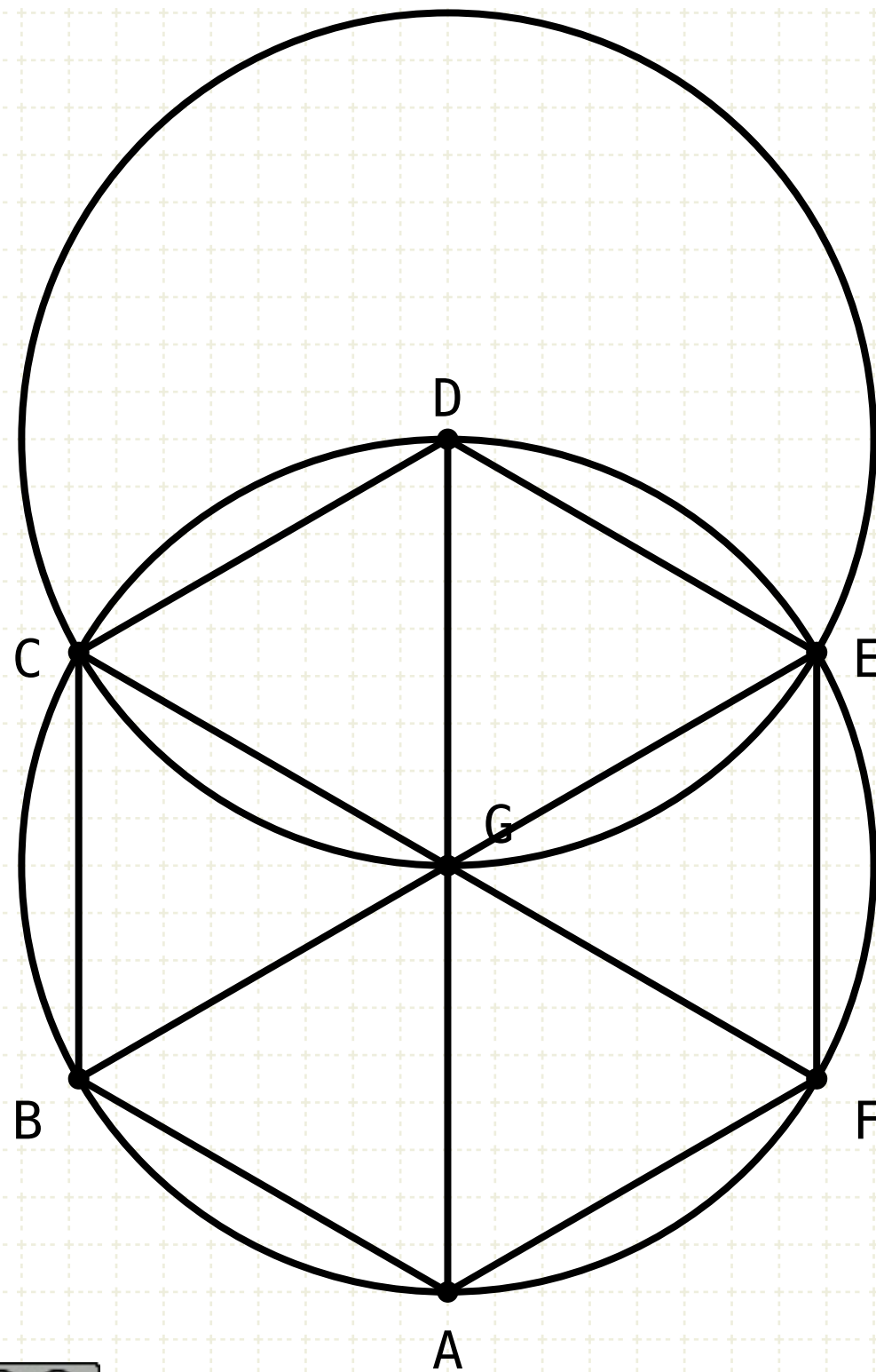


# Euclid's Elements

## Book IV



Philosophy (nature) is written in that great book which ever is before our eyes -- I mean the universe -- but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it - without which one wanders in vain through a dark labyrinth.

**Galileo Galilei**



# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



# Table of Contents, Chapter 4

1	Fit a given straight line into a given circle, if the line is less than the diameter	11	In a given circle to inscribe an equilateral and equiangular pentagon
2	In a given circle to inscribe a triangle equiangular with a given triangle	12	About a given circle to circumscribe an equilateral and equiangular pentagon
3	About a given circle to circumscribe a triangle equiangular with a given triangle	13	In a given pentagon, which is equilateral and equiangular, to inscribe a circle
4	In a given triangle, to inscribe a circle	14	About a given pentagon, which is equilateral and equiangular, to circumscribe a circle
5	About a given triangle to circumscribe a circle	<b>15</b>	<b>In a given circle to inscribe an equilateral and equiangular hexagon</b>
6	In a given circle to inscribe a square	16	In a given circle to inscribe a fifteen angled figure which shall be both equilateral and equiangular
7	About a given circle to circumscribe a square		
8	In a given square, to inscribe a circle		
9	About a given square, to circumscribe a circle		
10	To construct an isosceles triangle having each of the angles at the base double of the remaining one		



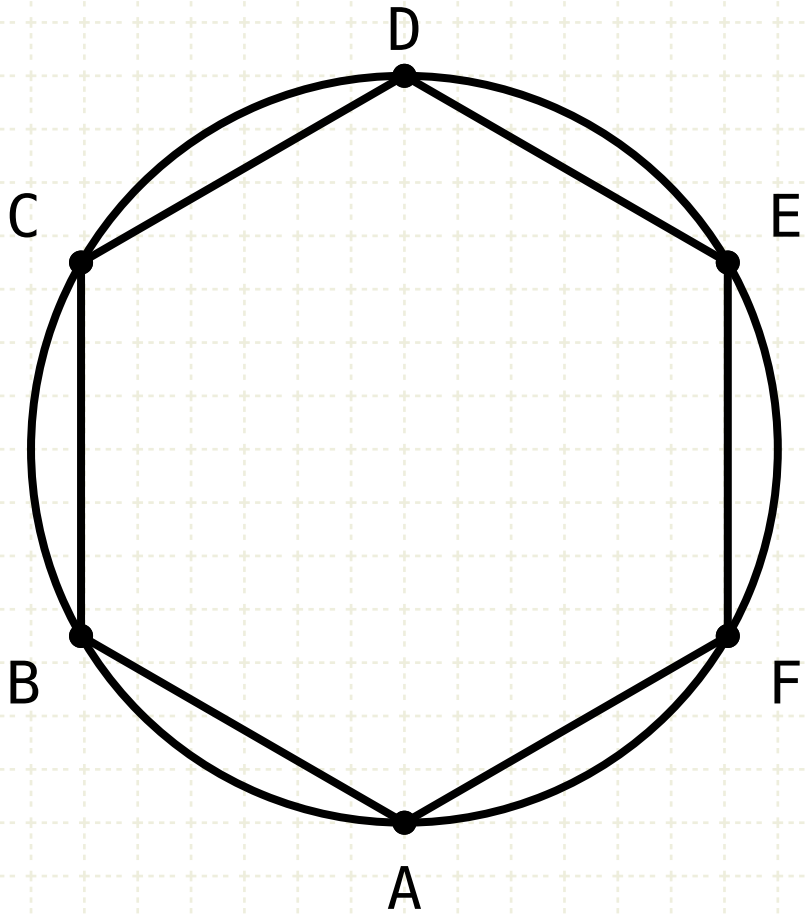
# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



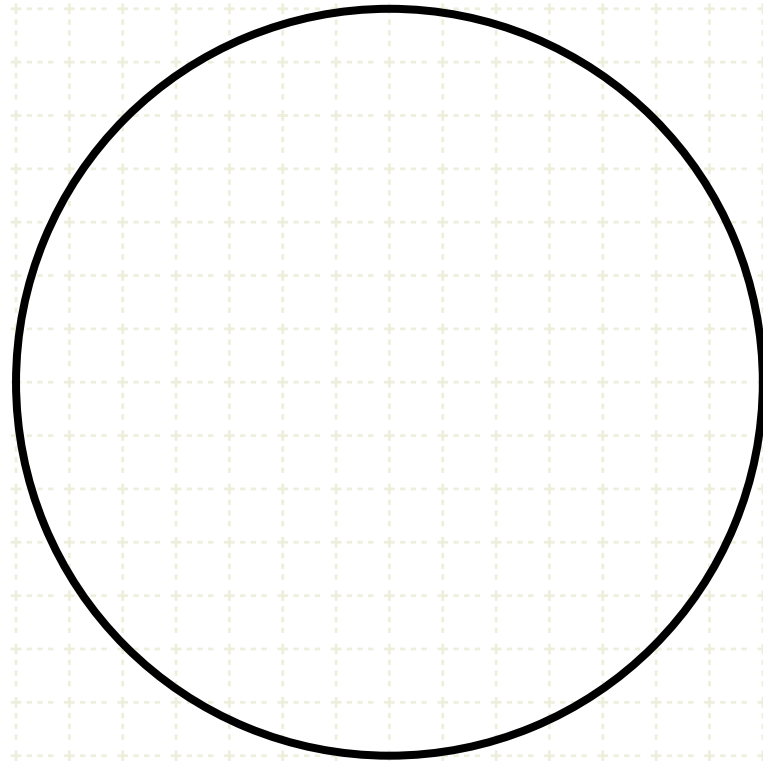
### In other words

Given a circle, draw a six sided polygon with equal sides and equal angles on the inside of the circle.

# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.

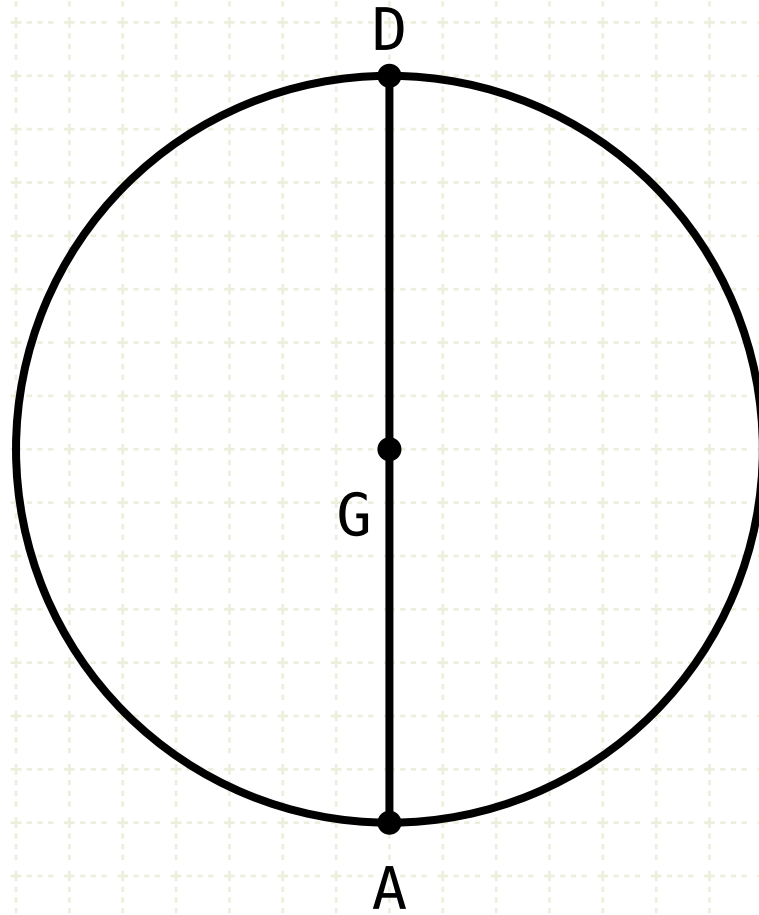
## Construction





## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.

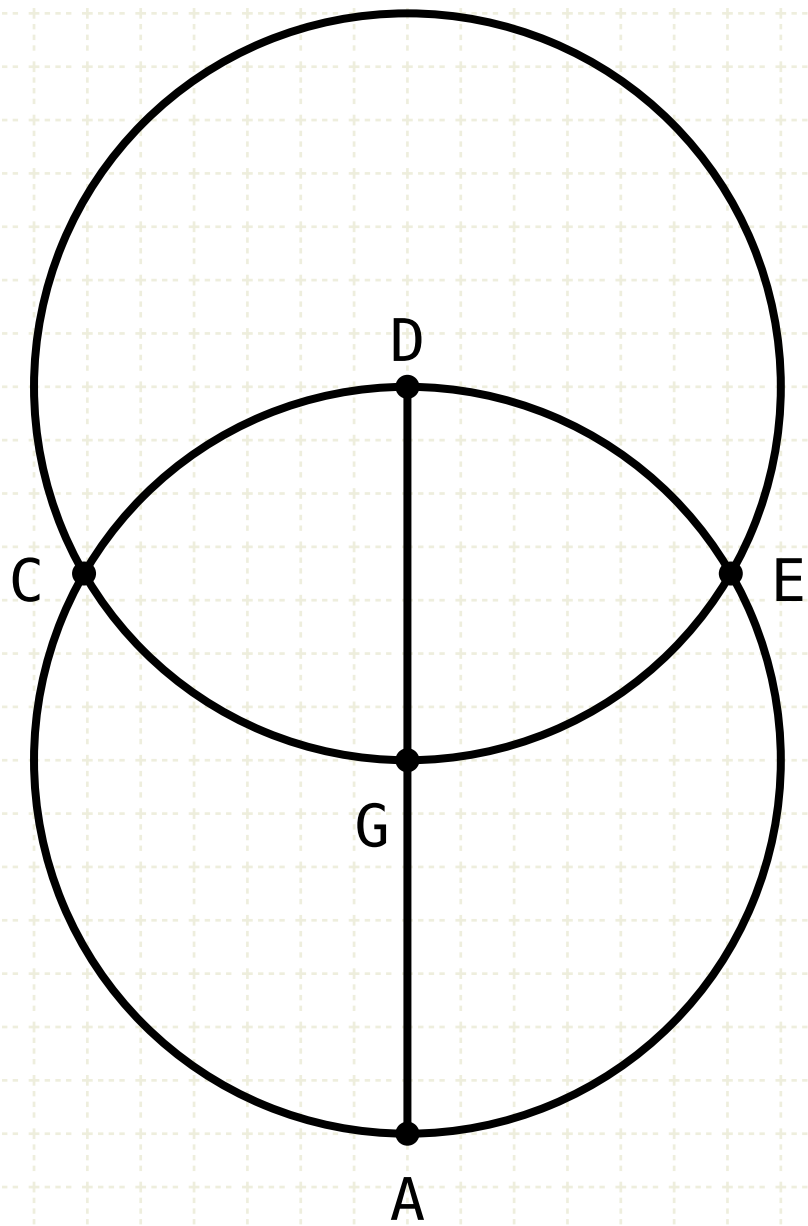


## Construction

Draw the diameter AD through the centre of the circle G

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



### Construction

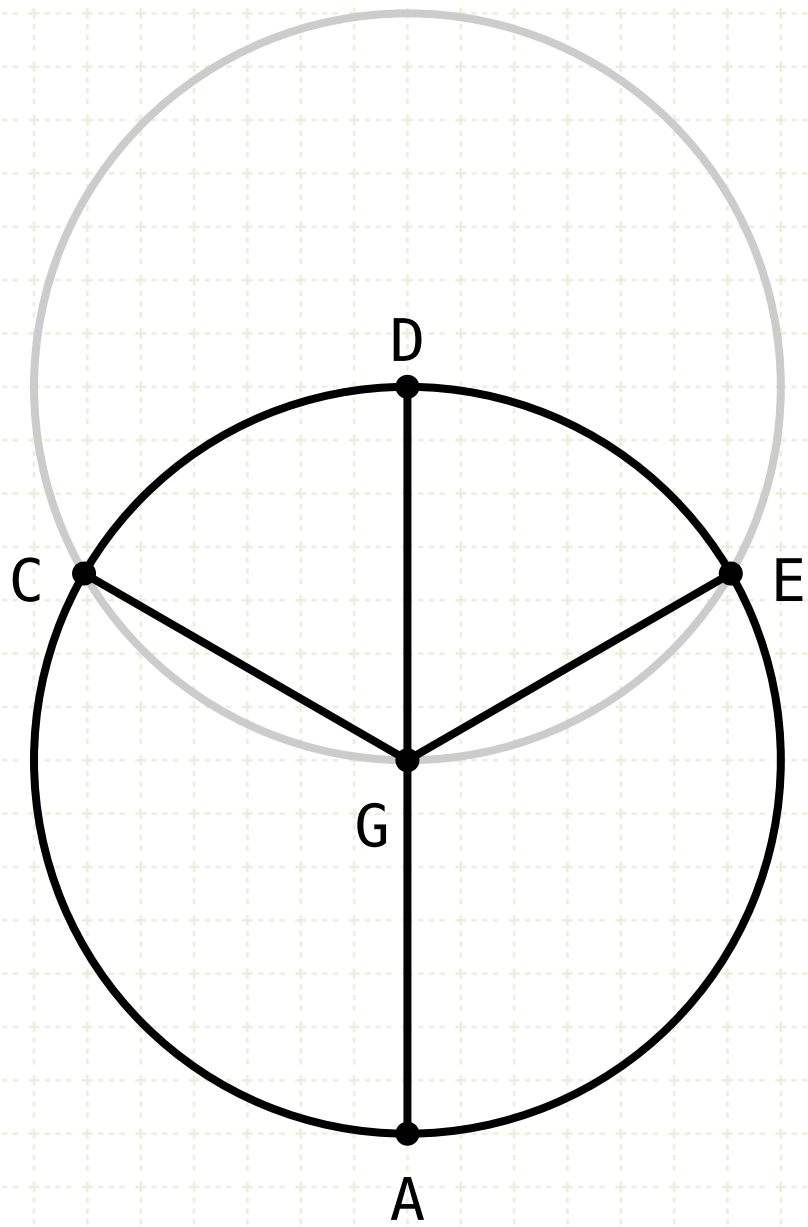
Draw the diameter AD through the centre of the circle G

Draw a circle with D as the centre, and DG as the radius



## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



### Construction

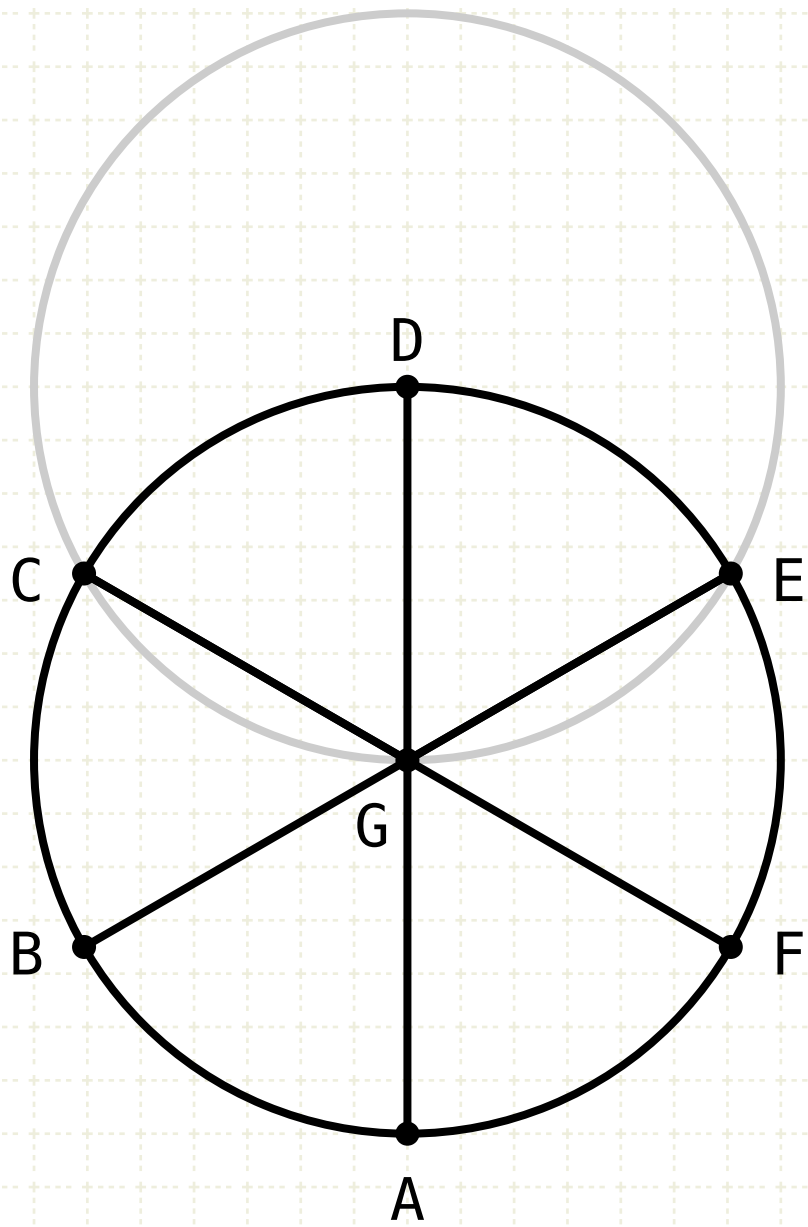
Draw the diameter AD through the centre of the circle G

Draw a circle with D as the centre, and DG as the radius

Draw the lines CG and EG

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



### Construction

Draw the diameter AD through the centre of the circle G

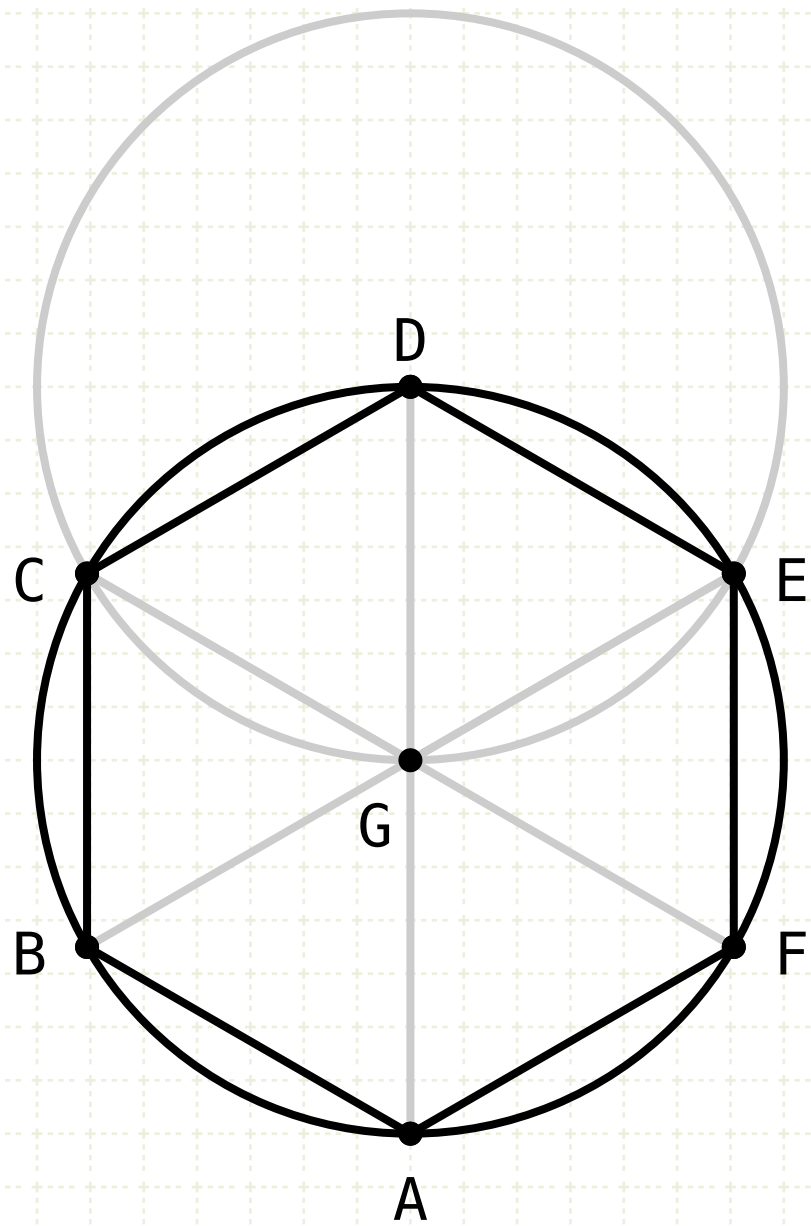
Draw a circle with D as the centre, and DG as the radius

Draw the lines CG and EG

Extend the lines CG and EG to the other side of the circle at points F and B respectively

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



### Construction

Draw the diameter AD through the centre of the circle G

Draw a circle with D as the centre, and DG as the radius

Draw the lines CG and EG

Extend the lines CG and EG to the other side of the circle at points F and B respectively

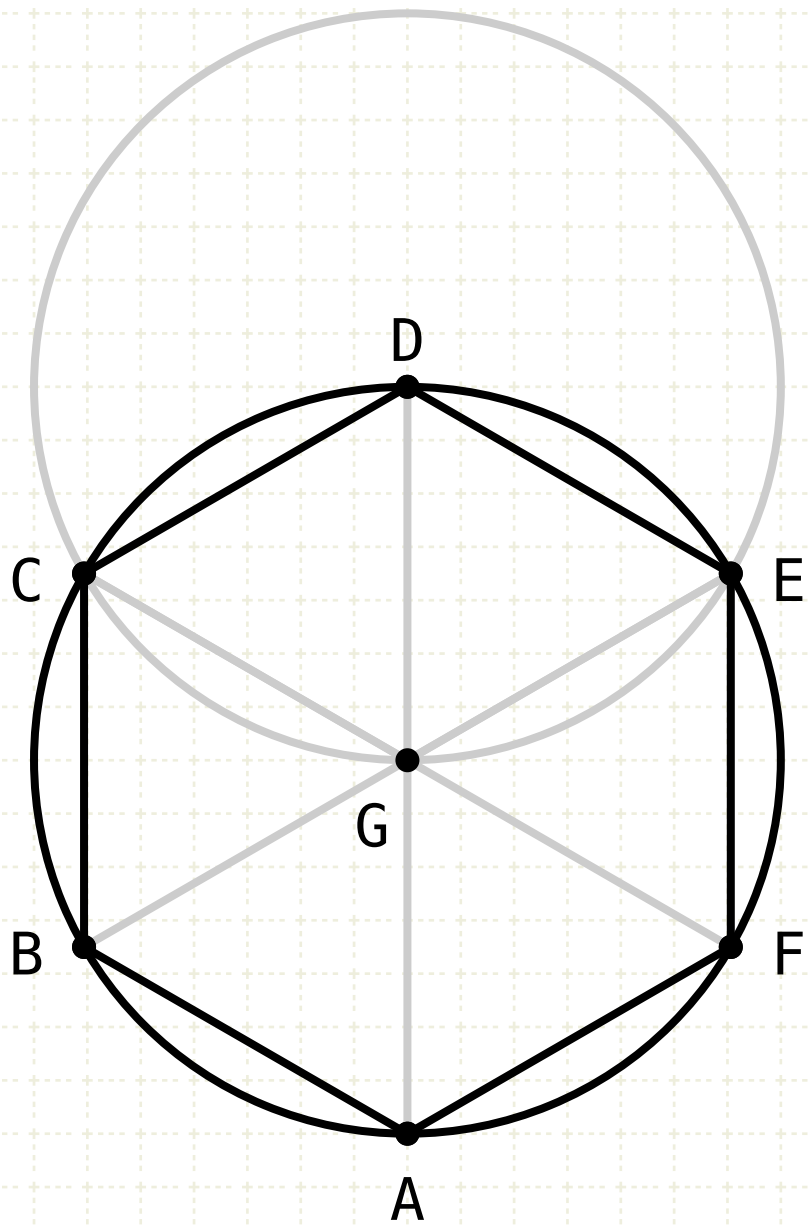
Draw lines AB, BC, CD, DE, EA

The resulting hexagon is equilateral and equiangular

# Proposition 15 of Book IV

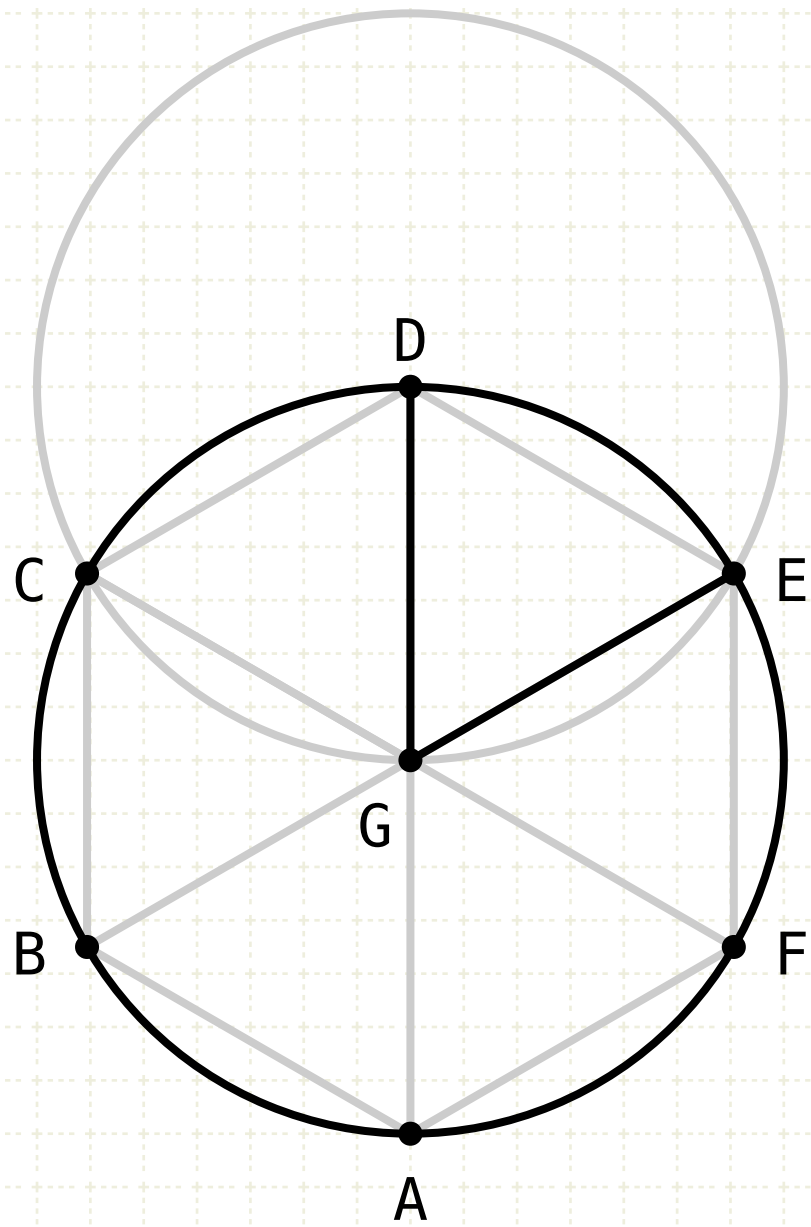
In a given circle to inscribe an equilateral and equiangular hexagon.

## Proof



## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



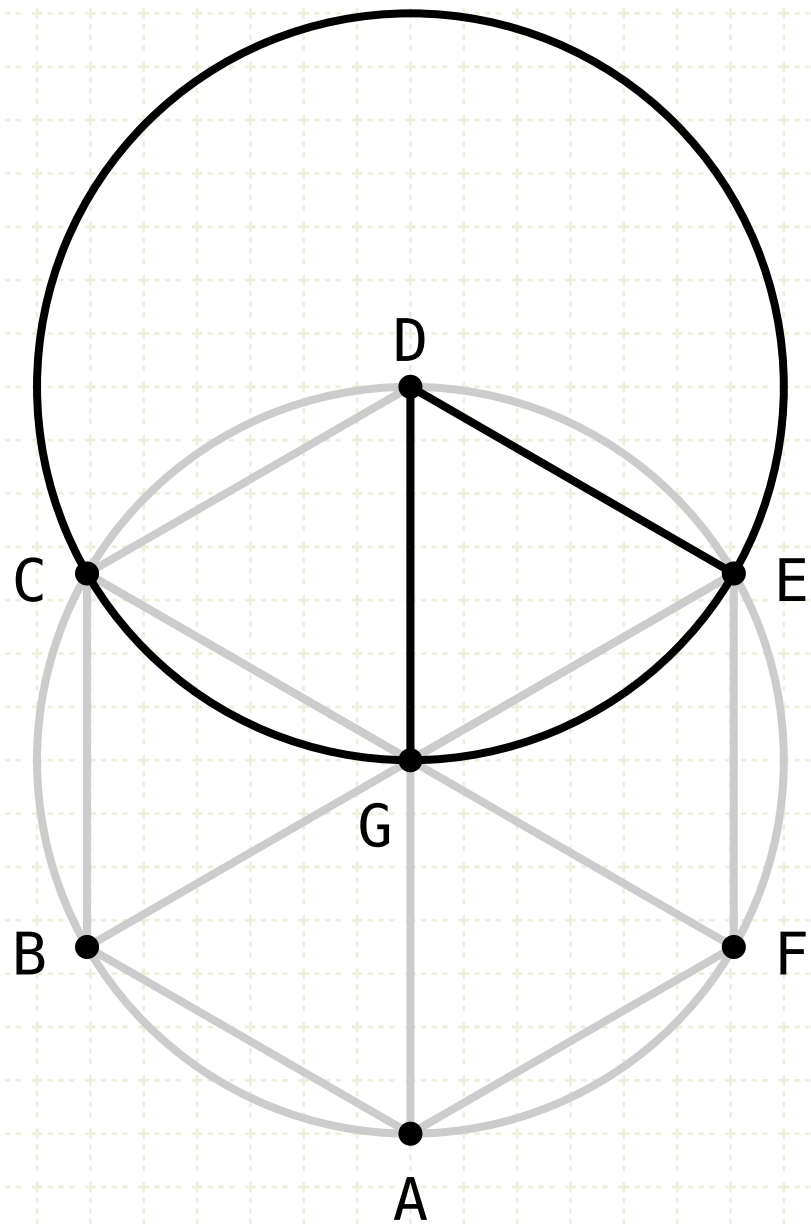
$$DG = EG$$

### Proof

The lines DG and EG are radii of the same circle, and thus are equal

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

### Proof

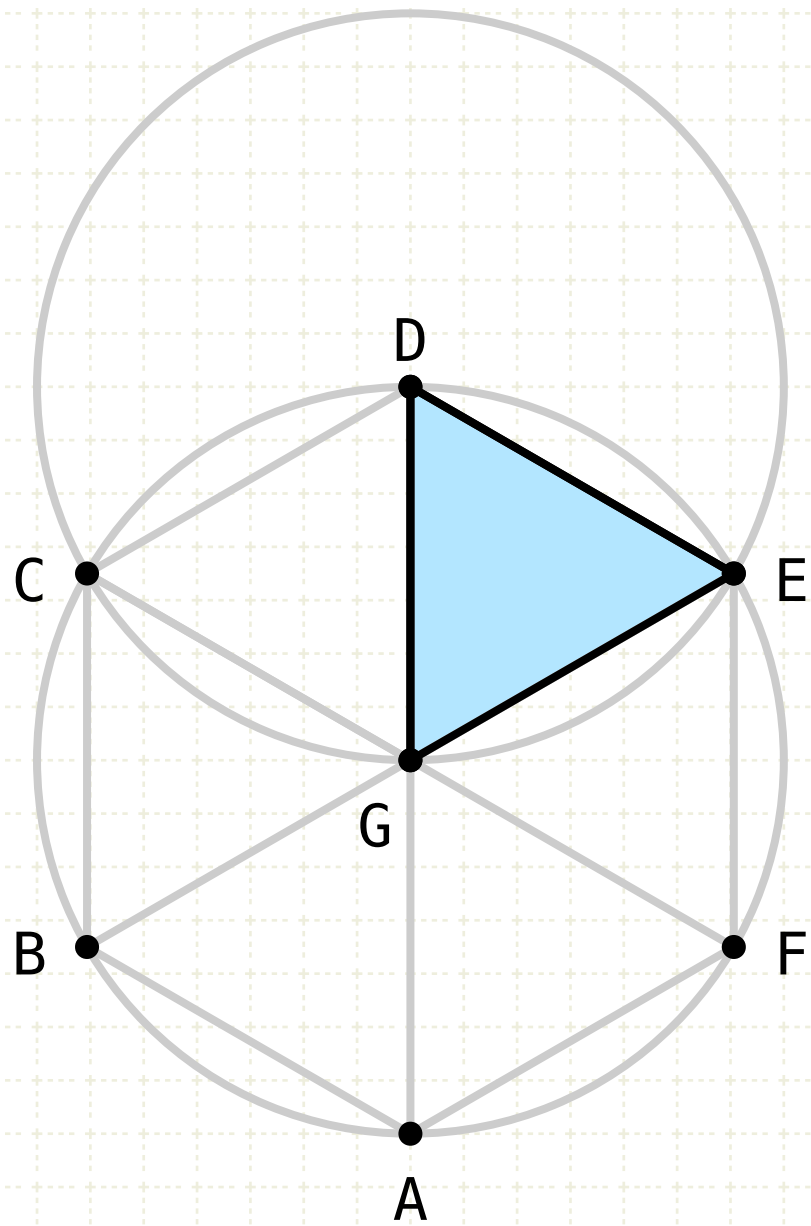
The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal



## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

### Proof

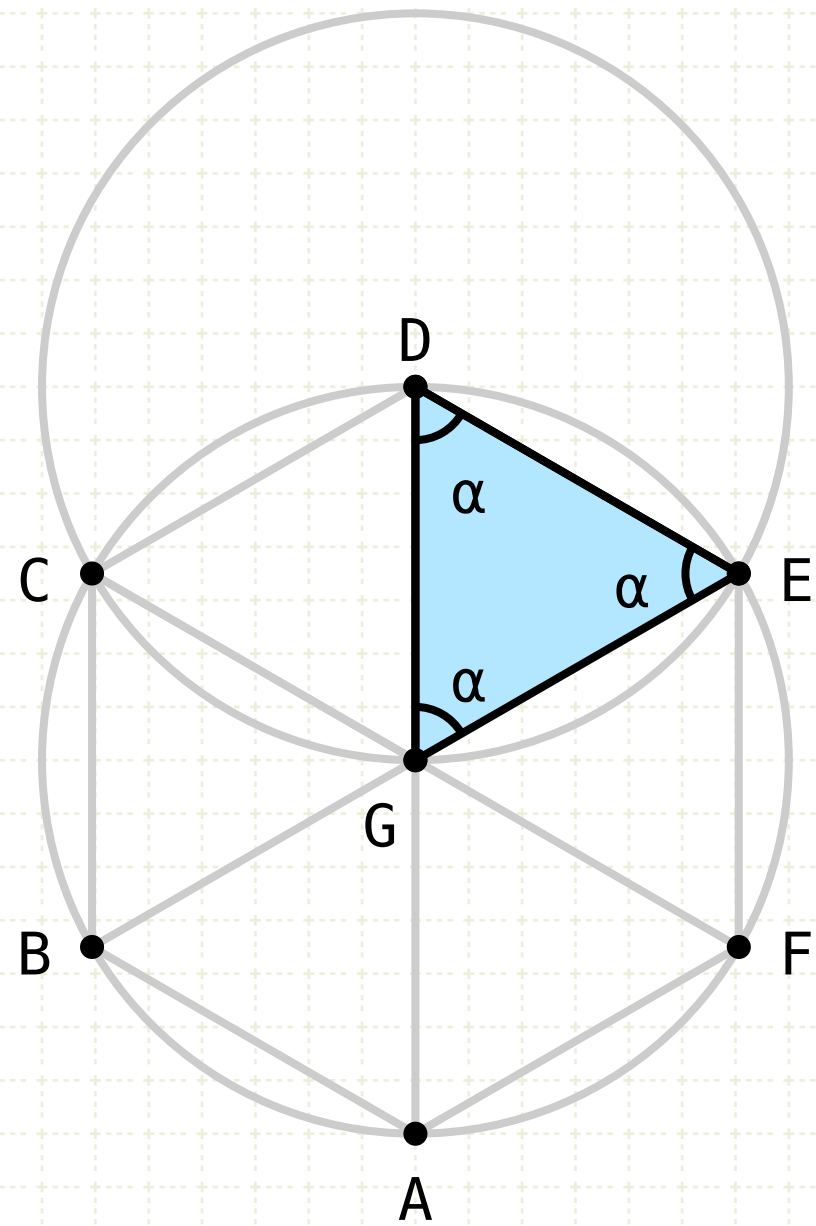
The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal

Therefore the triangle DGE is an equilateral triangle

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

### Proof

The lines DG and EG are radii of the same circle, and thus are equal

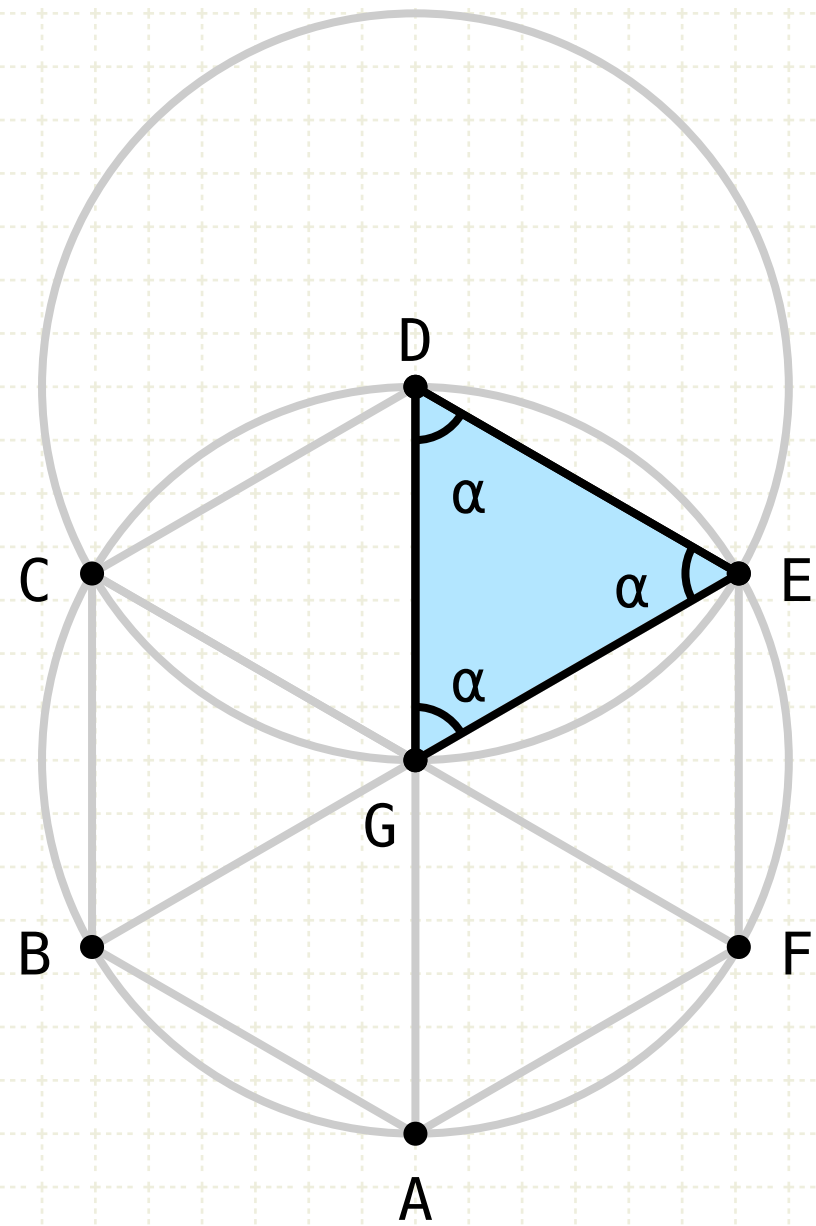
The lines DG and DE are radii of the same circle, and thus are equal

Therefore the triangle DGE is an equilateral triangle

An equilateral triangle is also an isosceles triangle, regardless of which side is chosen as its base, therefore, all the angles within the triangle are equal (I-5)

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

### Proof

The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal

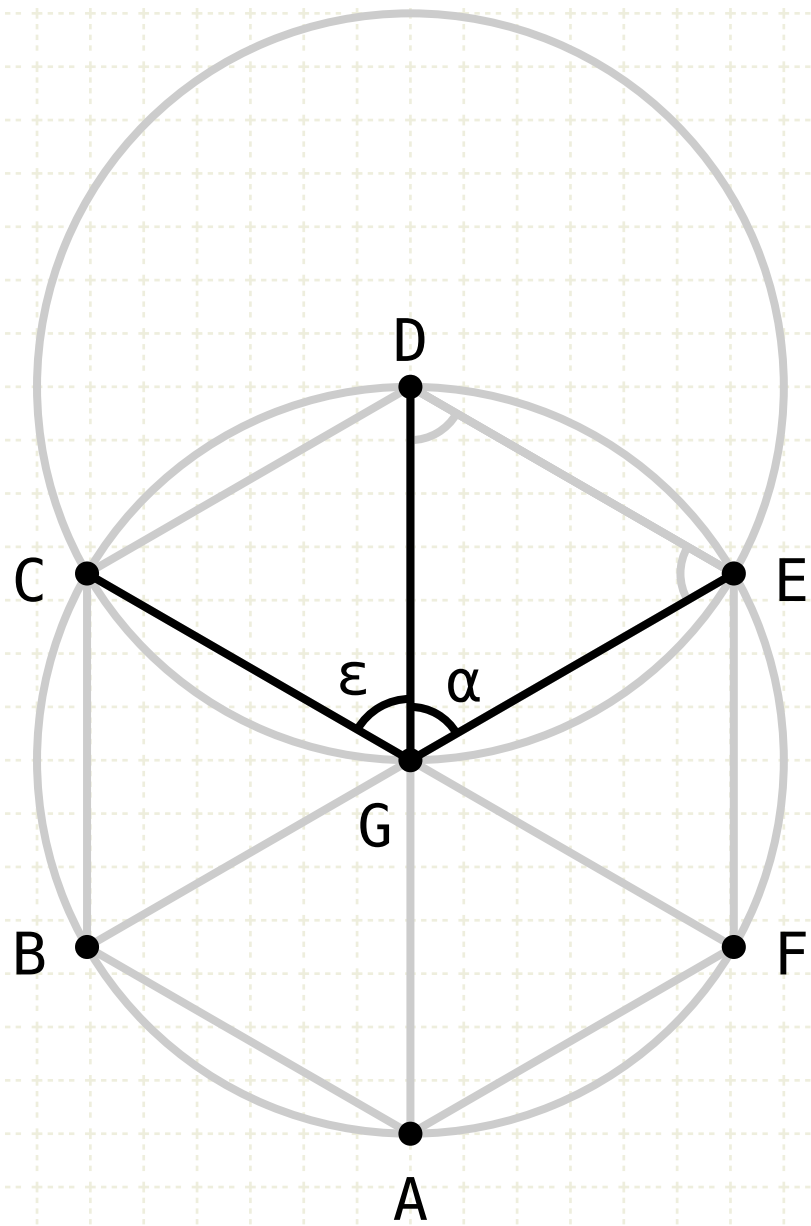
Therefore the triangle DGE is an equilateral triangle

An equilateral triangle is also an isosceles triangle, regardless of which side is chosen as its base, therefore, all the angles within the triangle are equal (I·5)

The sum of all the angles within a triangle is two right angles (I·32), therefore the angle EGD is one-third of two right angles

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

$$\angle CGD = \varepsilon = (1/3) \cdot 2L$$

### Proof

The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal

Therefore the triangle DGE is an equilateral triangle

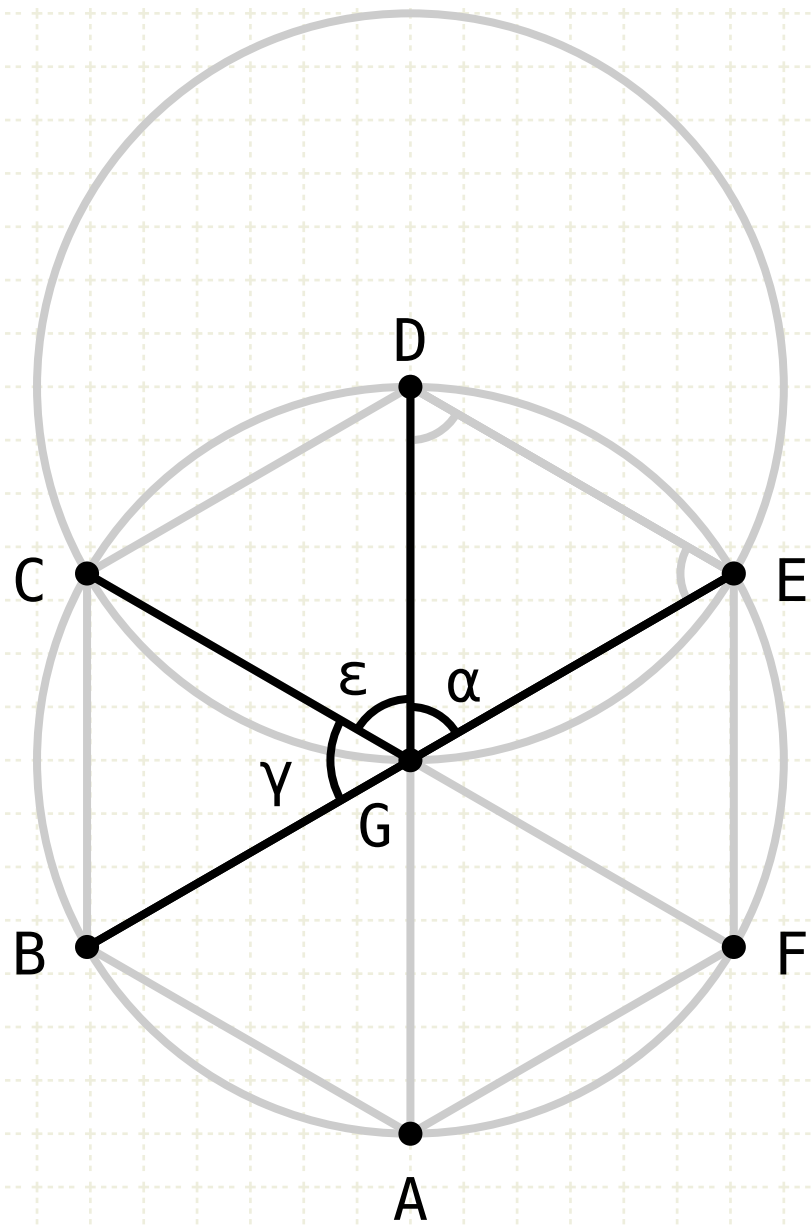
An equilateral triangle is also an isosceles triangle, regardless of which side is chosen as its base, therefore, all the angles within the triangle are equal (I-5)

The sum of all the angles within a triangle is two right angles (I-32), therefore the angle EGD is one-third of two right angles

Similarly, it can be shown that CGD is also one-third of two right angles

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2\angle$$

$$\angle CGD = \epsilon = (1/3) \cdot 2\angle$$

$$(\epsilon + \alpha) + \gamma = 2\angle$$

### Proof

The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal

Therefore the triangle DGE is an equilateral triangle

An equilateral triangle is also an isosceles triangle, regardless of which side is chosen as its base, therefore, all the angles within the triangle are equal (I·5)

The sum of all the angles within a triangle is two right angles (I·32), therefore the angle EGD is one-third of two right angles

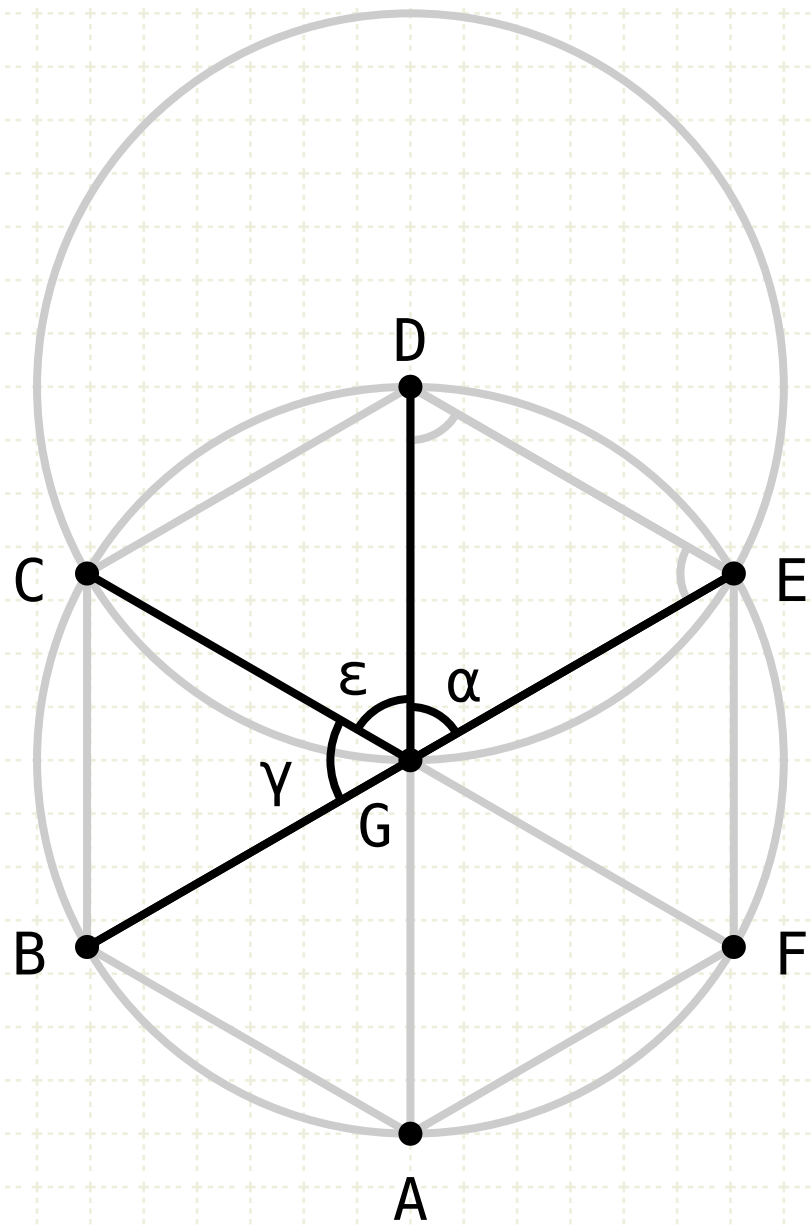
Similarly, it can be shown that CGD is also one-third of two right angles

The line CG cuts the straight line BE, therefore the angles CGE and CGB equal two right angles



## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

$$\angle CGD = \varepsilon = (1/3) \cdot 2L$$

$$(\varepsilon + \alpha) + \gamma = 2L$$

$$\gamma = \varepsilon = \alpha = (1/3) \cdot 2L$$

### Proof

The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal

Therefore the triangle DGE is an equilateral triangle

An equilateral triangle is also an isosceles triangle, regardless of which side is chosen as its base, therefore, all the angles within the triangle are equal (I·5)

The sum of all the angles within a triangle is two right angles (I·32), therefore the angle EGD is one-third of two right angles

Similarly, it can be shown that CGD is also one-third of two right angles

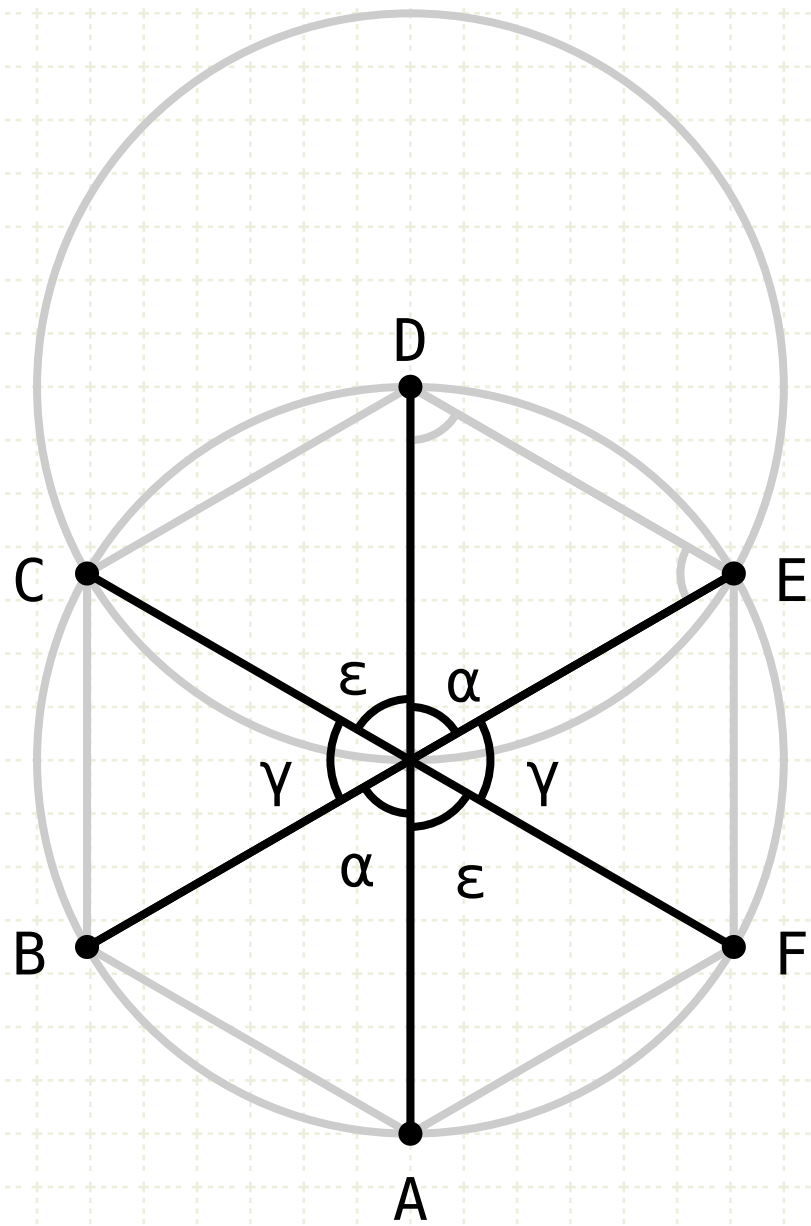
The line CG cuts the straight line BE, therefore the angles CGE and CGB equal two right angles

Therefore, angle BGC is also one third of two right angles and all of the angles are equal



# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$\begin{aligned}
 DG &= EG \\
 DG &= DE \\
 \angle DGE &= \alpha = (1/3) \cdot 2L \\
 \angle CGD &= \varepsilon = (1/3) \cdot 2L \\
 (\varepsilon + \alpha) + \gamma &= 2L \\
 \gamma &= \varepsilon = \alpha = (1/3) \cdot 2L
 \end{aligned}$$

## Proof

The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal

Therefore the triangle DGE is an equilateral triangle

An equilateral triangle is also an isosceles triangle, regardless of which side is chosen as its base, therefore, all the angles within the triangle are equal (I·5)

The sum of all the angles within a triangle is two right angles (I·32), therefore the angle EGD is one-third of two right angles

Similarly, it can be shown that CGD is also one-third of two right angles

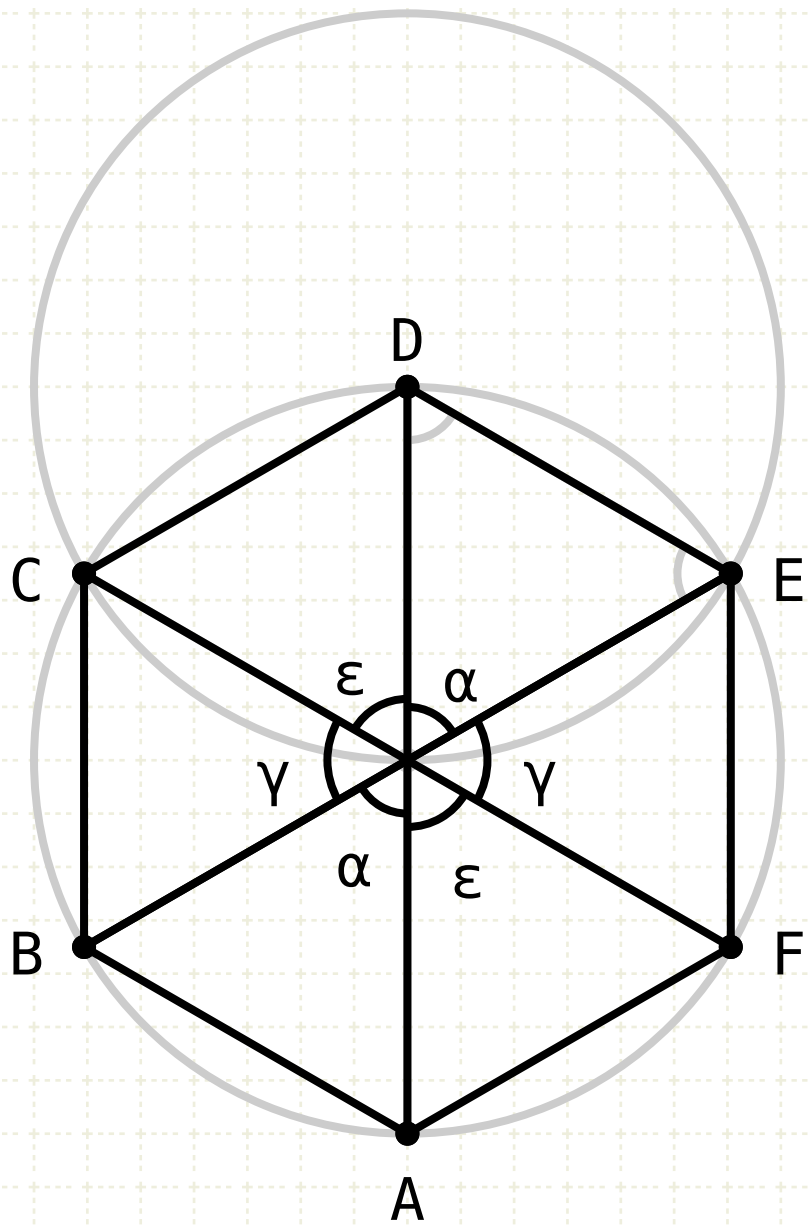
The line CG cuts the straight line BE, therefore the angles CGE and CGB equal two right angles

Therefore, angle BGC is also one third of two right angles and all of the angles are equal

The angles vertical to BGC, CGD, DGE are also equal (I·15)

## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

$$\angle CGD = \epsilon = (1/3) \cdot 2L$$

$$(\epsilon + \alpha) + \gamma = 2L$$

$$\gamma = \epsilon = \alpha = (1/3) \cdot 2L$$

$$AB = BC = CD = DE = EF$$

## Proof

The lines DG and EG are radii of the same circle, and thus are equal

The lines DG and DE are radii of the same circle, and thus are equal

Therefore the triangle DGE is an equilateral triangle

An equilateral triangle is also an isosceles triangle, regardless of which side is chosen as its base, therefore, all the angles within the triangle are equal (I·5)

The sum of all the angles within a triangle is two right angles (I·32), therefore the angle EGD is one-third of two right angles

Similarly, it can be shown that CGD is also one-third of two right angles

The line CG cuts the straight line BE, therefore the angles CGE and CGB equal two right angles

Therefore, angle BGC is also one third of two right angles and all of the angles are equal

The angles vertical to BGC, CGD, DGE are also equal (I·15)

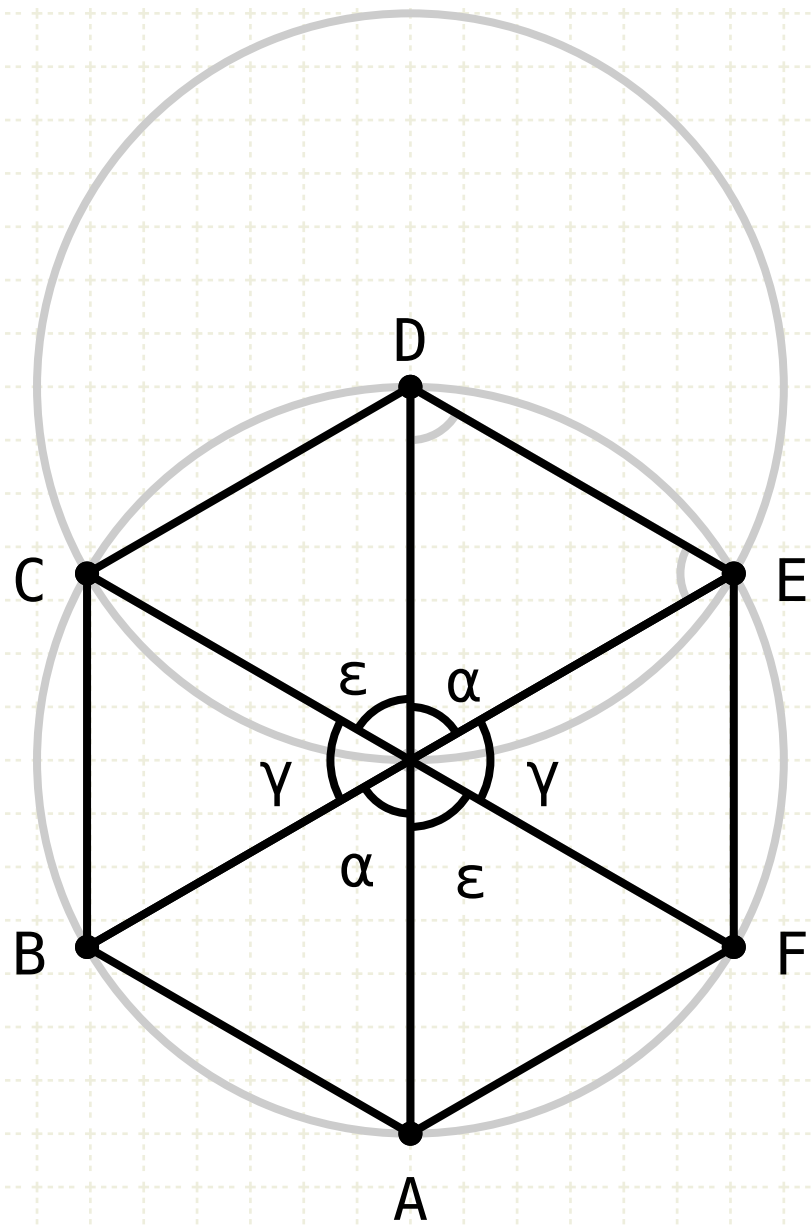
Equal angles subtend equal circumferences (III·26) and equal circumferences are subtended by equal straight lines (III·29)

Therefore the six lines are equal



## Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

$$\angle CGD = \varepsilon = (1/3) \cdot 2L$$

$$(\varepsilon + \alpha) + \gamma = 2L$$

$$\gamma = \varepsilon = \alpha = (1/3) \cdot 2L$$

$$AB = BC = CD = DE = EF$$

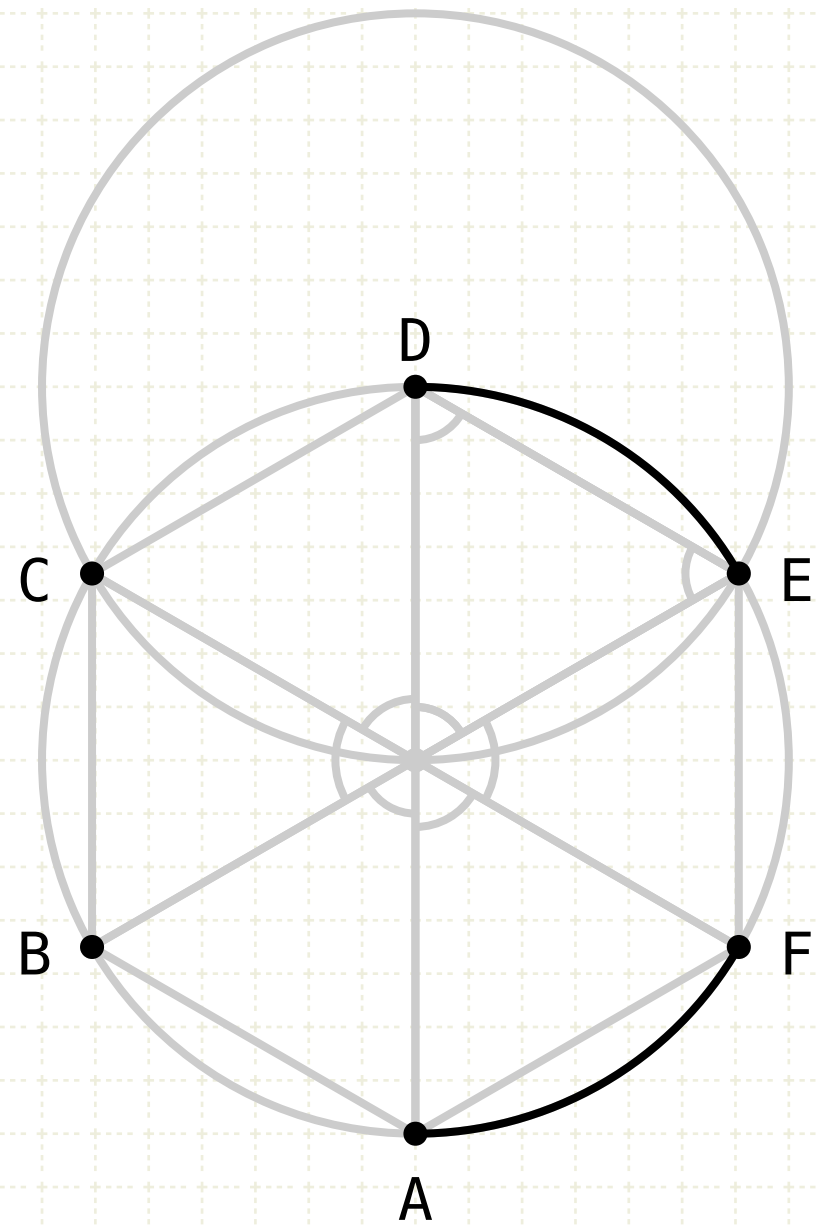
The hexagon was proven to be equilateral

It is also equiangular

### Proof (cont)

# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

$$\angle CGD = \varepsilon = (1/3) \cdot 2L$$

$$(\varepsilon + \alpha) + \gamma = 2L$$

$$\gamma = \varepsilon = \alpha = (1/3) \cdot 2L$$

$$AB = BC = CD = DE = EF$$

$$\text{arc } DE = \text{arc } FA$$

The hexagon was proven to be equilateral

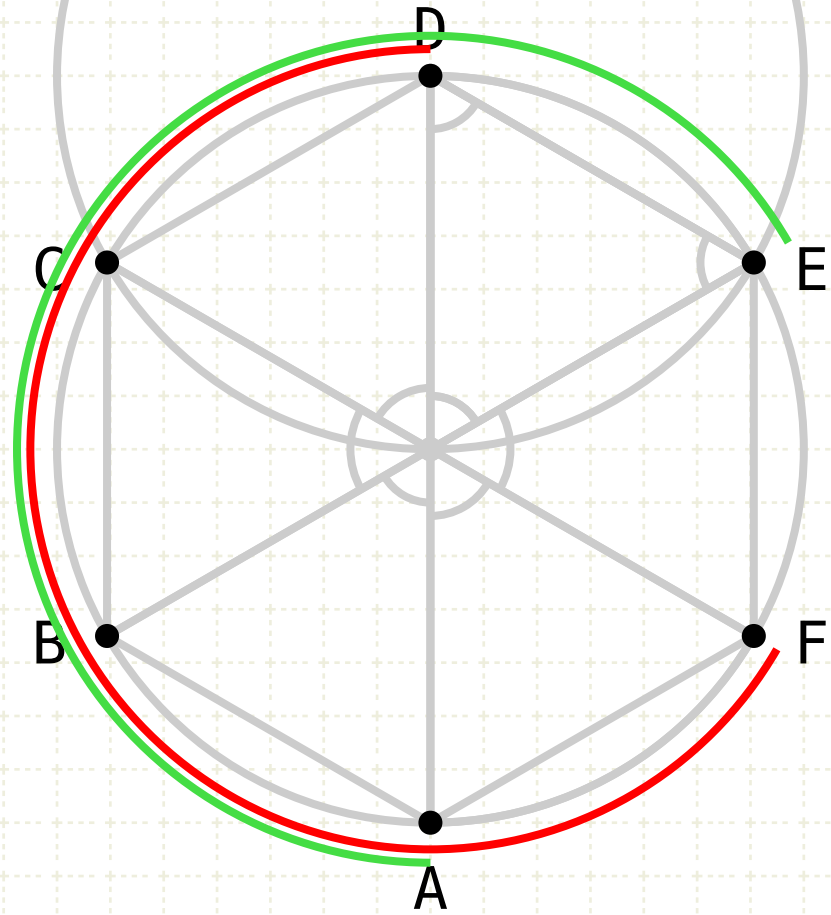
It is also equiangular

## Proof (cont)

The circumference FA is equal to the circumference ED

# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



The hexagon was proven to be equilateral  
It is also equiangular

## Proof (cont)

The circumference FA is equal to the circumference ED  
Let the circumference ABCD be added to each of FA and ED  
maintaining the equality

$$\begin{aligned} DG &= EG \\ DG &= DE \\ \angle DGE &= \alpha = (1/3) \cdot 2L \\ \angle CGD &= \varepsilon = (1/3) \cdot 2L \\ (\varepsilon + \alpha) + \gamma &= 2L \\ \gamma &= \varepsilon = \alpha = (1/3) \cdot 2L \\ AB &= BC = CD = DE = EF \end{aligned}$$

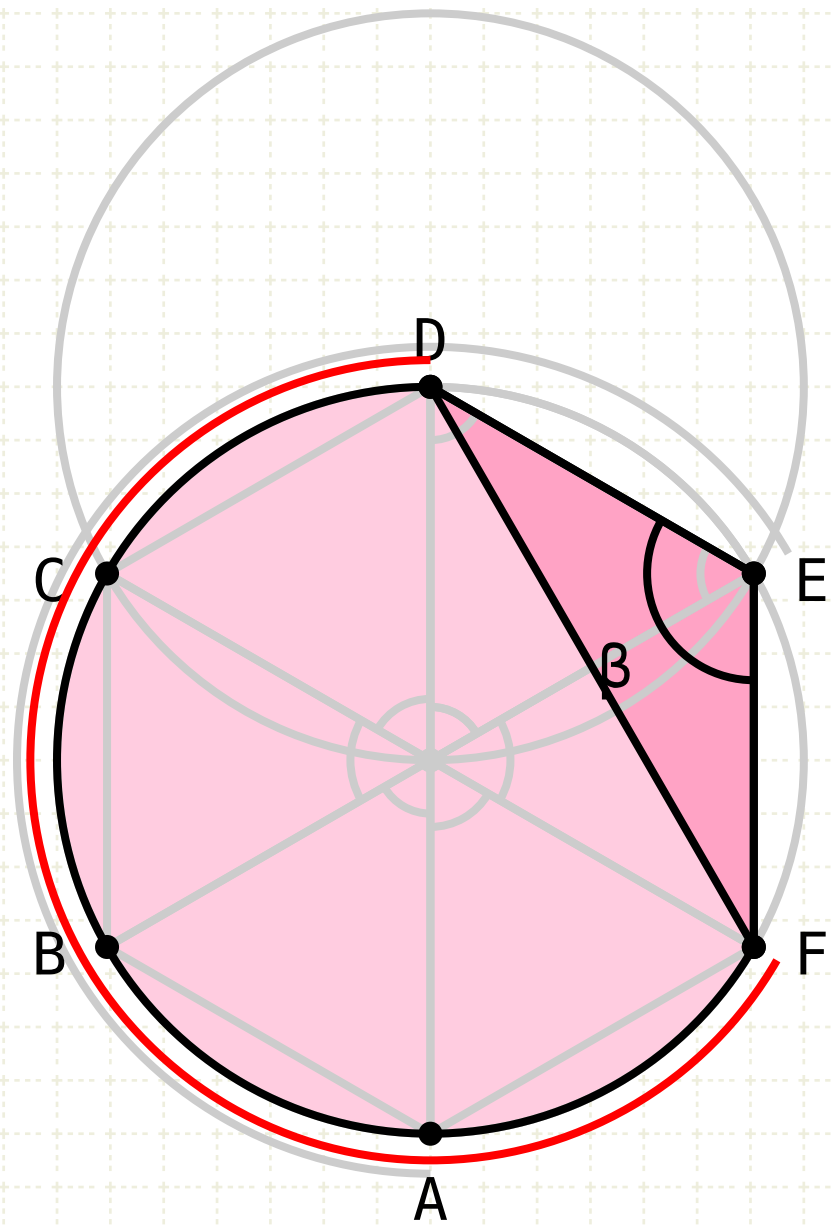
$$\begin{aligned} \text{circumference } DE &= \text{circumference } FA \\ \text{circumference } ABCD + \text{circumference } DE &= \text{circumference } FA + \text{circumference } ABCD \end{aligned}$$

$$\text{circumference } ABCDE = \text{circumference } FABCD$$



# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

$$\angle CGD = \epsilon = (1/3) \cdot 2L$$

$$(\epsilon + \alpha) + \gamma = 2L$$

$$\gamma = \epsilon = \alpha = (1/3) \cdot 2L$$

$$AB = BC = CD = DE = EF$$

$$\angle DE = \angle FA$$

$$\angle ABCD + \angle DE = \angle FA + \angle ABCD$$

$$\angle ABCDE = \angle FABCD$$

$$\angle DEF =$$

The hexagon was proven to be equilateral

It is also equiangular

## Proof (cont)

The circumference FA is equal to the circumference ED

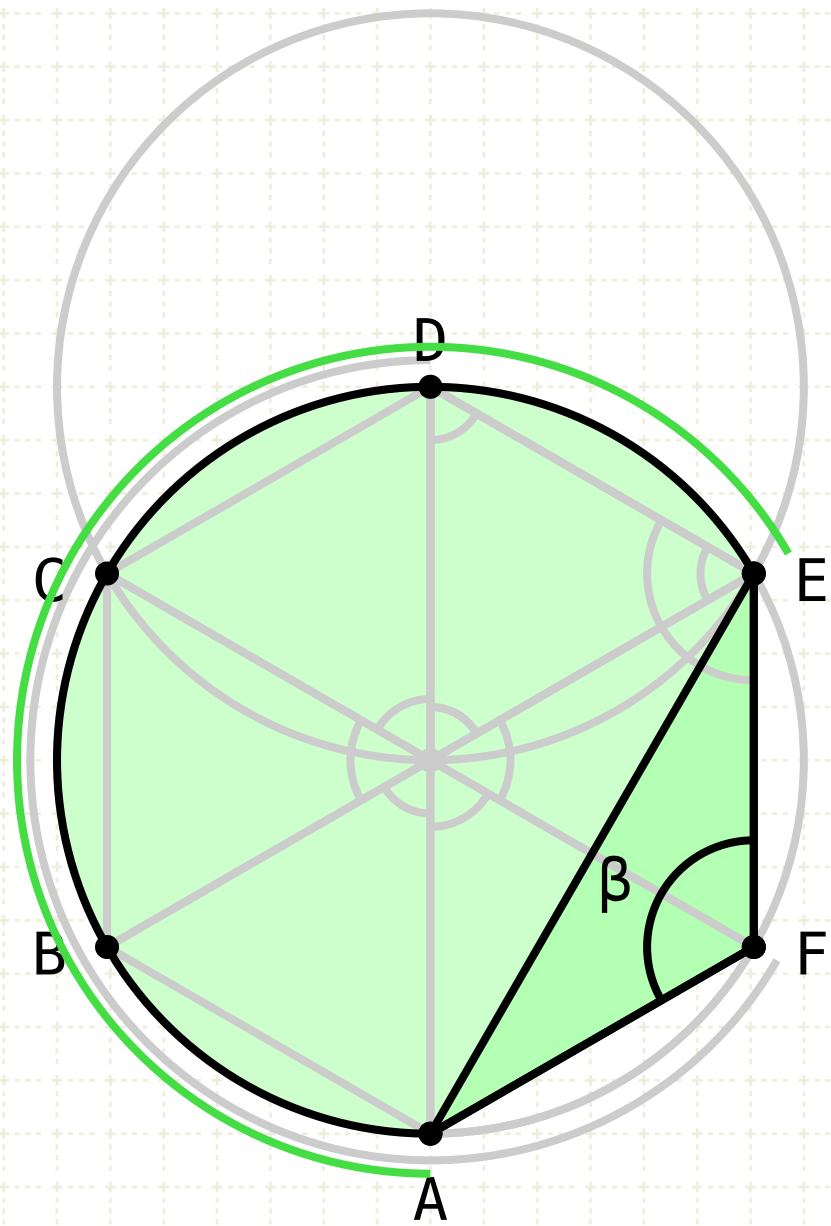
Let the circumference ABCD be added to each of FA and ED maintaining the equality

Equal circumferences have equal angles, therefore the angles are also equal (III·27)



# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$DG = EG$$

$$DG = DE$$

$$\angle DGE = \alpha = (1/3) \cdot 2L$$

$$\angle CGD = \epsilon = (1/3) \cdot 2L$$

$$(\epsilon + \alpha) + \gamma = 2L$$

$$\gamma = \epsilon = \alpha = (1/3) \cdot 2L$$

$$AB = BC = CD = DE = EF$$

$$\angle DE = \angle FA$$

$$\angle ABCD + \angle DE = \angle FA + \angle ABCD$$

$$\angle ABCDE = \angle FABCD$$

$$\angle DEF = \angle EFA$$

The hexagon was proven to be equilateral

It is also equiangular

## Proof (cont)

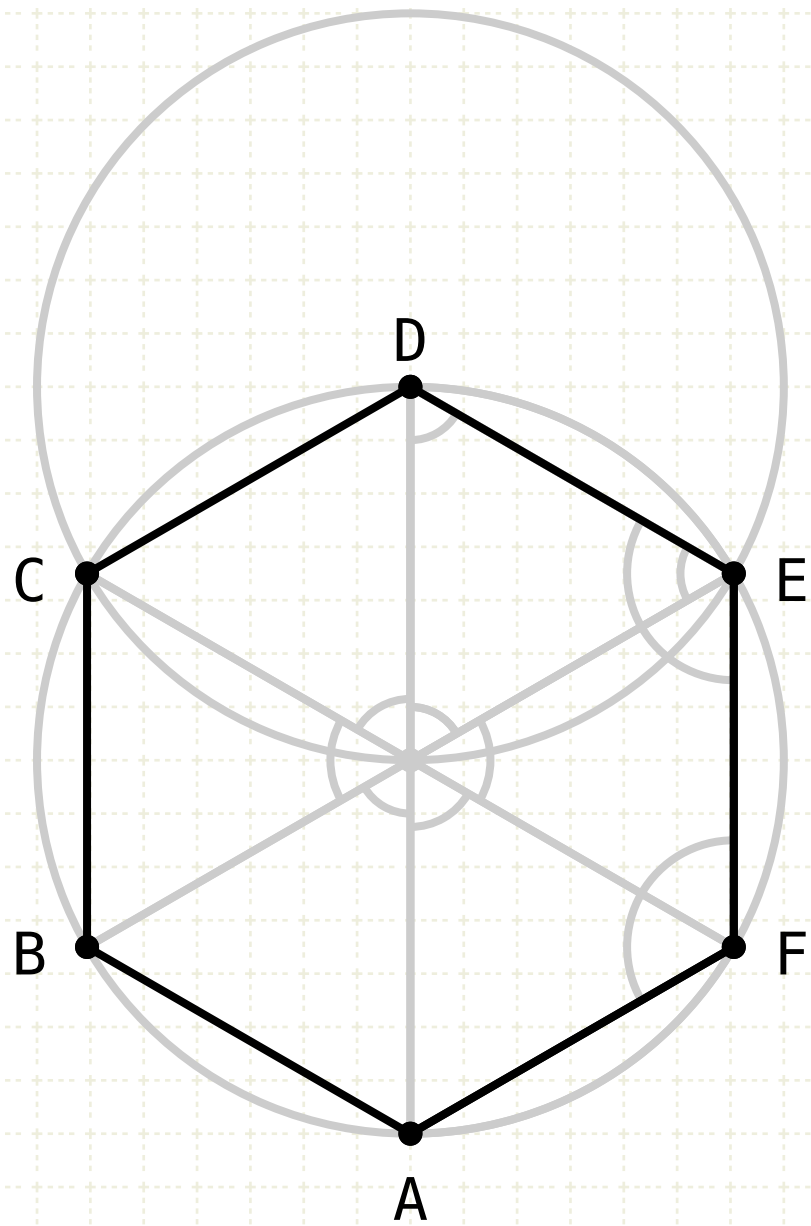
The circumference FA is equal to the circumference ED

Let the circumference ABCD be added to each of FA and ED maintaining the equality

Equal circumferences have equal angles, therefore the angles are also equal (III·27)

# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$\begin{aligned} DG &= EG \\ DG &= DE \\ \angle DGE &= \alpha = (1/3) \cdot 2L \\ \angle CGD &= \varepsilon = (1/3) \cdot 2L \\ (\varepsilon + \alpha) + \gamma &= 2L \\ \gamma &= \varepsilon = \alpha = (1/3) \cdot 2L \\ AB &= BC = CD = DE = EF \end{aligned}$$

$$\begin{aligned} \angle DE &= \angle FA \\ \angle ABCD + \angle DE &= \angle FA + \angle ABCD \end{aligned}$$

$$\begin{aligned} \angle ABCDE &= \angle FABCD \\ \angle DEF &= \angle EFA \\ \angle DCB &= \angle CBA = \angle BAF \\ &= \angle AFE = \angle FED = \angle EDC \end{aligned}$$

The hexagon was proven to be equilateral  
It is also equiangular

## Proof (cont)

The circumference FA is equal to the circumference ED

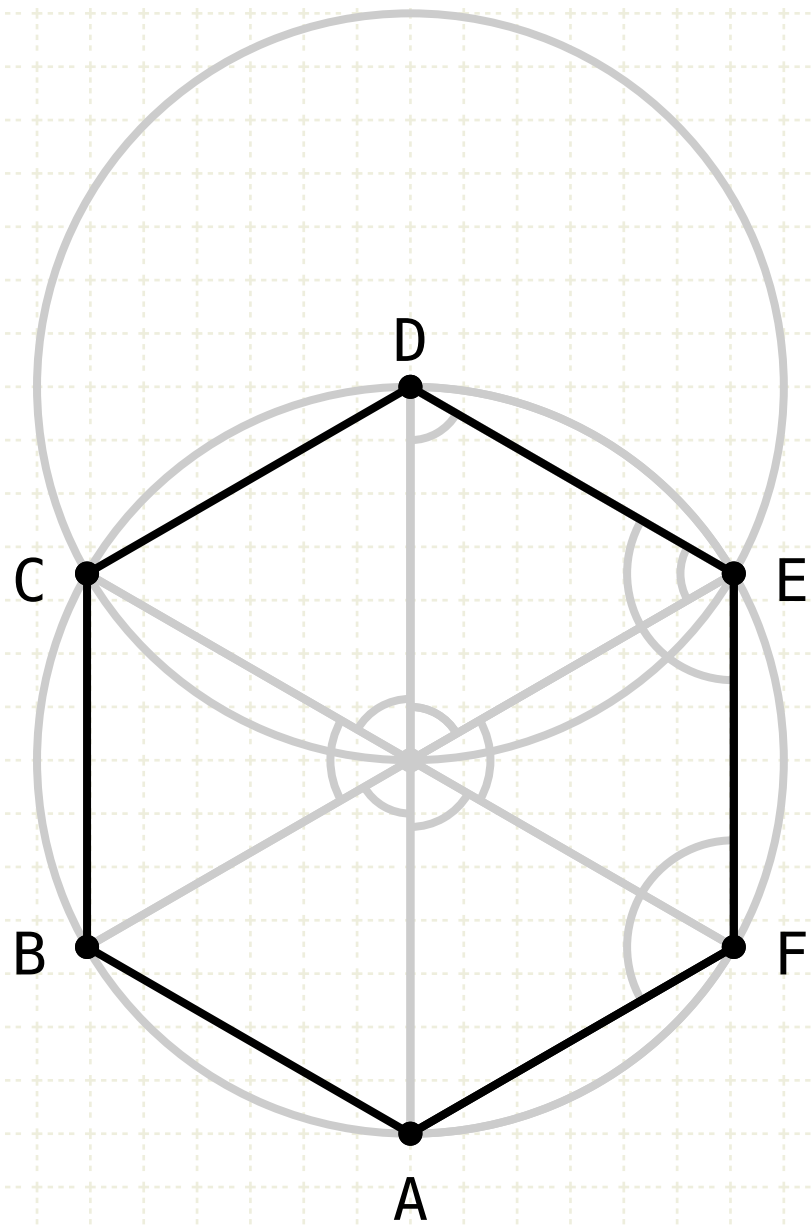
Let the circumference ABCD be added to each of FA and ED  
maintaining the equality

Equal circumferences have equal angles, therefore the angles  
are also equal (III·27)

Similarly, we can show that all the angles are equal

# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$\begin{aligned} DG &= EG \\ DG &= DE \\ \angle DGE &= \alpha = (1/3) \cdot 2L \\ \angle CGD &= \varepsilon = (1/3) \cdot 2L \\ (\varepsilon + \alpha) + \gamma &= 2L \\ \gamma &= \varepsilon = \alpha = (1/3) \cdot 2L \\ AB &= BC = CD = DE = EF \end{aligned}$$

$$\begin{aligned} \angle DE &= \angle FA \\ \angle ABCD + \angle DE &= \angle FA + \angle ABCD \end{aligned}$$

$$\begin{aligned} \angle ABCDE &= \angle FABCD \\ \angle DEF &= \angle EFA \\ \angle DCB &= \angle CBA = \angle BAF \\ &= \angle AFE = \angle FED = \angle EDC \end{aligned}$$

The hexagon was proven to be equilateral  
It is also equiangular

## Proof (cont)

The circumference FA is equal to the circumference ED

Let the circumference ABCD be added to each of FA and ED  
maintaining the equality

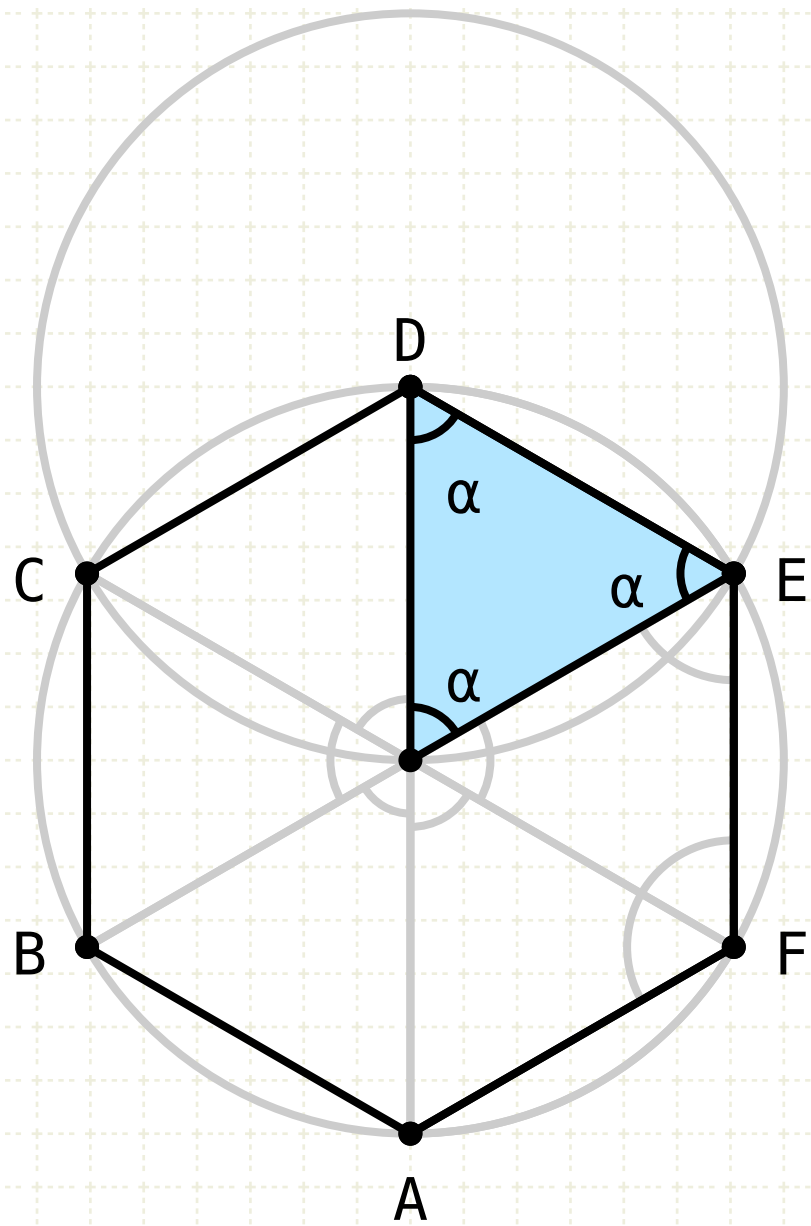
Equal circumferences have equal angles, therefore the angles  
are also equal (III·27)

Similarly, we can show that all the angles are equal

The hexagon is both equilateral and equiangular

# Proposition 15 of Book IV

In a given circle to inscribe an equilateral and equiangular hexagon.



$$\begin{aligned} DG &= EG \\ DG &= DE \\ \angle DGE &= \alpha = (1/3) \cdot 2L \\ \angle CGD &= \varepsilon = (1/3) \cdot 2L \\ (\varepsilon + \alpha) + \gamma &= 2L \\ \gamma &= \varepsilon = \alpha = (1/3) \cdot 2L \\ AB &= BC = CD = DE = EF \end{aligned}$$

$$\begin{aligned} \angle DE &= \angle FA \\ \angle ABCD + \angle DE &= \angle FA + \angle ABCD \end{aligned}$$

$$\begin{aligned} \angle ABCDE &= \angle FABCD \\ \angle DEF &= \angle EFA \\ \angle DCB &= \angle CBA = \angle BAF \\ &= \angle AFE = \angle FED = \angle EDC \end{aligned}$$

The hexagon was proven to be equilateral  
It is also equiangular

## Proof (cont)

The circumference FA is equal to the circumference ED

Let the circumference ABCD be added to each of FA and ED  
maintaining the equality

Equal circumferences have equal angles, therefore the angles  
are also equal (III·27)

Similarly, we can show that all the angles are equal

The hexagon is both equilateral and equiangular

## Note:

From the first part of the proof, it can be noted that the side of a  
hexagon is equal to the radius of the circle.



# Youtube Videos

<https://www.youtube.com/c/SandyBultena>

*Copyright © 2019 by Sandy Bultena.*



Except where otherwise noted, this work is licensed under  
<http://creativecommons.org/licenses/by-nc/3.0>