Euclid's Elements

Book VII

Definitions:

1 A unit is that by virtue of which each of the things that exist is called one

2 A number is a multitude composed of units. (not one)

3 A number is part of a number, the less of the greater, when it measures the greater

11 A prime number is that which is measured by a unit alone.

12 Numbers prime to one another are those which are measured by a unit alone as a common measure

15 A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

20 Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph-Louis Lagrange
(1736 to 1813)

Table of Contents, Chapter 7

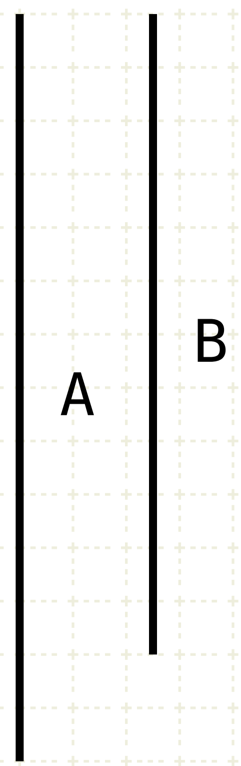
Table of Contents, Chapter 7

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



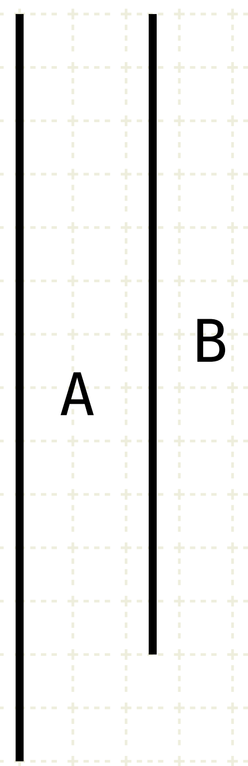
gcd(A,B) = 1

In other words

Let A and B be relatively prime.

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$
$(A,B) \in S$ such that $A \le x,\ B \le y,\ \forall (x,y) \in S$

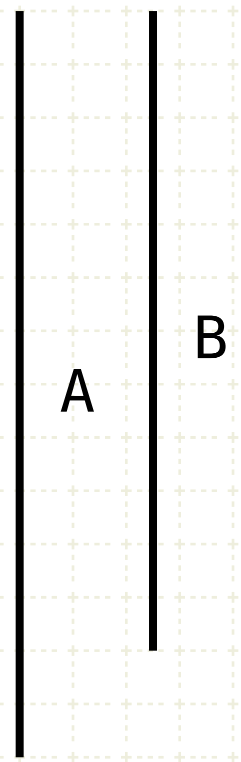
In other words

Let A and B be relatively prime.

Then A and B are the smallest whole numbers that can be used to create the same ratio equal to A to B

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.

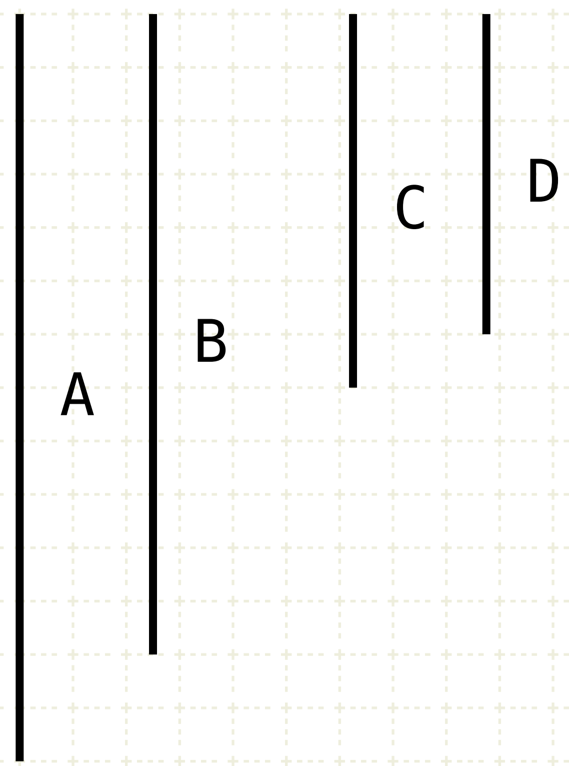


`gcd(A,B) = 1`

Proof by Contradiction

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



gcd(A,B) = 1

A:B = C:D

C < A

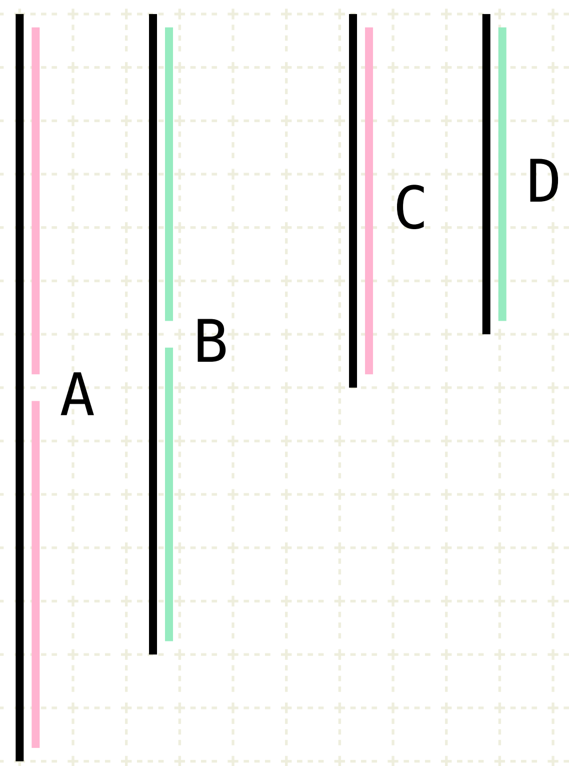
D < B

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$

$C < A$

$D < B$

$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$

$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$

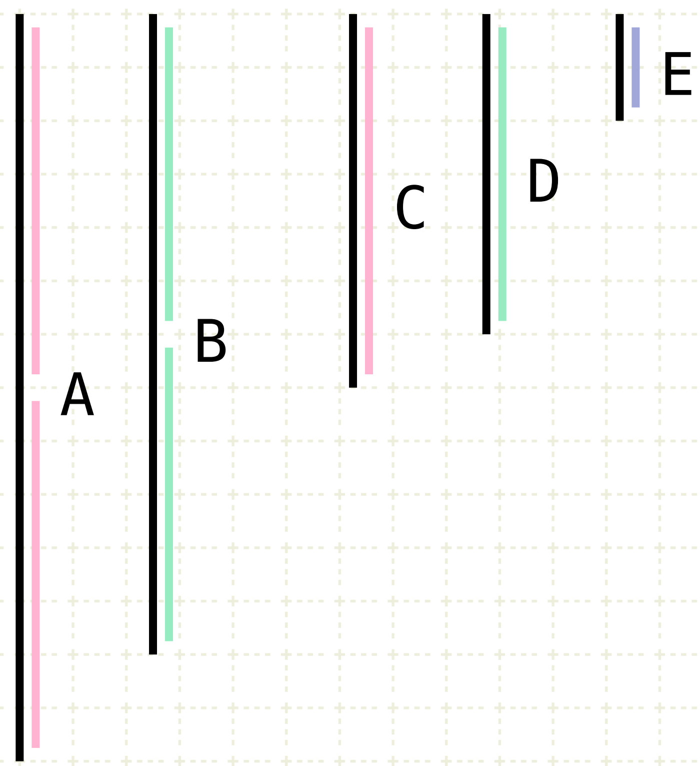
Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$

$C < A$

$D < B$

$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$

$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$

$A = E \cdot C,\ E \neq 1$

Proof by Contradiction

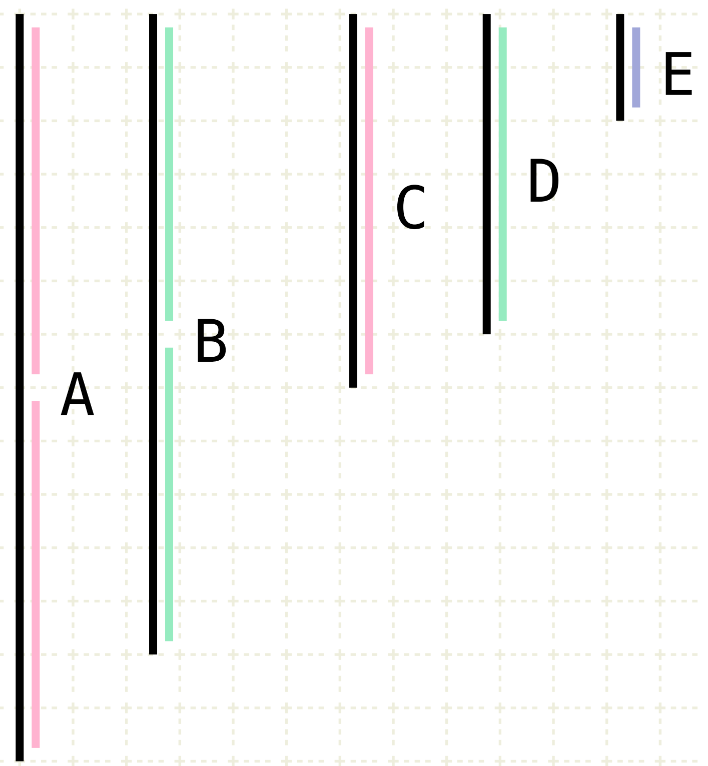
Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$

$C < A$

$D < B$

$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$

$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$

$A = E \cdot C,\ E \neq 1$

$B = E \cdot D,\ E \neq 1$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

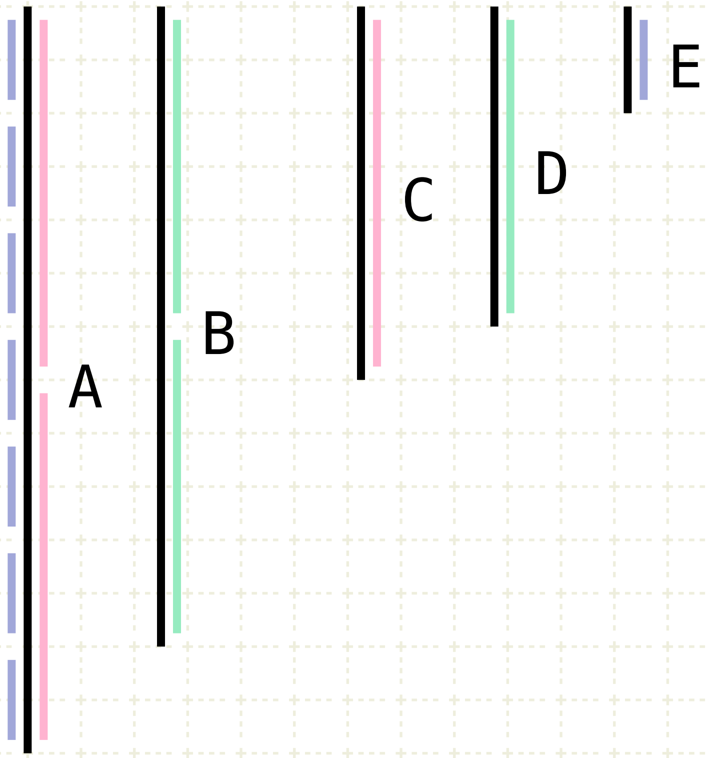
Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$

$C < A$

$D < B$

$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$

$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$

$A = E \cdot C, \ E \neq 1$

$B = E \cdot D, \ E \neq 1$

$A = C \cdot E$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

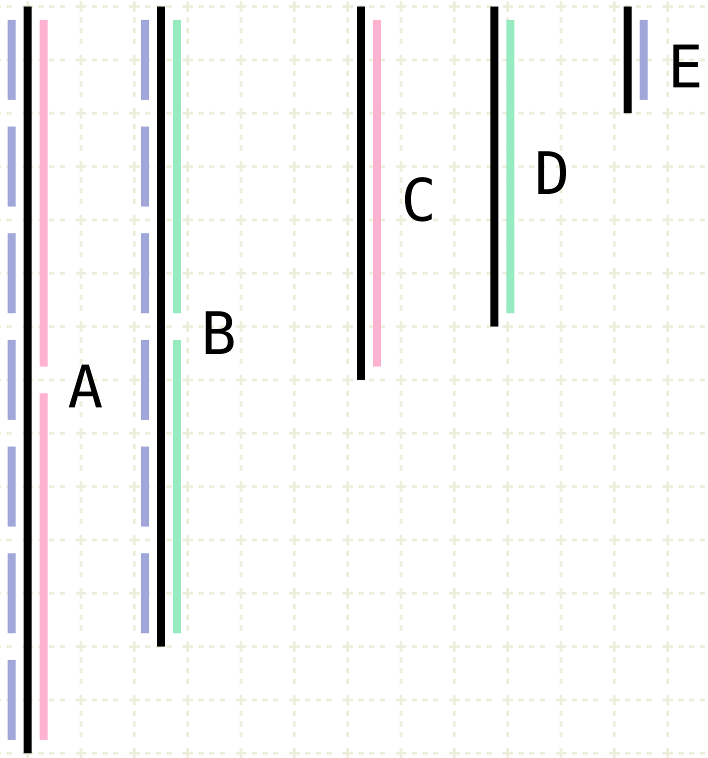
Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$

$C < A$

$D < B$

$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$

$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$

$A = E \cdot C,\ E \neq 1$

$B = E \cdot D,\ E \neq 1$

$A = C \cdot E$

$B = D \cdot E$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

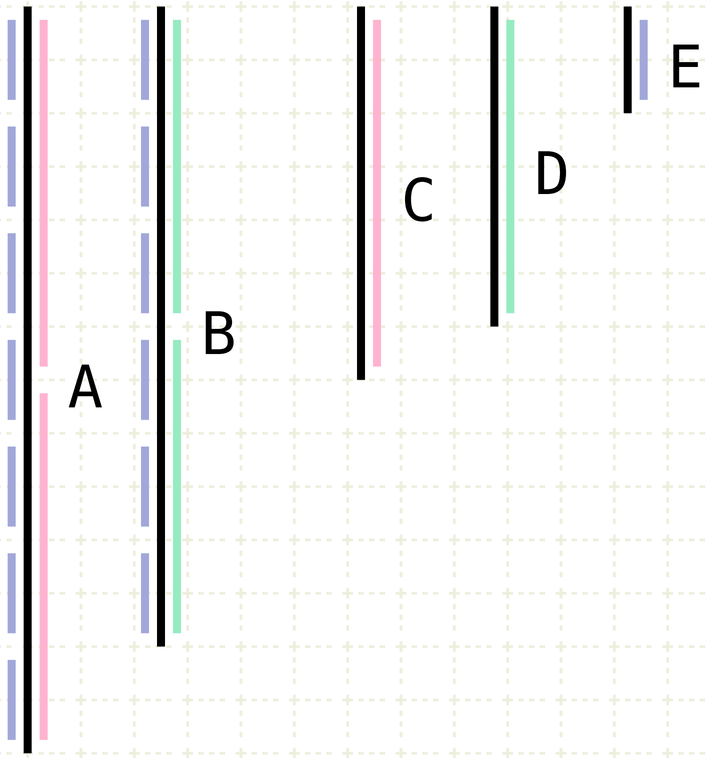
Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$

$C < A$

$D < B$

$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$

$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$

$A = E \cdot C,\ E \neq 1$

$B = E \cdot D,\ E \neq 1$

$A = C \cdot E$

$B = D \cdot E$

$\gcd(A,B) = E,\ E \neq 1$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

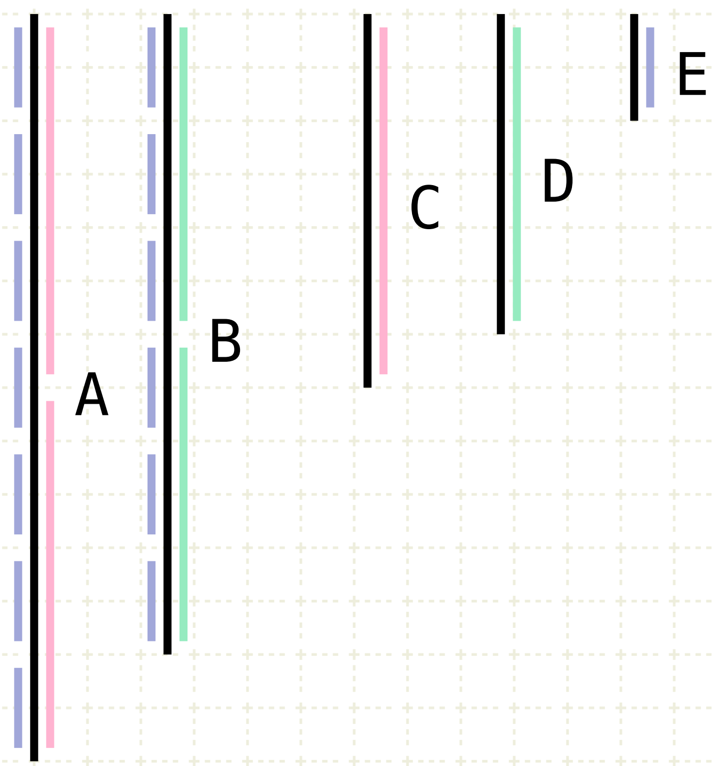
Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

Therefore A,B have a common divisor E, which is greater than one.

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$

$C < A$

$D < B$

$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$

$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$

$A = E \cdot C,\ E \neq 1$

$B = E \cdot D,\ E \neq 1$

$A = C \cdot E$

$B = D \cdot E$

$\gcd(A,B) = E,\ E \neq 1$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

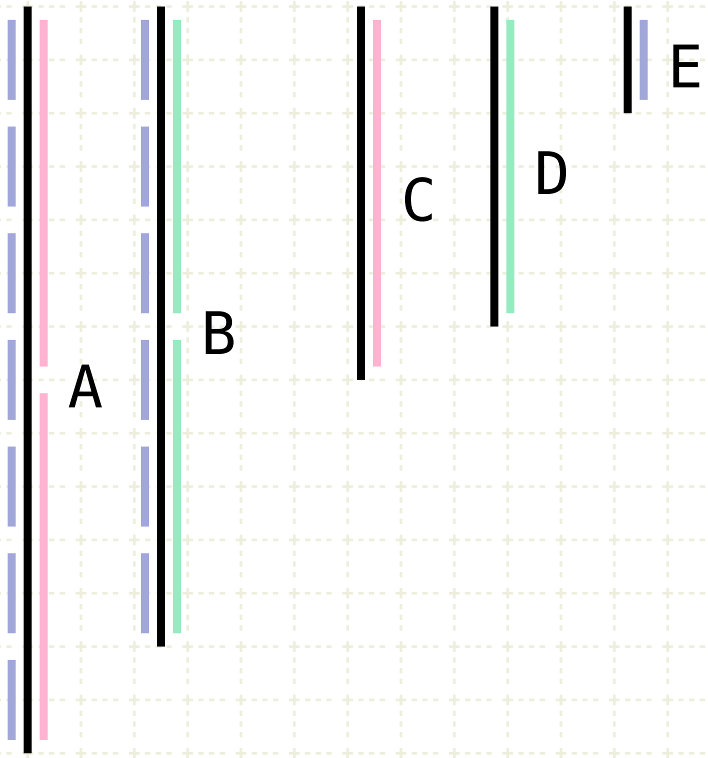
Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

Therefore A,B have a common divisor E, which is greater than one.

This is not possible since A and B are prime to one another (VII Def.12)

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$
$C < A$
$D < B$
$A = C_1 + C_2 + \dots + C_i + \dots C_n$
$B = D_1 + D_2 + \dots + D_i + \dots D_n$
$A = E\cdot C,\ E \neq 1$
$B = E\cdot D,\ E \neq 1$
$A = C\cdot E$
$B = D\cdot E$
$\gcd(A,B) = E,\ E \neq 1$

$S = \{ (x,y) \mid x\in\mathbb{N},\ y\in\mathbb{N},\ x:y=A:B \}$
$(A,B)\in S$ such that $A\leq x,\ B\leq y,\ \forall(x,y)\in S$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

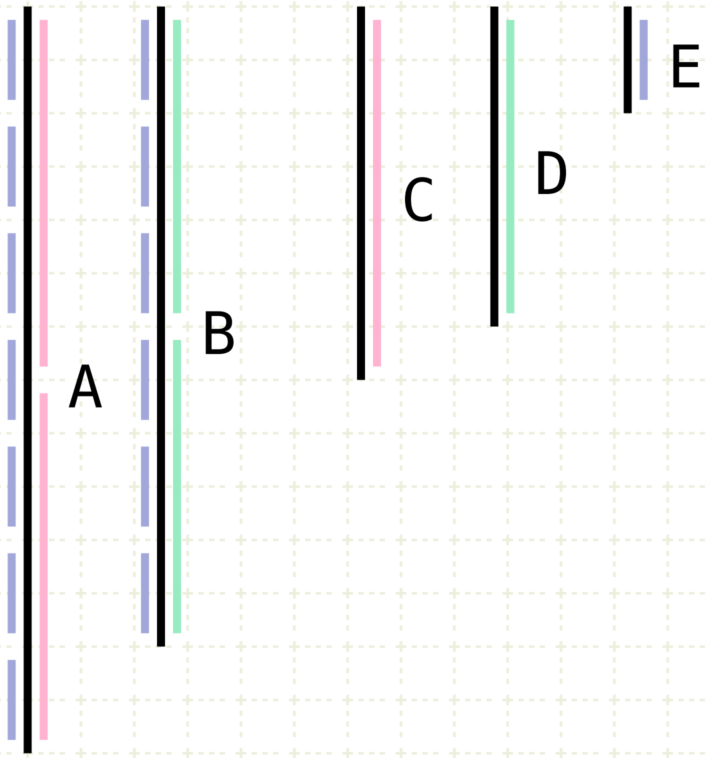
Therefore A,B have a common divisor E, which is greater than one.

This is not possible since A and B are prime to one another (VII Def.12)

Thus A and B are the lowest numbers that have the ratio A to B

Proposition 21 of Book VII

Numbers prime to one another are the least of those which have the same ratio with them.



$\gcd(A,B) = 1$

$A:B = C:D$
$C < A$
$D < B$
$A = C_1 + C_2 + \ldots + C_i + \ldots C_n$
$B = D_1 + D_2 + \ldots + D_i + \ldots D_n$
$A = E \cdot C,\ E \neq 1$
$B = E \cdot D,\ E \neq 1$
$A = C \cdot E$
$B = D \cdot E$
$\gcd(A,B) = E,\ E \neq 1$

$S = \{ (x,y) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ x:y = A:B \}$
$(A,B) \in S$ such that $A \le x,\ B \le y,\ \forall (x,y) \in S$

Proof by Contradiction

Let C and D be the lowest two numbers that have the same ratio as A to B, and let C,D be smaller than A,B

Therefore C measures A the same number of times that D measures B (VII·20)

Let E be equal to the number of times C measures A

Therefore D also measures B according to the units in E

Since C measures A according to the units in E, therefore E also measures A according to the units in C (VII·16)

Since B measures D according to the units in E, therefore E also measures B according to the units in D (VII·16)

Therefore A,B have a common divisor E, which is greater than one.

This is not possible since A and B are prime to one another (VII Def.12)

Thus A and B are the lowest numbers that have the ratio A to B

Youtube Videos

https://www.youtube.com/c/SandyBultena