

Euclid's Elements

Book VII

Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)



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1	Determine if two numbers are relatively prime	10	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$	21	If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B
2	Find the greatest common divisor for two numbers	11	If $A:B = C:D$, then $(A-C):(B-D) = A:B$	22	If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime
3	Find the largest common divisor for three numbers	12	If $A:B = C:D$, then $(A+C):(B+C) = A:B$	23	If A,B are relatively prime and if $A = n \cdot C$, then B,C are relatively prime
4	Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B	13	If $A:B = C:D$, then $A:C = B:D$	24	If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C
5	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, then $(B+D) = (1/q) \cdot (A+C)$	14	If $A:B = D:E$ and $B:C = E:F$, then $A:C = D:F$	25	If A,B are relatively prime then A^2, B are relatively prime
6	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, then $(B+D) = (p/q) \cdot (A+C)$	15	If $B = i \cdot 1$ and $E = i \cdot D$, and if $D = j \cdot 1$ then $E = j \cdot B$	26	If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$
7	If $B = A/q$ and $D = C/q$, $B > D$, then $(B-D) = (A-C)/q$	16	$A \times B = B \times A$	27	If A,B are relatively prime, then A^2, B^2 are relatively prime, and A^3, B^3 are relatively prime, and so on
8	If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, $B > D$, then $(B-D) = (p/q) \cdot (A-C)$	17	If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$		
9	If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$	18	If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$		
		19	If $A:B = C:D$ then $A \times D = B \times C$ If $A \times D = B \times C$ then $A:B = C:D$		
		20	Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$		



Table of Contents, Chapter 7

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| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime | 37 | If $A = p \cdot B$, then $A = q \cdot C$ where $C = p \cdot 1$ |
| 29 | If A is prime, and $B \neq n \cdot A$, then A,B are relatively prime | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$ |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a$, $1/b$, $1/c$ |
| 31 | If $A = B \times C$, then $A = j \cdot D$ where D is prime | | |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime | | |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C | | |
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Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



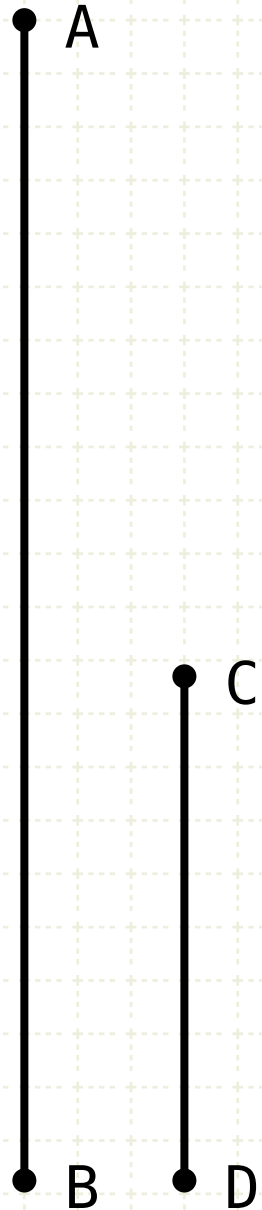
Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another

In other words

Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime



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Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

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Example

$$AB = 145, \quad CD = 63$$



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another

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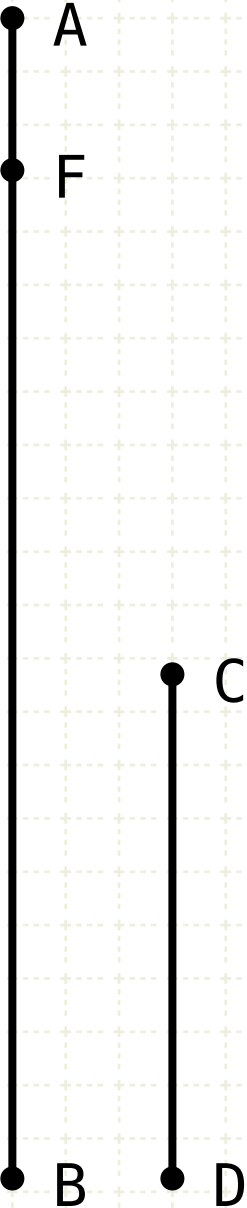
Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

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Example

Let CD measure BF with the remainder AF less than CD,

$AB = 145, CD = 63$



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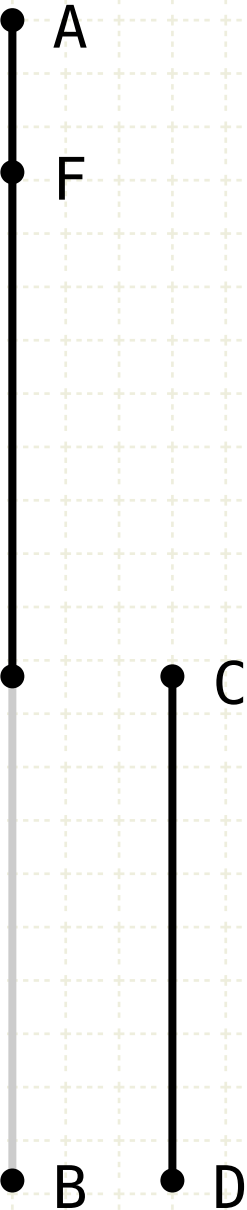
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Example

Let CD measure BF with the remainder AF less than CD,

$$\begin{aligned} AB &= 145, \quad CD = 63 \\ 145 - 63 &= 82 \end{aligned}$$



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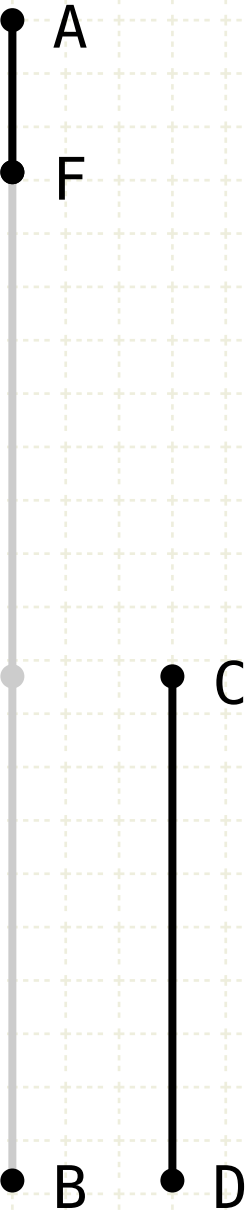
Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

$$\begin{aligned} AB &= 145, \quad CD = 63 \\ 145 - 63 &= 82 \\ 82 - 63 &= 19 \end{aligned}$$



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Example

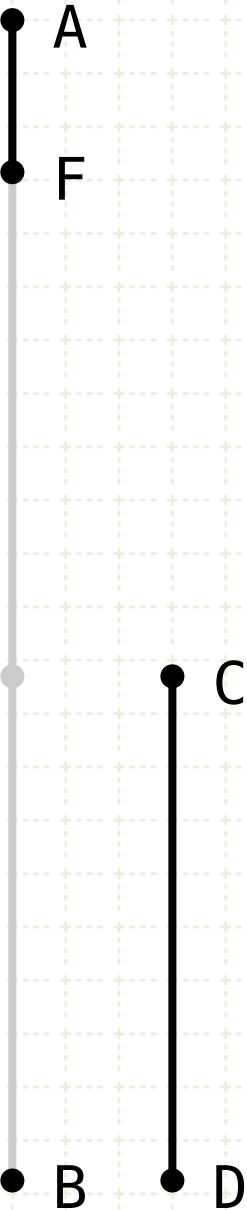
Let CD measure BF with the remainder AF less than CD,

$$AB = 145, CD = 63$$

$$145 - 63 = 82$$

$$82 - 63 = 19$$

$$AF=19$$



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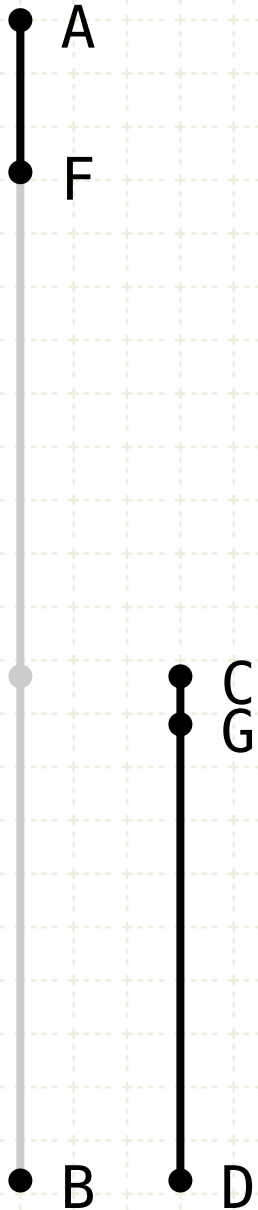
Let AF measure DG, with CG less than AF

$$AB = 145, CD = 63$$

$$145 - 63 = 82$$

$$82 - 63 = 19$$

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Example

Let CD measure BF with the remainder AF less than CD,

Let AF measure DG, with CG less than AF

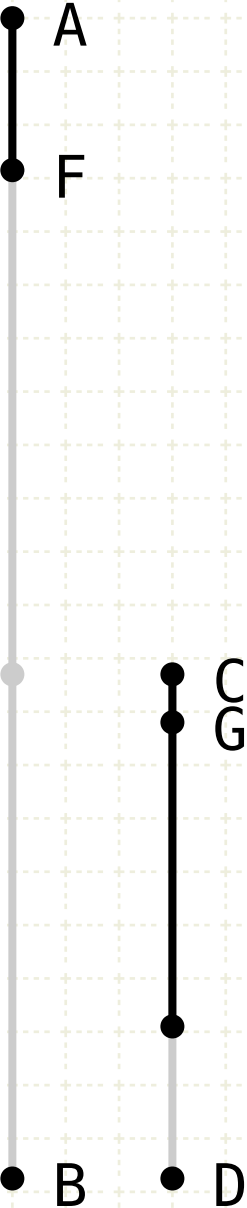
$$AB = 145, CD = 63$$

$$145 - 63 = 82$$

$$82 - 63 = 19$$

$$AF = 19$$

$$63 - 19 = 44$$



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another

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Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

Let AF measure DG, with CG less than AF

$$AB = 145, CD = 63$$

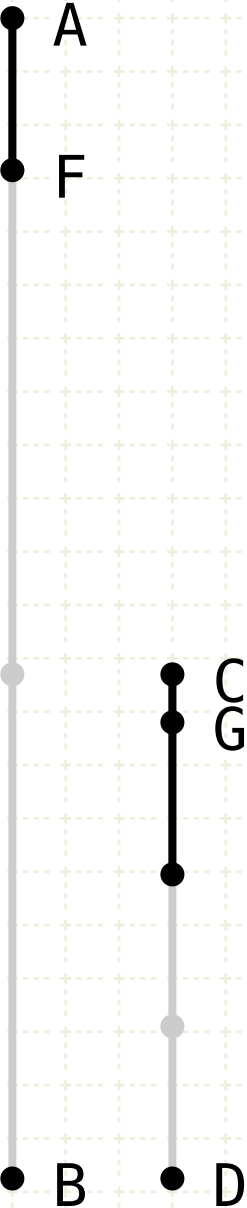
$$145 - 63 = 82$$

$$82 - 63 = 19$$

$$AF = 19$$

$$63 - 19 = 44$$

$$44 - 19 = 25$$



Proposition 1 of Book VII

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Example

Let CD measure BF with the remainder AF less than CD,

Let AF measure DG, with CG less than AF

$AB = 145, CD = 63$

$145 - 63 = 82$

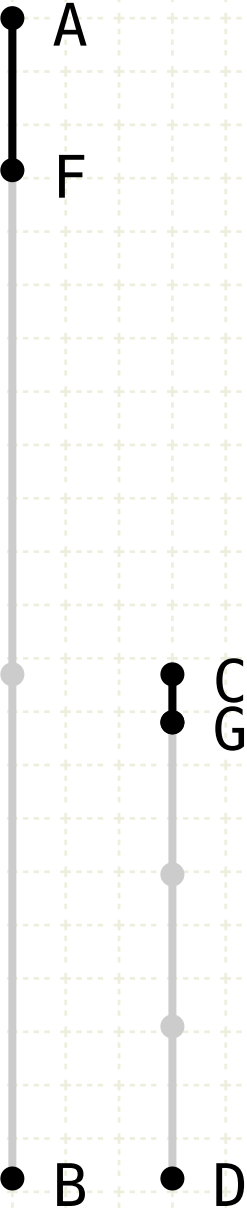
$82 - 63 = 19$

$AF = 19$

$63 - 19 = 44$

$44 - 19 = 25$

$25 - 19 = 6$



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another

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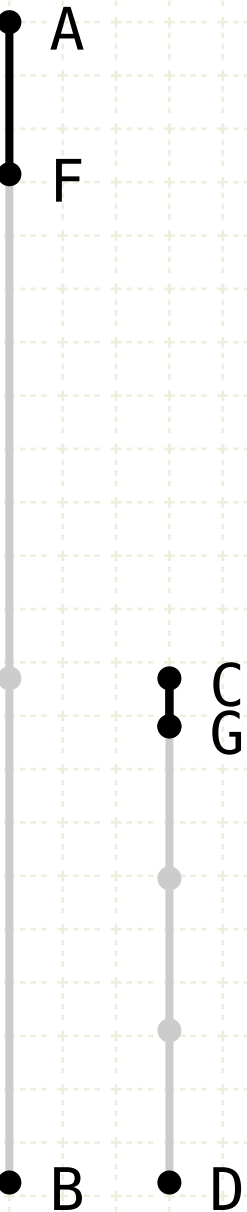
Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

Let AF measure DG, with CG less than AF



$$AB = 145, CD = 63$$

$$145 - 63 = 82$$

$$82 - 63 = 19$$

$$AF = 19$$

$$63 - 19 = 44$$

$$44 - 19 = 25$$

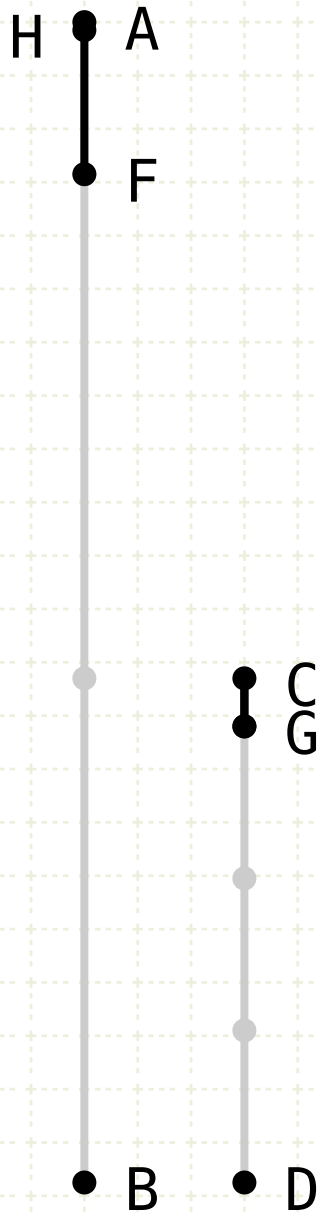
$$25 - 19 = 6$$

$$CG = 6$$



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$AB = 145, CD = 63$$

$$145 - 63 = 82$$

$$82 - 63 = 19$$

$$AF=19$$

$$63 - 19 = 44$$

$$44 - 19 = 25$$

$$25 - 19 = 6$$

$$CG=6$$

In other words

Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

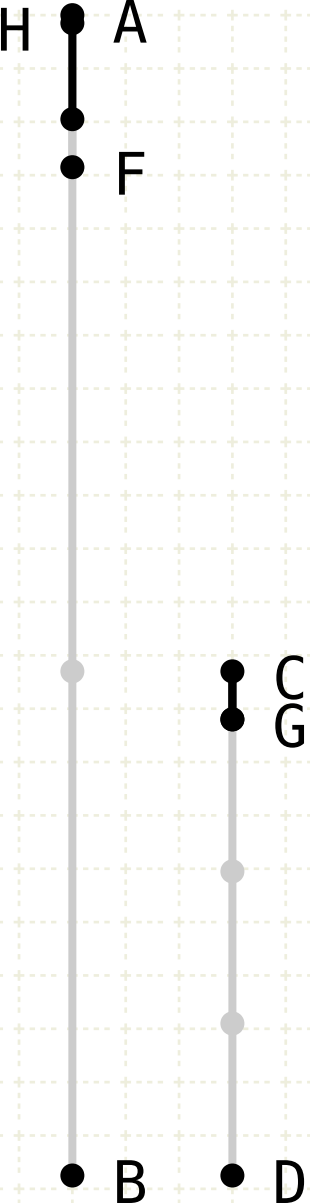
Let AF measure DG, with CG less than AF

And let CG measure FH ...



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$AB = 145, CD = 63$$

$$145 - 63 = 82$$

$$82 - 63 = 19$$

$$AF=19$$

$$63 - 19 = 44$$

$$44 - 19 = 25$$

$$25 - 19 = 6$$

$$CG=6$$

$$19 - 6 = 13$$

In other words

Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

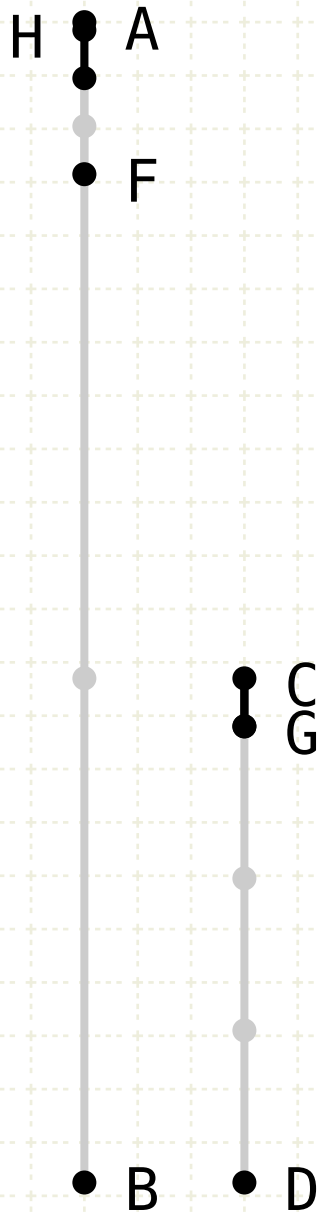
Let AF measure DG, with CG less than AF

And let CG measure FH ...



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$AB = 145, CD = 63$

$145 - 63 = 82$

$82 - 63 = 19$

$AF=19$

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$44 - 19 = 25$

$25 - 19 = 6$

$CG=6$

$19 - 6 = 13$

$13 - 6 = 7$

In other words

Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

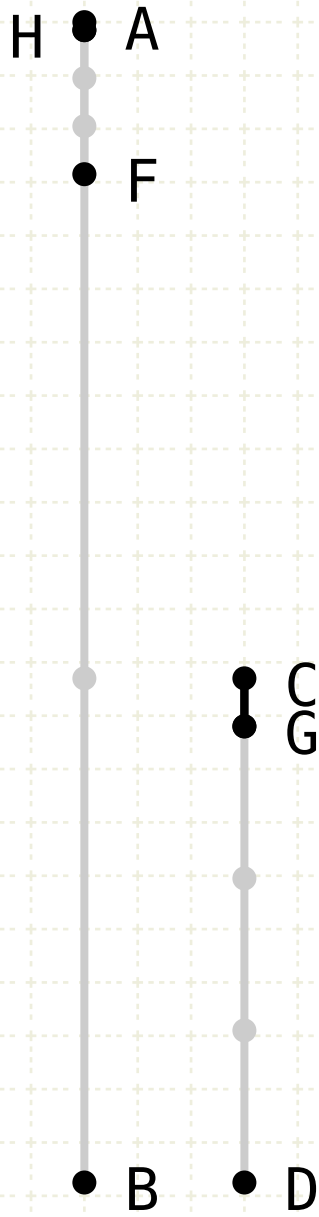
Let AF measure DG, with CG less than AF

And let CG measure FH ...



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$AB = 145, CD = 63$

$145 - 63 = 82$
 $82 - 63 = 19$

$AF = 19$

$63 - 19 = 44$
 $44 - 19 = 25$
 $25 - 19 = 6$

$CG = 6$

$19 - 6 = 13$
 $13 - 6 = 7$
 $7 - 6 = 1$

In other words

Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

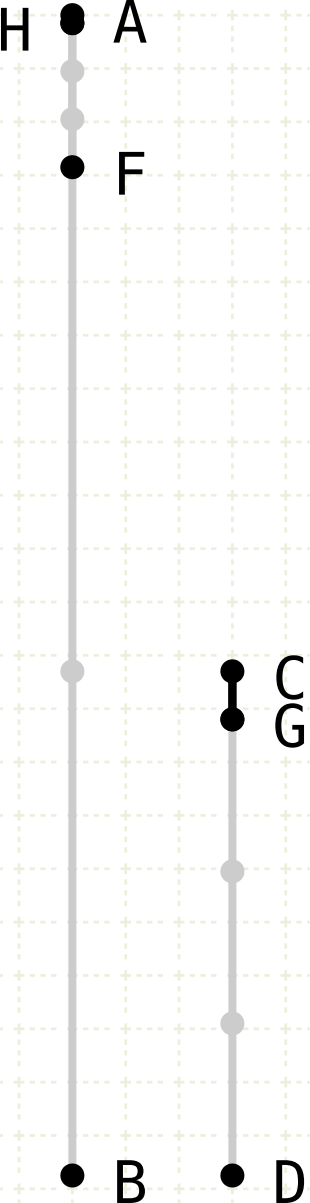
Let AF measure DG, with CG less than AF

And let CG measure FH ...



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$AB = 145, CD = 63$$

$$145 - 63 = 82$$

$$82 - 63 = 19$$

$$AF=19$$

$$63 - 19 = 44$$

$$44 - 19 = 25$$

$$25 - 19 = 6$$

$$CG=6$$

$$19 - 6 = 13$$

$$13 - 6 = 7$$

$$7 - 6 = 1$$

$$AH = 1$$

In other words

Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

Let AF measure DG, with CG less than AF

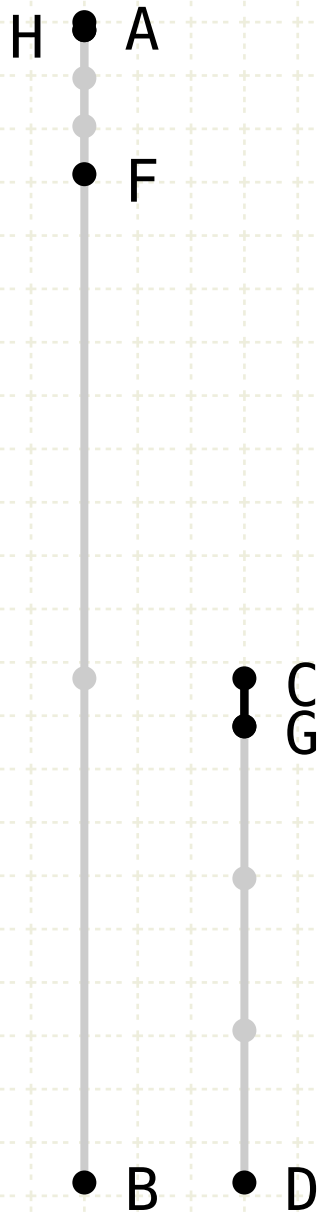
And let CG measure FH ...

... leaving a single unit as the remainder



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$AB = 145, CD = 63$

$145 - 63 = 82$
 $82 - 63 = 19$

$AF=19$

$63 - 19 = 44$
 $44 - 19 = 25$
 $25 - 19 = 6$

$CG=6$

$19 - 6 = 13$
 $13 - 6 = 7$
 $7 - 6 = 1$

$AH = 1$

$\gcd(145, 63) = 1$

In other words

Start with two unequal natural numbers, continuously subtract the smaller from the larger as long as one number is not a multiple of the other

If the resulting number is the number one, then the two numbers are relatively prime

Example

Let CD measure BF with the remainder AF less than CD,

Let AF measure DG, with CG less than AF

And let CG measure FH ...

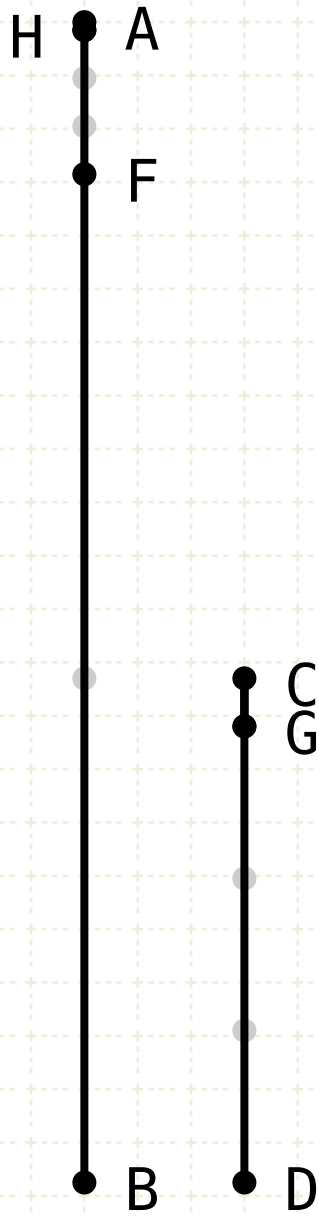
... leaving a single unit as the remainder

145 and 63 are prime to one another



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$BF = CD + CD + \dots = a \cdot CD, \quad AF < CD$
 $DG = AF + AF + \dots = b \cdot AF, \quad CG < AF$
 $FH = CG + CG + \dots = c \cdot CG$
 $AH = 1$

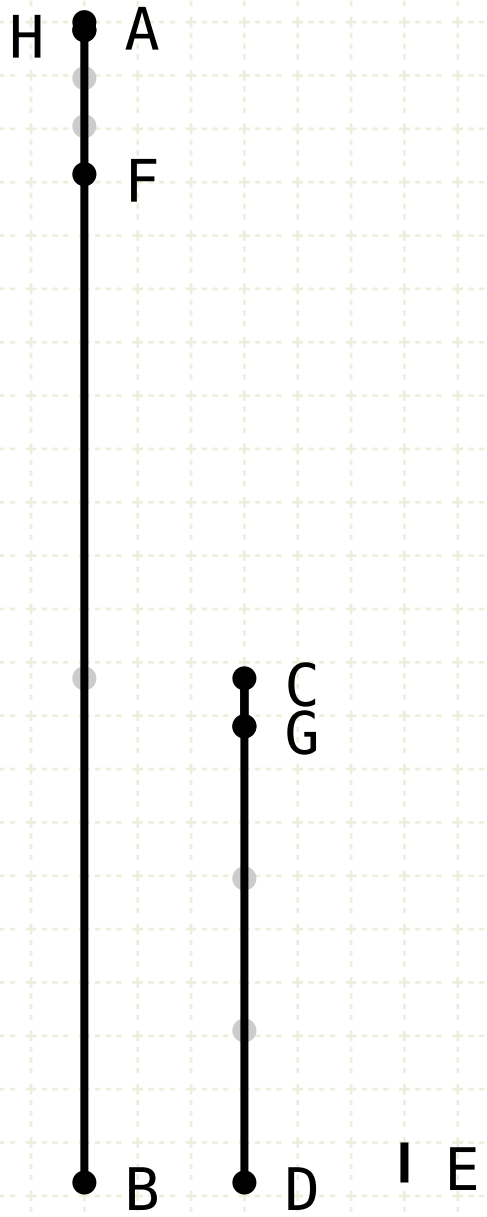
Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$\begin{aligned} BF &= CD + CD + \dots = a \cdot CD, \quad AF < CD \\ DG &= AF + AF + \dots = b \cdot AF, \quad CG < AF \\ FH &= CG + CG + \dots = c \cdot CG \\ AH &= 1 \end{aligned}$$

Let

$$\begin{aligned} AB &= p \cdot E \\ CD &= q \cdot E \\ E &\neq 1 \end{aligned}$$

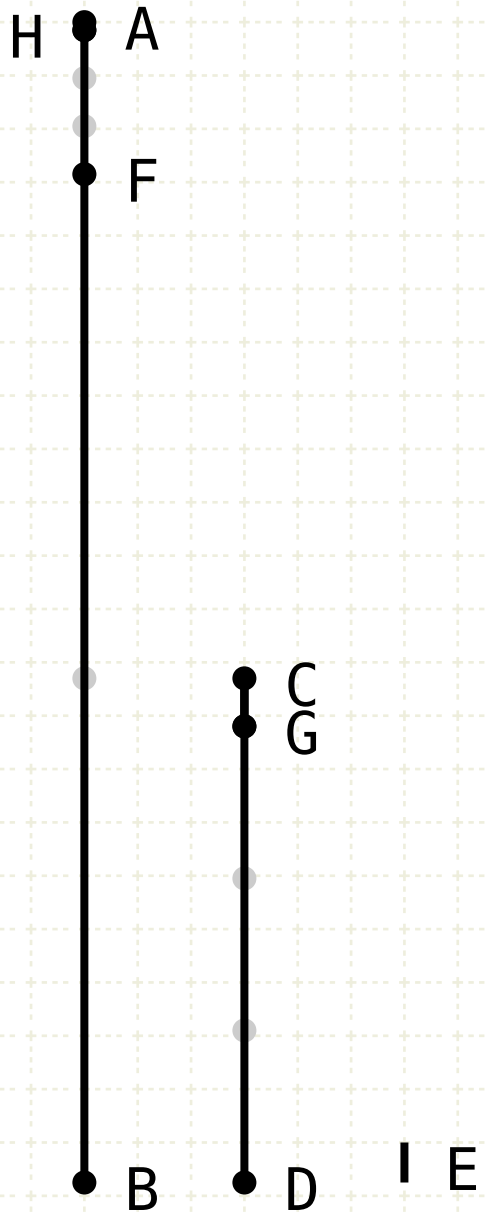
Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one
Assume that AB,CD are not relatively prime
Therefore there is some natural number 'E' which measures both AB and CD



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$\begin{aligned} BF &= CD + CD + \dots = a \cdot CD, \quad AF < CD \\ DG &= AF + AF + \dots = b \cdot AF, \quad CG < AF \\ FH &= CG + CG + \dots = c \cdot CG \\ AH &= 1 \end{aligned}$$

Let

$$\begin{aligned} AB &= p \cdot E \\ CD &= q \cdot E \\ E &\neq 1 \\ BF &= q \cdot E + q \cdot E + \dots = d \cdot E \end{aligned}$$

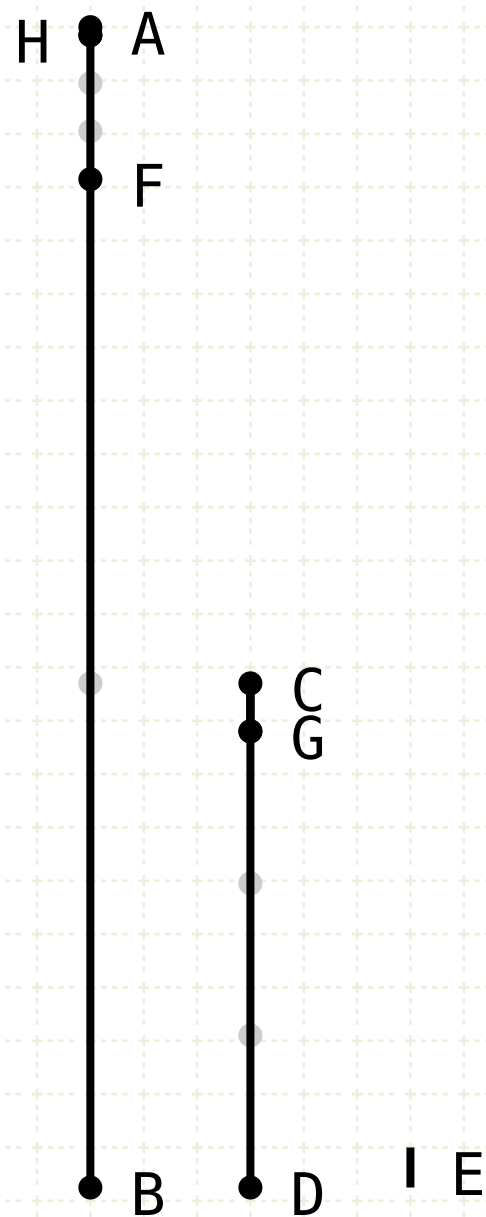
Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one
Assume that AB,CD are not relatively prime
Therefore there is some natural number 'E' which measures both AB and CD
Since E measures CD, and CD measures BF, E also measures BF



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$BF = CD + CD + \dots = a \cdot CD, \quad AF < CD$$

$$DG = AF + AF + \dots = b \cdot AF, \quad CG < AF$$

$$FH = CG + CG + \dots = c \cdot CG$$

$$AH = 1$$

Let

$$AB = p \cdot E$$

$$CD = q \cdot E$$

$$E \neq 1$$

$$BF = q \cdot E + q \cdot E + \dots = d \cdot E$$

$$AF = AB - BF = p \cdot E - d \cdot E = e \cdot E$$

Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,

And AF measure DG, with CG less than AF

And let CG measure FH, leaving AH equal to one

Assume that AB,CD are not relatively prime

Therefore there is some natural number 'E' which measures both AB and CD

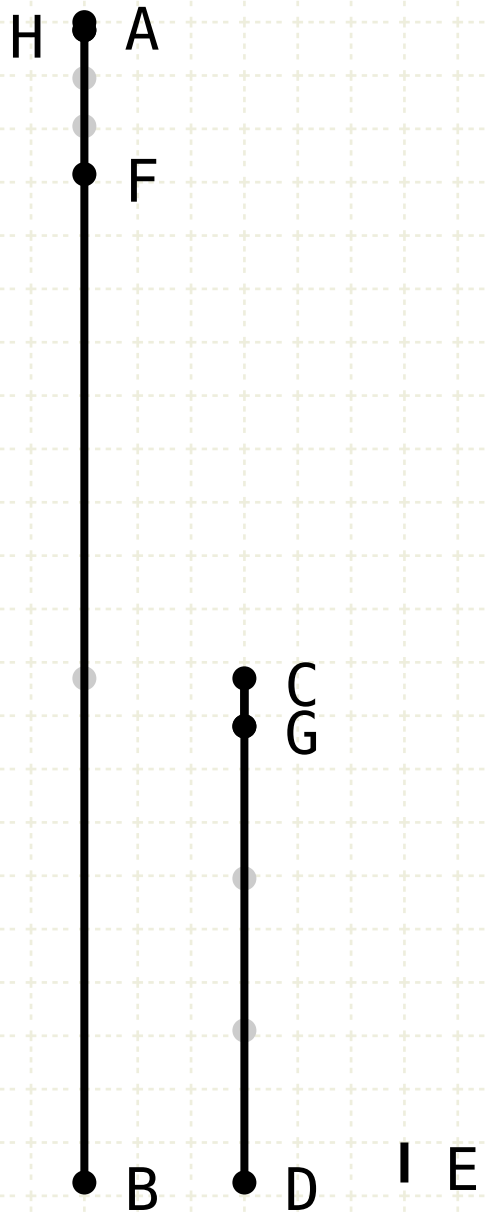
Since E measures CD, and CD measures BF, E also measures BF

But E also measures AB, therefore it also measures AF



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$\begin{aligned} BF &= CD + CD + \dots = a \cdot CD, \quad AF < CD \\ DG &= AF + AF + \dots = b \cdot AF, \quad CG < AF \\ FH &= CG + CG + \dots = c \cdot CG \\ AH &= 1 \\ \text{Let} \\ AB &= p \cdot E \\ CD &= q \cdot E \\ E &\neq 1 \\ BF &= q \cdot E + q \cdot E + \dots = d \cdot E \\ AF &= AB - BF = p \cdot E - d \cdot E = e \cdot E \\ DG &= e \cdot E + e \cdot E + \dots = f \cdot E \end{aligned}$$

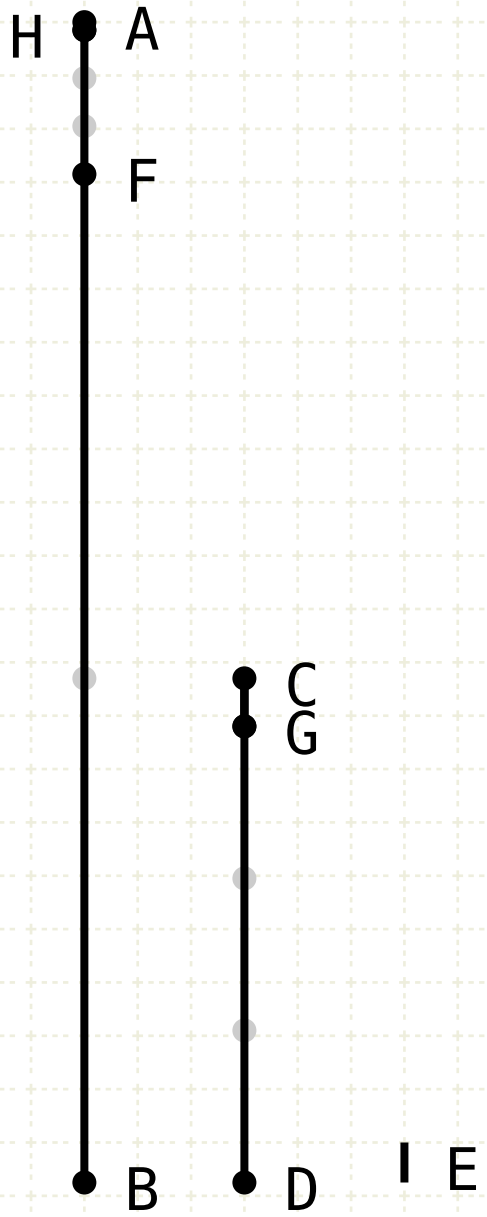
Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one
Assume that AB,CD are not relatively prime
Therefore there is some natural number 'E' which measures both AB and CD
Since E measures CD, and CD measures BF, E also measures BF
But E also measures AB, therefore it also measures AF
But AF measures DG, therefore E also measures DG



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$\begin{aligned} BF &= CD + CD + \dots = a \cdot CD, \quad AF < CD \\ DG &= AF + AF + \dots = b \cdot AF, \quad CG < AF \\ FH &= CG + CG + \dots = c \cdot CG \\ AH &= 1 \end{aligned}$$

Let

$$\begin{aligned} AB &= p \cdot E \\ CD &= q \cdot E \\ E &\neq 1 \\ BF &= q \cdot E + q \cdot E + \dots = d \cdot E \\ AF &= AB - BF = p \cdot E - d \cdot E = e \cdot E \\ DG &= e \cdot E + e \cdot E + \dots = f \cdot E \\ CG &= CD - DG = q \cdot E - f \cdot E = g \cdot E \end{aligned}$$

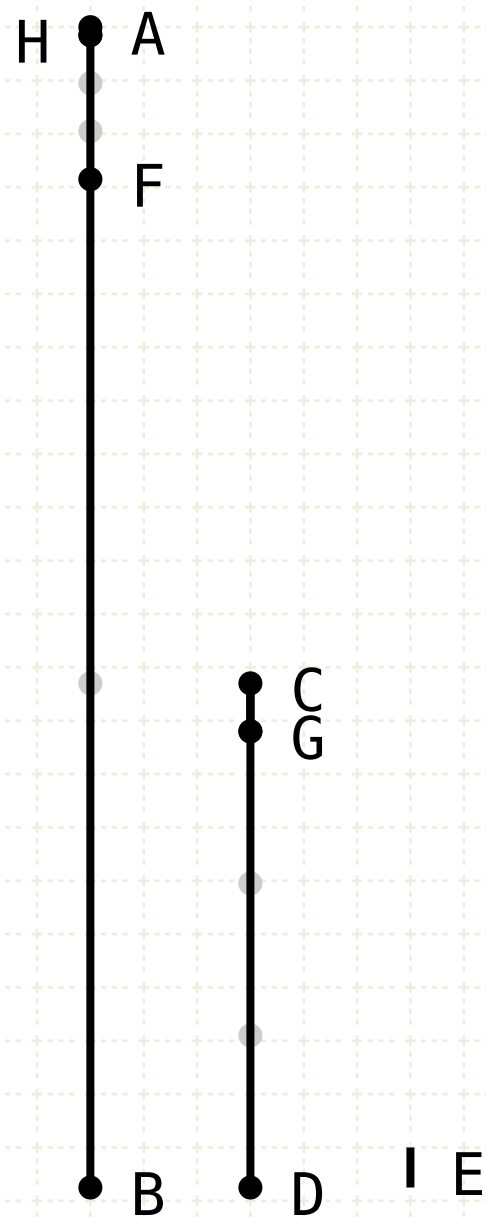
Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one
Assume that AB,CD are not relatively prime
Therefore there is some natural number 'E' which measures both AB and CD
Since E measures CD, and CD measures BF, E also measures BF
But E also measures AB, therefore it also measures AF
But AF measures DG, therefore E also measures DG
But it also measures the whole CD, therefore it also measures CG



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$BF = CD + CD + \dots = a \cdot CD, \quad AF < CD$$

$$DG = AF + AF + \dots = b \cdot AF, \quad CG < AF$$

$$FH = CG + CG + \dots = c \cdot CG$$

$$AH = 1$$

Let

$$AB = p \cdot E$$

$$CD = q \cdot E$$

$$E \neq 1$$

$$BF = q \cdot E + q \cdot E + \dots = d \cdot E$$

$$AF = AB - BF = p \cdot E - d \cdot E = e \cdot E$$

$$DG = e \cdot E + e \cdot E + \dots = f \cdot E$$

$$CG = CD - DG = q \cdot E - f \cdot E = g \cdot E$$

$$FH = g \cdot E + g \cdot E + \dots = h \cdot E$$

Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,

And AF measure DG, with CG less than AF

And let CG measure FH, leaving AH equal to one

Assume that AB,CD are not relatively prime

Therefore there is some natural number 'E' which measures both AB and CD

Since E measures CD, and CD measures BF, E also measures BF

But E also measures AB, therefore it also measures AF

But AF measures DG, therefore E also measures DG

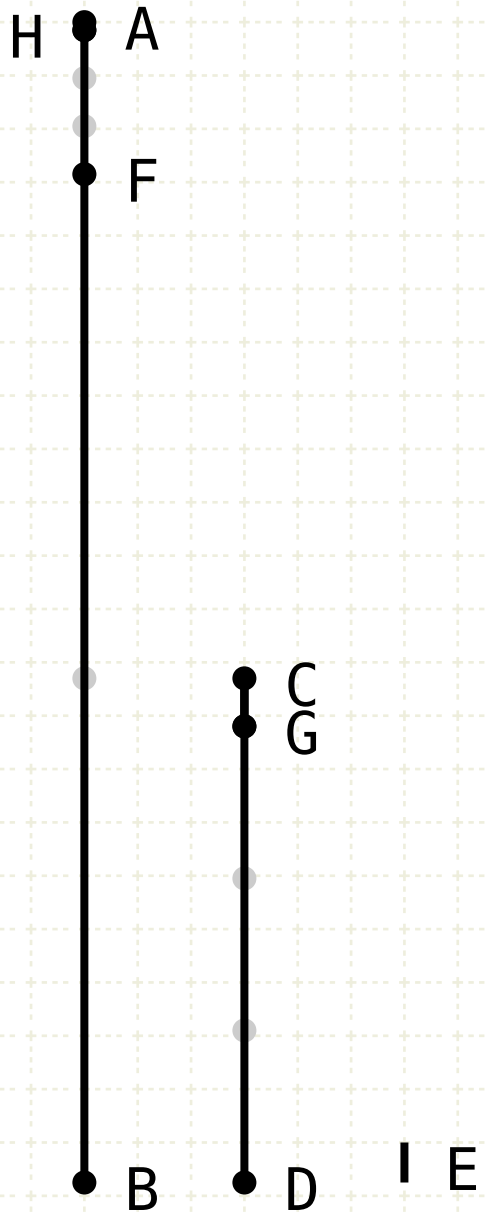
But it also measures the whole CD, therefore it also measures CG

But CG measures FH, therefore E also measures FH



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$\begin{aligned} BF &= CD + CD + \dots = a \cdot CD, \quad AF < CD \\ DG &= AF + AF + \dots = b \cdot AF, \quad CG < AF \\ FH &= CG + CG + \dots = c \cdot CG \\ AH &= 1 \\ \text{Let} \\ AB &= p \cdot E \\ CD &= q \cdot E \\ E &\neq 1 \\ BF &= q \cdot E + q \cdot E + \dots = d \cdot E \\ AF &= AB - BF = p \cdot E - d \cdot E = e \cdot E \\ DG &= e \cdot E + e \cdot E + \dots = f \cdot E \\ CG &= CD - DG = q \cdot E - f \cdot E = g \cdot E \\ FH &= g \cdot E + g \cdot E + \dots = h \cdot E \\ AH &= AF - FH = e \cdot E - h \cdot E = m \cdot E \end{aligned}$$

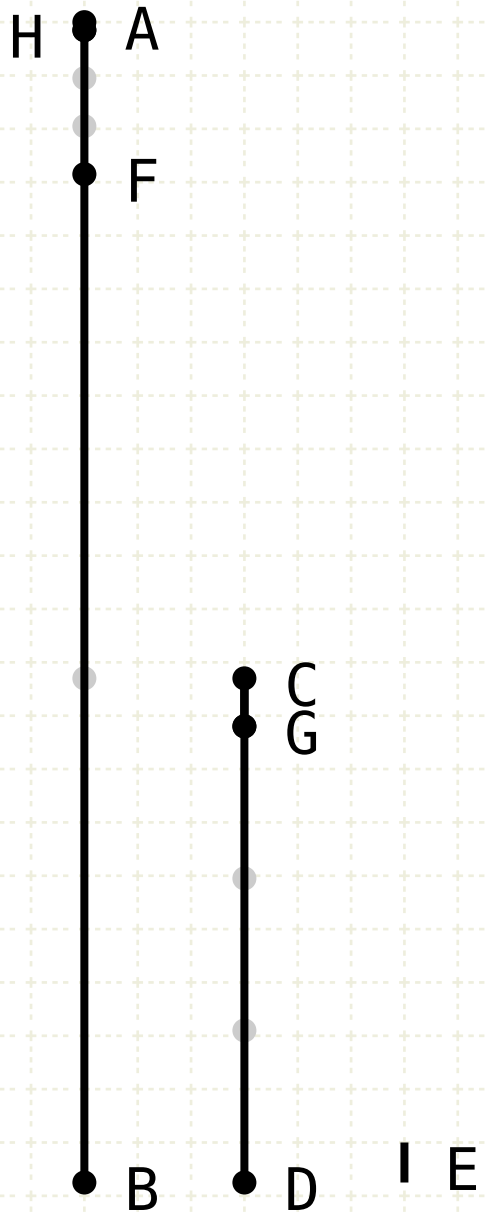
Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one
Assume that AB,CD are not relatively prime
Therefore there is some natural number 'E' which measures both AB and CD
Since E measures CD, and CD measures BF, E also measures BF
But E also measures AB, therefore it also measures AF
But AF measures DG, therefore E also measures DG
But it also measures the whole CD, therefore it also measures CG
But CG measures FH, therefore E also measures FH
But E also measures the whole of AF, therefore it will also measure the remainder AH



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$\begin{aligned} BF &= CD + CD + \dots = a \cdot CD, \quad AF < CD \\ DG &= AF + AF + \dots = b \cdot AF, \quad CG < AF \\ FH &= CG + CG + \dots = c \cdot CG \end{aligned}$$

$$AH = 1$$

Let

$$\begin{aligned} AB &= p \cdot E \\ CD &= q \cdot E \end{aligned}$$

$$E \neq 1$$

$$\begin{aligned} BF &= q \cdot E + q \cdot E + \dots = d \cdot E \\ AF &= AB - BF = p \cdot E - d \cdot E = e \cdot E \\ DG &= e \cdot E + e \cdot E + \dots = f \cdot E \\ CG &= CD - DG = q \cdot E - f \cdot E = g \cdot E \\ FH &= g \cdot E + g \cdot E + \dots = h \cdot E \end{aligned}$$

$$AH = AF - FH = e \cdot E - h \cdot E = m \cdot E$$

$$AH = m \cdot E = 1 \times$$

Proof by Contradiction

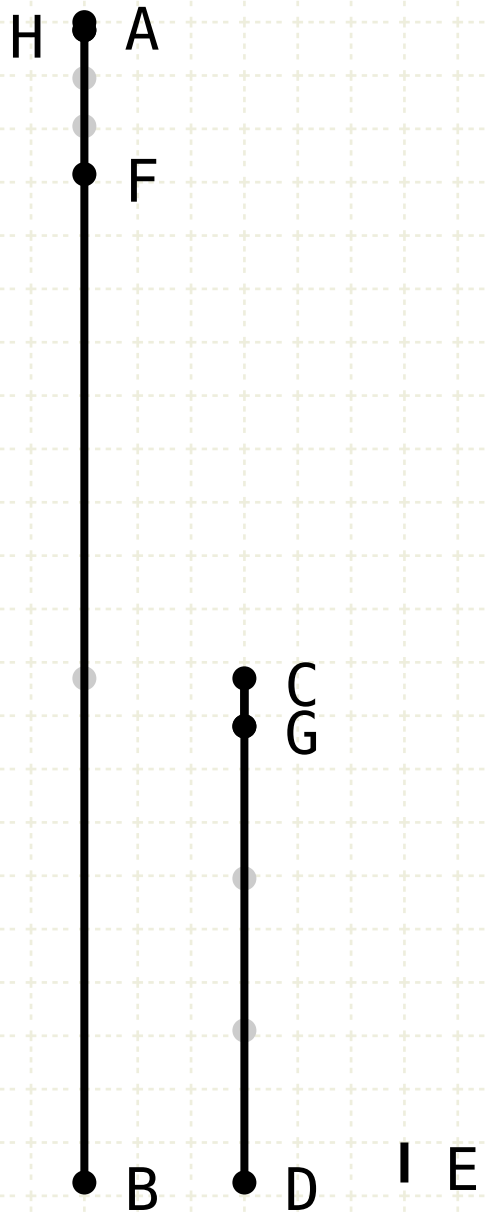
Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one
Assume that AB,CD are not relatively prime
Therefore there is some natural number 'E' which measures both AB and CD
Since E measures CD, and CD measures BF, E also measures BF
But E also measures AB, therefore it also measures AF
But AF measures DG, therefore E also measures DG
But it also measures the whole CD, therefore it also measures CG
But CG measures FH, therefore E also measures FH
But E also measures the whole of AF, therefore it will also measure the remainder AH

But... AH cannot be simultaneously be equal to one and a multiple of a number greater than one
Therefore we have a contradiction, and AB and CD must be relatively prime



Proposition 1 of Book VII

Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another



$$\begin{aligned} BF &= CD + CD + \dots = a \cdot CD, \quad AF < CD \\ DG &= AF + AF + \dots = b \cdot AF, \quad CG < AF \\ FH &= CG + CG + \dots = c \cdot CG \\ AH &= 1 \end{aligned}$$

Let

$$\begin{aligned} AB &= p \cdot E \\ CD &= q \cdot E \\ E &\neq 1 \end{aligned}$$

$$\begin{aligned} BF &= q \cdot E + q \cdot E + \dots = d \cdot E \\ AF &= AB - BF = p \cdot E - d \cdot E = e \cdot E \\ DG &= e \cdot E + e \cdot E + \dots = f \cdot E \\ CG &= CD - DG = q \cdot E - f \cdot E = g \cdot E \\ FH &= g \cdot E + g \cdot E + \dots = h \cdot E \\ AH &= AF - FH = e \cdot E - h \cdot E = m \cdot E \end{aligned}$$

$$AH = m \cdot E = 1 \quad x$$

Proof by Contradiction

Let CD measure BF with the remainder AF less than CD,
And AF measure DG, with CG less than AF
And let CG measure FH, leaving AH equal to one
Assume that AB,CD are not relatively prime
Therefore there is some natural number 'E' which measures both AB and CD
Since E measures CD, and CD measures BF, E also measures BF
But E also measures AB, therefore it also measures AF
But AF measures DG, therefore E also measures DG
But it also measures the whole CD, therefore it also measures CG
But CG measures FH, therefore E also measures FH
But E also measures the whole of AF, therefore it will also measure the remainder AH

But... AH cannot be simultaneously be equal to one and a multiple of a number greater than one
Therefore we have a contradiction, and AB and CD must be relatively prime



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