

Euclid's Elements

Book VII

Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)



Table of Contents, Chapter 7

1	Determine if two numbers are relatively prime	10	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$	21	If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B
2	Find the greatest common divisor for two numbers	11	If $A:B = C:D$, then $(A-C):(B-D) = A:B$	22	If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime
3	Find the largest common divisor for three numbers	12	If $A:B = C:D$, then $(A+C):(B+C) = A:B$	23	If A,B are relatively prime and if $A = n \cdot C$, then B,C are relatively prime
4	Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B	13	If $A:B = C:D$, then $A:C = B:D$	24	If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C
5	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, then $(B+D) = (1/q) \cdot (A+C)$	14	If $A:B = D:E$ and $B:C = E:F$, then $A:C = D:F$	25	If A,B are relatively prime then A^2, B are relatively prime
6	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, then $(B+D) = (p/q) \cdot (A+C)$	15	If $B = i \cdot 1$ and $E = i \cdot D$, and if $D = j \cdot 1$ then $E = j \cdot B$	26	If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$
7	If $B = A/q$ and $D = C/q$, $B > D$, then $(B-D) = (A-C)/q$	16	$A \times B = B \times A$	27	If A,B are relatively prime, then A^2, B^2 are relatively prime, and A^3, B^3 are relatively prime, and so on
8	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, $B > D$, then $(B-D) = (p/q) \cdot (A-C)$	17	If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$		
9	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$	18	If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$		
		19	If $A:B = C:D$ then $A \times D = B \times C$ If $A \times D = B \times C$ then $A:B = C:D$		
		20	Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$		



Table of Contents, Chapter 7

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|----|--|----|---|
| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime | 37 | If $A = p \cdot B$, then $A = q \cdot C$ where $C = p \cdot 1$ |
| 29 | If A is prime, and $B \neq n \cdot A$, then A,B are relatively prime | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$ |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a$, $1/b$, $1/c$ |
| 31 | If $A = B \times C$, then $A = j \cdot D$ where D is prime | | |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime | | |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C | | |
| 34 | Find the lowest common denominator of 2 numbers | | |
| 35 | If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$, then $C = i \cdot E$ | | |
| 36 | Find the least common multiple of 3 numbers | | |



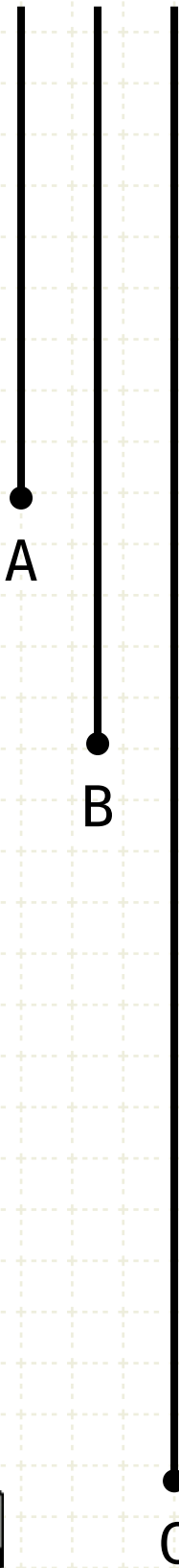
Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z\in\mathbb{N}, \ x:y:z=A:B:C\}$$

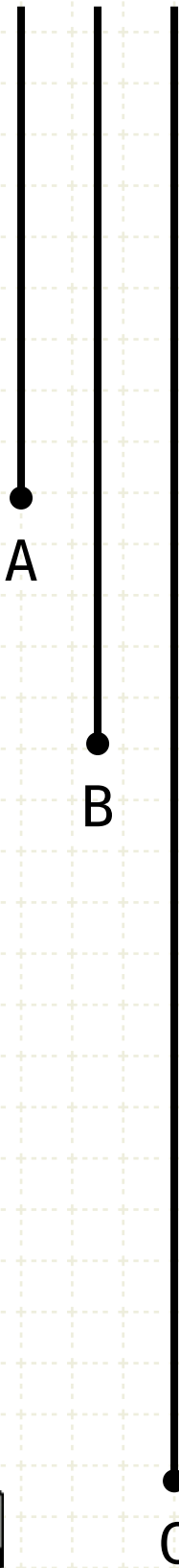
In other words

Given as many numbers as we please, say A,B,C



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z\in\mathbb{N}, \ x:y:z=A:B:C\}$$

Find
 $(X,Y,Z)\in S$
such that $X\leq x, \ Y\leq y, \ Z\leq z, \forall (x,y,z)\in S$

In other words

Given as many numbers as we please, say A,B,C
Find the least numbers X,Y,Z which are in the same ratio as
A,B,C

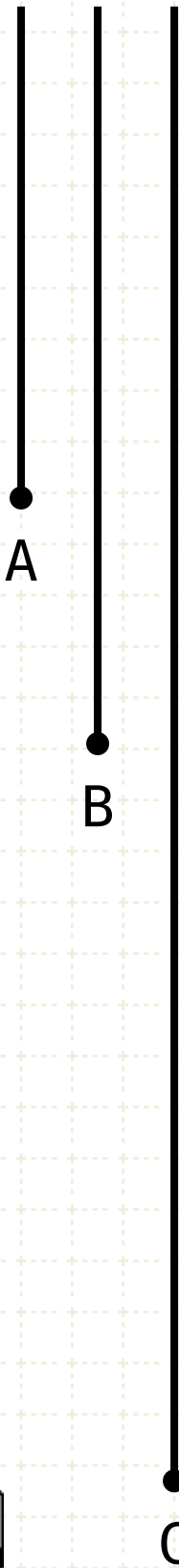


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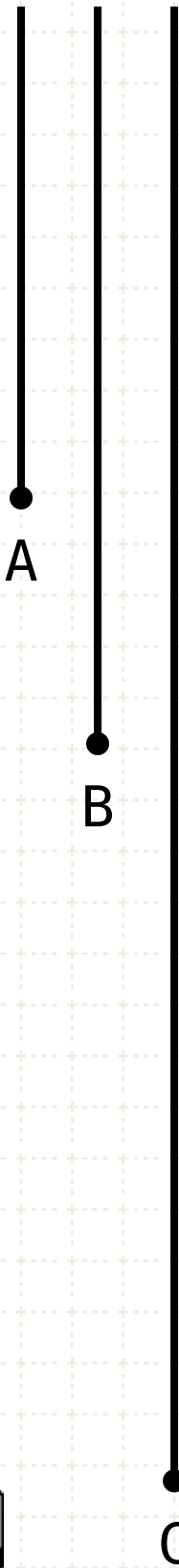
Method

$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \ x:y:z=A:B:C\}$$



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z\in\mathbb{N}, \ x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = 1$$

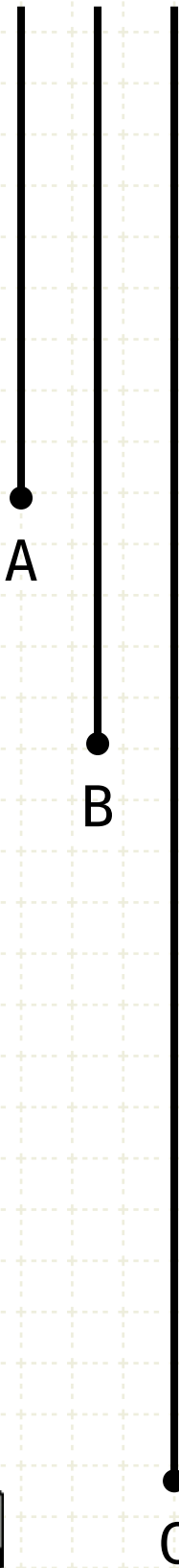
Method

Let A,B,C be prime to one another



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Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z\in\mathbb{N}, \ x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = 1$$

$$A\leq x, \ B\leq y, \ C\leq z, \ \forall (x,y,z)\in S$$

Method

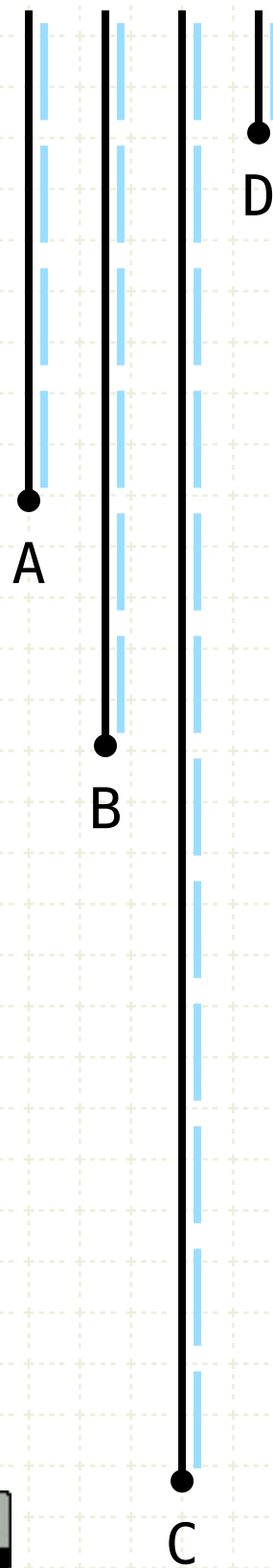
Let A,B,C be prime to one another

Then A,B,C are the least numbers with this ratio (VII·21)



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \ x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

Method

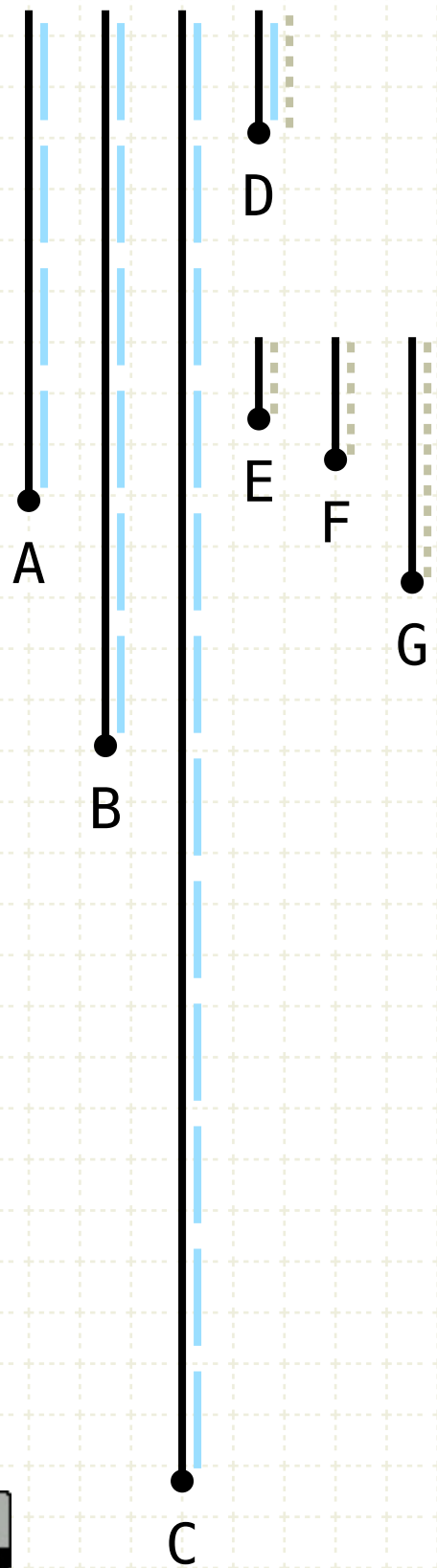
Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z\in\mathbb{N}, \ x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = E \cdot D$$

$$B = F \cdot D$$

$$C = G \cdot D$$

Method

Let A,B,C NOT be prime to one another

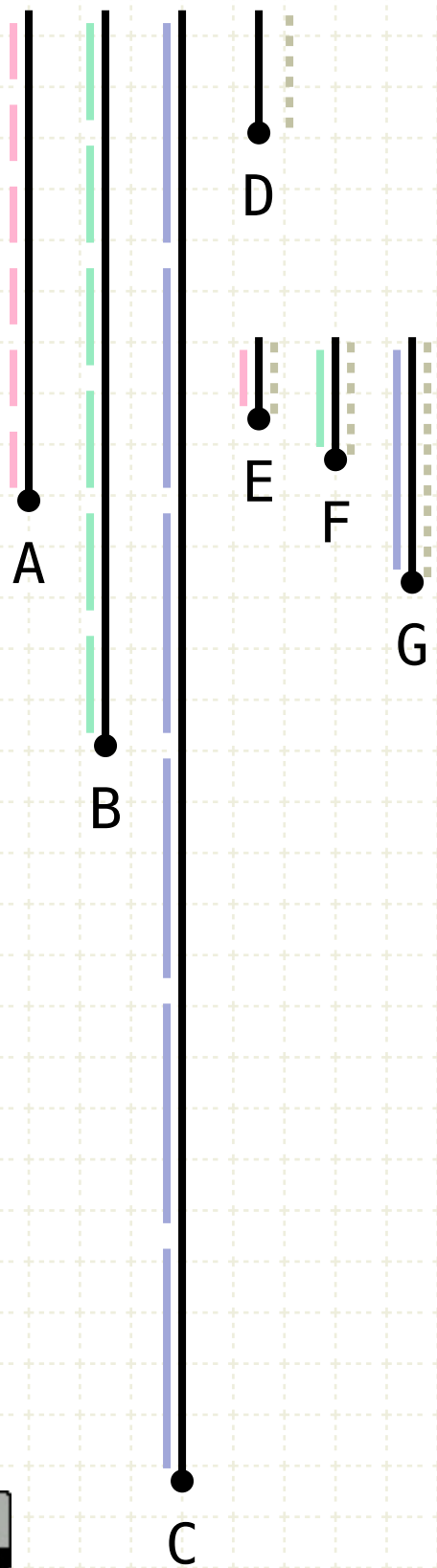
Let D be the greatest common divisor of A,B and C (VII-3)

and, as many times a D measure the number A,B,C, let there be the same number of units in E,F,G respectively



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z\in\mathbb{N}, \ x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = E \cdot D$$

$$B = F \cdot D$$

$$C = G \cdot D$$

$$A = D \cdot E$$

$$B = D \cdot F$$

$$C = D \cdot G$$

Method

Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)

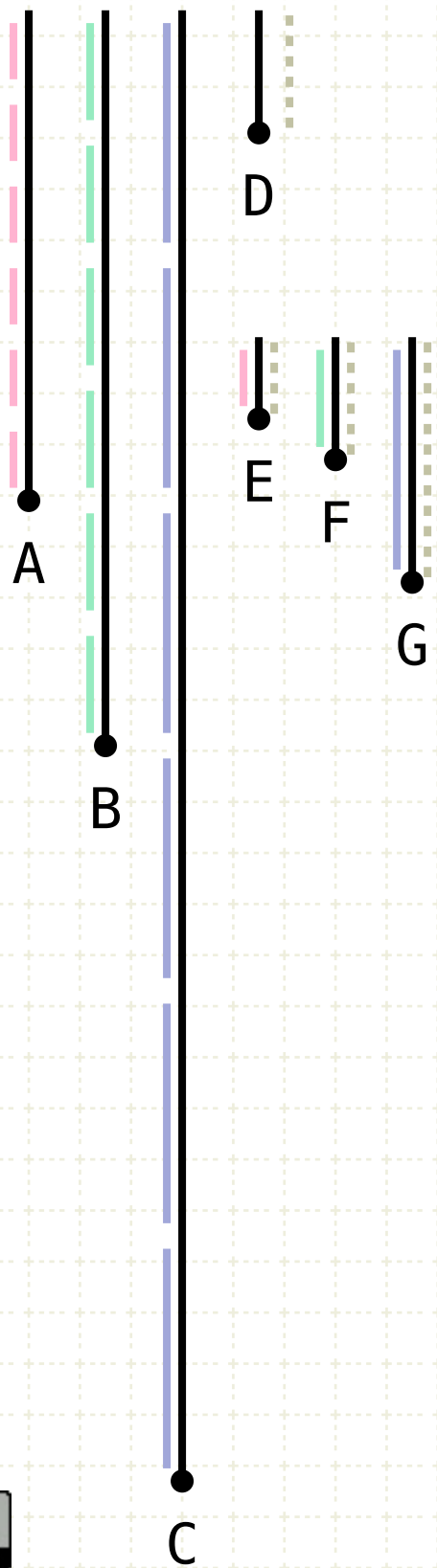
and, as many times a D measure the number A,B,C, let there be the same number of units in E,F,G respectively

Therefore E,F,G measure A,B,C according to the units in D (VII·16)



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$$A = E \cdot D$$

$$B = F \cdot D$$

$$C = G \cdot D$$

$$A = D \cdot E$$

$$B = D \cdot F$$

$$C = D \cdot G$$

$$E:F:G = A:B:C$$

Method

Let A,B,C NOT be prime to one another

Let D be the greatest common divisor of A,B and C (VII·3)

and, as many times a D measure the number A,B,C, let there be the same number of units in E,F,G respectively

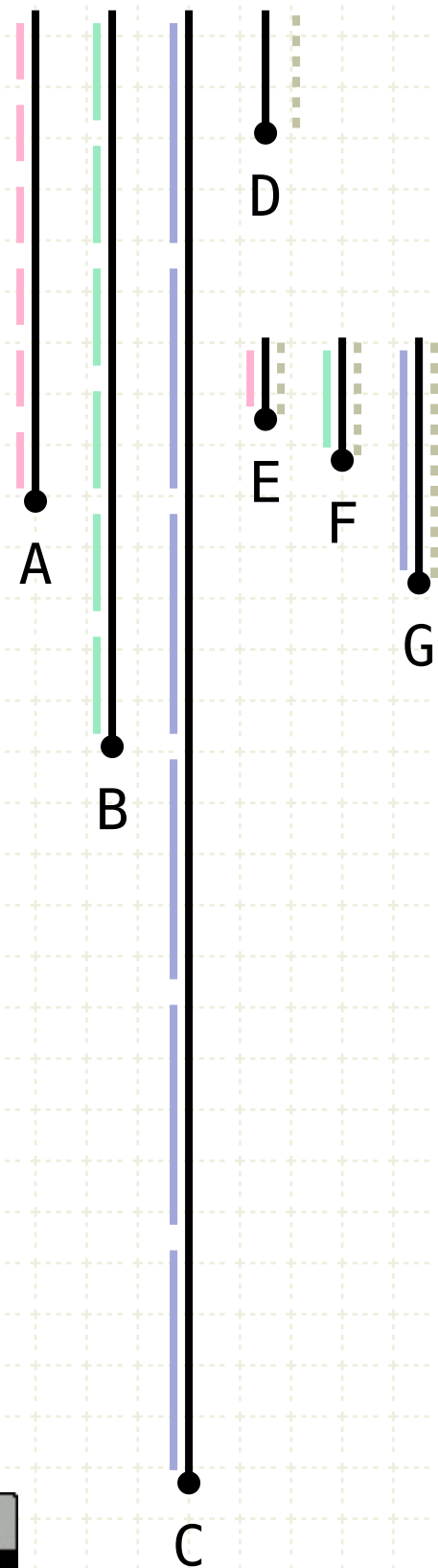
Therefore E,F,G measure A,B,C according to the units in D (VII·16)

Therefore E,F,G are in the same ratio as A,B,C (VII.Def.20)



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S = \{ (x, y, z) \mid x, y, z \in \mathbb{N}, x : y : z = A : B : C \}$$

$$\gcd(A, B, C) = D$$

$$A = E \cdot D$$

$$B = F \cdot D$$

$$C = G \cdot D$$

$$A = D \cdot E$$

$$B = D \cdot F$$

$$C = D \cdot G$$

$$E : F : G = A : B : C$$

$$E \leq x, F \leq y, G \leq z, \forall (x, y, z) \in S$$

Method

Let A, B, C NOT be prime to one another

Let D be the greatest common divisor of A, B and C (VII·3)

and, as many times a D measure the number A, B, C, let there be the same number of units in E, F, G respectively

Therefore E, F, G measure A, B, C according to the units in D (VII·16)

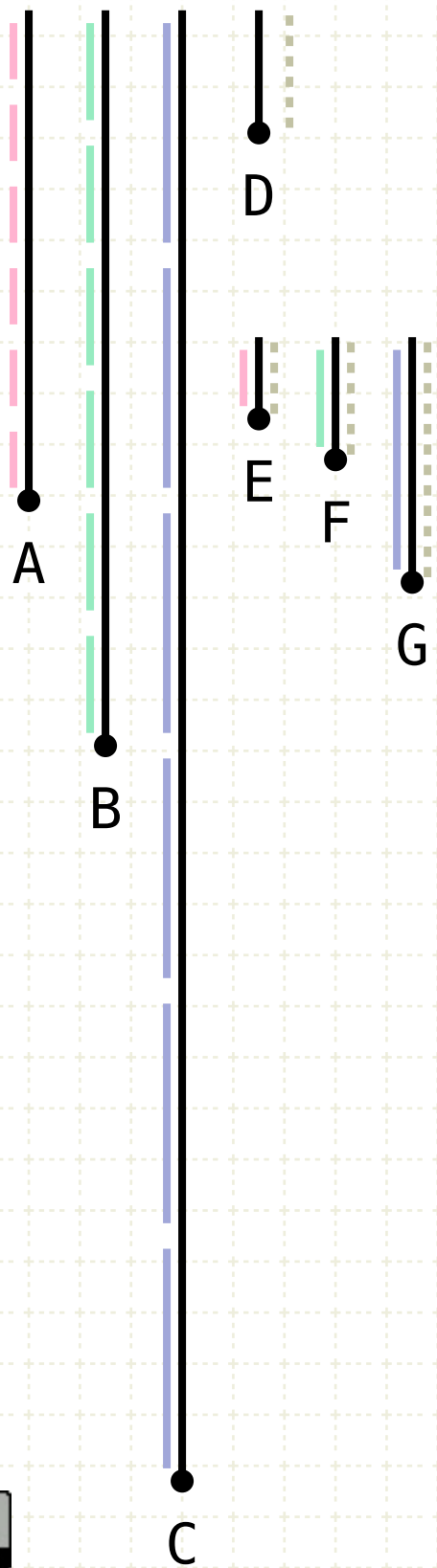
Therefore E, F, G are in the same ratio as A, B, C (VII.Def.20)

E, F, G are the smallest numbers that have the ratio of A, B, C



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Given as many numbers as we please, to find the least of those which have the same ratio with them.



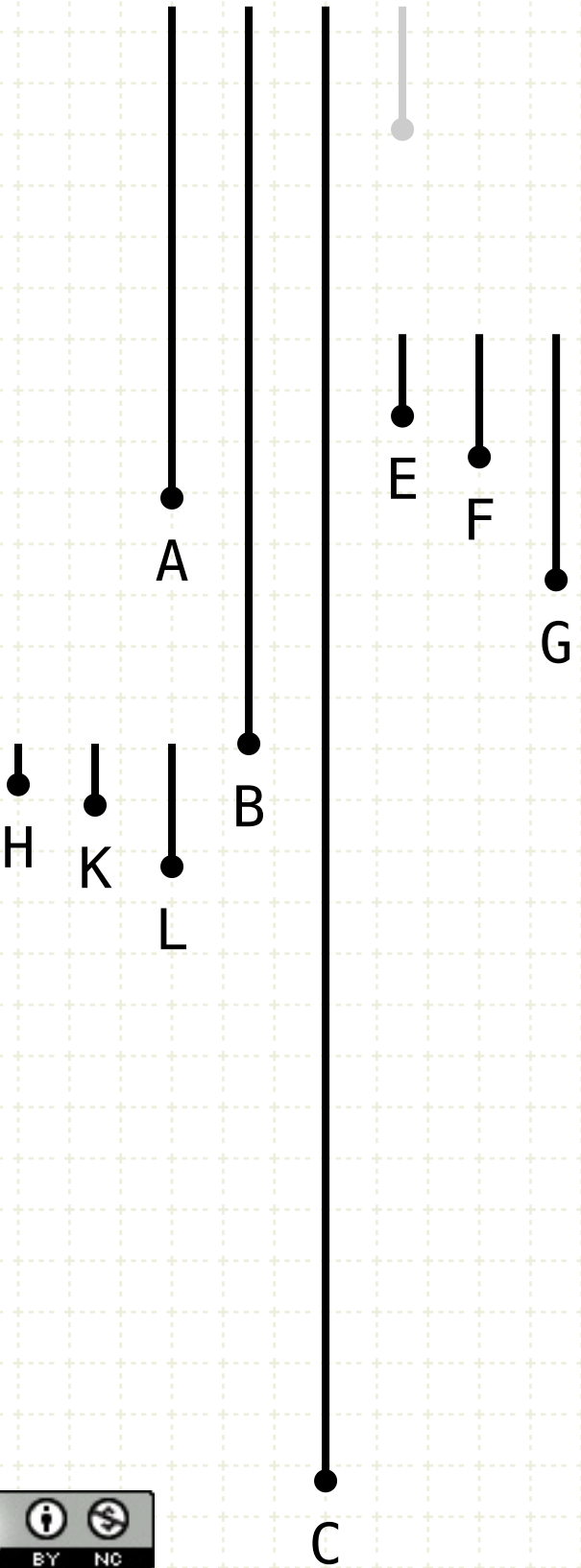
$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \ x:y:z=A:B:C\}$$
$$\gcd(A,B,C) = D$$
$$A = D \cdot E, \ B = D \cdot F, \ C = D \cdot G$$
$$E:F:G = A:B:C$$
$$E \leq x, \ F \leq y, \ G \leq z, \ \forall (x,y,z) \in S$$

Proof by Contradiction



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \ x:y:z=A:B:C\}$$

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$$A = D \cdot E, \ B = D \cdot F, \ C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \ F \leq y, \ G \leq z, \ \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \ K < F, \ L < G$$

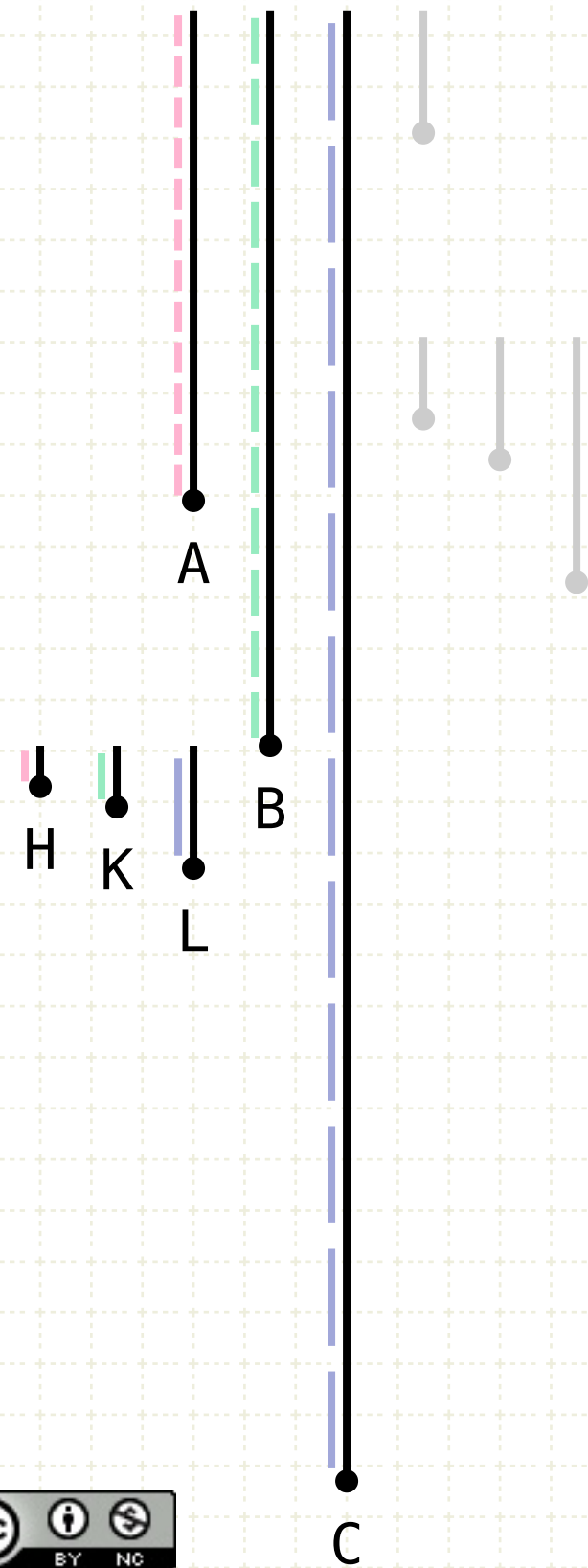
Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \ x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = D \cdot E, \ B = D \cdot F, \ C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \ F \leq y, \ G \leq z, \ \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \ K < F, \ L < G$$

$$A = M \cdot H, \ B = M \cdot K, \ C = M \cdot L$$

Proof by Contradiction

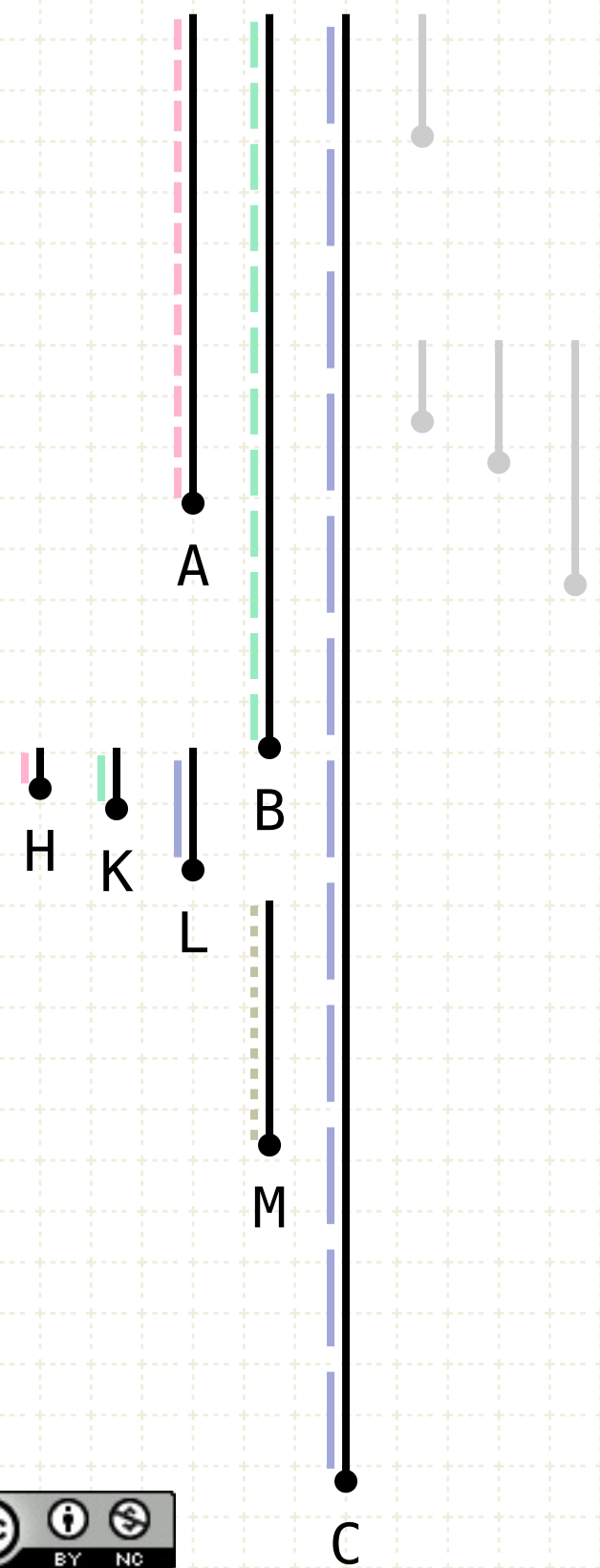
Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \ x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = D \cdot E, \ B = D \cdot F, \ C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \ F \leq y, \ G \leq z, \ \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \ K < F, \ L < G$$

$$A = M \cdot H, \ B = M \cdot K, \ C = M \cdot L$$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

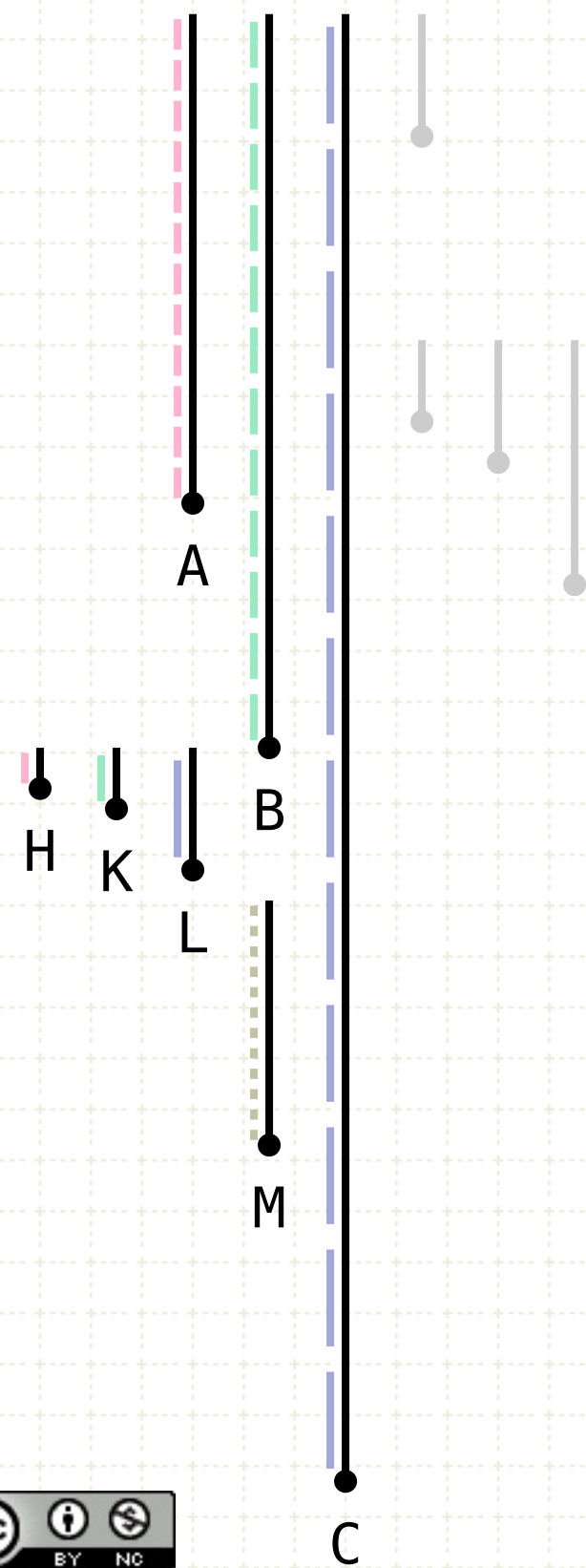
Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



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$$\gcd(A,B,C) = D$$

$$A = D \cdot E, \quad B = D \cdot F, \quad C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \quad F \leq y, \quad G \leq z, \quad \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \quad K < F, \quad L < G$$

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Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

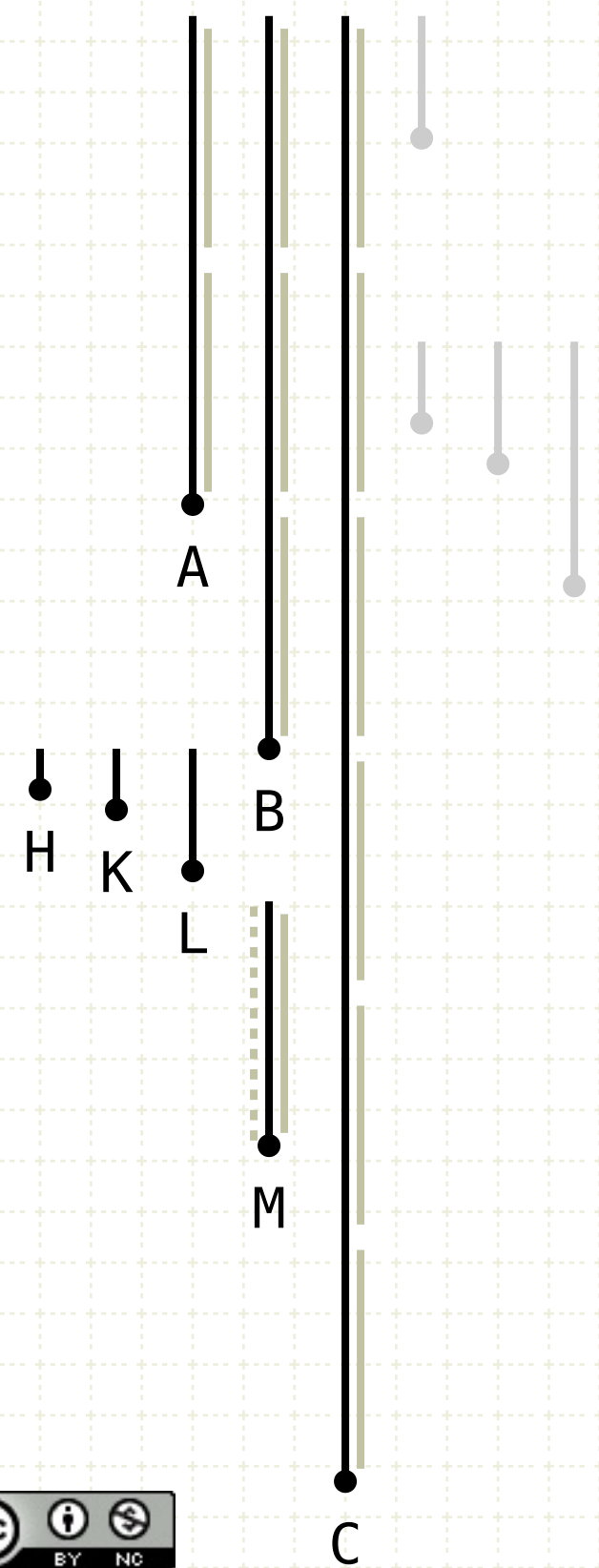
As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \quad x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

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$$E:F:G = A:B:C$$

$$E \leq x, \quad F \leq y, \quad G \leq z, \quad \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \quad K < F, \quad L < G$$

$$A = M \cdot H, \quad B = M \cdot K, \quad C = M \cdot L$$

$$A = H \cdot M, \quad B = K \cdot M, \quad C = L \cdot M$$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

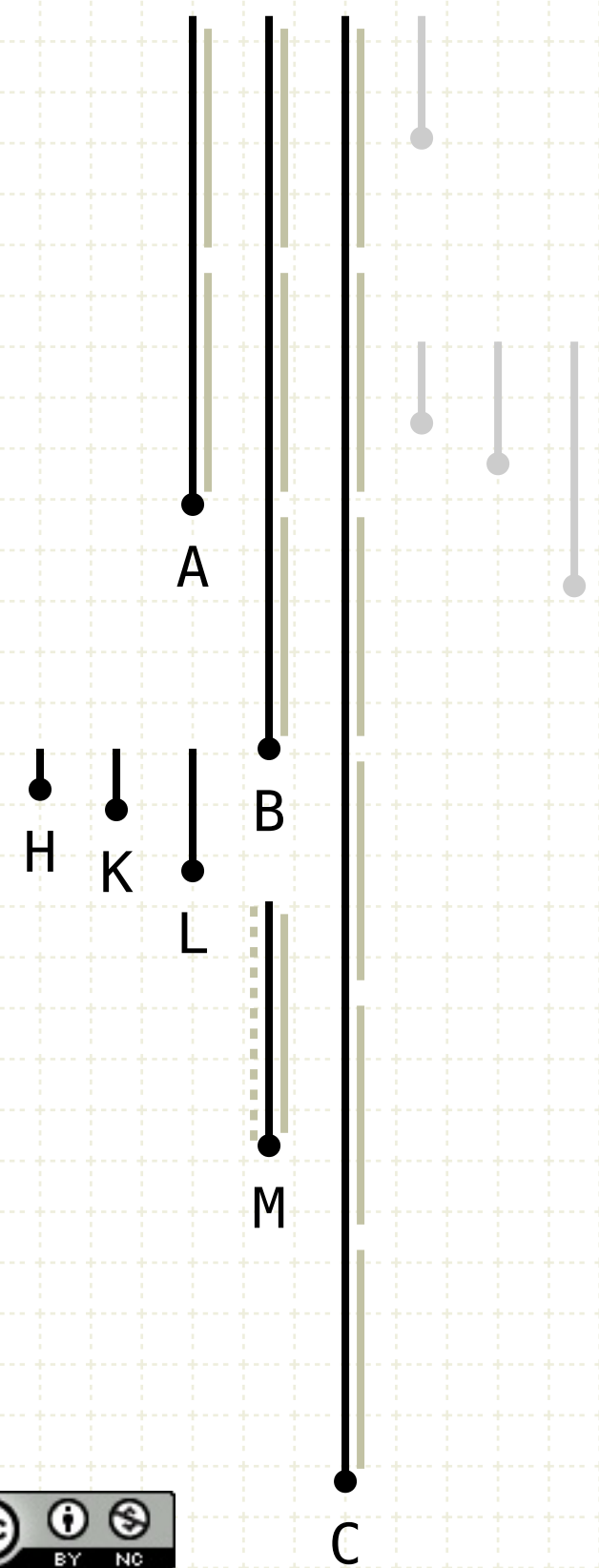
Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \quad x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = D \cdot E, \quad B = D \cdot F, \quad C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \quad F \leq y, \quad G \leq z, \quad \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \quad K < F, \quad L < G$$

$$A = M \cdot H, \quad B = M \cdot K, \quad C = M \cdot L$$

$$A = H \cdot M, \quad B = K \cdot M, \quad C = L \cdot M$$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

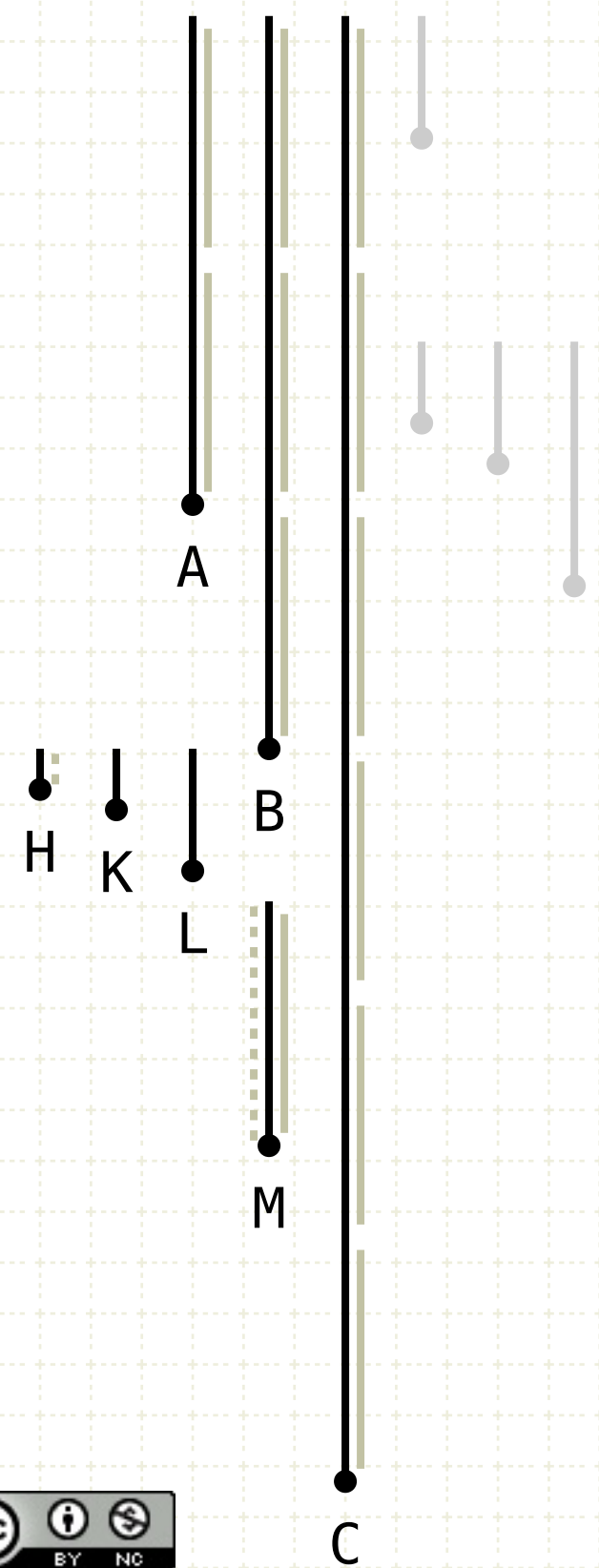
Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \quad x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = D \cdot E, \quad B = D \cdot F, \quad C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \quad F \leq y, \quad G \leq z, \quad \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \quad K < F, \quad L < G$$

$$A = M \cdot H, \quad B = M \cdot K, \quad C = M \cdot L$$

$$A = H \cdot M, \quad B = K \cdot M, \quad C = L \cdot M$$

$$A = M \times H$$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

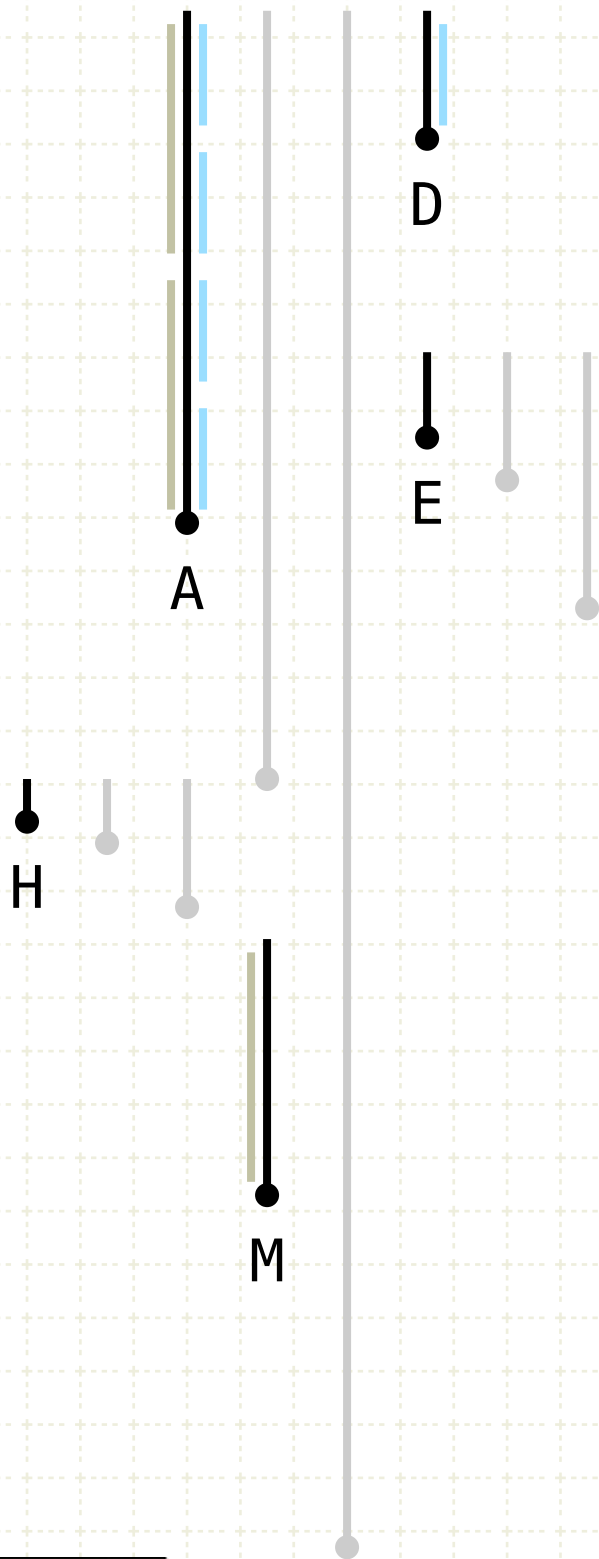
Therefore M measures A,B,C

Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \quad x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = D \cdot E, \quad B = D \cdot F, \quad C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \quad F \leq y, \quad G \leq z, \quad \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \quad K < F, \quad L < G$$

$$A = M \cdot H, \quad B = M \cdot K, \quad C = M \cdot L$$

$$A = H \cdot M, \quad B = K \cdot M, \quad C = L \cdot M$$

$$A = M \times H$$

$$A = M \times H = E \times D$$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C

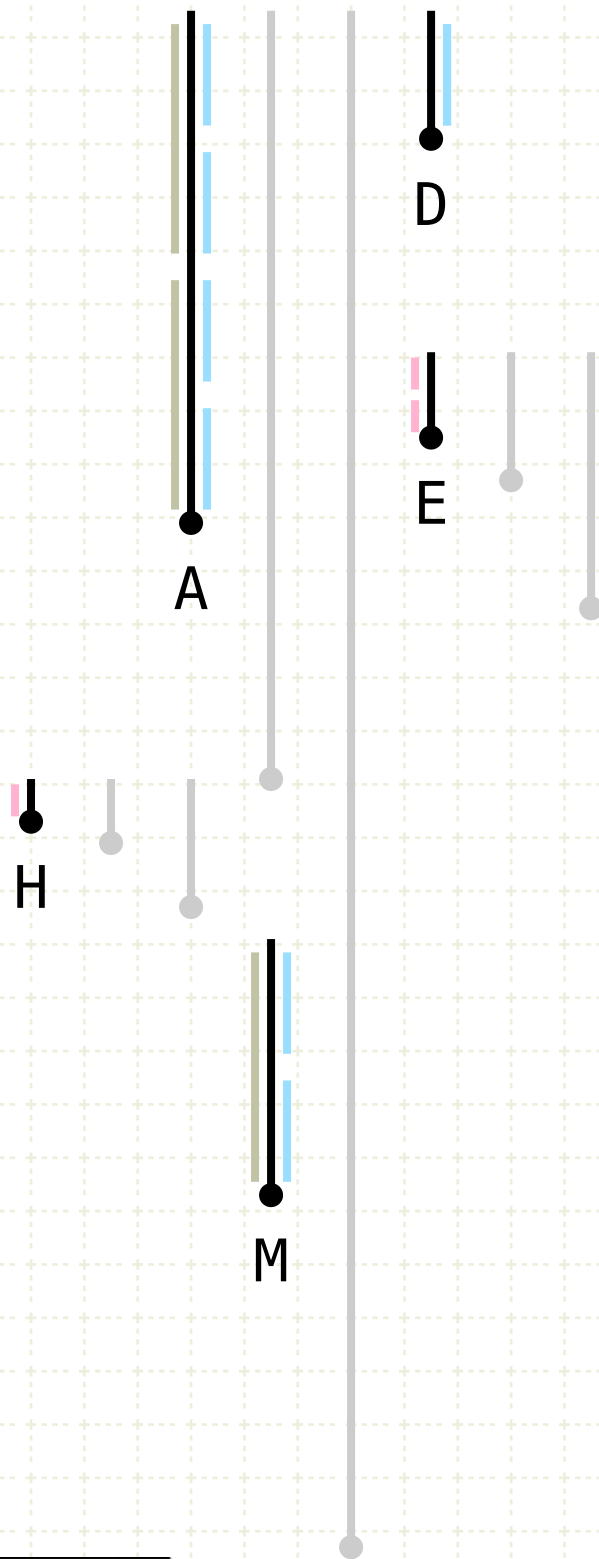
Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)

Therefore the product of E,D is equal to the product H,M



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z) \mid x,y,z \in \mathbb{N}, \quad x:y:z=A:B:C\}$$

$$\gcd(A,B,C) = D$$

$$A = D \cdot E, \quad B = D \cdot F, \quad C = D \cdot G$$

$$E:F:G = A:B:C$$

$$E \leq x, \quad F \leq y, \quad G \leq z, \quad \forall (x,y,z) \in S$$

$$H:K:L = E:F:G = A:B:C$$

$$H < E, \quad K < F, \quad L < G$$

$$A = M \cdot H, \quad B = M \cdot K, \quad C = M \cdot L$$

$$A = H \cdot M, \quad B = K \cdot M, \quad C = L \cdot M$$

$$A = M \times H$$

$$A = M \times H = E \times D$$

$$E:H = M:D$$

Proof by Contradiction

Let the number H,K,L be in the same ratio of A,B,C, and smaller than the numbers E,F,G, respectively

Thus, H,K,L measure A,B,C the same number of times

As many times as H measures A, let M have the same number of units

Therefore the numbers H,K,L measure the numbers A,B,C according to the units in M

Therefore M measures A according to the units in H, B according to the units in K, and C according to the units in L (VII·16)

Therefore M measures A,B,C

Since H measures A according to the units in M, M multiplied by H is equal to A (VII.Def.15)

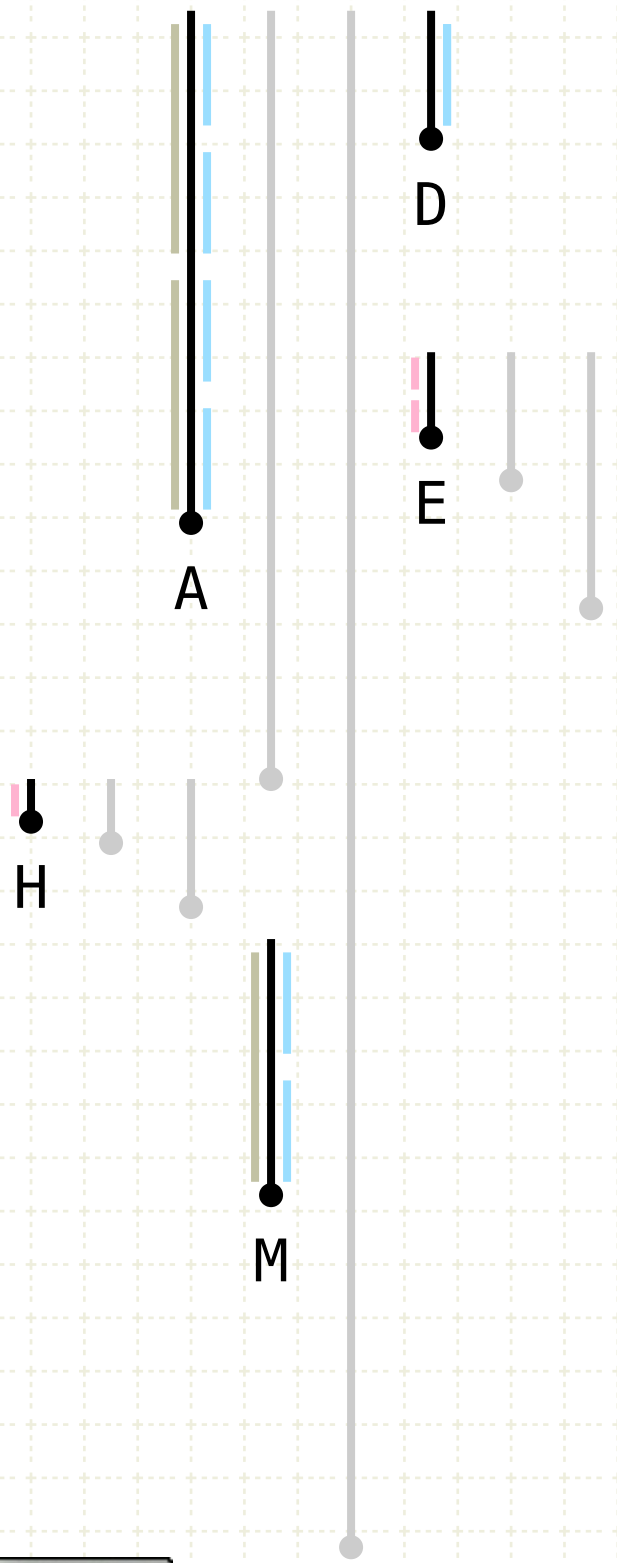
Therefore the product of E,D is equal to the product H,M

Therefore E is to H as M is to D (VII·19)



Proposition 33 of Book VII

Given as many numbers as we please, to find the least of those which have the same ratio with them.



$$S=\{(x,y,z)|x,y,z\in\mathbb{N},\ x:y:z=A:B:C\}$$

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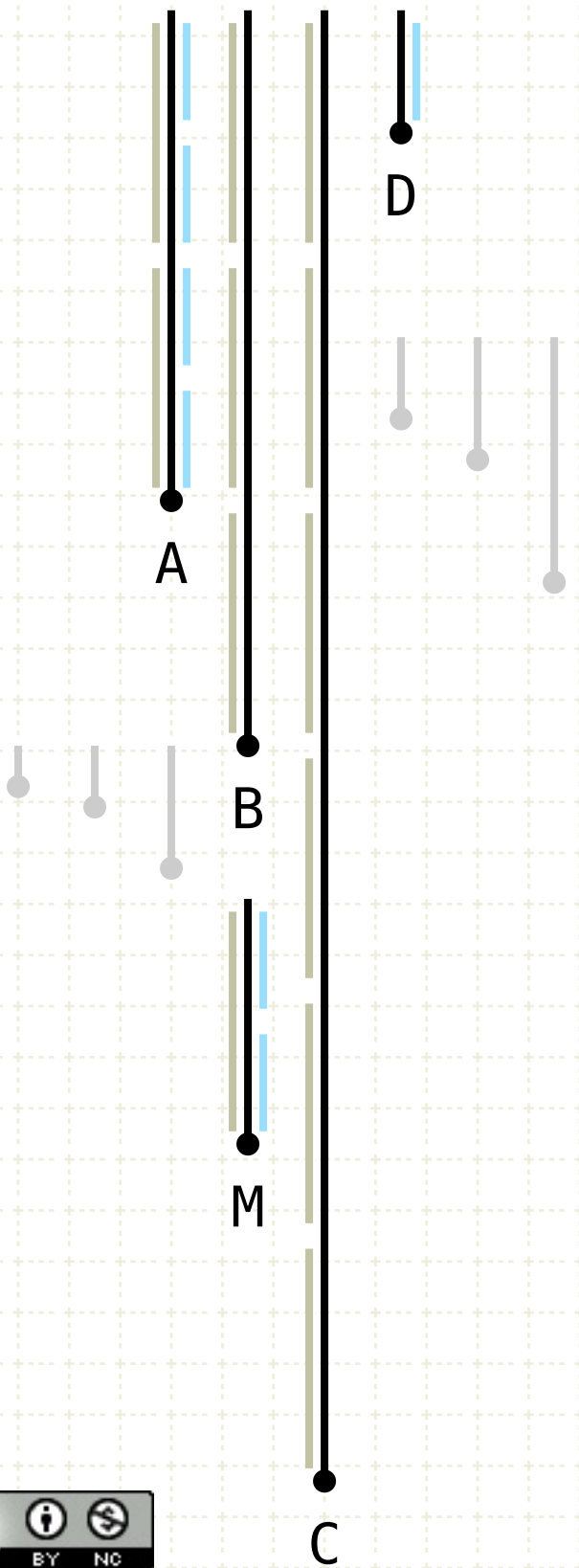
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But E is greater H, therefore M is also greater than D



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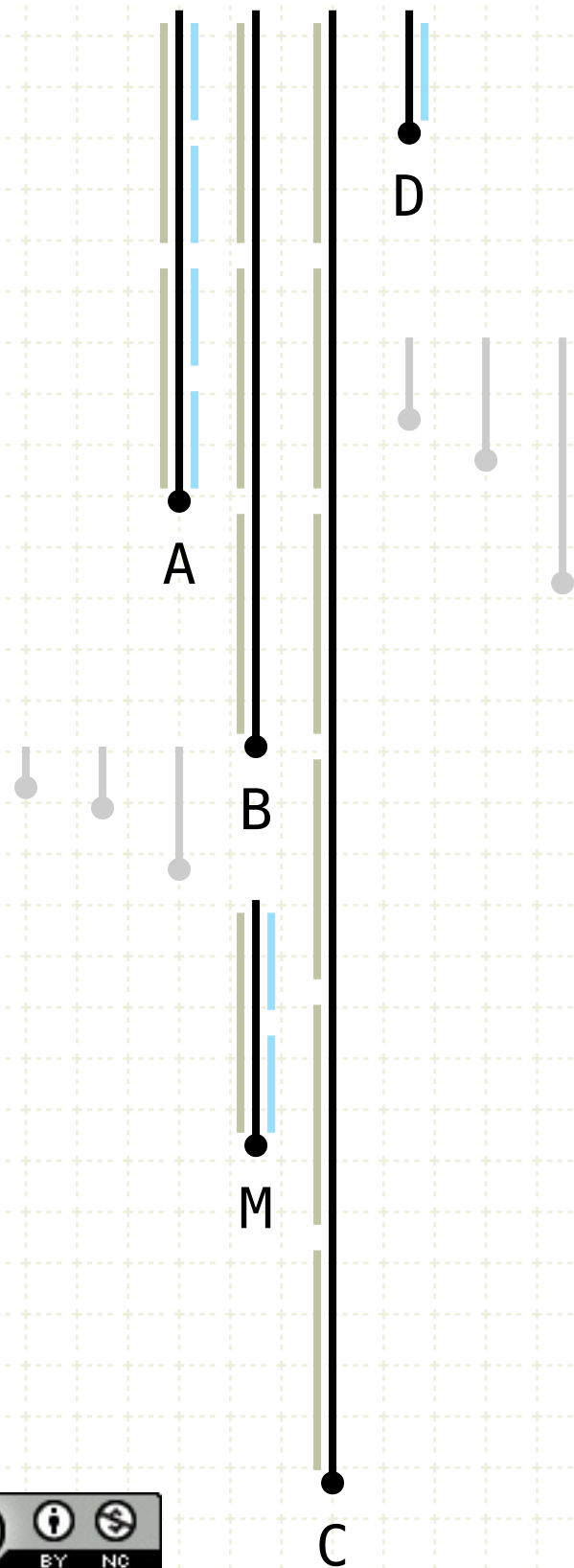
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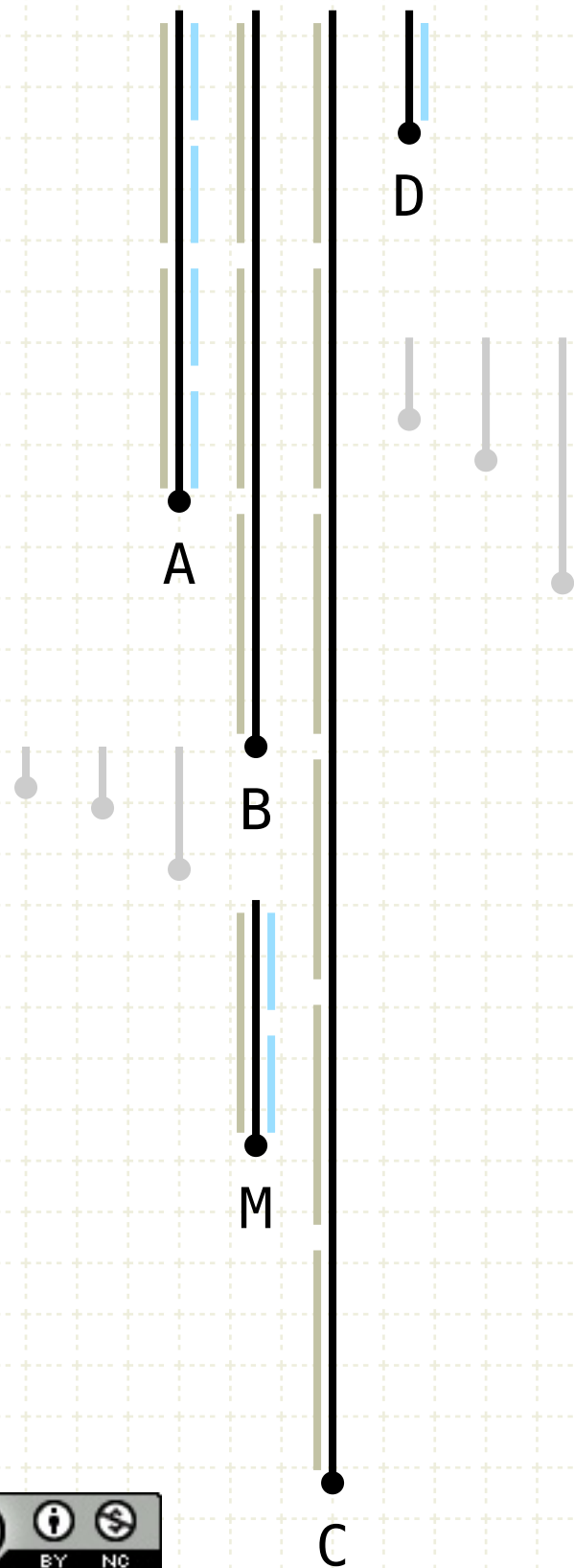
And it measures A,B,C

Which is impossible, because D is the greatest common divisor of A, B, C



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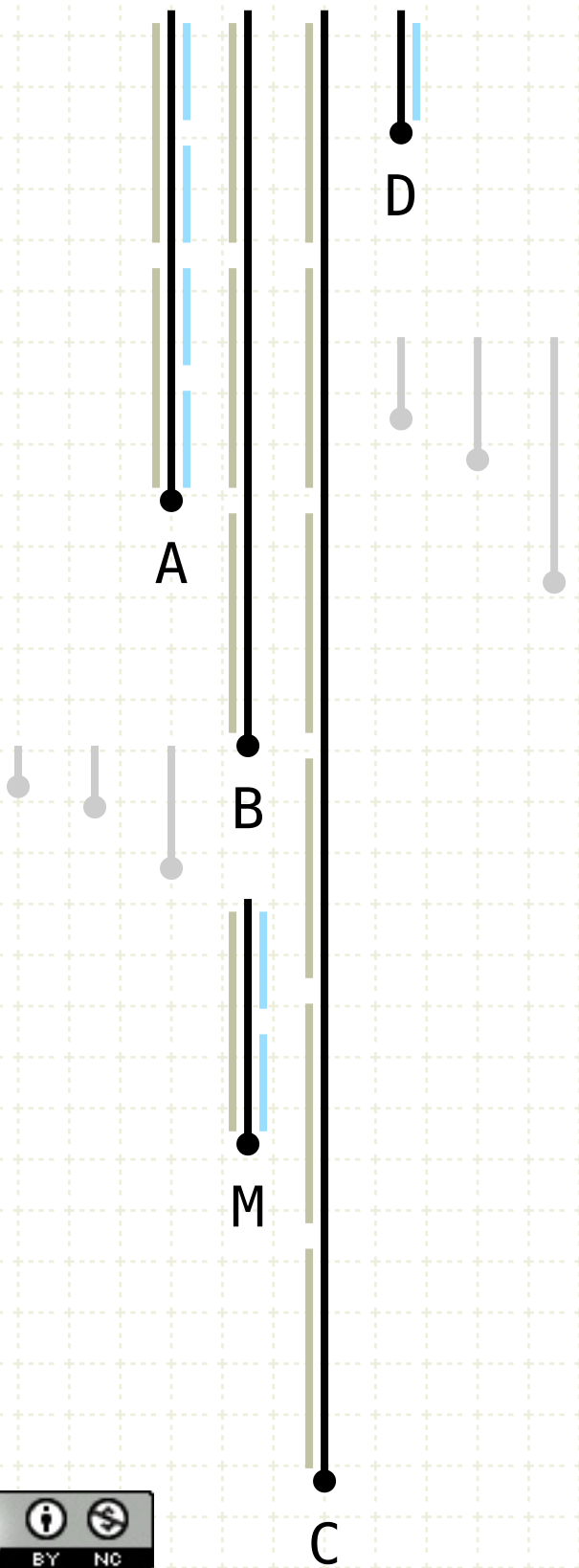
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Therefore, there cannot be any numbers less than E,F,G with the same ratio as A,B,C, which was the original assumption



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Therefore E is to H as M is to D (VII·19)

But E is greater H, therefore M is also greater than D

And it measures A,B,C

Which is impossible, because D is the greatest common divisor of A,B,C

Therefore, there cannot be any numbers less than E,F,G with the same ratio as A,B,C, which was the original assumption

Thus, E,F,G are the smallest numbers with the ratio A,B,C



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