

Euclid's Elements

Book VII

Definitions:

- 1 A unit is that by virtue of which each of the things that exist is called one
- 2 A number is a multitude composed of units. (not one)
- 3 A number is part of a number, the less of the greater, when it measures the greater
- 11 A prime number is that which is measured by a unit alone.
- 12 Numbers prime to one another are those which are measured by a unit alone as a common measure
- 15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
- 20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)



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| 1 | Determine if two numbers are relatively prime | 10 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$ | 21 | If A,B are relatively prime, then A,B are the smallest whole numbers that can be used to describe the ratio A:B |
| 2 | Find the greatest common divisor for two numbers | 11 | If $A:B = C:D$, then $(A-C):(B-D) = A:B$ | 22 | If A,B are the smallest whole numbers that can be used to describe the ratio A:B, then A,B are relatively prime |
| 3 | Find the largest common divisor for three numbers | 12 | If $A:B = C:D$, then $(A+C):(B+C) = A:B$ | 23 | If A,B are relatively prime and if $A = n \cdot C$, then B,C are relatively prime |
| 4 | Given two natural numbers, A and B, either B is part of A, or there exists a natural number (a part) that can measure both A and B | 13 | If $A:B = C:D$, then $A:C = B:D$ | 24 | If A,C are relatively prime and B,C are relatively prime then the $A \times B$ is relatively prime to C |
| 5 | If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, then $(B+D) = (1/q) \cdot (A+C)$ | 14 | If $A:B = D:E$ and $B:C = E:F$, then $A:C = D:F$ | 25 | If A,B are relatively prime then A^2, B are relatively prime |
| 6 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, then $(B+D) = (p/q) \cdot (A+C)$ | 15 | If $B = i \cdot 1$ and $E = i \cdot D$, and if $D = j \cdot 1$ then $E = j \cdot B$ | 26 | If A is relatively prime to C and D, and if B is also relatively prime to C and D, then $A \times B$ is relatively prime to $C \times D$ |
| 7 | If $B = A/q$ and $D = C/q$, $B > D$, then $(B-D) = (A-C)/q$ | 16 | $A \times B = B \times A$ | 27 | If A,B are relatively prime, then A^2, B^2 are relatively prime, and A^3, B^3 are relatively prime, and so on |
| 8 | If $B = (p/q) \cdot A$ and $D = (p/q) \cdot C$, $B > D$, then $(B-D) = (p/q) \cdot (A-C)$ | 17 | If $D = A \times B$ and $E = A \times C$ then $D:E = B:C$ | | |
| 9 | If $B = (1/q) \cdot A$ and $D = (1/q) \cdot C$, and If $B = (r/s) \cdot D$, then $A = (r/s) \cdot C$ | 18 | If $D = B \times A$ and $E = C \times A$ then $D:E = B:C$ | | |
| | | 19 | If $A:B = C:D$ then $A \times D = B \times C$ If $A \times D = B \times C$ then $A:B = C:D$ | | |
| | | 20 | Given the ratio A:B and C,D are the smallest numbers such that $A:B = C:D$ then $A = n \cdot C$ and $B = n \cdot D$ | | |



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|----|--|----|---|
| 28 | If A,B are relatively prime, then A,(A+B) are relatively prime | 37 | If $A = p \cdot B$, then $A = q \cdot C$ where $C = p \cdot 1$ |
| 29 | If A is prime, and $B \neq n \cdot A$, then A,B are relatively prime | 38 | If $A = (1/c) \cdot B$ and $C = c \cdot 1$ then $A = n \cdot C$ |
| 30 | If $C = A \times B$ and $C = i \cdot D$ where D is prime, then either $A = j \cdot D$ or $B = j \cdot D$ | 39 | Find the smallest number that has the fractions $1/a, 1/b, 1/c$ |
| 31 | If $A = B \times C$, then $A = j \cdot D$ where D is prime | | |
| 32 | If A is a number then it is either prime, or $A = j \cdot D$ where D is prime | | |
| 33 | Find the smallest numbers X,Y,Z where the ratio X:Y:Z is equal to the given ratio A:B:C | | |
| 34 | Find the lowest common denominator of 2 numbers | | |
| 35 | If E is the lowest common denominator of A,B, and if $C = n \cdot A = m \cdot B$, then $C = i \cdot E$ | | |
| 36 | Find the least common multiple of 3 numbers | | |



Proposition 20 of Book VII

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



Proposition 20 of Book VII

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less

$$A:B = C:D$$

if

$$A = (p/q)C$$

$$B = (p/q)D$$

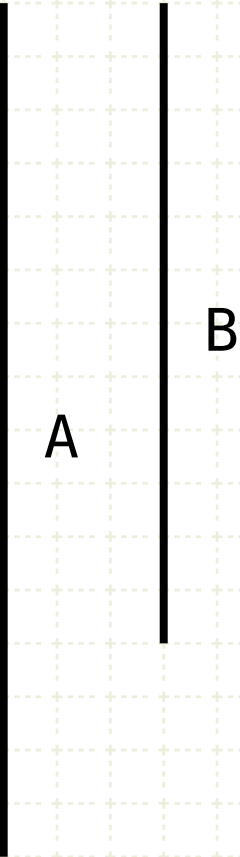
Definition 20

Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth



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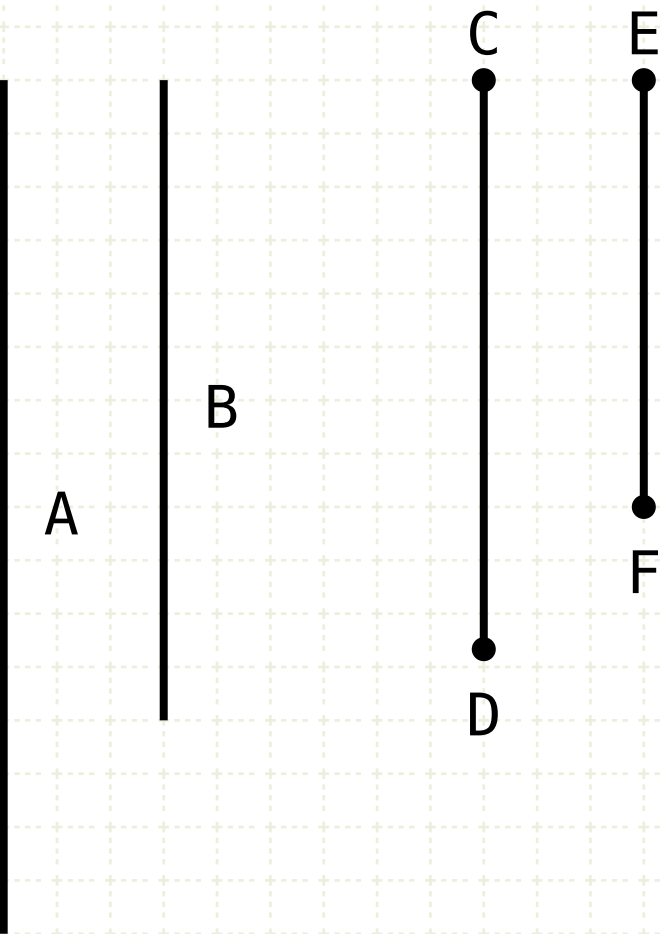
In other words

Given a ratio A to B



Proposition 20 of Book VII

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



$$A:B = CD:EF$$

$$S = \{ (x,y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x:y=A:B \}$$
$$(CD,EF) \in S \text{ such that } CD \leq x, EF \leq y, \forall (x,y) \in S$$

In other words

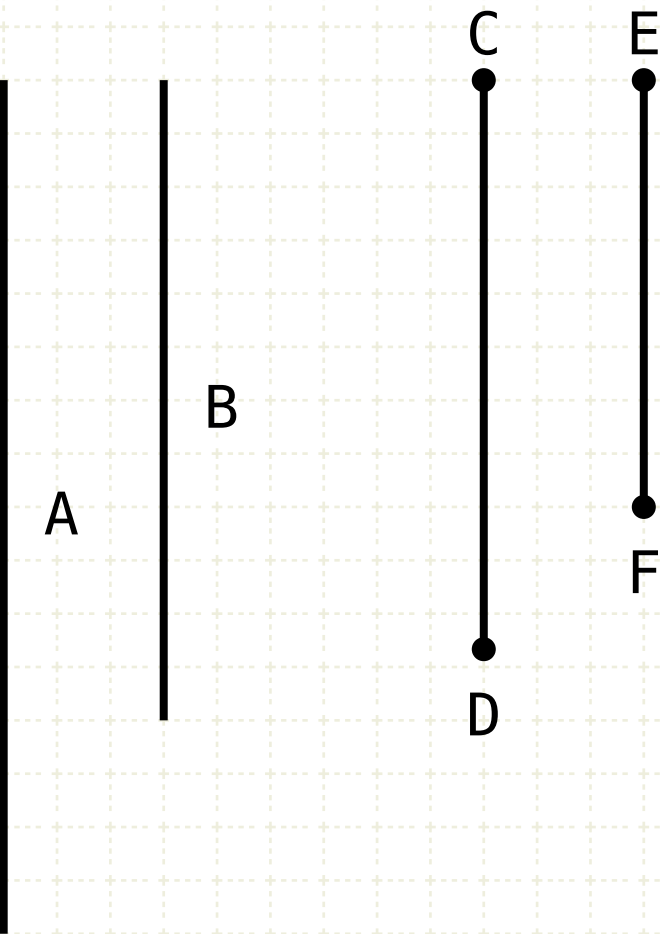
Given a ratio A to B

Let the numbers CD and EF be equal to the ratio A to B and ... and CD,EF are the smallest numbers whose ratio equals the ratio of A to B



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$$(CD,EF) \in S \text{ such that } CD \leq x, EF \leq y, \forall (x,y) \in S$$

$$\rightarrow A = n \cdot CD, B = n \cdot EF$$

In other words

Given a ratio A to B

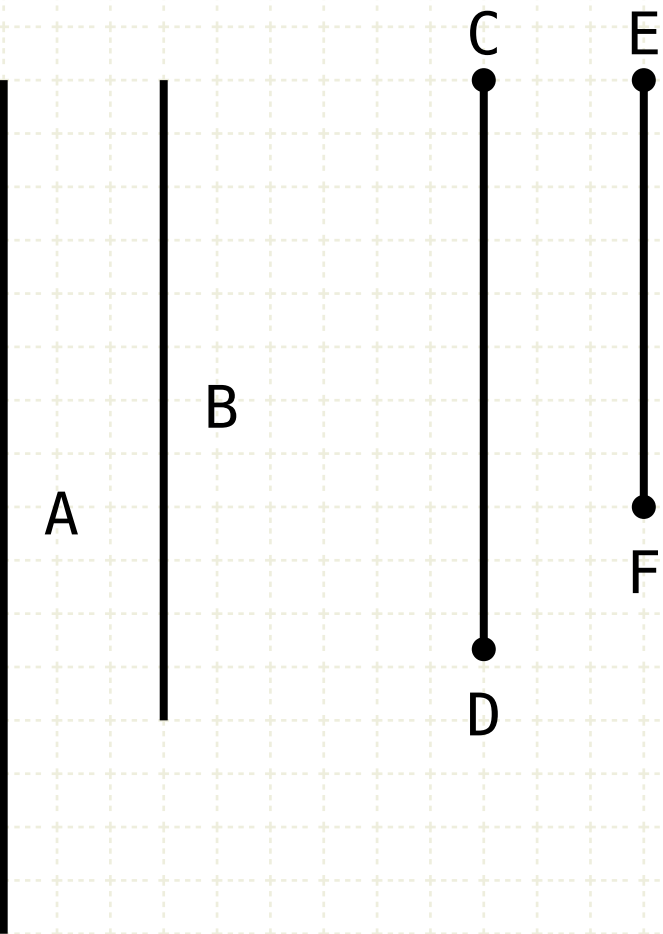
Let the numbers CD and EF be equal to the ratio A to B and ... and CD,EF are the smallest numbers whose ratio equals the ratio of A to B

Then, CD measures A the same number of times that EF measures B



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$$A:B = CD:EF$$

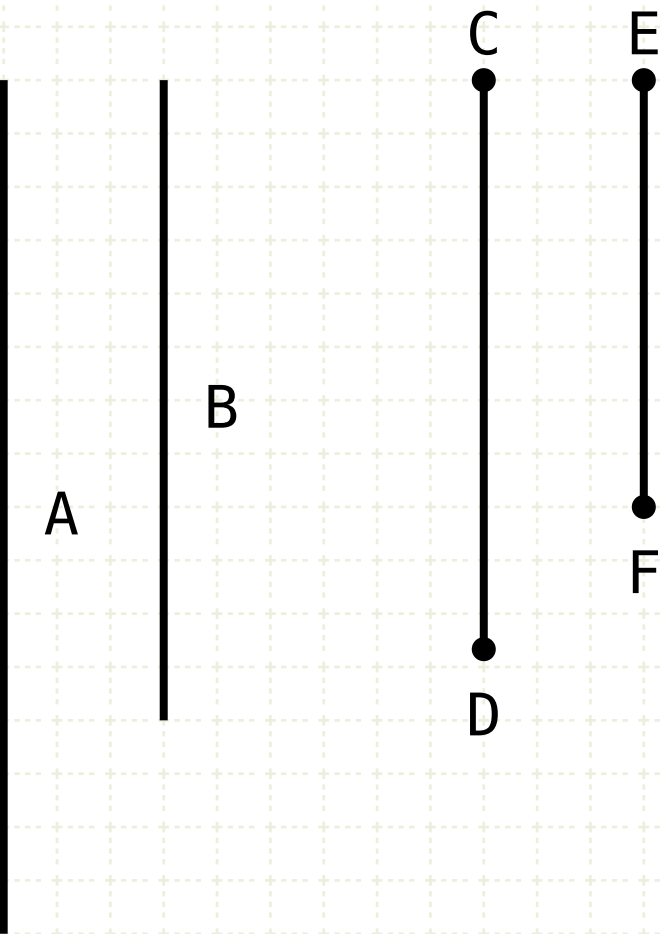
$$S = \{ (x,y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x:y=A:B \}$$
$$(CD,EF) \in S \text{ such that } CD \leq x, EF \leq y, \forall (x,y) \in S$$

Proof by Contradiction



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$$A:CD = B:EF$$

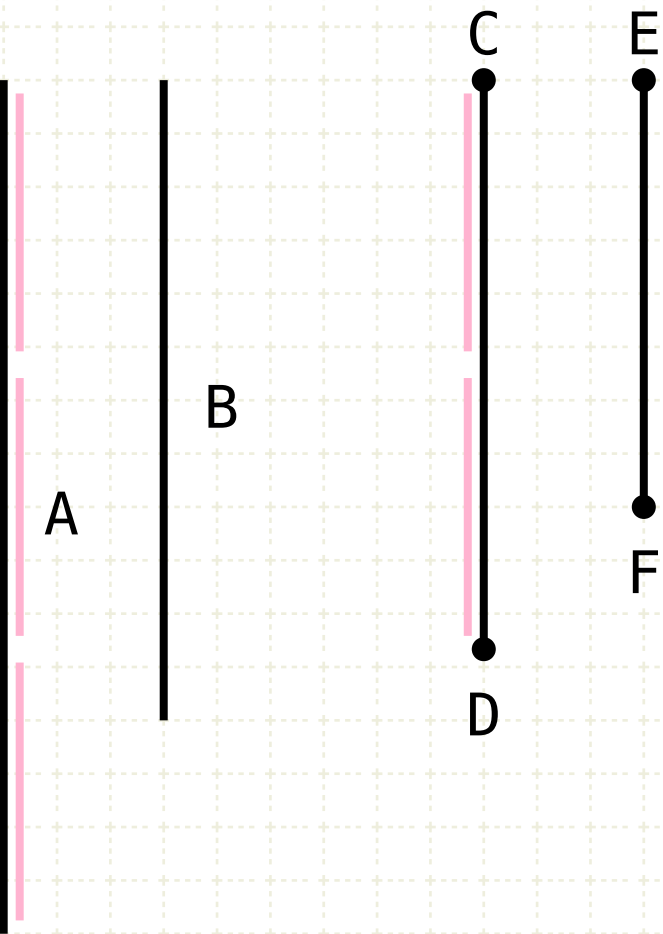
Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)



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$$A:B = CD:EF$$

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$$(CD,EF) \in S \text{ such that } CD \leq x, EF \leq y, \forall (x,y) \in S$$

$$A:CD = B:EF$$

$$CD = (p/q)A$$

Proof by Contradiction

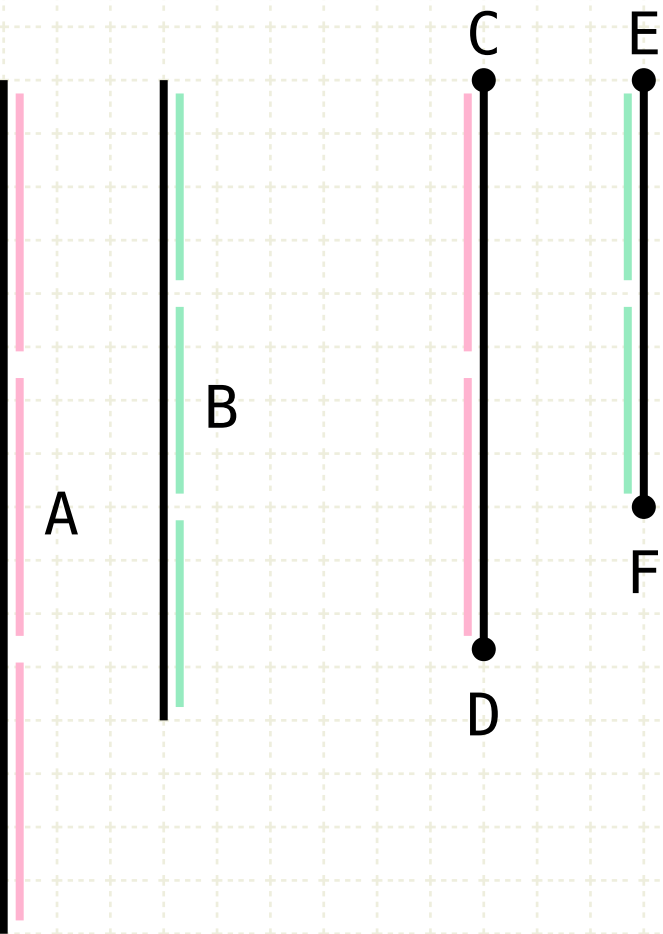
Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A



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The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



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$$A:CD = B:EF$$

$$CD = (p/q)A$$

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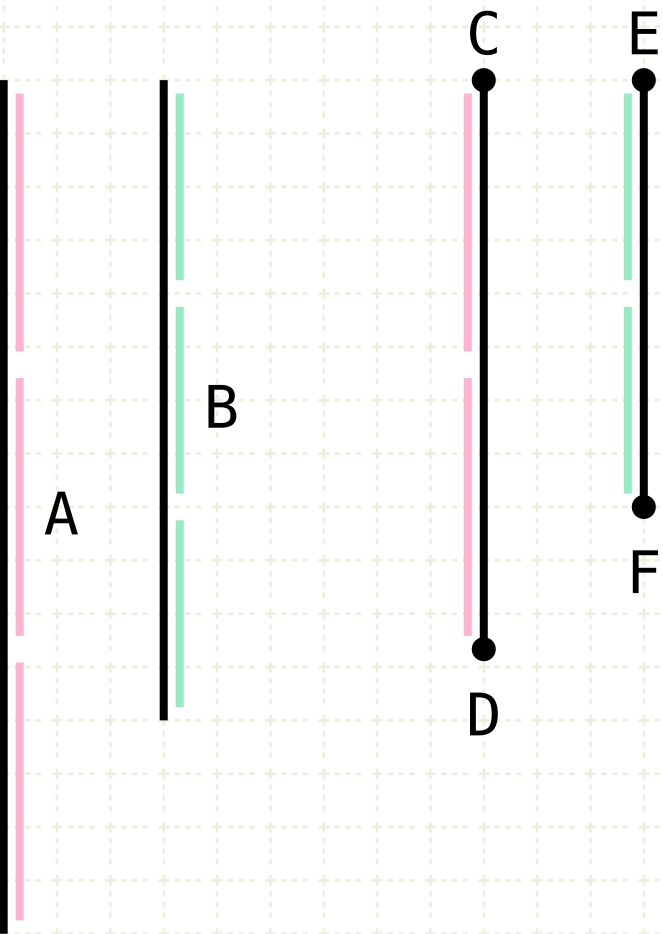
Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)



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The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



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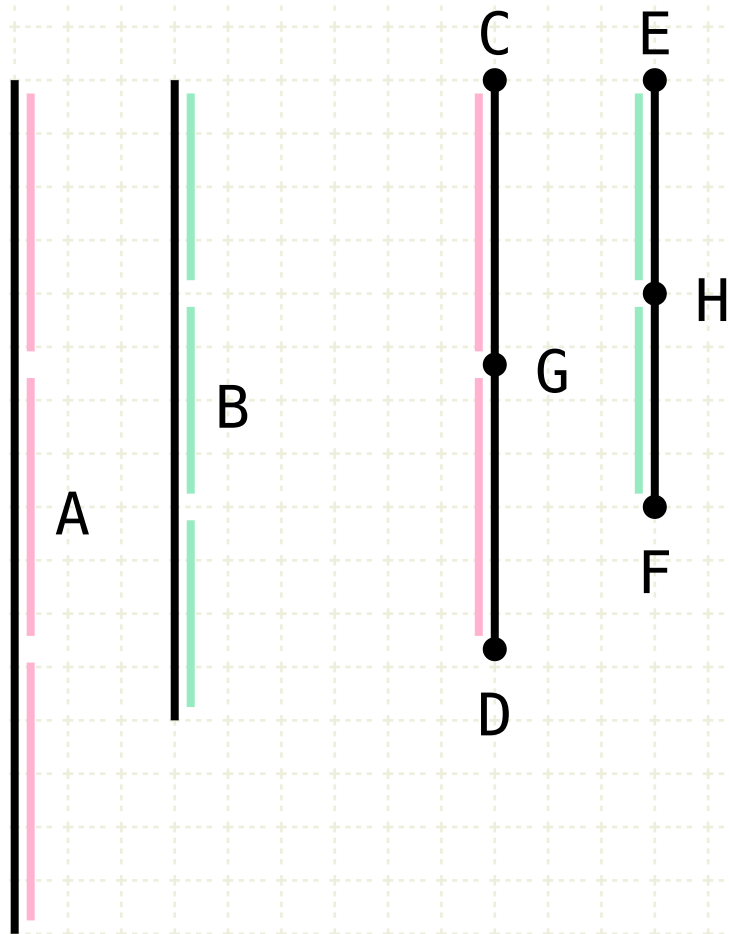
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Therefore as many parts there are of A in CD, so many parts there are of B in EF



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The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



$$A:B = CD:EF$$

$$S = \{ (x,y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x:y=A:B \}$$

$(CD,EF) \in S$ such that $CD \leq x, EF \leq y, \forall (x,y) \in S$

$$A:CD = B:EF$$

$$CD = (p/q)A$$

$$EF = (p/q)B$$

$$CG = GD = (1/q)A$$

$$EH = HF = (1/q)B$$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

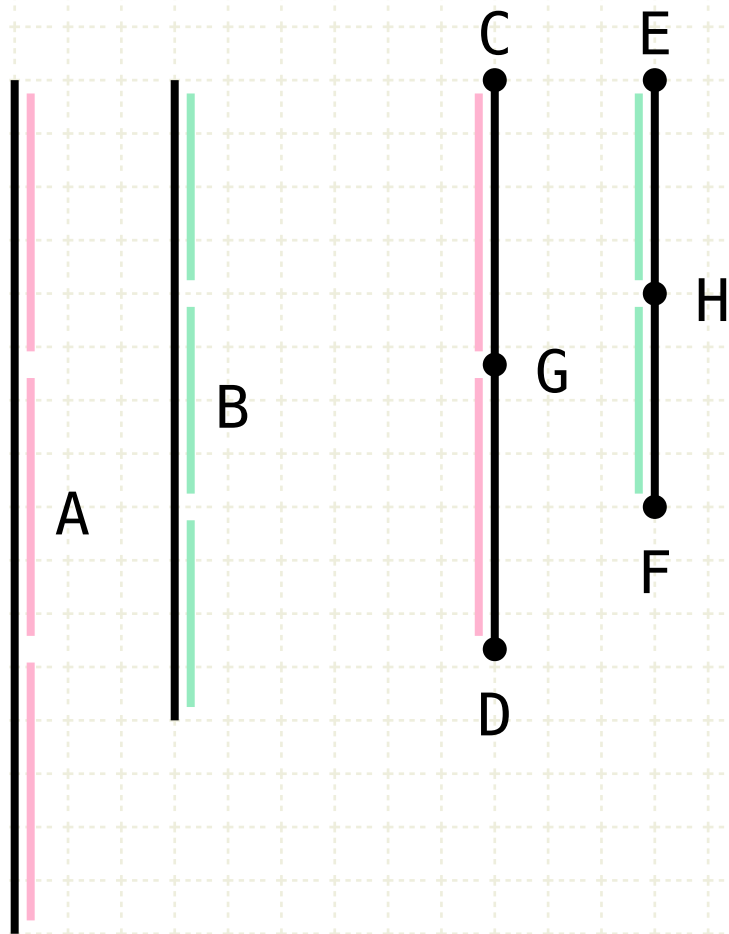
Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)



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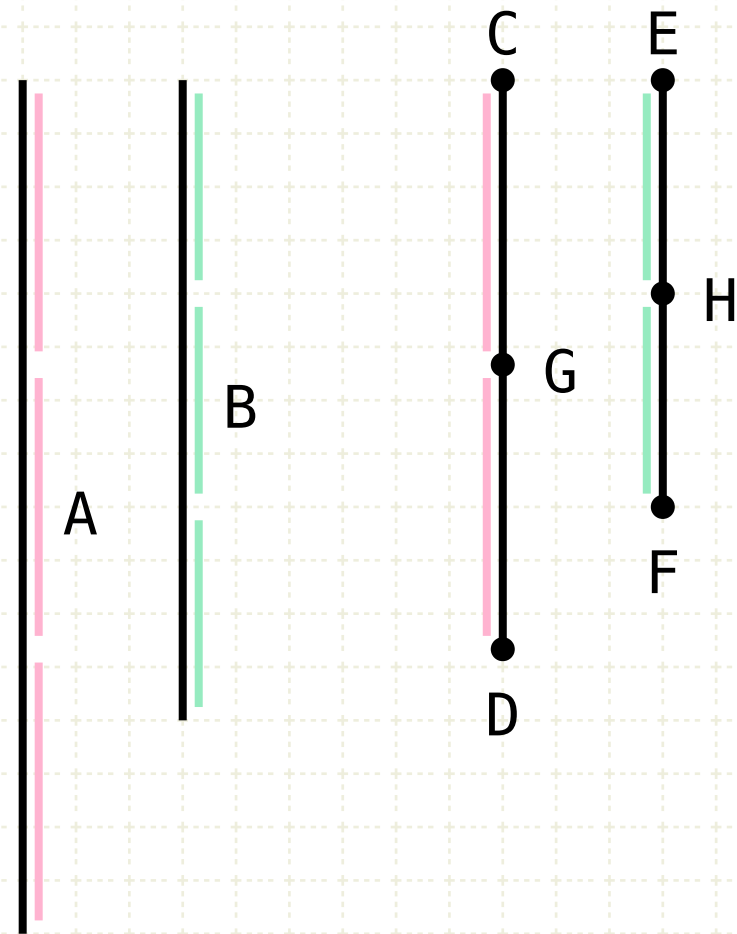
Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal



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$$EH = HF = (1/q)B$$

$$CG:EH = GD:HF$$

$$CG:EH = (CG+GD):(EH+HF) = CD:EF$$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

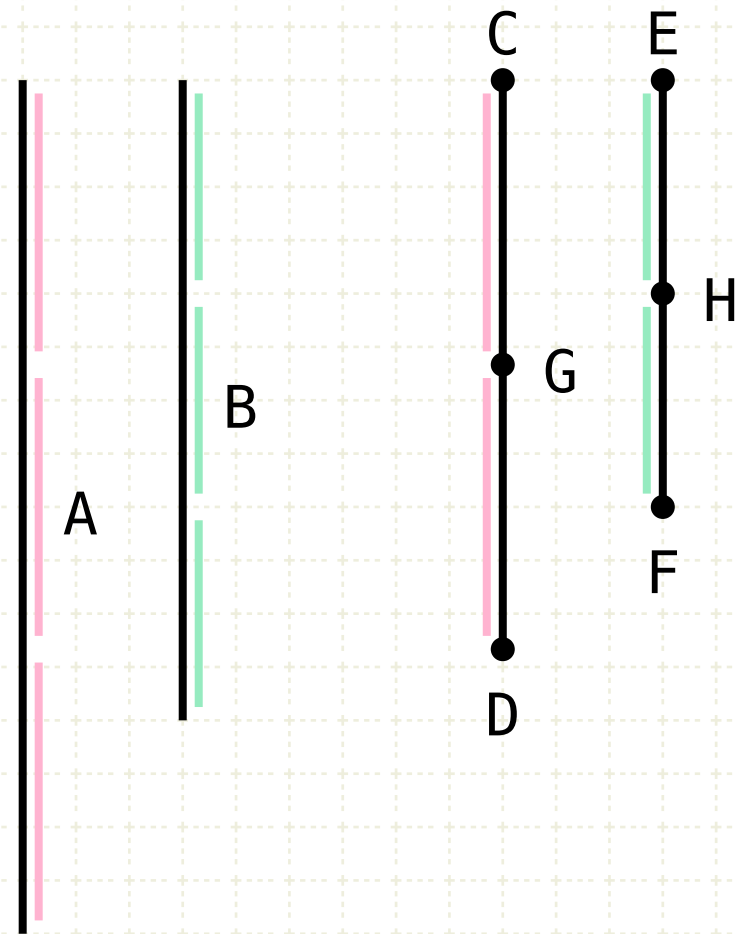
Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)



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The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



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$$A:CD = B:EF$$

$$CD = (p/q)A$$

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$$CG:EH = GD:HF$$

$$CG:EH = (CG+GD):(EH+HF) = CD:EF$$

$$CG < CD, EH < EF$$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

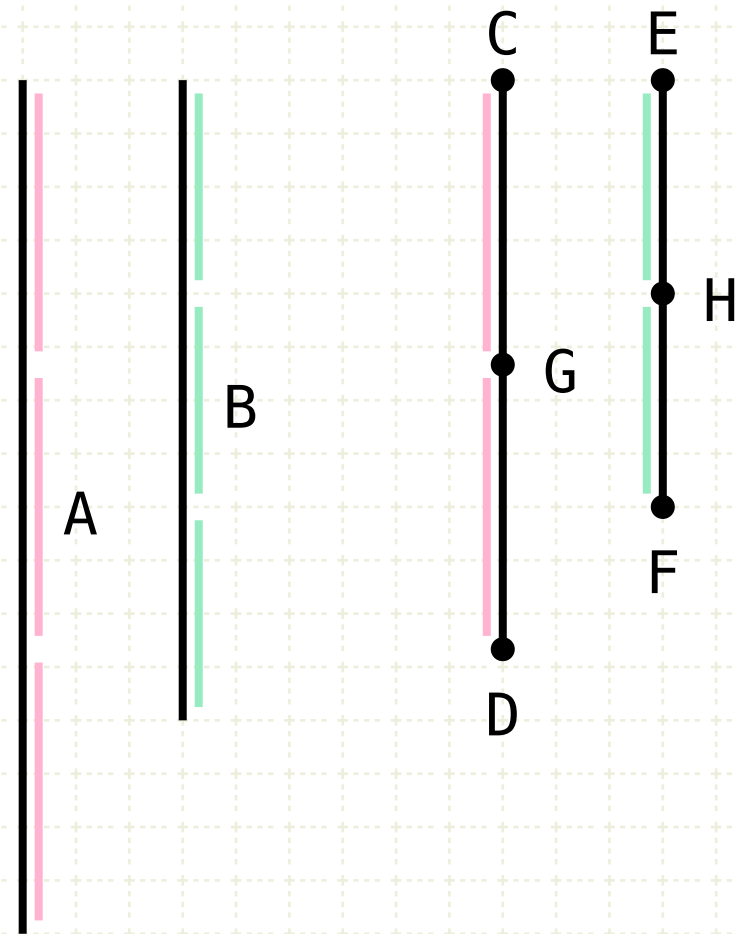
If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

Therefore, there is a ratio, CG to EH, which is equal to the ratio CD to EF, where CG is less than CD



Proposition 20 of Book VII

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less



$$A:B = CD:EF$$

$$S = \{ (x,y) \mid x \in \mathbb{N}, y \in \mathbb{N}, x:y=A:B \}$$

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$$A:CD = B:EF$$

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$$CG:EH = GD:HF$$

$$CG:EH = (CG+GD):(EH+HF) = CD:EF$$

$$CG < CD, EH < EF$$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

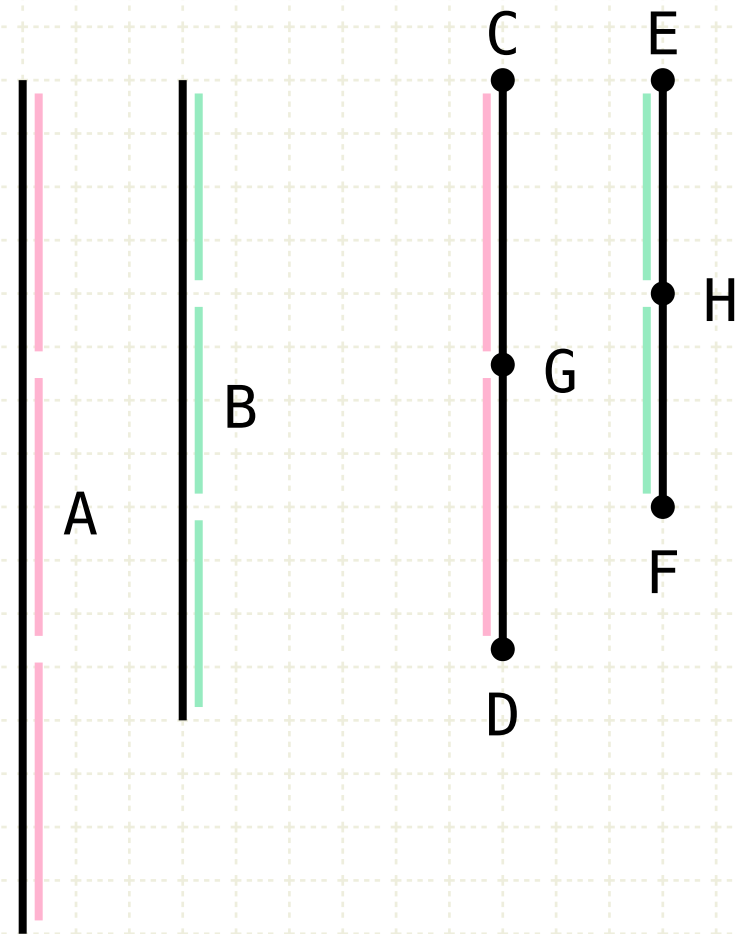
Therefore, there is a ratio, CG to EH, which is equal to the ratio CD to EF, where CG is less than CD

Which contradicts the original hypothesis that CD, EF are the lowest numbers that can have that ratio



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$$A:CD = B:EF$$

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$$CG:EH = (CG+GD):(EH+HF) = CD:EF$$

$$CG < CD, EH < EF$$

$$CD = (1/q)A, A = q \cdot CD$$

Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

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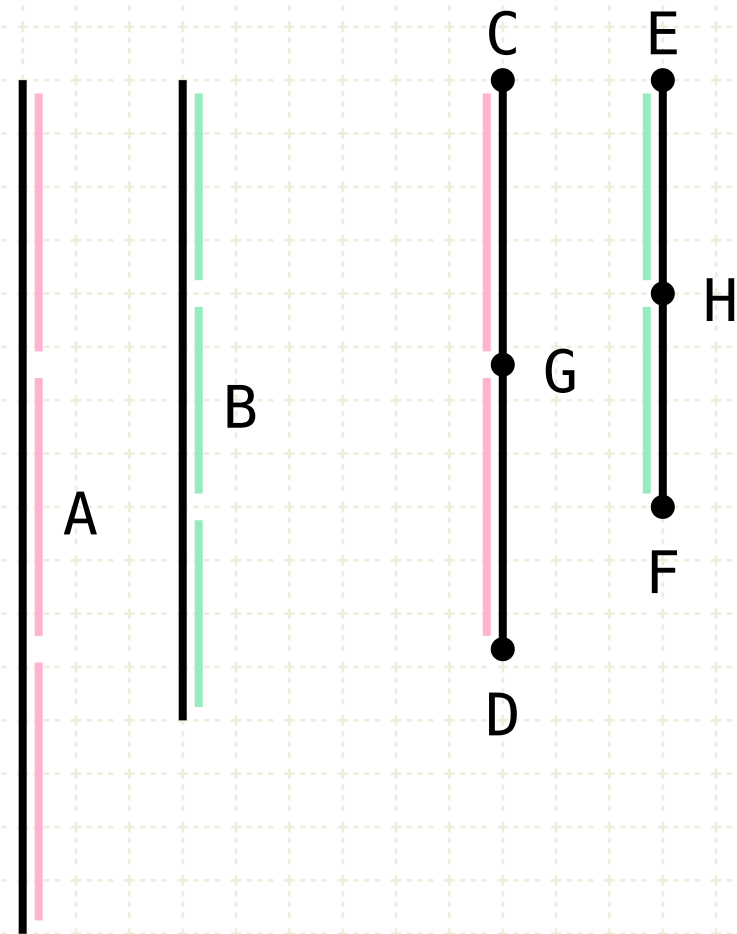
Which contradicts the original hypothesis that CD, EF are the lowest numbers that can have that ratio

Therefore, CD is not parts of A, therefore it is a part of A (VII·4)



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$$CD = (1/q)A, A = q \cdot CD$$

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Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

Thus EF is also the same parts of B that CD is of A (Def 20)

Therefore as many parts there are of A in CD, so many parts there are of B in EF

Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

Since CG,GD are equal and EH,HF are equal, the ratios CG to EH and GD to HF are also equal

If ratios are equal, the ratio of the sum of all the antecedents to the ratio of the sum of the consequents is also equal (VII·12)

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Which contradicts the original hypothesis that CD, EF are the lowest numbers that can have that ratio

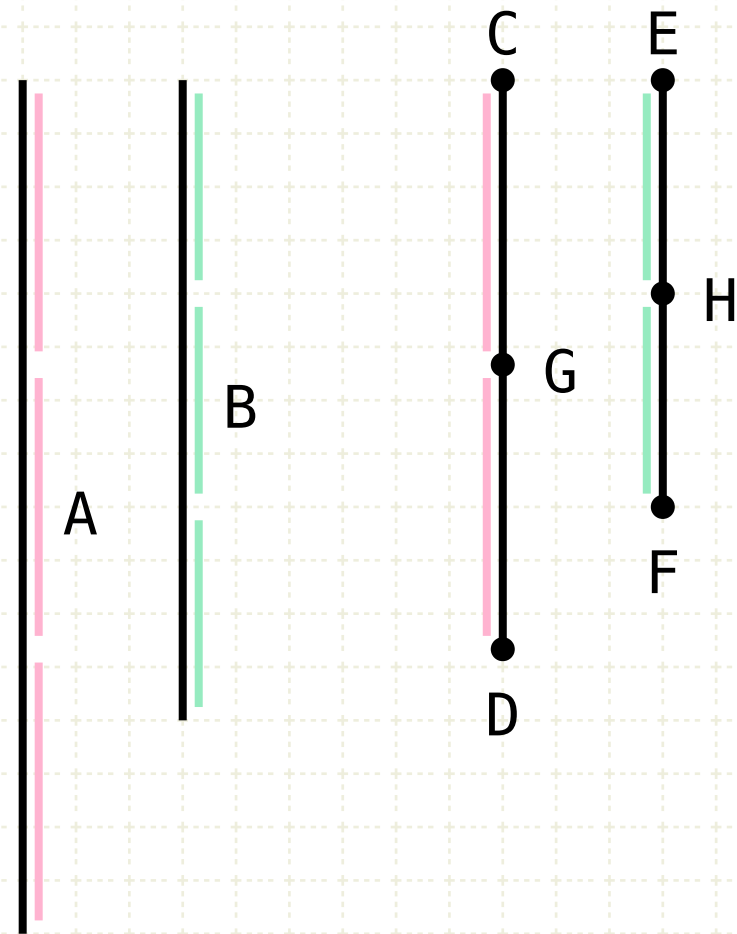
Therefore, CD is not parts of A, therefore it is a part of A (VII·4)

And CD measures A the same number of times that EF measures B (Def.20)



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$$CG < CD, EH < EF$$

$$CD = (1/q)A, A = q \cdot CD$$

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Proof by Contradiction

Firstly, the ratio of A to CD is equal to B to EF (VII·13)

Assume that CD does not measure A, therefore CD is parts of A

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Divide CD into the parts of A (CG,GD), and EF into the parts of B (EH,HF)

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Therefore, CD is not parts of A, therefore it is a part of A (VII·4)

And CD measures A the same number of times that EF measures B (Def.20)



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