

# Euclid's Elements

## Book VII

### Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

*As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.*

**Joseph-Louis Lagrange**  
**(1736 to 1813)**



# Table of Contents, Chapter 7

1	Determine if two numbers are relatively prime	10	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$	21	If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B
2	Find the greatest common divisor for two numbers	11	If $A:B = C:D$ , then $(A-C):(B-D) = A:B$	22	If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime
3	Find the largest common divisor for three numbers	12	If $A:B = C:D$ , then $(A+C):(B+C) = A:B$	23	If A,B are relatively prime and if $A = n \cdot C$ , then B,C are relatively prime
4	<b>Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B</b>	13	If $A:B = C:D$ , then $A:C = B:D$	24	If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C
5	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , then $(B+D) = (1/q) \cdot (A+C)$	14	If $A:B = D:E$ and $B:C = E:F$ , then $A:C = D:F$	25	If A,B are relatively prime then $A^2, B$ are relatively prime
6	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , then $(B+D) = (p/q) \cdot (A+C)$	15	If $B = i \cdot 1$ and $E = i \cdot D$ , and if $D = j \cdot 1$ then $E = j \cdot B$	26	If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$
7	If $B = A/q$ and $D = C/q$ , $B > D$ , then $(B-D) = (A-C)/q$	16	$A \times B = B \times A$	27	If A,B are relatively prime, then $A^2, B^2$ are relatively prime, and $A^3, B^3$ are relatively prime, and so on
8	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , $B > D$ , then $(B-D) = (p/q) \cdot (A-C)$	17	If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$		
9	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$	18	If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$		
		19	If $A:B = C:D$ then $A \times D = B \times C$ If $A \times D = B \times C$ then $A:B = C:D$		
		20	Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$		



## Table of Contents, Chapter 7

- |    |  |    |   |
|----|--|----|---|
| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime   | 37 | If $A = p \cdot B$ , then $A = q \cdot C$ where $C = p \cdot 1$       |
| 29 | If A is prime, and $B \neq n \cdot A$ , then A,B are relatively prime                                    | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$       |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a$ , $1/b$ , $1/c$ |
| 31 | If $A = B \times C$ , then $A = j \cdot D$ where D is prime  |    |   |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime                            |    |   |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C                  |    |   |
| 34 | Find the lowest common denominator of 2 numbers  |    |   |
| 35 | If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$ , then $C = i \cdot E$  |    |   |
| 36 | Find the least common multiple of 3 numbers  |    |   |



# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater



# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater

## Definitions

3. A number is a 'part' of a number, the less of the greater, when it measures the greater

$$A = 10, B = 2,$$

B is part of A

$$A = B + B + B + B + B$$

4. but 'parts' when it does not measure it

$$A=10, B=6$$

Let the part of A be 2

$$p = 2, A = p + p + p + p + p$$

B is a multiple of the part of A (B is parts of A)

$$B = p + p + p$$





# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater

$$\begin{aligned} (A, B) &\in \mathbb{N} \\ \exists (p, m, n) &\in \mathbb{N} \text{ such that} \\ A &= m \cdot p \\ B &= n \cdot p \end{aligned}$$

## In other words

Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B



# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater



## In other words

Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B

## Proof

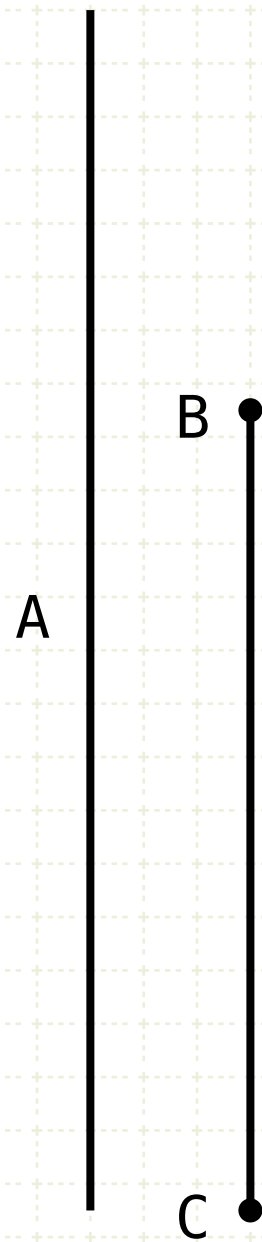
Either A, B are co-prime, or not



# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater

$$\gcd(A, BC) = 1$$



## In other words

Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B

## Proof

Either A, BC are co-prime, or not

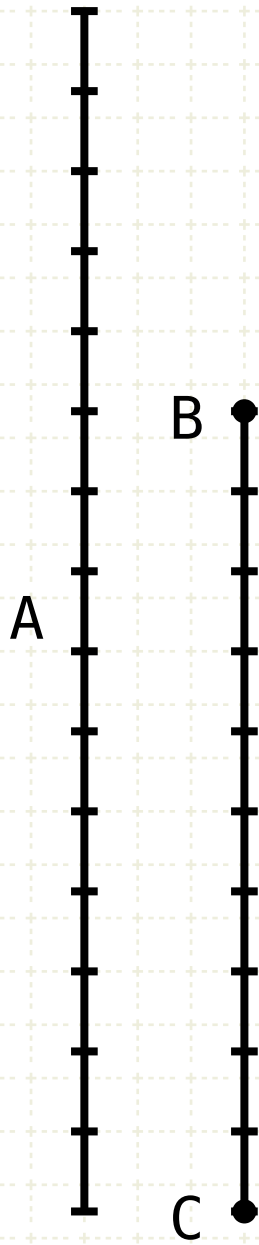
Assume A, BC are co-prime





# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater



$$\gcd(A, BC) = 1$$

$$u = 1$$

$$A = q \cdot u$$

$$BC = p \cdot u$$

## In other words

Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B

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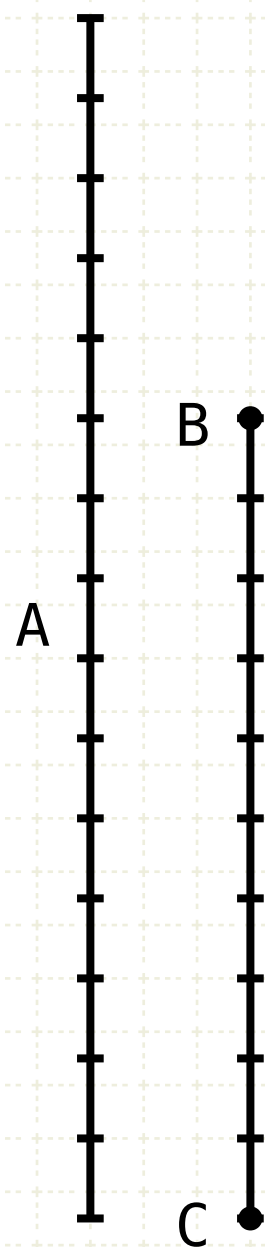
Assume A, BC are co-prime

Divide BC into individual units (the number 1)



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## Proof

Either A, BC are co-prime, or not

Assume A, BC are co-prime

Divide BC into individual units (the number 1)

Each unit in B will be some part of A, since '1' measures A, so BC will be some parts of A

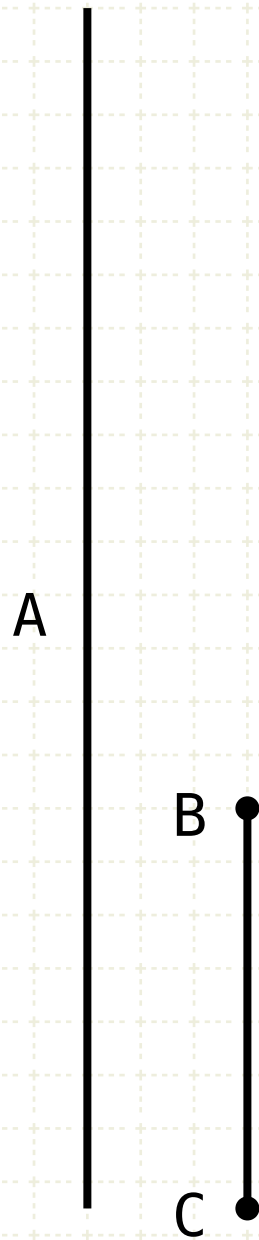


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Any number is either a part or parts of any number, the less of the greater

$$\gcd(A, BC) = BC$$

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## Proof

Either A, BC are co-prime, or not

Assume A, BC are not co-prime and BC measures A



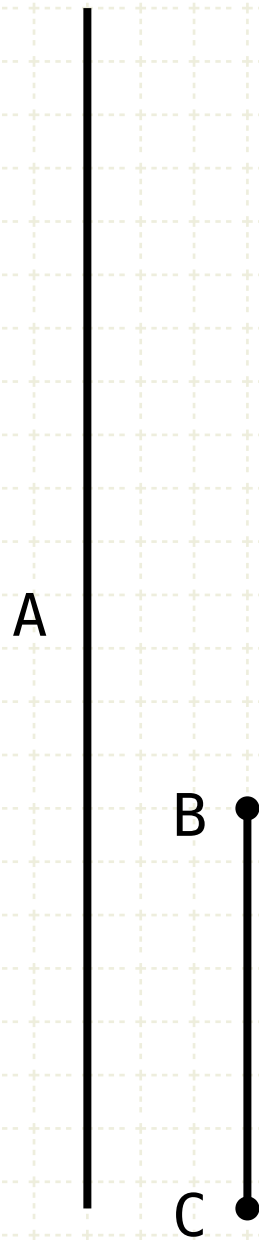
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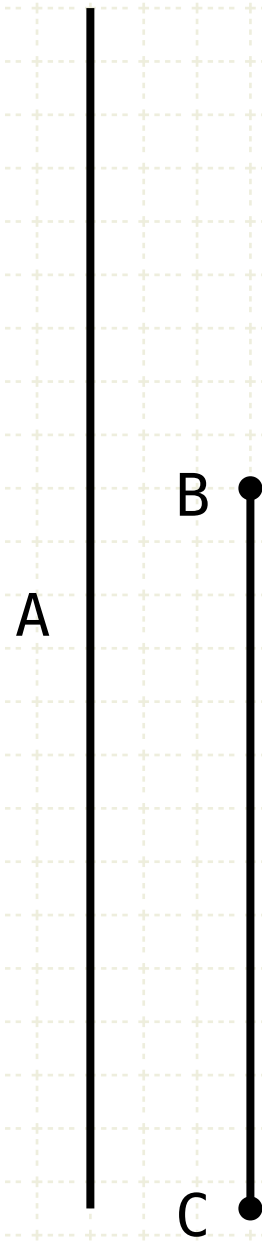
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If BC measures A, BC is a part of A



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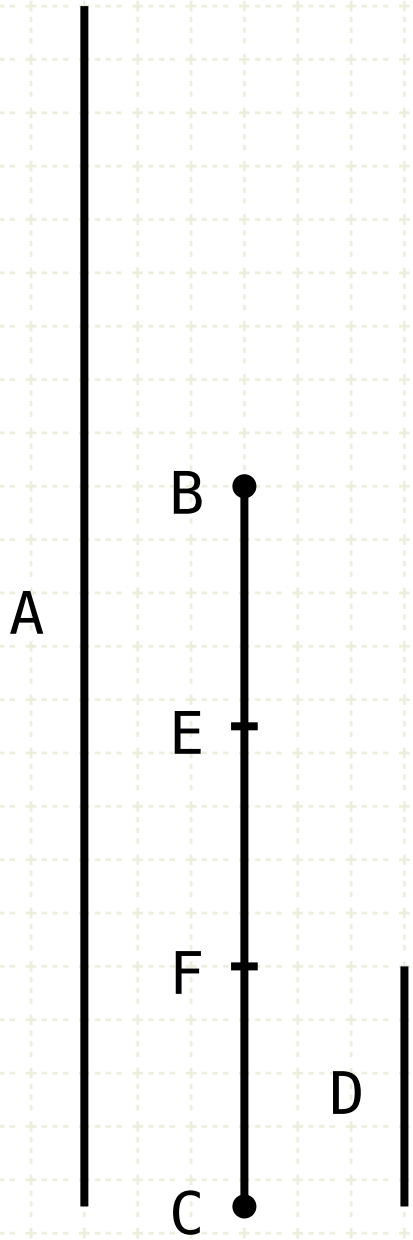
Assume A, BC are not co-prime and BC does not measure A





# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater



$$\gcd(A, BC) = D$$

$$BE = EF = FC = D$$

$$BC = BE + EF + FC$$

## In other words

Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B

## Proof

Either A, BC are co-prime, or not

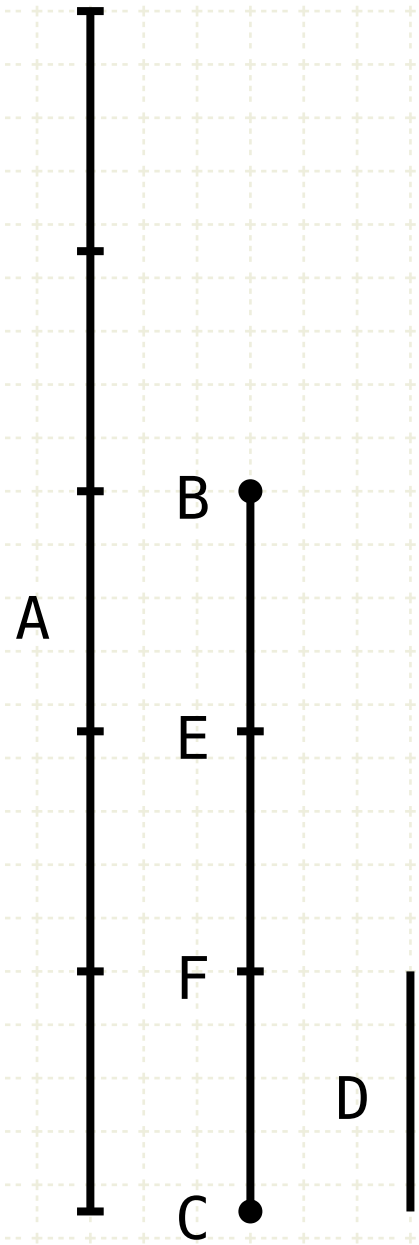
Assume A, BC are not co-prime and BC does not measure A

Find the largest common divisor D (VII·2), and divide BC into the numbers equal to D, namely BE, EF, FC



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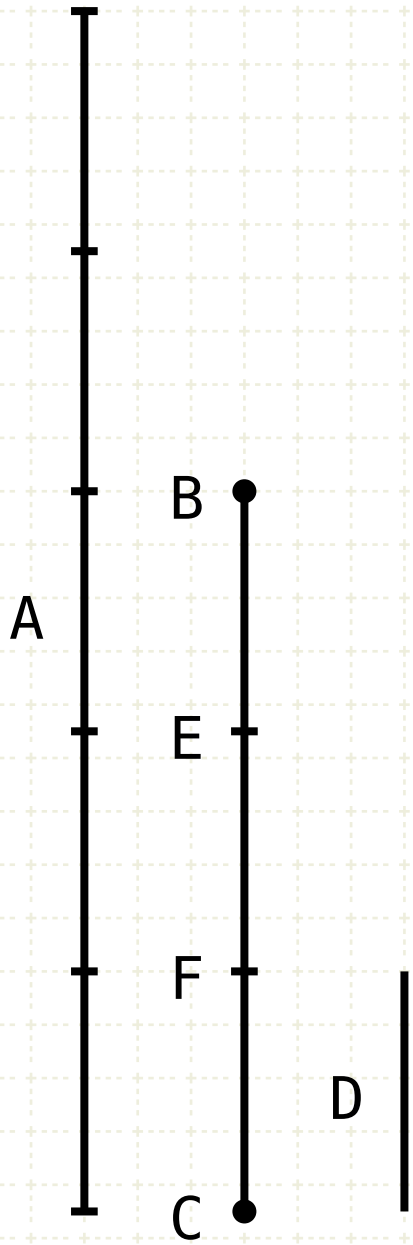
Find the largest common divisor D (VII·2), and divide BC into the numbers equal to D, namely BE, EF, FC

Since D measures A, D is a part of A



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$$\gcd(A, BC) = D$$

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Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B

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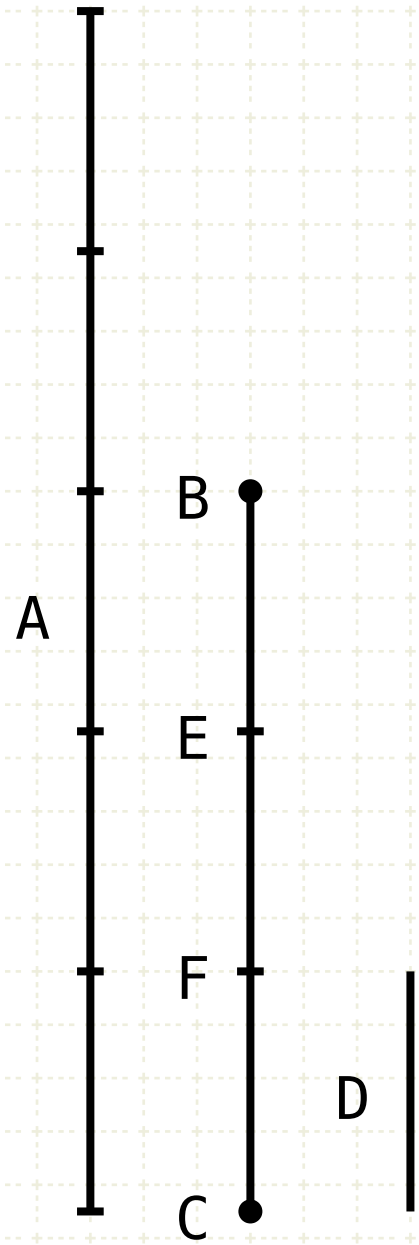
Since D measures A, D is a part of A

But BE,EF,FC also equal D



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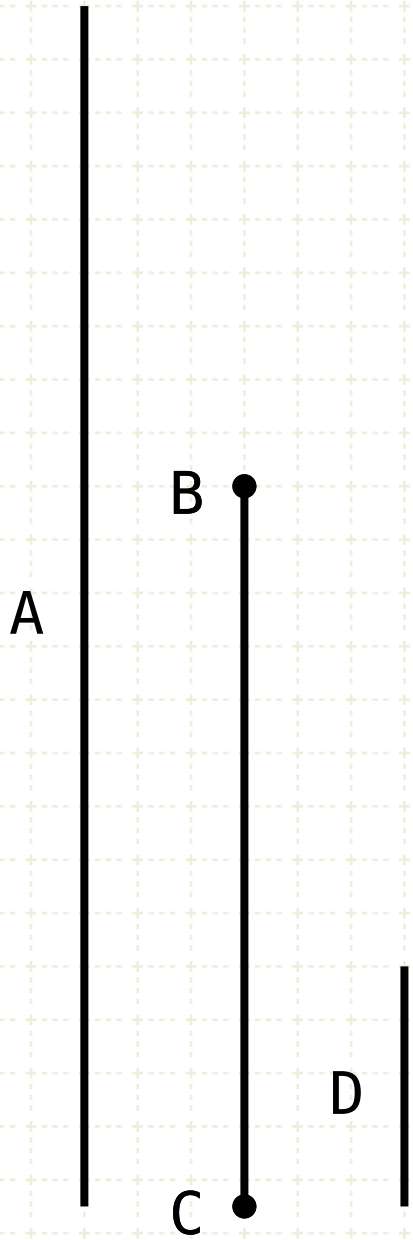
Since D measures A, D is a part of A

But BE, EF, FC also equal D

D is a part of A, so BC is a sum of the parts of A

# Proposition 4 of Book VII

Any number is either a part or parts of any number, the less of the greater



$$\gcd(A, BC) = 1$$

$$u = 1$$

$$A = q \cdot u$$

$$BC = p \cdot u$$

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Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B





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