

Euclid's Elements

Book VI

One can state, without exaggeration, that the observation of and the search for similarities and differences are the basis of all human knowledge.

Alfred Nobel



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2	If a line cuts a triangle, parallel to its base, it will cut the sides of the triangle proportionally			15	In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and vice versa
3	If an angle of a triangle is bisected and the straight line cutting the angle also cuts the base, the segments of the base will have the same ratio as the remaining sides of the triangle	8	If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another	16	If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means, and vice versa
4	If two triangles have equal angles, then the sides opposite the equal angles are proportional, as well, the sides of the triangles on either side of the equal angles are also proportional	9	From a given straight line to cut off a given fraction	17	If three straight lines are proportional, the rectangle contained by the extremes is equal to the square on the mean; and vice versa
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6	If two triangles have one angle equal to one angle and the sides about the equal angles are proportional, then the triangles will be equiangular	11	To two given straight lines to find a third proportional	19	Similar triangles are to one another in the duplicate ratio of the corresponding sides
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Table of Contents, Chapter 3

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21	Figures which are are similar to the same rectilineal figure are also similar to one another	27	Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar to a parallelogram drawn on half the said line, the largest will be one that is drawn on half of the straight line and is similar to the defect		
22	If four straight lines are proportional, similar rectilineal figures will also be proportional; and vice versa	28	To a given straight line, apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one		
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25	To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure				



Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



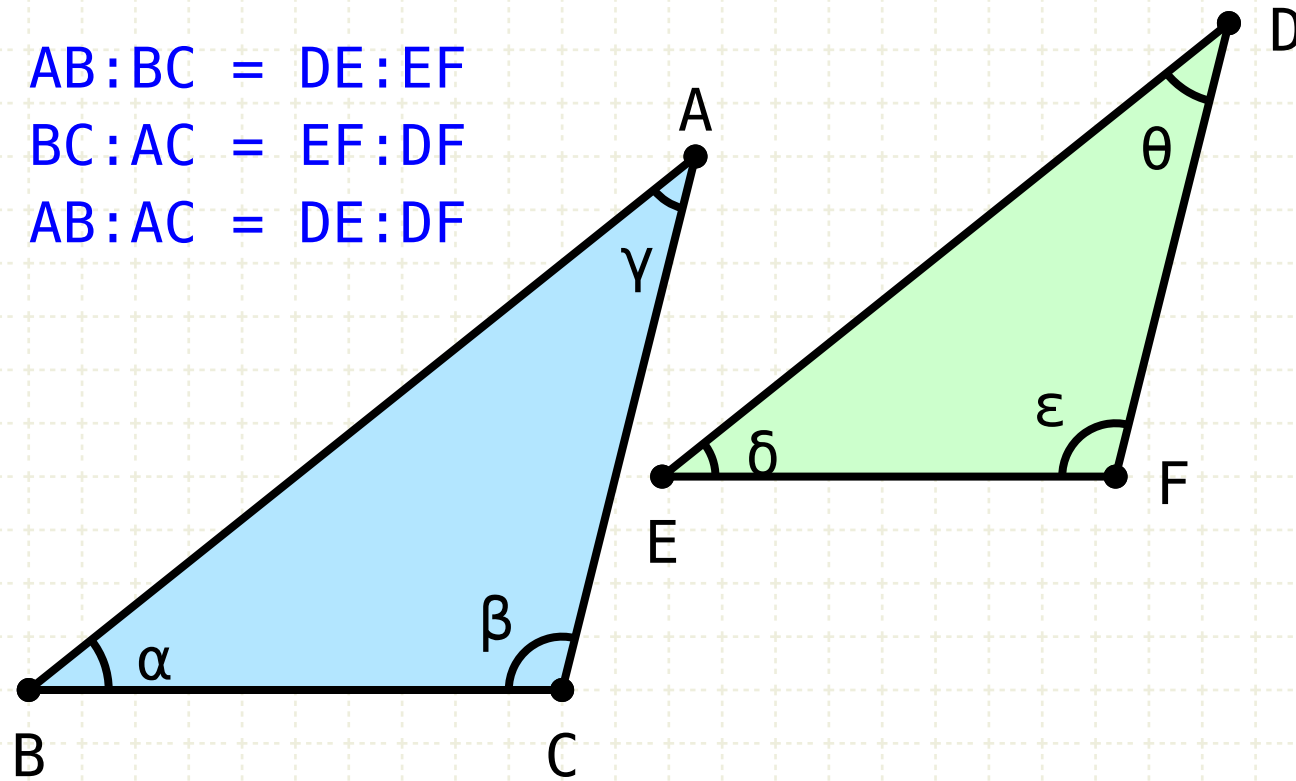
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$$AB:BC = DE:EF$$

$$BC:AC = EF:DF$$

$$AB:AC = DE:DF$$



$$\alpha = \delta, \beta = \epsilon, \gamma = \theta$$

In other words

If side 'a' is to side 'b' of one triangle, and is equal to side 'd' to 'e' of another, and similarly for all sides, then the angle between 'a' and 'b' will be equal to the angles between 'd' and 'e'

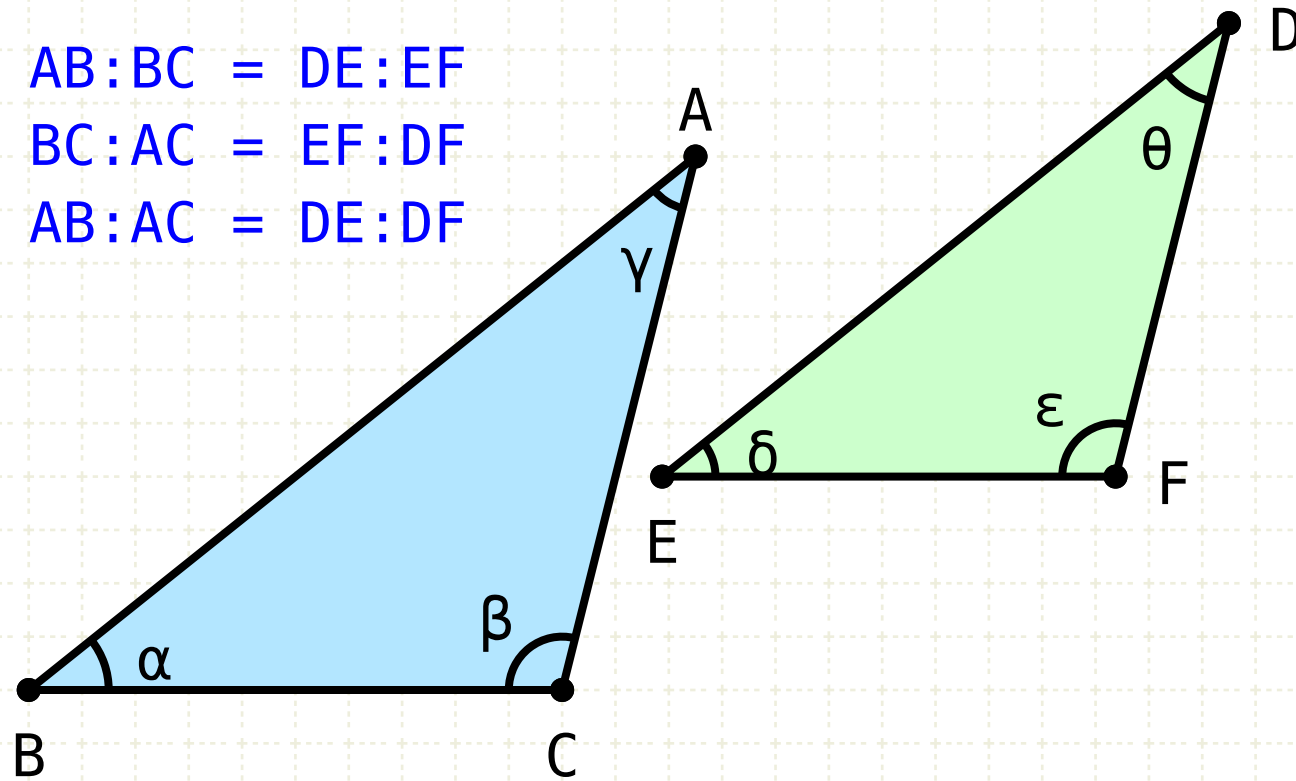
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Proof

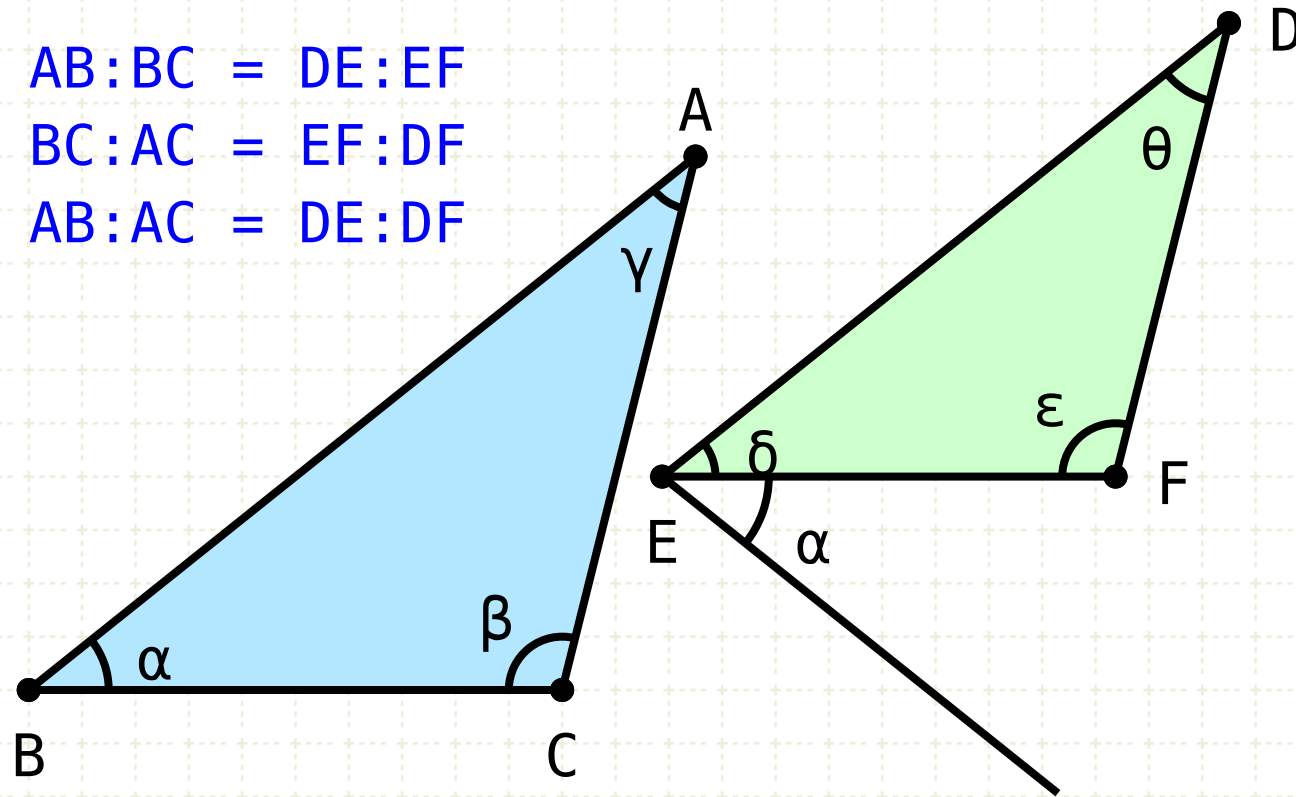
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Proof

On the point E, construct an angle FEG on the line EF equal to the angle α (I-23)

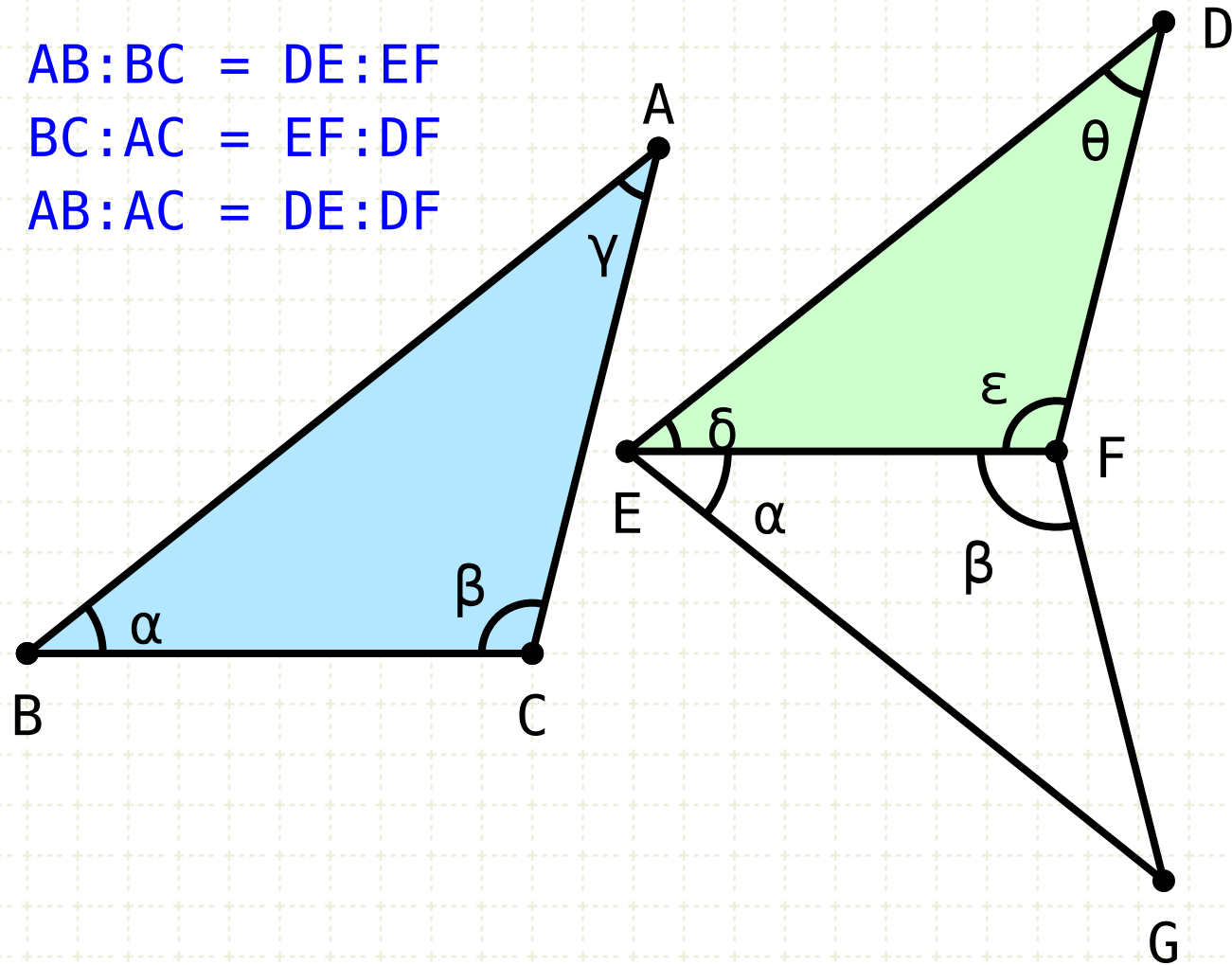
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Proof

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On the point F, construct an angle EFG on the line EF equal to the angle β (I-23)

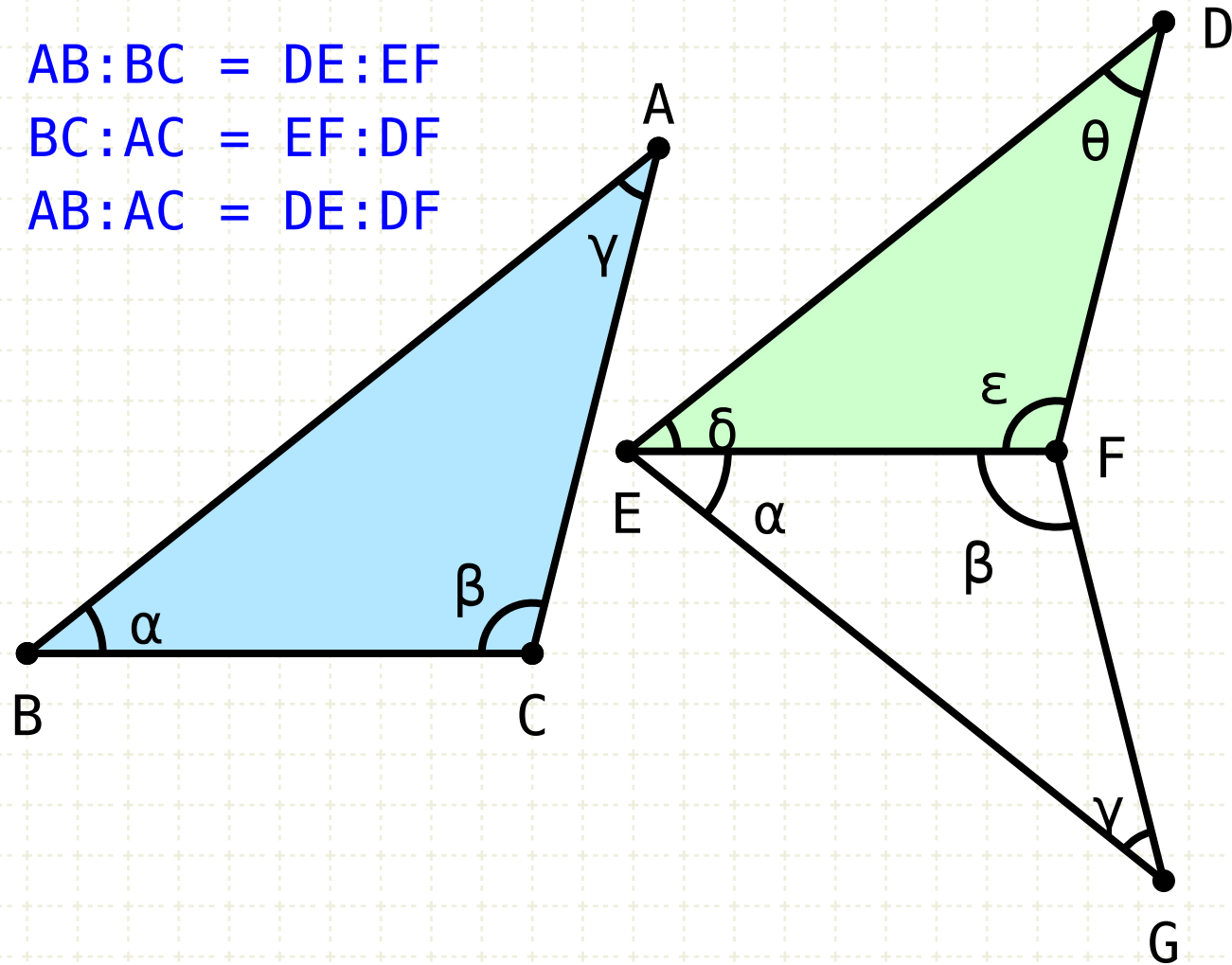
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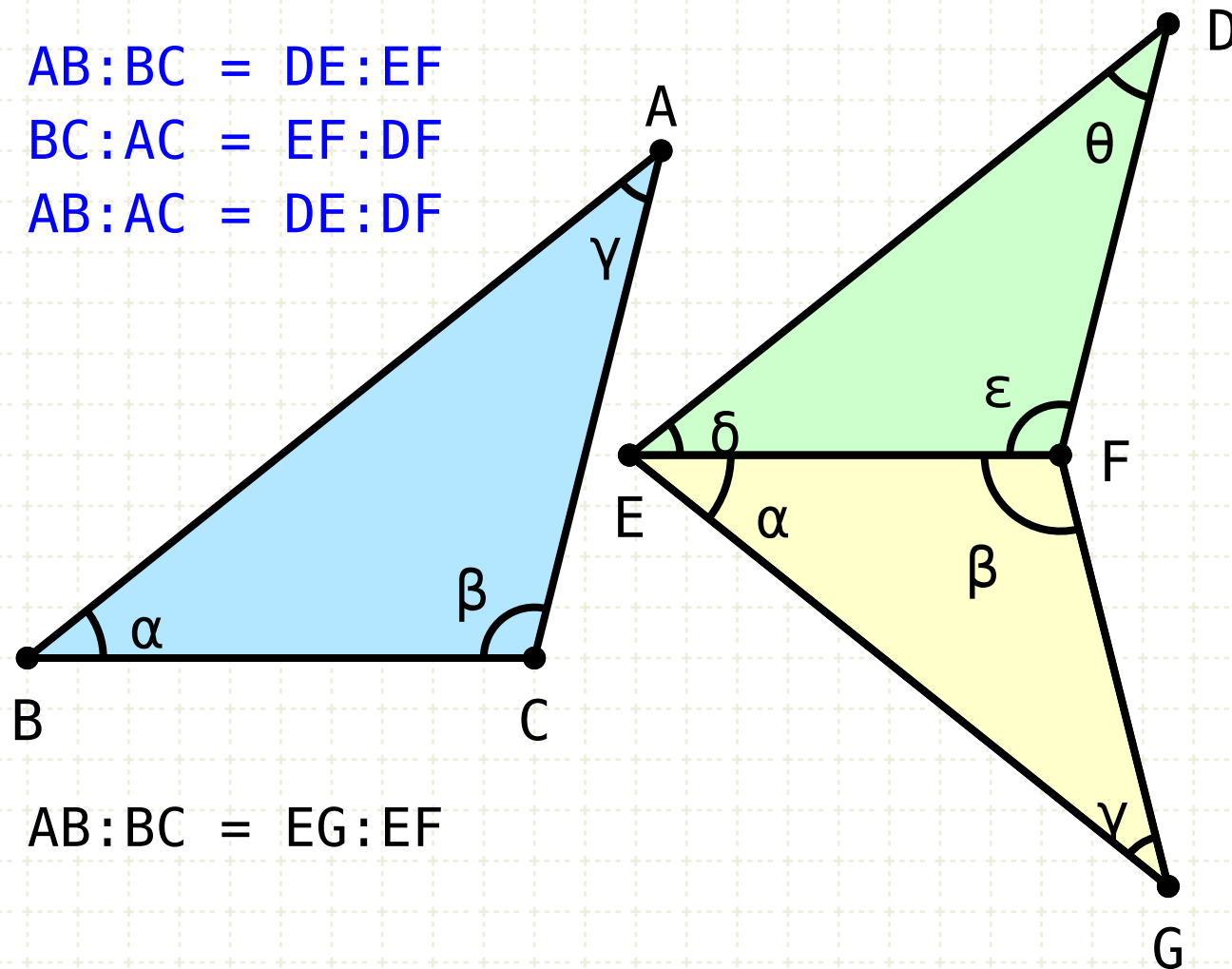
On the point E, construct an angle FEG on the line EF equal to the angle α (I·23)

On the point F, construct an angle EFG on the line EF equal to the angle β (I·23)

And thus, the angle at G will also be the angle at A (I·32)

Proposition 5 of Book VI

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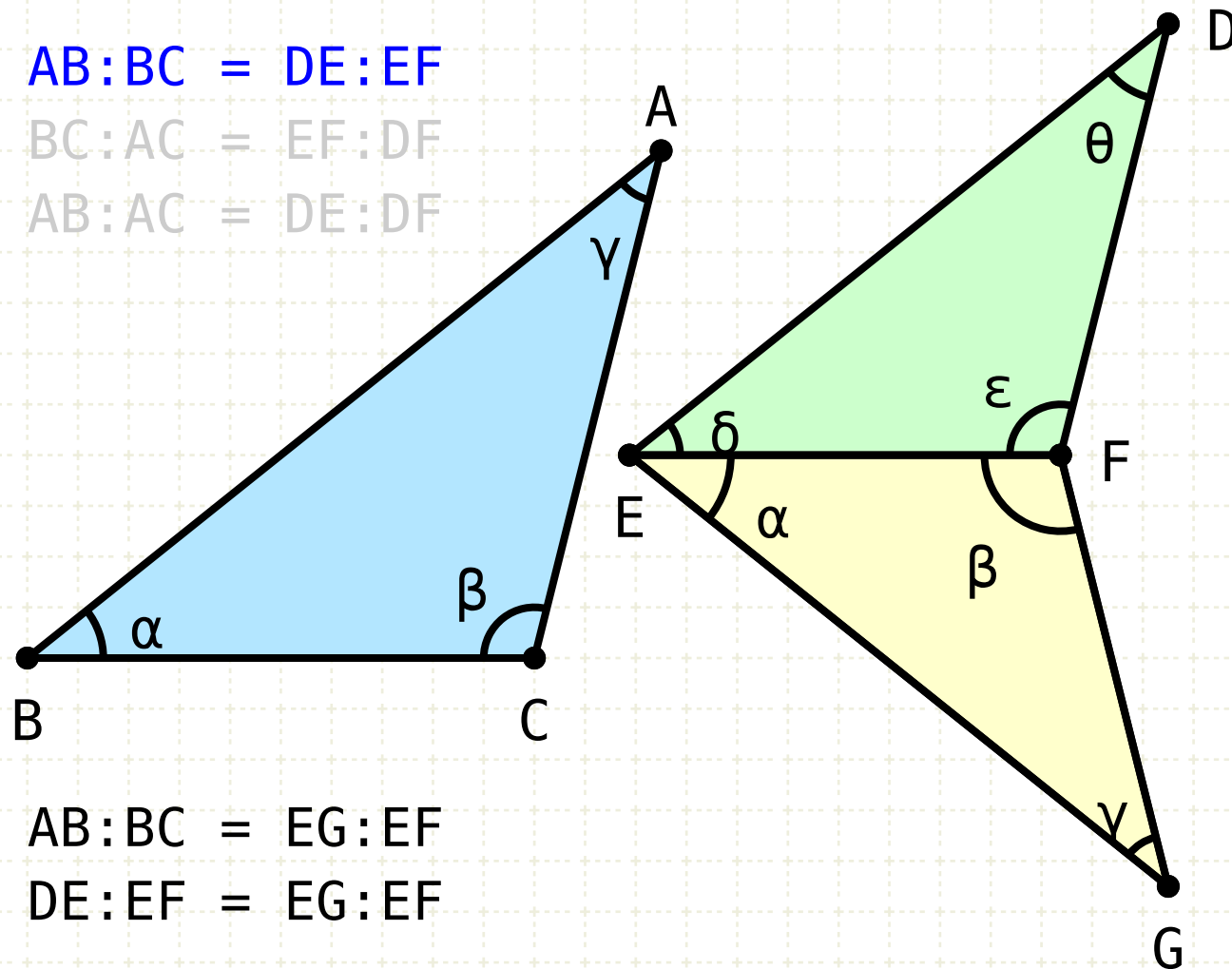
On the point F, construct an angle EFG on the line EF equal to the angle β (I·23)

And thus, the angle at G will also be the angle at A (I·32)

Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

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It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



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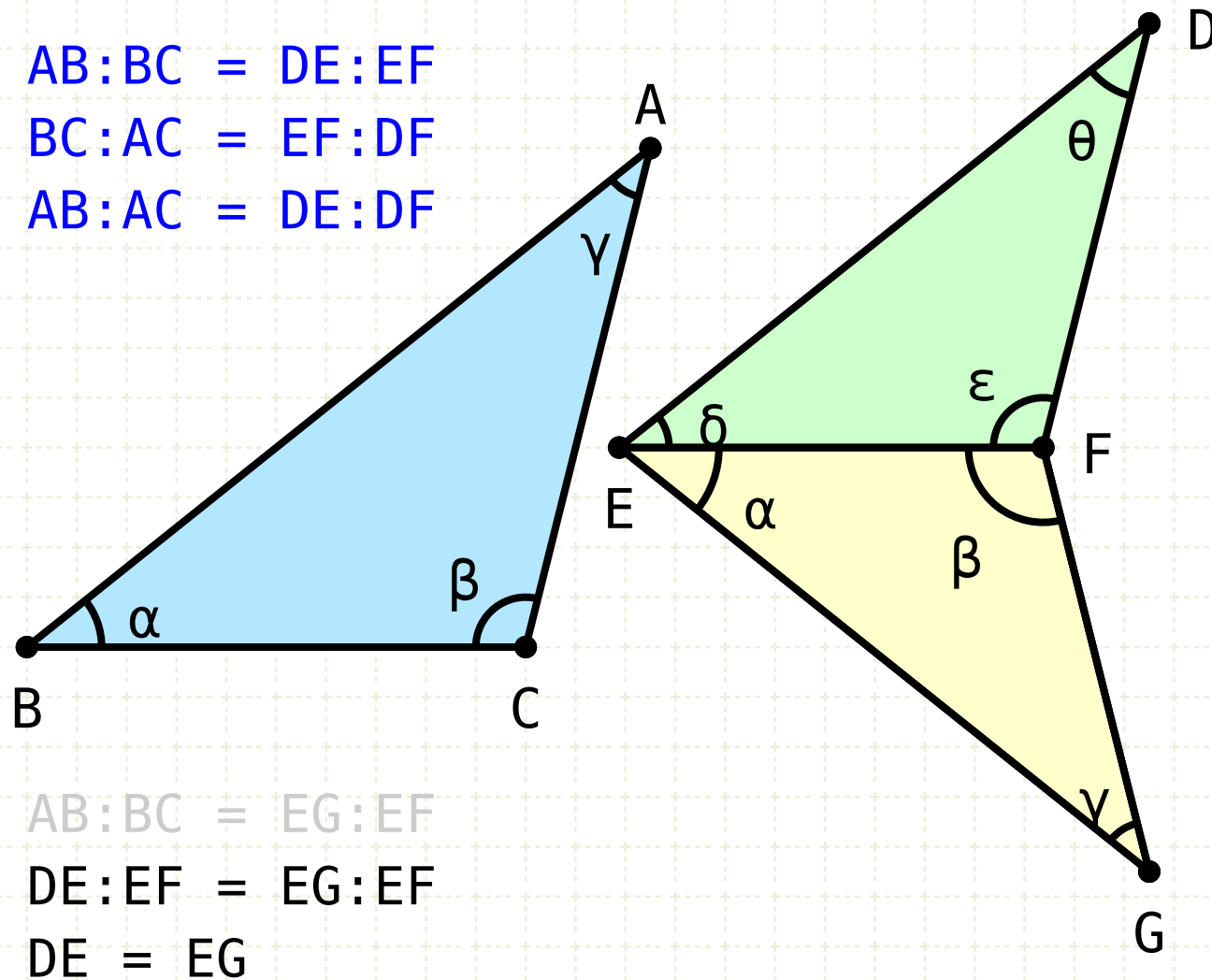
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Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

But the ratio AB to BC is equal to DE to EF, therefore the ratio DE to EF equals EG to EF (V·11)

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



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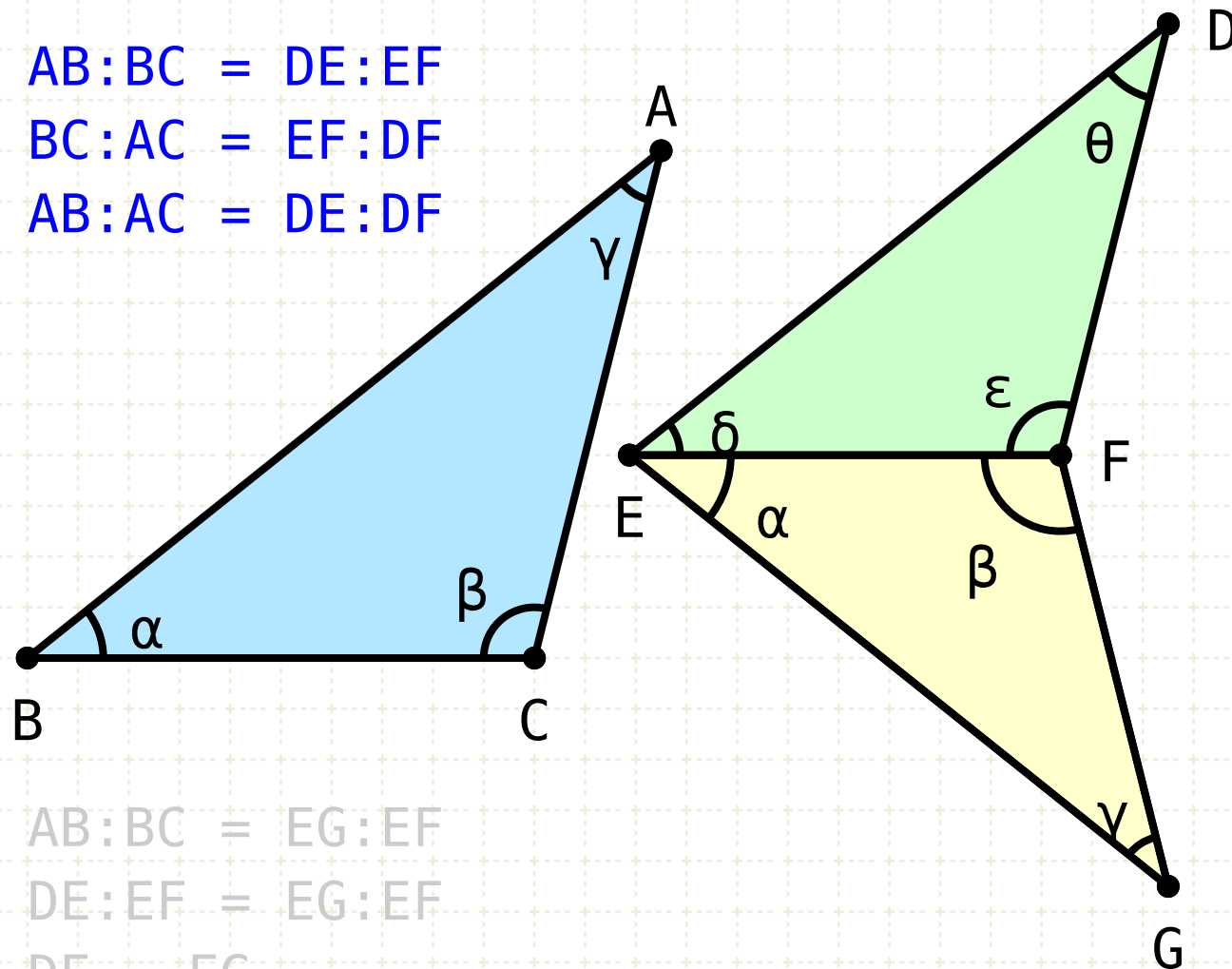
Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

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Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AB:BC &= DE:EF \\ BC:AC &= EF:DF \\ AB:AC &= DE:DF \end{aligned}$$

$$\begin{aligned} AB:BC &= EG:EF \\ DE:EF &= EG:EF \\ DE &= EG \\ DF &= FG \end{aligned}$$

Proof

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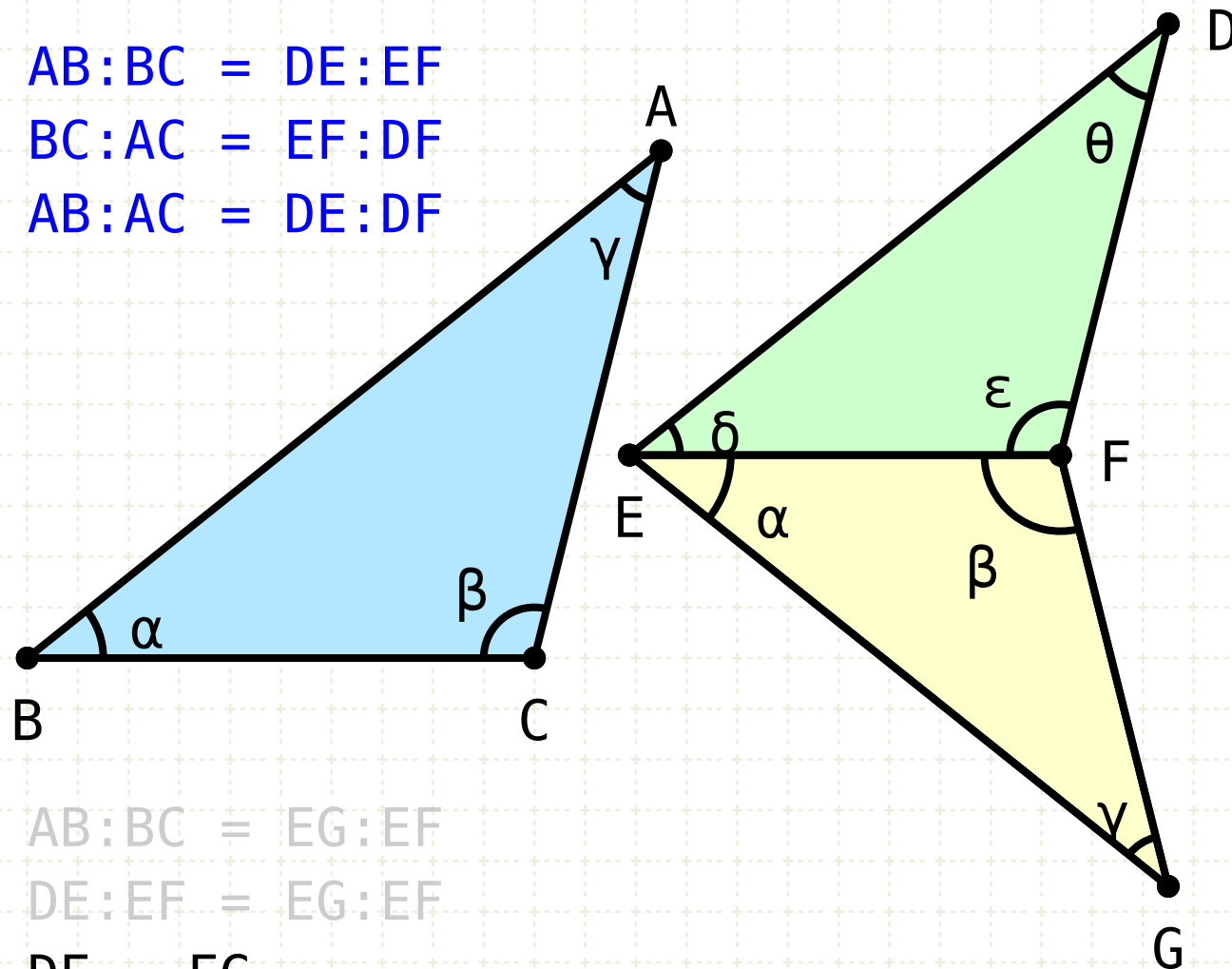
But the ratio AB to BC is equal to DE to EF, therefore the ratio DE to EF equals EG to EF (V·11)

Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

and for the same reason DF is also equal to FG

Proposition 5 of Book VI

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$$\begin{aligned} AB:BC &= EG:EF \\ DE:EF &= EG:EF \\ DE &= EG \\ DF &= FG \\ \alpha &= \delta \end{aligned}$$

Proof

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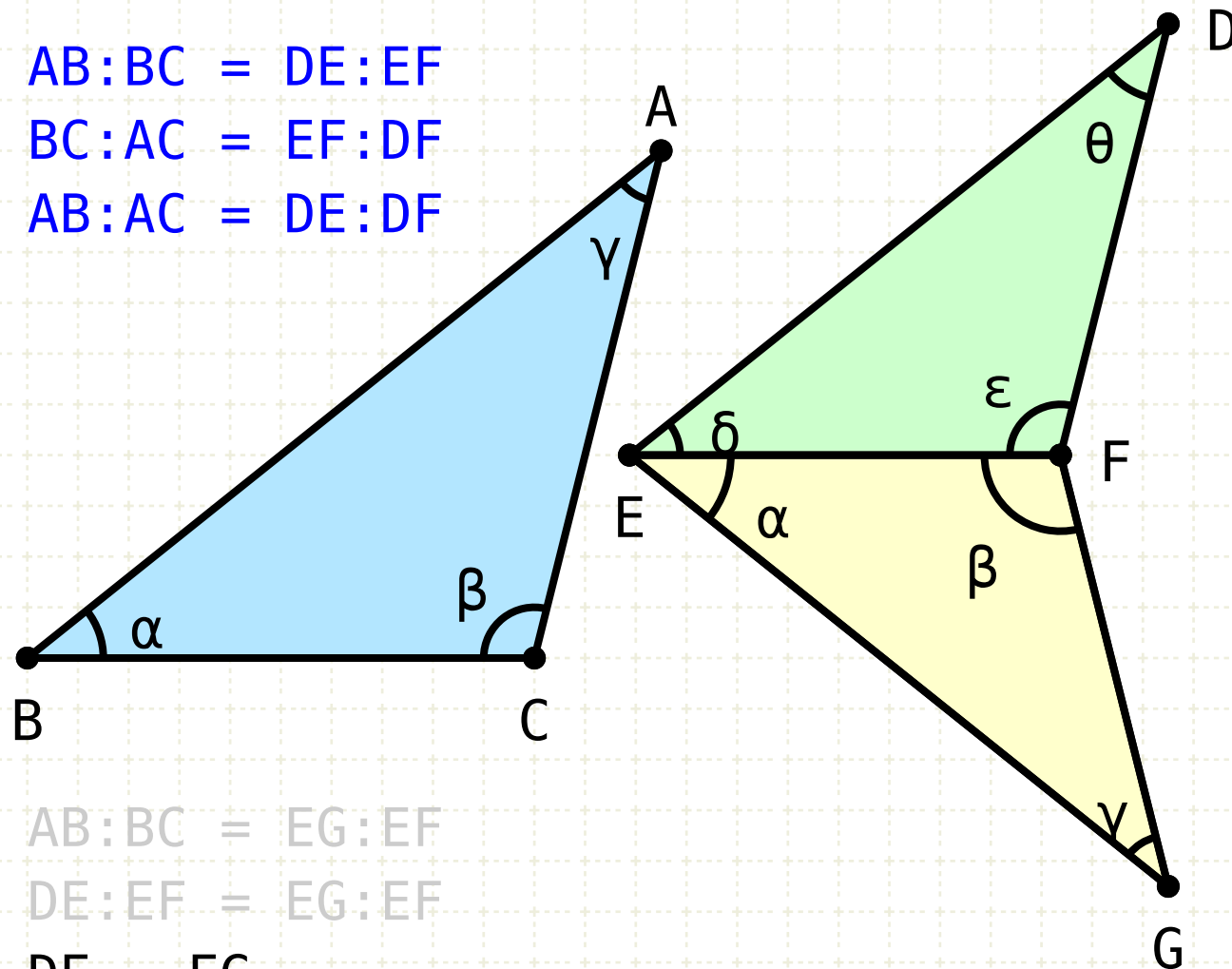
Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

and for the same reason DF is also equal to FG

Since DE is equal to EG, and DF equals FG, and there is a common base EF (three sides equal) then the angle DEF is equal to GEF (I·8),

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$$DE = EG$$

$$DF = FG$$

$$\alpha = \delta$$

$$\beta = \epsilon$$

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Proof

On the point E, construct an angle FEG on the line EF equal to the angle α (I·23)

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Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

But the ratio AB to BC is equal to DE to EF, therefore the ratio DE to EF equals EG to EF (V·11)

Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

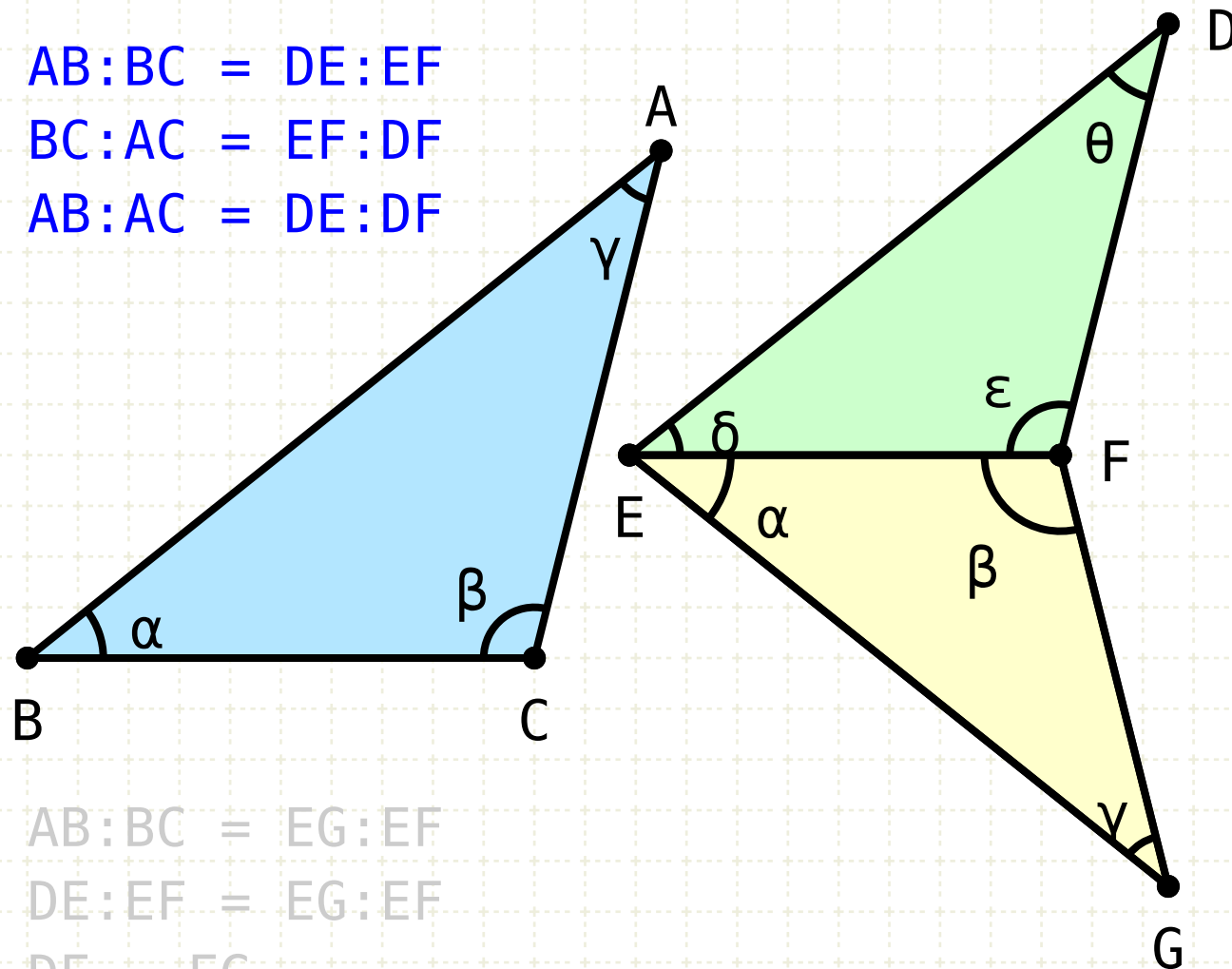
and for the same reason DF is also equal to FG

Since DE is equal to EG, and DF equals FG, and there is a common base EF (three sides equal) then the angle DEF is equal to GEF (I·8),

and finally, since there are two equal sides subtending an equal angle, both triangles DEF and EFG are equal (I·4)

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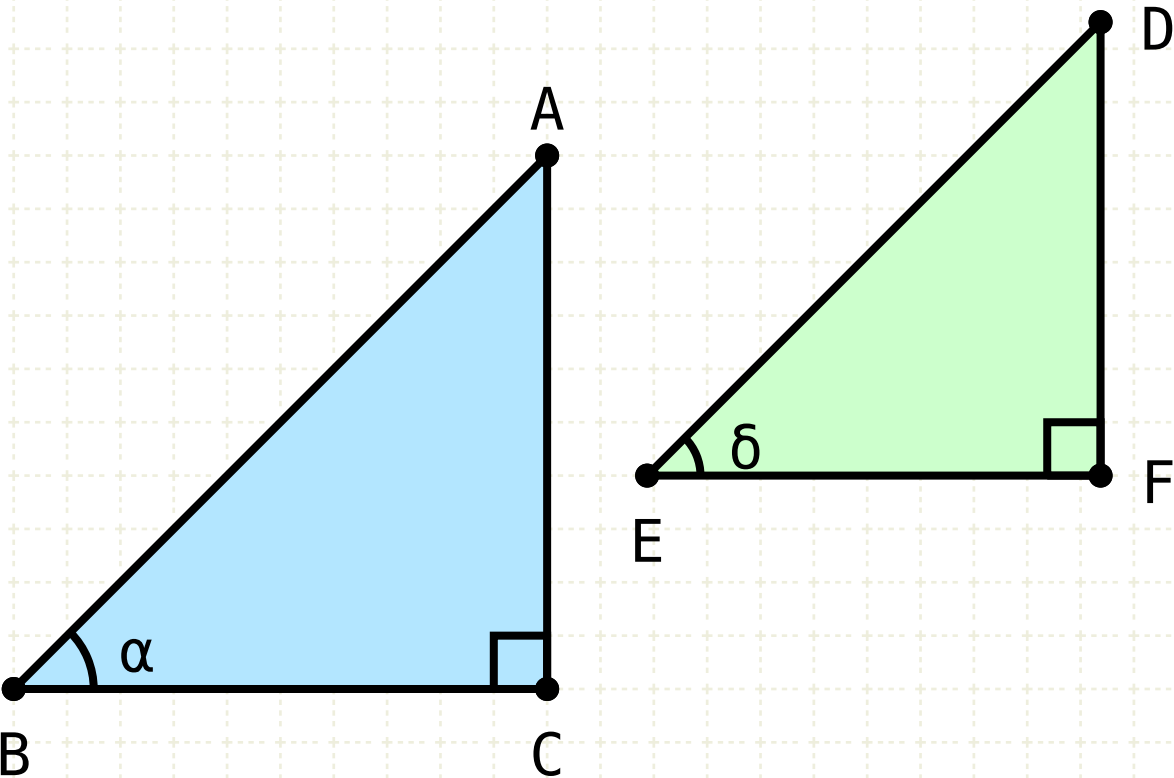
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and finally, since there are two equal sides subtending an equal angle, both triangles DEF and EFG are equal (I·4)

So finally, the triangle DEF is equiangular to triangle ABC

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend

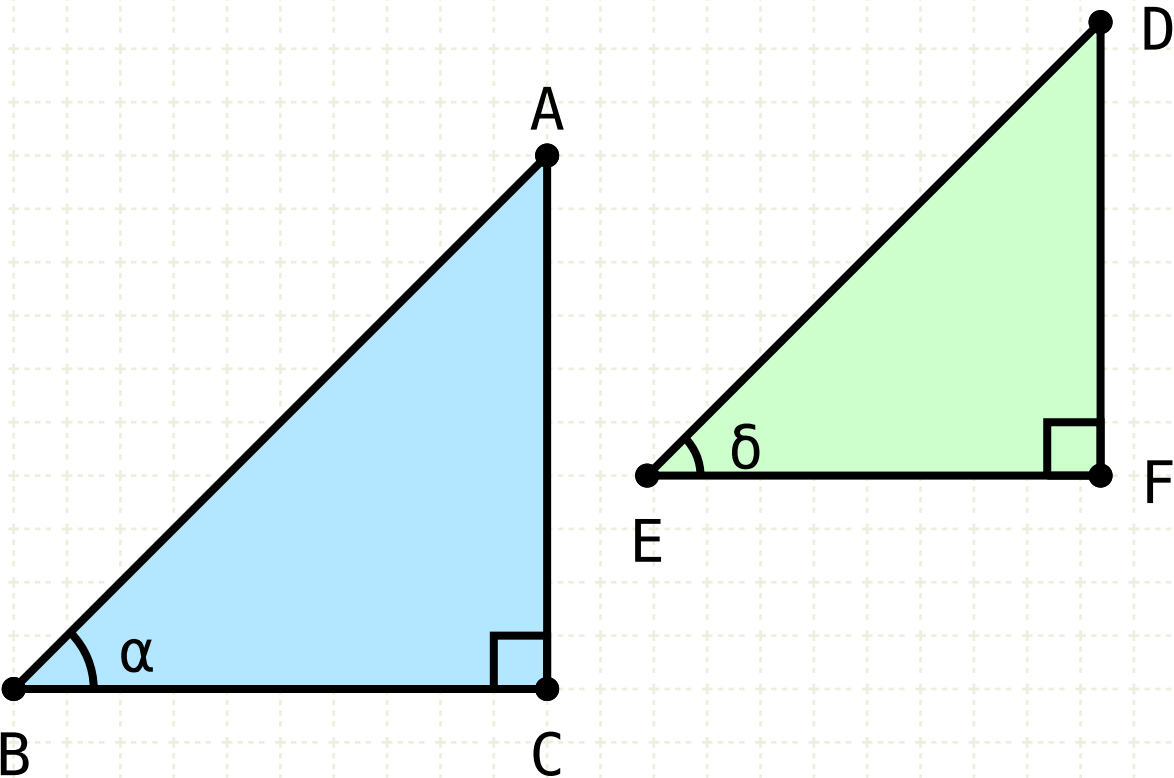


Aside - Trigonometry



Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$\alpha = \delta$

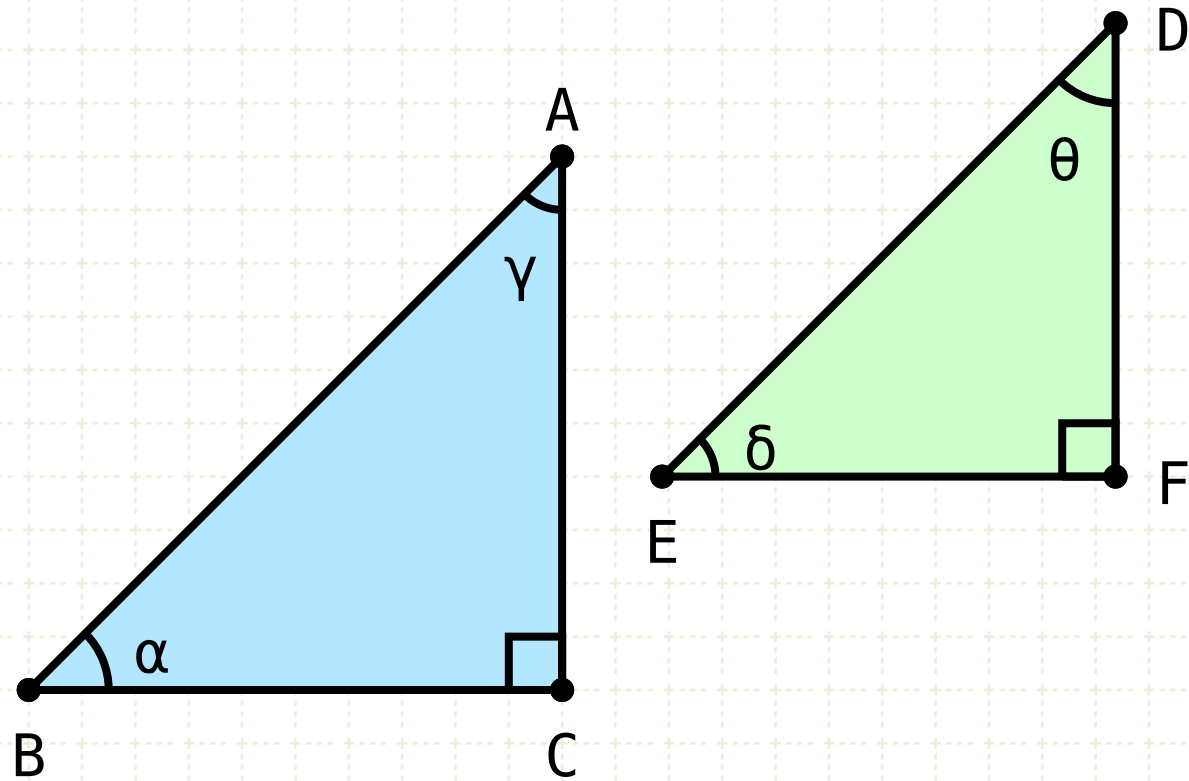
Aside - Trigonometry

Consider two right angle triangles where angle ABC equals angle DEF



Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\alpha = \delta$$
$$\gamma = \theta$$

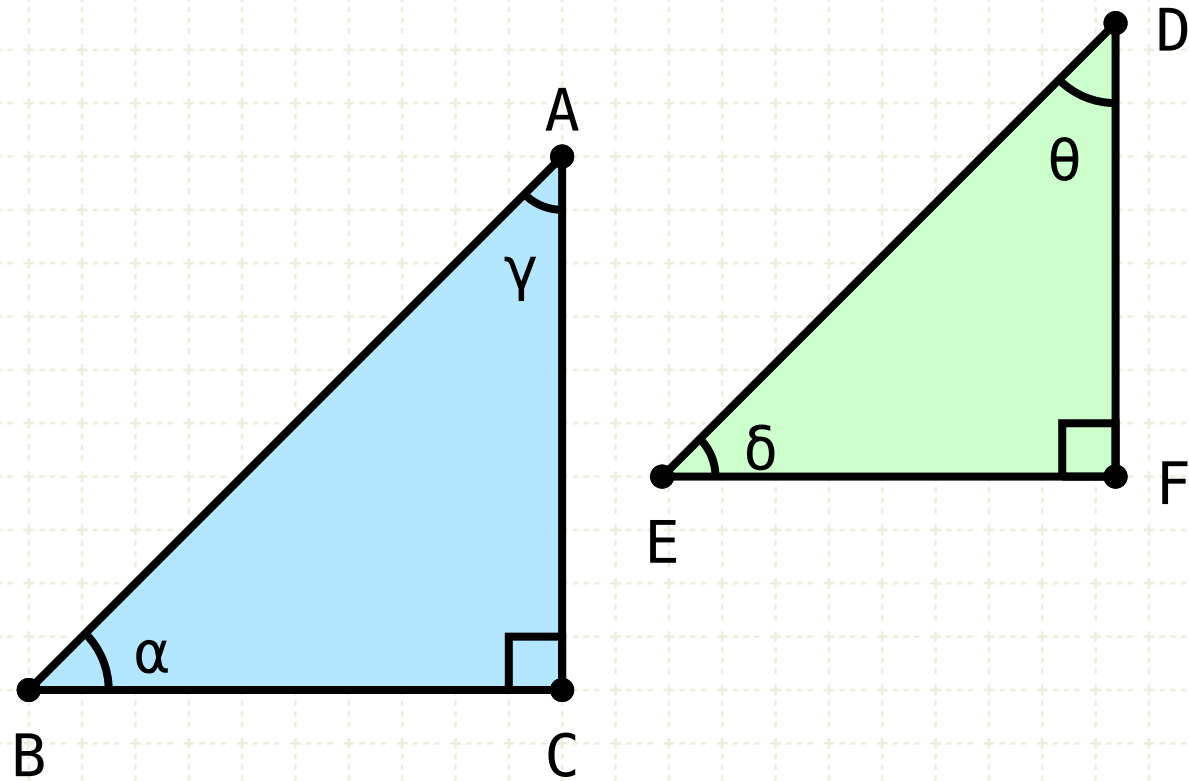
Aside - Trigonometry

Consider two right angle triangles where angle ABC equals angle DEF

Since two of the angles are equal in both triangles, the third must also be equal, hence angle BAC equals EDF

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\alpha = \delta$$

$$\gamma = \theta$$

$$AC:AB = DF:DE$$

Aside - Trigonometry

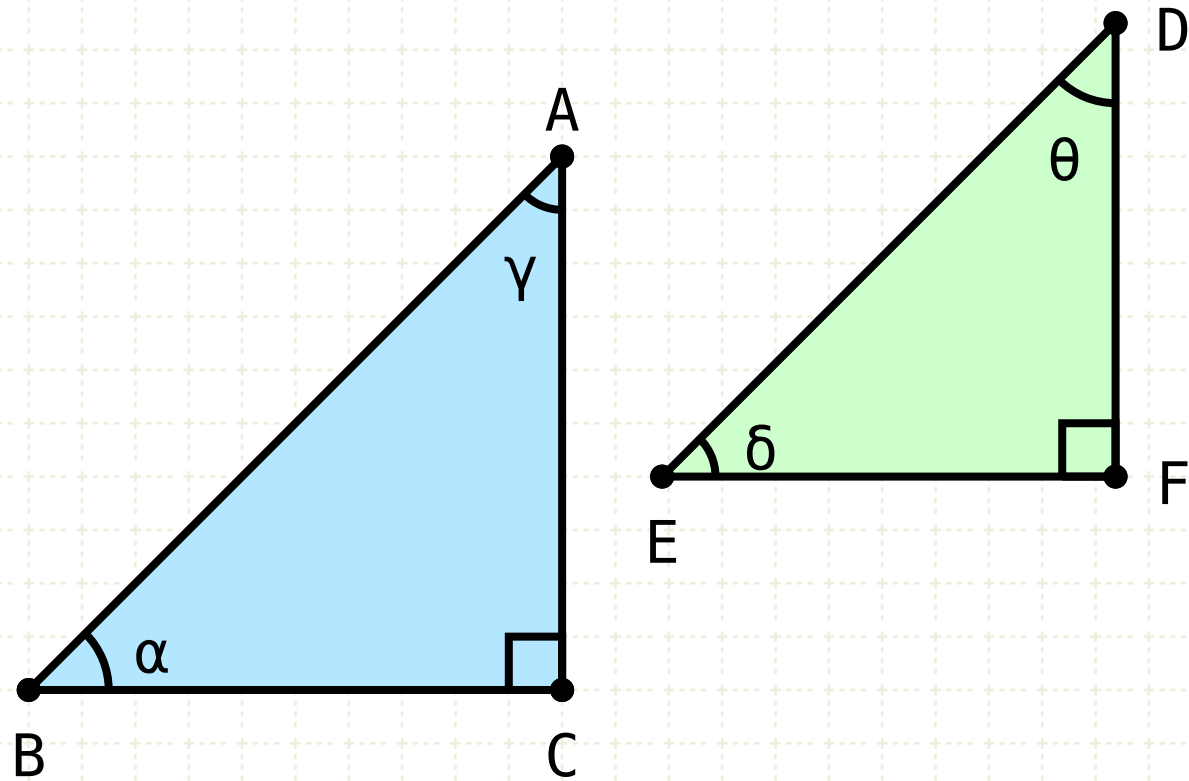
Consider two right angle triangles where angle ABC equals angle DEF

Since two of the angles are equal in both triangles, the third must also be equal, hence angle BAC equals EDF

Then, according to (VI.4), the ratio of the sides will be equal, in other words, AC to AB equals DF to DE, etc

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



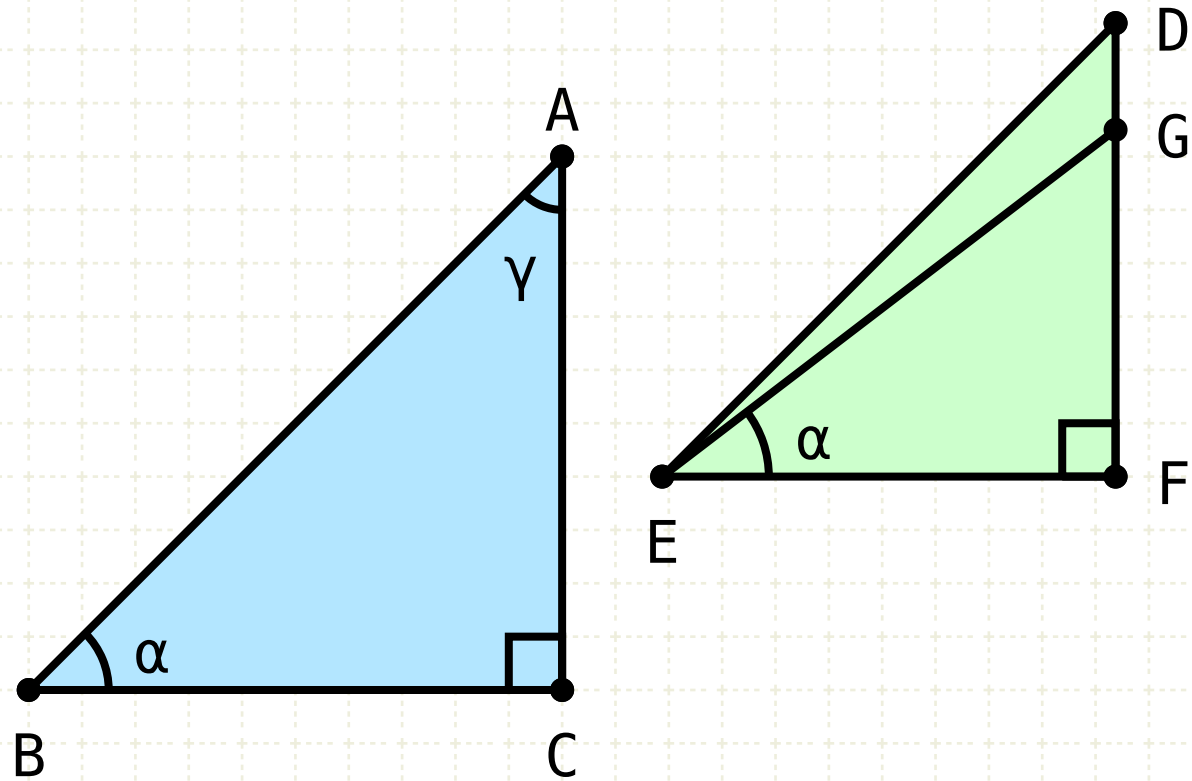
$$AC:AB = DF:DE$$

Aside - Trigonometry

Conversly, consider two right triangles where AC to AB equals DF to DE

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

Aside - Trigonometry

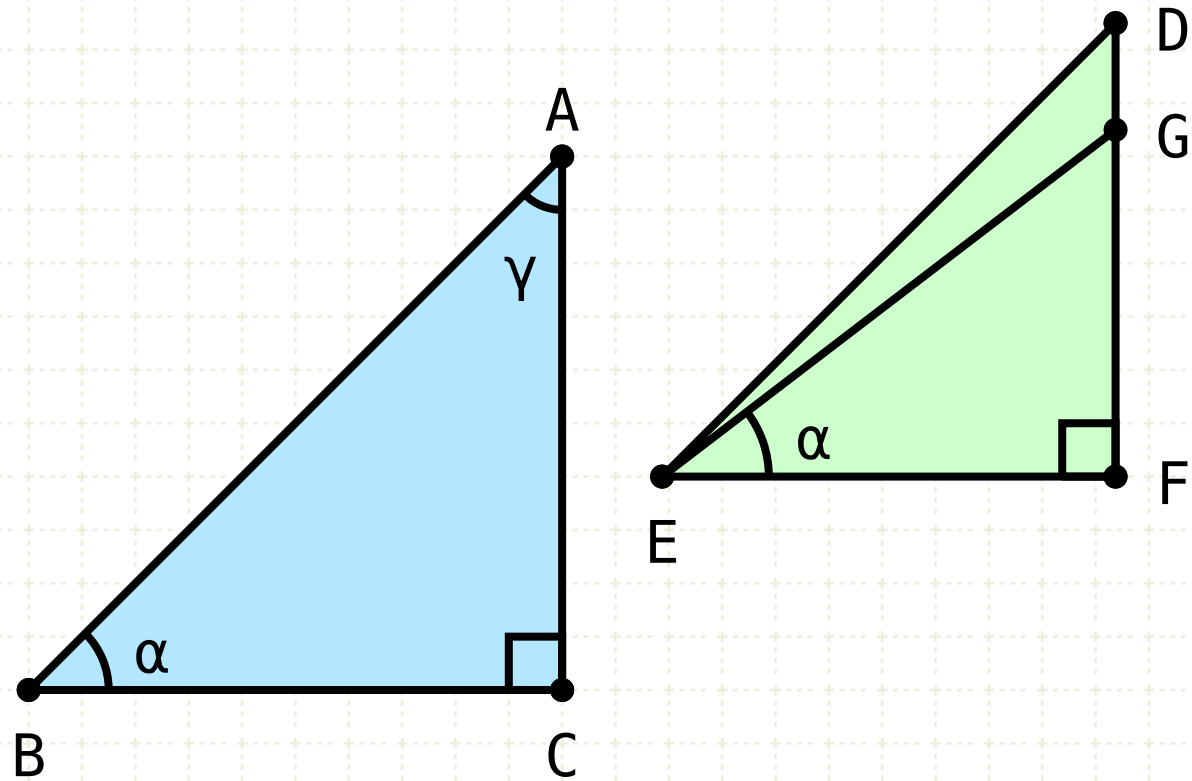
Conversly, consider two right triangles where AC to AB equals DF to DE

Assume that α is not equal to angle δ ...

Draw line EG, such that angle GEF equals α

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

$$FG:EG = AC:AB = DF:DE$$

Aside - Trigonometry

Conversly, consider two right triangles where AC to AB equals DF to DE

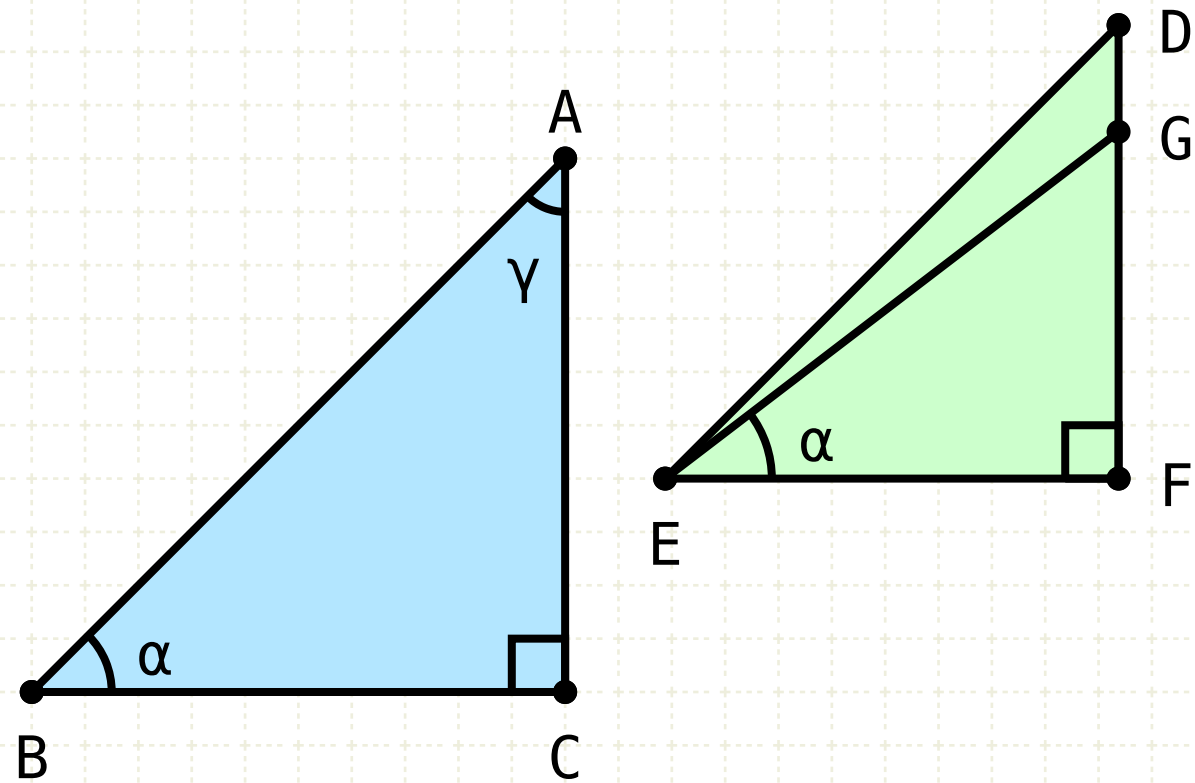
Assume that α is not equal to angle δ ...

Draw line EG, such that angle GEF equals α

Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

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Aside - Trigonometry

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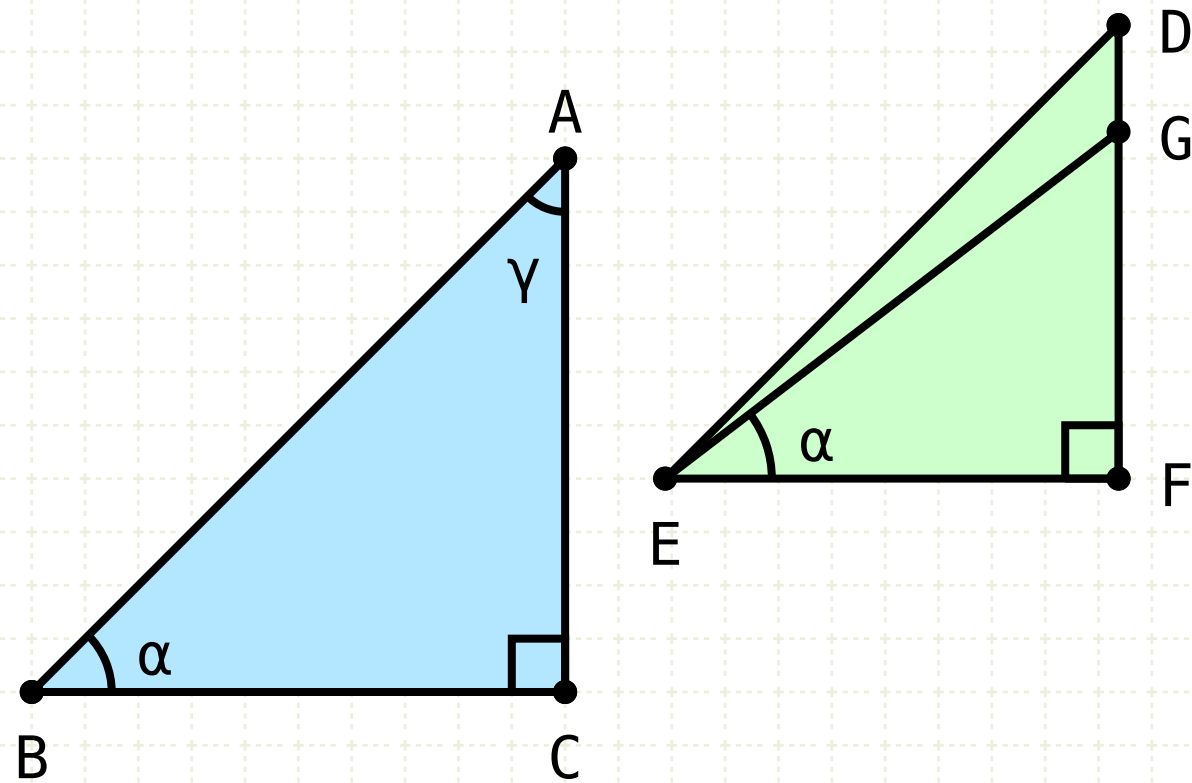
Draw line EG, such that angle GEF equals α

Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG

With a bit of math (pythagoras' theorem), it can be shown that the point G must be the same point as D

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$c = ED, b = DF, a = EF, d = FG, e = EG$$

$$FG:EG = AC:AB = DF:DE \rightarrow d/e = b/c$$

$$a^2 + b^2 = c^2; a^2 + d^2 = e^2$$

$$a^2 + (b \cdot (e/c))^2 = e^2$$

$$a^2 = e^2 \cdot (1 - (b^2/c^2))$$

$$a^2 = e^2 \cdot ((c^2 - b^2)/c^2)$$

$$a^2 = e^2 \cdot (a^2/c^2)$$

$$a^2 = a^2 \cdot (e/c)^2$$

$$\therefore e = c$$

Aside - Trigonometry

Conversly, consider two right triangles where AC to AB equals DF to DE

Assume that α is not equal to angle δ ...

Draw line EG, such that angle GEF equals α

Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG

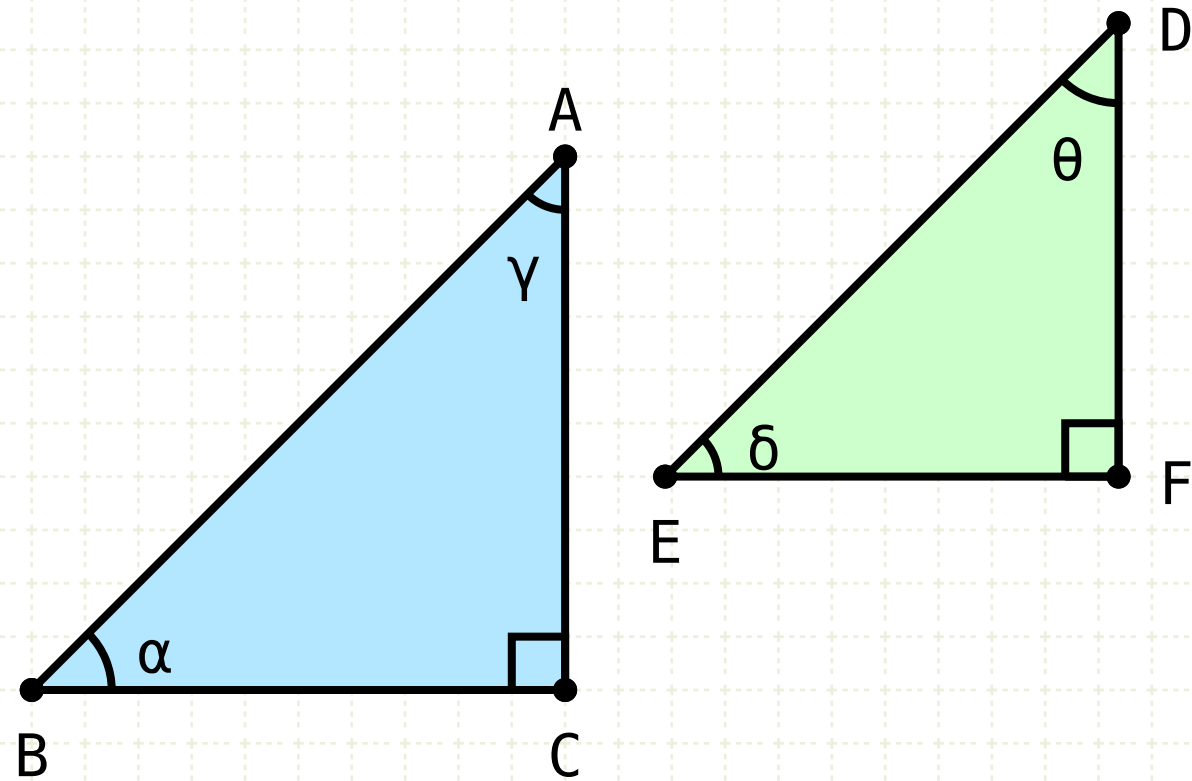
With a bit of math (pythagoras' theorem), it can be shown that the point G must be the same point as D

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Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

$$\delta = \alpha$$

Aside - Trigonometry

Conversly, consider two right triangles where AC to AB equals DF to DE

Assume that α is not equal to angle δ ...

Draw line EG, such that angle GEF equals α

Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG

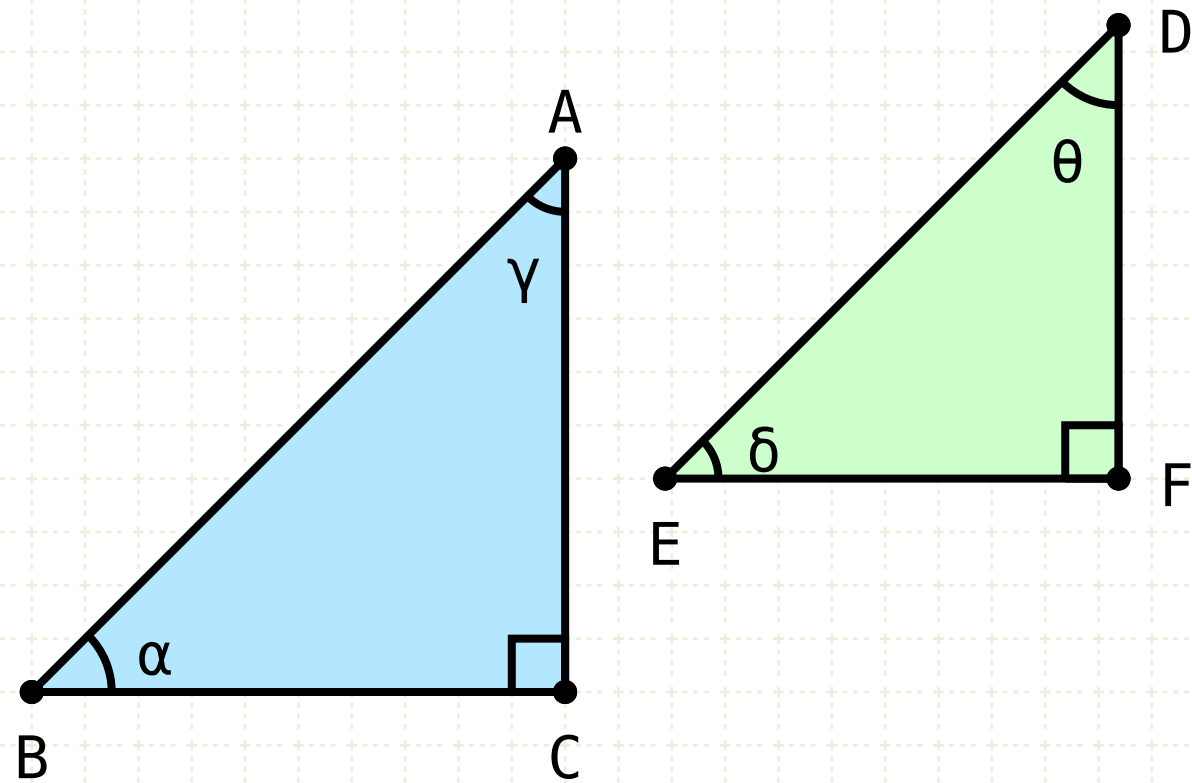
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So the angle α equals the angle δ

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE \rightarrow \alpha = \delta$$

$$\alpha = \delta \rightarrow AC:AB = DF:DE$$

Aside - Trigonometry

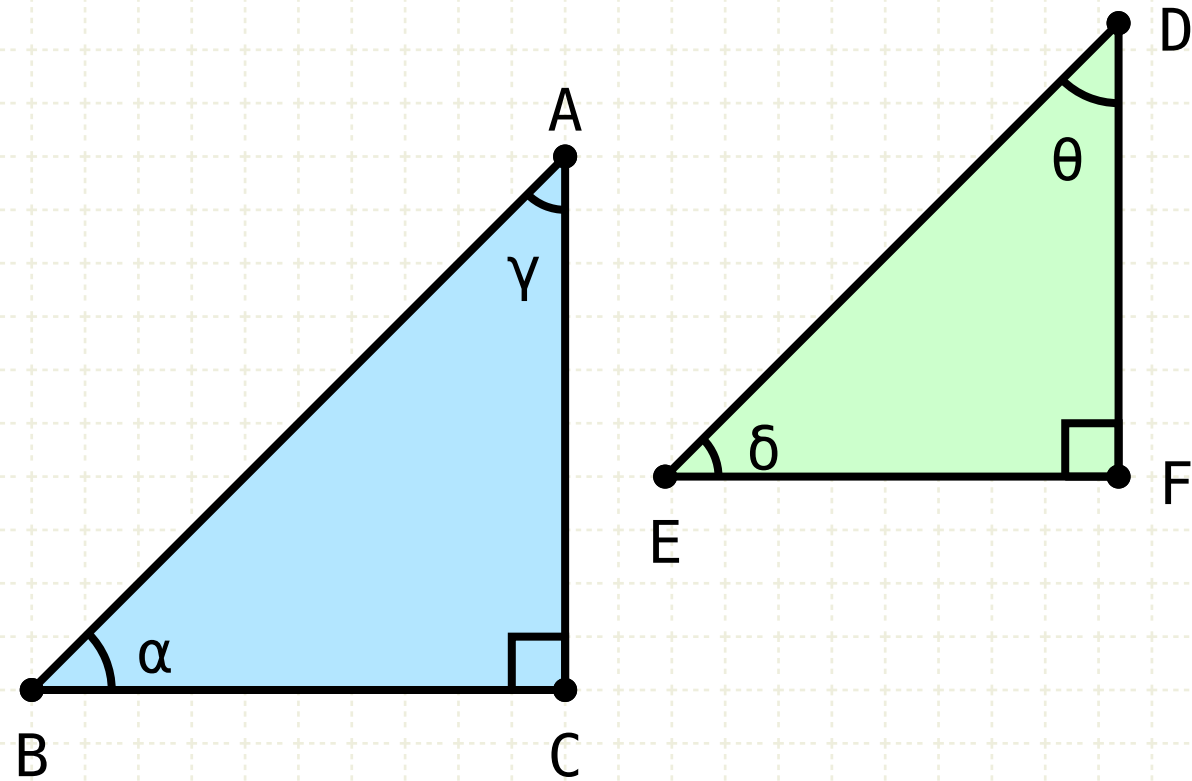
Conclusion:

Given two right triangles ABC and DEF

- * If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- * If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AC:AB &= DF:DE \rightarrow \alpha = \delta \\ \alpha = \delta &\rightarrow AC:AB = DF:DE \end{aligned}$$

Definition:

$$\sin(\alpha) = AC:AB$$

$$\sin(\delta) = DF:DE$$

Aside - Trigonometry

Conclusion:

Given two right triangles ABC and DEF

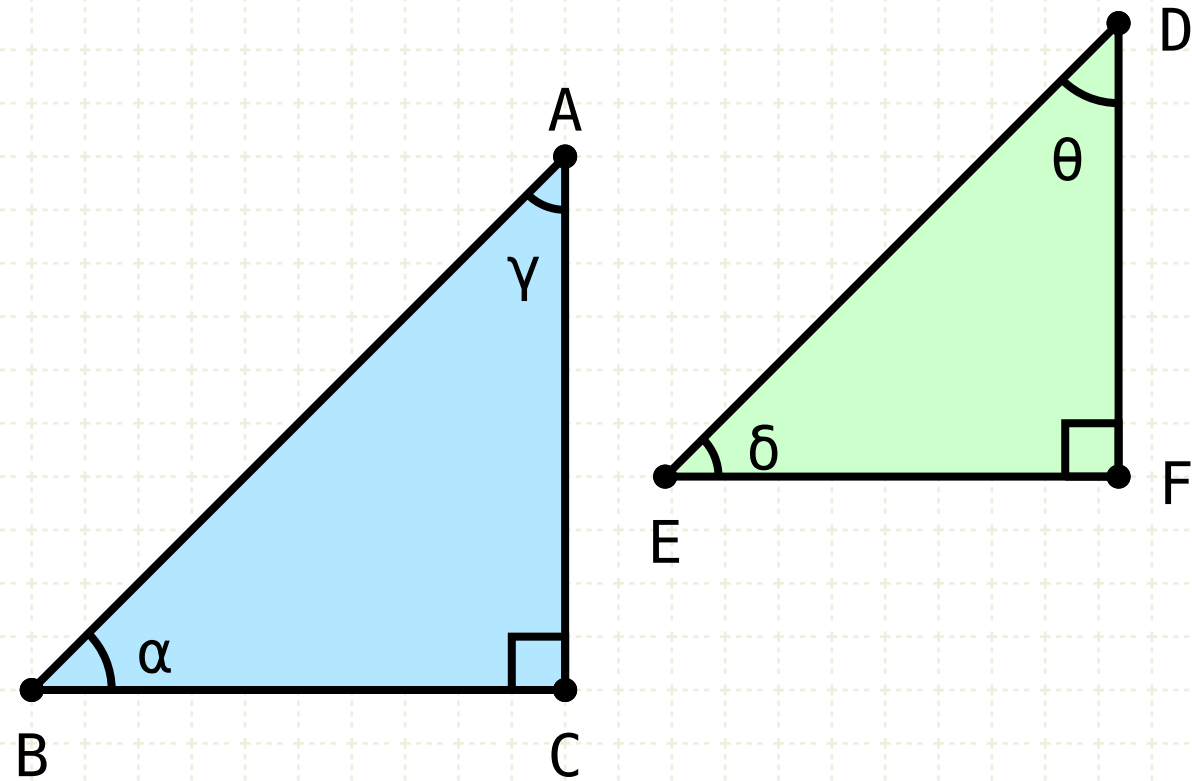
- * If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- * If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

So for every right angle triangle, the ratio of the sides (opposite over hypotenuse) is unique for every angle

Lets call this ratio, as a function of the angle, 'sine'

Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AC:AB &= DF:DE \rightarrow \alpha = \delta \\ \alpha = \delta &\rightarrow AC:AB = DF:DE \end{aligned}$$

Definition:

$$\begin{aligned} \sin(\alpha) &= AC:AB & \cos(\alpha) &= BC:AB \\ \sin(\delta) &= DF:DE & \cos(a) &= EF:DE \end{aligned}$$

Aside - Trigonometry

Conclusion:

Given two right triangles ABC and DEF

- * If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- * If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

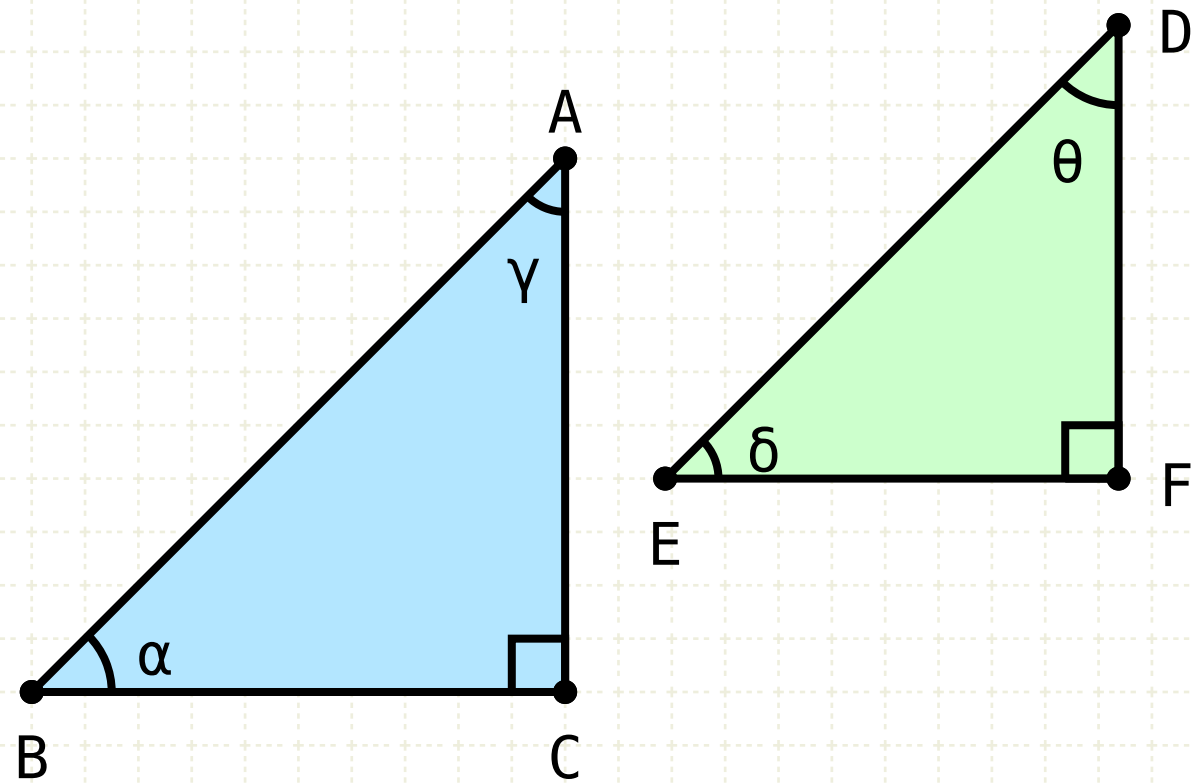
So for every right angle triangle, the ratio of the sides (opposite over hypotenuse) is unique for every angle

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We can use the same arguments to define the 'cosine' of an angle as the ratio of BC to AB

Proposition 5 of Book VI

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Definition:

$$\begin{aligned} \sin(\alpha) &= AC:AB & \cos(\alpha) &= BC:AB \\ \sin(\delta) &= DF:DE & \cos(a) &= EF:DE \end{aligned}$$

$$\begin{aligned} \sin^2(\alpha) + \cos^2(\alpha) &= (AC)^2/(AB)^2 + (BC)^2/(AB)^2 \\ &= ((AC)^2 + (BC)^2)/(AB)^2 \\ &= (AB)^2/(AB)^2 = 1 \end{aligned}$$

Aside - Trigonometry

Conclusion:

Given two right triangles ABC and DEF

- * If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- * If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

So for every right angle triangle, the ratio of the sides (opposite over hypotenuse) is unique for every angle

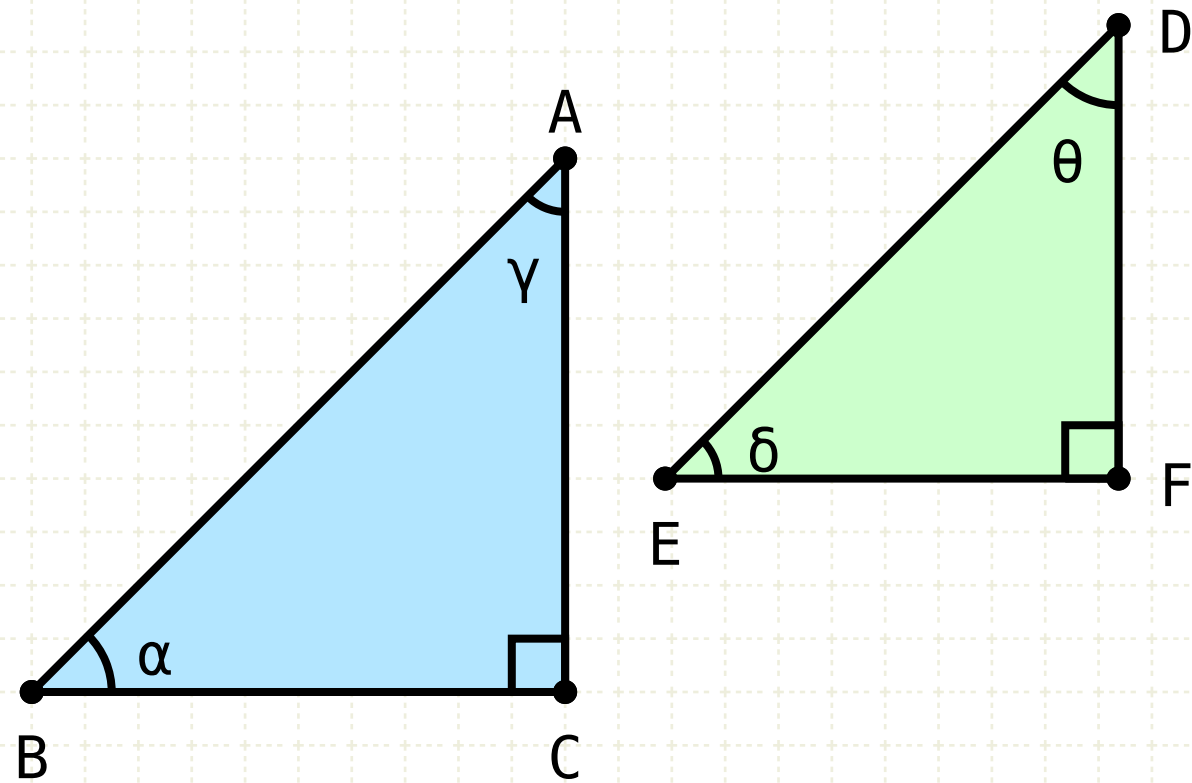
Lets call this ratio, as a function of the angle, 'sine'

We can use the same arguments to define the 'cosine' of an angle as the ratio of BC to AB



Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AC:AB &= DF:DE \rightarrow \alpha = \delta \\ \alpha = \delta &\rightarrow AC:AB = DF:DE \end{aligned}$$

Definition:

$$\begin{aligned} \sin(\alpha) &= AC:AB & \cos(\alpha) &= BC:AB \\ \sin(\delta) &= DF:DE & \cos(a) &= EF:DE \end{aligned}$$

$$\begin{aligned} \sin^2(\alpha) + \cos^2(\alpha) &= (AC)^2/(AB)^2 + (BC)^2/(AB)^2 \\ &= ((AC)^2 + (BC)^2)/(AB)^2 \\ &= (AB)^2/(AB)^2 = 1 \end{aligned}$$

Aside - Trigonometry

Conclusion:

Given two right triangles ABC and DEF

- * If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- * If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

So for every right angle triangle, the ratio of the sides (opposite over hypotenuse) is unique for every angle

Lets call this ratio, as a function of the angle, 'sine'

We can use the same arguments to define the 'cosine' of an angle as the ratio of BC to AB

Sine and cosine have been expanded to include definitions for angles larger than a right angle, and even negative angles, but these ratios shown above are the roots of trigonometry



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