

# Euclid's Elements

## Book VII

### Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

*As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.*

**Joseph-Louis Lagrange**  
**(1736 to 1813)**



# Table of Contents, Chapter 7

- |   |  |    |   |           |  |
|---|--|----|---|-----------|--|
| 1 | Determine if two numbers are relatively prime  | 10 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$              | 21        | If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B                          |
| 2 | Find the greatest common divisor for two numbers   | 11 | If $A:B = C:D$ , then $(A-C):(B-D) = A:B$   | <b>22</b> | <b>If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime</b>                   |
| 3 | Find the largest common divisor for three numbers  | 12 | If $A:B = C:D$ , then $(A+C):(B+C) = A:B$   | 23        | If A,B are relatively prime and if $A = n \cdot C$ , then B,C are relatively prime   |
| 4 | Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B | 13 | If $A:B = C:D$ , then $A:C = B:D$   | 24        | If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C                                  |
| 5 | If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , then $(B+D) = (1/q) \cdot (A+C)$  | 14 | If $A:B = D:E$ and $B:C = E:F$ , then $A:C = D:F$   | 25        | If A,B are relatively prime then $A^2, B$ are relatively prime   |
| 6 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , then $(B+D) = (p/q) \cdot (A+C)$  | 15 | If $B = i \cdot 1$ and $E = i \cdot D$ , and if $D = j \cdot 1$ then $E = j \cdot B$                                | 26        | If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$ |
| 7 | If $B = A/q$ and $D = C/q$ , $B > D$ , then $(B-D) = (A-C)/q$  | 16 | $A \times B = B \times A$   | 27        | If A,B are relatively prime, then $A^2, B^2$ are relatively prime, and $A^3, B^3$ are relatively prime, and so on                        |
| 8 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , $B > D$ , then $(B-D) = (p/q) \cdot (A-C)$  | 17 | If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$   |           |  |
| 9 | If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$                             | 18 | If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$   |           |  |
|   |  | 19 | If $A:B = C:D$ then $A \times D = B \times C$<br>If $A \times D = B \times C$ then $A:B = C:D$                      |           |  |
|   |  | 20 | Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$ |           |  |



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- |    |  |    |   |
|----|--|----|---|
| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime   | 37 | If $A = p \cdot B$ , then $A = q \cdot C$ where $C = p \cdot 1$       |
| 29 | If A is prime, and $B \neq n \cdot A$ , then A,B are relatively prime                                    | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$       |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a$ , $1/b$ , $1/c$ |
| 31 | If $A = B \times C$ , then $A = j \cdot D$ where D is prime  |    |   |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime                            |    |   |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C                  |    |   |
| 34 | Find the lowest common denominator of 2 numbers  |    |   |
| 35 | If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$ , then $C = i \cdot E$  |    |   |
| 36 | Find the least common multiple of 3 numbers  |    |   |



# Proposition 22 of Book VII

The least numbers of those which have the same ratio with them are prime to one another.



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A ●—————

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$$S = \{ (x,y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x:y=A:B \}$$
$$(A,B) \in S \text{ such that } A \leq x, B \leq y, \forall (x,y) \in S$$

$$\gcd(A,B) = 1$$

## In other words

If A and B are the smallest whole numbers that are equal to the ratio of A to B, then they are prime to one another





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## Proof by Contradiction



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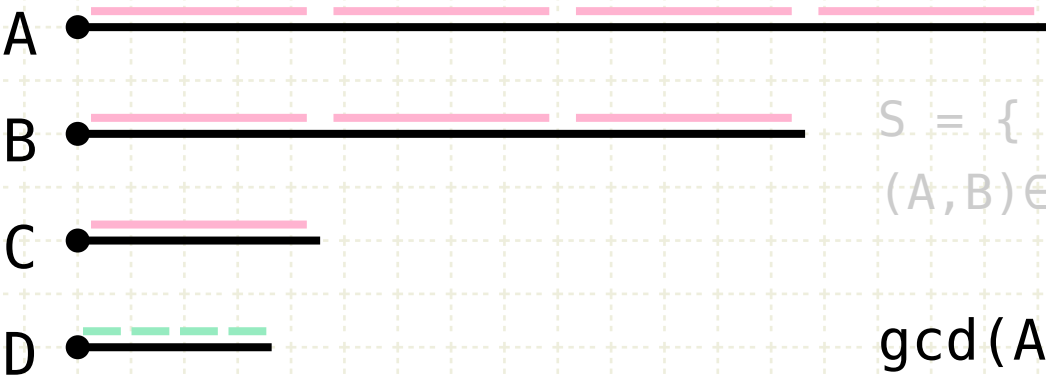
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$$A = d \cdot C$$
$$D = d \cdot 1$$

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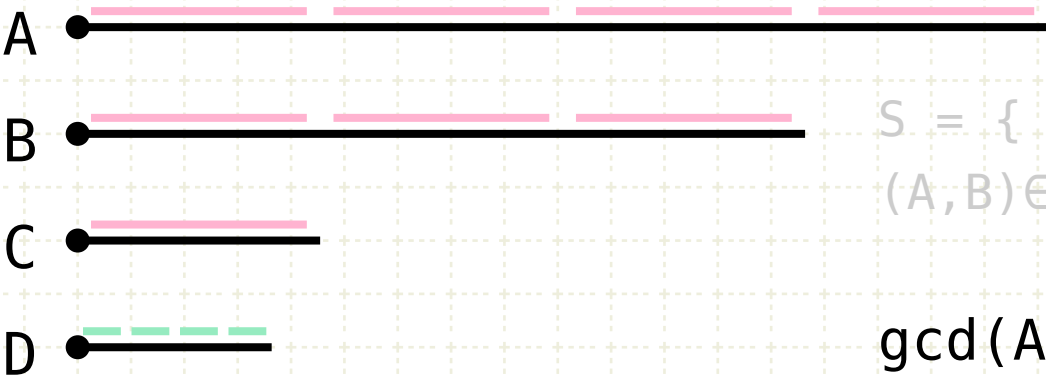
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For as many times that C measures A, let there be so many units in D





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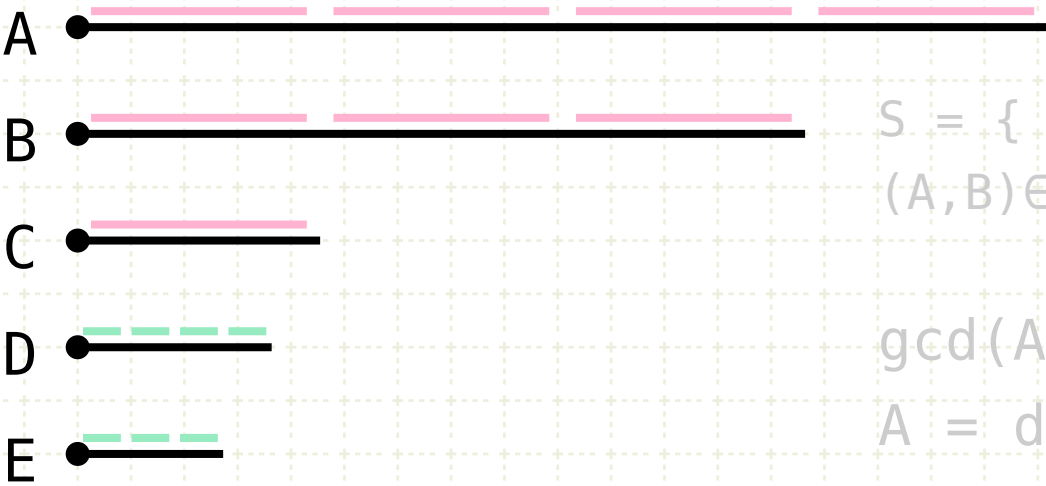
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C measures A according to the units in D, therefore A is equal to C times D (VII.Def.15)



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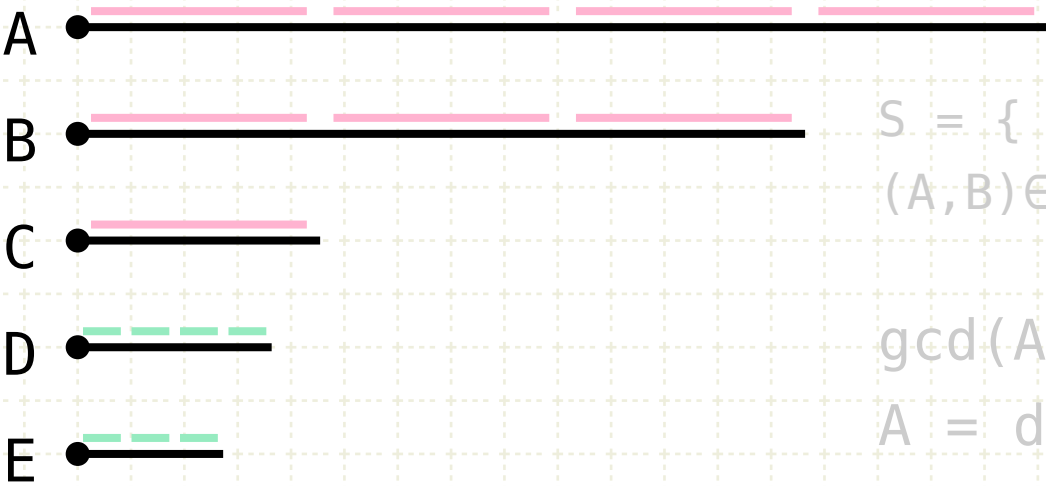
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$$D:E = A:B$$

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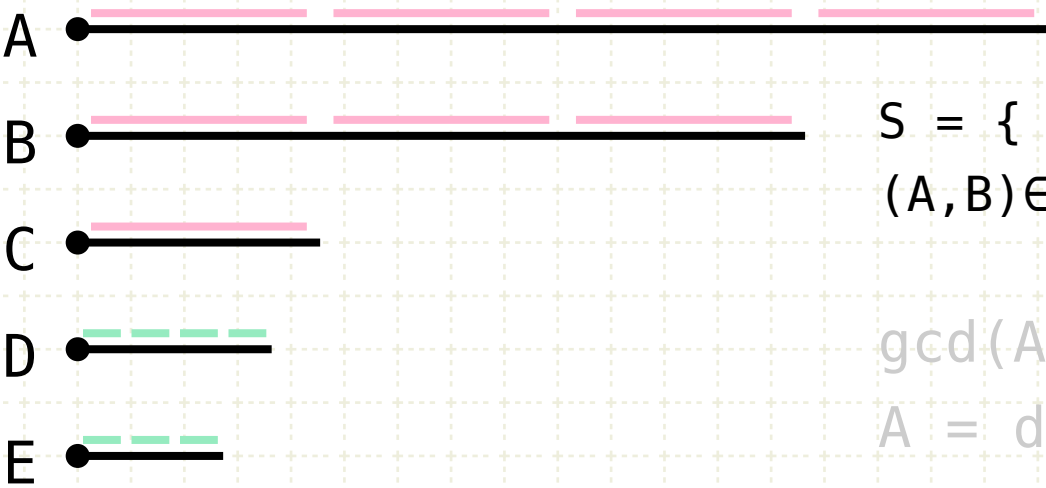
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Thus A,B are equal to C multiplied by D and E, respectively, therefore D is to E as A is to B (VII·17)



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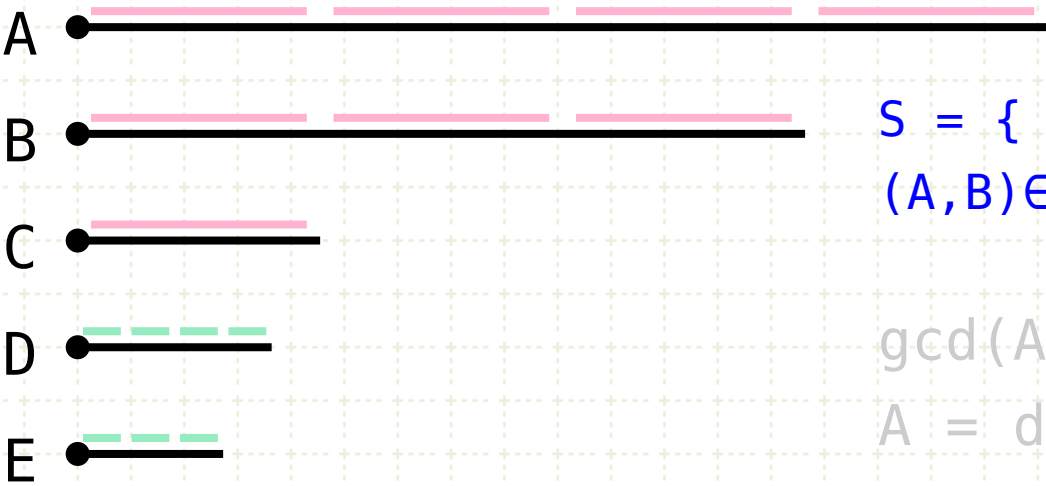
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But D and E are less than A and B, which violates the original hypothesis



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C measures A according to the units in D, therefore A is equal to C times D (VII.Def.15)

By the same argument, B is equal to C times E

Thus A,B are equal to C multiplied by D and E, respectively, therefore D is to E as A is to B (VII·17)

But D and E are less than A and B, which violates the original hypothesis

Therefore, A and B are relatively prime





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