

# Euclid's Elements

## Book VI

*One can state, without exaggeration, that the observation of and the search for similarities and differences are the basis of all human knowledge.*

**Alfred Nobel**



# Table of Contents, Chapter 6

1	If the height of two triangles are equal, then the ratio of the areas is equal to the ratio of the bases	7	If two triangles have one angle equal to one angle, and the sides about other angles are proportional, and the remaining angles either both less or both not less than a right angle, then triangles will be equiangular	14	In equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional; and vice versa
2	If a line cuts a triangle, parallel to its base, it will cut the sides of the triangle proportionally			15	In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and vice versa
3	If an angle of a triangle is bisected and the straight line cutting the angle also cuts the base, the segments of the base will have the same ratio as the remaining sides of the triangle	8	If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another	16	If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means, and vice versa
4	If two triangles have equal angles, then the sides opposite the equal angles are proportional, as well, the sides of the triangles on either side of the equal angles are also proportional	9	From a given straight line to cut off a given fraction	17	If three straight lines are proportional, the rectangle contained by the extremes is equal to the square on the mean; and vice versa
5	<b>If two triangles have proportional sides, the triangles will be equiangular</b>	10	To cut a given uncut straight line similarly to a given cut straight line	18	On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure
6	If two triangles have one angle equal to one angle and the sides about the equal angles are proportional, then the triangles will be equiangular	11	To two given straight lines to find a third proportional	19	Similar triangles are to one another in the duplicate ratio of the corresponding sides
		12	To three given straight lines to find a fourth proportional		
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# Table of Contents, Chapter 6

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| 20 | Similar polygons are divided into the same number of similar triangles, which have the same ratio as the wholes, and the polygons have duplicate ratios to their corresponding sides | 26 | If from a parallelogram a similar parallelogram with a common angle is subtracted, it is about the same diameter as the original  | 31 | In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle                                     |
| 21 | Figures which are are similar to the same rectilineal figure are also similar to one another   | 27 | Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar to a parallelogram drawn on half the said line, the largest will be one that is drawn on half of the straight line and is similar to the defect | 32 | If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangle will be in a straight line |
| 22 | If four straight lines are proportional, similar rectilineal figures will also be proportional; and vice versa   | 28 | To a given straight line, apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one   | 33 | In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences   |
| 23 | Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides   | 29 | To a given straight line, apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one   |    |   |
| 24 | In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another  | 30 | To cut a finite straight line in extreme ratio  |    |   |
| 25 | To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure   |    |   |    |   |



# Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend





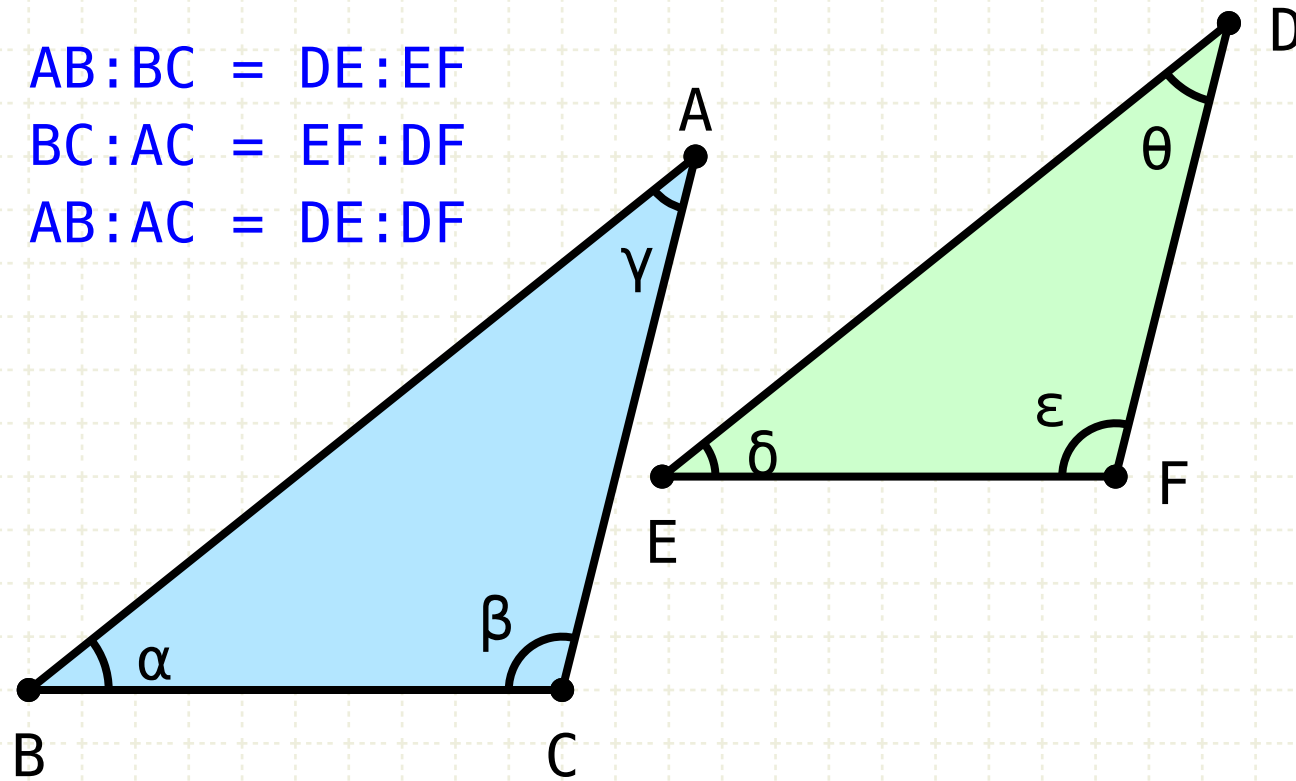
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It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend

$$AB:BC = DE:EF$$

$$BC:AC = EF:DF$$

$$AB:AC = DE:DF$$



$$\alpha = \delta, \beta = \epsilon, \gamma = \theta$$

### In other words

If side 'a' is to side 'b' of one triangle, and is equal to side 'd' to 'e' of another, and similarly for all sides, then the angle between 'a' and 'b' will be equal to the angles between 'd' and 'e'

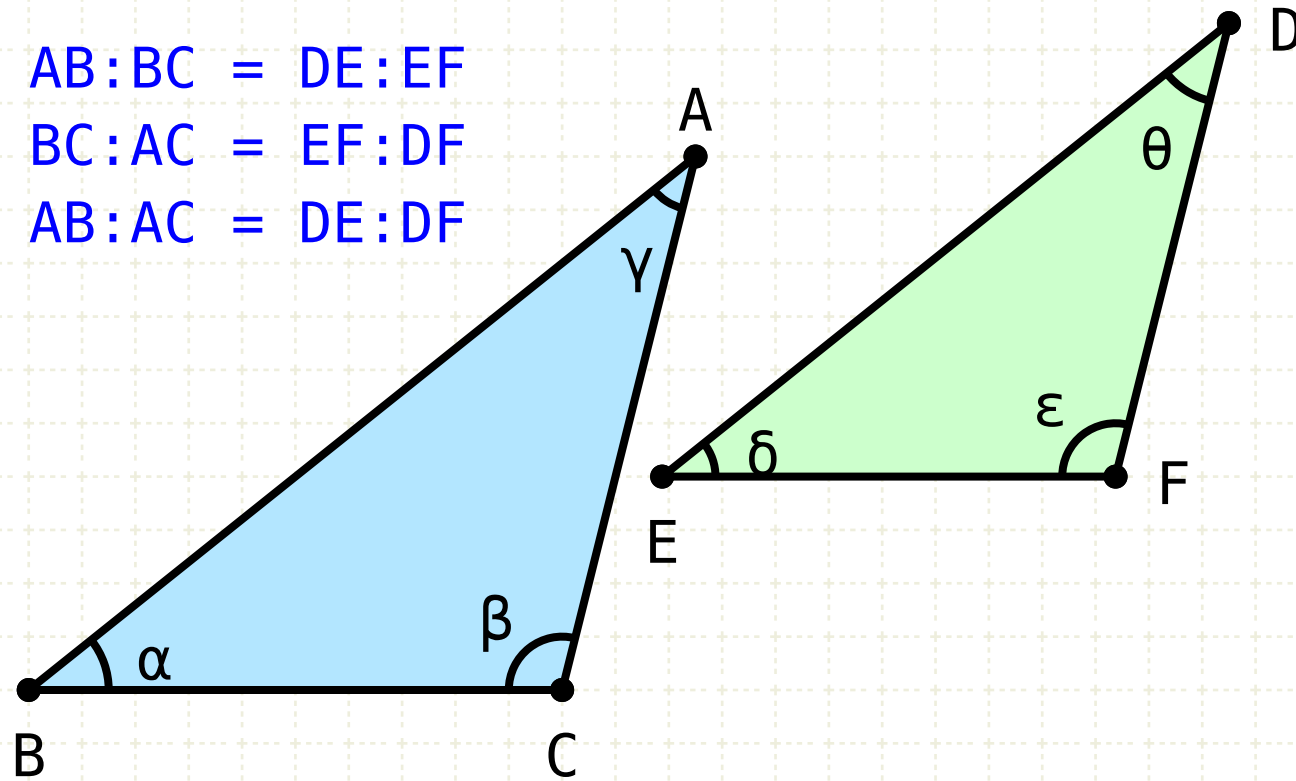
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**Proof**

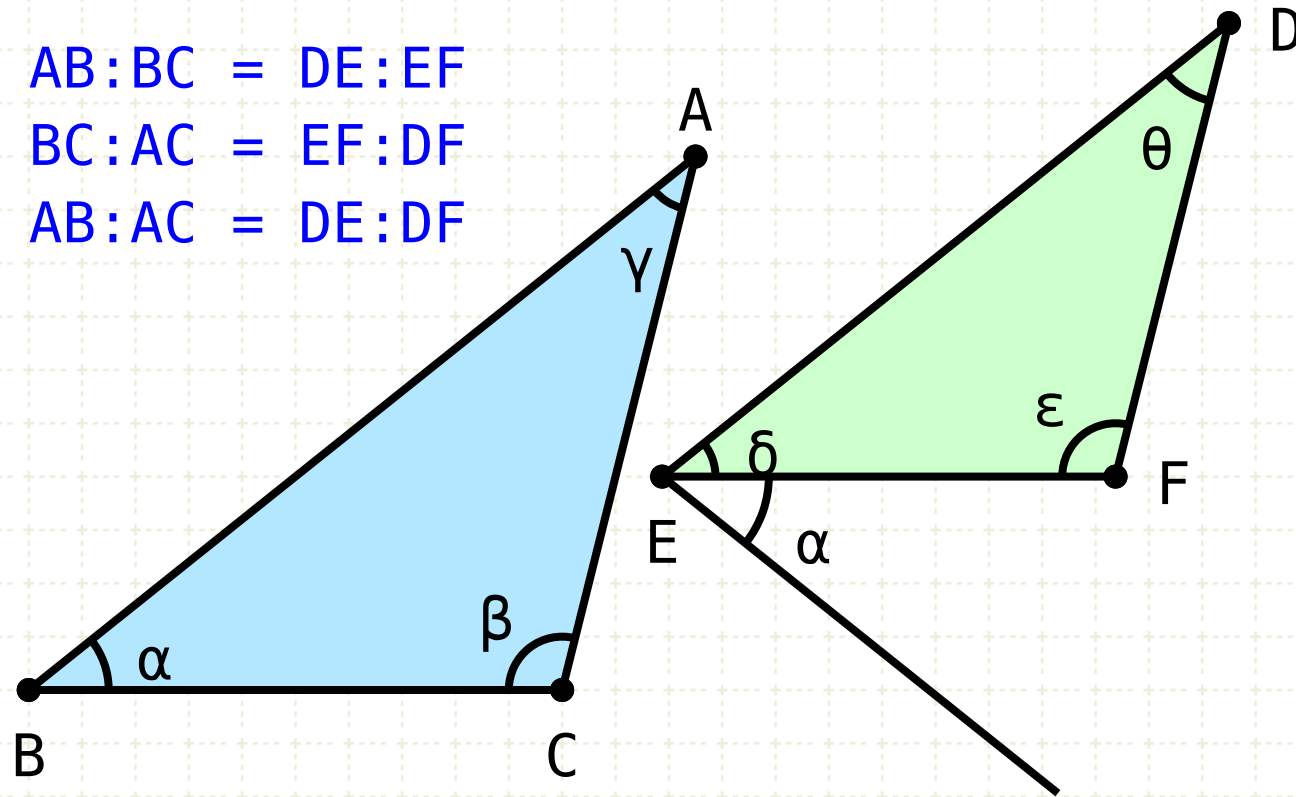
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$$AB:BC = DE:EF$$

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### Proof

On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I-23)

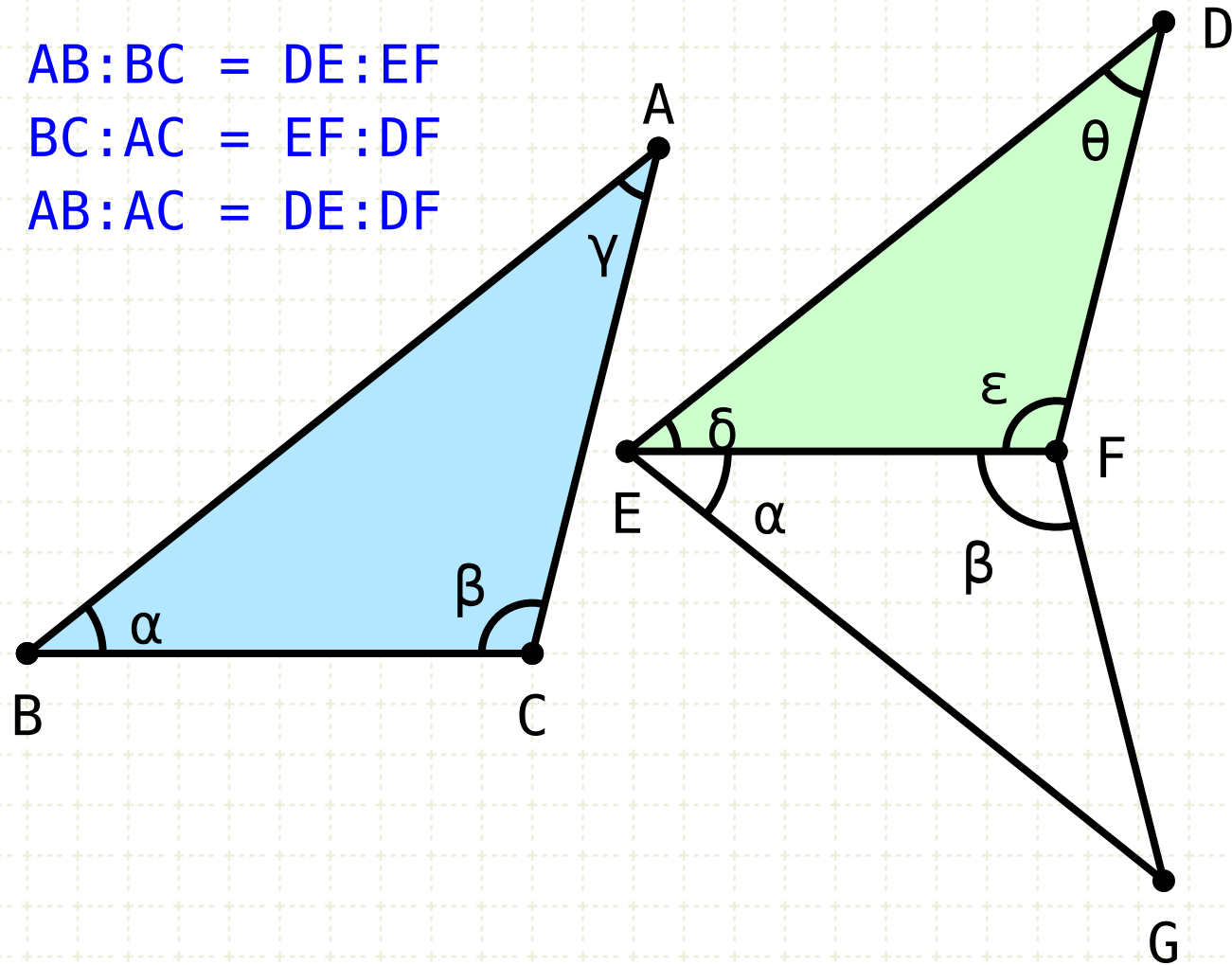
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$$AB:BC = DE:EF$$

$$BC:AC = EF:DF$$

$$AB:AC = DE:DF$$



### Proof

On the point  $E$ , construct an angle  $FEG$  on the line  $EF$  equal to the angle  $\alpha$  (I-23)

On the point  $F$ , construct an angle  $EFG$  on the line  $EF$  equal to the angle  $\beta$  (I-23)



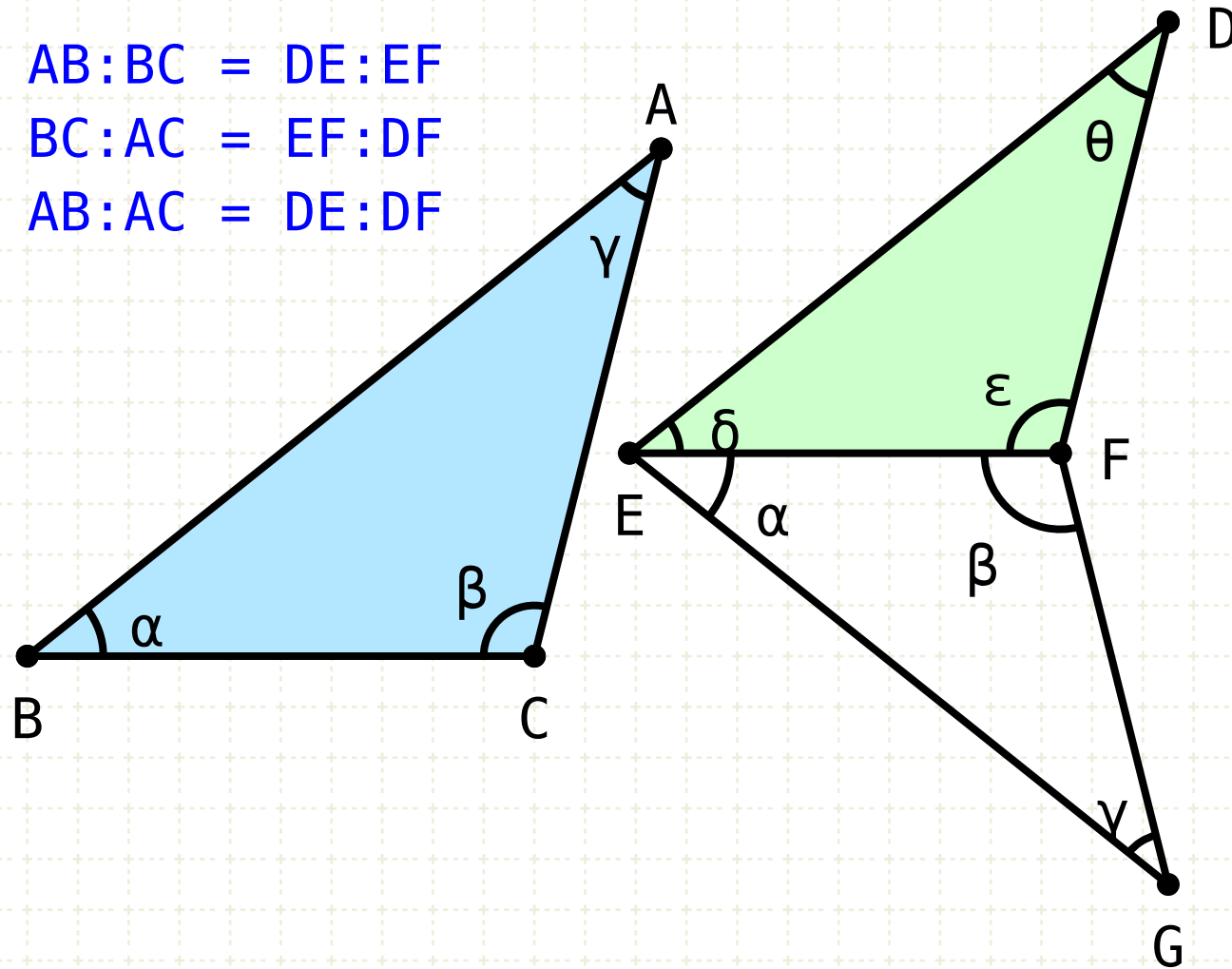
## Proposition 5 of Book VI

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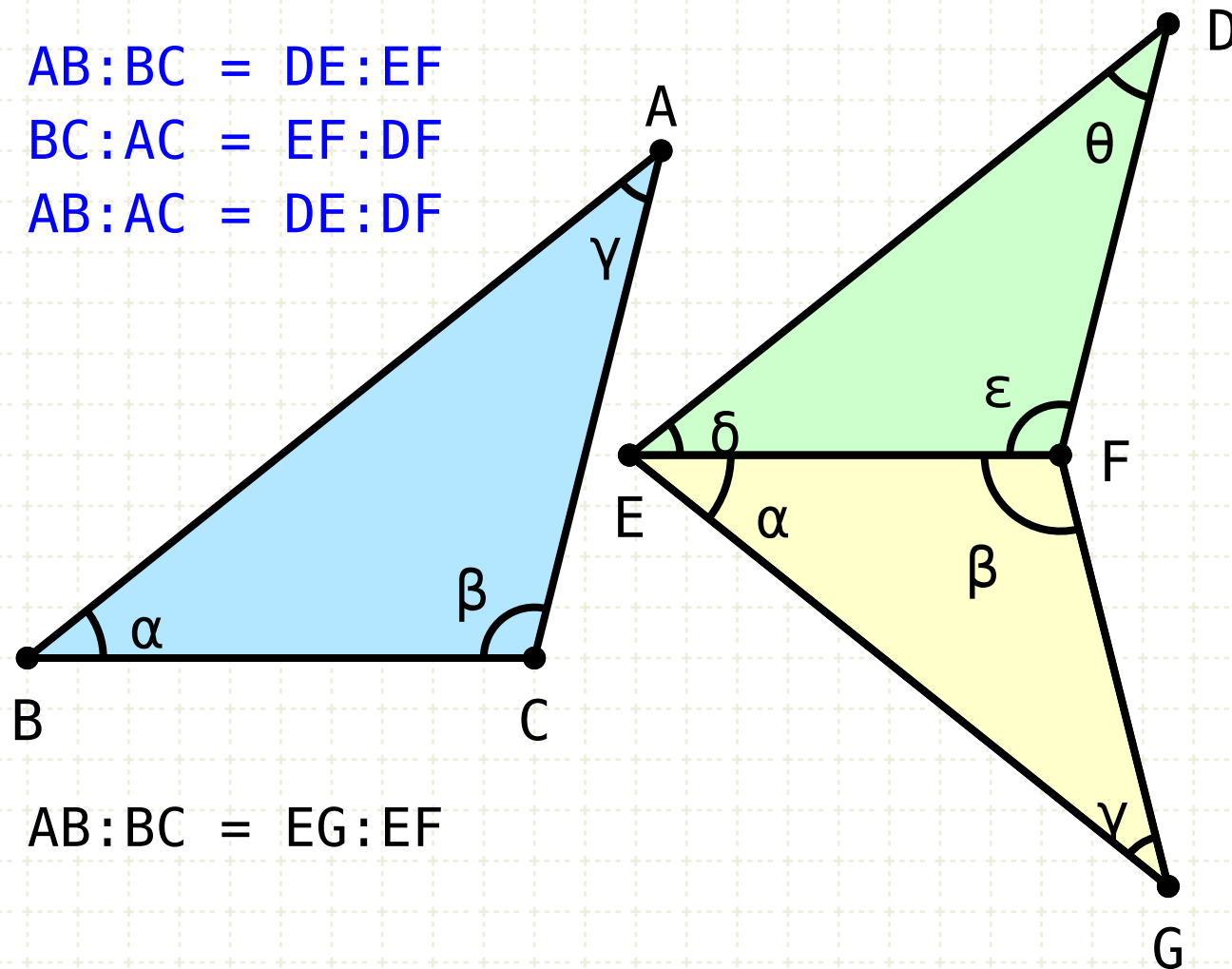
On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I·23)

On the point F, construct an angle EFG on the line EF equal to the angle  $\beta$  (I·23)

And thus, the angle at G will also be the angle at A (I·32)

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



### Proof

On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I·23)

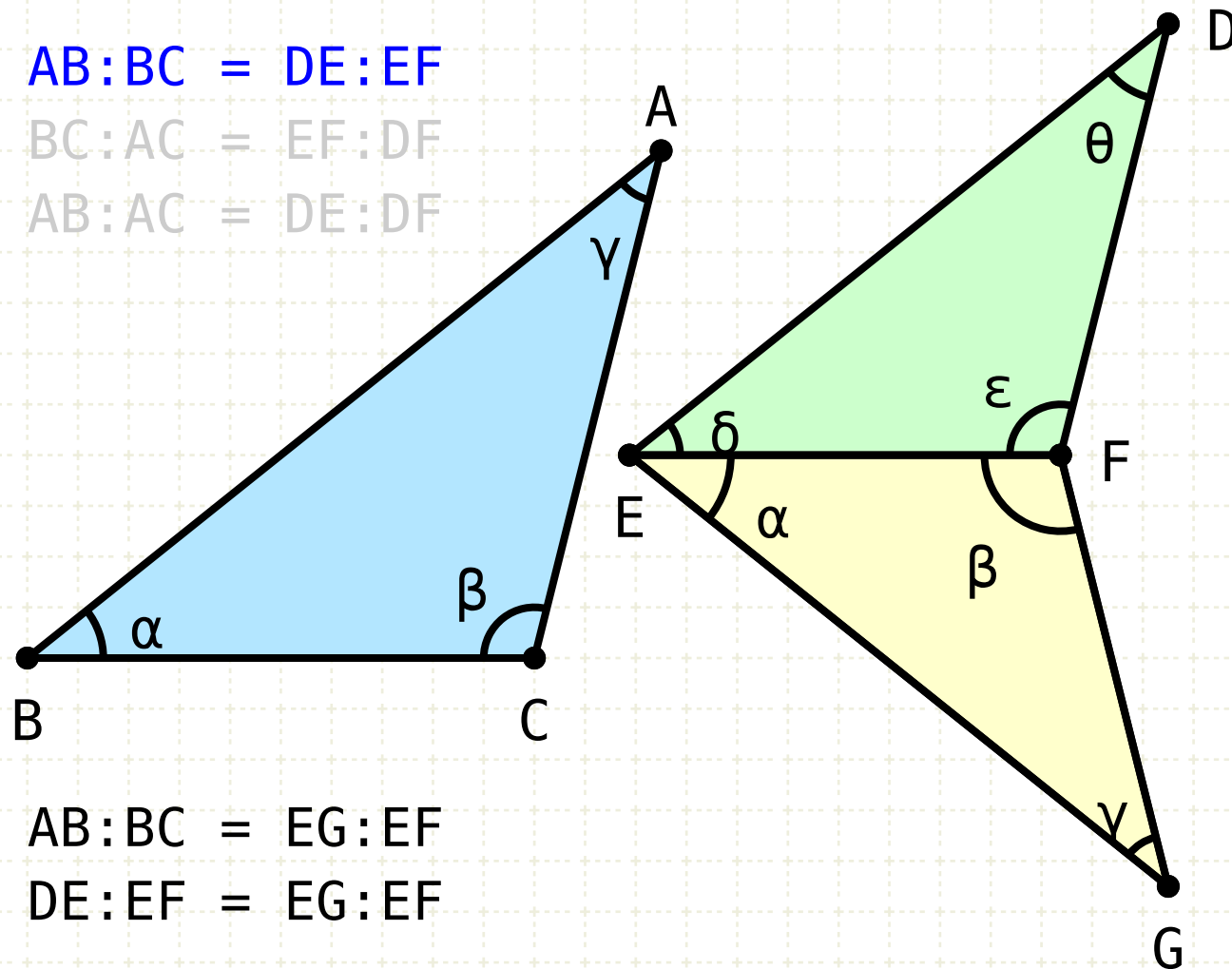
On the point F, construct an angle EFG on the line EF equal to the angle  $\beta$  (I·23)

And thus, the angle at G will also be the angle at A (I·32)

Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



### Proof

On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I·23)

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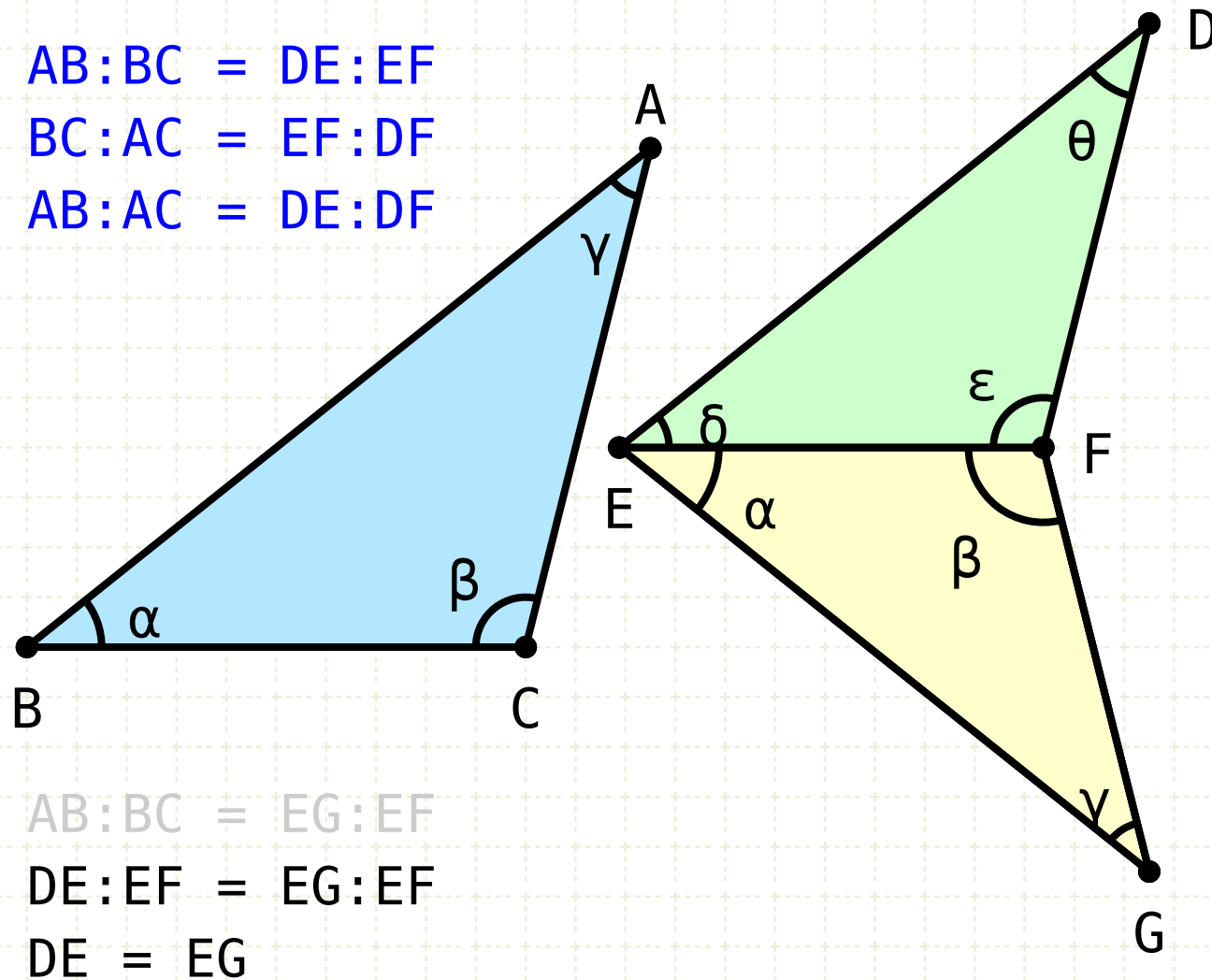
And thus, the angle at G will also be the angle at A (I·32)

Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

But the ratio AB to BC is equal to DE to EF, therefore the ratio DE to EF equals EG to EF (V·11)

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



### Proof

On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I·23)

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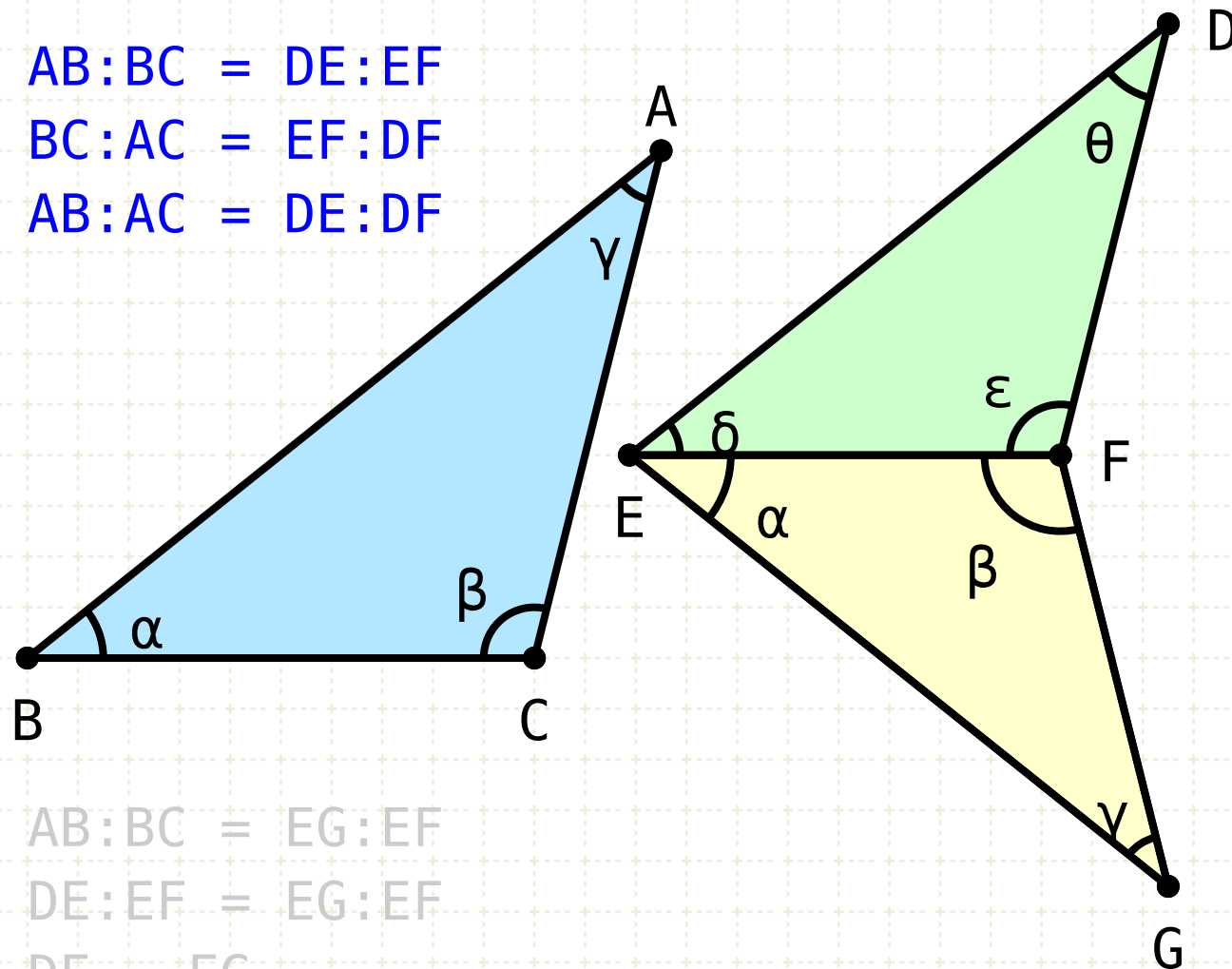
Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

But the ratio AB to BC is equal to DE to EF, therefore the ratio DE to EF equals EG to EF (V·11)

Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AB:BC &= DE:EF \\ BC:AC &= EF:DF \\ AB:AC &= DE:DF \end{aligned}$$

$$\begin{aligned} AB:BC &= EG:EF \\ DE:EF &= EG:EF \\ DE &= EG \\ DF &= FG \end{aligned}$$

### Proof

On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I·23)

On the point F, construct an angle EFG on the line EF equal to the angle  $\beta$  (I·23)

And thus, the angle at G will also be the angle at A (I·32)

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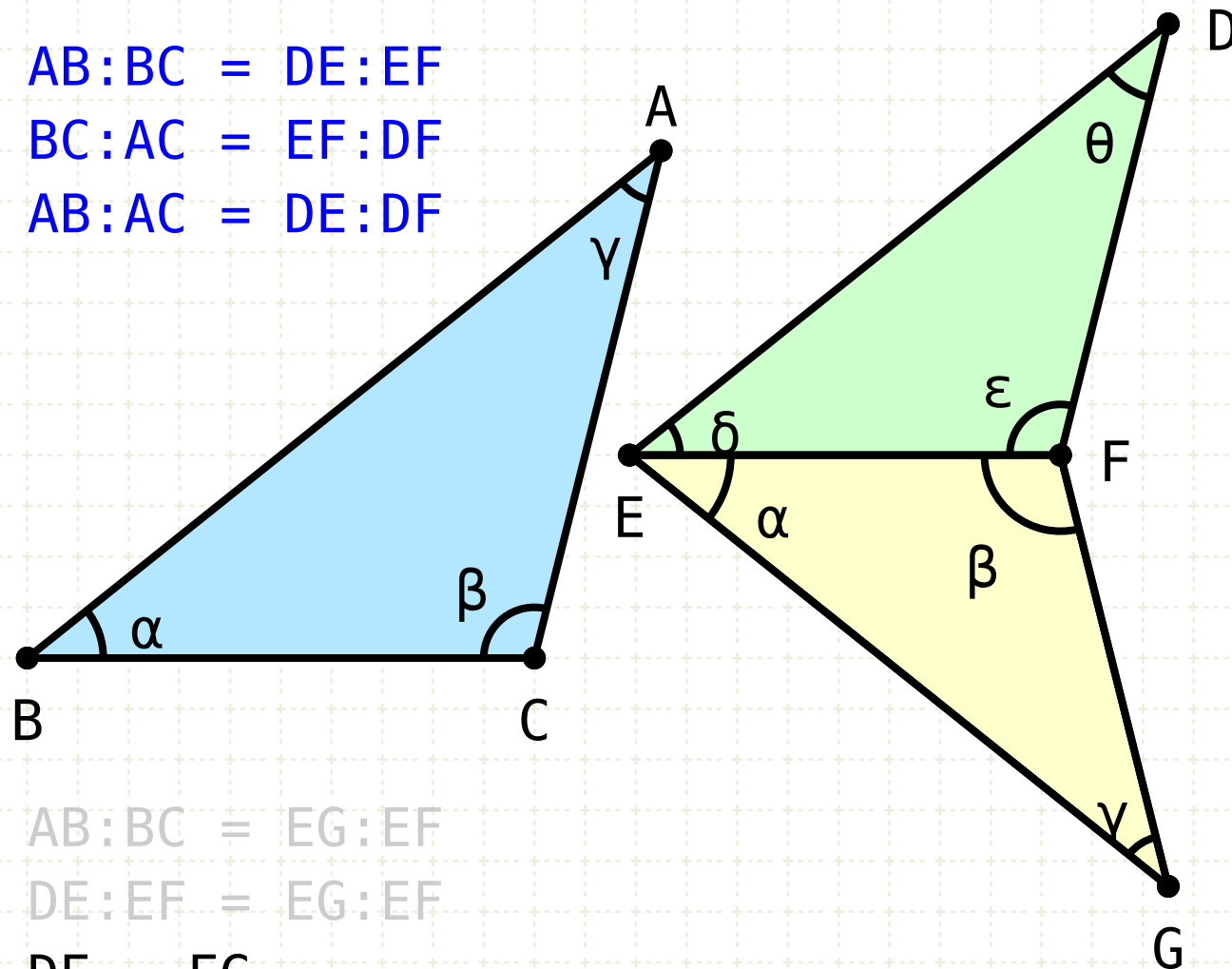
Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

and for the same reason DF is also equal to FG



## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AB:BC &= DE:EF \\ BC:AC &= EF:DF \\ AB:AC &= DE:DF \end{aligned}$$

$$\begin{aligned} AB:BC &= EG:EF \\ DE:EF &= EG:EF \\ DE &= EG \\ DF &= FG \\ \alpha &= \delta \end{aligned}$$

### Proof

On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I·23)

On the point F, construct an angle EFG on the line EF equal to the angle  $\beta$  (I·23)

And thus, the angle at G will also be the angle at A (I·32)

Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

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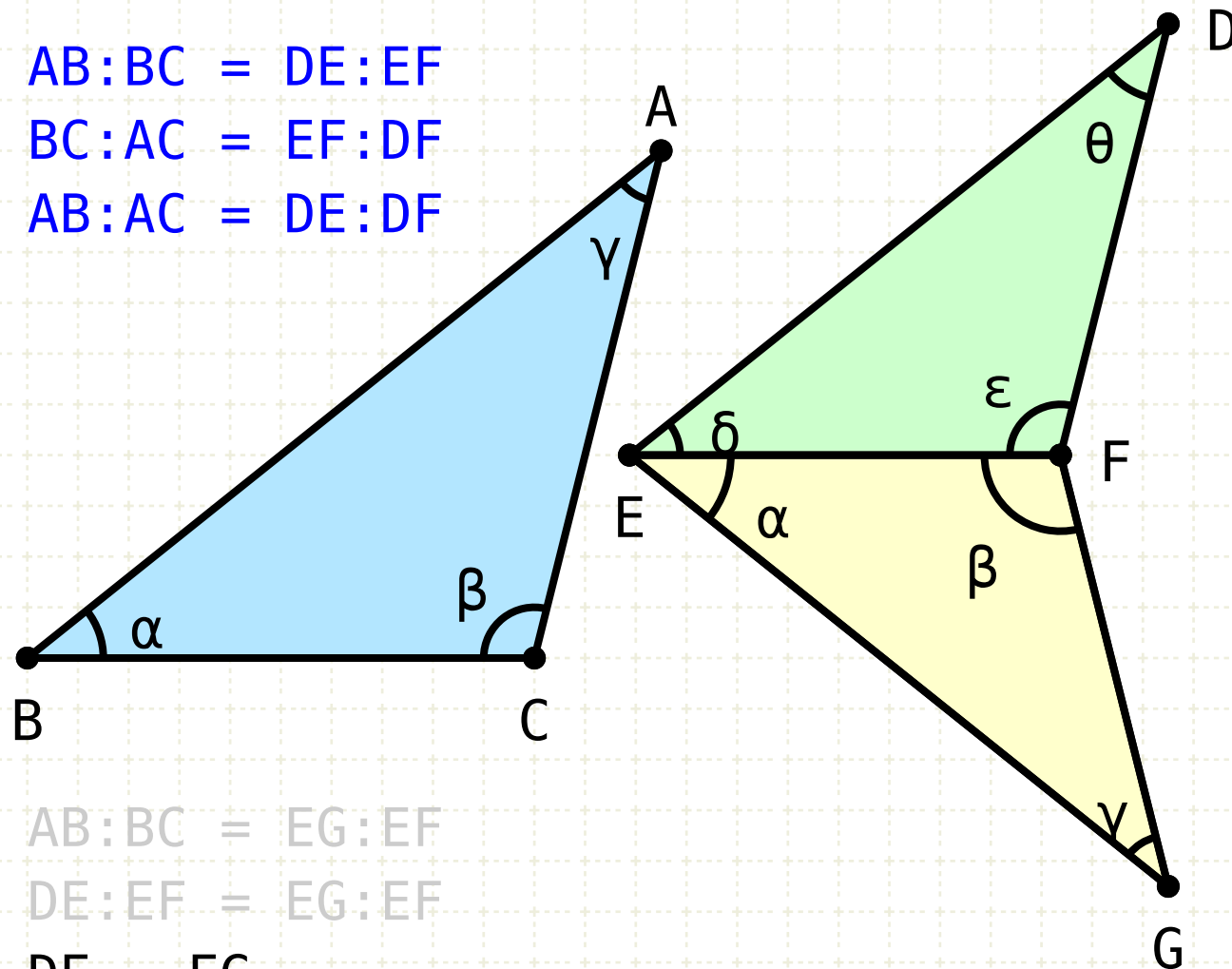
Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

and for the same reason DF is also equal to FG

Since DE is equal to EG, and DF equals FG, and there is a common base EF (three sides equal) then the angle DEF is equal to GEF (I·8),

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AB:BC &= DE:EF \\ BC:AC &= EF:DF \\ AB:AC &= DE:DF \end{aligned}$$

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$$DE = EG$$

$$DF = FG$$

$$\alpha = \delta$$

$$\beta = \epsilon$$

$$\gamma = \theta$$

### Proof

On the point E, construct an angle FEG on the line EF equal to the angle  $\alpha$  (I·23)

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Therefore the triangle ABC is equiangular to EFG, and as such, the edges surrounding the equal angles will be in proportion, i.e. AB is to BC as EG to EF (VI·4)

But the ratio AB to BC is equal to DE to EF, therefore the ratio DE to EF equals EG to EF (V·11)

Since DE and EG have the same ratio to EF, DE and EG are equal (V·9),

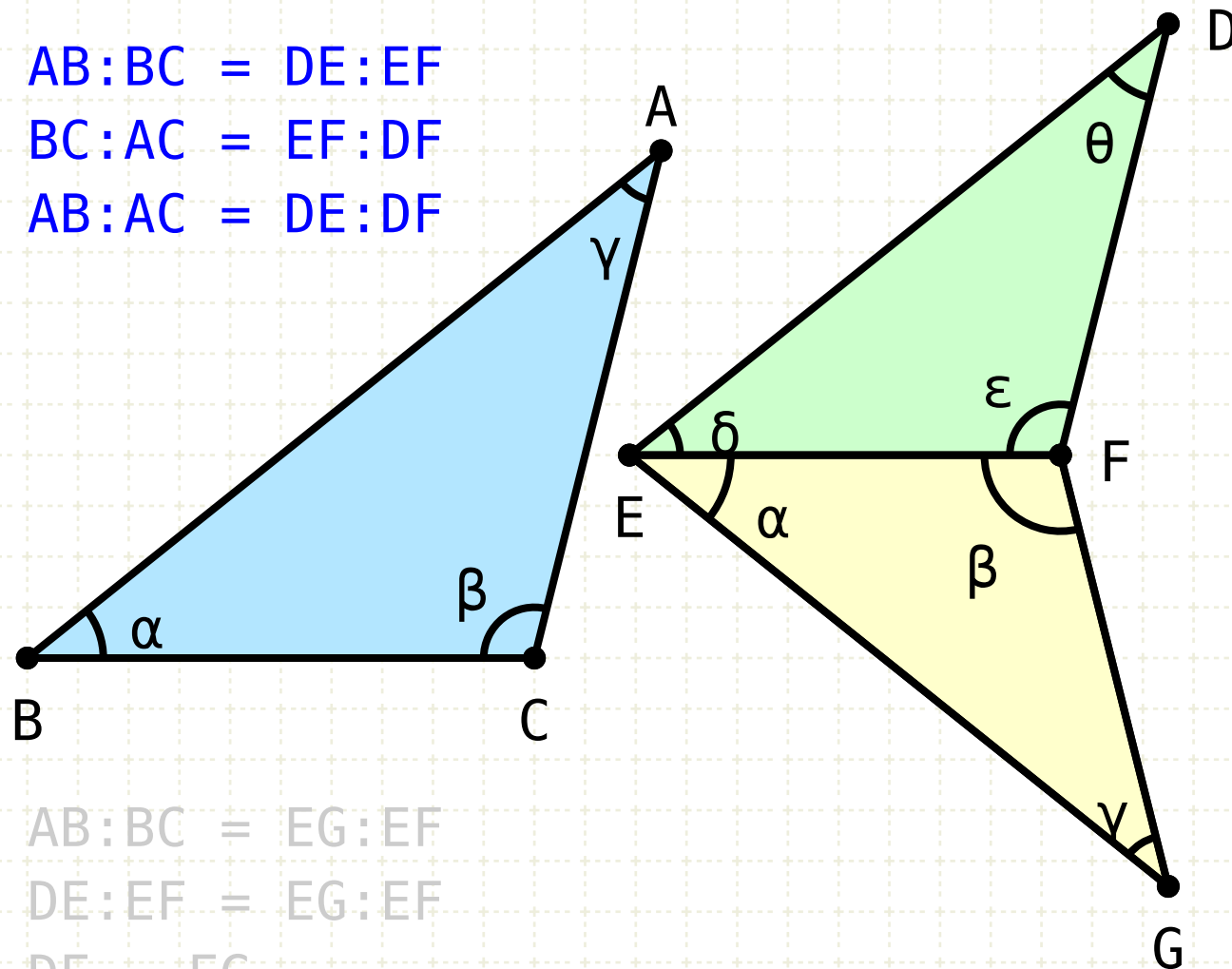
and for the same reason DF is also equal to FG

Since DE is equal to EG, and DF equals FG, and there is a common base EF (three sides equal) then the angle DEF is equal to GEF (I·8),

and finally, since there are two equal sides subtending an equal angle, both triangles DEF and EFG are equal (I·4)

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AB:BC &= DE:EF \\ BC:AC &= EF:DF \\ AB:AC &= DE:DF \end{aligned}$$

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But the ratio AB to BC is equal to DE to EF, therefore the ratio DE to EF equals EG to EF (V·11)

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and for the same reason DF is also equal to FG

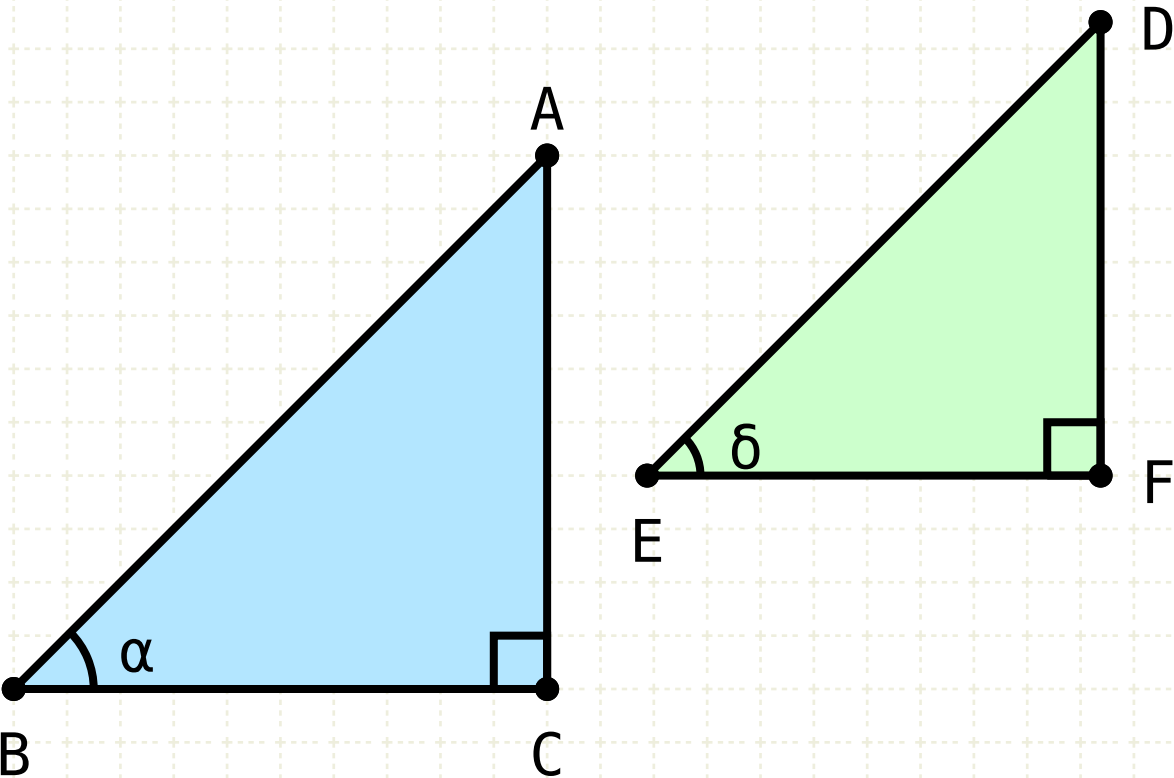
Since DE is equal to EG, and DF equals FG, and there is a common base EF (three sides equal) then the angle DEF is equal to GEF (I·8),

and finally, since there are two equal sides subtending an equal angle, both triangles DEF and EFG are equal (I·4)

So finally, the triangle DEF is equiangular to triangle ABC

# Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend

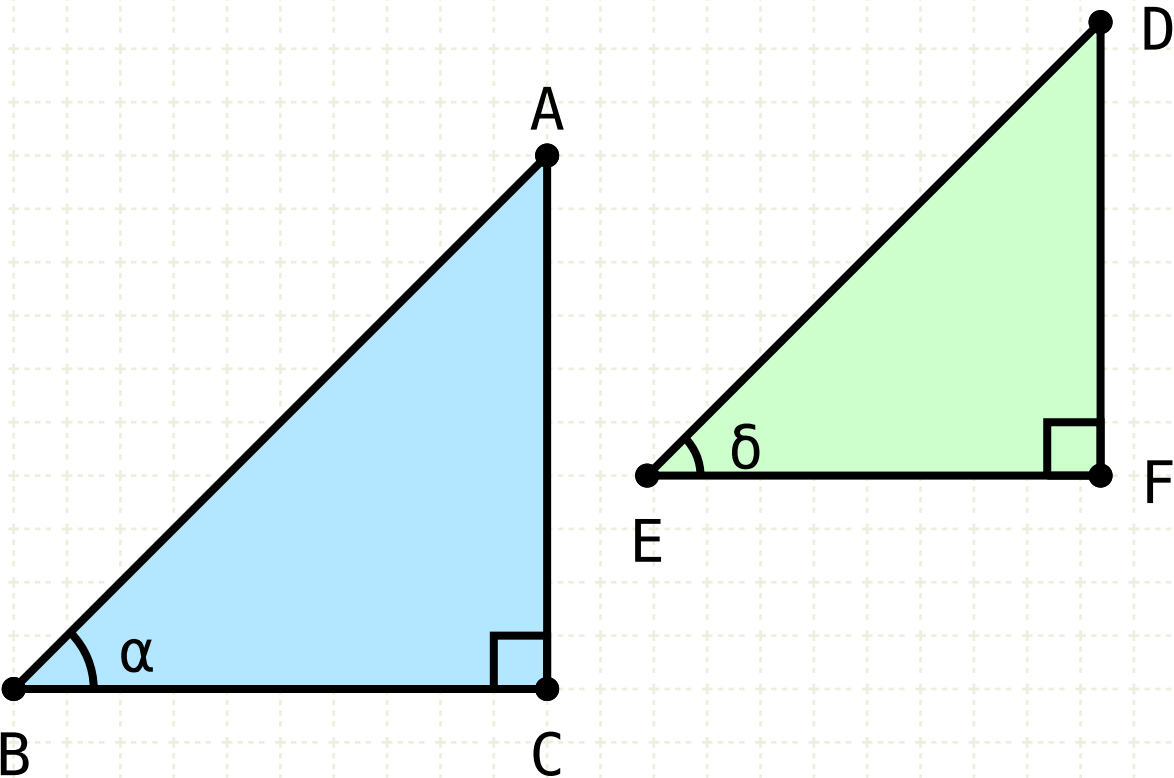


Aside - Trigonometry



# Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$\alpha = \delta$

## Aside - Trigonometry

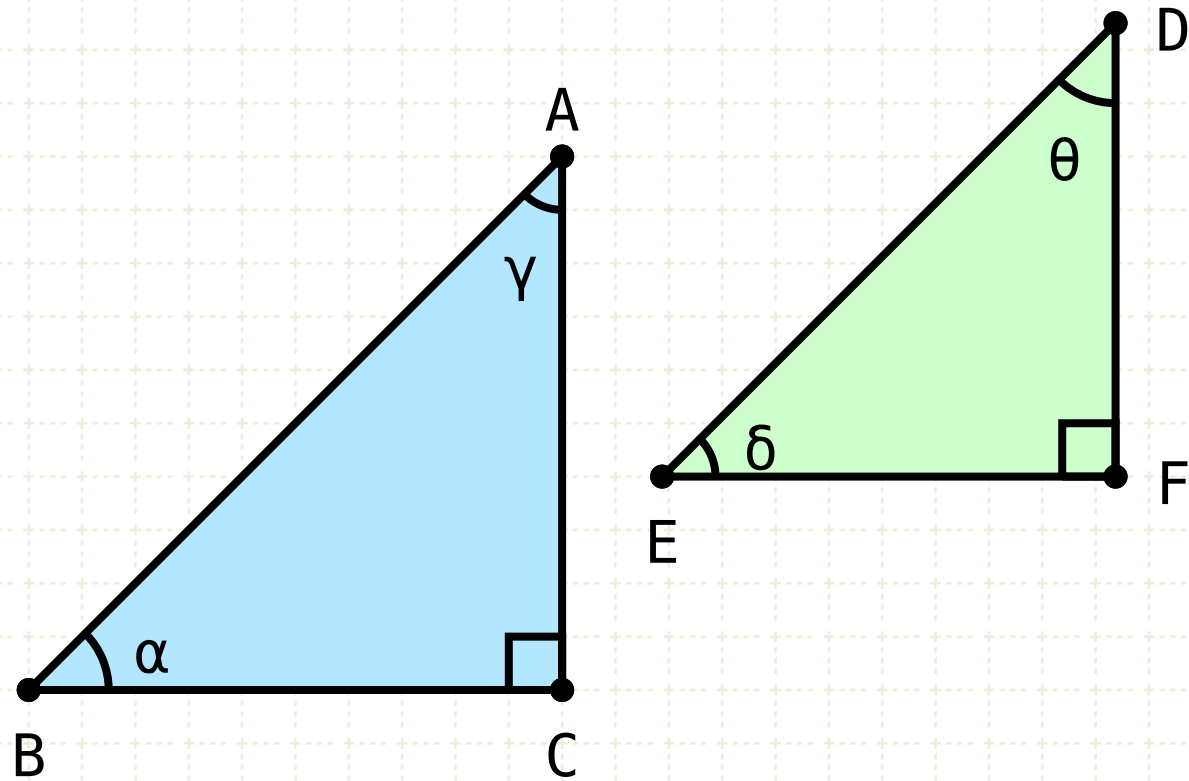
Consider two right angle triangles where angle ABC equals angle DEF





## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\alpha = \delta$$
$$\gamma = \theta$$

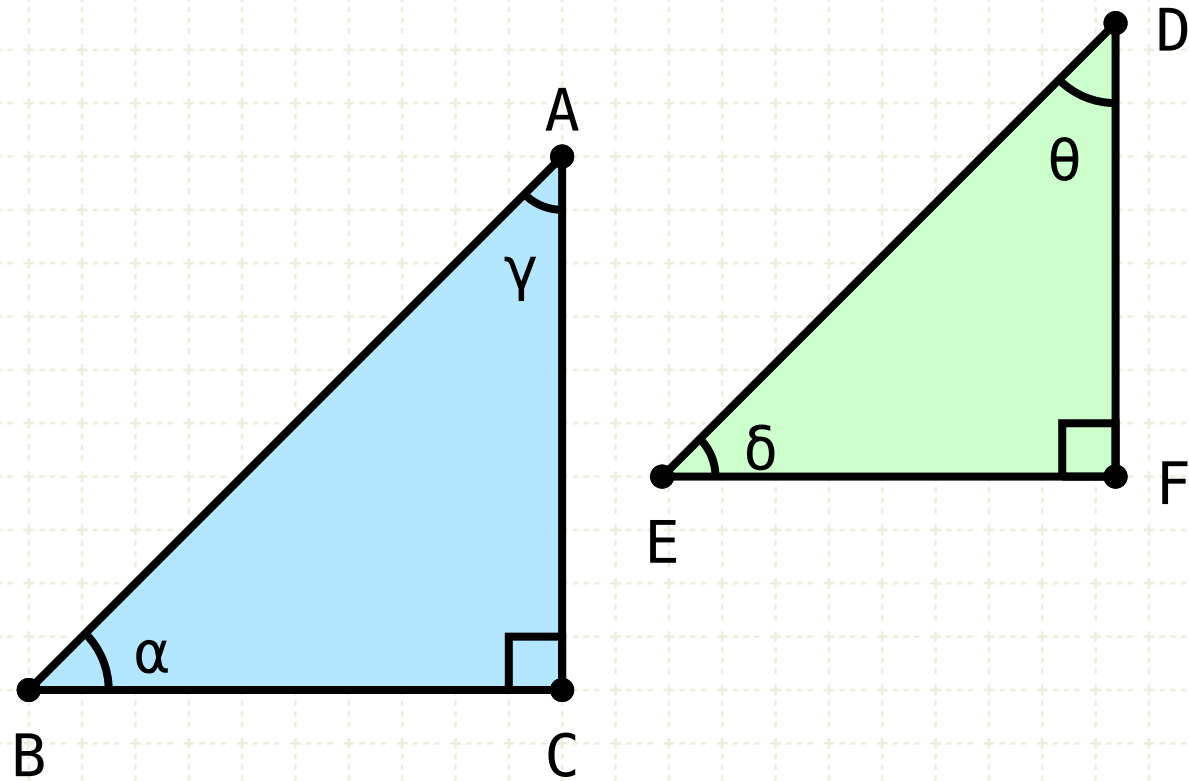
### Aside - Trigonometry

Consider two right angle triangles where angle ABC equals angle DEF

Since two of the angles are equal in both triangles, the third must also be equal, hence angle BAC equals EDF

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\alpha = \delta$$

$$\gamma = \theta$$

$$AC:AB = DF:DE$$

### Aside - Trigonometry

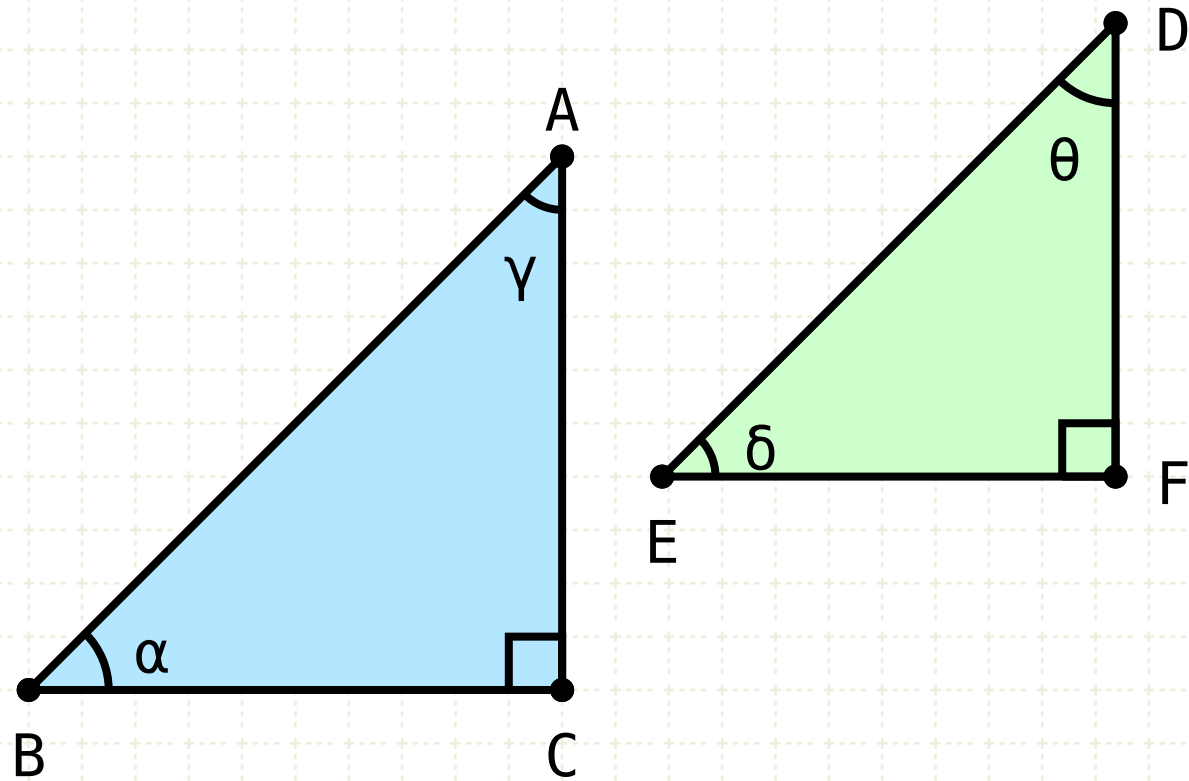
Consider two right angle triangles where angle ABC equals angle DEF

Since two of the angles are equal in both triangles, the third must also be equal, hence angle BAC equals EDF

Then, according to (VI·4), the ratio of the sides will be equal, in other words, AC to AB equals DF to DE, etc

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



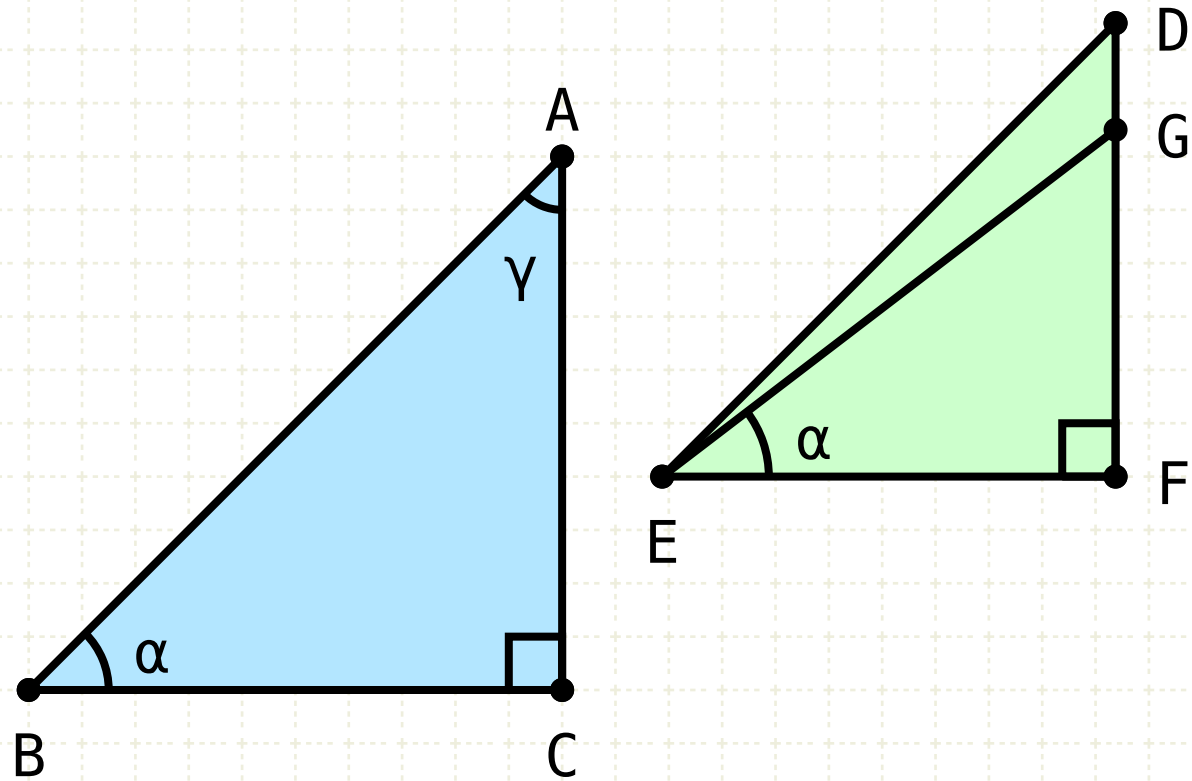
$$AC:AB = DF:DE$$

### Aside - Trigonometry

Conversly, consider two right triangles where  $AC$  to  $AB$  equals  $DF$  to  $DE$

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

### Aside - Trigonometry

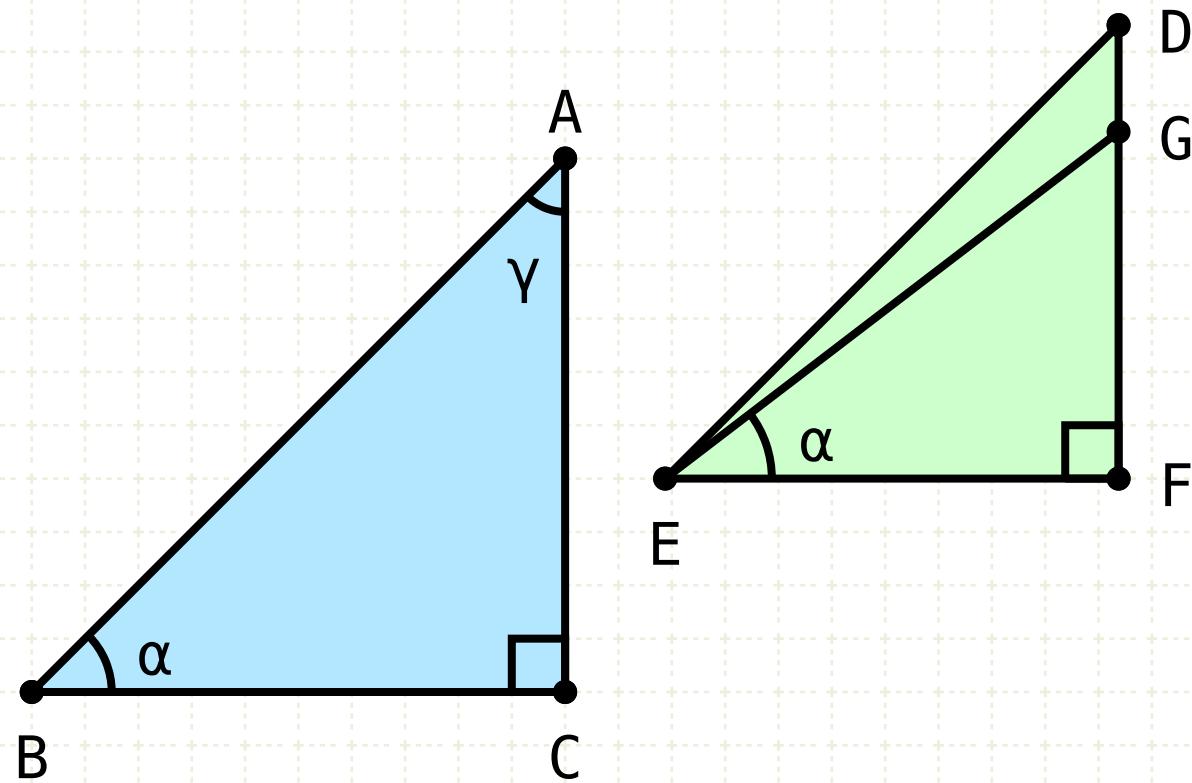
Conversly, consider two right triangles where AC to AB equals DF to DE

Assume that  $\alpha$  is not equal to angle  $\delta$ ...

Draw line EG, such that angle GEF equals  $\alpha$

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

$$FG:EG = AC:AB = DF:DE$$

### Aside - Trigonometry

Conversly, consider two right triangles where AC to AB equals DF to DE

Assume that  $\alpha$  is not equal to angle  $\delta$ ...

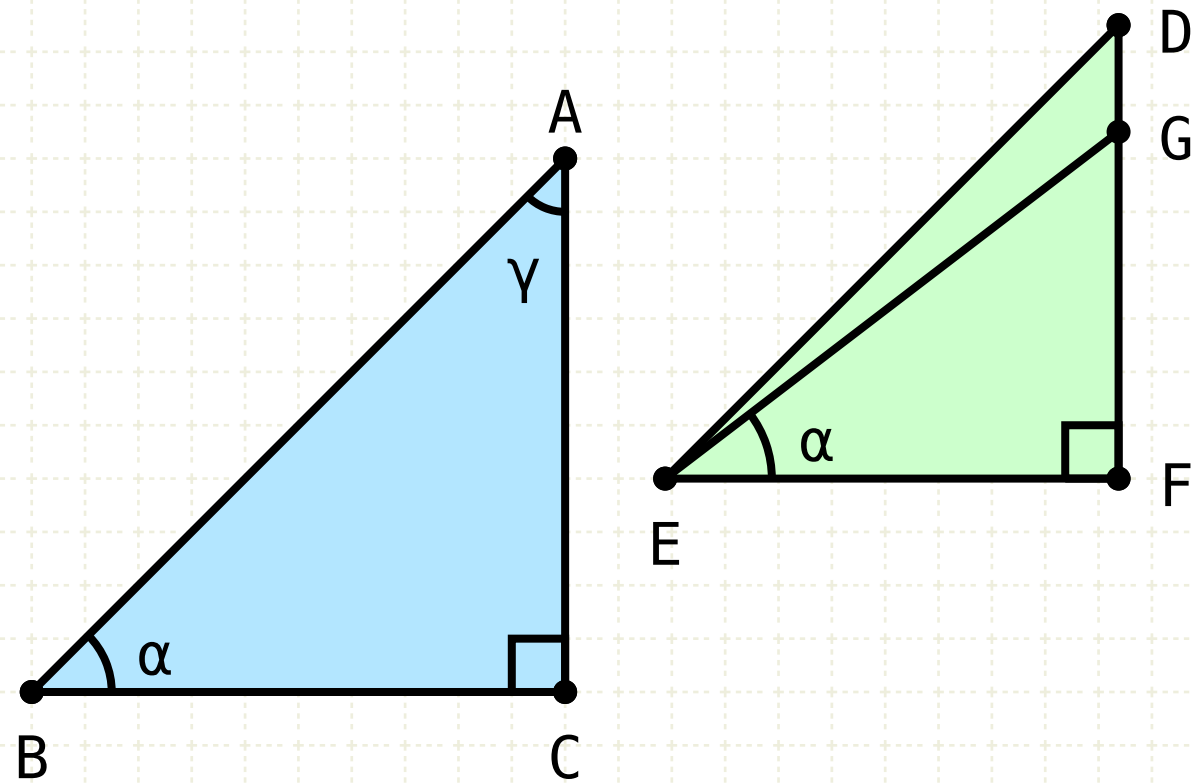
Draw line EG, such that angle GEF equals  $\alpha$

Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG



## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

$$FG:EG = AC:AB = DF:DE$$

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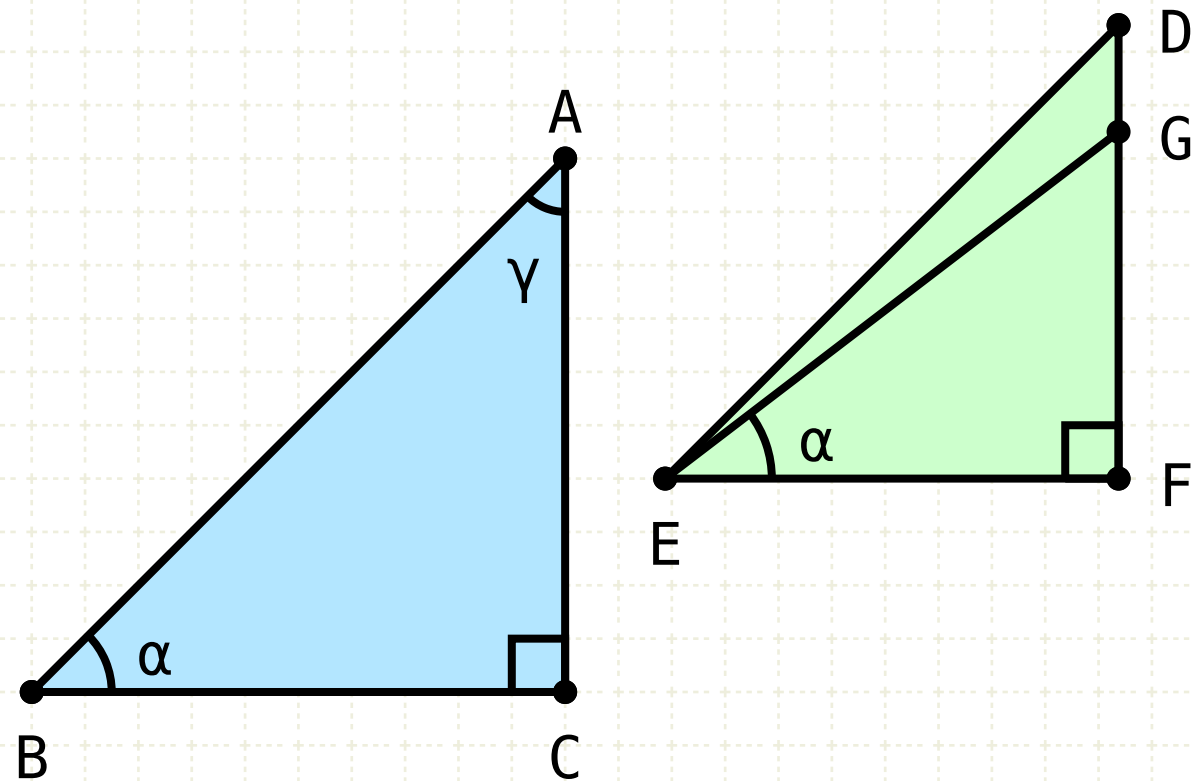
Draw line EG, such that angle GEF equals  $\alpha$

Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG

With a bit of math (pythagoras' theorem), it can be shown that the point G must be the same point as D

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$c = ED, b = DF, a = EF, d = FG, e = EG$$

$$FG:EG = AC:AB = DF:DE \rightarrow d/e = b/c$$

$$a^2 + b^2 = c^2; a^2 + d^2 = e^2$$

$$a^2 + (b \cdot (e/c))^2 = e^2$$

$$a^2 = e^2 \cdot (1 - (b^2/c^2))$$

$$a^2 = e^2 \cdot ((c^2 - b^2)/c^2)$$

$$a^2 = e^2 \cdot (a^2/c^2)$$

$$a^2 = a^2 \cdot (e/c)^2$$

$$\therefore e = c$$

## Aside - Trigonometry

Conversly, consider two right triangles where AC to AB equals DF to DE

Assume that  $\alpha$  is not equal to angle  $\delta$ ...

Draw line EG, such that angle GEF equals  $\alpha$

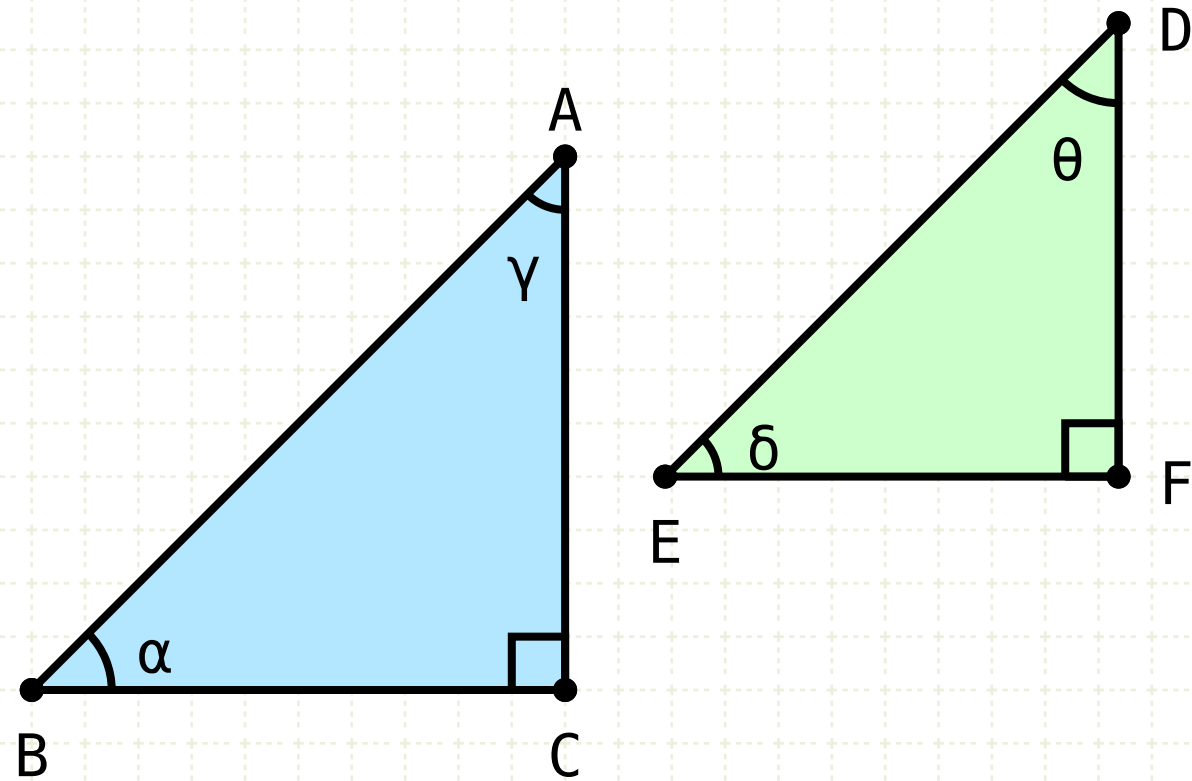
Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG

With a bit of math (pythagoras' theorem), it can be shown that the point G must be the same point as D



## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE$$

$$\delta = \alpha$$

### Aside - Trigonometry

Conversly, consider two right triangles where AC to AB equals DF to DE

Assume that  $\alpha$  is not equal to angle  $\delta$ ...

Draw line EG, such that angle GEF equals  $\alpha$

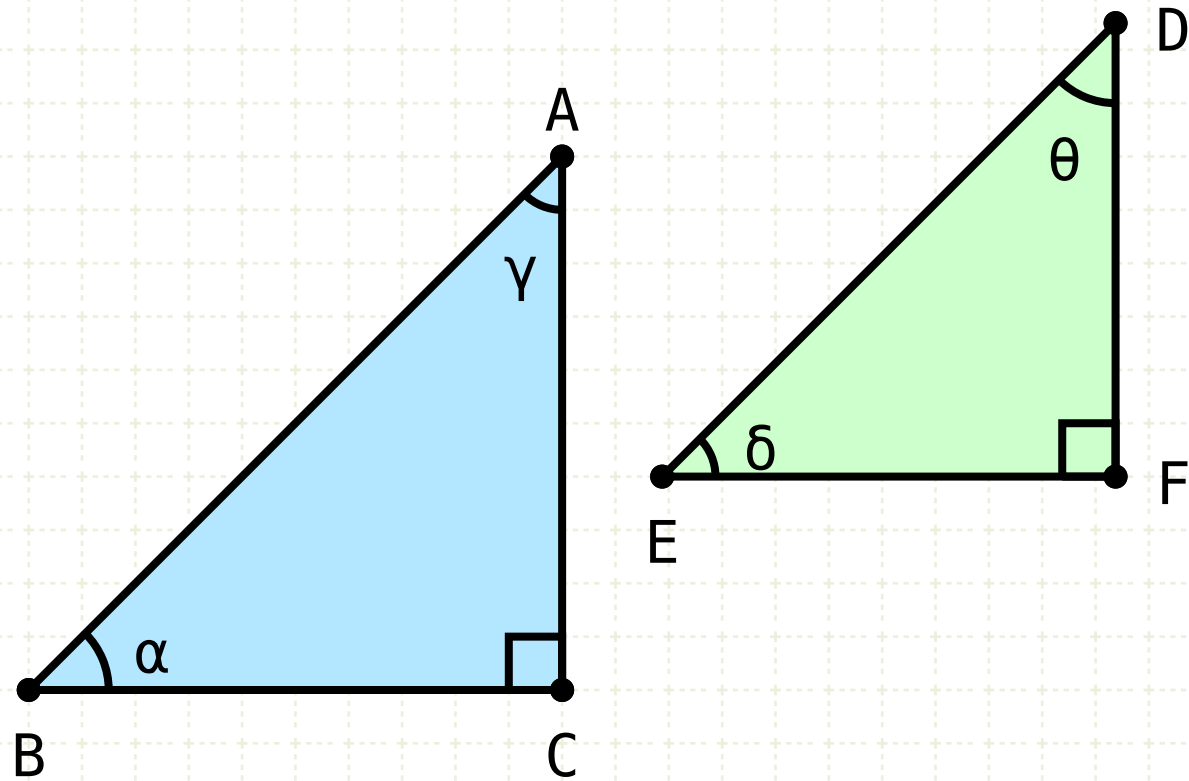
Triangle GEF is equiangular to ABC, so therefore AC to BE equals FG to EG

With a bit of math (pythagoras' theorem), it can be shown that the point G must be the same point as D

So the angle  $\alpha$  equals the angle  $\delta$

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$AC:AB = DF:DE \rightarrow \alpha = \delta$$

$$\alpha = \delta \rightarrow AC:AB = DF:DE$$

### Aside - Trigonometry

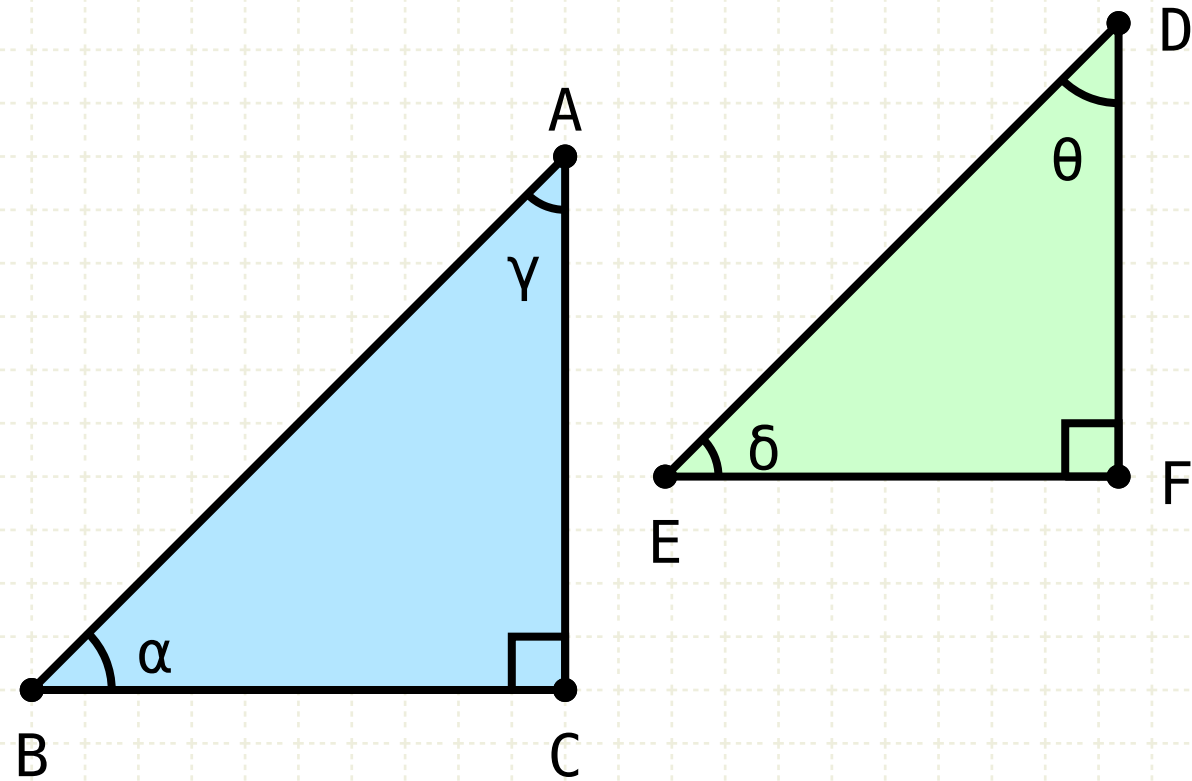
Conclusion:

Given two right triangles ABC and DEF

- \* If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- \* If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AC:AB &= DF:DE \rightarrow \alpha = \delta \\ \alpha = \delta &\rightarrow AC:AB = DF:DE \end{aligned}$$

Definition:

$$\sin(\alpha) = AC:AB$$

$$\sin(\delta) = DF:DE$$

## Aside - Trigonometry

Conclusion:

Given two right triangles ABC and DEF

- \* If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- \* If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

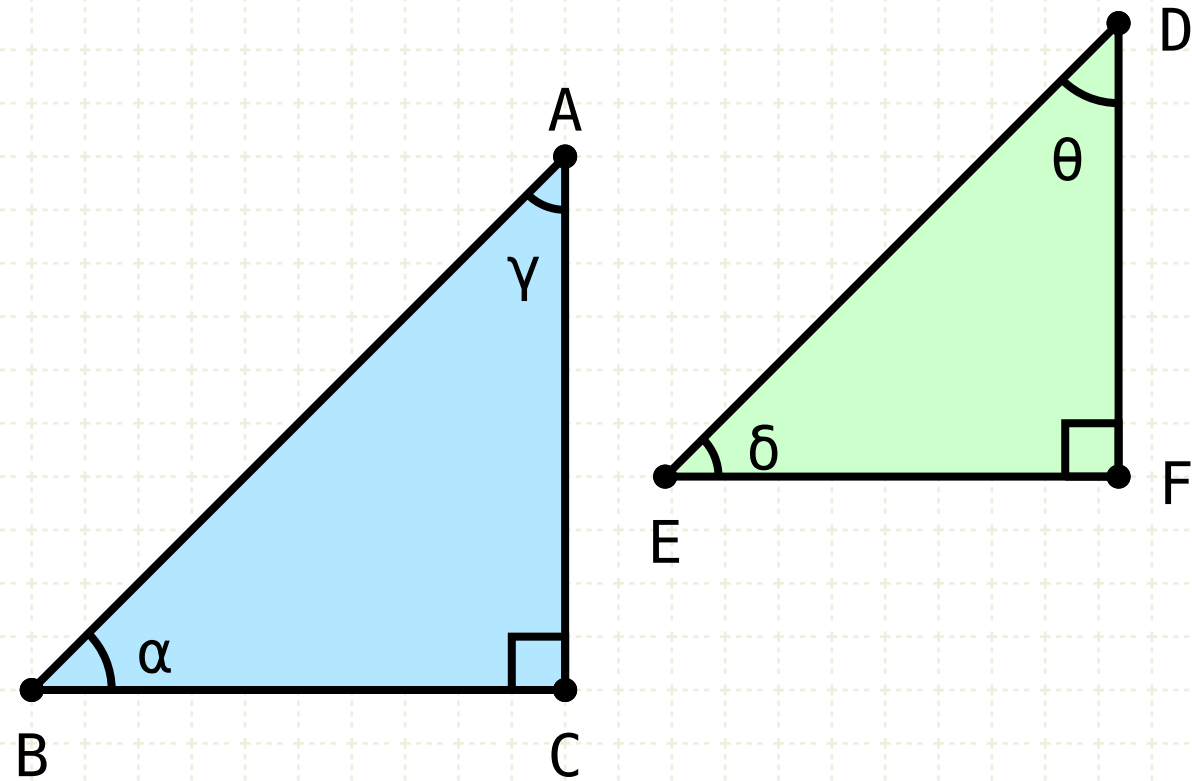
So for every right angle triangle, the ratio of the sides (opposite over hypotenuse) is unique for every angle

Lets call this ratio, as a function of the angle, 'sine'



## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AC:AB &= DF:DE \rightarrow \alpha = \delta \\ \alpha = \delta &\rightarrow AC:AB = DF:DE \end{aligned}$$

Definition:

$$\begin{aligned} \sin(\alpha) &= AC:AB & \cos(\alpha) &= BC:AB \\ \sin(\delta) &= DF:DE & \cos(a) &= EF:DE \end{aligned}$$

## Aside - Trigonometry

Conclusion:

Given two right triangles ABC and DEF

- \* If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- \* If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

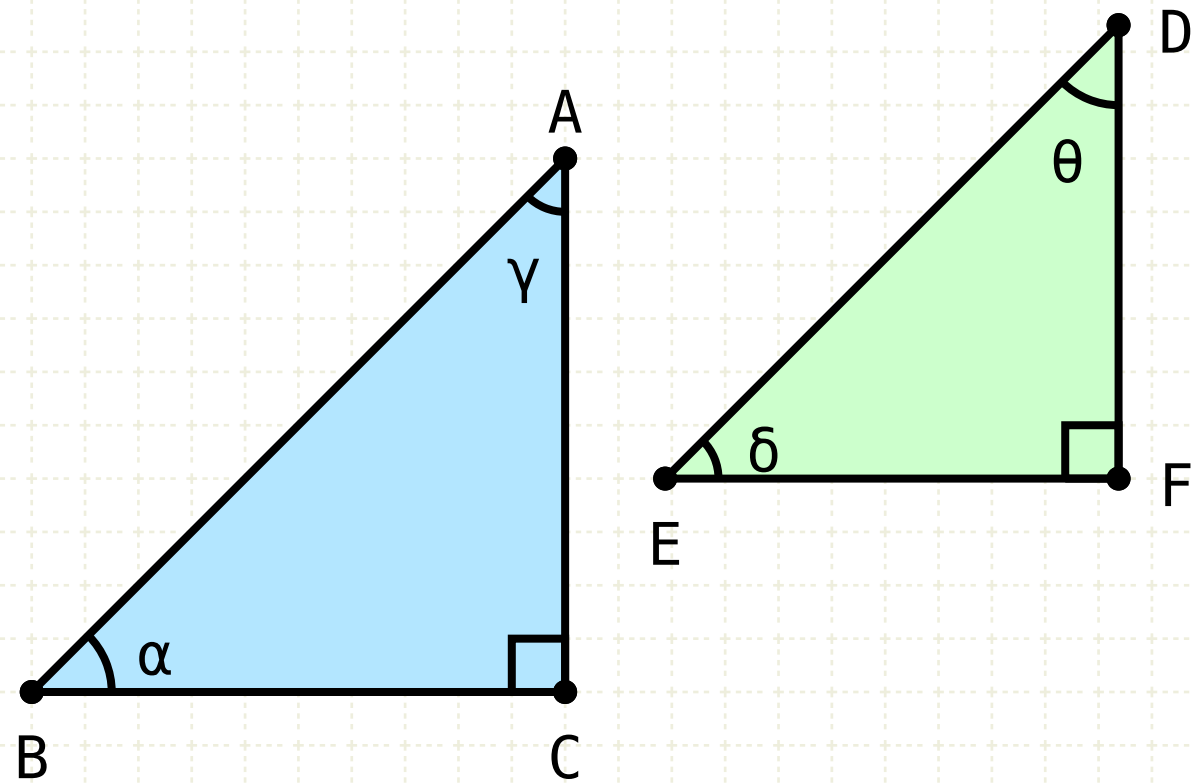
So for every right angle triangle, the ratio of the sides (opposite over hypotenuse) is unique for every angle

Lets call this ratio, as a function of the angle, 'sine'

We can use the same arguments to define the 'cosine' of an angle as the ratio of BC to AB

## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AC:AB &= DF:DE \rightarrow \alpha = \delta \\ \alpha = \delta &\rightarrow AC:AB = DF:DE \end{aligned}$$

Definition:

$$\begin{aligned} \sin(\alpha) &= AC:AB & \cos(\alpha) &= BC:AB \\ \sin(\delta) &= DF:DE & \cos(a) &= EF:DE \end{aligned}$$

$$\begin{aligned} \sin^2(\alpha) + \cos^2(\alpha) &= (AC)^2/(AB)^2 + (BC)^2/(AB)^2 \\ &= ((AC)^2 + (BC)^2)/(AB)^2 \\ &= (AB)^2/(AB)^2 = 1 \end{aligned}$$

## Aside - Trigonometry

Conclusion:

Given two right triangles ABC and DEF

- \* If the ratio of AC to AB equals DF to DE, then the angle ABC is equal to the angle DEF
- \* If the angle ABC is equal to the angle DEF, then the ratio of AC to AB equals DF to DE

So for every right angle triangle, the ratio of the sides (opposite over hypotenuse) is unique for every angle

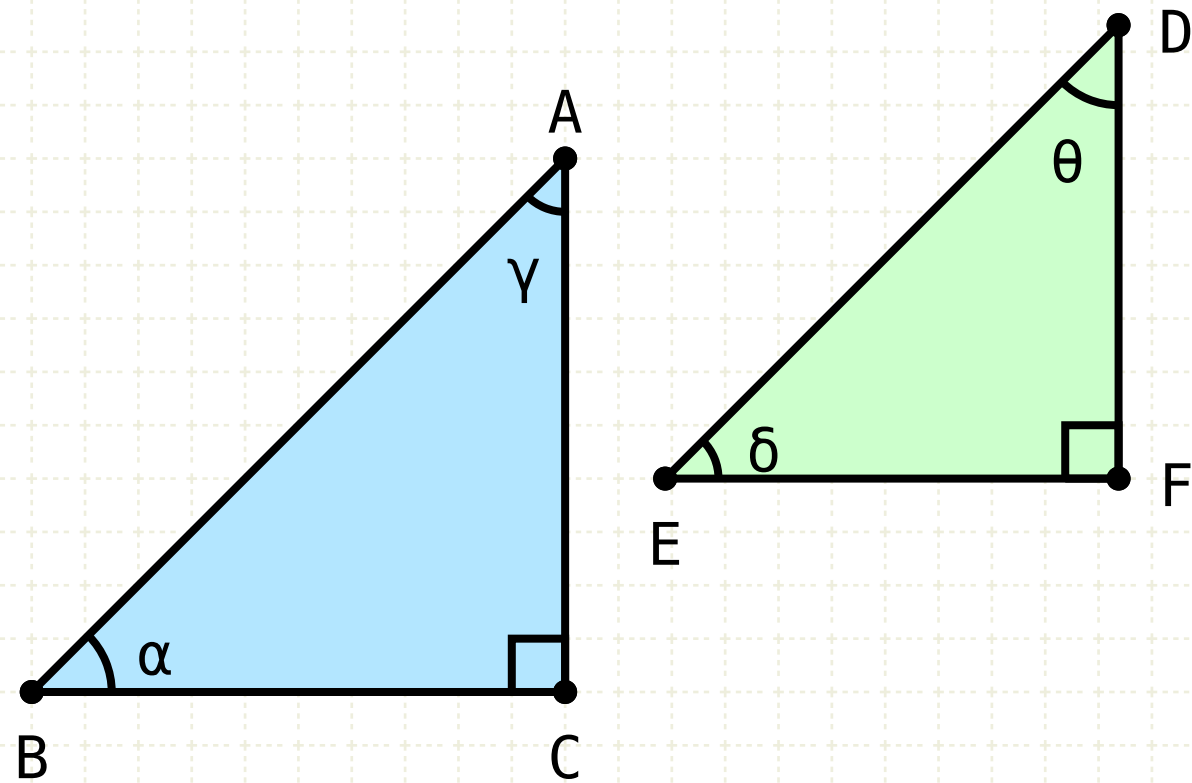
Lets call this ratio, as a function of the angle, 'sine'

We can use the same arguments to define the 'cosine' of an angle as the ratio of BC to AB



## Proposition 5 of Book VI

It two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend



$$\begin{aligned} AC:AB &= DF:DE \rightarrow \alpha = \delta \\ \alpha = \delta &\rightarrow AC:AB = DF:DE \end{aligned}$$

Definition:

$$\begin{aligned} \sin(\alpha) &= AC:AB & \cos(\alpha) &= BC:AB \\ \sin(\delta) &= DF:DE & \cos(a) &= EF:DE \end{aligned}$$

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Lets call this ratio, as a function of the angle, 'sine'

We can use the same arguments to define the 'cosine' of an angle as the ratio of BC to AB

Sine and cosine have been expanded to include definitions for angles larger than a right angle, and even negative angles, but these ratios shown above are the roots of trigonometry



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