

Euclid's Elements

Book VII

Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)



Table of Contents, Chapter 7

1	Determine if two numbers are relatively prime	10	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$	21	If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B
2	Find the greatest common divisor for two numbers	11	If $A:B = C:D$, then $(A-C):(B-D) = A:B$	22	If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime
3	Find the largest common divisor for three numbers	12	If $A:B = C:D$, then $(A+C):(B+C) = A:B$	23	If A,B are relatively prime and if $A = n \cdot C$, then B,C are relatively prime
4	Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B	13	If $A:B = C:D$, then $A:C = B:D$	24	If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C
5	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, then $(B+D) = (1/q) \cdot (A+C)$	14	If $A:B = D:E$ and $B:C = E:F$, then $A:C = D:F$	25	If A,B are relatively prime then A^2, B are relatively prime
6	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, then $(B+D) = (p/q) \cdot (A+C)$	15	If $B = i \cdot 1$ and $E = i \cdot D$, and if $D = j \cdot 1$ then $E = j \cdot B$	26	If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$
7	If $B = A/q$ and $D = C/q$, $B > D$, then $(B-D) = (A-C)/q$	16	$A \times B = B \times A$	27	If A,B are relatively prime, then A^2, B^2 are relatively prime, and A^3, B^3 are relatively prime, and so on
8	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, $B > D$, then $(B-D) = (p/q) \cdot (A-C)$	17	If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$		
9	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$	18	If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$		
		19	If $A:B = C:D$ then $A \times D = B \times C$ If $A \times D = B \times C$ then $A:B = C:D$		
		20	Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$		



Table of Contents, Chapter 7

28 **If A,B are relatively prime, then A,(A+B) are relatively prime**

29 If A is prime, and $B \neq n \cdot A$, then A,B are relatively prime

30 If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$

31 If $A = B \times C$, then $A = j \cdot D$ where D is prime

32 If A is a number then it is either prime, or $A = j \cdot D$ where D is prime

33 Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C

34 Find the lowest common denominator of 2 numbers

35 If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$, then $C = i \cdot E$

36 Find the least common multiple of 3 numbers

37 If $A = p \cdot B$, then $A = q \cdot C$ where $C = p \cdot 1$

38 If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$

39 Find the smallest number that has the fractions $1/a$, $1/b$, $1/c$



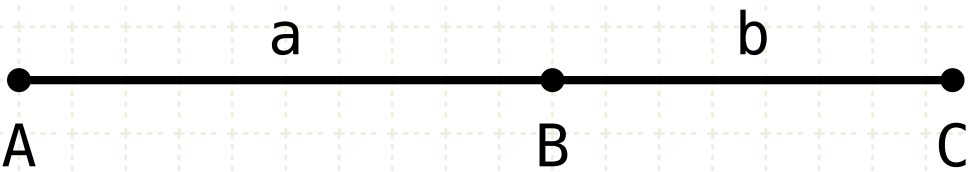
Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$\gcd(AB,BC) = 1$

$\gcd(a,b) = 1$

$AC = AB + BC$

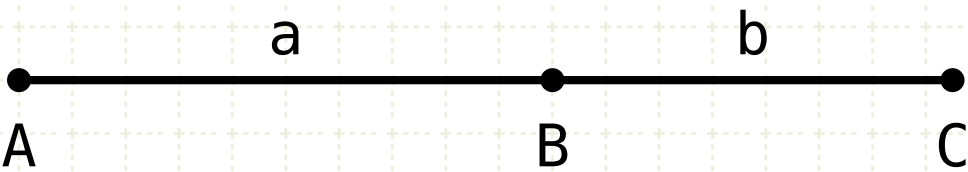
In other words (part a)

Let AB and BC be relatively prime



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB,BC) = 1 \qquad \gcd(a,b) = 1$$
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$$\gcd(AB,AC) = 1 \qquad \gcd(a,a+b) = 1$$
$$\gcd(BC,AC) = 1 \qquad \gcd(b,a+b) = 1$$

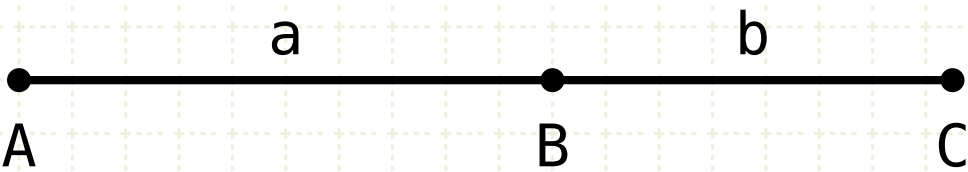
In other words (part a)

Let AB and BC be relatively prime
Then the sum AC is relatively prime to AB and BC



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



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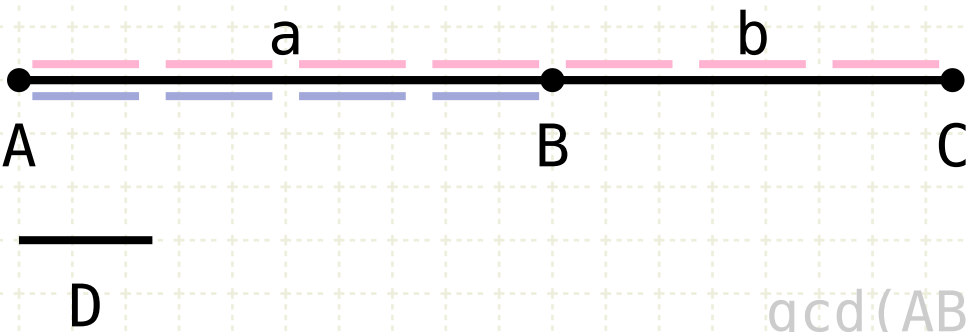
Let AB and BC be relatively prime
Then the sum AC is relatively prime to AB and BC

Proof by Contradiction



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB, BC) = 1 \qquad \gcd(a, b) = 1$$
$$AC = AB + BC$$

$$AC = p \cdot D$$
$$AB = q \cdot D$$

In other words (part a)

Let AB and BC be relatively prime
Then the sum AC is relatively prime to AB and BC

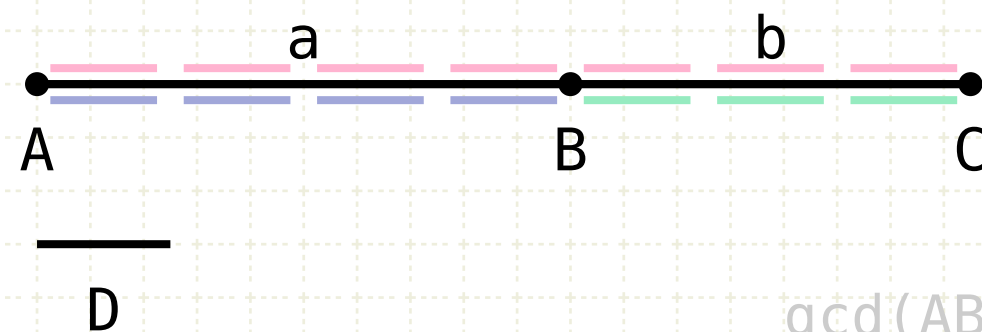
Proof by Contradiction

Assume AB and AC are not prime to each other. Let D be a common measure of AC, AB



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB, BC) = 1 \qquad \gcd(a, b) = 1$$

$$AC = AB + BC$$

$$AC = p \cdot D$$

$$AB = q \cdot D$$

$$BC = p \cdot D - q \cdot D$$

$$BC = r \cdot D$$

In other words (part a)

Let AB and BC be relatively prime

Then the sum AC is relatively prime to AB and BC

Proof by Contradiction

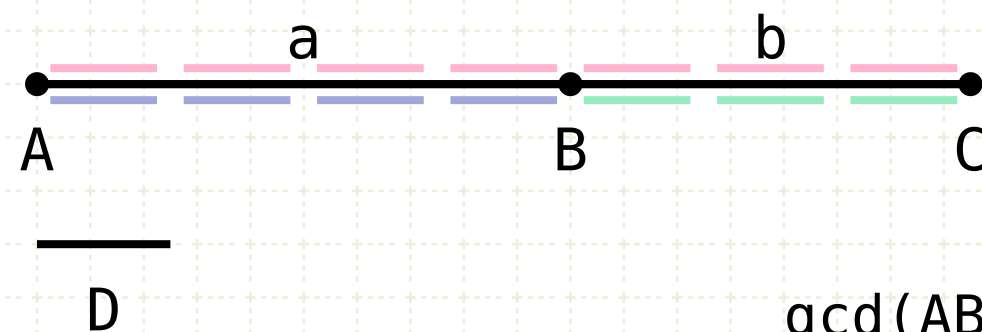
Assume AB and AC are not prime to each other. Let D be a common measure of AC, AB

Since D measures AC and AB, it will also measure the remainder BC



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB, BC) = 1 \qquad \gcd(a, b) = 1$$

$$AC = AB + BC$$

$$AC = p \cdot D$$

$$AB = q \cdot D$$

$$BC = p \cdot D - q \cdot D$$

$$BC = r \cdot D$$

In other words (part a)

Let AB and BC be relatively prime

Then the sum AC is relatively prime to AB and BC

Proof by Contradiction

Assume AB and AC are not prime to each other. Let D be a common measure of AC, AB

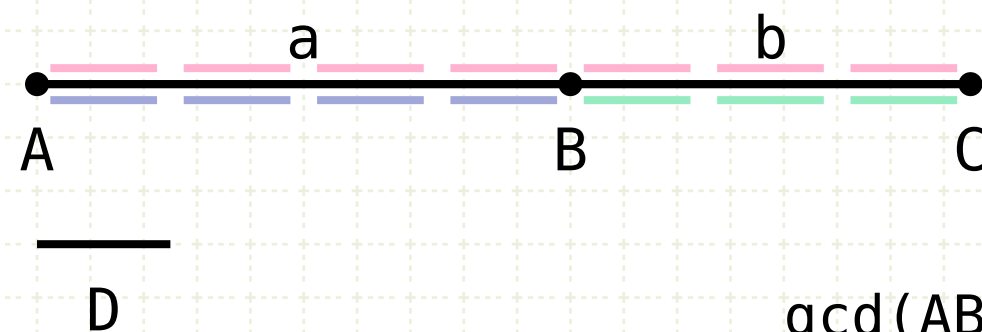
Since D measures AC and AB, it will also measure the remainder BC

Therefore D measures AB and BC, which are prime to one another, ...



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB, BC) = 1 \qquad \gcd(a, b) = 1$$

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Since D measures AC and AB, it will also measure the remainder BC

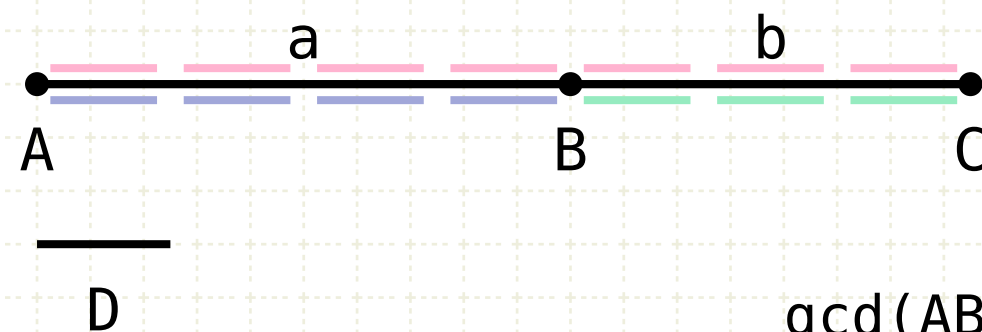
Therefore D measures AB and BC, which are prime to one another, ...

... which is impossible (VII.Def.12)



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$\gcd(AB, BC) = 1$ $\gcd(a, b) = 1$

$AC = AB + BC$

$AC = p \cdot D$ x

$AB = q \cdot D$ x

$BC = p \cdot D - q \cdot D$

$BC = r \cdot D$

In other words (part a)

Let AB and BC be relatively prime
Then the sum AC is relatively prime to AB and BC

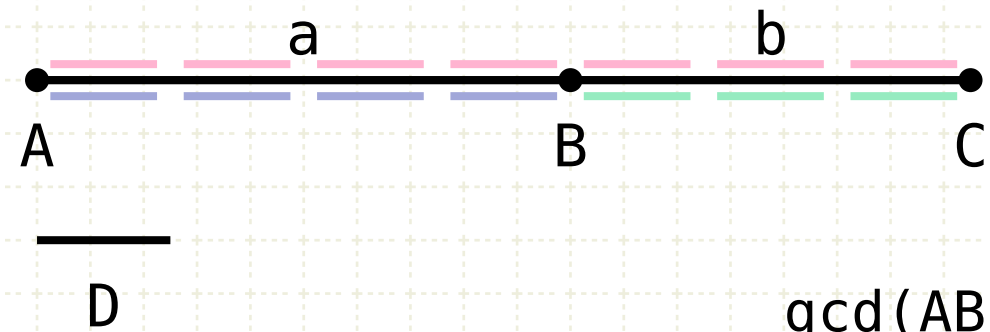
Proof by Contradiction

Assume AB and AC are not prime to each other. Let D be a common measure of AC, AB
Since D measures AC and AB, it will also measure the remainder BC
Therefore D measures AB and BC, which are prime to one another, ...
... which is impossible (VII.Def.12)
So, the original premise that there is a number D that measures AB,AC is invalid



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB, BC) = 1 \qquad \gcd(a, b) = 1$$

$$AC = AB + BC$$

$$AC = p \cdot D \quad x$$

$$AB = q \cdot D \quad x$$

$$BC = p \cdot D - q \cdot D$$

$$BC = r \cdot D$$

$$\gcd(AB, AC) = 1 \qquad \gcd(a, a+b) = 1$$

In other words (part a)

Let AB and BC be relatively prime

Then the sum AC is relatively prime to AB and BC

Proof by Contradiction

Assume AB and AC are not prime to each other. Let D be a common measure of AC, AB

Since D measures AC and AB, it will also measure the remainder BC

Therefore D measures AB and BC, which are prime to one another, ...

... which is impossible (VII.Def.12)

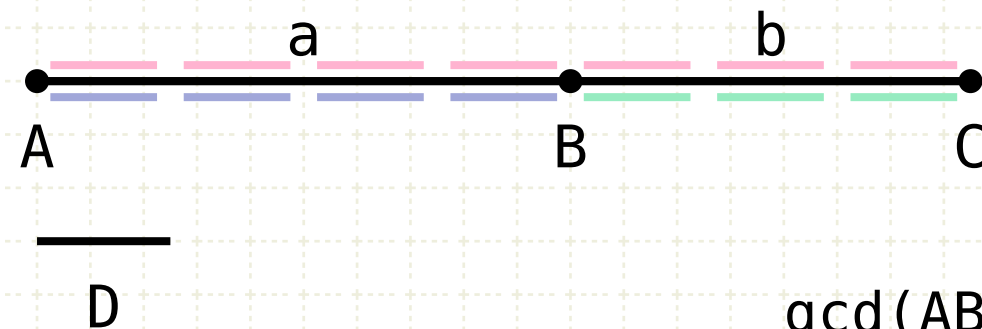
So, the original premise that there is a number D that measures AB, AC is invalid

Thus there is no number D that measures AB, AC, so they are prime to one another



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB, BC) = 1 \qquad \gcd(a, b) = 1$$

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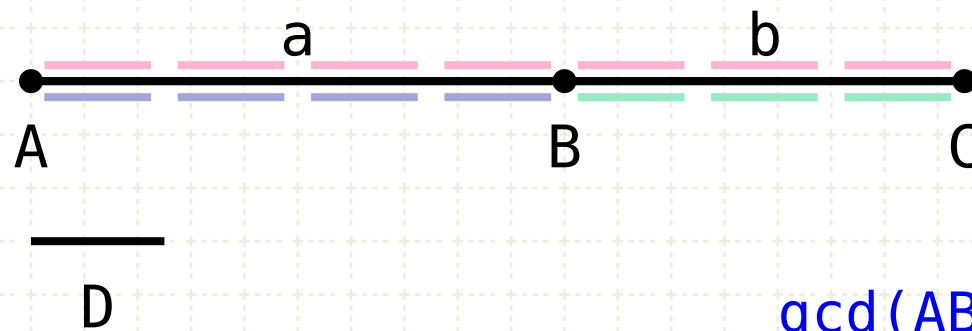
Thus there is no number D that measures AB,AC, so they are prime to one another

For the same reason, BC and AC are prime to each other



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AB, BC) = 1 \quad \gcd(a, b) = 1$$

$$AC = AB + BC$$

$$\begin{aligned} AC &= p \cdot D \quad x \\ AB &= q \cdot D \quad x \\ BC &= p \cdot D - q \cdot D \\ BC &= r \cdot D \end{aligned}$$

$$\begin{aligned} \gcd(AB, AC) &= 1 & \gcd(a, a+b) &= 1 \\ \gcd(BC, AC) &= 1 & \gcd(b, a+b) &= 1 \end{aligned}$$

In other words (part a)

Let AB and BC be relatively prime

Then the sum AC is relatively prime to AB and BC

Proof by Contradiction

Assume AB and AC are not prime to each other. Let D be a common measure of AC, AB

Since D measures AC and AB, it will also measure the remainder BC

Therefore D measures AB and BC, which are prime to one another, ...

... which is impossible (VII.Def.12)

So, the original premise that there is a number D that measures AB, AC is invalid

Thus there is no number D that measures AB, AC, so they are prime to one another

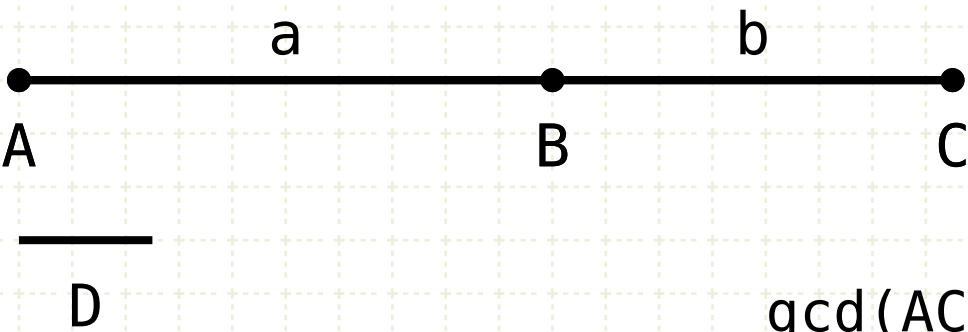
For the same reason, BC and AC are prime to each other

Therefore AB and BC are both prime to the sum AB, BC



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$\gcd(AC, AB) = 1$

$\gcd(a+b, a) = 1$

$AC = AB + BC$

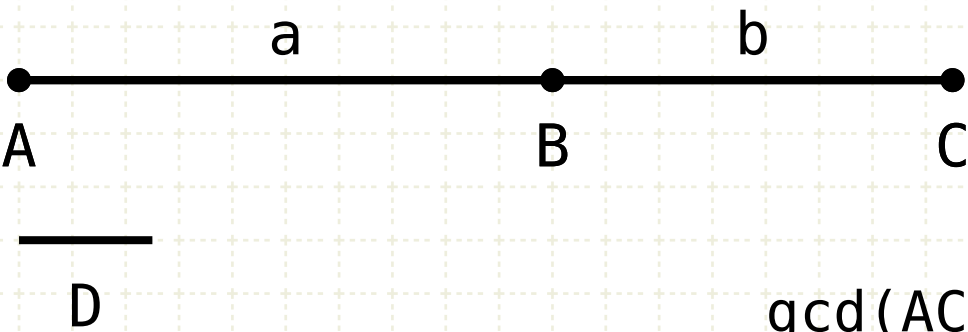
In other words (part b)

Let AB and AC be relatively prime



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$\gcd(AC, AB) = 1$

$AC = AB + BC$

$\gcd(a+b, a) = 1$

$\gcd(AB, BC) = 1$

$\gcd(a, b) = 1$

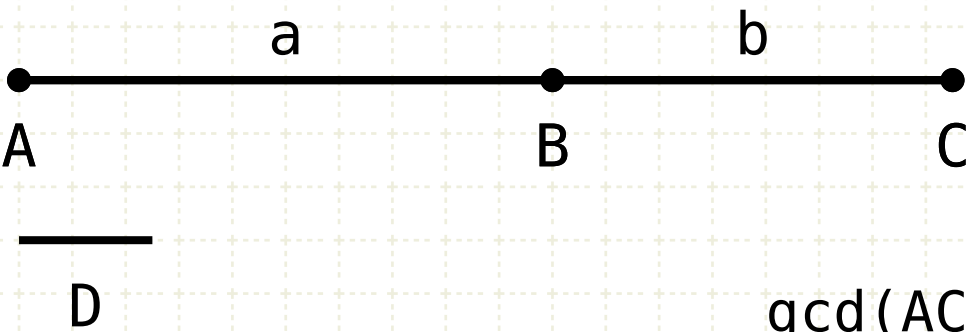
In other words (part b)

Let AB and AC be relatively prime
Then the parts AB and BC will be relatively prime



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If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$\gcd(AC, AB) = 1$ $\gcd(a+b, a) = 1$

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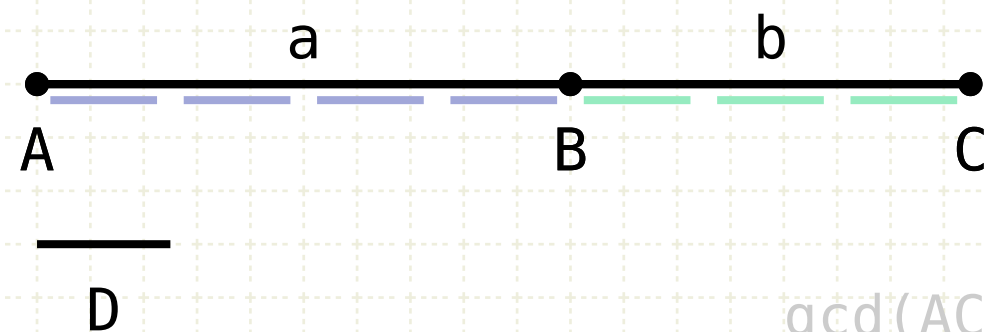
Let AB and AC be relatively prime
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Proof by Contradiction



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AC, AB) = 1 \qquad \gcd(a+b, a) = 1$$

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$$AB = p \cdot D$$

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In other words (part b)

Let AB and AC be relatively prime

Then the parts AB and BC will be relatively prime

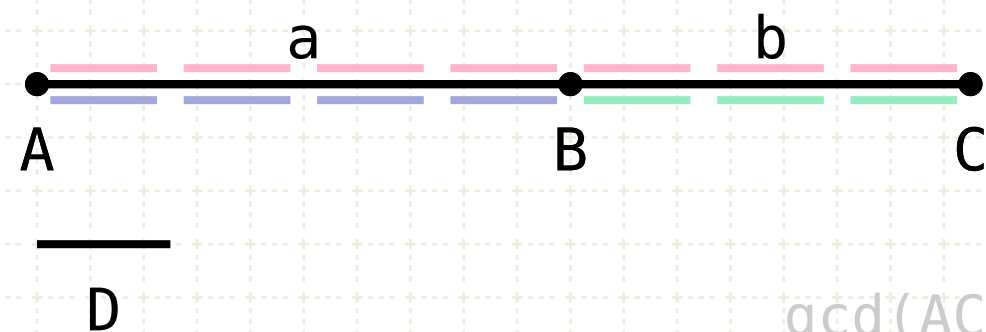
Proof by Contradiction

Assume AB and BC are not prime to each other. Let D be a common measure of AB, BC



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



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$$BC = q \cdot D$$

$$AC = p \cdot D + q \cdot D$$

$$AC = r \cdot D$$

In other words (part b)

Let AB and AC be relatively prime

Then the parts AB and BC will be relatively prime

Proof by Contradiction

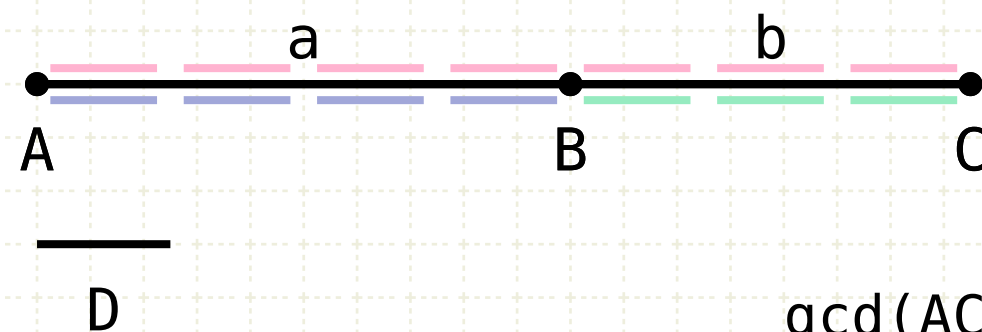
Assume AB and BC are not prime to each other. Let D be a common measure of AB, BC

Since D measures AB and BC, it will also measure the whole AC



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



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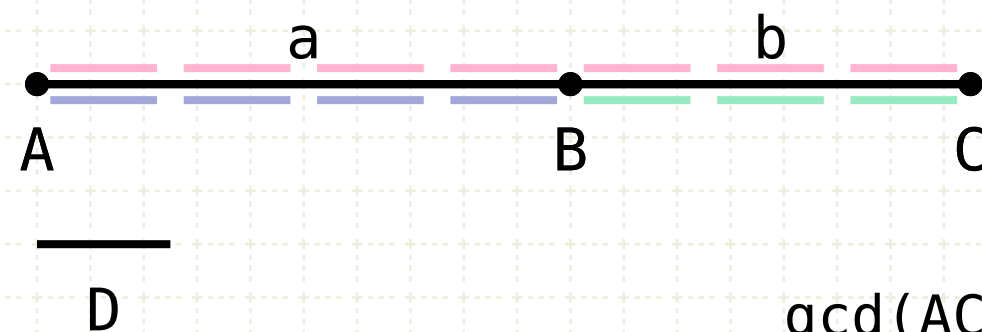
Since D measures AB and BC, it will also measure the whole AC

Therefore D measures AB and AC, which are prime to one another, ...



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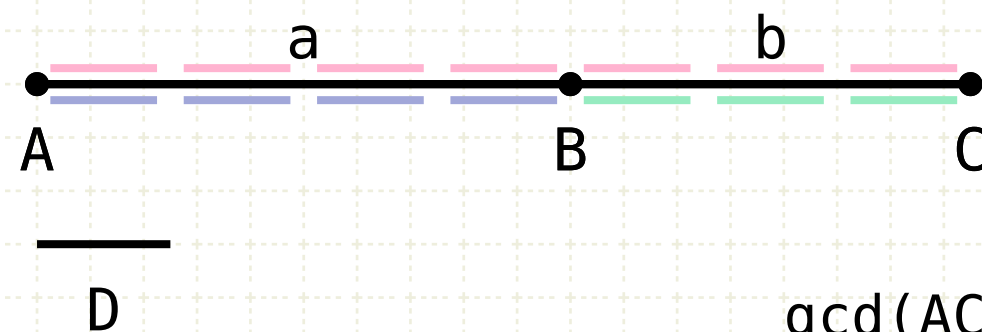
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$\gcd(AC, AB) = 1$ $\gcd(a+b, a) = 1$

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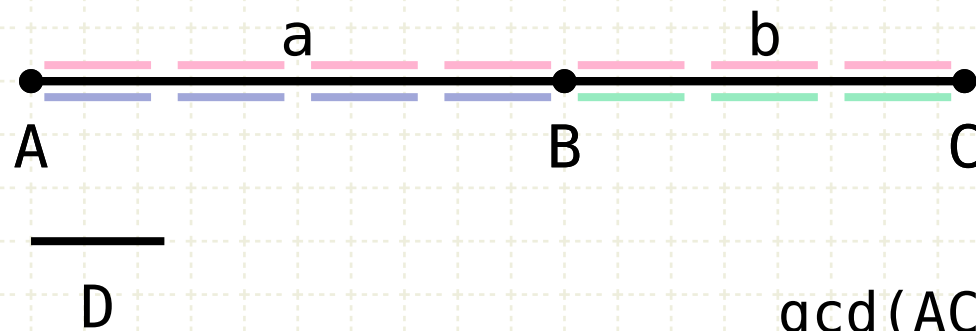
Assume AB and BC are not prime to each other. Let D be a common measure of AB, BC
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Therefore D measures AB and AC, which are prime to one another, ...
... which is impossible (VII.Def.12)
So, the original premise that there is a number D that measures AB,BC is invalid



Proposition 28 of Book VII

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another



$$\gcd(AC, AB) = 1 \quad \gcd(a+b, a) = 1$$

$$AC = AB + BC$$

$$AB = p \cdot D \quad x$$

$$BC = q \cdot D \quad x$$

$$AC = p \cdot D + q \cdot D$$

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$$\gcd(AB, BC) = 1 \quad \gcd(a, b) = 1$$

In other words (part b)

Let AB and AC be relatively prime

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Assume AB and BC are not prime to each other. Let D be a common measure of AB, BC

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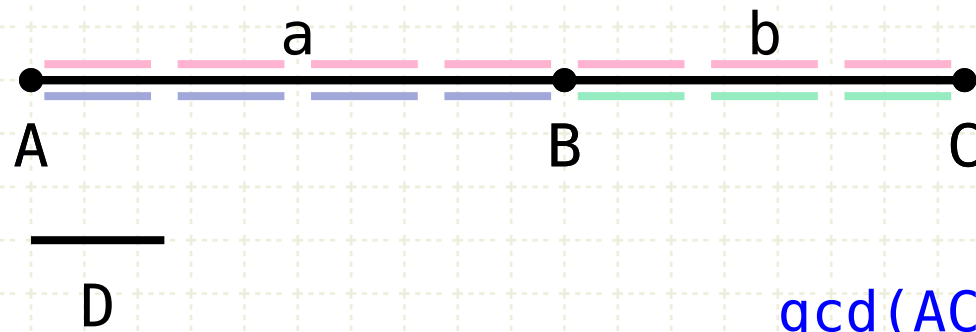
So, the original premise that there is a number D that measures AB,BC is invalid

Thus there is no number D that measures AB,BC, so they are prime to one another



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