Euclid's Elements

Book VII

Definitions:

1 A unit is that by virtue of which each of the things that exist is called one

2 A number is a multitude composed of units. (not one)

3 A number is part of a number, the less of the greater, when it measures the greater

11 A prime number is that which is measured by a unit alone.

12 Numbers prime to one another are those which are measured by a unit alone as a common measure

15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)

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Proposition 22 of Book VII

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$$(A,B) \in S \text{ such that } A \leq x, B \leq y,\ \forall (x,y) \in S$$

$$\gcd(A,B) = 1$$

In other words

If A and B are the smallest whole numbers that are equal to the ratio of A to B, then they are prime to one another

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Proof by Contradiction

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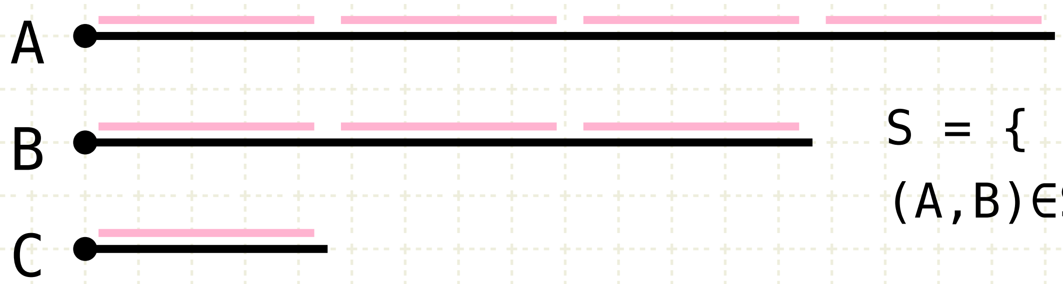

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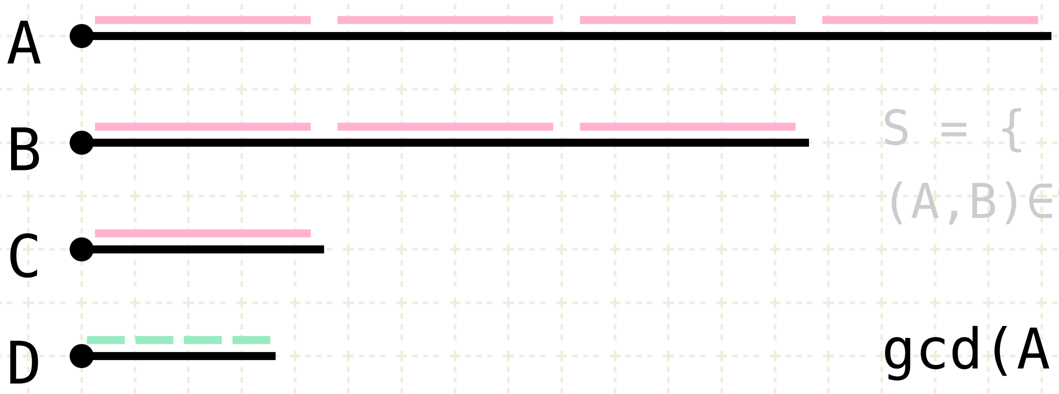

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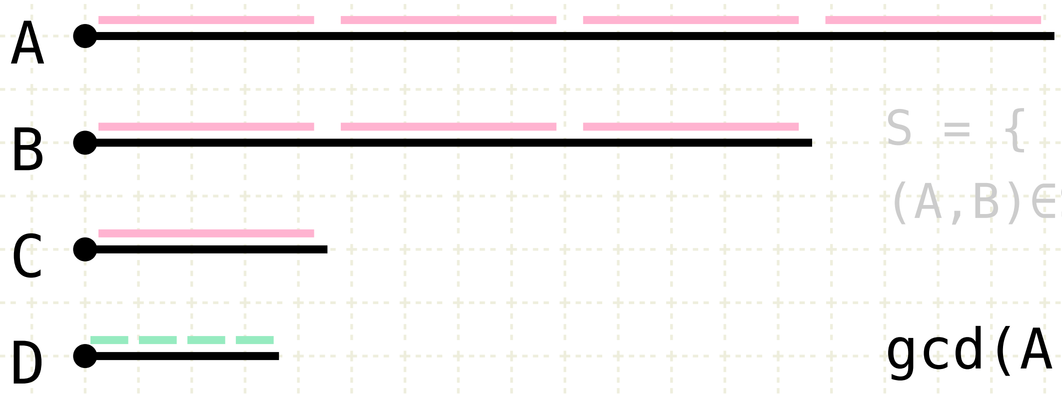

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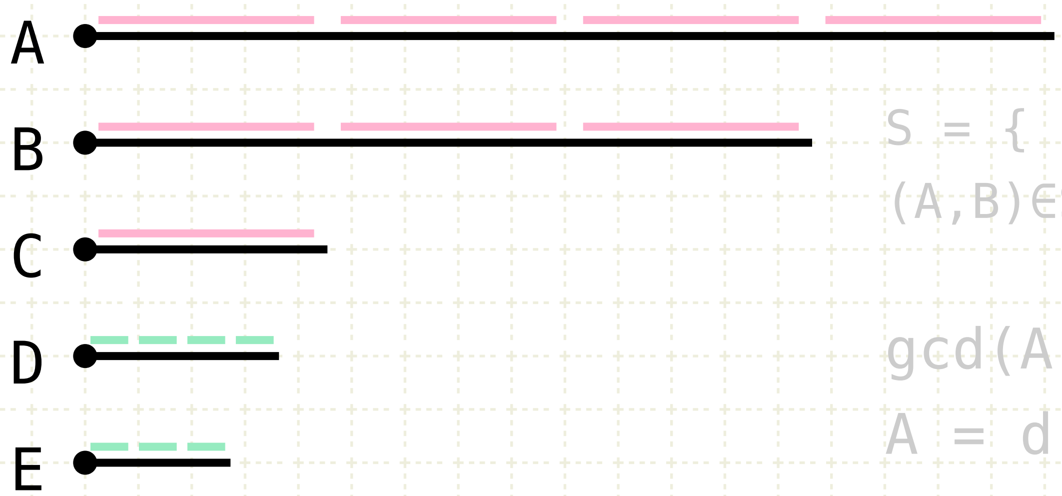

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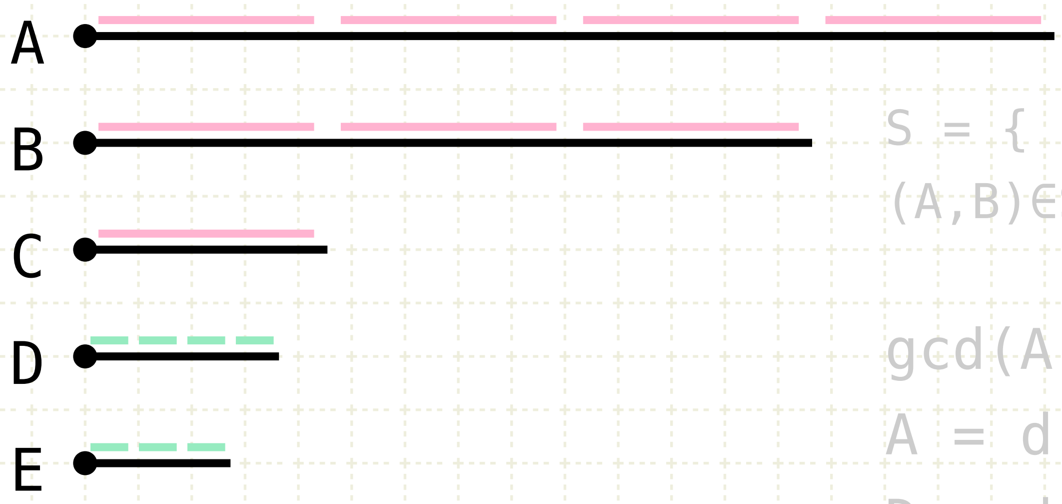

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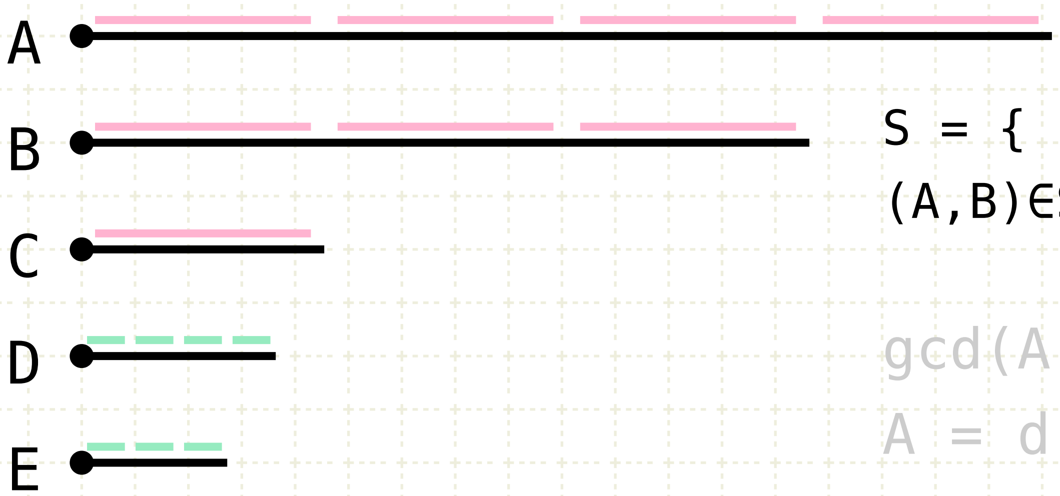

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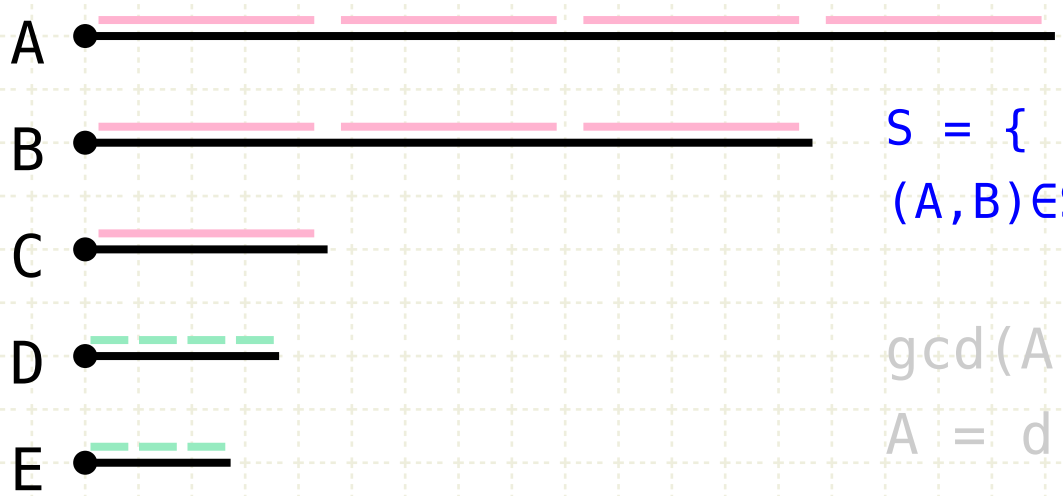

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But D and E are less than A and B, which violates the original hypothesis

Therefore, A and B are relatively prime

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