

Euclid's Elements

Book VII

Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)



Table of Contents, Chapter 7

- | | | | | | |
|---|--|----|---|----|--|
| 1 | Determine if two numbers are relatively prime | 10 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$ | 21 | If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B |
| 2 | Find the greatest common divisor for two numbers | 11 | If $A:B = C:D$, then $(A-C):(B-D) = A:B$ | 22 | If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime |
| 3 | Find the largest common divisor for three numbers | 12 | If $A:B = C:D$, then $(A+C):(B+C) = A:B$ | 23 | If A,B are relatively prime and if $A = n \cdot C$, then B,C are relatively prime |
| 4 | Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B | 13 | If $A:B = C:D$, then $A:C = B:D$ | 24 | If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C |
| 5 | If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, then $(B+D) = (1/q) \cdot (A+C)$ | 14 | If $A:B = D:E$ and $B:C = E:F$, then $A:C = D:F$ | 25 | If A,B are relatively prime then A^2, B are relatively prime |
| 6 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, then $(B+D) = (p/q) \cdot (A+C)$ | 15 | If $B = i \cdot 1$ and $E = i \cdot D$, and if $D = j \cdot 1$ then $E = j \cdot B$ | 26 | If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$ |
| 7 | If $B = A/q$ and $D = C/q$, $B > D$, then $(B-D) = (A-C)/q$ | 16 | $A \times B = B \times A$ | 27 | If A,B are relatively prime, then A^2, B^2 are relatively prime, and A^3, B^3 are relatively prime, and so on |
| 8 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, $B > D$, then $(B-D) = (p/q) \cdot (A-C)$ | 17 | If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$ | | |
| 9 | If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$ | 18 | If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$ | | |
| | | 19 | If $A:B = C:D$ then $A \times D = B \times C$
If $A \times D = B \times C$ then $A:B = C:D$ | | |
| | | 20 | Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$ | | |



Table of Contents, Chapter 7

- | | | | |
|----|--|----|---|
| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime | 37 | If $A = p \cdot B$, then $A = q \cdot C$ where $C = p \cdot 1$ |
| 29 | If A is prime, and $B \neq n \cdot A$, then A,B are relatively prime | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$ |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a$, $1/b$, $1/c$ |
| 31 | If $A = B \times C$, then $A = j \cdot D$ where D is prime | | |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime | | |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C | | |
| 34 | Find the lowest common denominator of 2 numbers | | |
| 35 | If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$, then $C = i \cdot E$ | | |
| 36 | Find the least common multiple of 3 numbers | | |



Proposition 21 of Book VII

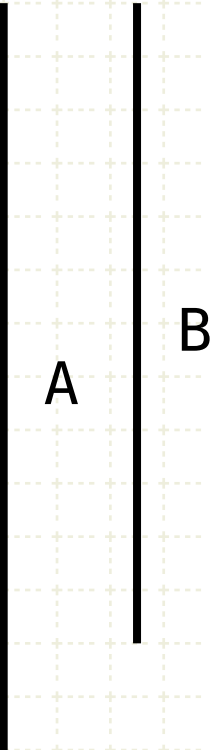
Numbers prime to one another are the least of those which have the same ratio with them.



Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.

$$\gcd(A,B) = 1$$



In other words

Let A and B be relatively prime.



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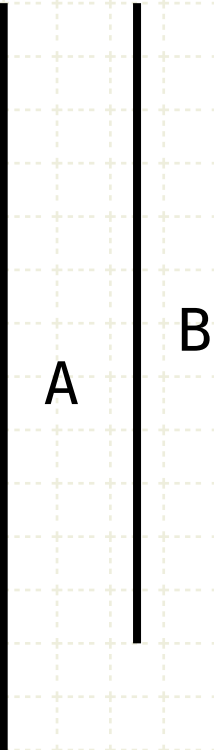
$$\gcd(A,B) = 1$$

$$S = \{ (x,y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x:y=A:B \}$$
$$(A,B) \in S \text{ such that } A \leq x, B \leq y, \forall (x,y) \in S$$

In other words

Let A and B be relatively prime.

Then A and B are the smallest whole numbers that can be used to create the same ratio equal to A to B

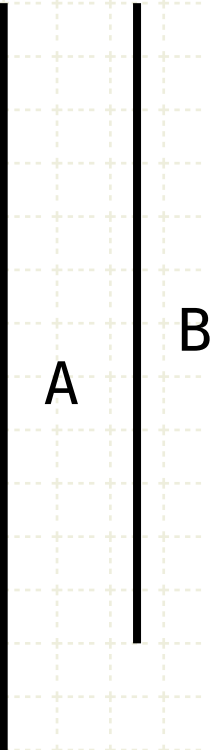


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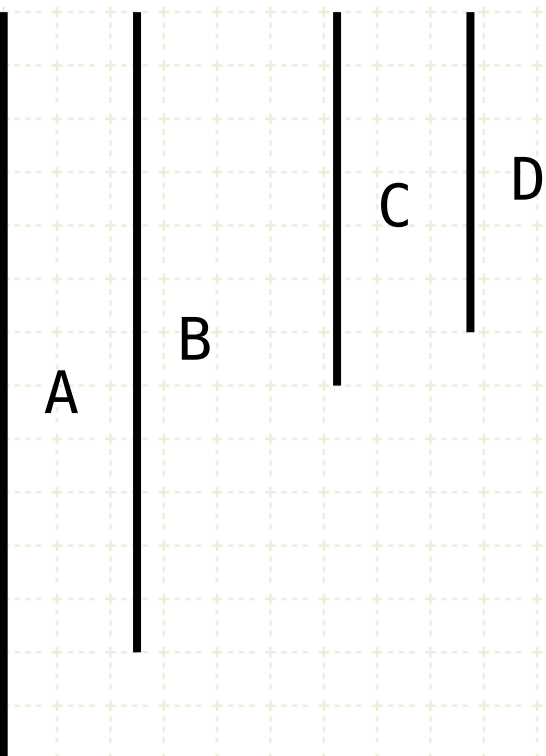
$$\gcd(A,B) = 1$$

Proof by Contradiction



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$$\gcd(A,B) = 1$$

$$A:B = C:D$$

$$C < A$$

$$D < B$$

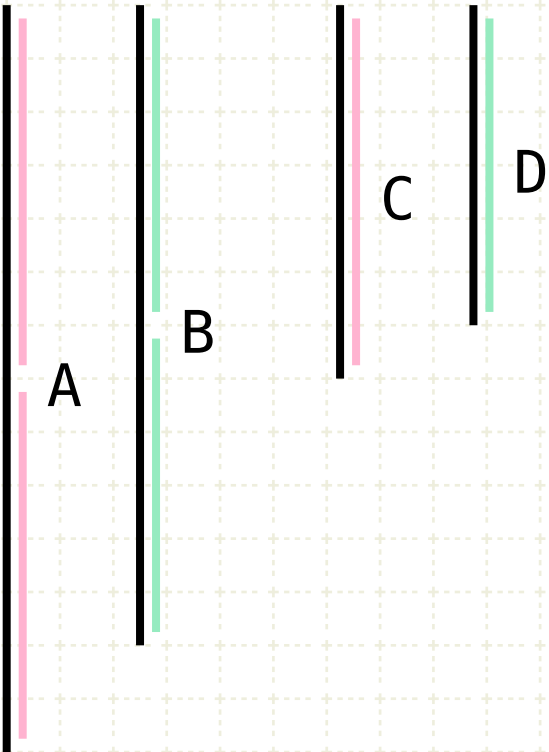
Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B



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$$B = D_1 + D_2 + \dots + D_i + \dots + D_n$$

Proof by Contradiction

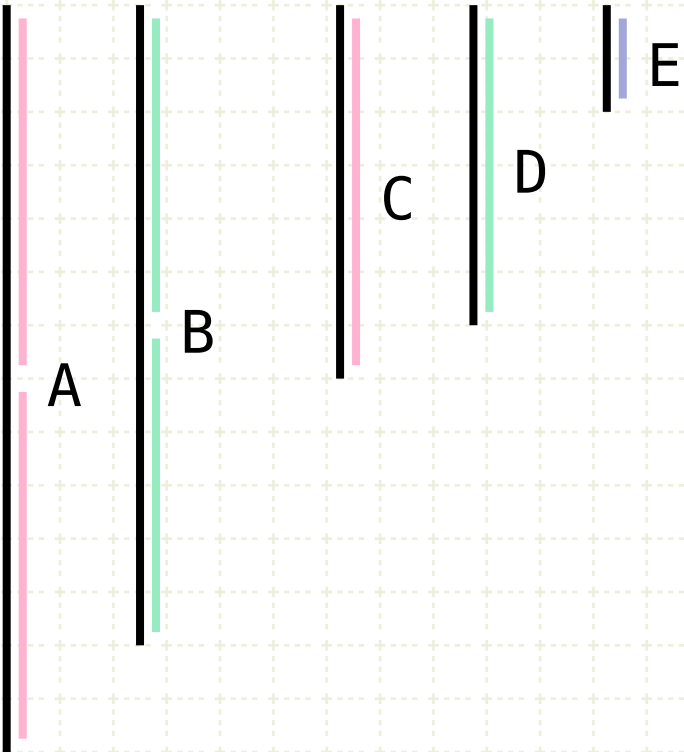
Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)



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$$A = E \cdot C, \quad E \neq 1$$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

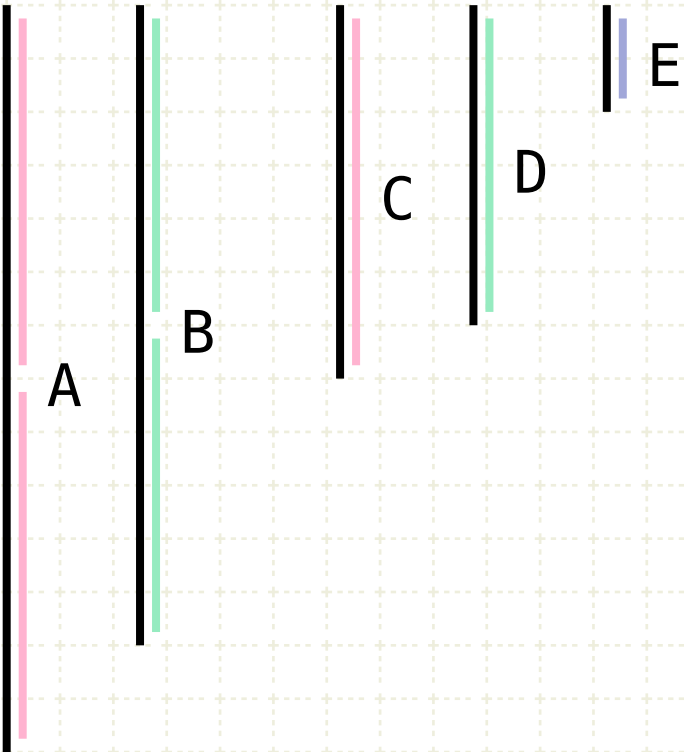
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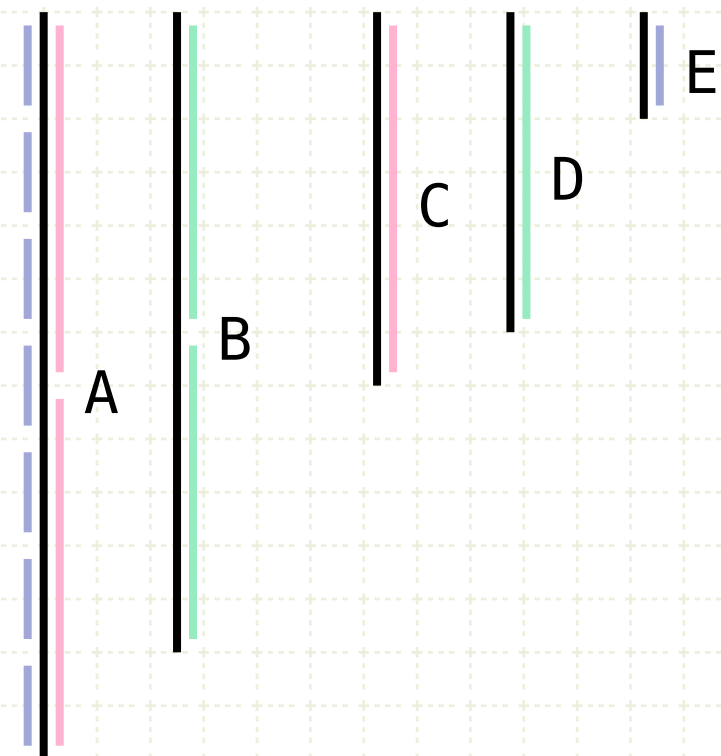
Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E



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Numbers prime to one another are the least of those which have the same ratio with them.



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$$A = C \cdot E$$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

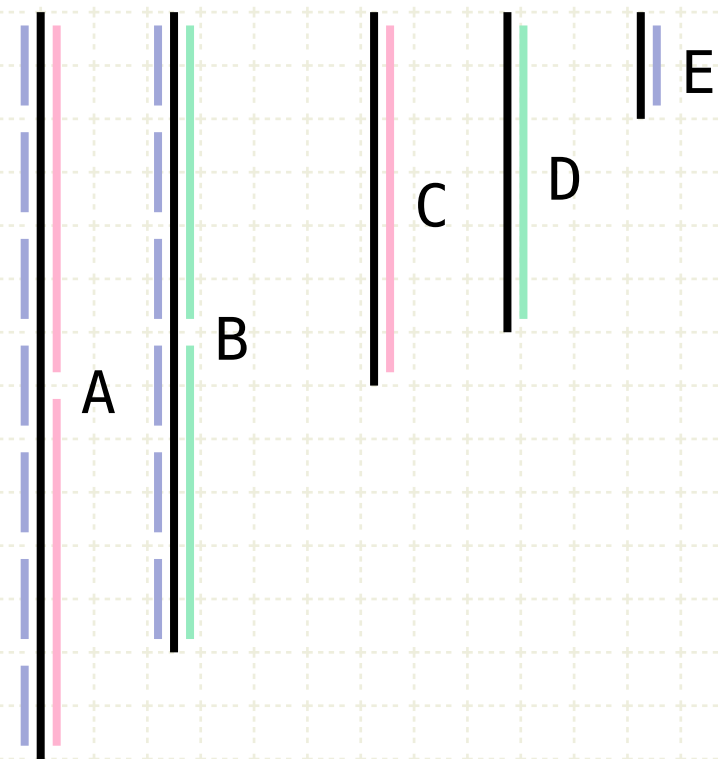
Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)



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$$A = E \cdot C, \quad E \neq 1$$

$$B = E \cdot D, \quad E \neq 1$$

$$A = C \cdot E$$

$$B = D \cdot E$$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

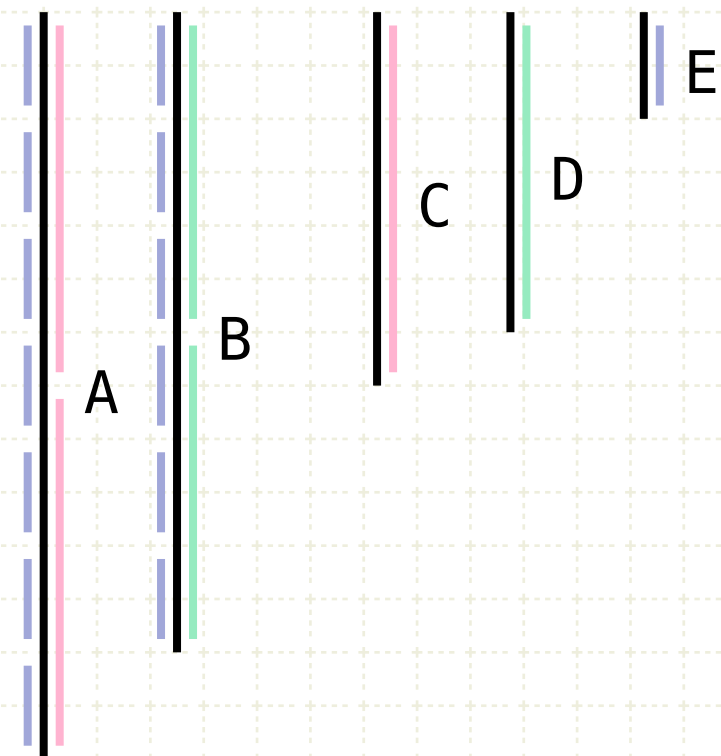
Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)



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$$B = E \cdot D, \quad E \neq 1$$

$$A = C \cdot E$$

$$B = D \cdot E$$

$$\gcd(A, B) = E, \quad E \neq 1$$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

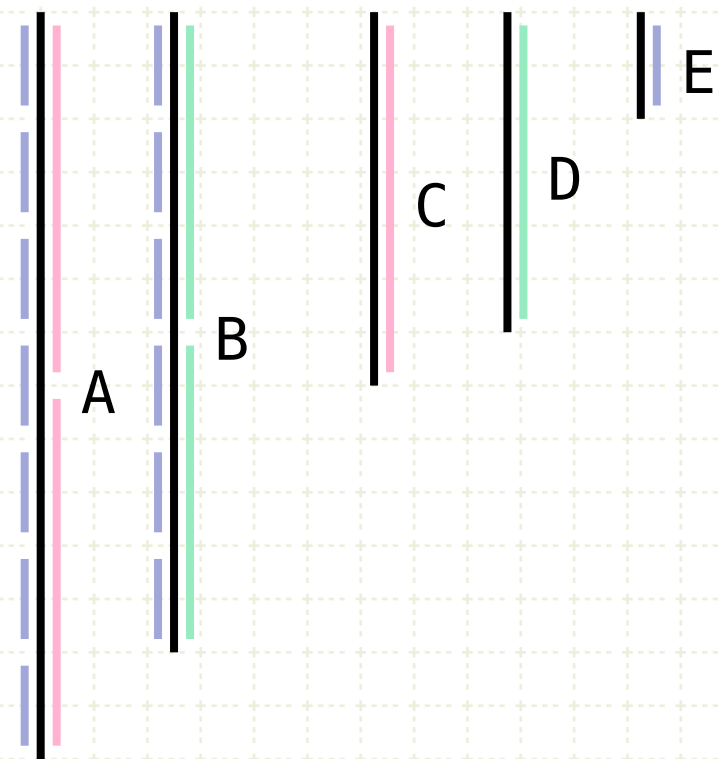
Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

Therefore A,B have a common divisor E, which is greater than one.



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Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

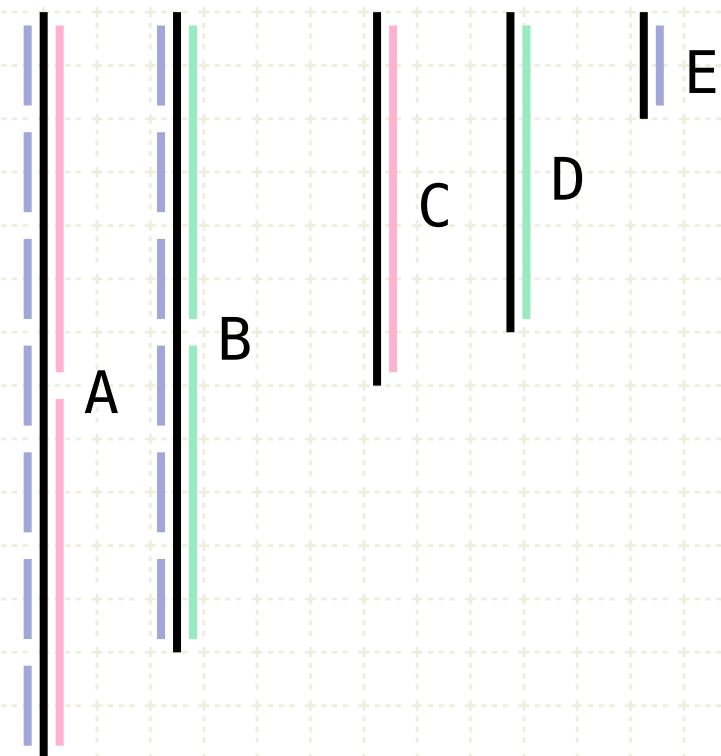
Therefore A,B have a common divisor E, which is greater than one.

This is not possible since A and B are prime to one another (VII Def.12)



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Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

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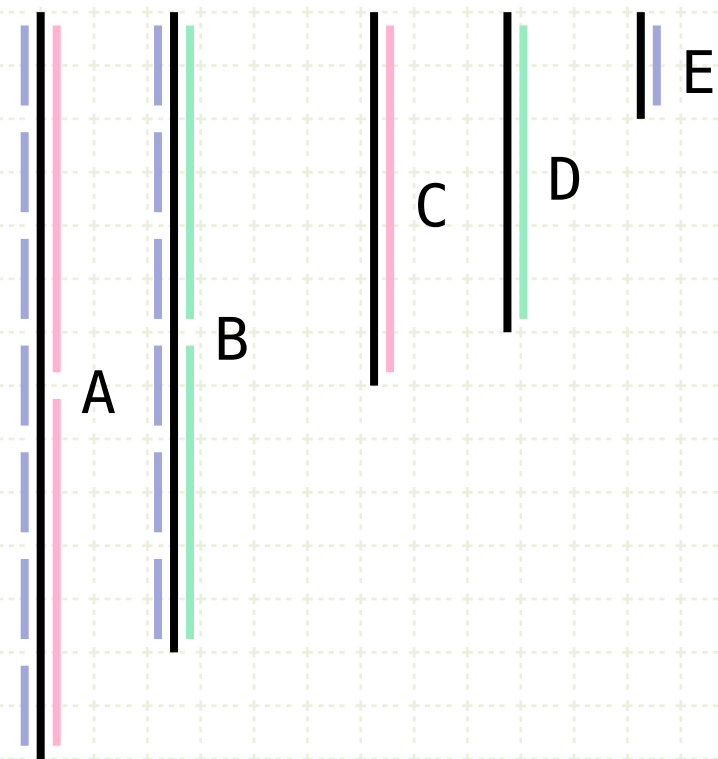
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Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

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