

# Euclid's Elements

## Book VII

### Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

*As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.*

**Joseph-Louis Lagrange**  
**(1736 to 1813)**



# Table of Contents, Chapter 7

1	Determine if two numbers are relatively prime	10	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$	21	If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B
2	Find the greatest common divisor for two numbers	11	If $A:B = C:D$ , then $(A-C):(B-D) = A:B$	22	If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime
3	<b>Find the largest common divisor for three numbers</b>	12	If $A:B = C:D$ , then $(A+C):(B+C) = A:B$	23	If A,B are relatively prime and if $A = n \cdot C$ , then B,C are relatively prime
4	Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B	13	If $A:B = C:D$ , then $A:C = B:D$	24	If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C
5	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , then $(B+D) = (1/q) \cdot (A+C)$	14	If $A:B = D:E$ and $B:C = E:F$ , then $A:C = D:F$	25	If A,B are relatively prime then $A^2, B$ are relatively prime
6	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , then $(B+D) = (p/q) \cdot (A+C)$	15	If $B = i \cdot 1$ and $E = i \cdot D$ , and if $D = j \cdot 1$ then $E = j \cdot B$	26	If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$
7	If $B = A/q$ and $D = C/q$ , $B > D$ , then $(B-D) = (A-C)/q$	16	$A \times B = B \times A$	27	If A,B are relatively prime, then $A^2, B^2$ are relatively prime, and $A^3, B^3$ are relatively prime, and so on
8	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$ , $B > D$ , then $(B-D) = (p/q) \cdot (A-C)$	17	If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$		
9	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$ , and If $B = (r/s) \cdot D$ , then $A = (r/s) \cdot C$	18	If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$		
		19	If $A:B = C:D$ then $A \times D = B \times C$ If $A \times D = B \times C$ then $A:B = C:D$		
		20	Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$		



# Table of Contents, Chapter 7

- |    |  |    |   |
|----|--|----|---|
| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime   | 37 | If $A = p \cdot B$ , then $A = q \cdot C$ where $C = p \cdot 1$       |
| 29 | If A is prime, and $B \neq n \cdot A$ , then A,B are relatively prime                                    | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$       |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a$ , $1/b$ , $1/c$ |
| 31 | If $A = B \times C$ , then $A = j \cdot D$ where D is prime  |    |   |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime                            |    |   |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C                  |    |   |
| 34 | Find the lowest common denominator of 2 numbers  |    |   |
| 35 | If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$ , then $C = i \cdot E$  |    |   |
| 36 | Find the least common multiple of 3 numbers  |    |   |



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

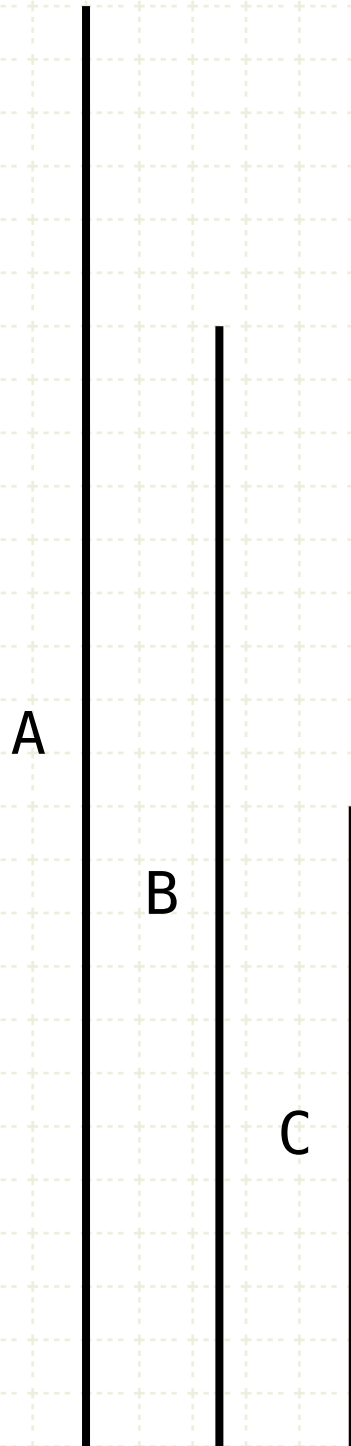


# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

## In other words

Find the largest common divisor for three numbers





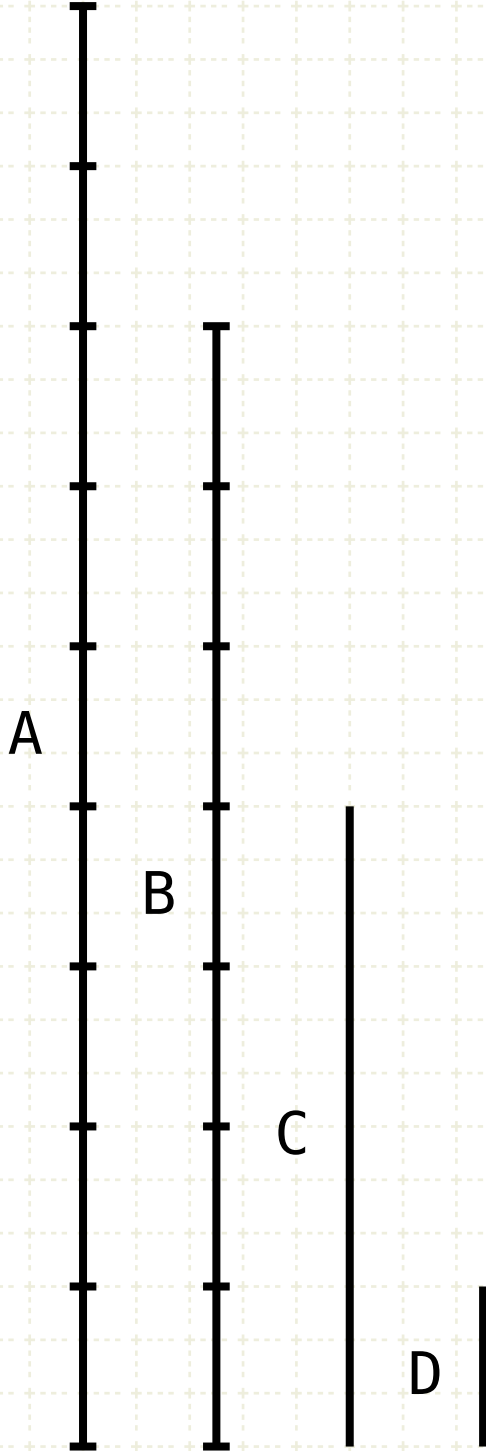
# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

## Method

Find D, the greatest measure of A and B (VII·2)

$$\gcd(A,B) = D$$



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

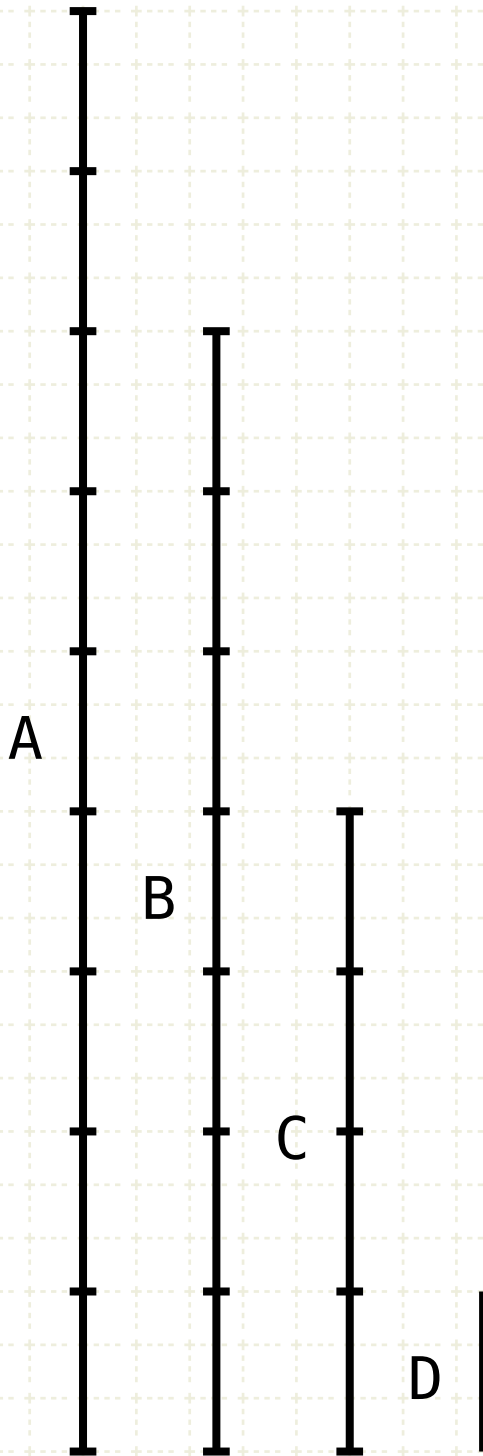
$$\gcd(A,B) = D$$

$$\text{let } C = q \cdot D$$

## Method

Find D, the greatest measure of A and B (VII·2)

If D measure C, then D is a common divisor for A,B,C



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Given three numbers not prime to one another, to find their greatest common measure.

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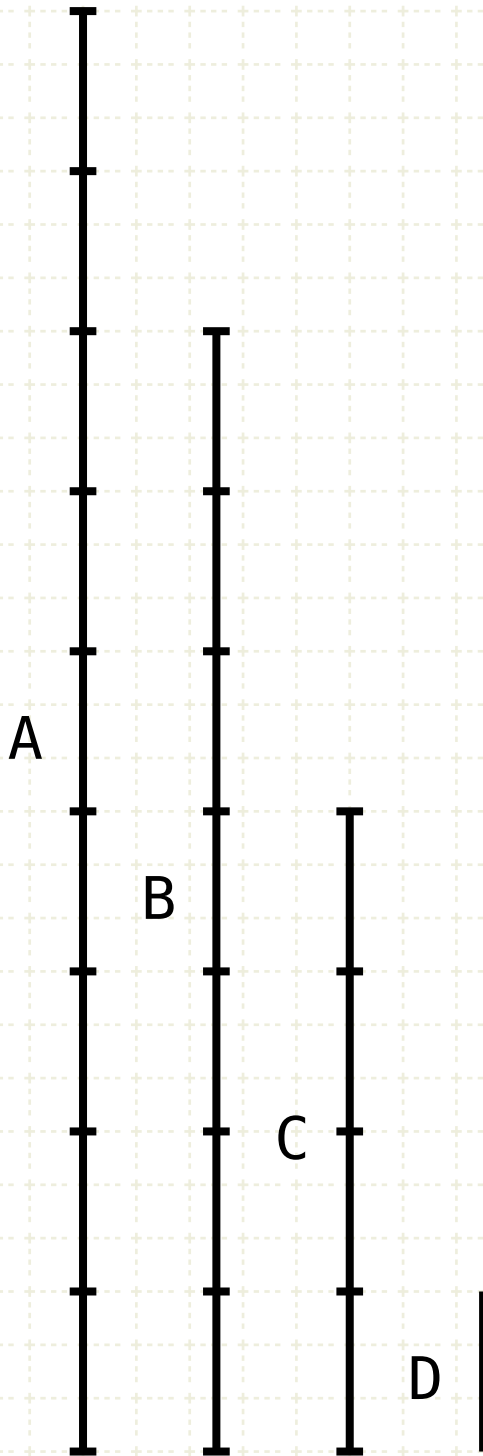
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## Method

Find D, the greatest measure of A and B (VII·2)

If D measure C, then D is a common divisor for A,B,C

D is the greatest common divisor



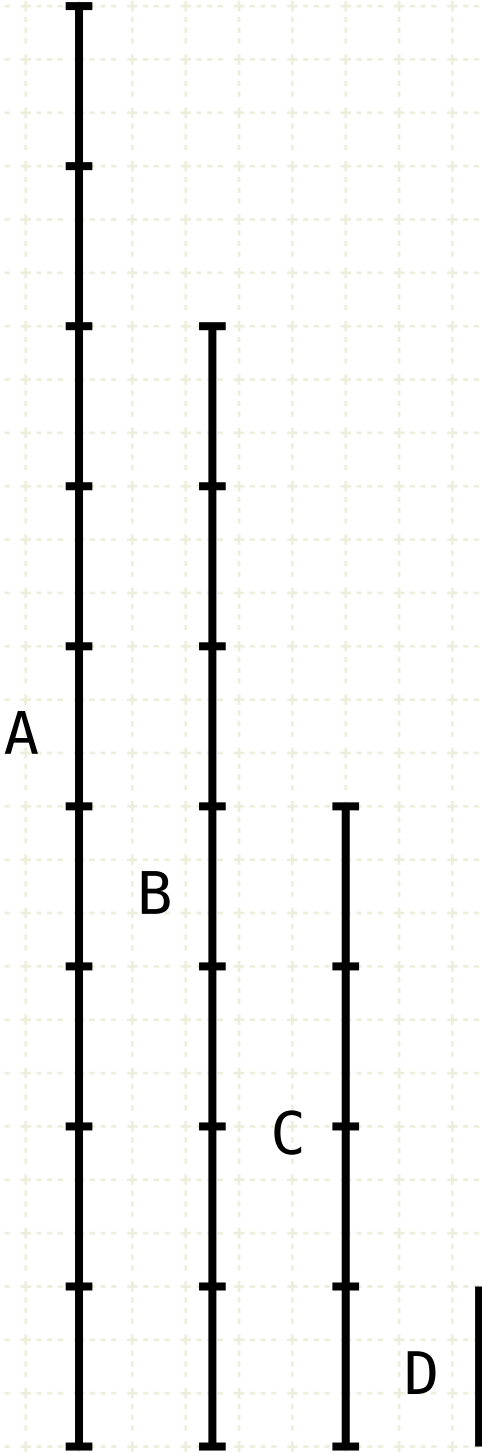


# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$$\gcd(A,B) = D$$

$$\text{let } C = q \cdot D$$



## Method

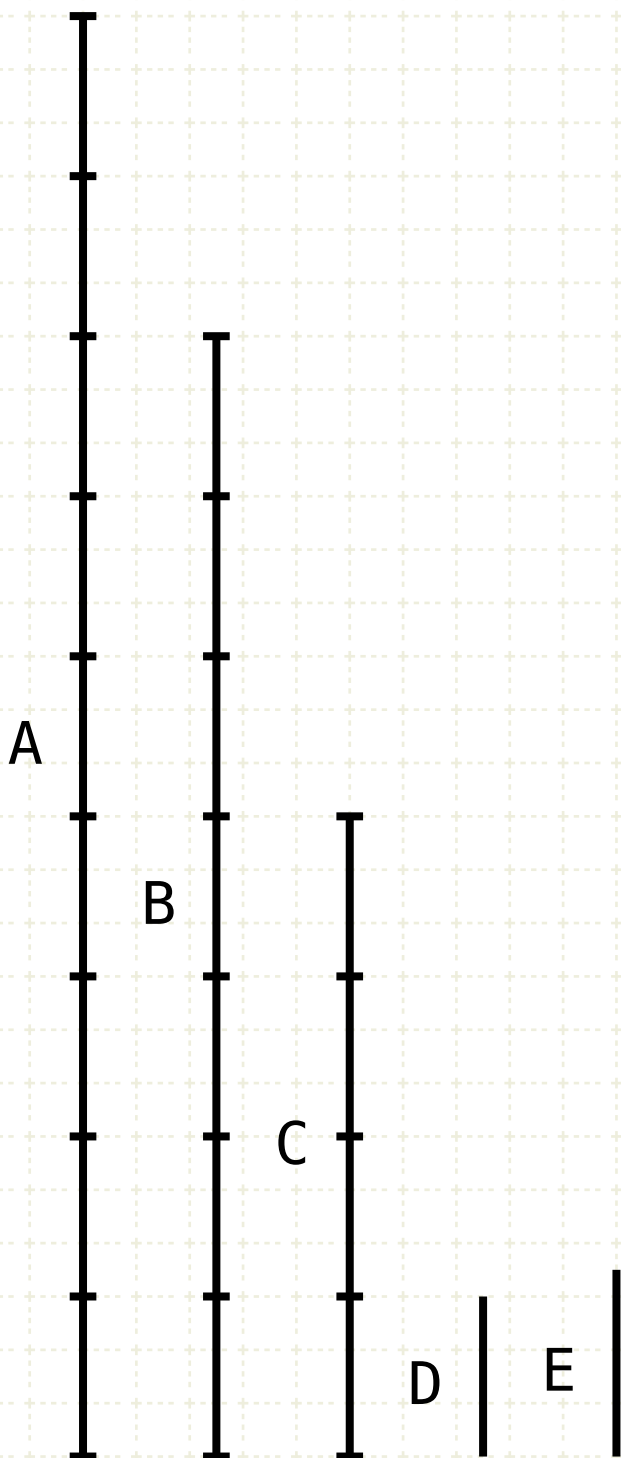
Find D, the greatest measure of A and B (VII·2)  
If D measure C, then D is a common divisor for A,B,C  
D is the greatest common divisor

## Proof by contradiction



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.



$$\gcd(A, B) = D$$

$$\text{let } C = q \cdot D$$

$$\text{let } E = \gcd(A, B, C)$$

$$D < E$$

$$A = p \cdot E$$

$$B = q \cdot E$$

$$C = r \cdot E$$

## Method

Find D, the greatest measure of A and B (VII·2)

If D measure C, then D is a common divisor for A,B,C

D is the greatest common divisor

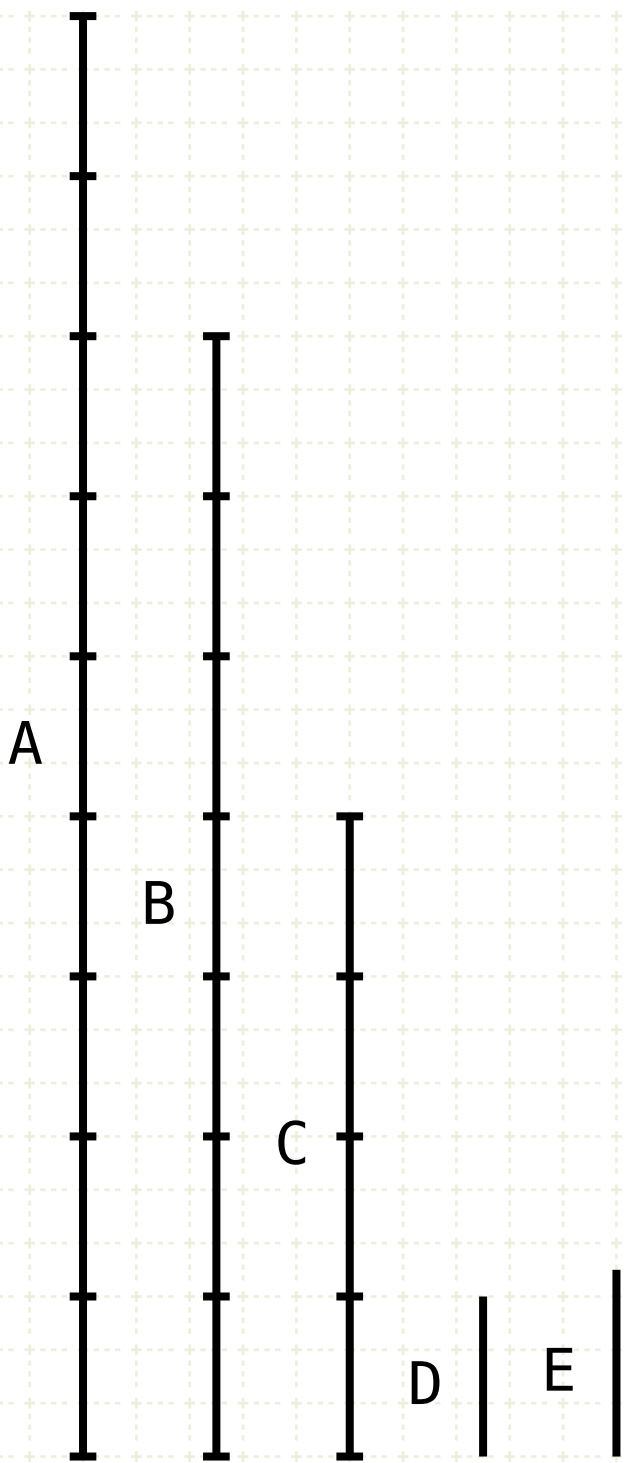
## Proof by contradiction

If D is not the greatest common divisor, let E be the greatest common divisor



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.



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$$B = q \cdot E$$

$$C = r \cdot E$$

$$D = s \cdot E$$

## Method

Find D, the greatest measure of A and B (VII·2)

If D measure C, then D is a common divisor for A,B,C

D is the greatest common divisor

## Proof by contradiction

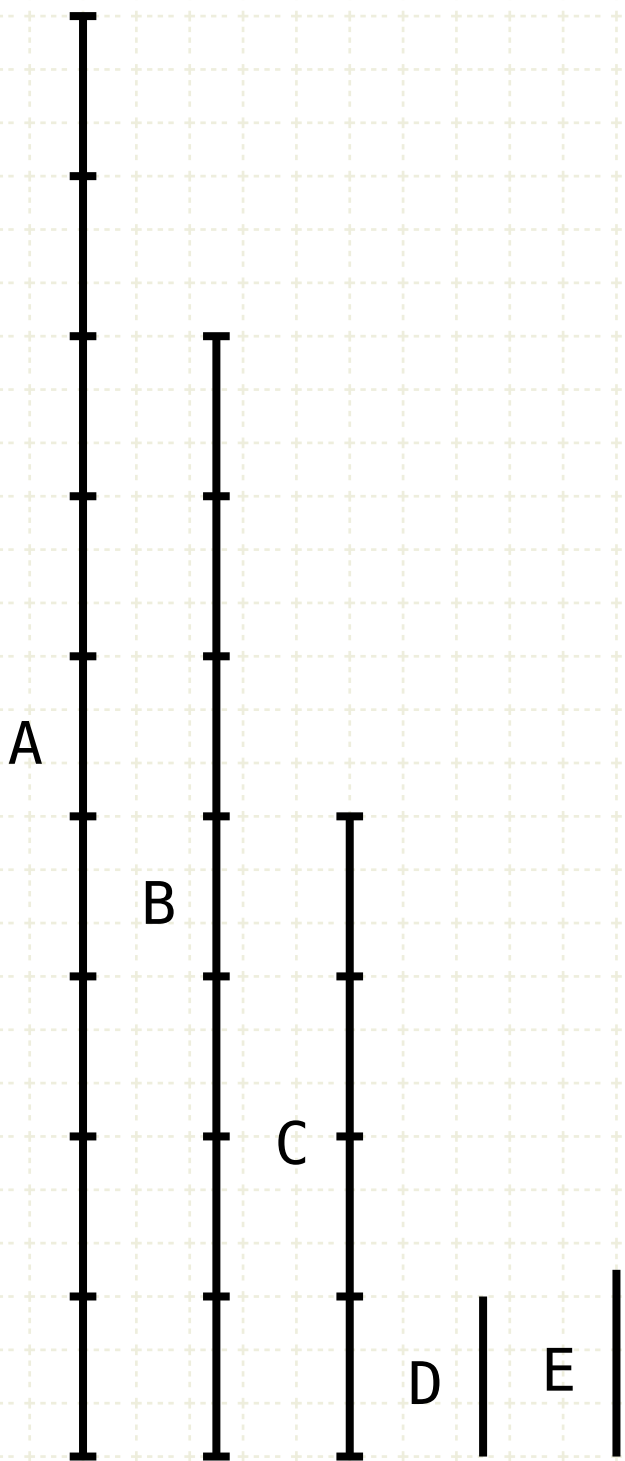
If D is not the greatest common divisor, let E be the greatest common divisor

Since E measures A,B,C, it will also measure the greatest common divisor of A,B, which is D (VII·2 Por)



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.



$$\gcd(A, B) = D$$

$$\text{let } C = q \cdot D$$

$$\text{let } E = \gcd(A, B, C)$$

$$D < E$$

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$$D = s \cdot E$$

## Method

Find D, the greatest measure of A and B (VII·2)

If D measure C, then D is a common divisor for A,B,C

D is the greatest common divisor

## Proof by contradiction

If D is not the greatest common divisor, let E be the greatest common divisor

Since E measures A,B,C, it will also measure the greatest common divisor of A,B, which is D (VII·2 Por)

But D is less than E, so it cannot be measured by E,



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$\text{gcd}(A,B) = D$

$\text{let } C = q \cdot D$

$\text{let } E = \text{gcd}(A,B,C)$

$D < E$

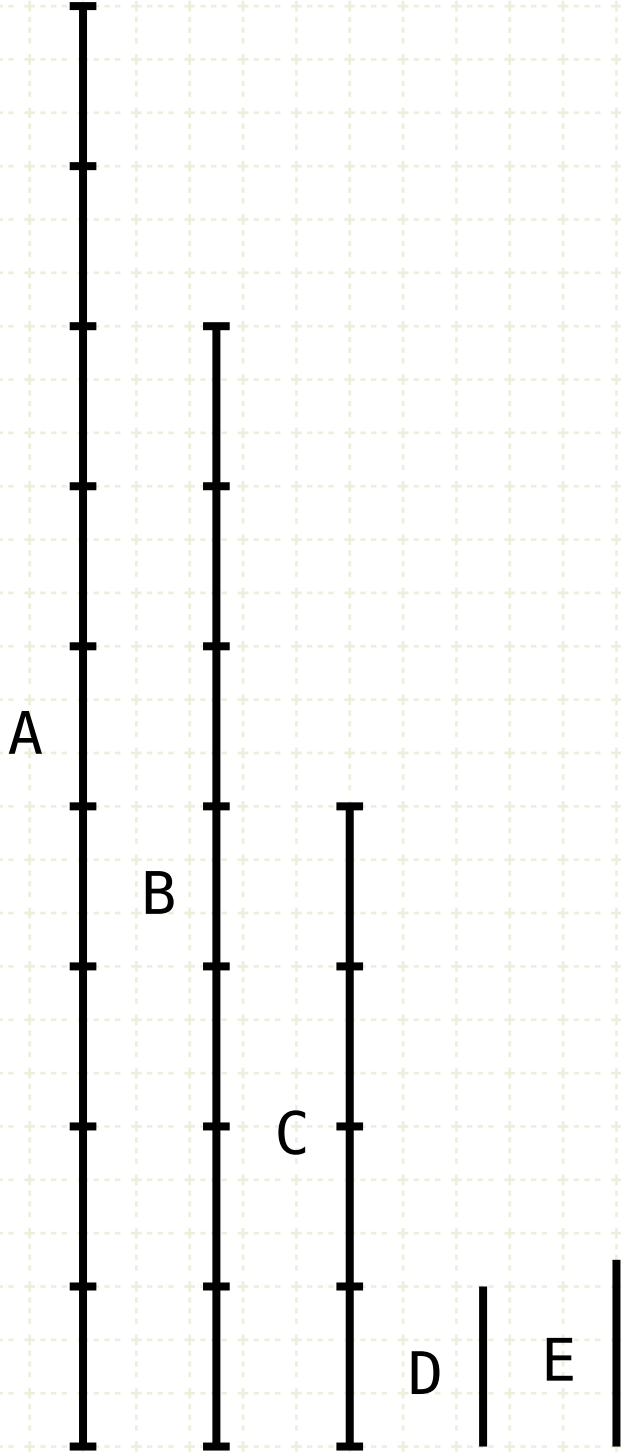
$A = p \cdot E$

$B = q \cdot E$

$C = r \cdot E$

$D = s \cdot E$

$\therefore \text{gcd}(A,B,C) = D$



## Method

Find D, the greatest measure of A and B (VII·2)

If D measure C, then D is a common divisor for A,B,C

D is the greatest common divisor

## Proof by contradiction

If D is not the greatest common divisor, let E be the greatest common divisor

Since E measures A,B,C, it will also measure the greatest common divisor of A,B, which is D (VII·2 Por)

But D is less than E, so it cannot be measured by E, therefore D is the greatest common divisor



# Proposition 3 of Book VII

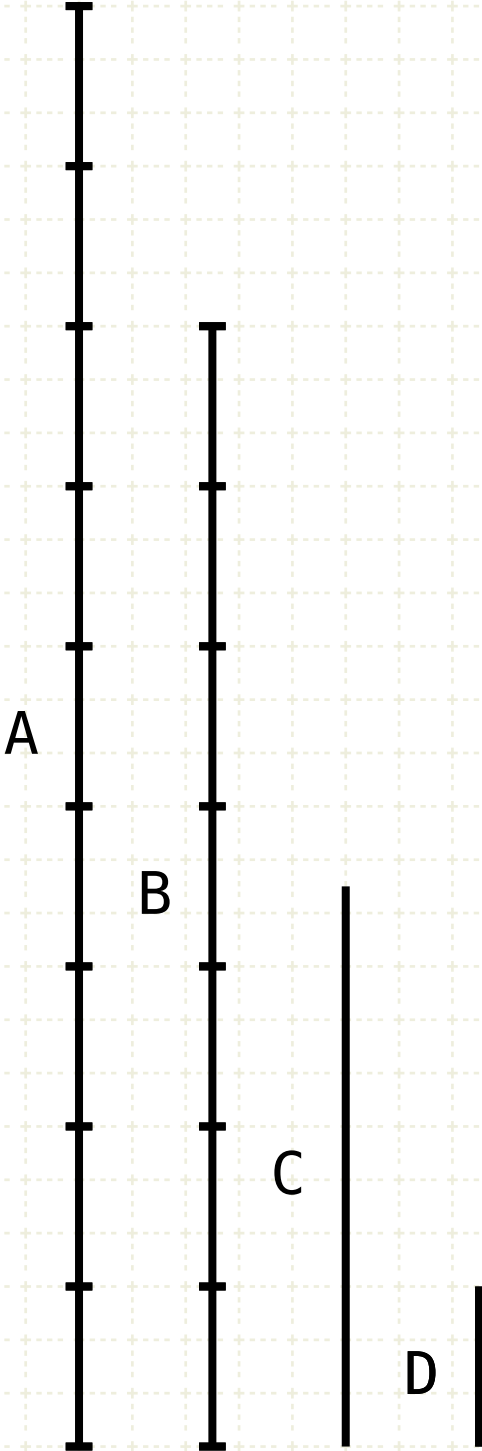
Given three numbers not prime to one another, to find their greatest common measure.

## Method

Find D, the greatest measure of A and B (VII·2)

$$\gcd(A,B) = D$$

$$\text{let } C \neq q \cdot D$$



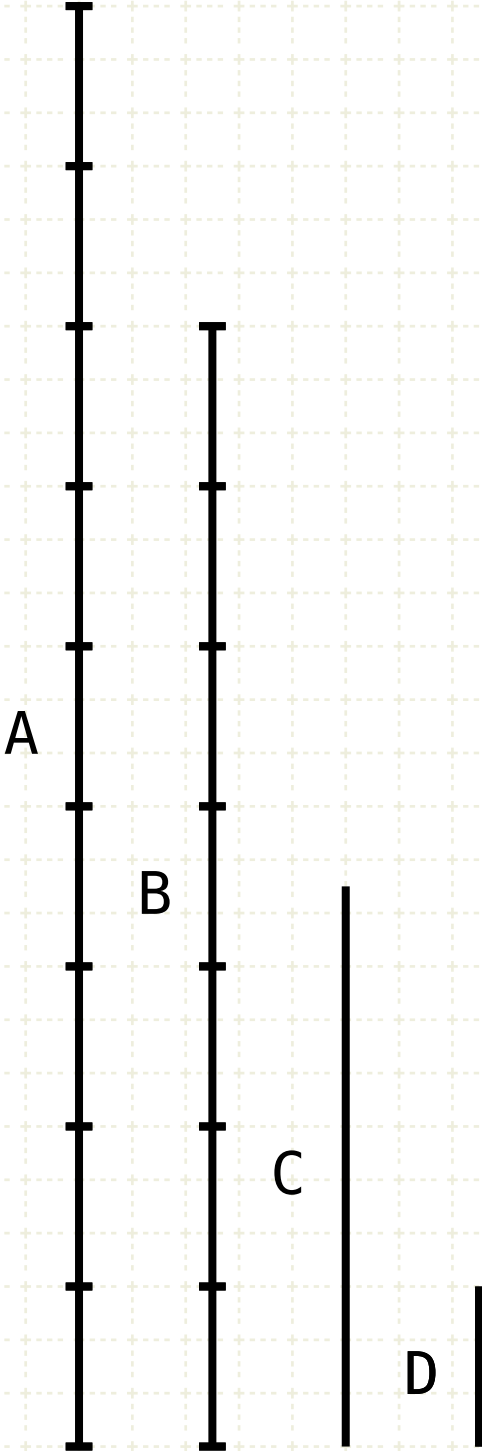


# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$\gcd(A,B) = D$

let  $C \neq q \cdot D$



## Method

Find D, the greatest measure of A and B (VII·2)

C and D are not prime to one another



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$$\gcd(A,B) = D$$

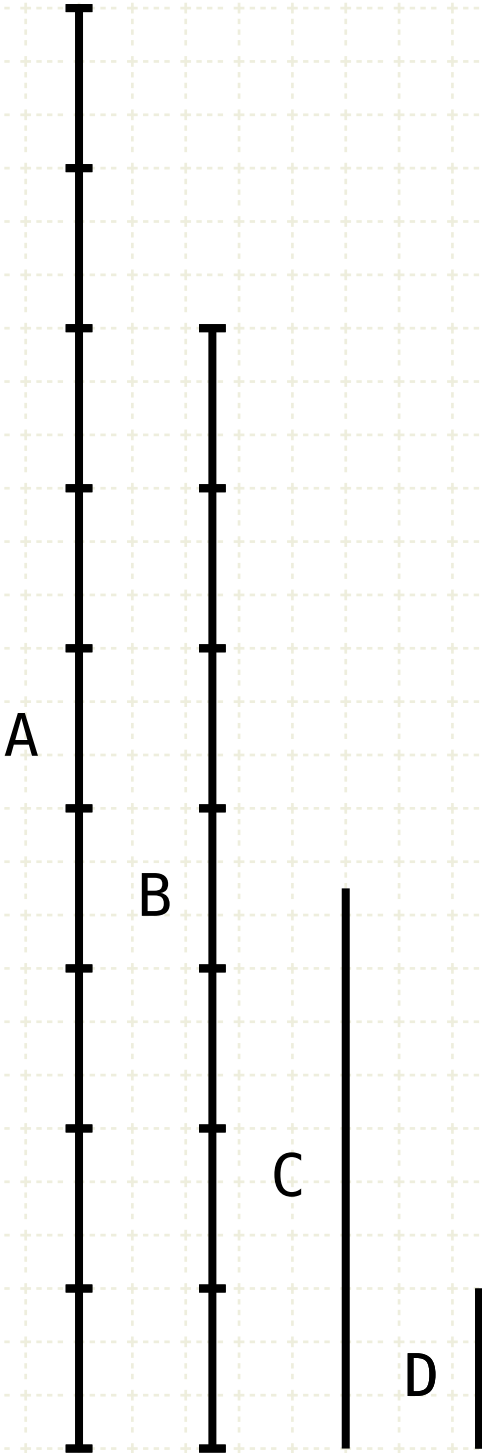
$$\text{let } C \neq q \cdot D$$

## Method

Find D, the greatest measure of A and B (VII·2)

C and D are not prime to one another

## Proof



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

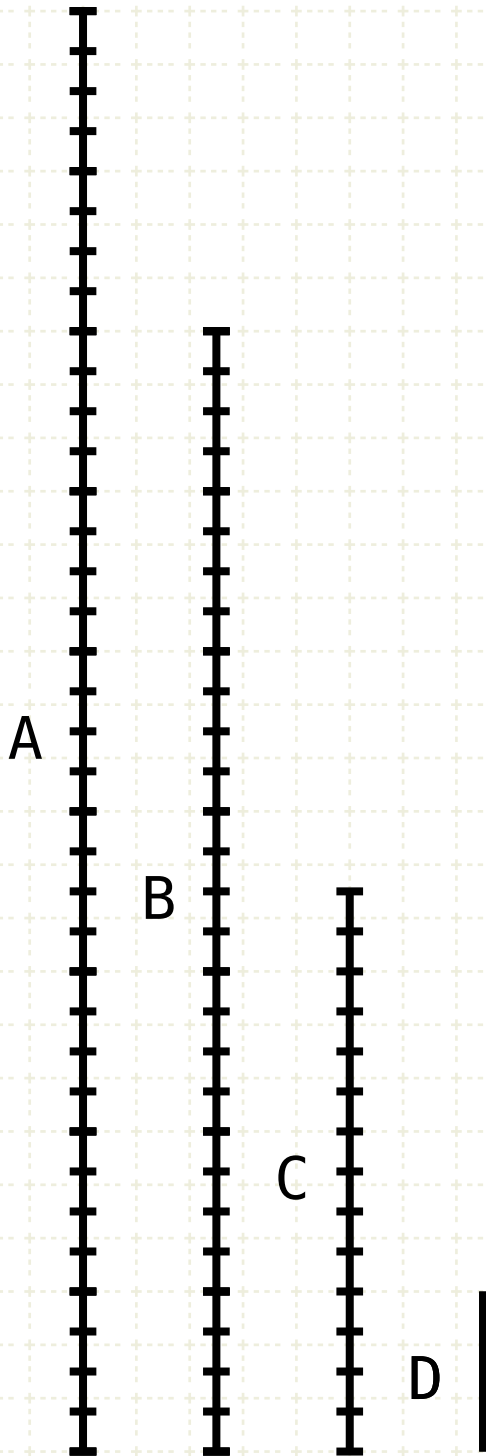
$$\gcd(A,B) = D$$

$$\text{let } C \neq q \cdot D$$

$$A = p \cdot x$$

$$B = q \cdot x$$

$$C = r \cdot x$$



## Method

Find D, the greatest measure of A and B (VII·2)

C and D are not prime to one another

## Proof

Since A,B,C are not prime to one another, there is a number x that measures A,B,C



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$$\gcd(A,B) = D$$

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$$A = p \cdot x$$

$$B = q \cdot x$$

$$C = r \cdot x$$

$$D = s \cdot x$$

## Method

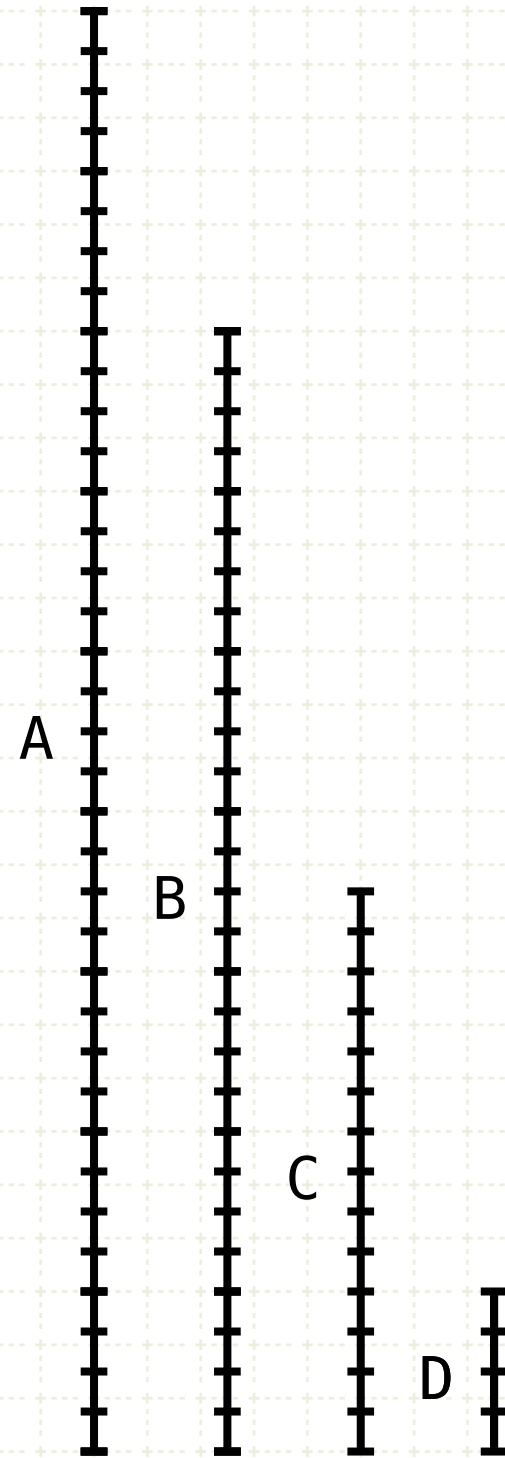
Find D, the greatest measure of A and B (VII·2)

C and D are not prime to one another

## Proof

Since A,B,C are not prime to one another, there is a number x that measures A,B,C

This number x will also measure D, the greatest common divisor of A,B (VII·2 Por)



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$$\gcd(A,B) = D$$

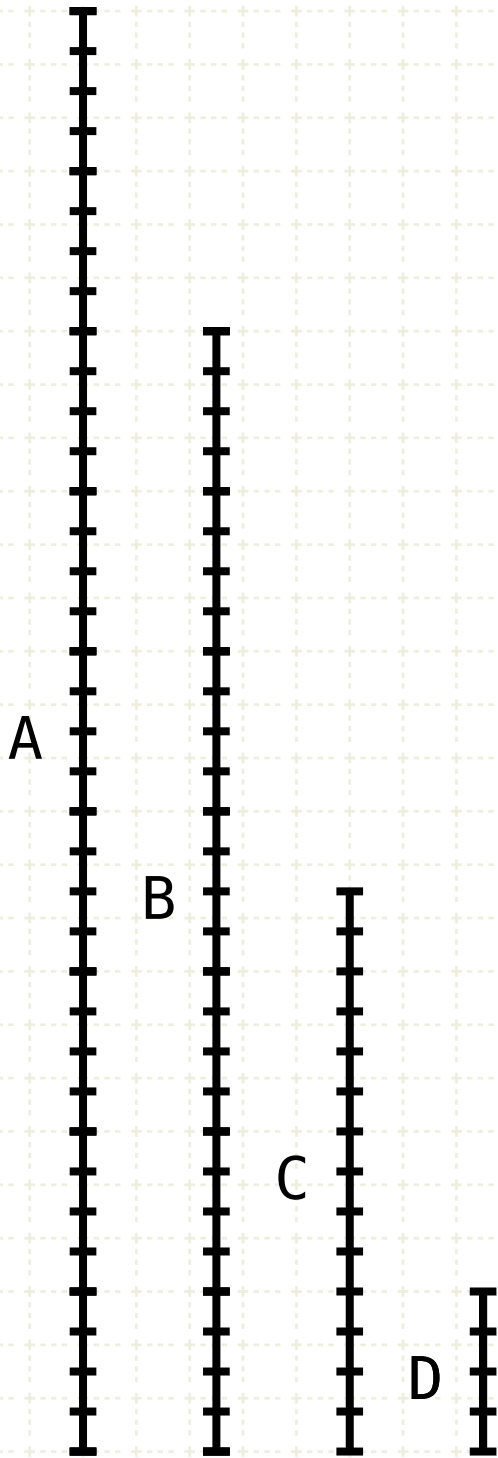
$$\text{let } C \neq q \cdot D$$

$$A = p \cdot x$$

$$B = q \cdot x$$

$$C = r \cdot x$$

$$D = s \cdot x$$



## Method

Find D, the greatest measure of A and B (VII·2)

C and D are not prime to one another

## Proof

Since A,B,C are not prime to one another, there is a number x that measures A,B,C

This number x will also measure D, the greatest common divisor of A,B (VII·2 Por)

Since x measures D and C, D and C are not prime to each other



# Proposition 3 of Book VII

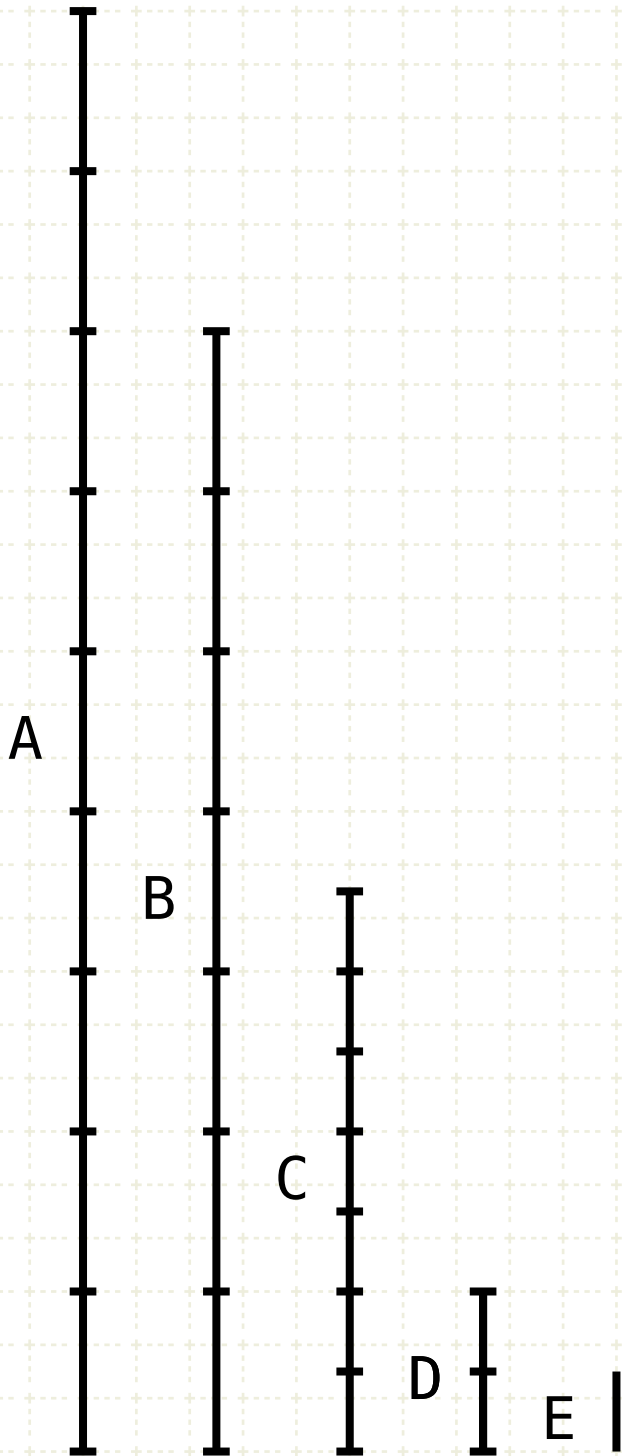
Given three numbers not prime to one another, to find their greatest common measure.

$\gcd(A, B) = D$   
 $\text{let } C \neq q \cdot D$   
 $\gcd(C, D) = E$

## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)





# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

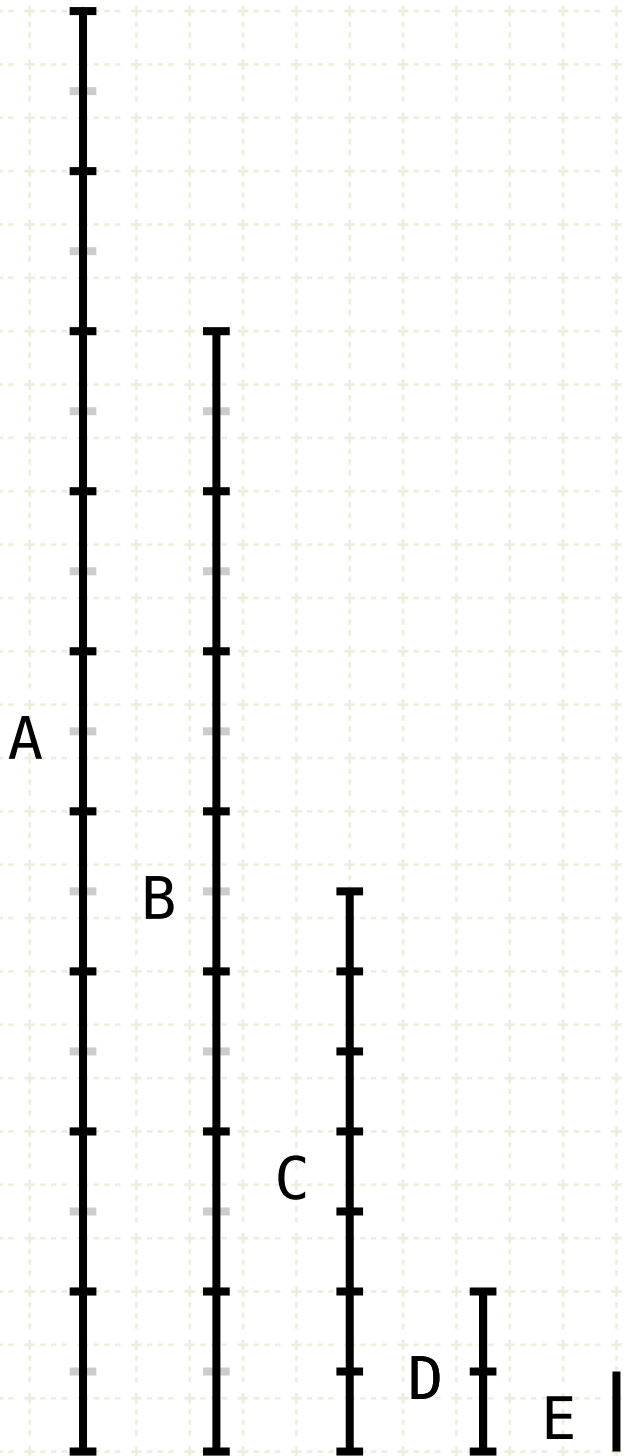
$$\gcd(A,B) = D$$

$$\text{let } C \neq q \cdot D$$

$$\gcd(C,D) = E$$

$$A = a \cdot E$$

$$B = b \cdot E$$



## Method

Find D, the greatest measure of A and B (VII-2)

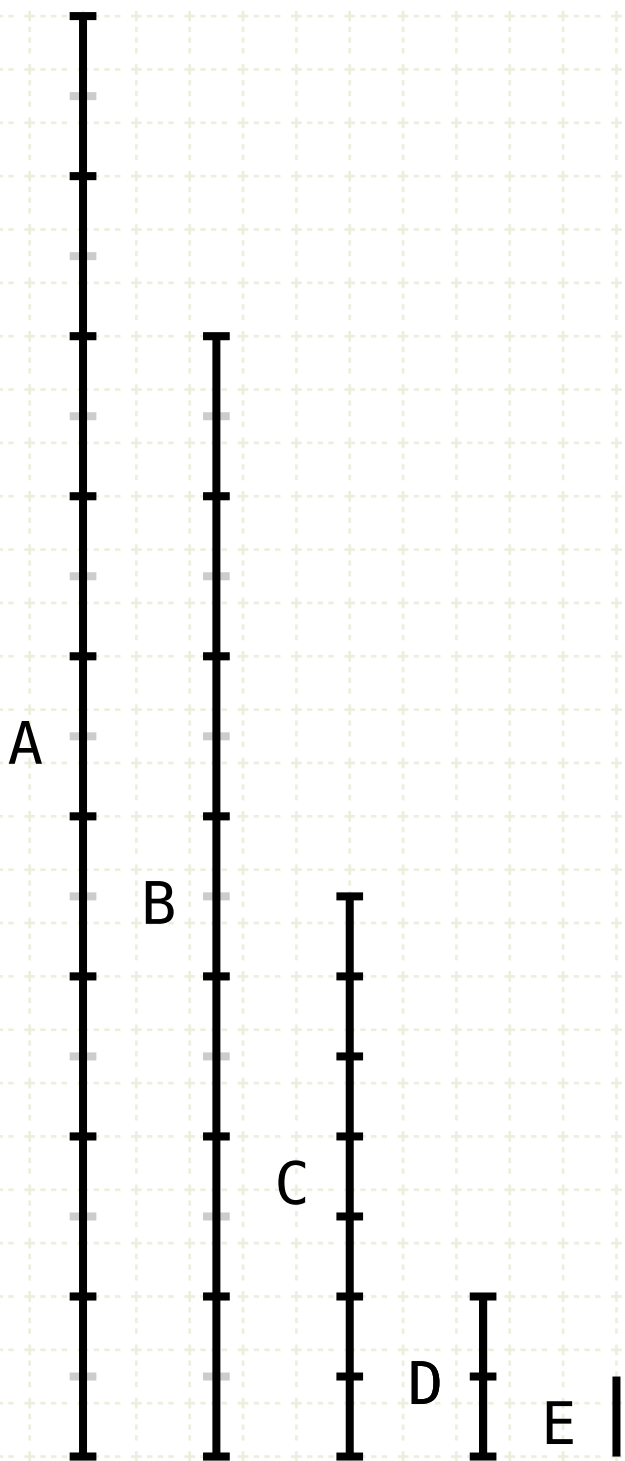
Find the greatest common divisor E for C and D (VII-2)

Since E is a measure of D, and since D is a measure of A,B, E is also a measure of A,B



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.



$$\gcd(A, B) = D$$

$$\text{let } C \neq q \cdot D$$

$$\gcd(C, D) = E$$

$$A = a \cdot E$$

$$B = b \cdot E$$

## Method

Find D, the greatest measure of A and B (VII-2)

Find the greatest common divisor E for C and D (VII-2)

Since E is a measure of D, and since D is a measure of A,B, E is also a measure of A,B

And since E also measures C, E is a common measure of A,B,C



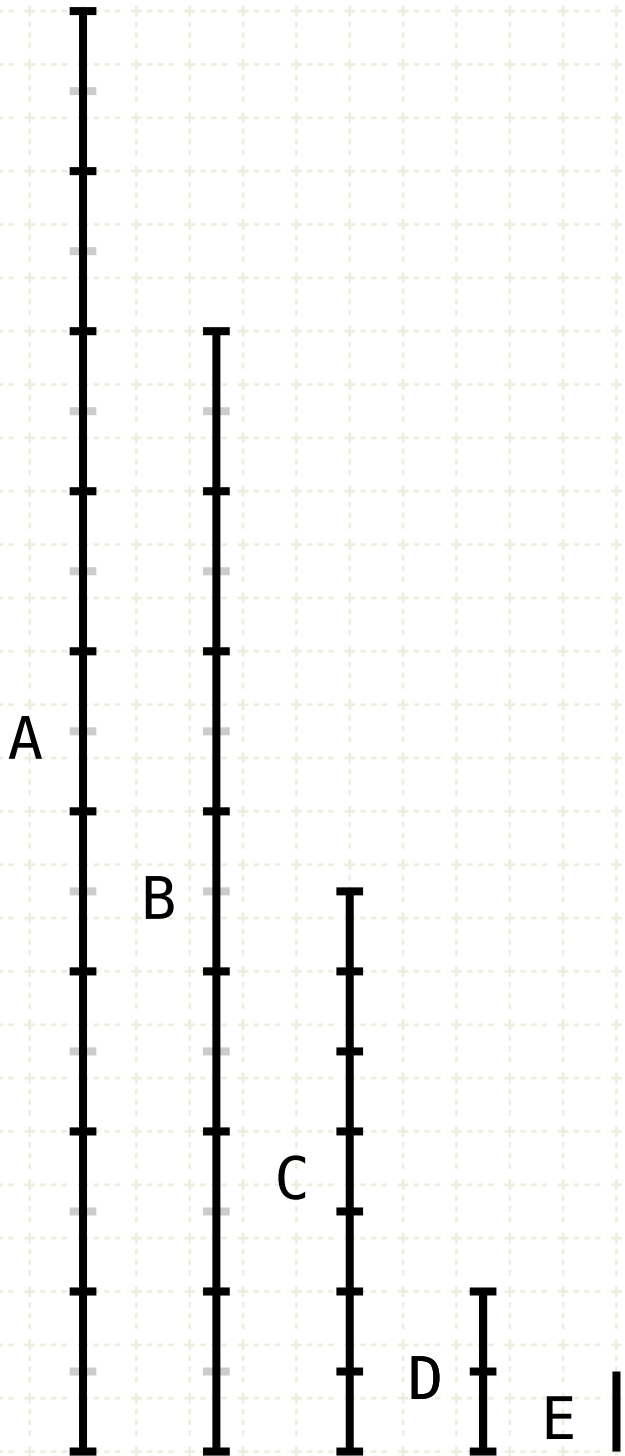
# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$\gcd(A,B) = D$   
 $\gcd(C,D) = E$

## Method

- Find D, the greatest measure of A and B (VII·2)
- Find the greatest common divisor E for C and D (VII·2)
- E is the Greatest Common Divisor of A,B,C

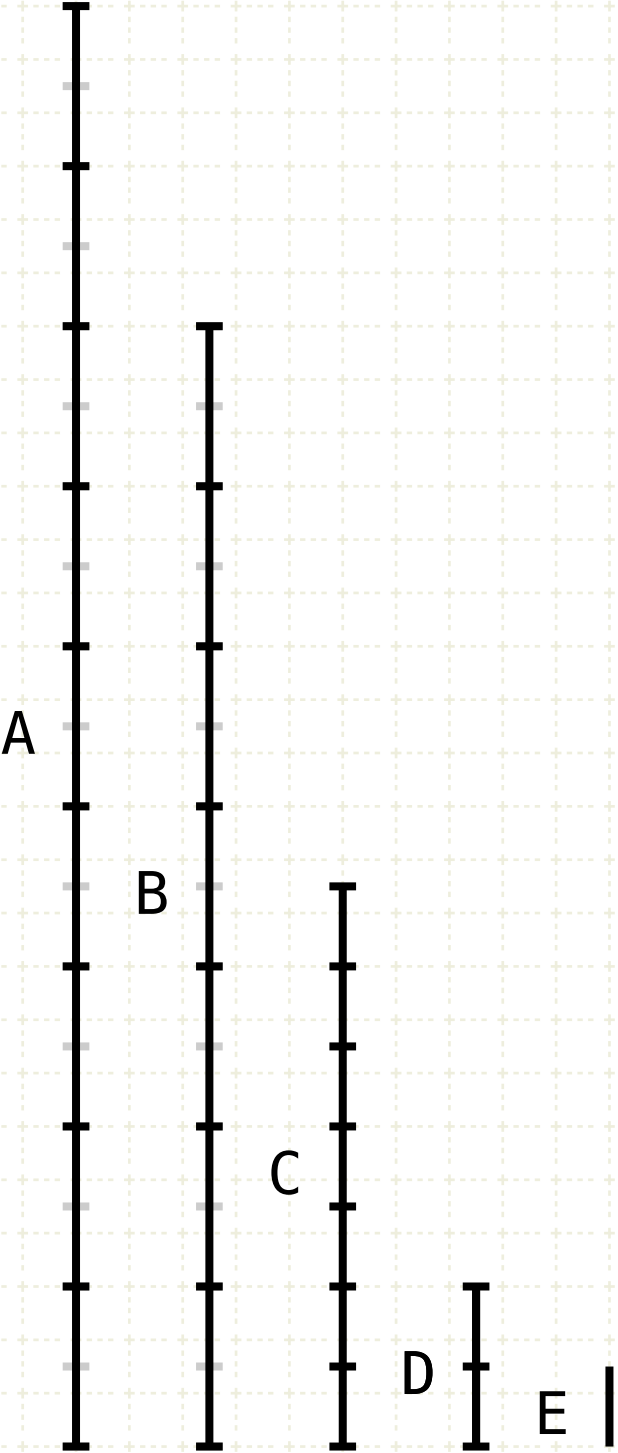


# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$\gcd(A,B) = D$

$\gcd(C,D) = E$



## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)

E is the Greatest Common Divisor of A,B,C

## Proof by Contradiction



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$$\gcd(A,B) = D$$

$$\gcd(C,D) = E$$

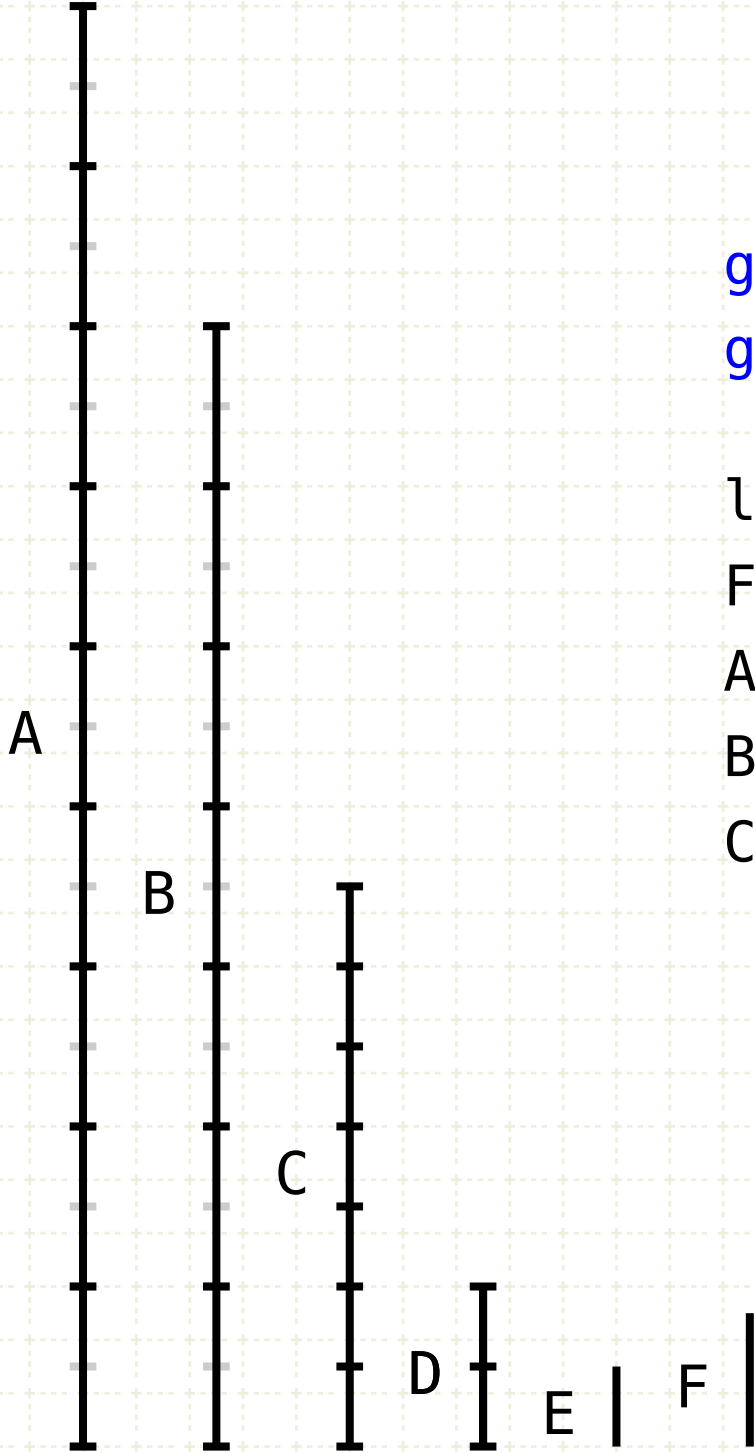
$$\text{let } F = \gcd(A,B,C)$$

$$F > E$$

$$A = p \cdot F$$

$$B = q \cdot F$$

$$C = r \cdot F$$



## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)

E is the Greatest Common Divisor of A,B,C

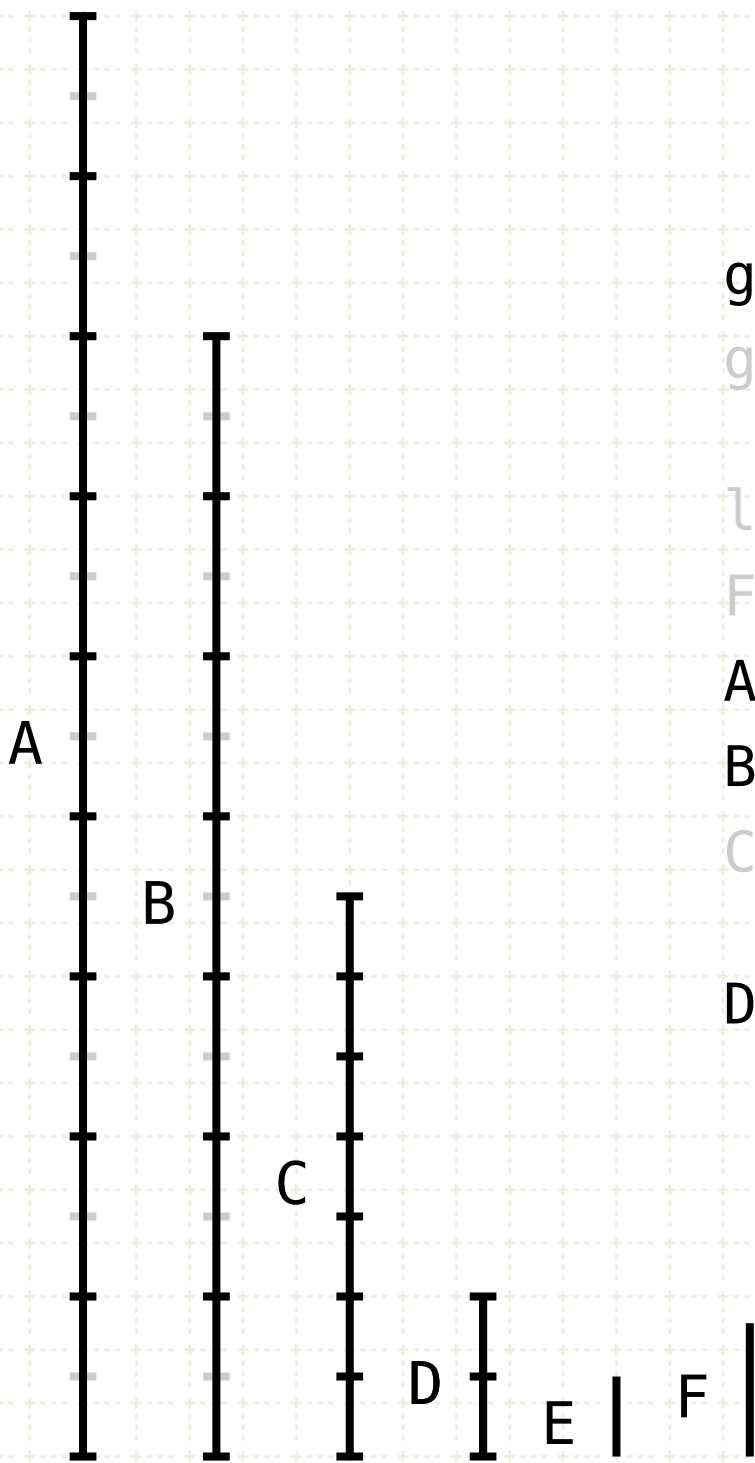
## Proof by Contradiction

Assume that F, which is greater than E, is the largest common divisor



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.



$$\gcd(A, B) = D$$

$$\gcd(C, D) = E$$

$$\text{let } F = \gcd(A, B, C)$$

$$F > E$$

$$A = p \cdot F$$

$$B = q \cdot F$$

$$C = r \cdot F$$

$$D = s \cdot F$$

## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)

E is the Greatest Common Divisor of A,B,C

## Proof by Contradiction

Assume that F, which is greater than E, is the largest common divisor

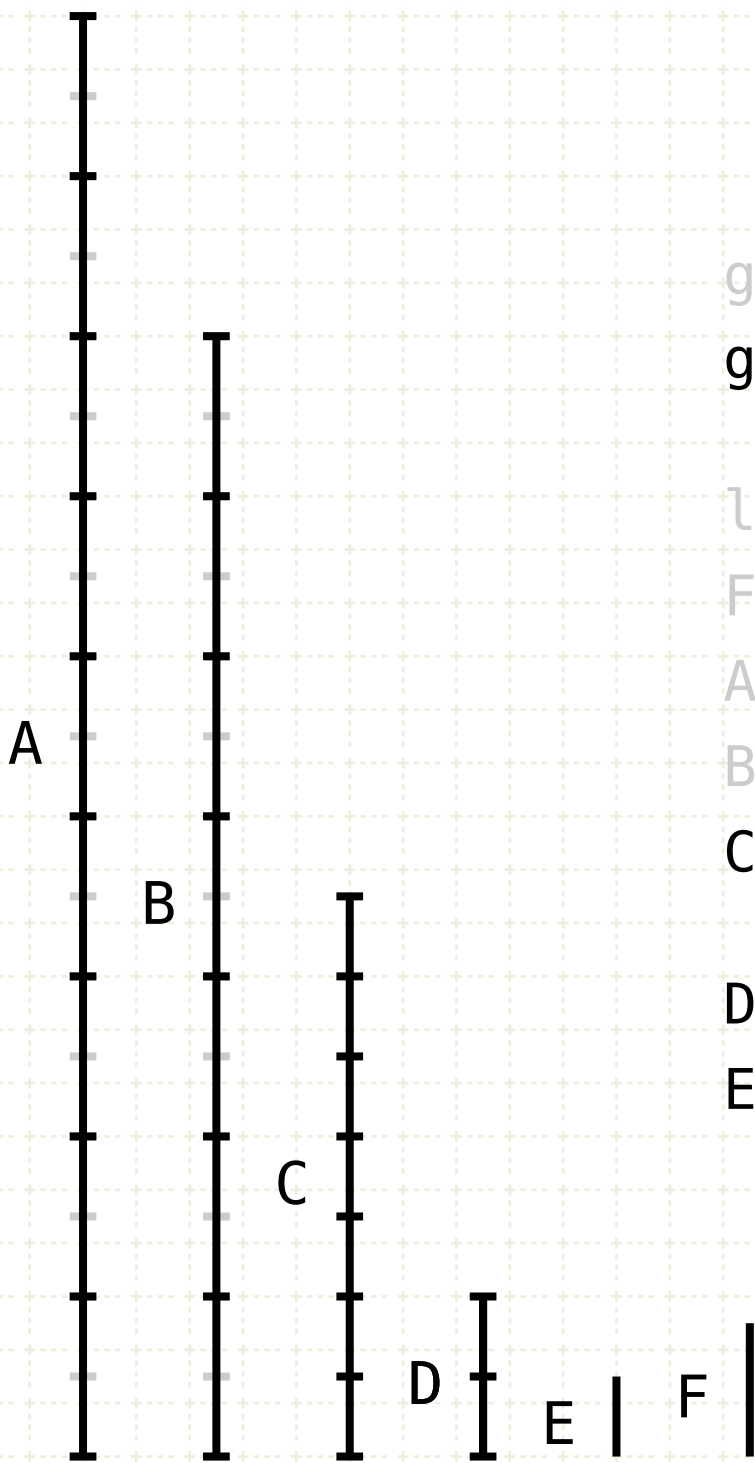
F measures A,B so it also measures D, the greatest common divisor of A,B (VII·2 Por)





# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.



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$$\text{let } F = \gcd(A, B, C)$$

$$F > E$$

$$A = p \cdot F$$

$$B = q \cdot F$$

$$C = r \cdot F$$

$$D = s \cdot F$$

$$E = n \cdot F$$

## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)

E is the Greatest Common Divisor of A,B,C

## Proof by Contradiction

Assume that F, which is greater than E, is the largest common divisor

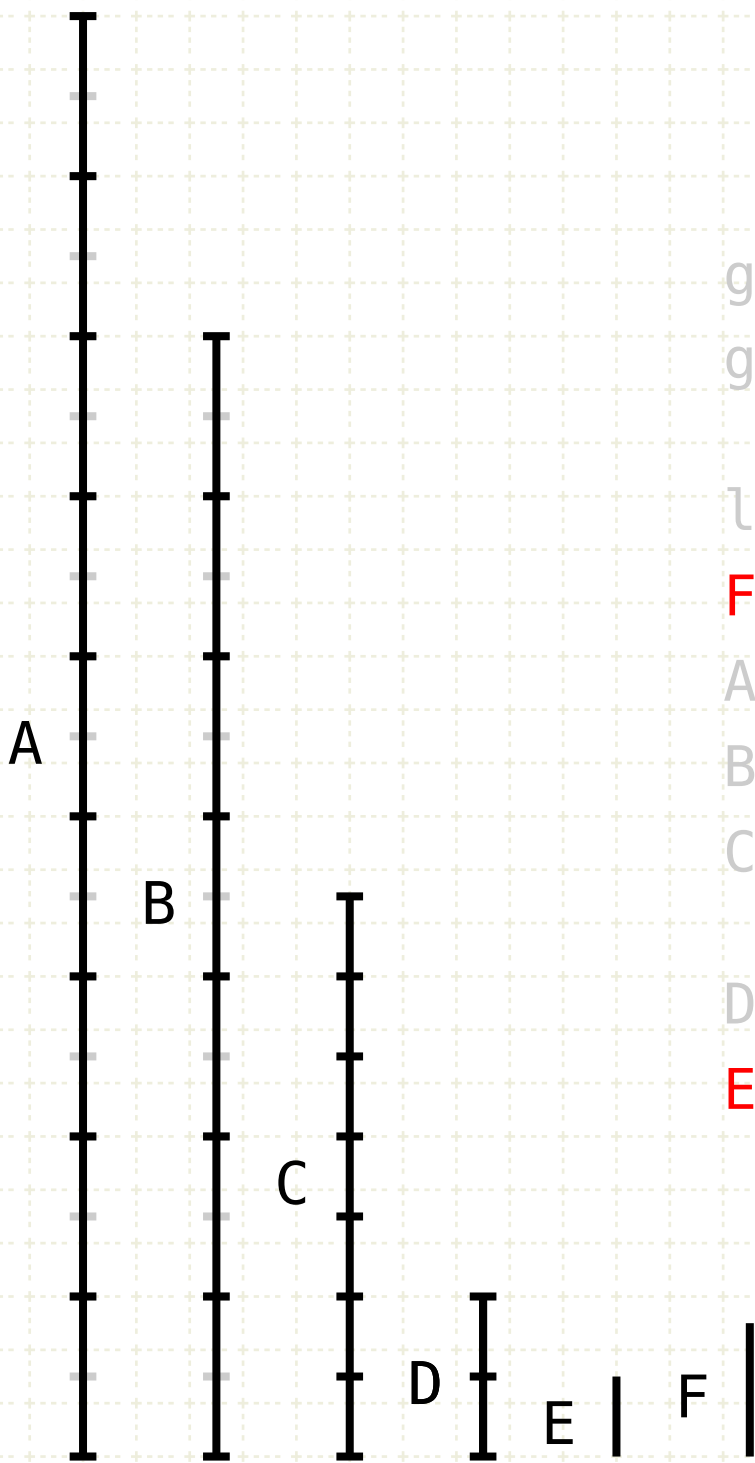
F measures A,B so it also measures D, the greatest common divisor of A,B (VII·2 Por)

F measures D,C so it also measures E, the greatest common divisor of D,C (VII·2 Por)



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.



$$\gcd(A, B) = D$$

$$\gcd(C, D) = E$$

$$\text{let } F = \gcd(A, B, C)$$

$$F > E$$

$$A = p \cdot F$$

$$B = q \cdot F$$

$$C = r \cdot F$$

$$D = s \cdot F$$

$$E = n \cdot F$$

## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)

E is the Greatest Common Divisor of A,B,C

## Proof by Contradiction

Assume that F, which is greater than E, is the largest common divisor

F measures A,B so it also measures D, the greatest common divisor of A,B (VII·2 Por)

F measures D,C so it also measures E, the greatest common divisor of D,C (VII·2 Por)

But F, being larger than E, cannot measure E, hence F cannot be larger than E



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$\gcd(A,B) = D$

$\gcd(C,D) = E$

let  $F = \gcd(A,B,C)$

$F > E$

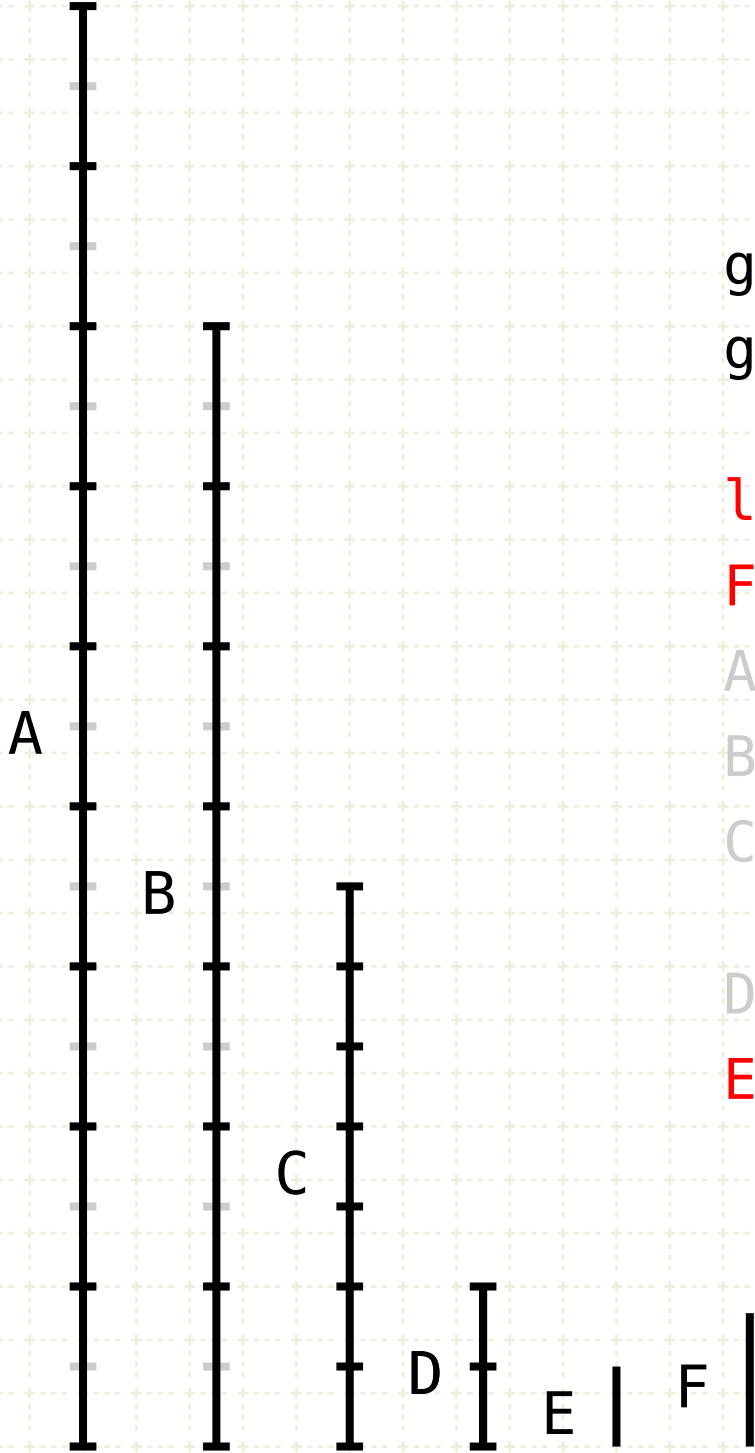
$A = p \cdot F$

$B = q \cdot F$

$C = r \cdot F$

$D = s \cdot F$

$E = n \cdot F$



## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)

E is the Greatest Common Divisor of A,B,C

## Proof by Contradiction

Assume that F, which is greater than E, is the largest common divisor

F measures A,B so it also measures D, the greatest common divisor of A,B (VII·2 Por)

F measures D,C so it also measures E, the greatest common divisor of D,C (VII·2 Por)

But F, being larger than E, cannot measure E, hence F cannot be larger than E



# Proposition 3 of Book VII

Given three numbers not prime to one another, to find their greatest common measure.

$\text{gcd}(A,B) = D$

$\text{gcd}(C,D) = E$

$\text{let } F = \text{gcd}(A,B,C)$

$F > E$

$A = p \cdot F$

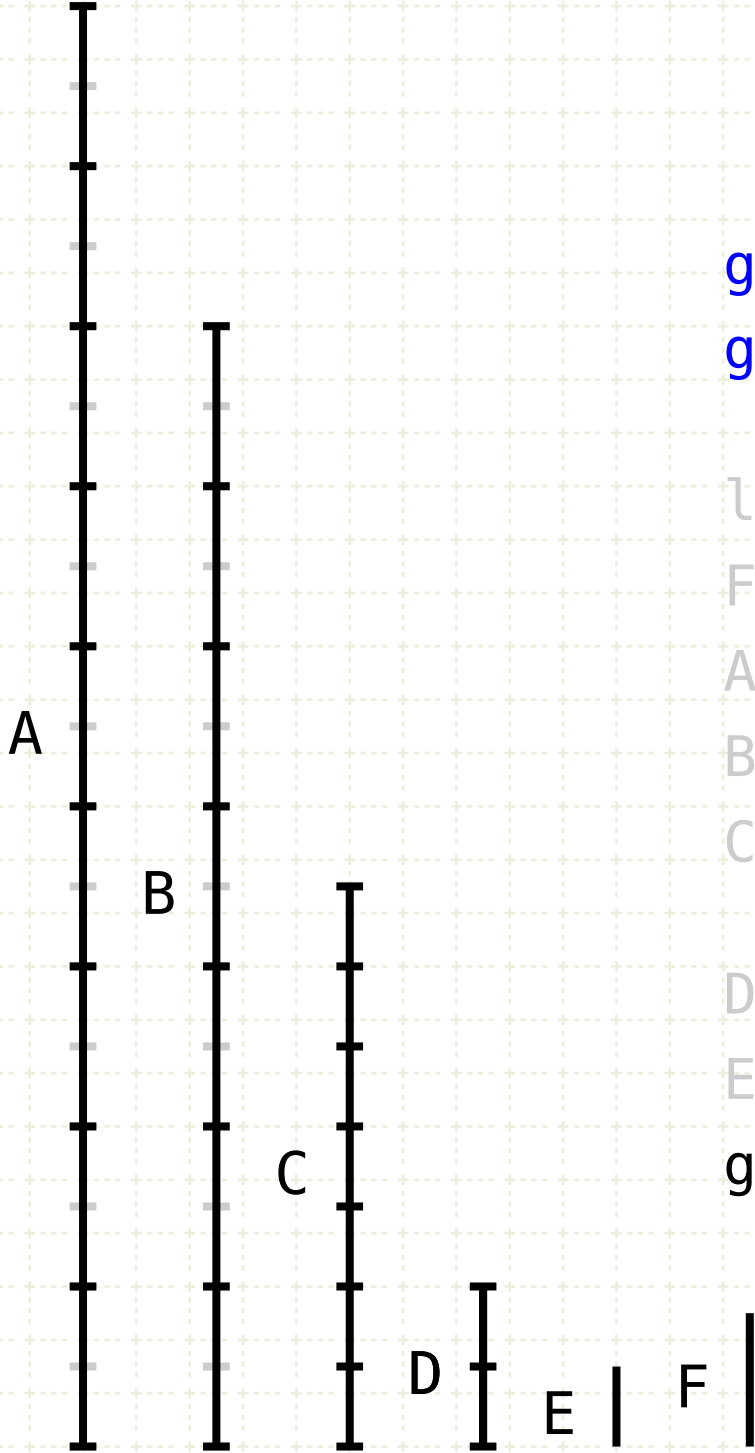
$B = q \cdot F$

$C = r \cdot F$

$D = s \cdot F$

$E = n \cdot F$

$\text{gcd}(A,B,C) = E$



## Method

Find D, the greatest measure of A and B (VII·2)

Find the greatest common divisor E for C and D (VII·2)

E is the Greatest Common Divisor of A,B,C

## Proof by Contradiction

Assume that F, which is greater than E, is the largest common divisor

F measures A,B so it also measures D, the greatest common divisor of A,B (VII·2 Por)

F measures D,C so it also measures E, the greatest common divisor of D,C (VII·2 Por)

But F, being larger than E, cannot measure E, hence F cannot be larger than E

Therefore E is the greatest common divisor



# Youtube Videos

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