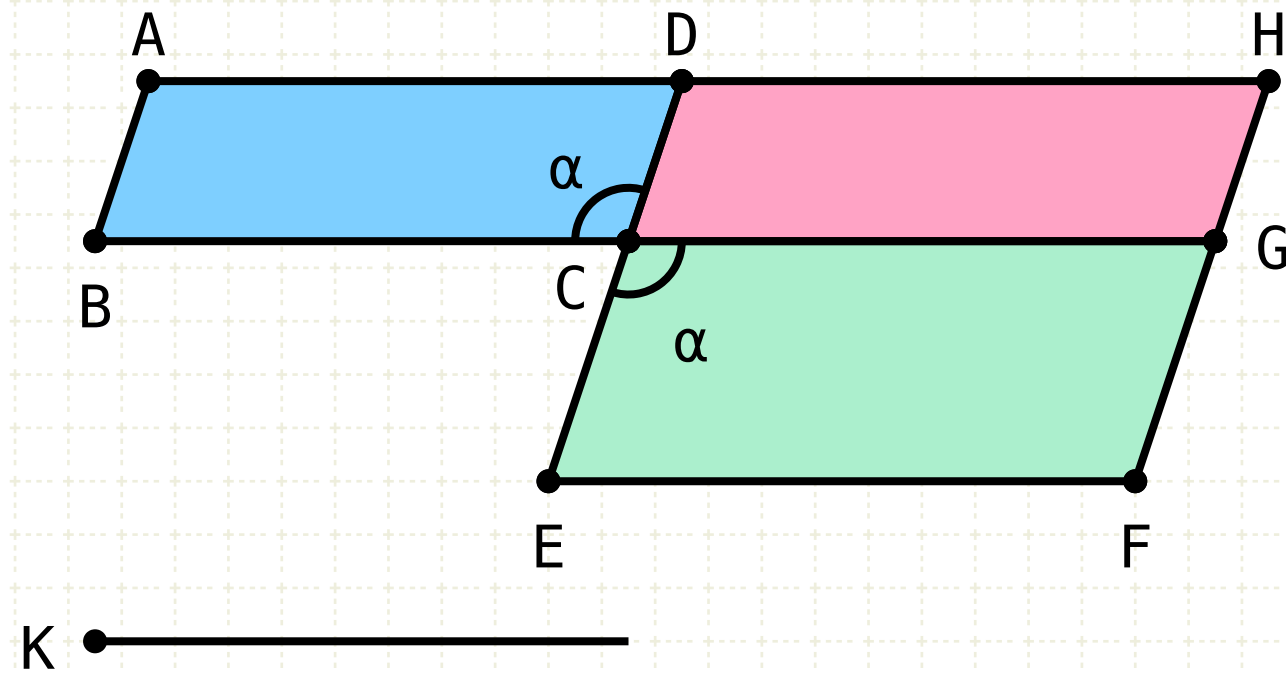
### Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

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Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



## Proof

Let the parallelograms be placed so that  $BC$  is in a straight line with  $CG$  therefore  $DC$  is also in a straight line with  $CE$

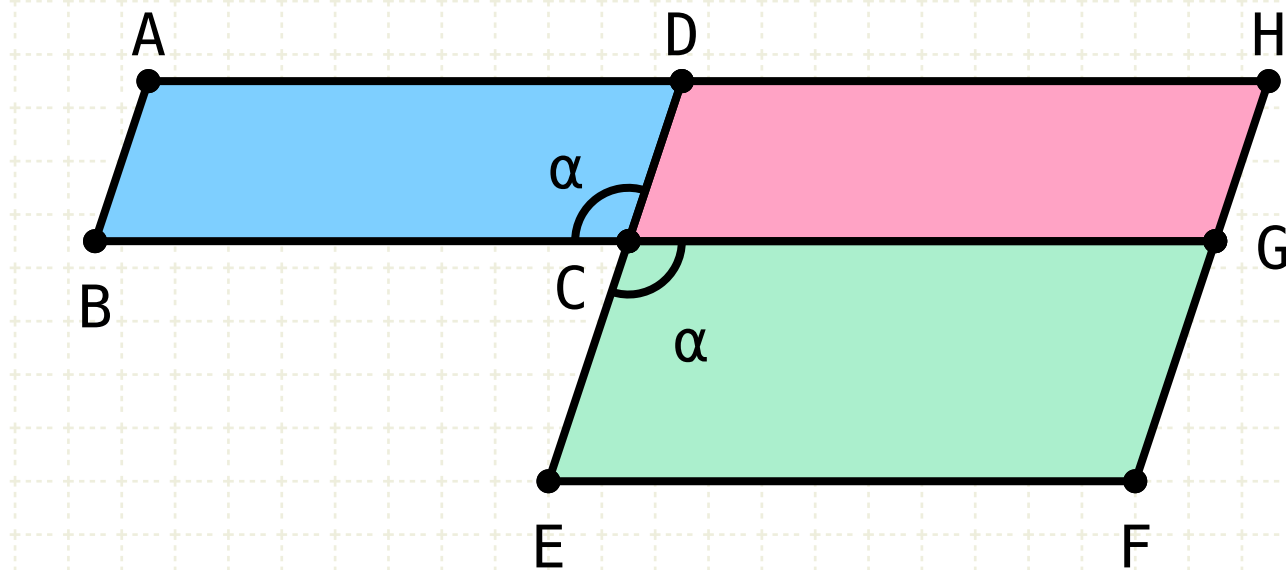
Draw the parallelogram  $DG$

Define an arbitrary line  $K$ , and draw another line  $L$  such that the ratio as  $BC$  is to  $CG$  so is  $K$  to  $L$  (VI·12)



# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$K:L = BC:CG$$

## Proof

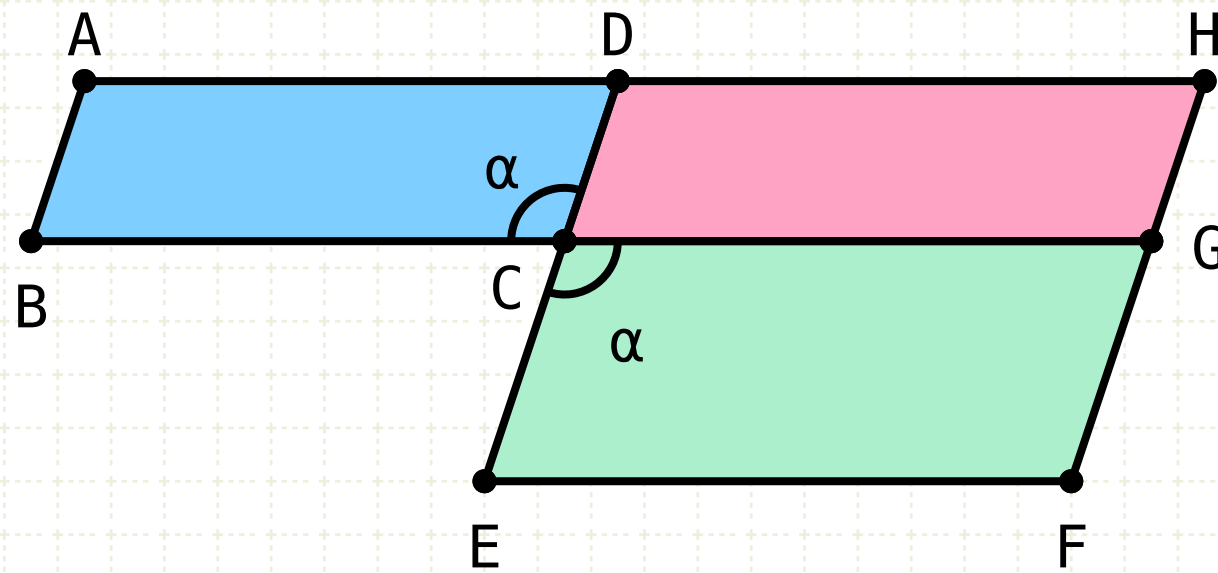
Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$K:L = BC:CG$$

$$L:M = CD:CE$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

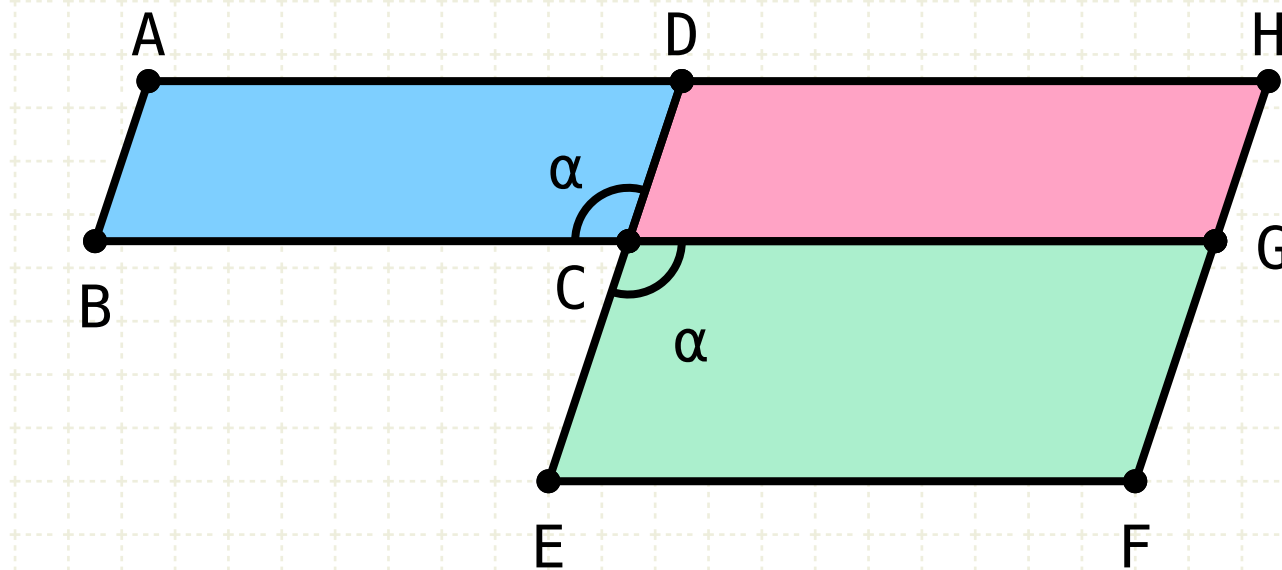
Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$K:L = BC:CG$$

$$L:M = CD:CE$$

$$K:M = (BC \cdot CD) : (CG \cdot CE)$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

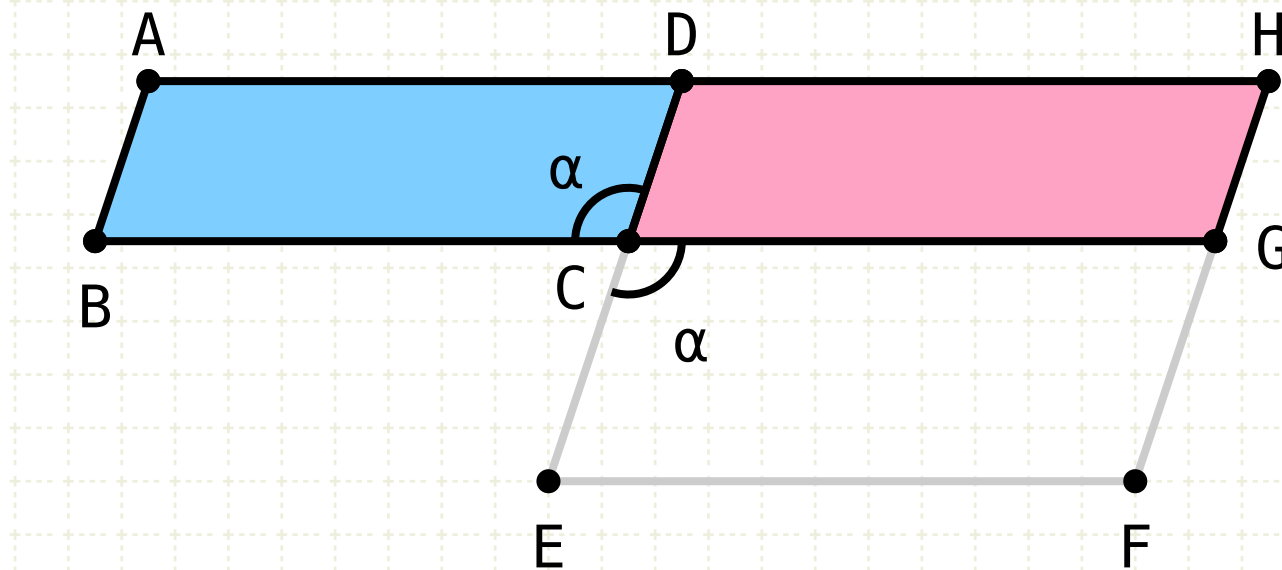
Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$K:L = BC:CG$$

$$L:M = CD:CE$$

$$K:M = (BC \cdot CD):(CG \cdot CE)$$

$$BC:CG = \square AC:\square CH$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

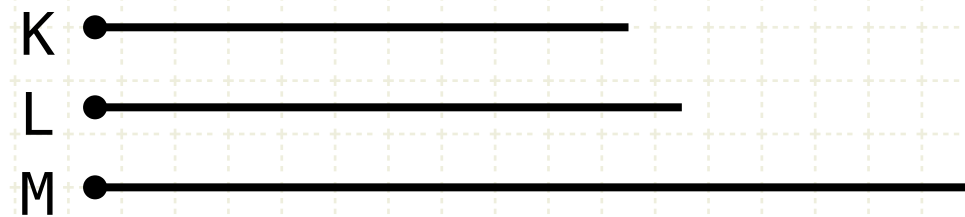
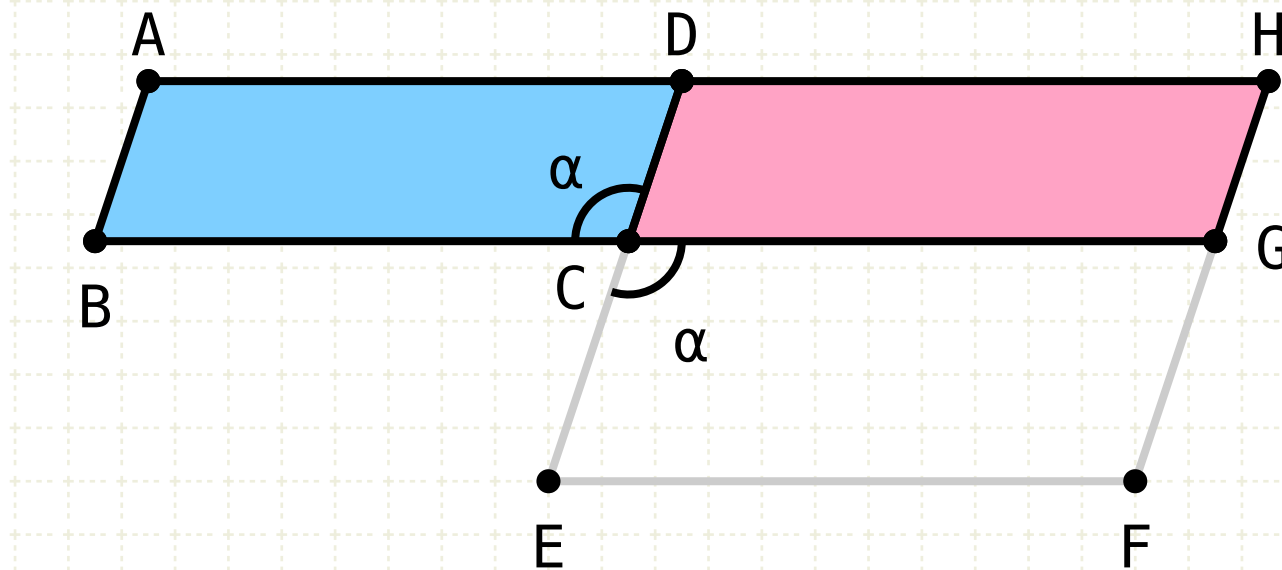
Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

The ratio of the sides BC to CG is equal to the ratio of the parallelograms AC,CH (VI·1)

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Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$K:L = BC:CG$$

$$L:M = CD:CE$$

$$K:M = (BC \cdot CD):(CG \cdot CE)$$

$$BC:CG = \square AC:\square CH$$

$$K:L = \square AC:\square CH$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

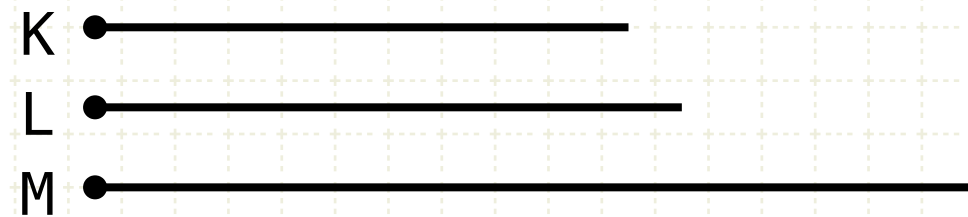
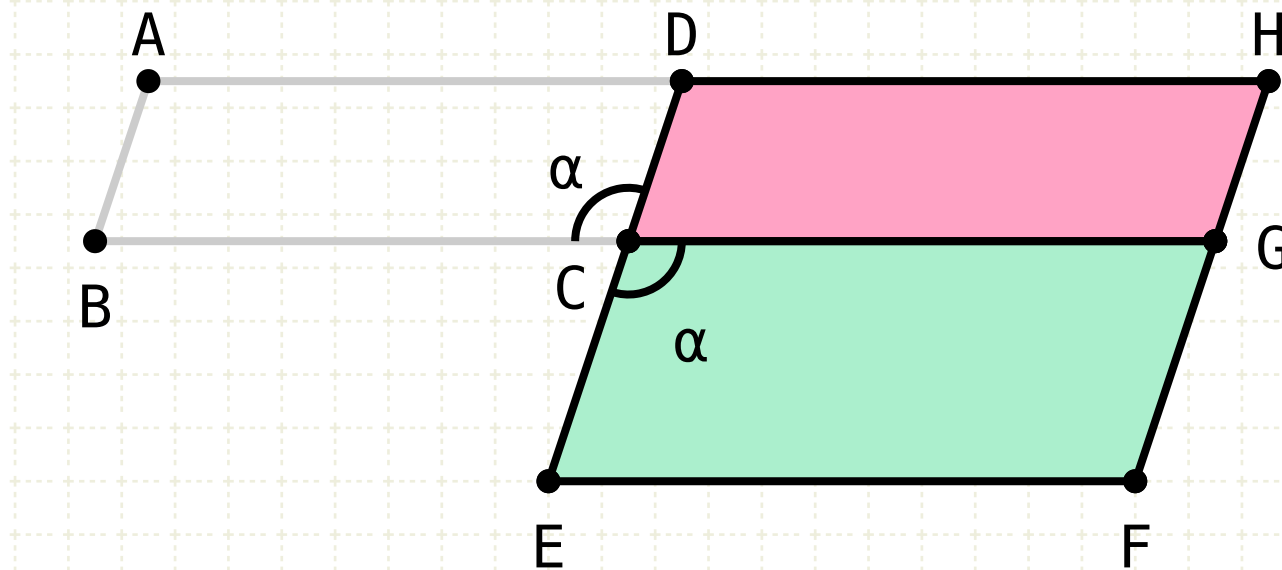
The ratio of the sides BC to CG is equal to the ratio of the parallelograms AC,CH (VI·1)

But K to L is also equal to BC to CG, so K to L is equal to the ratio of the parallelograms AC,CH (V·11)



# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$\begin{aligned} K:L &= BC:CG \\ L:M &= CD:CE \\ K:M &= (BC \cdot CD):(CG \cdot CE) \\ BC:CG &= \square AC:\square CH \\ K:L &= \square AC:\square CH \\ CD:CE &= \square CH:\square CF \end{aligned}$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

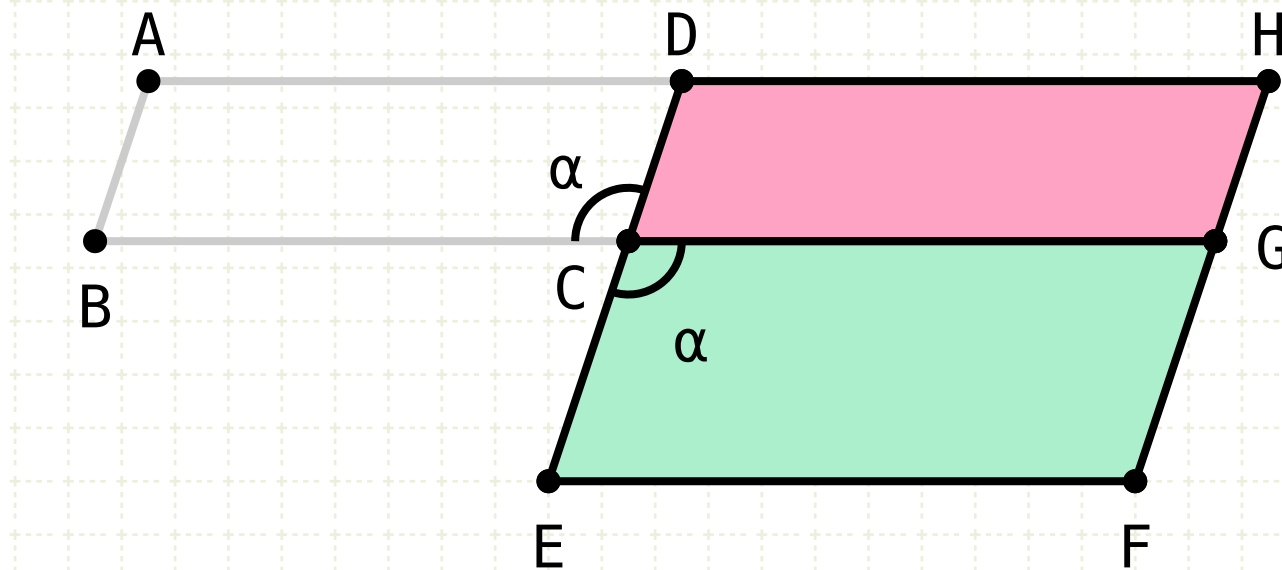
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The ratio of the sides DC to CE is equal to the ratio of the parallelograms CH,CF (VI·1)

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Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$K:L = BC:CG$$

$$L:M = CD:CE$$

$$K:M = (BC \cdot CD):(CG \cdot CE)$$

$$BC:CG = \square AC:\square CH$$

$$K:L = \square AC:\square CH$$

$$CD:CE = \square CH:\square CF$$

$$L:M = \square CH:\square CF$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

The ratio of the sides BC to CG is equal to the ratio of the parallelograms AC,CH (VI·1)

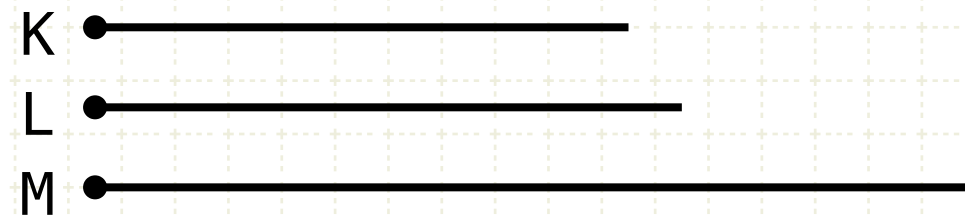
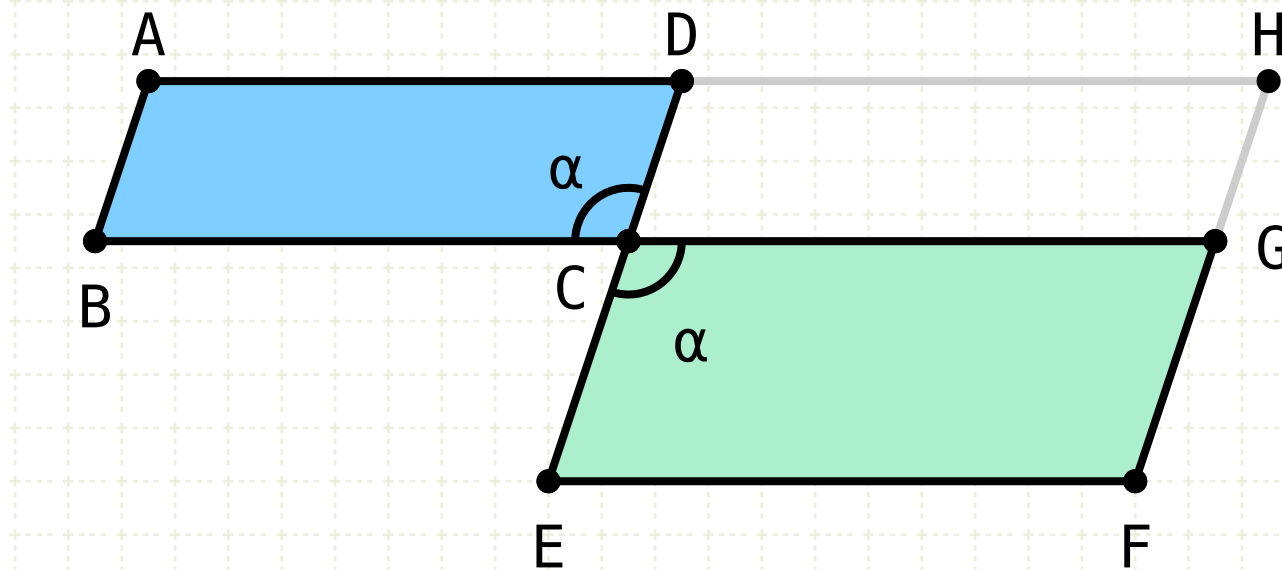
But K to L is also equal to BC to CG, so K to L is equal to the ratio of the parallelograms AC,CH (V·11)

The ratio of the sides DC to CE is equal to the ratio of the parallelograms CH,CF (VI·1)

Therefore L to M is equal to the ratio of the parallelograms CH,CF (V·11)

# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$\begin{aligned}
 K:L &= BC:CG \\
 L:M &= DC:CE \\
 K:M &= (BC \cdot CD):(CG \cdot CE) \\
 BC:CG &= \square AC:\square CH \\
 K:L &= \square AC:\square CH \\
 CD:CE &= \square CH:\square CF \\
 L:M &= \square CH:\square CF \\
 K:M &= \square AC:\square CF
 \end{aligned}$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

The ratio of the sides BC to CG is equal to the ratio of the parallelograms AC,CH (VI·1)

But K to L is also equal to BC to CG, so K to L is equal to the ratio of the parallelograms AC,CH (V·11)

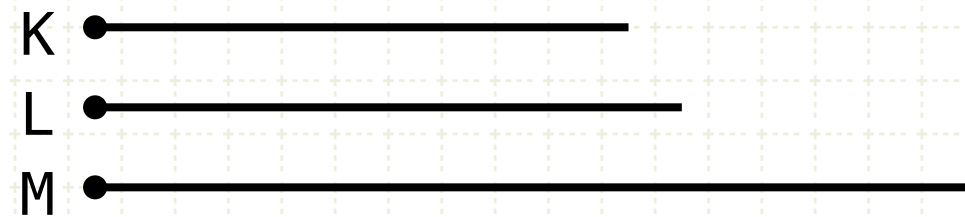
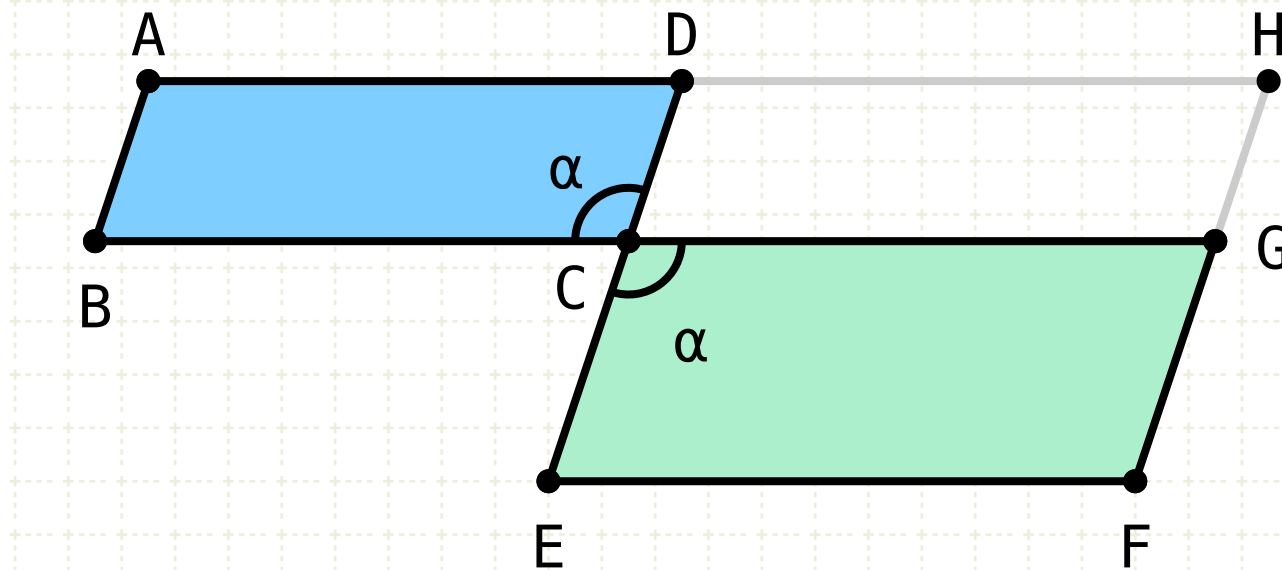
The ratio of the sides DC to CE is equal to the ratio of the parallelograms CH,CF (VI·1)

Therefore L to M is equal to the ratio of the parallelograms CH,CF (V·11)

Comparing the ratios of K to L and L to M it can be seen that the ratio K to M is equal (ex aequali) to the ratio of the parallelograms AC to CF (V·22)

# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$\begin{aligned}
 K:L &= BC:CG & \square AC:\square CF &= (BC \cdot CD):(CG \cdot CE) \\
 L:M &= CD:CE \\
 K:M &= (BC \cdot CD):(CG \cdot CE) \\
 BC:CG &= \square AC:\square CH \\
 K:L &= \square AC:\square CH \\
 CD:CE &= \square CH:\square CF \\
 L:M &= \square CH:\square CF \\
 K:M &= \square AC:\square CF
 \end{aligned}$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

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But K to L is also equal to BC to CG, so K to L is equal to the ratio of the parallelograms AC,CH (V·11)

The ratio of the sides DC to CE is equal to the ratio of the parallelograms CH,CF (VI·1)

Therefore L to M is equal to the ratio of the parallelograms CH,CF (V·11)

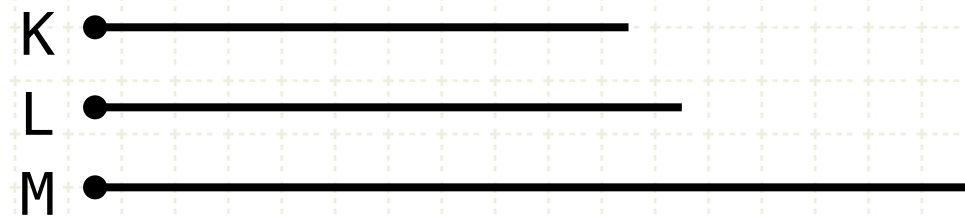
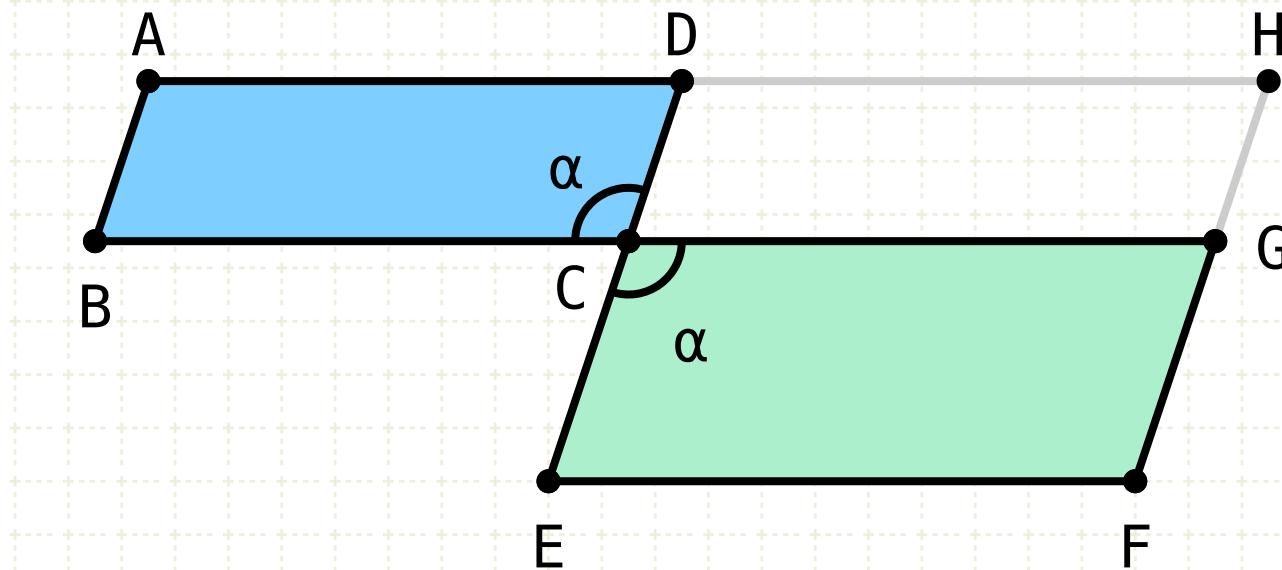
Comparing the ratios of K to L and L to M it can be seen that the ratio K to M is equal (ex aequali) to the ratio of the parallelograms AC to CF (V·22)

But K to M is equal to the compound ratio of the sides, therefore the compound ratio of the sides is also equal to the ratio of the parallelograms



# Proposition 23 of Book VI

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides



$$\begin{aligned}
 K:L &= BC:CG & \square AC:\square CF &= (BC \cdot CD):(CG \cdot CE) \\
 L:M &= CD:CE \\
 K:M &= (BC \cdot CD):(CG \cdot CE) \\
 BC:CG &= \square AC:\square CH \\
 K:L &= \square AC:\square CH \\
 CD:CE &= \square CH:\square CF \\
 L:M &= \square CH:\square CF \\
 K:M &= \square AC:\square CF
 \end{aligned}$$

## Proof

Let the parallelograms be placed so that BC is in a straight line with CG therefore DC is also in a straight line with CE

Draw the parallelogram DG

Define an arbitrary line K, and draw another line L such that the ratio as BC is to CG so is K to L (VI·12)

Draw another line M such that the ratio as DC is to CE so is L to M (VI·12)

K to M is the compounded ratio of K to L and L to M, therefore it is also equal to the compound ratio of the sides, BC to CG and CD to CE

The ratio of the sides BC to CG is equal to the ratio of the parallelograms AC,CH (VI·1)

But K to L is also equal to BC to CG, so K to L is equal to the ratio of the parallelograms AC,CH (V·11)

The ratio of the sides DC to CE is equal to the ratio of the parallelograms CH,CF (VI·1)

Therefore L to M is equal to the ratio of the parallelograms CH,CF (V·11)

Comparing the ratios of K to L and L to M it can be seen that the ratio K to M is equal (ex aequali) to the ratio of the parallelograms AC to CF (V·22)

But K to M is equal to the compound ratio of the sides, therefore the compound ratio of the sides is also equal to the ratio of the parallelograms



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