Euclid's Elements

Book VII

Definitions:

1 A unit is that by virtue of which each of the things that exist is called one

2 A number is a multitude composed of units. (not one)

3 A number is part of a number, the less of the greater, when it measures the greater

11 A prime number is that which is measured by a unit alone.

12 Numbers prime to one another are those which are measured by a unit alone as a common measure

15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)

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Proposition 20 of Book VII

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less

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A:B = C:D

if

A = (p/q)C

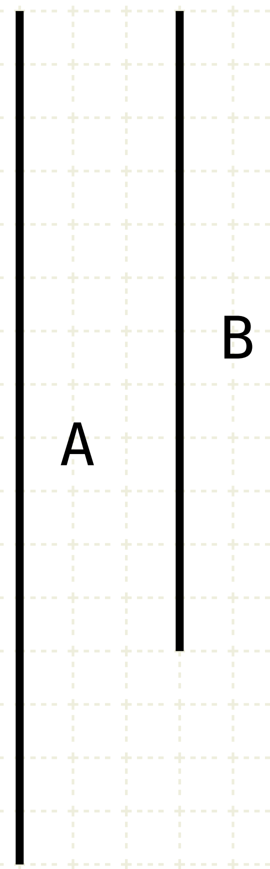
B = (p/q)D

Definition 20

Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth

Proposition 20 of Book VII

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



In other words

Given a ratio A to B

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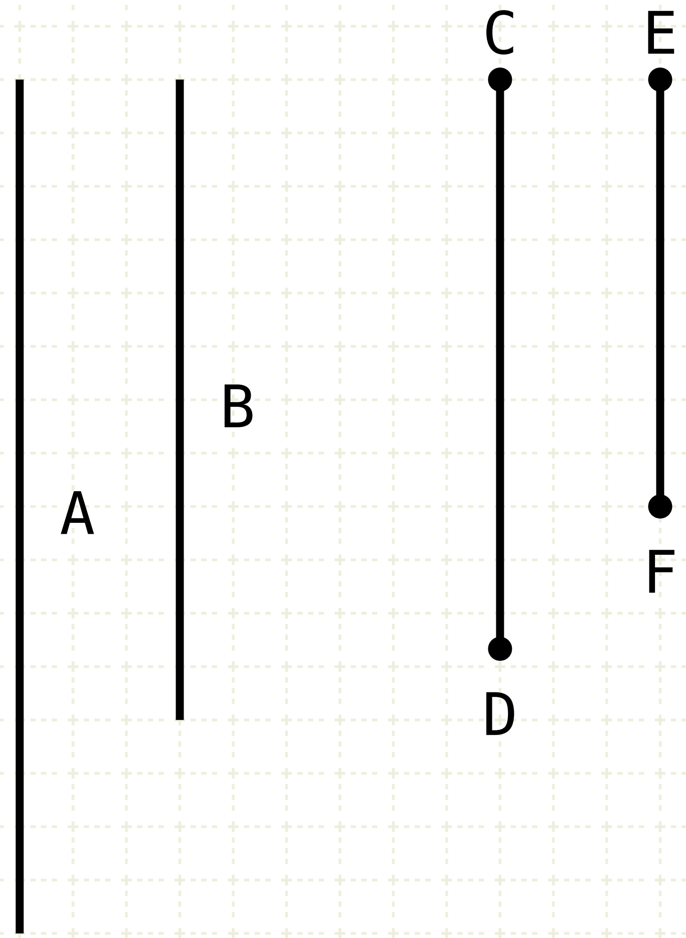

A:B = CD:EF

S = { (x,y) | x∈ℕ, y∈ℕ, x:y=A:B }

(CD,EF)∈S such thats CD≤x, EF≤y, ∀(x,y)∈S

In other words

Given a ratio A to B

Let the numbers CD and EF be equal to the ratio A to B and

… and CD,EF are the smallest numbers whose ratio equals the ratio of A to B

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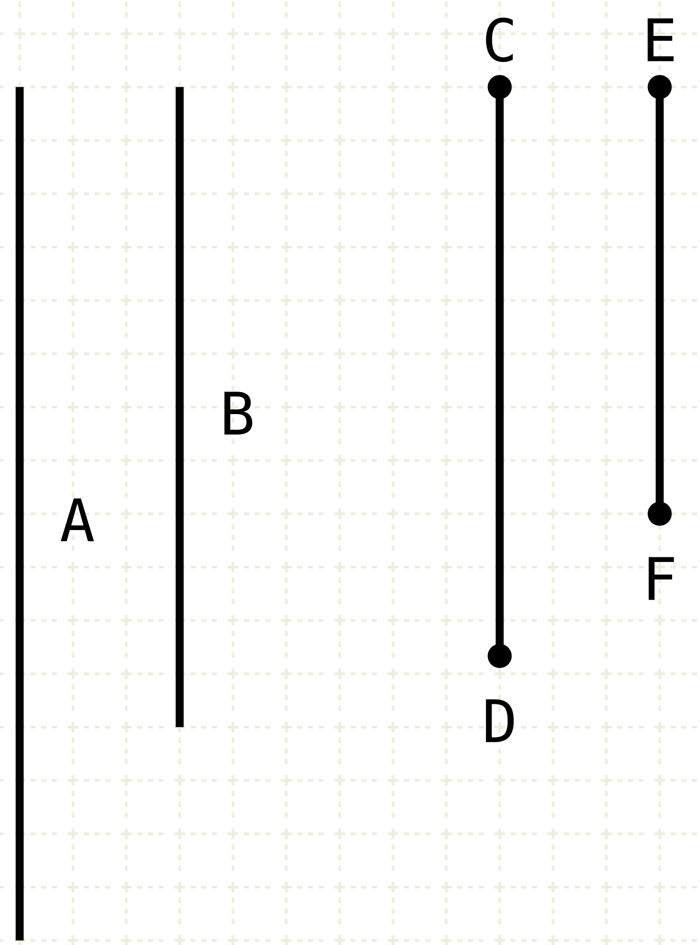

$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$

$(CD,EF) \in S$ such thats $CD \leq x,\ EF \leq y,\ \forall (x,y) \in S$

$\rightarrow A = n \cdot CD,\ B = n \cdot EF$

In other words

Given a ratio A to B

Let the numbers CD and EF be equal to the ratio A to B and

… and CD,EF are the smallest numbers whose ratio equals the ratio of A to B

Then, CD measures A the same number of times that EF measures B

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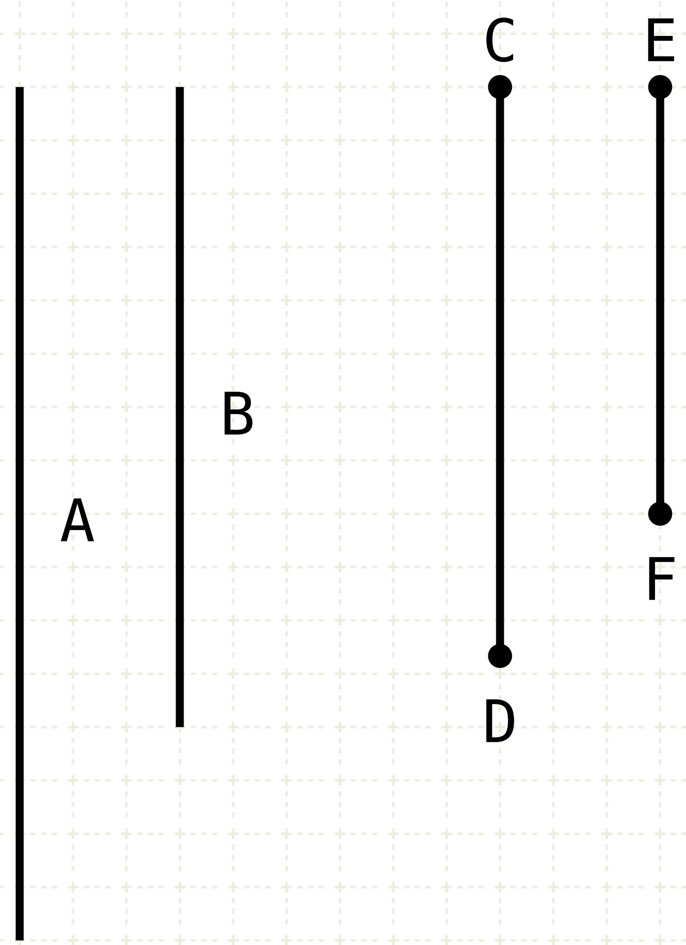

Proof by Contradiction

A:B = CD:EF

S = { (x,y) | x∈ℕ, y∈ℕ, x:y=A:B }

(CD,EF)∈S such that CD≤x, EF≤y, ∀(x,y)∈S

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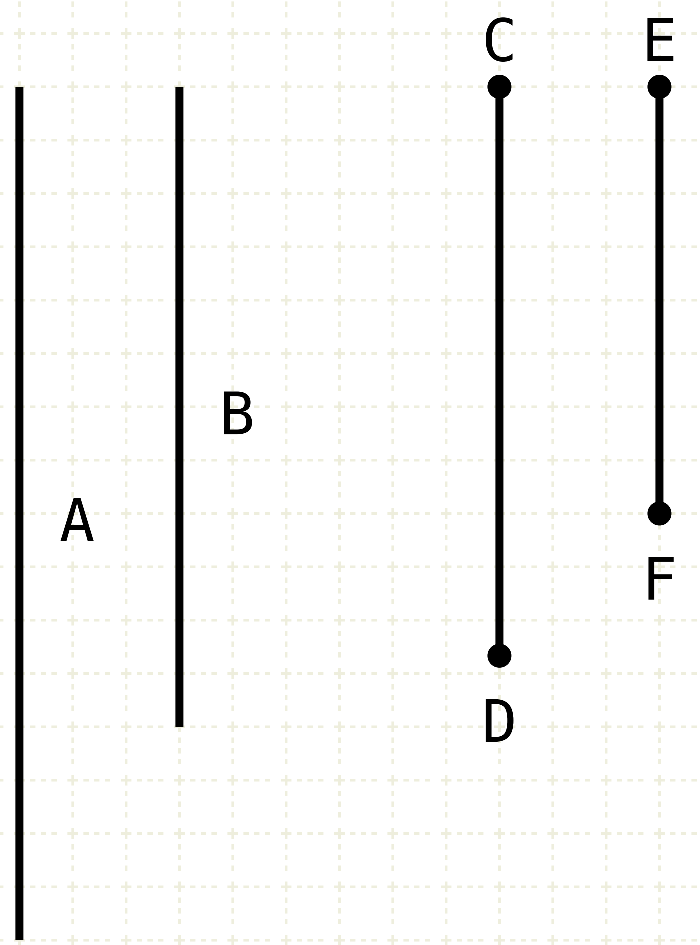

$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$

$(CD,EF) \in S$ such that $CD \le x,\ EF \le y,\ \forall (x,y) \in S$

$A:CD = B:EF$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

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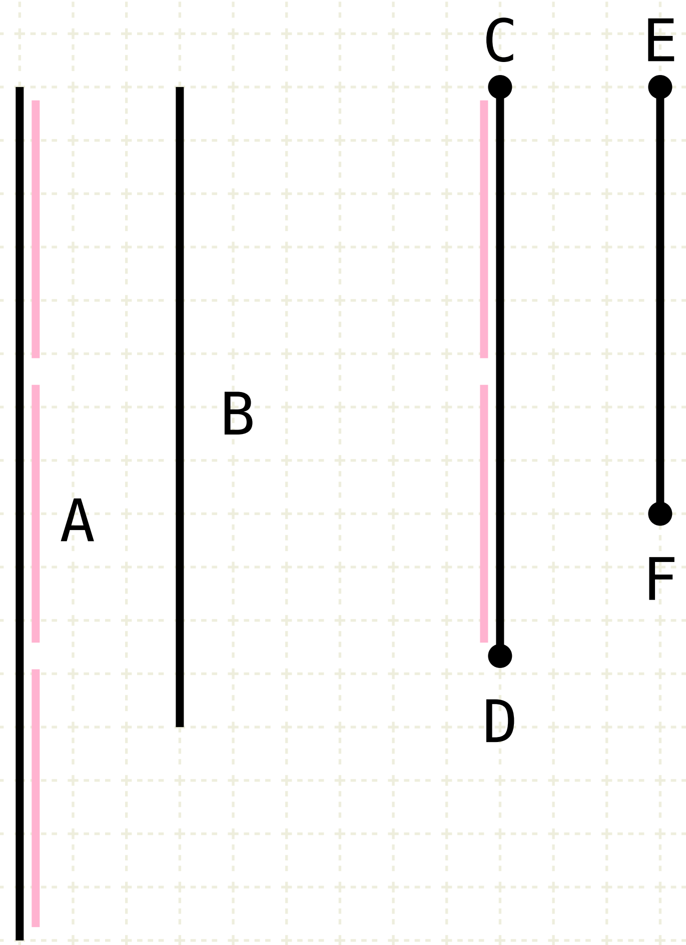

A:B = CD:EF

S = { (x,y) | x∈ℕ, y∈ℕ, x:y=A:B }

(CD,EF)∈S such that CD≤x, EF≤y, ∀(x,y)∈S

A:CD = B:EF

CD = (p/q)A

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

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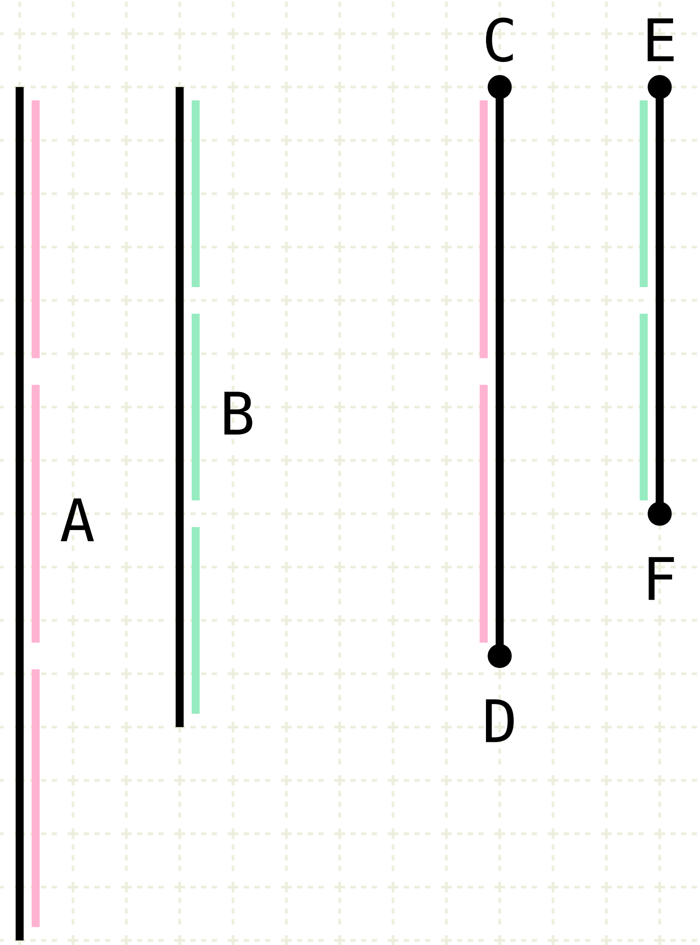

$$A:B = CD:EF$$

$$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$$

$$(CD,EF) \in S \text{ such that } CD \leq x,\ EF \leq y,\ \forall (x,y) \in S$$

$$A:CD = B:EF$$

$$CD = (p/q)A$$

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Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

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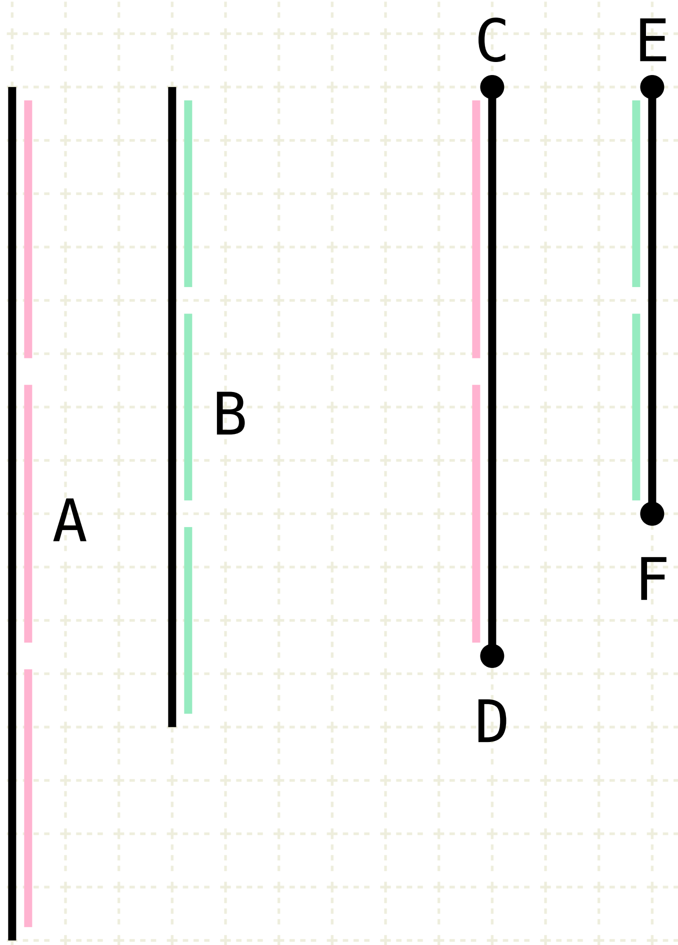

$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$
$(CD,EF) \in S$ such that $CD \le x,\ EF \le y,\ \forall (x,y) \in S$

$A:CD = B:EF$
$CD = (p/q)A$
$EF = (p/q)B$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

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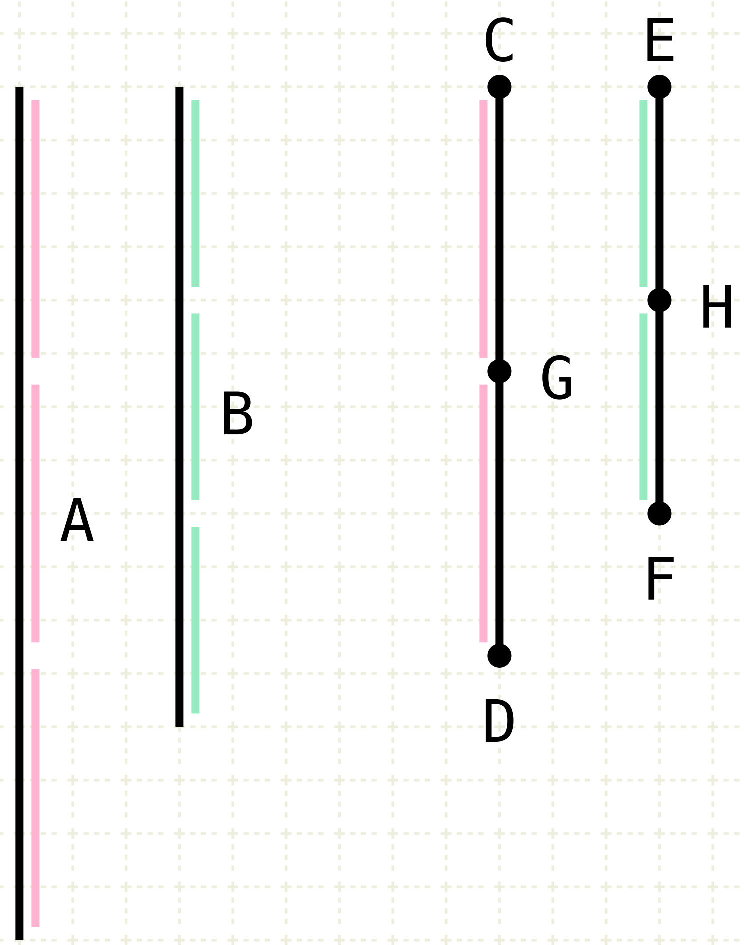
The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$

$(CD,EF) \in S$ such that $CD \le x,\ EF \le y,\ \forall (x,y) \in S$

$A:CD = B:EF$

$CD = (p/q)A$

$EF = (p/q)B$

$CG = GD = (1/q)A$

$EH = HF = (1/q)B$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

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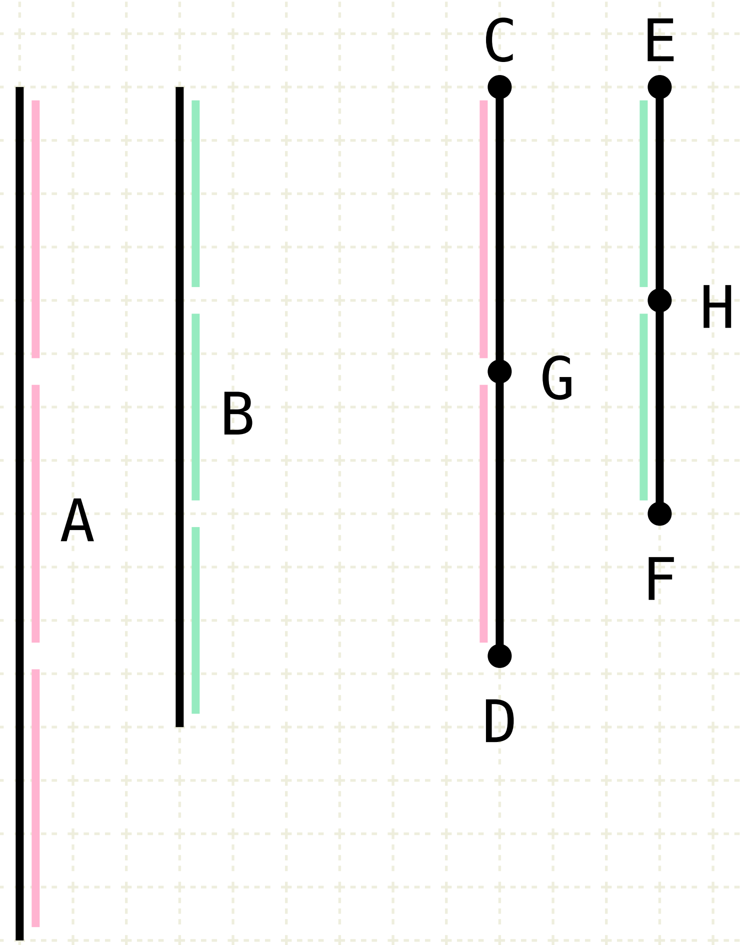
The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$

$(CD,EF) \in S$ such that $CD \leq x,\ EF \leq y,\ \forall (x,y) \in S$

$A:CD = B:EF$

$CD = (p/q)A$

$EF = (p/q)B$

$CG = GD = (1/q)A$

$EH = HF = (1/q)B$

$CG:EH = GD:HF$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

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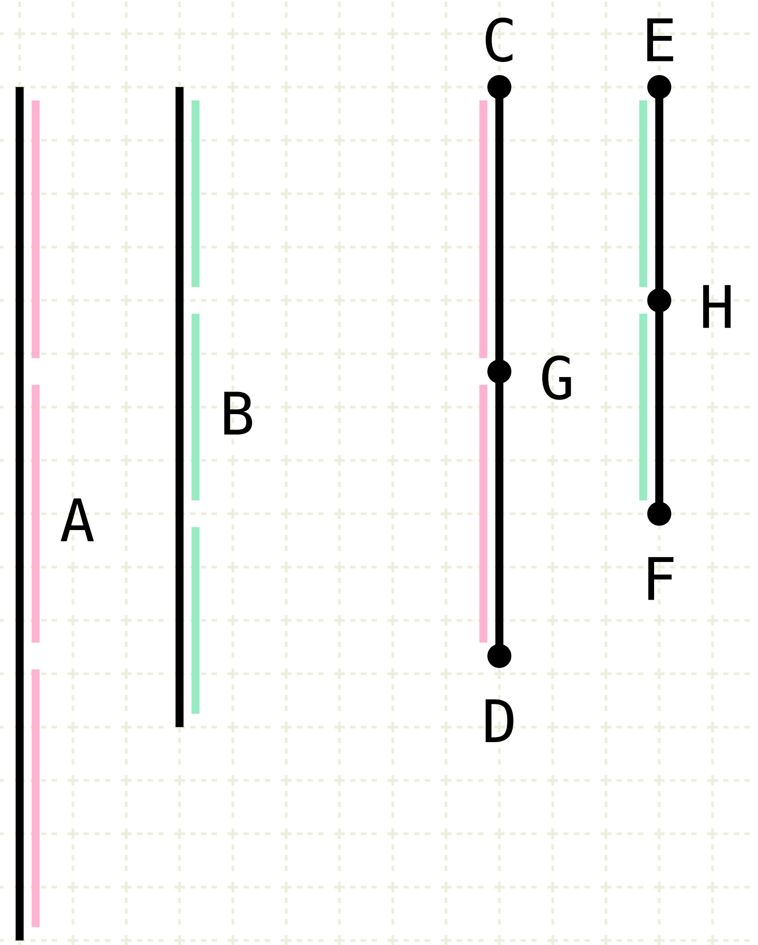
The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$
$(CD,EF) \in S$ such that $CD \le x,\ EF \le y,\ \forall (x,y) \in S$

$A:CD = B:EF$
$CD = (p/q)A$
$EF = (p/q)B$
$CG = GD = (1/q)A$
$EH = HF = (1/q)B$
$CG:EH = GD:HF$
$CG:EH = (CG+GD):(EH+HF) = CD:EF$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

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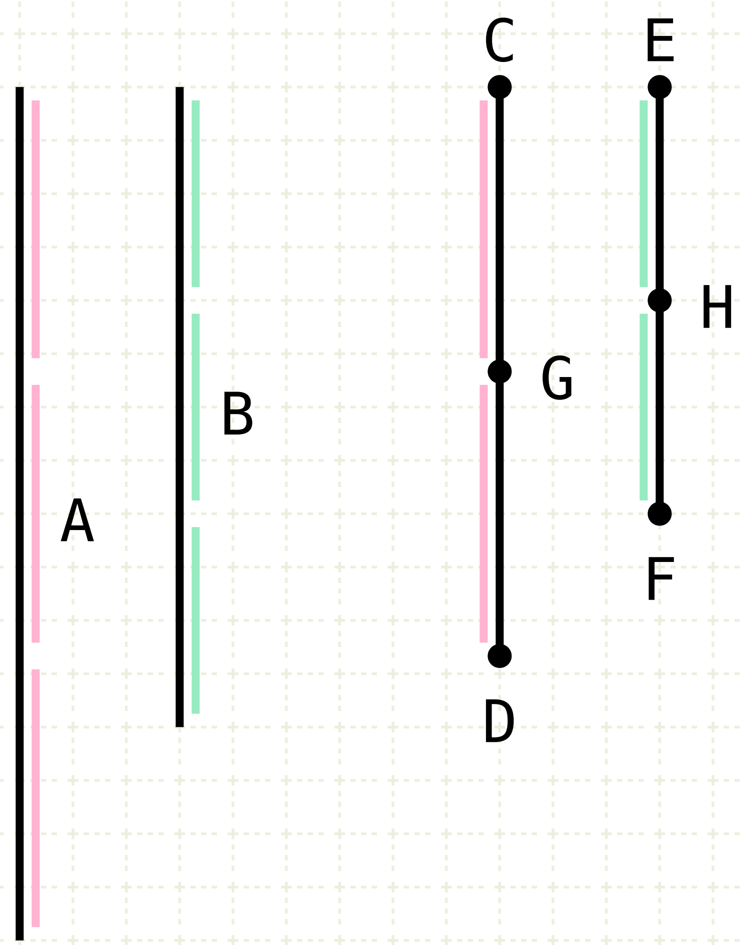

$$A:B = CD:EF$$

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$$(CD,EF) \in S \text{ such that } CD \le x,\ EF \le y,\ \forall (x,y) \in S$$

$$A:CD = B:EF$$

$$CD = (p/q)A$$

$$EF = (p/q)B$$

$$CG = GD = (1/q)A$$

$$EH = HF = (1/q)B$$

$$CG:EH = GD:HF$$

$$CG:EH = (CG+GD):(EH+HF) = CD:EF$$

$$CG < CD,\ EH < EF$$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

Therefore, there is a ratio, CG to EH, which is equal to the ratio CD to EF, where CG is less than CD

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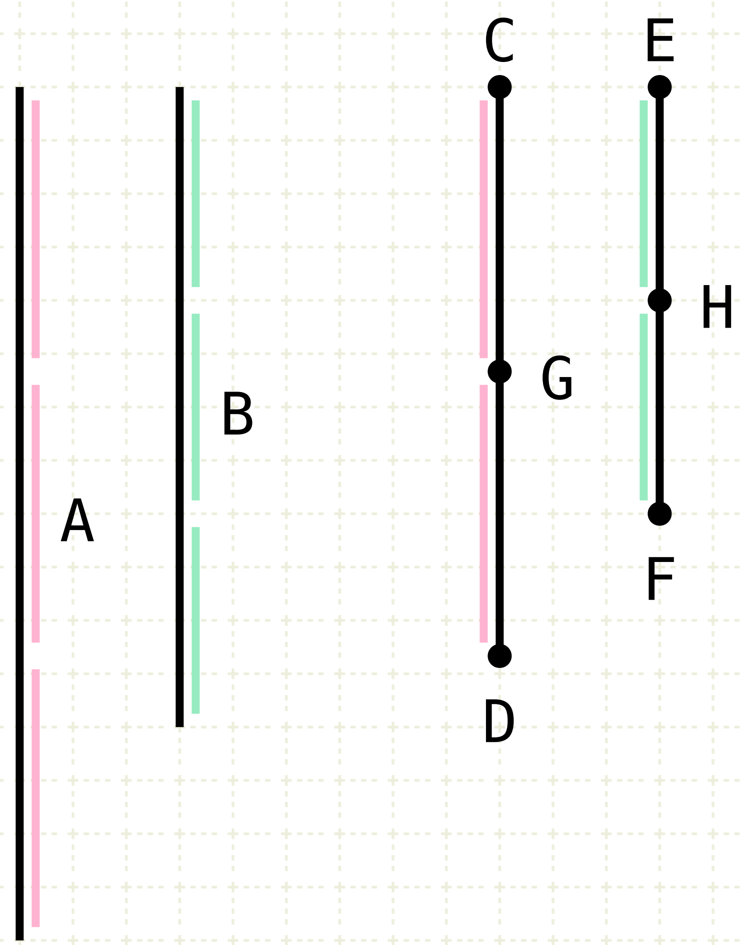

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A:CD = B:EF
CD = (p/q)A
EF = (p/q)B
CG = GD = (1/q)A
EH = HF = (1/q)B
CG:EH = GD:HF
CG:EH = (CG+GD):(EH+HF) = CD:EF
CG < CD, EH < EF

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

Therefore, there is a ratio, CG to EH, which is equal to the ratio CD to EF, where CG is less than CD

Which contradicts the original hypothesis that CD, EF are the lowest numbers that can have that ratio

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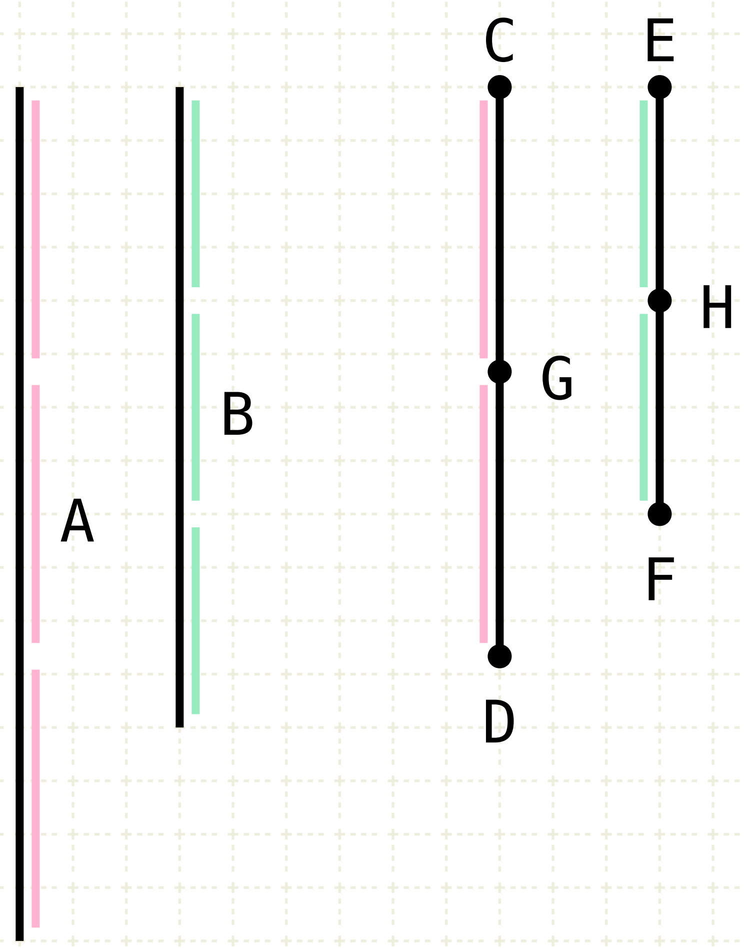
The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$

$(CD,EF) \in S$ such that $CD \leq x,\ EF \leq y,\ \forall (x,y) \in S$

$A:CD = B:EF$

$CD = (p/q)A$

$EF = (p/q)B$

$CG = GD = (1/q)A$

$EH = HF = (1/q)B$

$CG:EH = GD:HF$

$CG:EH = (CG+GD):(EH+HF) = CD:EF$

$CG < CD,\ EH < EF$

$CD = (1/q)A,\ A = q \cdot CD$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

Therefore, there is a ratio, CG to EH, which is equal to the ratio CD to EF, where CG is less than CD

Which contradicts the original hypothesis that CD, EF are the lowest numbers that can have that ratio

Therefore, CD is not parts of A, therefore it is a part of A (VII·4)

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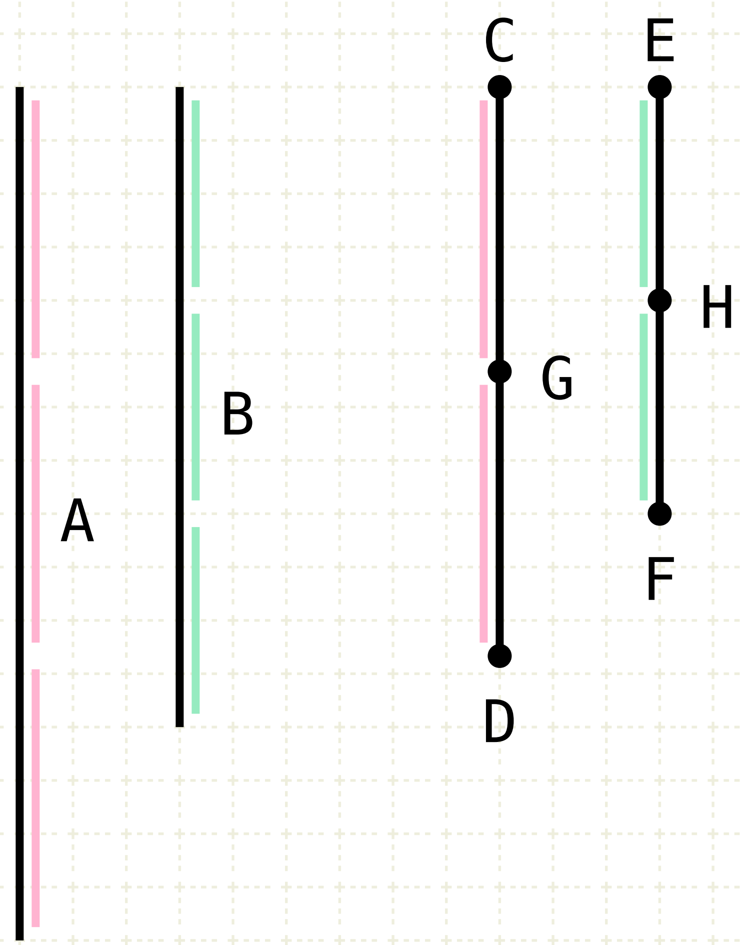

$A:B = CD:EF$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$

$(CD,EF) \in S$ such that $CD \leq x,\ EF \leq y,\ \forall (x,y) \in S$

$A:CD = B:EF$

$CD = (p/q)A$

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$CG = GD = (1/q)A$

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$CG:EH = GD:HF$

$CG:EH = (CG+GD):(EH+HF) = CD:EF$

$CG < CD,\ EH < EF$

$CD = (1/q)A,\ A = q \cdot CD$

$EF = (1/q)B,\ B = q \cdot EF$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

Therefore, there is a ratio, CG to EH, which is equal to the ratio CD to EF, where CG is less than CD

Which contradicts the original hypothesis that CD, EF are the lowest numbers that can have that ratio

Therefore, CD is not parts of A, therefore it is a part of A (VII·4)

And CD measures A the same number of times that EF measures B (Def.20)

Proposition 20 of Book VII

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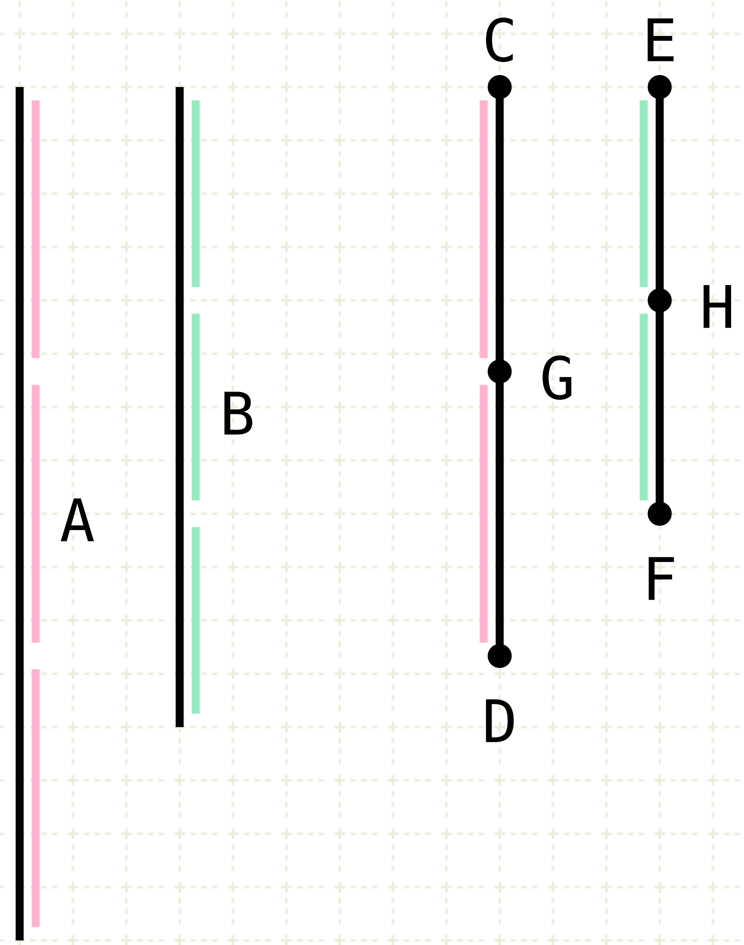

$A:B = CD:EF$

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$A:CD = B:EF$

$CD = (p/q)A$

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$CG:EH = GD:HF$

$CG:EH = (CG+GD):(EH+HF) = CD:EF$

$CG < CD,\ EH < EF$

$CD = (1/q)A,\ A = q \cdot CD$

$EF = (1/q)B,\ B = q \cdot EF$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

Therefore, there is a ratio, CG to EH, which is equal to the ratio CD to EF, where CG is less than CD

Which contradicts the original hypothesis that CD, EF are the lowest numbers that can have that ratio

Therefore, CD is not parts of A, therefore it is a part of A (VII·4)

And CD measures A the same number of times that EF measures B (Def.20)

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