

# Euclid's Elements

## Book VII

### Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

*As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.*

**Joseph-Louis Lagrange**  
**(1736 to 1813)**



# Table of Contents, Chapter 7

1	Determine if two numbers are relatively prime	10	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$	21	If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B
2	Find the greatest common divisor for two numbers	11	If $A:B = C:D$ , then $(A-C):(B-D) = A:B$	22	If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime
3	Find the largest common divisor for three numbers	12	If $A:B = C:D$ , then $(A+C):(B+C) = A:B$	23	If A,B are relatively prime and if $A = n \cdot C$ , then B,C are relatively prime
4	Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B	13	If $A:B = C:D$ , then $A:C = B:D$	<b>24</b>	<b>If A,C are relatively prime and B,C are relatively prime then the <math>A \times B</math> is relatively prime to C</b>
5	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , then $(B+D) = (1/q) \cdot (A+C)$	14	If $A:B = D:E$ and $B:C = E:F$ , then $A:C = D:F$	25	If A,B are relatively prime then $A^2, B$ are relatively prime
6	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , then $(B+D) = (p/q) \cdot (A+C)$	15	If $B = i \cdot 1$ and $E = i \cdot D$ , and if $D = j \cdot 1$ then $E = j \cdot B$	26	If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$
7	If $B = A/q$ and $D = C/q$ , $B > D$ , then $(B-D) = (A-C)/q$	16	$A \times B = B \times A$	27	If A,B are relatively prime, then $A^2, B^2$ are relatively prime, and $A^3, B^3$ are relatively prime, and so on
8	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , $B > D$ , then $(B-D) = (p/q) \cdot (A-C)$	17	If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$		
9	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$	18	If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$		
		19	If $A:B = C:D$ then $A \times D = B \times C$ If $A \times D = B \times C$ then $A:B = C:D$		
		20	Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$		



## Table of Contents, Chapter 7

- |    |  |    |   |
|----|--|----|---|
| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime   | 37 | If $A = p \cdot B$ , then $A = q \cdot C$ where $C = p \cdot 1$       |
| 29 | If A is prime, and $B \neq n \cdot A$ , then A,B are relatively prime                                    | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$       |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a$ , $1/b$ , $1/c$ |
| 31 | If $A = B \times C$ , then $A = j \cdot D$ where D is prime  |    |   |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime                            |    |   |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C                  |    |   |
| 34 | Find the lowest common denominator of 2 numbers  |    |   |
| 35 | If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$ , then $C = i \cdot E$  |    |   |
| 36 | Find the least common multiple of 3 numbers  |    |   |



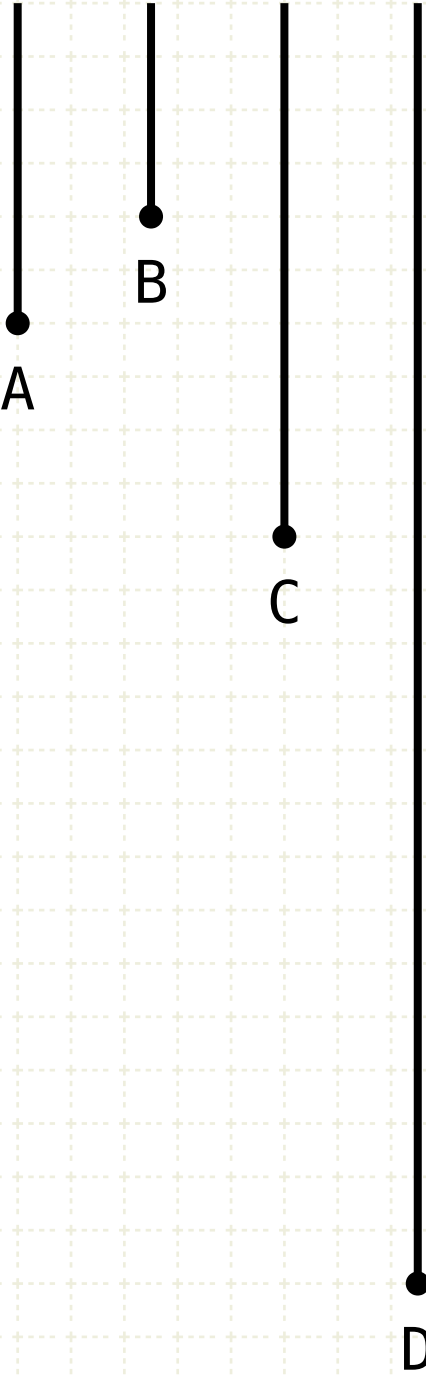
# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\begin{aligned} \gcd(A,C) &= 1 \\ \gcd(B,C) &= 1 \\ D &= A \times B \end{aligned}$$

## In other words

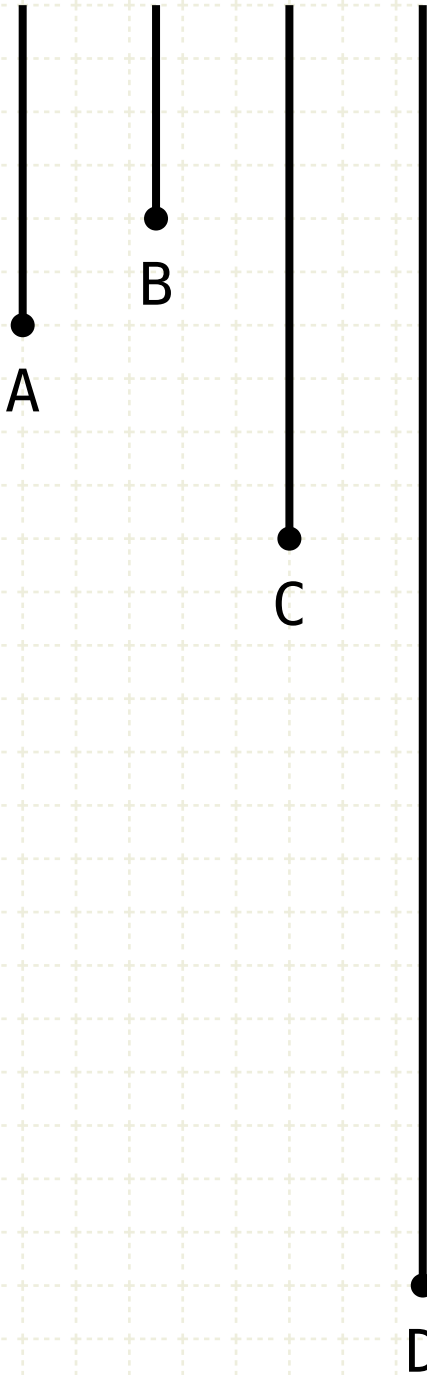
If A and B are both prime to C, and D is the product of A,B





# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A,C) = 1$$

$$\gcd(B,C) = 1$$

$$D = A \times B$$

$$\gcd(C,D) = 1$$

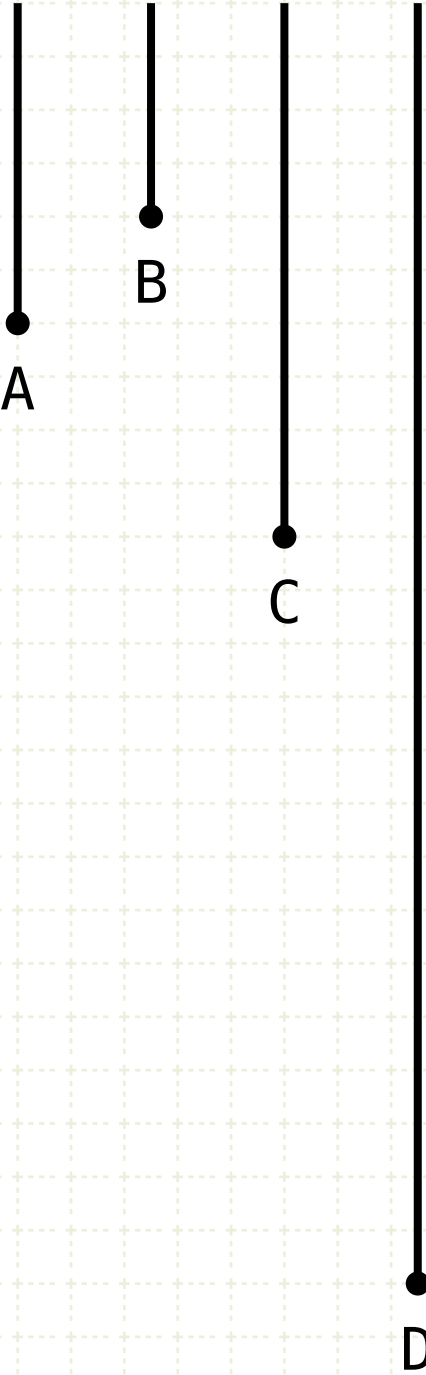
## In other words

If A and B are both prime to C, and D is the product of A,B  
Then C and D are prime to one another



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



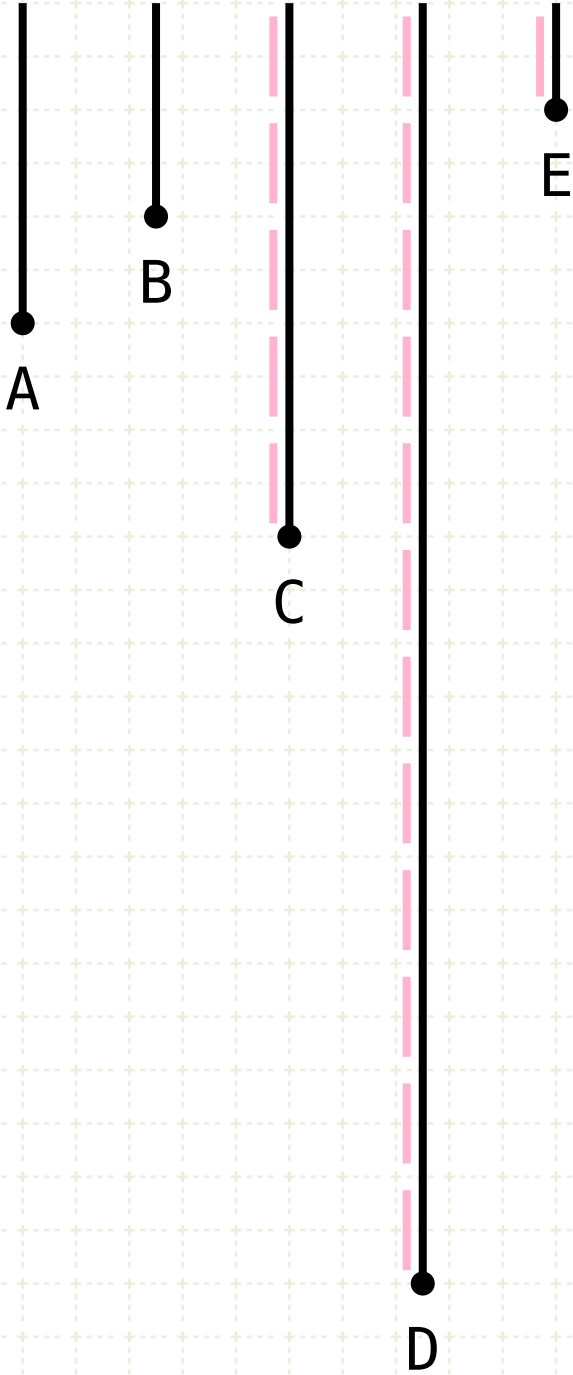
$$\begin{aligned} \gcd(A,C) &= 1 \\ \gcd(B,C) &= 1 \\ D &= A \times B \end{aligned}$$

## Proof by Contradiction



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

## Proof by Contradiction

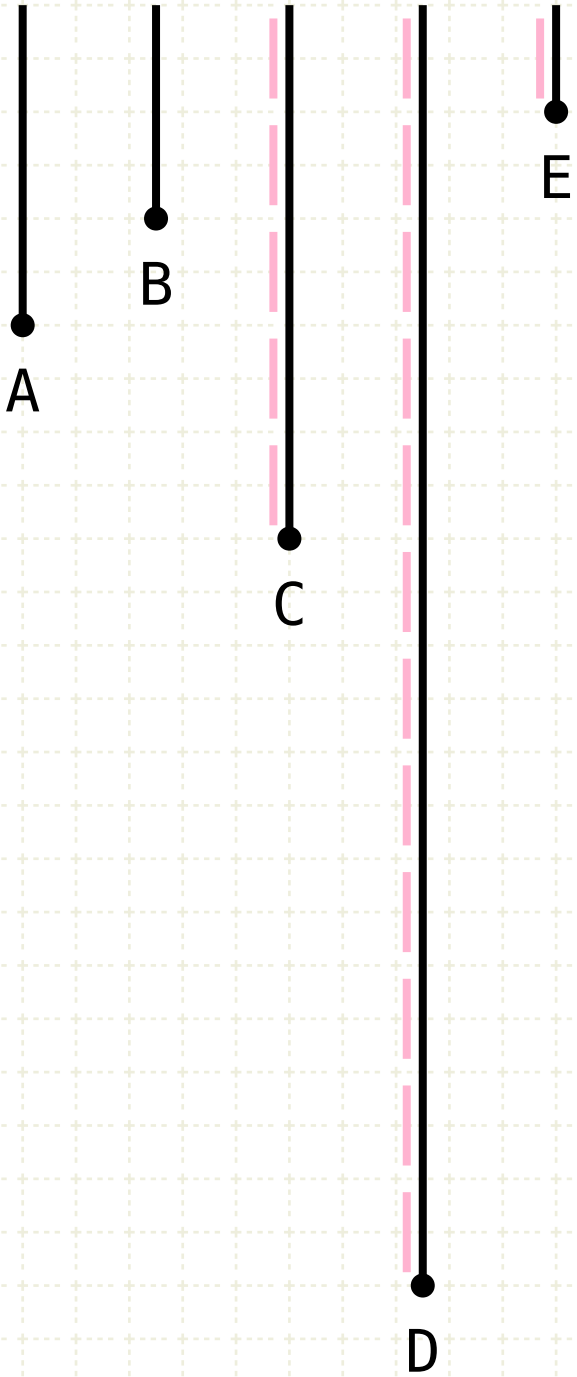
Assume that C and D are not prime to one another, and E measures both of them





# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A,C) = 1$$

$$\gcd(B,C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A,E) = 1$$

## Proof by Contradiction

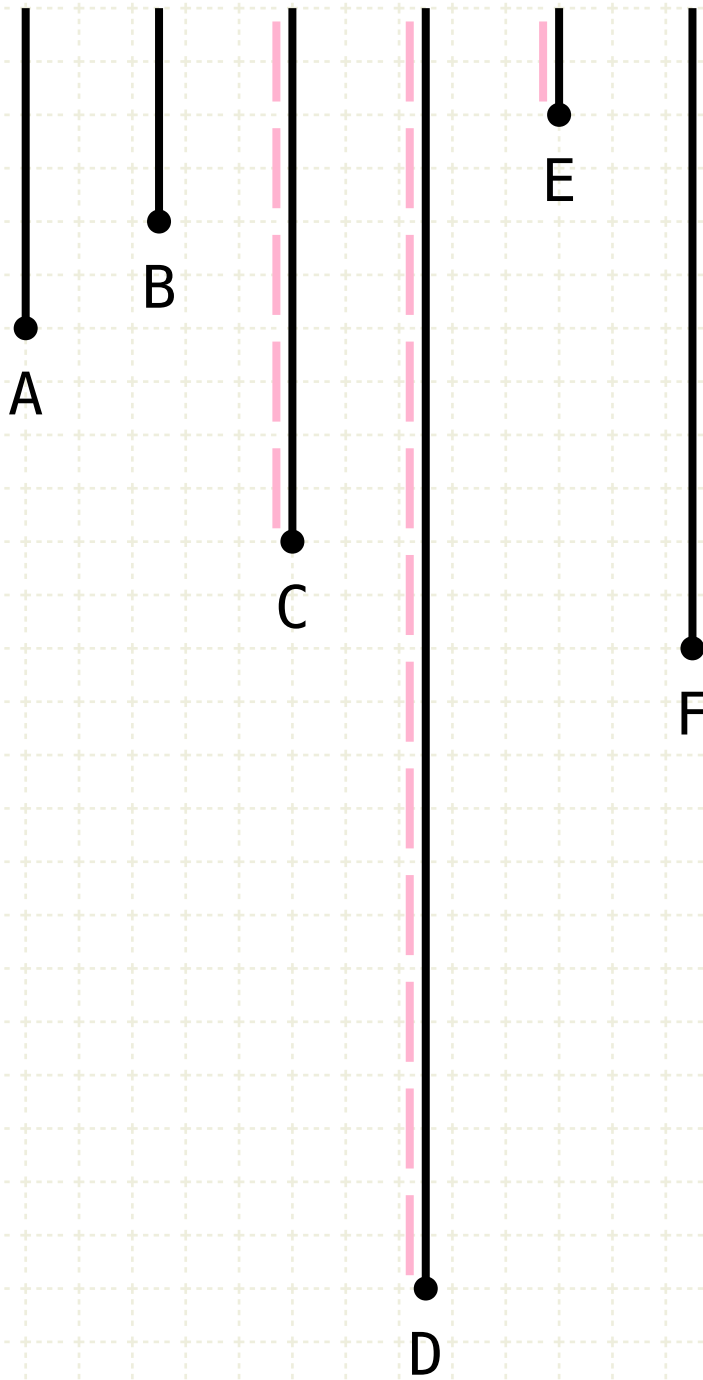
Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A,C) = 1$$

$$\gcd(B,C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A,E) = 1$$

$$F = f \cdot 1$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

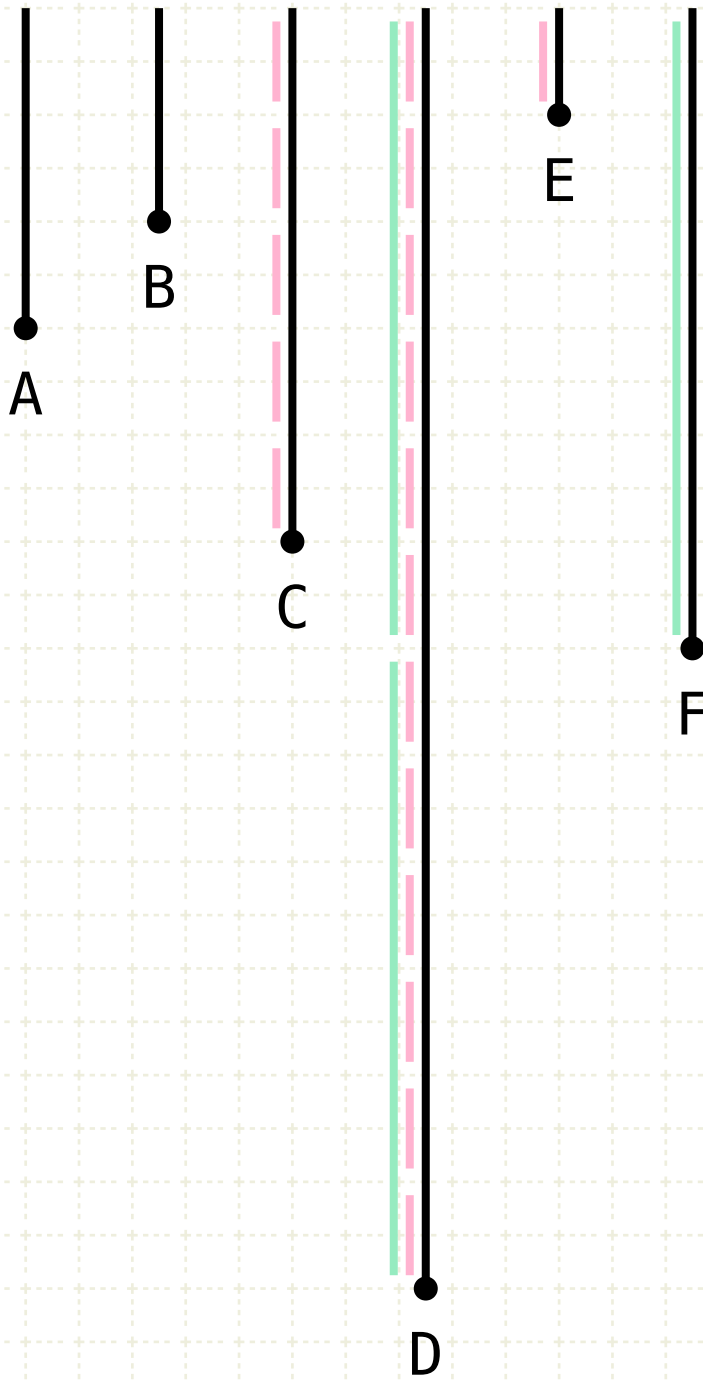
Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$F = f \cdot 1$$

$$E = e \cdot 1$$

$$D = e \cdot F$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

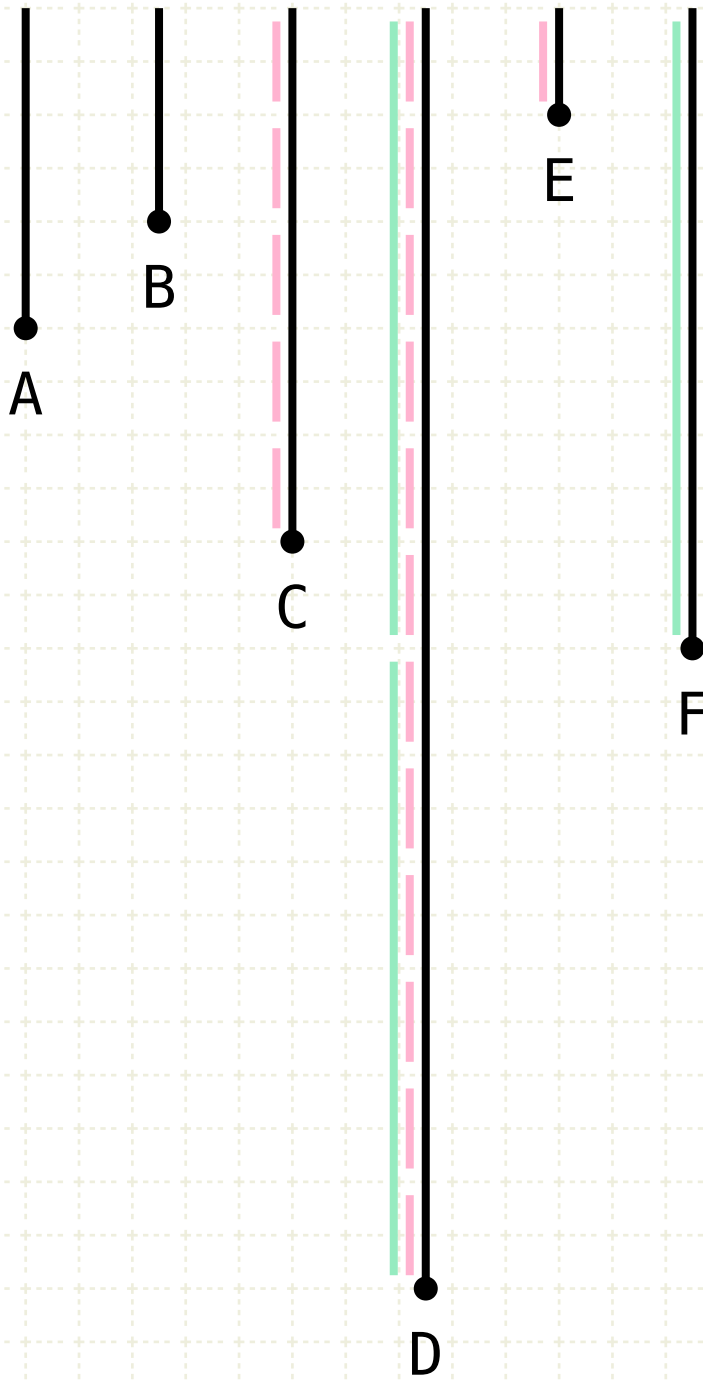
Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$F = f \cdot 1$$

$$E = e \cdot 1$$

$$D = e \cdot F$$

$$D = F \times E$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

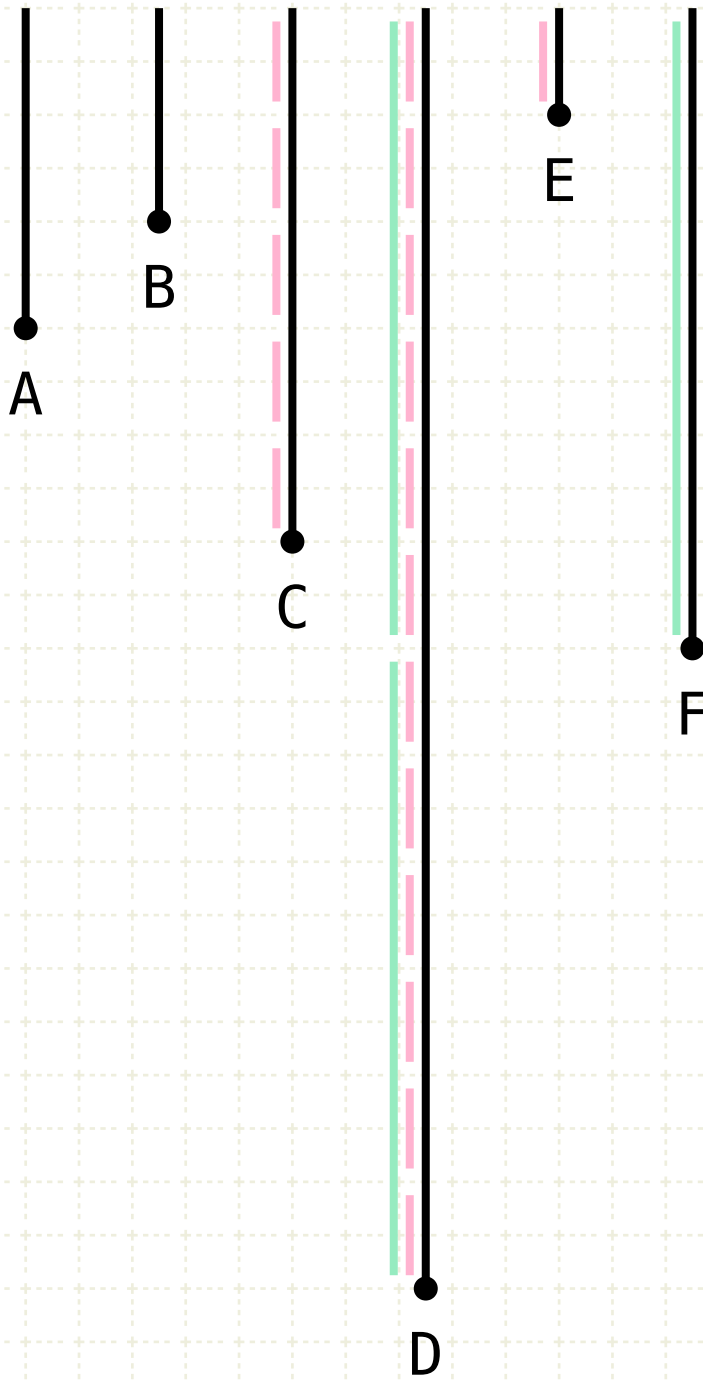
Therefore F also measures D according to the units in E (VII·16)

Therefore D equals F times E (VII.Def.15)



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$D = F \times E$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

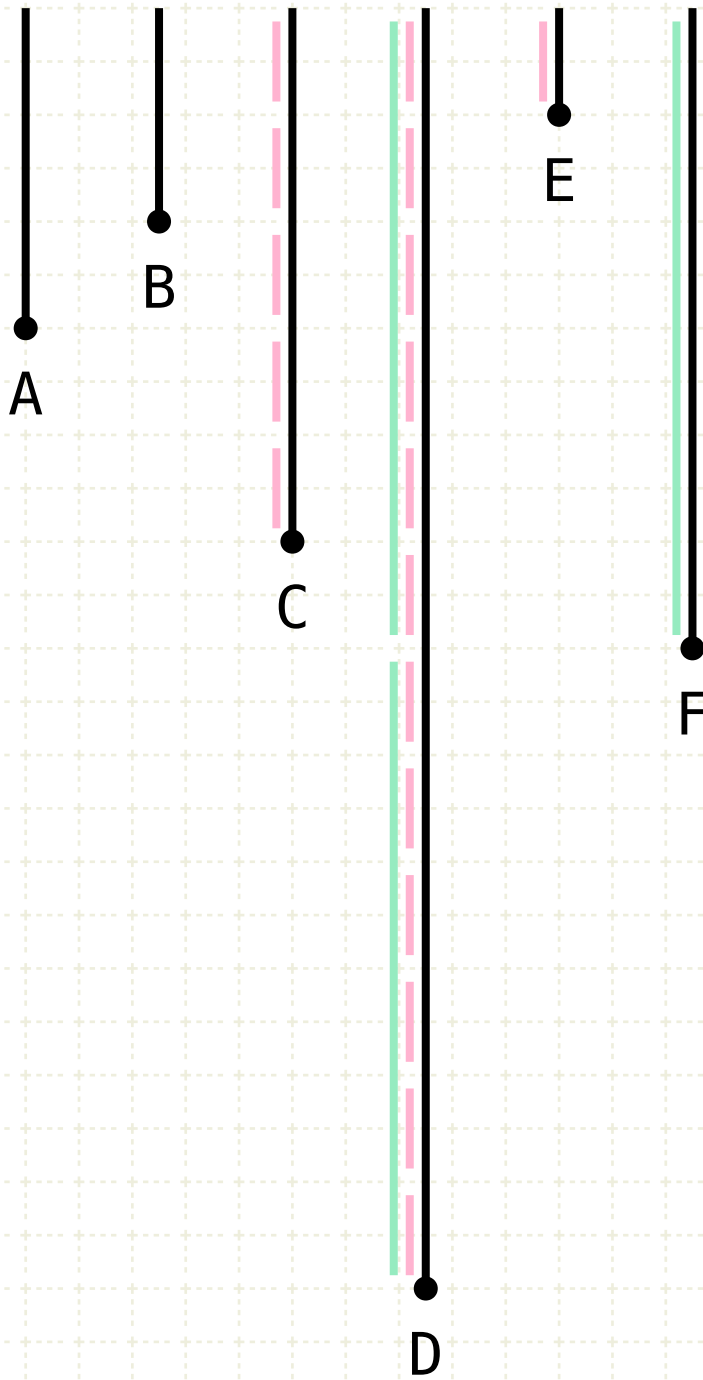
Therefore D equals F times E (VII.Def.15)





# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$D = F \times E$$

$$F \times E = A \times B$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

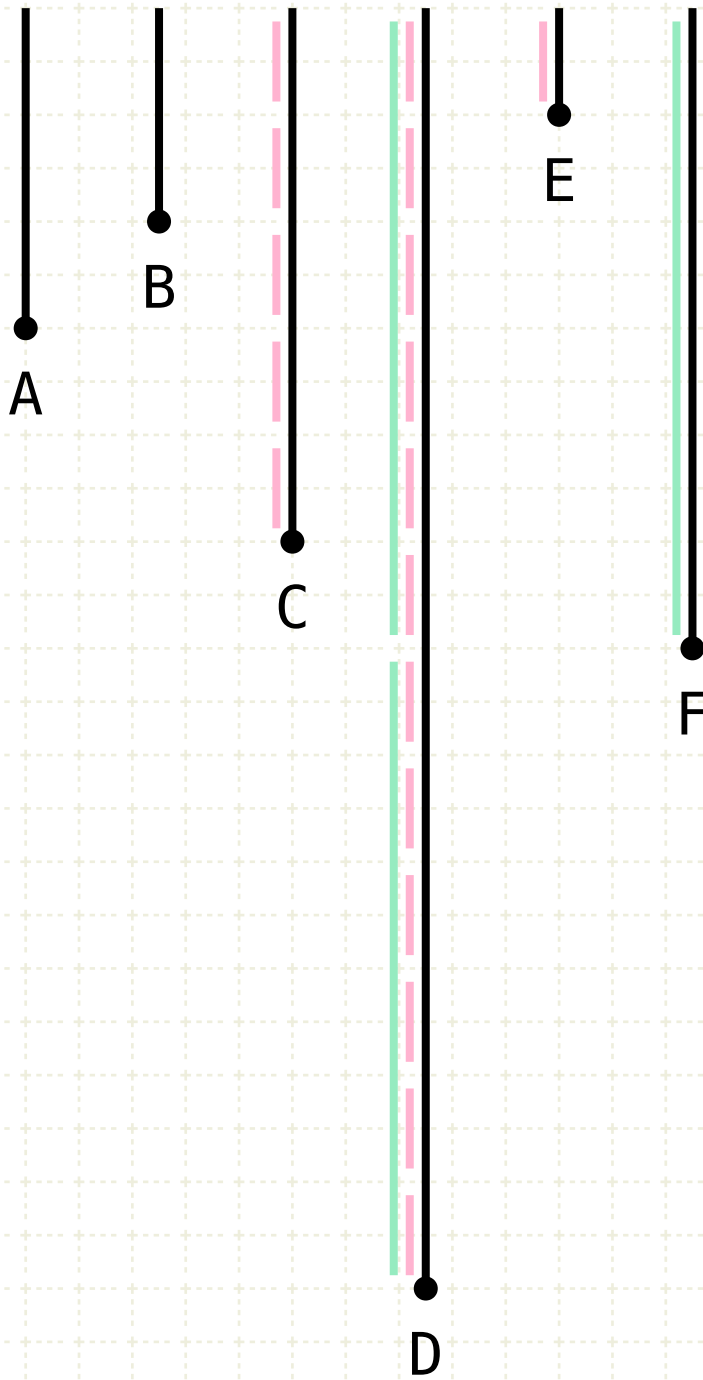
Therefore D equals F times E (VII.Def.15)

But D is also equal to A times B, therefore F times E equals A times B



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$D = F \times E$$

$$F \times E = A \times B$$

$$E : A = B : F$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

Therefore D equals F times E (VII.Def.15)

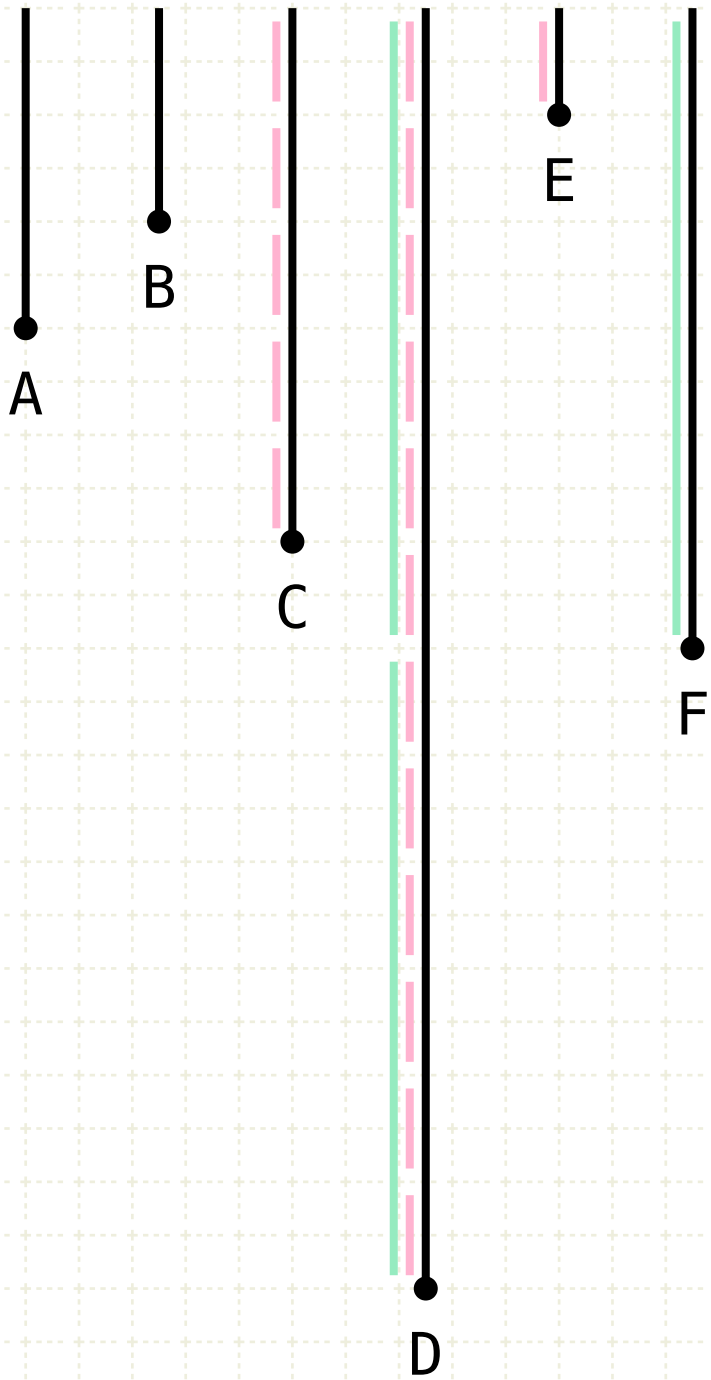
But D is also equal to A times B, therefore F times E equals A times B

If the product of the extremes are equal to that of the means, the four numbers are proportional, in other words E is to A as B is to F (VII·19)



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A,C) = 1$$

$$\gcd(B,C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A,E) = 1$$

$$D = F \times E$$

$$F \times E = A \times B$$

$$E:A = B:F$$

$$S = \{ (x,y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x:y=E:A \}$$
$$(E:A) \in S \text{ such that } E \leq x, A \leq y, \forall (x,y) \in S$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

Therefore D equals F times E (VII.Def.15)

But D is also equal to A times B, therefore F times E equals A times B

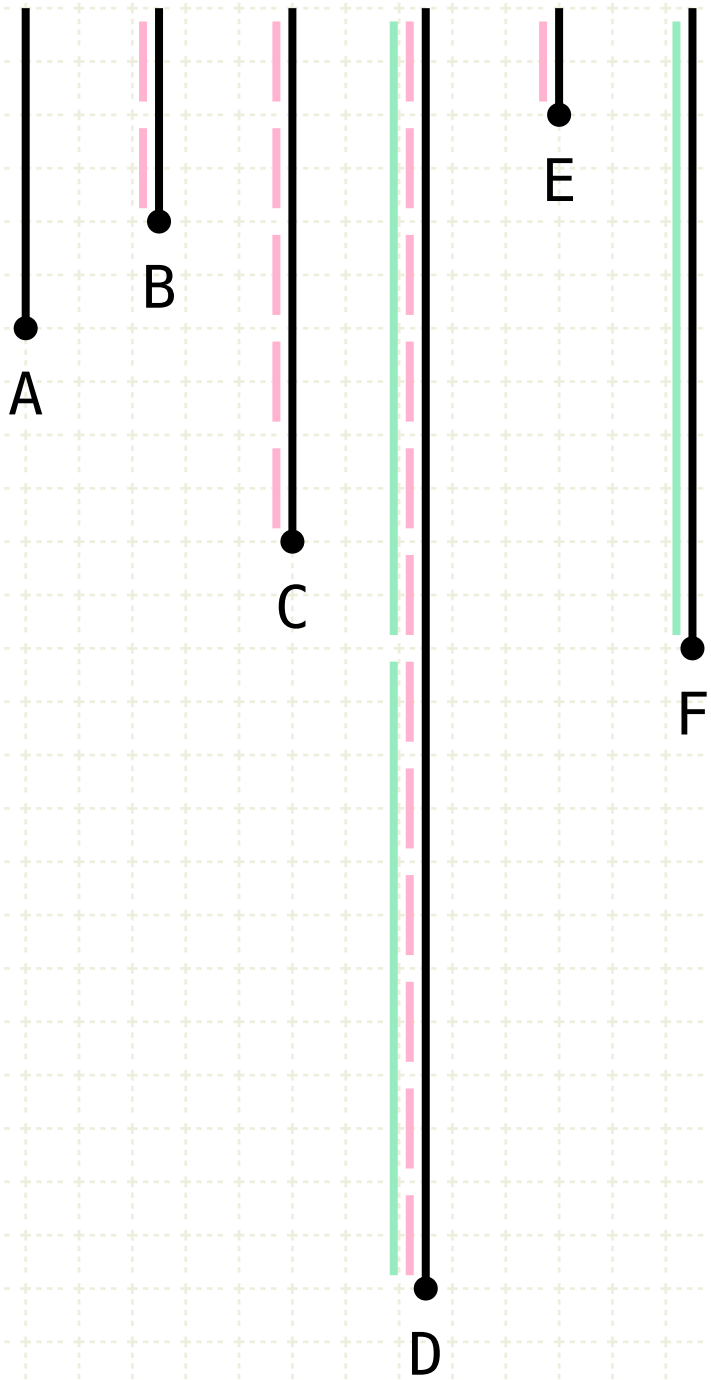
If the product of the extremes are equal to that of the means, the four numbers are proportional, in other words E is to A as B is to F (VII·19)

A and E are relatively prime, so therefore they are the smallest numbers which can represent the ratio E to A (VII·21)



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$D = F \times E$$

$$F \times E = A \times B$$

$$E : A = B : F$$

$$S = \{ (x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x : y = E : A \}$$
$$(E : A) \in S \text{ such that } E \leq x, A \leq y, \forall (x, y) \in S$$

$$B = r \cdot E$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

Therefore D equals F times E (VII.Def.15)

But D is also equal to A times B, therefore F times E equals A times B

If the product of the extremes are equal to that of the means, the four numbers are proportional, in other words E is to A as B is to F (VII·19)

A and E are relatively prime, so therefore they are the smallest numbers which can represent the ratio E to A (VII·21)

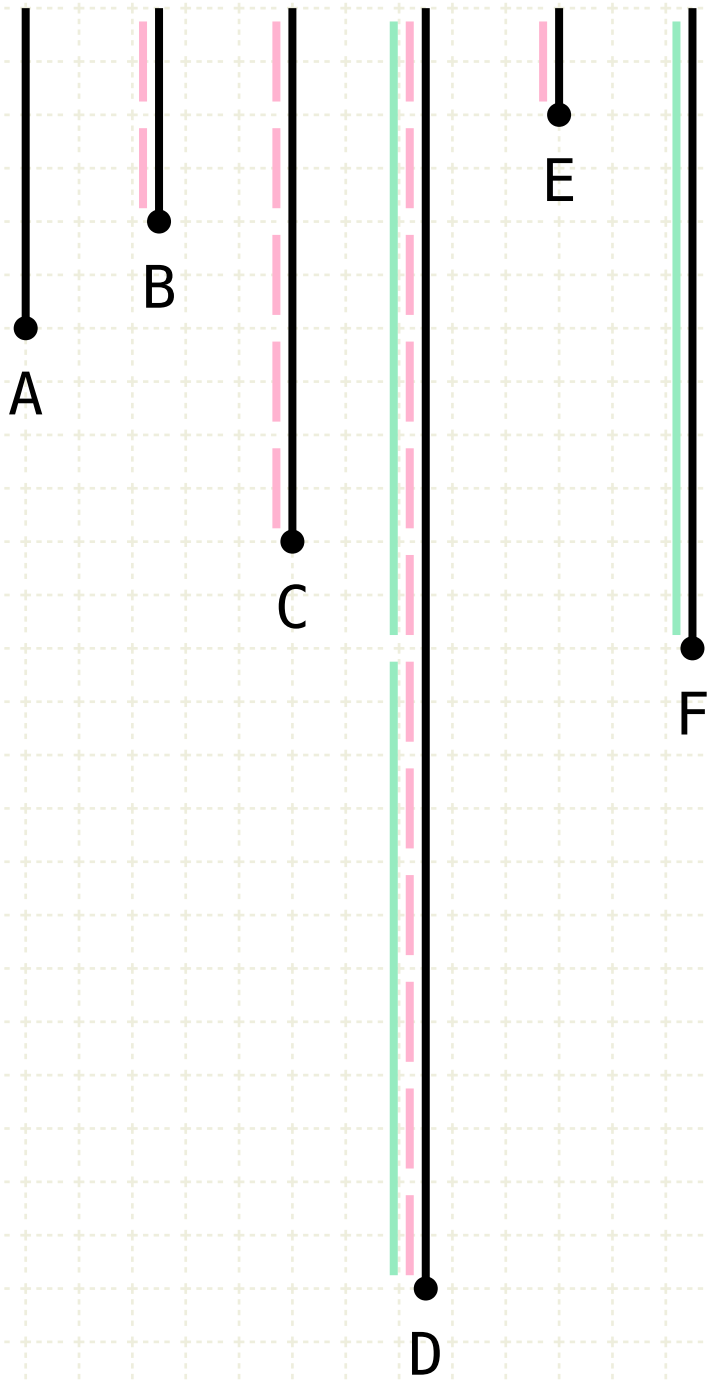
Since A and E are the smallest numbers in the ratio of B and F, E will measure B (and A will measure F) (VII·20)





# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$D = F \times E$$

$$F \times E = A \times B$$

$$E : A = B : F$$

$$S = \{ (x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x : y = E : A \}$$

$$(E : A) \in S \text{ such that } E \leq x, A \leq y, \forall (x, y) \in S$$

$$B = r \cdot E$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

Therefore D equals F times E (VII.Def.15)

But D is also equal to A times B, therefore F times E equals A times B

If the product of the extremes are equal to that of the means, the four numbers are proportional, in other words E is to A as B is to F (VII·19)

A and E are relatively prime, so therefore they are the smallest numbers which can represent the ratio E to A (VII·21)

Since A and E are the smallest numbers in the ratio of B and F, E will measure B (and A will measure F) (VII·20)

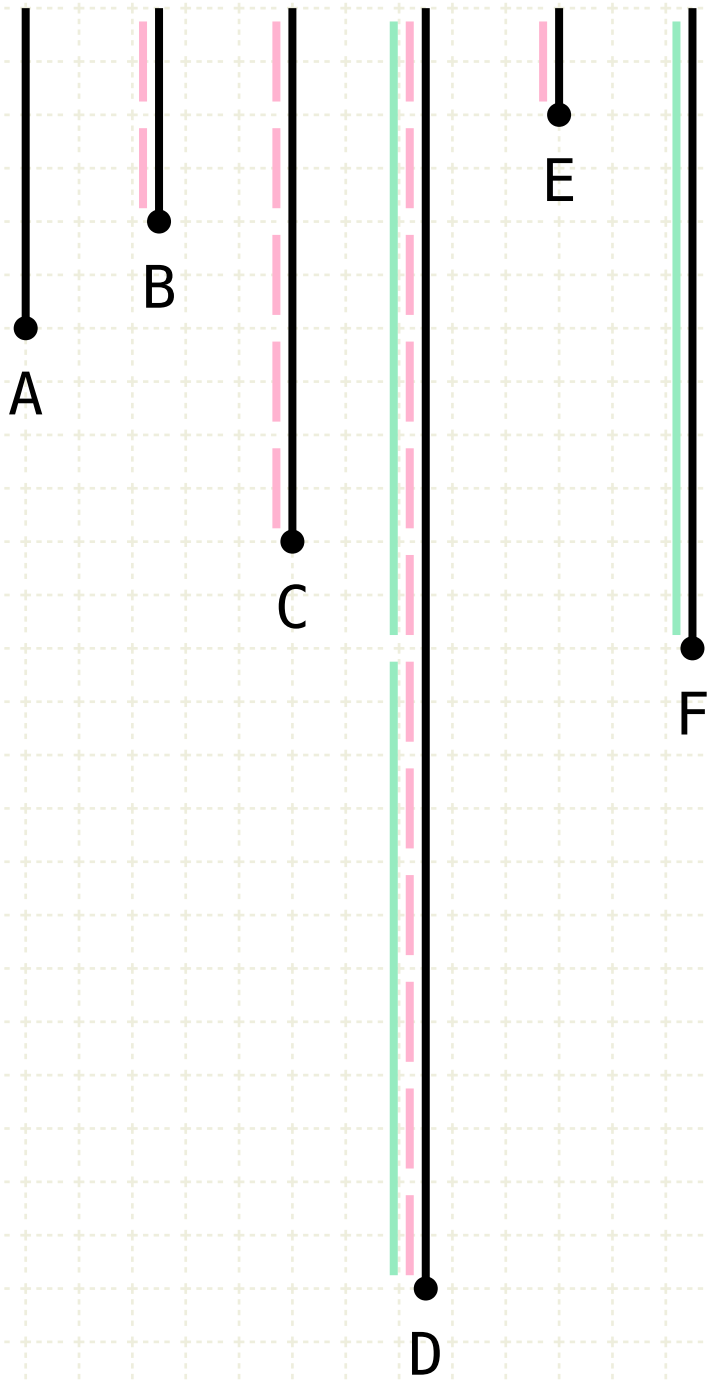
But E also measures C, which is impossible since B,C are relatively prime (VII.Def.12)





# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$D = F \times E$$

$$F \times E = A \times B$$

$$E : A = B : F$$

$$S = \{ (x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x : y = E : A \}$$

$$(E : A) \in S \text{ such that } E \leq x, A \leq y, \forall (x, y) \in S$$

$$B = r \cdot E$$

$$\gcd(C, D) = 1$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

Therefore D equals F times E (VII.Def.15)

But D is also equal to A times B, therefore F times E equals A times B

If the product of the extremes are equal to that of the means, the four numbers are proportional, in other words E is to A as B is to F (VII·19)

A and E are relatively prime, so therefore they are the smallest numbers which can represent the ratio E to A (VII·21)

Since A and E are the smallest numbers in the ratio of B and F, E will measure B (and A will measure F) (VII·20)

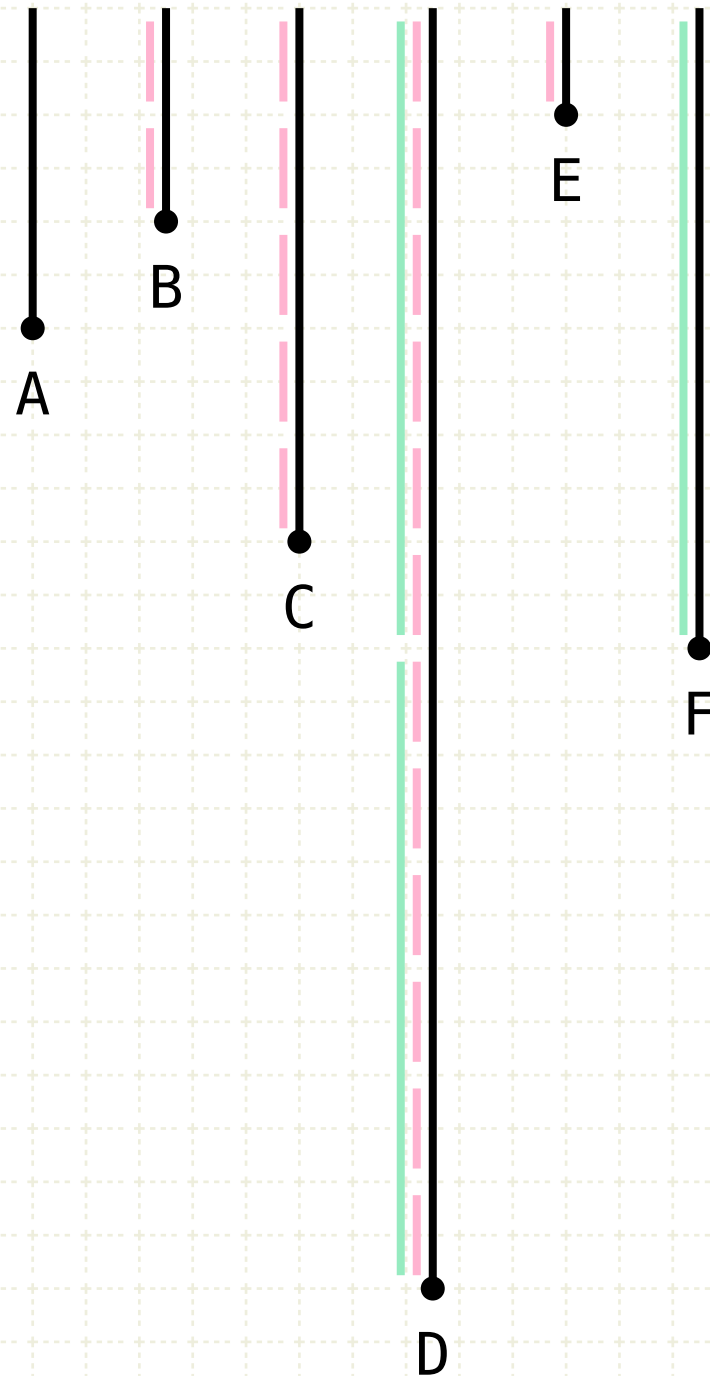
But E also measures C, which is impossible since B,C are relatively prime (VII.Def.12)

Therefore no number E measures C and D, which means that C and D are relatively prime



# Proposition 24 of Book VII

If two numbers be prime to any number, their product also will be prime to the same



$$\gcd(A, C) = 1$$

$$\gcd(B, C) = 1$$

$$D = A \times B$$

$$C = k \cdot E$$

$$D = f \cdot E$$

$$\gcd(A, E) = 1$$

$$D = F \times E$$

$$F \times E = A \times B$$

$$E : A = B : F$$

$$S = \{ (x, y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x : y = E : A \}$$

$$(E : A) \in S \text{ such that } E \leq x, A \leq y, \forall (x, y) \in S$$

$$B = r \cdot E$$

$$\gcd(C, D) = 1$$

## Proof by Contradiction

Assume that C and D are not prime to one another, and E measures both of them

Since A and C are relatively prime, and E measures C, then A and E are also relatively prime (VII·23)

Let F be equal to the number of times that E measures D

Therefore F also measures D according to the units in E (VII·16)

Therefore D equals F times E (VII.Def.15)

But D is also equal to A times B, therefore F times E equals A times B

If the product of the extremes are equal to that of the means, the four numbers are proportional, in other words E is to A as B is to F (VII·19)

A and E are relatively prime, so therefore they are the smallest numbers which can represent the ratio E to A (VII·21)

Since A and E are the smallest numbers in the ratio of B and F, E will measure B (and A will measure F) (VII·20)

But E also measures C, which is impossible since B, C are relatively prime (VII.Def.12)

Therefore no number E measures C and D, which means that C and D are relatively prime



# Youtube Videos

<https://www.youtube.com/c/SandyBultena>

*Copyright © 2019 by Sandy Bultena.*



Except where otherwise noted, this work is licensed under  
<http://creativecommons.org/licenses/by-nc/3.0>