

# Euclid's Elements

## Book III



*A circle is a round straight line with a hole in the middle.*

**Mark Twain**

quoting a schoolchild in "-English as She Is Taught-"

*If people stand in a circle long enough, they'll eventually begin to dance.*

**George Carlin, Napalm and Silly Putty (2001)**



## Table of Contents, Chapter 3

1	To find the centre of a circle	9	If three lines, starting at a point 'A' and touching the circle, are all equal, then 'A' is the centre of the circle	17	From a given point to draw a straight line touching a given circle
2	A chord of a circle always lies inside the circle	10	A circle does not cut a circle at more points than two	18	If line touches a circle, then it is perpendicular to the diameter that touches that point
3	A line through the centre of a circle bisects a chord, and vice versa	11	Point of contact between two internal circles, and their centres, are collinear	19	If line touches a circle, then the centre of the circle lies on a line perpendicular to the original
4	A line not through the centre of a circle does not bisect a chord	12	Point of contact between two external circles, and their centres, are collinear	20	The angle at the centre of a circle is twice that from an angle from the circumference
5	If two circles cut one another, they will not have the same center	13	A circle does not touch a circle at more points than one, whether it touch it internally or externally.	21	In a circle the angles in the same segment are equal to one another
6	If two circles touch one another, they will not have the same center	14	In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.	22	The opposite angles of quadrilaterals in circles are equal to two right angles
7	Consider two lines from a point inside a circle to the edge, the longer one will be the one closest to the longest part of the diameter passing through the original point	15	The longest line in a circle is its diameter, shorter the farther away from the diameter	23	On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side
8	Consider two lines from a point outside a circle to the edge, the line closest to the centre will be longer on the concave side and shorter on the convex side	16	<b>A line on the circle, perpendicular to the diameter, lies outside the circle</b>	24	Similar segments of circles on equal straight lines are equal to one another



## Table of Contents, Chapter 3

- |    |   |    |  |
|----|---|----|--|
| 25 | Given a segment of a circle, to describe the complete circle of which it is a segment.                          | 34 | Construct a circle segment on a given circle, such that the angle within the segment is equal to a given angle               |
| 26 | In equal circles equal angles stand on equal circumferences   | 35 | If two circle chords intersect, the segments on one multiplied together equals the segments of the other multiplied together |
| 27 | In equal circles angles standing on equal circumferences are equal to one another                               |    |  |
| 28 | In equal circles equal straight lines cut off equal circumferences  | 36 | Secant-tangent law   |
| 29 | In equal circles equal circumferences are subtended by equal straight lines                                     | 37 | Converse of the secant-tangent law   |
| 30 | To bisect a given circumference   |    |  |
| 31 | In a circle the angle in the semicircle is right ...  |    |  |
| 32 | The angle between a tangent and a straight line cutting a circle is equal to the angle in the alternate segment |    |  |
| 33 | Construct a circle segment on a given line, such that the angle within the segment is equal to a given angle    |    |  |



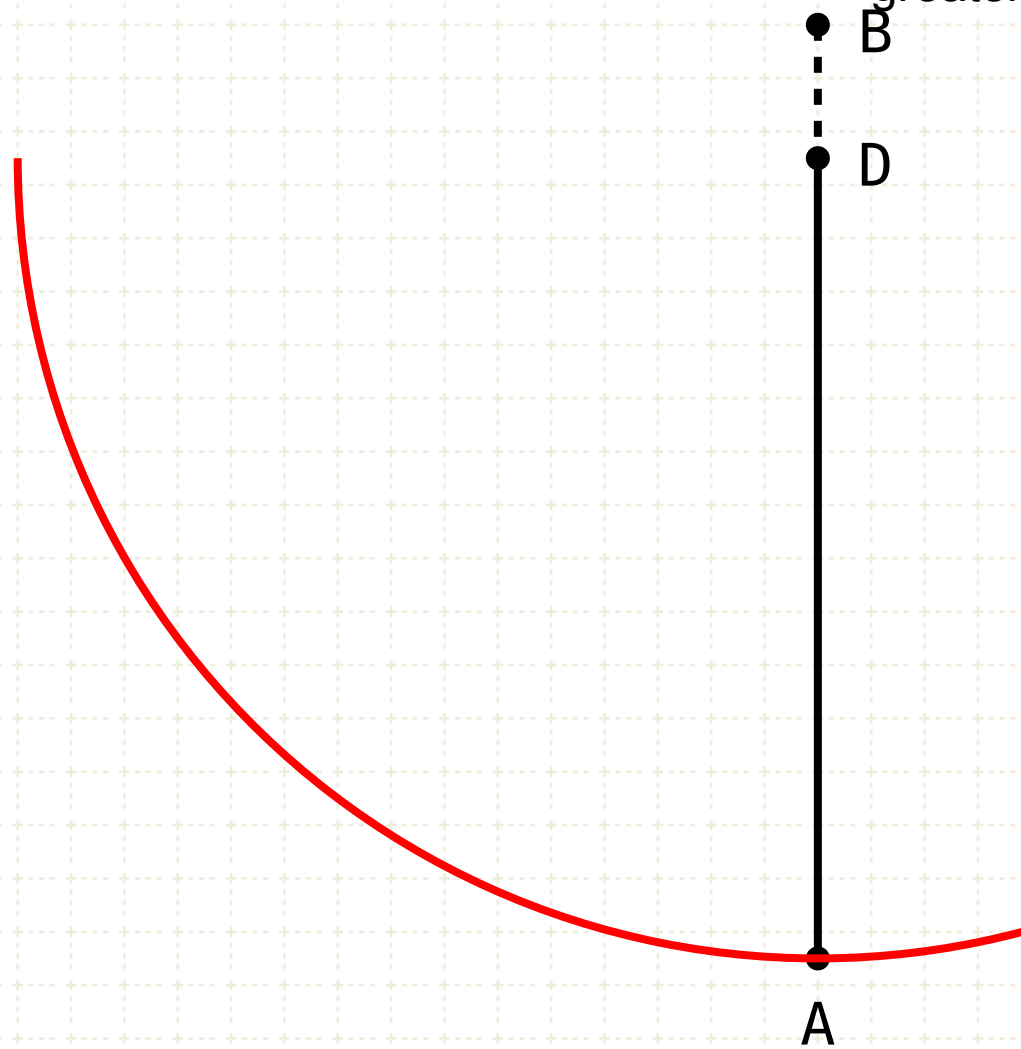
## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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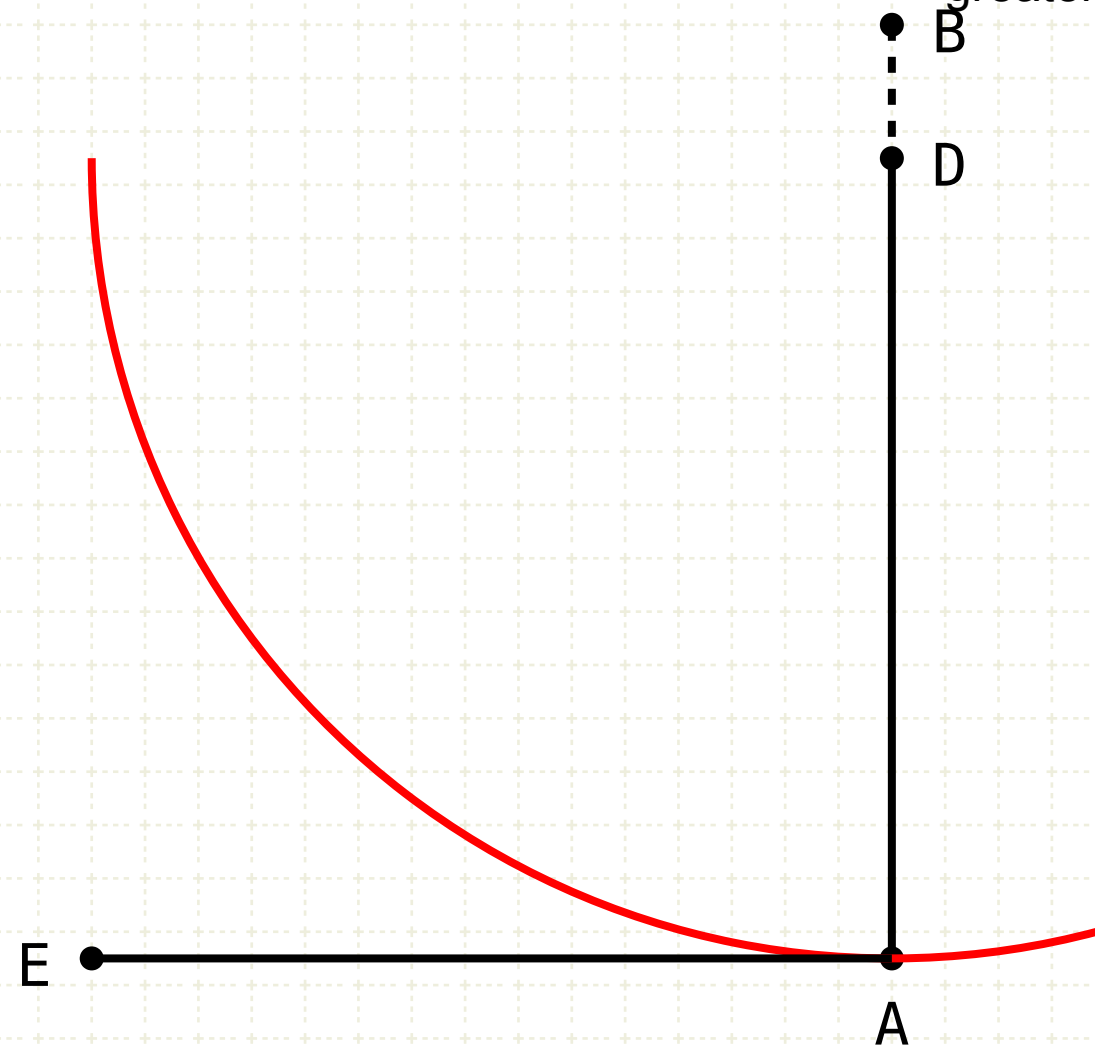
### In other words

Given a circle, with AB as a diameter, and D as the centre...



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The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



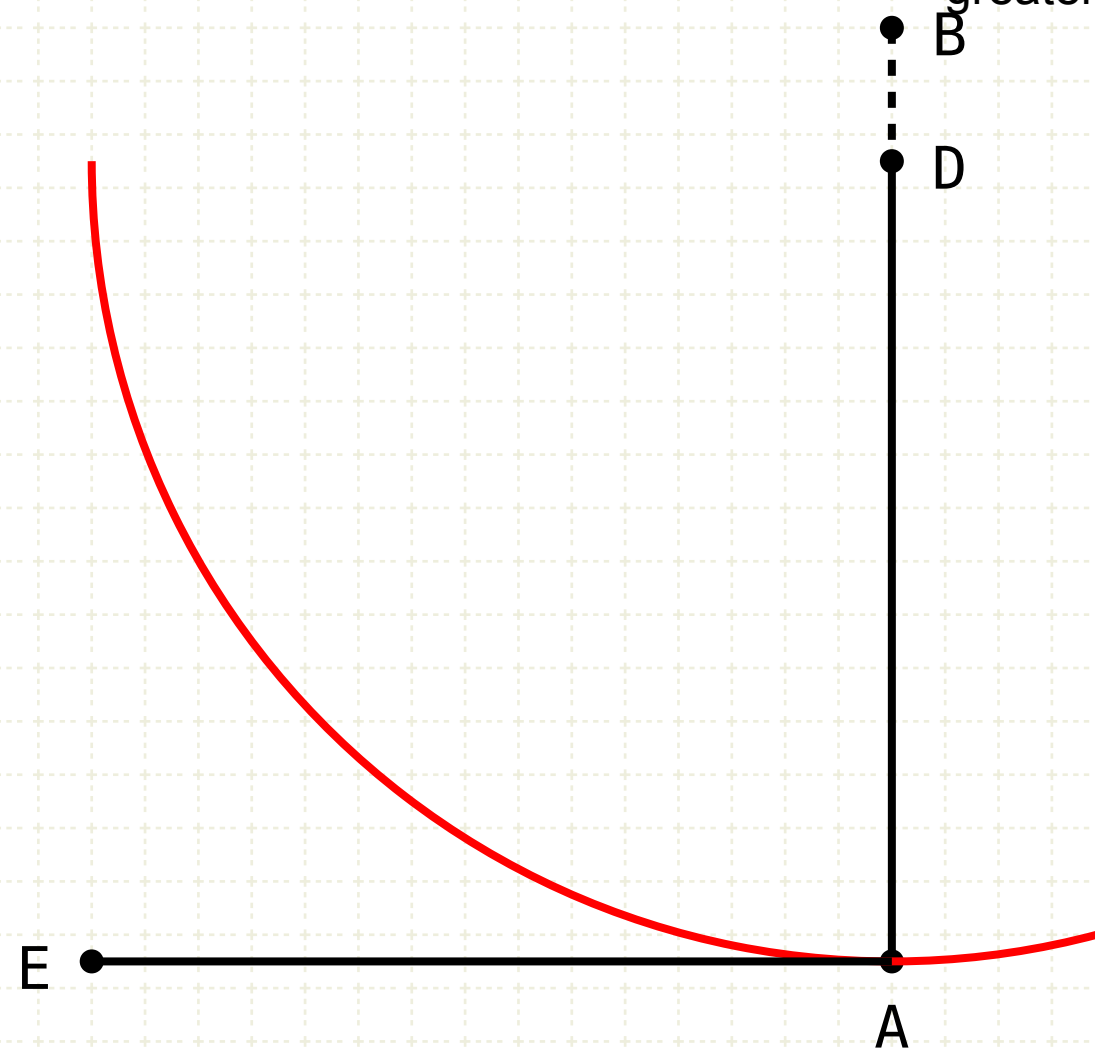
### In other words

Given a circle, with AB as a diameter, and D as the centre...

Draw a line AE from point A, perpendicular to AB

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The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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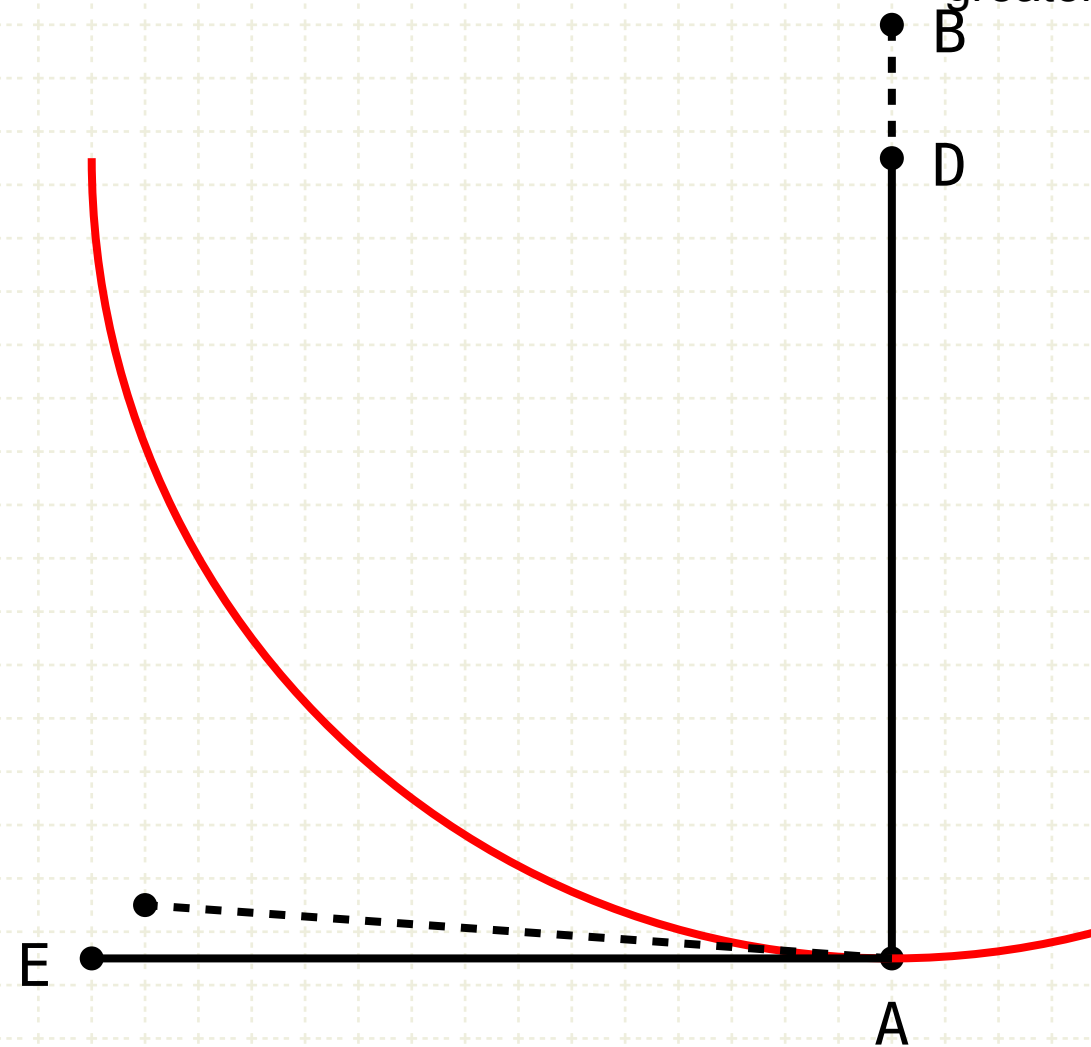
Given a circle, with AB as a diameter, and D as the centre...

Draw a line AE from point A, perpendicular to AB

(1) The Line AE falls outside of the circle

# Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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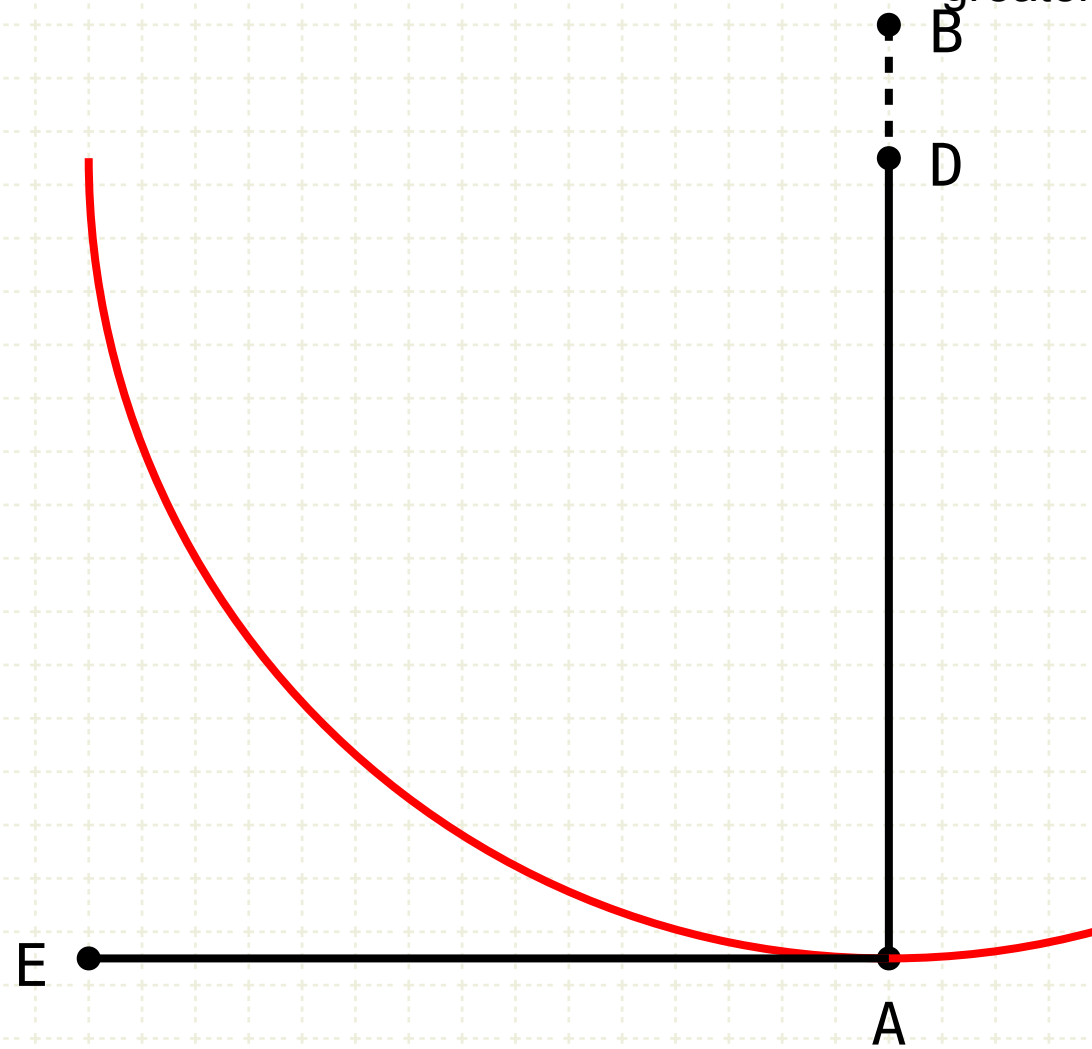
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- (1) The Line AE falls outside of the circle
- (2) No other line from point A can squeeze between the circumference, and the line AE



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The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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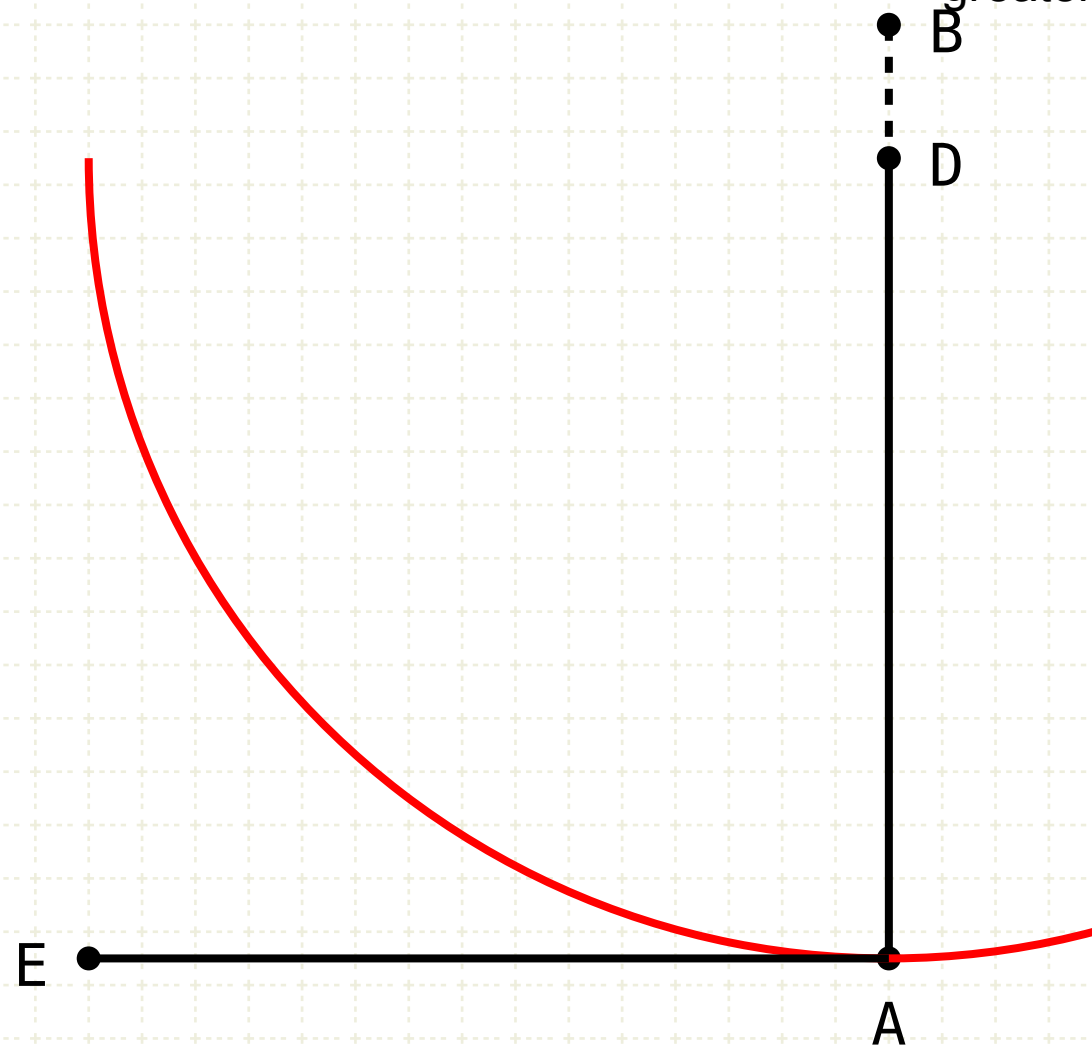
Given a circle, with AB as a diameter, and D as the centre...

Draw a line AE from point A, perpendicular to AB

- (1) The Line AE falls outside of the circle
- (2) No other line from point A can squeeze between the circumference, and the line AE
- (3) - The angle between AB and the circumference of the circle cannot be less than a right angle,  
- and the angle between AE and the circumference cannot be greater than zero

## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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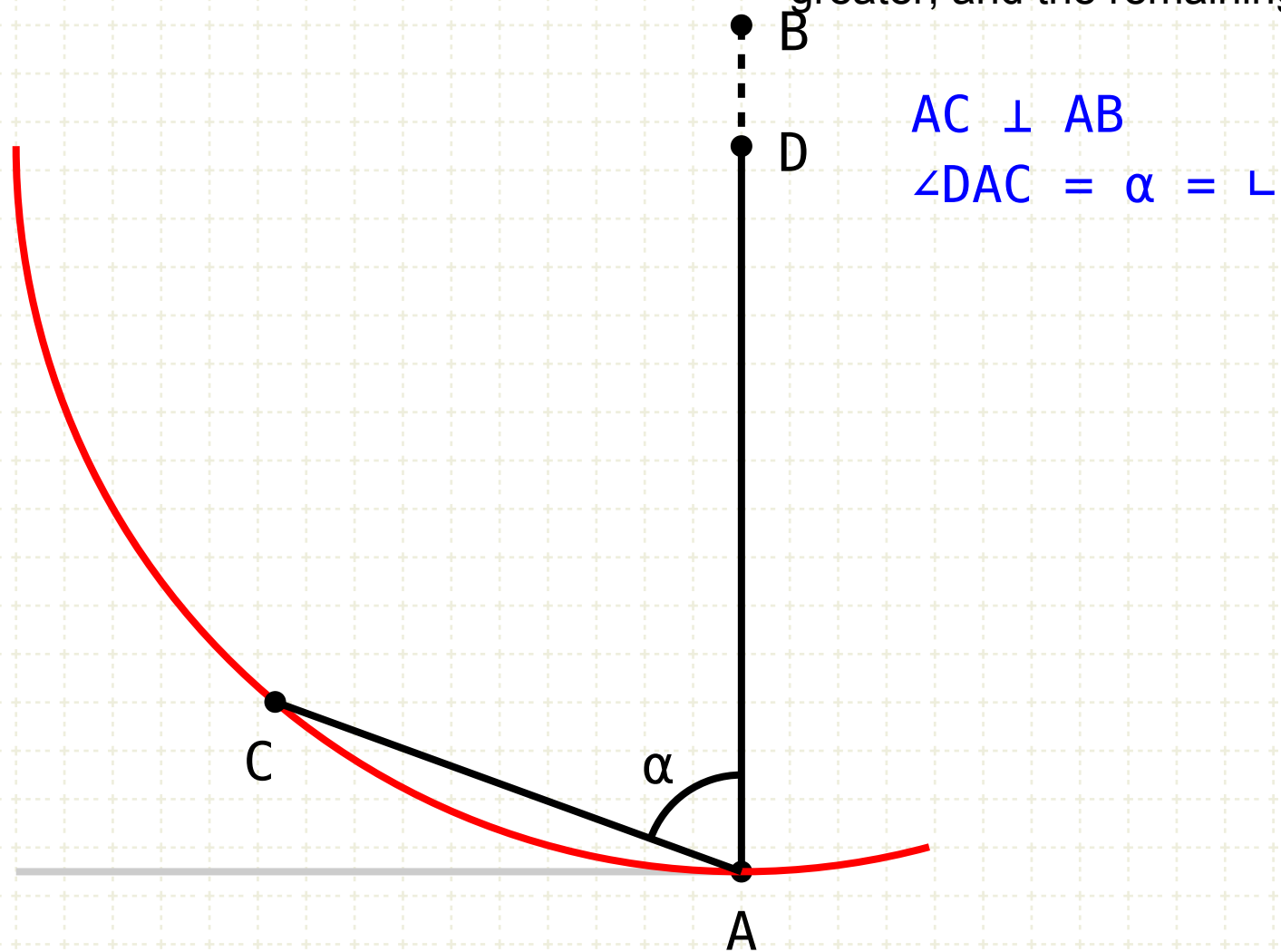
Given a circle, with AB as a diameter, and D as the centre...

(1) The Line AE falls outside of the circle

### Proof by Contradiction (1)

## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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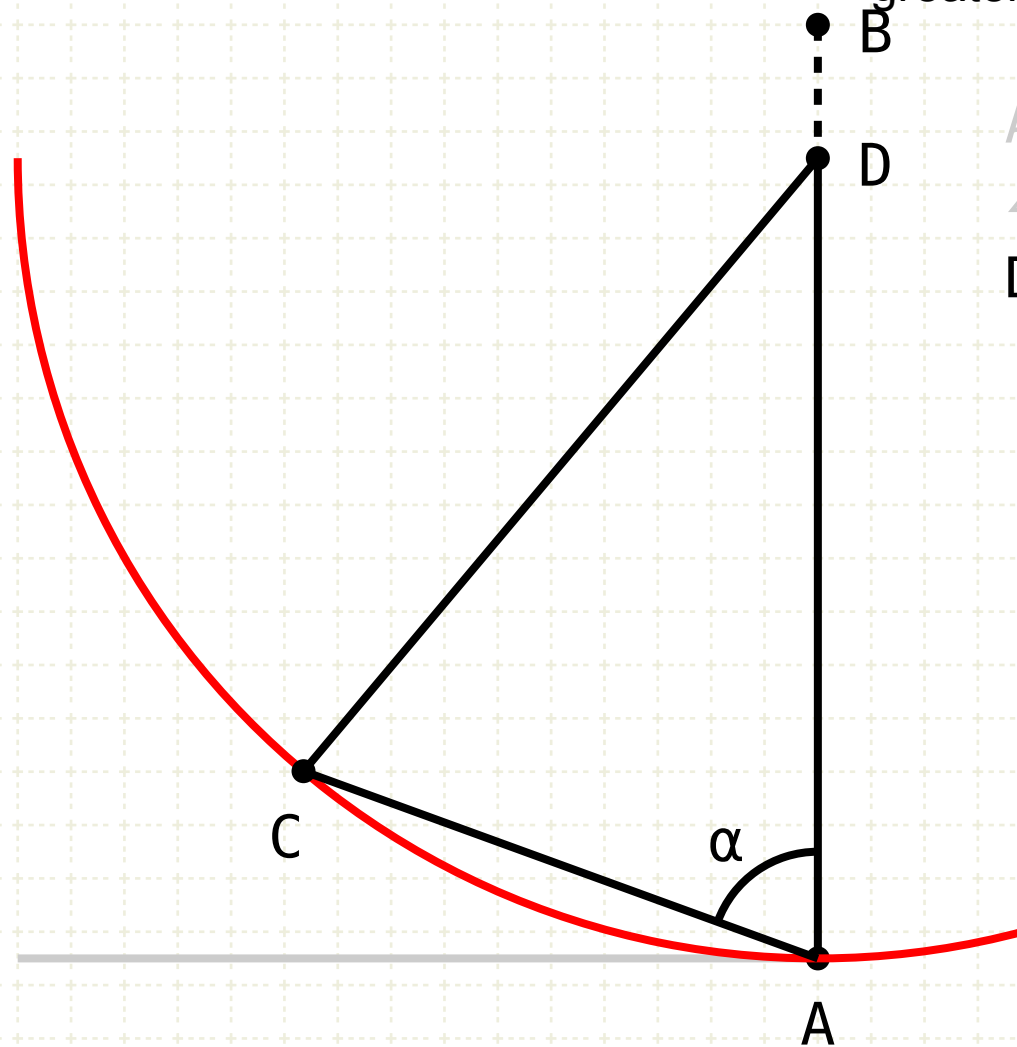
(1) The Line AE falls outside of the circle

### Proof by Contradiction (1)

Assume that the line perpendicular to AB lies within the circle, such as line AC

## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$\begin{aligned} AC &\perp AB \\ \angle DAC &= \alpha = \angle \\ DC &= AC \end{aligned}$$

### In other words

Given a circle, with AB as a diameter, and D as the centre...

(1) The Line AE falls outside of the circle

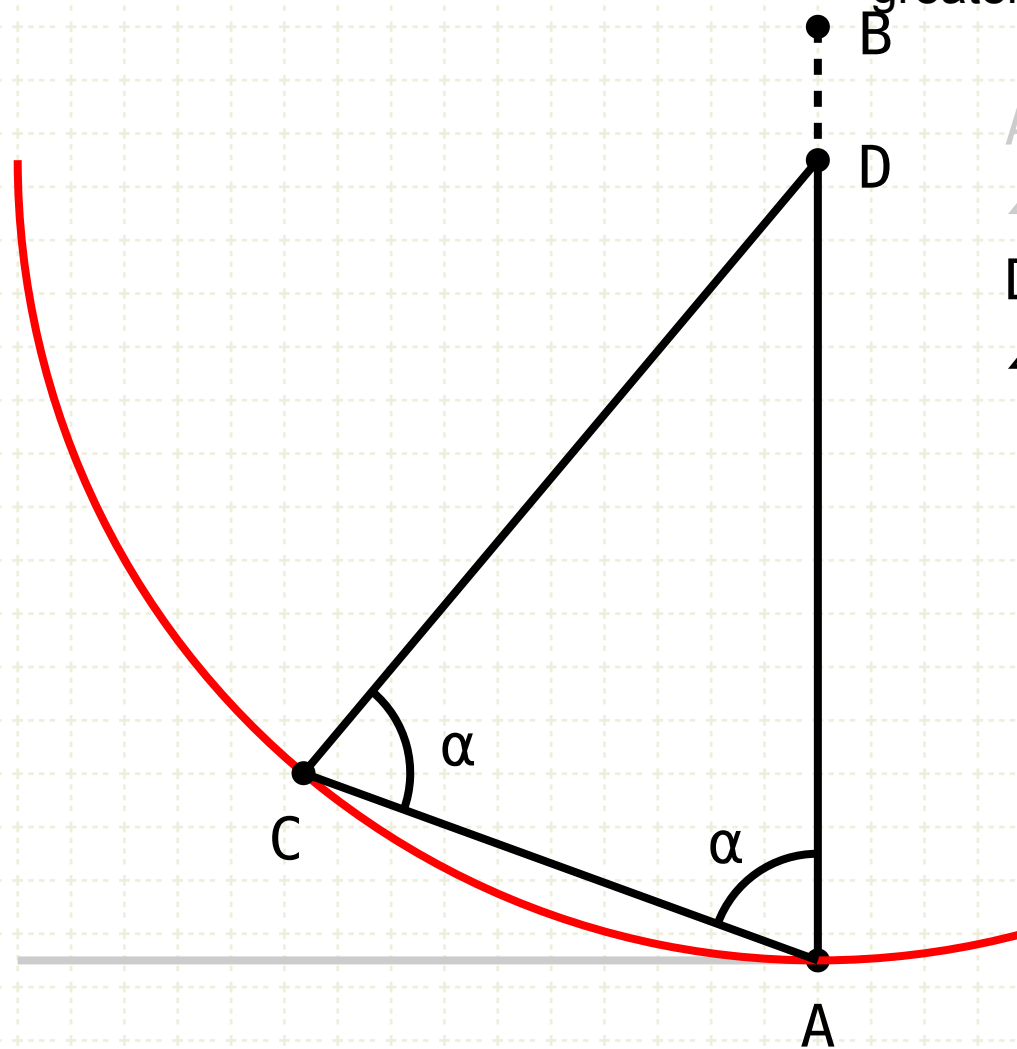
### Proof by Contradiction (1)

Assume that the line perpendicular to AB lies within the circle, such as line AC

Draw the line DC

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The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$AC \perp AB$$

$$\angle DAC = \alpha = \angle DCA$$

$$DC = AC$$

$$\angle DAC = \angle DCA = \alpha$$

### In other words

Given a circle, with AB as a diameter, and D as the centre...

(1) The Line AE falls outside of the circle

### Proof by Contradiction (1)

Assume that the line perpendicular to AB lies within the circle, such as line AC

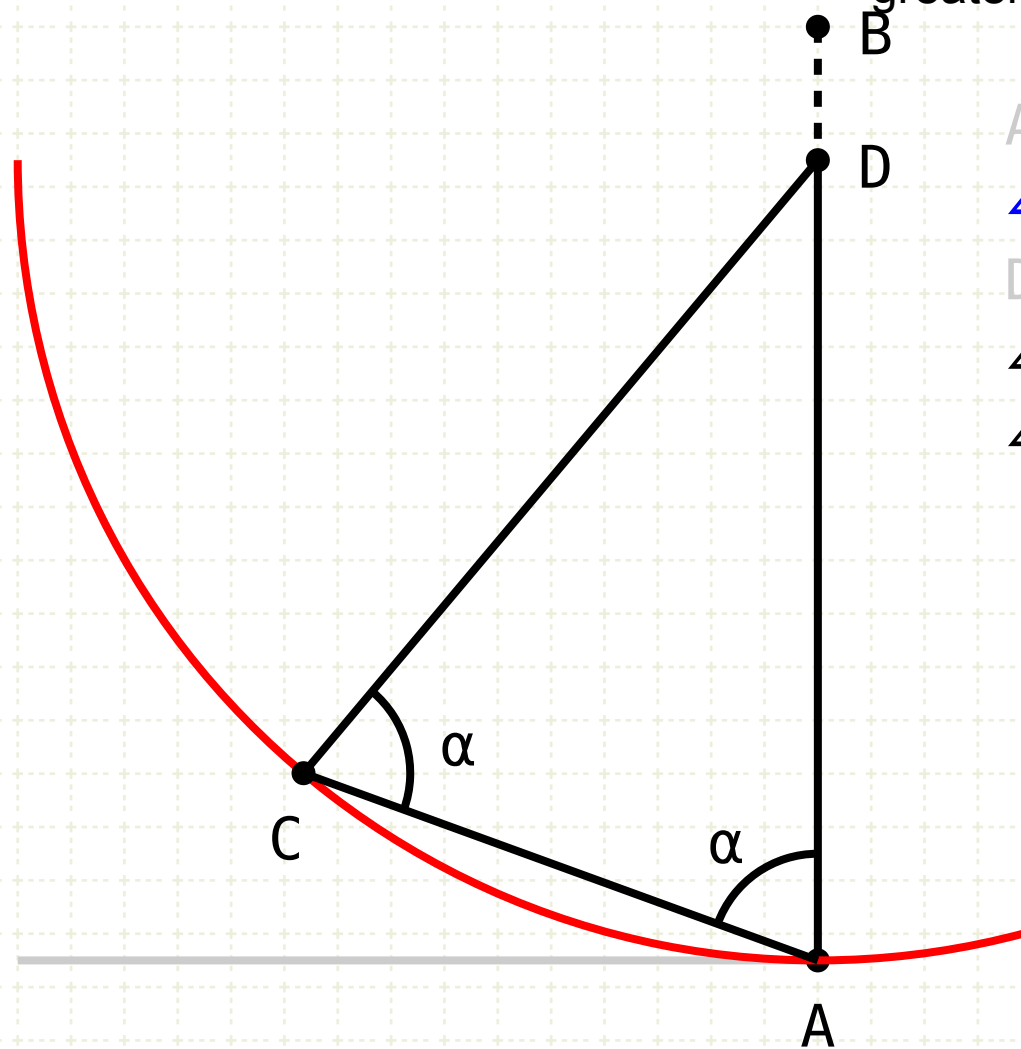
Draw the line DC

DC equals AC, therefore the triangle is an isosceles triangle, and the angles DAC and DCA are equal (I-5)



## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$AC \perp AB$$

$$\angle DAC = \alpha = L$$

$$DC = AC$$

$$\angle DAC = \angle DCA = \alpha$$

$$\angle DAC + \angle DCA = 2 \cdot L$$

### In other words

Given a circle, with AB as a diameter, and D as the centre...

(1) The Line AE falls outside of the circle

### Proof by Contradiction (1)

Assume that the line perpendicular to AB lies within the circle, such as line AC

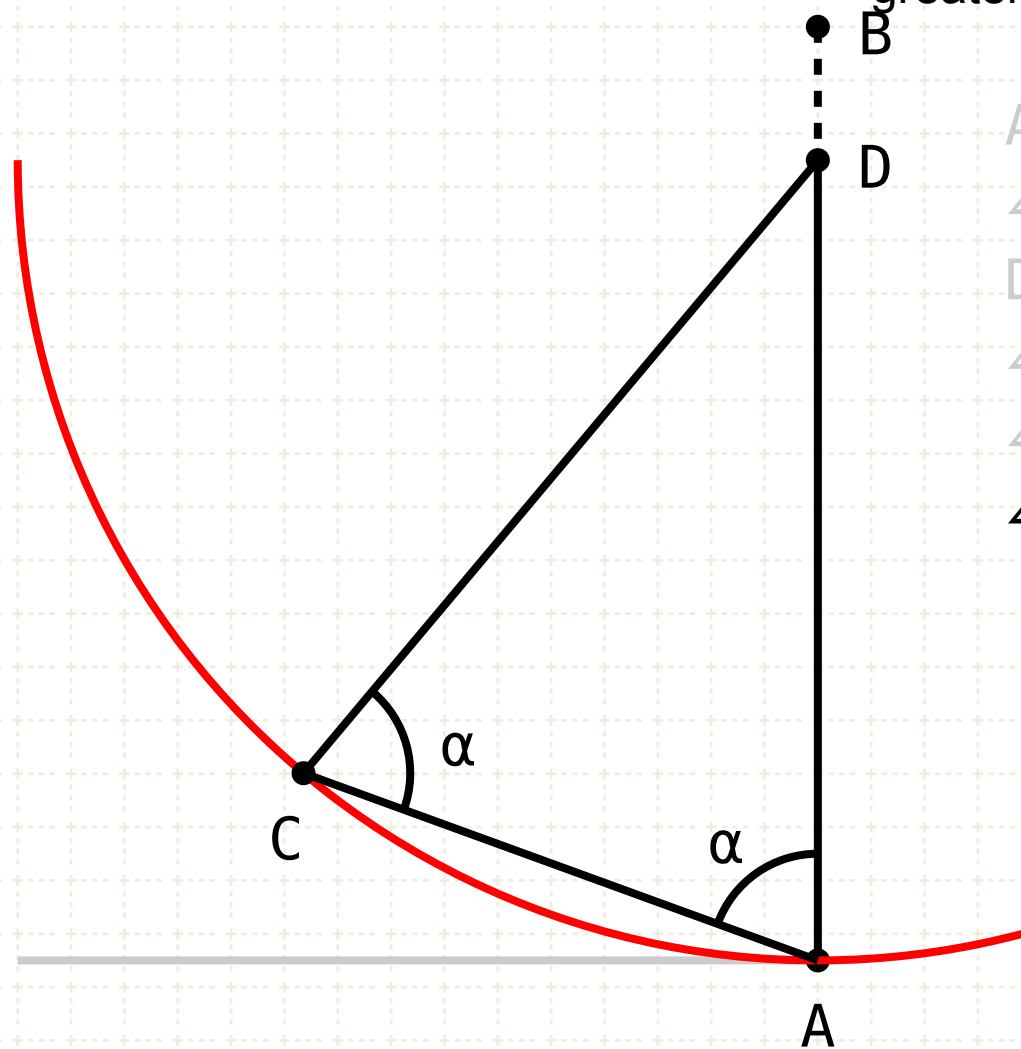
Draw the line DC

DC equals AC, therefore the triangle is an isosceles triangle, and the angles DAC and DCA are equal (I-5)

DAC is right, which means DCA is also right, and their sum is equal to two right angles

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$$AC \perp AB$$

$$\angle DAC = \alpha = L$$

$$DC = AC$$

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$$\angle DAC + \angle DCA = 2 \cdot L$$

$$\angle DAC + \angle DCA < 2 \cdot L$$

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Given a circle, with AB as a diameter, and D as the centre...

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Assume that the line perpendicular to AB lies within the circle, such as line AC

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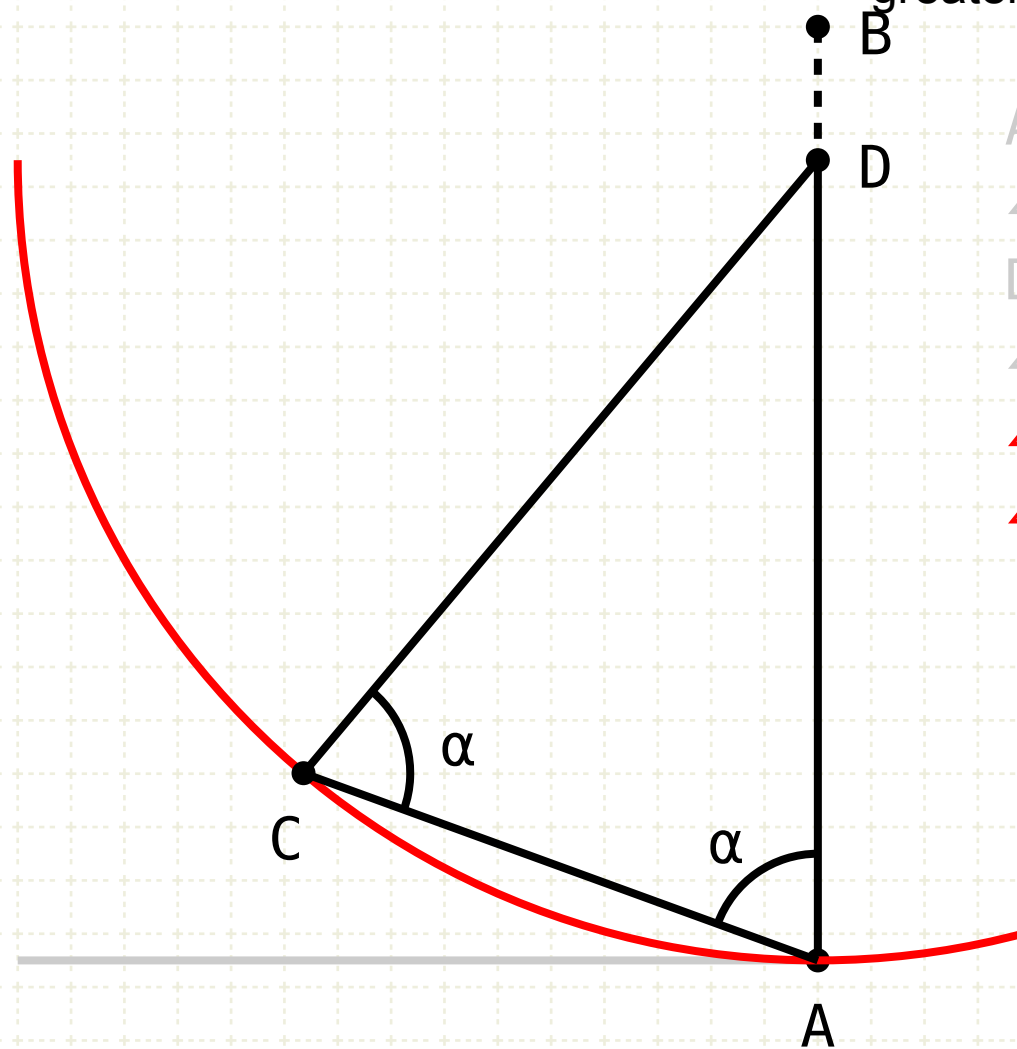
DC equals AC, therefore the triangle is an isosceles triangle, and the angles DAC and DCA are equal (I·5)

DAC is right, which means DCA is also right, and their sum is equal to two right angles

The sum of any two angles in a triangle must be less than two right angles (I·17)

## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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Given a circle, with AB as a diameter, and D as the centre...

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Assume that the line perpendicular to AB lies within the circle, such as line AC

Draw the line DC

DC equals AC, therefore the triangle is an isosceles triangle, and the angles DAC and DCA are equal (I·5)

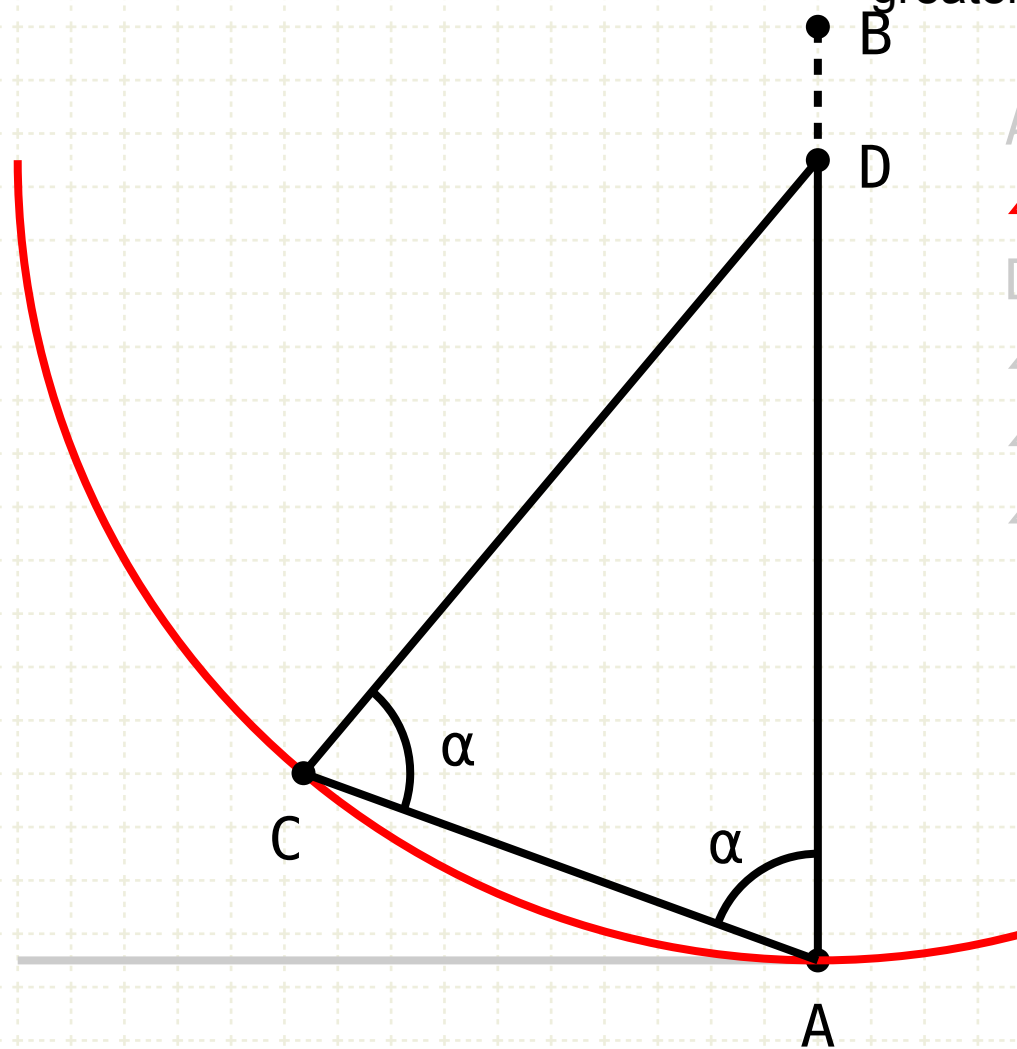
DAC is right, which means DCA is also right, and their sum is equal to two right angles

The sum of any two angles in a triangle must be less than two right angles (I·17)

Thus we have a contradiction

## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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Given a circle, with AB as a diameter, and D as the centre...

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Assume that the line perpendicular to AB lies within the circle, such as line AC

Draw the line DC

DC equals AC, therefore the triangle is an isosceles triangle, and the angles DAC and DCA are equal (I·5)

DAC is right, which means DCA is also right, and their sum is equal to two right angles

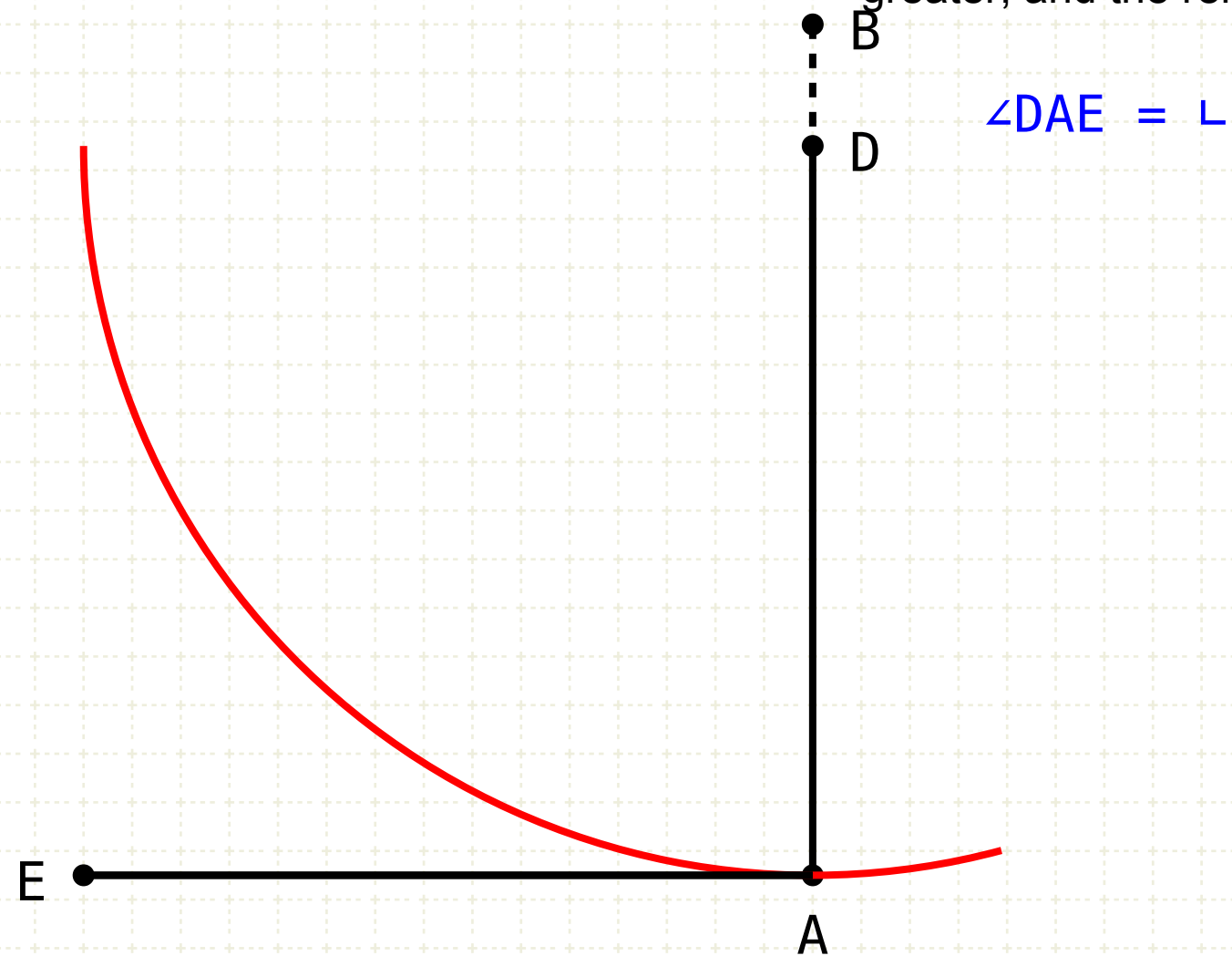
The sum of any two angles in a triangle must be less than two right angles (I·17)

Thus we have a contradiction

The angle DAC cannot be a right angle

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The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



## In other words

Given a circle, with  $AB$  as a diameter, and  $D$  as the centre...

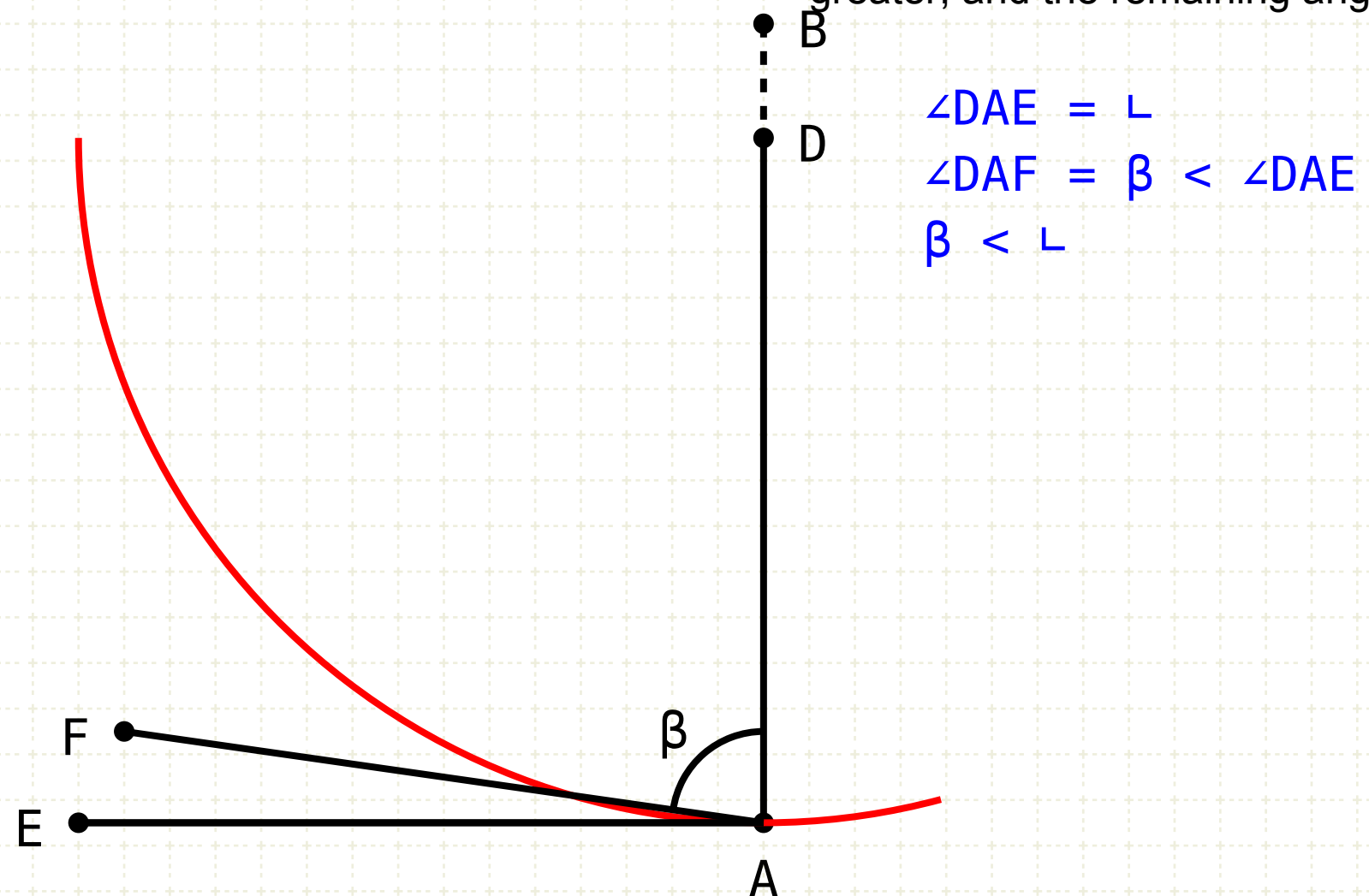
(2) No other line from point  $A$  can squeeze between the circumference, and the line  $AE$

## Proof by Contradiction (2)



## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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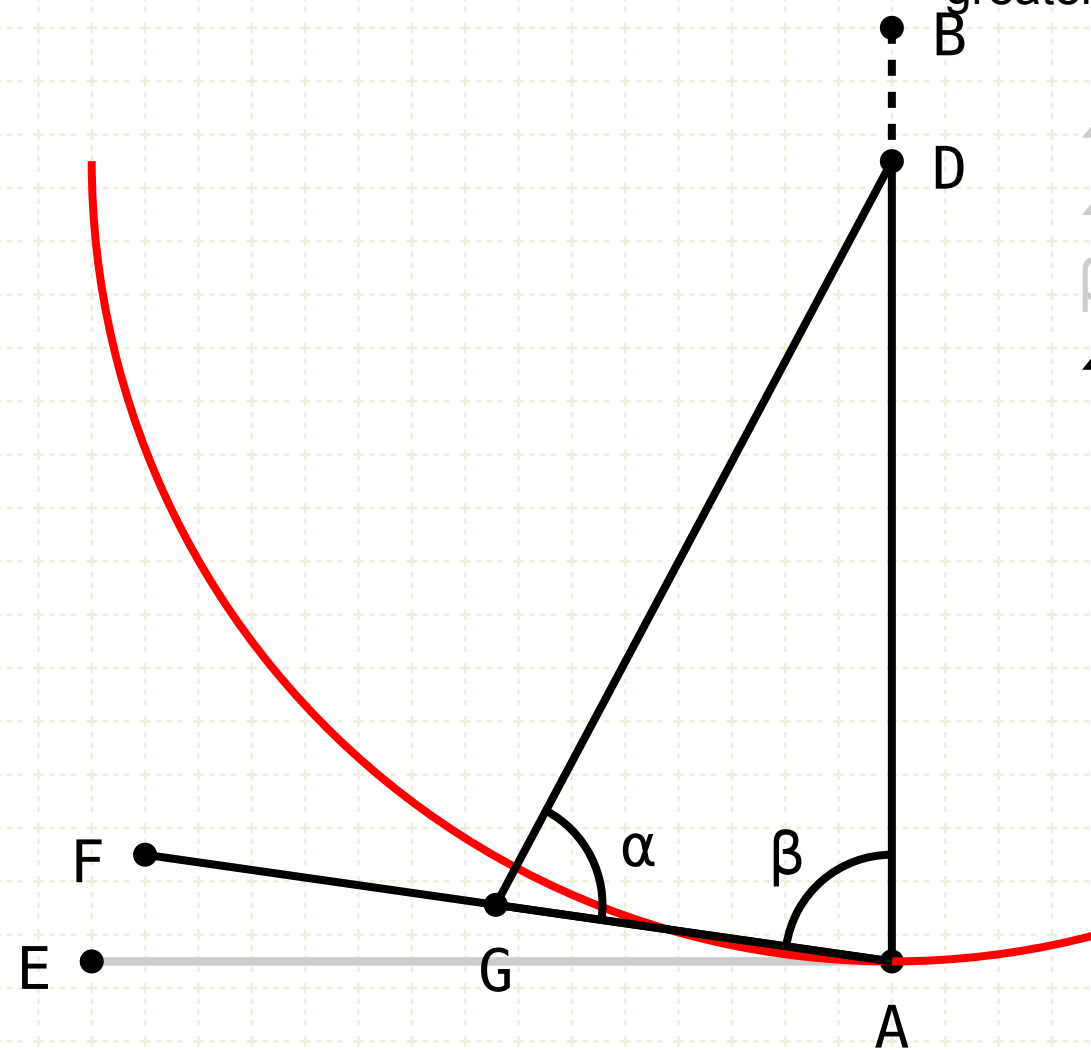
(2) No other line from point A can squeeze between the circumference, and the line AE

### Proof by Contradiction (2)

Assume that the line FA lies between the circumference of the circle and the line AE

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The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$\begin{aligned}\angle DAE &= L \\ \angle DAF &= \beta < \angle DAE \\ \beta &< L \\ \angle DGA &= \alpha = L\end{aligned}$$

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Given a circle, with AB as a diameter, and D as the centre...

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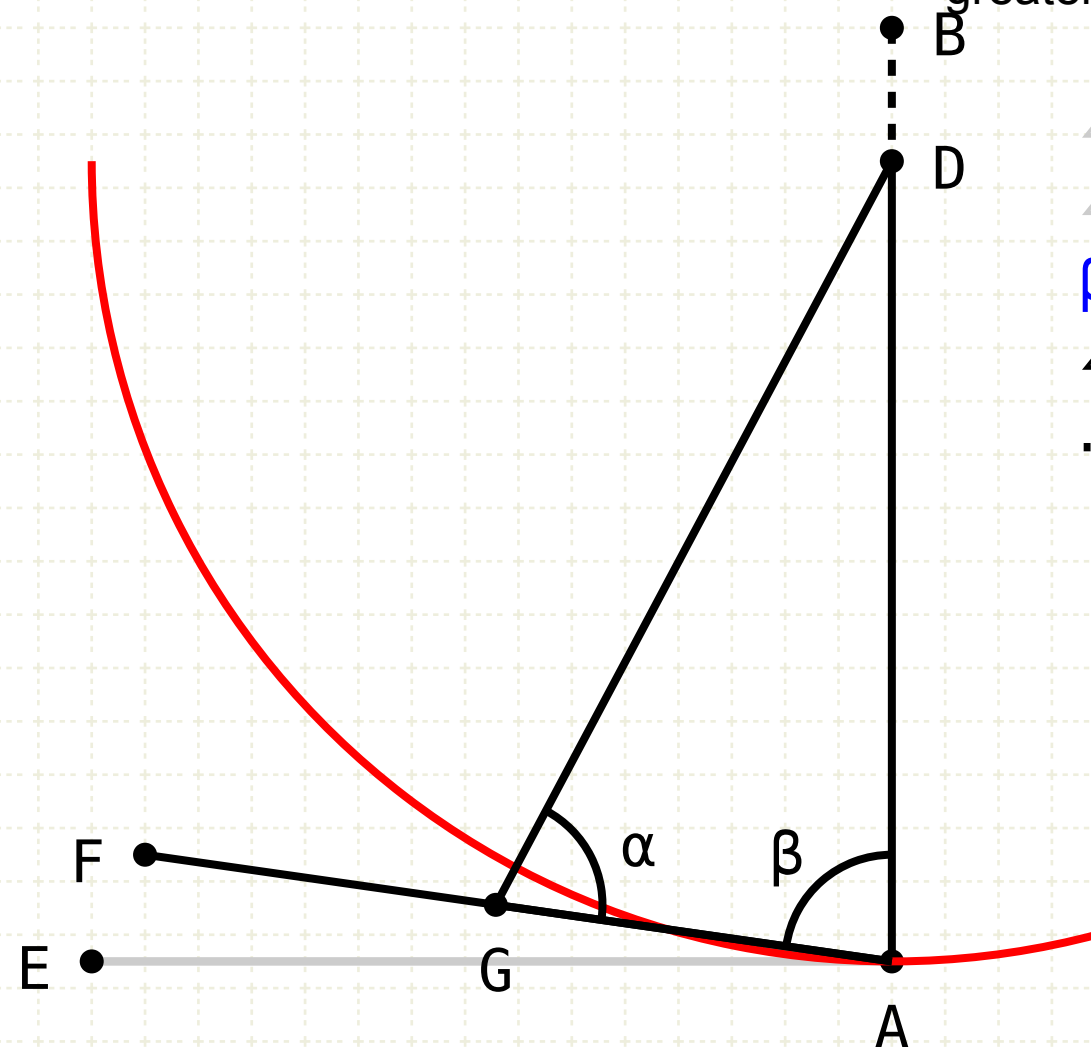
### Proof by Contradiction (2)

Assume that the line FA lies between the circumference of the circle and the line AE

Draw a perpendicular line from D to line FA, intersecting at the point G

# Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$\begin{aligned}\angle DAE &= L \\ \angle DAF &= \beta < \angle DAE \\ \beta &< L \\ \angle DGA &= \alpha = L \\ \therefore \alpha &> \beta\end{aligned}$$

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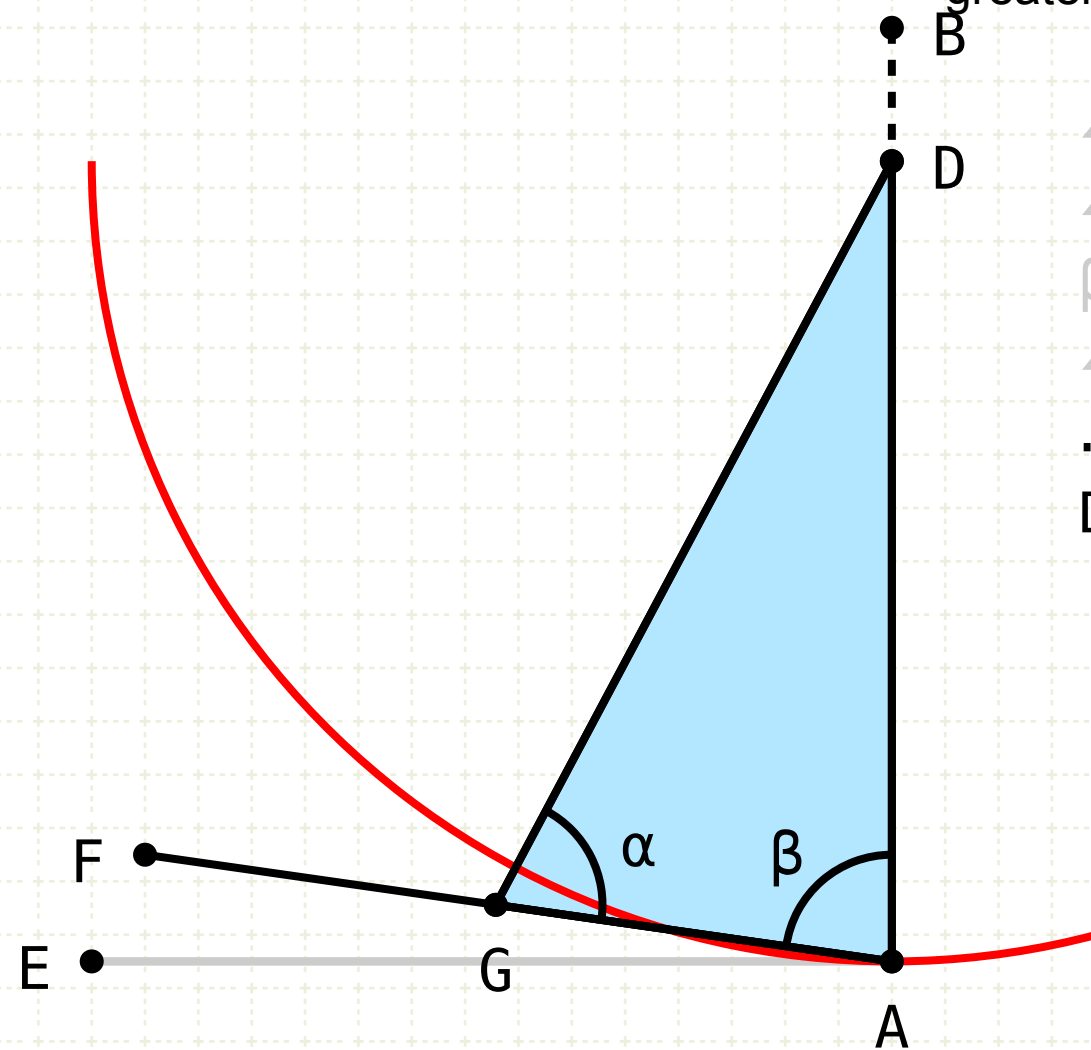
Assume that the line FA lies between the circumference of the circle and the line AE

Draw a perpendicular line from D to line FA, intersecting at the point G

Angle DGA ( $\alpha$ ) is right, and DAG ( $\beta$ ) is less than a right angle, so DGA is greater than DAG

## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$\begin{aligned}\angle DAE &= L \\ \angle DAF &= \beta < \angle DAE \\ \beta &< L \\ \angle DGA &= \alpha = L \\ \therefore \alpha &> \beta \\ DA &> DG\end{aligned}$$

### In other words

Given a circle, with AB as a diameter, and D as the centre...

(2) No other line from point A can squeeze between the circumference, and the line AE

### Proof by Contradiction (2)

Assume that the line FA lies between the circumference of the circle and the line AE

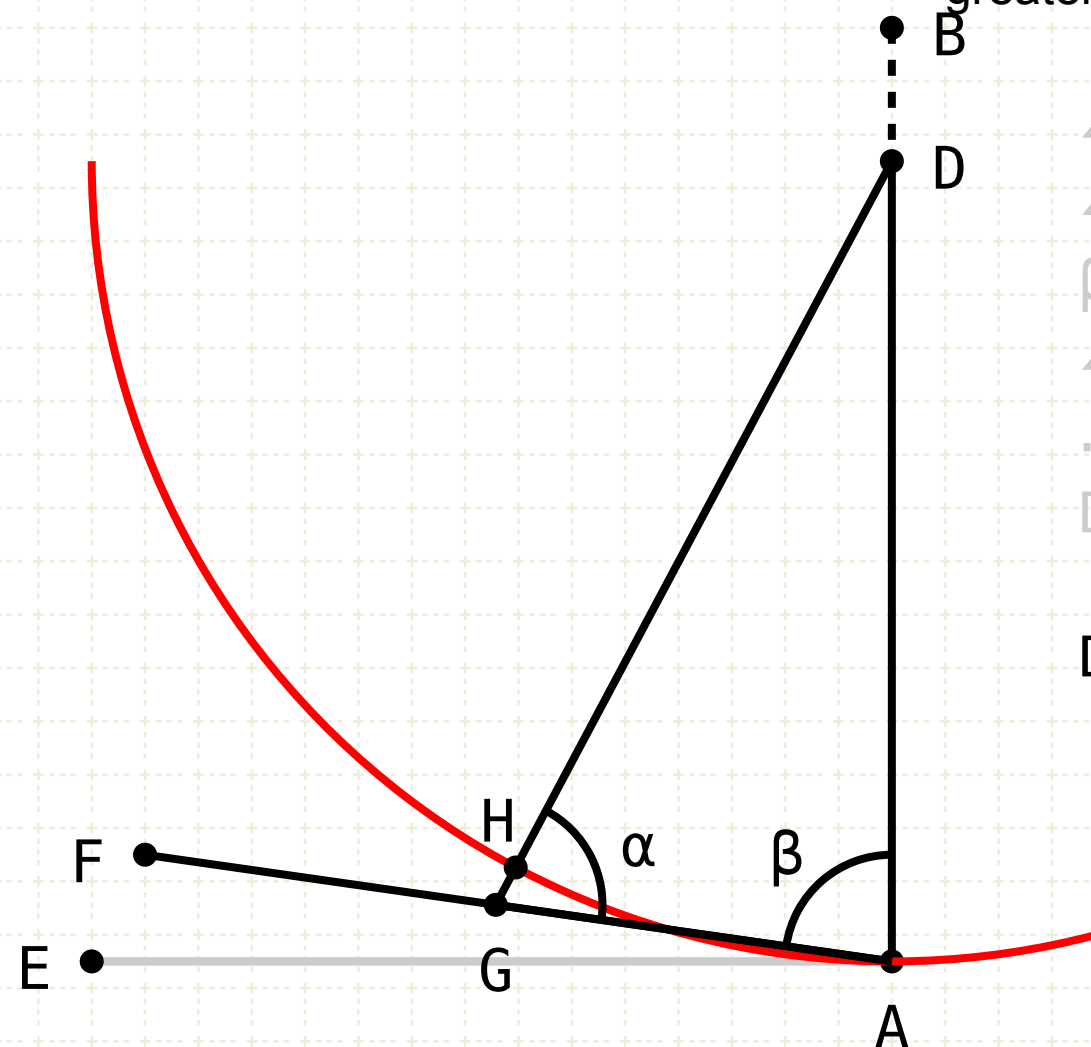
Draw a perpendicular line from D to line FA, intersecting at the point G

Angle DGA ( $\alpha$ ) is right, and DAG ( $\beta$ ) is less than a right angle, so DGA is greater than DAG

In a triangle, the larger line is opposite the larger angle, thus DA is larger than DG (I.19)

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The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$\begin{aligned} \angle DAE &= L \\ \angle DAF &= \beta < \angle DAE \\ \beta &< L \\ \angle DGA &= \alpha = L \\ \therefore \alpha &> \beta \\ DA &> DG \\ DG &> DH \end{aligned}$$

## In other words

Given a circle, with AB as a diameter, and D as the centre...

(2) No other line from point A can squeeze between the circumference, and the line AE

## Proof by Contradiction (2)

Assume that the line FA lies between the circumference of the circle and the line AE

Draw a perpendicular line from D to line FA, intersecting at the point G

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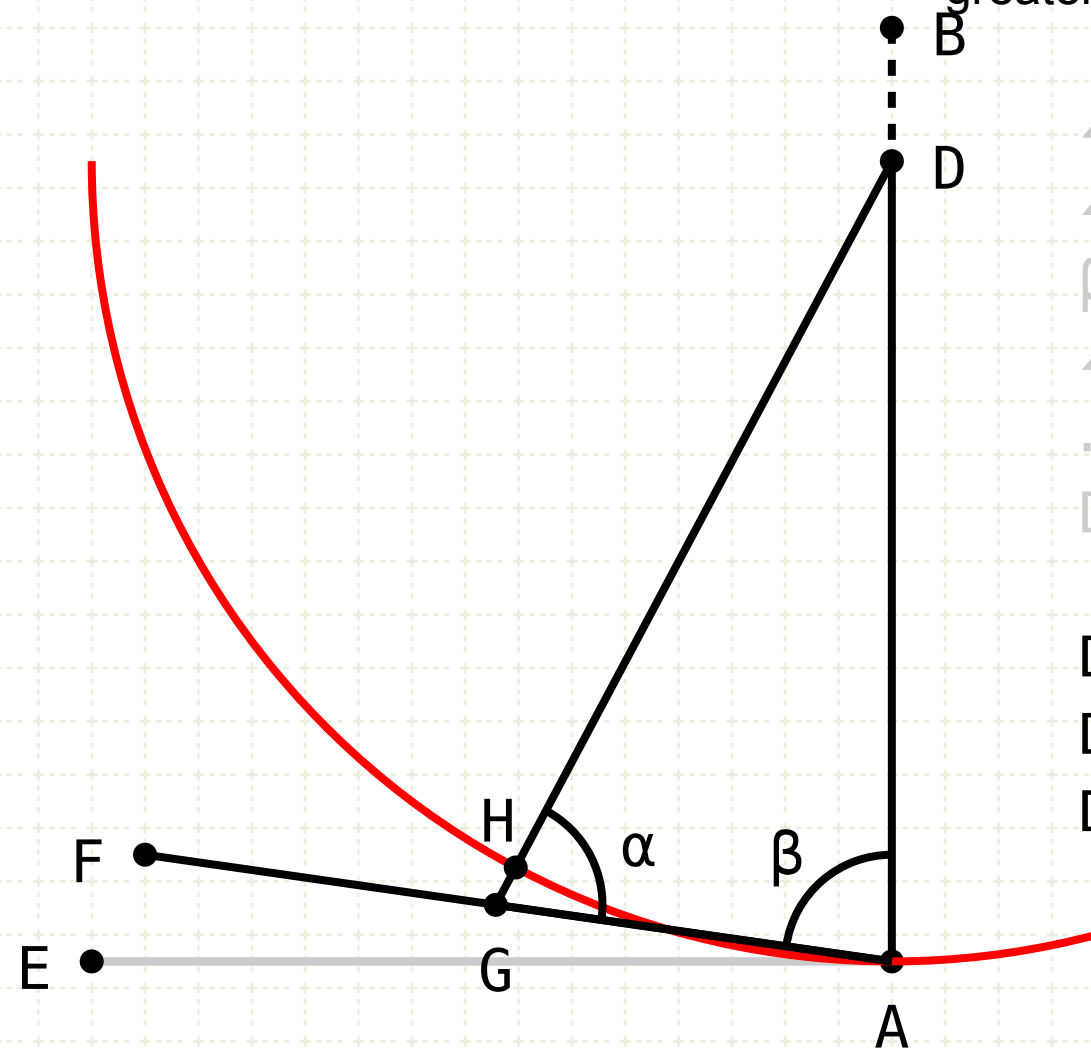
In a triangle, the larger line is opposite the larger angle, thus DA is larger than DG (I.19)

DG is larger than DH



## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$\begin{aligned}\angle DAE &= L \\ \angle DAF &= \beta < \angle DAE \\ \beta &< L \\ \angle DGA &= \alpha = L \\ \therefore \alpha &> \beta \\ DA &> DG\end{aligned}$$

DG > DH  
DA = DH  
DG > DA

## In other words

Given a circle, with AB as a diameter, and D as the centre...

(2) No other line from point A can squeeze between the circumference, and the line AE

## Proof by Contradiction (2)

Assume that the line FA lies between the circumference of the circle and the line AE

Draw a perpendicular line from D to line FA, intersecting at the point G

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In a triangle, the larger line is opposite the larger angle, thus DA is larger than DG (I.19)

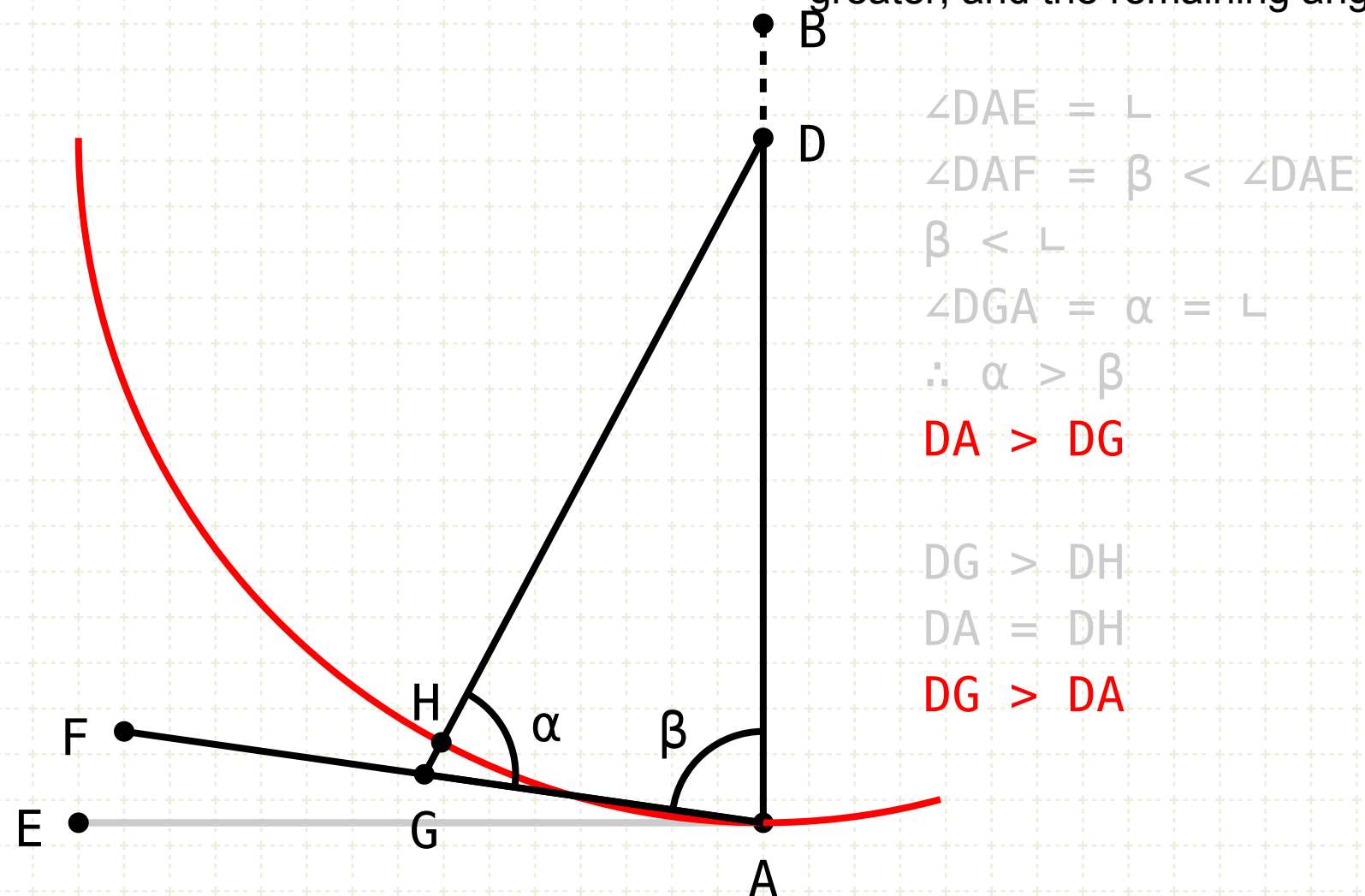
DG is larger than DH

DH and DA are equal (radii of the same circle), thus DG is larger than DA



## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



### In other words

Given a circle, with AB as a diameter, and D as the centre...

(2) No other line from point A can squeeze between the circumference, and the line AE

### Proof by Contradiction (2)

Assume that the line FA lies between the circumference of the circle and the line AE

Draw a perpendicular line from D to line FA, intersecting at the point G

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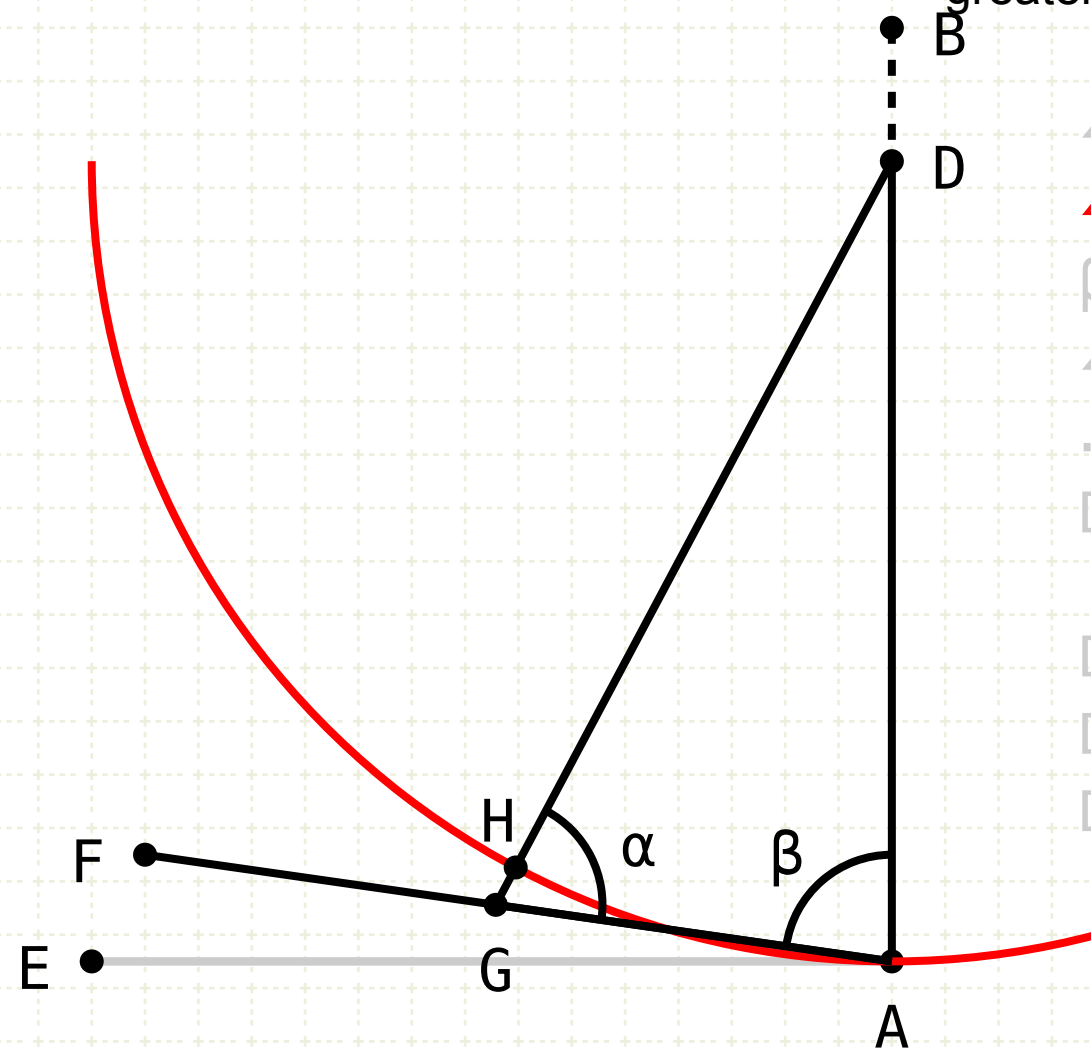
DG is larger than DH

DH and DA are equal (radii of the same circle), thus DG is larger than DA

Thus we have a contradiction

# Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



$$\begin{aligned} \angle DAE &= L \\ \angle DAF &= \beta < \angle DAE \\ \beta &< L \\ \angle DGA &= \alpha = L \\ \therefore \alpha &> \beta \\ DA &> DG \\ DG &> DH \\ DA &= DH \\ DG &> DA \end{aligned}$$

## In other words

Given a circle, with AB as a diameter, and D as the centre...

(2) No other line from point A can squeeze between the circumference, and the line AE

## Proof by Contradiction (2)

Assume that the line FA lies between the circumference of the circle and the line AE

Draw a perpendicular line from D to line FA, intersecting at the point G

Angle DGA ( $\alpha$ ) is right, and DAG ( $\beta$ ) is less than a right angle, so DGA is greater than DAG

In a triangle, the larger line is opposite the larger angle, thus DA is larger than DG (I.19)

DG is larger than DH

DH and DA are equal (radii of the same circle), thus DG is larger than DA

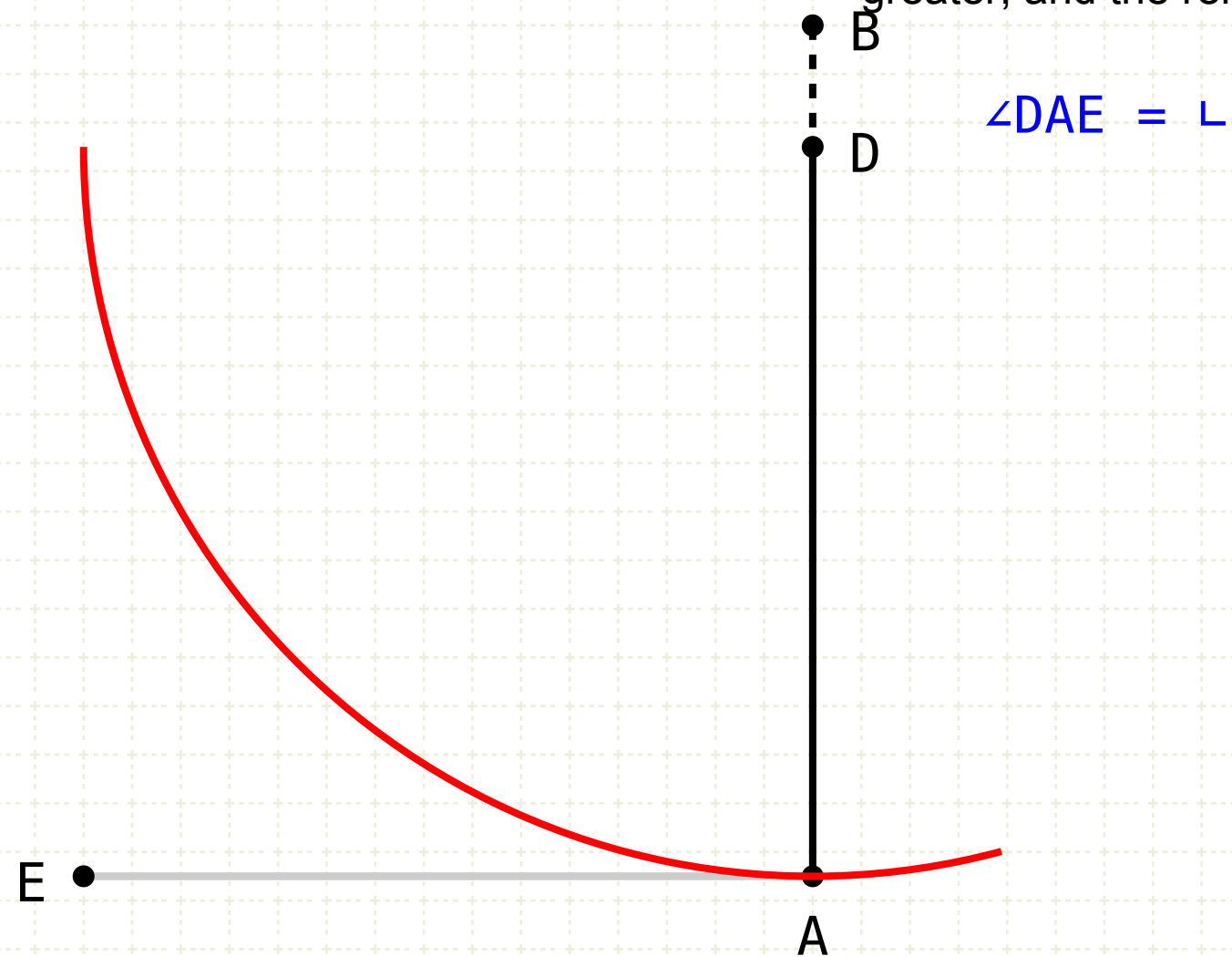
Thus we have a contradiction

The line FA cannot exist between the line AE and the circumference



## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



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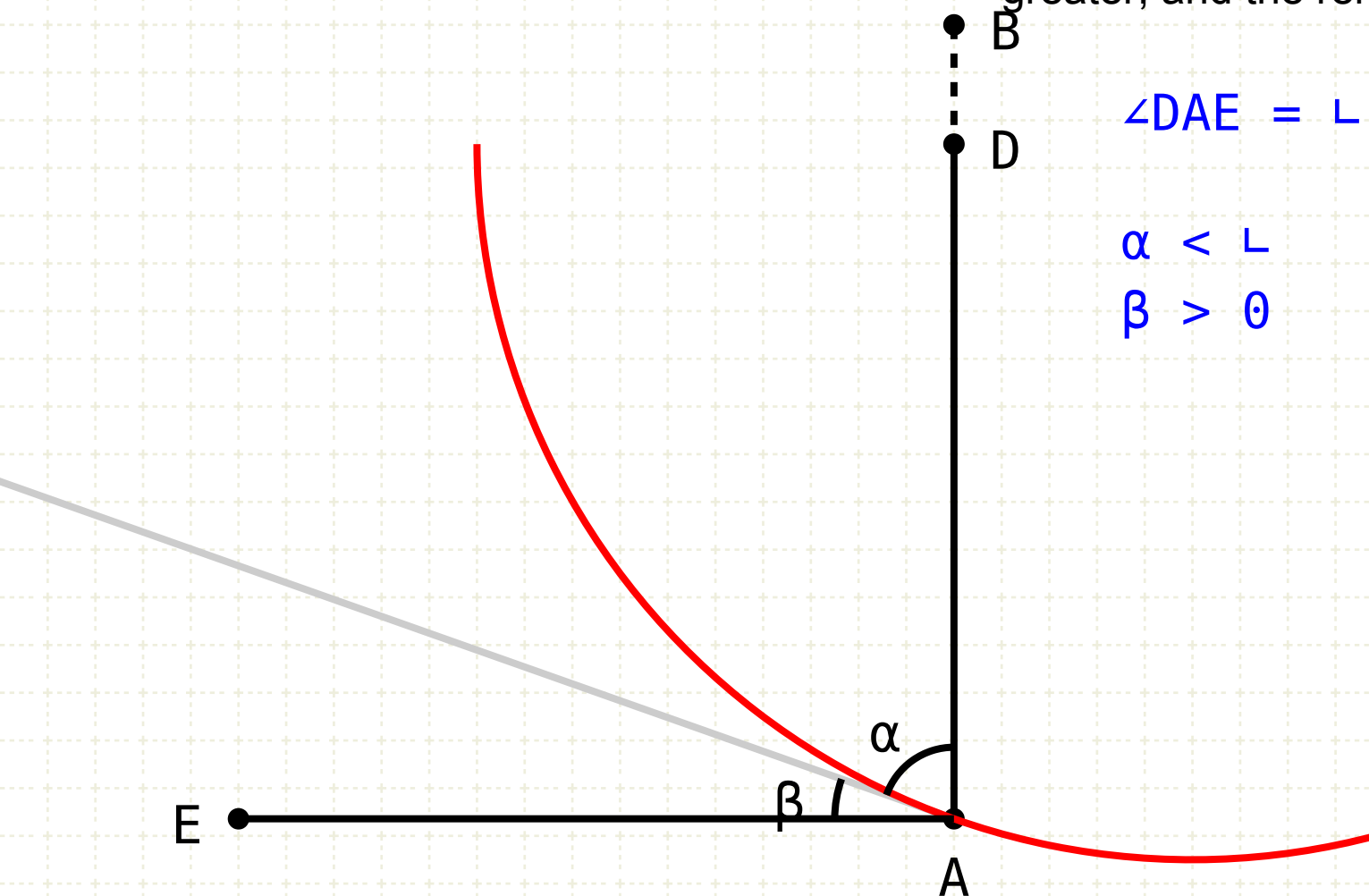
- (3) - The angle between  $AB$  and the circumference of the circle cannot be less than a right angle,  
- and the angle between  $AE$  and the circumference cannot be less than zero

### Proof by Contradiction (3)



# Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



## In other words

Given a circle, with AB as a diameter, and D as the centre...

- (3) - The angle between AB and the circumference of the circle cannot be less than a right angle,  
- and the angle between AE and the circumference cannot be less than zero

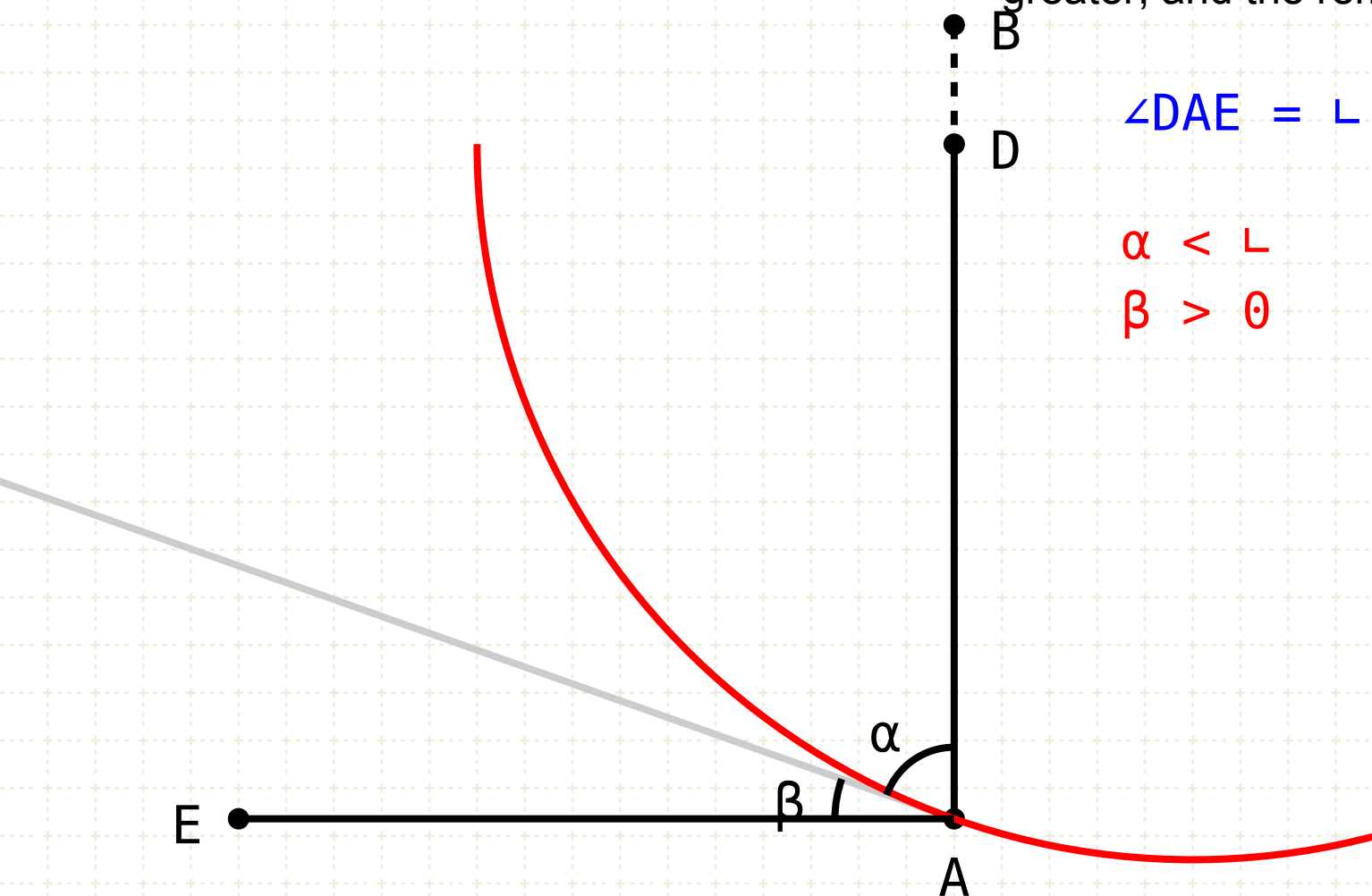
## Proof by Contradiction (3)

Assume that the angle between AB and the circumference is less than a right angle, and that the angle between AE and the circumference is greater than zero



## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



### In other words

Given a circle, with AB as a diameter, and D as the centre...

- (3) - The angle between AB and the circumference of the circle cannot be less than a right angle,  
- and the angle between AE and the circumference cannot be less than zero

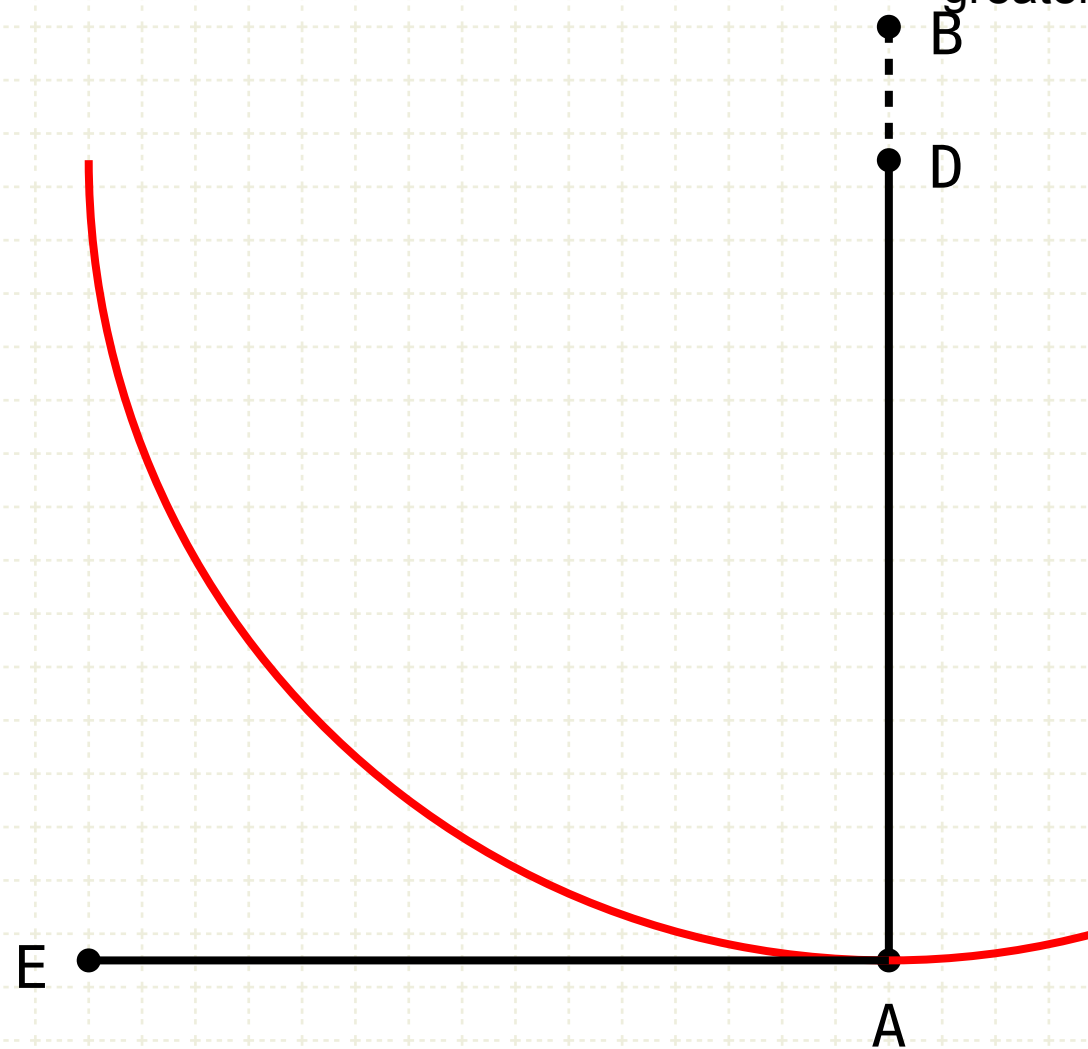
### Proof by Contradiction (3)

Assume that the angle between AB and the circumference is less than a right angle, and that the angle between AE and the circumference is greater than zero

Then the tangent (grey line) can be inserted between the circumference and the line EA, which was just proven to be impossible

## Proposition 16 of Book III

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



### In other words

Given a circle, with AB as a diameter, and D as the centre...

Draw a line AE from point A, perpendicular to AB

- (1) The Line AE falls outside of the circle
- (2) No other line from point A can squeeze between the circumference, and the line AE
- (3) - The angle between AB and the circumference of the circle cannot be less than a right angle,  
- and the angle between AE and the circumference cannot be greater than zero

### PORISM

A straight line drawn at right angles to the diameter of a circle, from one of the endpoints, touches the circle

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